Analytic Geometry

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Contents

PREFACE

This book is meant as a basic text book for a course in Analytic Geometry.

Throughout the book, the connections and interrelations between algebra and geometry are emphasized. The notions of Linear Algebra are introduced and applied simultaneously with the more traditional topics of Analytic Geometry. Some of the notions of Linear Algebra are used without mentioning them explicitely.

The material is separated into eight chapters and there are two appendices. In Chapter 1, we summarize certain elementary mathematical notions such as sets, relations, functions; and we state the Fundamental Principle of Analytic Geometry. We also emphasize the notion of an equivalence relation in this chapter. Chapter 2 introduces Cartesian Coordinates in the plane and in three space. The chapter includes discussions on polar coordinates, trigonometry, lines in the plane and change of coordinates in the plane. Chapter 3 studies vectors in the plane. Lines, line-segments, half-lines, half-planes and polygons are studied through vectors. In Chapter 4, we introduce the conic sections and discuss the general quadratic equation in two variables.

Chapter 5 studies vectors in three space. Lines and planes in three space are studied through vectors. At the end of the chapter, vectors in n-space ($n \geq 2$) and vectors with complex components are introduced.

In Chapter 6, we introduce matrices and we study solutions of systems of linear equations by matrices.

In Chapter 7, we introduce determinants and give the fundamental properties of determinants. We also study Cramer's Rule, the inverse of a matrix, and we introduce characteristic values and characteristic vectors of a 3×3 real symmetric matrix as a preparation for Chapter 8.

In Chapter 8, we introduce canonical equations of quadratic surfaces and give a discussion on the general quadratic equation of three variables.

Appendix A is a summary of real and complex numbers including the statements of the Principle of Mathematical Induction and the Fundamental Theorem of Algebra.

Appendix B gives the proof of the theorem about the expansion of determinants.

In writing this text I have drawn from my classroom experience and have been influenced by many sources which I cannot enumerate. I express my gratitude to T. Terzioğlu and M.Z. Orhon for their support and encouragement. I wish to express my gratitude to many colleagues who provided critical remarks and encouragement during the preparation of this text. In particular, I benefited much from the suggestions and comments of Ş. Alpay, A. Pamir and T. Önder. I am also grateful to Y. Sözen and Y.Z. Gürtaş for their assistance. Finally, I wish to thank Z. Öner, late Mrs. and G. İncesu and G. Gümüş for their competent typing of the manuscript.

Chapter 1

FUNDAMENTAL PRINCIPLE OF ANALYTIC GEOMETRY

Analytic Geometry is a branch of mathematics which studies geometry through the use of algebra. It was *Rene Descartes* (1596-1650) who introduced the subject for the first time. Analytic geometry is based on the observation that there is a one-to-one correspondence between the points of a straight line and the real numbers (See §5). This fact is used to introduce coordinate systems in the plane or in three space, so that a geometric object can be viewed as a set of pairs of real numbers or as a set of triples of real numbers.

In this chapter, we list notations, review set theoretic notions and give the fundamental principle of analytic geometry.

1.1 Set Theory

The word "set" means a "collection" or an "aggregate", but these are synonyms rather than definitions. We can illustrate the concept of a set by the examples: the set of men who are taller than two meters; the set of letters in the Turkish alphabet; the set of positive integers less than 6.

The objects that make up a set are called *elements* of the set. Given a set, an

object is either an element of the set or it is not an element of the set (but not both).

We denote sets, mostly, by capital Latin letters: $A, B, C, E, \ldots$, and elements by small letters: $a, b, c, x, y, z, \ldots$. If E is a set and x is an element of E, we write $x \in E$. If x is not an element of E, we write $x \notin E$. A set may have no elements –for example, the set of men who are taller than 10 meters. Such a set with no elements is called the *empty set*, and is denoted by ϕ.

In mathematics, one is particularly interested in sets of numbers. We denote the set of real numbers by $\mathbb{R}$, the set of rational numbers by $\mathbb{Q}$, the set of integers by $\mathbb{Z}$, and the set of natural numbers by $\mathbb{N}$. The reader may refer to Appendix A for a discussion about real numbers.

If a set contains a finite number of elements, then one can denote this set by listing all its elements in braces. For example, the set of positive integers less than 6 can be denoted as $\{1, 2, 3, 4, 5\}$. The set of all elements of a set E with property P is denoted by $\{x : x \in E, P(x)\}$. Thus the set of positive integers less than 6 can also be denoted by $\{x : x \in \mathbb{Z}, \quad 0 < x < 6\}$.

Given two sets E and B, if every element of E is also an element of B then we say that E *is a subset of* B, or E *is contained in* B. When E is a subset of B, we write $E \subseteq B$ or $B \supseteq E$. If E and B have the same elements, that is, if every element of E is an element of B and every element of B is an element of E, then we say that E *and* B *are equal* and we write $E = B$. Thus

$$E = B \Leftrightarrow E \subseteq B \quad \text{and} \quad B \subseteq E.$$

If $E \subseteq B$ and $E \neq B$ (E is not equal to B), then we say that E is a *proper subset* of B, and we write $E \subset B$. Note that the empty set is a proper subset of any non-empty set.

We assume each discussion in which a number of sets are involved takes place in the context of a single fixed set. This set is called the *universal set*, and is denoted by U. Every set is assumed to be a subset of U. The choice of the universal set is arbitrary. For instance, in discussions involving numbers we may choose the universal set to be the set of real numbers or the set of complex numbers or something else. If we choose the universal set to be the set of real numbers, then the equation $x^2 + 1 = 0$ has no solution in the universal set. What if you choose the universal set to be the set of complex numbers?

In the rest of this section we consider several ways of combining sets with one another, and we develop the basic properties of these combinations.

The first operation we discuss is that of forming unions. The *union* of two sets

E and B, written $E \cup B$, is defined to be the set of all elements that are in E or in B (including those which are in both). Thus

$$E \cup B = \{x : x \in E \quad \text{or} \quad x \in B\}.$$

It is customary to illustrate set operations by diagrams. We represent the universal set by a rectangle, a subset E by a circle inside the rectangle. The points inside the circle represent elements of the set E.

The *intersection*, $E \cap B$, of two sets E and B is the set of all elements which are in both E and B. In symbols,

$$E \cap B = \{x : x \in E \quad \text{and} \quad x \in B\}.$$

The *complement*, $E \setminus B$, of a set B with respect to a set E is the set of all elements in E which are not elements of the set B, i.e.,

$$E \setminus B = \{x : x \in E \quad \text{and} \quad x \notin B\}.$$

$E \setminus B$ is also called the difference of the sets E and B, and read as "E minus B". The difference $U \setminus B$ is called the complement of B and denoted by B'.

Example 1.1.1 Let $E = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$. Then

$$\begin{aligned} E \cup B &= \{1, 2, 3, 4, 5, 6, 8\} \\ E \cap B &= \{2, 4\} \\ E \setminus B &= \{1, 3, 5\} \end{aligned}$$

Taking $U = \mathbb{Z}$,

$$E' = \{x : x \in \mathbb{Z}, \quad x < 1 \quad \text{or} \quad x > 5\}.$$

The three operations - union, intersection, complement - satisfy the following properties:

Theorem 1.1.2 *Let E, B and C be sets. Then*

a) $E \cup E = E$; $E \cap E = E$

b) $E \cup U = U$; $E \cap U = E$

c) $E \cup \phi = E$; $E \cap \phi = \phi$

d) $E \cup B = B \cup E$; $E \cap B = B \cap E$

e) $E \cup (B \cup C) = (E \cup B) \cup C$; $E \cap (B \cap C) = (E \cap B) \cap C$

f) $E \subseteq B \Leftrightarrow E \cup B = B \Leftrightarrow E \cap B = E$

g) $E \cap (B \cup C) = (E \cap B) \cup (E \cap C)$

h) $E \cup (B \setminus E) = E \cup B$; $E \cap (B \setminus E) = \phi$

i) $E' \cup E = U$; $E' \cap E = \phi$

j) $(E \cup B)' = E' \cap B'$; $(E \cap B)' = E' \cup B'$

k) $(E')' = E$.

Proof. Each of these properties can be observed by drawing Venn-diagrams. We prove (g) and leave the others to exercises.

$$\begin{aligned} x \in E \cap (B \cup C) &\Leftrightarrow x \in E \text{ and } x \in B \cup C \\ &\Leftrightarrow x \in E \text{ and } (x \in B \text{ or } x \in C) \\ &\Leftrightarrow (x \in E \text{ and } x \in B) \text{ or } (x \in E \text{ and } x \in C) \\ &\Leftrightarrow x \in E \cap B \text{ or } x \in E \cap C \\ &\Leftrightarrow x \in (E \cap B) \cup (E \cap C). \end{aligned}$$

This completes the proof of (g). □

Another operation on sets is the formation of cartesian products. The *cartesian product*, $E \times B$, of two sets E and B is the set of all ordered pairs (x, y) where $x \in E$ and $y \in B$:

$$E \times B = \{(x, y) : x \in E \quad \text{and} \quad y \in B\}.$$

Note that, in general, $E \times B \neq B \times E$.

Example 1.1.3 Let $E = \{1, 2\}$, $B = \{a, b, c\}$. Then

$$\begin{aligned} E \times B &= \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}, \\ B \times E &= \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}. \end{aligned}$$

The cartesian product of three sets E, B and C can be defined as

$$E \times B \times C = \{(x, y, z) : x \in E \quad \text{and} \quad y \in B \quad \text{and} \quad z \in C\}.$$

We can consider the cartesian product of a set E by itself:

$$E \times E = \{(x_1, x_2) : x_1 \in E \quad \text{and} \quad x_2 \in E\}.$$

We write $E \times E = E^2$, $E \times E \times E = E^3$.

Exercises

1. Given the sets $E = \{1,2,3\}$, $B = \{3,4\}$, $C = \{2,3,4,5,6\}$,
   a) Find $E \cup B$ and $E \cap B$,
   b) Find $E \cup C$ and $E \cap C$,
   c) Find $B \cup C$ and $B \cap C$,
   d) Find $E \setminus B$ and $B \setminus C$,
   e) Find $C \setminus (E \cup B)$ and $(E \cup B) \setminus C$,
   f) Find $(E \cup B) \setminus (E \cap B)$.
2. Show that for any two sets E and B, we have
   a) $E \subseteq E \cup B$,
   b) $E \cap B \subseteq B$,
   c) $E \cap B \subseteq E \cup B$.
3. E, B and C being sets, draw Venn-diagrams for each of the following situations:
   a) $E \subset B, \quad C \subset B, \quad E \cap C = \phi$,
   b) $E \subset B, \quad C \not\subset B, \quad E \cap C = \phi$,
   c) $E \subset B, \quad C \subset B, \quad E \cap C \neq \phi$,
   d) $E \subset B, \quad C \not\subset B, \quad E \cap C \neq \phi$,
   e) $E \subset (B \cap C), \quad B \subset C$.
4. Find all elements of the set
$$\{x : x \in \mathbb{R},\ x^2 - 1 = 0\} \cap \{x : x \in \mathbb{R},\ x^2 - 2x + 1 = 0\}.$$
5. a) Prove that $E \cup \phi = E$.
   b) If $E \cup B = \phi$ does it follow that $E = B = \phi$?
   c) If $E \cup B = E$ does it follow that $B = \phi$?
   d) Prove that $E \cap \phi = \phi$.
   e) If $E \cap B = \phi$, does it follow that $E = \phi$ or $B = \phi$?
6. Prove the statements in Theorem 1.1.1.
7. Find $E \times B$, $E \times C$, $C \times B$, B^2 and $B \times C$ for the sets E, B and C in Ex. 1.

1.2 Relations

The word relation implies, an association of some elements of one set to some elements of another set according to a property possessed by them. For example, "... is the father of ..." is a relation which associates an element of the set of all men some elements of the set of all human beings in the world. Thus a relation, in everyday life, consists of two sets E and B and a statement that is either true or false for any ordered pair (x, y) in $E \times B$. This leads to the following mathematical definition of relations:

Any subset R of $E \times B$ is called a *relation* from the set E *to* the set B. For an ordered pair $(x, y) \in E \times B$, if $(x, y) \in R$ then we say that x *is related to* y by the relation R, and we write xRy. If $(x, y) \notin R$, we say that x *is not related to* y, and we write $x \not R\, y$.

Example 1.2.1 Let $E = \{1, 2\}$, $B = \{a, b, c\}$. Then

$$R = \{(1, a), (1, c), (2, b)\}$$

is a relation from E to B. We have $1Ra, 1b, 1Rc, 2a, 2Rb, 2c$. Write down other relations from E to B. How many distinct relations are there from E to B?

Example 1.2.2 Let E be an arbitrary set. The usual *equality*, "=", is a relation from E to E. In terms of the above definition

$$\text{“} = \text{”} = \{(x, x) : x \in E\}.$$

As the previous example shows, we may have relations from a set E to the same set (E itself). If R is a relation from E to E, we say that R is a *relation in* E. Hence equality, $=$, is a relation in E for any set E.

Example 1.2.3 Let U be a universal set and let us denote the collection of all subsets of U by 2^U. Then "being subset of" is a relation in 2^U:

$$\text{“} \subseteq \text{”} = \{(A, B) : A \in 2^U,\ B \in 2^U \quad \text{and} \quad A \subseteq B\}.$$

From here on we are interested in relations in a set E.

A relation R in a set E is called *reflexive* if xRx for every $x \in E$. In other words, R is reflexive if $(x, x) \in R$ for any $x \in E$. The reader can easily verify that both of the relations in Examples 1.2.2 and 1.2.3 are reflexive.

Example 1.2.4 Let E be the set of all lines in the plane and let $//$ be the relation

$$// = \{(\ell, k) : \ell \in E \text{ and } k \in E \text{ and } \ell \text{ is parallel to or coincides with } k\}.$$

This relation is obviously reflexive, i.e., $\ell // \ell$ for every $\ell \in E$. On the other hand, the relation

$$\perp = \{(\ell, k) : \ell \in E \text{ and } k \in E \text{ and } \ell \text{ is perpendicular to } k\}$$

is not reflexive since $\ell \not\perp \ell$ for any $\ell \in E$.

A relation R in a set E is called *symmetric* if whenever xRy then yRx for $x, y \in E$. In other words, R is symmetric if for any $x, y \in E$

$$(x, y) \in R \Rightarrow (y, x) \in R.$$

The relations $=, //$, and $\perp$ are symmetric, but $\subseteq$ is not symmetric.

A relation R in E is called *anti-symmetric* if xRy and yRx imply $x = y$. Thus, if x and y are elements of E such that $x \neq y$, then possibly xRy or yRx but never both. The relations $=$ and $\subseteq$ above are anti-symmetric. The relations $//$ and $\perp$ are not anti-symmetric.

A relation R in E is called *transitive* if xRy and yRz imply xRz. Thus a subset R of $E \times E$ is a transitive relation if for $x, y, z \in E$.

$$(x, y) \in R \text{ and } (y, z) \in R \Rightarrow (x, z) \in R.$$

The relations $=$, and $//$ above are transitive. The relation $\perp$ is not transitive.

A relation R in a set E is called an *equivalence relation* if it is reflexive, symmetric and transitive. The usual equality, $=$, is an equivalence relation in any set E. The relation $\subseteq$ in 2^U, and the relation $\perp$ in the set of all lines in the plane are not equivalence relations, because $\subseteq$ is not symmetric and $\perp$ is not reflexive. Since the relation $//$ is reflexive, symmetric and transitive in the set of all lines in the plane, it is an equivalence relation.

Let R be an equivalence relation in a set E, and let $x \in E$. The subset $[x]_R$ of E consisting of all elements which are related to x by the relation R, i.e.,

$$[x]_R = \{y : y \in E \text{ and } yRx\}$$

is called an *equivalence class* of E with respect to the relation R. The element x is said to be a *representative* of the equivalence class $[x]_R$. $[x]_R$ is also called the *equivalence class containing* x or the *equivalence class represented by* x.

Example 1.2.5 Let E be the set of all points in the plane and let M be a fixed point in E. For any point $P \in E$, let $|PM|$ denote the distance from P to M. Let

$$\rho = \{(P, Q) : P \in E, Q \in E \text{ and } |PM| = |QM|\}.$$

It is easy to verify that ρ is an equivalence relation in E. $P\rho Q$ if P and Q are equidistant from the fixed point M. For any point N, other than M, the equivalence class $[N]_\rho$ containing N consists of all points lying on the circle that is centered at M and passing through N. The equivalence class $[M]_\rho$ has only one point, namely M.

We notice, in the above example, that equivalence classes are concentric circles and therefore distinct equivalence classes have no element in common, i.e., they are *disjoint*. Two points P and Q are related if and only if they lie on the same circle, i.e., $[P]_\rho = [Q]_\rho$. The following two lemmata show that these properties are shared by any equivalence relation.

Lemma 1.2.6 *Let R be an equivalence relation in a set E. Then for any $x, y \in E$*

$$xRy \Leftrightarrow [x]_R = [y]_R.$$

Proof. Assume that xRy. Take $z \in [x]_R$. Then zRx. By transitivity, zRy. That is $z \in [y]_R$. This shows that $[x]_R \subseteq [y]_R$. Similarly, one can show under the assumption xRy, that $[y]_R \subseteq [x]_R$. Hence $[x]_R = [y]_R$.

Conversely, assume that $[x]_R = [y]_R$. Then by reflexivity, xRx whence $[x]_R = [y]_R$. By definition of $[y]_R, xRy$. □

Lemma 1.2.6 shows that any element of an equivalence class $[x]_R$ is a representative for this class.

Lemma 1.2.7 *Let R be an equivalence relation in a set E. Then two equivalence classes of E are either identical or have no element in common.*

Proof. Let $[x]_R$ and $[y]_R$ be two equivalence classes of E and suppose that $[x]_R$ and $[y]_R$ have an element z in common. Then zRx and zRy. By Lemma 1.2.6, we have $[x]_R = [z]_R = [y]_R$. □

Exercises

1. Given the set $E = \{1, 2, 3\}$. Find relations R_1, R_2, R_3, R_4, R_5 and R in E such that

a) R_1 is reflexive,

b) R_2 is reflexive and symmetric,

c) R_3 is transitive,

d) R_4 is transitive but not reflexive,

e) R_5 is reflexive and transitive but not symmetric,

f) R is an equivalence relation. (Determine the equivalence classes).

2. State which of the following is an equivalence relation. If it is not, tell which property is not satisfied. If it is, describe the equivalence classes. In each case, E is the set of elements to which the given relation applies.

   a) $E = \{t : t \text{ is a triangle in the plane }\}$, $t_1 R t_2$ if triangle t_1 is congruent to triangle t_2.

   b) $E = \{\ell : \ell \text{ is a line in the plane }\}, \ell_1 R \ell_2$ if ℓ_1 and ℓ_2 are coincident or intersecting.

   c) $E = \mathbb{Z}$, the set of integers, mRn if $m - n$ is even.

   d) $E = \mathbb{Z}, mRn$ if $m - n$ is odd.

   e) $E = \mathbb{Z}, mRn$ if $m - n$ is divisible by 3.

   f) $E = \mathbb{R}$, the set of real numbers, xRy if $|x| = |y|$.

   g) $E = \mathbb{R}, xRy$ if $|x - y| = 1$.

1.3 Functions

In this section we shall study relations that have a certain uniqueness property.

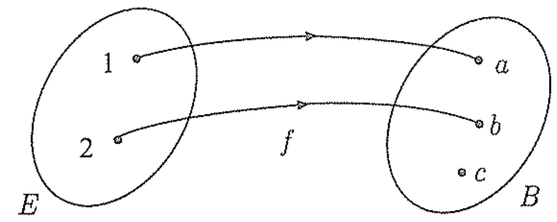
Let E and B be two sets. A relation f from E to B is called a *function* if for any element x in E there is exactly one (one and only one) pair $(x, y) \in f$. Thus a function f from E to B can be viewed as a rule which assigns a unique element of B for each element of E.

The relation R of Example 3 is not a function, because $(1, a) \in R$ and $(1, c) \in R$ and $a \neq c$. If we define f to be the relation $f = \{(1, a), (2, b)\}$ then f is a function from $E = \{1, 2\}$ to $B = \{a, b, c\}$.

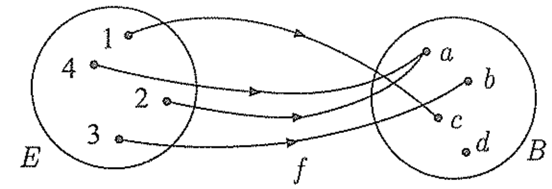
If f is a function from E to B, then E is called the *domain* of f, and B is called the codomain of f the *range of f*. If $(x, y) \in f$, we also write $y = f(x)$ and call y the *image* of x under f, or the value of f at x.

The expression "f is a function from E to B" will often be abbreviated as "$f : E \to B$" or as "$f : E \to B, x \to f(x)$".

It may be helpful to illustrate functions by diagrams. For instance, the above function $f : \{1, 2\} \to \{a, b, c\}$ can be illustrated as



Example 1.3.1 The diagram



illustrates the function $f = \{(1, c), (2, a), (3, b), (4, a)\}$ from $E = \{1, 2, 3, 4\}$ to $B = \{a, b, c, d\}$.

Let $f : E \to B$ be a function and $S \subseteq E$. Then

$$f|_S = \{(x, y) | x \in S \quad \text{and} \quad (x, y) \in f\}$$

is a function from S to B, and it is called the restriction of f to S.

Example 1.3.2 Let E, B and f be as in Example 1.3.1, and let $S = \{1, 3\} \subseteq E$. Then

$$f|_S = \{(1, c), (3, b)\}.$$

Let $f : E \to B$ and $g : B \to C$ be two functions. Then one can combine f and g to obtain a function from E to C. Consider

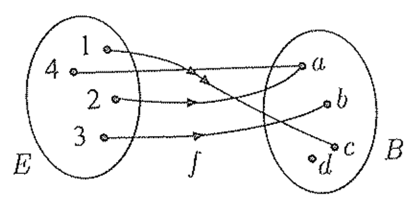
$$g \circ f = \{(x, y) : x \in E, (f(x), y) \in g\}.$$

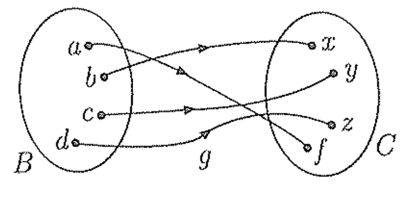
$g \circ f$ is a relation from E to C, and for any $x \in E$ there is one and only one $(x, y) \in g \circ f$. For; given $x \in E$, there is one and only one $f(x) \in B$, and for $f(x)$ there is one and only one $y \in C$. Hence $g \circ f$ is a function from E to C. $g \circ f$ is called the *composition* of f *with* g. Note that,

$$g \circ f(x) = g(f(x))$$

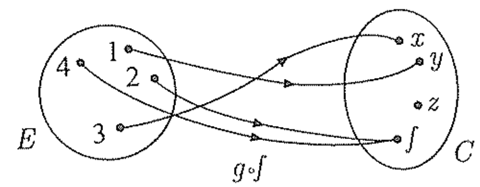
for any $x \in E$.

Example 1.3.3 Let f and g be the functions defined by





Then $g \circ f$ can be represented by the diagram



A function $f : E \to B$ is said to be *one-to-one* (or injective) if no two distinct elements of E have the same image under f. Thus $f : E \to B$ is one-to-one if

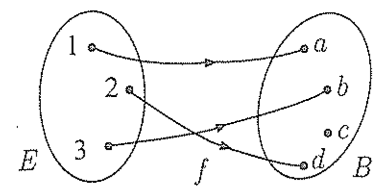
$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

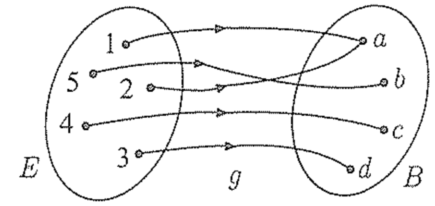
or equivalently

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

for $x_1, x_2 \in E$.

Example 1.3.4 Consider the functions f and g defined by the diagrams below. The function f is one-to-one because the three elements of the set E have three distinct images under f. The function g is not one-to-one because $1 \neq 2$ but $g(1) = g(2) = a$.





For a function $f : E \to B$, we define $f(E) = \{f(x) : x \in E\}$, and call this set the *image* of f. We say that f is *onto* (or surjective) if $f(E) = B$. Thus f is onto if every element of B is the image of at least one element of E under f.

Of the two functions f and g of Example 11, g is onto but f is not onto.

If a function $f : E \to B$ is one-to-one and onto, we say that f is a *one-to-one correspondence* between E and B.

Example 1.3.5 For any set E, $1_E = \{(x, x) : x \in E\}$ is a function. $1_E : E \to E, x \to x$ is called the identity function on E. Clearly, 1_E is a one-to-one correspondence.

Let $f : E \to B$ be an arbitrary function and consider

$$f^{-1} = \{(y, x) : (x, y) \in f\}.$$

Clearly, f^{-1} is a relation from B to E. If f is onto, then for every $y \in B$ there exists $(y, x) \in f^{-1}$. If f is also one-to-one, then for every $y \in B$ there is one and only one $(y, x) \in f^{-1}$. Thus, *if f is a one-to-one correspondence between E and B then f^{-1} is a function from B to E.* Then f^{-1} is called the *inverse* of f. One can easily prove that, if $f : E \to B$ is a one-to-one correspondence then

$$f^{-1} \circ f = 1_E \quad \text{and} \quad f \circ f^{-1} = 1_B.$$

Remark. We notice, once more, that a function $f : E \to B$ has *inverse* $f^{-1} : B \to E$ if f is one to one and onto. However, if f is one-to-one (but perhaps not onto), we may consider f as a one-to-one and onto function from E to $f(E)$ and consider the inverse $f^{-1} : f(E) \to E$.

Exercises

1. Which of the following sets define a function and which describe only a relation?

a) $\{(a,b),(b,c),(c,d),(d,e)\}$

b) $\{(a,b),(a,c),(a,d),(a,e)\}$

c) $\{(x,y); x \in \mathbb{R}, y \in \mathbb{R}, y = x-3\}$

d) $\{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, |x|+|y| = 1\}$

e) $\{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, x+|y| = 1\}$

f) $\{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, |x|+y = 1\}$

g) $\{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, x^2+y^2+1 = 0\}$

h) $\{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, x^2+y+1 = 0\}$

i) $\{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, x+y^2+1 = 0\}$

j) $\{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, x+y+1 = 0\}$

k) $\{(x,y) : x \in \mathbb{R} \setminus \{1\}, y \in R, y = \frac{x+1}{x-1}\}$.

2. Determine whether or not each of the functions (if any) in Exercise 1 is one-to-one or onto.

3. The following define functions from certain subsets of $\mathbb{R}$, the set of real numbers, to $\mathbb{R}$. Determine the largest possible domain for each function

a) $x \to x$ b) $x \to |x|$, c) $x \to x^2$ d) $x \to \sqrt{9-x^2}$

e) $x \to \frac{x}{|x|+x}$ f) $x \to \frac{1}{1-x}$ g) $x \to \frac{x}{x^2+1}$ h) $x \to \frac{x}{x^2-1}$ i) $x \to \frac{1}{x}$

4. Let $f : E \to B$ be a function and consider subsets $S, S_1, S_2 \subseteq E$ and $T, T_1, T_2 \subseteq B$. We define $f(S) = \{f(x) : x \in S\}, f^{-1}(T) = \{x \in E : f(x) \in T\}$

a) Show that $f(S) \subseteq B$ and $f^{-1}(T) \subseteq E$.

b) Show that if $S_1 \subseteq S_2$ then $f(S_1) \subseteq f(S_2)$.

c) Show that if $T_1 \subseteq T_2$ then $f^{-1}(T_1) \subseteq f^{-1}(T_2)$.

d) Show that $f(f^{-1}(T)) \subseteq T$ and $f^{-1}(f(S)) \supseteq S$.

e) Show that f is onto $\Leftrightarrow f(f^{-1}(B)) = B$.

f) Show that f is one-to-one $\Leftrightarrow$ for any subset $S \subseteq E$ we have $f^{-1}(f(S)) = S$.

5. Let $f : E \to B$ be a function. Prove that the relation R defined by $x_1 R x_2$ if $f(x_1) = f(x_2)$ is an equivalence relation in E. Describe the equivalence classes.

1.4 Families of Sets

Let I and F be two sets. Assume that there is a function $f : I \to F$ which is onto. Put $f(i) = x_i$ for any $i \in I$. Then F can be represented as

$$F = \{x_i : i \in I\}.$$

The set I is called an *index set*. If the elements of F are sets, then F is called a *family of sets*.

If $F = \{E_i : i \in I\}$ is a family of sets, then the *union*, $\cup_{E \in F} E$, and the *intersection*, $\cap_{E \in F} E$, are defined by

$$\begin{aligned} \bigcup_{E \in F} E &= \bigcup_{i \in I} E_i = \{x : x \in E_i \text{ for } \textit{at least one } i \in I\} \\ \bigcap_{E \in F} E &= \bigcap_{i \in I} E_i = \{x : x \in E_i \text{ for } \textit{each } i \in I\} \end{aligned}$$

Example 1.4.1 Let $E_n = \{1, 2, \ldots, n\}$ for any natural number $n \in N$. Then

$$\bigcup_{n \in \mathbb{N}} E_n = \bigcup_{n=1}^{\infty} E_n = \mathbb{N} \text{ and } \bigcap_{n \in \mathbb{N}} E_n = \bigcap_{n=1}^{\infty} E_n = \{1\}.$$

Example 1.4.2 Let R be an equivalence relation in a set E. Then

$$\bigcup_{x \in E} [x]_R = E.$$

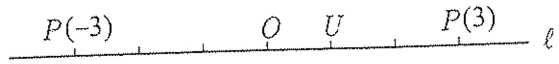
1.5 Fundamental Principle of Analytic Geometry

As we have already pointed out, analytic geometry is based on the idea that a one-to-one correspondence can be established between the set of points of a straight line and the set of all real numbers. This means that to every point on the straight line there corresponds *exactly one* (one and only one) real number, and conversely, to every real number there corresponds *exactly one* point on the line. Furthermore, this correspondence is such that the orderings of the numbers and of the points of the straight line correspond. We give the precise formulation of the Fundamental Principle in the next paragraph.

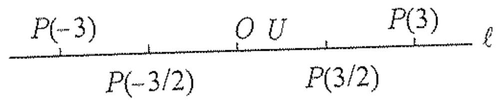
Let ℓ be a straight line. We denote the set of all points on ℓ again by the letter ℓ. There exists a function $P : \mathbb{R} \to \ell$, $x \to P(x)$, which satisfies the following properties:

a) If $P(0) = O$ and $P(1) = U$, then $O \neq U$.

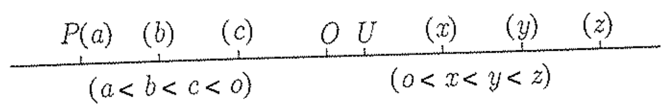
b) For any positive integer n, the point $P(n)$ lies on the same side of O as U and is n times as far from O as U. The point $P(-n)$ lies on the opposite side of O with respect to U and is n times as far from O as U.



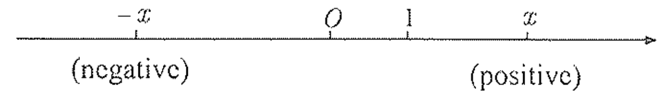
c) If p/q is a positive rational number where p and q are positive integers, then $P(p/q)$ lies on the same side of O as U and such that $P(p)$ is q times as far from O as $P(p/q)$, i.e., the line segment from O to $P(p)$ is divided into q equal parts and $P(p/q)$ is the one end point of the part whose other end point is O. The point $P(-p/q)$ is located on the opposite side of O with respect to U such that $P(-p)$ is q times as far from O as $P(-p/q)$.



d) If x and y are real numbers such that $x < y$, i.e., x is less than y, then the direction from $P(x)$ to $P(y)$ is the same as the direction from O to U. Thus if $x < y < z$ then $P(y)$ lies between $P(x)$ and $P(z)$.



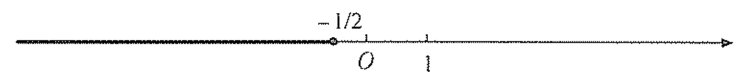
A straight line ℓ together with a function P as above is called a *number axis*. The points $P(0) = O$ and $P(1) = U$ are called the *origin* and the *unit point*, respectively. The distance from O to U is called the *unit length* on the number axis. The direction from O to U is called the *direction* of the number axis. The number x corresponding to the point $P(x)$ on the number axis is called the coordinate of that point. Coordinates of O and U are 0 and 1, respectively. From now on, we shall also write 0 for O, 1 for U and x for $P(x)$ on the number axis and we shall refer to 0, 1 or x as points as well as numbers.



The above established correspondence between real numbers and points of a line makes it possible to represent any set of numbers by a graph on the line, and conversely, any graph on the line as a set of real numbers. For instance, the origin O divides the number axis into two disjoint parts. One of these consists of the points with positive coordinates and it is called the positive axis. The other part consists of the points with negative coordinates and it is called the negative axis.

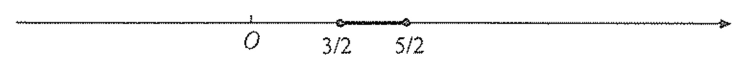
Example 1.5.1 Represent the set $A = \{x : x \in \mathbb{R}, 2x + 1 < 0\}$ as a subset of the number axis.

Solution. Let $x \in \mathbb{R}$. Then $x \in A \Leftrightarrow 2x + 1 < 0 \Leftrightarrow 2x < -1 \Leftrightarrow x < -\frac{1}{2}$. These real numbers can be represented on the number axis as follows:



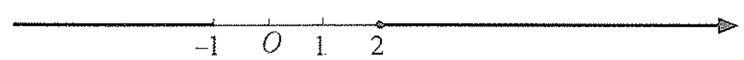
Example 1.5.2 Describe the set $B = \{x : x \in \mathbb{R}, |x - 2| < \frac{1}{2}\}$ on the number axis.

Solution. $x \in B \Leftrightarrow |x - 2| < \dfrac{1}{2} \Leftrightarrow -\dfrac{1}{2} < x - 2 < \dfrac{1}{2} \Leftrightarrow (\dfrac{3}{2} < x < \dfrac{5}{2})$.

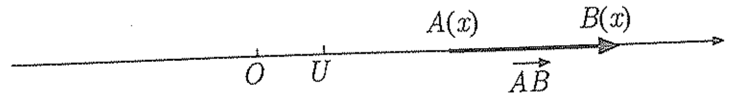


Example 1.5.3 Describe the set $C = \{x : x \in \mathbb{R}, 2x + 1 \geq 5 \text{ or } 3x + 1 < -2\}$ on the number axis.

Solution. $x \in C \Leftrightarrow 2x + 1 \geq 5$ or $3x + 1 < -2 \Leftrightarrow x \geq 2$ or $x < -1$.



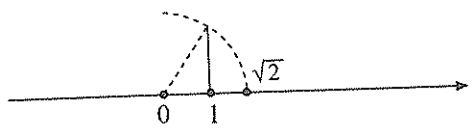
An ordered pair (A, B) of points on a number axis is called a *directed segment* from A to B. The point A is called the initial point and B is called the terminal point of the directed segment from A to B. The directed segment from A to B can be visualized as an arrow directed from A towards B and it is denoted by $\overrightarrow{AB}$.



If A has coordinate x and B has coordinate y, then $AB = y - x$ is called the value of the directed segment $\overrightarrow{AB}$, or it is called the *directed distance* from A to B. Thus, the value of $\overrightarrow{OU}$ is 1. It is clear that $AB = -BA$. For any two points $A(x)$ and $B(y)$ on the number axis, the length of the directed segment $\overrightarrow{AB}$ is the absolute value of the directed distance from A to B, therefore we denote it by $|AB|$. We have $|AB| = |BA| = |y - x|$.

Exercises

1. Locate the following numbers on the number axis.
   a) 5 b) -4 c) 3/4 d) -9/5 e) -4.3 (decimal)
   f) $\sqrt{2}$ g) $3\sqrt{2}$ h) $\sqrt{3}$ i) $2\sqrt{3}$ j) $\sqrt{8}$
   (Hint: To locate $\sqrt{2}$, you can make use of the following figure)



2. Describe the following sets of real numbers on the number axis:
   a) $\{x : 1 \le x \le 4\}$
   b) $\{x : |x - \frac{1}{2}| < 3\}$
   c) $\{x : (x-1)x(x+1) < 0\}$
   d) $\{x : |x - 1| \ge \frac{3}{2}\}$
   e) $\{x : x + |x - 1| < 1\}$
   f) $\{x : x^2 < x\}$.
3. Determine whether or not each of the sets in the preceeding exercise is bounded above and find the supremum of each set (see Appendix A).

4. If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are two directed segments on a number axis, define $\overrightarrow{AB} \cong \overrightarrow{CD}$ if $AB = CD$.
   a) Show that $\cong$ is an equivalence relation in the set of all directed segments on the number axis.
   b) Determine the equivalence classes.
   c) Show that for any directed segment $\overrightarrow{AB}$, there exists a unique point P such that $\overrightarrow{AB} \cong \overrightarrow{OP}$.
   d) Find P such that $\overrightarrow{AB} \cong \overrightarrow{OP}$ when $A(-4), B(-3)$.
5. Prove that for any three points A, B, C on the number axis, $AB + BC = AC$.
6. Is $|AB| + |BC| = |AC|$ true for any three points on the number axis?

Chapter 2

CARTESIAN COORDINATES

Fundamental Principle of Analytic Geometry states that there is a one-to-one correspondence between points on a straight line and real numbers. We will use this to establish a one-to-one correspondence between the set of all points in plane and the set of ordered pairs of real numbers; and a one-to-one correspondence between the set of all points in three-space and the set of all triples of real numbers. Then, we are able to express relationships among points or sets of points in plane or in space algebraically.

2.1 Cartesian Coordinates in the Plane

Given a plane, we choose two number axes in that plane with origin $O = O'$ and unit points U, U', respectively, in such a way that they meet at the point $O = O'$ and they are perpendicular. For convenience, we assume that distance from O to U is the same as the distance from O to U'.

Now, we are ready to describe the one-to-one correspondence between the set of all points in the plane and the set

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$$

of ordered pairs of real numbers.

Given a point P in the plane, the ordered pair assigned to P is determined as follows: We drop a perpendicular from P to the X-axis, we obtain the point P_x, called the projection of P on X-axis. Since any point on X-axis represents a (unique) real number, the projection P_x of P on X-axis represents a real number, say x. Similarly, the projection P_y of P on Y-axis represents a (unique) real number, say y. We assign the ordered pair (x, y) to P. x and y are called, respectively, *X-coordinate* and *Y-coordinate* of the point P.

Given an ordered pair (x, y) of real numbers, the point corresponding to (x, y) is determined by reversing the above process. Through the point P_x corresponding to the real number x on the X-axis we draw a line perpendicular to X-axis, and through the point P_y corresponding to the real number y on Y-axis we draw a line perpendicular to Y-axis. The intersection of these two lines is the point P corresponding to the ordered pair (x, y).

The fact that the correspondence established in this manner is a one-to-one correspondence is obvious from elementary geometry. When a one-to-one correspondence has been established between the points of a plane and the ordered pairs of real numbers (in the above manner), we say that a Cartesian Coordinate System on the plane has been determined. A plane together with a Cartesian Coordinate system on it will be called the *Cartesian Plane*.

Note that the common origin of the coordinate axes (X- and Y-axes) in a Cartesian coordinate system corresponds to the pair (0, 0), and it is called the *origin* of the system.

From now on, if a point P corresponds to the ordered pair (x, y) i.e., P has coordinates (x, y) then we will write $P(x, y)$. Thus $P(x, y)$ and $Q(a, b)$ are the same point if and only if $x = a$ and $y = b$.

In practice, the axes are usually chosen so that the X-axis is horizontal with the unit point right to the origin, while the Y-axis is vertical with the unit point above the origin. For this reason, a line parallel to X-axis is called *horizontal* and a line parallel to Y-axis is called *vertical*.

Once a cartesian coordinate system is given in the plane, to any set of points in the plane corresponds a set of ordered pairs of real numbers, i.e., to any relation R from $\mathbb{R}$ to $\mathbb{R}$, corresponds a set of points in the plane. The set of points in the plane corresponding to a relation R from $\mathbb{R}$ to $\mathbb{R}$ is called the *graph* of the relation R. For example, the graph of the relation

$$R = \{(x, y) : x \in \mathbb{R}, y = 3\}$$

is the horizontal line through the point corresponding to the number 3 on Y-axis. We will have more examples of graphs in Section 3.

The coordinate axes determine four subsets of the plane called *quadrants* and numbered according to the following rule: The first quadrant is the set of points having positive X-coordinates and positive Y-coordinates; the second quadrant is the set of points having negative X-coordinates and positive Y-coordinates; the third quadrant is the set of points having negative X-coordinates and negative Y-coordinates; and, finally, the fourth quadrant is the set of points having positive X-coordinates and negative Y-coordinates (see Fig. 2.1).

Let $P(x,y)$ be a point in the plane. From the foregoing it follows that

if $x>0, y>0$, P lies in the first quadrant;
if $x<0, y>0$, P lies in the second quadrant;
if $x<0, y<0$, P lies in the third quadrant;
if $x>0, y<0$, P lies in the fourth quadrant.

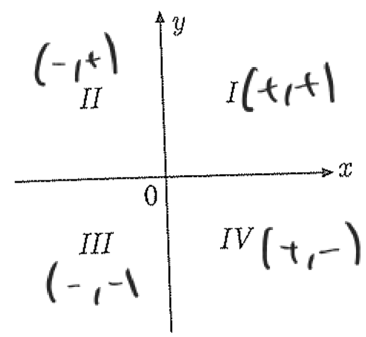


Fig. 2.1

Consideration of the quadrants is useful because it permits an easy orientation as to the position of the given points by the signs of their coordinates.

Given two points, P, Q in the plane, we denote the distance between P and Q by $|PQ|$. We have

Theorem 2.1.1 *Let $P(x,y)$ and $Q(a,b)$ be any two points in the plane. Then*

$$|PQ| = \sqrt{(x-a)^2+(y-b)^2}. \tag{2.1.1}$$

distance

Proof. There are three cases.

Case 1. Assume that P and Q lie on the same horizontal line (see Fig. 2.3). Then $y=b$ and

$$|PQ| = |P_xQ_x| = |x-a| = \sqrt{(x-a)^2}.$$

Hence (2.1.1) is valid in this case.

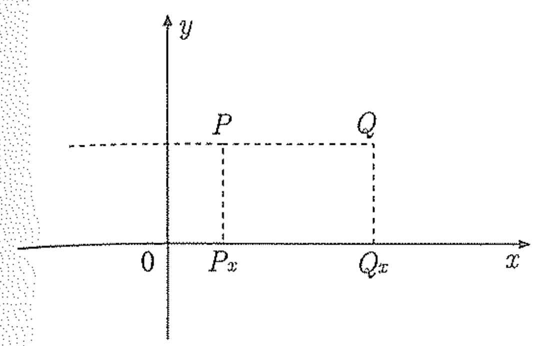


Fig. 2.2.

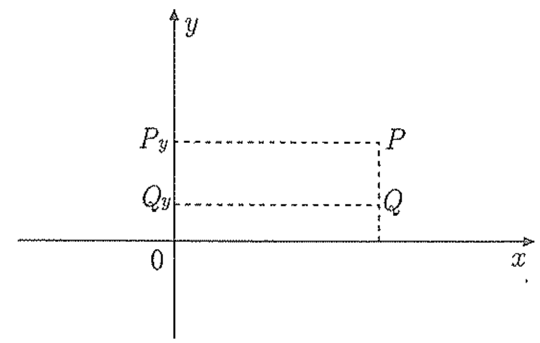


Fig. 2.3.

Case 2. Assume that P and Q lie on the same vertical line (see Fig. 2.4). Then $x=a$ and

$$|PQ| = |P_yQ_y| = |y-b| = \sqrt{(y-b)^2},$$

proving (2.1.1).

Case 3. Assume that P and Q do not lie on the same vertical or horizontal line (see Fig. 2.4). Then by Pythagorian theorem,

$$|PQ|^2 = |PA|^2 + |QA|^2, \quad \text{or} \quad |PQ| = \sqrt{|PA|^2+|QA|^2}.$$

Furthermore,

$$\begin{aligned} |PA|^2 &= |P_xQ_x|^2 = |x-a|^2 = (x-a)^2 \\ |QA|^2 &= |P_yQ_y|^2 = |y-b|^2 = (y-b)^2. \end{aligned}$$

Hence $|PQ| = \sqrt{(x-a)^2+(y-b)^2}$.

$$|PQ| = \sqrt{(x-a)^2+(y-b)^2}$$

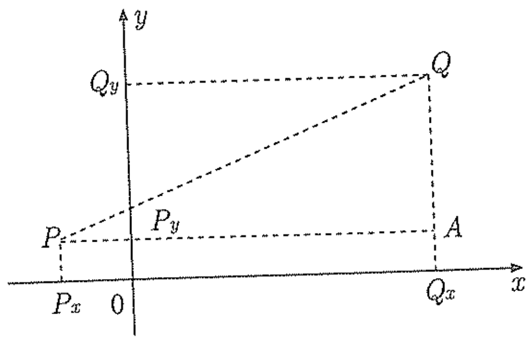


Fig. 2.4.

This completes the proof of the theorem. □

Example 2.1.2 Find $|PQ|$ if $P = (1,5)$ and $Q = (-2,1)$.

Solution. $|PQ| = \sqrt{(1-(-2))^2 + (5-1)^2} = \sqrt{3^2+4^2} = 5$.

Example 2.1.3 Given $P = (5,1)$ and $Q = (x,-2)$. Determine x so that $|PQ| = 5$.

Solution.

$$\begin{aligned} |PQ| &= \sqrt{(5-x)^2+(3)^2} = \sqrt{25-10x+x^2+9} = \sqrt{(x^2-10x+9)+25} \\ &= \sqrt{(x-1)(x-9)+25}. \end{aligned}$$

Hence $|PQ| = 5 \Leftrightarrow (x-1)(x-9) = 0 \Leftrightarrow x = 1$ or $x = 9$.

Exercises

1. Plot each of the following pairs of points in the Cartesian Plane, and in each case, find the distance $|PQ|$.

   a) $P(-1,2), Q(3,-5)$
   b) $P(\frac{1}{2},1), Q(\frac{3}{2},-2)$
   c) $P(\frac{1}{2},\frac{1}{3}), Q(-1,\frac{2}{3})$

2. Show that the points $A(1,2), B(4,-2)$ and $C(5,5)$ are vertices of an isosceles triangle in the plane. (*Hint*: Compute $|AB|, |AC|$, and $|BC|$.)

3. Recall that a *circle* in the plane is the set of all points in the plane at a given distance (the *radius*) from a given point (the center). Using Cartesian coordinates, describe the circle of radius r centered at $C(x_0,y_0)$ as a subset of $\mathbb{R}^2$.

4. For what values of x does the point $A(x,-2)$ lie on the circle of radius 5 centered at $C(-1,1)$?

5. For what values of x does the point $(x,1)$ lie inside the circle of radius 5 centered at $C(1,-2)$?

6. For what values of y does the point $(-3,y)$ lie outside the circle of radius $\sqrt{17}$ centered at $C(-2,1)$?

7. Prove that $|P_1P_2| + |P_2P_3| \geq |P_1P_3|$ for any three points in the plane. When does the equality occur? Interpret the inequality geometrically.

2.2 Lines in Plane

Let ℓ be a (straight) line in the Cartesian plane. Then, one and only one of the following three cases is possible: ℓ is vertical, ℓ is horizontal, or ℓ is inclined (neither horizontal nor vertical).

If ℓ_v is a vertical line (see Fig. 2.5) then the X-coordinate of any point on ℓ_v is equal to a fixed real number a. Hence

$$\ell_v = \{(a,y) : y \in \mathbb{R}\} = \{(x,y) : x = a\}.$$

Vertical - dikey

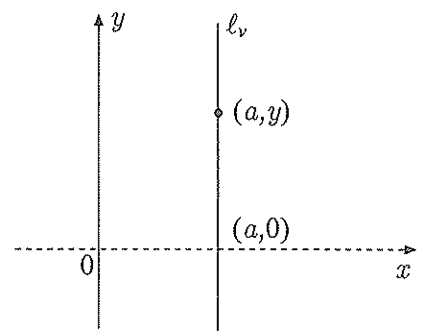


Fig. 2.5.

If ℓ_h is a horizontal line (see Fig. 2.6), then the Y-coordinate of any point on ℓ_h is equal to a fixed real number b. Hence

$$\ell_h = \{(x, b) : x \in \mathbb{R}\} = \{(x, y) : y = b\}.$$

horizontal → yatay

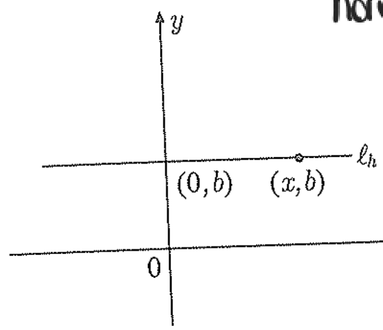


Fig. 2.6.

Now let ℓ be an inclined line so that ℓ intersects the X-axis at $A(a, 0)$ ~~ad~~ and the Y-axis at $B(0, b)$ (see Fig. 2.7). Consider a point $M(x_1, y_1)$ on ℓ other than $A(a, 0)$ and define

$$m = \frac{M_y O}{M_x A} = \frac{y_1}{x_1 - a}.$$

If $N(x_2, y_2)$ is another point on ℓ, then we see from the similarity of the triangles MAM_x and NAN_x that

$$m = \frac{y_1}{x_1 - a} = \frac{y_2}{x_2 - a}.$$

From the elementary properties of ratios of real numbers, we conclude that

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{2.2.1}$$

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

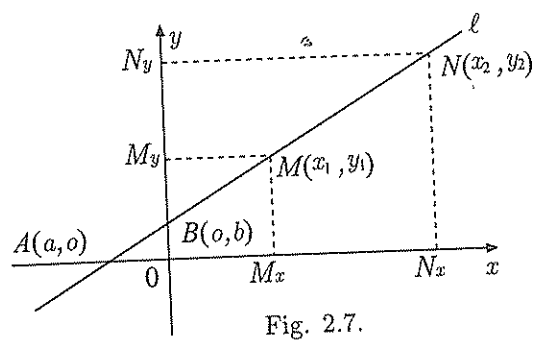


Fig. 2.7.

Hence the ratio defined in (2.2.1) is the same for any two distinct points $M(x_1, y_1)$ and $N(x_2, y_2)$ on the line. It is called the *slope* of the line ℓ. It also follows from the above observation that a point $P(x, y)$ is on the line ℓ if and only if

slope

$$\boxed{m = \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},}$$

or

$$y = m(x - x_1) + y_1.$$

In set theoretical notations,

$$\ell = \{(x, y) : y = m(x - x_1) + y_1\}$$

where m is the slope of ℓ and $M(x_1, y_1)$ is a point on ℓ.

Remark. It is clear that the slope of a vertical line is undefined. For, if $M(x_1, y_1)$ and $N(x_2, y_2)$ are two points on a vertical line then $x_1 = x_2$ and the ratio in (2.2.1) is undefined. However, if M and N are two points on a horizontal line, then the ratio in (2.2.1) is defined and equal to zero. Hence we can say, *the slope of a vertical line is undefined and the slope of a horizontal line is zero.*

We call the equations $x = a, y = b$, and $y = m(x - x_1) + y_1$ the *defining equations* of the lines ℓ_v, ℓ_h, and ℓ, respectively.

An equation of the form $Ax + By + C = 0$ where A, B and C are real numbers, not all zero, is called a linear equation in x and y.

We have thus proved

Theorem 2.2.1 *The defining equation of any straight line is a linear equation.*

The converse of this theorem is also true:

Theorem 2.2.2 *Any linear equation $Ax + By + C = O$ is the defining equation of a straight line.*

Proof. We consider the cases $B = 0$ and $B \neq 0$. If $B = 0$, then $A \neq 0$. For, otherwise C also would be zero. Hence, in this case, the equation reduces to $x = -C/A$ which is a defining equation of a (vertical) straight line. If $B \neq 0$, then the equation $Ax + By + C = 0$ reduces to $y = -(A/B)x - (C/B)$ which is a defining equation of the straight line having slope $-A/B$ (possibly equal to zero) and which contains the point $(0, -C/B)$. This completes the proof. □

Example 2.2.3 Write a defining equation of the straight line which passes through the points $P_1(-3,5)$ and $P_2(-5,1)$.

Solution. The slope of the line is

y'ler farkı / x'ler farkı

$$m = \frac{1-5}{-5-(-3)} = \frac{-4}{-2} = 2.$$

Since the line contains the point $P_2(-5,1)$,

$$y = 2(x+5)+1 \quad \text{or} \quad y = 2x+11$$

is a defining equation of the line. Note that if we take $P_1(-3,5)$ instead of $P_2(-5,1)$, we obtain $y = 2(x+3)+5$ or $y = 2x+11$ the same equation.

If the X-coordinate or the Y-coordinate of a point on a line ℓ is given, we can use the defining equation of ℓ to find the other coordinate of that point. For example, for the line with defining equation $y = 2x+11$, the point whose X-coordinate is 1 has Y-coordinate 13.

Any non-vertical line contains a point whose X-coordinate is 0. The Y-coordinate of this point is called the *Y-intercept* of the line. Similarly, any non-horizontal line contains a point whose Y-coordinate is 0. The X-coordinate of this point is called the *X-intercept* of the line.

It is clear that any non-vertical line has a defining equation of the form

$$\boxed{y = mx+b}$$

where m is the slope and b is the Y-intercept of the line. Any non-horizontal and non-vertical line has a defining equation of the form

$$x = \frac{1}{m}y + a$$

where m is the slope and a is the X-intercept of the line.

Example 2.2.4 Find the slope and the Y-intercept of the line defined by the linear equation $3x+2y-5=0$.

Solution. Dividing by 2, we obtain the equivalent equation

$$y = -\frac{3}{2}x + \frac{5}{2}.$$

Hence $m = -\frac{3}{2}$ is the slope and $\frac{5}{2}$ is the Y-intercept.

y'yi yalnız bırakınca x önündeki katsayı eğimi, kalan sayı y-intercept olur.

Remark. Given two lines ℓ_1 and ℓ_2 in the plane, either they *coincide* or they *intersect* at a point or they are *parallel*. If they intersect at a point, the point of intersection is the point whose coordinates satisfy the defining equations of both ℓ_1 and ℓ_2. Thus, finding the point of intersection of two lines amounts to solving a system of two linear equations.

Example 2.2.5 Let ℓ_1 be the line which passes through the points $P_1(4,1)$ and $Q_1(2,-1)$, and let ℓ_2 be the line which passes through $P_2(2,0)$ and $Q_2(4,4)$. Find the point of intersection of these two lines.

Solution. Defining equations of the lines ℓ_1 and ℓ_2 can be written, respectively, as

$$y = \frac{-1-1}{2-4}(x-2)-1 \quad \text{and} \quad y = \frac{4-0}{4-2}(x-4)+4$$

$$y = x-3 \quad \text{and} \quad y = 2x-4.$$

Substitute $y = x-3$ in the second equation. Then $x-3 = 2x-4$ $x = 1$. This is the X-coordinate of the point of intersection. The Y-coordinate can be obtained by substituting $x = 1$ in the defining equation of ℓ_1 or ℓ_2. Hence $y = 2.1-4 = -2$ and $P(1,-2)$ is the point of intersection.

kesişimlerini denklemleri eşitleyerek bul. (x nok) sonra birinde yerine yazarak y-component bul.

One can use linear equations to characterize parallel, coincident or perpendicular lines. We first consider coincident lines.

Theorem 2.2.6 *Let* $\ell_1 = \{(x,y) : Ax+By+C=0\}$ *and* $\ell_2 = \{(x,y) : ax+by+c=0\}$ *be two lines in the plane. Then* ℓ_1 *and* ℓ_2 *are coincident if and only if there exists* $k \neq 0$ *such that* $A = ka, B = kb$ *and* $C = kc$.

Proof. If there is $k \neq 0$ such that $A = ka, B = kb$ and $C = kc$ then, for a point (x,y), $Ax+By+C = 0 \iff ax+by+c = 0$ and therefore the two lines are coincident.

To prove the converse, assume first that $\ell_1 = \ell_2$ is vertical. Then $B = b = 0, A \neq 0$. Take $k = A/a$. It is easy to see then that $A = ka, B = kb$ and $C = kc$. Assume, now, that $\ell_1 = \ell_2$ is not vertical. Then $B \neq 0$ and $b \neq 0$, and we have

$$Ax+By+C = 0 \iff y = -\frac{A}{B}x - \frac{C}{B}$$

$$ax+by+c = o \iff y = -\frac{a}{b}x - \frac{c}{b}$$

Since the slope and the Y-intercept of a line are uniquely determined real numbers, we have

$$\frac{A}{B} = \frac{a}{b}, \quad \frac{C}{B} = \frac{c}{b}$$

For $k = B/b$, we have $A = ka, B = kb, C = kc$. □

Theorem 2.2.7 *Let $\ell_1 = \{(x,y) : Ax+By+C=0\}$ and $\ell_2 = \{(x,y) : ax+by+c=0\}$ be two (distinct) lines in the plane. Then*

a) ℓ_1 and ℓ_2 are parallel $\Longleftrightarrow$ there exists $k \neq 0$ such that $A = ka, B = kb$ and $C \neq kc$,

b) ℓ_1 and ℓ_2 are perpendicular $\Longleftrightarrow$ $Aa + Bb = 0$.

dik

Proof. a) Let $\tilde{\ell_1} = \{(x,y) : Ax+By=0\}$. Then ℓ_1 and $\tilde{\ell_1}$ are coincident if $C=0$, and otherwise they are parallel (See Fig. 2.8). Similarly ℓ_2 and $\tilde{\ell_2} = \{(x,y) : ax+by=0\}$ are either coincident or they are parallel. Since $\tilde{\ell_1}$ and $\tilde{\ell_2}$ are lines through the origin, ℓ_1 and ℓ_2 are parallel $\Longleftrightarrow$ $\tilde{\ell_1}$ and $\tilde{\ell_2}$ are coincident $\Longleftrightarrow$ there exists $k \neq 0$ such that $A = ka$ and $B = kb$. Note that $C \neq kc$, because $\ell_1 \neq \ell_2$.

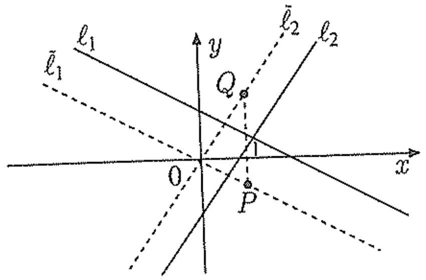


Fig. 2.8

b) Let $\tilde{\ell_1}$ and $\tilde{\ell_2}$ be defined as above (See Fig. 2.8). Then ℓ_1 and ℓ_2 are perpendicular if and only if $\tilde{\ell_1}$ and $\tilde{\ell_2}$ are perpendicular. Assume that neither ℓ_1 nor ℓ_2 is vertical, i.e., $Bb \neq 0$. Consider the points $P(1, -\frac{A}{B})$ and $Q(1, -\frac{a}{b})$ on the lines $\tilde{\ell_1}$ and $\tilde{\ell_2}$, respectively. By Pythagorian theorem, $\tilde{\ell_1}$ and $\tilde{\ell_2}$ are perpendicular if and only if $|0P|^2 + |OQ|^2 = |PQ|^2$. The last identity yields

$$\begin{aligned} (A/B)^2 + 1 + (a/b)^2 + 1 &= [(A/B) - (a/b)]^2] = (A/B)^2 + (a/b)^2 - 2(Aa)/(Bb) \\ 2 &= -2(Aa)/(Bb) \\ Aa + Bb &= 0. \end{aligned}$$

The assertion is trivial for the cases when ℓ_1 and ℓ_2 is vertical. □

Corollary 2.2.8 *Let ℓ_1 and ℓ_2 be two non-vertical lines with slopes m_1 and m_2, respectively. Then*

a) ℓ_1 and ℓ_2 are parallel or coincident $\Longleftrightarrow$ $m_1 = m_2$,

b) ℓ_1 and ℓ_2 are perpendicular $\Longleftrightarrow$ $m_1 m_2 = -1$.

Proof. Let $y = m_1 x + b_1$ and $y = m_2 x + b_2$ be the defining equations of ℓ_1 and ℓ_2, respectively. The statements a) and b) can be deduced directly from Theorem 2.2.6 and Theorem 2.2.7. □

Exercises

1. Write a defining equation for each of the lines through $M(1,1)$ and
   a) with slope 2, b) with Y-intercept 7 c) with X- intercept 2
   d) with slope -2 e) through $P(-1,2)$ f)through $O(0,0)$.

2. In each case below, determine $\ell_1 \cap \ell_2$
   a) $\ell_1 = \{(x,y) : 2x + y - 1 = 0\}$, $\ell_2 = \{(x,y) : y - 3x - 1 = 0\}$
   b) $\ell_1 = \{(x,y) : 2x - y + 1 = 0\}$, $\ell_2 = \{(x,y) : 4x - 2y + 2 = 0\}$
   c) $\ell_1 = \{(x,y) : 3x + 2y - 15\}$, $\ell_2 = \{(x,y) : 6x + 4y - 15 = 0\}$.

3. Determine the coordinates of the vertices of the triangle determined by the lines whose defining equations are $2x - 3y = 0, x - y = 1$ and $2x - y + 3 = 0$.

4. Let $\ell_1 = \{(x,y) : 2x + \frac{y}{3} = 1\}$ and $\ell_2 = \{(x,y) : kx + 2y = -2\}$.
   a)Determine k if $\ell_1 // \ell_2$ b) Determine k if $\ell_1 \perp \ell_2$.

5. Given $\ell_1 = \{(x,y) : x + By + 1 = 0\}, \ell_2 = \{(x,y) : ax + y + 1 = 0\}$
   a) Determine a and B for which ℓ_1 and ℓ_2 are coincident.
   b) Determine a and B for which ℓ_1 and ℓ_2 are perpendicular.
   c) Determine all a and B for which $\ell_1 \cap \ell_2 \neq \phi$.

6. Write a defining equation for each of the lines through $M(-1,3)$ and
   a) parallel to $\{(x,y) : y = 3x - 1\}$,
   b) perpendicular to $\{(x,y) : y = 3x - 1\}$,
   c) parallel to $\{(x,y) : 3x - 5y = 1\}$,
   d) perpendicular to $\{(x,y) : 3x - 5y = 2\}$.

7. Prove that $A(5,1), B(1,-3)$ and $C(3,3)$ are the vertices of a right triangle
   a) by using the distance formula (2.1.1) and Pythagorian Theorem.
   b) by considering the slope of the line passing through each pair of points.

8. Given n points $P_1, P_2, ..., P_n$ in the plane, $n \geq 2$, we say that they are co-linear if all of them lie on the same line. Show that the four points $P_1(0,1), P_2(3,3), P_3(-3,-1)$ and $P_4(6,5)$ are collinear. Find an equation of the line which contains all these points.

Hatalı

9. Given the equations

$$\ell_1 : 8x + 3y + 1 = 0, \ell_2 : 2x + y - 1 = 0$$

of two sides of a parallelogram and the equation

$$d : \quad 3x + 2y + 3 = 0$$

of one of its diagonals. Determine the coordinates of the vertices of the parallelogram.

10. The sides of a triangle lie on the lines

$$x + 5y - 7 = 0, \quad 3x - 2y - 4 = 0, \quad 7x + y + 19 = 0.$$

Calculate the area S of the triangle.

11. The area S of a triangle is 8 square units; two of its vertices are the points $A(1,-2)$ and $B(2,3)$, and the third vertex C lies on the line

$$2x + y - 2 = 0.$$

Find the coordinates of the vertex C.

12. Given the midpoints $M_1(2,1), M_2(5,3), M_3(3,-4)$ of the sides of a triangle. Write the equations of its sides.

13. Find the equations of the sides of a triangle ABC with $A(1,3)$ as a vertex, if

$$x - 2y + 1 = 0 \quad \text{and} \quad y - 1 = 0$$

are the equations of two of its medians.

14. Find the equations of the sides of a triangle having $B(-4,-5)$ as a vertex, if

$$5x + 3y - 4 = 0 \quad \text{and} \quad 3x + 8y + 13 = 0$$

are the equations of two of its altitudes.

15. Find the equations of the sides of a triangle having $A(4,-1)$ as a vertex, if

$$x - 1 = 0 \quad \text{and} \quad x - y - 1 = 0$$

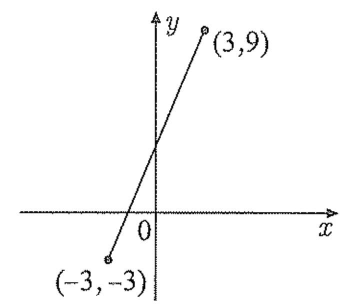
are the equations of two bisectors of its angles.

2.3 Graphs of Relations from $\mathbb{R}$ to $\mathbb{R}$

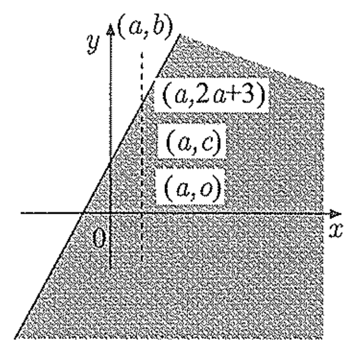
Recall that a relation from **R** to **R** is a subset of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Hence, in view of section 1, there is a one-to -one correspondence between relations from $\mathbb{R}$ to $\mathbb{R}$ and subsets of the Cartesian plane. The subset of the plane which corresponds to the ordered pairs of a relation is called the graph of the relation.

We have seen above (Theorem 2.2.2) that the graph of a relation defined by a linear equation $Ax + By + C = 0$ is always a line. In Chapter 4, we deal with graphs of quadratic equations. Now, we give a few more examples.

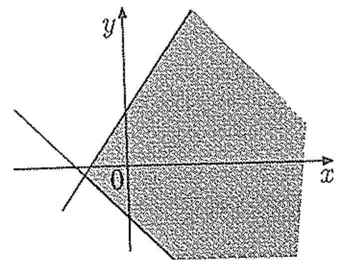
Example 2.3.1 The graph of the relation $R = \{(x,y) : y = 2x + 3, -3 \leq x \leq 3\}$ is the line segment with end points $(-3,-3)$ and $(3,9)$.



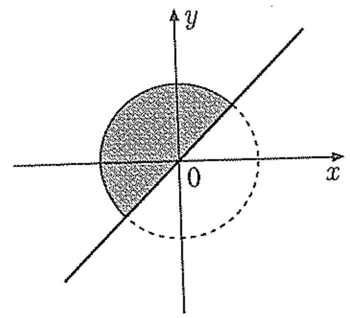
Example 2.3.2 Let us consider the graph of $R = \{(x,y) : y \leq 2x+3\}$. The equality $y = 2x + 3$ defines a line ℓ. If (a,b) is a point above the line ℓ then $b > 2a + 3$ so that $(a,b) \notin R$. However if (a,c) is on or below ℓ then $c \leq 2a + 3$ and $(a,c) \in R$. Hence the graph of R consists of the points on or below the line ℓ.



Example 2.3.3 Consider $R = \{(x, y) : y \leq 2x + 3 \text{ and } y \geq -x - 2\}$. By discussions as in the previous example, we see that the graph of R consists of those points which are on or below $\ell_1 = \{(x, y) : y = 2x + 3\}$ and on or above $\ell_2 = \{(x, y) : y = -x - 2\}$.



Example 2.3.4 Sketch the graph of $R = \{(x, y) : x < y \text{ and } x^2 + y^2 \leq 1\}$. The graph is the intersection of the graph of the relation defined by $x < y$ and the graph of the relation defined by $x^2 + y^2 \leq 1$. The first one is the set of points above the line $y = x$ and the second one is the set of all points on or inside the circle $x^2 + y^2 = 1$.

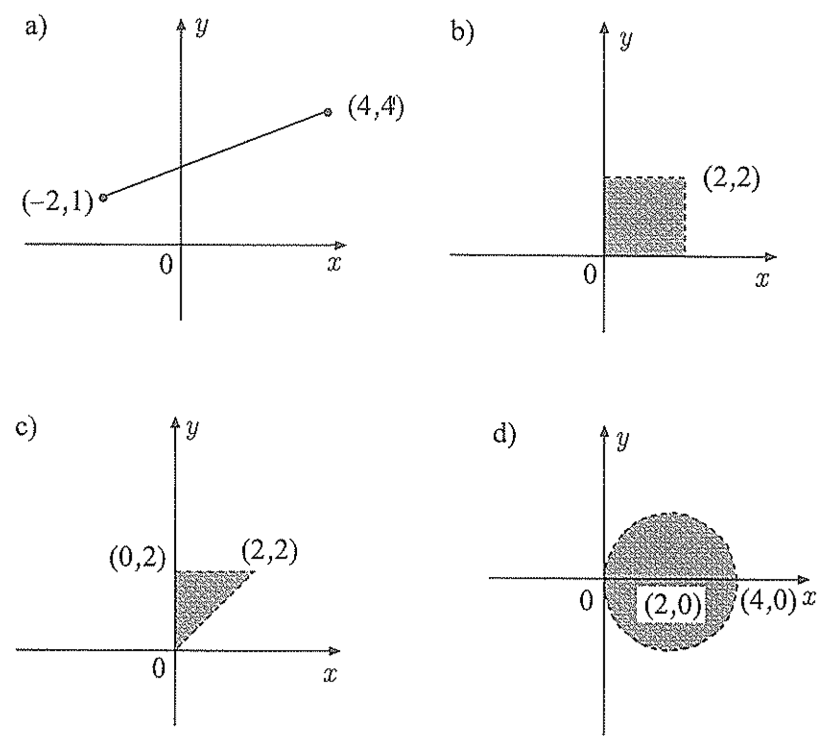


Remark. Recall that a function is a relation which contains no two distinct pairs with the same first member. Thus *a relation from $\mathbb{R}$ to $\mathbb{R}$ is a function if and only if any vertical line intersects its graph at most at one point.* Similarly, *a function from $\mathbb{R}$ to $\mathbb{R}$ is one-to-one if and only if any horizontal line intersects its graph at most at one point.*

Exercises

1. Sketch the graph of each of the following relations (from $\mathbb{R}$ to $\mathbb{R}$).
   a) $\{(x, y) : x > 0\}$ b) $\{(x, y) : 0 \leq x \leq 1\}$.
   c) $\{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$ d) $\{(x, y) : 0 < x < y\}$.
   e) $\{(x, y) : 0 \leq x \leq y \leq 1\}$ f) $\{(x, y) : x^2 + y^2 - 2x < 0\}$.

2. In each case below, determine the relation (from $\mathbb{R}$ to $\mathbb{R}$) whose graph is given.



3. Graph the following relations.
   a) $\{(x, y) : 2y = x + |x|\}$ b) $\{(x, y) : 2y < x + |x|\}$.
   c) $\{(x, y) : |x| = |y|\}$ d) $\{(x, y) : |x| < |y|\}$.
   e) $\{(x.y) : |x| + |y| = 0\}$, f) $\{(x, y) : |x| + |y| = 1\}$.
   g) $\{(x, y) : |x| + |y| < 1\}$, h) $\{(x, y,) : |x| + |y| > 1\}$.

2.4 Review of Trigonometry

We assume that the reader is already familiar with trigonometric functions *sine* and *cosine*.

The circle of radius 1 centered at the origin is called the *trigonometric circle*. We define a function from the set of real numbers *onto* the set of points on the trigonometric circle in the following way. To the real number 0, zero, we assign the point $P_0(1, 0)$ on the trigonometric circle. To find the point which is assigned to a real number t; starting at the point $P_0(1,0)$, we proceed $\mid t \mid$ units along the circumference of the trigonometric circle *counterclockwise if* $t \geq 0$, *clockwise if* $t < 0$. The point P_t that we reach is assigned to the real number t (See Fig. 2.9). Thus the point $(0,1)$ is assigned to the real number $\pi/2$ and the point $(0,-1)$ is assigned to $-\pi/2$.

Note that if P_t is the point assigned to the real number t, then the *radian* measure of the angle P_tOP_0 is t.

