**Quantum Matter**
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# Tutorial 2: More 1D models

## 1 TFIM by Kramers-Wannier duality

In this problem we will study the transverse field Ising model by yet another duality transformation: namely the Kramers-Wannier duality [1], filling in the details to what Chong presented in the last lecture.

We begin with the familiar Hamiltonian,

$$H = \sum_i \left[ -J\sigma_i^z \sigma_{i+1}^z - g\sigma_i^x \right]. \tag{1}$$

Let us try to change variables from the $\sigma$ basis which describes the spins to ones which describe domain walls. A complete basis of states is the set of eigenstates of $\sigma_i^z$ for all $i$. Each element of the basis is specified by a string of $N$ bits, with $N$ sites, with each bit given by state of $\sigma_i^z$. Now for each such string, i.e. classical arrangement of Ising spins, we can determine whether each bond, connecting neighboring spins, is satisfied or not. If it is satisfied, i.e. the two spins are parallel, we say there is no domain wall there. If it is not, i.e. the two spins are anti-parallel, we say there is a domain wall there. We will be working with the model on periodic boundary conditions, i.e., the qubits are on a ring.

a) Argue that we can assign a "dual" spin $\tau^x_{i+\frac{1}{2}} = \pm 1$ to each bond according to

$$\tau^x_{i+\frac{1}{2}} = \sigma_i^z \sigma_{i+1}^z, \tag{2}$$

where $\tau^x = +1$ if there is no domain wall, and $\tau^x = -1$ is there is a domain wall. First, show that given $\sigma_i^z$ values we uniquely determine the $\tau_a^x$ values. Is the converse true? Are all $\tau_a^x$ values allowed? For an allowed set of $\tau_a^x$ variables, how many linearly

[1] Adapted from notes by Leon Balents at `https://spinsandelectrons.com/wp-content/uploads/2016/04/notes.pdf`.

independent physical spin states are allowed? Show that the domain wall variables must satisfy the constraint,

$$\prod_a \tau_a^x = \prod_{i=1}^{N} \tau_{i+\frac{1}{2}}^x = 1. \tag{3}$$

*Hint: For a given set of $\tau_a^x$ variables, imagine fixing the first physical spin $\sigma_1^z$ arbitrarily, and then consider how the other spins are fixed.*

b) So we have seen that a set of variables $\tau_a^z$ does not uniquely label a physical spin state. Argue that we can add an additional quantum number, namely the eigenvalue of the Ising symmetry generator ($U = \prod_i \sigma_i^x$) to make the set of domain wall variables complete.

c) Now let us look at the conjugate variables. The operator $\sigma_i^x$ flips the $i$th spin, which therefore creates or annihilates domain walls on two bonds which share site $i$. Thus we expect that

$$\sigma_i^x = \mu_i \tau_{i-\frac{1}{2}}^z \tau_{i+\frac{1}{2}}^z, \tag{4}$$

where $\mu_i = \pm 1$ is a c-number, not an operator. Independent of the choice of $\mu_i$, this achieves the desired sign flip since $\tau_a^z$ anticommutes with $\tau_a^x$. The parameters $\mu_i$ are almost arbitrary and just represent a convention.

However, the parameters must still satisfy the symmetry constraint, set by the symmetry quantum number we have chosen. Explain how that constrains the allowed values of $\mu_i$.

d) Check that the newly defined $\tau$ operators satisfy the same operator algebra as the Paulis $\sigma$.

e) A somewhat inconvenient aspect of the duality map is that it expresses $\sigma_i^x$ in terms of $\tau_a^z$ and not the other way around. Argue that in general a single $\tau_a^z$ operators is not a physical operator, in that it does not satisfy the required domain wall constraint. Argue that this is related to the fact that there is no physical way to create a single domain wall. What is the operator dictionary for creating two domain walls at $a$ and $b$?

