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Homework 1

Deadline: April 18. (Pass/Fail deadline May 2.)

1 Kitaev Chain

Consider the Kitaev chain model, which is physically a one dimensional superconducting wire, with the Hamiltonian:

$$H = -\mu \sum_n c_n^\dagger c_n - t \sum_n (c_{n+1}^\dagger c_n + \text{h.c.}) + \Delta \sum_n (c_n c_{n+1} + \text{h.c.})$$

where μ is the chemical potential, t is the hopping parameter, and Δ is the superconducting pairing. Notice, the Hamiltonian has the natural fermion parity symmetry.

Part 1

Express the creation and annihilation operators $c_n^\dagger$ and c_n in terms of Majorana operators γ_{2n-1} and γ_{2n}. Then rewrite the Hamiltonian in the special case where using these Majorana operators.

Part 2

For a Kitaev chain with open boundary conditions, show that when $\Delta = t$ and $\mu = 0$, there are two Majorana modes γ_1 and γ_{2N} that do not appear in the Hamiltonian. What is the physical significance of these modes? From now on, we will consider $\Delta = t$ for the rest of the problem.

Part 3

Calculate the energy spectrum $E(k)$ of the bulk Kitaev chain (with periodic boundary conditions) and determine the conditions on μ for which the energy gap closes. At what values of k does this occur?

Hint: You can do this in two ways: either in the Majorana basis, or in Nambu basis by diagonalizing the so-called Bogoliubov de Gennes Hamiltonian. In the latter, you construct a vector out of the creation and annihilation operators as $C = (c_1, c_2, \cdots, c_L, c_1^\dagger, \cdots, c_L^\dagger)^T$ *(in the position basis), and write the Hamiltonian as a Matrix* $H = \frac{1}{2}C^\dagger H_{BdG} C$*. The matrix* H_{BdG} *has to be diagonalized. You would want to do this in the Fourier basis.*

Part 4

Near the phase transition at $\mu = -2t$, expand the Hamiltonian $H(k)$ around $k = 0$ to obtain an effective Dirac Hamiltonian. Identify the "mass" parameter m and explain how its sign relates to the topological phase of the system.

Part 5

Consider a Kitaev chain in the continuum limit with a spatially varying mass parameter $m(x)$ that changes sign at $x = 0$, creating a domain wall between topologically distinct regions. The effective Dirac Hamiltonian near this domain wall is given by:

$$H = -v\tau_y\partial_x + m(x)\tau_z \tag{1}$$

where the mass satisfies $m(x) \to +m_0$ as $x \to +\infty$ and $m(x) \to -m_0$ as $x \to -\infty$.

(a) Show that for a zero-energy solution of this Hamiltonian ($H\Psi = 0$), the wavefunction must satisfy:

$$\partial_x\Psi(x) = \frac{1}{v}m(x)\tau_x\Psi(x) \tag{2}$$

(b) For a specific mass profile $m(x) = m_0 \tanh(x/\xi)$, where ξ is the width of the domain wall:

- Find the explicit form of the normalized zero-energy bound state
- Explain why only one of the two linearly independent solutions is physically acceptable

(c) Compare this zero energy bound state with the Majorana modes at the endpoints of a finite Kitaev chain. What similarities and differences exist between these two cases?

2 Cluster chain as an example of Symmetry protected topological phase

So far we have seen how the imposition of symmetry in a quantum many-body Hamiltonian leads to interesting phases and phase transitions depending on whether the symmetry is broken spontaneously. The symmetric state, aka the "paramagnet" was arguably the more boring of the scenarios, where we have a unique short range correlated ground state. In this problem (and in the rest of this week's lectures), we will see that even the humble paramagnet can have hidden long range order and non-trivial ground state degeneracy (when studied on topologically non-trivial manifolds), as long as the protecting symmetry is not broken.

We will study the following 1d Hamiltonian $H = -\sum_j h_j$, where $h_j = Z_{j-1}X_jZ_{j+1}$, where X_i, Z_i are Pauli operators associated with a lattice site i. We will study this problem on both an open chain, where the sites run over $j = 2, \cdots, N-1$, and a ring with periodic boundary condition, where the sites run over $j = 1, \cdots, N$.

a) Confirm that this is a commuting projector Hamiltonian: $[h_j, h_{j'}] = 0$.

b) What are the eigenvalues of h_j? Describe the ground state in terms of the eigenvalues of h_j.

c) Put the system on a ring. What is the ground state degeneracy of this model?

