Quantum Matter
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Tutorial 6: Laughlin to LSM

In this tutorial we will warm up by seeing how the Laughlin flux threading argument from the lecture leads to the quantization of Hall conductance in Landau levels. Next, we will see a further example of how such a flux threading argument can be used to prove the Lieb Schultz Mattis theorem.

1 Quantization of Hall conductivity via Laughlin's flux threading argument

Consider 2D electron gas in an annulus in the presence of transverse magnetic field, as shown in Fig 1. In addition to the background magnetic field B which penetrates the sample, we thread an additional flux Φ through the centre of the ring. Inside the ring, which is pure gauge. Nonetheless, the flux can affect the quantum states of the electron as the Φ enters the Hamiltonian through the gauge potential A, which can be taken to be around the ring, $A_\phi = \frac{\Phi}{2\pi r}$ through minimal coupling.

Recall from Tutorial 4 that within a Landau level, the degenerate states are labeled by their angular momenta. The minimally coupled Hamiltonian in the polar coordinates is,

$$H[\Phi] = -\frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{2}\left(\frac{1}{r}\left(-i\partial_\phi - \frac{\Phi}{2\pi}\right) - \frac{rB}{2}\right)^2. \tag{1}$$

The important thing to notice is that the angular momentum along the ring, $L_z \equiv -i\partial_\phi$ is a conserved quantity, and its eigenvalues are quantized to integers. The lowest Landau level for the Hamiltonian $H[0]$, when the flux threaded is $\Phi = 0$, with angular momentum m is $|m, \Phi = 0\rangle$, whose amplitudes are given by,

$$\psi_m \sim e^{im\phi} r^m e^{-r^2/4l_B^2}, \quad l_B \sim \sqrt{1/B}. \tag{2}$$

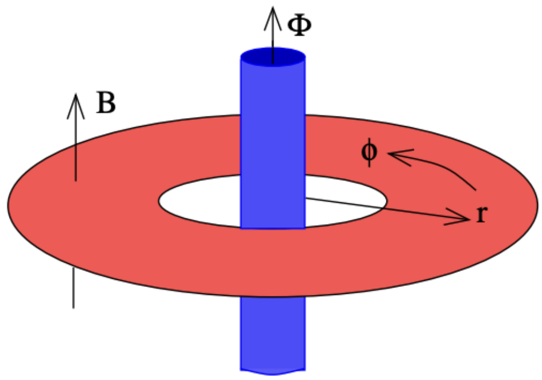

Figure 1: 2D electron gas in an annulus in the presence of transverse magnetic field. Figure from `https://www.damtp.cam.ac.uk/user/tong/qhe/qhe.pdf`

This state is localized at radius $r_m \sim \sqrt{2ml_B^2}$. All physical constants have been set to $1 = \hbar = e = c$.

Suppose that we slowly increase Φ from 0 to $\Phi_0 = 2\pi$ from time $t = 0$ to $t = 1$. This induces an electric field around the ring, leading to Hall response.

a) Argue that the angular momentum L_z remains a good quantum number for any threaded flux. So, the quantum number m does not change throughout the evolution.

   Now suppose we track the state $|\psi(t=0)\rangle = |m, \Phi = 0\rangle$ as we thread the flux slowly, leading to $|\psi(t)\rangle = U_t\,|m, \Phi = 0\rangle$. Argue that the time evolved state must also have angular momentum m and the final state at $t = 1$ must be proportional to the state $|m, \Phi = 2\pi\rangle$.

b) Consider now the Hamiltonian projected to a particular subspace with angular momentum m, denoted by $H_m[\Phi] \equiv P_m H[\Phi] P_m$. How is the projected Hamiltonian at $t = 1$ when the flux is $\Phi = 2\pi$, related to the projected Hamiltonian at $t = 0$, when the flux was $\Phi = 0$?

c) Argue that this implies,

$$|\psi(t=1)\rangle \sim |m-1, \Phi=0\rangle . \tag{3}$$

Note, this depends crucially on the fact the Landau level eigenstate at a fixed angular momentum is a unique gapped state.

