

Recall:

• Entanglement : $\begin{matrix} & A & & B \\ & 0 & 0 & 0 \end{matrix} \bigg| \begin{matrix} & 0 & 0 & 0 \end{matrix}$

$$|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\sum_{ij} \psi_{ij} |i\rangle_A |j\rangle_B$$

$$\rho_A = \sum_{i,i'} \psi_{i'j}^* \psi_{ij} |i'\rangle \langle i|_A = \text{tr}_B(|\psi\rangle\langle\psi|)$$

Entanglement entropy : $S = -\text{Tr}_A \rho_A \log \rho_A$

• Schmidt decomposition : \cong SVD

$$|\psi_A\rangle = \sum_{\alpha=1}^{\min(N_A, N_B)} \lambda_{\alpha} |\phi_{\alpha}\rangle_A |\phi_{\alpha}\rangle_B \quad \text{such that}$$

$$\langle \phi_{\alpha} | \phi_{\alpha'} \rangle = \delta_{\alpha\alpha'}$$

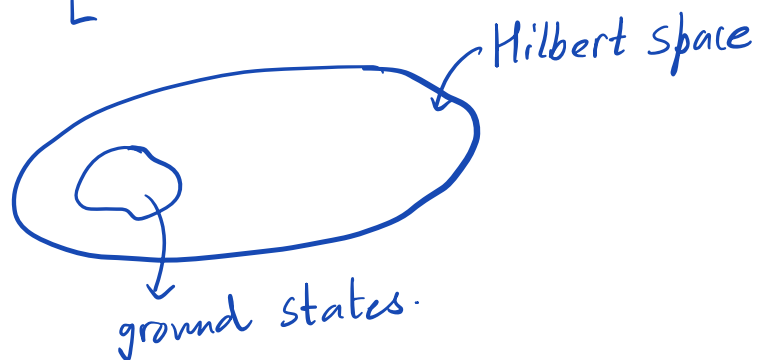
$$\rho_A = \sum_{\alpha} \lambda_{\alpha}^2 |\phi_{\alpha}\rangle_A \langle \phi_{\alpha}|_A$$

$$\Rightarrow S = - \sum_{\alpha} \lambda_{\alpha}^2 \log \lambda_{\alpha}^2$$

- Area law:

Ground states of (gapped) local Hamiltonians fulfill the area law $S \sim L^{D-1}$

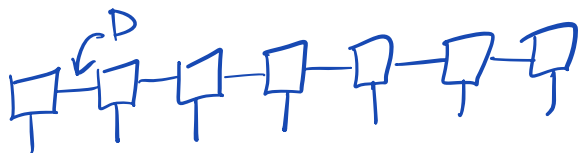
1D. $\underbrace{\begin{array}{c} L/2 \\ 0 \ 0 \ 0 \end{array} | \begin{array}{c} L/2 \\ 0 \ 0 \ 0 \end{array}}_L \quad S(L) = \text{const.} \quad (L > \xi)$



1D area law \Rightarrow Schmidt values decay quickly and thus we can approximate $|\psi\rangle$ by keeping $D = \text{const.}$, with Schmidt decomposition

$$|\psi\rangle \approx \sum_{\alpha=1}^D \lambda_{\alpha} |\alpha\rangle_A |\alpha\rangle_B$$

- Matrix product states: constant bond dimension D



Canonical form of MPS

MPS are not uniquely defined.

$$\boxed{A} \rightarrow \boxed{X} \boxed{A} \boxed{X^\dagger} \quad \leftarrow \text{represents same state.}$$

Bonds are directly related to Schmidt decomposition.

$$|\psi\rangle \stackrel{\text{SVD}}{=} \sum_{\alpha} \lambda_{\alpha} |\alpha\rangle_L |\alpha\rangle_R$$

$$= \begin{array}{c} \text{Singular value matrix} \\ \downarrow \\ \boxed{P} \boxed{\Lambda} \boxed{P} \boxed{\Lambda} \boxed{P} \boxed{\Lambda} \boxed{P} \dots \end{array}$$

this is a canonical representation.

