Assignment 6

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***Abstract*—This document explains the properties of conic sections and to trace the given curve.**

Download all python codes from

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https://github.com/subhasishsaikia22/EE5609−
    Matrix−theory
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and latex-tikz codes from

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https://github.com/subhasishsaikia22/EE5609−
    Matrix−theory
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1 Problem

Trace the following central conics:

$$40x^2 + 36xy + 25y^2 - 196x - 122y + 205 = 0 \tag{1.0.1}$$

2 Explanation

The general equation of a second degree can be expressed as:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

$$\implies \mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{2.0.3}$$

The given equation of the curve can be expressed as:

$$40x^2 + 2(18)xy + 25y^2 + 2(-98)x + 2(-61)y + 205 = 0 \tag{2.0.4}$$

Comparing (2.0.1),(2.0.3) and (2.0.4):

$$\mathbf{V} = \begin{pmatrix} 40 & \sqrt{18} \\ \sqrt{18} & 25 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -98 \\ -61 \end{pmatrix} and \quad f = 205 \tag{2.0.5}$$

$$\implies \left|\mathbf{V}\right| = 982 \quad and \quad b^2 - ac = 18 - 40.25 = -982 \tag{2.0.6}$$

Since $\left|\mathbf{V}\right| > 0$ and $b^2 < ac$, (2.0.4) represent an ellipse.
The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 40 & \sqrt{18} \\ \sqrt{18} & \lambda - 25 \end{vmatrix} = 0 \tag{2.0.7}$$

$$\implies \lambda^2 - 65\lambda + 982 = 0 \tag{2.0.8}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.0.8). The roots of (2.0.8) i.e the eigenvalues are given by

$$\lambda_1 = \frac{65 + \sqrt{297}}{2}, \lambda_2 = \frac{65 - \sqrt{297}}{2} \tag{2.0.9}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.10}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.0.11}$$

for $\lambda_1 = \frac{65+\sqrt{297}}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{\sqrt{297}-15}{2} & -\sqrt{18} \\ -\sqrt{18} & \frac{\sqrt{297}+15}{2} \end{pmatrix} \tag{2.0.12}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{2\sqrt{18}}{\sqrt{297}-15}R_1} \begin{pmatrix} \frac{\sqrt{297}-15}{2} & -\sqrt{18} \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

From (2.0.11) and (2.0.16)

$$\implies \mathbf{p_1} = \begin{pmatrix} \sqrt{18} \\ \frac{\sqrt{297}-15}{2} \end{pmatrix} \tag{2.0.14}$$

For $\lambda_2 = \frac{65-\sqrt{297}}{2}$

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{-\sqrt{297}-15}{2} & -\sqrt{18} \\ -\sqrt{18} & \frac{15-\sqrt{297}}{2} \end{pmatrix} \tag{2.0.15}$$

$$\xleftrightarrow[R_1 = -R_1]{R_2 = R_2 + \frac{2\sqrt{18}}{\sqrt{297}+15}R_1} \begin{pmatrix} \frac{\sqrt{297}+15}{2} & \sqrt{18} \\ 0 & 0 \end{pmatrix} \tag{2.0.16}$$

$$\implies \mathbf{p_2} = \begin{pmatrix} -\sqrt{18} \\ \frac{\sqrt{297}+15}{2} \end{pmatrix} \tag{2.0.17}$$

using the affine transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} + c' \tag{2.0.18}$$

such that

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D} \quad and \quad \mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix}, \quad \mathbf{P}^T = \mathbf{P}^{-1} \tag{2.0.19}$$

Where $\mathbf{D}$ is a diagonal matrix, we get

$$\mathbf{D} = \begin{pmatrix} \frac{65+\sqrt{297}}{2} & 0 \\ 0 & \frac{65-\sqrt{297}}{2} \end{pmatrix} \tag{2.0.20}$$

Now (2.0.2) can be written as,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \quad \left|\mathbf{V}\right| \neq 0 \tag{2.0.21}$$

