Assignment 8

Subhasish Saikia
AI20MTECH14001

Abstract

This document explains the condition for diagonalization of a matrix.

Download latex-tikz codes from

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https://github.com/subhasishsaikia22/EE5609−Matrix−theory
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1 Problem

Which of the following matrices is not diagonalizable over $\mathbb{R}$?

1) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
2) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
3) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
4) $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

2 Solution

Definition	Let $\mathbf{W_i}$ be the eigenspace corresponding to eigenvalue λ_i of a n by n matrix $\mathbf{A}$. then the characteristic polynomial of $\mathbf{A}$ can be expressed as: $$(x-\lambda_1)^{d_1}(x-\lambda_2)^{d_2}\dots(x-\lambda_k)^{d_k} \quad (2.0.1)$$ where d_i is the algebraic multiplicity of eigenvalue λ_i and $\dim\left(W_i\right)$is the geometric multiplicity of the corresponding eigenvalueλ_i. Matrix $\mathbf{A}$ is said to be diagonalizable if and only if $$dim\left(W_i\right) = d_i \quad (2.0.2)$$ i.e geometric multiplicity=algebraic multiplicity
Explanation	$\dim\left(W_i\right)$, the geometric multiplicity of the corresponding eigen value λ_i for a matrix $\mathbf{A}$ is the null space of the matrix $(A-\lambda_i I)$ which is the nullity of the matrix $(A-\lambda_i I)$ From Rank nullity theorem, Nullity of a matrix= no of variables− Rank of the matrix.
Option 1	$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
solution	The given matrix is a 3 by 3 triangular matrix and as such the eigen values of the matrix are:

	$\lambda_1 = 2, \lambda_2 = 3 \quad and \quad \lambda_3 = 2 \qquad (2.0.3)$ The characteristic polynomial of the matrix: $(x-2)^2(x-3)^1 \qquad (2.0.4)$ Thus the algebraic multiplicity d_1 of eigenvalue=2 is 2. For $\lambda_1 = 2$ $(A-\lambda_1 I) = (A-2I) = \begin{pmatrix} 2-2 & 0 & 1 \\ 0 & 3-2 & 0 \\ 0 & 0 & 2-2 \end{pmatrix} \qquad (2.0.5)$ $\implies \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.0.6)$ Thus the rank of $(A-2I)$ is 2 and nullity of $(A-2I)$, $\dim\left(W_1\right)$ is $3-2$=1 Since $\dim\left(W_1\right) \neq \mathrm{d}_1$, **The given matrix is not diagonalizable**
Option 2	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
solution	The given matrix is a 2 by 2 matrix and as such the eigen values of the matrix are: $\left\lvert \mathbf{A} - \lambda I \right\rvert = 0 \qquad (2.0.7)$ $\implies \lambda^2 - 2\lambda = 0 \qquad (2.0.8)$ $\implies \lambda_1 = 0 \quad and \quad \lambda_2 = 2 \qquad (2.0.9)$ The characteristic polynomial of the matrix: $x(x-2) \qquad (2.0.10)$ Thus the algebraic multiplicity d_2 of the matrix is 1 For $\lambda_2 = 2$ $(A-\lambda_2 I) = (A-2I) = \begin{pmatrix} 1-2 & 1 \\ 1 & 1-2 \end{pmatrix} \qquad (2.0.11)$ $\implies \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \xleftrightarrow[R_2=R_2+R_1]{} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \qquad (2.0.12)$ Thus the rank of $(A-2I)$ is 1 and nullity of $(A-2I)$, $\dim\left(W_2\right)$ is $2-1$=1 Since $\dim\left(W_2\right) = d_2$, **The given matrix is diagonalizable**
Option 3	$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
solution	The given matrix is a 3 by 3 triangular matrix and as such the eigen values of the matrix are: $\lambda_1 = 2, \lambda_2 = 3 \quad and \quad \lambda_3 = 3 \qquad (2.0.13)$

	The characteristic polynomial of the matrix: $(x-2)(x-3)^2 \quad (2.0.14)$ Thus the algebraic multiplicity d_2 of the matrix is 2 For $\lambda_2 = 3$ $(A-\lambda_2 I) = (A-3I) = \begin{pmatrix} 2-3 & 1 & 0 \\ 0 & 3-3 & 0 \\ 0 & 0 & 3-3 \end{pmatrix} \quad (2.0.15)$ $\implies \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.16)$ Thus the rank of $(A-3I)$ is 1 and nullity of $(A-3I)$, $\dim\left(W_2\right)$ is $3-1$=2 Since $\dim\left(W_2\right) = d_2$, **The given matrix is diagonalizable**
Option 4	$\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$
solution	The given matrix is a 2 by 2 matrix and as such the eigen values of the matrix are: $\left\lvert \mathbf{A} - \lambda I \right\rvert = 0 \quad (2.0.17)$ $\implies \lambda^2 - 5\lambda + 6 = 0 \quad (2.0.18)$ $\implies \lambda(\lambda-3) - 2(\lambda-3) = 0 \quad (2.0.19)$ $\implies \lambda_1 = 2 \quad and \quad \lambda_2 = 3 \quad (2.0.20)$ The characteristic polynomial of the matrix: $(x-2)(x-3) \quad (2.0.21)$ Thus the algebraic multiplicity d_2 of the matrix is 1 For $\lambda_2 = 3$ $(A-\lambda_2 I) = (A-3I) = \begin{pmatrix} 1-3 & -1 \\ 2 & 4-3 \end{pmatrix} \quad (2.0.22)$ $\implies \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \xleftrightarrow[R_2=R_2+R_1]{} \begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix} \quad (2.0.23)$ Thus the rank of $(A-3I)$ is 1 and nullity of $(A-3I)$, $\dim\left(W_2\right)$ is $2-1$=1 Since $\dim\left(W_2\right) = d_2$, **The given matrix is diagonalizable**

TABLE 1: Explanation