What does this mean for radial position of the final state?

d) Argue why this implies that when we start with the completely filled lowest Landau level, one electron gets moved from the outer edge to the inner edge after the flux is threaded, and that this implies the Hall conductivity is quantized.

2 Lieb-Schultz-Mattis-Oshikawa-Hastings theorem from flux threading

A similar flux threading argument can be used to prove a deep theorem: A quantum lattice system with translation with conserved particle number can not have a unique gapped symmetric groundstate if the number of particles in each unit cell is not an integer.

We will follow an argument in Oshikawa's paper (`https://arxiv.org/pdf/cond-mat/9911137`) that proves this statement using the flux threading argument.

Consider a $1d$ system of particles hopping in a periodic lattice of L sites and lattice constant a, with translation symmetry generated by the translation operator $T : r \to r + a$, with $T^L = 1$, and such that particle number is conserved (which generates the $U(1)$ symmetry). The lattice can thus be taken to be a ring of L sites. The onsite particle number operator is denoted by $\hat{n}_r$. Note, the theorem is independent of dimension and persists in the $L \to \infty$ limit, but we will use this simplified setup for the proof.

The argument goes as follows: we introduce a background $U(1)$ gauge field that couples to the particles (imagine the particles carry a fictitious electric charge, and we thread a fictitious magnetic field through the ring). Assume the flux threading is adiabatic, so the evolved state is in the groundspace of the evolved Hamiltonian. This effect is generated by adding a gauge potential $A_{j,j+1}$ along each bond of the lattice between sites j and $j+1$, such that the sum of the gauge potential along the ring reproduces the threaded flux,

$$\sum_j A_{j,j+1} = \Phi, \tag{4}$$

much like the flux threading process in the 2d electron gas we discussed in the last problem. In a simple gauge choice, the gauge potential can be taken to be uniform $A_{j,j+1} = \Phi/L$. Let the Hamiltonian at any Φ be denoted by $H[\Phi]$. We will not make use of any Hamiltonian explicitly, but you can keep the concrete example of a free particle hopping on the ring,

$$H[\Phi] = -\sum_j e^{iA_{j,j+1}} a_j^\dagger a_{j+1} + h.c., \tag{5}$$

in the back of your mind.

The translation operator can be represented in terms of the crystal momentum P such that $T = e^{iP}$, and the Hamiltonian commutes with the crystal momentum for any threaded flux, $[H[0], P] = 0$.

Suppose we now start with a groundstate $|\psi_0\rangle$ of $H[0]$, and adiabatically turn on a flux from $\Phi = 0$ to $\Phi = 2\pi$. We also choose the initial state to be an eigenstate of the crystal momentum, with eigenvalue P_0.

a) Argue that in the uniform gauge choice, the crystal momentum is still a good quantum number. What will be the eigenvalue of the crystal momentum of the evolved state as we thread flux?

b) Once the flux 2π is threaded, the spectrum of the Hamiltonian must come back to the original spectrum. However, say the state has transformed from $|\psi_0\rangle \to |\psi_0'\rangle$.

   Show that the effect of the 2π flux can actually be removed by a large gauge transformation,

$$U = e^{i\sum_j \theta_j \hat{n}_j}, \quad A_{j,j+1} \to A_{j,j+1} + \theta_j - \theta_{j+1}, \tag{6}$$

   for a particular choice of θ_j.

c) Now, let us study the state after the gauge transformation, $|\psi_0''\rangle = U|\psi_0'\rangle$. Use the identity,

$$U^{-1}TU = T\exp\left(2\pi i \sum_r \hat{n}_r/L\right), \tag{7}$$

   to show that $|\psi_0''\rangle$ is an eigenstate of the momentum operator, and compute its eigenvalue. You can set $\nu = \sum \hat{n}_r/L$ to be the filling, or the number of particles per unit cell.

   Then argue that $|\psi_0''\rangle$ must be a distinct groundstate when the filling is not an integer $\nu \notin \mathbb{Z}$, thereby proving the theorem.