Tutorial 04 – More space

Exercise 1. L, NL

Let $\mathsf{L} := \mathsf{DSPACE}(\log n)$ and $\mathsf{NL} := \mathsf{NSPACE}(\log n)$.

1. Show that $\textsc{Even} = \{x \mid x \text{ has an even number of 1s}\}$ is in L.
2. Show that the language of balanced parentheses (with only one kind of parenthesis) is in L.
3. Show that $\textsc{Path} = \{\langle G = (V, E), x \in V, y \in V\rangle \mid \text{There exists a path between } x \text{ and } y \text{ in } G\}$ is in NL.

Exercise 2. NL-*completeness*

Recall that $\mathsf{L} = \mathsf{DSPACE}(\log n)$ and $\mathsf{NL} = \mathsf{NSPACE}(\log n)$. We would like to study problems that are complete for the class NL, and therefore we need to define a notion of NL-*completeness*:

Definition. A language A is *logspace reducible* to a language B ($A \leqslant_l B$), if there exists a function $f : \{0,1\}^* \to \{0,1\}^*$ that can be computed in logspace such that $x \in A$ iff $f(x) \in B$ for every $x \in \{0,1\}^*$.
A language B is NL-*complete* if it is in NL and for every $A \in \mathsf{NL}$, $A \leqslant_l B$.

1. Prove that $\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P}$.
2. Prove that if $A \leqslant_l B$ and $B \leqslant_l C$, then $A \leqslant_l C$. Furthermore, prove that if $A \leqslant_l B$ and $B \in \mathsf{NL}$, then $A \in \mathsf{NL}$.
3. Prove that $\textsc{Path} = \{\langle G = (V, E), s \in V, t \in V\rangle \mid \text{There exists a path between } x \text{ and } y \text{ in } G\}$ is NL-complete.
4. Prove that $\textsc{StrongConn} = \{G = (V, E) \mid G \text{ is a strongly connected graph}\}$ is NL-complete.

Exercise 3. $\mathsf{NL} = \mathsf{coNL}$

Let coNL be the class of languages whose complements belong to NL. We want to show that $\mathsf{NL} = \mathsf{coNL}$ (Immerman–Szelepcsényi theorem).

1. Show that $\textsc{coPath} = \{(G, s, t) : \text{ there is no path from } s \text{ to } t \text{ in } G\}$ is coNL-complete (by logspace reductions).

Given a graph G and its vertex s, we define S_i as the set of vertices of G that are accessible from s in at most i steps.

2. Propose an NDTM M_0 in NL that takes on input (G, s, u, i) and accepts iff $u \in S_i$.
3. Propose a NDTM M_1 in NL that has at least one accepting path on input $(G, s, u, i, |S_i|)$ and, when M_1 accepts this input, then it outputs 1 if $u \in S_i$ and 0 otherwise. (We suppose that i and $|S_i|$ are given in binary.)
4. Propose a NDTM M_2 in NL that has at least one accepting path on input $(G, s, i, |S_i|)$ and, when M_2 accepts this input, then $|S_{i+1}|$ is written on the output tape.

5. Construct a NDTM M in NL with input (G, s) such that $M(G, s)$ has at least one accepting path and, when if $M(G, s)$ accepts, the number of vertices accessible from s is written in the output tape.

6. Prove that $\text{co}\textsc{Path} \in \mathsf{NL}$.

7. Deduce that $\mathsf{NL} = \mathsf{coNL}$.

Exercise 4. *Little space is no space at all!*

We are going to show that a language that can be recognized in $o(\log \log n)$ space can in fact be recognized in $O(1)$ space. This implies, e.g., that $\mathsf{DSPACE}(\sqrt{\log \log n}) = \mathsf{DSPACE}(1)$.

1. Let $\mathcal{L} \notin \mathsf{DSPACE}(1)$ and suppose that M is a Turing machine that recognizes $\mathcal{L}$ in space $o(\log \log n)$. For every $k \in \mathbb{N}$ show that there exists $x \in \mathcal{L}$ such that $M(x)$ uses at least k cells on the work tape during the computation. Furthermore, if $x_k \in \mathcal{L}$ denotes the shortest word with this property, then $\lim_{k \to \infty} |x_k| \to \infty$.

2. At every step of the computation of $M(x)$ we define the current *internal configuration* of M as the tuple (q, y), where q is the current state of M and y encodes the current contents of the work tapes of M and the positions of the work heads. Show that M has $o(\log |x|)$ different internal configurations during the computation of $M(x)$.

3. We define the ith *crossing sequence* $C_i(x)$ of a M on an input x, $1 \leq i \leq |x| - 1$, as a vector $(c_1, c_2, \ldots, c_m)$, where c_1 is the internal configuration of M when the input head first crossed from a cell i to a cell $i + 1$ on the input tape, c_2 is the internal configuration of M when the input head first crossed from a cell $i + 1$ to i, etc. (so that the odd elements denote configurations of M after crossing ith cell left-to-right, and the even ones – after crossing right-to-left from $(i + 1)$th cell to ith). Show that there are $o(|x|)$ different crossing-sequences that appear on an input x.

4. Show that the crossing sequences of M on x_k are pairwise different. Conclude.