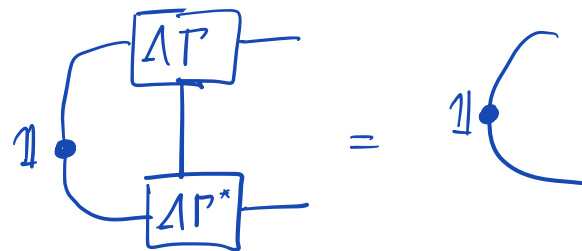
Because of SVD : orthonormality of "Right" vectors

$$\delta_{\alpha\alpha'} = {}_R \langle \alpha | \alpha' \rangle_R$$

$$= \begin{array}{c} \alpha \\ \alpha \end{array} \begin{array}{c} \boxed{P} \boxed{\Lambda} \boxed{P} \boxed{\Lambda} \boxed{P} \\ \boxed{P^*} \boxed{\Lambda} \boxed{P^*} \boxed{\Lambda} \boxed{P^*} \end{array} \quad \leftarrow \text{Transfer matrix}$$

$$= \begin{array}{c} \boxed{P\Lambda} \\ \boxed{P^*\Lambda} \end{array} \bullet \mathbb{1} = \bigcup \bullet \mathbb{1}$$

[Similar for the left] \rightsquigarrow



\Leftrightarrow Transfer matrices have left/right eigenvalue 1 with eigenvector $\mathbb{1}$

\Leftrightarrow uniquely defines MPS upto a $U(1)$ phase $\sim e^{i\theta}$, that does not change the global state.

Very convenient to evaluate expectation values:

$$\begin{aligned}
 \langle \psi | O_i | \psi \rangle &= \text{Diagram of a matrix product state (MPS) with a central operator } O_i. \\
 &= \text{Diagram showing the contraction of the MPS with its adjoint, with red boxes highlighting the left and right parts. Below the boxes is a red vector \mathbb{1} with a dot, indicating the contraction of the indices.} \\
 &= \text{Diagram showing the final simplified result: a central operator } O_i \text{ with four legs, each labeled } \Lambda^2, \text{ and a red vector \mathbb{1} with a dot.} \rightarrow \text{Easy to compute!}
 \end{aligned}$$

Time Evolving Block Decimation (TEBD) [Vidal '03]

Algorithm to find ground state given a local Hamiltonian H .

Time evolution in imaginary time yields the ground state.

Start from an initial state $|\psi_i\rangle$

$$|\psi_i\rangle = \sum_{\alpha} \psi_i^{\alpha} |\alpha\rangle$$

\uparrow energy eigenstate.

$$H|\alpha\rangle = E_{\alpha}|\alpha\rangle$$

\uparrow energy \uparrow ground state.

$E_1 \leq E_2 \leq E_3 \leq \dots$

$$e^{-\tau H} |\psi_i\rangle = \sum_{\alpha} (\psi_i^{\alpha}) e^{-\tau E_{\alpha}} |\alpha\rangle$$

If we take $\tau \rightarrow \infty$, only the lowest energy will

be remaining: $e^{-\tau H} |\psi_i\rangle \xrightarrow{\tau \rightarrow \infty} \underbrace{\psi_i^{[\alpha=1]}}_{\text{this shouldn't be zero}} e^{-\tau E_{\alpha}} \underbrace{|\alpha=1\rangle}_{\downarrow \text{ground state}}$

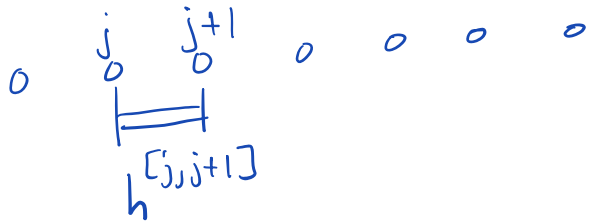
$$|gs\rangle = \lim_{\tau \rightarrow \infty} \frac{e^{-H\tau} |\psi_i\rangle}{\|e^{-H\tau} |\psi_i\rangle\|}$$

"Trotterization"

Assume the Hamiltonian has the form $H = \sum_j h^{[j,j+1]}$

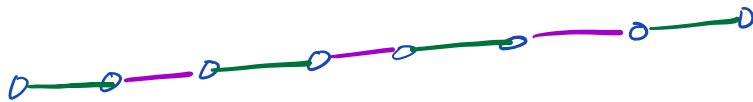
(Recall the Transverse field Ising model :)

$$H = - \sum_j Z_j Z_{j+1} - g \sum_i X_i$$



Decompose the Hamiltonian $H = F + G$

$$\underline{F} = \sum_{\text{even } j} h^{[j,j+1]} \quad ; \quad \underline{G} = \sum_{\text{odd } j} h^{[j,j+1]}$$



Observe $[F^i, F^j] = [G^i, G^j] = 0$

but $[G, F] \neq 0$

$$e^{\varepsilon(A+B)} \neq e^{\varepsilon A} \cdot e^{\varepsilon B} \quad \text{if } [A, B] \neq 0$$

Baker Campbell Hausdorff formula

$$e^{\varepsilon A} \cdot e^{\varepsilon B} = e^{\varepsilon(A+B) + \frac{\varepsilon^2}{2} [A, B] + \dots}$$

