Assignment 9

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Abstract

This document explains the properties of inner product on $\mathbb{R}^n$

Download latex-tikz codes from

https://github.com/subhasishsaikia22/EE5609−Matrix−theory

1 Problem

Let $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ denote the standard inner product on $\mathbb{R}^n$. For a non zero $\mathbf{w} \in \mathbb{R}^n$, define $T_{\mathbf{w}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T_{\mathbf{w}}(\mathbf{v}) = \mathbf{v} - \frac{2\langle \mathbf{v},\mathbf{w}\rangle}{\langle \mathbf{w},\mathbf{w}\rangle}\mathbf{w}, \mathbf{v} \in \mathbb{R}^n$. Which of the following are true?

1) det $(T_{\mathbf{w}})$=1
2) $\langle T_{\mathbf{w}}(\mathbf{v_1}), T_{\mathbf{w}}(\mathbf{v_2})\rangle = \langle \mathbf{v_1}, \mathbf{v_2}\rangle \quad \forall \mathbf{v_1}$
3) $T_{\mathbf{w}} = T_{\mathbf{w}}^{-1}$
4) $T_{2\mathbf{w}} = 2T_{\mathbf{w}}$

2 Explanation

Inner Product	Let two vectors $\mathbf{u}$ and $\mathbf{v}$ be defined as: $$\mathbf{u} = \begin{pmatrix} u_1 \\ \mathbf{u_2} \\ . \\ . \\ . \\ . \\ \mathbf{u_n} \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ \mathbf{v_2} \\ . \\ . \\ . \\ . \\ \mathbf{v_n} \end{pmatrix} \in \mathbb{R}^n \quad (2.0.1)$$ then the inner product of $\mathbf{u}$ and $\mathbf{v}$ on $\mathbb{R}^n$: $$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T\mathbf{v} = \mathbf{u_1}\mathbf{v_1} + \mathbf{u_2}\mathbf{v_2} + ... + \mathbf{u_n}\mathbf{v_n} \quad (2.0.2)$$
Inner Product Property used	$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x} = \langle \mathbf{y}, \mathbf{x}\rangle \quad (2.0.3)$$ linearity property: $$\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w}\rangle = a\langle \mathbf{u}, \mathbf{w}\rangle + b\langle \mathbf{v}, \mathbf{w}\rangle \quad (2.0.4)$$

TABLE 1: Definition and properties used

3 Solution

Given	For n=2, $$T_{\mathbf{w}} : \mathbb{R}^2 \to \mathbb{R}^2, T_{\mathbf{w}}(\mathbf{v}) = \mathbf{v} - \frac{2\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w} \quad (3.0.1)$$ Let the standard basis vectors of $\mathbb{R}^2$: $\mathbf{e_1} = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{e_2} = \begin{pmatrix}0\\1\end{pmatrix}$ and non zero vector $\mathbf{w} = \begin{pmatrix}1\\1\end{pmatrix}$
Statement 1	det $(T_{\mathbf{w}})$=1
solution	$$T_{\mathbf{w}}(\mathbf{e_1}) = \mathbf{e_1} - \frac{2\langle \mathbf{e_1}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w} = \begin{pmatrix}1\\0\end{pmatrix} - \frac{2\left\langle\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\rangle}{\left\langle\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\rangle}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} - \frac{2.1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} \quad (3.0.2)$$ $$T_{\mathbf{w}}(\mathbf{e_2}) = \mathbf{e_2} - \frac{2\langle \mathbf{e_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w} = \begin{pmatrix}0\\1\end{pmatrix} - \frac{2\left\langle\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\rangle}{\left\langle\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\rangle}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} - \frac{2.1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} \quad (3.0.3)$$ $$T_{\mathbf{w}} = \begin{pmatrix}T_{\mathbf{w}}(\mathbf{e_1}) & T_{\mathbf{w}}(\mathbf{e_2})\end{pmatrix} = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix} \quad (3.0.4)$$ $$\implies \left|T_{\mathbf{w}}\right| = \begin{vmatrix}0 & -1\\-1 & 0\end{vmatrix} = 0 - 1 = -1 \quad (3.0.5)$$ **The given statement is false**
Statement 2	$\left\langle T_{\mathbf{w}}(\mathbf{v_1}), T_{\mathbf{w}}(\mathbf{v_2})\right\rangle = \langle \mathbf{v_1}, \mathbf{v_2}\rangle \quad \forall \mathbf{v_1}$