Obviously, the function defined above is *onto* but *not one-to-one*. If the point P_t is assigned the real number t, then the same point P_t is assigned the numbers

$$t \pm 2\pi, t \pm 4\pi, ..., t \pm 2k\pi, \ldots.$$

Namely, $P_t = P_{t+2k\pi}$ for any $k \in \mathbb{Z}$.

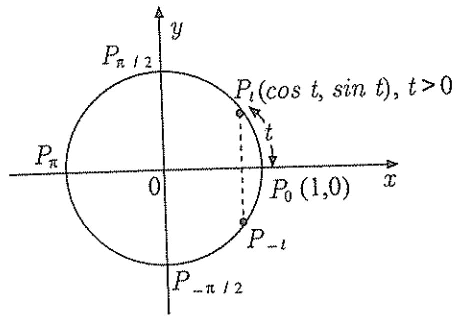


Fig. 2.9.

Now we are ready to give the definitions of *cosine* and *sine* functions. If the number t is assigned the point P_t on the trigonometric circle, then the *X-coordinate of* P_t *is the number* $\cos t$ and *Y-coordinate of* P_t *is the number* $\sin t$.

Example 2.4.1 Find the numbers $cos(\frac{\pi}{2}), sin(\pi), sin(-\frac{\pi}{2})$ and $cos(\frac{5\pi}{4})$.

Solution. See Fig.2.10 for the location of the points $P_{\pi/2}, P_{-\pi/2}, P_\pi, P_{5\pi/4}$. We have $P_{\pi/2}(0,1), P_\pi(-1,0)$ and $P_{-\pi/2}(0,-1)$. Hence $cos\frac{\pi}{2} = 0$, $sin\pi = 0, sin(-\frac{\pi}{2}) = -1$. As for $P_{5\pi/4}(x,y)$, we see that $x = y < 0$. Since it is a point on the trigonometric circle, $x^2 + y^2 = 1$. Thus $x^2 + x^2 = 1$, or $x = y = \frac{-1}{\sqrt{2}}$. Hence $\cos(\frac{5\pi}{4}) = -\frac{1}{\sqrt{2}}$.

For any real number t, the point $P_t(\cos t, \sin t)$ is a point on the trigonometric circle. Thus

$$\boxed{cos^2 t + sin^2 t = 1.}$$

We know that for any real number t and for any integer k, the numbers t and $t + 2k\pi$ are assigned the same point P_t on the trigonometric circle. Thus $\cos(t + 2k\pi) = \cos t$ $\sin(t + 2k\pi) = \sin t$ for any integer k.

The remaining trigonometric functions are defined by the equations

$$\boxed{\tan t = \frac{\sin t}{\cos t}, \quad \cot t = \frac{\cos t}{\sin t} \quad \sec t = \frac{1}{\cos t}, \quad cosec\ t = \frac{1}{\sin t}.}$$

The trigonometric functions are also defined in terms of angles and right triangles. We can relate the two definitions in the following way (See Fig. 2.10). Given an angle AOB, we can choose the cartesian coordinate system in the plane in such a way that O is the origin and the line jointing O to B and oriented from O to B is the X-axis.

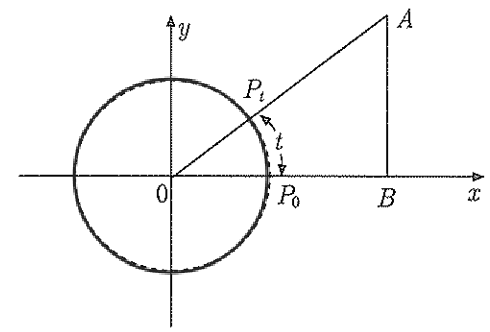


Fig. 2.10

Then, the point P_t of intersection of the trigonometric circle with the line joining O to A is the point which is assigned to the radian measure t of the angle AOB. If A has coordinates (x, y) and $\mid OA \mid = \sqrt{x^2 + y^2} = r$, then using similar triangles we get

$$\cos t = \frac{x}{r}, \quad \sin t = \frac{y}{r} \quad \tan t = \frac{y}{x}, \quad \cot t = \frac{x}{y} \quad \sec t = \frac{r}{x}, \quad cosec\ t = \frac{r}{y}.$$

Exercises

1. Use a sketch of the trigonometric circle to determine each of the following numbers

   a) $sin\frac{\pi}{6}$, b) $cos\frac{\pi}{6}$, c) $sin\frac{\pi}{4}$, d) $cos\frac{\pi}{4}$, e) $sin\frac{\pi}{3}$
   f) $cos\frac{\pi}{3}$, g) $sin\frac{25\pi}{4}$, h) $cos\frac{25\pi}{4}$, i) $sin\frac{-25\pi}{3}$, j) $cos\frac{-25\pi}{4}$,
   k) $tan\frac{26\pi}{6}$.

2. Which of the following inequalities are true?
   a) $sin2 > 0$, b) $sin(-2) > 0$, c) $cos2 > 0$, d) $cos(-2) > 0$
   e) $sin2 < sin3$, f) $cos2 < cos3$, g) $sin2 < cos2$, h) $cos2 < sin3$
   i) $sin2 < cos3$.

3. Use a sketch of the trigonometric circle to prove the following identities
   a) $cos(-x) = cosx$, b) $sin(-x) = -sinx$, c) $cos(\frac{\pi}{2} - x) = sinx$
   d) $sin(\frac{\pi}{2} - x) = cosx$, e) $cos(\pi + x) = -cosx$, f) $sin(\pi + x) = -sinx$
   g) $cos(\pi - x) = -cosx$, h) $sin(\pi - x) = sinx$.
   i) $cos(x - y) = cosxcosy + sinxsiny$, j)$sin(x + y) = sinxcosy + sinycosx$.

4. Prove a) $sin2x = 2sinxcosx$, b) $cos2x = cos^2x - sin^2x$.

5. Complete the following table

t	cost	sint	tant
0	1	0	0
$\pi/6$			
$\pi/4$			
$\pi/3$			
$\pi/2$			
$2\pi/3$			
$3\pi/4$			
$5\pi/6$			

6. For natural numbers $n = 1, 2, \ldots$ prove that there exist polynomials $p_n(x), q_n(x)$, such that $\sin n\theta = p_n(\tan\theta)\cos^n\theta$ and $\cos n\theta = q_n(\tan\theta)\cos^n\theta$.

7. Then for $n > 1$, prove that one can establish the following identities: $p_n'(x) = nq_{n-1}(x)$ and $q_n'(x) = -np_{n-1}(x)$.

Kartezyen Denk.

$x+y=5$

Geçilirken;

x yerine $r\cos\theta$

y yerine $r\sin\theta$ yazılır.

$x^2+y^2=r^2$

$\to r\cos\theta + r\sin\theta = 5$

Kutupsal Denklem

$r.\cos\theta + \sin^2\theta = 1$

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ yazılır.

$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

$r = \sqrt{x^2+y^2}$

$\theta = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$

2.5 Polar Coordinates

In section 2.1, we have introduced cartesian coordinates in the plane. This was achieved by establishing a one-to-one correspondence between points of the plane and ordered pairs of real numbers. Now, we give another correspondence between points of the plane and pairs of real numbers. This time the correspondence is *no longer one-to-one.*

In the plane, we take a number axis and call it the *polar axis.* A polar coordinate system in the plane consists of a number axis, namely the polar axis. The origin O of this number axis is called the *pole* of the polar coordinate system (See Fig. 2.11).

Once we choose a polar coordinate system in the plane, to every pair of real numbers we may assign a unique point in the plane and to every point in the plane we may assign pairs of real numbers, called *polar coordinates* of the point.

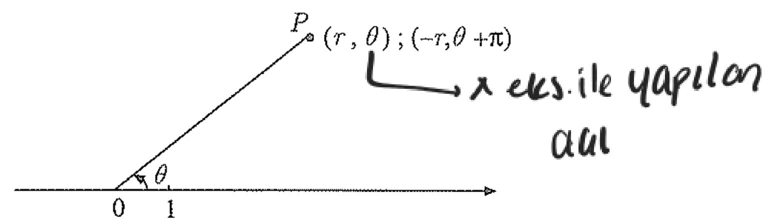


Fig. 2.11.

Given a point P in the plane, the ordered pairs of real numbers assigned to P are determined as follows: If $P \neq O$, the pole, then we let r be the distence between O and P, and let θ be the radian measure of the angle between the segment OP and the positive sense of the polar axis, measured from the polar axis towards the segment OP in counterclockwise direction. We assign to P any one of the pairs

$$(r, \theta + 2k\pi), k = 0, \pm 1, \pm 2, \ldots \quad \text{or} \tag{2.5.1}$$

$$(-r, \theta + (2k+1)\pi), k = 0, \pm 1, \pm 2, \ldots \tag{2.5.2}$$

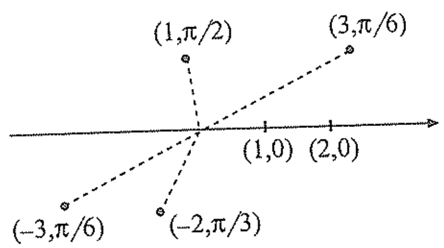
To the pole O, we assign any one of the pairs $(0, \theta)$, where θ is an arbitrary real number.

By reversing the above process we see that any pair of real numbers is the polar coordinates of a unique point in the plane. By definition, every point in the plane has infinitely many polar coordinates. However, for a point $P \neq O$, if one of the polar coordinates, say (r, θ), is known, then any other polar coordinates of P is of the form $(r, \theta + 2k\pi)$ or $(-r, \theta + (2k+1)\pi)$, where $k \in \mathbb{Z}$.

Example 2.5.1 Locate the points, with polar coordinates,

$$(3, \frac{\pi}{6}), (-3, \frac{\pi}{6}), (2, 0) \text{ and } (-2, \frac{\pi}{3})$$

in the polar coordinate system.



Note that there are two classes of polar coordinates of a point $P \neq O$ in the plane, given by (2.5.1) and (2.5.2). Any polar coordinate of the class in (2.5.1) can be obtained from (r, θ) and any polar coodinate of the class in (2.5.2) can be obtained from $(-r, \theta + \pi)$, in the obvious manner.

It is advantageous to consider the polar coordinate system and the Cartesian coordinate system, in the plane, together by taking the polar axis as the X-axis. Then we can convert from polar coordinates of a point P to cartesian coordinates and vice versa. Given (r, θ) or $(-r, \theta + \pi)$ as polar coordinates of a point P, the cartesian coordinates (x, y) of P satisfies the equations (See Fig. 2.12)

$$\boxed{\begin{aligned} x &= r\cos\theta (= -r\cos(\theta + \pi)), \\ y &= r\sin\theta (= -r\sin(\theta + \pi)). \end{aligned}}$$

Conversely, given (x, y) as Cartesian coordinates of a point $P \neq O$, polar coordinates (r, θ) or $(-r, \theta + \pi)$ of P satisfy the equations

$$\boxed{x^2 + y^2 = r^2, \qquad \frac{x}{r} = \cos\theta, \qquad \frac{y}{r} = \sin\theta.}$$

By definition, $r = 0$ and θ is arbitrary for the origin $O(0, 0)$ of the Cartesian coodinate system.

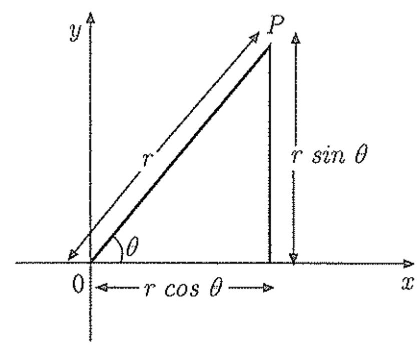


Fig. 2.12

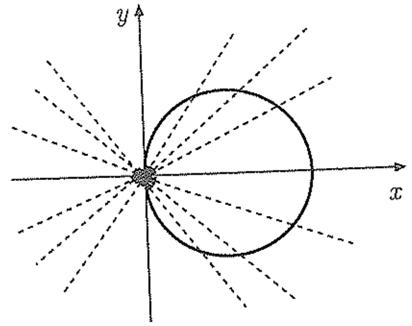
From now on, in any discussion involving both Cartesian and polar coordinates, we will assume that the polar axis is the positive X-axis.

The *graph* of an equation $F(r, \theta) = 0$, involving polar coordinates, is the set of all points P in the plane such that *at least one of* the polar coordinates of P satisfies the equation. Here the situation is a little different from the situation for the graph of an equation $F(x, y) = 0$. There may be points P such that some of its polar coordinates satisfy a given equation while the others do not.

To sketch the graph of a polar equation $F(r, \theta) = 0$, we select a succession of convenient values of θ and use them in the equation to obtain the corresponding values of r. We thus get a sequence of points; and by constructing a smooth curve through the successive points in order of increasing values of θ, we obtain a sketch of the graph.

Example 2.5.2 Let us consider the equation $r = 2cos\theta$. We construct a table of values and use it to sketch the graph.

θ	r
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/4$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2
$5\pi/4$	$-\sqrt{2}$
$3\pi/2$	0
$7\pi/4$	$\sqrt{2}$
2π	2



As we plot these points we see that $\theta > 2\pi$ yield no new points.

In drawing graph of a polar equation, we may make use of symmetry. The graph of an equation $F(r,\theta) = 0$ is *symmetric about X-axis* (or the polar axis) if whenever $F(r,\theta) = 0$ for a point (r,θ), we have $F(r,-\theta) = 0$ or $F(-r,\pi-\theta) = 0$. Similarly, as we can observe from Fig.2.13, the graph of $F(r,\theta) = 0$ is *symmetric about Y-axis* if whenever $F(r,\theta) = 0$ for a point (r,θ), we have

$$F(-r,-\theta) = 0 \text{ or } F(r,\pi-\theta) = 0.$$

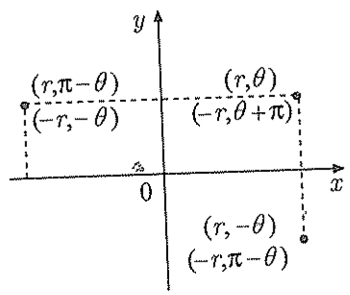


Fig. 2.13.

If the graph of an equation $F(r,\theta) = 0$ is symmetric about X-axis, then in sketching the graph, it is enough to consider the values of θ only between 0 and π.

Similarly, if there is symmetry about Y-axis, then it suffices to consider $-\frac{\pi}{2} \leq \theta \leq \pi/2$. If the graph is symmetric about both X-axis and Y-axis, then it is enough to consider $0 \leq \theta \leq \pi/2$. In each case, we sketch the graph for the specified values of θ, and obtain the rest of the graph by symmetry.

Example 2.5.3 Sketch the graph of $r = 2cos2\theta$.

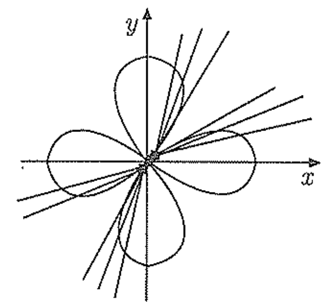
Solution. If we let $F(r,\theta) = r - 2cos2\theta$, we see that

$$\begin{aligned} F(r,-\theta) &= r - 2cos(-2\theta) = r - 2cos2\theta = F(r,\theta) \\ F(r,\pi-\theta) &= r - 2cos(2(\pi-\theta)) = r - 2cos2\theta = F(r,\theta). \end{aligned}$$

Thus the graph is symmetric about both X- and Y-axis. We first sketch the graph for $0 \leq \theta \leq \pi/2$:

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2

The rest of the graph is obtained by symmetry.



Example 2.5.4 Sketch the graph of $r = 2 + 2cos\theta$.

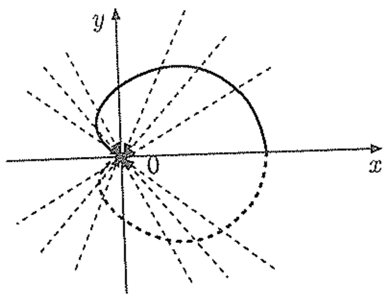
Solution. Let $F(r,\theta) = r - 2 - 2cos\theta$.
Then

$$F(r,-\theta) = r - 2 - 2cos(-\theta) = r - 2 - 2cos\theta = F(r,\theta).$$

Hence the graph is symmetric about X-axis. One can check that for $F(r,\theta) = 0$, $F(-r,-\theta) \neq 0$ and $F(r,\pi-\theta) \neq 0$; therefore the graph is not symmetric about Y-axis. We first sketch the graph for $0 \leq \theta \leq \pi$. For this we construct the table of values

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	4	$2+\sqrt{3}$	$2+\sqrt{2}$	3	2	1	$2-\sqrt{2}$	$2-\sqrt{3}$	0

and obtain the part of the graph by plotting these points. The rest of the graph is obtained by taking symmetry.



Exercises

1. Plot each of the following points, given in polar coordinates.

   a) $(3, \frac{\pi}{6})$, b) $(3, -\frac{5\pi}{4})$, c)$(-3, \frac{17\pi}{6})$, d) $(3, \pi)$
   e) $(3, -\pi)$, f) $(-3, \pi)$, g)$(-3, -\pi)$, h) $(2, \frac{\pi}{3})$
   i) $(2, -\frac{\pi}{3})$, j) $(-2, \frac{\pi}{3})$, k)$(-2, -\frac{\pi}{3})$.

2. Recall that if the radian measure of an angle is t and the degree measure is A, then $\frac{A}{180} = \frac{t}{\pi}$. Plot the following points where the second (polar) coordinates are given in degrees.

   a)$(3, 30°)$, b)$(3, -30°)$, c) $(-3, 30°)$
   d) $(-3, -30°)$, e) $(3, 960°)$, f)$(-3, -390°)$.

3. Find the cartesian coordinates of the points in exercises 1 and 2.

4. Find $\mid PQ \mid$ where P and Q are given in polar coordinates:

   a) $P(1, \pi), Q(1, \frac{\pi}{2})$, b)$P(-1, \pi), Q(1, \theta)$
   c)$P(1, \frac{\pi}{6}), Q(-2, \frac{\pi}{3})$, d)$P(1, 120°), Q(-2, 210°)$
   (Hint: Plot the given points.)

5. Sketch the graph of the following polar equations:

   a) $r = -2cos\theta$, b) $r = 2sin\theta$, c)$r = 2 - 2sin\theta$, d) $r = 2sin(2\theta)$,
   e) $r = 1 - sin2\theta$, f) $r = 2cos(3\theta)$, h)$r = 2cosec\theta$,
   i) $r = \frac{1}{2-cos\theta}$, j) $rcos\theta = 4$, l) $r = 1 + 2cos\theta$.
   m) $r^2 = 4cos2\theta$, n) $r = cos4\theta$.

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2.6 Change of Coordinates: Rotation and Translation

orijin sabit oynar.

A Cartesian coordinate system, in the plane, makes it possible to describe a geometric problem algebraically, and vice versa. In problems of analytic geometry, it may sometimes be useful to replace the given coordinate system by another coordinate system which is thought to be more convenient. When we consider two coordinate systems in the plane, the coordinates of a point in one system is, in general, different from its coordinates in the other system. Therefore, when making use of two coordinate systems in a single problem, one would like to know how the coordinates of an arbitrary point in one system are related to its coordinates in the other system.

In this section, we study two examples of change of coordinates in the plane. Any change of coordinates can be, essentially, obtained by a repeated application of these two (See the remark at the end of this section).

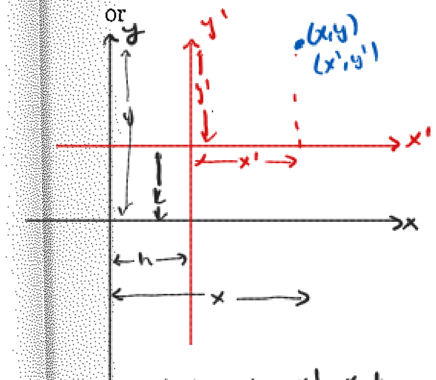
Let us first assume that $\tilde{X}\tilde{Y}$-coordinate system is obtained from XY-coordinate system by moving the origin of XY-coordinate system to a new position without changing the direction and the unit length of the axes (See Fig. 2.14). Then we say that $\tilde{X}\tilde{Y}$- coordinate system is a translation of XY-coordinate system. If the origin $\tilde{O}$ of $\tilde{X}\tilde{Y}$-coordinate system has XY-coordinates (h, k), then the origin O of XY-coordinate system has $\tilde{X}\tilde{Y}$-coordinates $(-h, -k)$.

More generally, XY-coordinates (x, y) and $\tilde{X}\tilde{Y}$-coordinates $(\tilde{x}, \tilde{y})$ of an arbitrary point P in the plane are related by

$$x = \tilde{x} + h, y = \tilde{y} + k$$

or

$$\tilde{x} = x - h, \tilde{y} = y - k.$$



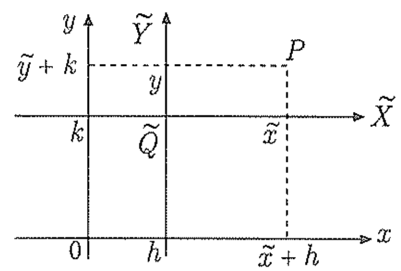


Fig. 2.14

$X = h + x'$ $x' = x - h$
$y = k + y'$ $y' = y - k$

This result can be formulated as follows: *When XY system is translated by* *an amount h in the direction of X-axis and by an amount k in the direction of Y-axis, then the coordinates of an arbitrary point in the new system are obtained by subtracting h from the X-coordinate and k from the Y-coordinate.*

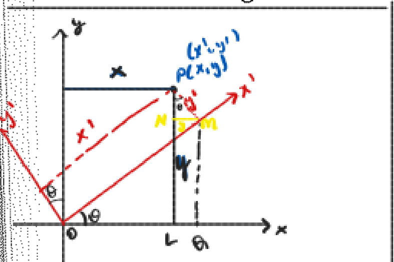
Example 2.6.1 Let $\tilde{X}\tilde{Y}$-coordinate system be the translation of XY-coordinate system such that the origin $\tilde{O}$ of $\tilde{X}\tilde{Y}$-coordinate system has XY-coordinates $(3, 2)$. Let ℓ be the line defined by the equation $2x - y - 4 = 0$ in XY-coordinates. Describe ℓ in $\tilde{X}\tilde{Y}$-coordinates.

Solution. Since $x = \tilde{x} + 3$ and $y = \tilde{y} + 2$, we have

$$2x - y - 4 = 0 \Longleftrightarrow 2(\tilde{x} + 3) - (\tilde{y} + 2) - 4 = 0 \Longleftrightarrow 2\tilde{x} - \tilde{y} = 0 \Longleftrightarrow \tilde{y} = 2\tilde{x}.$$

Hence $\tilde{y} = 2\tilde{x}$ is the equation of ℓ in $\tilde{X}\tilde{Y}$-coordinates.

Now, let $\overline{XY}$-coordinate system be obtained from XY-coordinate system by turning both axes in the same direction and through the same angle, without changing the origin and the unit length of the axes (See Fig.2.15). The position of the $\overline{X}$- and $\overline{Y}$-axes relative to the XY system is determined by giving the angle of rotation which brings the X-and Y-axes into coincidence with $\overline{X}$- and $\overline{Y}$-axes, respectively. This angle will be denoted by α; the positive direction of rotation will be defined as in section 4. Then we say that $\overline{XY}$ system is obtained from XY system by a rotation through α.



OL = x , PL = y

Om = x' , Pm = y'

OL = OQ − QL

= Omcosθ − Pmsinθ

x = x'cosθ − y'sinθ

PL = PN + NL

= PN + mQ

= Pmcosθ + Omsinθ

= y'cosθ + x'sinθ

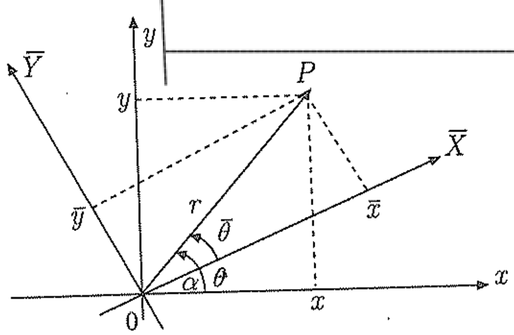


Fig. 2.15.

Let P be an arbitrary point in the plane with XY-coordinates (x, y) and $\overline{XY}$-coordinates $(\overline{x}, \overline{y})$. Corresponding to these two Cartesian coordinate systems in the plane we may consider two polar coordinate systems: the X-axis and the $\overline{X}$-axis. Let

x = x'cosθ − y'sinθ x(−sinθ) ⇒ −xsinθ = −x'cosθsinθ + y'sin²θ

y = x'sinθ + y'cosθ x(cosθ) ⇒ ycosθ = x'cosθsinθ + y'cos²θ

−xsinθ + ycosθ = y'(cos²θ + sin²θ)

y' = −xsinθ + ycosθ

x = x'cosθ − y'sinθ x cosθ ⇒ xcosθ = x'cos²θ − y'sinθcosθ

y = x'sinθ + y'cosθ x sinθ ⇒ ysinθ = x'sin²θ + y'cosθsinθ

xcosθ + ysinθ = x'(cos²θ + sin²θ)

x' = xcosθ + ysinθ

(r, θ) be the polar coordinates of P when X-axis is taken as the polar axis, and let $(r, \overline{\theta})$ be the polar coordinates of the same point when $\overline{X}$-axis is taken as the polar axis. Note that $r = | OP |$ in both case, and we have $\theta = \overline{\theta} + \alpha$. Cartesian coordinates and polar coordinates are related by

$$\boxed{\begin{array}{ll} x = r\cos\theta & \overline{x} = r\cos\overline{\theta} \\ y = r\sin\theta & \overline{y} = r\sin\overline{\theta}. \end{array}}$$

Hence

$$\begin{aligned} x &= r\cos\theta = r\cos(\overline{\theta} + \alpha) = r(\cos\overline{\theta}\cos\alpha - \sin\overline{\theta}\sin\alpha) = \overline{x}\cos\alpha - \overline{y}\sin\alpha \\ y &= r\sin\theta = r\sin(\overline{\theta} + \alpha) = r(\sin\overline{\theta}\cos\alpha + \cos\overline{\theta}\sin\alpha) = \overline{x}sin\alpha + \overline{y}\cos\alpha. \end{aligned}$$

Thus, XY-coordinates and $\overline{XY}$-coordinates of P are related by

$$\begin{cases} x = \overline{x}\cos\alpha - \overline{y}\sin\alpha \\ y = \overline{x}\sin\alpha + \overline{y}\cos\alpha \end{cases} \tag{2.6.1}$$

We may consider (2.6.1) as a system of linear equations in $\overline{x}$ and $\overline{y}$, and by solving $\overline{x}$ and $\overline{y}$, we can express $\overline{x}$ and $\overline{y}$ in terms of x, y, and α. However such an expression can also be obtained by the following argument: If $\overline{XY}$-system is obtained from XY-system by a rotation through α then XY-system is obtained from $\overline{XY}$-system by a rotation through $(-\alpha)$. Therefore, if we replace α by $(-\alpha)$ and interchange x and $\overline{x}, y$ and $\overline{y}$, respectively, then we obtain

$$\begin{cases} \overline{x} = xcos\alpha + ysin\alpha \\ \overline{y} = -xsin\alpha + ycos\alpha \end{cases} \tag{2.6.2}$$

which are the required equations.

Example 2.6.2 Let $\overline{XY}$-system be obtained from XY-system by a rotation through $\pi/4$, and let

$$C = \{(x, y) : 4x^2 + 4y^2 + 2\sqrt{2x} - 6\sqrt{2}y + 1 = 0\}$$

in XY-coordinates. Describe the set C in $\overline{X}\,\overline{Y}$-coordinates.

Solution. We have

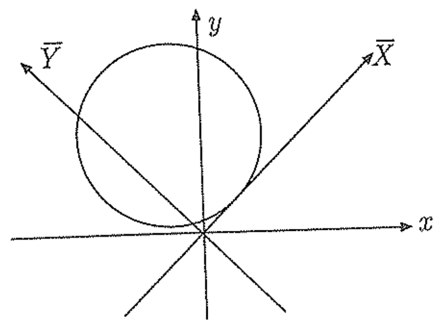
$$\begin{aligned} x &= \overline{x}cos\pi/4 - \overline{y}sin\pi/4 = \frac{\overline{x} - \overline{y}}{\sqrt{2}} \\ y &= \overline{x}sin\pi/4 + \overline{y}cos\pi/4 = \frac{\overline{x} + \overline{y}}{\sqrt{2}}. \end{aligned}$$

	x'	y'
x	cosθ	−sinθ
y	sinθ	cosθ

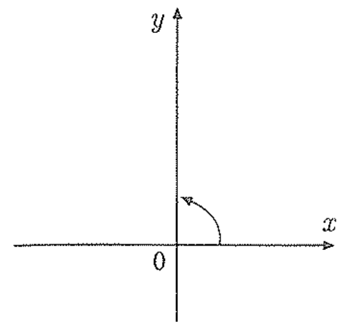
Hence

$$
\begin{aligned}
4x^2+4y^2+2\sqrt{2x}-6\sqrt{2y}+1=0 \quad &\Leftrightarrow \quad 4\left(\frac{\overline{x}-\overline{y}}{\sqrt{2}}\right)^2+4\left(\frac{\overline{x}+\overline{y}}{\sqrt{2}}\right)^2 \\
&\qquad +2\sqrt{2}\left(\frac{\overline{x}-\overline{y}}{\sqrt{2}}\right)-6\sqrt{2}\left(\frac{\overline{x}+\overline{y}}{\sqrt{2}}\right)+1=0 \\
&\Leftrightarrow \quad 2(\overline{x}-\overline{y})^2+2(\overline{x}+\overline{y})^2 \\
&\qquad +2(\overline{x}-\overline{y})-6(\overline{x}+\overline{y})+1=0 \\
&\Leftrightarrow \quad 4\overline{x}^2+4\overline{y}^2-4\overline{x}-8\overline{y}+1=0 \\
&\Leftrightarrow \quad \overline{x}^2+\overline{y}^2-\overline{x}-2\overline{y}+\frac{1}{4}=0 \\
&\Leftrightarrow \quad (\overline{x}-\frac{1}{2})^2+(\overline{y}-1)^2-\frac{1}{4}-1+\frac{1}{4}=0 \\
&\Leftrightarrow \quad (\overline{x}-\frac{1}{2})^2+(\overline{y}-1)^2=1.
\end{aligned}
$$

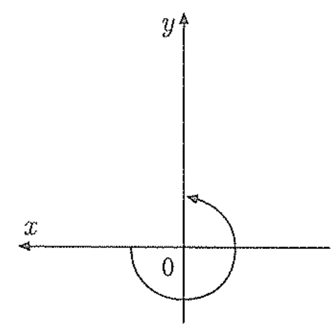
Therefore $C=\{(\overline{x},\overline{y}):(\overline{x}-\frac{1}{2})^2+(\overline{y}-1)^2=1\}$ in $\overline{XY}$-coordinates.



Remark. In all of our discussions, so far, we have chosen the position of the coordinate axes in such a way that the positive Y-axis can be obtained from the positive X-axis by rotating the X-axis (about O) through $\frac{\pi}{2}$ in the counterclockwise direction. Such coordinate systems are called *right-handed*. Sometimes, however, use is made of a system whose axes are positioned in a different manner. Namely, the axes may be positioned in such a way that the positive Y-axis is obtained from the positive X-axis by rotating the X-axis throug $\frac{3\pi}{2}$ in the counterclockwise direction. Such coordinate systems are called *left-handed*.



(right-handed)



(left-handed)

Consider two cartesian coordinate systems in the plane. If they are both right-handed, or both left-handed, then these systems can be obtained from each other by means of a translation followed by a rotation through a suitable angle. Hence, if $X'Y'$-coordinate system is obtained from the XY-system by means of, first, a translation of the origin O to O' with XY-coordinates (h,k) and then a rotation through α (of the obtained system (See Fig.2.16)), then $X'Y'$- coordinates (x',y') and XY-coordinates (x,y) of an arbitrary point are related by

$$
\begin{cases} x=x'\cos\alpha-y'\sin\alpha+h \\ y=x'\sin\alpha+y'\cos\alpha+k \end{cases} \tag{2.6.3}
$$

$$
\begin{cases} x'=(x-h)\cos\alpha-(y-k)\sin\alpha \\ y'=-(x-h)\sin\alpha+(y-k)\cos\alpha \end{cases} \tag{2.6.4}
$$

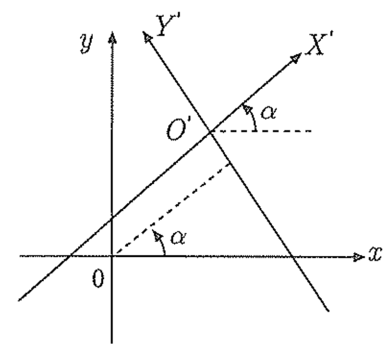


Fig. 2.16

One can obtain similar relations among the coordinates of a point in a right-handed system and its coordinates in a left-handed system (See Exercise 6, 7).

Exercises

1. Solve equation (2.6.1) to obtain (2.6.2).

2. Give a proof of (2.6.4) by using (2.6.2).

3. Consider the line ℓ with equation $2y - x = 3$. a) tranlate the XY-coordinates into a suitable $\overline{XY}$-coordinates such that ℓ passes through the origin in the new system.
b) rotate the $\overline{XY}$-coordinate system through a suitable angle (to obtain $\overline{\overline{XY}}$-coordinate system such that ℓ is a horizontal line in $\overline{\overline{XY}}$ -coordinates.
c) Find the distance from $P(1,4)$ to ℓ.
Draw a figure.

4. Use the idea in the previous exercise to find the distance between the given point and the given line:
a) $(2,-6)$ and $4x - 3y + 4 = 0$, b) $(2,5)$ and $3x + y - 6 = 0$,
c) $(3,4)$ and $x + y + 1 = 0$, d)$(-3,-4)$ and $x + y - 1 = 0$.

5. Find the distance between the lines $x + y - 1 = 0$ and $3x + 3y - 2 = 0$.

6. Let XY-coordinate system be right handed and $X'Y'$ system be left handed and assume that the two systems have a common origin, Y- and Y'- axis have the same direction, but X- and X'-axis have opposite direction. Show that XY-coordinates (x,y) and $X'Y'$-coordinates (x', y') of an arbitrary point are related by $x = -x'$, $y = y'$.

7. Find relations, similar to (2.6.3) and (2.6.4) among the coordinates of a point in a right-handed system and those in a left-handed system.

2.7 Cartesian Coordinates in 3-space

In this section, we show how to establish a one-to-one correspondence between points in 3-space and ordered triples of real numbers,i.e., elements of $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Let us consider three number axes with common origin O, unit points U, U', U'', respectively, and that $|OU| = |OU'| = |OU''|$. Let us further assume that these three number axes are mutually perpendicular at O (See Fig.2.17). By convention, we call these axes X -,Y-, and Z -axes, and choose the orientation for these axes as in Fig.2.18. We assume that the positive X-axis extends directly out from the page. This is just a convention, other orientations are also commonly used.

Let us assume that we have chosen the X-Y- and Z- axes as above. The plane determined by the X-axis and the Y-axis is called the *XY-plane.* The plane determined by the X-axis and the Z-axis is called the XZ-plane. YZ-plane is defined similarly.

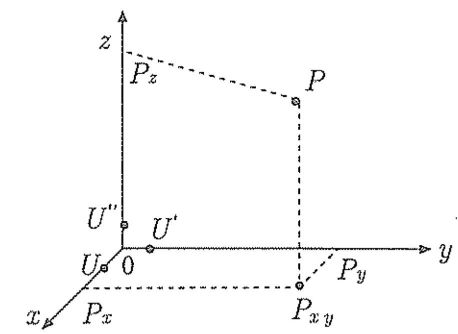


Fig. 2.17

Given a point P in 3-space, the ordered triple of real numbers assigned to P is determined as follows: We drop a perpendicular from P to XY-plane so that we obtain a unique point P_{xy} on XY-plane. (We have $P = P_{xy}$ if P is already on XY-plane). The projection P_x of P_{xy} on X-axis represents a real number - say x. Similarly, the projection P_y of P_{xy} on Y-axis represents a real number - say y. Finally, drop a perpendicular from P to Z-axis; the projection P_z represents a real number - say z. We assign the ordered triple (x,y,z) to P. The numbers x, y, and z are called, respectively, the X-, Y-, *and Z-coordinates* of the point P.

By reversing the above process, given a triple (x,y,z) of real numbers we can find a point whose X-coordinate is x, Y- coordinate is y, and Z-coordinate is z.

It is clear that the correspondence thus established is a one-to-one correspondence. When such a one-to-one correspondence has been established between the points of 3-space and the triples of real numbers, we say that a Cartesian coordinate system in 3-space has been determined. The X-, Y-, and Z-coordinates of a point are called its Cartesian coordinates.

From now on, if a point P has Cartesian coordinates (x,y,z) we will write $P(x,y,z)$. Thus, in Fig.2.17, $P_{xy}(x,y,0)$, $P_x(x,0,0)$, $P_y(0,y,0)$, $P_z(0,0,z)$, $O(0,0,0)$, $U(1,0,0)$, $U'(0,1,0)$, and $U''(0,0,1)$.

As before, the distance between two points P and Q in 3-space is denoted by $|PQ|$.

Once a Cartesian coordinate system has been set up in 3-space, relationship among points can be expressed algebraically in terms of their coordinates.

Theorem 2.7.1 *If $P(x, y, z)$ and $Q(a, b, c)$ are two points in 3-space then*

$$|PQ| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

Proof. We refer to Fig.2.17. By Pythagorian Theorem,

$$|PQ| = \sqrt{|PA|^2 + |QA|^2}.$$

On the other hand

$$\begin{aligned} |PA| &= |P_{xy}Q_{xy}| = \sqrt{(x-a)^2 + (y-b)^2} \\ |QA| &= |z-c| = \sqrt{(z-c)^2}. \end{aligned}$$

Hence

$$|PQ| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

□

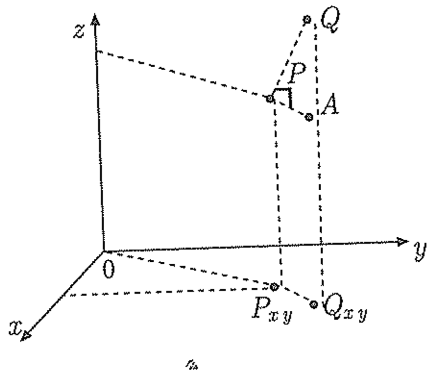


Fig. 2.18

Another use of Cartesian coordinate system in 3-space is that relations from $\mathbb{R}^2$ to $\mathbb{R}$ can be represented as sets of points in 3-space. For, a relation from $\mathbb{R}^2$ to $\mathbb{R}$ is nothing but a subset of $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and Cartesian coordinate system establishes a one-to-one correspondence between $\mathbb{R}^3$ and the set of points in 3-space. The set of points in space wich corresponds to a relation R from $\mathbb{R}^2$ to $\mathbb{R}$ is called the *graph of R*.

Example 2.7.2 The graph of the relation $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ consists of the points each of which is at one unit distance from the origin O. Hence it is the sphere of radius one centered at O.

Example 2.7.3 The graph of $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ consists of the points on or inside the unit cube.

Exercises

1. Describe each of the following sets in space
   a) $\{(x, y, z) : z = 0\}$ b) $\{(x, y, z) : y = 0\}$ c) $\{(x, y, z) : x = 0\}$
   d) $\{(x, y, z) : x = 2\}$ e) $\{(x, y, z) : y = 2\}$ f) $\{(x, y, z) : x = y = 1\}$
   g) $\{(x, y, z) : x = 1, y = 2, z = 3\}$ h) $\{(x, y, z) : x = y\}$.
2. Given the relation $\phi = \{(x, y, z) : x + y + z = 1\}$,
   a) Sketch the intersection of ϕ with XY-plane
   b) Sketch the intersection of ϕ with XZ-plane
   c) Sketch the intersection of ϕ with YZ-plane
   d) Describe ϕ.
3. Same question for $\delta = \{(x, y, z) : x^2 + y^2 = 1\}$.
4. Find $|PQ|$ if a) $P(-1, 0, 4), Q(3, 0, -1)$, b) $P(-2, -1, 3), Q(3, 4, 2)$
5. Describe a method for determining whether three points in space are collinear. (Hint: Use distance formula).

Vectors in the Plane

If $\vec{v}$ is a vector whose initial point is (0,0) and terminal point is (v_1, v_2), then
$\vec{v} = \langle v_1, v_2 \rangle$ is the components form of $\vec{v}$

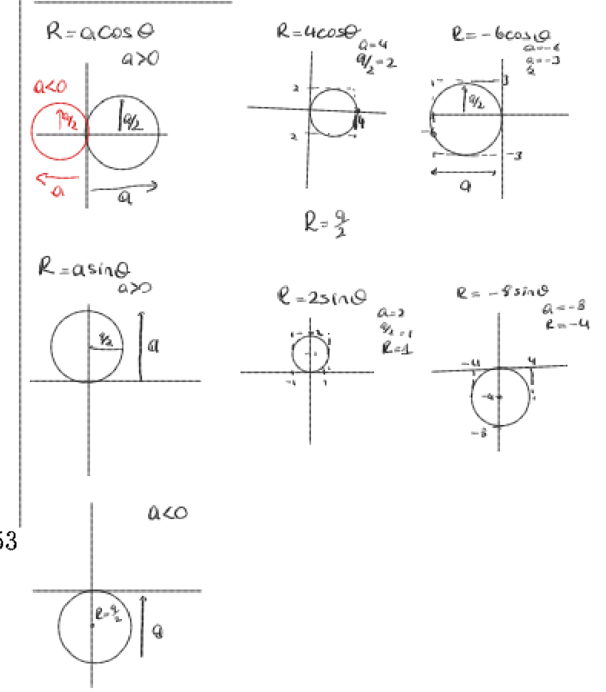
Since the initial point is (0,0), we say the vector is in standard position.

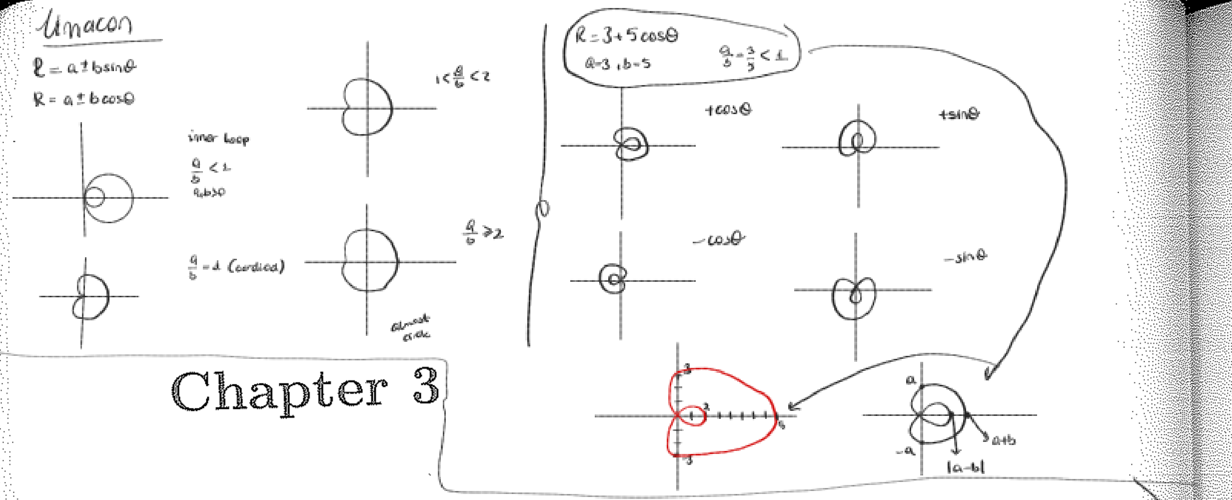
NOTE: The vector with initial point and terminal point (0,0) is called the zero vector. $\vec{0} = \langle 0, 0 \rangle$

$\vec{v}$ = terminal - initial;
$\vec{v} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle$

The length/magnitude of a vector;
$|\vec{v}| = \sqrt{v_1^2 + v_2^2}$
If $|\vec{v}| = 1$, the $\vec{v}$ is a unit vector.

Graph Polar Equations





Chapter 3

VECTORS IN THE PLANE

In physics, a vector quantity is described as a quantity that has both magnitude (length) and direction. Quantities such as velocity and force are represented by vectors. Geometrically, a vector is represented as an arrow in 3-space. The length of the arrow is the length of the vector and the head of the arrow indicates the direction of the vector. Two arrows which have the same length and the same direction are regarded to represent the same vector.

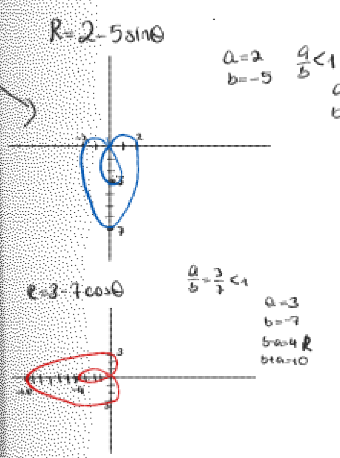
In this chapter, we study vectors which lie in a plane and use them to study plane geometry. Vectors in 3-space will be treated in Chapter Five.

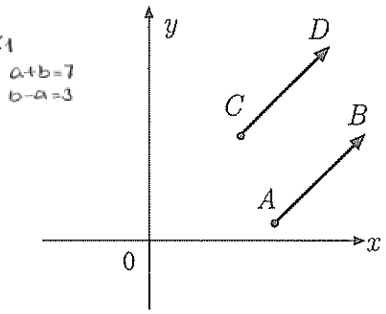
3.1 Directed Segments and Vectors

We assume that a Cartesian coordinate system has been chosen in the plane. An ordered pair (A, B) of points in the plane is called a *directed segment* from A to B. The point A is called the *initial point* and B is called the *terminal point* of the directed segment. The directed segment from A to B is denoted by $\overrightarrow{AB}$, and it is visualized as an arrow joining A to B and directed from A towards B.

Given two directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ with $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$, $D(d_1, d_2)$; we say that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are *equivalent* and write $\overrightarrow{AB} \cong \overrightarrow{CD}$ if $b_1 - a_1 = d_1 - c_1$ and $b_2 - a_2 = d_2 - c_2$. Thus, geometrically, $\overrightarrow{AB} \cong \overrightarrow{CD}$ if we can move $\overrightarrow{AB}$ parallel to itself or along the line joining A and B in such a way that A coincides with C and B coincides with D. In other words, $\overrightarrow{AB} \cong \overrightarrow{CD}$ if they have the same length and the same direction (See Fig. 3.1).

We have thus defined a relation in the set of all directed segments in the plane. This relation is reflexive and symmetric (Exercise 2). The relation ($\cong$) is also transitive. To see this, consider the points $A(a_1, a_2), B(b_1, b_2), C(c_1, c_2), D(d_1, d_2), E(e_1, e_2), F(f_1, f_2)$, in the plane and assume that $\overrightarrow{AB} \cong \overrightarrow{CD}$ and $\overrightarrow{CD} \cong \overrightarrow{EF}$. Then $b_1 - a_1 = d_1 - c_1 = f_1 - e_1$ and $b_2 - a_2 = d_2 - c_2 = f_2 - e_2$ which proves $\overrightarrow{AB} \cong \overrightarrow{EF}$.





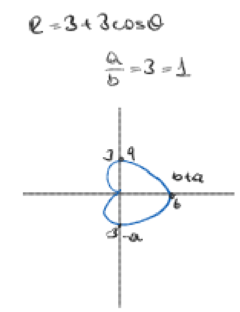


Fig. 3.1

It follows that $\cong$ is an equivalence relation in the set of all directed segments in the plane. An equivalence class of directed segments in the plane with respect to the relation $\cong$ is called a vector in the plane.

Example 3.1.1 Let $A(a_1, a_2)$ be an arbitrary but a fixed point in the plane, and consider the directed segment $\overrightarrow{AA}$. Then, for a directed segment $\overrightarrow{CD}$ with $C(c_1, c_2)$, we have

$$\overrightarrow{AA} \cong \overrightarrow{CD} \iff d_1 - c_1 = a_1 - a_1 = 0 \quad \text{and} \quad d_2 - c_2 = a_2 - a_2 = 0 \iff C = D.$$

Thus the vector (equivalence class) represented by $\overrightarrow{AA}$ consists of all directed segments of the form $\overrightarrow{CC}$. We call this vector the *zero vector* and denote it by $\vec{0}$.

Example 3.1.2 Let $A(2, -1), B(-3, 0)$ and $C(1, 1)$. Find a point D such that the directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector.

Solution. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector if, and only if, $\overrightarrow{AB} \cong \overrightarrow{CD}$ (See Lemma 1.2.6). Let $D(x, y)$. Thus $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector if and only

Ex. Find the dot product $\vec{u}\cdot\vec{v}$ given $|\vec{u}|=6$, $|\vec{v}|=3$, angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{6}$.

$\cos\frac{\pi}{6} = \frac{u\cdot v}{|u||v|} = \frac{\vec{u}\cdot\vec{v}}{18} = \frac{\sqrt{3}}{2} = 9\sqrt{3} = \vec{u}\cdot\vec{v}$

if,

$$-3-2 = x-1 \quad \text{and} \quad 0-(-1) = y-1 \quad \text{or} \quad x=-4 \quad \text{and} \quad y=2$$

and $D(-4,2)$ is the required point.

Let $A(a_1,a_2)$ and $B(b_1,b_2)$ be two points in the plane and consider the point $P(b_1-a_1, b_2-a_2)$. It is easily seen that $\overrightarrow{AB} \cong \overrightarrow{OP}$ where $O(0,0)$ is the origin of the coordinate system. Furthermore, P is the only point having this property (See Fig. 3.2). We express this result in

Ex: Find the magnitude of the vector $\vec{V} = \underbrace{2i+3j}_{\langle 2,3\rangle}$

$|\vec{V}| = \sqrt{a^2+b^2}$, where $\vec{V} = \langle a,b\rangle$

$|\vec{V}| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

Ex: Find the vector $\vec{V}$ with the given magnitude and same direction as $\vec{u}$.

$|\vec{V}| = 4$, $\vec{u} = \langle 0,3\rangle$

First, we will find a unit vector in the same direction as $\vec{u}$.

Then, multiply our vector by 4

$\frac{\vec{u}}{|\vec{u}|} = \frac{1}{|\vec{u}|}\cdot\vec{u} = \frac{1}{\sqrt{9}}\cdot\langle 0,3\rangle = \frac{1}{3}\cdot\langle 0,3\rangle$

$= \langle 0,1\rangle$

Finally, $4\cdot\langle 0,1\rangle = \langle 0,4\rangle$

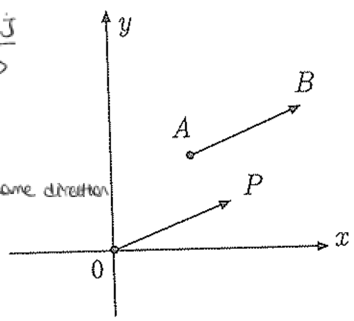


Fig. 3.2

Ex. Find the angle θ between the vectors.

$\vec{u} = \langle 3,1\rangle$, $\vec{V} = \langle 2,-1\rangle$

$\cos\theta = \frac{\vec{u}\cdot\vec{V}}{|\vec{u}||\vec{V}|} = \frac{5}{\sqrt{10}\cdot\sqrt{5}} = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\vec{u}\cdot\vec{V} = 6 + 1\cdot(-1)$

$\vec{u}\cdot\vec{V} = 6-1 = 5$

$|\vec{u}| = \sqrt{9+1} = \sqrt{10}$

$|\vec{V}| = \sqrt{4+1} = \sqrt{5}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\arccos\frac{\sqrt{2}}{2} = \theta$

$\theta = \frac{\pi}{4}$

Ex: Find the dot product $\vec{u} = (3,6)$, $\vec{V} = (-2,4)$

$\vec{u}\cdot\vec{V} = 3\cdot(-2) + 6\cdot 4$

$\vec{u}\cdot\vec{V} = -6 + 24 = 18$

Theorem 3.1.3 *Every vector in the plane has a unique representative of the form $\overrightarrow{OP}$ where O is the origin of the coordinate system.*

Because of the uniqueness of the representative $\overrightarrow{OP}$ for any vector, we identify the directed segment $\overrightarrow{OP}$ with the vector that it represents. Let us also notice that there is a one-to-one correspondence between vectors $\overrightarrow{OP}$ and points P in the plane. For this reason, for any point $P(x,y)$ we write $\overrightarrow{OP} = \vec{P} = (x,y)$ and call x and y, respectively, X-component and Y-component of the vector (represented by) $\overrightarrow{OP}$.

Given the points $A(a_1,a_2)$ and $B(b_1,b_2)$ in the plane, the vector represented by the directed segment $\overrightarrow{AB}$ has X-component b_1-a_1 and Y-component b_2-a_2.

We will denote vectors by symbols like $\vec{u}, \vec{v}, \vec{w}, \vec{M}, \vec{N}, \ldots$, etc.

For example, if $A(2,-1)$ and $B(-1,3)$ then the vector $\vec{u}$ represented by $\overrightarrow{AB}$ is $\vec{u} = (-3,4)$.

It is clear that for two vectors $\vec{u} = (x,y)$ and $\vec{v} = (a,b)$,

$$\vec{u} = \vec{v} \iff x = a \quad \text{and} \quad y = b. \tag{3.1.1}$$

In words, *two vectors are equal if their corresponding components are equal.*

Given a vector $\vec{u} = (x,y)$, the *length* (or the *magnitude*), $|\vec{u}|$, of $\vec{u}$ is defined as

$$|\vec{u}| = \sqrt{x^2+y^2}.$$

It follows that the length of the vector $\vec{u}$ represented by the directed segment $\overrightarrow{AB}$, where $A(a_1,a_2)$ and $B(b_1,b_2)$, is equal to

$$|\vec{u}| = |AB| = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2},$$

the distance between A and B. Note that if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector then $|AB| = |CD|$. Hence we can define the length of a vector to be *the distance between the initial point and the terminal point of any directed segment representing that vector.*

Remark. By definition, $|\vec{v}| \geq 0$ for any $\vec{v}$. Furthermore, $|\vec{v}| = 0 \iff \vec{v} = \vec{0}$. For, $x^2+y^2 = 0 \iff x = y = 0$.

Exercises

1. Determine the unknown cooridnate in each case if $\overrightarrow{AB} \cong \overrightarrow{CD}$.
   a) $A(3,5)$, $B(4,6)$, $C(-2,5)$, $D(\overset{-1}{x},\overset{6}{y})$.
   b) $A(-1,1)$, $B(3,5)$, $C(x,y)$, $D(2x,1)$.
   c) $A(3,-\frac{1}{3})$, $B(2,8)$, $C(3,y)$, $D(x,-10)$.

2. Prove that the relation $\cong$ is reflexive and symmetric in the set of all directed segments.

3. In each case below, determine the vector for which $\overrightarrow{AB}$ is a representative, i.e., find P such that $\overrightarrow{AB} \cong \overrightarrow{OP}$.
   a)$A(2,4), B(3,5)$ b)$A(-1,-3), B(2,4)$.
   c)$A(x,y), B(x+2,y-1)$, d)$A(a,b), B(0,0)$.

4. Prove that if $\overrightarrow{AB} \cong \overrightarrow{CD}$ then $\overrightarrow{BA} \cong \overrightarrow{DC}$.

5. In each case below, determine the unknown coordinates so that $\overrightarrow{AB}$ is a representative for the vector $\vec{u}$.
   a) $\vec{u} = (-4,1)$, $A(2,6)$, $B(x,y)$.
   b) $\vec{u} = (1,1)$, $A(x,y)$, $B(3,5)$.
   c) $\vec{u} = (0,2)$, $A(x,0)$, $B(2x+1,y)$.

6. In each case below, find the X- and Y-components of the vector represented by the directed segment $\overrightarrow{AB}$.
   a) $A(-1,5)$, $B(2,7)$, $b)A(1,1)$, $B(-2,5)$.
   c) $A(2,1)$, $B(\pi,p)$, $d)A(0,-1)$, $B(1,0)$.

7. Compute the length of each vector in Exercise 5 and Exercise 6.

3.2 Algebra of Vectors

In this section we define addition of vectors and multiplication of a vector by a scalar. Here a *scalar* means just a real number.

Given two vectors $\vec{u} = (x,y)$ and $\vec{v} = (a,b)$, we define the *sum*, $\vec{u}+\vec{v}$, by

$$\vec{u}+\vec{v} = (x+a, v+b).$$

Thus the sum of two vectors is again a vector, each of whose components is the sum of the corresponding components of the summands.

Example 3.2.1 If $\vec{u} = (-1,3)$ and $\vec{v} = (2,-1)$ then $\vec{u}+\vec{v} = (-1+2, 3+(-1)) = (1,2)$.

One can see, geometrically or algebraically, that the sum, $\vec{u}+\vec{v}$, of two vectors can be represented as the diagonal of a parallelogram, two adjacent sides of which represent the vectors $\vec{u}$ and $\vec{v}$. In fact (See Fig. 3.3), the slope of the line joining $O(0,0)$ to $Q(a,b)$ is $m = \frac{b}{a}$ and the slope of the line joining P to R is $\frac{(y+b)-y}{(x+a)-x} = \frac{b}{a}$. Thus $OQ//PR$. Similarly, $OP//QR$, proving that $OQRP$ is a parallelogram.

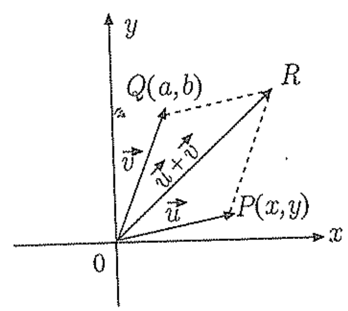


Fig. 3.3

The *multiplication* of a vector $\vec{u} = (x,y)$ *by a scalar* c is defined by

$$c\vec{u} = (cx, cy).$$

Thus $c\vec{u}$ is the vector obtained from $\vec{u} = (x,y)$ by multiplying each component of $\vec{u}$ by the scalar c. Note that

$$|\, c\vec{u} \,| = \sqrt{(cx)^2 + (cy)^2} = |\, c \,|\ |\, \vec{u} \,|$$

where $|\, c \,|$ denotes the absolute value of c. Hence, geometrically, if $c > 0$ then $c\vec{u}$ is a vector in the same direction as $\vec{u}$ and the length of $c\vec{u}$ is $|\, c \,|$ times the length of $\vec{u}$, if $c < 0$ then $c\vec{u}$ and $\vec{u}$ have opposite directions and the length of $c\vec{u}$ is again $|\, c \,|$ times the length of $\vec{u}$ (See Fig. 3.4). What can you say if $c = 0$?

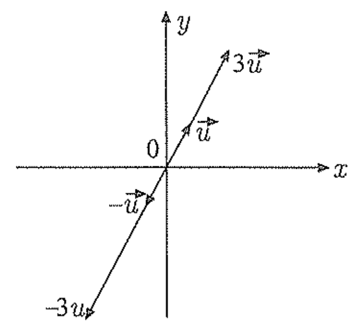
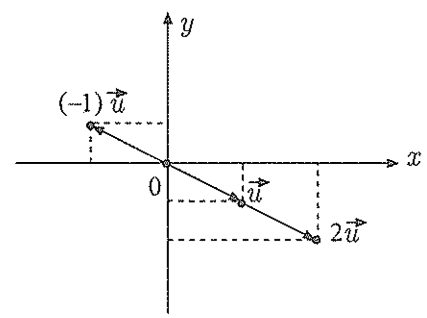


Fig. 3.4

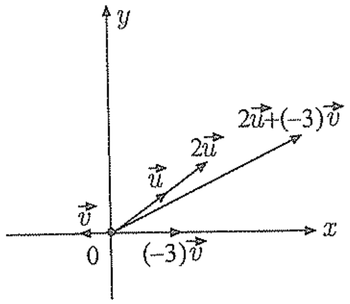
The above observations suggest the following definitions. Two vectors $\vec{u}$ and $\vec{v}$ are like-directed if $\vec{v} = cu$ for some scalar $c > 0$. We say that $\vec{u}$ and $\vec{c}$ are opposite-directed if $\vec{v} = c\vec{u}$ for some $c < 0$. The vectors $\vec{u}$ and and $\vec{v}$ are said to be *parallel* if they are like - or opposite - directed, i.e., if $\vec{v} = c\vec{u}$ for some $c \neq 0$.

Example 3.2.2 Let $\vec{u} = (2,1)$. Then $2\vec{u} = (4,-2)$ and $(-1)\vec{u} = (-2,1)$.



Example 3.2.3 Given $\vec{u} = (2, 1)$, $v = (-1, 0)$; find $2\vec{u} + (-3)\vec{v}$.

Solution. $2\vec{u} + (-3)\vec{v} = (4, 2) + (3, 0) = (4 + 3,\ 2 + 0) = (7, 2)$.



Example 3.2.4 Let $\vec{u} = (2, -1)$ and let $\vec{v}$ be the vector represented by the directed segment $\vec{AB}$ where $A(-1, 3)$, $B(2, 5)$. Find a point D such that $\overrightarrow{AD}$ represents $\vec{u} + \vec{v}$.

Solution. Since $\vec{v}$ is represented by $\overrightarrow{AB}, \vec{v} = (2 - (-1), 5 - 3) = (3, 2)$. $\vec{u} + \vec{v} = (2, -1) + (3, 2) = (5, 1)$. Let $D(x, y)$. Then $\overrightarrow{AD}$ represents $\vec{u} + \vec{v}$ if

$$x - (-1) = 5 \quad \text{and} \quad y - 3 = 1 \quad \text{or} \quad x = 4 \quad \text{and} \quad y = 4.$$

Hence $D(4, 4)$ is the required point.

For vectors $\vec{u} = (x, y)$ and $\vec{v} = (a, b)$, we write $(-1)\vec{v} = -\vec{v}$ and $\vec{u} + (-1)\vec{v} = \vec{u} - \vec{v}$. Hence $-\vec{v} = (-a, -b)$ and $\vec{u} - \vec{v} = (x - a, y - b)$. Thus if $\vec{u}$ is the vector represented by the directed segment $\overrightarrow{AB}$ with $A(a_1, a_2), B(b_1, b_2)$ then $\vec{u} = \vec{B} - \vec{A} = (b_1 - a_1, b_2 - a_2)$. For a geometric interpretation, see Fig. 3.5. below.

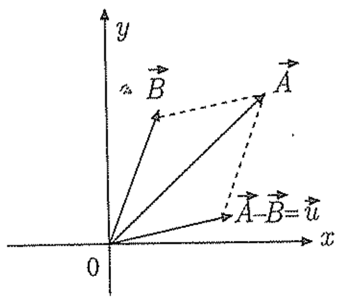


Fig. 3.5.

Operations on vectors satisfy the following properties.

Theorem 3.2.5 *Let $\vec{u}, \vec{v}, \vec{w}$ be vectors (in the plane), $\vec{O} = (0, 0)$, the zero vector, and c, d scalars. Then*

(a) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

(b) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(c) $\vec{0} + \vec{u} = \vec{u}$ *and* $\vec{0}$ *is the only vector with this property.*

(d) $\vec{u} + (-\vec{u}) = \vec{0}$, *and for any* $\vec{u}$, $(-\vec{u})$ *is the only vector with this property.*

(e) $c(d\vec{u}) = (cd)\vec{u}$

(f) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$

(g) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

(h) $1.\vec{u} = \vec{u}$.

Proof. Each of these rules can be proved by using the definitions about vectors and properties of real numbers. For example, part (b) is proved as follows. Let $\vec{u} = (x, y), \vec{v} = (a, b)$. Then

$$\begin{aligned} \vec{u} + \vec{v} &= (x + a, y + b) \ (definition) \\ &= (a - x, b + y) \ (property\ of \mathbb{R}) \\ &= \vec{v} + \vec{u} \ (definition) \end{aligned}$$

We leave the rest of the proof to the reader (Exercise 3). □

By part (a), we can write $\vec{u} + \vec{v} + \vec{w} = (x + a + c, y + b + d)$ for three vectors $\vec{u} = (x, y)$, $\vec{v} = (a, b)$ and $\vec{w} = (c, d)$.

Example 3.2.6 Given $\vec{u} = (1, -1), \vec{v} = (0, 2), \vec{w} = (-2, 3)$. Compute $3\vec{u} - \vec{v} + 1\vec{w}$.

Solution.

$$\begin{aligned} 3\vec{u} - \vec{v} + 2\vec{w} &= (3, -3) - (0, 2) + (-4, 6) \\ &= = (3 + 0 - 4, -3 - 2 + 6) = (-1, 1). \end{aligned}$$

Given n vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ and n scalars $c_1, c_2, ..., c_n$, the expression

$$c_1\vec{u_1} + c_2\vec{u_2} + ... + c_n\vec{u_n}$$

is called a *linear combination* of the vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$.

Example 3.2.7 Let $\vec{i} = (1,0), \vec{j} = (0,1)$. Let $\vec{u} = (x,y)$ be an arbitrary vector in the plane. Then we have

$$\vec{u} = (x,y) = (x,0) + (0,y) = x(1,0) + y(0,1) = x\vec{i} + y\vec{j}$$

Thus *any vector in the plane is a linear combination of the vectors* $\vec{i}$ *and* $\vec{j}$. The vectors $\vec{i}$ and $\vec{j}$ have the additional properties that $|\vec{u}| = 1$, $|\vec{j}| = 1$ and for any $c_1, c_2 \in \mathbb{R}$, $c_1\vec{i} + c_2\vec{j} = \vec{0} \Longrightarrow c_1 = c_2 = 0$. (The vectors $\vec{i}$ and $\vec{j}$ are called *basic unit vectors* in the plane).

A vector $\vec{u}$ is called a unit vector if $|\vec{u}| = 1$. Thus $\vec{i}$ and $\vec{j}$ are unit vectors. If $\vec{v}$ is any vector with $\vec{v} \neq \vec{0}$ then $\vec{u} = \vec{v}/|\vec{v}|$ is a unit vector, and the vectors $\vec{v}$ and $\vec{v}/|\vec{v}|$ are like-directed.

Let $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ be n vectors in the plane. We say that $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ are *linearly dependent* if there exist scalars $c_1, c_2, ..., c_n \in \mathbb{R}$, not all zero, such that $c_1\vec{u_1} + c_2\vec{u_2} + ... + c_n\vec{u_n} = \vec{0}$. If $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ are not linearly dependent then we say that they are *linearly independent*. In other words, $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ are linearly independent if

$$c_1\vec{u}_1 + c_2\vec{u}_2 + ... + c_n\vec{u}_n = \vec{0} \Longrightarrow c_1 = c_2 = ... = c_n = 0.$$

We have observed above that the basic unit vectors $\vec{i}$ and $\vec{j}$ are linearly independent.

Example 3.2.8 Let $\vec{u}_1 = (2,7)$ and $\vec{u}_2 = (-1,3)$. Assume that $c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$. Then

$$\vec{0} = c_1\vec{u}_1 + c_2\vec{u}_2 = (2c_1, 7c_1) + (-c_2, 3c_2) = (2c_1 - c_2, 7c_1 + 3c_2).$$

Since $\vec{0} = (0,0)$, the last identity implies $2c_1 - c_2 = 0$, $7c_1 + 3c_2 = 0$. Hence $2c_1 = c_2$ and $3c_1 = 0$; which yields $c_1 = c_2 = 0$. Therefore $\vec{u}_1$ and $\vec{u}_2$ are linearly independent.

Example 3.2.9 Let $\vec{u}_1 = (-1,3)$ and $\vec{u}_2 = (2,-6)$. Let us see if there exist scalars c_1, c_2 such that $c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$ with $c_1 \neq 0$ or $c_2 \neq 0$. We have

$$c_1\vec{u}_1 + c_2\vec{u}_2 = (-c_1, 3c_1) + (2c_2, -6c_2) = (-c_1 + 2c_2, 3c_1 - 6c_2).$$

Thus $c_1\vec{u}_1 + c_2\vec{u}_2 = 0 \Longrightarrow -c_1 + 2c_2 = 0$ and $3c_1 - 6c_2 = 0 \Longrightarrow c_1 = 2c_2$. It follows instance, that for $c_2 = 1$ and $c_1 = 2, 2\vec{u}_1 + \vec{u}_2 = \vec{0}$. Therefore $\vec{u}_1 = (-1,3)$ and $\vec{u}_2 = (2,-6)$ are linearly dependent.

Remark. If two non-zero vectors $\vec{u}_1$ and $\vec{u}_2$ are linearly dependent, then $c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$ with $c_1 \neq 0$ and $c_2 \neq 0$; and therefore $\vec{u}_1 = -\frac{c_2}{c_1}u_2$. Thus, two non-zero vectors $\vec{u}_1$

and $\vec{u}_2$ are linearly dependent $\Longleftrightarrow$ one is a scalar multiple of the other. Note also that if at least one of $\vec{\alpha}_1, \vec{\alpha}_2, ..., \vec{\alpha}_n$ is the zero vector, then $\vec{\alpha}_1, \vec{\alpha}_2, ..., \vec{\alpha}_n$ are linearly dependent.

Exercises

1. Given the vectors $\vec{u} = (-4,3)$ and $\vec{v} = (2,5)$, compute each of the following vectors and plot them in the Cartesian plane.
   (4,-3) (-4,-10
   a) $-\vec{u}$ b) $-2\vec{v}$ c) $\vec{u} + \vec{v}$ d) $\vec{u} - \vec{v}$
   e) $3\vec{u} + 2\vec{v}$ f) $2\vec{v} - 3\vec{u}$ g) $|\vec{v}|\vec{u} + |\vec{u}|\vec{v}$

2. Let $\vec{u} = (4,-3)$ and let $\vec{v}$ be the vector represented by $\vec{AB}$ where $A(2,-1)$ and $B(-1,3)$.
   a) Find a point D such that $\overrightarrow{AD}$ is a representative for $\vec{u} + \vec{v}$.
   b) Find a point M such that $\overrightarrow{MB}$ is a representative for $\vec{u} - \vec{v}$.
   c) Find a scalar c so that $\vec{u} + c\vec{v} = (1,1)$.
   d) Compute $|2\vec{u} - \vec{v}|$.

3. Prove each of the rules stated in Theorem 3.2.5.

4. Prove that every vector in the plane can be written as a linear combination of the vectors $\vec{u}_1 = (-1,0)$ and $\vec{u}_2 = (0,1)$.

5. Prove that every vector in the plane can be written as a linear combination of the vectors $\vec{v}_1 = (1,2)$ and $\vec{v}_2 = (1,3)$.

6. Show that
   a) $\vec{u}_1 = (1,2)$ and $\vec{u}_2 = (-1,2)$ are linearly independent.
   b) $\vec{u}_1 = (1,2)$ and $\vec{u}_2 = (3,6)$ are linearly dependent.
   c) $\vec{u}_1 = (1,2)$ and $\vec{u}_2 = (1,0), \vec{u}_3 = (0,1)$ are linearly dependent.

7. Prove that any three vectors in the plane are linearly dependent. What can you say about four vectors, five vectors?

8. If $\vec{u} = k\vec{v}, k \in \mathbb{R}$, we say that $\vec{u}$ is a *scalar multiple* of $\vec{v}$. Is the relation"...is a scalar multiple of..." an equivalence relation in the set of all vectors in the plane? Justify your answer.

9. Let R be the relation, in the set of all vectors in the plane, defined by $\vec{u}R\vec{v} \Longleftrightarrow |\vec{u}| = |\vec{v}|$. Is R an equivalence relation? Justify your answer.

10. Let $\vec{u}$ and $\vec{v}$ be two fixed vectors in the plane. Describe the following sets geometrically.
a) $S_1 = \{\vec{u} + k\vec{v} : k \geq 0\}$ b) $S_2 = \{k\vec{u} + (1-k)\vec{v} : 0 \leq k \leq 1\}$
c) $S_3 = \{\vec{u} + k\vec{v} : k \leq 0\}$ d) $S_4 = \{\vec{u} + k\vec{v} : k \in \mathbb{R}\}$
e) $S_5 = \{P = (x,y) : \mid \vec{P} - \vec{u} \mid = 1\}$ f) $S_6 = \{P = (x,y) : 1 < \mid \vec{P} - \vec{u} \mid \leq 2\}$
(Draw figures).