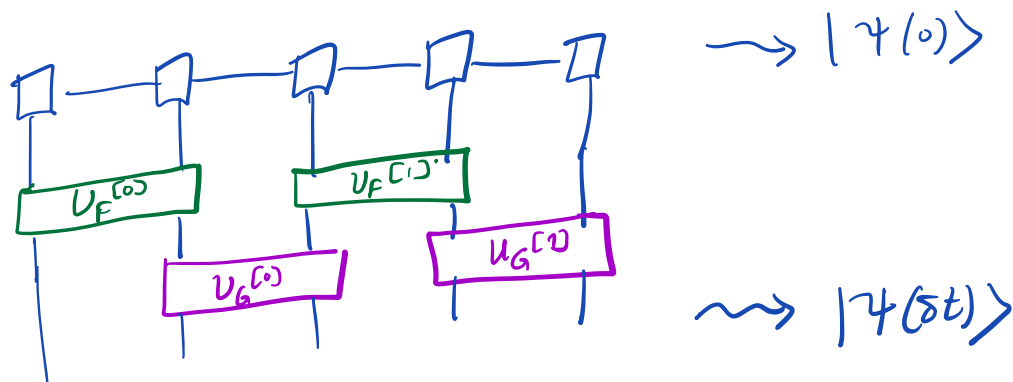
Decompose time evolution :

$$\exp(-iHt) = \left[\exp(-iH \underbrace{\frac{t}{N}}_{=\delta t}) \right]^N$$

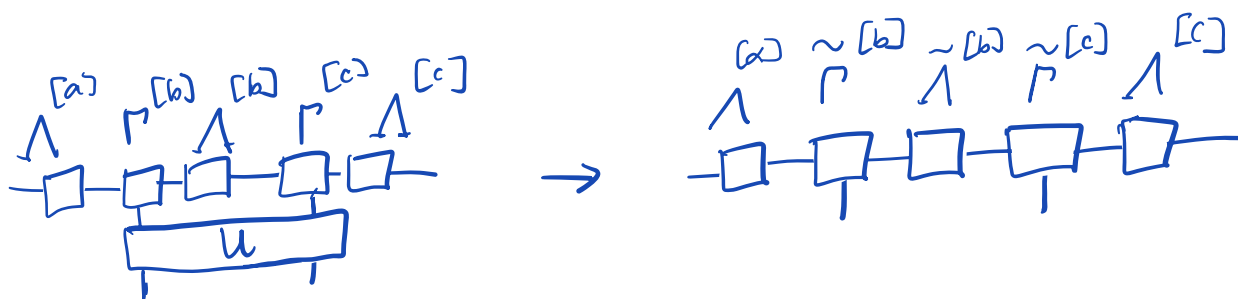
$$e^{-i\delta t (F+G)} \approx \underbrace{e^{-i\delta t F}}_{U_F} \underbrace{e^{-i\delta t G}}_{U_G} + \mathcal{O}(\delta t^2)$$

$$U_F = \prod_{\text{even } j} e^{-iF^{[j]} \delta t} \quad ; \quad U_G = \prod_{\text{odd } j} e^{-iG^{[j]} \delta t}$$

Evolution of an MPS for one time step:



Need an algorithm to project back to MPS form



TEBD algorithm

1. "Apply U "

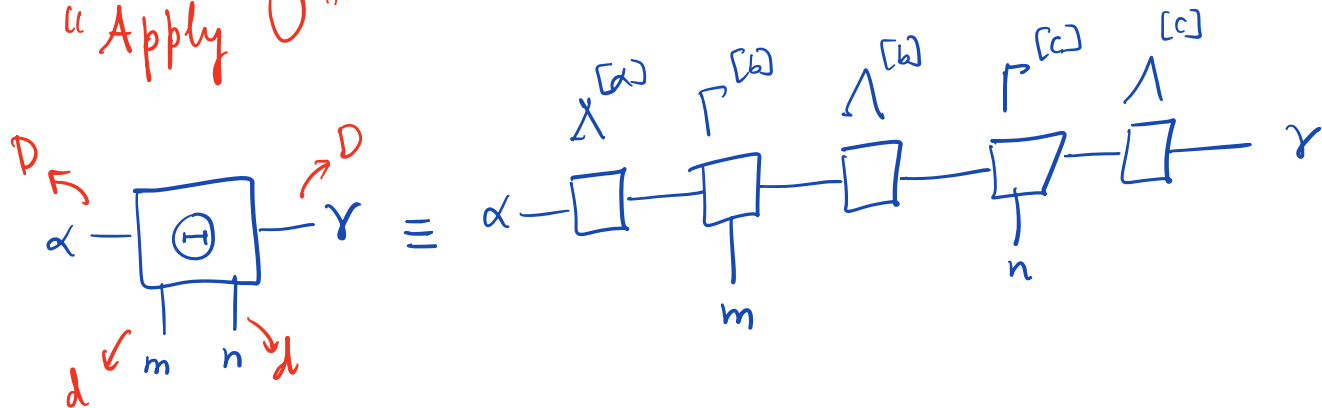
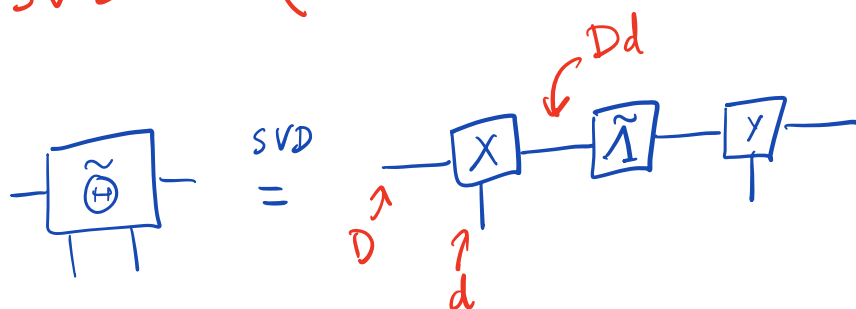