<table>
<tr><td>solution</td><td>

$$\left\langle T_{\mathbf{w}}\left(\mathbf{v_1}\right), T_{\mathbf{w}}\left(\mathbf{v_2}\right)\right\rangle = \left\langle \mathbf{v_1} - \frac{2\langle \mathbf{v_1}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}, \mathbf{v_2} - \frac{2\langle \mathbf{v_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}\right\rangle \tag{3.0.6}$$

$$= \left\langle \mathbf{v_1}, \mathbf{v_2} - \frac{2\langle \mathbf{v_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}\right\rangle + \left\langle -\frac{2\langle \mathbf{v_1}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}, \mathbf{v_2} - \frac{2\langle \mathbf{v_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}\right\rangle \tag{3.0.7}$$

$$= \langle \mathbf{v_1}, \mathbf{v_2}\rangle - \frac{2\langle \mathbf{v_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\langle \mathbf{v_1}, \mathbf{w}\rangle + \left\langle -\frac{2\langle \mathbf{v_1}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}, \mathbf{v_2}\right\rangle + \left\langle -\frac{2\langle \mathbf{v_1}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}, -\frac{2\langle \mathbf{v_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}\right\rangle \tag{3.0.8}$$

$$= \langle \mathbf{v_1}, \mathbf{v_2}\rangle - \frac{2\langle \mathbf{v_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\langle \mathbf{v_1}, \mathbf{w}\rangle - \frac{2\langle \mathbf{v_1}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\langle \mathbf{w}, \mathbf{v_2}\rangle + \frac{4\langle \mathbf{v_1}, \mathbf{w}\rangle\langle \mathbf{v_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle\langle \mathbf{w}, \mathbf{w}\rangle}\langle \mathbf{w}, \mathbf{w}\rangle \tag{3.0.9}$$

$$= \langle \mathbf{v_1}, \mathbf{v_2}\rangle - \frac{2\langle \mathbf{v_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\langle \mathbf{v_1}, \mathbf{w}\rangle - \frac{2\langle \mathbf{v_1}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\langle \mathbf{v_2}, \mathbf{w}\rangle + \frac{4\langle \mathbf{v_1}, \mathbf{w}\rangle\langle \mathbf{v_2}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle} \tag{3.0.10}$$

$$\implies \left\langle T_{\mathbf{w}}\left(\mathbf{v_1}\right), T_{\mathbf{w}}\left(\mathbf{v_2}\right)\right\rangle = \langle \mathbf{v_1}, \mathbf{v_2}\rangle \tag{3.0.11}$$

The given statement is correct
</td></tr>
<tr><td>Statement 3</td><td>$T_{\mathbf{w}} = T_{\mathbf{w}}^{-1}$</td></tr>
<tr><td>solution</td><td>

Suppose:

$$T_{\mathbf{w}}\left(\mathbf{v}\right) = \mathbf{u} \implies T_{\mathbf{w}}^{-1}\left(\mathbf{u}\right) = \mathbf{v} \tag{3.0.12}$$

Then :

$$T_{\mathbf{w}}\left(\mathbf{v}\right) = \mathbf{u} \tag{3.0.13}$$

$$\implies \mathbf{v} - \frac{2\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w} = \mathbf{u} \tag{3.0.14}$$

$$\implies \left\langle \mathbf{v} - \frac{2\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}, \mathbf{w}\right\rangle = \langle \mathbf{u}, \mathbf{w}\rangle \tag{3.0.15}$$

$$\implies \langle \mathbf{v}, \mathbf{w}\rangle - \frac{2\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\langle \mathbf{w}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle \tag{3.0.16}$$

$$\implies -\langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle \tag{3.0.17}$$

using (3.0.14) and (3.0.17)

$$T_{\mathbf{w}}^{-1}\left(\mathbf{u}\right) = \mathbf{v} = \mathbf{u} + \frac{2\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w} \tag{3.0.18}$$

$$\implies T_{\mathbf{w}}^{-1}\left(\mathbf{u}\right) = \mathbf{u} - \frac{2\langle \mathbf{u}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w} \tag{3.0.19}$$

$$\implies T_{\mathbf{w}}^{-1}\left(\mathbf{u}\right) = T_{\mathbf{w}}\left(\mathbf{u}\right) \tag{3.0.20}$$

$$\implies T_{\mathbf{w}}^{-1} = T_{\mathbf{w}} \tag{3.0.21}$$

The given statement is correct
</td></tr>
<tr><td>Statement 4</td><td>$T_{2\mathbf{w}} = 2T_{\mathbf{w}}$</td></tr>
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solution	$$T_{\mathbf{2w}}(\mathbf{v}) = \mathbf{v} - \frac{2\langle \mathbf{v}, \mathbf{2w}\rangle}{\langle \mathbf{2w}, \mathbf{2w}\rangle}\mathbf{2w} \quad (3.0.22)$$ $$= \mathbf{v} - \frac{2.2.2\langle \mathbf{v}, \mathbf{w}\rangle}{2.2\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w} \quad (3.0.23)$$ $$= \mathbf{v} - \frac{2\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w} \quad (3.0.24)$$ $$2T_{\mathbf{w}}(\mathbf{v}) = 2\left(\mathbf{v} - \frac{2\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}\right) \quad (3.0.25)$$ $$\implies 2T_{\mathbf{w}}(\mathbf{v}) \neq 2T_{\mathbf{2w}}(\mathbf{v}) \quad (3.0.26)$$ **The given statement is false**

TABLE 2: solution