11. Let $\vec{u} = (4,-3)$ and let $\vec{v} = v(\vec{AB})$ where $A = (2,-1)$, $B = (-1,3)$, and $v(\overrightarrow{AB})$ denotes the vector represented by the directed segment $\overrightarrow{AB}$

    a) Find a point D such that $v(\overrightarrow{AD}) = \vec{u} + \vec{v}$.

    b) Find a point M such that $\overrightarrow{MB}$ is a representative for $\vec{u} - \vec{v}$.

    c) Find a scalar c so that $\vec{u} + c\vec{v} = (1,1)$.

12. Suppose that $\vec{u} = v(\overrightarrow{AB})$, $v = v(\overrightarrow{AC})$, D is on AB 2/3 of the way from A to B, and E is the midpoint of AC. Find $v(\overrightarrow{DE})$ in terms of u and v.

13. Let $\vec{u} = v(\overrightarrow{AB}) = v(\overrightarrow{CD})$, $\vec{v} = v(\overrightarrow{AC}) = v(\overrightarrow{BD})$, E is 2/3 of the way from B to D, $v(\overrightarrow{CF}) = \frac{1}{2}v(\overrightarrow{CD})$. Find $v(\overrightarrow{EF})$ in terms of $\vec{u}$ and $\vec{v}$.

14. Suppose that on the sides of $\Delta(ABC)$, $v(\overrightarrow{BD}) = \frac{2}{3}v(\overrightarrow{BC})$, $v(\overrightarrow{CE}) = \frac{2}{3}v(\overrightarrow{CA})$, $v(\overrightarrow{AF}) = (\frac{2}{3})v(\overrightarrow{AB})$.

    a) Show that $v(\overrightarrow{AD}) + v(\overrightarrow{BE}) + v(\overrightarrow{CF}) = 0$

    b) Show that the equation in (a) is true if the fraction 2/3 is replaced by any real number h.

15. Let $\vec{a} = v(\overrightarrow{OA})$, $b = v(\overrightarrow{OB})$, $c = v(\overrightarrow{OE})$. Show that the medians of ΔABC meet at a point P, and express $v(\overrightarrow{OP})$ in terms of a, b, c. Draw a figure.

3.3 Scalar Product, Angle Between Two Vectors

In this section, we define an operation on vectors such that when this operation is applied to a pair of vectors, the result is a scalar.

Let $\vec{u} = (a,b)$, $\vec{v} = (x,y)$ be two vectors in the plane. The *scalar product*, $\vec{u} \circ \vec{v}$, of $\vec{u}$ and $\vec{v}$ is defined by

$$\vec{u} \circ \vec{v} = ax + by.$$

Example 3.3.1 Let $\vec{u} = (-1,-2)$, $\vec{v} = (-4,3)$. Then

$$\vec{u} \circ \vec{v} = (-1)\cdot(-4) + (-2)\cdot 3 = -2.$$

The scalar product has the following properties.

Theorem 3.3.2 *Let $\vec{u} = (a,b)$, $\vec{v} = (x,y)$, $\vec{w} = (s,t)$ be vectors in the plane and let $c \in \mathbb{R}$ be a scalar. Then*
a) $\vec{u} \circ \vec{v} = \vec{v} \circ \vec{u}$ *b)* $(c\vec{u}) \circ \vec{v} = \vec{u} \circ (c\vec{v}) = c(\vec{u} \circ \vec{v})$
c) $\vec{u} \circ \vec{0} = 0$ *d)* $\vec{u} \circ (\vec{v} + \vec{w}) = \vec{u} \circ \vec{v} + \vec{u} \circ \vec{w}$
e) $\vec{u} \circ \vec{u} = \mid \vec{u} \mid^2$ *f)* $\mid \vec{u} \circ \vec{v} \mid \leq \mid \vec{u} \mid\mid \vec{v} \mid$.

Proof. We give the proof of part f) and leave the proof of the rest to the exercises. For the proof of part f), we first observe that the inequality is trivially satisfied if $\mid \vec{u} \mid = 0$. Thus, we may assume that $\mid \vec{u} \mid \neq 0$. We consider the vector $\vec{z} = (k\vec{u} + \vec{v})$ where k is any scalar for the time being. Then $\vec{z} = (ka + x, kb + y)$, and

$$\begin{aligned} 0 \leq \mid \vec{z} \mid^2 &= (ka+x)^2 + (kb+y)^2 = (a^2+b^2)k^2 + 2(ax+by)k + (x^2+y^2) \\ &= \mid u \mid^2 \mid k \mid^2 + 2(\vec{u} \circ \vec{v})k + \mid \vec{v} \mid^2 . \end{aligned}$$

Now, we let

$$k = -\frac{(\vec{u} \circ \vec{v})}{\mid u \mid^2}.$$

Then

$$0 \leq \frac{(\vec{u} \circ \vec{v})^2}{\mid u \mid^2} - 2\frac{\vec{u} \circ \vec{v})^2}{\mid u \mid^2} + \mid \vec{v} \mid^2$$

which reduces to

$$\mid \vec{u} \circ \vec{v} \mid \leq \mid \vec{u} \mid\mid \vec{v} \mid .$$

This completes the proof of (f). □

The inequality in part (f) is known as *Cauchy-Schwartz* inequality.

Corollary 3.3.3 *For any two vectors $\vec{u}$ and $\vec{v}$,*

$$\mid \vec{u} + \vec{v} \mid \leq \mid \vec{u} \mid + \mid \vec{v} \mid . \tag{3.3.1}$$

Proof. We have

$$\begin{aligned} \mid \vec{u} + \vec{v} \mid^2 &= (\vec{u} + \vec{v}) \circ (\vec{u} + \vec{v}) = \mid \vec{u} \mid^2 + \vec{u} \circ \vec{v} + \vec{v} \circ \vec{u} + \mid \vec{v} \mid^2 \\ &= \mid \vec{u} \mid^2 + 2(\vec{u} \circ \vec{v}) + \mid \vec{v} \mid^2 \\ &\leq \mid \vec{u} \mid^2 + 2\mid \vec{u} \mid\mid \vec{v} \mid + \mid \vec{v} \mid^2 \text{ (By Cauchy-Schwarz Inequality).} \end{aligned}$$

Thus $|\vec{u}+\vec{v}|^2 \le (|\vec{u}|+|\vec{v}|)^2$. □

The inequality (3.3.1) is known as the Triangle Inequality. In fact, each of $|\vec{u}|, |\vec{v}|$ and $|\vec{u}+\vec{v}|$ measures the length of one side of a triangle (See Fig.3.6). The inequality (3.3.1) states that *the length of one side of a triangle is less than the sum of lengths of the other two sides.*

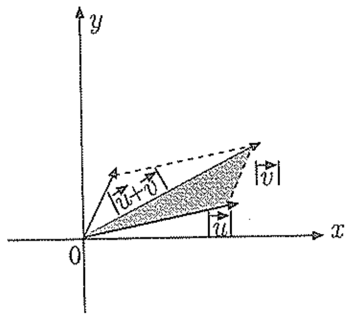


Fig. 3.6.

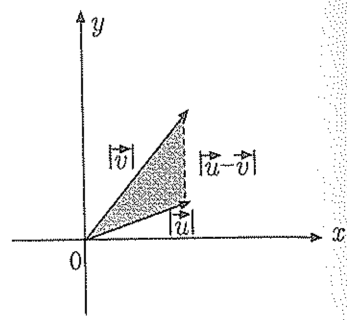


Fig. 3.7.

Recall that there is a similar statement about the difference of lengths of two sides of a triangle: *The difference of lengths of two sides of a triangle is less than the length of the third side.* In vector notation, this amounts to (See Fig. 3.7.)

Corollary 3.3.4 *For any two vectors $\vec{u}$ and $\vec{v}$,*

$$|\vec{u}-\vec{v}| \ge ||\vec{u}|-|\vec{v}|| \qquad (3.3.2)$$

Proof. We have

$$|\vec{u}| = |(\vec{u}-\vec{v})+\vec{v}| \le |\vec{u}-\vec{v}|+|\vec{v}|.$$

Hence $|\vec{u}|-|\vec{v}| \le |\vec{u}-\vec{v}|$. Similarly, $|\vec{v}|-|\vec{u}| \le |\vec{u}-\vec{v}|$, and thus $||\vec{u}|-|\vec{v}|| \le |\vec{u}-\vec{v}|$. □

It is clear intuitively what we mean by the angle between two vectors in the plane. However, we will define this concept mathematically by means of Cauchy-Schwartz inequality. We will also observe that this definition coincides with our intuitive notion.

Let $\vec{u}$ and $\vec{v}$ be two vectors in the plane. If one (or both) of the vectors $\vec{u}$ and $\vec{v}$ is zero, then the *angle between vectors $\vec{u}$ and $\vec{v}$* is defined to be any angle with radian

measure any real number θ. If $\vec{u} \ne \vec{0}$ and $\vec{v} \ne \vec{0}$, then Cauchy-Schwartz inequality implies

$$\frac{|\vec{u}\circ\vec{v}|}{|\vec{u}||\vec{v}|} \le 1,$$

or equivalently,

$$-1 \le \frac{\vec{u}\circ\vec{v}}{|\vec{u}||\vec{v}|} \le 1.$$

Thus, there exists $\theta \in [0,\pi]$ such that

$$cos\theta = \frac{\vec{u}\circ\vec{v}}{|\vec{u}||\vec{v}|}. \qquad (3.3.3)$$

In this case, the angle whose radian measure is θ radians, $0 \le \theta \le \pi$, and satisfies (3.3.3) is called the *angle between $\vec{u}$ and $\vec{v}$.*

To see that this definition coincides with our intuitive notion of the angle between two vectors, locate the vectors $\vec{u}$ and $\vec{v}$ as in Fig.3.8, and apply *Cosine Rule* to the triangle formed by $\vec{u}$ and $\vec{v}$ for the angle θ at O. Thus we get

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\,cos\theta.$$

On the other hand, $|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\circ\vec{v}$.

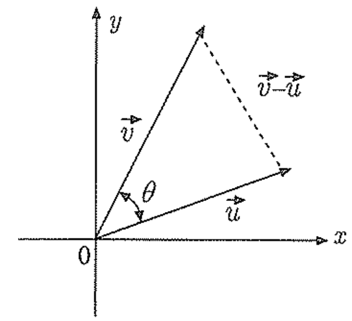


Fig. 3.8

Comparing the two equations, we obtain $\vec{u}\circ\vec{v} = |\vec{u}||\vec{v}|\,cos\theta$ and this gives

$$cos\theta = \frac{\vec{u}\circ\vec{v}}{|\vec{u}||\vec{v}|}.$$

Hence the angle θ at O is the same as the angle given by (3.3.3).

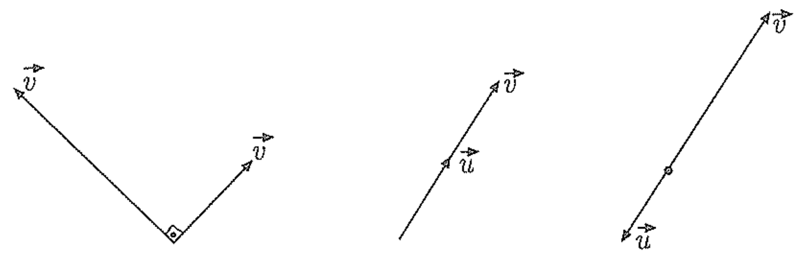
Example 3.3.5 Find the angle θ between the vectors $\vec{u} = (3,1)$ and $\vec{v} = (-1,-2)$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{-3+(-2)}{\sqrt{10}\cdot\sqrt{5}} = \frac{-\cancel{5}}{\cancel{5}\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\theta = \frac{3\pi}{4}$$

Solution. $\vec{u} \circ \vec{v} = 3.(-1) + 1.(-2) = -5$,
$|\vec{u}| = \sqrt{9+1} = \sqrt{10}, |\vec{v}| = \sqrt{1+4} = \sqrt{5}$.
Thus
$$cos\theta = \frac{-5}{\sqrt{10}\sqrt{5}} = \frac{-1}{\sqrt{2}}, \quad 0 \le \theta \le \theta = \frac{3\pi}{4}.$$

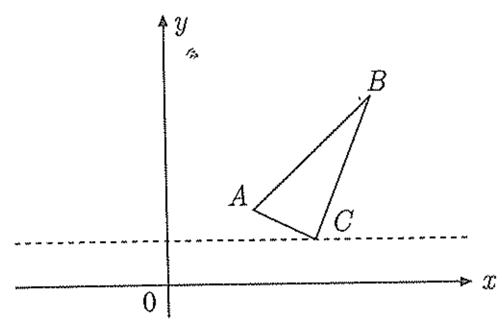
Two vectors $\vec{u}$ and $\vec{v}$ are said to be *perpendicular* if $\vec{u} \circ \vec{v} = 0$. It is clear that two nonzero vectors $\vec{u}$ and $\vec{v}$ are perpendicular if and only if the angle between $\vec{u}$ and $\vec{v}$ is a right angle. Similarly, $\vec{u}$ and $\vec{v}$ are like directed (respectively, opposite-directed) if and only if the angle between $\vec{u}$ and $\vec{v}$ is 0 (respectively π) radians.



Perpendicular ($\theta = \frac{\pi}{2}$) *like – directed* ($\theta = 0$) *opposite – directed* ($\theta = \pi$)

Example 3.3.6 Determine the number x so that the points $A(3,2), B(7,5)$ and $C(x,1)$ form a right triangle with $\vec{AB}$ as its hypothenuse.

Solution. The angle ACB, which is the angle between the vectors represented by $\vec{CA}$ and $\vec{CB}$, must be a right angle. Therefore
$0 = (\vec{A} - \vec{C}) \circ (\vec{B} - \vec{C}) = (3-x, 2-1) \circ (7-x, 5-1) = (3-x)(7-x) + 4 = x^2 - 10x + 21 + 4 = (x-5)^2 \Longrightarrow x = 5$.



Remark. Given a vector $\vec{u} = (a,b)$, the vector $\vec{v} = (-b,a)$ is perpendicular to $\vec{u}$. For, $\vec{u} \circ \vec{v} = -ab + ab = 0$.

Exercises

1. Given the vectors $\vec{u} = (3,-1), \vec{v} = (2,3), \vec{w} = (4,-5)$, compute the following
a) $\vec{u} \circ \vec{v}$, b)$(|\vec{w}|\vec{u}) \circ \vec{v}$, c) $\vec{u} \circ \vec{w}$, d)$|\vec{u} \circ \vec{w}|$
e)$|\vec{u} + \vec{v}|$, f) $|\vec{u}| + |\vec{v}|$, g)$|\vec{v} - \vec{w}|$, h) $|\vec{v}| - |\vec{w}|$.

2. Prove each of the properties in Theorem 3.3.1.

3. Determine the cosine of the angle between the vectors $\vec{u}$ and $\vec{v}$:
a) $\vec{u} = (2,-1), \vec{v} = (-1,3)$ b) $\vec{u} = (1,2), \vec{v} = (-1,-2)$
c) $\vec{u} = (4, 2\sqrt{3}), \vec{v} = (1,4)$
d) $\vec{u} = (2,-1), \vec{v} = (1,-3)$.

4. Determine the radian measure of the angle between $\vec{u}$ and $\vec{v}$:
a) $\vec{u} = (2, 2\sqrt{3}), \vec{v} = (-3\sqrt{3}, -3)$ b) $\vec{u} = (4, 2\sqrt{3}), \vec{v} = (2,4)$
c) $\vec{u} = (4, 2\sqrt{3}), \vec{v} = (-2,4)$
d) $\vec{u} = (-4, 2\sqrt{3}), \vec{v} = (-1,2)$.

5. In each case below, determine whether the vectors represented by $\vec{AB}$ and $\vec{CD}$ are like-directed, opposite-directed, perpendicular or neither.
a) $A(5,12), B(-3,8), C(-2,-1), D(6,1)$
b) $A(5,12), B(6,10), C(1+2\sqrt{2}, \sqrt{2}+1), D(-1,1)$
c) $A(5,12), B(6,10), C(\sqrt{2}+1, 1+2\sqrt{2}), D(1,-1)$
d) $A(5,12), B(6,10), C(-1,-2), D(6,1)$.

6. Prove the following:
a)$(\vec{u}+\vec{v}) \circ (\vec{u}+\vec{v}) = |\vec{u}|^2 + 2\vec{u} \circ \vec{v} + |\vec{v}|^2$, b) $(\vec{u}-\vec{v}) \circ (\vec{u}+\vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$
c) $|\vec{u}+\vec{v}| = |\vec{u}-\vec{v}| \Longleftrightarrow \vec{u} \circ \vec{v} = 0$. d) $|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 \Longleftrightarrow \vec{u} \circ \vec{v} = 0$.
(Compare d) with Pythagorian Theorem.)

7. Let $\vec{i}$ and $\vec{j}$ be the basic unit vectors,i.e., $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$. Let $\vec{v} = (a,b)$ be an arbitrary vector. Denote the angle between $\vec{i}$ and $\vec{v}$ by α, and the angle between $\vec{j}$ and $\vec{v}$ by β. These angles are called the direction angles of $\vec{v}$. The numbers $cos\alpha$ and $cos\beta$ are called the direction cosines of the vector $\vec{v}$.
a) Show that $cos\alpha = a/|\vec{v}|$ and $cos\beta = b/|\vec{v}|$.
b) Find the direction cosines of $\vec{v} = (-3,4)$.

c) Find a unit vector $\vec{v}$ whose direction cosines are $cos\alpha = -\frac{\sqrt{3}}{2}$ and $cos\beta = -\frac{1}{2}$. How many such vectors are there?

8. Given the following conditions, determine the unknowns:
a) $\vec{v} = x(\frac{3}{2}, -2)$ and $|\vec{v}| = \frac{1}{2}$, b) $\vec{v} = (x, 6)$ and $\vec{v} \circ (2,1) = 0$.

9. The successive vertices of a quadrilateral are $A(2,3), B(-1,4)$, $C(-8,11)$ and $D(3,9)$. Show that the diagonals of the quadrilateral are perpendicular.

10. Two vectors with length 4 and 5 make an angle of $\pi/3$ radians with each other. Find their scalar product.

11. $\vec{u}$ and $\vec{v}$ are vectors with $|\vec{u}| = 6, |\vec{v}| = 8$ and $|\vec{u} - \vec{v}| = 4$. Find $\vec{u} \circ \vec{v}$.

12. Show that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

13. Find the cosines of the interior angles of the triangle with vertices $A(-1,-1), B(3,1)$ and $C(1,5)$.

14. Given a nonzero vector $\vec{v}$ in the plane, every vector $\vec{u}$ can be expressed as the sum
$$\vec{u} = \vec{v}' + \vec{v}^{\perp}$$
of a vector $\vec{v}'$ parallel to $\vec{v}$ and a vector $\vec{v}^{\perp}$ perpendicular to $\vec{v}$.
The vector $\vec{v}'$ is called the *projection of* $\vec{u}$ *on* $\vec{v}$ and it is denoted by $proj_{\vec{v}}(\vec{u})$. Show that
$$proj_{\vec{v}}(\vec{u}) = \left(\frac{\vec{v} \circ \vec{u}}{|\vec{v}|^2}\right)\vec{v}.$$
The length of $proj_{\vec{v}}(\vec{u})$ is called the component of $\vec{u}$ *in* $\vec{v}$ *direction*, and it is denoted by $comp_{\vec{v}}(\vec{u})$. Thus
$$comp_{\vec{v}}(\vec{u}) = \frac{\vec{v} \circ \vec{u}}{|\vec{v}|}.$$
Find $proj_{\vec{v}}(\vec{u}), proj_{\vec{v}}(\vec{w}), comp_{\vec{v}}(\vec{u})$ and $comp_{\vec{v}}(\vec{w})$ for the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ in exercise 1 above.

15. Let $\vec{u}, \vec{v}$ are vectors with $\|\vec{u}\| = 6, \|\vec{v}\| = 8$ and $\|\vec{u} - \vec{v}\| = 4$. Find $\vec{u}\,\vec{v}$.

16. Using vector methods show that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of lengths of the sides.

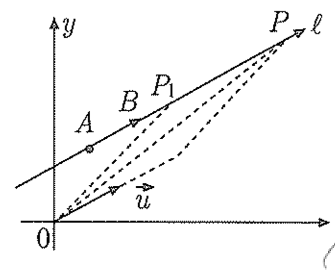
17. Find the cosines of the interior angles of the triangle with vertices $A(-1,-1), B(3,1)$ and $C(1,5)$.

$\overline{AB}$

Ray => $\overrightarrow{CD}$
Segment => $\overline{AB}$
Point => $\cdot A$ A.
Line => $\overleftrightarrow{AB}$

3.4 Lines, Half-Lines and Line Segments

Consider a line ℓ in the plane and assume that $P_1(x_1, y_1)$ is a given point on the line. Take any directed segment $\vec{AB}$ on the line which represents the vector $\vec{u} = (a,b) \neq \vec{0}$ (See Fig. 3.9). Then for any point $P(x,y)$ on the line, the vector $\vec{u}$ and the vector represented by $(\vec{P_1P})$ are parallel (like-or opposite- directed). That is,
$$\vec{P} - \vec{P_1} = t\vec{u}, \quad t \in \mathbb{R}$$
or
$$P = \vec{P_1} + t\vec{u}, \quad t \in \mathbb{R} \qquad (3.4.1)$$



$\hat{BAC} = \overrightarrow{AB} \cup \overrightarrow{AC}$

Fig. 3.9.

Conversely, if for a point $P(x,y)$ in the plane the vector $\vec{u}$ and the vector represented by $\vec{P_1P}$ are parallel, that is, if (3.4.1) holds for some $t \in \mathbb{R}$, then P is necessarily on the line ℓ. Thus, $P(x,y)$ is on ℓ if and only if $\vec{P}$ satisfies the equation (3.4.1) for some $t \in \mathbb{R}$. For this reason, the equation (3.4.1) is called a *vector-equation of the line through* P_1 *in the direction of* $\vec{u}$. The vector $\vec{u}$ is called a *direction vector* of the line.

Note that different vectors may be direction vectors of the same line. Two non-zero vectors are direction vectors of the same line if they are parallel.

Example 3.4.1 Find a vector-equation of the line through $P_1(2,-1)$ and in the direction of $\vec{u} = (3,2)$.

Solution. A vector-equation for this line is
$$\vec{P} = (2,-1) + t(3,2), t \in \mathbb{R} \text{ or } (x,y) = (3t+2, 2t-1), t \in \mathbb{R}.$$

One can easily see that the linear equation $2x - 3y - 7 = 0$ is the defining equation of the line in example 14. Furthermore, the vector $\vec{N} = (2,-3)$ formed by the coefficients of x and y in the linear equation is perpendicular to the direction vector $\vec{u} = (3,2)$ of the line. We shall see below that this is the special case of a more general fact.

If $Ax + By + C = 0$ is the defining equation of a line ℓ, then the vector $\vec{N} = (A, B)$ is called a normal vector of the line ℓ. We have

$$\ell = \{(x,y) : Ax + By + C = 0\} = \{P : \vec{N} \circ \vec{P} + C = 0\}$$

where $\vec{N} = (A,B)$ and $\vec{P} = (x,y)$. The equation $\vec{N} \circ \vec{P} + C = 0$ is called a normal equation for ℓ.

Theorem 3.4.2 *Normal vector and direction vector of a line are perpendicular.*

Proof. Let $\vec{N} \circ \vec{P} + C = 0$ and $\vec{P} = \vec{P_1} + t\vec{u}$ be normal equation and vector-equation of a line ℓ. Then

$$\vec{N} \circ \vec{P} + C = \vec{N} \circ (\vec{P_1} + t\vec{u}) + C = \vec{N} \circ \vec{P_1} + t(\vec{N} \circ u) + C = 0.$$

Since P_1 is a point on ℓ , $\vec{N} \circ \vec{P_1} + C = 0$. Therefore $t(\vec{N} \circ \vec{u}) = 0$ for any $t \in \mathbb{R}$. Hence $\vec{N} \circ \vec{u} = 0$, and $\vec{N}$ and $\vec{u}$ are perpendicular. □

Corollary 3.4.3 *Any nonzero vector perpendicular to the normal vector of a line is a direction vector for that line.*

Example 3.4.4 Find a vector-equation of the line passing through $P_1(1,3)$ and perpendicular to the line (with normal equation) $3x - 2y - 4 = 0$.

Solution. $\vec{N_1} = (3,-2)$ is a normal vector of the line $3x - 2y - 4$. Therefore $\vec{N_1} = (3,-2)$ is perpendicular to the normal vector of the required line. That is, $\vec{N_1}$ is a direction vector of the required line. Hence

$$\vec{P} = \vec{P_1} + t\vec{N_1}, \quad t \in \mathbb{R} \quad \text{or} \quad (x,y) = (3t+1, -2t+3), \quad t \in \mathbb{R}$$

is a vector-equation of that line.

Example 3.4.5 Find the cosine of the acute angle between the lines $3x - 4y - 2 = 0$ and $2x - y + 3 = 0$.

Solution. $\vec{N} = (3,-4)$ and $\vec{M} = (2,-1)$ are perpendicular to the two lines. Hence the angle θ between $\vec{M}$ and $\vec{N}$ is either the acute or the obtuse angle between the two lines. In either case cosine of the acute angle between them is

$$| \cos\theta | = \left| \frac{\vec{N} \circ \vec{M}}{| \vec{N} || \vec{M} |} \right| = \left| \frac{6+4}{\sqrt{25}\sqrt{5}} \right| = \frac{2}{\sqrt{5}}.$$

Different equations may describe the same line. In general, two equations are said to be *equivalent* if they have the same graph. In Theorem 2.2.6, we proved that two linear equations $Ax + By + C = 0$ and $ax + by + c = 0$ are equivalent if and only if there exists $k \neq 0$ such that $A = ka, B = kb$ and $C = kc$. Using this result one can prove (See Exercise 4).

Theorem 3.4.6 *Two vector equations*

$$(x,y) = (x_1,y_1) + t(a_1,b_1) \ and (x,y) = (x_2,y_2) + t(a_2,b_2)$$

are equivalent $\iff$ *there exists* $k \neq 0$ *such that* $b_2 = kb_1, a_2 = ka_1$ *and* $b_1(x_1 - x_2) = a_1(y_1 - y_2)$.

Now, let $P_1(x_1,y_1)$ and $= P_2(x_2,y_2)$ be two distinct points in the plane and consider the line passing through these two points. Then $\vec{u} = \vec{P_2} - \vec{P_1}$ is a direction vector for that line. Hence a vector-equation for the line passing through P_1 and P_2 is

$$\vec{P} = \vec{P_1} + t(\vec{P_2} - \vec{P_1}), \quad t \in \mathbb{R}$$

or

$$\vec{P} = t\vec{P_2} + (1-t)P_1, \quad t \in \mathbb{R}. \tag{3.4.2}$$

Let us denote the (undirected) line segment joining P_1 to P_2 by $[P_1P_2]$. More precisely, if ℓ is the line passing through P_1 and P_2 then $[P_1P_2]$ consists of all points of ℓ between P_1 and P_2 (See Fig. 3.10).

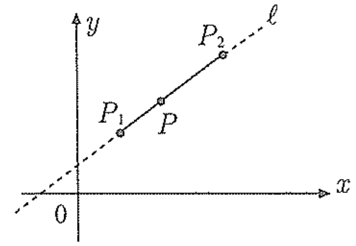


Fig. 3.10

A point P of ℓ is on the segment if and only if

$$|\vec{P}-\vec{P}_1| \leq |\vec{P}_2-\vec{P}_1| \text{ and } |\vec{P}-\vec{P}_2| \leq |\vec{P}_2-\vec{P}_1|.$$

Such a point satisfies the equation (3.4.2) for some t. Thus

$$\begin{aligned} |\vec{P}-\vec{P}_1| &= |t||\vec{P}_2-\vec{P}_1| \\ |\vec{P}-\vec{P}_2| &= |1-t| \ |\vec{P}_2-\vec{P}_1|. \end{aligned}$$

It follows that P is on $[P_1P_2]$ if and only if $|t| \leq 1$ and $|1-t| \leq 1$ or equivalently $0 \leq t \leq 1$. In set theoretic notations,

$$[P_1P_2] = \{P : \vec{P} = t\vec{P}_2 + (1-t)\vec{P}_1, \quad 0 \leq t \leq 1\}.$$

In terms of Cartesian Coordinates,

$$[P_1P_2] = \{(x,y) : x = tx_2 + (1-t)x_1 \text{ and } y = ty_2 + (1-t)y_1, \quad 0 \leq t \leq 1\}.$$

Another interesting subset of a line is a half-line. Given a non-zero vector $\vec{u} = (a,b)$ and a point $P_1(x_1,y_1)$ in the plane, the set

$$\ell^+ = \{P : \vec{P} = \vec{P}_1 + t\vec{u}, \quad t \geq 0\}$$

is called the *half-line from P_1 in the direction of $\vec{u}$.* The set

$$\ell^- = \{P : \vec{P} = \vec{P}_1 + t\vec{u}, \quad t \leq 0\}$$

is called the *half-line from P_1 in the opposite-direction of $\vec{u}$.* Clearly, we have (See Fig.3.11)

$$\ell = \{P : \vec{P}_1 + t\vec{u}, \quad t \in \mathbb{R}\} = \ell^- \cup \ell^+.$$

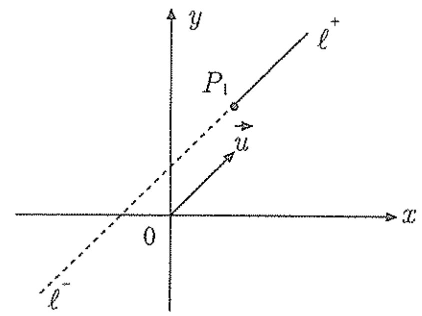


Fig. 3.11

Example 3.4.7 Let $P_1(-1,2)$ be a point and $\vec{u} = (1,1)$ be a vector in the plane. Verify whether or not the points $A(5,8)$, $B(4,5)$, $C(-2,1)$ are on the half-line ℓ^+ from P_1 in the direction of $\vec{u}$.

Solution. We observe that

$$\begin{aligned} \vec{A} &= (5,8) = (-1,2) + 6(1,1) = \vec{P}_1 + 6\vec{u} \\ \vec{B} &= (4,5), \vec{B} \neq \vec{P}_1 + t\vec{u} \text{ for any } t \in \mathbb{R} \\ \vec{C} &= (-2,1) = (-1,2) + (-1)(1,1) = \vec{P}_1 + (-1)\vec{u}. \end{aligned}$$

Hence $A \in \ell^+, B \notin \ell^+, C \notin \ell^+$.

Exercises

1. Write a vector-equation for each of the following lines
   a) the line through $P_1(2,1)$ in the direction of $\vec{u} = (4,5)$
   b) the line through $P_1(2,1)$ with slope -3.
   c) the line through $P_1(2,1)$ parallel to $2x - y - 6 = 0$.
   d) the line through $P_1(2,1)$ perpendicular to $2x - 6 - y = 0$.
   e) the line through $P_1(2,1)$ and $P_2(1,3)$.

2. Find normal vectors and write down normal equations of the lines in the previous exercise.

3. Find the cosine of the acute angle between the lines
   a) $\ell_1 = \{(x,y) : 2x - y - 6 = 0\}$, $\quad \ell_2 = \{(x,y) : 3x + 4y - 1 = 0\}$
   b) $\ell_1 = \{(x,y) : (x,y) = (3t-1, -t+2)\}$, $\quad \ell_2 = \{(x,y) : (x,y) = (t, 1-t)\}$.

4. Prove Theorem 3.4.6.

5. Given the points $A(1,3), B(2,1)$ and $C(4,5)$,
   a)Find the point $P \in [AB]$ such that $|AP| = \frac{1}{3}|BP|$
   b) Find a point Q on the half-line from A in the direction of $\vec{u} = \vec{C} - \vec{B}$ such that $|AQ| = 3$.
   c) Find a ponit D on the half-line from A in the opposite-direction of $\vec{u} = \vec{C} - \vec{B}$ such that $|AD| = 1$.

3.5 More About Lines: Distance, Bisectors, Symmetry

Consider a line ℓ and a point $P_\circ(x_\circ, y_\circ)$ in the plane. By the *distance* $|P_\circ\ell|$ *from* $P_\circ$ *to* ℓ we mean

$$|P_\circ\ell| = min\{|P_\circ P| : P \in \ell\}.$$

Thus, for a point P_1 on ℓ, $|P_\circ\ell| = |P_\circ P_1|$ if and only if the vector $\vec{P}_1 - \vec{P}_\circ$, represented by $P_\circ\vec{P}_1$, is a normal vector for ℓ (See Fig. 3.12). If ℓ is given as

$$\ell = \{(x,y) : Ax + By + C = 0\},$$

then $\vec{P}_1 - \vec{P}_\circ$ is parallel to the normal vector $\vec{N} = (A, B)$, and for any point $P(x,y)$ on ℓ, we have $|P_\circ\ell| = |P_\circ P|\, cos\theta$ and

$$|\vec{N} \circ (\vec{P} - \vec{P}_\circ)| = |\vec{N}||P_\circ P|\, cos\theta$$

$$|Ax_\circ + By_\circ + C| = (\sqrt{A^2 + B^2})\, |P_\circ\ell|$$

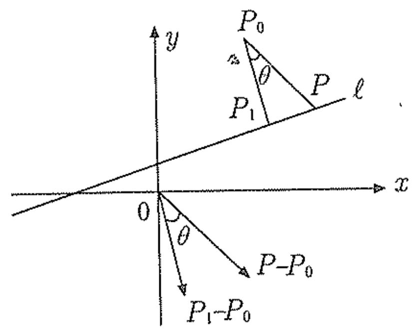


Fig. 3.12

Hence

$$|P_\circ\ell| = \frac{|Ax_\circ + By_\circ + C|}{\sqrt{A^2 + B^2}}.$$

Example 3.5.1 The distance from $P_\circ(-1, 2)$ to the line $\ell = \{(x,y) : 3x - 4y + 6 = 0\}$ is

$$|P_\circ\ell| = \frac{|3 \cdot (-1) - 4 \cdot 2 + 6|}{\sqrt{3^2 + 4^2}} = 1.$$

Example 3.5.2 Find the distance from $P_\circ(-1, 1)$ to the line

$$\ell = \{P : \vec{P} = (2, 0) + t(4, 3)\}.$$

$x = 4t-2$
$y = 3t-0 = 3t$

Solution. The line is given by its vector-equation. We use the vector-equation to find a linear equation for ℓ.

$$\vec{P} = (x,y) = (-2, 0) + t(4,3) \iff x = 4t - 2, \quad y = 3t.$$
$$\iff 3x - 4y = -6 \iff 3x - 4y + 6 = 0.$$

Therefore $\ell = \{(x,y) : 3x - 4y + 6 = 0\}$ and $|P_\circ\ell| = \frac{|3\cdot(-1)-4\cdot 2+6|}{\sqrt{3^2+4^2}} = 1$.

There are other ways of finding the distance from a point to a line; see Exercise 3.

Now, consider a line segment $[P_\circ Q_\circ]$. By the *perpendicular bisector* of $[P_0Q_0]$ we mean the line which passes through the midpoint of $[P_0Q_0]$ and which is perpendicular to $[P_0Q_0]$ (See Fig. 3.13). The midpoint M of $[P_\circ Q_\circ]$ is given by

$$\vec{M} = \frac{1}{2}\vec{P}_\circ + \frac{1}{2}Q_\circ$$

Since the perpendicular bisector is perpendicular to $[P_\circ Q_\circ]$, $\vec{N} = \vec{Q}_\circ - \vec{P}_\circ$ is a normal vector, and any vector $\vec{u}$ which is perpendicular to $\vec{N}$ is a direction vector for it.

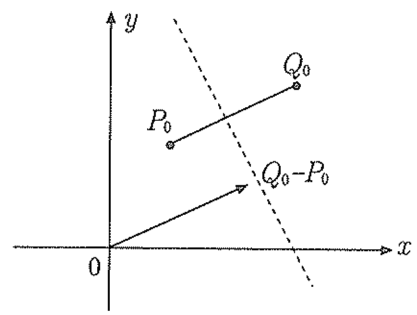


Fig. 3.13.

Example 3.5.3 Find the perpendicular bisector of the segment joining $P_\circ(-1,2)$ and $Q_\circ(3,4)$.

Solution. The midpoint of $[P_\circ Q_\circ]$ is $M(1,3)$, $\vec{N} = \vec{Q}_\circ - \vec{P}_\circ = (4,2)$. We may take $\vec{u} = (-2,4)$ as a direction vector for the perpendicular bisector. Thus a vector equation for the perpendicular bisector is

$$\vec{P} = \vec{M} + t\vec{u}, \quad t \in \mathbb{R} \quad \text{or} \quad (x,y) = (-2t+1, 4t+3), \quad t \in \mathbb{R}.$$

Using this vector equation, one may also obtain the normal equation $2x + y - 5 = 0$.

In studying graphs of relations in $\mathbb{R}$, it is very helpful to know about symmetry of the graph about a line or about a point. Given a line ℓ, two points P and Q are said to be *symmetric partners* of each other *about* ℓ if ℓ is the perpendicular bisector of $[PQ]$. Given a point M, two points P and Q are said to be *symmetric partners* of each other *about* M if M is the midpoint of $[PQ]$ (See Fig.3.13).

Note that symmetric partner of a point (x,y) about X-axis is $(x,-y)$, about Y-axis is $(-x,y)$, and about the origin is $(-x,-y)$.

Example 3.5.4 Find the symmetric partner of $P(1,2)$ about the line $\ell = \{(x,y) : 2x - 3y - 9 = 0\}$.

Solution. (See the figure below). Consider the line ℓ_1 passing through $(1,2)$ and perpendicular to the line ℓ. The normal vector $(2,-3)$ of ℓ is a direction vector for ℓ_1. Hence $(x,y) = (1,2) + t(2,-3) = (2t+1, -3t+2)$ is a vector-equation for ℓ_1. Consider the point M at which ℓ and ℓ_1 meet. Since $M \in \ell_1$, M has coordinates $(2t+1, -3t+2)$ for some $t \in \mathbb{R}$; and since $M \in \ell$,

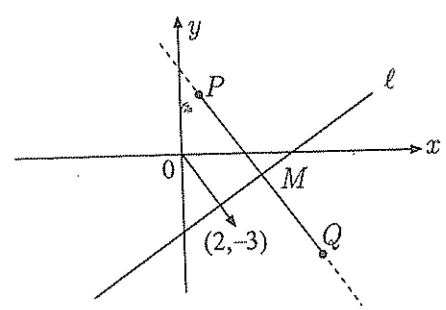


Fig. 3.14 .

$2(2t+1) - 3(-3t+2) - 9 = 0$ which yields $13t - 13 = 0$ or $t = 1$.

Thus M has coordinates $(2\cdot1+1, -3\cdot1+2) = (3,-1)$. If $Q(a,b)$ is the symmetric partner of $P(1,2)$ about ℓ then ℓ is the perpendicular bisector of $[PQ]$, and therefore M is the midpoint of $[PQ]$. It follows that

$$\frac{a+1}{2} = 3, \quad \frac{b+2}{2} = -1 \quad \text{or} \quad a = 5 \quad \text{and} \quad b = -4.$$

$Q(5,-4)$ is the symmetric partner of $P(1,2)$ about ℓ.

A set S of points, in the plane, is said to be *symmetric about* a given *line* (or *about* a given *point*) if S contains the symmetric partner of each of its points about that line (respectively about that point).

Thus a set S is symmetric about X-axis if whenever $(x,y) \in S$ then $(x,-y) \in S$. Similarly, S is symmetric about Y-axis if $(x,y) \in S \Longrightarrow (-x,y) \in S$; and S is symmetric about the origin if $(x,y) \in S \Longrightarrow (-x,-y) \in S$.

Exercises

1. Find the distance from $P_\circ(3,-2)$ to each of the following lines
   a) $\ell = \{P : \vec{P} = (2,4) + t(3,1)\}$, b) $\ell = \{(x,y) : x + 4y - 2 = 0\}$.
2. Determine the unknown coordinate of the point $M(1, y_\circ)$ so that $\mid M\ell \mid = 1$ for the line $\ell = \{(x,y) : 3x + 4y - 1 = 0\}$.
3. Given the point $P_\circ(-1,2)$ and the line ℓ with vector-equation $(x,y) = (2t-2, 3t)$.
   a) Write down the vector-equation of the line ℓ_1 which passes through $P_\circ$ and which is perpendicular to ℓ.
   b) Find the coordinates of the point M at which ℓ and ℓ_1 meet.
   c) Find $\mid P_\circ M \mid$. What is $\mid P_\circ \ell \mid$?
   d) Find the symmetric partner of $P_\circ$ about ℓ.
4. Find the symmetric partner of $(3,2)$ about the line $\ell = \{P : \vec{P} = (4t-2, -t)\}$.
5. Write down a normal equation of the perpendicular bisector of $[P_\circ Q_\circ]$ where $P_\circ(3,2), Q_\circ(-2,4)$.
6. Find the equation of the straight line passing through the point $P(3,5)$ and equidistant from the points $A(-7,3)$ and $B(11,-15)$.
7. Find the point M_1 symmetric to the point $M_2(8,-9)$ with respect to the straight line which passes through the points $A(3,-4)$ and $B(-1,-2)$.

3.6 Convex Sets, Half-Planes and Polygons

Let E be a set of points in the plane. We say that E is *convex* if

$$P_1, P_2 \in E \Longrightarrow [P_1P_2] \subseteq E.$$

Equivalently, E is convex if

$$(x_1, y_1), (x_2, y_2) \in E \Longrightarrow k(x_2, y_2) + (1-k)(x_1, y_1) \in E$$

for any $0 \le k \le 1$.

Clearly, any line segment or any half-line or any line is convex. We have

Theorem 3.6.1 *If E_1 and E_2 are two convex sets, then $E_1 \cap E_2$ is convex.*

Proof. Let $P_1, P_2 \in E_1 \cap E_2$. Then $[P_1P_2] \subseteq E_1$ and $[P_1P_2] \subseteq E_2$. Therefore $[P_1P_2] \subseteq E_1 \cap E_2$, proving that $E_1 \cap E_2$ is convex. □

Corollary 3.6.2 *If each of $E_1, E_2, ..., E_n$ is convex, then $E_1 \cap E_2 \cap ... \cap ... \cap E_n$ is convex.*

Now, let ℓ be the line

$$\ell = \{P : \vec{N} \circ \vec{P} + C = 0\}$$

where $\vec{N} = (A, B)$ is the normal vector and C is a scalar. The sets

$$\mathcal{H} = \{P : \vec{N} \circ \vec{P} + C > 0\} \quad \text{and } \mathcal{U} = \{P : \vec{N} \circ \vec{P} + C < 0\}$$

are called half-planes determined by ℓ (See Examples 2.3.2 and 9 in Chapter 2.3.3).

Let us consider a point $P_\circ$ on ℓ, and consider the half-line

$$\ell_\circ^+ = \{P : \vec{P} = \vec{P}_\circ + t\vec{N}, t \ge 0\}$$

from $P_\circ$ in the direction of $\vec{N}$.

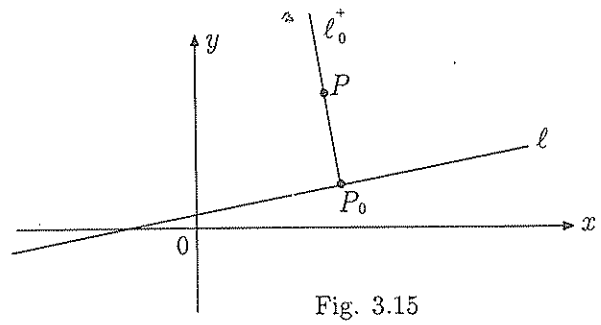


Fig. 3.15

Then, for any $P \ne P_\circ$ on $\ell_\circ^+$, we have $\vec{P} = \vec{P}_\circ + t\vec{N}, t > 0$ and

$$\vec{N} \circ \vec{P} + C = \vec{N} \circ (P_\circ + t\vec{N}) + C = \vec{N} \circ \vec{P}_\circ + C + t \mid \vec{N} \mid^2 .$$

Since $P_\circ \in \ell$ and $t > 0$, we get

$$\vec{N} \circ \vec{P} + C = t \mid \vec{N} \mid^2 > 0$$

for any $P \ne P_\circ$ on $\ell_\circ^+$. Therefore

$$\vec{P} = \vec{P}_\circ + t\vec{N}, t > 0 \Longrightarrow P \in \mathcal{H}.$$

Conversely, if $P \in \mathcal{H}$, then we can find $P_\circ \in \ell$ and $t > 0$ such that $\vec{P} = P_\circ + t\vec{N}$. To see this, note first that the scalar t and the point $P_\circ$ that we are looking for should satisfy

$$\vec{P} = \vec{P}_\circ + t\vec{N} \text{ and } \vec{N} \circ \vec{P} + C = t \mid \vec{N} \mid^2 .$$

Thus, we let

$$t = \frac{\vec{N} \circ \vec{P} + C}{\mid \vec{N} \mid^2} \text{ and } \vec{P}_\circ = \vec{P} - t\vec{N}.$$

Then $t > 0$ since $\vec{N} \circ \vec{P} + C > 0$, and $P_\circ \in \ell$ since $\vec{N} \circ \vec{P}_\circ + C = 0$. Consequently, we can write

$$\mathcal{H} = \{P : \vec{P} = \vec{P}_\circ + t\vec{N} \text{ for some } t > 0 \text{ and } P_\circ \in \ell\},$$

Similar discussions show that

$$\mathcal{U} = \{P : \vec{P} = \vec{P}_\circ + t\vec{N} \text{ for some } t < 0 \text{ and } P_\circ \in \ell\}.$$

Remark. Descriptions of $\mathcal{H}$ and $\mathcal{U}$ in this way can be interpreted as follows. If $P_\circ$ is a point on ℓ then any point other than $P_\circ$ of the half-line $\ell_\circ^+$ from $P_\circ$ in the direction of $\vec{N}$ is in $\mathcal{H}$ (See Fig.3.14). Conversely, any point of $\mathcal{H}$ is on such an half-line. Thus, *the graph of $\mathcal{H}$ consists of all points above (or else below) the line ℓ.* Similarly, the graph of $\mathcal{U}$ consists of all points below (or else above) the line ℓ. Finally, we can write

$$\mathbb{R}^2 = \mathcal{H} \cup \ell \cup \mathcal{U}.$$

Example 3.6.3 Sketch the graph of the set

$$E = \{(x, y) : 2x - 3y < 1 \text{ and } x + 2y > -2\}.$$

Solution. Let $\vec{N} = (2, -3)$ and $\vec{M} = (1, 2)$ Then

$$\begin{aligned} E &= \{P : \vec{N} \circ \vec{P} - 1 < 0 \text{ and } \vec{M} \circ \vec{P} + 2 > 0\} \\ E &= \{P : \vec{N} \circ \vec{P} - 1 < 0\} \cap \{P : \vec{M} \circ \vec{P} + 2 > 0\}. \end{aligned}$$

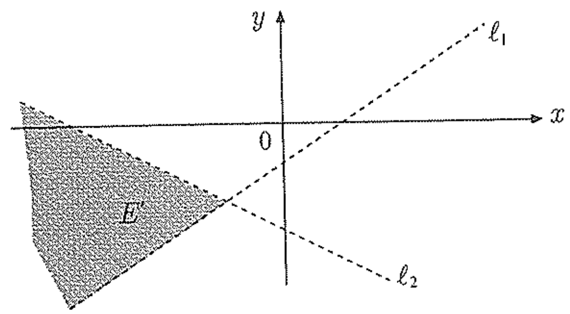
Of these two sets, appearing on the right hand side of the last identity, the first one is the (upper) half-plane determined by the line

$$\ell_1 = \{P : \vec{N} \circ P - 1 = 0\} = \{(x, y) : 2x - 3y - 1 = 0\}$$

and the second one is the (lower) half-plane determined by the line

$$\ell_2 = \{P : \vec{M} \circ \vec{P} + 2 = 0\} = \{(x, y) : x + 2y + 2 = 0\}.$$

Hence E consists of all points above ℓ_1 and below ℓ_2.



If $\mathcal{H}$ and $\mathcal{U}$ are the half-planes determined by the line ℓ then

$$\bar{\mathcal{H}} = \mathcal{H} \cup \ell \text{ and } \bar{\mathcal{U}} = \mathcal{U} \cup \ell$$

are called the *closed half-planes* determined by ℓ.

Theorem 3.6.4 *Every half-plane is convex.*

Proof. We give an algebraic proof for a closed half plane $\bar{\mathcal{H}} = \{P : \vec{N} \circ \vec{P} + C \geq 0\}$. Take $P_1, P_2 \in \bar{\mathcal{H}}$, and take $P \in [P_1 P_2]$. Then $\vec{P} = k\vec{P}_2 + (1-k)\vec{P}_1$ for some $0 \leq k \leq 1$. Thus

$$\begin{aligned} \vec{N} \circ \vec{P} + C &= k(\vec{N} \circ \vec{P}_2) + (1-k)(\vec{N} \circ \vec{P}_1) + C \\ &= k(\vec{N} \circ \vec{P}_2 + C) + (1-k)(\vec{N} \circ \vec{P}_1 + C) \geq 0, \end{aligned}$$

because $0 \leq k \leq 1$ and $P_1, P_2 \in \bar{\mathcal{H}}$. Therefore $[P_1 P_2] \subseteq \bar{\mathcal{H}}$. This completes the proof. □

The intersection of a finite number of closed half-planes is called a polygon. The subsets of the lines which determine the half planes that are contained in the polygon are called the *sides* of the polygon. A side of a polygon is either a line or a half-line or a line segment. The intersection of two sides of a polygon is called a *vertex*. If a polygon has m sides each of which is a line segment then it is called a *finite m-sided polygon.*

From Theorem 3.6.1 and Theorem 3.6.4 we deduce the

Corollary 3.6.5 *Every polygon is convex.*

Finite m-sided polygons can be characterized by means of their vertices:

Theorem 3.6.6 *Let*

$$\bar{\mathcal{H}}_1 = \{P : \vec{N}_1 \circ \vec{P} + C_1 \geq 0\}, \quad \bar{\mathcal{H}}_2 = \{P : \vec{N}_2 \circ \vec{P} + C_2 \geq 0\}, \ldots, \bar{\mathcal{H}}_m = \{P : \vec{N}_m \circ \vec{P} + C_m \geq 0\}.$$

be closed half-planes, and assume that their intersection

$$S = \bar{\mathcal{H}}_1 \cap \bar{\mathcal{H}}_2 \cap \ldots \cap \bar{\mathcal{H}}_m$$

is a finite m-sided polygon with vertices $V_1, V_2, \ldots, V_m$. Then a point $P_\circ$ belongs to S if and only if there exist scalars $k_1 \geq 0, k_2 \geq 0, \ldots, k_m \geq 0$ such that $k_1 + k_2 + \ldots + k_m = 1$ and $\vec{P}_\circ = k_1\vec{V}_1 + k_2\vec{V}_2 + \ldots + k_m\vec{V}_m$.

Proof. Assume that $P_\circ \in S$. If $P_\circ$ is on a side $[V_i V_{i+1}]$, then $P_\circ = k\vec{V}_i + (1-k)\vec{V}_{i+1}$ for some $0 \leq k \leq 1$; thus taking $k_i = k, k_{i+1} = 1 - k$ and $k_h = 0$ for indices h other than i and $i + 1$, we get $\vec{P}_\circ = k_1\vec{V}_1 + k_2\vec{V}_2 + \ldots + k_m\vec{V}_m$ with $k_1 + k_2 + \ldots + k_m = 1$. If $P_\circ$ is not on any side of the poligon (See Fig.3.15) then consider the half-line ℓ_1^+ from V_1 in the direction of the directed segment $\vec{V_1 P_\circ}$. This half- line contains $P_\circ$ and intersects one of the sides, say $[V_i V_{i+1}]$ in a point P_i. For otherwise the polygon would not be finite.

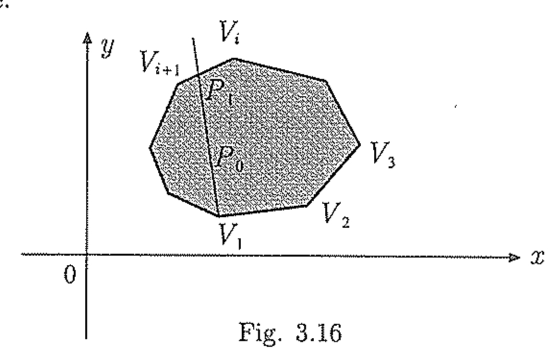


Fig. 3.16

Since $P_i \in [V_iV_{i+1}]$, we can write $\vec{P_i} = t\vec{V_i} + (1-t)\vec{V}_{i+1}, 0 \le t \le 1$. On the other hand, $P_\circ \in [V_1P_i]$ implies that $\vec{P_\circ} = s\vec{V_1} + (1-s)\vec{P_i}, 0 \le s \le 1$. Hence

$$\begin{aligned}\vec{P_\circ} &= s\vec{V_1} + (1-s)[t\vec{V_i} + (1-t)\vec{V}_{i+1}]\\ \vec{P_\circ} &= s\vec{V_1} + (1-s)t\vec{V_i} + (1-s)(1-t)\vec{V}_{i+1}, \quad 0 \le s \le 1, 0 \le t \le 1.\end{aligned}$$

We take $k_1 = s$, $k_i = (1-s)t$, $k_{i+1} = (1-s)(1-t)$ and $k_h = 0$ for indices other than 1 ,i and $i+1$. Then we have

$$k_1 + k_2 + \ldots + k_m = s + (1-s)t + (1-s)(1-t) = 1$$

and

$$\vec{P_\circ} = k_1\vec{V_1} + k_2\vec{V_2} + \ldots + k_m\vec{V_m}.$$

Conversely, assume that there exist scalars $k_1 \ge 0, k_2 \ge 0, \ldots, k_m \ge 0$ such that $k_1 + k_2 + \ldots + k_m = 1$ and

$$\vec{P_\circ} = k_1\vec{V_1} + k_2\vec{V_2} + \ldots + k_m\vec{V_m}.$$

Then, for any $j = 1, 2, \ldots, m$, we have

$$\vec{N_j} \circ \vec{P_\circ} = k_1(\vec{N_j} \circ \vec{V_1}) + k_2(\vec{N_j} \circ \vec{V_2}) + \ldots + k_m(\vec{N_j} \circ \vec{V_m}).$$

On the other hand, each vertex V_i is a point in the polygon S, i.e. $V_i \in \mathcal{H}_j$ for each i and j. Therefore $\vec{N_j} \circ \vec{V_i} + C_j \ge 0$, and this implies that $k_i(\vec{N_j} \circ \vec{V_i}) \ge -k_iC_j$ for each $i = 1, \ldots, m$. Thus

$$\vec{N_j} \circ \vec{P_\circ} \ge -k_1C_j - \ldots - k_mC_j = -(k_1 + k_2 + \ldots + k_m)C_j = -C_j,$$

that is,

$$\vec{N_j} \circ \vec{P_\circ} + C_j \ge 0 \text{ and } P_\circ \in \mathcal{H}_j$$

for each $j = 1, \ldots, m$. This shows that $P_\circ \in S$ and completes the proof of the theorem. □

Remark. According to the theorem we have just proved, a finite m-sided polygon S with vertices $V_1, V_2, \ldots, V_m$ can also be defined as

$$S = \{P : \vec{P} = k_1\vec{V_1} + \ldots + k_m\vec{V_m}, k_1 \ge 0, \ldots, k_m \ge 0, \sum_{i=1}^{m} k_i = 1\}.$$

Exercises

1. Determine whether or not the following sets are convex.
   a) $\{(x,y) : x^2 + y^2 = 4\}$, b) $\{(x,y) : x^2 + y^2 < 4\}$
   c) $\{(x,y) : x^2 + y^2 \le 4\}$, d) $\{(x,y) : x^2 + y^2 > 4\}$
   e) $\{(x,y) : x^2 + y^2 \ge 4\}$, f) $\{(x,y) : 1 < x^2 + y^2 < 4\}$
   g) $\{P :\mid \vec{P} - (1,1) \mid < \sqrt{2}\}$, h) $\{P :\mid \vec{P} - (1,1) \mid > \sqrt{2}\}$.

2. Graph the following half-planes. a)$\mathcal{H} = \{(x,y) : x + y - 1 > 0\}$,
   b) $\bar{\mathcal{U}} = \{(x,y) : 2x - 2y - 1 \le 0\}$
   c) $\bar{\mathcal{H}} = \{(x,y) : x - 2y - 2 \ge 0\}$, d) $\mathcal{U} = \{(x,y) : x + y - 4 < 0\}$.

3. Graph the following polygons:
   a) $S = \{(x,y) : x + y - 1 \ge 0$ and $2x - y - 1 \ge 0\}$
   b) $S = \{(x,y) : x + y - 1 \ge 0$ and $x - 2y - 2 \ge 0\}$
   c) $S = \{(x,y) : x + y - 1 \ge 0$ and $x + y \ge 0\}$
   d) $S = \{(x,y) : x + y - 1 \ge 0,\ 2x - y - 1 \le 0, x - 2y - 2 \ge 0$ and $x + y - 1 \le 0\}$.

4. Graph the polygon determined in each case.
   a) $x + 3y + 9 \ge 0$ and $3x - 2y + 6 \ge 0$ and $2x + y - 4 \ge 0$.
   b) $x + 3y + 9 \ge 0$ and $2x + 6y + 5 \ge 0$ and $2x + y - 4 \ge 0$.
   c) $3y - x - 9 \ge 0$ and $2y - 3x + 6 \le 0$ and $2x + 4y - 1 \ge 0$.
   d) $x - y \ge 0$ and $x - 1 \ge 0$ and $x + y - 1 \ge 0$ and $y - 4 \le 0$ and $x - 3 \le 0$.

M1

Chapter 4

CONIC SECTIONS

In this chapter, we study geometric objects in the plane called *conic sections*. They are called conic sections, because each of them can be described as the intersection of a plane and a right circular cone in 3-space. This is apparent, geometrically, from Fig. 4.1.

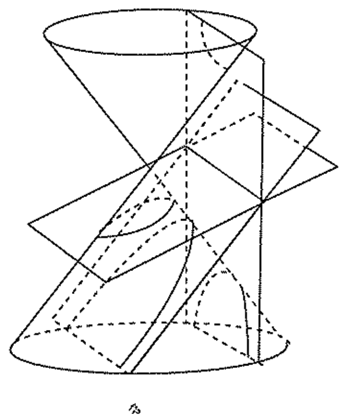


Fig. 4.1.

We will see that *any conic section is the graph of a second degree equation in x and y*, and that *the graph of any second degree equation is a conic section or a degenerate form of a conic section.*

4.1 Fundamental Definitions

Let F be a fixed point, d a fixed line in the plane; and let e be a positive real number. Then the set of all points P, in the plane, such that

$$\frac{|PF|}{|Pd|} = e$$

is called the conic section or simply the *conic* with *focus* F, *directrix* d, and *eccentricity* e. Symbolically, the conic with focus F, directrix d, and eccentricity e is the set

$$\zeta = \left\{ P : \frac{|PF|}{|Pd|} = e \right\} \tag{4.1.1}$$

As in Fig. 4.2., we consider the line ℓ which is perpendicular to the directrix d and passes through the focus F. This line ℓ is called the *axis* of the conic. An immediate observation is the following theorem:

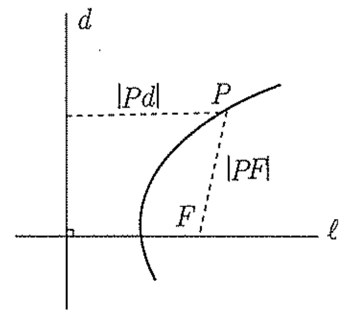


Fig. 4.2

Theorem 4.1.1 *Every conic is symmetric about its axis ℓ.*

Proof. Let P be a point on the conic and let Q be its symmetric partner about ℓ (See Fig.4.3.).Then ℓ is the perpendicular bisector of $[PQ]$, whence
$|PF| = |QF|$ and $|Pd| = |Qd|$. □

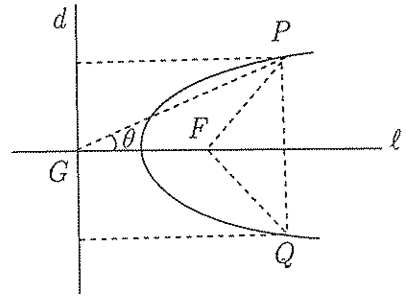


Fig. 4.3.

We denote the point of intersection of ℓ and d by G. It is clear from Fig 4.3. that

$$| Pd |=| PG || cos\theta |=| \vec{P}-\vec{G} || cos\theta |$$

where θ is the angle between the vectors $(\vec{P}-\vec{G})$ and $(\vec{F}-\vec{G})$. Then we also have

$$| cos\theta |= \frac{|(\vec{P}-\vec{G})\circ(\vec{F}-\vec{G})|}{|\vec{P}-\vec{G}||\vec{F}-\vec{G}|}.$$

Hence

$$| Pd |= \frac{|(\vec{P}-\vec{G})\circ(\vec{F}-\vec{G})|}{|\vec{F}-\vec{G}|} \quad \text{and}$$

$$\frac{| PF |}{| Pd |} = \frac{|\vec{P}-\vec{F}|}{| Pd |} = \frac{|\vec{P}-\vec{F}||\vec{F}-\vec{G}|}{|(\vec{P}-\vec{G})\circ(\vec{F}-\vec{G})|}.$$

Thus, the conic (4.1.1) can also be described as

$$\zeta = \{P : | \vec{P}-\vec{F} || \vec{F}-\vec{G} |= e\,|(\vec{P}-\vec{G})\circ(\vec{F}-\vec{G})|\}. \qquad (4.1.2)$$

Remark. In the foregoing discussions we have implicitely assumed that F does not lie on the directrix d. What happens if F lies on d? See Exercise 4.

The points at which the conic intersects its axis are called *vertices* of the conic. *A vertex* of the conic is either on the segment $[GF]$ or it is outside the segment $[GF]$. If V is a vertex on the segment $[GF]$ then

$$\vec{V} = k\vec{F} + (1-k)\vec{G}, 0 \leq k \leq 1.$$

Since V is on the conic

$$e = \frac{| VF |}{| Vd |} = \frac{|\vec{F}-\vec{V}|}{|\vec{G}-\vec{V}|} = \frac{|1-k||\vec{F}-\vec{G}|}{|k||\vec{F}-\vec{G}|} = \frac{1-k}{k}.$$

This yields $k = 1/(1+e)$. Hence the point V which is given by

$$\vec{V} = \frac{\vec{F}+e\vec{G}}{1+e} \qquad (4.1.3)$$

is a vertex of the conic on the segment $[GF]$. If V' is a vertex outside the segment $[GF]$ then

$$\vec{V}' = s\vec{F} + (1-s)\vec{G}, s < 0 \text{ or } s > 1.$$

We also have

$$e = \frac{| V'F |}{| V'd |} = \frac{|\vec{F}-\vec{V}'|}{|\vec{G}-\vec{V}'|} = \frac{|1-s|}{|s|} = \frac{s-1}{s}, s < 0 \text{ or } s > 1.$$

If $e = 1$, then there is no $s < 0$ or $s > 1$ satisfying the equation $e = (s-1)/s$. Therefore, if $e = 1$ then the conic has only one vertex V, which is on the segment $[GF]$. However, if $e \neq 1$, then $s = 1/(1-e)$ satisfies the equation $e = (s-1)/s$ and the point V' that is given by

$$\vec{V}' = \frac{\vec{F}-e\vec{G}}{1-e} \qquad (4.1.4)$$

is a vertex of the conic outside the segment $[GF]$. Location of the vertices relative to the points G and F is completely determined by the eccentricity e (See Fig. 4.4).

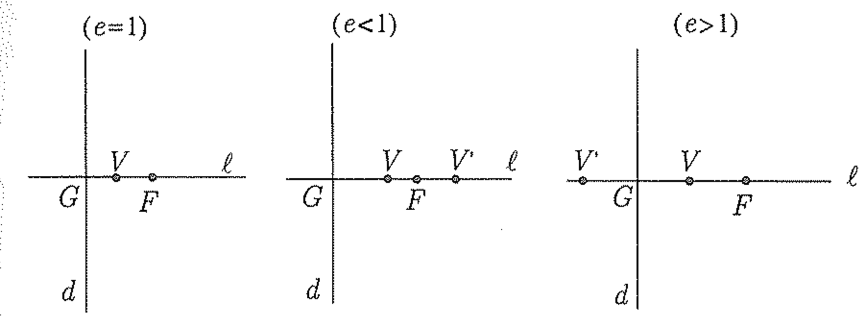


Fig. 4.4.

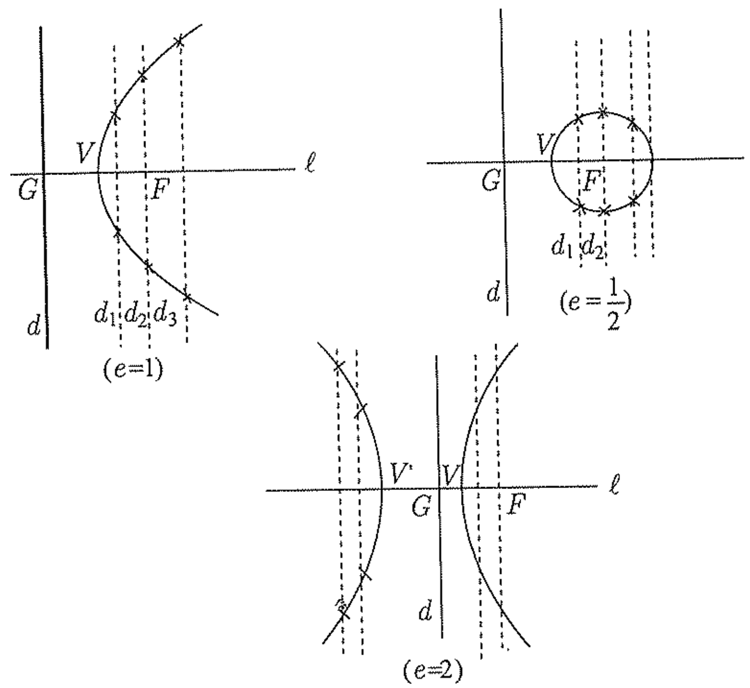
A conic with eccentricity $e = 1$ is called a parabola. If $e < 1$, the conic is called an ellipse, and if $e > 1$, the conic is called an hyperbola.

We have proved above the following theorem:

Theorem 4.1.2 *A parabola has one vertex, an ellipse and a hyperbola have two vertices.*

This theorem shows that compared with the parabola, the ellipse and the hyperbola have some similarities. They have, for instance, two vertices while a parabola has only one vertex. If V and V' are the vertices of an ellipse or a hyperbola then the midpoint C of the segment $[VV']$ is called the center of the conic. Since this concept is defined for the ellipse and the hyperbola but not for the parabola, the ellipse and the hyperbola are also called central conics. We will study central conics in section 3. Now, a few words about sketching conic sections.

The graph of a conic section with focus F, directrix d, and eccentricity e can be obtained as follows. Sketch lightly a line d_1 parallel to d; it intersects the axis of the conic at, say D_1. Set a compass equal to the distance $e \mid D_1G \mid$. From F, swing arcs which intersect d_1. The points you obtain are on the conic with focus F, directrix d, and eccentricity e. Repeating this process we get more points on the conic, and by joining these points we obtain the graph. The following figures illustrate this method.



Another method for sketching conic sections is to find an equation of the conic section in coordinate form.

Example 4.1.3 Find an equation, in coordinate form, and sketch the graph of the conic section with focus $F(\frac{3}{2}, \frac{3}{2})$ directrix $d = \{(x, y) : x + y = 0\}$, and eccentricity

$e = \frac{1}{2}$.

Solution. Since $e = \frac{1}{2} < 1$, the conic section is an ellipse. A point $P(x, y)$ is on the ellipse if and only if $\mid PF \mid = e \mid Pd \mid$. We have

$$\mid PF \mid = \sqrt{(x - \frac{3}{2})^2 + (y - \frac{3}{2})^2}, \quad \mid Pd \mid = \frac{\mid x + y \mid}{\sqrt{2}}.$$

Hence, $\mid PF \mid = e \mid Pd \mid$ means

$$\sqrt{(x - \frac{3}{2})^2 + (y - \frac{3}{2})^2} = \frac{1}{2}\frac{\mid x + y \mid}{\sqrt{2}} \quad \text{or } (x - \frac{3}{2})^2 + (y - \frac{3}{2})^2 = \frac{1}{8}(x + y)^2 \quad \text{or}$$

$$x^2 - 3x + \frac{9}{4} + y^2 - 3y + \frac{9}{4} = \frac{1}{8}x^2 + \frac{1}{4}xy + \frac{1}{8}y^2 \qquad 7x^2 - 2xy + 7y^2 - 24x - 24y + 36 = 0.$$

This is the equation, in coordinate form, of the ellipse. The axis of the ellipse is $\ell = \{(x, y) : x - y = 0\}$. Hence the vertices can be obtained by substituting $y = x$ in the equation of the ellipse:

$$7x^2 - 2xy + 7y^2 - 24x - 24y + 36 = 0, y = x \implies 12x^2 - 48x + 36 = 0$$
$$\implies x^2 - 4x + 3 = 0 \implies x = 1 \text{ and } x = 3.$$

Hence $V(1, 1)$ and $V'(3, 3)$ are the vertices. (The vertices can also be found by using the formulae (4.1.3) and (4.1.4)). To sketch the ellipse, we try to find more and more points on the ellipse. For example, substituting $x = 1$ in the equation we obtain

$$7y^2 - 26y + 19 = 0 \Longrightarrow y = 1 \text{ or } y = \frac{19}{7} \simeq 2.7.$$

Substituting $x = 3$, in the equation, we get

$$7y^2 - 30y + 27 = 0 \Longrightarrow y = 3 \text{ or } y = \frac{9}{7} \simeq 1.3.$$

Hence $(1, 1), (1, 2.7), (3, 3), (3, 1.3)$ are points on the ellipse. Similarly (or by symmetry about the line ℓ), $(2.7, 1)$ and $(1.3, 3)$ also are on the ellipse.

Exercises

1. Find the equation, in coordinate form, of the conic section with the given focus, directrix and eccentricity.
   a) $F(2, 0), d : x = -2, e = \frac{1}{2}$, b) $F(2, 0), d : x = -2, e = 1$,
   c) $F(2, 0), d : x = -2, e = 2$, d) $F(1, 1), d : y = x, e = 2$,
   e) $F(1, 2), d : y = -2, e = \frac{1}{3}$, f) $F(1, 2), d : x = -2, e = 3$.

2. Find the equation, in coordinate form, and sketch the graph of the ellipse with $F(2,0), d = \{(x,y) : x = -4\}$ and eccentricity
a) $e = \frac{1}{8}$, b) $e = \frac{1}{4}$, c) $e = \frac{1}{3}$ d) $e = \frac{1}{2}$, e) $e = \frac{3}{4}$.

3. Tell how the shape of an ellipse changes as the eccentricity increases from near zero to near one.

4. What can you say about the set $\zeta = \{P :\mid PF \mid= e \mid Pd \mid\}$ if F is a point on the line d and
a) $0 < e < 1$, b) $e = 1$, c) $e > 1$.

