Matrices

Numbers are used to give information on a quantity. Matrices can be used to give information of two or more quantities such an occasion given below.

Example:

Three students named A, B & C got prized for their performances in the examination. There were,

6 exercise books and 3 pens in A's parcel

4 exercise books and 2 pens in B's parcel

3 exercise books in C's parcel

We can show this information in a table.

	Exercise books	**Pens**
A	6	3
B	4	2
C	3	0

This information can be forwarded as below.

$$\begin{pmatrix} 6 & 3 \\ 4 & 2 \\ 3 & 0 \end{pmatrix} \longleftarrow \text{Matrix}$$

Definition of a Matrix

A matrix is a rectangular array of numbers. If the array has 'm' number of rows and 'n' number of columns, then it is a $m \times n$ matrix. The number of m and n are called dimensions of the matrix.

$$\begin{pmatrix} 6 & 3 \\ 4 & 2 \\ 3 & 0 \end{pmatrix}_{3\times 2}$$

This matrix contains 3 rows and 2 columns

Notations:

Matrices are commonly written in box brackets or parentheses. Matrices are usually symbolized using upper case letters, while lower case letters representing the entries/elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_n \\ . & . & & & . \\ . & . & & & . \\ . & . & & & . \\ a_{m1} & a_m & . & . & a_{mn} \end{bmatrix}_{m\times n}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_n \\ . & . & & & . \\ . & . & & & . \\ . & . & & & . \\ a_{m1} & a_m & . & . & a_{mn} \end{pmatrix}_{m\times n}$$

A = $\left(a_{ij}\right)_{m\times n}$

where,

i is the row number

j is the column number

We will use A_{ij} to represent the entry in i^{th} row and j^{th} column of matrix A.

Types of Matrices

1. **Null Matrix**

   If all the elements of a matrix are zero then it is called as a null matrix or zero matrix.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2\times 3}$$

2. **Row Matrix**

   If a matrix has just one row, then it is called as a row matrix.

$$\begin{pmatrix} 6 & 4 & 5 \end{pmatrix}_{1\times 3}$$

3. **Column Matrix**

   If a matrix has just one column, then it is called as a column matrix.

$$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}_{3\times 1}$$

4. **Square Matrix**

A square matrix is a matrix with the same number of rows and columns.

$$\begin{pmatrix} 4 & 2 & 7 \\ 3 & 6 & 1 \\ 1 & 5 & 2 \end{pmatrix}_{3\times 3}$$

5. **Diagonal Matrix**

A diagonal matric is a matrix having non zero elements only in the diagonal running from upper left to the lower right.

Note: Every diagonal matrix is a square matrix.

$$\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}_{2\times 2} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}_{3\times 3}$$

If all entries of A below the main diagonal are zero, then A is called as an upper triangular matrix.

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix}_{3\times 3}$$

If all entries of A above the main diagonal are zero, then A is called as a lower triangular matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 5 & 2 & 4 \end{pmatrix}_{3\times 3}$$

6. **Identity Matrix**

The identity matrix I_n of size n is the n-by-n matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to zero.

Note: Every identity matrix is a square matrix.

$$I_1 = [1]_{1\times 1} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \qquad I_n = \begin{bmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & & . \\ . & . & & . & . \\ 0 & 0 & . & . & 1 \end{bmatrix}_{n\times n}$$

7. Equal Matrices

Two matrices are said to be equal if all three of the following conditions are met.

- Each matrix has same number of rows.
- Each matrix has same number of columns
- Corresponding elements within each matrix are equal.

Example:

$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

When

$a = p$

$b = q$

$c = r$

$d = s$

Then X and Y are equal matrices. i.e X = Y.

<u>Matrix Operations</u>

1. Addition of matrices

Two matrices can be added if and only if they are of the same order. A matrix of the same order is obtained as the sum of two matrices by adding the corresponding elements.