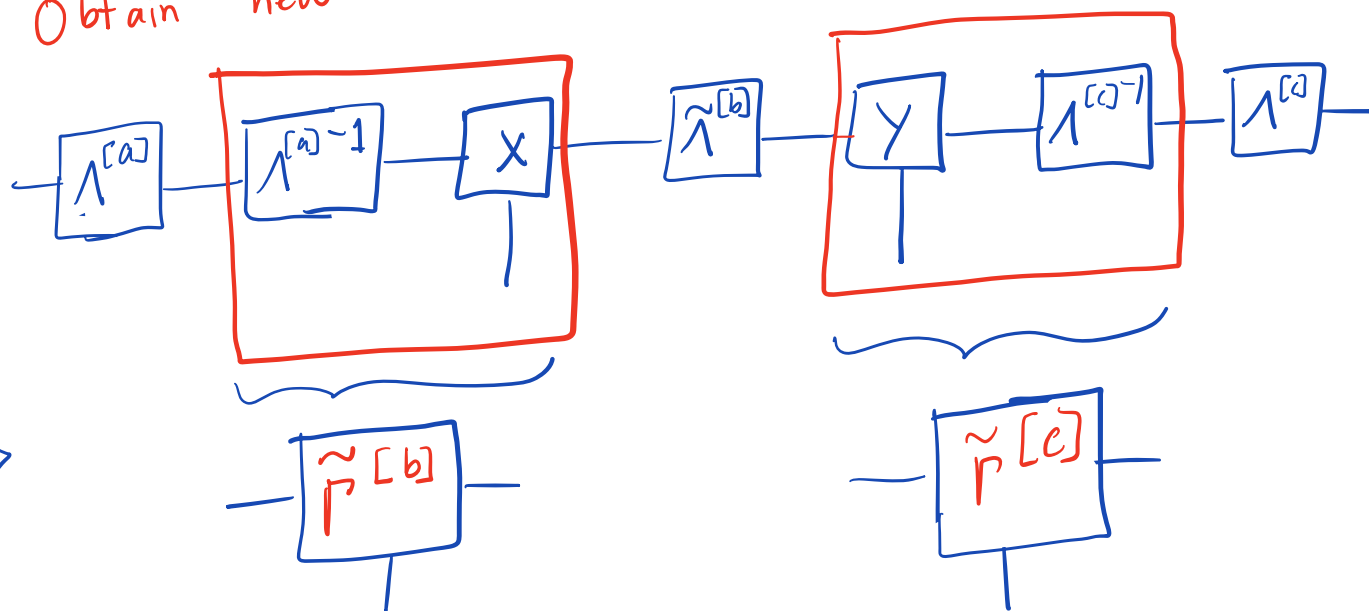
Diagram showing the decomposition of a tensor into a contraction of two tensors, one with a circle and plus sign, and another labeled U .

$$\left(\tilde{\Theta}^{mn}_{\alpha\gamma} = U^{mn}_{m'n'} \Theta^{m'n'}_{\alpha\gamma} \right)$$

2. "SVD" ($dD \times dD$ matrix)



3. Obtain new MPS.



4. Truncate

Discard smallest Schmidt values

$$Dd \rightarrow D$$

(i.e. keep only D rows / columns of tensors)

Applying this algorithm iteratively to even/odd bonds: we obtain the time evolution!

Computational time scales as $O(Ld^3D^3)$

Errors:

1. Truncation error \rightarrow take larger D
2. Trotter error \rightarrow take smaller δt .