Tutorial 06 – Time hierarchy, oracles, and advice

Exercise 1. *Deterministic time hierarchy*

In this exercise we consider the *time hierarchy theorem* that states that allowing Turing machines more computation time strictly increases the set of languages that they can decide.

Theorem. *If f, g are functions satisfying $f(n)\log f(n) = o(g(n))$ and g is time constructible, then* $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$.

Let $\mathcal{U}$ be the universal deterministic Turing machine constructed on TD1 (i.e., a machine that halts within $C_\alpha T \log T$ steps if $M_\alpha(x)$ halts within T steps.) Consider the language

$$\mathcal{L} := \{(\alpha, x) \colon \mathcal{U}\big(\alpha, (\alpha, x)\big) \text{ rejects in at most } g(|(\alpha, x)|) \text{ steps}\}\,.$$

1. Show that $\mathcal{L} \in \mathrm{DTIME}(g(n))$.
2. Show that if M_α is a machine that runs in time $cf(n)$ for some $c > 0$, then the language recognized by M_α is not equal to $\mathcal{L}$, i.e., $\mathcal{L} \notin \mathrm{DTIME}(f(n))$.
3. Prove that $\mathsf{P} \subsetneq \mathsf{EXP}$.

Exercise 2. *Some Oracles*

1. Let C be one of the classes P, NP, EXP, NEXP. Show that $\mathsf{C}^{\mathsf{P}} = \mathsf{C}$.
2. Show that $\mathsf{P}^{\mathsf{PSPACE}} = \mathsf{NP}^{\mathsf{PSPACE}} = \mathsf{PSPACE}$.
3. Show that $\mathsf{NEXP} \subseteq \mathsf{EXP}^{\mathsf{NP}}$.
4. Show that if $\mathsf{NP} = \mathsf{coNP}$, then $\mathsf{NP}^{\mathsf{NP}} = \mathsf{NP}$.

Exercise 3. *Advice*

1. Prove that $\mathsf{NP} \subseteq \mathsf{P}/\mathsf{log}$ implies $\mathsf{P} = \mathsf{NP}$.
2. Prove that every language $\mathcal{L} \subset \Sigma^*$ belongs to $\mathsf{P}/|\Sigma|^n$.
3. Show that there exists a *decidable* unary language that does not belong to P.
   A language $L \subseteq \{0,1\}^*$ is called sparse if there exists a polynomial p such that $|L \cap \{0,1\}^n| \leqslant p(n)$ for all n
4. Show that every sparse language is in P/poly

Exercise 4. *Nondeterministic time hierarchy*

We will prove the time hierarchy theorem for nondeterministic TMs:

Theorem (Nondeterministic Time Hierarchy Theorem)**.** *If f, g are functions satisfying $f(n+1) = o(g(n))$, and g is nondecreasing, time constructible, and satisfies $g(n) \geq n$, then* $\mathsf{NTIME}(f(n)) \subsetneq \mathsf{NTIME}(g(n))$.

To prove the theorem, we use the following fact: there exists a universal nondeterministic TM $\hat{\mathcal{U}}$ such that if $M_\alpha(x)$ halts within T steps, then $\hat{\mathcal{U}}(\alpha, x)$ halts within $C_\alpha T$ steps (and $\hat{\mathcal{U}}(\alpha, x)$ has an accepting path iff $M_\alpha(x)$ does).

Theorem (Nondeterministic Universal Turing Machine)**.** *There exists a NDTM $\mathcal{U}_N$ such that for every $x, \alpha \in \{0,1\}^*, \mathcal{U}_N(\alpha, x) = N_\alpha(x)$ where N_α denotes the NDTM represented by α. Moreover, if N_α halts on input x within T steps, then $\mathcal{U}_N(\alpha, x)$ halts within cT steps for some constant c.*

Now, construct a NDTM V that takes on input $\langle \alpha, 1^k, y \rangle$ (where we insert some new input 1^k to artificially increase the size), and again we want V to accept iff $M_\alpha(\langle \alpha, y \rangle)$ rejects.

- if $|y| \geqslant g(|\alpha| + k)$, then accept iff $M_\alpha(\langle \alpha, 1^k, \epsilon \rangle)$ rejects on computation path y. Note that if M_α does not halt on this computation path, then V rejects.
- if $|y| \geqslant g(|\alpha| + k)$, it runs $M_\alpha(\langle \alpha, 1^k, y0 \rangle)$ and $M_\alpha(\langle \alpha, 1^k, y1 \rangle)$ for $\leqslant g(n)$ steps each and accepts iff both accept

1. Show that V runs in time $O(g(n))$.
2. Show that there is no machine N such that $N(\langle \alpha, 1^i, x \rangle) = V(\langle \alpha, 1^i, x \rangle)$ for all α, i and n that runs in time $\leqslant cf(n)$.