Derivation of EM Algorithm.

E-Step : Expectation of $\log(P(x/z))$

$$E_{z/x,\theta^{(t)}}[\log(P(x,Z/\theta))] =$$

$$\sum_{z} \log(P(x,Z/\theta)).$$

$P(Z/x,\theta^{(t)})$

$i \rightarrow$ image $j \rightarrow j^{th}$ Annotator

let $A(i) \in$ set of Annotators for i^{th} image

we know μ, σ, β of previous iteration $\theta-1$

The likelihood function is defined as,

$$L(x, m, \theta) = \log\left(\prod_{i}\prod_{j \in A(i)} P(x_{ij}, m_j/\theta)\right)$$

$$= \sum_{i}\sum_{j \in A(i)} \log P(x_{ij}/m_j,\theta) \cdot P(m_j/\theta)$$

$\hookrightarrow$ 1 (using conditional probability)

Where θ denotes the set of parameters in previous iteration $\in [\mu_i, \sigma, \beta]$.

We know,

$$\underbrace{P(m_j = 1/\theta) = \beta}_{\text{Good Annotator}} \quad \underbrace{P(m_j = 0/\theta) = 1-\beta}_{\text{Bad Annotator}}$$

E-step.

$\therefore E[L/x, \theta^{t-1}] =$

from ①,

$$\sum_{i} \sum_{j \in A(i)} E\left[\log\left(P(x_{i,j}/m_j, \theta^{t-1}) \cdot P(m_j/\theta^{t})\right) / x, \theta^{t-1}\right]$$

Expanding the above eqn.,

$$= \sum_{i} \sum_{j \in A(i)} P(m_j = 1/x_{ij}, \theta^{t-1}) \log\left(\left(P(x_{ij}/m_j = 1, \theta\right) \cdot \beta\right)$$

$$+ P(m_j = 0/x_{ij}, \theta^{t-1}) \log\left(\left(P(x_{ij}/m_j = 0, \theta\right)(1-\beta)\right)$$

$$= \sum_{i} \sum_{j \in A(i)} c_{ij} \log\left(\underbrace{\frac{\beta}{\sqrt{2\pi}\,\sigma} \exp - \frac{(x_{ij} - \mu_i)^2}{2\sigma^2}}_{\text{Gaussian distribution}}\right) + (1 - c_{ij}) \log\left(\frac{1-\beta}{10}\right) \quad ②$$

where $c_{ij} = P(m_j = 1/x_{ij}; \theta^{t-1})$

$$= \frac{P(x_{ij}/m_j = 1, \theta^{t-1})\, P(m_j = 1/\theta^{t-1})}{P(x_{ij}/\theta^{t-1})}$$

$$\because P(A/B) = \frac{P(B/A)\, P(A)}{P(B)}$$

substituting the mean and variance corresponding to iteration t-1 in (2)

$$= \frac{P(x_{ij} / m_j = 1, \theta^{t-1}) \, P(m_j = 1 / \theta^{t-1})}{\sum_{m_j} P(x_{ij} / m_j, \theta^{t-1}) \, p(m_j / \theta^{t-1})}$$

$$= \frac{\frac{1}{\sqrt{2\pi}\sigma(t-1)} \exp\left(-\frac{(x_{i,j} - \mu(t-1)_i)^2}{2\sigma(t-1)^2}\right) \beta(t-1)}{\frac{1-\beta(t-1)}{10} + \frac{1}{\sqrt{2\pi}\,\sigma(t-1)} \exp\left(-\frac{(x_{ij} - \mu(t-1)_i)^2}{2\sigma(t-1)^2}\right) \cdot \beta(t-1)}$$

M-step:

To solve and maximize the expectation,

$$\theta^{t+1} = \underset{\theta}{\operatorname{argmax}} \sum \log(P(x, Z/\theta)) \, P(Z/x, \theta^t)$$

We maximize by taking partial derivatives,

$$\frac{\partial L}{\partial \mu_i} = k \sum_{j \in A_i} c_{ij} (x_{ij} - \mu_i) = 0$$

Solving for μ_i

$$\sum_i \sum_j x_{ij} c_{ij} - \sum_i \sum_j c_{ij} \mu_i = 0$$

$$\mu_i = \sum_{j \in A_i} (x_{ij} c_{ij}) \cdot \frac{1}{\sum_{j \in A_i} c_{ij}}$$

Maximizing w.r.t variance,

$$\frac{\partial L}{\partial \sigma} = \sum_i \sum_{j \in A(i)} \left[\left(-\frac{1}{\sigma} \right) + \frac{(x_{ij} - \mu_i)^2}{\sigma^3} \right) \cdot c_{ij} \Big] = 0$$

$$\sum_i \sigma^2 \sum_j c_{ij} = \sum_i \left(\sum_j c_{ij} (x_{ij} - \mu_i)^2 \right]$$

$$\sum_i \sigma^2 \sum_j c_{ji} = \sum_i \sum_j c_{ij} (x_{ij} - \mu_i)^2$$

$$\sigma^2 = \frac{\sum_i \sum_j (x_{ij} - \mu_i)^2 c_{ij}}{\sum_i \sum_j c_{ij}}$$

where $j \in A(i)$ $A(i) \Rightarrow$ Annotators for i^{th} img.

Maximizing wrt β,

$$\frac{\partial L}{\partial \beta} = \sum_i \sum_j \left[\frac{c_{ij}}{\beta} - \frac{1}{1-\beta} c_{ji} \right] = 0$$

$$\Rightarrow \sum_i \sum_j \frac{c_{ij}}{\beta} - \sum_i \sum_j \frac{c_{ij}}{1-\beta} = 0$$

$$\sum_i \sum_j (1-\beta) c_{ji} - \sum_i \sum_j \beta - c_{ij} = 0$$

$$\sum_i \sum_j (c_{ij}) \beta = \sum_i \sum_j c_{ji}$$

$$= \sum_{i} \sum_{j \in A(i)} \left(\frac{c_{ji}}{\beta} - \frac{(1 - c_{ij})}{1 - \beta} \right) = 0$$

$$\boxed{\therefore \quad \beta_j = \frac{\sum_{i \in S_j} c_{ij}}{N_j}}$$

Where $N_j \rightarrow$ total number of Annotations by user j

S_j = set of images labeled by user j.