5. Find the vertices (or the vertex) of the conic sections in exercise 1.

4.2 The Parabola

By definition, $e = 1$ and by (4.1.2), the defining equation of the parabola is

$$[\mid \vec{P} - \vec{F} \mid\mid \vec{F} - \vec{G} \mid]^2 = [(\vec{P} - \vec{G}) \circ (\vec{F} - \vec{G})]^2$$

The parabola has a vertex which is the midpoint of $[GF]$:

$$\vec{V} = \frac{\vec{G} + \vec{F}}{2}$$

In order to obtain a simple equation, in coordinate form, for the parabola, we choose the (Cartesian) coordinate system in the plane so that V is the origin of the coordinate system and the axis ℓ is the Y-axis (See Fig.4.5). Then the focus F is on Y-axis, say $F(0,c)$. On the other hand, $V(0,0)$ is the midpoint of $[GF]$. Therefore $G(O,-c)$, that is to say, the directrix of the parabola is $d = \{(x,y) : y + c = 0\}$. Taking $\vec{P} = (x,y)$ in (4.1.2), the defining equations becomes

$$[x^2 + (y-c)^2]4c^2 = [(x, y+c) \circ (0, 2c)]^2$$

$$[x^2 + (y-c)^2]4c^2 = 4c^2(y+c)^2 \quad \text{or}$$

$$x^2 + (y-c)^2 = (y+c)^2 \quad \text{or} \quad x^2 = 4cy \quad \text{or} \quad y = x^2/4c, \quad (c \neq 0).$$

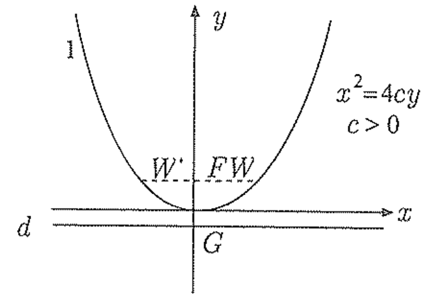


Fig. 4.5

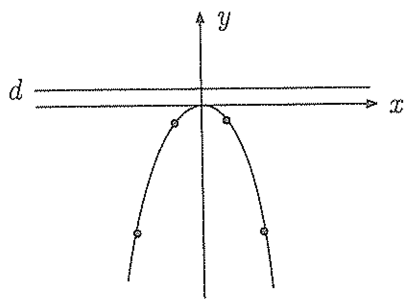
Thus we have proved the following theorem:

Theorem 4.2.1 *The equation of the parabola with vertex at the origin and focus $F(0,c)$ is $x^2 = 4cy$. The directrix of the parabola is the horizontal line $d = \{(x,y) : y = -c\}$.*

Remark. Obviously, the converse of Theorem 4.2.1 is also true. Namely, $x^2 = 4cy$ is the equation of the parabola with vertex at the origin and focus $F(0,c)$. As in Fig.4.5, the parabola opens upward if $c > 0$. The parabola opens downward if $c < 0$. To draw the graph of the parabola we find few more points on the parabola. It is customary to consider the points of intersection of the parabola and the line passing through the focus and perpendicular to the axis (of the parabola). The chord of the parabola passing through the focus and perpendicular to the axis of the parabola is called the *latus rectum* of the parabola. Thus, the endpoints of the latus rectum of the parabola $x^2 = 4cy$ are $W(2c,c)$ and $W'(-2c,c)$. (See Exercise 2).

Example 4.2.2 Find the focus, directrix, vertex and length of the latus rectum of the parabola $x^2 = -\frac{1}{2}y$ and sketch the graph.

Solution. By the Theorem, $x^2 = 4(-\frac{1}{8})y$ is the equation of the parabola with vertex $V(0,0)$, focus $F(0,-1/8)$ and directrix $d = \{(x,y) : y = 1/8\}$. The endpoints of the latus rectum are $(-1/4,-1/8)$ and $(1/4,-1/8)$. The length of the latus rectum is $(1/4) + (1/4) = 1/2$. The graph is given below.

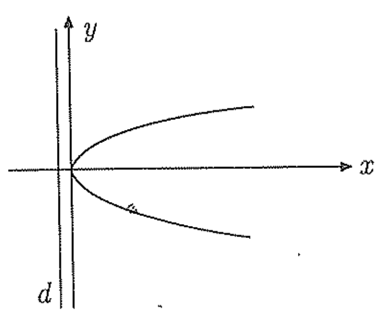


If we choose the coordinate system so that V is at the origin and the axis of the parabola is X-axis, we again obtain a simple equation for the parabola. One can easily prove

Theorem 4.2.3 *The equation of the parabola with vertex at the origin and focus $F(c,0)$ is $y^2 = 4cx$. The directrix of the parabola is the vertical line $d = \{(x,y) : x = -c\}$.*

Example 4.2.4 Find the focus, directrix and the length of the latus rectum of the parabola $y^2 = \frac{1}{2}x$.

Solution. $4c = \frac{1}{2}, c = \frac{1}{8}$. Thus $F(\frac{1}{8},0), V(0,0), d = \{(x,y) : x = -\frac{1}{8}\}$ The endpoints of the latus rectum are $(\frac{1}{8},\frac{1}{4})$ and $(\frac{1}{8},-\frac{1}{4})$. Thus the length of the latus rectum is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. The graph is given below.



Remark. Note that if $c > 0$, the parabola $y^2 = 4cx$ lies on the right hand side of Y-axis. If $c < 0$, it lies on the left hand side of Y-axis. If $c = 0$, then $y^2 = 4cx$ reduces to the X-axis. Similarly, $x^2 = 4cy$ reduces to the Y-axis. These two are called degenerate forms of a parabola. From Theorem 4.2.1 and Theorem 4.2.3, we

conclude that the equations $Ax^2 + 2Ey = 0$ and $Cy^2 + 2Dx = 0$ represent parabolas or degenerate form of parabolas each with vertex at the origin and focus on one of the coordinate axes.

Exercises

1. Find the equation and sketch the graph of the parabola with focus $F(2,0)$ and vertex $V(0,0)$.
2. Show that the endpoints of the latus rectum of the parabola $x^2 = 4cy$ are $(2c,c)$ and $(-2c,c)$.
3. Find the equation, in coordinate form, of the parabola that satisfies the given conditions:
   -4
   a) $d = \{(1\ y) : x = -4\}, V(0,0)$,
   b) $d = \{(x,y) : y = x\}, V(-1,1)$,
   c) $F(1,4), V(0,0)$,
   d) F lies on $x + y = 4$, axis is $y = x, V(0,0)$.
4. Determine all points in the plane which are equidistant from the point. $(1,2)$ and the line $4x - 3y = 12$.
5. Find the directrix of the parabola with
   a) equation $2x^2 + 5y = 0$, b) vertex $V(0,4)$, focus $F(0,3)$
   c) vertex $V(0,0)$, axis $y = 0$, and focus lies on $x + 4y = 12$.
6. Find the equation of a parabola with vertex at the origin, if;
   1) the parabola is symmetrically situated with respect to the axis Ox and passes through the point $A(9,6)$;
   2) the parabola is symmetrically situated with respect to the axis Ox and passes through the point $B(-1,3)$;
   3) the parabola is symmetrically situated with respect to the axis Oy and passes through the point $C(1,1)$;
   4) the parabola is symmetrically with respect to the axis Oy and passes through the point $D(4,-8)$.
7. Write the equation of the parabola which has the focus $F(0,-3)$ and passes through the origin, and whose axis coincides with the y-axis.
8. Find the focus F and the equation of the directrix of the parabola $y^2 = 24x$.

9. Write the equation of a parabola, it its focus is $F(-7,0)$ and the equation of the directrix is $x-7=0$.

10. Find the equation of the parabola with focus $F(7,2)$ and directrix $x-5=0$.

11. Find the equation of the parabola whose focus is $F(4,3)$ and whose directrix is $y+1=0$.

4.3 Central Conics

By definition, a central conic is a conic with eccentricity $e \neq 1$, and by (4.1.2) the defining equation is

$$| \vec{P} - \vec{F} |^2 | \vec{F} - \vec{G} |^2 = e^2 [(\vec{P} - \vec{G}) \circ (\vec{F} - \vec{G})]^2.$$

As we have observed before, a central conic has two vertices given by

$$\vec{V} = \frac{\vec{F} + e\vec{G}}{1+e}, \quad V' = \frac{\vec{F} - e\vec{G}}{1-e}.$$

The midpoint C of the segment $[VV']$ is the center of the conic. Thus

$$\vec{C} = \frac{1}{2}(\vec{V} + \vec{V}').$$

Let us choose a coordinate system, in the plane, with C as the origin of the coordinate system. Then

$$\vec{C} = \vec{0} = \frac{1}{2}(\vec{V} + \vec{V}'),$$

and therefore $\vec{V}' = -\vec{V}$. If we write this more explicitely,

$$\frac{\vec{F} + e\vec{G}}{1+e} = -\frac{\vec{F} - e\vec{G}}{1-e}.$$

We obtain

$$\vec{F} = e^2\vec{G}, \quad \vec{V} = e\vec{G}, \quad \vec{F} = e\vec{V}. \tag{4.3.1}$$

Hence the defining equation of the central conic with $C(0,0)$ can be written as

$$| \vec{P} - \vec{F} |^2 | \vec{F} - \frac{1}{e^2}\vec{F} |^2 = e^2 [(\vec{P} - \frac{1}{e^2}\vec{F}) \circ (\vec{F} - \frac{1}{e^2}\vec{F})]^2.$$

Simplfying this equation, we get

$$e^2(| \vec{P} |^2 | \vec{F} |^2) - e^4(\vec{P} \circ \vec{F})^2 = (1-e^2) \quad | \vec{F} |^4. \tag{4.3.2}$$

Thus, the above equation is the equation of a central conic centred at the origin $(C(0,0))$. Note that the equation involves only e and F.

We have proved before that every conic is symmetric about its axis. Now, we prove

Theorem 4.3.1 *Every central conic is symmetric about its center.*

Proof. Choose a coordinate system so that the center C of the conic is the origin. Then (4.3.2) is a defining equation of the conic; the symmetric partner of any point $P(x,y)$ about $C(0,0)$ is $P'(-x,-y)$. Since $| \vec{P} |=| \vec{P}' |$ and $(\vec{P} \circ \vec{F})^2 = (\vec{P}' \circ \vec{F})^2$, the point P satisfies the equation (4.3.2) if and only if P' satisfies it. That is, P is on the conic if and only if P' is on the conic. Hence the conic is symmetric about C. □

As in Fig 4.6., we consider the line ℓ' through the center C which is perpendicular to the axis ℓ. Combining Theorem 4.1.1 and Theorem 4.3.1, we obtain

Corollary 4.3.2 *Every central conic is symmetric about the line ℓ'.*

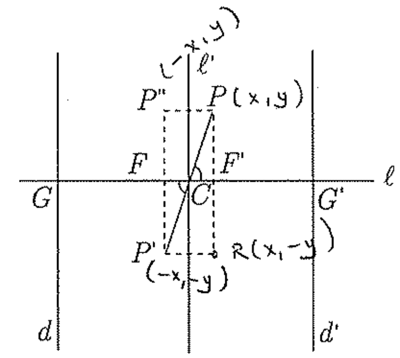


Fig.4.6.

Remark. Let F' be the symmetric partner of the focus F about the line ℓ'. Let also d' be the line which is symmetric with the directrix d about the line ℓ' (See Fig. 4.6). Then for any point P,

$$\frac{| PF |}{| Pd |} = \frac{| P'F |}{| P'd |} = \frac{| PF' |}{| Pd' |}.$$

Therefore, the conic with focus F', directrix d' and eccentricity e coincides with our former conic with focus F, directrix d and eccentricity e. In this sense, we can state the following

Corollary 4.3.3 *Every central conic has another focus F' and another directrix d'.*

The equation (4.3.2) involves e and F. Using the identities in (4.3.1), we can obtain equivalent equations for a central conic centred at the origin:

$$| \vec{P'} |^2 | \vec{V} |^2 - e^2(\vec{P} \circ \vec{V})^2 = (1 - e^2) | V |^4 \tag{4.3.3}$$

$$| \vec{P} |^2 | \vec{G} |^2 - e^2(\vec{P} \circ \vec{G})^2 = e^2(1 - e^2) | \vec{G} |^4 \tag{4.3.4}$$

We may thus conclude that a central conic centred at the origin is completely determined by any one of the pairs "e and F", "e and V" or "e and G".

Example 4.3.4 Find the equation, in coordinate form of the hyperbola with focus $F(1,1)$, eccentricity $e = 2$ and center $C(0,0)$.

Solution. The hyperbola is centred at the origin. Thus by (4.3.2), a point $P(x,y)$ is on the hyperbola if and only if,

$$e^2 | \vec{P}^2 | \vec{F} |^2 - e^4(\vec{P} \circ \vec{F})^2 = (1 - e^2) | \vec{F} |^4,$$

$$4(x^2 + y^2)(1 + 1) - 16(x + y)^2 = (1 - 4) \circ (1 + 1)^2$$

$$2x^2 + 2y^2 - 4x^2 - 8xy - 4y^2 = -3$$

$$2x^2 + 8xy + 2y^2 - 3 = 0.$$

In the above discussions, we only assume that the center of the conic is located at the origin of the coordinate system. The positions of the coordinate axes are arbitrary. If we further assume that one of the coordinate axes coincides with the axis ℓ of the conic, that is, if the vertices lie on one of the coordinate axes, then the defining equation of the conic gets a simpler form.

If V is on X-axis, say $V(a,0), a > 0$; then $F(ae,0)$ is the focus and $G(\frac{a}{e},0)$ is the point of intersection of the axis and the directrix (See (4.3.1)). The defining equation (4.3.3) becomes

$$(x^2 + y^2)(a^2) - e^2(ax)^2 = (1 - e^2)(a^2)$$

$$x^2 + y^2 - e^2x^2 = (1 - e^2)a^2$$

$$(1 - e^2)x^2 + y^2 = (1 - e^2)a^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{(1 - e^2)a^2} = 1.$$

We summarize these results in

Theorem 4.3.5 *The equation of a central conic with center at the origin and vertex $V(a,0), a > 0$, is*

$$\frac{x^2}{a^2} + \frac{y^2}{(1 - e^2)a^2} = 1.$$

The other vertex is $V'(-a,0)$, the foci are $F(ae,0), F'(-ae,0)$ and the directrices are $d = \{(x,y) : x = \frac{a}{e}\}, d' = \{(x,y) : x = -\frac{a}{e}\}$.

Similarly, if V is on Y-axis, we get

Theorem 4.3.6 *The equation of a central conic with center at the origin and vertex $V(0,a), a > 0$, is*

$$\frac{x^2}{(1 - e^2)a^2} + \frac{y^2}{a^2} = 1.$$

The other vertex is $V'(0,-a)$, the foci are $F(0,ae), F'(0,-ae)$ and the directrices are $d = \{(x,y) : y = \frac{a}{e}\}, d' = \{(x,y) : y = -\frac{a}{e}\}$.

4.4 The Ellipse

If e is the eccentricity of an ellipse then

$$0 < e < 1 \Rightarrow 0 < e^2 < 1 \Rightarrow 0 < 1 - e^2 < 1.$$

Hence $(1 - e^2)$ is a positive real number. When the ellipse is centered at the origin and has vertex $V(a,0), a > 0$, on X-axis, we define

$$b = a\sqrt{1 - e^2} \tag{4.4.1}$$

Then $b^2 = a^2(1 - e^2)$ and Theorem 4.3.5 can be restated for an ellipse as follows

Theorem 4.4.1 *The equation of an ellipse with center at the origin and vertex $V(a,0), a > 0$, is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (b = a\sqrt{1 - e^2}) \tag{4.4.2}$$

The other vertex is $V'(-a,0)$, the foci are $F(ae,0), F'(-ae,0)$, and the directrices are $d = \{(x,y) : x = \frac{a}{e}\}, d' = \{(x,y) : x = -\frac{a}{e}\}$.

Remark. Since $1 - e^2 < 1$, we have $b < a$. As for the parabola, the chord of the ellipse through a focus perpendicular to the axis is called a latus rectum. Each of $[L_1L_2]$ and $[L_1'L_2']$ is a latus rectum for the ellipse in Fig.4.7. The chord $[VV']$ of the ellipse on the axis ℓ is called the major diameter and the chord $[WW']$ on ℓ' is called the minor diameter of the ellipse. For the ellipse (4.4.2) in Theorem 4.4.1, we have $L_1(ae, a(1-e^2)), L_2(ae, -a(1-e^2)), L_1'(-ae, a(1-e^2), L_2'(-ae, -a(1-e^2)), W(0,b), W'(0,-b)$. These points, together with the vertices, are enough to sketch the ellipse.

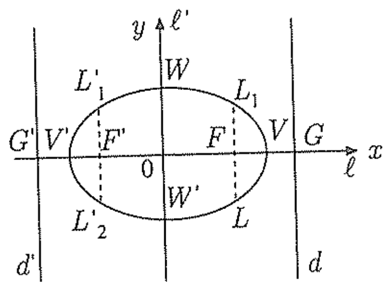


Figure 4.7.

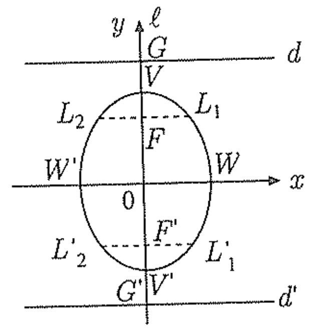


Figure 4.8.

Corresponding to Theorem 4.3.6, we have

Theorem 4.4.2 *The equation of an ellipse with center at the origin and vertex $V(0,a), a > 0$, is*

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad (b = a\sqrt{1-e^2}) \tag{4.4.3}$$

The other vertex is $V'(0,-a)$, the foci are $F(0, ae), F'(0, -ae)$, and the directrices are $d = \{(x,y) : y = \frac{a}{e}\}, d' = \{(x,y) : y = -\frac{a}{e}\}$.

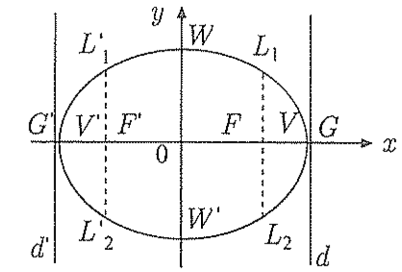
Figure 4.8 illustrates the graph of an ellipse with center at the origin and vertex on Y-axis.

Example 4.4.3 Find the vertices, foci, eccentricity, directrices, and length of the latus rectum, major and minor diameters of the ellipse whose equation is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Solution. Since $4 < 9$, this is the equation of an ellipse centered at the origin with vertices $V(3,0)$, $V'(-3,0)$ on X-axis. Thus $a = 3, b = 2$. Major diameter is $2a = 6$ units and minor diameter is $2b = 4$ units long. Since $b = a\sqrt{1-e^2}$, $2 = 3\sqrt{1-e^2} \Longrightarrow 1 - e^2 = \frac{4}{9} \Longrightarrow e = \frac{\sqrt{5}}{3}$. The foci are $F(\sqrt{5}, 0), F(-\sqrt{5}, 0)$.. The directrices are

$$d = \{(x,y) : x = \frac{-9}{\sqrt{5}}\}, d' = \{(x,y) : x = \frac{9}{\sqrt{5}}\}.$$

Hence the length of a latus rectum is $2a(1-e^2) = \frac{8}{3}$ units.



Remark. We have thus seen that an equation of the form $Ax^2 + Cy^2 = 1$ with $A > 0, C > 0$ and $A \neq C$ represents an ellipse with center at the origin and vertices on one of the coordinate axes. Note that if $A = C$ then the equation represents a circle. If $A < 0$ and $C < 0$ then the graph of the equation is empty. These two forms will be considered as *degenerate forms* of an ellipse.

Exercises

1. Find the vertices, foci, directrices and length of the latus rectum of the ellipse whose equation is

   a) $9x^2 + 4y^2 = 36$, b) $25x^2 + 4y^2 = 25$,
   c) $4x^2 + 25y^2 = 25$, d) $4x^2 + 25y^2 = 100$,
   e) $25x^2 + 4y^2 = 100$, f) $3x^2 + 5y^2 = 1$.

2. In each of the following cases write down the equation and sketch the graph of the ellipse with the given information.
   a) Center $C(0,0)$, major diameter along X-axis and 4 units long, minor diameter 1 unit long.
   b) Foci $F(2,0), F'(-2,0)$, and major diameter 8 units long.
   c) Vertices $V(0,5), V'(0,-5)$, and minor diameter 6 units long.
   d) Focus $F(4,0)$, directrix $d = \{(x,y) : x = 9\}, e = \frac{2}{3}$.

3. What are the nearest points of an ellipse to its focus?

4. The earth moves in an elliptical orbit with major diameter 92.9 miles long, eccentricity $e = 0.017$, and the sun is at one of the foci. How close does the earth come to the sun?

5. Find the points of intersection (if any) of the line $2x - 3y = 2$ with each of the ellipses in exercise 1 and exercise 2.

6. Discuss the graph of the following relations

a) $\{(x,y) : \frac{x^2}{25} + \frac{y^2}{16} = 1\}$

b) $\{(x,q) : (x,y) : \frac{x^2}{25} + \frac{y^2}{16} = 1\}$

c) $\{(x,y) : sin(\pi\sqrt{4x^2 + 9y^2}) = 0\}$

d) $\{(x,y) : 4x^2 < (2-y)(2+y)\}$.

7. Write the equation of the ellipse satisfying the following conditions:
1) the major axis equals 26, and the foci are $F_1(-10,0)$ and $F_2(14,0)$;
2) the minor axis equals 2, and the foci are $F_1(-1,-1), F_2(1,1)$;
3) the foci are $F_1(=2,\frac{3}{2}), F_2(2,-\frac{3}{2})$, and the eccentricity $e = \frac{\sqrt{2}}{2}$;
4) the foci are $F_1(1,3), F_2(3,1)$, and the distance between the directrices is $12\sqrt{2}$.

8. Find the equation of an ellipse, given the eccentricity $e = \frac{2}{3}$, one focus $F(2,1)$, and the equation $x - 5 = 0$ of the directrix corresponding to this focus.

9. Find the equation of an ellipse, given the eccentricity $e = \frac{1}{2}$, one focus $F(-4,1)$, and the equation $y + 3 = 0$ of the directrix corresponding to this focus.

10. The point $A(-3,-5)$ lies on an ellipse which has a focus $F(-1,-4)$ and whose corresponding directrix is given by the equation

$$x - 2 = 0.$$

Write the equation of the ellipse.

11. Determine the values of m for which the line $y = -x + m$:
1) cuts the ellipse $\frac{x^2}{20} + \frac{y^2}{5} = 1$; 2) touches the ellipse; 3) passes outside the ellipse.

12. Write the equations of the tangent lines to the ellipse

$$\frac{x^2}{10} + \frac{2y^2}{5} = 1$$

which are parallel to the line

$$3x + 2y + 7 = 0.$$

13. From the point $A(\frac{10}{3}, \frac{5}{3})$, tangent lines are drawn to the ellipse

$$\frac{x^2}{20} + \frac{y^2}{5} = 1.$$

Write their equations.

14. Find the equation of the ellipse whose axes are coincident with the coordinate axes and which touches the two lines $3x - 2y - 20 = 0$, $x + 6y - 20 = 0$.

4.5 The Hyperbola

If e is the eccentricity of an hyperbola, then $e > 1 \Rightarrow e^2 - 1 > 0$. Hence $(e^2 - 1)$ is a positive real number. When the hyperbola is centered at the origin and has vertex $V(a,0), a > 0$, on X-axis, we define $b = a\sqrt{e^2-1}$. Then $b^2 = a^2(e^2-1)$ and Theorem 4.3.5 can be restated for an hyperbola as follows

Theorem 4.5.1 *The equation of an hyperbola with center at the origin and vertex $V(a,0), a > 0$, is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (b = a\sqrt{e^2-1}).$$

The other vertex is $V'(-a,0)$, the foci are $F(ae,0), F'(-ae,0)$, and the directrices are $d = \{(x,y) : x = \frac{a}{e}\}$, $d' = \{(x,y) : x = -\frac{a}{e}\}$.

The graph of an hyperbola with center at the origin and vertex on X-axis is illustrated in Fig. 4.9.

Corresponding to Theorem 4.3.6, we have

Theorem 4.5.2 *The equation of an hyperbola with center at the origin and vertex $V(0,a), a > 0$, is*

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad (b = a\sqrt{e^2-1}).$$

The other vertex is $V'(0,-a)$, the foci are $F(0,ae)$, $F'(0,-ae)$ and the directices are $d = \{(x,y) : y = \frac{a}{e}\}$, $d' = \{(x,y) : y = -\frac{a}{e}\}$.

The graph of an hyperbola with center at the origin and vertex on Y-axis is illustrated in Fig. 4.10.

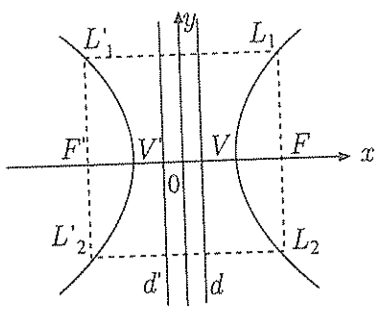


Fig. 4.9.

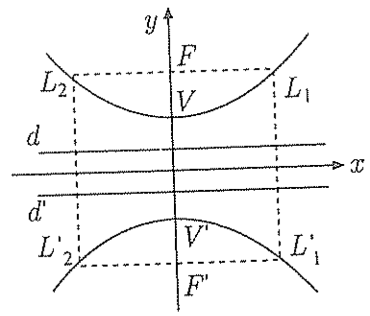
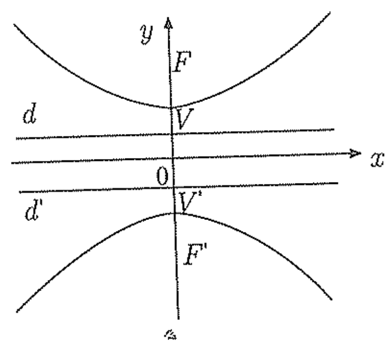


Fig. 4.10.

As for the other conics, the chord of the hyperbola through a focus perpendicular to the axis of the hyperbola is called a *latus rectum.* Each of $[L_1L_2]$ and $[L_1'L_2']$ is a latus rectum for the hyperbola in Fig.4.9. or in Fig.4.10. Note that the hyperbola never intersects the line ℓ'. Note also that the number $b = a\sqrt{e^2-1}$ may be greater than or less than or equal to the number a according as $e < \sqrt{2}$ or $e > \sqrt{2}$ or $e = \sqrt{2}$. For an ellipse we always have $b < a$.

Example 4.5.3 Find the vertices, foci, eccentricity, directrices and length of the latus rectum of the hyperbola with equation $\frac{y^2}{9} - \frac{x^2}{16} = 1$. Sketch its graph.



Solution. By Theorem 4.5.2, this is an hyperbola with center at the origin, vertices $V(0,3), V'(0,-3)$. We have $a = 3$ and $b = 4$. Since $b^2 = a^2(e^2-1)$.

$$16 = 9(e^2-1) \Longrightarrow 9e^2 = 25 \Longrightarrow e = \frac{5}{3}$$

Hence the foci are $F(0,5), F'(0,-5)$. The directrices are given by $y = 9/5, y = -9/5$. The endpoints of the latus rectum corresponding to F are $L_1(\frac{16}{3},5), L_2(-\frac{16}{3},5)$. Therefore the length of the latus rectum is $\dfrac{16}{3} + \dfrac{16}{3} = \dfrac{32}{3}$.

Remark. Thus, an equation of the form $Ax^2 - Cy^2 = 1$ with $AC > 0$ represents an hyperbola with center at the origin and vertices on one of the coordinate axes.

Exercises

1. Find the vertices, foci, eccentricity, directrices and the length of the latus rectum of the hyperbola whose equation is
   a) $\frac{x^2}{4} - \frac{y^2}{9} = 1$, d) $25x^2 - 4y^2 = 25$,
   b) $\frac{x^2}{9} - \frac{y^2}{4} = 1$, e) $25y^2 - 4x^2 = 100$,
   c) $\frac{y^2}{9} - \frac{x^2}{4} = 1$ f) $3x^2 - 5y^2 = 1$.

2. In each of the following cases, find the equation and sketch the graph of the hyperbola with the given information
   a) Center $C(0,0)$, vertex $V(3,0), e = 2$.
   b) Focus $F(-6,0)$, directrix $d = \{(x,y) : x = -2\}, e = 3$.
   c) Center $C(0,0)$, focus $F(5,0)$, length of the latus rectum is $\frac{32}{3}$ units long.
   d) Center $C(0,0)$, focus $F(5,5), e = 2$.

3. Find the points of intersection (if any) of the line $3x + 4y = 0$ with each of the hyperbolas in exercise 1 and exercise 2.

4. Discuss the graph of the following relations
   a) $\{(x,y) : \frac{x^2}{25} - \frac{y^2}{16} \leq 1\}$, d) $\{(x,y) : x^2 - y^2 - 1 \leq 0\}$,
   b) $\{(x,y) : \frac{x^2}{25} - \frac{y^2}{16} > 1\}$, e) $\{(x,y) : x(x - \mid x \mid) + 2y(y + \mid y \mid) = 4\}$
   c) $\{(x,y) : \frac{x^2}{25} - \frac{y^2}{25} + 1 \leq 0\}$,

5. Write the equation of the hyperbola whose foci are symmetrically situated on the x-axis with respect to the origin, and which satisfies the following conditions:
   a) the axes $2a = 10$ and $2b = 8$;
   b) the distance between the foci $2c = 10$, and the axis $2b = 8$;
   c) the distance between the foci $2c = 6$, and the eccentricity $e = \frac{3}{2}$;
   d) the axis $2a = 16$ and the eccentricity $e = \frac{5}{4}$;
   e) the equations of the asymptoties are

$$y = \pm\frac{4}{3}x,$$

   and the distance between the foci $2c = 20$;
   f) the distance between the directrices is equal to $22\frac{2}{13}$, and the distance between

the foci $2c = 26$;
g) the distance between the directrices is $\frac{32}{5}$, and the axis $2b = 6$;
h) the distance between the directrices is $\frac{8}{3}$, and the eccentricity $e = \frac{3}{2}$;
i) the equations of the asymptotes are $y = \pm\frac{3}{4}x$, and the distance between the directrices is $12\frac{4}{5}$.

6. Find the equation of the hyperbola whose foci are symmetrically situated on the y-axis with respect to the origin, and which satisfies the following conditions:
a) the semi-axes $a = 6, b = 18$ (the latter a denotes here the semi-axis of the hyperbola lying on the x-axis);
b) the distance between the foci $2c = 10$, and the eccentricity $e = \frac{5}{3}$;
c) the equations of the asymptotes are

$$y = \pm\frac{12}{5}x,$$

and the distance between the vertices equals 48;
d) the distance between the directrices is $7\frac{1}{7}$, and the eccentricity $e = \frac{7}{5}$;
e) the equations of the asymptotes are $y = \pm\frac{4}{3}x$, and the distance between the directrices is $6\frac{2}{5}$.

7. Write the equation of a hyperbola whose foci are symmetrically situated on the x-axis with respect to the origin, given:
a) the points $M_1(6, -1)$ and $M_2(-8, 2\sqrt{2})$ of the hyperbola;
b) the point $M_1(= 5, 3)$ of the hyperbola and the eccentricity $e = \sqrt{2}$;
c) the point $M_1(\frac{9}{2}, -1)$ of the hyperbola and the equations $y = \pm\frac{2}{3}x$ of the asymptotes;
d) the point $M_1(-3, \frac{5}{2})$ of the hyperbola and the equations $x = \pm\frac{4}{3}$ of the directrices;
e) the equations $y = \pm\frac{3}{4}x$ of the asymptotes and the equations $x = \pm\frac{16}{5}$ of the directrices.

8. Write the equation of the hyperbola satisfying the following conditions:
1) the distance between its vertices is 24, and the foci are $F_1(-10, 2), F_2(16, 2)$;
2) the foci are $F_1(3, 4), F_2(-3, -4)$, and the distance between the directrices equals 3.5;
3) the angle between the asymptotes is $90°$, and the foci are $F_1(4, -4), F_2(-2, 2)$.

9. Find the equation of a hyperbola, given its eccentricity $e = \frac{5}{4}$, one focus $F(5, 0)$, and the equation $5x - 16 = 0$ of the directrix associated with this focus.

10. Find the equation of a hyperbola, given its eccentricity $e = \frac{13}{12}$, one focus $F(0, 13)$, and the equation $13y - 144 = 0$ of the directrix associated with this focus.

11. Determine the values of m for which the line $y = \frac{5}{2}x + m$:
1) cuts the hyperbola $\frac{x^2}{9} - \frac{y^2}{36} = 1$; 2) touches this hyperbola; 3) passes outside the hyperbola.

12. Find the equation of the hyperbola whose axes are coincident with the coordinate axes and which touches the two lines $5x - 6y = 0$, $13x - 10y - 48 = 0$.

13. Show that the points of intersection of the ellipse $\frac{x^2}{20} + \frac{y^2}{5} = 1$ and the hyperbola $\frac{x^2}{12} - \frac{y^2}{3} = 1$ are the vertices of a rectangle, and find the equations of the sides of this rectangle.

4.6 The Asymptotes of an Hyperbola

Consider an hyperbola centered at the origin and having vertices, say $V(a, 0), V'(-a, 0)$ on X-axis. We know that the equation of such an hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b = a\sqrt{e^2 - 1}$ and $e > 1$ is the eccentricity of the hyperbola.

We have already noticed that Y-axis, which is the line ℓ' in our former notations, does not intersect the hyperbola. Now, the following question may be of interest. *Under what conditions does a line $Ax + By = 0$ through the origin intersects the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$?* To answer this question, we solve the equation of the line and the equation of the hyperbola simultaneously. Since we may exclude Y-axis, we may assume that $B \neq 0$. Substituting $y = -\frac{A}{B}x$ in the equation of the hyperbola, we obtain

$$\frac{x^2}{a^2} - \frac{[(A/B)x]^2}{b^2} = 1$$

$$(b^2B^2 - a^2A^2)x^2 = B^2a^2b^2.$$

This equation has a solution for x if and only if

$$(b^2B^2 - a^2A^2) > 0,$$

and this is the case if and only if

$$\left| \frac{A}{B} \right| < \frac{b}{a}.$$

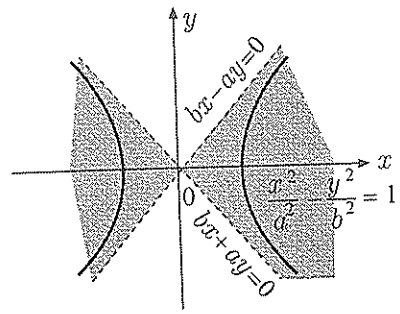


Fig. 4.11.

Thus the line $Ax + By = 0$ with $B \neq 0$ intersects the hyperbola if and only if the slope $-\frac{A}{B}$ of the line satisfies

$$-\frac{b}{a} < -\frac{A}{B} < \frac{b}{a}.$$

If the slope of the line is positive, i.e., if $-\frac{A}{B} > 0$ then the smallest value $-\frac{A}{B}$ can take on without the line intersect the hyperbola is $\frac{b}{a}$; but if $-\frac{A}{B} < 0$ then the largest value $-\frac{A}{B}$ can take on without the line intersect the hyperbola is $-\frac{b}{a}$. This means that the graph of the hyperbola lies in the region which is the set of all points below the line $y = \frac{b}{a}x$ and above the line $y = -\frac{b}{a}x$ (the shaded region in Fig.4.11). These two lines

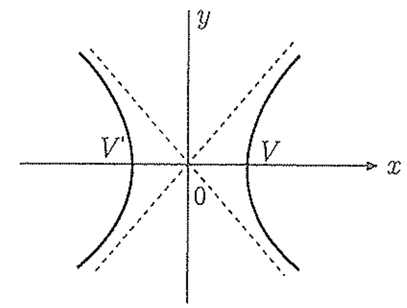
$$y = \frac{b}{a}x \text{ and } y = -\frac{b}{a}x \quad \text{or} \quad bx - ay = 0 \text{ and } bx + ay = 0$$

are called the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Similar discussions can be given for an hyperbola with equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Then the *asymptotes* are the lines

$$y = \frac{a}{b}x \text{ and } y = -\frac{a}{b}x \quad \text{or} \quad ax - by = 0 \text{ and } ax + by = 0.$$

Example 4.6.1 Find the asymptotes of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ and sketch its graph.



Solution. $a = 3$ and $b = 4$. Thus the asymptotes are

$$4x - 3y = 0 \text{ and } 4x + 3y = 0.$$

The vertices are $V(3,0)$. $V'(-3,0)$. The eccentricity is obtained by

$$b = 4 = a\sqrt{e^2-1} = 3\sqrt{e^2-1} \Rightarrow e^2 - 1 = \frac{16}{9} \Longrightarrow e = \frac{5}{3}.$$

The foci are $F(5,0), F'(-5,0)$, and the end points of latus rectum are $L_1(5, \frac{16}{3}), L_2(5, -\frac{16}{3}), L_1'(-5, \frac{16}{3}), L_2'(-5, -\frac{16}{3})$.

Exercises

1. Find the asymptotes of each of the following hyperbolas
   a) $\frac{x^2}{4} - \frac{y^2}{9} = 1$, b) $\frac{y^2}{9} - \frac{x^2}{4} = 1$, c) $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

2. A practical way for finding the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the following
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} \Longrightarrow (\frac{x}{a} - \frac{y}{b})(\frac{x}{a} + \frac{y}{b}) = 0 \Longrightarrow y = \frac{b}{a}x \text{ or } y = -\frac{b}{a}x.$$
   Use this method to find the asymptotes of $4x^2 - 25y^2 = 100$.

3. Find the equation and sketch the graph of the hyperbola which contains the point $(4,1)$ and whose asymptotes are $x + 2y = 0$ and $x - 2y = 0$.

4.7 The General Quadratic Equation

It was proved in Chapter 2 that any linear equation $Ax + By + C = 0$ is the defining equation of a straight line in the plane. One way of generalizing the linear equation is to increase its degree and consider all possible second degree terms along with those of the first degree. Such an equation might be written as

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0. \tag{4.7.1}$$

An equation as above, where not all of the coefficients A, B, C are zero, is called a *quadratic equation* in x and y. (The coefficients 2 are used for convenience).

We have seen in the preceding sections that the defining equation of any conic section is a quadratic equation in a suitable coordinate system. The aim of this section to show that the graph of any quadratic equation is a conic section or a degenerate form of a conic section, such as two intersecting lines. Here by the graph of (4.7.1) we mean the set of all points $P(x, y)$ in XY-plane satisfying that equation. If we change coordinates, the same graph turns out to be the graph of some other quadratic equation. Therefore by suitable change of coordinates one may expect to simplify the general quadratic equation (4.7.1). In simplifying (4.7.1), it is required to

a) *eliminate the xy-term,*

b) *reduce the number of first degree terms to a minimum* (remove them completely, if possible),

c) *remove the constant term, if possible.*

If the coefficient $2B$ is different from zero, in (4.7.1), then we rotate the XY-coordinate system through an angle $\alpha, 0 < \alpha < \frac{\pi}{2}$, to obtain $\overline{XY}$-coordinate system. Thus the XY-coordinates (x, y) and $\overline{XY}$-co-ordinates $(\overline{x}, \overline{y})$ of a point are related by

$$\begin{aligned} x &= \overline{x}cos\alpha - \overline{y}sin\alpha \\ y &= \overline{x}sin\alpha + \overline{y}cos\alpha. \end{aligned}$$

Substituting these in (4.7.1), we get

$$\begin{aligned} Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F &= \overline{A}\overline{x}^2 + 2\overline{B}\overline{x}\overline{y} + \overline{C}\overline{y}^2 + 2\overline{D}\overline{x} + 2\overline{E}\overline{y} + \overline{F} \\ &= 0. \end{aligned} \tag{4.7.2}$$

where

$$\begin{aligned} \overline{A} &= Acos^2\alpha + Bsin2\alpha + Csin^2\alpha \\ 2\overline{B} &= 2Bcos2\alpha - (A - C)sin2\alpha \\ \overline{C} &= Asin^2\alpha - Bsin2\alpha + Ccos^2\alpha \end{aligned}$$

$$\begin{aligned} 2\overline{D} &= 2Ecos\alpha - 2Dsin\alpha \\ 2\overline{E} &= 2E\cos\alpha - 2D\sin\alpha \\ \overline{F} &= F. \end{aligned}$$

Using trigonometric identities, one can prove (See Exercise 2) the following

$$\overline{A} + \overline{C} = A + C \text{ and } \overline{AC} - \overline{B^2} = AC - B^2. \tag{4.7.3}$$

Since $0 < \alpha < \frac{\pi}{2}$, we have $sin\alpha > 0, cos\alpha > 0$ and $sin2\alpha > 0$. Thus in (4.7.2),

$$\frac{\overline{B}}{Bsin2\alpha} = cot2\alpha - \frac{A - C}{2B}.$$

Therefore, $\overline{B}$ will be zero if we choose α so that

$$cot2\alpha = \frac{A - C}{2B}, \qquad 0 < \alpha < \frac{\pi}{2}. \tag{4.7.4}$$

We *can* choose α in this way, because $cot2\alpha$ takes all values in $\mathbb{R}$ when $0 < 2\alpha < \pi$.

Hence we can eliminate the xy-term (if it exists) from (4.7.1) *by rotating the coordinate axes through the angle* α *satisfying* (4.7.4).

Example 4.7.1 Eliminate the xy-term from

$$8x^2 - 4xy + 5y^2 - 36 = 0.$$

Solution. We have $A = 8$, $B = -2, C = 5, D = E = 0, F = -36$; and

$$\begin{aligned} cot2\alpha &= \frac{8-5}{-4} = -\frac{3}{4}, \quad 0 < \alpha < \frac{\pi}{2} \\ sin2\alpha &= \frac{1}{\sqrt{1 + cot^2 2\alpha}} = \frac{1}{\sqrt{1 + 9/16}} = \frac{4}{5} \\ cos2\alpha &= cot2\alpha . sin2\alpha = -\frac{3}{4}.\frac{4}{5} = -\frac{3}{5}. \end{aligned}$$

Hence

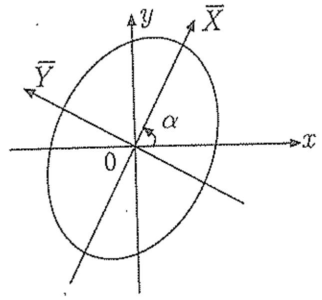
$$cos\alpha = \sqrt{\frac{1 + cos2\alpha}{2}} = \sqrt{\frac{1 - 3/5}{2}} = \frac{1}{\sqrt{5}} \quad sin\alpha = \sqrt{\frac{1 - cos2\alpha}{2}} = \sqrt{\frac{1 + 3/5}{2}} = \frac{2}{\sqrt{5}}.$$

We rotate the coordinate axes through α. Then

$$\begin{aligned} x &= \overline{x}cos\alpha - \overline{y}sin\alpha = \frac{\overline{x} - 2\overline{y}}{\sqrt{5}} \\ y &= \overline{x}sin\alpha + \overline{y}cos\alpha = \frac{2\overline{x} + \overline{y}}{\sqrt{5}} \end{aligned}$$

and the given equation reduces to

$$4\overline{x}^2 + 9\overline{y}^2 - 36 = 0 \quad \text{or} \quad \frac{\overline{x}^2}{9} + \frac{\overline{y}^2}{4} = 1$$



which is the defining equation of an ellipse in $\overline{XY}$-coordinates. The coefficients in the last equation may be computed directly by substituting the values of x and y in terms of $\overline{x}$ and $\overline{y}$ in the original equation or by using (4.7.2).

Now, we explain how to reduce the number of linear terms (in a general quadradic equation) to a minimum. Suppose that we have eliminated the xy-term from (4.7.1) and obtained

$$\overline{Ax^2} + \overline{Cy^2} + 2\overline{Dx} + 2\overline{Ey} + \overline{F} = 0. \tag{4.7.5}$$

Let us translate the $\overline{XY}$-coordinate system to $\overline{\overline{XY}}$-coordinate system by

$$\overline{x} = \overline{\overline{x}} + h, \overline{y} = \overline{\overline{y}} + k$$

where

$$h = \begin{cases} -\overline{D}/\overline{A} & if \quad \overline{A} \neq 0 \\ 0 & if \quad \overline{A} = 0 \end{cases} \qquad k = \begin{cases} -\overline{E}/\overline{C} & if \quad \overline{C} \neq 0 \\ 0 & if \quad \overline{C} = 0. \end{cases}$$

Such a translation transforms the equation (4.7.5) to

$$\overline{A}\overline{\overline{x}}^2 + \overline{C}\overline{\overline{y}}^2 + 2\overline{\overline{D}}\overline{\overline{x}} + 2\overline{\overline{E}}\overline{\overline{y}} + \overline{\overline{F}} = 0 \tag{4.7.6}$$

where

$$\overline{\overline{D}} = \overline{A}h + \overline{D} = \begin{cases} 0 & if \quad \overline{A} \neq 0 \\ \overline{D} & if \quad \overline{A} = 0, \end{cases}$$

$$\overline{\overline{E}} = \overline{C}k + \overline{E} = \begin{cases} 0 & if \quad \overline{C} \neq 0 \\ \overline{E} & if \quad \overline{C} = 0. \end{cases}$$

$$\overline{\overline{F}} = \overline{A}h^2 + \overline{C}k^2 + \overline{F}.$$

We can thus conclude that the equation (4.7.6), which is obtained from (4.7.1) by a rotation and a translation of the coordinate axes, satisfies the requirements a) and b) stated in the beginning of this section. Therefore, we may get information about the general quadratic equation (4.7.1) by analysing (4.7.6).

For the equation (4.7.6), we have three cases:

$$\overline{A}\,\overline{C} = 0 \text{ or } \overline{A}\,\overline{C} > 0 \text{ or } \overline{A}\,\overline{C} < 0.$$

By (4.7.3), these cases correspond to

$$AC - B^2 = 0 \text{ or } AC - B^2 > 0 \text{ or } AC - B^2 < 0.$$

Case 1. $\overline{A}\,\overline{C} = 0$. We first note that $\overline{A}$ and $\overline{C}$ can not both be zero. For, otherwise, by (4.7.3), $A + C = 0$ and $AC = B^2$. This implies $(A+C)^2 = A^2 + 2AC + C^2 = A^2 + 2B^2 + C^2 = 0$, which is impossible by our assumption that not all of A, B or C are zero. Then (4.7.6) reduces to

$$\overline{A}\overline{\overline{x}}^2 + 2\overline{\overline{E}}\overline{\overline{y}} + \overline{\overline{F}} = 0 \text{ or } \overline{C}\overline{\overline{y}}^2 + 2\overline{\overline{D}}\overline{\overline{x}} + \overline{\overline{F}} = 0$$

according as $\overline{A} \neq 0$ or $\overline{C} \neq 0$. Let us consider the case $\overline{A} \neq 0$ (the first equation). If $\overline{\overline{E}} \neq 0$, that equation can be written as

$$\overline{\overline{x}}^2 + 2(\overline{\overline{E}}/\overline{A})(\overline{\overline{y}} + \overline{A}\,\overline{\overline{F}}/2\overline{\overline{E}}) = 0.$$

We perform the translation $\overline{\overline{x}} = \tilde{x}$, $\overline{\overline{y}} = \tilde{y} - \overline{A}\overline{\overline{F}}/2\overline{\overline{E}}$. Then the last equation reduces to $\tilde{x}^2 + 2(\overline{\overline{E}}/\overline{A})\tilde{y} = 0$, which is the defining equation of a parabola in $\tilde{X}\tilde{Y}$- coordinates. If $\overline{\overline{E}} = 0$, then the equation that we consider can be written as

$$\overline{\overline{x}}^2 + \overline{\overline{F}}/\overline{A} = 0.$$

It is clear that this equation represents the X-axis if $\overline{\overline{F}} = 0$, no points if $\overline{\overline{F}}/\overline{A} > 0$ and it represents two vertical lines if $\overline{\overline{F}}/\overline{A} < 0$. All these are considered as *degenerate forms* of a parabola.

The case $\overline{C} \neq 0$ can be discussed similarly. We leave it to the reader to show that the equation corresponding to the case $\overline{A} = 0, \overline{C} \neq 0$ represents a parabola or a degenerate form of a parabola (Exercise 3).

Case 2. $\overline{A}\,\overline{C} > 0$. Then $\overline{A}$ and $\overline{C}$ are either both positive or both negative, and (4.7.6) reduces to

$$\overline{A}\overline{\overline{x}}^2 + \overline{C}\overline{\overline{y}}^2 + \overline{\overline{F}} = 0.$$

If $\overline{\overline{F}} \neq 0$ and has opposite sign as $\overline{A}$ (or $\overline{C}$), then the above equation can be written as

$$\frac{\overline{\overline{x}}^2}{(-\overline{\overline{F}}/\overline{A})} + \frac{\overline{\overline{y}}^2}{-\overline{\overline{F}}/\overline{C})} = 1$$

where $(-\overline{\overline{F}}/\overline{A}) > 0$ and $(-\overline{\overline{F}}/\overline{C}) > 0$. Hence it represents an ellipse (or possibly a circle) in $\overline{\overline{XY}}$-plane. If $\overline{\overline{F}} \neq 0$ and has the same sign as $\overline{A}$ (or $\overline{C}$), then there is no point in $\overline{\overline{XY}}$-plane satisfying the equation $\overline{A}\overline{\overline{x}}^2 + \overline{C}\overline{\overline{y}}^2 + \overline{\overline{F}} = 0$. If $\overline{\overline{F}} = 0$, then the graph of $\overline{A}\overline{\overline{X}}^2 + \overline{C}\overline{\overline{y}}^2 + \overline{\overline{F}} = 0$ consists of the origin in $\overline{\overline{XY}}$- plane. The last two cases are considered as degenerate forms of an ellipse.

Case 3. $\overline{AC} < 0$. Then $\overline{A}$ and $\overline{C}$ have opposite sign, and (4.7.6) reduces to

$$\overline{A}\overline{\overline{x}}^2 + \overline{C}\overline{\overline{y}}^2 + \overline{\overline{F}} = 0.$$

If $\overline{\overline{F}} \neq 0$, we write this equation in the form

$$\frac{\overline{\overline{x}}^2}{(-\overline{\overline{F}}/\overline{A})} + \frac{\overline{\overline{y}}^2}{(-\overline{\overline{F}}/\overline{C})} = 1$$

and we see that it represents an hyperbola, because $(\overline{\overline{F}}/\overline{A})$ and $(\overline{\overline{F}}/\overline{C})$ have opposite sign. If $\overline{\overline{F}} = 0$, then

$$\overline{A}\overline{\overline{x}}^2 + \overline{C}\overline{\overline{y}}^2 = 0 \quad \text{or} \quad \overline{\overline{y}} = \pm\sqrt{\overline{A}/\overline{C}}\,\overline{\overline{x}}$$

which represents two lines through the origin in $\overline{\overline{XY}}$-plane. The latter is called a degenerate form of an hyperbola.

Thus, using the fact that $\overline{AC} = AC - B^2$, we can state

Theorem 4.7.2 *The graph of the equation (4.7.1) is*
a) a parabola (or a degenerate parabola) if $AC - B^2 = 0$,
b) an ellipse (or a degenerate ellipse) if $AC - B^2 > 0$,
c) an hyperbola (or a degenerate hyperbola) if $AC - B^2 < 0$.

Example 4.7.3 Identify and sketch the graph of

$$4x^2 + 24xy + 11y^2 - 48x - 14y - 45 = 0.$$

Solution. $AC - B^2 = 44 - 144 < 0$. Hence the graph is an hyperbola or a degenerate hyperbola. We have

$$cot2\alpha = \frac{4-11}{24} = \frac{-7}{24}, \quad 0 < \alpha < \frac{\pi}{2}.$$

Thus

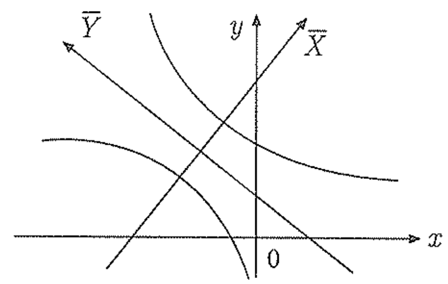
$$sin2\alpha = \frac{1}{\sqrt{1 + cot^2 2\alpha}} = \frac{1}{\sqrt{1 + \frac{49}{576}}} = \frac{24}{25},$$

$$cos2\alpha = -\sqrt{(1 - sin^2 2\alpha)^2} = \frac{-7}{25} \text{ (since } cot2\alpha < 0),$$

$$cos\alpha = \sqrt{\frac{1 + cos2\alpha}{2}} = \sqrt{\frac{1 - 7/25}{2}} = \frac{3}{5}, \quad sin\alpha = \frac{4}{5}.$$

We substitute $x = \frac{3\overline{x} - 4\overline{y}}{5}$, $y = \frac{4\overline{x} + 3\overline{y}}{5}$ in the given equation and we obtain $4\overline{x}^2 - \overline{y}^2 - 8\overline{x} + 6\overline{y} - 9 = 0$. We translate by $\overline{x} = \overline{\overline{x}} + 1$, $\overline{y} = \overline{\overline{y}} + 3$. Then the last equation reduces to $4\overline{\overline{x}}^2 - \overline{\overline{y}}^2 - 4 = 0$ or $\frac{\overline{\overline{x}}^2}{1} - \frac{\overline{\overline{y}}^2}{4} = 1$.

The graph is as follows:



Exercises

1. Identify and sktech the graph of the following equations
   a) $x^2 + 24xy - 6y^2 - 28x - 30y + 40 = 0$
   b) $x^2 - 24xy - 6y^2 + 10x - 8y - 20 = 0$
   c) $4x^2 - 12xy - 3y^2 + x + y - 1 = 0$
   d) $15x^2 - 24xy - 10y^2 - 60x + 70y + 100 = 0$
   e) $5x^2 - xy + 5y^2 - 3x + y - 1 = 0.$

2. Prove the identities in (4.7.3).

3. Consider the quadratic equation $Cy^2 + 2Dx + F = 0$ where $C \neq 0$. Describe its graph for all possible values of C, D and F.

4. Identify and sketch the graph of the following equations.
   a) $x^2 + 2xy + y^2 - 4x - 4y + 5 = 0$

b) $x^2 + 2xy + y^2 - 4x - 4y = 0$
c) $x^2 + 2xy + y^2 - 4x - 4y - 4 = 0$
d) $6x^2 - 7xy - 20y^2 - 7x + 7y - 3 = 0$
e) $4x^2 - 4xy + y^2 + 4x - 2y + 1 = 0$.

5. Show that if $AC - B^2 \neq 0$ in the general quadratic equation (4.7.1) then it is possible to reduce the number of linear terms to a minimum by performing directly a suitable translation. (*Hint:* Substitute $x = \tilde{x} + h; y = \tilde{y} + k$ in (4.7.1) and determine suitable values for h and k.)

6. In each of the following, determine the type of the given equation;[1] reduce the equation to its simplest form by a translation of the coordinate axes; determine the geometric object represented by the equation and draw this object, showing both the old and the new coordinate axes.
a) $4x^2 + 9y^2 - 40x + 36y + 100 = 0$;
b) $9x^2 - 16y^2 - 54x - 64y - 127 = 0$;
c) $9x^2 + 4y^2 + 18x - 8y + 49 = 0$;
d) $4x^2 - y^2 + 8x - 2y + 3 = 0$;
e) $2x^2 + 3y^2 + 8x - 6y + 11 = 0$.

7. In each of the following, reduce the given equation to its simplest form; determine the type of the equation; determine the geometric object represented by the equation and draw this object, showing both the old and the new coordinate axes.
a) $32x^2 + 52xy - 7y^2 + 180 = 0$;
b) $5x^2 - 6xy + 5y^2 - 32 = 0$;
c) $17x^2 - 12xy + 8y^2 = 0$;
d) $5x^2 + 24xy - 5y^2 = 0$;
e) $5x^2 - 6xy + 5y^2 + 8 = 0$.

8. In each of the following, reduce the given equation to the canonical form; determine the type of the equation and the geometric object represented by the equation; draw this geometric object, showing the original, auxiliary and new coordinate axes.
a) $3x^2 + 10xy + 3y^2 - 2x - 14y - 13 = 0$;
b) $25x^2 - 14xy + 25y^2 + 64x - 64y - 224 = 0$;
c) $4xy + 3y^2 + 16x + 12y - 36 = 0$;
d) $7x^2 + 6xy - y^2 + 28x + 12y + 28 = 0$;
e) $19x^2 + 6xy + 11y^2 + 38x + 6y + 29 = 0$;
f) $5x^2 - 2xy + 5y^2 - 4x + 20y + 20 = 0$.

[1] That is, determine whether the equation is of the elliptic, hyperbolic, or parabolic type.

9. Without transforming the coordinates, show that each of the following equations represents a single point (a degenerate ellipse) and find its coordinates:
a) $5x^2 - 6xy + 2y^2 - 2x + 2 = 0$;
b) $x^2 + 2xy + 2y^2 + 6y + 9 = 0$;
c) $5x^2 + 4xy + y^2 - 6x - 2y + 2 = 0$;
d) $x^2 - 6xy + 10y^2 + 10x - 32y + 26 = 0$.

10. Without transforming the coordinates, show that each of the following equations represents a hyperbola and find the values of its semi-axes:
a) $4x^2 + 24xy + 11y^2 + 64x + 42y + 5x = 0$;
b) $12x^2 + 26xy + 12y^2 - 52x - 48y + 73 = 0$;
c) $3x^2 + 4xy - 12x + 16 = 0$;
d) $x^2 - 6xy - 7y^2 + 10x - 30y + 23 = 0$.

11. Without transforming the coordinates, show that each of the following equations represents a pair of intersecting straight lines (a degenerate hyperbola) and find their equations:
a) $3x^2 + 4xy + y^2 - 2x - 1 = 0$;
b) $x^2 - 6xy + 8y^2 - 4y - 4 = 0$;
c) $x^2 - 4xy + 3y^2 = 0$;
d) $x^2 + 4xy + 3y^2 - 6x - 12y + 9 = 0$.

4.8 Tangents and Polars

In this section, we study some general properties of the general quadratic equation (4.7.1). *These properties apply, of course to all conic sections.*

It is convenient to introduce the following notations. For two points $P_1(x_1, y_1), P_2(x_2, y_2)$ and scalars A, B, C, D, E, F where at least one of A, B, C is not zero, we let

$$f(\vec{P}_1, \vec{P}_2) = Ax_1x_2 + B(x_1y_2 + x_2y_1) + Cy_1y_2 + D(x_1 + x_2) + E(y_1 + y_2) + F \quad (4.8.1)$$

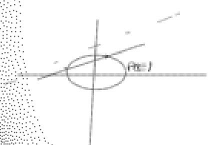
Note that $f(\vec{P}_1, \vec{P}_2) = f(\vec{P}_2, \vec{P}_1)$ and that for any $\vec{P} = (x, y)$,

$$f(\vec{P}, \vec{P}) = 0 \iff Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

the general quadratic equation (4.7.1).

In general, one would like to know about the intersection of a line ℓ and the graph of the general quadratic equation (4.7.1) which can now be written as

Question: Given a line in the plane, what is $\ell \cap P(E)$?



FACT: A line in the plane intersects the graph of a quadratic eqn. at at most two points

$f(\vec{P},\vec{P}) = 0$. The points of intersection (if any) of a line ℓ and the graph of the equation $f(\vec{P},\vec{P}) = 0$ is obtained by solving the equation of the line and the equation $f(\vec{P},\vec{P}) = 0$ simultaneously. For example, if $\vec{P} = (x(t), y(t))$ is a vector-equation for ℓ then we get the points of intersection by solving the equation

$$f(\vec{P},\ \vec{P}) = f((x(t), y(t)), (x(t), y(t))) = 0 \tag{4.8.2}$$

for t, and substituting the obtained values of t in $\vec{P} = (x(t), y(t))$.

The equation (4.8.2) is a quadratic equation in t. Thus, a line intersects the graph of the general quadratic equation in at most two points.

Example 4.8.1 Find the points of intersection of the line $\ell = \{P : \vec{P} = (t+2, \frac{-t+1}{2})\}$ and the ellipse $2x^2 + 4y^2 = 9$.

Solution. We have $f(\vec{P},\vec{P}) = 2x^2 + 4y^2 - 9 = 0$. If P is a point of intersection then $\vec{P} = (t+2, \frac{-t+1}{2})$ for some t and $f(\vec{P},\vec{P}) = 0$. Then

$$\begin{aligned} f(\vec{P},\vec{P}) &= 2(t+2)^2 + 4(\frac{-t+1}{2})^2 - 9 = 0 \\ & \quad 2t^2 + 8t + 8 + t^2 - 2t + 1 - 9 = 0 \\ & \quad 3t^2 + 6t = 0; t = 0 \quad \text{or} \quad t = -2. \end{aligned}$$

Hence $P_1(0, \frac{1}{2})$ and $P_2(-2, \frac{3}{2})$ are the points of intersection.

Def

If the two points of intersection of a line and the graph of $f(\vec{P},\vec{P}) = 0$ coincide, then the line is called a *tangent* to the graph of $f(\vec{P},\vec{P}) = 0$. The point of intersection of the tangent with the graph of $f(\vec{P},\vec{P}) = 0$ is called the *point of contact* of the tangent. If a line ℓ is a tangent with point of contact $P_\circ$, then we also say that ℓ *is tangent to the graph at* $P_\circ$.

Consider an arbitrary but fixed point $Q_\circ(a_\circ, b_\circ)$ in the plane and consider the equation

$$f(\vec{Q}_\circ, \vec{P}) = 0.$$

We observe that

$$f(\vec{Q}_\circ, \vec{P}) = (Aa_\circ + Bb_\circ + D)x + (Ba_\circ + Cb_\circ + E)y + (Da_\circ + Eb_\circ + F).$$

Thus $f(\vec{Q}_\circ, \vec{P}) = 0$ is a linear equation in x and y; in other words, it is the equation of a line, say $\ell_\circ$. The line $\ell_\circ$ described in this way is called the *polar* of $Q_\circ$ with respect to $f(\vec{P},\vec{P}) = 0$. The point $Q_\circ$ is called the *pole* of the line $\ell_\circ$ with respect to $f(\vec{P},\vec{P}) = 0$.

Theorem 4.8.2 *If $P_\circ(x_\circ, y_\circ)$ is a point on the graph of $f(\vec{P},\vec{P}) = 0$, then the polar of $P_\circ$ with respect to $f(\vec{P},\vec{P}) = 0$ is tangent to the graph of $f(\vec{P},\vec{P}) = 0$ with point of contact $P_\circ$.*

Proof. Let $\ell_\circ$ be the polar of $P_\circ$ with respect to $f(\vec{P},\vec{P}) = 0$. Then a linear equation for $\ell_\circ$ is

$$f(\vec{P}_\circ, \vec{P}) = (Ax_\circ + By_\circ + D)x + (Bx_\circ + Cy_\circ + E)y + (Dx_\circ + Ey_\circ + F) = 0.$$

We let $\alpha = (Ax_\circ + By_\circ + D)$, $\beta = (Bx_\circ + Cy_\circ + E)$. Then $\vec{N} = (\alpha, \beta)$ and $\vec{u} = (\beta, -\alpha)$ are normal and direction vectors of $\ell_\circ$, respectively. Since $P_\circ$ is a point on $\ell_\circ$,

$$\ell_\circ = \{P : \vec{P} = (\beta t + x_\circ, -\alpha t + y_\circ)\}.$$

The points of intersection of $\ell_\circ$ and the graph of $f(\vec{P},\vec{P}) = 0$ are obtained by solving $\vec{P} = (\beta t + x_\circ, -\alpha t + y_\circ)$ and $f(\vec{P},\vec{P}) = 0$ simultaneously for t. We have

$$\begin{aligned} f(\vec{P},\vec{P}) &= A(\beta t + x_\circ)^2 + 2B(\beta t + x_\circ)(-\alpha t + y_\circ) + C(-\alpha t + y_\circ)^2 \\ & \quad +2D(\beta t + x_\circ) + 2E(\alpha t + y_\circ) + F \\ &= (A\beta^2 - 2B\alpha\beta + C\alpha^2)t^2 + 2(Ax_\circ\beta - Bx_\circ\alpha + By_\circ\beta \\ & \quad -Cy_\circ\alpha + D\beta - E\alpha)t \\ & \quad +(Ax_\circ^2 + 2Bx_\circ y_\circ + Cy_\circ^2 + 2Dx_\circ + 2Ey_\circ + F) \\ &= (A\beta^2 - 2B\alpha\beta + C\alpha^2)t^2 + 2[(Ax_\circ By_\circ + D)\beta \\ & \quad -(Bx_\circ + Cy_\circ + E)\alpha]t + f(\vec{P}_\circ, \vec{P}_\circ) \\ &= (A\beta^2 - 2B\alpha\beta + C\alpha^2)t^2 + 2[\alpha\beta - \beta\alpha]t + f(\vec{P}_\circ, \vec{P}_\circ) \\ &= (A\beta^2 - 2B\alpha\beta + C\alpha^2)t^2 + f(\vec{P}_\circ, \vec{P}_\circ). \end{aligned}$$

The point $P_\circ$ is on the graph, therefore $f(\vec{P}_\circ, \vec{P}_\circ) = 0$, and we thus obtain

$$f(\vec{P},\vec{P}) = (A\beta^2 - 2B\alpha\beta + C\alpha^2)t^2 = 0.$$

Hence $\vec{P} = (\beta t + x_\circ, -\alpha t + y_\circ)$ is a point of intersection if and only if $t = 0$ if and only if $P = P_\circ$. This shows that the polar $\ell_\circ$ of $P_\circ$ intersects the graph only at $P_\circ$. Therefore $\ell_\circ$ is tangent to the graph of $f(\vec{P},\vec{P}) = 0$ with point of contact $P_\circ$. □

Example 4.8.3 Find the equation of the tangent to the circle $x^2 + y^2 = 2$ at the point $P_\circ(1,1)$.

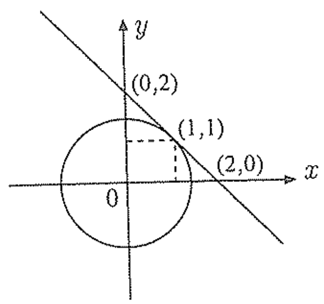
Solution. First note that $P_\circ(1,1)$ is on the circle. For $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, let

$$f(\vec{P}_1, \vec{P}_2) = x_1x_2 + y_1y_2 - 2$$

so that

$$f(\vec{P},\vec{P}) = x^2 + y^2 - 2 = 0$$

is the equation of the given circle. Then $f(\vec{P}_\circ,\vec{P}) = x + y - 2 = 0$ is the equation of the tangent at $P_\circ(1,1)$.



The following theorem establishes the relation between polars and tangents in general.

Theorem 4.8.4 *Let $Q_\circ(a_\circ, b_\circ)$ be a point in the plane and let $\ell_\circ = \{P : f(\vec{Q}_\circ, \vec{P}) = 0\}$ be its polar with respect to $f(\vec{P},\vec{P}) = 0$. If a line through $Q_\circ$ is tangent to the graph of $f(\vec{P},\vec{P}) = 0$, then the polar $\ell_\circ$ passes through the point of contact of the tangent.*

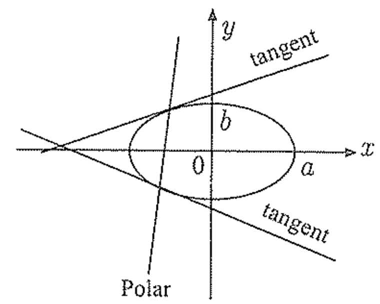
Proof. Let ℓ be a line through $Q_\circ(a_\circ, b_\circ)$ which is tangent to the graph of $f(\vec{P},\vec{P}) = 0$ with point of contact $P_\circ(x_\circ, y_\circ)$. Then $\ell = \{P : f(\vec{P}_\circ, \vec{P}) = 0\}$. Since $Q_\circ$ is on this line, $f(\vec{P}_\circ, \vec{Q}_\circ) = 0$. On the other hand, $f(\vec{Q}_\circ, \vec{P}_\circ) = f(\vec{P}_\circ, \vec{Q}_\circ) = 0$. Hence $P_\circ$ lies on $\ell_\circ$. In other words $\ell_\circ$ intersects the graph of $f(\vec{P},\vec{P}) = 0$ at $P_\circ$. □

Corollary 4.8.5 *Let $Q_\circ(a_\circ, b_\circ)$ be a point in the plane. If there are two tangents to the graph of $f(\vec{P},\vec{P}) = 0$ passing through $Q_\circ$, then the polar of $Q_\circ$ is the (unique) line passing through the two points of contact.*

Corollary 4.8.6 *Given a point $Q_\circ$ in the plane, there are at most two tangents to the graph of $f(\vec{P},\vec{P}) = 0$ through the point $Q_\circ$.*

The figure given below illsutrates the situation in the above corollaries for a point $Q_\circ$ and an ellipse

$$b^2x^2 + a^2y^2 - a^2b^2 = 0.$$



Example 4.8.7 Find the points of contact of the tangents to the ellipse $2x^2 + 3y^2 = 18$ through the point $Q_\circ(-3, 4)$. Find the equations of these tangents.

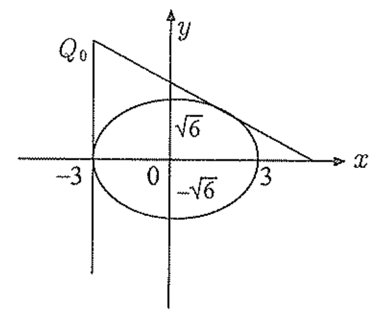
Solution. It is enough to find the points of intersection of the polar of $Q_\circ(-3,4)$ and the ellipse $2x^2 + 3y^2 = 18$. We let

$$f(\vec{P}_1, \vec{P}_2) = 2x_1x_2 + 3y_1y_2 - 18 \quad \text{so that} \quad f(\vec{P},\vec{P}) = 2x^2 + 3y^2 - 18 = 0$$

is the equation of the given ellipse. The polar of $Q_\circ$ has the equation

$$f(\vec{Q}_\circ, \vec{P}) = 2(-3)x + 3(4)y - 18 = 0 \quad \text{or} \quad x - 2y + 3 = 0.$$

Substitute $x = 2y - 3$ in the equation of the ellipse. Then



$$2(2y-3)^2 + 3y^2 - 18 = 0$$

$$28y^2 - 24y + 18 + 3y^2 - 18 = 0$$

$$11y^2 - 24y = 0; \quad y = 0 \text{ or } y = \frac{24}{11}.$$

Thus the points of contact are $Q_1(-3,0)$ and $Q_2(\frac{15}{11},\frac{24}{11})$. The equations of tangents at these points can be obtained by

$$\begin{aligned} f(\vec{Q}_1,\vec{P}) &= 2(-3)x + 3(0)y - 18 = 0; x = -3 \\ f(\vec{Q}_2,\vec{P}) &= 2(\frac{15}{11})x + 3(\frac{24}{11})y - 18 = 0; 5x + 12y = 33. \end{aligned}$$

Exercises

1. Find the points of intersection (if any) of the given line and the given conic section.
   a) $3x + 4y = 6, 2x^2 + 5y = 0,$ b) $2x - 3y = 2, 4x^2 + 9y^2 - 36 = 0$
   c) $2x - 3y = 2, 4x^2 - 9y^2 = 36,$ d) $2x + 3y = 0, 4x^2 - 9y^2 - 36 = 0.$

2. In each case below, write down the equation of the polar of the given point with respect to the given equation
   a) $(1,2), 2x^2 + 3y^2 = 5,$ b) $(-1,1), x^2 + xy = 1$
   c)$(1,2), x^2 - 3xy - y^2 + 2x + 3y + 1 = 0,$ d)$(1,1), 4x^2 + 9y^2 = 36$
   e)$(4,4), 4x^2 + 9y^2 = 36,$ f)$(3,3), x^2 - y^2 = 3.$

3. In each case below, write down the equation of the tangent to the graph of the given equation at the given point. (*Hint:* Note that the given point is on the graph. Express the given equation in the form $f(\vec{P},\vec{P}) = 0$).
   a)$(-1,1), 2x^2 + 3y^2 = 5,$ b)$(1,0), x^2 + xy = 1$
   c)$(-1,2), x^2 - 3xy - y^2 + 2x + 3y = 1,$ d)$(2,1), x^2 - y^2 = 3$
   e)$(1\frac{3\sqrt{3}}{2}), 9x^2 + 4y^2 = 36.$

4. Find the points of contact of the tangents through the given point to the graph of the given equation.
   a) $(-1,1), 2x^2 + 3y^2 = 5,$ b)$(3,4), 16x^2 + 9y^2 = 288$
   c)$(6,8), 16x^2 + 9y^2 = 288,$ d)$(5,0), x^2 - y^2 = 1$
   e)$(6,8), 16x^2 - 9y^2 = 144,$ f)$(-1,1), x^2 - 4xy - y^2 + 2x + 2y + 1 = 0.$

5. Given an equation $f(\vec{P}_1,\vec{P}_2) = 0$ as in (4.8.1), two points Q_1 and Q_2 in the plane, we say that Q_1 is *harmonic conjugate to* Q_2 if $f(\vec{Q}_1,\vec{Q}_2) = 0$. Prove that

   a) If Q_1 is harmonic conjugate to Q_2 then Q_2 is harmonic conjugate to Q_1.

   b) Q_1 is harmonic conjugate to $Q_1 \iff Q_1$ lies on the graph of $f(\vec{P},\vec{P}) = 0$.