   Hint: Count the number of independent h_j (aka, the stabilizers).

d) Argue that the ground state can be described as:

$$|\psi\rangle = \prod_e V_e|+\rangle^{\otimes N} \tag{3}$$

   where e refer to an edge connecting the two sites e_1 and e_2, and $V_e = \frac{1}{2}(1 + Z_{e_1} + Z_{e_2} - Z_{e_1}Z_{e_2})$ is the CZ gate.

   Check that $[V_e, V_{e'}] = 0$. Also check that $V_e^\dagger V_e = \mathbb{I}$.

   What is the depth of $(\prod_e V_e)$ as a unitary circuit? Confirm that this is a depth 1 circuit, and hence in fact the ground state is just the transformed paramagnet state $|+\rangle^{\otimes N}$ with a depth 1 unitary circuit.

e) Consider now the model on open boundary conditions (OBC). What is the ground state degeneracy now?

f) We will argue that there will be 4 degenerate edge states, protected by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is generated by:

$$\tilde{X}_{\text{odd}} = \prod_{j=\text{odd}} X_j \tag{4}$$

$$\tilde{X}_{\text{even}} = \prod_{j=\text{even}} X_j \tag{5}$$

First confirm that the Hamiltonian has this symmetry (among many others! However, we will investigate the effects of breaking most other symmetries while preserving this one.)

g) Next, consider the action of the symmetry generators on the groundstates. To do this explicitly, consider a spin chain on even number of sites $2N$ with open boundary conditions. Show that we have,

$$\tilde{X}_{\text{odd}} |gs\rangle = \underbrace{(X_1 Z_2)}_{U_L^1} \underbrace{Z_{2N}}_{U_R^1} |gs\rangle \tag{6}$$

$$\tilde{X}_{\text{even}} |gs\rangle = \underbrace{Z_1}_{U_L^2} \underbrace{(Z_{2N-1} X_{2N})}_{U_R^2} |gs\rangle \, . \tag{7}$$

h) To interpret the relations above, prove that $U_L^{1,2}$ are two operators localized on the the left end, with the mutual anticommutation relation,

$$U_L^1 U_L^2 = -U_L^2 U_L^1, \tag{8}$$

and similarly for the operators on the right.

Argue that this implies that there is a degree of freedom localized on the left which has a Hilbert space dimension of 2. Does this explain the observed ground state degeneracy of the model on an open chain? Now argue why the ground state degeneracy will be a robust feature even when we move away from this fixed point Hamiltonian as long as we preserve the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. We call such paramagnetic states symmetry protected topological (SPT) phases.

i) Now, we consider the signature of the SPT-ness of the ground state through correlation functions. Argue that by definition, (any) ground state will have the following non-trivial string correlation function:

$$\langle gs| \, Z_{2k-1} X_{2k} X_{2k+2} X_{2k+4} \cdots X_{2j-2} X_{2j} Z_{2j+1} \, |gs\rangle = 1 \text{ for } k < j. \tag{9}$$

This is a "hidden" order even in the seemingly "disordered" paramagnetic state.

We can interpret the SPT phase in yet another useful way. Imagine the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT arising from two $\mathbb{Z}_2$ symmetric spin chains on the even and odd sites respectively. Recall, that $\mathrm{DW}_x = \prod_{j \in \text{even}, -\infty \leq j < x} X_j$ is the domain wall creation operator on the even chain at site x (and similarly for the odd chain). Recall from lecture that condensation of the domain wall operator ($\langle \mathrm{DW}_x \mathrm{DW}_y \rangle = O(1)$) leads to the conventional symmetric state, and the condensation of the Z operator ($\langle Z_x Z_y \rangle = O(1)$) leads to the SSB phase. Argue that the above non-trivial string order parameter implies that the bound state of the domain wall operator on the even chain and the Z operator on the odd chain are condensed. This is a general approach towards constructing certain SPT phases, referred to as domain wall decoration.

j) Lastly, confirm that the circuit in (d) $\prod_e V_e$ that connects the trivial paramagnetic state to the non-trivial SPT state does not preserve the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry in the sense: the entire circuit $\prod_e V_e$ commutes with th generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$, but the individual gates V_e do not commute with the symmetry.