<u>Example</u>

1. If $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}_{2\times2}$ and $B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}_{2\times2}$, then find A+B.

$$A + B = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}_{2\times2} + \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}_{2\times2} = \begin{pmatrix} 2+3 & 4+2 \\ 1+1 & 3+0 \end{pmatrix}_{2\times2} = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}_{2\times2}$$

2. If $A = \begin{pmatrix} 2 & 8 & 3 & -4 \\ 1 & 2 & 5 & 0 \end{pmatrix}_{2\times4}$ and $= \begin{pmatrix} 1 & -3 & 2 & 7 \\ 5 & -6 & 3 & 1 \end{pmatrix}_{2\times4}$, then find A+B.

$$A + B = \begin{pmatrix} 2 & 8 & 3 & -4 \\ 1 & 2 & 5 & 0 \end{pmatrix}_{2\times4} + \begin{pmatrix} 1 & -3 & 2 & 7 \\ 5 & -6 & 3 & 1 \end{pmatrix}_{2\times4}$$

$$= \begin{pmatrix} 2+1 & 8+(-3) & 3+2 & (-4)+7 \\ 1+5 & 2+(-6) & 5+3 & 0+1 \end{pmatrix}_{2\times4} = \begin{pmatrix} 3 & 5 & 5 & 3 \\ 6 & -4 & 8 & 1 \end{pmatrix}_{2\times4}$$

3. If $A = \begin{pmatrix} 1 & 2 & 4 \\ 5 & 0 & -3 \end{pmatrix}_{2\times 3}$ and $B = \begin{pmatrix} 1 & 2 & 6 \\ 7 & 8 & 0 \\ 3 & 2 & 1 \end{pmatrix}_{3\times 3}$ then find A+B.

2. Subtraction of matrices

Two matrices can be subtracted if and only if they are of the same order. A matrix of the same order is obtained as the difference of two matrices by subtracting the corresponding elements.

Example:

1. If $A = \begin{pmatrix} 2 & 4 \\ 1 & 8 \end{pmatrix}_{2\times 2}$ and $= \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}_{2\times 2}$, then find A-B.

$$A - B = \begin{pmatrix} 2 & 4 \\ 1 & 8 \end{pmatrix}_{2\times 2} - \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 2-1 & 4-2 \\ 1-0 & 8-4 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}_{2\times 2}$$

2. If $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 6 & 9 \end{pmatrix}_{2\times 3}$ and $= \begin{pmatrix} -2 & 0 & -5 \\ -3 & 4 & 7 \end{pmatrix}_{2\times 3}$, then find A-B.

$$A - B = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 6 & 9 \end{pmatrix}_{2\times 3} - \begin{pmatrix} -2 & 0 & -5 \\ -3 & 4 & -7 \end{pmatrix}_{2\times 3}$$

$$= \begin{pmatrix} 1-(-2) & -1-0 & 3-(-5) \\ 0-(-3) & 6-4 & 9-7 \end{pmatrix}_{2\times 3}$$

$$= \begin{pmatrix} 3 & -1 & 8 \\ 3 & 2 & 2 \end{pmatrix}_{2\times 3}$$

3. Multiplication by a scalar

When a matrix is multiplied by an integer, all elements should be multiplied by the particular integer.

Example:

If A = $\begin{pmatrix} 1 & 5 & 3 \\ 8 & 1 & 0 \end{pmatrix}_{2\times 3}$, then find 5A.

$$5A = 5\begin{pmatrix} 1 & 5 & 3 \\ 8 & 1 & 0 \end{pmatrix}_{2\times 3} = \begin{pmatrix} 5\times 1 & 5\times 5 & 5\times 3 \\ 5\times 8 & 5\times 1 & 5\times 0 \end{pmatrix}_{2\times 3} = \begin{pmatrix} 5 & 25 & 15 \\ 40 & 5 & 0 \end{pmatrix}_{2\times 3}$$

Exercise 01

Given that $A = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 1 & 2 \end{pmatrix}$ $C = \begin{pmatrix} 4 & 3 \\ 0 & 6 \end{pmatrix}$

$D = \begin{pmatrix} 5 & -3 & 4 \end{pmatrix}$ $E = \begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix}$ $F = \begin{pmatrix} 2 & 4 \\ 1 & 5 \\ 3 & 2 \end{pmatrix}$

$G = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 3 \end{pmatrix}$ $H = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 1 \end{pmatrix}$

I. Write down the order of the matrices given above.

II. Write down the pairs of matrices conformable for addition.

III. Find, if it exists

a. A+D

b. B+G

c. C+E

d. F+H

IV. If a_{ij} , b_{ij} , c_{ij} , … are typical elements of A, B, C, … write down the values of a_{12} , b_{23} , c_{22} , d_{13} , e_{11}, f_{31}, g_{12}, h_{32}.

