

Assignment 7

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Abstract—This document explains whether two lines meet at a point and to find out the distance of the point from any plane using singular valued decomposition and to verify the result using least squares.

Download all python codes from

<https://github.com/subhasishsaikia22/EE5609-Matrix-theory>

and latex-tikz codes from

<https://github.com/subhasishsaikia22/EE5609-Matrix-theory>

augmented matrix are both 2.
If the two given lines intersect:

$$\mathbf{x}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (2.0.5)$$

$$\Rightarrow \begin{pmatrix} m_1 & m_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \quad (2.0.6)$$

$$\Rightarrow \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \quad (2.0.7)$$

where

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \quad (2.0.8)$$

$$\mathbf{M} = \begin{pmatrix} m_1 & m_2 \end{pmatrix} \quad (2.0.9)$$

(2.0.7) can be expressed as the matrix equation:

$$\begin{pmatrix} -2 & 1 \\ 2 & -4 \\ -3 & 3 \end{pmatrix} \lambda = \begin{pmatrix} -4 \\ 4 \\ -6 \end{pmatrix} \quad (2.0.10)$$

the augmented matrix of (2.0.7)

$$\begin{pmatrix} -2 & 1 & -4 \\ 2 & -4 & 4 \\ -3 & 3 & -6 \end{pmatrix} \xleftrightarrow{R_2=R_2+R_1} \begin{pmatrix} -2 & 1 & -4 \\ 0 & -3 & 0 \\ -3 & 3 & -6 \end{pmatrix} \quad (2.0.11)$$

$$\xleftrightarrow{R_3=R_3+\frac{3}{2}R_2} \begin{pmatrix} -2 & 1 & -4 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \end{pmatrix} \xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} -2 & 1 & -4 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.12)$$

Since both coefficient matrix and the augmented matrix has $rank = 2$, the given equation of lines meet at a point.

From (2.0.12)

$$\begin{pmatrix} -2 & 1 & -4 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_1=R_1+R_2]{R_2=\frac{1}{3}R_2} \begin{pmatrix} -2 & 0 & -4 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.13)$$

$$\xleftrightarrow{R_1=\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.14)$$

1 PROBLEM

Show that the lines

$$x = 4 - 2t, \quad y = -3 + 2t, \quad z = 5 - 3t \quad (1.0.1)$$

$$x = t, \quad y = 1 - 4t, \quad z = -1 + 3t \quad (1.0.2)$$

meet in a point. Determine the coordinates of this point. Find the distance of the point from any one of the planes using single valued decomposition and verify the result using least squares.

2 POINT OF INTERSECTION OF TWO LINES

Let the two lines

$$L_1 : x = 4 - 2t, \quad y = -3 + 2t, \quad z = 5 - 3t \quad (2.0.1)$$

$$\Rightarrow L_1 : \mathbf{x} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} \quad (2.0.2)$$

$$L_2 : x = t, \quad y = 1 - 4t, \quad z = -1 + 3t \quad (2.0.3)$$

$$\Rightarrow L_2 : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \quad (2.0.4)$$

The two lines will meet at a point if and only if the rank of the coefficient matrix and the rank of the

From(2.0.7) and(2.0.14),

$$\lambda_1 = 2, \lambda_2 = 0 \quad (2.0.15)$$

Therefore the point of intersection of the given lines:

$$L_1 : \mathbf{x} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} \quad (2.0.16)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (2.0.17)$$

putting a=1 and b=0 we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (3.0.10)$$

putting a=0 and b=1 we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (3.0.11)$$

Now we solve the equation:

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (3.0.12)$$

To find the distance of the point of intersection of the given lines (2.0.17) from any plane. We need to find the foot of the perpendicular from the particular point to the given plane. Let us consider an arbitrary plane

$$(1 \ 2 \ -2)\mathbf{x} = 9 \quad (3.0.1)$$

and the point of intersection is

$$\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (3.0.2)$$

The plane is in the form of

$$\mathbf{n}^T \mathbf{x} = c \quad (3.0.3)$$

from(3.0.1),where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad (3.0.4)$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (3.0.5)$$

$$c = 9 \quad (3.0.6)$$

Let \mathbf{m}_1 and \mathbf{m}_2 be the orthogonal vectors to the normal vector \mathbf{n} . Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (3.0.7)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0 \quad (3.0.8)$$

$$\Rightarrow a + 2b - 2c = 0 \quad (3.0.9)$$

In order to solve (3.0.12),have to perform single valued decomposition on \mathbf{M} as follows:

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (3.0.13)$$

Where the columns of \mathbf{V} are the eigen vectors of $\mathbf{M}^T \mathbf{M}$, the columns of \mathbf{U} are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T \mathbf{M}$.