We can imagine "pushing" one domain wall off to $-\infty$ and show that we can write a formula for a single domain wall operator in an infinite system,

$$\tau_a^z \text{“=”} \prod_{i<a} \mu_i \sigma_i^x. \tag{5}$$

It captures the physical statement: to create a single domain wall we need to flip a semi-infinite string of spins to the left of the wall.

f) Now we will study the ground state of the dual model. We expect the ground state will be in the even parity sector. So let's assume $U = +1$ and proceed – this only captures half the states. Then we can take $\mu_i = +1$ and simplify the calculations. Show that we can rewrite the TFIM Hamiltonian, $H_\pm = P_\pm H P_\pm$, with $P_\pm = (1 \pm U)/2$ the projection operator onto the $U = \pm 1$ sector. We obtain

$$H_+ = \sum_{a \in \text{links}} \left[ -J\tau_a^x - g\tau_a^z \tau_{a+1}^z \right]. \tag{6}$$

Remarkably, this has an identical form to $H$ written in terms of the original spins, but with $g \leftrightarrow J$. We say that the transformation from $\sigma$ to $\tau$ variables is a duality transformation, and the one dimensional quantum Ising model is self-dual.

The duality implies that if the model with parameters $(J, g)$ is critical, then so is the model with parameters $(g, J)$. Assuming there is only one quantum phase transition, what is the value of the critical $g/J$?

g) Notice that this duality (dubbed Kramers-Wannier duality) maps the disordered phase to the ordered phase, and vice versa. Can you argue why this implies that the duality map is necessarily non-local?

# 2 Cluster chain as an example of Symmetry protected topological phase

So far we have seen how the imposition of symmetry in a quantum many-body Hamiltonian leads to interesting phases and phase transitions depending on whether the symmetry is broken spontaneously. The symmetric state, aka the "paramagnet" was arguably the more boring of the scenarios, where we have a unique short range correlated ground state. In this problem (and in the rest of this week's lectures), we will see that even the humble paramagnet can have hidden long range order and non-trivial ground state degeneracy (when studied on topologically non-trivial manifolds), as long as the protecting symmetry is not broken.

We will study the following 1d Hamiltonian $H = -\sum_j h_j$, where $h_j = Z_{j-1} X_j Z_{j+1}$, where $X_i, Z_i$ are Pauli operators associated with a lattice site $i$. We will study this problem on both an open chain, where the sites run over $j = 2, \cdots, N-1$, and a ring with periodic boundary condition, where the sites run over $j = 1, \cdots, N$.

a) Confirm that this is a commuting projector Hamiltonian: $[h_j, h_{j'}] = 0$.

b) What are the eigenvalues of $h_j$? Describe the ground state in terms of the eigenvalues of $h_j$.

c) Put the system on a ring. What is the ground state degeneracy of this model?

   *Hint: Count the number of independent $h_j$ (aka, the stabilizers).*

d) Argue that the ground state can be described as:

$$|\psi\rangle = \prod_e V_e |+\rangle^{\otimes N} \tag{7}$$

   where $e$ refer to an edge connecting the two sites $e_1$ and $e_2$, and $V_e = \frac{1}{2}(1 + Z_{e_1} + Z_{e_2} - Z_{e_1} Z_{e_2})$ is the $CZ$ gate.

   Check that $[V_e, V_{e'}] = 0$. Also check that $V_e^\dagger V_e = \mathbb{I}$.

   *Hint: It is most convenient to show this using the stabilizer formalism. Note,the $|+\rangle^{\otimes N}$ is stabilized by $\{X_j\}$. How does each $X_j$ transform under the unitary evolution $\prod_e V_e$? If you are not familiar with the stabilizer formalism, you can skip the proof during the tutorial.*

   What is the depth of $(\prod_e V_e)$ as a unitary circuit? Confirm that this is a depth 1 circuit, and hence in fact the ground state is just the transformed paramagnet state $|+\rangle^{\otimes N}$ with a depth 1 unitary circuit.