V. Find

a. 2A

b. -2C

c. 3E

d. 4G

e. -3H

VI. Find

a. 2B+ 3G

b. 2E- 3C

c. 3F+5H

Transpose of a Matrix

The transpose of $m \times n$ matrix A, written as A^T, is defined as the $n \times m$ matrix whose i^{th} row is the i^{th} column of A and j^{th} column is the j^{th} row of A. This means that, if $A = (a_{ij})_{m\times n}$, then $A^T = (a_{ji})_{n\times m}$.

Example:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}, \text{then} \quad A^T = \begin{pmatrix} 1 & 4 \\ 3 & 5 \\ 2 & 6 \end{pmatrix}_{3\times 2}$$

Note: square matrix A is said to be symmetric if $A = A^T$.

Example:

If $B = \begin{pmatrix} 2 & 4 \\ 1 & 5 \\ 3 & 2 \end{pmatrix}$, then $B^T = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 5 & 2 \end{pmatrix}$

If $C = \begin{pmatrix} 5 & -3 & 4 \end{pmatrix}$, then $C^T = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$

If $D = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, then $D^T = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$

Find the transpose of following matrices and hence identify which matrices can be considered as symmetric matrices.

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}_{2\times 2} \qquad B = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 0 & 5 \\ 3 & 5 & 2 \end{pmatrix}_{3\times 3}$$

Matrix Multiplication

Let A and B be two matrices. If the number of columns in A is equal to the number of rows in B; we say that A and B are conformable for the matrix product AB.

If A is an order $m \times n$ and B is of order $n \times p$, then the product AB is defined and is a matrix of order $m \times p$.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$. Then the product AB is defined as follows.

$$AB = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]$$

That is if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

Then AB = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

In general $AB \neq BA$.

Example:

Let $A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \\ 3 & 2 & 5 \end{pmatrix}$. Since A is a 2×3 and B is a 3×3 AB is defined and is given by,

$$AB$$
$$= \begin{pmatrix} 3\times 1 + 2\times(-1) + 1\times 3 & 3\times 2 + 2\times 0 + 1\times 2 & 3\times 0 + 2\times 3 + 1\times 5 \\ 4\times 1 + 5\times(-1) + 6\times 3 & 4\times 2 + 5\times 0 + 6\times 2 & 4\times 0 + 5\times 3 + 6\times 5 \end{pmatrix}$$
$$= \begin{pmatrix} 3-2+3 & 6+0+2 & 0+6+5 \\ 4-5+18 & 8+0+12 & 0+15+30 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 8 & 11 \\ 17 & 20 & 45 \end{pmatrix}$$

<u>Exercise 02</u>

Let $A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

I. Obtain product AB and BA.
II. Check whether AB = BA or not.

Determinant of a square matrix

Every square matrix can be associated with a real number called its determinant.

Definition of the Determinant of a 2×2 matrix

The determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

det(A) = $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ab - cd$

Note that the determinant is the difference of the products of the two diagonals of the matrix.

Exercise 03

Find the determinant of each matrix.

I. $A = \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}$

II. $B = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

III. $C = \begin{pmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{pmatrix}$

Definition of the determinant of a 3×3 matrix.

Consider 3×3 matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

Its determinant is calculated as follows.

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

Example

Find the determinant of each matrix.

$$A = \begin{pmatrix} 5 & 7 & 3 \\ 4 & 8 & 2 \\ 4 & 6 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 4 & -6 \\ 2 & -1 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 7 & 3 \\ 4 & 8 & 2 \\ 4 & 6 & 3 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 5 & 7 & 3 \\ 4 & 8 & 2 \\ 4 & 6 & 3 \end{vmatrix} = 5(24-12) - 7(12-8) + 3(24-32) = 5 \times 12 - 7 \times 4 + 3 \times (-8)$$

= 60-28-24

= 8

Adjoint Matrix

Transpose of the matrix including cofactors is called as the Adjoint matrix.

Minors and Cofactor of a square matrix

If A is a square matrix, the minor M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A.