6. Find the harmonic conjugate(s) of $Q_1(5,0)$ with respect to the circle $x^2 + y^2 = 1$.

7. Prove that the tangents at the endpoints of the latus rectum of an hyperbola intersect on the directrix. Same question for an ellipse and a parabola.

Chapter 5

VECTORS IN THREE SPACE

In this chapter, we introduce vectors which lie in 3-space and use them to study geometry in 3-space. We study lines and planes in 3-space through vectors.

Vectors in 3-space are defined in the same way as vectors in the plane, as equivalence classes of directed segments in 3-space.

At the end of the chapter, sections 9 and 10, vectors in higher dimension and vectors with complex components are introduced.

5.1 Directed Segments and Vectors

An ordered pair (A, B) of points in 3-space is called a *directed segment* from A to B. The point A is called the *initial point* and B is called the *terminal point* of the directed segment from A to B. The directed segment from A to B is visualized as an arrow in 3-space with tail at A and head at B, and it is denoted by $\overrightarrow{AB}$.

Given two directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ with

$$A(a_1, a_2, a_3),\ B(b_1, b_2, b_3),\ C(c_1, c_2, c_3),\ D(d_1, d_2, d_3)$$

in a certain Cartesian coordinate system, we say that $\overrightarrow{AB}$ is *equivalent* to $\overrightarrow{CD}$ and

write $\overrightarrow{AB} \cong \overrightarrow{CD}$ if

$$b_1 - a_1 = d_1 - c_1 \quad\text{and}\quad b_2 - a_2 = d_2 - c_2 \quad\text{and}\quad b_3 - a_3 = d_3 - c_3.$$

As for directed segments in the plane, we see that $\overrightarrow{AB} \cong \overrightarrow{CD}$ if and only if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same length and the same direction (see Fig. 5.1).

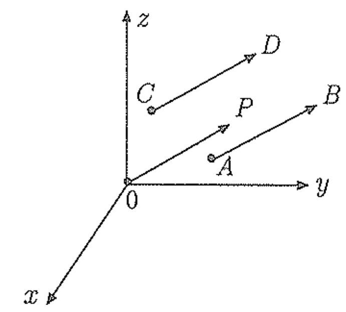


Fig. 5.1.

One can easily prove that "being equivalent to", $\cong$, is an equivalence relation in the set of all directed segments in 3-space. An equivalence class of directed segments, with respect to the relation $\cong$, is called a *vector* in 3-space. If $\overrightarrow{AB} \cong \overrightarrow{CD}$ then we say that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector.

Example 5.1.1 Given the points $A(2, -1, 3), B(-3, 0, 4)$ and $C(1, 1, 1)$, find a point D such that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector.

Solution. Let $D(x, y, z)$. Then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector if and only if $\overrightarrow{AB} \cong \overrightarrow{CD}$, i.e.,

$$-3 - 2 = x - 1 \text{ and } 0 - (-1) = y - 1 \text{ and } 4 - 3 = z - 1,$$

or

$$x = -4 \quad\text{and}\quad y = 2 \quad\text{and}\quad z = 2.$$

Thus $D(-4, 2, 2)$ is the required point.

Let A be any point in 3-space. The vector represented by the directed segment $\overrightarrow{AA}$ consists of all directed segments $\overrightarrow{CC}$ of the same type. This vector is called the *zero vector* and it is denoted by $\vec{0}$.

For any two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ in 3-space there exists a unique point P such that $\overrightarrow{OP} \cong \overrightarrow{AB}$ where $0(0,0,0)$ is the origin of the coordinate system. The point P is uniquely determined by the points A and B. Namely, $P(b_1 - a_1,\ b_2 - a_2,\ b_3 - a_3)$. Thus corresponding to Theorem 3.1.3, we have

Theorem 5.1.2 *Every vector in 3-space has a unique representative of the form $\overrightarrow{OP}$ where O is the origin of the coordinate system.*

Because of the uniqueness of the representative $\overrightarrow{OP}$ for any vector, we identify the directed segment $\overrightarrow{OP}$ with the vector that it represents. If $P(x, y, z)$ then we write

$$\overrightarrow{OP} = \vec{P} = (x, y, z)$$

and call x, y, and z the *X-component, Y-component* and *Z-component* of the vector (represented by) $\overrightarrow{OP}$, respectively. Note that the components of $\overrightarrow{OP} = \vec{P}$ are the coordinates of the point P. Note also that the X-, Y-, and Z-components of the vector represented by a directed segment $\overrightarrow{AB}$ with $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ are $b_1 - a_2, b_2 - a_2$, and $b_3 - a_3$.

We will denote vectors in 3-space by symbols like $\vec{u}, \vec{v}, \vec{w}, \vec{A}, \vec{P}, \ldots$ etc.

Given a vector $\vec{u} = (x, y, z)$, the *length*, $|\vec{u}|$, of $\vec{u}$ is defined as

$$|\vec{u}| = \sqrt{x^2 + y^2 + z^2}. \tag{5.1.1}$$

Remark. The length of the vector $\vec{u}$ represented by the directed segment AB with $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is

$$|\vec{u}| = |AB| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2},$$

the distance from A to B. Note here that if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector u then $|AB| = |CD| = |\vec{u}|$. Hence we can define the length of a vector to be *the distance between the initial point and the terminal point of any directed segment representing that vector.*

Exercises

1. Determine the unknown coordinates for each case if $\overrightarrow{AB} \cong \overrightarrow{CD}$.

   a) $A(3,5,2),\ B(4,6,-1),\ C(-2,5,7),\ D(x,y,z)$.

   b) $A(-1,1,x),\ B(3,5,-2),\ C(3,y,2),\ D(z,-1,0)$.

2. Prove that $\cong$ is an equivalence relation in the set of all directed segments in 3-space.

3. Prove that if $\overrightarrow{AB} \cong \overrightarrow{CD}$ then $\overrightarrow{BA} \cong \overrightarrow{DC}$.

4. In each case below, determine the unknown coordinates so that $\overrightarrow{AB}$ is a representative for the vector $\vec{u}$.

   a) $\vec{u} = (-4,1,3),\ A(2,y,6),\ B(z,-1,x)$

   b) $\vec{u} = (1,1,1),\ A(x,-1,z),\ B(2x,y,3z)$.

5. In each case below, determine the X-, Y-, and Z-components of the vector represented by $\overrightarrow{AB}$, and find the length of the corresponding vector.

   a) $A(-1,5,7),\ B(2,-3,4)$; b) $A(0,-1,1),\ B(1,-1,0)$;

   c) $A(-1,2\pi,4),\ B(\pi,2,3)$; d) $A(\sqrt{2},1,\sqrt{3}),\ B(1+\sqrt{2},2,\sqrt{3}+1)$.

6. Prove that $|\vec{u}| \geq 0$ for any vector in 3-space, and that $|\vec{u}| = 0 \Leftrightarrow \vec{u} = \vec{0}$.

5.2 Algebra of Vectors in 3-space

Addition of vectors and multiplication of a vector (in 3-space) by a scalar are defined *componentwise*, just as for vectors in the plane. Here we give the definitions and state the main properties.

Let us first notice that for $\vec{u} = (x, y, z)$ and $\vec{v} = (a, b, c)$,

$$\vec{u} = \vec{v} \Leftrightarrow x = a \text{ and } y = b \text{ and } z = c. \tag{5.2.1}$$

Given two vectors $\vec{u} = (x, y, z)$ and $\vec{v} = (a, b, c)$, we define the *sum*, $\vec{u} + \vec{v}$, of $\vec{u}$ and $\vec{v}$ by

$$\vec{u} + \vec{v} = (x + a, y + b, z + c). \tag{5.2.2}$$

The *multiplication of a vector $\vec{u} = (x, y, z)$ by a scalar* c is defined by

$$c\vec{u} = (cx, cy, cz). \tag{5.2.3}$$

These can be interpreted geometrically (see Fig. 5.2 and Fig. 5.3). The sum $\vec{u} + \vec{v}$ is represented as the diagonal of the parallelogram, two adjecent sides of which represent the vectors $\vec{u}$ and $\vec{v}$.

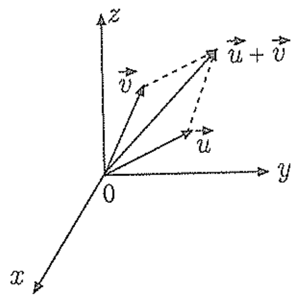


Fig. 5.2.

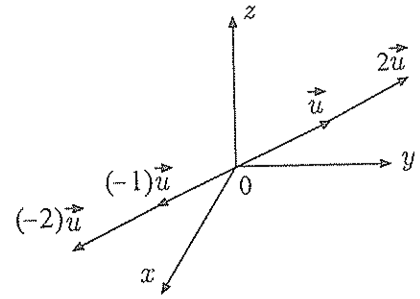


Fig. 5.3.

Example 5.2.1 If $\vec{u} = (-2,1,3)$ and $\vec{v} = (-1,0,1)$, then

$$\vec{u}+\vec{v} = (-2+(-1),\ 1+0, 3+1) = (-3,1,4),$$

$$2\vec{u} = (-4,2,6),\ (-3)\vec{v} = (3,0,-3),$$

$$2\vec{u}+(-3)\vec{v} = (-1,2,3).$$

Let $\vec{u}$ and $\vec{v}$ be two vectors. We write

$$(-)\vec{v} = -\vec{v} \quad \text{and} \quad \vec{u}+(-\vec{v}) = \vec{u}-\vec{v}.$$

Thus if $\vec{u} = (x,y,z)$ and $\vec{v} = (a,b,c)$, then

$$-\vec{v} = (-a,-b,-c) \quad \text{and} \quad \vec{u}-\vec{v} = (x-a, y-b, z-c).$$

If $\vec{u}$ is the vector represented by $\vec{AB}$, then we have $\vec{u} = \vec{B} - \vec{A}$.

Example 5.2.2 Let $\vec{u} = (-2,1,3)$ and let $\vec{v}$ be the vector represented by $\vec{AB}$ with $A(-1,3,5)$ and $B(2,5,-2)$. Find a point D such that AD represents $\vec{u}+\vec{v}$.

Solution. We have $\vec{v} = \vec{B} - \vec{A} = (3,2,-7)$; $\vec{u}+\vec{v} = (1,3,-4)$. Let $D(x,y,z)$ be the point such that AD represents $\vec{u}+\vec{v}$. Then $\vec{u}+\vec{v} = \vec{D}-\vec{A} = (x+1, y-3, z-5) = (1,3,-4)$. Hence $x=0, y=6, z=1$; and $D(0,6,1)$ is the required point.

Operations on vectors in 3-space satisfy the same properties as operations on vectors in the plane.

Theorem 5.2.3 *Let $\vec{u},\vec{v},\vec{w}$ be vectors in 3-space, $\vec{O} = (0,0,0)$ the zero vector, and c,d scalars. Then*

(a) $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$.

(b) $\vec{u}+\vec{v} = \vec{v}+\vec{w}$.

(c) $\vec{u}+\vec{O} = \vec{O}+\vec{u} = \vec{u}$, *and* $\vec{O}$ *is the only vector with this property.*

(d) $\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{O}$, *and* $(-\vec{u})$ *is the only vector with this property.*

(e) $c(d\vec{u}) = (cd)\vec{u}$.

(f) $(c+d)\vec{u} = c\vec{u}+d\vec{u}$.

(g) $c(\vec{u}+\vec{v}) = c\vec{u}+d\vec{v}$.

(h) $1\vec{u} = \vec{u}$.

Proof. We prove part (g) and leave the rest of the proof to exercises. Let $\vec{u} = (x_1,y_1,z_1), \vec{v} = (x_2,y_2,z_2)$ and let c be a scalar. Then

$$\begin{aligned} c(\vec{u}+\vec{v}) &= (c(x_1+x_2), c(y_1+y_2), c(z_1+z_2)) \ \text{(definition)} \\ &= (cx_1+cx_2, cy_1+cy_2, cz_1+cz_2) \ \text{(property of } \mathbb{R}\text{)} \\ &= c\vec{u}+c\vec{v} \ \text{(definition)}\ . \end{aligned}$$

□

By part (a), we can write

$$\vec{u}+\vec{v}+\vec{w} = (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$$

for the sum of three vectors $\vec{u} = (x_1,y_1,z_1), \vec{v} = (x_2,y_2,z_2), \vec{w} = (x_3,y_3,z_3)$. Given n vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ and n scalars $c_1, c_2, \ldots, c_n$, the expression

$$c_1\,\vec{u}_1 + c_2\,\vec{u}_2 + \cdots + c_n\,\vec{u}_n$$

is called a *linear combination* of the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$.

Example 5.2.4 Let $\vec{\mathbf{i}} = (1,0,0)$, $\vec{\mathbf{j}} = (0,1,0)$ and $\vec{\mathbf{k}} = (0,0,1)$, and let $\vec{u} = (x,y,z)$ be any vector in 3-space. Then we can write

$$\begin{aligned} \vec{u} &= (x,y,z) = (x,0,0)+(0,y,0)+(0,0,z) \\ &= x(1,0,0)+y(0,1,0)+z(0,0,1) = x\,\vec{\mathbf{i}}+y\,\vec{\mathbf{j}}+z\,\vec{\mathbf{k}}. \end{aligned}$$

Thus, *any vector in 3-space can be expressed as a linear combination of the vectors* $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$ and $\vec{\mathbf{k}}$.

A vector $\vec{u}$ is called a *unit vector* if $|\vec{u}| = 1$. One can easily observe that the vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$ and $\vec{\mathrm{k}}$ of Example 4 are unit vectors. Furthermore, for any vector $\vec{v}$, the vector

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

is a unit vector in the direction of $\vec{v}$.

The definition of *linearly dependent* or *linearly independent* vectors given in Chapter 3 are valid for vectors in 3-space, too. Namely, the vectors $\vec{u}_1\vec{u}_2, \ldots, \vec{u}_n$ (in 3-space) are *linearly independent* if for any scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$ we have

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = \vec{O} \Rightarrow c_1 = c_2 = \cdots = c_n = 0.$$

If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are not linearly independent then they are *linearly dependent.*

The vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$ and $\vec{\mathrm{k}}$ of Example 4 are linearly independent. They are called *basic unit vectors* in 3-space.

Example 5.2.5 Let $\vec{u}_1 = (2,7,3), \vec{u}_2 = (-1,3,4)$ and $\vec{u}_3 = (1,5,-6)$. Let us try to find out if $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{O}$ for some c_1, c_2, c_3. The identity

$$\vec{O} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = (2c_1 - c_2 + c_3, 7c_1 + 3c_2 + 5c_3, 3c_1 + 4c_2 - 6c_3)$$

is equivalent to the system of linear equations

$$\begin{cases} 2c_1 - c_2 + c_3 = 0 \\ 7c_1 + 3c_2 + 5c_3 = 0 \\ 3c_1 + 4c_2 - 6c_3 = 0. \end{cases}$$

We solve this system of linear equations, by elimination, for c_1, c_2 and c_3. For instance, adding (-5) times the first equation to the second and (6) times the first equation to the last one we obtain

$$\begin{cases} 2c_1 - c_2 + c_3 = 0 \\ -3c_1 + 8c_2 = 0 \\ 15c_1 - 2c_2 = 0. \end{cases}$$

Now, adding (5) times the second equation to the last one, we get

$$\begin{cases} 2c_1 - c_2 + c_3 = 0 \\ -3c_1 + 8c_2 = 0 \\ 38c_2 = 0. \end{cases}$$

It follows that $c_1 = c_2 = c_3 = 0$, and that $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are linearly independent.

Example 5.2.6 Show that the vectors $\vec{u}_1 = (2,7,3), \vec{u}_2 = (-1,3,4)$ and $\vec{u}_3 = (7,5,-6)$ are linearly dependent.

Solution. We should find c_1, c_2, c_3, not all zero, such that $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{O}$. We have

$$\vec{O} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = (2c_1 - c_2 + 7c_3, 7c_1 + 3c_2 + 5c_3, 3c_1 + 4c_2 - 6c_3).$$

The corresponding system of linear equation can be solved as follows:

$$\begin{cases} 2c_1 - c_2 + 7c_3 = 0 \\ 7c_1 + 3c_2 + 5c_3 = 0 \\ 3c_1 + 4c_2 - 6c_3 = 0 \end{cases} \Leftrightarrow \begin{cases} 2c_1 - c_2 + 7c_3 = 0 \\ 13c_1 + 26c_3 = 0 \\ 11c_1 + 22c_3 = 0 \end{cases} \Leftrightarrow \begin{cases} 2c_1 - c_2 + 7c_3 = 0 \\ c_1 + 2c_3 = 0 \end{cases}$$

Hence $c_3 = t$, $c_1 = -2t$, $c_2 = 3t$ ($t \in \mathbb{R}$) is a solution for the system. Taking $c_3 = 1$, we get $c_2 = 3$, $c_1 = -2$ and $-2\vec{u}_1 + 3\vec{u}_2 + \vec{u}_3 = \vec{O}$. This shows that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly dependent.

Exercises

1. Given $\vec{u} = (-4,3,1)$ and $\vec{v} = (2,5,-3)$, compute the following

   a) $\vec{u} + \vec{v}$ b) $2\vec{v}$ c) $-\vec{v}$

   d) $\vec{u} - 2\vec{v}$ e) $3\vec{u} - 2\vec{v}$ f) $|\vec{v}|\vec{u} - |\vec{u}|\vec{v}$

2. Prove each of the rules stated in Theorem 5.2.3.

3. Prove that every vector in 3-space can be written as a linear combination of $\vec{v}_1 = (-1,0,0)$, $\vec{v}_2 = (0,-1,0)$, $\vec{v}_3 = (0,0,-1)$.

4. In each case below, check whether or not the given vectors are linearly dependent.

   a) $\vec{u}_1 = (1,2,1)$, $\vec{u}_2 = (-1,2,1)$, $\vec{u}_3 = (1,2-1)$
   b) $\vec{u}_1 = (2,1,2)$, $\vec{u}_2 = (-2,1,2)$, $\vec{u}_3 = (2,1,-2)$
   c) $\vec{u}_1 = (1,1,1)$, $\vec{u}_2 = (-3,5,7)$, $\vec{u}_3 = (-3,13,17)$
   d) $\vec{u}_1 = (\pi,\sqrt{2},7)$, $\vec{u}_2 = (2,1,3)$, $\vec{u}_3 = (\pi,1+\sqrt{2},-4)$
   e) $\vec{u}_1 = (\pi,\sqrt{2},7)$, $\vec{u}_2 = (2,1,3)$, $\vec{u}_3 = (\pi,1+\sqrt{2},-4)$
   $\vec{u}_4 = (\pi,1-\sqrt{2},1)$.

5. Prove that any four vectors in 3-space are linearly dependent. What can you say about five vectors?

5.3 Scalar Product, Angle Between Two Vectors

Let $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$ be two vectors in 3-space. We define the *scalar product*, $\vec{u} \circ \vec{v}$, of $\vec{u}$ and $\vec{v}$ by

$$\vec{u} \circ \vec{v} = x_1x_2 + y_1y_2 + z_1z_2.$$

Example 5.3.1 If $\vec{u} = (1, -2, 3)$ and $\vec{v} = (3, -1, 2)$ then

$$\vec{u} \circ \vec{v} = 1 \cdot 3 = (-2) \cdot (-1) + 3 \cdot 2 = 3 + 2 + 6 = 11.$$

The scalar product has the following properties:

Theorem 5.3.2 *Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in 3-space and let c be a scalar. Then*

(a) $\vec{u} \circ \vec{v} = \vec{v} \circ \vec{u}$

(b) $(c\vec{u}) \circ \vec{v} = \vec{u} \circ (c\vec{v}) = c(\vec{u} \circ \vec{v})$

(c) $\vec{u} \circ \vec{u} = |\vec{u}|^2$

(d) $\vec{u} \circ \vec{O} = \vec{O} \circ \vec{u} = 0$

(e) $\vec{u} \circ (\vec{v} + \vec{w}) = \vec{u} \circ \vec{v} + \vec{u} \circ \vec{w}$

(f) $|\vec{u} \circ \vec{v}| \leq |\vec{u}|\,|\vec{v}|$.

Proof. See the proof of Theorem 3.3.2. □

The inequality in part (f) of the theorem is known as the *Cauchy-Schwarz Inequality.* Using Cauchy-Schwarz Inequality one can further prove.

Corollary 5.3.3 *For any two vectors $\vec{u}$ and $\vec{v}$ in 3-space,*

$$|\,|\vec{u}| - |\vec{v}|\,| \leq |\vec{u} - \vec{v}| \leq |\vec{u}| + |\vec{v}|.$$

The above inequalities are known as *triangle inequalities.*

Cauchy-Schwartz inequality can also be used to define angle between two vectors in 3-space. If $\vec{u}$ and $\vec{v}$ are two nonzero vectors in 3-space, the Cauchy-Schwartz inequality implies that

$$-1 \leq \frac{\vec{u} \circ \vec{v}}{|\vec{u}|\,|\vec{v}|} \leq 1.$$

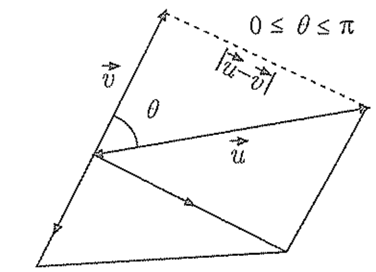


Fig. 5.4.

The angle which has radian measure θ, and satisfies

$$\cos\theta = \frac{\vec{u} \circ \vec{v}}{|\vec{u}|\,|\vec{v}|}, \quad 0 \leq \theta \leq \pi \tag{5.3.1}$$

is called the *angle between $\vec{u}$ and $\vec{v}$.* One can use the Cosine Rule and argue as before and see that this definition coincides with our intuitive notion of angle between two vectors (see Fig. 5.4). If $\vec{u} = 0$ or $\vec{v} = 0$, then the angle between $\vec{u}$ and $\vec{v}$ is defined to be any angle with radian measure any $\theta \in \mathbb{R}$.

Example 5.3.4 Find the angle θ between the vectors $\vec{u} = (1, 2, -2)$ and $\vec{v} = (1, 0, -1)$.

Solution. $\vec{u} \circ \vec{v} = 1 + 0 + 2 = 3$, $|\vec{u}| = \sqrt{1 + 4 + 4} = 3$, $|\vec{v}| = \sqrt{1 + 0 + 1} = \sqrt{2}$. Thus

$$\cos\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}, \quad 0 \leq \theta \leq \pi.$$

The restrictions on θ imply that $\theta = \pi/4$.

Let $\vec{u}$ and $\vec{v}$ be two non-zero vectors in 3-space, and let θ be the angle between them, $0 \leq \theta \leq \pi$. Then $\vec{u}$ and $\vec{v}$ are *like-directed* if $\theta = 0$, they are *opposite-directed* if $\theta = \pi$, they are *perpendicular* if $\theta = \pi/2$. It is clear from (5.3.1) that $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u} \circ \vec{v} = 0$.

Exercises

1. Compute $\vec{u} \circ \vec{v}$ if

   a) $\vec{u} = (2, -1, 2)$, $\vec{v} = (2, -3, 6)$

b) $\vec{u} = (2, -1, 2)$, $\vec{v} = (1, 2, 1)$

c) $\vec{u} = (2, -1, 2)$, $\vec{v} = (-2, 1, -2)$

d) $\vec{u} = (2, -1, 2)$, $\vec{v} = (1, 4, 1)$.

2. Find the cosine of the angle between each pair of vectors in Exercise 1.

3. Prove the statement in Theorem 5.3.2.

4. Prove the following

a) $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + 2\vec{u} \circ \vec{v} + |\vec{v}|^2$

b) $(\vec{u} + \vec{v}) \circ (\vec{u} - \vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$

c) $|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}| \Leftrightarrow \vec{u} \circ \vec{v} = 0$.

5. Find a unit vector which is perpendicular to both $\vec{u} = (1, 4, 1)$ and $\vec{v} = (0, -1, 1)$.

6. If $\vec{u}$ and $\vec{v}$ are like-directed non-zero vectors, show that $\vec{u} = c\vec{v}$ for some scalar $c > 0$.

5.4 Cross Product

In this section, we define a new operation in the set of vectors in 3-space: the cross product. *This operation is defined only for vectors in 3-space.*

Let $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$ be two vectors in 3-space. The *cross product*, $\vec{u} \times \vec{v}$, of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} = (y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_1 y_2 - x_2 y_1). \tag{5.4.1}$$

Remark. If the reader is familiar with 2×2 and 3×3 determinants, then he may make use of the notation

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{\mathrm{i}} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{\mathrm{j}} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{\mathrm{k}}.$$

Example 5.4.1 If $\vec{u} = (-1, 2, 3)$ and $\vec{v} = (2, 7, -1)$, then

$$\vec{u} \times \vec{v} = (2 \cdot (-1) - 7 \cdot 3, 3 \cdot 2 - (-1) \cdot (-1), (-1), (-1) \cdot 7 - 2 \cdot 2) = (-23, 5, -11).$$

Using determinants

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ -1 & 2 & 3 \\ 2 & 7 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 7 & -1 \end{vmatrix} \vec{\mathrm{i}} - \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} \vec{\mathrm{j}} + \begin{vmatrix} -1 & 2 \\ 2 & 7 \end{vmatrix} \vec{\mathrm{k}} \\ &= (2(-1) - 7 \cdot 3)\vec{\mathrm{i}} - ((-1)(-1) - 3 \cdot 2)\vec{\mathrm{j}} + ((-1) \cdot 7 - 2 \cdot 2)\vec{\mathrm{k}} = \\ &\quad (-23, +5, -11). \end{aligned}$$

The cross product has the following properties:

Theorem 5.4.2 *Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in 3-space and let c be a scalar. Then*

(a) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

(b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(c) $(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$

(d) $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$

(e) $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \circ \vec{w})\vec{v} - (\vec{v} \circ \vec{w})\vec{u}$

(f) $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \circ \vec{v})^2$

(g) $|\vec{u} \times \vec{v}| = |\vec{u}|\ |\vec{v}| \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

Proof. Each of these properties can be proved directly by using the definition (5.4.1). For instance, if $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$, then

$$\begin{aligned} \vec{u} \times \vec{v} &= (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1) \\ &= -(y_2 z_1 - y_1 z_2, z_2 x_1 - z_1 x_2, x_2 y_1 - x_1 y_2) = -(\vec{v} \times \vec{u}), \end{aligned}$$

proving (a). We prove (e) and leave the proof of the rest to exercises. For this, let $\vec{u}$ and $\vec{v}$ be as above and let $\vec{w} = (x_3, y_3, z_3)$. Let also $\vec{u} \times \vec{v} = (a, b, c)$. Namely,

$$y_1 z_2 - y_2 z_1 = a, \quad z_1 x_2 - z_2 x_1 = b, \quad x_1 y_2 - x_2 y_1 = c.$$

Let $(\vec{u} \times \vec{v}) \times \vec{w} = (x, y, z)$. Then

$$\begin{aligned} x &= bz_3 - y_3 c = z_1 x_2 z_3 - z_2 x_1 z_3 - y_3 x_1 y_2 + y_3 x_2 y_1 \\ y &= cx_3 - z_3 a = x_1 y_2 x_3 - x_2 y_1 x_3 - z_3 y_1 z_2 + z_3 y_2 z_1 \\ z &= ay_3 - x_3 b = y_1 z_2 y_3 - y_2 z_1 y_3 - x_3 z_1 x_2 + x_3 z_2 x_1. \end{aligned}$$

We thus have

$$\begin{aligned} x &= (z_1z_3 + y_1y_3)x_2 - (z_2z_3 + y_2y_3)x_1 \\ &= (x_1x_3 + y_1y_3 + z_1z_3)x_2 - (x_2x_3 + y_2y_3 + z_2z_3)x_1 \\ x &= (\vec{u} \circ \vec{w})x_2 - (\vec{v} \circ \vec{w})x_1. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} y &= (\vec{u} \circ \vec{w})y_2 - (\vec{v} \circ \vec{w})y_1 \\ z &= (\vec{u} \circ \vec{w})z_2 - (\vec{v} \circ \vec{w})z_1. \end{aligned}$$

Hence

$$\begin{aligned} (\vec{u} \times \vec{v}) \times \vec{w} &= (x, y, z) = (\vec{u} \circ \vec{w})(x_2, y_2, z_2) - (\vec{v} \circ \vec{w})(x_1, y_1, z_1) \\ &= (\vec{u} \circ \vec{w})\vec{v} - (\vec{v} \circ \vec{w})\vec{u}, \end{aligned}$$

proving (e). □

From part (g), we deduce

Corollary 5.4.3 *Let $\vec{u}$ and $\vec{v}$ be two non-zero vectors in 3-space and let θ be the angle between $\vec{u}$ and $\vec{v}$. Then*

$$\theta = O \quad or \quad \theta = \pi \Leftrightarrow \vec{u} \times \vec{v} = \vec{O}.$$

In other words, $\vec{u}$ and $\vec{v}$ are parallel (like - or opposite - directed) if and only if $\vec{u} \times \vec{v} = 0$.

Theorem 5.4.2 (d) provides us with a practical way of finding a vector which is perpendicular to two vectors.

Example 5.4.4 Find a vector $\vec{w}$ which is perpendicular to both $\vec{u} = (1, 4, 1)$ and $\vec{v} = (0, -1, 1)$.

Solution. $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. Thus,

$$\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ 1 & 4 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 5\vec{\mathrm{i}} - \vec{\mathrm{j}} - \vec{\mathrm{k}} = (5, -1, -1).$$

Exercises

1. Compute $\vec{u} \times \vec{v}$ if

   a) $\vec{u} = (2, -1, 2)$, $\vec{v} = (2, -3, 6)$,
   b) $\vec{u} = (-2, 1, 2)$, $\vec{v} = (1, 2, 1)$
   c) $\vec{u} = (2, -1, 2)$, $\vec{v} = (-2, 1, -2)$
   d) $\vec{u} = (2, -1, 2)$, $\vec{v} = (1, 4, 1)$.

2. Compute the following

   a) $\vec{\mathrm{i}} \times \vec{\mathrm{j}}$ b) $\vec{\mathrm{j}} \times \vec{\mathrm{k}}$ c) $\vec{\mathrm{k}} \times \vec{\mathrm{i}}$ d) $\vec{\mathrm{i}} \circ \vec{\mathrm{j}}$
   e) $\vec{\mathrm{j}} \circ \vec{\mathrm{k}}$ f) $\vec{\mathrm{k}} \circ \vec{\mathrm{i}}$ g) $(\vec{\mathrm{i}} \times \vec{\mathrm{j}}) \times \vec{\mathrm{k}}$ h) $(\vec{\mathrm{i}} \times \vec{\mathrm{j}}) \circ \vec{\mathrm{k}}$.

3. Complete the proof of Theorem 5.4.2.

4. Given $\vec{u} = (-2, 1, 2)$, $\vec{v} = (-2, 3, 6)$, $\vec{w} = (1, 1, -1)$. Compute
   a) $(\vec{u} \times \vec{v}) \times \vec{w}$ b) $\vec{u} \times (\vec{v} \times \vec{w})$ c) $(\vec{u} \times \vec{v}) \circ \vec{w}$ d) $\vec{u} \circ (\vec{v} \times \vec{w})$.

5. If $\vec{u}, \vec{v}$ and $\vec{w}$ are arbitrary vectors in 3-space, is there any difference between $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$?

6. Is there any difference between $(\vec{u} \times \vec{u}) \times \vec{v}$ and $\vec{u} \times (\vec{u} \times \vec{v})$?

7. In each case below, find a non-zero vector which is perpendicular to both $\vec{u}$ and $\vec{v}$.

   a) $\vec{u} = (1, 2, 3)$, $\vec{u} = (1, -1, 0)$, b) $\vec{u} = (0, 1, -2)$, $\vec{v} = (1, 0, 1)$
   c) $\vec{u} = (\sqrt{2}, 0, \sqrt{3})$, $\vec{v} = (\sqrt{3}, 1, \sqrt{2})$ d) $\vec{u} = (1, 1, 1)$, $\vec{v} = (1, 0, 1)$.

5.5 Lines in 3-Space

Let ℓ be a line in 3-space and consider two distinct points A and B on ℓ. Then $\vec{AB}$ represents a non-zero vector $\vec{u} = (m_1, m_2, m_3)$, i.e., $\vec{u} = \vec{B} - \vec{A}$. For any pair of points C and D on ℓ, the vector represented by $\vec{CD}$ is parallel to (like-directed or opposite-directed as) the vector $\vec{u}$. Thus if $P_1(x_1, y_1, z_1)$ is a given point on ℓ, then for any point $P(x, y, z)$ on ℓ, the vector represented by $\vec{P_1P}$ is parallel to $\vec{u}$ (see Fig. 5.5). That is,

$$\vec{P} - \vec{P_1} = t\vec{u}, \quad t \in \mathbb{R} \quad \text{or} \quad \vec{P} = \vec{P_1} + t\vec{u}, \quad t \in \mathbb{R}. \tag{5.5.1}$$

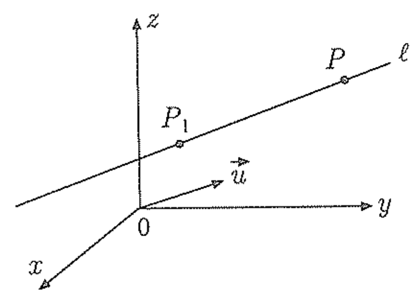


Fig. 5.5.

Conversely, if a point $P(x, y, z)$ satisfies (5.5.1) for some $t \in \mathbb{R}$, then P is on ℓ.

In set theoretical notations,

$$\ell = \{P : \vec{P} = \vec{P_1} + t\vec{u}, \quad t \in \mathbb{R}\}.$$

The equation (5.5.1) is called a *vector-equation for the line ℓ through P_1 in the direction of $\vec{u}$.* The non-zero vector $\vec{u}$ is called a *direction vector* for ℓ.

Note that different vectors may be direction vectors for the same line and thus the line may have different vector-equations (see Exercises 2).

Example 5.5.1 The discussions in the beginning show that if we are given two points A and B on a line ℓ, then $\vec{u} = \vec{B} - \vec{A}$ is a direction vector for ℓ. Hence

$$\vec{P} = A + t(\vec{B} - \vec{A}) \quad \text{or} \quad \vec{P} = t\vec{B} + (1-t)\vec{A}, \quad t \in \mathbb{R}$$

is a vector-equation for ℓ.

Example 5.5.2 Find a vector-equation of the line through $P_1(2, -1, 1)$ and perpendicular to both of the vectors $\vec{v} = (1, -1, 1)$ and $\vec{w} = (0, 1, -2)$.

Solution. A direction vector for this line will be perpendicular to both $\vec{v}$ and $\vec{w}$. We know that $\vec{u} = \vec{v} \times \vec{w}$ is such a vector. Thus $\vec{u} = \vec{v} \times \vec{w} = (1, 2, 1)$ is a direction vector and

$$\vec{P} = (2, -1, 1) + t(1, 2, 1) \text{ or } (x, y, z) = (t+2, 2t-1, t+1), \quad t \in \mathbb{R}$$

is a vector-equation for that line.

The equation (5.5.1), written more explicitely,

$$(x, y, z) = (m_1 t + x_1,\ m_2 t + y_1,\ m_3 t + z_1), \quad t \in \mathbb{R}$$

leads to the system of linear equations

$$\begin{cases} x = m_1 t + x_1 \\ y = m_2 t + y_1 \\ z = m_3 t + z_1, \quad t \in \mathbb{R}. \end{cases}$$

These equations are called parametric equations of the line. Solving for t from these equations, we obtain

$$\frac{x - x_1}{m_1} = \frac{y - y_1}{m_2} = \frac{z - z_1}{m_3} \quad (= t)$$

which is called a symmetric equation of the line. The symbol t in all these equations is called a *parameter.*

Example 5.5.3 The following are the parametric equations and the symmetric equation of the line in Example 12.

$$\begin{cases} x = t + 2 \\ y = 2t - 1 \\ z = t + 1, \end{cases} \qquad \frac{x-2}{1} = \frac{y+1}{2} = \frac{z-1}{1}.$$

Let ℓ_1 and ℓ_2 be two lines in three space. To determine $\ell_1 \cap \ell_2$, we solve the corresponding equations of ℓ_1 and ℓ_2 simultaneously. If

$$(x, y, z) = (m_1 t + x_1, m_2 t + y_1, m_3 t + z_1), (x, y, z) = (p_1 t + x_2, p_2 t + y_2, p_3 t + z_2)$$

are vector-equations for ℓ_1 and ℓ_2, respectively, then a point $P(x, y, z)$ belongs to $\ell_1 \cap \ell_2$ if and only if

$$(x, y, z) = (m_1 t_1 + x_1, m_2 t_1 + y_1, m_3 t_1 + z_1) \quad \text{and} \quad (x, y, z) = (p_1 t_2 + x_2, p_2 t_2 + y_2, p_3 t_2 + z_2)$$

for suitable values t_1 and t_2 of the parameter t. In general $t_1 \neq t_2$. Therefore, to determine $\ell_1 \cap \ell_2$ we use different parameters in the equations of ℓ_1 and ℓ_2. We write, for instance,

$$(x, y, z) = (m_1 t + x_1, m_2 t + y_1, m_3 t + z_1)$$

for the equation of ℓ_1, and

$$(x, y, z) = (p_1 s + x_2, p_2 s + y_2, p_3 s + z_2)$$

for the equation of ℓ_2. Then $\ell_1 \cap \ell_2$ is obtained by solving the system of linear equations

$$\begin{cases} m_1 t + x_1 = p_1 s + x_2 \\ m_2 t + y_1 = p_2 s + y_2 \\ m_3 t + z_1 = p_3 s + z_2 \end{cases} \quad \text{or} \quad \begin{cases} m_1 t - p_1 s = x_2 - x_1 \\ m_2 t - p_2 s = y_2 - y_1 \\ m_3 t - p_3 s = z_2 - z_1 \end{cases}$$

for s and t. If this system has a solution, then either $\ell_1 \cap \ell_2 = \ell_1 = \ell_2$ (the lines are *coincident*) or $\ell_1 \cap \ell_2$ consists of one single point (they are *interesting*). If the system has no solution, then $\ell_1 \cap \ell_2 = \phi$. The lines ℓ_1 and ℓ_2 are *parallel* if their direction vectors are parallel and $\ell_1 \cap \ell_2 = \phi$. We say that ℓ_1 and ℓ_2 are *skew* if $\ell_1 \cap \ell_2 = \phi$ and they are not parallel.

Thus there are four possibilities for two lines in 3-space. They are coincident or intersecting or parallel or skew.

Example 5.5.4 Find $\ell_1 \cap \ell_2$ if

$$\begin{aligned} \ell_1 &= \{P : \vec{P} = (-t+2, t-1, -t+1)\} \\ \ell_2 &= \{P : \vec{P} = (-2s+4, s-3, 3s+3)\}. \end{aligned}$$

Solution. We solve the system

$$\begin{cases} -t+2 = -2s+3 \\ t-1 = s-3 \\ -t+1 = 3s+3 \end{cases} \quad \text{or} \quad \begin{cases} -t+2s = 1 \\ t-s = -2 \\ -t-3s = 2. \end{cases}$$

for s and t. We get $s = 0$ and $t = -2$. We substitute $t = -2$ in the equation of ℓ_1 (or $s = 0$ in the equation of ℓ_2), and we obtain $\vec{P} = (4, -3, 3)$. Thus $\ell_1 \cap \ell_2 = \{(4, -3, 3)\}$.

Example 5.5.5 Show that the lines

$$\ell_1 = \{P : \vec{P} = (2t+1, t+2, 2t-1)\} \text{ and } \ell_2 = \{P : \vec{P} = (3s+2, s+3, 2s+2)\}$$

are skew.

Solution. We first notice that ℓ_1 and ℓ_2 are not parallel, because $\vec{u}_1 = (2, 1, 2)$ and $\vec{u}_2 = (3, 1, 2)$ are their direction vectors which are not parallel. It remains to show that $\ell_1 \cap \ell_2 = \phi$. For this, it is enough to show that the system

$$\begin{cases} 2t+1 = 3s+2 \\ t+2 = s+3 \\ 2t-1 = 2s+2 \end{cases} \quad \text{or} \quad \begin{cases} 2t-3s = 1 \\ t-s = 1 \\ 2t-2s = 3 \end{cases}$$

has no solution. In fact, if $t - s = 1$ is satisfied for some t and s, then $2t - 2s \neq 3$. Thus ℓ_1 and ℓ_2 are skew.

Two intersecting lines ℓ_1 and ℓ_2 determine an acute and an obtuse angle. One of these angles is the angle θ between the direction vectors $\vec{u}_1$ and $\vec{u}_2$ for ℓ_1 and ℓ_2.

Thus if α is the *acute angle* between ℓ_1 and ℓ_2, then $\alpha = \theta$ if $\theta \leq \frac{\pi}{2}$ and $\alpha = \pi - \theta$ if $\theta > \frac{\pi}{2}$. Therefore

$$\cos\alpha = |\cos\theta| = \frac{|\vec{u}_1 \circ \vec{u}_2|}{|\vec{u}_1|\,|\vec{u}_2|}.$$

Example 5.5.6 Find the cosine of the acute angle between the lines given in Example 5.5.4.

Solution. Direction vectors of the lines are

$$\vec{u}_1 = (-1, 1, -1), \quad \vec{u}_2 = (-2, 1, 3).$$

Therefore

$$\cos\alpha = \frac{|\vec{u}_1 \circ \vec{u}_2|}{|\vec{u}_1|\,|\vec{u}_2|} = \frac{|2+1-3|}{\sqrt{3}\sqrt{14}} = 0.$$

Hence $\alpha = \frac{\pi}{2}$ and the lines are perpendicular.

Exercises

1. Find a vector-equation of the line
   a) through (1, 2, 3) and in the direction of $\vec{u} = (1, -1, 3)$
   b) through the points (1, 2, 3) and (1, -1, 3).
2. Show that the vectors $\vec{u}_1 = (1, -3, -2)$ and $\vec{u}_2 = (-2, 6, 4)$ are direction vectors for the same line and that

   $$\vec{P} = (t-1, -3t+1, -2t+2) \text{ and } \vec{P} = (-2t+1, 6t-5, 4t-2)$$

   are vector-equations for the same line.
3. Let P_1 and P_2 be two distinct points in 3-space.
   a) Show that $\vec{P} = t\vec{P}_1 + (1-t)\vec{P}_2$ is a vector-equation of the line passing through P_1 and P_2.
   b) Show that the *line segment* joining P_1 to P_2 is given by

   $$[P_1P_2] = \{P : t\vec{P}_1 + (1-t)\vec{P}_2, \quad 0 \leq t \leq 1\}.$$
4. Find parametric equations of the line through (1, 2, 3) and parallel to the line through (1, -1, 3) and (2, -2, 4).

5. Find a symmetric equation of the line through (1, 1, 1) and perpendicular to the two lines

$$\begin{aligned} \ell_1 &= \{(x,y,z) : (x,y,z) = (t-1, -t+1, t)\}, \\ \ell_2 &= \{(x,y,z) : (x,y,z) = (t-2, 2t-1, -3t+3)\}. \end{aligned}$$

6. Find $\ell_1 \cap \ell_2$ for the lines in Exercise 5.

7. Find the cosine of the acute angle between the lines in Exercise 5.

8. Given

$$\ell = \left\{(x,y,z) : \frac{x-1}{2} = \frac{y+2}{-1} = \frac{z}{3}\right\},$$

find a line ℓ' through (1, 2, 3) such that ℓ and ℓ' are skew.

9. Write the canonical equations of the straight line passing through the point $M_1(2,0,-3)$ and parallel;
a) to the vector $a = \{2,-3,5\}$;
b) to the straight line $\dfrac{x-1}{5} = \dfrac{y+2}{2} = \dfrac{z+1}{-1}$;
c) to the axis Ox;
d) to the axis Oy;
e) to the axis Oz.

10. Write the canonical equations of the straight line passing through the two given points:
a) $(1,-2,1), (3,1,-1)$; c) $(3,-1,0), (1,0,-3)$;
b) $(0,-2,3), (3,-2,1)$; d) $(1,2,-4), (-1,2,-4)$.

11. Write the parametric equations of the straight line passing through the point $M_1(1,-1,-3)$ and parallel;
a) to the vector $a = \{2,-3,4\}$;
b) to the line $\dfrac{x-1}{2} = \dfrac{y+2}{5} = \dfrac{z-1}{0}$;
c) to the line $x = 3t-1,\ y = -2t+3,\ z = 5t+2$.

12. Giver the vertices $A(3,6,-7), B(-5,2,3), C(4,-7,-2)$ of a triangle. Write the parametric equations of the median drawn from the vertex C.

13. Given the vertices $A(1,-2,-4), B(3,1,-3), C(5,1,-7)$ of a triangle. Find the parametric equations of the altitude drawn from the vertex B.

14. In each of the following, prove that the given lines are mutually perpendicular:
a) $\dfrac{x}{1} = \dfrac{y-1}{-2} = \dfrac{z}{3}$ and $\begin{cases} 3x+y-5z+1=0, \\ 2x+3y-8z+3=0; \end{cases}$
b) $x = 2t+1,\ y = 3t-2,\ z = -6t+1$ and

$$\begin{cases} 2x+y-4z+2=0, \\ 4x-y-5z+4=0; \end{cases}$$

c) $\begin{cases} x+y-3z-1=0, \\ 2x-y-9z-2=0 \end{cases}$ and $\begin{cases} 2x+y+2z+5=0, \\ 2x-2y-z+2=0. \end{cases}$

15. Find the acute angle between the lines

$$\frac{x-3}{1} = \frac{y+2}{-1} = \frac{z}{\sqrt{2}}, \quad \frac{x+2}{1} = \frac{y-3}{1} = \frac{z+5}{\sqrt{2}}.$$

16. Find the obtuse angle between the lines

$$x = 3t-2, \quad y = 0, \quad z = -t+3;$$

$$x = 2t-1, \quad y = 0, \quad z = t-3.$$

17. Prove that the lines represented by the parametric equations $x = 2t-3, y = 3t-2, z = -4t+6$ and $x = t+5, y = -4t-1, z = t-4$ intersect.

18. Show that the plane $x - 2 = 0$ intersects the ellipsoid

$$\frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1$$

in an ellipse; find the axes and vertices of this ellipse.

19. Which of the following points on the line $x = 3+2t,\ y = -2+3t,\ z = 4-3t$
(a) $(1,1,1)$ (b) $(1,-1,0)$ (c) $(1,0,-2)$ (d) $(4,-\frac{1}{2},\frac{5}{2})$

20. Which of the following pairs of lines are perpendicular?

a) $\begin{aligned} x &= 2+2t \\ y &= -3-3t \\ z &= 4+4t \end{aligned}$ and $\begin{aligned} x &= 2+t \\ y &= 4-t \\ z &= 5-t \end{aligned}$

b) $\begin{aligned} x &= 3-t \\ y &= 4+t \\ z &= 2+2t \end{aligned}$ and $\begin{aligned} x &= 2t \\ y &= 3-2t \\ z &= 4+2z \end{aligned}$

21. Show that the following parametric equation define the same line

$$\begin{array}{lcl} x = 2 + 3t & & x = -1 - 9t \\ y = 3 - 2t & \text{and} & y = 5 + 6t \\ z = -1 + 4t & & z = -5 - 12t \end{array}$$

22. Show that the lines

$$\ell : \frac{x-2}{2} = \frac{y+1}{-3} = \frac{z-1}{4}, \quad \ell : \frac{x-2}{-3} = \frac{y+1}{2} = \frac{z-1}{3}$$

are perpendicular.

23. Determine whether the following points lie on a line $A = (3,1,0), B = (2,2,2), C = (0,4,6)$

24. Given $\ell = \{(x,y,z) : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}\}$, find a line ℓ' through $(1,2,3)$ such that ℓ and ℓ' are skew.

25. Find the acute angle between the diagonal of a cube and one of its edges.

5.6 Planes

In this section, we will see that planes in 3-space can be described by (scalar) equations similar to the equations of lines in the plane.

Let $\mathcal{P}$ be a plane in 3-space, let $P_0(x_0, y_0, z_0)$ be a point on $\mathcal{P}$, and let $\vec{N} = (A, B, C)$ be a vector which is perpendicular to $\mathcal{P}$ (see Fig. 5.6.).

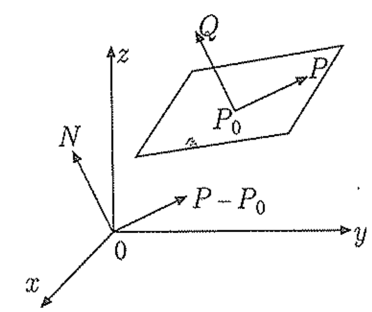


Fig. 5.6.

Saying $\vec{N}$ is perpendicular to $\mathcal{P}$, we mean that any line in the direction of $\vec{N}$ is perpendicular to any line which lies on $\mathcal{P}$. Then, for any point $P(x,y,z)$ on $\mathcal{P}$, the vector $\vec{P} - \vec{P_0}$ represented by the directed segment $\vec{P_0P}$ is perpendicular to $\vec{N}$. Conversely, if $\vec{P} - \vec{P_0}$ is perpendicular to $\vec{N}$ then P lies on the plane $\mathcal{P}$. Thus

$$\mathcal{P} = \{P : \vec{N} \circ (\vec{P} - \vec{P_0}) = 0\}.$$

The equation

$$\vec{N} \circ (\vec{P} - \vec{P_0}) = 0$$

is called an equation *of the plane $\mathcal{P}$ passing through P_0 and perpendicular to $\vec{N}$*. This equation involves vectors, but obviously it is a scalar equation. We can also write it as

$$\begin{aligned} A(x - x_0) + B(y - y_0) + C(z - z_0) &= 0 \quad \text{or} \\ Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) &= 0. \end{aligned}$$

The vector $\vec{N} = (A, B, C)$ is called a *normal vector* for the plane $\mathcal{P}$.

Any equation of the form $Ax + By + Cz + D = 0$ where not all of A, B, C, D are zero, is an equation of a plane with normal vector $\vec{N} = (A, B, C)$. In fact, if, for instance, $B \neq 0$ then it is the equation of the plane through the point $P_0(0, -\frac{D}{B}, 0)$ and perpendicular to $\vec{N}$.

Example 5.6.1 An equation of the plane through $P_0(1,2,3)$ perpendicular to $\vec{N} = (2,-4,6)$ is

$$\vec{N} \circ (\vec{P} - \vec{P_0}) = (2,-4,6) \circ (x-1, y-2, z-3) = 0$$
$$2x - 4y + 6z - 12 = 0.$$

Note that $x - 2y + 3z - 6 = 0$ is an equation of the same plane (see Exercise 8).

Example 5.6.2 Find an equation of the plane through the three points $P_0(1,2,3), P_1(-1,2,1), P_2(-2,1,4)$.

Solution. We need to find a normal vector for the plane. The vector

$$\vec{N} = (\vec{P_2} - \vec{P_0}) \times (\vec{P_1} - \vec{P_0})$$

is perpendicular to both $\vec{P_2} - \vec{P_0}$ and $\vec{P_1} - \vec{P_0}$. Hence $\vec{N}$ is perpendicular to the plane through P_0, P_1 and P_2. We have

$$\vec{N} = (-3,-1,1) \times (-2,0,-2) = (2,-8,-2).$$

Thus

$$\vec{N} \circ (\vec{P} - \vec{P}_0) = (2, -8, -2) \circ (x - 1, y - 2, z - 3) = 0$$

$$2x - 8y - 2z + 20 = 0 \quad \text{or} \quad x - 4y - z + 10 = 0$$

is an equation for the plane through P_0, P_1, P_2.

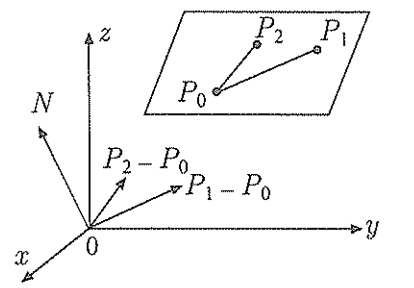


Fig. 5.7.

Consider two planes $\mathcal{P}_1$ and $\mathcal{P}_2$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are not coincident then they are either parallel or they intersect in a line; they are parallel if and only if their normal vectors are parallel.

Example 5.6.3 Find the vector-equation for the line of intersection of the planes

$$\mathcal{P}_1 = \{(x, y, z) : x + 2y + 3z + 4 = 0\} \text{ and } \mathcal{P}_2 = \{(x, y, z) : x - 2y + 2z - 1 = 0\}.$$

Solution. $\vec{N}_1 = (1, 2, 3)$ and $\vec{N}_2 = (1, -2, 2)$ are normal vectors for $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Let us denote the line of intersection by ℓ. Both $\vec{N}_1$ and $\vec{N}_2$ are perpendicular to ℓ. Therefore $\vec{u} = \vec{N}_1 \times \vec{N}_2$ is a direction vector for ℓ:

$$\vec{u} = \vec{N}_1 \times \vec{N}_2 = (10, 1, -4).$$

We can write down a vector-equation for ℓ if we know a point P_0 on ℓ, i.e., a point which is on both $\mathcal{P}_1$ and $\mathcal{P}_2$. Such a point is found by solving the equations of $\mathcal{P}_1$ and $\mathcal{P}_2$ simultaneously.

$$\begin{cases} x + 2y + 3z + 4 = 0 \\ x - 2y + 2z - 1 = 0 \end{cases} \Rightarrow \begin{cases} x + 2y + 3z + 4 = 0 \\ 2x + 5z + 3 = 0. \end{cases}$$

Let $x = 1$. Then by the second equation on the right $z = -1$, and by the first equation $y = -1$. hence $P_0(1, -1, -1)$ is on ℓ, and

$$\vec{P} = (1, -1, -1) + t(10, 1, -4) \text{ or } (x, y, z) = (10t + 1, t - 1, -4t - 1)$$

is a vector equation for the line of intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$.

The *acute angle* between two intersecting planes is defined to be the angle between two lines drawn in the two planes perpendicular to the line of intersection of the planes from a point on that line (see Fig. 5.8.).

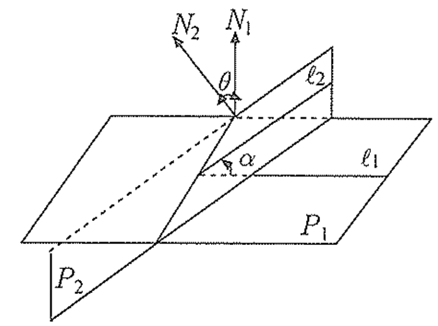


Fig. 5.8.

It is clear that if $\vec{N}_1$ and $\vec{N}_2$ are normals to the two planes, and if α is the acute angle between these planes, then

$$\cos\alpha = |\cos\theta| = \frac{|\vec{N}_1 \circ \vec{N}_2|}{|\vec{N}_1|\,|\vec{N}_2|}$$

where θ is the angle between $\vec{N}_1$ and $\vec{N}_2$.

Cosine of the acute angle α between the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ of Example 19 is

$$\cos\alpha = \frac{|\vec{N}_1 \circ \vec{N}_2|}{|\vec{N}_1|\,|\vec{N}_2|} = \frac{|1 - 4 + 6|}{\sqrt{14}\sqrt{9}} = \frac{1}{\sqrt{14}}.$$

One may also consider the intersection of a plane and a line (in 3-space). Given a plane

$$\mathcal{P} = \{(x, y, z) : Ax + By + Cz + D = 0\}$$

and a line

$$\ell = \{(x, y, z) : (x, y, z) = (m_1 t + x_1, m_2 t + y_1, m_3 t + z_1\},$$

a point of ℓ lies on $\mathcal{P}$ if and only if

$$A(m_1 t + x_1) + B(m_2 t + y_1) + C(m_3 t + z_1) + D = 0.$$

Thus $\ell \cap \mathcal{P}$ can be determined by solving this equation for t, and substituting the obtained value of t in the equation of ℓ. If the left hand side of the above equation is identically zero in t, this means that ℓ lies on $\mathcal{P}$, i.e., $\ell \subseteq \mathcal{P}$. If it has no solution for t, then $\ell \cap \mathcal{P} = \phi$. If it has a solution, then ℓ intersects $\mathcal{P}$ in a point.

Example 5.6.4 Determine $\ell \cap \mathcal{P}$ if ℓ and $\mathcal{P}$ have equations

$$(x, y, z) = (t+1, t-2, 3t+1) \quad \text{and} \quad x - y + 2z + 1 = 0.$$

Solution. If $(x, y, z) = (t+1, t-2, 3t+1)$ is a point of intersection, then

$$(t+1) - (t-2) + 2(3t+1) + 1 = 0 \Rightarrow 6t + 6 = 0 \Rightarrow t = -1.$$

Hence $\ell \cap \mathcal{P} = \{(0, -3, -2)\}$.

Exercises

1. Find an equation of the plane

   a) through (1, 2, 3) and perpendicular to the line

   $$\ell = \left\{(x, y, z)\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z}{3}\right\}.$$

   b) through (1, 2, 3), (1, 1, 1) and (-1, 2, -3).

   c) through (1, 2, 3) and

   $$\ell = \left\{(x, y, z) : \frac{x-1}{2} = \frac{y+2}{-1} = \frac{z}{3}\right\}.$$

   d) determined by the lines

   $$\ell_1 = \left\{(x, y, z) : x + 1 = \frac{y-1}{-1} = z\right\}$$

   and

   $$\ell_2 = \{(x, y, z) : (x, y, z) = (t-2, 2t-1, -3t+3)\}.$$

   e) through (1, 2, 3) and parallel to the plane $\mathcal{P} = \{(x, y, z) : 2x - 5y + z - 1 = 0\}$.

2. Show that $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is an equation of the plane passing through the points $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$.

3. Find the cosine of the acute angle between the two planes with equations

   a) $x + y + z - 1 = 0, \quad 2x - y + 2z + 4 = 0$

   b) $x + 2y + z = 1, \quad 4x - 2y - z - 3 = 0$

   c) $x - 6y + 2z - 1 = 0, \quad x - 2y + 2z - 3 = 0.$

4. In each case in Exercise 3, find a vector-equation of the line of intersection of the two planes.

5. Determine $\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_3$ if $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ are the planes with equations

   a) $x + y + z - 1 = 0, \ 2x - y + 2z + 4 = 0, \ 3x - 5y + z + 1 = 0$

   b) $x + 2y + z = 1, \ 3x + 6y + 3z - 3 = 0, \ 4x - 2y - z = 0$

   c) $x - 6y + 2z - 1 = 0, \ x - 4y + 4z - 6 = 0, \ x - 2y - 2z + 3 = 0.$

6. Determine $\ell \cap \mathcal{P}$ if $\mathcal{P}$ and ℓ have equations

   a) $(x, y, z) = (-t-1, t+2, 2t-1), \ x - 2y + 3z + 4 = 0$

   b) $\frac{x-1}{-1} = \frac{y+2}{2} = \frac{z}{3}, \ -x + 2y + 3z - 4 = 0$

   c) $\frac{x}{3} = \frac{y-2}{1} = \frac{z+1}{-1}, \ x - 2y + z - 4 = 0$

   d) $(x, y, z) = (10t + 6, t - 1, -4t - 3), x - 2y + 2z - 1 = 0.$

7. Given a line ℓ with direction vector $\vec{u}$ and a plane $\mathcal{P}$ with normal vector $\vec{N}$, verify that $\ell \cap \mathcal{P} = \ell$ or $\ell \cap \mathcal{P} = \phi$ if and only if $\vec{N} \circ \vec{u} = 0$.

8. Determine the conditions under which the equations $Ax + By + Cz + D = 0$ and $A'x + B'y + C'z + D' = 0$ represent the same plane.

9. In each of the following, find the coordinates of a vector normal to the given plane and write the general expression for the coordinates of its arbitrary normal vector:

   $a) 2x - y - 2z + 5 = 0; \quad b) x + 5y - z = 0;$

   $c) 3x - 2y - 7 = 0; \quad d) 5y - 3z = 0;$

   $e) x + 2 = 0; \quad f) y - 3 = 0.$

10. Determine which of the following pairs of equations represent parallel planes:

    a) $2x - 3y + 5z - 7 = 0, \quad 2x - 3y + 5z + 3 = 0;$

    b) $4x + 2y - 4z + 5 = 0, \quad 2x + y + 2z - 1 = 0;$

    c) $x - 3z + 2 = 0, \quad 2x - 6z - 7 = 0.$

11. Determine which of the following pairs of equations represent perpendicular planes:
a) $3x - y - 2z - 5 = 0, \quad x + 9y - 3z + 2 = 0;$
b) $2x + 3y - z - 3 = 0, \quad x - y - z + 5 = 0;$
c) $2x - 5y + z = 0, \quad x + 2z - 3 = 0.$

12. Determine the values of l and m for which the following pairs of equations represent parallel planes:
a) $2x + ly + 3z - 5 = 0, \quad mx - 6y - 6z + 2 = 0;$
b) $3x - y + lz - 9 = 0, \quad 2x + my + 2z - 3 = 0;$
c) $mx + 3y - 2z - 1 = 0, \quad 2x - 5y - lz = 0.$

13. Find the equation of the plane passing through the origin and perpendicular to the two planes
$$2x - y + 3z - 1 = 0, \quad x + 2y + z = 0.$$

14. Find the equation of the plane which passes through the two points $M_1(1, -1, -2)$ and $M_2(3, 1, 1)$ and is perpendicular to the plane $x - 2y + 3z - 5 = 0$.

15. Prove that the three planes $7x+4y+7z+1 = 0, 2x-y-z+2 = 0, x+2y+3z-1 = 0$ pass through the same straight line.

16. Prove that the three planes $2x-y+3z-5 = 0, 3x+y+2z-1 = 0, 4x+3y+z+2 = 0$ intersect in three distinct and parallel lines.

17. Determine the values of a and b for which the planes
$$2x - y + 3z - 1 = 0, \quad x + 2y - z + b = 0, \quad x + ay - 6z + 10 = 0;$$
a) have a common point;
b) pass through the same straight line;
c) intersect in three distinct and parallel lines.

18. What angle does the line of intersection of to planes $2x+y-z = 0$ and $x+y+2z = 0$ make with x-axis?

19. Find the angle between the diagonal of a cube and a diagonal of one of its edges.

5.7 Distance From a Point to a Plane or to a Line

Consider a plane
$$\mathcal{P} = \{(x, y, z) : Ax + By + Cz + D = 0\}$$

and a point $P_0(x_0, y_0, z_0)$. By the *distance*, $|P_0\mathcal{P}|$, *from* P_0 *to* $\mathcal{P}$, we mean the shortest distance from P_0 to $\mathcal{P}$, i.e.,
$$|P_0\mathcal{P}| = \min\{|P_0P| : P \in \mathcal{P}\}.$$

We know from elementary geometry that $|P_0\mathcal{P}| = |P_0H|$ where H is the point on $\mathcal{P}$ such that the vector $\vec{P}_0 - \vec{H}$, represented by $\vec{HP_0}$, is perpendicular to $\mathcal{P}$. In other words, H is the point on $\mathcal{P}$ for which $\vec{P}_0 - \vec{H}$ is parallel to the normal vector of $\mathcal{P}$ (see Fig. 5.9.).

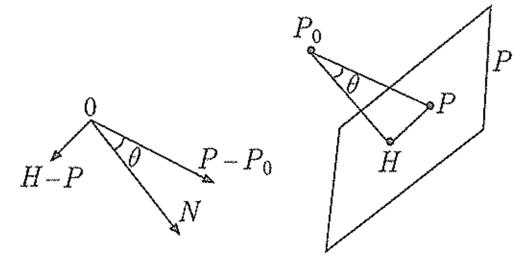


Fig. 5.9.

As we see from Fig. 5.9,
$$|P_0\mathcal{P}| = |P_0P|\,|\cos\theta|, |\cos\theta| = \frac{|\vec{N} \circ (\vec{P} - \vec{P}_0)|}{|\vec{N}|\,|\vec{P} - \vec{P}_0|}$$
where $\vec{N} = (A, B, C)$ is a normal vector for $\mathcal{P}$ and P is any point on $\mathcal{P}$. Thus
$$|P_0\mathcal{P}| = \frac{|\vec{N} \circ (\vec{P} - \vec{P}_0)|}{|\vec{N}|} = \frac{|Ax + By + Cz + D - (Ax_0 + By_0 + Cz_0 + D)|}{\sqrt{A^2 + B^2 + C^2}}.$$
Since $P \in \mathcal{P}, Ax + By + Cz + D = 0$, and we get
$$|P_0\mathcal{P}| = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Remark. Notice the similarity between the formula for the distance from a point to a line in the plane and the formula for the distance from a point to a plane in 3-space. An equation $Ax + By + C = 0$ may be viewed as the equation of a plane in 3-space!

Example 5.7.1 Find the distance from $P_0(3, -3, -2)$ to the plane $\mathcal{P} = \{(x, y, z) : x + 2y - 2z + 8 = 0\}$.

Solution. $|P_0\mathcal{P}| = \dfrac{|3-2\cdot 3+2\cdot 2+8|}{\sqrt{1+4+4}} = \dfrac{9}{3} = 3.$

Consider now the problem of finding the *distance*, $|P_0\ell|$, *from a point* P_0 *to a line*

$$\ell = \{(x,y,z) : (x,y,z) = (x_1,y_1,z_1) + t(m_1,m_2,m_3)\}.$$

Here $|P_0\ell| = |P_0H|$ for the point H on ℓ for which $\vec{P}_0 - \vec{H}$ is perpendicular to ℓ, or equivalently, to the direction vector $\vec{u} = (m_1,m_2,m_3)$ of ℓ. We see from Fig. 5.10. that for any point P on ℓ,

$$\begin{aligned}|P_0\ell| &= |P_0P|\,|\sin\theta| = |P_0P|\frac{|\vec{u}\times(\vec{P}_0-\vec{P})|}{|\vec{u}|\,|\vec{P}_0-\vec{P}|}\\ |P_0\ell| &= \frac{|\vec{u}\times(\vec{P}_0-\vec{P})|}{|\vec{u}|}.\end{aligned}$$

This expression is independent of the choice of $P \in \ell$.

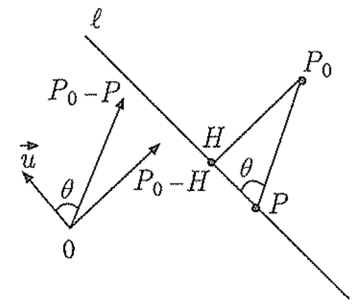


Fig. 5.10.

Example 5.7.2 Find the distance from $P_0(1,0,1)$ to the line

$$\ell = \{(x,y,z) : (x,y,z) = (2t-1,-2t+2,t+1)\}.$$

Solution. $\vec{u} = (2,-2,1)$, (taking $t = 0$) $P(-1,2,1)$ is a point on ℓ, $\vec{P}_0 - \vec{P} = (2,-2,0)$. Thus

$$|P_0\ell| = \frac{|\vec{u}\times(\vec{P}_0-\vec{P})|}{|\vec{u}|} = \frac{|(2,-2,1)\times(2,-2,0)|}{\sqrt{4+4+1}} = \frac{|(2,2,0)|}{3} = \frac{2\sqrt{2}}{3}.$$

We could also start with an arbitrary point $P(2t-1,-2t+2,t+1)$ on ℓ. Then we would have

$$|P_0\ell| = \frac{|(2,-2,1)\times(-2t+2,2t-2,-t)|}{\sqrt{4+4+1}} = \frac{|(2,-2,0)|}{3} = \frac{2\sqrt{2}}{3}.$$

Remark. Given a line ℓ and a plane $\mathcal{P}$ in 3-space, one may consider *the distance* $|\ell\mathcal{P}|$ *from the line* ℓ *to the plane*

$$|\ell\mathcal{P}| = \min\{|PQ| : P \in \ell,\ Q \in \mathcal{P}\}.$$

If $\ell\cap\mathcal{P} \neq \phi$, then $|\ell\mathcal{P}| = 0$. If $\ell\cap\mathcal{P} = \phi$, i.e., ℓ does not intersect $\mathcal{P}$, then we say that ℓ is *parallel* to $\mathcal{P}$; then $|\ell\mathcal{P}| = |P_0\mathcal{P}|$ for any $P_0 \in \ell$. Similarly, one may consider the *distance* $|\ell_1\ell_2|$ *between two lines two lines or the distance* $|\mathcal{P}_1\mathcal{P}_2|$ *between two planes.* (See Exercises 3, 4).

Example 5.7.3 Find $|\ell\mathcal{P}|$ if ℓ and $\mathcal{P}$ have equations

$$(x,y,z) = (2t-1,t+2,-t+1) \quad\text{and}\quad x-4y-2z-1=0.$$

Solution. We first check that $\ell\cap\mathcal{P} = \phi$. To see this, we substitute $x = 2t-1$, $y = t+2$, $z = -t+1$ in the equation of:

$$(2t-1)-4(t+2)-2(-t+1)-1 = 0 \Rightarrow -12 = 0.$$

Hence $\ell\cap\mathcal{P} = \phi$. Next, we find a point $P_0 \in \ell$ and compute $|P_0\mathcal{P}|$. $P_0(-1,2,1) \in \ell$. Thus

$$|\ell\mathcal{P}| = |P_0P| = \frac{|-1-4\cdot 2-2\cdot 1-1|}{\sqrt{1+16+4}} = \frac{12}{\sqrt{21}}.$$

Exercises

1. Find the distance from (1, 2, 3) to

   a) the line $\ell = \left\{(x,y,z) : \dfrac{x-2}{1} = \dfrac{y+1}{-2} = \dfrac{z-3}{1}\right\}$

   b) the plane $\mathcal{P} = \{(x,y,z) : x+y+z-1 = 0\}$.

2. Find $|\ell\mathcal{P}|$ if ℓ and $\mathcal{P}$ have equations

   a) $\dfrac{x-1}{3} = \dfrac{y+2}{-1} = \dfrac{z+1}{1}, \quad x+2y-z-1=0$

   b) $\dfrac{x-1}{-1} = \dfrac{y+2}{2} = \dfrac{z}{3}, \quad -x+2y+3z-4=0$

   c) $\dfrac{x-6}{10} = \dfrac{y+1}{1} = \dfrac{z+3}{-4}, \quad x-2y+2z-1=0.$

3. Find $|\ell_1\ell_2|$ if ℓ_1 and ℓ_2 have equations

a) $\dfrac{x-1}{3}=\dfrac{y+2}{-1}=\dfrac{z+1}{1}\quad \dfrac{2x+3}{3}=\dfrac{y+2}{-2}=\dfrac{z+2}{2}$

b) $\dfrac{x-1}{3}=\dfrac{y+2}{-1}=\dfrac{z+1}{1}\quad \dfrac{x+2}{1}=\dfrac{y-2}{-1}=\dfrac{z}{3}.$

(*Hint*: if $\ell_1\cap\ell_2\neq\phi$ then $|\ell_1\ell_2|=0$, if ℓ_1 and ℓ_2 are parallel then $|\ell_1\ell_2|=|P_1\ell_2|$ for any $P_1\in\ell_1$, if ℓ_1 and ℓ_2 are skew then $|\ell_1\ell_2|=|P_1\mathcal{P}_2|$ where P_1 is any point on ℓ_1 and $\mathcal{P}_2$ is a plane passing through ℓ_2 and parallel to ℓ_1.).

4. Find $|\mathcal{P}_1\mathcal{P}_2|$ if $\mathcal{P}_1$ and $\mathcal{P}_2$ have equations

a) $x-2y+z=3,\quad 2x-4y+2z-13=0$

b) $x-2y+z=3,\quad 2x-4y+2z-6=0$

c) $x-2y+z=3,\quad 2x-y+z-1=0.$

5. Find the length of the altitude from A of the triangle with vertices $A(1,-3,4)$, $B(3,2,1)$ and $C(2,-2,1)$.

6. Calculate the distance d of the point $P(-1,1,-2)$ from the plane passing through the three points $M_1(1,-1,1), M_2(-2,1,3)$ and $M_3(4,-5,-2)$.