And,

$$\mathbf{c}' = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.0.22}$$

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{2.0.23}$$

The centre of the conic section in (2.0.4) is given by $\mathbf{c}'$ in (2.0.22). We compute $\mathbf{V}^{-1}$ as follows,

$$\left(\begin{matrix} 40 & \sqrt{18} \\ \sqrt{18} & 25 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right) \xleftrightarrow[R_1=\frac{1}{40}R_1]{R_2=R_2-\frac{\sqrt{18}}{40}R_1} \left(\begin{matrix} 1 & \frac{\sqrt{18}}{40} \\ 0 & \frac{982}{40} \end{matrix}\middle|\begin{matrix} \frac{1}{40} & 0 \\ -\frac{\sqrt{18}}{40} & 1 \end{matrix}\right) \tag{2.0.24}$$

$$\xleftrightarrow[R_1=R_1-\frac{\sqrt{18}}{40}R_2]{R_2=\frac{40}{982}R_2} \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} \frac{25}{982} & -\frac{\sqrt{18}}{982} \\ -\frac{\sqrt{18}}{982} & \frac{40}{982} \end{matrix}\right) \tag{2.0.25}$$

Hence $\mathbf{V}^{-1}$ is given by,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{25}{982} & -\frac{\sqrt{18}}{982} \\ -\frac{\sqrt{18}}{982} & \frac{40}{982} \end{pmatrix} \tag{2.0.26}$$

Now $\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}$ is given by,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} = \frac{1}{982}\begin{pmatrix} -98 & -61 \end{pmatrix}\begin{pmatrix} 25 & -\sqrt{18} \\ -\sqrt{18} & 40 \end{pmatrix}\begin{pmatrix} -98 \\ -61 \end{pmatrix} \tag{2.0.27}$$

$$= 344.4203 \tag{2.0.28}$$

And, $\mathbf{V}^{-1}\mathbf{u}$ is given by,

$$\mathbf{V}^{-1}\mathbf{u} = \frac{1}{982}\begin{pmatrix} 25 & -\sqrt{18} \\ -\sqrt{18} & 40 \end{pmatrix}\begin{pmatrix} -98 \\ -61 \end{pmatrix} \tag{2.0.29}$$

$$\tag{2.0.30}$$

By putting the value of (2.0.29), the center of the ellipse is given by (2.0.22) as follows,

$$\mathbf{c}' = \begin{pmatrix} 2.231 \\ 2.061 \end{pmatrix} \tag{2.0.31}$$

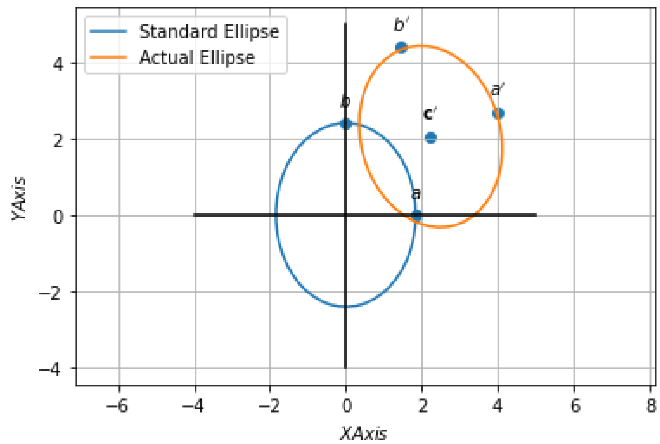

Fig. 1: Graphical representation of the actual curve $40x^2 + 36xy + 25y^2 - 196x - 122y + 205 = 0$, which represent an ellipse.

Also the semi-major axis (a) and semi-minor axis (b) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = 1.8414 \tag{2.0.32}$$

$$b = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = 2.416 \tag{2.0.33}$$

Finally from (2.0.21), the equation of ellipse is given by,

$$\mathbf{y}^T\begin{pmatrix} 41.116 & 0 \\ 0 & 23.883 \end{pmatrix}\mathbf{y} = 139.4203 \tag{2.0.34}$$