   By observation, we thus have $V_{1,2} X_1 V_{1,2} = X_1 Z_2$, and thus, $(\prod_e V_e) X_i (\prod_e V_e) = Z_{i-1} X_i Z_{i+1}$. Thus the state is exactly stabilized by the Hamiltonian terms.

e) Consider now the model on open boundary conditions (OBC). What is the ground state degeneracy now?

   *Hint: You can use the fact that $2^k$ linearly independent pure $N-$qubit states can be stabilized by $N - k$ independent stabilizers. Why is this true?*

f) We will argue that there will be 4 degenerate edge states, protected by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is generated by:

$$\tilde{X}_{\text{odd}} = \prod_{j=\text{odd}} X_j \tag{8}$$

$$\tilde{X}_{\text{even}} = \prod_{j=\text{even}} X_j \tag{9}$$

First confirm that the Hamiltonian has this symmetry (among many others! However, we will investigate the effects of breaking most other symmetries while preserving this one.)

g) Next, consider the action of the symmetry generators on the groundstates. To do this explicitly, consider a spin chain on even number of sites $2N$ with open boundary conditions. Show that we have,

$$\tilde{X}_{\text{odd}} |gs\rangle = \underbrace{(X_1 Z_2)}_{U_L^1} \underbrace{Z_{2N}}_{U_R^1} |gs\rangle \tag{10}$$

$$\tilde{X}_{\text{even}} |gs\rangle = \underbrace{Z_1}_{U_L^2} \underbrace{(Z_{2N-1} X_{2N})}_{U_R^2} |gs\rangle . \tag{11}$$

h) To interpret the relations above, prove that $U_L^{1,2}$ are two operators localized on the the left end, with the mutual anticommutation relation,

$$U_L^1 U_L^2 = -U_L^2 U_L^1, \tag{12}$$

and similarly for the operators on the right.

Argue that this implies that there is a degree of freedom localized on the left which has a Hilbert space dimension of 2. Does this explain the observed ground state degeneracy of the model on an open chain? Now argue why the ground state degeneracy will be a robust feature even when we move away from this fixed point Hamiltonian as long as we preserve the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. We call such paramagnetic states symmetry protected topological (SPT) phases.

Another way of interpreting these edge degrees of freedom is as projective representations of the protecting symmetry in one lower dimension (i.e. in this case, at a point). Recall (or look up) the definition of projective representations and argue the mutual anticommutation relations among $U_L^1$ and $U_L^2$ imply a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the edge of the system.

i) Now, we consider the signature of the SPT-ness of the ground state through correlation functions. Argue that by definition, (any) ground state will have the following non-trivial string correlation function:

$$\langle gs| Z_{2k-1} X_{2k} X_{2k+2} X_{2k+4} \cdots X_{2j-2} X_{2j} Z_{2j+1} |gs\rangle = 1 \text{ for } k < j. \tag{13}$$

This is a “hidden” order even in the seemingly “disordered” paramagnetic state.

j) We will interpret the SPT phase in yet another useful way. Imagine the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT arising from two $\mathbb{Z}_2$ symmetric spin chains on the even and odd sites respectively. Recall, that $\mathrm{DW}_x = \prod_{j \in \mathrm{even}, -\infty \leq j < x} X_j$ is the domain wall creation operator on the even chain at site $x$ (and similarly for the odd chain). Recall from lecture that condensation of the domain wall operator ($\langle \mathrm{DW}_x \mathrm{DW}_y \rangle = O(1)$) leads to the conventional symmetric state, and the condensation of the $Z$ operator ($\langle Z_x Z_y \rangle = O(1)$) leads to the SSB phase. Argue that the above non-trivial string order parameter implies that the bound state of the domain wall operator on the even chain and the $Z$ operator on the odd chain are condensed. This is a general approach towards constructing certain SPT phases, referred to as domain wall decoration.

k) Lastly, confirm that the circuit in (d) $\prod_e V_e$ that connects the trivial paramagnetic state to the non-trivial SPT state does not preserve the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.