7. On the axis Oy, find a point situated at a distance $d=4$ from the plane $x+2y-2z-2=0$.

8. On the axis Oz, find a point equidistant from the point $M(1,-2,0)$ and the plane $3x-2y+6z-9=0$.

9. In each of the following, find the equation of the locus of points equidistant from the two parallel planes;

a) $4x-y-2z-3=0,\quad 4x-y-2z-5=0,$

b) $3x+2y-z+3=0,\quad 3x+2y-z-1=0$

c) $5x-3y+z+3=0,\quad 10x-6y+2z+7=0.$

10. Determine whether the origin lies inside the acute or the obtuse angle formed by the two planes $x-2y+3z-5=0\ \ 2x-y-z+3=0$.

5.8 Families of Planes or Lines

Lines and planes are sets of points. One may also consider sets whose elements are lines or planes. A set of lines is called a *family of lines* and a set of planes is called a *family of planes*. Sometimes one is able to describe a family of lines or planes by means of one single equation. For example, the family of all planes, in 3-space, which are parallel to the XY-plane can be characterized by the equation $z=k$ where k denotes an arbitrary real number. The letter k in such an equation is called a *parameter*.

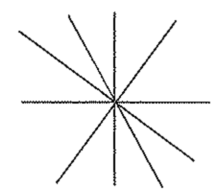
Example 5.8.1 Describe the family of all lines in the plane which are perpendicular to the line $2x-y+2=0$.

Solution. Any line in that family will have $\vec{N}=(1,2)$ as a normal vector, and the point of intersection of this line with the line $2x-y+2=0$ will be of the form $P_k(k,2k+2)$. Thus the family consists of the lines

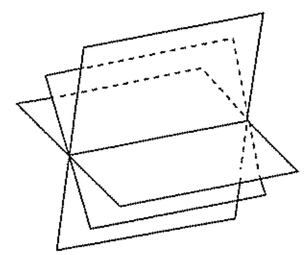
$$\begin{aligned}\ell_k &= \{P:\vec{N}\circ(\vec{P}-\vec{P}_k)=0\}\\ &= \{(x,y):(1,2)\circ(x-k,y-2k-2)=0\}\\ \ell_k &= \{(x,y):x+2y-5k-4=0\}\end{aligned}$$

where k denotes any real number.

The family of planes passing through a fixed line ℓ_0 is called a pencil of planes through ℓ_0. The family of lines passing through a fixed point P_0 is called a pencil of lines through P_0.



(pencil of lines)



(pencil of planes)

The pencil of planes through a given line ℓ_0 can be characterized by any two distinct planes in that pencil. More precisely,

Theorem 5.8.2 *If $\mathcal{P}_1 = \{(x,y,z) : Ax + By + Cz + D = 0\}$ and $\mathcal{P}_2 = \{(x,y,z) : A'x + B'y + C'z + D' = 0\}$ are two distinct planes passing through the line ℓ_0, then the pencil of the planes through ℓ_0 consists of all planes of the form*

$$\mathcal{P} = \{(x,y,z) : k(Ax + By + Cz + D) + m(A'x + B'y + C'z + D') = 0\} \qquad (5.8.2)$$

where k and m are parameters not both zero.

Proof. That the plane $\mathcal{P}$ in (5.8.1) passes through ℓ_0 is obvious, because $\mathcal{P}_1$ and $\mathcal{P}_2$ pass through ℓ_0. Conversely, assume that $\mathcal{P}$ is a plane passing through ℓ_0. We show that $\mathcal{P}$ can be expressed as in (5.8.1). If $\mathcal{P} = \mathcal{P}_1$ or $\mathcal{P} = \mathcal{P}_2$, then there is nothing to prove. Thus, we may assume that $\mathcal{P} \neq \mathcal{P}_1$ and $\mathcal{P} \neq \mathcal{P}_2$. Hence $\mathcal{P} \cap \mathcal{P}_1 = \mathcal{P} \cap \mathcal{P}_2 = \ell_0$. Choose $P_0(x_0, y_0, z_0) \in \mathcal{P}$ such that $P_0 \notin \ell_0$. Then $P_0 \notin \mathcal{P}_1$ which implies $Ax_0 + By_0 + Cz_0 + D \neq 0$. Similarly, $P_0 \notin \mathcal{P}_2$ and this implies $A'x_0 + B'y_0 + C'z_0 + D' \neq 0$. Now, let $m = -(Ax_0 + By_0 + Cz_0 + D)$ and $k = (A'x_0 + B'y_0 + C'z_0 + D')$ and consider the equation

$$k(Ax + By + Cz + D) + m(A'x + B'y + C'z + D') = 0. \qquad (*)$$

The point (x_0, y_0, z_0) satisfies this equation. Furthermore, any point on ℓ_0 satisfies this equation because for any $(x_1, y_1, z_1) \in \ell$,

$$Ax_1 + By_1 + Cz_1 + D = A'x_1 + B'y_1 + C'z_1 + D' = 0.$$

It follows that, $(*)$ is an equation of the plane passing through P_0 and ℓ_0. There is one and only one such plane and it is $\mathcal{P}$ by our assumption. This completes the proof of the theorem. □

Example 5.8.3 Find an equation of the plane $\mathcal{P}$ which passes through the line of intersection of the planes

$$\mathcal{P}_1 = \{(x,y,z) : x - 2y + 2z - 1 = 0\} \text{ and } \mathcal{P}_2 = \{(x,y,z) : 2x + y - 4z + 1 = 0\}$$

and which is perpendicular to the plane $\mathcal{P}_0 = \{(x,y,z) : x - y + z = 0\}$.

Solution. The plane $\mathcal{P}$ sought is an element of the pencil of planes containing $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus, by Theorem 5.8.1,

$$\mathcal{P} = \{(x,y,z) : k(x - 2y + 2z - 1) + m(2x + y - 4z + 1) = 0\}$$

$$\mathcal{P} = \{(x,y,z) : (k + 2m)x + (-2k + m)y + (2k - 4m)z + (-k + m) = 0\}.$$

Hence, a normal vector for $\mathcal{P}$ is of the form $\vec{N} = (k + 2m,\ -2k + m,\ 2k - 4m)$. Normal vector for $\mathcal{P}_0$ is $\vec{N}_0 = (1, -1, 1)$. Since $\mathcal{P}$ will be perpendicular to $\mathcal{P}_0$,

$$\begin{aligned} \vec{N} \circ \vec{N}_0 = 0 \quad &\to \quad (k + 2m) - (-2k + m) + (2k - 4m) = 0 \\ &\to \quad 5k - 3m = 0. \end{aligned}$$

Choosing $m = 5, k = 3$, we obtain

$$\begin{aligned} \mathcal{P} &= \{(x,y,z) : 3(x - 2y + 2z - 1) + 5(2x + y - 4z + 1) = 0\} \\ &= \{(x,y,z) : 13x - y - 14z + 2 = 0\}. \end{aligned}$$

Remark. Theorem 5.8.2, can be stated for pencils of lines in the plane. For, lines in the plane can be considered as planes parallel to z-axis in 3-space.

Exercises

1. In each of the following cases write down the equation of the family of all lines satisfying the given condition.
   a) all lines in the plane parallel to $2x - y = 4$
   b) all lines in the plane perpendicular to $2x - y = 4$
   c) all lines in 3-space parallel to $2x - y + z = 4$
   d) all lines in 3-space perpendicular to $2x - y + z = 4$.
2. Write down the equation of the pencil of lines
   a) through $(1, 2, 3)$
   b) through the point of intersection of the lines
   $$\frac{x-1}{2} = \frac{y}{-3} = \frac{z+1}{1} \quad \text{and} \quad \frac{x-3}{1} = \frac{y+3}{2} = \frac{z}{3}.$$
3. Write down the equation of the pencil of planes through
   a) the line $\dfrac{x-1}{2} = \dfrac{y}{-3} = \dfrac{z+1}{1}$
   b) the line of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 2$.
4. Find an equation of the plane through (1, 2, 3) and through the line of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 2$.
5. Find the equations of the planes that bisect the angles between the two planes with equations $x + y + z = 1$ and $2x - 3y + z + 1 = 0$.

5.9 Vectors in n-Space

We have defined *vectors* (in the plane or in 3-space) as *equivalence classes of directed segments*. However, we have observed that there is a one-to-one correspondence between vectors and points. The *coordinates* of the point corresponding to a given vector are called *components* of that vector. Equality of vectors, the operations of addition and multiplication by a scalar, etc., of vectors are all defined in terms of the components. Therefore we can identify a vector by its corresponding point and *we can define a vector in the plane as an ordered pair of real numbers* (i.e., a point), *a vector in 3-space as a triple of real numbers* (i.e., a point).

For any integer $n \geq 1$, the set

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

is called *Euclidean n-space* or simply *n-space*. The elements of $\mathbb{R}^n$, which are n-tuples of real numbers, are called *points* in the n-space. If $(x_1, x_2, \ldots, x_n)$ is a point in n-space, the numbers $x_1, x_2, \ldots, x_n$ are called, respectively, the *first, second,* ... , *n-th* coordinates of that point.

Thus we may regard 1-space as the number axis, and 2-space as the Cartesian plane. The 3-space defined in the above manner fits with our intuitive concept of "3-space".

Our aim in this section is to define vectors in n-space for any $n \geq 1$. We could achieve this by adopting the definitions given for $n = 2$ and $n = 3$, but for $n \geq 4$ geometric intuition disappears. For this reason and in view of the preceding remarks, we define a vector in n-space as follows.

The elements (points) of $\mathbb{R}^n$ are also called *vectors in n-space.* It is customary to use the notation

$$\vec{P} = (x_1, x_2, \ldots, x_n)$$

when a point $P(x_1, x_2, \ldots, x_n)$ is considered as a vector, and call $x_1, x_2, \ldots, x_n$ the *components* of $\vec{P}$. We have done so, $n = 2, 3$. However, it is clear that there is no danger in using the same notation for points and vectors. Therefore

"From here on, we omit arrows in the notations for vectors and use the same notation for points and vectors".

We will use, as before, the letters $u, v, w, \ldots, A, B, \ldots, P, Q, \ldots$ etc. (without arrows) to denote vectors.

Let $P = (x_1, x_2, \ldots, x_n)$, $Q = (y_1, y_2, \ldots, y_n)$ be two vectors in n-space. We

have

$$P = Q \Leftrightarrow x_1 = y_1 \quad \text{and} \quad x_2 = y_2 \quad \text{and} \quad \cdots \quad \text{and} \quad x_n = y_n.$$

The *sum,* $P + Q$, of P and Q is defined to be the vector

$$P + Q = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

The *multiplication* of the vector P *by a scalar* c is defined as

$$cP = (cx_1, cx_2, \ldots, cx_n).$$

We write $-P$ for $(-1)P$ and $P - Q$ for $P + (-Q)$. The vector $O = (O, O, \ldots, O)$ is called the *zero vector.*

Theorem 5.9.1 *Let A, B, C be arbitrary vectors and let $O = (O, O, \ldots, O)$ be the zero vector in n-space. Let c and d be scalars. Then*

a) $A + (B + C) = (A + B) + C$

b) $A + B = B + A$

c) $A + O = A$, *and O is the only vector with this property.*

d) $A + (-A) = 0$, *and $(-A)$ is the only vector with this property.*

e) $c(dA) = (cd)A$

f) $(c + d)A = cA + dA$

g) $c(A + B) = cA + cB$

h) $1 \cdot A = A$.

Proof. See the proofs of Theorem 3.2.5 and Theorem 5.2.3 □

The *length,* $|P|$, of a vector $P = (x_1, x_2, \ldots, x_n)$ is defined as

$$|P| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

It is clear that $|P| \geq 0$ for any P, and $|P| = 0 \Leftrightarrow P = 0$.

Linearly dependent or *linearly independent* vectors are defined as before (see §2.).

Example 5.9.2 The vectors

$$e_1 = (1,0,\ldots,0),\ e_2 = (0,1,0,\ldots,0),\ldots,e_n = (0,\ldots,0,1)$$

are linearly independent in n-space. For,

$$\begin{aligned} c_1e_1 + c_2e_2 + \cdots + c_ne_n = 0 \quad &\Rightarrow \quad (c_1,c_2,\ldots,c_n) = 0 \\ &\Rightarrow \quad c_1 = c_2 = \cdots = c_n = 0. \end{aligned}$$

Any vector $P = (x_1,x_2,\ldots,x_n)$ in n-space can be written uniquely as a linear combination of these vectors:

$$P = (x_1,x_2,\ldots,x_n) = x_1e_1 + x_2e_2 + \cdots + x_ne_n.$$

The *scalar product*, $P \circ Q$, of two vectors $P = (x_1,x_2,\ldots,x_n)$ and $Q = (y_1,y_2,\ldots,y_n)$ in n-space is defined to be the scalar

$$P \circ Q = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i.$$

The scalar product has the following properties:

Theorem 5.9.3 *Let A, B, C be vectors in n-space and let c be a scalar. Then*

a) $A \circ B = B \circ A$

b) $(cA) \circ B = c(A \circ B) = A \circ (cB)$

c) $A \circ O = 0$

d) $A \circ A = |A|^2$

e) $A \circ (B + C) = A \circ B + A \circ C$

f) $|A \circ B| \leq |A|\,|B|$.

The last inequality in Theorem 5.9.3 is called the Cauchy-Schwarz inequality. Using this inequality one can prove the *triangle inequalities*

$$|\,|A| - |B|\,| \leq |A - B| \leq |A| + |B|.$$

The proof of the theorem and the triangle inequalities is left to the exercises.

Remark. For a vector A in n-space with components $a_1, a_2, \ldots, a_n$ we have been using the notation

$$A = (a_1, a_2, \ldots, a_n).$$

It is sometimes convenient to denote the same vector by

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

When we use the first notation, we refer to A as a *row vector*. If we use the second notation then we refer to A as a *column vector*. If $A = (a_1, a_2, \ldots, a_n)$ is a row vector and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is a column vector (in n-space), we still write

$$A \circ B = \sum_{i=1}^{n} a_ib_i = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

Exercises

1. Given the vectors $A = (1,2,3,4)$, $B = (-1,2,2,-3)$ and $C = (1,-2,3,-4)$ compute the following.
   a) $A + B$ b) $A - B$ c) $2A + 3B + C$ d) $A \circ B$
   e) $(A \circ B)C$ f) $|A|B - |B|A$ g) $|A + C|$ h) $(A \circ B)C - (A \circ C)B$.

2. Prove Theorem 5.9.1.

3. Prove Theorem 5.9.3 and the triangle inequalities.

4. Determine whether the following vectors are linearly dependent or linearly independent.

   a) $u_1 = (1,-1,1,2)$ $u_2 = (2,3,-1,1)$, $u_3 = (3,2,1,1)$.

   b) $u_1 = (1,-1,1,2)$, $u_2 = (1,-4,2,3)$ $u_3 = (2,1,-1,3)$.

5.10 Vectors with Complex Components

Components of the vectors that we have considered so far are real numbers. However, one can think of vectors whose components are complex numbers. The reader can find a discussion about complex numbers in Appendix A. The set of complex numbers is denoted by $\mathbb{C}$. For any integer $n \geq 1$, the set

$$\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) : z_1, z_2, \ldots, z_n \in \mathbb{C}\}$$

is called the *complex n-space*. The elements of $\mathbb{C}^n$, which are n-tuples of complex numbers, are called *vectors* in complex n-space or they are called *vectors with complex components*. If $P = (z_1, z_2, \ldots, z_n)$ is a vector in complex n-space, then $z_1, z_2, \ldots, z_n$ are called, respectively, the *first, second,* $\ldots$, *n-th* components of the vector P. Note that vectors in Euclidean n-space can be considered as vectors with complex components.

Let $P = (z_1, z_2, \ldots, z_n)$, $Q = (a_1, a_2, \ldots, a_n)$ be two vectors in complex n-space and let $\lambda \in \mathbb{C}$. Then $P = Q \Leftrightarrow z_1 = a_1, \ z_2 = a_2, \ldots, z_n = a_n$. We define $P + Q = (z_1 + a_1, z_2 + a_2, \ldots, z_n + a_n)$, $\lambda P = (\lambda z_1, \lambda z_2, \ldots, \lambda z_n)$. Thus, addition of vectors with complex components and multiplication of a vector with complex components by a complex number are defined componentwise. One can easily observe that Theorem 5.9.1 is still true if A, B, C are vectors with complex components and c, d are complex numbers.

Let $P_1, P_2, \ldots, P_r$ be vectors with complex components (in complex n-space for a fixed n), and let $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{C}$. The expression

$$\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_r P_r$$

is called a *linear combination* of the vectors $P_1, P_2, \ldots, P_r$.

For any $P = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, we define

$$\overline{P} = (\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_n)$$

where $\overline{z}$ denotes the complex conjugate of z. The vector $\overline{P}$ is called the *complex conjugate* of P.

The *scalar product*, $P \circ Q$, of two vectors $P = (z_1, \ldots, z_n)$ and $Q = (a_1, \ldots, a_n)$ with complex components is defined as

$$P \circ Q = z_1 a_1 + z_2 a_2 + \cdots + z_n a_n = \sum_{j=1}^{n} z_j a_j.$$

The *length*, $|P|$, of $P = (z_1, z_2, \ldots, z_n)$ is defined by

$$|P| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}.$$

It is easy to see that all the properties, except (d), in Theorem 5.9.3 hold true if we let A, B, C be vectors in complex n-space and c be a complex number. Corresponding to Theorem 5.9.2 (d), we have

$$P \circ \overline{P} = |P|^2$$

for any $P \in \mathbb{C}^n$. In fact,

$$P \circ \overline{P} = \sum_{j=1}^{n} z_j \overline{z}_j = \sum_{j=1}^{n} |z_j|^2 = |P|^2.$$

The vector $P = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ may also be denoted by

$$P = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

We refer to P as a *row vector* or a *column vector* according as we choose the first or the second notation. If A is a row vector and B is a column vector in complex n-space, we still write

$$A \circ B = \sum_{j=1}^{n} a_j b_j$$

where $a_1, \ldots, a_n$ are components of A and $b_1, \ldots, b_n$ are components of B.

Exercises

1. Given the vectors $A = (1, i-1, i+1, 2i)$, $B = (-i, 0, 2i-1, i)$ and $C = (1+i, i+2, 0, 2)$, compute the following
   a) $A + B$
   b) $A - B$
   c) $2A + iB - (1-i)C$
   d) $\overline{A} + 2iB - \overline{C}$
   e) $\overline{B} \circ A$

f) $|A|B - |B|A$

g) $(A \circ B)C + (A \circ C)B$

h) $\overline{A} \circ C - \overline{B} \circ B$.

2. State and prove analogs of Theorem 5.9.1 and Theorem 5.9.3 for vectors with complex components.

3. Let $P_1, P_2, \ldots, P_r$ be vectors with complex components. We say that $P_1, P_2, \ldots, P_r$ are *linearly dependent over* $\mathbb{C}$ if there exist $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{C}$, not all zero, such that $\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_r P_r = 0$. We say that $P_1, P_2, \ldots, P_r$ are *linearly dependent over* $\mathbb{R}$ if there exist $c_1, c_2, \ldots, c_r \in \mathbb{R}$, not all zero, such that $c_1 P_1 + c_2 P_2 + \cdots + c_r P_r = 0$. If $P_1, P_2, \ldots, P_r$ are not linearly dependent over $\mathbb{C}$ (over $\mathbb{R}$) then we say that they are *linearly independent over* $\mathbb{C}$ (respectively over $\mathbb{R}$). Prove that the vectors

$$P_1 = (i, 1+i), \quad P_2 = (1-i, i), \quad P_3 = (i+2, -1)$$

are linearly dependent over $\mathbb{C}$, and that they are linearly independent over $\mathbb{R}$.

4. Determine whether the following vectors are linearly dependent or linearly independent over $\mathbb{C}$.

a) $P_1 = (i, i+1, 1), \quad P_2 = (1-i, i, 2i-1), \quad P_3 = (3i-1, 4i, 1-i)$

b) $P_1 = (i, i+1, 1), \quad P_2 = (1-i, i, 2i-1), \quad P_3 = (1, 1-i, -i)$

c) $P_1 = (1, 2, 3), \quad P_2 = (-2, 1, -1), \quad P_3 = (1, -1, -2)$.

5. Determine whether the vectors in Exercise 4 are linearly dependent or linearly independent over $\mathbb{R}$.

Chapter 6

SYSTEMS OF LINEAR EQUATIONS AND MATRICES

We come up with systems of linear equations when we investigate the intersection of lines or planes, and when we test the linear dependence or linear independence of vectors. In this chapter, we introduce matrices which are used for solving systems of linear equations.

A system of linear equations is a set of equations like

$$\begin{cases} 3x - 5y + 4z = 1 \\ 5x + 2y - 6z = 0. \end{cases} \qquad (*)$$

If we let

$$\alpha_1 = (3, -5, 4), \quad \alpha_2 = (5, 2, -6), \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

then the system $(*)$ can be expressed as

$$\begin{cases} \alpha_1 \circ X = 1 \\ \alpha_2 \circ X = 0. \end{cases} \quad (*)$$

On the other hand, the system (*) is completely determined by the array of real numbers

$$\tilde{A} = \left[\begin{array}{rrr|l} 3 & -5 & 4 & 1 \\ 5 & 2 & -6 & 0. \end{array}\right]$$

Such an array of real numbers is called a matrix. In order to solve the system (*), we operate with the matrix $\tilde{A}$ rather than the variables x, y and z.

6.1 Matrices

Let p and q be two integers, $p \geq 1$ and $q \geq 1$. An array of real numbers of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ . & . & \dots & . \\ a_{p1} & a_{p2} & \dots & a_{pq} \end{bmatrix} \tag{6.1.1}$$

is called a $p \times q$ *matrix* over the set of real numbers. It consists of p rows and q columns. The rows can be considered as row vectors in q-space, and the columns as column vectors in p-space. For example, the first row of the matrix A is

$$\alpha_1 = (a_{11}, a_{12}, ..., a_{1q}),$$

the second column is

$$A_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ . \\ a_{p2} \end{bmatrix}.$$

In general, for any $i = 1, 2, ..., p$

$$\alpha_i = (a_{i1}, a_{i2}, ..., a_{iq})$$

denotes the *i-th row* of A; and for any $j = 1, 2, ..., q$

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ . \\ a_{pj} \end{bmatrix}.$$

denotes the *j-th column* of A.

The real number a_{ij} which is common to the i-th row and the j-th column is called the ij-*entry* of the matrix A.

Example 6.1.1 The matrix

$$A = \begin{bmatrix} 3 & -5 & 4 \\ 5 & 2 & -6 \end{bmatrix}$$

is a 2×3 matrix. It rows are

$$\alpha_1 = (3, -5, 4), \quad \alpha_2 = (5, 2, -6),$$

and its columns are

$$A_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 4 \\ -6 \end{bmatrix}.$$

Example 6.1.2 For any $p \geq 1$ and $q \geq 1$, one may consider the $p \times q$ matrix all of whose entries are zero. We call this matrix the $p \times q$ *zero matrix* and denote it by O. Thus

$$O = \left.\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \underbrace{. \quad . \quad \dots \quad 0}_{q} \end{bmatrix}\right\} p.$$

Each row vector of the $p \times q$ zero matrix is the zero vector in q-space. Each column vector of it is the zero vector in p-space.

An abbreviation for the notation in (6.1.1) is

$$A = [a_{ij}], i = 1, 2, ..., p \text{ and } j = 1, 2, ..., q. \tag{6.1.2}$$

Here i denotes the row and j denotes the column to which the entry a_{ij} belongs. For instance, $a_{23} = -6$ in Example 6.1.1. One may also abreviate the notation for the matrix A as follows:

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_p \end{bmatrix} = [A_1, A_2, ..., A_q].$$

Remark. Any row vector in n-space can be considered as a $1 \times n$ matrix. Similarly any column vector in n-space can be considered as an $n \times 1$ matrix. We refer to these, respectively, as *row matrices* and column matrices 1×1 matrices are nothing but real numbers.

If A is a $p \times q$ matrix, we also say that A is of *size* $p \times q$. If $p = q$, i.e., if the number of columns of A is the same as the number of rows of A, then A is called a *square matrix.*

Example 6.1.3 The matrices

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1, \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 4 & 7 \\ 12 & 6 & 3. \end{bmatrix}$$

are both square matrices. A is of size 2×2 and B is of size 3×3.

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be *equal* if they have the same size and if the corresponding entries are the same. Thus

$$A = B \Longleftrightarrow a_{ij} = b_{ij} \text{ for all } i \text{ and } j;$$

It follows that

$$A = B \Longleftrightarrow \alpha_i = \beta_i \text{ for all } i \Longleftrightarrow A_j = B_j \text{ for all } j.$$

(β_i denotes the i-th row of the matrix B.)

6.2 Algebra of Matrices

We shall now define *addition* of two matrices, *multiplication* of a matrix by a *scalar* and *multiplication* of two matrices.

We define addition of matrices only when they have the same size. Let $p \geq 1$ and $q \geq 1$ be two fixed integers; and let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $p \times q$ matrices. The *sum*, $A+B$ of A and B is defined to be the $p \times q$ matrix whose ij-entry is $a_{ij} + b_{ij}$. Thus

$$A + B = [a_{ij} + b_{ij}], \quad i = 1, 2, ..., p \quad \text{and} \quad j = 1, 2, ..., q.$$

In terms of row and column vectors,

$$A + B = \begin{bmatrix} \alpha_1 & + & \beta_1 \\ \alpha_2 & + & \beta_2 \\ & . & \\ & . & \\ & . & \\ \alpha_p & + & \beta_p \end{bmatrix} = [A_1 + B_1, A_2 + B_2, ..., A_q + B_q]$$

Example 6.2.1 Let

$$A = \begin{bmatrix} 3 & -5 & 4 \\ 5 & 2 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 8 & 1 & -1 \\ -3 & 12 & 4 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} 3+8 & (-5)+1 & 4+(-1) \\ 5+(-3) & 2+12 & 6+4 \end{bmatrix} = \begin{bmatrix} 11 & -4 & 3 \\ 2 & 14 & 10 \end{bmatrix}.$$

Let c be a scalar and $A = [a_{ij}]$ be a $p \times q$ matrix. We define the *multiplication* cA of A *by the scalar* c as follows:

$$cA = [c\, a_{ij}].$$

Thus, cA is again a $p \times q$ matrix whose ij-entry is $c\, a_{ij}$, the product of c and the ij-entry of A. In terms of row and column vectors:

$$cA = \begin{bmatrix} c\alpha_1 \\ c\alpha_2 \\ . \\ . \\ . \\ c\alpha_p \end{bmatrix} = [cA_1, cA_2, ..., cA_q].$$

Example 6.2.2 Let A, B be as in Example 6.2.1 and $c = 3$. Then

$$3A = \begin{bmatrix} 9 & -15 & 12 \\ 15 & 6 & 18 \end{bmatrix}, \quad 3B = \begin{bmatrix} 24 & 3 & -3 \\ -9 & 36 & 12 \end{bmatrix}.$$

We write

$$-A = (-1)A \text{ and } A - B = A + (-B)$$

for any $p \times q$ matrices A and B.

Theorem 6.2.3 *Let $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}]$ be all $p \times q$ matrices, let O denote the $p \times q$ zero matrix and let c and d be scalars. Then*

a) $(A + B) + C = A + (B + C)$

b) $A + B = B + A$

c) $O + A = A$, and O is the only matrix with this property.

d) $A + (-A) = 0$, and $(-A)$ is the only matrix with this property.

e) $c(A + B) = (cA) + (cB)$

f) $(c + d)A = (cA) + (dA)$

g) $c(dA) = (cd)A$

h) $1A = A$.

Proof. All of them follow from the definitions. We prove part a) and leave the others to the reader as exercises. We have

$$(A + B) = C = [(A_1 + B_1) + C_1, ..., (A_q + B_q) + C_q]$$

Applying Theorem 5.2.3 (a), to the column vectors of $(A + B) + C$, we obtain

$$\begin{aligned} (A + B) + C &= [A_1 + (B_1 + C_1), ..., A_q + (B_q + C_q)] \\ &= A + (B + C). \end{aligned}$$

This completes the proof of a). □

We shall now define the multiplication of matrices. We define the *product*, AB, of two matrices A and B only when A is a $p \times q$ matrix and B is a $q \times r$ matrix. In other words, *AB is defined only when the number of columns of A is equal to the number of rows of B.*

Let $A = [a_{ij}]$, $B = [b_{jk}]$, $i = 1, 2, ..., p$, $j = 1, 2, ..., q$ and $k = 1, 2, ..., r$ be two matrices. We represent A and B in terms of their rows and columns, respectively, as

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_p \end{bmatrix}, \quad B = [B_1, B_2, ..., B_r].$$

The product, AB, of A and B is defined as

$$AB = [\alpha_i \circ B_k], \quad i = 1, 2, ..., p \text{ and } \quad k = 1, 2, ..., r.$$

Thus the ik-entry of AB is $\alpha_i \circ B_k$, the scalar product of the i-th row of A and the k-th column of B. AB is a $p \times r$ matrix.

Example 6.2.4

$$A = \begin{bmatrix} 3 & -5 & 4 \\ 5 & 2 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Then

$$\alpha_1 = (3, -5, 4), \quad \alpha_2 = (5, 2, 6), \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

and

$$AB = \begin{bmatrix} \alpha_1 \circ B_1 & \alpha_1 \circ B_2 & \alpha_1 \circ B_3 \\ \alpha_2 \circ B_1 & \alpha_2 \circ B_2 & \alpha_2 \circ B_3 \end{bmatrix} = \begin{bmatrix} (3 + 0 + 4) & (-3 - 5 + 0) & (-5 - 4) \\ (5 + 0 + 6) & (-5 + 2 + 0) & (2 - 6) \end{bmatrix}.$$

$$AB = \begin{bmatrix} 7 & -8 & -9 \\ 11 & -3 & -4 \end{bmatrix}.$$

Note that BA is undefined in this example.

Example 6.2.5 Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then

$$\alpha_1 = (1, 1), \quad \alpha_2 = (0, 1), \qquad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$AB = \begin{bmatrix} \alpha_1 \circ B_1 & \alpha_1 \circ B_2 \\ \alpha_2 \circ B_1 & \alpha_2 \circ B_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

In this example, BA is also defined. We have

$$B_1 = (1, 0), \quad B_2 = (1, 1), \qquad A_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$BA = \begin{bmatrix} \beta_1 \circ A_1 & \beta_1 \circ A_2 \\ \beta_2 \circ A_1 & \beta_2 \circ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Remark. The above examples show that BA may not be defined although AB is defined, and even if both are defined AB *need not be equal to* BA.

Example 6.2.6 Given

$$A = \begin{bmatrix} 3 & 5 \\ -5 & 2 \\ 4 & 6 \end{bmatrix} B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} C = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

compute $A(BC)$ and $A(B+C)$.

Solution. We first compute $B+C$ and BC.

$$B + C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

$$BC = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Thus

$$A(BC) = \begin{bmatrix} 3 & 5 \\ -5 & 2 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -12 & 7 \\ 2 & 2 \end{bmatrix}.$$

$$A(B+C) = \begin{bmatrix} 3 & 5 \\ -5 & 2 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -12 & 9 \\ 2 & 8 \end{bmatrix}.$$

Theorem 6.2.7 *Let A, B and C be matrices such that in each of the following properties the indicated operations can be performed; and let c be a scalar. Then*

(a) $A(BC) = (AB)C$

(b) $A(B+C) = AB + AC$

(c) $c(AB) = (cA)B = A(cB)$.

Proof. We prove the first part and leave the rest of the proof to the reader as exercise. Let $A = [a_{ij}]$ be a $p \times q$ matrix. Since the indicated operations are defined, $B = [b_{jk}]$ is a $q \times r$ matrix and $C = [c_{kh}]$ is an $r \times s$ matrix for some r and s. Then both $A(BC)$ and $(AB)C$ are $p \times s$ matrices. Let us denote the ik-entry of AB by u_{ik}, ih-entry of

$(AB)C$ by x_{ih}, jh-entry of BC by v_{jh} and ih-entry of $A(BC)$ by y_{ih}. Then u_{ik} is the scalar product of the i-th row of A and k-th column of B, i.e.,

$$u_{ik} = \sum_{k=1}^{q} a_{ij}b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{iq}b_{qk}.$$

Similarly,

$$x_{ih} = \sum_{k=1}^{r} u_{ik}c_{kh} = \sum_{k=1}^{r}\left(\sum_{j=1}^{q} a_{ij}b_{jk}\right)c_{kh}$$

$$x_{ih} = \sum_{k=1}^{r}\sum_{j=1}^{q} a_{ij}b_{jk}c_{kh} = \sum_{j=1}^{q}\sum_{k=1}^{r} a_{ij}b_{jk}c_{kh}$$

and

$$v_{jh} = \sum_{k=1}^{r} b_{jk}c_{kh}$$

$$y_{ih} = \sum_{j=1}^{q} a_{ij}v_{jh} = \sum_{j=1}^{q} a_{ij}(\sum_{k=1}^{r} b_{jk}c_{kh})$$

$$y_{ih} = \sum_{j=1}^{q}\sum_{k=1}^{r} a_{ij}b_{jk}c_{kh} = x_{ih}.$$

We see that the corresponding entries of $A(BC)$ and $(AB)C$ are equal. Therefore $A(BC) = A(BC)$. □

Consider a $p \times p$ square matrix $M = [m_{ij}]$. The p-tuple $(m_{11}, m_{22}, \ldots, m_{pp})$ is called the main diagonal of M. A $p \times p$ matrix is called *the $p \times p$ identity matrix* if each of its entries along the main diagonal is 1 and all the other entries are zero. The $p \times p$ identity matrix is denoted by $I_{p\times p}$ or simply by I.

$$M = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ . & . & \ldots & . \\ a_{p1} & a_{p2} & \ldots & a_{pp} \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & 1 \end{bmatrix}.$$

The k-th row (or the k-th column) of the $p \times p$ identity matrix is the basic unit vector e_k in p-space. It follows that for any $p \times q$ matrix A, we have

$$I_{p\times p}A = A \text{ and } AI_{q\times q} = A.$$

Now we define another notion related to a matrix. Let $A = [a_{ij}]$ be a $p \times q$ matrix. The $q \times p$ matrix $B = [b_{ij}]$ such that $b_{ij} = a_{ji}$ for each $i = 1, \ldots, q$ and

$j = 1, ..., p$ is called the *transpose* of A, and is also denoted by ${}^tA$. Note that the i-th row of ${}^tA$ is the i-th column of A, and the j-th column of ${}^tA$ is the j-th row of A for any $i = 1, ..., q$ and $j = 1, ..., p$. In other words, rows of A become columns of ${}^tA$ and columns of A become rows of ${}^tA$. If A is the matrix written in (6.1.1) then

$${}^tA = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{p1} \\ a_{12} & a_{22} & \dots & a_{p2} \\ . & . & \dots & . \\ a_{1q} & a_{2q} & \dots & a_{pq} \end{bmatrix}.$$

Example 6.2.8 The following special cases illustrate the transpose.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 4 \end{bmatrix} \Longrightarrow {}^tA = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 4 \end{bmatrix}, \quad A = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Longrightarrow {}^tA = [2 \ \ -1 \ \ 1].$$

Theorem 6.2.9 *Let A and B be matrices such that in each of the following properties the indicated operations can be performed. Let c be a scalar. Then*

$$\begin{array}{llll} (a) & {}^t({}^tA) = A & (b) & {}^t(A+B) = {}^tA + {}^tB. \\ (c) & {}^t(cA) = c({}^tA) & (d) & {}^t(AB) = {}^tB\,{}^tA. \end{array}$$

Proof. The first three properties are trivial. We prove the last property. Let A be a $p \times q$ matrix with rows $\alpha_1, \alpha_2, ..., \alpha_{1p}$ and B be $q \times r$ matrix with columns $B_1, B_2, ..., B_r$. Then ${}^tB$ is an $r \times q$ matrix with rows $B_1, B_2, ..., B_r$ and ${}^tA$ is a $q \times p$ matrix with columns $\alpha_1, \alpha_2, ..., \alpha_p$. Therefore ${}^tB{}^tA$ is an $r \times p$ matrix with ij-entry $B_i \circ \alpha_j$ i.e.,

$${}^tB{}^tA = [B_i \circ \alpha_j], \quad i = 1, ..., r; \quad j = 1, ..., q.$$

On the other hand, ij-entry of ${}^t(AB)$ is the ji-entry of AB which is $\alpha_j \circ B_i$. Thus

$${}^t(AB) = [B_i \circ \alpha_j], \quad i = 1, ..., r; \quad j = 1, ..., q.$$

proving d). □

Exercises

1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & -2 & 5 \end{bmatrix}.$$

Find $A + B, 2A, 3B, 2A + 3B, -A, 3B - A, A - B$, and $B - A$.

2. Determine the unknowns x, y and z if

$$\begin{bmatrix} x & -1 \\ 0 & y \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ z & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3. Determine the unknowns x, y and z if

$$\begin{bmatrix} x & -1 \\ 0 & y-1 \end{bmatrix} + \begin{bmatrix} 2x & 1 \\ z+1 & 3x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

4. Write down the row vectors $\alpha_1, \alpha_2, \beta_1, \beta_2$ and the column vectors $A_1, A_2, A_3, B_1, B_2, B_3$ in exercise 1.

5. Given

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ -1 & 0 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -3 & -2 \\ 2 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ -2 & -1 \\ 1 & 0 \end{bmatrix}.$$

a) Compute $(AB)C$ and $A(BC)$,
b) Compute BA and compare with AB.
c) Is CB defined?
d) Is $(BA)C$ defined?
e) Is AC or CA defined?
f) Is $A + B$ defined?

6. If A is a square matrix, we write $A^2 = AA$ and $A^n = A(A^{n-1})$ for any $n > 2$. Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

a) Compute $A^2, B^2, C^2,\ A^3, B^3, C^3,\ A^{10}, B^{10}$,
b) Generalize the above results to 4×4 matrices.

7. If all the entries except possibly those on the main diagonal of square matrix are zero, then that matrix is called a *diagonal matrix*. Let $A = [a_{ij}]$ be a $p \times p$ matrix and D be a diagonal $p \times p$ matrix:

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ . & . & \dots & d_p \end{bmatrix}.$$

a) Show that j-th column of AD is d_jA_j.
b) Show that i-th column of AD is $(\alpha_i \circ (d_1, ..., d_p)) = (d_1a_{i1}, ..., d_pa_{ip})$.

c) Is $AD = DA$ true in general?

d) Is $AD = DA$ true if A is also a diagonal matrix?

8. Let
$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \\ 5 & 2 & 7 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

a) Find AX, $(A^2)X$, $({}^tA)X$, $({}^tX)AX$.

b) Find a 3×3 matrix B such that $BA = I$, the 3×3 identity matrix.

9. Let
$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ -4 & 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

a) Compute BA, AB, AX, BAX, BC,

b) Determine x, y and z such that $AX = C$.

10. Let
$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

a) Determine x and y so that $AX = 0$,

b) Determine x and y so that $AX = B$,

c) Determine x and y so that $AB = X$,

d) Determine x and y so that $A^2X = B$.

11. Given
$$A = \begin{bmatrix} -1 & 4 \\ 3 & -1 \end{bmatrix} B = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} C = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}.$$

a) Compute AB, BC, ABC, AC and ACB.

b) Compute $({}^tA)A, ({}^tB)B$ and $({}^tC)C$.

12. Let A be any matrix. We say that A is symmetric if ${}^tA = A$.

a) Show that $A + {}^tA$ is symmetric for any square matrix A.

b) Determine whether or not the matrices A, B, C of exercise 11 are symmetric.

13. Let X, A, B and C be $m \times n$ matrices. Solve the equation $(X + A) + (B - C) = 3X + 4A$ i.e.. Find X in terms of A, B and C.

14. If A, B, X, and Y are $m \times n$ matrices, solve the equations
$$\begin{aligned} 3X + 3Y &= A \\ X + 2Y &= B \end{aligned}$$
i.e., find X and Y in terms of A and B.

15. Let A and B be matrices. Suppose that AB and BA are defined and $AB = BA$. Show that A and B are square matrices of the same order.

16. If B is a 2×2 matrix such that $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} B = 0$, show that $B = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ for some numbers a and b.

17. Show that a 2×2 matrix A commutes with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ iff $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ for some numbers a and b.

18. Show by induction or otherwise that
$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & nx & ny + \frac{n(n-1)}{2}x^2 \\ 0 & 1 & nx \\ 0 & 0 & 1 \end{bmatrix}.$$

19. Let A and B be $n \times n$ matrices. Show that $(A - B)(A + B) = A^2 - B^2$ if and only if A and B commute.

20. Let D be a 3×3 diagonal matrix and f a polynomial. If
$$D = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$
Show that
$$f(D) = \begin{bmatrix} f(a_1) & 0 & 0 \\ 0 & f(a_2) & 0 \\ 0 & 0 & f(a_3) \end{bmatrix}.$$

21. Show that sums and products of diagonal matrices are again diagonal matrices. Show that any pair of diagonal matrices commutes.

22. If A and B are commuting square matrices, show that A^m and B^n commute, where m, n are positive integers.

Find examples that illustrate the following properties of matrix multiplication

(a) It may be that $AB = 0$ even though $A \neq 0, B \neq 0$

(b) It may be that $A^2 = 0$ even though $A \neq 0$

(c) It is possible for $A^3 = 0$ while $A \neq 0$ and $A^2 \neq 0$

(d) For A different from 0 and I, it is still possible that $A^2 = I$

(e) For A different from I and $-I$, it is still possible that $A^2 = I$.

6.3 Systems of Linear Equations

Let $a_{ij}, i = 1, ..., p; j = 1, ..., q$ be pq real numbers and also $c_1, c_2, ..., c_p \in \mathbb{R}$. Then a set of equation of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2q}x_q = c_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{p1}x_1 + a_{p2}x_2 + \ldots + a_{pq}x_q = c_p \end{cases} \tag{6.3.1}$$

is called a *system* of p *linear equation* in q unknowns $x_1, x_2, ..., x_q$. Such a system defines, uniquely, a $p \times q$ matrix A, a $q \times 1$ matrix X and a $p \times 1$ matrix C:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_q \end{bmatrix} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_p \end{bmatrix}.$$

The matrix A is called the *coefficient matrix* of the system of linear equations (6.3.1) can be written in matrix form as

$$AX = C \tag{6.3.2}$$

Conversely, it is clear that any matrix equation of the above form leads to a system of linear equations.

Together with the coefficient matrix A, we also consider the $p \times (q+1)$ matrix

$$\tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1q} & c_1 \\ a_{21} & a_{22} & \cdots & a_{2q} & c_2 \\ . & . & \cdots & . & \\ . & . & \cdots & . & \\ . & . & \cdots & . & \\ a_{p1} & a_{p2} & \cdots & a_{pq} & c_p \end{array}\right].$$

This matrix is called the *augmented matrix* of the system (6.3.1). Any $p \times (q+1)$ matrix is the augmented matrix of a system of p linear equations in q unknowns, in the natural way.

Example 6.3.1 The augmented matrix of the system (∗) in the first paragraph of this chapter is

$$\left[\begin{array}{ccc|c} 3 & -5 & 4 & 1 \\ 5 & 2 & -6 & 0 \end{array}\right].$$

Example 6.3.2 The 3×4 matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 3 & -4 & -1 \\ -1 & 0 & 7 & 2 \end{array}\right].$$

is the augmented matrix of the system

$$\begin{cases} x_1 - x_2 &= 1 \\ 2x_1 + 3x_2 - 4x_3 &= -1 \\ -x_1 + 7x_3 &= 2. \end{cases}$$

A vector $\xi = (\xi_1, \xi_2, ..., \xi_q)$ in $\mathbb{R}^q$ is called a *solution* of the system of linear equations (6.3.1) if each equation in that system is satisfied for $x_1 = \xi_1, \quad x_2 = \xi_2, ..., x_q = \xi_q$; i.e.,

$$A({}^t\xi) = C.$$

Two systems of linear equations are said to be *equivalent* if their sets of solutions in $\mathbb{R}^q$ are the same.

The set of solutions of a system of linear equations is determined by reducing that system to an equivalent but a simpler system. For this purpose, we make use of the following operations (See Example 5.5.4 and 5.5.5 of Chapter 5):

(Er_1) *Multiplication of both sides of any equation in the system by a non-zero scalar.*

(Er_2) *The interchange of any two equations in the system.*

(Er_3) *Addition of any equation in the system to another equation in the system.*

These operations are called *row operations* on systems of linear equations.

It is clear that if we apply a finite number of these operations to a system of linear equations then we obtain a system which has the same set of solutions as the system in the beginning. Furthermore, for each of these operations there exists naturally an inverse operation. Thus we have the

Theorem 6.3.3 *Two systems of linear equations are equivalent if and only if each of these systems can be obtained from the other by applying a finite number of row operations (of the form* $(r_1), (r_2), (r_3)$*).*

The row operations $(r_1), (r_2)$ *and* (r_3) *on a system of linear equations induce the following operations on its augmented matrix.*

(Er_1) Multiplication of any row of the matrix by a non-zero scalar.

(Er_2) The interchange of any two rows of the matrix.

(Er_3) Addition of any row of the matrix to another row of the matrix.

These operations are called elementary row operations on matrices. If a matrix N can be obtained from another matrix M by a finite sequence of elementary row operations then we say that M is row-equivalent to N, and we write $M \stackrel{\sim}{\to} N$.

It is clear that to each elementary row operation on matrices corresponds, naturaly, an inverse operation. Therefore if $M \stackrel{\sim}{\to} N$ then $N \stackrel{\sim}{\to} M$ and instead of saying "M is row equivalent to N", we can also say "*M and N are row-equivalent*". Morevover, it is a direct consequence of the definition that if $M \stackrel{\sim}{\to} N$ and $N \stackrel{\sim}{\to} P$, then $M \stackrel{\sim}{\to} P$.

We can thus restate the theorem above in terms of the augmented matrices:

Theorem 6.3.4 *Two systems of linear equations are equivalent if and only if their augmented matrices are row-equivalent.*

As a consequence of the above discussions, in order to determine the set of solutions of a system of linear equations, we take the augmented matrix of the system and put it into a "simpler" form by elementary row operations. We describe now what we intend by "simpler" form of matrices.

Let $M = [m_{ij}]$ be an $s \times k$ matrix with rows $\mu_1, \mu_2, ..., \mu_s$. If μ_i is a non-zero row of M, the first non-zero component (entry) of μ_i is called the *leading entry* of μ_i.

For example, the leading entry of the first row of the matrix

$$M = \begin{bmatrix} 0 & 0 & 5 \\ 2 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

is 5, the leading entry of the second row is 2, and the last row is zero.

We say that the $s \times t$ matrix M is *row-reduced* if the following two conditions are satisfied:

(E_1) *the leading entry of every non-zero row of* M *is* 1.

(E_2) *Every column containing such a leading entry* 1 *has all its other entries equal to zero.*

Example 6.3.5 The following are examples of row-reduced matrices:

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 6.3.6 The matrix

$$A = \begin{bmatrix} 0 & 0 & 5 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

is not row-reduced. However we have

$$A = \begin{bmatrix} 0 & 0 & 5 \\ 2 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \stackrel{\sim}{\to} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \stackrel{\sim}{\to} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1/2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \stackrel{\sim}{\to} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

where the last matrix is row-reduced. The following theorem shows that this is the special case of a more general fact.

Theorem 6.3.7 *Any matrix* M *is row-equivalent to a row-reduced matrix.*

Proof. Let $M = [m_{ij}]$ be an $s \times t$ matrix with rows $\mu_1, \mu_2, ..., \mu_s$. Suppose that μ_k is a non-zero row of M having the leading entry m_{kh}. Multiply the k-th row of M by (m_{kh}^{-1}). The leading entry of the k-th row of the obtained matrix becomes 1. Now, subtract m_{ih} times the k-th row from any other non-zero row in the obtained matrix. This reduces every other entry in the h-th column to zero, so that conditions (E_1) and (E_2) are satisfied as regards the k-th row. The proof follows by induction on the number of non-zero rows. □

Given a row-reduced matrix N, we can find a row-reduced matrix M which is row-equivalent to N and which satisfies the following additional conditions:

(E_3) *Every zero-row of M comes below every non-zero row of it*

(E_4) *Suppose that μ_h and μ_k are any two non-zero rows of M with $h < k$, and suppose that the leading entry of μ_h occurs in the n-th column and the leading entry of μ_k occure in the r-th column. Then $n < r$.*

A row-reduced matrix satisfying the additional conditions (E_3) and (E_4) is called a *row-reduced echelon* matrix.

Corollary 6.3.8 *Any matrix is row-equivalent to a row-reduced echelon matrix.*

Example 6.3.9 Consider the row-reduced matrices M, N and P of example 12. They are not row-reduced echelon matrices. We have

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tilde{\rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad N \tilde{\rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad P \tilde{\rightarrow} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

which are row-reduced echelon.

Remark. To determine the set of solutions of a system of linear equations, we find a row-reduced echelon matrix which is row - equivalent to the augmented matrix of the system. The system of linear equations corresponding to that row- reduced echelon matrix is of simpler form than the original system, in the sense that we can easily determine the set of solutions. Namely we have

Theorem 6.3.10 *Let the augmented matrix $\tilde{A}$ of a system of linear equations in q unknowns be row-equivalent to a row-reduced echelon matrix M. Then we have two cases:*

a) If M contains a row of the form $(0...0|1)$, then the system has no solution. In this case we say that the system is inconsistent.

b) If M contains r non-zero rows and no row of the form $(0...0|1)$, then the system has solutions. A general solution can be given by using $(q-r)$ parameters.

Remark. If we are in case b) of the Theorem 6.3.5, a general solution of the system will be of the form

$$\xi = (d_{10} + d_{11}t_1 + \dots + d_{1q-r}t_{q-r}, \dots, d_{qo} + dq_1 t_1 + \dots + d_{qq-r}t_{q-r})$$

where d_{ij} are fixed real numbers $1 \geq i \geq q, 0 \geq j \geq q-r$; and $t_1, \dots, t_{q-r}$ are parameters.

Example 6.3.11 Consider the system of linear equations

$$\begin{cases} x_1 + x_2 - x_4 = 1 \\ 2x_1 + 3x_2 + x_3 - 2x_4 = 1 \\ x_1 - x_3 - x_4 = 2. \end{cases}$$

Its augmented matrix is

$$\tilde{A} = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 2 & 3 & 1 & -2 & 1 \\ 1 & 0 & -1 & -1 & 2 \end{array}\right].$$

We have

$$\tilde{A} \tilde{\rightarrow} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 1 \end{array}\right] \quad (add(-2)\alpha_1 \text{ to } \alpha_2 \text{ and } -\alpha_1 \text{ to } \alpha_3).$$

$$\tilde{\rightarrow} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (\text{ add } -\alpha_2 \text{ to } \alpha_1 \text{ and } \alpha_2 \text{ to } \alpha_3)$$

which is row-reduced echelon. Thus the given system is equivalent to

$$x_1 - x_3 - x_4 = 2$$
$$x_2 + x_3 = 1.$$

Letting $x_3 = t_1$ and $x_4 = t_2$, we obtain $x_1 = t_1 + t_2 + 2$ and $x_2 = -t_1 + 1$. Thus the set of solutions of the system is

$$S = \{(t_1 + t_2 + 2, -t_1 + 1, t_1, t_2) : t_1, t_2 \in \mathbb{R}\}.$$

Here, $\xi = (t_1 + t_2 + 2, -t_1 + 1, t_1, t_2)$ is referred as the *general solution*. A *particular solution* is obtained by assigning values to the parameters t_1 and t_2. For example, if we assign the values $t_1 = 1, t_2 = 2$ then we get the particular solution $(5, 0, 1, 2)$

Example 6.3.12 Determine the set of solutions of the system

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 4x_2 + x_3 = -1 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ x_1 + 2x_2 + 2x_3 = 2. \end{cases}$$

Solution. The augmented matrix of the system is

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & -1 \\ -1 & -2 & 3 & 1 \\ 1 & 2 & 2 & 2 \end{array}\right] \stackrel{\sim}{\to} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 1 \end{array}\right] \stackrel{\sim}{\to} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right].$$

$$\stackrel{\sim}{\to} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 4 \end{array}\right] \stackrel{\sim}{\to} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \stackrel{\sim}{\to} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The last matrix in the above sequence is row-reduced echelon. The existence of the rox $(0\ 0\ 0\ 1)$ shows that the system is inconsistent. It has no solution, $S = \phi$.

Exercises

1. Find a row-reduced echelon matrix which is row-equivalent to

$$a) \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad b) \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad c) \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad e) \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix}.$$

2. Show that being row-equivalent, $\stackrel{\sim}{\to}$ is an equivalence relation in the set of all $p \times q$ matrices.

3. How many row-reduced echelon 2×2 matrices can you write? How many row-reduced echelon 2×2 matrices can you write such that each entry is either 1 or 0 ?

4. In each case below, find a row-reduced echelon matrix which is row-equivalent to the given matrix.

$$a) \begin{bmatrix} 1 & 2 & 3 & -3 \\ 2 & 4 & 5 & -3 \\ 3 & 6 & 7 & -4 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 2 & -1 \end{bmatrix} \qquad b) \begin{bmatrix} 2 & 1 & 3 & -1 \\ 1 & 0 & -1 & 2 \\ -6 & -2 & -8 & 2 \\ 0 & 1 & 3 & -3 \\ 3 & 2 & 3 & 0 \end{bmatrix}.$$

$$c) \begin{bmatrix} -1 & 1 & 2 & 3 \\ 2 & -2 & -4 & -6 \\ -6 & 6 & 12 & 18 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -2 & -3 \end{bmatrix} \qquad d) \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 5 & -1 \\ 3 & 0 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 3 & 3 & 0 \end{bmatrix}.$$

5. In each case below, determine the set of solutions of the given system of linear equations by putting the augmented matrix into row-reduced echelon form.

$$a) \begin{cases} x_1 + 2x_2 + 3x_3 = -3 \\ 2x_1 + 4x_2 + 5x_3 = -3 \\ 3x_1 + 3x_2 + 6x_3 = -4 \\ x_1 + 2x_2 + x_3 = 0 \\ x_1 + 2x_2 + 2x_3 = -1 \end{cases} \qquad b) \begin{cases} 2x_1 + x_2 + 3x_3 = -1 \\ x_1 + x_3 = 2 \\ -6x_1 - 2x_2 - 8x_3 = 2 \\ x_2 + 3x_3 = -3 \\ 3x_1 + 2x_2 + 3x_3 = 0 \end{cases}$$

$$c) \begin{cases} -x_1 + x_2 + 2x_3 = 3 \\ 2x_1 + 2x_2 + 4x_3 = -6 \\ -6x_1 + 6x_2 + 12x_3 = 18 \\ 0x_1 + 0x_2 + 0x_3 = 0 \\ x_1 + x_2 + 2x_3 = -3 \end{cases} \qquad d) \begin{cases} x_1 + x_2 + 2x_3 = 1 \\ x_1 + 2x_2 + 5x_3 = -1 \\ 3x_1 + 2x_3 = 1 \\ 2x_2 + 3x_3 = 1. \end{cases}$$

6. Determine the set of solutions of the following systems of linear equations.
a) $x_1 + 2x_2 + 3x_3 = 4$, b) $0x_1 + 0x_2 + 0x_3 = 0$, c) $0x_1 + 0x_2 + 0x_3 = 1$

$$d) \begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ x_2 + x_3 = 1 \end{cases} \qquad e) \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

$$f) \begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 0x_1 + 0x_2 + 0x_3 = 1 \end{cases} \qquad g) \begin{cases} x + 2y + 3z = 4 \\ 2x - y + 4z = 1 \end{cases}$$

$$h) \begin{cases} x + 2y + 3z = 4 \\ 2x - y + 4z = 1 \\ 3x - 2y + z = -3 \end{cases} \qquad i) \begin{cases} x + 2y + 3z = 3 \\ 2x - y + 4z = 2 \\ 3x + 2y + z = 1. \end{cases}$$

7. In each case below, determine the constants c_1, c_2, c_3 so that the system has a solution

$$a)\ \begin{cases} x+y=c_1 \\ 2x+3y=c_2 \end{cases} \qquad b)\ \begin{cases} x+y=c_1 \\ 2x+2y=c_2 \end{cases}$$

$$c)\ \begin{cases} x+y=c_1 \\ x-y=c_2 \end{cases} \qquad d)\ \begin{cases} x+y+z=c_1 \\ 2x+y+z=c_2 \end{cases}$$

$$e)\ \begin{cases} x+y+z=c_1 \\ x+2y+2z=c_2 \\ 2x+2y+z=c_3 \end{cases} \quad f)\ \begin{cases} c_1x+y+z=0 \\ x+c_2y+z=0 \\ x+y+c_3z=c_1 \end{cases}$$

8. Determine the set of solutions of

$$\begin{cases} x_1+2x_2-x_3+x_4-x_5 &= 4 \\ x_1-x_2+x_3+2x_4+3x_5 &= 3 \\ 2x_1+x_2+2x_3+x_4-x_5 &= 2 \\ x_1-x_2+x_3+2x_5 &= 1 \\ 3x_1+2x_3-x_4 &= 0 \end{cases}$$

9. Determine whether a solution exists and what the solution is, if it exists interpret geometrically

$$\begin{array}{llllll} (a) & 2x+y=7 & (b) & x+y=3 & (c) & 4x-3y=8 \\ & x-3y=2 & & 3x+2y=8 & & 8x-6y=24 \\ (d) & x-y=8 & (e) & x+2y=3 & (f) & 2x-6y=7 \\ & 2x+8y=6 & & 2x+4y=6 & & 3x-9y=4 \end{array}$$

10. Determine a necessary and sufficient condition on a, b, and c so that the following system of equations admit solutions

$$\begin{array}{llll} (a) & 2x+3y-2z=a & (b) & x+4y-2z=a \\ & x-2y+z=b & & 2x-2y+3z=b \\ & x-9y+5z=c & & x-6y+5z=c \end{array}$$

6.4 Intersection of Three Planes

Let

$$\begin{aligned} \mathcal{P}_1 &: A_1x+B_1y+C_1z+D_1=0 \\ \mathcal{P}_2 &: A_2x+B_2y+C_2z+D_2=0 \\ \mathcal{P}_3 &: A_3x+B_3y+C_3z+D_3=0 \end{aligned}$$

be three planes in 3-space, and let us try to determine $\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_3$. The intersection is nothing but the set of solutions of the system of linear equations

$$\begin{cases} A_1x+B_1y+C_1z=-D_1 \\ A_2x+B_2y+C_2z=-D_2 \\ A_3x+B_3y+C_3z=-D_3 \end{cases}$$

The coefficient matrix

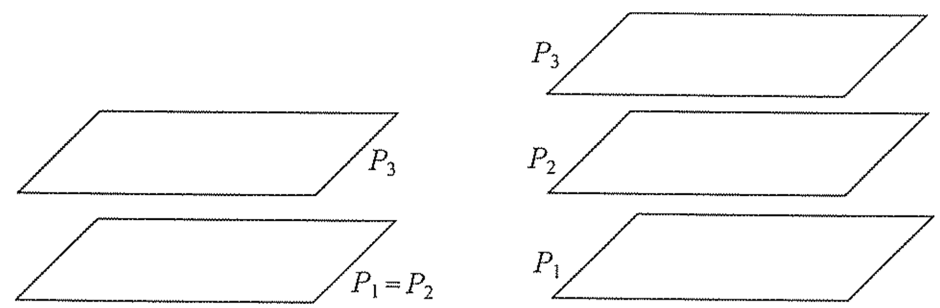
$$A=\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}.$$

is a 3×3 matrix. Let M be the 3×4 row-reduced echelon matrix which is row-equivalent to the augmented matrix

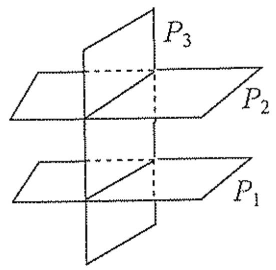
$$\tilde{A}=\left[\begin{array}{ccc|c} A_1 & B_1 & C_1 & -D_1 \\ A_2 & B_2 & C_2 & -D_2 \\ A_3 & B_3 & C_3 & -D_3 \end{array}\right].$$

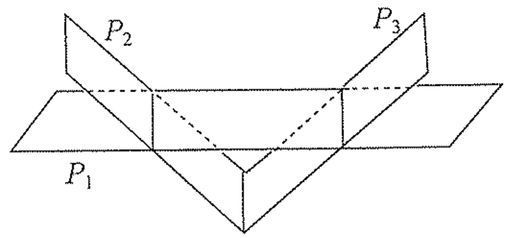
There are four cases (recall Theorem 6.3.5):

Case 1. M contains a row of the form $(0\ \ 0\ \ 0 \mid 1)$. Then the system has no solution, i.e., $\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_3 = \phi$. The possibilities in that case are shown in the following figures



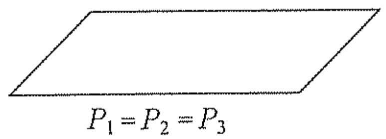
($\mathcal{P}_1 = \mathcal{P}_2$ and $\mathcal{P}_1$ is parallel to $\mathcal{P}_3$) ($\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, are parallel to each other)





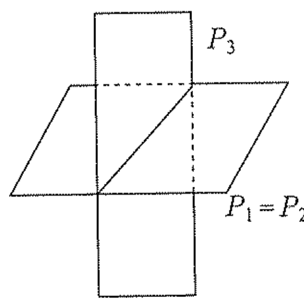
($\mathcal{P}_1$ is parallel to $\mathcal{P}_2$ and $\mathcal{P}_3$ intersect them) (the planes mutually intersect but $\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_3 = \phi$)

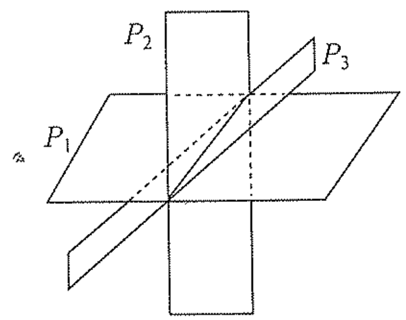
Case 2. M contains no row of the form $(0 \ \ 0 \ \ 0 \mid 1)$ and contains only one non-zero row. Then the system has solutions. A general solution can be given by using $3 - 1 = 2$ parameters. In this case all the planes coincide.



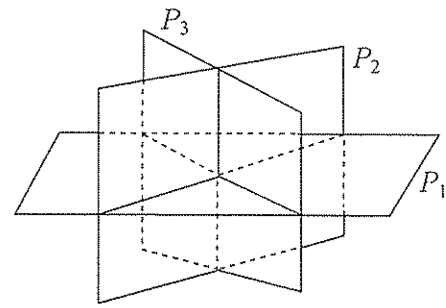
$$\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3$$

Case 3. M contains no row of the form $(0 \ \ 0 \ \ 0 \mid 1)$ and contains two non-zero rows. Then the system has solutions. A general solution can be given by using $3 - 2 = 1$ parameter. In that case $\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_3$is a line, the possibilities are shown in the following figures





Case 4. M contains no row of the form $(0 \ \ 0 \ \ 0 \mid 1)$ and contains three non-zero rows. Then the system has a unique solution, because the general solution involves no parameter. Thus the three planes intersect at a point.



Remark. In cases 1 or 3, the exact positions of the planes can be determined by considering $\mathcal{P}_1 \cap \mathcal{P}_2, \mathcal{P}_1 \cap \mathcal{P}_3$ and $\mathcal{P}_2 \cap \mathcal{P}_3$ separately.

Exercises

1. In each case below, determine the intersection of the given planes.

$$a) \begin{cases} x + 2y + 3z = 3 \\ -x + y - z = 4 \\ x - y + z = 5 \end{cases} \quad b) \begin{cases} x - 2y + z = 0 \\ -3x + 6y + z = 1 \\ 2x - 4y - z = 2 \end{cases} \quad c) \begin{cases} 4x + y - z = 1 \\ -2x + y - z = -1 \\ 3x - y + z = 0 \end{cases}$$

$$d) \begin{cases} x + 2y + 3z = 0 \\ 2x + 3y + 4z = 1 \\ 3x + 4y + 5z = 2 \\ 4x + 5y + 6z = 3 \end{cases} \quad e) \begin{cases} x + 2y + 3z = 0 \\ 2x + 3y + 4z = 0 \\ x + y + z = 0. \end{cases}$$

6.5 Homogeneous Systems of Linear Equations

If the scalars $c_1, c_2, ..., c_p$ in the system (6.3.1) are all zero, then the system is called an *homogeneous system* of linear equation. Thus an homogeneous system of p equations in q unknowns looks like

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1q}x_q = 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2q}x_q = 0 \\ \quad \cdots \\ a_{p1}x_1 + a_{p2}x_2 + ... + a_{pq}x_q = 0 \end{cases} \tag{6.5.1}$$

The corresponding matrix equation is $AX = 0$ where $A = [a_{ij}]$ is the coefficient matrix and 0 denotes the $p \times 1$ zero matrix.

The zero vector $0 = (0, 0, ..., 0)$ in q-space is always a solution of the homogeneous system (6.5.1). It is called the *trivial solution* of the system. Every other solution (if any) is called a non-trivial solution.

Remark. The entries in the last column of the augmented matrix of an homogeneous system are all zero. Therefore, an homogeneous system of linear equations is completely determined by its coefficient matrix. To determine the set of solutions of a homogeneous system, we find a row-reduced echelon matrix which is row-equivalent to the coefficient matrix of the given system. Then the given system and the (homogeneous) system corresponding to the obtained row-reduced echelon matrix are equivalent. We have

Theorem 6.5.1 *Let the coefficient matrix A of an homogeneous system of linear equations in q unknowns be row-equivalent to a row-reduced echelon matrix which has n non-zero rows. Then*

a) If $n \geq q$, the system has only the trivial solution,

b) If $n < q$, the system has non-trivial solutions and a general solution can be given by using $(q - n)$ parameters. In particular, if the number of equations in the system is less than the number of unknowns then the system has non-trivial solutions.

Example 6.5.2 Determine the set of solutions of the homogeneous system

$$\begin{cases} x + 2y + 2z &= 0 \\ -x - 3y + 4z &= 0 \\ x + y + 8z &= 0 \\ 2x + 3y + 10z &= 0. \end{cases}$$

Solution. The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -3 & 4 \\ 1 & 1 & 8 \\ 2 & 3 & 10 \end{bmatrix} \stackrel{\sim}{\rightarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 6 \\ 0 & -1 & 6 \\ 0 & -1 & 6 \end{bmatrix} \stackrel{\sim}{\rightarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \stackrel{\sim}{\rightarrow} \begin{bmatrix} 1 & 0 & 14 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are non-trivial solutions depending on the homogeneous system of linear equations corresponding to the row-reduced echelon matrix is

$$x + 14z = 0$$
$$y - 6z = 0.$$

Thus a general solution of the system is $\xi = (-14t, 6t, t)$. The set of solutions is $S = \{(-14t, 6t, t) : t \in \mathbb{R}\}$.

Remark. Suppose that the number of equations and the number of unknowns are the same, i.e., $p = q$, in an homogeneous system. Then the coefficient matrix of the system is a square matrix. Using Theorem 6.5.1, we can easily observe the following

Corollary 6.5.3 *Let A be a square matrix. Then $AX = 0$ has only the trivial solution $\Leftrightarrow A \stackrel{\sim}{\rightarrow} I$, the identity matrix.*

Proof. A square matrix which is row-reduced and echelon has no zero row if and only if it is the identity matrix. □

Corollary 6.5.4 *Let A be a square matrix. Then $A \stackrel{\sim}{\rightarrow} I$ if and only if $AX = C$ has a unique solution for any C.*

Proof. $A \stackrel{\sim}{\rightarrow} I \Leftrightarrow [A|C] \stackrel{\sim}{\rightarrow} [I|D]$ for some D. □

In testing linear dependence of vectors in p-space, we may use Theorem 6.5.1. For this purpose, we consider vectors in p-space as column vectors. Let

$$u_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{p1} \end{bmatrix}, \quad u_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{p2} \end{bmatrix}, ..., u_q = \begin{bmatrix} a_{1q} \\ a_{2q} \\ \cdot \\ \cdot \\ \cdot \\ a_{pq} \end{bmatrix}.$$

be q vectors in p-space. They are linearly dependent $\Leftrightarrow$ there exist scalars $\xi_1, \xi_2, ..., \xi_q$, not all zero, such that

$$\xi_1 u_1 + \xi_2 u_2 + ... + \xi_q u_q = 0.$$

This is the case if and only if the homogeneous system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1q}x_q = 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2q}x_q = 0 \\ \cdots\cdots \\ a_{p1}x_1 + a_{p2}x_2 + ... + a_{pq}x_q = 0 \end{cases}$$

has a non-trivial solution. Hence we have the

Corollary 6.5.5 *Let $u_1, ..., u_q$ be q vectors in p-space and let A be the $p \times q$ matrix whose columns are $u_1, u_2, ..., u_q$. Then $u_1, u_2, ..., u_q$ are linearly dependent $\Leftrightarrow$ A is row-equivalent to a row reduced echelon matrix which contains at most $q-1$ non-zero rows. In particular, if $p < q$ then $u_1, u_2, ..., u_q$ are linearly dependent.*

Example 6.5.6 Determine whether or not the following vectors are linearly dependent.

$$v_1 = (1, -1, 0, 2), \quad v_2 = (2, -3, 5, 4), \quad v_3 = (2, 4, 10, 4).$$

Solution. $p = 4$ and $q = 3$.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -3 & 4 \\ 0 & 5 & 10 \\ 2 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 6 \\ 0 & 5 & 10 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are $3 > q - 1$ non-zero rows in the row-reduced echelon matrix. Thus, v_1, v_2, v_3 are linearly independent.

Exercises

1. The system (6.5.1) is also called the *homogeneous system associated with the non-homogeneous system* (6.3.1). In matrix notation, $AX = 0$ is called the *homogeneous equation associated with* $AX = C$. For each of the systems of linear equations in exercise 5 of the previous section, discuss the set of solutions of the homogeneous system associated with that system.

2. Find the set of solutions of the following systems of linear equations

a) $\begin{cases} x + 2y = 0 \\ 2x + y = 0 \\ x - y = 0 \end{cases}$ b) $\begin{cases} 2x - y + z = 0 \\ x - y + z = 0 \\ y + z = 0 \end{cases}$

c) $\begin{cases} x + 3y - z + w = 0 \\ y + w = 0 \\ -x - 2y + 3z = 0 \end{cases}$ d) $\begin{cases} x + 3y - z + w = 0 \\ y + w = 0 \\ -x - 2y + 3z = 0 \\ 3x + y - 4w = 0 \end{cases}$

3. Prove that

   a) if ξ and ξ' are two solutions of the nonhomogeneous system (6.3.1) then $\zeta = \xi - \xi'$ is a solution of the associated homogeneous system (6.5.1).

   b) if ξ is a solution of (6.3.1) and ζ is a solution of (6.5.1) then $\xi' = \xi + \zeta$ is a solution of (6.3.1).