The cofactor C_{ij} of the entry a_{ij} is ,

$$C_{ij} = (-1)^{i+j} \ M_{ij}$$

For example if $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ and the minor of b_2 is $\begin{vmatrix} a_2 & c_1 \\ a_3 & c_3 \end{vmatrix}$

Note: sign pattern for cofactors

1. 3×3 matrix $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$
2. 4×4 matrix $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

For 3×3 matrix $\begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$

For 4× 4 matrix $\begin{bmatrix} M_{11} & -M_{12} & M_{13} & -M_{14} \\ -M_{21} & M_{22} & -M_{23} & M_{24} \\ M_{31} & -M_{32} & M_{33} & -M_{34} \\ -M_{41} & M_{42} & -M_{43} & M_{44} \end{bmatrix}$

Example:

Find the adjoint matrix of A where $A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$

To find the minor M_{11}, delete the first row and first column of A and evaluate the determinant of the resulting matrix.

$$\begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$$

$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)(1) - 2 \times 0 = -1 - 0 = -1$

Similarly, to find M_{12} , delete the first row and second column.

$$\begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$$

$M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = (3 \times 1) - (4 \times 2) = 3 - 8 = -5$

By continuing this pattern, you will obtain the minors as,

$M_{11} = -1$ $M_{12} = -5$ $M_{13} = 4$

$M_{21} = 2$ $M_{22} = -4$ $M_{23} = -8$

$M_{31} = 5$ $M_{32} = -3$ $M_{33} = -6$

Now, to find the cofactors, combine these minors with the pattern of signs for a 3×3

$C_{11} = -1$ $C_{12} = 5$ $C_{13} = 4$

$C_{21} = -2$ $C_{22} = 4$ $C_{23} = 8$

$C_{31} = 5$ $C_{32} = 3$ $C_{33} = -6$

Therefore adjoint matrix of A,

$$adj(A) = \begin{bmatrix} -1 & 5 & 4 \\ -2 & -4 & 8 \\ 5 & 3 & -6 \end{bmatrix}^T = \begin{bmatrix} -1 & -2 & 5 \\ 5 & -4 & 3 \\ 4 & 8 & -6 \end{bmatrix}$$

<u>Exercise 04</u>

Find all minors and cofactors of matrix B. Hence obtain $adj(B)$ where

$$B = \begin{bmatrix} 4 & 0 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

<u>Inverse of a matrix</u>

Let A be a $n \times n$ square matrix and let I be the $n \times n$identity matrix. If there exist a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A.$$

Then A^{-1} is called as the inverse of A.

Finding inverse of a matrix

- If a matrix A has an inverse, A is called invertible (or non- singular); otherwise A is called as non-invertible (or singular).
- A non-singular matrix cannot have an inverse.

Example:

Find the inverse of $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$. To find the inverse of A try to solve the matrix equation $AA^{-1} = I$.

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a+4c & b+4d \\ -a-3c & -b-3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a + 4c = 1$ ---------------------01

$b + 4d = 0$---------------------02

$-a - 3c = 0$-------------------03

$-b - 3d = 1$-------------------04

By solving above equations;

$a = -3$

$b = -4$

$c = 1$

$d = 1$

Therefore $A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$.

For a 2×2 matrix, many people prefer to use a formular for the inverse. This simple formular which is only for 2×2 matrix is explained as follows.

If A is a 2 ×2 matrix given by A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then Ais invertible if and only if

$|A| = ad - bc \neq 0$

The inverse is given by

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ or } \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Exercise 05

Find the inverse of each matrix if exists.

A= $\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ B= $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$ C= $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ D= $\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

For any $n \times n$ matrix A: $A^{-1} = \frac{adj\,(A)}{\det(A)}$

Example:

Find the inverse of A = $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

$\det(A) = |A| = 1(0-2) - (-1)(-3+6) + 0\,(-2-0)$ = -2+3 = $1 \neq 0$

Therefore A^{-1} exists.

$C_{11} = +(-2) = -2$ $\quad$ $C_{12} = -(-3+6) = -3$ $\quad$ $C_{13} = (-2-0) = -2$

$C_{21} = -(3-0) = -3$ $\quad$ $C_{22} = (-3-0) = -3$ $\quad$ $C_{23} = -(-2+6) = -4$

$C_{31} = (1-0) = 1$ $\quad$ $C_{32} = -(-1-0) = 1$ $\quad$ $C_{33} = (0+1) = 1$

$$adj(A) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \begin{bmatrix} -2 & -3 & -2 \\ -3 & -3 & -4 \\ 1 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{adj\,(A)}{\det(A)} = \frac{1}{1}\begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