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \quad (3.0.14)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{5}{4} \end{pmatrix} \quad (3.0.15)$$

From (3.0.12) putting (3.0.13) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (3.0.16)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (3.0.17)$$

Where \mathbf{S}_+ is Moore-Penrose Pseudo-Inverse of \mathbf{S} . Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$|\mathbf{M}\mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (3.0.18)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & 1 \\ \frac{1}{2} & 1 & \frac{5}{4}-\lambda \end{vmatrix} = 0 \quad (3.0.19)$$

$$\Rightarrow -4\lambda^3 + 13\lambda^2 - 9\lambda = 0 \quad (3.0.20)$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad (3.0.21)$$

$$\lambda_2 = 1 \quad (3.0.22)$$

$$\lambda_3 = 0 \quad (3.0.23)$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{2}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix} \quad (3.0.24)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{2}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \quad (3.0.25)$$

Hence we obtain \mathbf{U} of (3.0.13) as follows,

$$\begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{5}{\sqrt{45}} & 0 & \frac{2}{3} \end{pmatrix} \quad (3.0.26)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get \mathbf{S} of (3.0.13) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.0.27)$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (3.0.28)$$

$$\Rightarrow \begin{pmatrix} \frac{5}{4} - \lambda & \frac{1}{2} \\ \frac{1}{2} & 2 - \lambda \end{pmatrix} = 0 \quad (3.0.29)$$

$$\Rightarrow \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \quad (3.0.30)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{9}{4} \quad (3.0.31)$$

$$\lambda_5 = 1 \quad (3.0.32)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (3.0.33)$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.0.34)$$

Hence we obtain \mathbf{V} of (3.0.13) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.0.35)$$

From (3.0.13) we get the Singular Value Decomposition of \mathbf{M} ,

$$\mathbf{M} = \begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{5}{\sqrt{45}} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}^T \quad (3.0.36)$$

Moore-Penrose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (3.0.37)$$

From (3.0.16) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\frac{1}{\sqrt{45}} \\ \frac{1}{\sqrt{5}} \\ \frac{-4}{3} \end{pmatrix} \quad (3.0.38)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\frac{2}{3\sqrt{45}} \\ \frac{\sqrt{1}}{\sqrt{5}} \end{pmatrix} \quad (3.0.39)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\frac{4}{9} \\ \frac{1}{9} \end{pmatrix} \quad (3.0.40)$$

4 VERIFICATION USING LEAST SQUARE

Verifying the solution of (3.0.40) using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (4.0.1)$$

Evaluating the R.H.S in (4.0.1) we get,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} \quad (4.0.2)$$

$$\Rightarrow \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} \quad (4.0.3)$$

Solving the augmented matrix of (4.0.3) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 2 & 0 \end{pmatrix} \xrightarrow{R_2=R_2-\frac{2}{5}R_1} \begin{pmatrix} \frac{5}{4} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{9}{5} & \frac{1}{5} \end{pmatrix} \quad (4.0.4)$$

$$\xrightarrow[R_1=\frac{4}{5}R_2]{R_2=\frac{5}{9}R_2} \begin{pmatrix} 1 & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{9} \end{pmatrix} \xrightarrow{R_1=R_1-\frac{2}{5}R_2} \begin{pmatrix} 1 & 0 & -\frac{4}{9} \\ 0 & 1 & \frac{1}{9} \end{pmatrix} \quad (4.0.5)$$

From equation (4.0.5), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{4}{9} \\ \frac{1}{9} \end{pmatrix} \quad (4.0.6)$$

Comparing results of \mathbf{x} from (3.0.40) and (4.0.6), we can say that the solution is verified

5 DISTANCE FROM A POINT TO A PLANE

Let the point of intersection of the lines be represented by \mathbf{P} and the foot of the perpendicular from the particular point \mathbf{P} to the given plane be \mathbf{Q} .

From (2.0.17) and (3.0.40) ,

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (5.0.1)$$

$$\mathbf{Q} = \begin{pmatrix} \frac{-4}{9} \\ \frac{1}{9} \\ 0 \end{pmatrix} \quad (5.0.2)$$

The distance between the point of intersection of the lines from the plane is:

$$\|\mathbf{P} - \mathbf{Q}\| = \sqrt{\left(0 - \frac{-4}{9}\right)^2 + \left(1 - \frac{1}{9}\right)^2 + (-1 - 0)^2} \quad (5.0.3)$$

$$\Rightarrow \|\mathbf{P} - \mathbf{Q}\| = \frac{\sqrt{161}}{9} = 1.409 \quad (5.0.4)$$