4. In each case below, test the given vectors for linear dependence.

$$\begin{array}{llll} a) & v_1 = (1,2,3,4,5) & b) & v_1 = (1,-1,2,3) \\ & v_2 = (2,3,4,5,6) & & v_2 = (2,1,2,1) \\ & v_3 = (3,4,5,6,7) & & v_3 = (0,1,1,0) \\ & v_4 = (4,5,6,7,8) & & v_4 = (1,0,1,1). \end{array}$$

5. Answer Exercise 5.2.5, in view of the Corollary 6.5.5 of Theorem 6.5.1.

6.6 Elementary Matrices

In this section, we show that elementary row operations on a $p \times q$ matrix A can be realized as multiplications of A from the left by certain $p \times p$ matrices, called elementary matrices. For example, the elementary row operation (Er_1), multiplication of a row of A by a non-zero scalar c, can be realized by multiplying A from the left by the matrix which is obtained from the identity matrix by multiplying the corresponding row of the identity matrix by c:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{(Er_1)} \begin{bmatrix} a_{11} & a_{12} \\ ca_{21} & ca_{22} \end{bmatrix} = A'$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} = E$$

$$EA = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ ca_{21} & ca_{22} \end{bmatrix} = A'.$$

If a $p \times p$ matrix E is obtained from the $p \times p$ identity matrix by one single elementary row operation, then E is called an *elementary $p \times p$ matrix.*

Example 6.6.1 The following are examples of elementary 3×3 matrices:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The main theorem of this section is

Theorem 6.6.2 *Any elementary row operation on a matrix A can be realized by multiplying A from the left by an elementary matrix, and conversely multiplying A from the left by an elementary matrix induces an elementary row operation on A.*

Proof. We give the proof for the elementary row operation (r_2). Let A be a $p \times p$ matrix and let A' be the matrix obtained from A by interchanging the i-th row and the k-th row. If we denote the rows of A by $\alpha_1, \alpha_2, ..., \alpha_p$ and those of A' by $\alpha'_1, \alpha'_2, ..., \alpha'_p$, then

$$\alpha'_i = \alpha_k, \quad \alpha'_k = \alpha_i, \quad \text{and} \quad \alpha'_h = \alpha_h, \quad \text{or} \quad h \neq i, k.$$

Let E be the elementary $p \times p$ matrix obtained from the identitiy matrix by interchanging the i-th row and the k-th column. The i-th row of the product EA is

$$(e_k \circ A_1,\ e_k \circ A_2, ..., e_k \circ A_q) = (a_{k1}, a_{k2}, ..., a_{kq}) = \alpha'_i,$$

its k-th row is

$$(e_i \circ A_1,\ e_i \circ A_2, ..., e_i \circ A_q) = (a_{i1}, a_{i2}, ..., a_{iq}) = \alpha'_k$$

and the h-th row (for $h \neq i, k$) is

$$(e_h \circ A_1,\ e_h \circ A_2, ..., e_h \circ A_q) = (a_{h1}, a_{h2}, ..., a_{hq}) = \alpha'_h.$$

Thus $EA = A'$, and this completes the proof for (Er_2). □

We have the following obvious corollaries to Theorem 6.6.2:

Corollary 6.6.3 *Let A and B be two $p \times q$ matrices. Then $A \stackrel{r}{\to} B$ if and only if there exists a $p \times p$ matrix M such that $B = MA$ and M product of finitely many elementary matrices.*

Corollary 6.6.4 *For any elementary matrix E we can find an elementary matrix E^* such that $EE^* = E^*E = I$.*

Corollary 6.6.5 *Let A be a square matrix. Then, $A \stackrel{r}{\to} I$ if and only if A is a product of finitely many elementary matrices.*

Exercises

1. Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$
   a) Show that $A \stackrel{r}{\to} I$.
   b) Express A as a product of elementary matrices.
2. Display all the possible elementary 2×2 matrices.
3. Display all the possible elementary 3×3 matrices.
4. Describe E^* of Corollary 6.6.4 of Theorem 6.6.2 for each of the matrices in Example 6.6.1.
5. Express each of the following matrices as a product of elementary matrices:
$$a) \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad b) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad c) \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

6.7 Invertible Matrices

Let A be a $p \times p$ matrix. If there exists a $p \times p$ matrix B such that

$$AB = BA = I$$

then A is said to be *invertible* (or *non singular*).

Example 6.7.1 For any p, the $p \times p$ identity matrix I is invertible. By Corollary 6.6.4 of Theorem 6.6.2, *every elementary matrix is invertible.*

Theorem 6.7.2 *If A is an invertible $p \times p$ matrix, then there is one and only one matrix B such that $AB = BA = I$.*

Proof. By definition, there exists B such that $AB = BA = I$. Suppose there is another matrix B' such that $AB' = B'A = I$. Then multiplying the last identity from the left by B, we obtain

$$B(AB') = BI \quad (BA)B' = B \quad B' = B.$$

This completes the proof. $\square$

If A is invertible, the unique matrix B for which $AB = BA = I$ is called the *inverse* of A. We write $B = A^{-1}$. Thus $AA^{-1} = A^{-1}A = I$.

Theorem 6.7.3 *If A and B are invertible $p \times p$ matrices, so is AB and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. This can be observed by direct computation:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

$\square$

Remark. The converse of this theorem is also true. See Corollary 6.7.8 of Theorem 6.7.6.

Corollary 6.7.4 *Let A and B be two $p \times p$ matrices. If $A \overset{\sim}{\to} B$, then $B = MA$ where M is an invertible matrix.*

Proof. If $A \overset{\sim}{\to} B$, then $B = MA$ where M is a product of elementary matrices (Corollary 6.6.5 of Theorem 6.6.2). M is invertible since every elementary matrix is invertible. $\square$

Corollary 6.7.5 *Let A and B be row-equivalent matrices. Then A is invertible $\Leftrightarrow$ B is invertible.*

Proof. Let $A \overset{\sim}{\to} B$ and assume that A is invertible. Then $B = MA$ for some invertible matrix M. Then $B = MA$ is invertible by Theorem 6.7.3. $\square$

Theorem 6.7.6 *For a $p \times p$ matrix A, the following are equivalent:*

a) A is invertible

b) $AX = C$ has a unique solution for every $p \times 1$ matric C

c) $AX = 0$ has only the trivial solution

d) $A \overset{\sim}{\to} I$

e) A is a product of elementary matrices.

Proof. Assume that A is invertible. Then $X = A^{-1}C$ is the unique solution of $AX = C$. Hence a) implies b).The statement c) is a special case of b). Therefore b) implies c). By Corollary 6.5.3 of Theorem 6.5.1, c) implies d). The statement d) implies e) by Corollary 6.6.5 of Theorem 6.6.2. Finally e) implies a) by Theorem 6.7.3. $\square$

Corollary 6.7.7 *Let A be a $p \times p$ matrix. If there is a $p \times p$ matrix N such that $NA = I$, then A is invertible and $N = A^{-1}$.*

Proof. If $NA = I$, then $AX = C$ has the unique solution $X = NC$ for any C. Hence A is invertible and $N = N(AA^{-1}) = (NA)A^{-1} = A^{-1}$. $\square$

Corollary 6.7.8 *Let A and B be $p \times p$ matrices. If AB is invertible, then both A and B are invertible.*

Proof. If AB is invertible, then $((AB)^{-1}A)B = I$. Hence by Corollary 6.7.7, B is invertible and $B^{-1} = (AB)^{-1}A$. Similarly, one can see that $A^{-1} = B(AB)^{-1}$. $\square$

Remark. The preceding theorem can be used to compute the inverse A^{-1} of an invertible matrix A. In fact, if A is invertible, then by Theorem 6.7.6 (d), $A \overset{\sim}{\to} I$ and by Corollary 6.6.3 of Theorem 6.6.2.

$$I = E_1E_2...E_mA \tag{6.7.1}$$

where $E_1, E_2, ..., E_m$ are elementary matrices. Hence taking inverse,

$$I = A^{-1}E_m^{-1}...E_2^{-1}E_1^{-1}.$$

Multiplying the last identity from the right by $E_1E_2...E_m$ we obtain

$$E_1E_2...E_m = A^{-1}. \tag{6.7.2}$$

The identities (6.7.1) and (6.7.2) can be interpreted as follows: *If I is obtained from A by a sequence of elementary row operations, then A^{-1} is obtained from I by applying the same sequence of elementary row operations.*

Thus we obtain a practical way of computing the inverse of a matrix. We write the matrix A and the identity matrix I side by side, and view this as a $p \times p$ matrix. We apply elementary row operations, so that the part corresponding to A reduces to I. Then the part corresponding to I reduces to A^{-1}.

Example 6.7.9 Find A^{-1} if

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}.$$

Solution.

$$[A/I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{\sim}{\to} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 2 & -1 & -2 & 0 & 1 \end{array}\right].$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -4 & -2 & 1 \end{array}\right] \overset{\sim}{\to} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & -3 & -1 & 1 \\ 0 & 0 & 1 & -4 & -2 & 1 \end{array}\right] = [I|A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ -4 & -2 & 1 \end{bmatrix}.$$

Example 6.7.10 Consider the system of linear equations

$$\begin{cases} x_1 - x_2 + x_3 = 2 \\ -x_1 + 2x_2 - 2x_3 = 1 \\ 2x_1 + x_3 = 3. \end{cases}$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}.$$

We have seen in Example 6.7.9 that A is invertible and

$$A^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ -4 & -2 & 1 \end{bmatrix}.$$

Considering the system as a matrix equation $AX = C$, we see that it has unique solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}C = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}.$$

Exercises

1. Show that if A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

2. In each case below, use the method of example 6.7.9 to find the inverse of the given matrix.

$$a)\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad b)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad c)\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

3. In each case below, test whether or not the given matrix is invertible; if it is invertible, find the inverse by elementary row operations.

$$a)\begin{bmatrix} 2 & -2 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad b)\begin{bmatrix} 2 & -1 & 1 \\ 1 & 4 & -1 \\ 3 & -3 & 2 \end{bmatrix} \quad c)\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$d)\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & -2 & 2 & 0 \\ 3 & 1 & 0 & -4 \end{bmatrix} \quad e)\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & -2 \\ -1 & -1 & 0 & 5 \\ -1 & 0 & 1 & -2 \end{bmatrix}.$$

4. Determine the set of solutions of the following systems of linear equations

$$a)\ \begin{array}{l} 2x_1 - 2x_2 + 5x_3 = 1 \\ x_1 + x_3 = -1 \\ 2x_1 + x_2 + 3x_3 = 3 \end{array} \qquad b)\ \begin{array}{l} 2x - y + z = 1 \\ x + 4y - z = -1 \\ 3x - 3y + 2z = 0 \end{array}$$

$$c)\ \begin{array}{l} x + 2y + 3z = 0 \\ -2x + y + z = 0 \\ x + z = 0 \end{array} \qquad d)\ \begin{array}{l} x + 3y - z + w = 1 \\ y + w = -1 \\ -x - 2y + 2z = 1 \\ 3x + y + -w = -1 \end{array}$$

$$e)\ \begin{array}{l} x - z = 10 \\ 2x + y - z - 2w = 5 \\ -x - y + 5w = 10 \\ -x + z - 2w = 5 \end{array}$$

5. Show that if A is invertible, so is A^m, and $(A^m)^{-1} = (A^{-1})^m$.

6. If A and B are square matrices and A is invertible, show that

$$(A+B)A^{-1}(A-B) = (A-B)A^{-1}(A+B).$$

7. If A, B are invertible square matrices, show that the following are equivalent
   (a) A commutes with B
   (b) A commutes with B^{-1}
   (c) A^{-1} commutes with B^{-1}.

6.8 Matrices with Complex Entries

In the preceding sections of this chapter we have considered matrices of the form $A = [a_{ij}]$ where the entries a_{ij} are all real numbers. If we let a_{ij} be complex numbers, then $A = [a_{ij}]$ is called a *matrix with complex entries.* Since the set of real numbers is a subset of the set of complex numbers, the set of matrices with real entries is a subset of the set of matrices with complex entries.

All the definitions that are given for matrices with real entries have obvious analogs for matrices with complex entries. Up to now, we have used the word "scalar" for a real number. *From now on, by a "scalar" we shall mean a real number if we are studying in the context of real numbers, but if we are studying in the context of complex numbers, then a "scalar" will mean a complex number.*

Addition, multiplication by a scalar (complex number), and multiplication of matrices with complex entries are defined exactly in the same way as they are defined for matrices with real entries. All the properties that are stated about addition, multiplication by a scalar and multiplication of matrices with real entries hold true for matrices with complex entries.

Example 6.8.1 Let

$$A = \begin{bmatrix} i-1 & 2i & 3 \\ -1 & i & 0 \\ 3 & i+2 & -i \end{bmatrix} B = \begin{bmatrix} i+1 & -i \\ 1 & 2i \\ i & -3 \end{bmatrix} C = \begin{bmatrix} 1-i & 2i \\ i & 0 \\ 1 & 2i+2 \end{bmatrix}.$$

Then

$$B+C = \begin{bmatrix} 2 & i \\ 1+i & 2i \\ 1+i & 2i-1 \end{bmatrix} \quad iA = \begin{bmatrix} -1-i & -2 & 3i \\ -i & -1 & 0 \\ 3i & -1+2i & 1 \end{bmatrix}.$$

$$\begin{aligned} AB &= \begin{bmatrix} (i-1)(i+1)+2i+3i & (i-1)(-i)+(2i)(2i)-9 \\ -i-1+i & i+i(2i) \\ 3(1+i)+i+2+(-i)(i) & -3i+(i+2)(2i)+3i \end{bmatrix} \\ &= \begin{bmatrix} 5i-2 & i-12 \\ -1 & i-2 \\ 4i+6 & 4i-2 \end{bmatrix}. \end{aligned}$$

The three elementary row operations Er_1, Er_2 and Er_3 apply to matrices with complex entries, too. Row-equivalence of matrices with complex entries and row-reduced echelon matrices with complex entries are defined in exactly the same way as for matrices with real entries. Every matrix with complex entries is row-equivalent to a row-reduced echelon matrix.

In short, we can say that every property we have obtained about matrices in the preceding sections is true also for matrices with complex entries. Furthermore, we can consider systems of linear equations with complex coefficients and investigate their solutions (in the complex q-space if there are q unknowns) by using the techniques developped in section 3.

Example 6.8.2 Determine the set of solutions (in $\mathbb{C}^3$) of the following system of linear equations

$$\begin{cases} -ix_1 + (2-i)x_2 + (2+i)x_3 = 1-i \\ x_1 + (1+i)x_2 + 2ix_3 = 2 \\ (1-i)x_2 + ix_3 = 2+i. \end{cases}$$

Solution. The augmented matrix of the system is

$$\tilde{A} = \left[\begin{array}{ccc|c} -i & 2-i & 2+i & 1-i \\ 1 & 1+i & 2i & 2 \\ 0 & 1-i & i & 2+i \end{array}\right] \stackrel{\sim}{\to} \left[\begin{array}{ccc|c} 0 & 1 & i & 1+i \\ 1 & 1+i & 2i & 2 \\ 0 & 1-i & i & 2+i \end{array}\right]$$

$$\stackrel{\sim}{\to} \left[\begin{array}{ccc|c} 0 & 1 & i & 1+i \\ 1 & 0 & 1+i & 2+2i \\ 0 & 0 & -1 & i \end{array}\right] \stackrel{\sim}{\to} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1-i \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 & -i \end{array}\right].$$

Hence there is a unique solution $x_1 = 1-i, x_2 = i, x_3 = -i$.

If $A = [a_{ij}]$ is a $p \times q$ matrix with complex entries, then we let

$$\overline{A} = [\overline{a}_{ij}] = \begin{bmatrix} \overline{a}_{11}\ \overline{a}_{12}...\overline{a}_{1q} \\ \overline{a}_{12}\ \overline{a}_{22}...\overline{a}_{2q} \\ \cdots \\ \overline{a}_{p1}\ \overline{a}_{p2}...\overline{a}_{pq} \end{bmatrix}$$

where $\overline{a}_{ij}$ denotes the complex conjugate of a_{ij}. Thus A is a matrix with real entries if and only if $\overline{A} = A$.

Exercises

Given the matrices

$$A = \begin{bmatrix} i & 2i+5 & 4i-1 \\ i+3 & 3i-1 & 2i \\ 1 & 0 & 1-i \end{bmatrix}, \quad B = \begin{bmatrix} -i & 1+i & 0 \\ 3i & 4i-2 & 1+i \\ 2i-2 & 0 & 4i \end{bmatrix}, \quad C = \begin{bmatrix} i \\ 1+i \\ 2-i \end{bmatrix}.$$

1. Find the following

   a) $A+B$ b) $2A+3B$ c) $iA-iB$ d) AB

   e) BA f) $A+\overline{A}$ g) $A-\overline{A}$ h) AC.

2. Express A as $A = A' + iA''$ where A' and A'' are matrices with real entries; and show that $\overline{A} = A' - iA''$.
3. Find a row-reduced echelon matrix which is row-equivalent to A.
4. Same equation for the matrix B.
5. Determine whether or not the matrix A is invertible. If it is invertible, find A^{-1}.
6. Find B^{-1} if it exists.
7. Determine the set of solutions of $AX = C$.
8. Determine the set of solutions of $BX = C$.
9. Find ${}^t(\overline{A})$ and $({}^tA)$.

Chapter 7

DETERMINANTS

In this chapter, we give the definition and fundamental properties of determinants. The discussions are given for determinants of arbitrary (real or complex) matrices. For convenience, examples are given in the context of real numbers. By a "scalar" we still understand a complex number if we study in the context of complex numbers.

7.1 Determinant of a Matrix

For every square matrix A, we are going to assign a number, called the *determinant* of A. The definition of the determinant will be given inductively.

The determinant of a 1×1 matrix $[a]$ is defined to be the number a. In practice, we are concerned with determinants of $p \times p$ matrices where $p \geq 2$. The determinant of a $p \times p$ matrix $A = [a_{ij}]$ is denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdot & \cdot & \cdots & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{vmatrix}$$

For $p = 2$, the determinant of (a 2×2 matrix)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is defined by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

For $p > 2$, we introduce the following terminology. Given a $p \times p$ matrix $A = [a_{ij}]$, the $(p-1) \times (p-1)$ matrix A_{ij} obtained from A by removing the i-th row and the j-th column is called the *minor* of the entry a_{ij}. For instance, if $p = 3$,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \quad A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \quad A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

are the *minors* of a_{11}, a_{12}, a_{13}, respectively.

Now, for a $p \times p$ matrix $A = [a_{ij}]$, $p > 2$, we define

$$|A| = \sum_{j=1}^{p} (-1)^{1+j} a_{1j}|A_{1j}| = a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{p+1} a_{1p}|A_{1p}| \qquad (7.1.1)$$

For each $i, j = 1, \ldots, p$, the expression

$$(-1)^{i+j}|A_{ij}|$$

is called the *cofactor* of a_{ij}. Thus the cofactor of a_{ij} is $(-1)^{i+j}$ times the determinant of the minor of a_{ij}.

Example 7.1.1 Let A be a 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

$$\begin{aligned} |A| &= a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22}). \end{aligned}$$

In particular

$$\begin{aligned} \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & 0 \\ 3 & -1 & 4 \end{vmatrix} &= 1\begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix} - (-2)\begin{vmatrix} -1 & 0 \\ 3 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} \\ &= 4 - 8 - 6 = -10. \end{aligned}$$

The rows (or columns) of a square matrix A are also called rows (or columns) of its determinant $|A|$. The determinant of a $p \times p$ matrix is also called a $p \times p$ determinant.

Theorem 7.1.2 *Let $A = [a_{ij}]$ be a $p \times p$ matrix. Then for any $i = 1, \ldots, p$, we have*

$$\sum_{j=1}^{p} (-1)^{i+j} a_{ij}|A_{ij}| = |A|. \qquad (7.1.2)$$

Proof. See Appendix B. □

The expression (7.1.2) is called the *expansion of $|A|$ with respect to the i-th row.* Thus, Theorem 7.1.2, states that expansion of $|A|$ with respect to any row is the same.

The following example may illustrate the use of (7.1.2).

Example 7.1.3 Compute $|A|$ if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 4 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix}.$$

Solution. Expand $|A|$ with respect to the third row:

$$|A| = (-1)^{3+1}1 \cdot \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 4 \\ 0 & 2 & 0 \end{vmatrix} + (-1)^{3+2} \cdot 0 + (-1)^{3+3} \cdot 0 + (-1)^{3+4} \cdot 0$$

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 4 \\ 0 & 2 & 0 \end{vmatrix} \quad \text{(expand w.r. to the 3-rd row)} \\ &= (-1)^{3+2} \cdot 2\begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = -2(8-4) = -8. \end{aligned}$$

Determinants have the following four *fundamental properties*:

Theorem 7.1.4 *Let $A = [a_{ij}]$ be a $p \times p$ matrix with rows $\alpha_1, \alpha_2, \ldots, \alpha_p$, and let I denote the $p \times p$ identity matrix. Then*

(D_1) $|I| = 1$.

(D_2) Suppose that for a fixed $i = 1, \ldots, p$, the i-th row of A is $\alpha_i = \beta_i + \gamma_i$ where β_i and γ_i are vectors in the p-space. Let B be the matrix obtained from A by replacing the i-th row of A with β_i, and let C be the matrix obtained from A by replacing the i-th row of A with γ_i. Then

$$|A| = |B| + |C|.$$

(D_3) Let c be a scalar and A' be the matrix obtained from A by multiplying the i-th row of A by c for a fixed $i = 1, \ldots, p$. Then

$$|A'| = c|A|.$$

(D_4) Let A'' be the matrix obtained from A by interchanging two rows of A. Then

$$|A''| = -|A|.$$

Remark. The following may clarify the statements in (D_2), (D_3) and (D_4).

$$\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad C = \begin{bmatrix} a_{11} & a_{12} \\ c_{21} & c_{22} \end{bmatrix} \\
\Rightarrow |A| &= a_{11}(b_{22}+c_{22}) - (b_{21}+c_{21})a_{12} \\
&= (a_{11}b_{22} - b_{21}a_{12}) + (a_{11}c_{22} - c_{21}a_{12}) = |B| + |C|.
\end{aligned}$$

$$\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A' = \begin{bmatrix} a_{11} & a_{12} \\ ca_{21} & ca_{22} \end{bmatrix} \\
&\Rightarrow \quad |A'| = a_{11}(ca_{22}) - (ca_{21})a_{12} = c(a_{11}a_{22} - a_{21}a_{12}) = c|A|. \\
A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\
A'' &= \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} \\
&\Rightarrow \quad |A''| = a_{21}a_{12} - a_{11}a_{22} = -(a_{11}a_{22} - a_{21}a_{12}) = -|A|.
\end{aligned}$$

Proof of the Theorem: (D_1). (induction on p). For $p = 2$,

$$|I| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Assume that (D_1) is true for the $(p-1) \times (p-1)$ identity matrix. Then, for the $p \times p$ identity matrix, expansion with respect to the first row yields

$$|I| = 1 \cdot |I_{11}| = |I_{11}| = 1,$$

since I_{11} is the $(p-1) \times (p-1)$ identity matrix.

(D_2). Let A, B and C be as in the theorem. For the fixed i, we have $A_{ij} = B_{ij} = C_{ij}$ for any $j = 1, \ldots, p$. Expansion with respect to the i-th row yields

$$\begin{aligned}
|A| &= \sum_{j=1}^{p} (-1)^{i+j} a_{ij} |A_{ij}| \\
&= \sum_{j=1}^{p} (-1)^{i+j} (b_{ij} + c_{ij}) |A_{ij}| \\
&= \sum_{j=1}^{p} (-1)^{i+j} b_{ij} |B_{ij}| + \sum_{j=1}^{p} (-1)^{i+j} c_{ij} |C_{ij}| \\
&= |B| + |C|.
\end{aligned}$$

(D_3). Let A, c and A' be as in the theorem. Then for the fixed i, $A_{ij} = A'_{ij}$ for $j = 1, \ldots, p$. Expansion with respect to the i-th row yields

$$\begin{aligned}
|A'| &= \sum_{j=1}^{p} (-1)^{i+j} c a_{ij} |A'_{ij}| = \sum_{j=1}^{p} (-1)^{i+j} c a_{ij} |A_{ij}| \\
&= c \left(\sum_{j=1}^{p} (-1)^{i+j} a_{ij} |A_{ij}| \right) = c|A|.
\end{aligned}$$

(D_4). We prove this part by induction on p. It is easily verified if $p = 2$. For $p > 2$, assume that A'' is obtained from A by interchanging the i-th and the k-th rows of A. Then, for any h other than i and k, A''_{hj} is obtained from A_{hj} by interchanging two rows of A_{hj}. Expansion with respect to the h-th row and the induction assumption yields

$$|A''| = \sum_{j=1}^{p} (-1)^{h+j} a_{hj} |A''_{hj}|$$

$$= \sum_{j=1}^{p}(-1)^{h+j}a_{hj}(-|A_{hj}|)$$
$$= -\sum_{j=1}^{p}(-1)^{h+j}a_{hj}|A_{hj}| = -|A|.$$

This completes the proof of the theorem.

□

Remark. The $p \times p$ determinant can be considered as a function from the set of all $p \times p$ matrices with real (or complex) entries to the set of real numbers, $\mathbb{R}$ (or to the set of complex numbers, $\mathbb{C}$). One can prove that *there is one and only one* function (from the set of all $p \times p$ matrices to $\mathbb{R}$) *satisfying the four properties in Theorem 7.1.4.* The proof of this fact can be found in any book on linear algebra.

Determinants have the following additional properties:

Theorem 7.1.5 *Let $A = [a_{ij}]$ be a $p \times p$ matrix. Then*

(D_5) If any row of A is 0, the zero vector, then $|A| = 0$.

(D_6) If any two rows of A are identical, then $|A| = 0$.

(D_7) For any $i, k = 1, \ldots, p$ and $i \neq k$, we have

$$\sum_{j=1}^{p}(-1)^{k+j}a_{ij}|A_{kj}| = 0 \qquad (7.1.3)$$

(D_8) If c is a scalar and B is the matrix obtained from A by adding c times a row of A to another row of A, then $|B| = |A|$.

(D_9) $|{}^tA| = |A|$, i.e., the value of a determinant is unchanged if its rows and columns are interchanged.

Proof. (D_5). If the i-th row is 0, then expansion with respect to the i-th row yields the result.

(D_6). If the identical rows are interchanged, then A does not change. By (D_4) of Theorem 7.1.4, $|A| = -|A|$ which implies $|A| = 0$.

(D_7). Let B be the matrix obtained from A by replacing the k-th row of A by the i-th row (the i-th row remains the same). Then by (D_6), $|B| = 0$. Expansion

with respect to the k-th row yields

$$0 = |B| = \sum_{j=1}^{p}(-1)^{k+j}b_{kj}|B_{kj}| = \sum_{j=1}^{p}(-1)^{k+j}a_{ij}|A_{kj}|.$$

(D_8). Suppose that B is obtained from A by adding c times the i-th row of A to the k-th row of A. Then, expansion with respect to the k-th row gives

$$\begin{aligned} |B| &= \sum_{j=1}^{p}(-1)^{k+j}b_{kj}|B_{kj}| = \sum_{j=1}^{p}(-1)^{k+j}(a_{kj} + ca_{ij})|A_{kj}| \\ &= \sum_{j=1}^{p}(-1)^{k+j}a_{kj}|A_{kj}| + c\sum_{j=1}^{p}(-1)^{k+j}a_{ij}|A_{kj}| \\ &= |A|. \end{aligned}$$

(D_9). We proceed by induction on p. For $p = 2$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |{}^tA| = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} = |A|.$$

Now, assume that the assertion is true for $(p-1) \times (p-1)$ matrices, and consider the $p \times p$ matrix A. For the sake of notation, we put $B = {}^tA$. Then, expansion with respect to the first row gives:

$$\begin{aligned} |{}^tA| &= |B| = \sum_{j=1}^{p}(-1)^{j+1}b_{1j}|B_{1j}| \\ &= \sum_{j=1}^{p}(-1)^{j+1}a_{j1}|B_{1j}|. \end{aligned}$$

Here B_{1j} is the matrix obtained from ${}^tA$ by removing the first row and the j-th column. Hence $B_{1j} = {}^t(A_{j1})$, and by induction assumption $|B_{1j}| = |A_{j1}|$. It follows that

$$|{}^tA| = \sum_{j=1}^{p}(-1)^{j+1}a_{j1}|A_{j1}| = a_{11}|A_{11}| + \sum_{j=2}^{p}(-1)^{j+1}a_{j1}|A_{j1}|.$$

Now, for any $j \geq 2$, and $k \geq 2$, let $A_{j1,1k}$ denote the minor of the entry a_{1k} (of A_{j1}). Namely, $A_{j1,1k}$ is obtained from A by removing the j-th and the first row together with the first and the k-th column. Then $A_{j1,1k} = a_{1k,j1}$, the matrix obtained by removing the first and the j-th row of A together with the k-th and the first column. Thus

$$|A_{j1}| = \sum_{k=2}^{p}(-1)^{k}a_{1k}|A_{j1,1k}| = \sum_{k=2}^{p}(-1)^{k}a_{1k}|A_{1k,j1}|.$$

Therefore

$$
\begin{aligned}
|A| &= a_{11}|A_{11}| + \sum_{j=2}^{p}(-1)^{j+1}a_{j1}\left(\sum_{k=2}^{p}(-1)^{k}a_{1k}|A_{1k,j1}|\right) \\
&= a_{11}|A_{11}| + \sum_{k=2}^{p}(-1)^{k+1}a_{1k}\left(\sum_{j=2}^{p}(-1)^{j}a_{j1}|A_{1k,j1}|\right).
\end{aligned}
$$

Again by induction assumption,

$$\sum_{j=2}^{p}(-1)^{j}a_{j1}|A_{1k,j1}| = |A_{1k}|.$$

Thus

$$|\,{}^{t}A| = a_{11}|A_{11}| + \sum_{k=2}^{p}(-1)^{k+1}a_{1k}|A_{1k}| = |A|.$$

This completes the proof of the theorem. □

Remark. By (D_9), for any $p \times p$ matrix $A = [a_{ij}]$ and for any $j = 1, 2, \ldots, p$,

$$|A| = \sum_{h=1}^{p}(-1)^{h+j}a_{hj}|A_{hj}|. \tag{7.1.4}$$

This expression is called the *expansion of* $|A|$ *with respect to the* j*-th column.* Furthermore, we may conclude that the properties listed above about rows of determinants are also true for columns. Namely, we have

Theorem 7.1.6 *Let* $A = [a_{ij}]$ *be a* $p \times p$ *matrix. Then*

(D_2') Suppose that for a fixed $j = 1, \ldots, p$ *the* j*-th column of* A *is*

$$A_j = \begin{bmatrix} b_{1j} + c_{1j} \\ \vdots \\ b_{pj} + c_{pj} \end{bmatrix} = B_j + C_j;$$

and let B *be the matrix obtained from* A *by replacing the* j*-th column* A_j *with* B_j*, let* C *be the matrix obtained by replacing the* j*-th column* A_j *with* C_j*. Then* $|A| = |B| + |C|$.

(D_3') Suppose that c *is a scalar and for a fixed* $j = 1, \ldots, p$, A' *is the matrix obtained from* A *by multiplying the* j*-th column of* A *by* c*. Then*

$$|A'| = c|A|.$$

(D_4') Suppose that A'' *is the* $p \times p$ *matrix obtained from* A *by interchanging two columns of* A*. Then* $|A''| = -|A|$.

(D_5') If any column of A *is 0, then* $|A| = 0$.

(D_6') If any two columns of A *are identical, then* $|A| = 0$.

(D_7') If c *is a scalar and* B *is the matrix obtained from* A *by adding* c *times a column of* A *to another column, then* $|B| = |A|$.

Remark. You may notice that the properties (D_3), (D_4) and (D_8) involve the elementary row operations $(Er_1), (Er_2)$ and (Er_3). These properties imply that if $A \xrightarrow{r} B$, then $|A|$ and $|B|$ differ by a non-zero constant multiple. In particular *if* $A \xrightarrow{r} B$, *then* $|A| = 0 \Leftrightarrow |B| = 0$.

Remark. Corresponding to the elementary row operations on matrices, we have also the *elementary column operations*:

(Ec_1) Multiplication of any column of the matrix by a non-zero scalar.

(Ec_2) The interchange of any two columns of the matrix.

(Ec_3) Addition of any column of the matrix to another column of the matrix.

Properties (D_3'), (D_4') and (D_8') show that if B is a matrix obtained from A by elementary column operations, then $|A|$ and $|B|$ differ by a non-zero constant multiple.

We make use of the elementary row or column operations in computing determinants.

Example 7.1.7 Compute

$$|A| = \begin{vmatrix} 1 & -2 & 0 & 4 \\ 3 & 4 & 1 & -2 \\ 3 & 0 & 6 & -9 \\ -1 & 8 & 6 & 3 \end{vmatrix}.$$

Solution.

$$|A| = 3\begin{vmatrix} 1 & -2 & 0 & 4 \\ 3 & 4 & 1 & -2 \\ 1 & 0 & 2 & -3 \\ -1 & 8 & 6 & 3 \end{vmatrix} \quad \text{(3 is factorized from } \alpha_3\text{)}$$

$$= 3.2 \begin{vmatrix} 1 & -1 & 0 & 4 \\ 3 & 2 & 1 & -2 \\ 1 & 0 & 2 & -3 \\ -1 & 4 & 6 & 3 \end{vmatrix} \text{ (2 is factorized from } A_2)$$

$$= 6 \begin{vmatrix} 1 & 0 & 0 & 4 \\ 3 & 5 & 1 & -2 \\ 1 & 1 & 2 & -3 \\ -1 & 3 & 6 & 3 \end{vmatrix} \text{ (} A_1 \text{ is added to } A_2)$$

$$= 6 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 5 & 1 & -14 \\ 1 & 1 & 2 & -7 \\ -1 & 3 & 6 & 7 \end{vmatrix} \text{ (-4 } A_1 \text{ is added to } A_4)$$

$$= 42 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 5 & 1 & -2 \\ 1 & 1 & 2 & -1 \\ -1 & 3 & 6 & 1 \end{vmatrix} \text{ (7 is factorized from } A_4)$$

$$= 42 \begin{vmatrix} 5 & 1 & -2 \\ 1 & 2 & -1 \\ 3 & 6 & 1 \end{vmatrix} \text{ (expansion with respect to the first row)}$$

$$|A| = 42|B| = 42 \begin{vmatrix} 5 & 1 & -2 \\ 1 & 2 & -1 \\ 3 & 6 & 1 \end{vmatrix}$$

$$= 42 \begin{vmatrix} 5 & 1 & -2 \\ 1 & 2 & -1 \\ 0 & 0 & 4 \end{vmatrix} \text{ (-3 } \beta_2 \text{ is added to } \beta_3)$$

$$= 42 \cdot (-1)^{3+3} \cdot 4 \cdot \begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix} = 42 \cdot 4 \cdot (10 - 1) = 1512.$$

Remark. The following notation provides a practical way for computing the cross product of two vectors in Euclidean 3-space. If $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ are two vectors in 3-space, we write

$$u \times v = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} i - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k$$

$$= (y_1 z_2 - y_2 z_1)i - (x_1 z_2 - x_2 z_1)j + (x_1 y_2 - x_2 y_1)k$$

$$= (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1).$$

We conclude this section by some more properties of determinants.

Theorem 7.1.8 *If E is an elementary $p \times p$ matrix, then for any $p \times p$ matrix A we have*

$$|EA| = |E|\,|A|.$$

Proof. If E is obtained from the $p \times p$ identity matrix I by (Er_1) then EA is obtained from A by (Er_1) so that by (D_3), $|EA| = c|A| = |E|\,|A|$. If E is obtained from I by (Er_2), then EA is obtained from A by (Er_2), and by (D_4), we have $|EA| = -|A| = |E|\,|A|$. Finally, if E is obtained from I by (Er_3), then EA is obtained from A by (Er_3), and by (D_8), $|EA| = |A| = |E|\,|A|$. This completes the proof. □

Corollary 7.1.9 *If E is an elementary matrix, then $|E| \neq 0$.*

Proof. For any elementary matrix E, there exists E^* (the inverse of E) such that $EE^* = I$. Hence

$$|EE^*| = |E|\,|E^*| = |I| = 1.$$

which shows that $|E| \neq 0$. □

Corollary 7.1.10 *For any $p \times p$ matrix A, $A \xrightarrow{} I$ if and only if $|A| \neq 0$.*

Proof. If $A \xrightarrow{} I$ then $A = E_1\, E_2 \cdots E_m$ where $E_1, E_2, \ldots, E_m$ are elementary matrices. Hence $|A| = |E_1|\,|E_2| \cdots |E_m| \neq 0$. Conversely, assume that $|A| \neq 0$. Let M be a row-reduced echelon matrix such that $A \xrightarrow{} M$. Then $A = E_1\, E_2 \cdots E_m M$ where $E_1, E_2, \ldots, E_m$ are elementary matrices. This shows that $|M| \neq 0$. Then M does not contain a zero row. Therefore $M = I$, and $A \xrightarrow{} I$. □

Corollary 7.1.11 *For any $p \times p$ matrix A the following are equivalent:*

a) A is invertible

b) $AX = C$ has a unique solution for any $p \times 1$ matrix C

c) $AX = 0$ has only the trivial solution $X = 0$

d) $A \rightarrow I$

e) *A is a product of finitely many elementary matrices*

f) $|A| \neq 0$.

Proof. This is just a combination of Corollary 6.7.8 above and Theorem 6.7.6. □

Corollary 7.1.12 *For any* $p \times p$ *matrices* A *and* B,

$$|AB| = |A|\,|B|.$$

Proof. If A is not invertible, then AB is not invertible, and then $|A| = |AB| = 0$ by Corollary 7.1.11 above. On the other hand, if A is invertible, then $A = E_1\, E_2 \cdots E_m$ where $E_1, E_2, \ldots, E_m$ are elementary matrices, and then $|A| = |E_1|\,|E_2| \cdots |E_m|$ and

$$|AB| = |E_1\, E_2 \cdots E_m B| = |E_1|\,|E_2| \cdots |E_m|\,|B| = |A|\,|B|.$$

□

Corollary 7.1.13 *If* A *is invertible, then* $|A^{-1}| = 1/|A|$.

Proof. $A^{-1}A = I \Rightarrow |A^{-1}|\,|A| = 1 \Rightarrow |A^{-1}| = 1/|A|$. □

Exercises

1. Let A and B be the matrices

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -i & 1+i \\ -4i & 2 & 1 \end{bmatrix}.$$

   a) Write down $A_{11}, A_{12}, A_{21}, B_{23}, B_{32}, B_{33}$.

   b) Compute $|A|, |B|, |AB|$.

2. Compute the following determinants

$$\text{a)} \begin{vmatrix} x & y \\ 2y & 2x \end{vmatrix} \quad \text{b)} \begin{vmatrix} x & y \\ 2x & 2y \end{vmatrix} \quad \text{c)} \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$\text{d)} \begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \quad \text{e)} \begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ x-1 & x-2 & x-3 \end{vmatrix} \quad \text{f)} \begin{vmatrix} 3 & 2 & 1 & 0 \\ -1 & 6 & 5 & 4 \\ 0 & -2 & -2 & -1 \\ 3 & 4 & 1 & 0 \end{vmatrix}.$$

3. If $A = [a_{ij}]$ is a $p \times p$ matrix such that $a_{ij} = 0$ for each $i > j$, then A is called an *upper triangular* matrix. Show that if A is an upper triangular matrix, then $|A| = a_{11}a_{22} \cdots a_{pp}$, the product of the entries on the main diagonal of A.

4. Show, by using determinant notation, that $u \times v = -v \times u$ for $u, v \in \mathbb{R}^3$.

5. a) Prove

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

   b) Generalize this to 4×4 determinants.

   c) Generalize this to $p \times p$ determinants.

6. Let $u = (x_1, y_1, z_1), v = (x_2, y_2, z_2)$ and $w = (x_3, y_3, z_3)$ be three vectors in 3-space. Show that

$$u \circ (v \times w) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (u \times v) \circ w.$$

7. Let $P_0(x_0, y_0, z_0), P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be three noncollinear points in three space. Show that

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_1 - x_0 & y_1 - y_0 & z_1 - z_0 \\ x_2 - x_0 & y_2 - y_0 & z_2 - z_0 \end{vmatrix} = 0$$

   is an equation of the plane passing through P_0, P_1 and P_2. Use this result to find an equation of the plane passing through $(1, 2, 3), (-1, 3, -2)$ and $(2, -1, 0)$.

8. Let v_1, v_2, v_3 be vectors in 3-space and let A be the matrix whose rows are v_1, v_2 and v_3. Show that the following statements are equivalent

   a) v_1, v_2, v_3 are linearly dependent

   b) v_1, v_2, v_3 lie on one and the same plane through the origin

   c) $(v_1 \times v_2) \circ v_3 = 0$

   d) $|A| = 0$.

9. Let $u_1 = (a_1, a_2), u_2 = (b_1, b_2)$ be two linearly independent vectors in the plane. Show that the area A of the parallelogram determined by u_1 and u_2 is

$$\begin{aligned} A &= |u_1|\,|u_2| \sin\theta \\ &= \left(\left| \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right| \right). \end{aligned}$$

10. Find the area of the triangle, in the plane, determined by the points $P(1,2), Q(1,-1)$ and $S(2,1)$.

11. Let $u_1 = (a_1, a_2, a_3)$, $u_2 = (b_1, b_2, b_3)$ be two linearly independent vectors in 3-space. Show that the area A of the parallelogram determined by u_1 and u_2 is

$$A = |u_1 \times u_2|.$$

Show also that if B is the matrix

$$B = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

Then

$$A = (\|B\ {}^tB\|)^{\frac{1}{2}}$$

where $\|B\ {}^tB\|$ denotes the absolute value of $|B\ {}^tB|$.

12. Find the area of the parallelogram, in 3-space, determined by the vectors $u = (1,2,3)$ and $v = (-1,1,1)$.

13. If $A = G_1G_2\cdots G_n$ is invertible, show that $G_1, G_2, \ldots, G_n$ are invertible.

14. If $\det A \neq 0$ and $AB = AC$, show that $B = C$.

15. Suppose $P^2 = P$ if $\lambda \neq 1$, prove $I - \lambda P$ is invertible and $(I-\lambda P)^{-1} = I + (\frac{\lambda}{1-\lambda})P$.

16. Find all values of x for which

(a) $\begin{vmatrix} x-1 & 1 & 1 \\ 0 & x-4 & 1 \\ 0 & 0 & x-2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} = 0.$

17. Let A be a square matrix of order n. Show that $\det(\alpha A) = \alpha^n \cdot \det A$.

18. Show that

$$\begin{vmatrix} x & 0 & 0 & \cdots & 0 & -1 \\ 0 & x & 0 & \cdots & 0 & 0 \\ & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & 0 & \cdots & & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 0 \\ -1 & 0 & 0 & \cdots & 0 & x \end{vmatrix} = x^{n-2}(x^2-1).$$

19. Show that

$$\begin{vmatrix} 1 & -(x_1+x_2) & x_1x_2 & 0 \\ 0 & 1 & -(x_1+x_2) & x_1x_2 \\ 1 & -(y_1+y_2) & y_1y_2 & 0 \\ 0 & 1 & -(y_1+y_2) & y_1y_2 \end{vmatrix} = (x_1-y_1)(x_1-y_2)(x_2-y_1)(x_2-y_2).$$

7.2 The Inverse of a Matrix, Cramer's Rule

In this section, we give another method for computing the inverse of an invertible matrix. As a result, we obtain a rule for solving systems of linear equations, called Cramer's Rule.

For a $p \times p$ matrix $A = [a_{ij}]$, the $p \times p$ matrix

$$adj(A) = \left[(-1)^{i+j}|A_{ji}|\right] = {}^t\left[(-1)^{i+j}|A_{ij}|\right]$$

is called the adjoint of the matrix A. Thus the ij-entry of the adjoint of A is the cofactor of the ji-entry a_{ji} of A. For $p = 3$, the adjoint of A looks like

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad adj(A) = \begin{bmatrix} |A_{11}| & -|A_{21}| & |A_{31}| \\ -|A_{12}| & |A_{22}| & -|A_{32}| \\ |A_{13}| & -|A_{23}| & |A_{33}| \end{bmatrix}.$$

Theorem 7.2.1 *For any $p \times p$ matrix A,*

$$A(adj(A)) = (adj(A))A = |A|I.$$

Proof. The ik-entry of $A(adj(A))$ is

$$\sum_{j=1}^{p} a_{ij}(-1)^{j+k}|A_{kj}|.$$

If $i = k$, then this is equal to $|A|$. If $i \neq k$, then this is equal to zero by by property (D_7). This proves $A(adj(A)) = |A|I$. Similarly $(adj(A))A = |A|I$. □

Corollary 7.2.2 *If A is an invertible $p \times p$ matrix, then*

$$A^{-1} = \frac{1}{|A|}adj(A).$$

Written more explicitely,

$$A^{-1} = {}^t\left[\frac{(-1)^{i+j}}{|A|}|A_{ij}|\right] = \left[\frac{(-1)^{i+j}}{|A|}|A_{ji}|\right]. \tag{7.2.1}$$

Example 7.2.3 Find the inverse of

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}.$$

Solution. We have

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = 1.$$

$$|A_{11}| = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} = 2\ |A_{21}| = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1,\ |A_{31}| = \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} = 0.$$

$$|A_{12}| = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = 3,\ |A_{22}| = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1,\ |A_{32}| = \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -1.$$

$$|A_{13}| = \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} = -4,\ |A_{23}| = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 2,\ |A_{33}| = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1.$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{|A_{11}|}{|A|} & -\frac{|A_{21}|}{|A|} & \frac{|A_{31}|}{|A|} \\ -\frac{|A_{12}|}{|A|} & \frac{|A_{22}|}{|A|} & -\frac{|A_{32}|}{|A|} \\ \frac{|A_{13}|}{|A|} & -\frac{|A_{23}|}{|A|} & \frac{|A_{33}|}{|A|} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ -4 & -2 & 1 \end{bmatrix}$$

In Theorem 6.7.6, we have seen that if A is an invertible matrix, then the system of linear equations (6.3.1), or the equation $AX = C$, has the unique solution $X = A^{-1}C$. Now, combining this with (7.2.1), we see that for each $i = 1, \ldots, p$,

$$x_i = \frac{1}{|A|} \left(\sum_{j=1}^{p} (-1)^{i+j} |A_{ji}| c_j \right).$$

The sum on the right hand-side of this identity can be realized as a determinant. It is the determinant of the matrix obtained from A by replacing the i-th column of A by the column C (*expansion with respect to the i-th column!*). Let Δ_i denote the determinant obtained from $|A|$ by replacing the i-th column of $|A|$ by C. Then

$$x_i = \frac{\Delta_i}{|A|}, \quad i = 1, \ldots, p.$$

We state this result in

Theorem 7.2.4 (Cramer's Rule). *If the coefficient matrix A of the system*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p = c_2 \\ \cdots \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pp}x_p = c_p \end{cases}$$

is invertible, then it has a unique solution given by

$$\xi = \left(\frac{\Delta_1}{|A|}, \frac{\Delta_2}{|A|}, \ldots, \frac{\Delta_p}{|A|} \right),$$

where $\Delta_1, \Delta_2, \ldots, \Delta_p$ are the determinants described above.

Example 7.2.5 Use Cramer's Rule to find the solution of the system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -x_1 + 2x_2 - 2x_3 &= 2 \\ 2x_1 + x_3 &= 3. \end{aligned}$$

Solution.

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ -1 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3.$$

$$\Delta_1 = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 4 & 0 & 2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = -2.$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 2 \\ -1 & 2 & -2 \\ 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -9.$$

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & 2 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 4 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = -5.$$

$$x_1 = \frac{\Delta_1}{|A|} = \frac{-2}{-3} = \frac{2}{3}, \quad x_2 = \frac{\Delta_2}{|A|} = 3, \quad x_3 = \frac{\Delta_3}{|A|} = \frac{5}{3}.$$

$$\xi = (2/3,\ 3,\ 5/3).$$

Exercises

1. Use Cramer's Rule, if possible, to solve the following systems of linear equations

a) $\begin{cases} x+2y=1 \\ -2x+y=-1 \end{cases}$ b) $\begin{cases} x+2y=1 \\ 3x+6y=2 \end{cases}$

c) $\begin{cases} 2x-y+z=1 \\ 2y+z=0 \\ x-y+z=2 \end{cases}$ d) $\begin{cases} x-y+z=2 \\ 2y+z=0 \\ x+y+2z=2 \end{cases}$

e) $\begin{cases} x+2y+3z=1 \\ -2x+y+z=2 \\ x+z=3 \end{cases}$ f) $\begin{cases} x+3y-z+w=2 \\ y+w=-1 \\ -x-2y+3z=1 \\ 3x+y-4w=0 \end{cases}$

2. For each of the systems above, find the inverse of the coefficient matrix.

7.3 Characteristic Values and Vectors

We now come to another important application of determinants. If $A=[a_{ij}]$ is a $p\times p$ matrix and u is a non-zero column vector in (real or complex) p-space, in general the vectors Au and u are linearly independent. However it can happen exceptionally that

$$Au=\lambda u \tag{7.3.1}$$

where λ is some scalar. If this is the case, then u is called a characteristic vector of A, and λ is called the corresponding characteristic value. The characteristic vector u and the characteristic value λ are said to *belong to each other*. There are physical and mathematical problems which are closely connected with the problem of finding characteristic vectors and characteristic values. The next chapter contains a geometric problem solved by means of characteristic vectors and characteristic values. In fact, the rest of the present chapter is just a preparation for Chapter 8.

The equation (7.3.1) can be written as

$$(\lambda I-A)u=0 \tag{7.3.2}$$

where I is the $p\times p$ identity matrix and 0 is $n\times 1$ zero matrix. This equation can be interpreted as a system of homogeneous equations where the number of equations is

equal to the number of unknowns. By Corollary 7.1.11 of Theorem 7.1.8, the equation (7.3.2) is satisfied by a non-zero vector u if and only if the scalar λ satisfies

$$|\lambda I-A|=0. \tag{7.3.3}$$

Thus characteristic values of A consist of the scalars λ satisfying the equation (7.3.3). This equation is called the *characteristic equation* of the matrix A. The left hand side of the characteristic equation is a polynomial of degree p in λ:

$$|\lambda I-A|=\lambda^p+c_{p-1}\lambda^{p-1}+\cdots+c_1\lambda+c_0=0$$

where $c_0,c_1,\ldots,c_{p-1}$ are scalars which are sums of product of different entries of A. We have, for instance,

$$c_0=(-1)^p|A|,\quad c_{p-1}=-(a_{11}+a_{22}+\cdots+a_{pp}).$$

By the Fundamental Theorem of Algebra (See Appendix A), *any $p\times p$ matrix has at least one and at most p distinct characteristic values.* The characteristic values are real or complex numbers.

Remark. Let $A=[a_{ij}]$ be a $p\times p$ matrix. By what we have observed above, there is a non-zero vector u in complex p-space such that $Au=\lambda u$ if and only if λ is a characteristic value of A. If A is a matrix with real entries, then there is a non-zero vector u in Euclidean p-space such that $Au=\lambda u$ if and only if λ is a real characteristic value of A. The last assertion can be proven as follows. If λ is a real characteristic value of A (i.e. $\lambda\in\mathbb{R}$), then let v be a non-zero vector in (perhaps complex) p-space such that $Av=\lambda v$. Then we can write $v=u+iu'$ where u and u' are in Euclidean p-space. It is immediately seen that $Au=\lambda u$. Conversely, if u is a non-zero vector in Euclidean p-space such that $Au=\lambda u$, then λ must be a real number because both Au and u have real components.

Example 7.3.1 Find all characteristic values and the corresponding characteristic vectors of

$$A=\begin{bmatrix} 2 & -5 & -1 \\ -5 & 2 & -1 \\ -1 & -1 & -4 \end{bmatrix}.$$

Solution. The characteristic equation of A is

$$|\lambda I-A|=\begin{vmatrix} \lambda-2 & 5 & 1 \\ 5 & \lambda-2 & 1 \\ 1 & 1 & \lambda+4 \end{vmatrix}=(\lambda-7)(\lambda+2)(\lambda+5)=0.$$

Thus $\lambda_1 = 7, \lambda_2 = -2, \lambda_3 = -5$ are the characteristic values of A. They are real characteristic values therefore the corresponding characteristic vectors can be obtained in Euclidean p-space. A characteristic vector belonging to $\lambda_1 = 7$ can be computed as follows. Such a vector, say u_1, will be a solution of the equation $(\lambda_1 I - A)u_1 = 0$. The coefficient matrix of this equation is

$$\lambda_1 I - A = \begin{bmatrix} 5 & 5 & 1 \\ 5 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix} \overset{\sim}{\to} \begin{bmatrix} 1 & 1 & 11 \\ 0 & 0 & -54 \\ 0 & 0 & -54 \end{bmatrix} \overset{\sim}{\to} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, a general solution for the above equation depends on one parameter;

$$u_1 = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}.$$

Taking $t = 1$, we obtain the particular solution

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

which is a characteristic vector of A belonging to the characteristic value $\lambda_1 = 7$. Similarly, one can see that

$$u_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad u_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

are characteristic vectors of A belonging to the characteristic values $\lambda_2 = -2$ and $\lambda_3 = -5$, respectively.

Example 7.3.2 Find all characteristic values and the corresponding characteristic vectors of

$$A = \begin{bmatrix} 1 & -2 & -4 \\ -1 & -1 & 0 \\ 1 & -1 & -2 \end{bmatrix}.$$

Solution. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 2 & 4 \\ 1 & \lambda + 1 & 0 \\ -1 & 1 & \lambda + 2 \end{vmatrix} = (\lambda + 2)(\lambda^2 + 1) = (\lambda + 2)(\lambda - i)(\lambda + i) = 0.$$

Thus $\lambda_1 = -2, \lambda_2 = i, \lambda_3 = -i$ are the characteristic values. The corresponding characteristic vectors are computed as follows:

$$\begin{aligned}
\lambda_1 I - A &= \begin{bmatrix} -3 & 2 & 4 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \overset{\sim}{\to} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}. \\
\lambda_2 I - A &= \begin{bmatrix} i-1 & 2 & 4 \\ 1 & i+1 & 0 \\ -1 & 1 & i+2 \end{bmatrix} \overset{\sim}{\to} \begin{bmatrix} 1 & 0 & -i-1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} i+1 \\ -1 \\ 1 \end{bmatrix}. \\
\lambda_3 I - A &= \begin{bmatrix} -i-1 & 2 & 4 \\ 1 & -i+1 & 0 \\ -1 & 1 & -i+2 \end{bmatrix} \overset{\sim}{\to} \begin{bmatrix} 1 & 0 & i-1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -i+1 \\ -1 \\ 1 \end{bmatrix}.
\end{aligned}$$

Remark. Both of the matrices in Example 7.3.1 and Example 7.3.2 have real entries but the characteristic values of the matrix in Example 7.3.1 are all real while the matrix of Example 7.3.2 has complex characteristic values. We may notice that the matrix A of Example 7.3.1 is *symmetric*, i.e., ${}^tA = A$.

Theorem 7.3.3 *Let A be a $p \times p$ matrix with real entries. If A is symmetric, then its characteristic values are all real.*

Proof. Let λ be a characteristic value of A and let $u \neq 0$ be a characteristic vector belonging to λ. Then $Au = \lambda u$. This implies $A\overline{u} = \overline{A}\overline{u} = \overline{\lambda u}$. Now, we compute ${}^t\overline{u}Au$ in two ways:

$$\begin{aligned}
{}^t\overline{u}Au &= {}^t\overline{u}(Au) = {}^t\overline{u}(\lambda u) = \lambda\, {}^t\overline{u}u = \lambda|u|^2 \\
{}^t\overline{u}Au &= ({}^t\overline{u}A)u = ({}^t\overline{u}\, {}^tA)u = {}^t(A\overline{u})u = {}^t(\overline{\lambda}\overline{u})u = \overline{\lambda}|u|^2.
\end{aligned}$$

Then

$$\lambda|u|^2 = \overline{\lambda}|u|^2, \quad u \neq 0 \Rightarrow \lambda = \overline{\lambda}, \quad \lambda \in \mathbb{R}.$$

This completes the proof. □

Remark. We know that two row or column vectors u and v in 3-space are perpendicular if and only if $u \circ v = 0$. We note that if u and v are column vectors, then $u \circ v = {}^tu\, v = {}^tv\, u$. If u and v are two column vectors in p-space, then we say that they are perpendicular if ${}^tu\, v = 0$.

Theorem 7.3.4 *Let A be a $p \times p$ matrix and let u be a non-zero characteristic vector of A belonging to the characteristic value λ. If a vector v in p-space is perpendicular to u, then it is also perpendicular to Au.*

Proof. If v is perpendicular to u, then ${}^t v u = 0$, and then ${}^t v(Au) = {}^t v(\lambda u) = \lambda({}^t vu) = 0$. Thus v is perpendicular to Au. □

Theorem 7.3.5 *Let A be a $p \times p$ symmetric matrix with real entries. Then characteristic vectors belonging to distinct characteristic values of A are mutually perpendicular.*

Proof. Let λ_1 and λ_2 be two distinct characteristic values of A, and let u_1 and u_2 be characteristic vectors belonging to λ_1 and λ_2, respectively. Then λ_1 and λ_2 are real numbers, and we may assume that u_1 and u_2 are in Euclidean p-space. We have

$$\begin{aligned} {}^t u_1 A u_2 &= {}^t u_1(Au_2) = {}^t u_1(\lambda_2 u_2) = \lambda_2({}^t u_1 u_2) \\ &= ({}^t u_1 A)u_2 = {}^t(Au_1)u_2 = {}^t(\lambda_1 u_1)u_2 = \lambda_1({}^t u_1 u_2). \end{aligned}$$

Therefore

$$\lambda_1({}^t u_1 u_2) = \lambda_2({}^t u_1 u_2), \quad (\lambda_1 - \lambda_2)({}^t u_1 u_2) = 0.$$

Since $\lambda_1 \neq \lambda_2$, we get ${}^t u_1 u_2 = 0$, proving that u_1 and u_2 are perpendicular. □

In Example 7.3.1, the matrix A is a symmetric matrix with real entries and it has three distinct characteristic values 7, -2 and -5. It is easily checked that u_1, u_2 and u_3 are mutually perpendicular.

Exercises

1. In each case below, find all the characteristic values and the corresponding characteristic vectors of the given matrix

   a) $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

   e) $\begin{bmatrix} 1 & 1-i \\ i+i & 2 \end{bmatrix}$ f) $\begin{bmatrix} i & 1-i \\ 1+i & 2i \end{bmatrix}$ g) $\begin{bmatrix} i & 1-i \\ \frac{1-i}{2} & 2 \end{bmatrix}$.

2. Same question for the matrices

   a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 6 & 5 & 3 \\ 0 & 1 & 1 \\ -3 & -3 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

3. In each case below, find three mutually perpendicular characteristic vectors of the given matrix

   a) $\begin{bmatrix} 1 & -5 & -1 \\ -5 & 1 & -1 \\ -1 & -1 & -5 \end{bmatrix}$ b) $\begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

   c) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -5 \\ -3 & -5 & -4 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

4. Let A be a $p \times p$ matrix, λ a scalar, and let u and v be non-zero vectors in p-space. Prove that

   a) If λ is a characteristic value of A, then 0 is a characteristic value of $(\lambda I - A)$.

   b) If u is a characteristic vector of A with characteristic value λ, then for any $c \neq 0$, cu is a characteristic vector of A with the same characteristic value λ.

   c) If u and v are both characteristic vectors of A with characteristic value λ, then for any scalar c and d not both zero, $cu + dv$ is a characteristic vector of A with characteristic value λ.

5. Let A be a $p \times p$ matrix (with real or complex entries). We say that A is hermitian if ${}^t\overline{A} = A$. We say that A is skew-hermitian if ${}^t\overline{A} = -A$.

   a) Prove that Theorem 7.3.3 and Theorem 7.3.5 are true if A is an hermitian matrix with complex entries.

   b) Prove that if A is skew hermitian, then characteristic values of A are all pure imaginary.

7.4 Symmetric Matrices with Real Entries

The main result of this section is Theorem 7.4.3 and its corollary. This result is also true for $p \times p$ symmetric matrices with real entries but the proof is beyond the scope of this text.

It will be convenient to denote a 3×3 symmetric matrix with real entries by

$$A = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}.$$

By Theorem 7.3.3, characteristic values of A are all real. In this section, we obtain a few more facts about characteristic values and characteristic vectors of A. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - A & -D & -E \\ -D & \lambda - B & -F \\ -E & -F & \lambda - C \end{vmatrix} = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0.$$

where

$$c_0 = -|A|, \quad c_1 = BC + CA + AB - D^2 - E^2 - F^2, \quad c_2 = -(A+B+C).$$

Theorem 7.4.1 *If all characteristic values of A are zero, then $A = 0$, the zero matrix.*

Proof. If all characteristic values of A are zero, then the characteristic equation of A is $|\lambda I - A| = \lambda^3 = 0$. Then $c_0 = c_1 = c_2 = 0$. We then have

$$c_2^2 - 2c_1 = A^2 + B^2 + C^2 + 2D^2 + 2E^2 + 2F^2 = 0,$$

which implies $A = B = C = D = E = F = 0$. □

Corollary 7.4.2 *If all characteristic values of A are equal, say λ_0, then $A = \lambda_0 I$ where I is the 3×3 identity matrix.*

Proof. If all characteristic values of A are equal, say λ_0, then all characteristic values of the matrix $(\lambda_0 I - A)$ are zero. Since $(\lambda_0 I - A)$ is symmetric, we conclude from the theorem that $\lambda_0 I - A = 0$. That is, $A = \lambda_0 I$. □

It follows from the above corollary that there are three possibilities for a 3×3 real symmetric matrix:

Case 1. *A has three distinct characteristic values.*

Case 2. *A has two distinct characteristic values.*

Case 3. *$A = \lambda_0 I$ for some $\lambda_0 \in \mathbb{R}$.*

If A has three distinct characteristic values, then by Theorem 7.3.5, we can find, three characteristic vectors u_1, u_2, u_3 in Euclidean 3-space; which are mutually perpendicular, i.e., ${}^t u_i u_j = 0$ if $i \neq j$. We now assert that the same is true even if the characteristic values are not distinct.

Theorem 7.4.3 *For any 3×3 symmetric matrix $A \neq 0$ with real entries we can find three mutually perpendicular characteristic vectors u_1, u_2, u_3 in Euclidean 3-space.*

Proof. Case 1 has already been treated. We show that the assertion is true for Case 2 and Case 3. Assume that λ_1, λ_2 and λ_3 are the characteristic values of A with $\lambda_1 \neq \lambda_2$ and $\lambda_2 = \lambda_3$. Let u_1 and u_2 be characteristic vectors (in Euclidean 3-space) belonging to λ_1 and λ_2, respectively, and consider the vector $u_3 = u_1 \times u_2$. We know that u_1 and u_2 are perpendicular by Theorem 7.3.5. Therefore $u_3 \neq 0$ and it is perpendicular to both u_1 and u_2. We may conclude that u_1, u_2 and u_3, being mutually perpendicular, are linearly independent (see Exercise 4). Since any four vectors in 3-space are linearly dependent, Au_3, u_1, u_2, u_3 are linearly dependent. Thus we can write

$$Au_3 = a_1u_1 + a_2u_2 + a_3u_3; \quad a_1, a_2, a_3 \in \mathbb{R}.$$

On the other hand, by Theorem 7.3.2, Au_1 and Au_2 are both perpendicular to u_3. Thus ${}^t(Au_1)u_3 = {}^t u_1(Au_3) = 0 = {}^t(Au_2)u_3 = {}^t u_2(Au_3)$, and

$$\begin{aligned} 0 &= {}^t u_1(Au_3) = {}^t u_1(a_1u_1 + a_2u_2 + a_3u_3) = a_1|u_1|^2 \\ 0 &= {}^t u_2(Au_3) = {}^t u_2(a_1u_1 + a_2u_2 + a_3u_3) = a_2|u_2|^2 \end{aligned}$$

showing that $a_1 = a_2 = 0$. Hence $Au_3 = a_3u_3$. By our assumption about characteristic values of A, $a_3 = \lambda_2 = \lambda_3$ and u_3 is a characteristic vector belonging to λ_2. This proves the assertion for the Case 2. As for Case 3, if $A = \lambda_0 I$ then one can easily see that

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are characteristic vectors of $\lambda_0 I$ with characteristic value λ_0 and they are mutually perpendicular. □

Remark. If $u \neq 0$ is a characteristic vector of A belonging to the characteristic value λ, then $\frac{u}{|u|}$ is also a characteristic vector of A with the same characteristic value λ. Therefore, *we may assume that the mutually perpendicular characteristic vectors u_1, u_2, u_3 of A are unit vectors.* Then

$${}^t u_i u_j = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j. \end{cases}$$

Let R be the 3×3 matrix whose columns are the characteristic vectors u_1, u_2, u_3 of A which are mutually perpendicular unit vectors in 3-space: $R = [u_1, u_2, u_3]$. Let also $\lambda_1, \lambda_2, \lambda_3$ be the characteristic values of A belonging to u_1, u_2, u_3, respectively. Then we have

$${}^t RAR = {}^t R[\lambda_1u_1, \lambda_2u_2, \lambda_3u_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

The matrix R has also the property

$$^tRR = I = R\,^tR. \tag{7.4.1}$$

Any matrix R with real entries which satisfies (7.4.1) is called an orthogonal matrix.

We summarize the above results in a

Corollary 7.4.4 *For any 3×3 symmetric matrix $A \neq 0$ with real entries we can find a 3×3 orthogonal matrix R such that*

$$^tRAR = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the characteristic values of A.

Example 7.4.5 Given

$$A = \begin{bmatrix} 2 & -5 & -1 \\ -5 & 2 & -1 \\ -1 & -1 & -4 \end{bmatrix},$$

find an orthogonal matrix R such that ^tRAR is a diagonal matrix.

Solution. We have seen in Example 7.3.1 that A has characteristic values $\lambda_1 = 7, \lambda_2 = -2, \lambda_3 = -5$. The unit vectors

$$v_1 = \frac{u_1}{|u_1|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad v_2 = \frac{u_2}{|u_2|} = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad v_3 = \frac{u_3}{|u_3|} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

are mutually perpendicular characteristic vectors belonging to the characteristic values $7, -2, -5$, respectively. Thus

$$R = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}.$$

is a matrix which satisfies the requirements. In fact, R is orthogonal and

$$^tRAR = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

Example 7.4.6 Find an orthogonal matrix R such that ^tRAR is a diagonal matrix:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & -2 \\ -1 & -2 & 3. \end{bmatrix}.$$

Solution. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda-3 & -2 & 1 \\ -2 & \lambda-6 & 2 \\ 1 & 2 & \lambda-3 \end{vmatrix} = \begin{vmatrix} \lambda-2 & 0 & \lambda-2 \\ 0 & \lambda-2 & 2\lambda-4 \\ 1 & 2 & \lambda-3 \end{vmatrix} = (\lambda-8)(\lambda-2)^2 = 0.$$

Thus A has characteristic values $\lambda_1 = 8, \lambda_2 = \lambda_3 = 2$. We have

$$8I - A = \begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 6 & 12 \\ 0 & -12 & -24 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2I - A = \begin{bmatrix} -1 & -2 & 1 \\ -2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that

$$v_1 = \begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -2/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}$$

are characteristic vectors belonging to 8 and 2, respectively. The third characteristic vector (which belongs to 2 and perpendicular to v_1 and v_2) is obtained by

$$v_3 = \frac{v_1 \times v_2}{|v_1 \times v_2|} = \begin{pmatrix} -1/\sqrt{30} \\ -2/\sqrt{30} \\ -5/\sqrt{30} \end{pmatrix}.$$

Thus

$$R = \begin{bmatrix} -1/\sqrt{6} & -2/\sqrt{3} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{3} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{bmatrix} = \frac{1}{\sqrt{30}} \begin{bmatrix} -\sqrt{5} & -2\sqrt{10} & -1 \\ -2\sqrt{5} & \sqrt{10} & -2 \\ \sqrt{5} & 0 & 5 \end{bmatrix}$$

is an orthogonal matrix for which ^tRAR is diagonal.

Exercises

1. For each of the following matrices find an orthogonal matrix R such that ${}^tRAR$ is diagonal.

a) $A = \begin{bmatrix} 1 & -5 & -1 \\ -5 & 1 & -1 \\ -1 & -1 & -5 \end{bmatrix}$ b) $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$ d) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$.

2. For each of the following matrices find three characteristic vectors which are mutually perpendicular unit vectors.

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

3. If R is an orthogonal matrix show that $R^{-1} = {}^tR$ and $|R| = \pm 1$.

4. If u_1, u_2, u_3 are mutually perpendicular unit vectors in 3-space, show that they are linearly independent (*Hint*: If $a_1u_1 + a_2u_2 + a_3u_3 = 0$ then consider ${}^tu_1(a_1u_1 + a_2u_2 + a_3u_3)$).

Chapter 8

SURFACES

A set of points in Euclidean 3-space that is the graph of a relation determined by a single equation $f(x, y, z) = 0$ in the variables x, y and z is called a surface. An intermediate example of a surface is a plane. In Chapter 5, we showed that a plane is the graph of an equation of first degree in x, y and z. In the present chapter, we study some elementary types of surfaces such as surfaces of revolution and quadratics. We show that any equation of the form

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + 2Gx + 2Hy + 2Kz + L = 0 \quad (8.0.1)$$

where at least one of A, B, C, D, E or F is different from zero, is a quadratic surface (or a degenerate form of a quadratic surface). For this, we make use of change of coordinates in 3-space. The equation (8.0.1) is called the *general quadratic equation* in three variables.

In graphing a surface it is adventageous to consider the intersection of the surface and various planes. The set of points that constitutes the intersection of a surface S and a plane P is called a *plane section* of S. If P is perpendicular to a line ℓ, then we say that the plane section is perpendicular to ℓ. Note that the plane section of a surface S by a plane parallel to one of the coordinate planes is perpendicular to one of the coordinate axes. If S is a given surface with equation $f(x, y, z) = 0$, then a plane section perpendicular to one of the coordinate axes is the graph of one of the relations

$$\begin{aligned} S_1 &= \{(x, y, z) : f(x, y, z) = 0 \text{ and } x = x_0\} \\ S_2 &= \{(x, y, z) : f(x, y, z) = 0 \text{ and } y = y_0\} \\ S_3 &= \{(x, y, z) : f(x, y, z) = 0 \text{ and } z = z_0\} \end{aligned}$$

where x_0, y_0 and z_0 are fixed real numbers. The plane section S_1 is the intersection of S and the plane with equation $x = x_0$. S_2 and S_3 can be described similarly.

8.1 Spheres and Cylinders

Geometrically, a *sphere* in 3-space is the set of all points that are equidistant from a fixed point called the center of the sphere. The distance between the center and any point on the sphere is called the radius of the sphere. Thus the sphere with radius r and center $C(h, k, \ell)$ is the surface determined by the equation

$$(x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2.$$

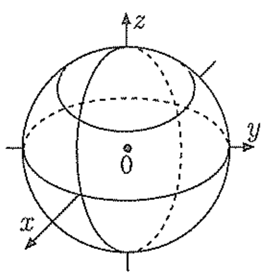


Fig. 8.1

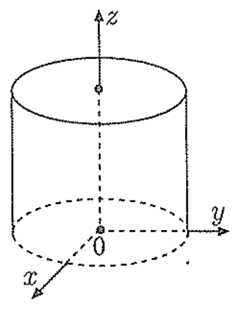


Fig. 8.2

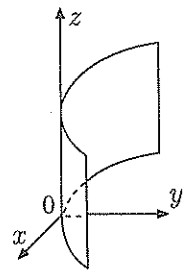


Fig. 8.3

One can easily observe that any plane section of a sphere perpendicular to a coordinate axis is either empty or a circle in 3-space. In fact, if S is the sphere described above, then the plane sections of S that are perpendicular to X-axis are of the form

$$\begin{aligned} S_1 &= \{(x,y,z) : (x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2 \text{ and } x = x_0\} \\ &= \{(x,y,z) : (y-k)^2 + (z-\ell)^2 = r^2 - (x_0-h)^2 \text{ and } x = x_0\}. \end{aligned}$$

S_1 is empty if $r^2 < (x_0-h)^2$ and it is a circle on the plane $x = x_0$ if $r^2 \geq (x_0-h)^2$. (When $r^2 = (x_0-h)^2$, the circle reduces to a point.) Plane sections perpendicular to Y- or Z-axes can be described similarly. Fig. 8.1. shows the graph and some of the plane sections of a sphere centered at the origin.

We now consider a plane P and a curve C that lies in the plane P. We consider the set of all lines in 3-space intersecting the curve C and perpendicular to P. The

set S of all points lying on these lines is called a right cylindrical surface, or briefly, a *cylinder*. The curve C is called the *directrix* of the cylinder. A cylinder is often named after its directrix. For instance, a cylinder whose directrix is a circle is called a circular cylinder; a cylinder whose directrix is a parabola is called a parabolic cylinder (see Fig. 8.2 and Fig. 8.3.).

Let $f(x,y)$ be an expression containing no variables other than x and y. Then the graph of

$$S_z = \{(x,y,z) : f(x,y) = 0\}$$

is a cylinder whose directrix is the graph of $\{(x,y) : f(x,y) = 0\}$ in XY-plane. Similarly, the graph of

$$S_x = \{(x,y,z) : f(y,z) = 0\}$$

is a cylinder whose directrix is the graph of $\{(y,z) : f(y,z) = 0\}$ in YZ-plane, and the graph of

$$S_y = \{(x,y,z) : f(x,z) = 0\}$$

is a cylinder in XZ-plane whose directrix is the graph of $\{(x,z) : f(x,z) = 0\}$.