<u>Exercise 06</u>

Find the inverse of A if exists.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & -6 \\ 2 & -1 & 3 \end{bmatrix}$$

Applications of matrices and determinants

Cramer's Rule

This rule uses determinants to write the solution of a system of linear equations. If a system of n linear equations in n variables has a coefficient matrix A with a non-zero determinant$|A|$, the solution of the system is ,

$$X_i = \frac{\det(A_i)}{\det(A)} \; ; i = 1,2,3, \dots, n$$

Therefore $X_1 = \frac{|A_1|}{|A|}$ $\quad X_2 = \frac{|A_2|}{|A|}$, ... , $X_n = \frac{|A_n|}{|A|}$

Where the i^{th} column of A_i is the column of constants in the system of equation. If the determinant of the coefficient matrix is zero, the system has either no or infinitely many solutions.

Example:

Use Cramer's rule to solve the system of linear equations.

$$4x - 2y = 10$$

$$3x - 5y = 11$$

Coefficient matrix $\quad A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$

Solution matrix $\quad b = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$

First, find the determinant of the coefficient matrix.

$|A| = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = 4 \times (-5) - (-2) \times 3 = -20 + 6 = -14 \neq 0$

Therefore the linear system has finite solutions.

Apply the Cramer's Rule.

$|A_x| = \begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix} = -50 + 22 = -28$ $\qquad x = \frac{|A_x|}{|A|} = \frac{-28}{-14} = 2$

$|A_y| = \begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix} = 44 - 30 = 14$ $\qquad y = \frac{|A_y|}{|A|} = \frac{14}{-14} = -1$

Exercise 07

Use Cramer's Rule to solve the system of linear equations.

01.
$$\begin{array}{rcrcrcr} -x & + & 2y & - & 3z & = & 1 \\ 2x & & & + & z & = & 0 \\ 3x & - & 4y & + & 4z & = & 2 \end{array}$$

02.
$$\begin{array}{rcrcrcr} -x & + & 2y & + & 3z & = & -7 \\ -4x & - & 5y & + & 6z & = & -13 \\ 7x & - & 8y & - & 9z & = & 39 \end{array}$$

Solutions

$$\begin{array}{rcrcrcr} -x & + & 2y & - & 3z & = & 1 \\ 2x & \square & \square & + & z & = & 0 \\ 3x & - & 4y & + & 4z & = & 2 \end{array}$$

$$|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \qquad B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$|A_X| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8 \qquad x = \frac{|A_X|}{|A|} = \frac{8}{10}$$

$$|A_y| = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15 \qquad y = \frac{|A_y|}{|A|} = \frac{-15}{10}$$

$$|A_z| = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16 \qquad z = \frac{|A_z|}{|A|} = \frac{-16}{10}$$

$$(x, y, z) = \left(\frac{4}{5}, \frac{-3}{2}, \frac{-8}{5}\right)$$

$$\begin{array}{rcrcrcr} -x & + & 2y & + & 3z & = & -7 \\ -4x & - & 5y & + & 6z & = & -13 \\ 7x & - & 8y & - & 9z & = & 39 \end{array}$$

$$|A| = \begin{vmatrix} -1 & 2 & 3 \\ -4 & -5 & 6 \\ 7 & -8 & -9 \end{vmatrix} = 120 \qquad B = \begin{pmatrix} -7 \\ -13 \\ 39 \end{pmatrix}$$

$$|A_X| = \begin{vmatrix} -7 & 2 & 3 \\ -13 & -5 & 6 \\ 39 & -8 & -9 \end{vmatrix} = 480 \qquad x = \frac{|A_X|}{|A|} = \frac{480}{120} = 4$$

$$|A_y| = \begin{vmatrix} -1 & -7 & 3 \\ 4 & -13 & 6 \\ 7 & 39 & -9 \end{vmatrix} = -120 \qquad y = \frac{|A_y|}{|A|} = \frac{-120}{120} = -1$$

$$|A_z| = \begin{vmatrix} -1 & 2 & -7 \\ -4 & -5 & -13 \\ 7 & -8 & 39 \end{vmatrix} = -40 \qquad z = \frac{|A_z|}{|A|} = \frac{-40}{120} = = -\frac{1}{3}$$

$$(x, y, z) = \left(4, -1, \frac{-1}{3}\right)$$