Given a cylinder in 3-space, we can always chose a coordinate system so that the directrix of the cylinder lies in one of the coordinate planes. Thus any cylinder in 3-space can be described as S_x, S_y or S_z as above in a suitable coordinate system.

Exercises

1. Construct the graph of each of the following relations

   a) $\{(x,y,z) : (x-1)^2 + (y-2)^2 + (z-3)^2 = 1\}$
   b) $\{(x,y,z) : (x-1)^2 + (y-2)^2 + (z-3)^2 = 0\}$
   c) $\{(x,y,z) : (x-1)^2 + (y-2)^2 = 1\}$
   d) $\{(x,y,z) : (x-1)^2 + (y-2)^2 = 0\}$
   e) $\{(x,y,z) : (x-1)^2 = 1\}$
   f) $\{(x,y,z) : (x-1)^2 = 0\}$
   g) $\{(x,y,z) : x^2 - y^2 = 4\}$
   h) $\{(x,y,z) : x^2 - y^2 = 0\}$
   i) $\{(x,y,z) : x^2 - y - 2x = 0\}$
   j) $\{(x,y,z) : 4y^2 - 9z^2 = 36\}$.

2. Construct the graph of each of the following relations

a) $\{(x, y, z) : x^2 + y^2 = 4 \text{ and } z = 0\}$

b) $\{(x, y, z) : x^2 + y^2 = 4 \text{ and } z = 2\}$

c) $\{(x, y, z) : x^2 + y^2 = 4 \text{ and } 0 \le z \le 1\}$

d) $\{(x, y, z) : y = \sin x, 0 \le x \le 2\pi\}$

e) $\{(x, y, z) : xy = 1 \text{ and } z = -1\}$

f) $\{(x, y, z) : xy = 1 \text{ and } z = 1\}$

g) $\{(x, y, z) : xy = 1 \text{ and } -1 \le z \le 1\}$.

3. In each case below, construct the graph of the relation defined by the given equation in 3-space

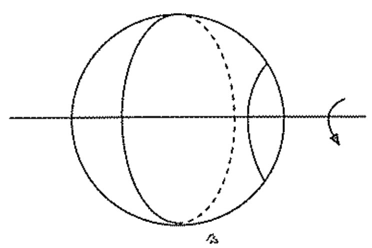
a) $y - x^2 - 4x = 0$ b) $x^2 + y^2 - 4y = 0$ c) $z^2 - 2z + y^2 = 0$

d) $x^2 + y^2 + z^2 = 9$ e) $x^2 + y^2 + z^2 + 2x = 9$ f) $4y^2 + 9z^2 = 36$

g) $y - \sqrt{x} = 0$ h) $z = \sqrt{1 - x^2 - y^2}$ i) $x + y + z = 1$.

8.2 Surfaces of Revolution

Consider a plane P and a curve C that lies in the plane P. Consider also a line ℓ which lies in the same plane P. If C is revolved about the line ℓ, then it generates a surface which is called a surface of revolution. For example, if a circle is revolved about a diameter, the surface generated is a sphere.



If a curve in the XY-plane is revolved about the X-axis, then each point of the curve describes a circle centered on the X-axis. Consider the curve defined by an equation of the form $y = r(x)$ in the XY-plane, and consider the surface S obtained by revolving that curve about the X-axis. A point $P(x, y, z)$ is on the surface of revolution if and only if $y^2 + z^2 = [r(x)]^2$ (See Fig. 8.4). Thus

$$S = \{(x, y, z) : y^2 + z^2 = [r(x)]^2\}.$$

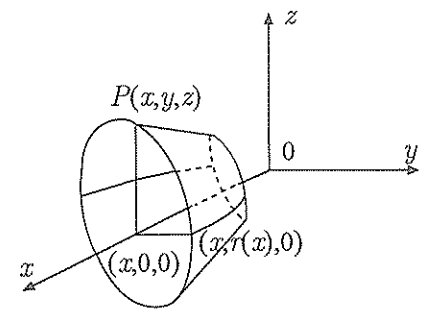
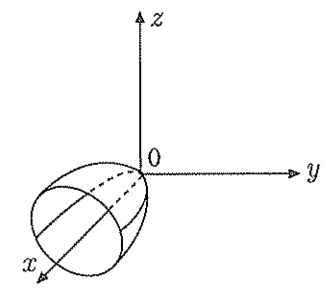


Fig. 8.4

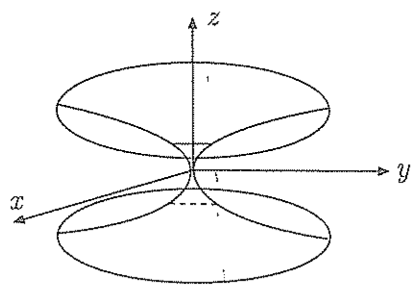
Hence *an equation of the surface is obtained from an equation of the curve* (in the XY-plane) *on replacing* y^2 *by* $(y^2 + z^2)$. Similarly, an equation of the surface of revolution of a curve in XZ-plane about X-axis is obtained from an equation of the curve on replacing z^2 by $z^2 + y^2$; and an equation of a surface of revolution of a curve in YZ-plane about Y-axis is obtained from an equation of the curve on replacing z^2 by $z^2 + x^2$.

Example 8.2.1 An equation of the surface of revolution of the parabola $y^2 = x$ about the X-axis is given by



$$y^2 + z^2 = x.$$

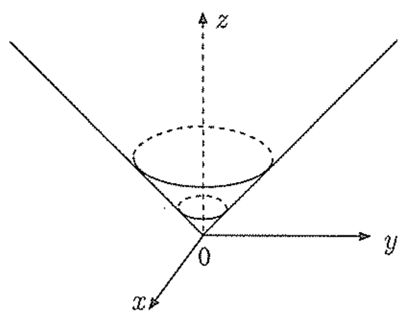
An equation of the surface obtained by revolving the parabola $y = z^2$ about the z-axis is given by



$$y^2 + x^2 = z^4$$

Remark. If an equation of a surface is given in the form $y^2 + z^2 = r(x)^2$ (or $x^2 + z^2 = r(y)^2$ or $x^2 + y^2 = r(z)^2$) then the surface can always be considered as a surface of revolution of a curve about one of the coordinate axes.

Example 8.2.2 The surface determined by the equation $x^2 + y^2 + z^2 = 0$ can be considered as the surface obtained by rotating the line $y = z$ (or $y = -z$) about Z-axis.



Exercises

1. In each case below, find an equation and sketch the graph of the surface obtained by revolving the given curve about the given coordinate axis.
   a) $2x - 3y + 12 = 0$, X-axis b) $2x - 3y + 12 = 0$, Y-axis
   c) $x^2 + y^2 = 1$, X-axis d) $x^2 + y^2 = 1$, Y-axis
   e) $(x-1)^2 + y^2 = 1$, X-axis f) $(x-1)^2 + y^2 = 1$, Y-axis
   g) $(x-2)^2 + y^2 = 1$, X-axis h) $(x-2)^2 + y^2 = 1$, Y-axis.

2. In each case below, interprete the given equation as an equation of a surface of revolution and graph the surface.
   a) $9x^2 + 4y^2 + 4z^2 = 36$ b) $9x^2 + 9y^2 + 4z^2 = 36$
   c) $4x^2 + 9y^2 + 9z^2 = 36$ d) $4x^2 + 4y^2 + 9z^2 = 36$
   e) $9x^2 - 4y^2 - 4z^2 = 36$ f) $9x^2 + 9y^2 - 4z^2 = 36$
   g) $4x^2 - 9y^2 - 9z^2 = 36$ h) $4x^2 + 4y^2 - 9z^2 = 36$
   i) $9x^2 - 4y^2 - 4z^2 = 0$ j) $9x^2 + 9y^2 - 4z^2 = 0$.
3. Find the equation of the surface generated by revolving the ellipse
$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \\ z = 0 \end{cases}$$
   about the axis Ox.
4. Find the equation of the surface generated by revolving the hyperbola
$$\begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \\ y = 0 \end{cases}$$
   about the axis Oz.

8.3 Canonical Equations of the Quadric Surfaces

In our study of conic sections we have shown that the graph of a quadratic equation in two variables is a conic section or a degenerate form of a conic section. In section 5 of this chapter, we will show that the graph of the general quadratic equation (8.0.1) is a -so called- quadratic surface or a degenerate form of a quadratic surface. In the present section, we give canonical equations of quadratic surfaces.

The graph of the relation defined by an equation, in a suitable coordinate system, of the form
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{8.3.1}$$
where a, b, c are positive real numbers, is called an *ellipsoid*. Plane sections of an ellipsoid by the coordinate planes are ellipses. If $a = b = c$ then the ellipsoid is just a sphere of radius a centered at the origin. Fig. 8.5. describes the graph of the equation (8.3.1).

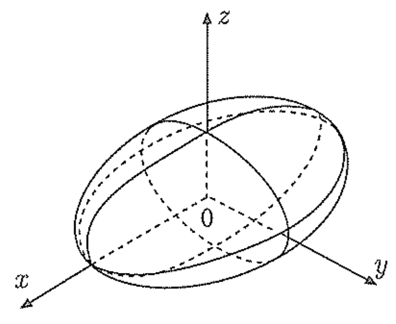


Fig. 8.5.

The graph of the relation defined by an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \tag{8.3.2}$$

where a, b, c are positive real numbers, is called an *elliptic cone.* Plane sections of an elliptic cone by planes parallel to XY-plane are ellipses in space (See Fig. 8.6).

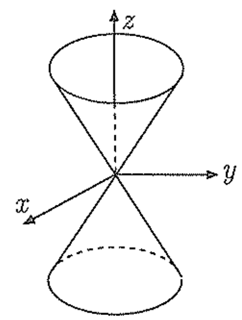


Fig. 8.6.

The surface determined by an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{8.3.3}$$

where a, b, c are positive real numbers, is called an *hyperboloid of one sheet.* It intersects the XY-plane in an ellipse, and the other coordinate planes in hyperbolas. The plane section determined by any plane parallel to XY-plane is an ellipse while the plane section determined by a plane parallel to YZ-plane or XZ-plane is an hyperbola. Fig. 8.7. shows the graph of an hyperboloid of one sheet.

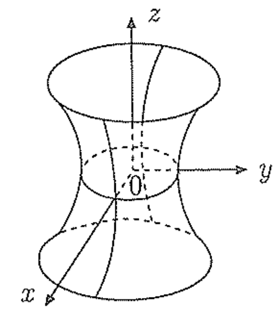


Fig. 8.7.

The surface determined by an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{8.3.4}$$

where a, b, c are positive real numbers is called an *hyperboloid of two sheets.* It intersects XY-plane and XZ-plane in hyperbolas (see Fig. 8.8).

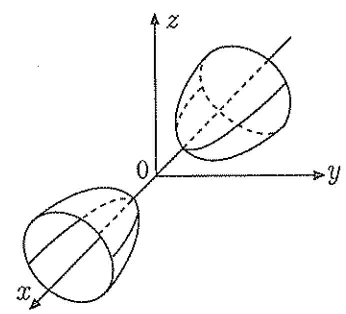


Fig. 8.8.

It does not intersect the YZ-plane. However, if we consider the plane section of this surface by a plane $x = k$, with $k^2 > a^2$, we get an ellipse on the plane $x = k$. In fact, such a plane section is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ and } x = k$$

or by

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1 \text{ and } x = k.$$

The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0 \tag{8.3.5}$$

where a and b are positive real numbers, is called an *elliptic paraboloid.* We notice that XZ-plane intersects the surface in the parabola whose equation is $z = x^2/a^2$, and the YZ-plane intersects it in the parabola whose equation is $z = y^2/b^2$. The XY-plane intersects the surface at the origin. Each plane $z = k$, with $k > 0$, intersects the surface in an ellipse whose equation is

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{and} \quad z = k$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k \quad \text{and} \quad z = k.$$

Fig. 8.9. shows the graph of an elliptic paraboloid.

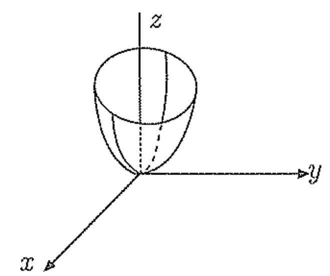


Fig. 8.9.

Remark. If the plane sections of an ellipsoid, a cone, an hyperboloid, or an elliptic paraboloid are circles then the surface is merely a surface of revolution.

The graph of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + z = 0 \tag{8.3.6}$$

where a and b are positive real numbers, is called an *hyperbolic paraboloid.* We notice that this surface does not resemble to a surface of revolution in any way. XZ-plane intersects the surface in the parabola whose equation is $z = -x^2/a^2$, YZ-plane intersects it in the parabola whose equation is $z = y^2/b^2$ (see Fig. 8.10).

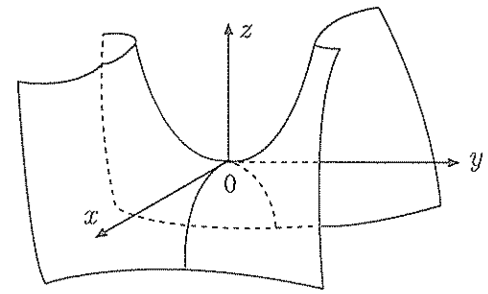


Fig. 8.10.

The plane section of this surface by a plane $z = k$, parallel to XY-plane, is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + z = 0 \quad \text{and} \quad z = k$$

or by

$$\frac{x^2}{ka^2} - \frac{y^2}{kb^2} = -1 \quad \text{and} \quad z = k.$$

A cylindrical surface whose directrix is a conic section is called a *quadratic cylinder.* A quadratic cylinder is called an elliptic cylinder, parabolic cylinder or hyperbolic cylinder according as its directrix is an ellipse, a parabola or a hyperbola. The following are canonical equations of quadratic cylinders with directrices in XY-plane.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{8.3.7}$$

$$\frac{x^2}{4c} - y = 0 \quad \text{or} \quad \frac{y^2}{4c} - x = 0 \tag{8.3.8}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \tag{8.3.9}$$

By a *quadratic surface* we mean a surface which is the graph of one of the canonical equations (8.3.1) to (8.3.9) in a suitable coordinate system.

The following is a list of the quadratic surfaces whose canonical equations are given above:

Quadric Surface	Canonical Equation	Graph
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Fig. 8.5.
Elliptic Cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	Fig. 8.6.
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Fig. 8.7.
Hyperboloid of two sheets	$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Fig. 8.8.
Elliptic Paraboloid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$	Fig. 8.9.
Hyperbolic Paraboloid	$\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$	Fig. 8.10.
Elliptic Cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Fig. 8.2.
Parabolic Cylinder	$\frac{x^2}{4c}-y=0$ or $\frac{y^2}{4c}-x=0$	Fig. 8.3.
Hyperbolic Cylinder	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	

Exercises

1. In each case below, identify and construct the graph of the given equation in 3-space.

   a) $4x^2+3y^2+2z^2=24$
   b) $4(x-1)^2+3(y-2)^2+2(z-3)^2=24$
   c) $4x^2+3y^2-2z^2=0$
   d) $4(x-1)^2+3(y-2)^2-2(z-3)^2=0$
   e) $4x^2+3y^2-2z^2=24$
   f) $4(x-1)^2+3(y-2)^2-2(z-3)^2=24$
   g) $4x^2-3y^2-2z^2=24$
   h) $4(x-1)^2-3(y-2)^2-2(z-3)^2=24$
   i) $4x^2+3y^2-12z=0$
   j) $4(x-1)^2+3(y-2)^2-12z=-36$
   k) $4x^2-3y^2+12z=0$
   l) $4(x-1)^2-3(y-2)^2+12z=36.$

2. The plane sections of the surface whose equation is $\frac{x^2}{4}-\frac{y^2}{9}-z=0$ determined by the following planes are familiar curves. Name these curves and sketch the surface.

   a) XY-plane b) YZ-plane
   c) the plane $y=3$ d) the plane $z=-1$
   e) the plane $x=4$.

3. In each case below, determine the numbers A, B, C and D such that the graph of the equation $Ax^2+By^2+Cz^2+D=0$ is a quadratic surface which contains the given points. Name these surfaces.

   a) $(1,0,0),(1/3,4/3,3),(1/2,2,0)$ b) $(1,-1,1),(2,1,0),(5,5,3)$
   c) $(1,-1,2),(0,0,1),(2,1,-1)$ d) $(1,-2,1),(0,1,0),(2,1,3)$.

4. *Symmetric partners* about a point or about a line in the plane were considered in Chapter 3. One can extend these definitions for points in 3-space in the following way. Let M be a point, I be a line and P be a plane in 3-space. Two points P and Q are said to be symmetric partners of each other *about the point* M if M is the midpoint of the segment $[PQ]$. P and Q are said to be *symmetric partners about the line* I if I is a perpendicular bisector of the segment $[PQ]$. P and Q are said to be *symmetric partners about the plane* P if P is perpendicular to the segment $[PQ]$ and bisects it. Show that

   a) $P(-x,-y,-z)$ and $Q(x,y,z)$ are symmetric partners about the origin.
   b) $P(-x,y,z)$ and $Q(x,y,z)$ are symmetric partners about YZ-plane.
   c) $P(x,-y,z)$ and $Q(x,y,z)$ are symmetric partners about XZ-plane.
   d) $P(x,y,z)$ and $Q(x,y,-z)$ are symmetric partners about XY-plane.

5. A set S of points in 3-space is said to be *symmetric about a point* M (or *a plane* P) if S contains the symmetric partner of each of its points about M (or, respectively, about P). Show that

   a) the quadratic surfaces defined by the canonical equations (8.3.1) to (8.3.4) are symmetric about XY-plane.
   b) all the quadratic surfaces listed in the table at the end of this section are symmetric about YZ-plane, except the cylinder $\frac{y^2}{4c}=x$.

6. Show that the plane $x-2=0$ intersects the ellipsoid
$$\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$$
in an ellipse; find the axes and vertices of this ellipse.

7. Show that the plane $z+1=0$ intersects the hyperboloid of one sheet
$$\frac{x^2}{32}-\frac{y^2}{18}+\frac{z^2}{2}=1$$
in a hyperbola; find the semi-axes and vertices of the hyperbola.

8. Show that the plane $y+6=0$ intersects the hyperbolic paraboloid

$$\frac{x^2}{5}-\frac{y^2}{4}=6z$$

9. Find the values of m for which the plane $x+mz-1=0$ intersects the hyperboloid of two sheets

$$x^2+y^2-z^2=-1$$

a) in an ellipse; b) in a hyperbola.

10. Find the values of m for which the plane $x+my-2=0$ intersects the elliptic paraboloid

$$\frac{x^2}{2}+\frac{z^2}{3}=y$$

a) in an ellipse; b) in a parabola.

11. Prove that the elliptic paraboloid

$$\frac{x^2}{9}+\frac{z^2}{4}=2y$$

has a point in common with the plane

$$2x-2y-z-10=0,$$

and find the coordinates of that point.

12. Prove that the hyperboloid of two sheets

$$\frac{x^2}{3}+\frac{y^2}{4}-\frac{z^2}{25}=-1$$

has a point in common with the plane

$$5x+2z+5=0,$$

and find the coordinates of that point.

13. Prove that the ellipsoid

$$\frac{x^2}{81}+\frac{y^2}{36}+\frac{z^2}{9}=1$$

has a point in common with the plane

$$4x-3y+12z-54=0,$$

and find the coordinates of that point.

14. Write the equation of the plane which is perpendicular to the vector $n=\{2,-1,-2\}$ and touches the elliptic paraboloid

$$\frac{x^2}{3}+\frac{y^2}{4}=2z.$$

15. In each of the following, find the points of intersection of the surface and the straight line:

a) $\frac{x^2}{81}+\frac{y^2}{36}+\frac{z^2}{9}=1$ and $\frac{x-3}{3}=\frac{y-4}{-6}=\frac{z+2}{4}$;
b) $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{4}=1$ and $\frac{x}{4}=\frac{y}{-3}=\frac{z+2}{4}$;
c) $\frac{x^2}{5}+\frac{y^2}{3}=z$ and $\frac{x+1}{2}=\frac{y-2}{-1}=\frac{z+3}{-2}$;
d) $\frac{x^2}{9}-\frac{y^2}{4}=z$ and $\frac{x}{3}=\frac{y-2}{-2}=\frac{z+1}{2}$.

16. Write the equation of the cone whose vertex is at the origin and whose directing curve is given by the equations:

a) $\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\\ z=c;\end{cases}$ b) $\begin{cases}\frac{x^2}{a^2}+\frac{z^2}{c^2}=1,\\ y=b;\end{cases}$ c) $\begin{cases}\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\\ x=a.\end{cases}$

17. Prove that the equation

$$z^2=xy$$

represents a cone with vertex at the origin.

18. Find the equation of the cone whose vertex is at the origin and whose directing curve is given by the equations

$$\begin{cases}x^2-2z+1=0,\\ y-z+1=0.\end{cases}$$

19. Find the equation of the cone whose vertex is at the point $(0,0,c)$ and whose directing curve is given by the equations

$$\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\\ z=0.\end{cases}$$

20. Find the equation of the cone whose vertex is at the point $(3,-1,-2)$ and whose directing curve is given by the equations

$$\begin{cases}x^2+y^2-z^2=1,\\ x-y+z=0.\end{cases}$$

21. A cylinder with elements perpendicular to the $x+y-2z-5=0$ plane is circumscribed about the sphere $x^2+y^2+z^2=1$. Find the equation of this cylinder.

8.4 Change of Coordinates in 3-Space

In this section, we derive the equations connecting the coordinates of a point in 3-space with reference to two different coordinate systems.

We first consider the case where the two coordinate systems are such that the corresponding coordinate axes are parallel as in Fig. 8.11.

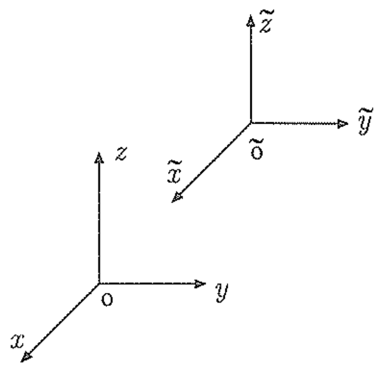


Fig. 8.11.

The origin $\tilde{O}$ of the $\tilde{X}\tilde{Y}\tilde{Z}$-coordinate system is the point having coordinates (x_0, y_0, z_0) with respect to the XYZ-system. More generally, the XYZ-coordinates (x, y, z) and the $\tilde{X}\tilde{Y}\tilde{Z}$-coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ of a point P in 3-space are related by

$$\begin{aligned} \tilde{x} = x - x_0; \quad & x = \tilde{x} + x_0 \\ \tilde{y} = y - y_0; \quad & y = \tilde{y} + y_0 \\ \tilde{z} = z - z_0; \quad & z = \tilde{z} + z_0 \end{aligned} \tag{8.4.1}$$

The $\tilde{X}\tilde{Y}\tilde{Z}$-system above is called a translation of the XYZ-system. If the expressions for x, y and z in the second set of equations (8.4.1) are substituted in an equation in x, y and z, one obtains the corresponding equation in $\tilde{x}, \tilde{y}$ and $\tilde{z}$; the two equations have the same graph.

Next we consider the case where the two coordinate systems are such that one is obtained from the other by a rotation. Namely, suppose that $\overline{XYZ}$-system is obtained by rotating the XYZ-system about a line which passes through the origin of the XYZ-system (see Fig. 8.12).

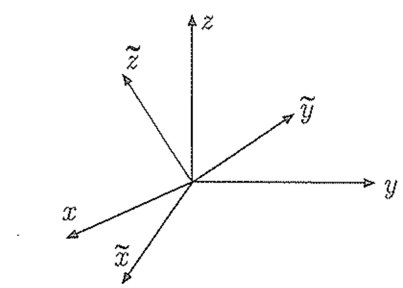


Fig. 8.12

Thus the origin of the $\overline{XYZ}$-system is the origin of the XYZ-system. Let $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ denote the basic unit vectors in XYZ-system and let $\bar{\bar{i}}, \bar{\bar{j}}$ and $\bar{\bar{k}}$ denote the basic unit vectors in $\overline{XYZ}$-system. Thus

$$\begin{aligned} \mathbf{i} &= (1,0,0),\ \mathbf{j} = (0,1,0),\ \mathbf{k} = (0,0,1) \text{ in } XYZ\text{-system}, \\ \bar{\bar{\mathbf{i}}} &= (1,0,0),\ \bar{\bar{\mathbf{j}}} = (0,1,0),\ \bar{\bar{\mathbf{k}}} = (0,0,1) \text{ in } \overline{XYZ}\text{-system}. \end{aligned}$$

The vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are mutually perpendicular unit vectors in 3-space, and so are the vectors $\bar{\bar{\mathbf{i}}}, \bar{\bar{\mathbf{j}}}$ and $\bar{\bar{\mathbf{k}}}$. Let

$$\bar{\bar{\mathbf{i}}} = (p_1, p_2, p_3),\ \bar{\bar{\mathbf{j}}} = (q_1, q_2, q_3),\ \bar{\bar{\mathbf{k}}} = (r_1, r_2, r_3)$$

in XYZ-system, and let

$$R = [\,{}^t\bar{\bar{\mathbf{i}}},\ {}^t\bar{\bar{\mathbf{j}}},\ {}^t\bar{\bar{\mathbf{k}}}\,] = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix}.$$

Since $\bar{\bar{\mathbf{i}}}, \bar{\bar{\mathbf{j}}}$ and $\bar{\bar{\mathbf{k}}}$ are mutually perpendicular unit vectors, R is an orthogonal matrix, i.e., ${}^tRR = I$ or ${}^tR = R^{-1}$.

If P is a point having XYZ-coordinates (x, y, z) and $\overline{XYZ}$-coordinates $(\overline{x}, \overline{y}, \overline{z})$, then

$$\begin{aligned} P &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \text{ in } XYZ - \text{coordinates}, \\ P &= \overline{x}\bar{\bar{\mathbf{i}}} + \overline{y}\bar{\bar{\mathbf{j}}} + \overline{z}\bar{\bar{\mathbf{k}}} \text{ in } \overline{XYZ} - \text{coordinates}. \end{aligned}$$

Thus

$$\begin{aligned} x\mathbf{i} + y\mathbf{j} + z\mathbf{k} &= \overline{x}\bar{\bar{i}} + \overline{y}\bar{\bar{j}} + \overline{z}\bar{\bar{k}} = (p_1\overline{x} + q_1\overline{y} + r_1\overline{z})\mathbf{i} \\ &\quad + (p_2\overline{x} + q_2\overline{y} + r_2\overline{z})\mathbf{j} + (p_3\overline{x} + q_3\overline{y} + r_3\overline{z})\mathbf{k}. \end{aligned}$$

This vector-equation yields the following system of linear equations

$$\left.\begin{aligned} x &= p_1\overline{x} + q_1\overline{y} + r_1\overline{z} \\ y &= p_2\overline{x} + q_2\overline{y} + r_2\overline{z} \\ z &= p_3\overline{x} + q_3\overline{y} + r_3\overline{z} \end{aligned}\right\} \tag{8.4.2}$$

where $\overline{x}, \overline{y}, \overline{z}$ are regarded as indeterminates. The coefficient matrix is the orthogonal matrix R defined above.

The system (8.4.2) can be written in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix} \quad \text{or} \quad P = R\overline{P}.$$

Since R is orthogonal, ${}^tR = R^{-1}$ and the above equation implies

$$\begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix} = {}^tR \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or} \quad \overline{P} = {}^tRP.$$

Here, P and $\overline{P}$ represent column vectors (or matrices) such that the components of P represent the coordinates of a point in XYZ-system while the components of $\overline{P}$ represent the coordinates of the same point in $\overline{XYZ}$-system.

We sum-up our results in

Theorem 8.4.1 *Let $\overline{XYZ}$-system be obtained by rotating the XYZ-system about a line through the origin. Let (x, y, z) and $(\overline{x}, \overline{y}, \overline{z})$ be the XYZ-coordinates and $\overline{XYZ}$-coordinates of a point in 3-space, respectively. Then there exists an orthogonal matrix R such that*

$$P = R\overline{P} \quad \textit{and} \quad \overline{P} = {}^tRP \tag{8.4.3}$$

where

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \textit{and} \quad \overline{P} = \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix}..$$

Remark. Let us notice that any orthogonal matrix R determines a change of coordinates. In fact, given XYZ-coordinates of a point, we define the $\overline{XYZ}$-coordinates by the equation (8.4.3). The column vectors of R are the basic unit vectors in $\overline{XYZ}$-system whose components are given with respect to XYZ-system.

Example 8.4.2 Consider the orthogonal matrix

$$R = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

If XYZ-coordinates of a point are (x, y, z), define $\overline{XYZ}$-coordinates by

$$\overline{P} = {}^tRP, \quad \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix} = {}^tR \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then

$$\overline{x} = \frac{1}{3}(x + 2y + 2z), \quad \overline{y} = \frac{1}{3}(2x + y - 2z), \quad \overline{z} = \frac{1}{3}(2x - 2y + z).$$

The plane whose equation is $x + 2y + 2z = 0$ in XYZ-system has equation $\overline{x} = 0$, namely, it is the $\overline{YZ}$-plane in $\overline{XYZ}$-system.

Remark. Rotation of coordinates in the plane can be described by 2×2 orthogonal matrices in view of the above discussions. Rotation through an angle α is determined by the orthogonal matrix

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Exercises

1. Find a suitable translation of the XYZ-coordinate system so that the equation corresponding to $x^2 + 2y^2 + 3z^2 - 4x - 16y + 12z + 20 = 0$ in the new system contains no terms of the first degree. Is this possible for the equation $xy + 4z - 1 = 0$?

2. Show that by a suitable rotation of the XYZ-system, the planes $x + 2y + 2z = 0$, $2x + y - 2z = 0$ and $2x - 2y + z = 0$ may be used as coordinate planes of a new coordinate system. Find the corresponding orthogonal matrix.

3. Show that a suitable rotation and then a translation of the XYZ-system, the three planes $x + 2y + 2z = 1$, $2x + y - 2z = 2$ and $2x - 2y + z = 3$ may be used as coordinate planes of a new coordinate system.

4. Show that each of the following matrices is an orthogonal matrix, and determine the corresponding change of coordinates.

a) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

8.5 The General Quadratic Equation

In section 3 of this chapter we have seen that canonical equation of any quadratic surface is a quadratic equation in x, y and z. Now, in this section, we shall show that the graph of any quadratic equation in x, y and z is a quadratic surface or a degenerate form of a quadratic surface. We prove this by showing that the general quadratic equation (8.0.1) can be reduced to one of the canonical equations (8.3.1) to (8.3.9) or a degenerate form of these by suitable change of coordinates. Let us write down here the general quadratic equation (8.0.1) once more.

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + 2Gx + 2Hy + 2Kz + L = 0$$

(A, B, C, D, E, F are not all zero).

Let us define

$$A = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix} \quad K = \begin{bmatrix} G \\ H \\ K \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then the general quadratic equation can be written in matrix form as

$$^tXAX + 2\,{}^tKX + L = 0. \tag{8.5.1}$$

Here the matrix A is a non-zero real symmetric matrix. It is called the *matrix of the equation* (8.0.1). Thus by section 7.4, A has real characteristic values, say $\lambda_1, \lambda_2, \lambda_3$, at least one of the characteristic values is different from zero; and there exists an orthogonal matrix R such that

$$^tRAR = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Consider the change of coordinates

$$X = R\overline{X}, \qquad \overline{X} = {}^tRX$$

determined by the orthogonal matrix R. Then the equation (8.5.3) transforms to the equation

$$\begin{aligned} {}^t(R\overline{X})A(R\overline{X}) + 2\,{}^tK(R\overline{X}) + L &= 0 \\ {}^t\overline{X}(\,{}^tRAR)\overline{X} + 2(\,{}^tKR)\overline{X} + L &= 0. \end{aligned}$$

Written explicitely,

$$\lambda_1\overline{x}^2 + \lambda_2\overline{y}^2 + \lambda_3\overline{z}^2 + 2\overline{G}\overline{x} + 2\overline{H}\overline{y} + 2\overline{K}\overline{z} + L = 0, \tag{8.5.2}$$

where

$$\overline{G} = {}^tKu_1, \quad \overline{H} = {}^tKu_2, \quad \overline{K} = {}^tKu_3$$

u_1, u_2, u_3 being the first, second and the third columns of R, respectively.

In equation (8.5.2), we consider the three cases, when $\lambda_1, \lambda_2, \lambda_3$ are different from zero, when one of $\lambda_1, \lambda_2, \lambda_3$ is equal to zero, and when two of them are equal to zero. Moreover, we can order the characteristic values in such a way that in the second case $\lambda_3 = 0$ and in the third case $\lambda_2 = \lambda_3 = 0$. (No other possibility arises because A is a non-zero symmetric matrix).

Case 1. $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$. We translate the $\overline{XYZ}$-system to $\overline{\overline{XYZ}}$-system by

$$\overline{\overline{x}} = \overline{x} + \frac{\overline{G}}{\lambda_1}, \quad \overline{\overline{y}} = \overline{y} + \frac{\overline{H}}{\lambda_2}, \quad \overline{\overline{z}} = \overline{z} + \frac{\overline{K}}{\lambda_3}.$$

Then the equation (8.5.1) reduces to

$$\lambda_1\overline{\overline{x}}^2 + \lambda_2\overline{\overline{y}}^2 + \lambda_3\overline{\overline{z}}^2 = k \tag{8.5.3}$$

where

$$k = \frac{\overline{G}^2}{\lambda_1} + \frac{\overline{H}^2}{\lambda_2} + \frac{\overline{K}^2}{\lambda_3} - L.$$

If $k \neq 0$ and $\lambda_1, \lambda_2, \lambda_3$ have the same sign, the graph of (8.5.3) is either an ellipsoid or it contains no point. The latter is called an imaginary (degenerate) ellipsoid. If $k = 0$ and $\lambda_1, \lambda_2, \lambda_3$ have the same sign then the graph reduces to a point; and it is considered as a degenerate form of an ellipsoid.

If $k \neq 0$ and two of the numbers λ_1/k, λ_2/k, λ_3/k are positive and one is negative, then the equation (8.5.3) is either of the form (8.3.3) or one of the forms

$$-\frac{\overline{\overline{x}}^2}{a^2} + \frac{\overline{\overline{y}}^2}{b^2} + \frac{\overline{\overline{z}}^2}{c^2} = 1, \quad \frac{\overline{\overline{x}}^2}{a^2} - \frac{\overline{\overline{y}}^2}{b^2} + \frac{\overline{\overline{z}}^2}{c^2} = 1$$

and consequently its graph is an hyperboloid of one sheet (see Exercise 3). If $k \neq 0$ and one of the numbers $\lambda_1/k, \lambda_2/k, \lambda_3/k$ is positive and the other two are negative, then the equation (8.5.3) is either of the form (8.3.4) or one of the forms

$$-\frac{\overline{\overline{x}}^2}{a^2} - \frac{\overline{\overline{y}}^2}{b^2} + \frac{\overline{\overline{z}}^2}{c^2} = 1, \quad \frac{\overline{\overline{x}}^2}{a^2} + \frac{\overline{\overline{y}}^2}{b^2} - \frac{\overline{\overline{z}}^2}{c^2} = 1$$

and consequently the surface is an hyperboloid of two sheets.

If $k = 0$ and one or two of $\lambda_1, \lambda_2, \lambda_3$ are positive and the others negative, then the surface is an elliptic cone.

Case 2. $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$. We translate the XYZ-system to $\overline{\overline{XYZ}}$-system by

$$\overline{\overline{x}} = \overline{x} + \frac{\overline{G}}{\lambda_1}, \quad \overline{\overline{y}} = \overline{y} + \frac{\overline{H}}{\lambda_2}, \quad \overline{\overline{z}} = \overline{z}.$$

Then the equation (8.5.1) reduces to

$$\lambda_1\overline{\overline{x}}^2 + \lambda_2\overline{\overline{y}}^2 + 2\overline{K}\overline{\overline{z}} + h = 0 \tag{8.5.4}$$

where

$$h = L - \frac{\overline{G}^2}{\lambda_1} - \frac{\overline{H}^2}{\lambda_2}.$$

If $\overline{K} \neq 0$, we translate coordinates once more by

$$\tilde{x} = \overline{\overline{x}}, \quad \tilde{y} = \overline{\overline{y}}, \quad \tilde{z} = \overline{\overline{z}} + \frac{h}{2\overline{K}}.$$

Then the equation (8.5.3) reduces to

$$\lambda_1\tilde{x}^2 + \lambda_2\tilde{y}^2 + 2\overline{K}\tilde{z} = 0$$

which is one of the form (8.3.5) or (8.3.6) according as λ_1 and λ_2 have the same or opposite signs. Consequently, the surface is a paraboloid.

If $\overline{K} = 0$ in (8.5.4), then we have

$$\lambda_1\overline{\overline{x}}^2 + \lambda_2\overline{\overline{y}}^2 + h = 0.$$

In this case, the surface is an elliptic or hyperbolic cylinder if $h \neq 0$; it consists of two intersecting planes if $h = 0$ and λ_1, λ_2 have opposite signs; and it consists of the $\overline{\overline{Z}}$-axis if $h = 0$ and λ_1, λ_2 have the same sign.

Case 3. $\lambda_1 \neq 0, \lambda_2 = \lambda_3 = 0$. We then translate the $\overline{XYZ}$-system to $\overline{\overline{XYZ}}$-system by

$$\overline{\overline{x}} = \overline{x} + \frac{\overline{G}}{\lambda_1}, \quad \overline{\overline{y}} = \overline{y}, \quad \overline{\overline{z}} = \overline{z}.$$

Then the equation (8.5.2) reduces to

$$\lambda_1\overline{\overline{x}}^2 + 2\overline{H}y + 2\overline{K}\overline{\overline{z}} + d = 0 \tag{8.5.5}$$

where

$$d = L - \frac{\overline{G}^2}{\lambda_1}.$$

If $\overline{H} = \overline{K} = 0$, the surface is degenerate. It consists of two parallel planes if $d/\lambda_1 < 0$, it has no point if $d/\lambda_1 > 0$, and it is a plane if $d = 0$.

If $\overline{H} \neq 0$ or $\overline{K} \neq 0$, we change coordinates by the orthogonal matrix

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \overline{H}/\sqrt{\overline{H}^2 + \overline{K}^2} & \overline{K}/\sqrt{\overline{H}^2 + \overline{K}^2} \\ 0 & \overline{K}/\sqrt{\overline{H}^2 + \overline{K}^2} & \overline{H}/\sqrt{\overline{H}^2 + \overline{K}^2} \end{bmatrix}.$$

We obtain $\tilde{X}\tilde{Y}\tilde{Z}$-coordinate system by

$$\tilde{X} = {}^tT\overline{\overline{X}} \quad \text{or} \quad \overline{\overline{X}} = T\tilde{X}.$$

Then (8.5.5) reduces to

$$\lambda_1\tilde{x}^2 + 2\overline{H}\left(\frac{\overline{H}\tilde{y} + \overline{K}\tilde{z}}{\sqrt{\overline{H}^2 + \overline{K}^2}}\right) + 2\overline{K}\left(\frac{\overline{K}\tilde{y} - \overline{H}\tilde{z}}{\sqrt{\overline{H}^2 + \overline{K}^2}}\right) + d = 0$$

$$\lambda_1\tilde{x}^2 + 2\sqrt{\overline{H}^2 + \overline{K}^2}\,\tilde{y} + d = 0.$$

Now, translate the $\tilde{X}\tilde{Y}\tilde{Z}$-system to $\tilde{\tilde{X}}\tilde{\tilde{Y}}\tilde{\tilde{Z}}$-system by

$$\tilde{\tilde{x}} = \tilde{x}, \quad \tilde{\tilde{y}} = \tilde{y} + d/\sqrt{\overline{H}^2 + \overline{K}^2}, \quad \tilde{\tilde{z}} = \tilde{z}.$$

Then (8.5.5) reduces to

$$\lambda_1\tilde{\tilde{x}}^2 + 2\sqrt{\overline{H}^2 + \overline{K}^2}\,\tilde{\tilde{y}} = 0.$$

Consequently the graph is a parabolic cylinder.

Hence we can state

Theorem 8.5.1 *The graph of the general quadratic equation (8.0.1) is a quadratic surface or a degenerate form of a quadratic surface.*

Example 8.5.2 Find a canonical equation of the surface $2x^2 + 2y^2 - 4z^2 - 10xy - 2xz - 2yz + 2x + 2y + 4z + 1 = 0$ in a suitable coordinate system. Identify the quadratic.

Solution. The matrix A corresponding to the given equation is

$$A = \begin{bmatrix} 2 & -5 & -1 \\ -5 & 2 & -1 \\ -1 & -1 & -4 \end{bmatrix}$$

which is the matrix of Example 7.3.1 in Chapter 7. Its characteristic values are $\lambda_1 = 7, \lambda_2 = -2, \lambda_3 = -5$ and the corresponding mutually perpendicular characteristic unit vectors are

$$u_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \quad u_3 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}.$$

Hence if we change coordinates by the orthogonal matrix

$$R = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}, \quad X = R\overline{X}$$

then the given equation reduces to

$$7\overline{x}^2 - 2\overline{y}^2 - 5\overline{z}^2 + 2\sqrt{6}\overline{z} + 1 = 0.$$

Now perform the translation

$$\overline{\overline{x}} = \overline{x}, \quad \overline{\overline{y}} = \overline{y}, \quad \overline{\overline{z}} = \overline{z} - \frac{\sqrt{6}}{5}.$$

We then obtain

$$7\overline{\overline{x}}^2 - 2\overline{\overline{y}}^2 - 5\overline{\overline{z}}^2 - \frac{6}{5} + 1 = 0$$

$$35\overline{\overline{x}}^2 - 10\overline{\overline{y}}^2 - 25\overline{\overline{z}}^2 = 1.$$

This is the canonical equation of an hyperboloid of two sheets in $\overline{\overline{XYZ}}$-coordinates.

Example 8.5.3 Discuss the graph of

$$x^2 + 4y^2 + 4z^2 + 4xy - 4xz - 8yz + 2x + 4y - 4z + 1 = 0.$$

Solution. The matrix of this quadratic equation is

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}.$$

The characteristic equation of A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & -2 & 2 \\ -2 & \lambda - 4 & 4 \\ 2 & 4 & \lambda - 4 \end{vmatrix} = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ -2 & \lambda - 4 & 4 \\ 0 & \lambda & \lambda \end{vmatrix} = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ -2\lambda & \lambda & 0 \\ 0 & \lambda & \lambda \end{vmatrix} \\ &= \lambda^2 \begin{vmatrix} \lambda - 1 & -2 & 2 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \lambda^2 \begin{vmatrix} \lambda - 1 & 0 & 4 \\ -2 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \lambda^2(\lambda - 9) = 0. \end{aligned}$$

Hence $\lambda_1 = 9, \lambda_2 = \lambda_3 = 0$. We compute the corresponding mutually orthogonal characteristic vectors.

$$[9I - A] = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 5 \\ 0 & 9 & 9 \\ 0 & -18 & -18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; u_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}.$$

$$[oI - A] = \begin{bmatrix} -1 & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; u_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \; u_3 = \begin{pmatrix} 4/3\sqrt{2} \\ -1/3\sqrt{2} \\ 1/3\sqrt{2} \end{pmatrix}.$$

We change coordinates by the orthogonal matrix

$$R = \begin{bmatrix} 1/3 & 0 & 4/3\sqrt{2} \\ 2/3 & 1/\sqrt{2} & -1/3\sqrt{2} \\ -2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \end{bmatrix} \quad X = R\overline{X}.$$

Then the given equation reduces to

$$\begin{aligned} 9\overline{x}^2 + 6\overline{x} + 1 &= 0 \\ 9(\overline{x}^2 + \frac{2}{3}\overline{x} + \frac{1}{9}) &= 0 \\ (\overline{x} + \frac{1}{3})^2 &= 0. \end{aligned}$$

Thus the graph is a degenerate quadratic surface which consists of a plane. The equation of the plane is $\overline{x} + \frac{1}{3} = 0$ in $\overline{XYZ}$-coordinates, or $x + 2y - 2z + 1 = 0$ in XYZ-coordinates.

Remark. The method of this section can also be used to identify the graph of a quadratic equation in two variables x and y. In fact, one can regard such an equation as a quadratic equation in x, y, z where coefficients of the terms involving the variable z are all zero. Then the matrix of such an equation of the form

$$A = \begin{bmatrix} A & B & 0 \\ B & C & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore in discussing the graph of an equation of the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

where at least one of A, B, C is different from zero, we consider the symmetric matrix

$$A = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

and its characteristic values λ_1, λ_2. We find mutually perpendicular unit vectors u_1 and u_2 which are characteristic vectors of A belonging to λ_1 and λ_2, respectively. Then we change coordinates in XY-plane by the orthogonal matrix R whose columns are u_1 and u_2:

$$R = [u_1, u_2], \quad X = R\overline{X}.$$

Then the xy-term of the quadratic equation is eliminated and the equation reduces to an equation of the form

$$\lambda_1 \overline{x}^2 + \lambda_2 \overline{y}^2 + 2\overline{D}\overline{x} + 2\overline{E}\overline{y} + F = 0.$$

We decide about the graph by performing a suitable translation of coordinate axes as in section 4.7.

Example 8.5.4 Discuss the graph of

$$2x^2 + 4xy + 5y^2 - 3x + 3y = 0.$$

Solution. The matrix of the equation is

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

and its characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 5 \end{vmatrix} = \lambda^2 - 7\lambda + 6 = 0.$$

Hence $\lambda_1 = 6$ and $\lambda_2 = 1$ are the characteristic values. The corresponding mutually perpendicular unit characteristic vectors are found as follows:

$$\begin{aligned} I - A &= \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \\ 6I - A &= \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1/\sqrt{2} \\ 0 & 0 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}. \end{aligned}$$

We change coordinates by

$$R = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}, \quad X = R\overline{X} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\overline{x} - 2\overline{y}}{\sqrt{5}} \\ \frac{2\overline{x} + \overline{y}}{\sqrt{5}} \end{bmatrix}.$$

This is just a rotation of coordinates through α with $cot 2\alpha = \frac{-3}{4}$, $0 < \alpha < \frac{\pi}{2}$. The equation reduces to

$$6\overline{x}^2 + \overline{y}^2 - \frac{3}{\sqrt{5}}\overline{x} + \frac{9}{\sqrt{5}}\overline{y} = 0$$

$$6\left(\overline{x} - \frac{1}{4\sqrt{5}}\right)^2 + \left(\overline{y} + \frac{9}{2\sqrt{5}}\right)^2 - \frac{3}{40} - \frac{9}{20} = 0.$$

Thus, translating coordinates by

$$\overline{\overline{x}} = \overline{x} - \frac{1}{4\sqrt{5}}, \quad \overline{\overline{y}} = \overline{y} + \frac{9}{2\sqrt{5}}$$

we get

$$\begin{aligned} 6\overline{\overline{x}}^2 + \overline{\overline{y}}^2 - \frac{21}{40} &= 0 \\ \frac{120}{7}\overline{\overline{x}}^2 + \frac{40}{21}\overline{\overline{y}}^2 &= 1, \end{aligned}$$

which is the equation of an ellipse.

Exercises

1. In each case below, reduce the given quadratic equation to a canonical equation of a quadratic surface by performing suitable change of coordinates

   a) $2x^2 + 2y^2 - 4z^2 - 10xy - 2xz - 2yz + 16x - 8y + 20z - 10 = 0$

   b) $x^2 + y^2 - 5z^2 - 10xy - 2xz - 2yz + 16x - 8y + 20z - 10 = 0$

   c) $2x^2 + y^2 - 4xy + 4yz - 6x + 8y - 12z + 9 = 0$

d) $x^2 + y^2 + z^2 + 4xy - 4xz + 4yz - 2x + y - 3z = 0$

e) $x^2 - 2y^2 - 2z^2 + 4xy + 4xz + yz - 2x + y - 3z - 1 = 0$.

2. Show that the graph of

$$x^2 + 4y^2 + 9z^2 + 4xy - 6xz - 12yz - 2x - 4y + 6z + 1 = 0$$

is a degenerate quadric which consists of a plane.

3. Solve the exercises of section 4.7 by using matrices.

4. Discuss the graph of each of the following equations

a) $2x^2 + y^2 + 2z^2 + 2xy - 4x - 2y + 4z = 0$

b) $x^2 + y^2 + 4z^2 - 2xy + 4xz - 4yz + 4x - 8z + 7 = 0$

c) $xy + 3xz + 14x + 2 = 0$

d) $x^2 + y^2 + z^2 + xy + xz + yz = 0$.

e) $y^2 - 4z^2 + 2yz + 6y + 4z + 1 = 0$.

Appendix A

REAL AND COMPLEX NUMBERS

We shall assume that the reader is intuitively familiar with real numbers and some of its uses. Rather than entering into the theoretical background for the construction of number systems, we shall give here a detailed review of the formal properties of these systems.

A.1 The Real Numbers

We denote the set of real numbers by $\mathbb{R}$. This set can be characterized as the set which satisfies the following three axioms: Axioms $\underline{F}, \underline{O}$ and $\underline{C}$.

Axiom F. (Field Axiom). $\mathbb{R}$ has two binary operations, [1] binary operations addition (+) and multiplication (.) such that

F.1. $(a + b) + c = a + (b + c); (a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c \in \mathbb{R}$.

F.2. There exists uniquely determined elements 0 and 1 in $\mathbb{R}$ such that

$$a + 0 = 0 + a = a; \quad a \cdot 1 = 1 \cdot a = a$$

for any $a \in \mathbb{R}$.

[1]By a binary operation $*$ in a set E we mean a function $* : E \times E \to E$, $(x, y) \to x * y$. For example, $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \to x + y$.

F.3. For any $a \in \mathbb{R}$ there exists $(-a) \in \mathbb{R}$ such that

$$a + (-a) = (-a) + a = 0;$$

and for any $a \in \mathbb{R} \setminus \{0\}$ there exists $a^{-1} \in \mathbb{R}$ such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

F.4. $a + b = b + a$; $a \cdot b = b \cdot a$ for any $a, b \in \mathbb{R}$.

F.5. $a \cdot (b + c) = a \cdot b + a \cdot c$ for any $a, b, c \in \mathbb{R}$.

The axiom F.2. asserts the existence of certain real numbers. These are the *additive identity* zero, O, and the *multiplicative identity* one, 1.

Instead of $a \cdot b$ we also write ab, instead of $a + (-b)$ we write $a - b$, and instead of a^{-1} we write $1/a$.

Axiom O. (Order Axiom) There is a relation $<$ in $\mathbb{R}$ such that

O.1. For any pair of elements $a, b \in \mathbb{R}$, one and only one of the following is true:

$$a < b \quad \text{or} \quad a = b \quad \text{or} \quad b < a.$$

O.2. $<$ is transitive, i.e., $a < b$ and $b < c$ imply $a < c$.

O.3. If $a, b, c \in \mathbb{R}$ and $a < b$, then $a + c < b + c$.

O.4. If $a, b, c \in \mathbb{R}$, $a < b$ and $0 < c$, then $ac < bc$.

If $a < b$, then we say that a *is less than* b or b *is grater than* a. If $0 < a$, we say that a is *positive*; if $a < 0$, we say that a is *negative.* It is clear that a real number is either positive or negative or zero.

All the familiar properties of real numbers involving order can now be proved from the above set of axioms. We prove the following:

Theorem. *Let $a, b, c, d \in \mathbb{R}$. Then*

i) $a < b \Leftrightarrow a - b < 0$.

ii) $a < b,\ c < d \Rightarrow a + c < b + d$.

Proof. (i) Assume that $a < b$. By axiom O.3, $a + (-b) < b + (-b)$. By axiom F.3, $a - b < 0$. Conversely, assume that $a - b < 0$. By axiom O.3, $(a - b) + b < 0 + b$. Using axioms F.1, F.2, and F.3, we get $a < b$.

(ii) Assume that $a < b$ and $c < d$. By axiom O.3, we can write $a + c < b + c$ and $c + b < d + b$. By axiom F.4, $c + b = b + c$ and $d + b = b + d$. Thus we have $a + c < b + c$ and $b + c < b + d$. Now by axiom O.2, $a + c < b + d$. This completes the proof. □

In order to give the last axiom which characterizes the set $\mathbb{R}$ of real numbers, we need the concepts of *supremum* and infimum for a subset of $\mathbb{R}$.

Let $a, b \in \mathbb{R}$. We write $a \leq b$ (read as a is less than or equal to b) if $a < b$ or $a = b$. Let S be a subset of $\mathbb{R}$. We say that a real number u is an *upper bound* for S if $x \leq u$ for all $x \in S$. Similarly, $\ell \in \mathbb{R}$ is called a *lower bound* for S if $x \geq \ell$ for all $x \in S$. If there is an upper bound (a lower bound) for S then we say that S is bounded above (bounded below) An upper bound b of S is called the *supremum* (or the *least upper bound*) of S if no upper bound of S is less than b. Thus b is the supremum of S if

i) b is an upper bound for S

ii) for any upper bound u of S we have $b \leq u$.

Similarly, a lower bound a of S is called the *infimum* (or the *greatest lower bound*) of S if no lower bound of S is larger than a. Note that S may or may not have a supremum (infimum) but if it has one then it is unique (See Exercise 4). Note also that the supremum of S, if it exists, need not be an element of S.

Axiom C. (Completeness Axiom). Every non-empty subset S of $\mathbb{R}$ which has an upper bound has a supremum.

Among the real numbers, the natural numbers $1, 1+1 = 2, 2+1 = 3, \ldots$ are the numbers used to denote the *size* of finite sets. A set is "finite" if it can be exhausted by throwing out its elements one at a time, labeling them 1, 2, 3, etc., and the last label used denotes the size (or the number of elements) of the set. The empty set ϕ has no elements at all. The real number 0 is the *number* of elements of the empty set.

We denote the set of natural numbers by $\mathbb{N}$: $\mathbb{N} = \{1, 2, 3, \ldots\}$. The *integers* are obtained by adjoining 0 and the negatives, $-n$, of natural numbers, n, to the set $\mathbb{N}$. We denote the set of integers by $\mathbb{Z} : \{0, \pm 1, \pm 2, \ldots\}$. The rational numbers are the fractions $\frac{m}{n}$, where m and n are integers and $n \neq 0$. Recall that, two fractions $\frac{m}{n}$ and

$\frac{r}{s}$ represent the same rational number, *i.e.*, $\frac{m}{n} = \frac{r}{s}$, if and only if $ms = nr$. We denote the set of rational numbers by $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\right\}$. The set $\mathbb{Z}$ of integers is regarded as a subset of $\mathbb{Q}$ by identifying each integer m with the fraction $\frac{m}{1}$.

One can easily observe that the rational numbers satisfy the Axiom F and Axiom O, but not the Axiom C. A well-known example is that the set $S = \{a : a \in \mathbb{Q},\ a \cdot a < 2\}$ of rational numbers has upper bounds in $\mathbb{Q}$ but its supremum, $\sqrt{2}$, is not rational. Thus, roughly speaking, the system of real numbers can be obtained by adjoining the missing suprema to the set of rational numbers. Actually one can prove that every real number is the supremum of a non-empty set of rational numbers.

The real number system is extended by adjoining the symbols *plus infinity*, $+\infty$ and the *minus infinity*, $-\infty$. These symbols satisfy the following properties:

(i) For any $a \in \mathbb{R}$, we have $-\infty < a < +\infty$, $a + \infty = +\infty$, $a + (-\infty) = -\infty$, and $\frac{a}{-\infty} = \frac{a}{\infty} = 0$.

(ii) For any positive real number a, we have $a \cdot (\infty) = \infty$, $a \cdot (-\infty) = -\infty$.

(iii) For any negative real number a, we have $a \cdot (+\infty) = -\infty$, $a \cdot (-\infty) = +\infty$.

(iv) $(+\infty) + (+\infty) = +\infty$, $(-\infty) + (-\infty) = -\infty$, $(+\infty) \cdot (+\infty) = +\infty$, $(-\infty) \cdot (+\infty) = -\infty$, $(-\infty) \cdot (-\infty) = +\infty$.

Remark. As for numbers, we write ∞ instead of $+\infty$. Let us also notice that the expressions

$$\frac{\infty}{\infty},\quad 0 \cdot \infty,\quad \infty - \infty$$

are undefined.

Remark. Let S be a non-empty subset of $\mathbb{R}$. If S does not have any upper bound then we define ∞ to be the supremum of S. Similarly, if S does not have any lower bound then we define $-\infty$ to be the infimum of S. Thus, in the extended real number system, $\mathbb{R} \cup \{+\infty, -\infty\}$, every non-empty subset has a supremum and an infimum. This is the main reason for extending the real number system.

We close this section with a theorem.

Theorem. *Given $x, y \in \mathbb{R}$ with $0 < x < y$. We have*

i) We can find an integer $n \in \mathbb{Z}$ such that $y < n$.

ii) We can find a rational number $r \in \mathbb{Q}$ such that $0 < r < x$.

iii) We can find a real number z such that $x < z < y$.

Proof. (i) Let $y \in \mathbb{R}$. Suppose that there is no integer n satisfying $y < n$. Then, the real number y is an upper bound for $\mathbb{Z}$, the set of integers. By the completeness axiom, then, $\mathbb{Z}$ has a supremum, say y_0. By definition $y_0 \leq y$, and there exists at least one integer k such that $y_0 - 1 < k$. But then by axiom 0.3, $y_0 < k + 1 \in \mathbb{Z}$. This contradiction shows that there is an integer n satisfying $y < n$.

(ii) Let $x \in \mathbb{R}$, $0 < x$. Then, we also have $0 < \frac{1}{x}$. By (i), there exists $n \in \mathbb{Z}$ with $\frac{1}{x} < n$. By axiom 0.4, we get $0 < \frac{1}{n} < x$, $\frac{1}{n} \in \mathbb{Q}$.

(iii) Take $z = \frac{x+y}{2}$. □

Let a and b be real numbers the larger of these numbers is called the *maximum* of a and b, and it is denoted by $\max\{a, b\}$. Similarly, *minimum* of a and b, denoted $\min\{a, b\}$, is the smaller of a and b. Thus, if $a \leq b$, then $\max\{a, b\} = b$ and $\min\{a, b\} = a$.

For any real number a, we define the absolute value, $|a|$, of a by

$$|a| = \max\{a, -a\}.$$

Hence

$$|a| = \begin{cases} a & \text{if} \quad a > 0 \\ 0 & \text{if} \quad a = 0 \\ -a & \text{if} \quad a < 0. \end{cases}$$

For example,

$$\begin{aligned} |2| &= \max\{2, -2\} = 2 \\ |-2| &= \max\{-2, 2\} = 2 \end{aligned}$$

The absolute value can be considered as a function $|\ | : \mathbb{R} \to \mathbb{R}$, $a \to |a|$. The image of this function is the set of all non-negative real numbers.

We list some properties of the absolute value:

(1) $|a| \geq 0$ for any $a \in \mathbb{R}$

(2) $|a| = 0 \Leftrightarrow a = 0$

(3) $|a \cdot b| = |a|\ |b|$

(4) $|a - b| \leq |a| + |b|$ (Triangle inequality).

All these properties can be proved by using the definition of absolute value (See Exercises 6, 7, 8).

Exercises

1. Using the order axioms 0.1, 0.2, 0.3, and 0.4, prove the following

   a) if $a < b$ and $c < 0$ then $ac > bc$

   b) if $a > 0$ then $\frac{1}{a} > 0$

   c) if $a > b$ and $c > d > 0$ then $ac > bd$.

   d) if $a > b > 0$ then $a^2 > b^2$

   e) if $a < b < 0$ then $a^2 > b^2$

   f) if $a > 0$ and $b > 0$ and $a^2 > b^2$ then $a > b$.

2. Give an example of

   a) a set of real numbers which is bounded above and below

   b) a set of real numbers which is bounded above but contains none of its upper bounds.

   c) a set of real numbers which is bounded above and contains one of its upper bounds.

3. Determine the supremum and infimum of the following sets of real numbers

   a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ b) $\{x : x \in \mathbb{N} \text{ and } |x - 5| < 5\}$

   c) $\{x : x \in \mathbb{Z} \text{ and } |x - 5| < 5\}$ d) $\{x : x \in \mathbb{R} \text{ and } |x - 5| < 5\}$

   e) $\mathbb{N}$ f) $\mathbb{Z}$.

4. Prove that if a set of real numbers S has a supremum then it has only one.

5. Prove that every real number is the supremum of a set of rational numbers.

6. Prove the four properties of the absolute value given in the text.

A.2 The Principle of Mathematical Induction

The principle of mathematical induction is one of the most powerful techniques of proof in arithmetic. It can be stated as follows.

The Principle of Mathematical Induction. If a statement concerning natural numbers n is known to be true for $n = 1$, and if the assumed truth of this statement for any $n = k$ implies its truth for $n = k+1$, then the statement is true for all natural numbers n.

Remark. Let there be given a statement concerning natural numbers and let M be the set of all natural numbers n for which this statement is true. Then the principle of mathematical induction can be formulated as follows. If $1 \in M$, and if $k+1 \in M$ for any $k \in M$, then $M = \mathbb{N}$.

Example A.2.1 Prove that

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$$

for any natural number n.

Solution. The left hand side of the given identity reduces to $1^{1-1} = 1^0 = 1$ for $n = 1$. This shows that the statement is true for $n = 1$. Assuming that it is true for $n = k$, i.e.,

$$1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1,$$

we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k &= (1 + 2 + 2^2 + \cdots + 2^{k-1}) + 2^k \\ &= (2^k - 1) + 2^k = 2^{k+1} - 1. \end{aligned}$$

Hence the assumed truth of the property for $n = k$ implies its truth for $n = k + 1$. Therefore it is true for any natural number n.

Remark. One can deduce, from the principle of induction, the following. *If a statement concerning natural numbers is true for $n = s$ and if the assumed truth of the statement for any $n = k$, $k \geq s$ implies its truth for $n = k+1$, then the statement is true for all natural numbers $n \geq s$.*

Example A.2.2 Assume that $x_1, x_2, \ldots, x_p; y_1, y_2, \ldots, y_p$ are real numbers for any natural number $p \geq 2$. Prove that

$$\sum_{j=2}^{p}\sum_{k=1}^{j-1} x_j y_k = \sum_{k=1}^{p-1}\sum_{j=k+1}^{p} x_j y_k$$

for any $p \geq 2$.

Solution. For $p = 2$, the given identity reduces to $x_2 y_1 = x_2 y_1$. Assume that the statement is true for $p = r$, i.e.,

$$\sum_{j=2}^{r}\sum_{k=1}^{j-1} x_j y_k = \sum_{k=1}^{r-1}\sum_{j=k+1}^{r} x_j y_k.$$

Then, for $p = r + 1$,

$$\begin{aligned}
\sum_{j=2}^{r+1}\sum_{k=1}^{j-1} x_j y_k &= \sum_{k=1}^{r} x_{r+1} y_k + \sum_{j=1}^{r}\sum_{k=1}^{j-1} x_j y_k \\
&= \sum_{k=1}^{r} x_{r+1} y_k + \sum_{k=1}^{r-1}\sum_{j=k+1}^{r} x_j y_k \\
&= x_{r+1} y_r + \sum_{k=1}^{r-1}\left(x_{r+1} y_k + \sum_{j=k+1}^{r} x_j y_k\right) \\
&= x_{r+1} y_r + \sum_{k=1}^{r-1}\sum_{j=k+1}^{r+1} x_j y_k \\
&= \sum_{k=r}^{r}\sum_{j=k+1}^{r+1} x_j y_k + \sum_{k=1}^{r-1}\sum_{j=k+1}^{r+1} x_j y_k = \sum_{k=1}^{r}\sum_{j=k+1}^{r+1} x_j y_k.
\end{aligned}$$

proving that the identity is true for all $p \geq 2$.

Exercises

1. Prove that each of the following properties is true for any natural number n.

   a) $\displaystyle\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ b) $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

   c) $\displaystyle\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2$ d) $1^n = 1$.

2. Let $x_1, x_2, \ldots, x_n, \ldots$ denote real numbers. Prove

   a) $x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$ for any $n \geq 1$.

   b) $x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \Leftrightarrow x_1 = x_2 = \cdots = x_n = 0$.

3. Prove that $n(n+1)(n+2)$ is divisible by 6 for any natural number n.

A.3 The Complex Numbers

The study of the solution of algebraic equations such as

$$x^2 + 1 = 0 \quad x^2 + 2x + 3 = 0$$

leads one to introduce complex numbers of the form $a + bi$, where a and b are real numbers and the symbol i represents a new quantity called the *imaginary unit.*

Thus, the set $\mathbb{C}$ of complex numbers is defined as $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$. If $z = a + bi \in \mathbb{C}$, then a is called the *real part* and b is called the *imaginary* part of the complex number z.

For two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, we define

$$z_1 = z_2 \Leftrightarrow a_1 = a_2 \text{ and } b_1 = b_2,$$

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i,$$

$$z_1\, z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i.$$

One can easily observe that addition and multiplication, defined as above, are binary operations in $\mathbb{C}$ and the field axiom (*Axiom F*) of real numbers is also satisfied by complex numbers. Moreover, by writing

$$a = a + 0 \cdot i$$

for any real number $a \in \mathbb{R}$, we may consider any real number as a complex number. Then addition or multiplication of real numbers when considered as complex numbers coincide with the usual ones.

The imaginary unit $i = 0 + 1 \cdot i$ has the property that $i^2 = -1$. If $z = a + bi$ is a non-zero complex number then its multiplicative inverse is given by

$$\frac{1}{z} = z^{-1} = \frac{a - bi}{a^2 + b^2}.$$

Example A.3.1 Find the real and imaginary parts of the complex number

$$z = (2 + 3i) + \frac{1 + 2i}{2 - 2i} + 1 - 2i.$$

Solution. We have

$$\frac{1 + 2i}{2 - 2i} = (1 + 2i)\frac{1}{2 - 2i} = (1 + 2i)\frac{2 + 2i}{8} = \frac{1}{8}(-2 + 6i) = \frac{-1 + 3i}{4}.$$

Hence
$$z = (2+3i) + (-\frac{1}{4} + \frac{3}{4}i) + (1-2i) = (3 - \frac{1}{4}) + (1 + \frac{3}{4})i,$$
z has real part $11/4$ and imaginary part $7/4$.

For any complex number $z = a + bi$, the number
$$\overline{z} = a - bi$$
is called the complex conjugate of z. One can easily prove that for any $z, z_1, z_2 \in \mathbb{C}$,

a) $\overline{z} = 0 \Leftrightarrow z = 0$ b) $\overline{(z_1 + z_2)} = \overline{z}_1 + \overline{z}_2$

c) $\overline{z_1\, z_2} = \overline{z}_1 \overline{z}_2$ d) $z + \overline{z}$ and $z\,\overline{z}$ are real numbers

e) z is real $\Leftrightarrow z = \overline{z}$.

The property (4) can be proved as follows. Let $z = a + bi$. Then $\overline{z} = a - bi$, and
$$\begin{aligned} z + \overline{z} &= (a+a) + (b-b)i = 2a \in R, \\ z\,\overline{z} &= (a^2+b^2) = (ab-ab)i = a^2 + b^2 \in R. \end{aligned}$$
We see that $z\,\overline{z}$ is always larger or equal to zero. The real number
$$|z| = \sqrt{z\,\overline{z}} = \sqrt{a^2+b^2}$$
is called the *absolute value* (or *modulus*) of the complex number $z = a + bi$. When z is a real number the absolute value here coincides with the usual one. The absolute value of complex numbers satisfies the four properties that are given in section 1 about the absolute value of real numbers.

The most important property of the set of complex numbers is the following.

Fundamental Theorem of Algebra. *Let $a_0, a_1, a_2, \ldots, a_{n-1}$ be complex numbers, $n \geq 2$. Then the equation*
$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1\, z + a_0 = 0$$
has at least one (and at most n) solutions in the set of complex numbers.

This theorem implies also that all solutions of the equation are contained in $\mathbb{C}$.

Exercises

1. Express each of the following complex numbers in the form $a + bi$.
   a) $(1+2i)(2-3i)$ b) $\frac{1+i}{1-i}$ c) $(1+i)\overline{(1-i)}$
   d) $(1+i)^2$ e) $(1+i)^3$ f) $i^5 + i^2 - i$
   g) $\frac{1}{i}$ h) $\frac{1}{1+i}$ i) $\overline{(2-3i)}(5+4i)$.
2. Prove the properties of the complex conjugate given in the text.
3. Find the absolute value of
   a) i b) $-i$ c) $1+i$ d) $-1-i$ e) -2.
4. Solve the following equations for z
   a) $z^2 + 1 = 0$ b) $z^2 + 2z + 3 = 0$ c) $z^2 - 1 = 0$
   d) $z^2 - 2z + 3 = 0$ e) $z^2 - 2z - 3 = 0$ f) $z^2 + 2z - 3 = 0$.

Appendix B

EXPANSION OF DETERMINANTS

We give here the proof of Theorem 7.1.2, concerning expansion of determinants with respect to any row. We restate the theorem:

Theorem. *Let $A = [a_{ij}]$ be a $p \times p$ matrix. Then for any $i = 1, 2, \ldots, p$, we have*

$$|A| = \sum_{j=1}^{p} (-1)^{i+j} a_{ij} |A_{ij}|.$$

Proof. If $i = 1$, then the statement coincides with the definition. For $i > 1$, we prove the assertion by induction on p. It is trivial for $p = 1$ or $p = 2$. Now, we assume that the assertion is true for $p - 1$, and show that it is then true for p. For this, let $A_{ij,1k}$ denote the $(p-2) \times (p-2)$ matrix obtained from A by first removing the i-th row and the j-th column and secondly removing the first row and the k-th column, where $i \geq 1$ and $j \neq k$. Clearly, $A_{ij,1k} = A_{1k,ij}$. We have, by the induction assumption,

$$\begin{aligned}
|A_{1k}| &= \sum_{\substack{j=1 \\ (j\neq k)}}^{p} (-1)^{i+j} a_{ij} |A_{1k,ij}| \\
|A_{1k}| &= \sum_{j=1}^{k-1} (-1)^{i+j+1} a_{ij} |A_{1k,ij}| + \sum_{j=k+1}^{p} (-1)^{i+j} a_{ij} |A_{1k,ij}|.
\end{aligned}$$

We also have,

$$\begin{aligned}
|A_{ij}| &= \sum_{\substack{k=1 \\ (k\neq j)}}^{p} (-1)^{k+1} a_{1k} |A_{ij,1k}| \\
|A_{ij}| &= \sum_{k=1}^{j-1} (-1)^{k+1} a_{1k} |A_{1k,ij}| + \sum_{k=j+1}^{p} (-1)^{k} a_{1k} |A_{1k,ij}|.
\end{aligned}$$

Hence using also the idea in Example 2 of Appendix A, we get

$$\begin{aligned}
|A| &= \sum_{k=1}^{p} (-1)^{k+1} a_{1k} |A_{1k}| \\
&= \sum_{k=1}^{p} (-1)^{k+1} a_{1k} \left(\sum_{j=1}^{k-1} (-1)^{i+j+1} a_{ij} |A_{1k,ij}| + \sum_{j=k+1}^{p} (-1)^{i+j} a_{ij} |A_{1k,ij}| \right) \\
&= \sum_{k=1}^{p} \sum_{j=1}^{k-1} (-1)^{i+j+k} a_{ij} a_{1k} |A_{1k,ij}| + \sum_{k=1}^{p} \sum_{j=k+1}^{p} (-1)^{i+j+k+1} a_{ij} a_{1k} |A_{1k,ij}| \\
&= \sum_{j=1}^{p} \sum_{k=j+1}^{p} (-1)^{i+j+k} a_{ij} a_{1k} |A_{1k,ij}| + \sum_{j=1}^{p} \sum_{k=1}^{j-1} (-1)^{i+j+k+1} a_{ij} a_{1k} |A_{1k,ij}| \\
|A| &= \sum_{j=1}^{p} (-1)^{i+j} a_{ij} |A_{ij}|.
\end{aligned}$$

This completes the proof. □

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