Ex. 2(*a*) *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *satisfying the conditions* $u(0, t) = u(l, t) = 0$ and $u(x, 0) = lx - x^2$.

[Himachal 2007, 12; Meerut 2010; Delhi 2007]

(*b*) *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *satisfying the conditions* $u(0, t) = u(l, t) = 0$ and $u(x, 0) = x$ when $0 \le x \le l/2$; $u(x, 0) = l - x$ when $l/2 \le x \le l$.

[Meerut 1998, 2000]

Sol. We can prove that the solution of heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \text{... (1)}$$

subject to the boundary conditions $\qquad u(0, t) = u(a, t) = 0$ for all t ... (2)

and the initial condition $\qquad u(x, 0) = f(x),\ 0 < x < a$ ... (3)

is given by

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a) e^{-C_n^2 t} \qquad \text{... (4)}$$

where

$$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx, n = 1,2,3,... \qquad \text{... (5)}$$

and

$$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \text{... (6)}$$

Part (*a*) : Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and $f(x) = lx - x^2$. Hence, (5) reduces to

$$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left\{\frac{-\cos(n\pi x)/l}{(n\pi)/l}\right\} - (l - 2x)\left\{\frac{-\sin(n\pi x)/l}{(n\pi)^2/l^2}\right\} + (-2)\left\{\frac{\cos(n\pi x)/l}{(n\pi)^3/l^3}\right\}\right]_0^l$$

[Using the chain rule of integration by parts]

$$= (2/l)\{(-2l^3/n^3\pi^3)\cos n\pi + (2l^3/n^3\pi^3)\} = (4l^2/n^3\pi^3)\{1 - (-1)^n\}$$

$$\therefore \quad E_n = \begin{cases} (8l^2)/(2m-1)^3\pi^3, \text{ if } n = 2m-1 \text{ (odd) and } m = 1,2,3,... \\ 0, \text{ if } n = 2m \text{ (even) where } m = 1, 2, 3,... \end{cases}$$

Then, by (6), $C_n^2 = \{(2m-1)^2\pi^2 k\}/l^2$ and so from (4) the required solution is given by

$$u(x,t) = \frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2 kt\}/l^2}.$$

Part (b) : Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and

$$f(x) = \begin{cases} x, & \text{when } 0 \le x \le l/2 \\ l - x, & \text{when } l/2 \le x \le l \end{cases} \qquad \text{... (6)}$$

$$\therefore (5) \Rightarrow E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}dx + \int_{l/2}^l f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \int_0^{l/2} \frac{2x}{l} \sin \frac{n\pi x}{l} dx + \int_{l/2}^{l} \frac{2}{l}(l-x) \sin \frac{n\pi x}{l} dx$$

$$= \left[\left(\frac{2x}{l}\right)\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right) - \left(\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)}{(n\pi)^2/l^2}\right)\right]_0^{l/2} + \left[\left(\frac{2(l-x)}{l}\right)\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right) - \left(-\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)/l}{(n\pi)^2/l^2}\right)\right]_{l/2}^{l}$$

[Using chain rule of integration by parts]

$$= -(l/n\pi)\cos(n\pi/2) + (2l/n^2\pi^2)\sin(n\pi/2) + (l/n\pi)\cos(n\pi/2) + (2l/n^2\pi^2)\sin(n\pi/2)$$

$$\therefore \qquad E_n = \frac{4l}{n^2\pi^2} \sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 4l/(2m-1)^2\pi^2, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

Then, $\qquad (6) \Rightarrow C_n^2 = \{(2m-1)^2\pi^2 k\}/l^2$

$\therefore$ From (4),

$$u(x,t) = \frac{4l}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \sin\frac{(2m-1)\pi x}{l} e^{-\{(2m-1)^2\pi^2 kt\}/l^2}$$

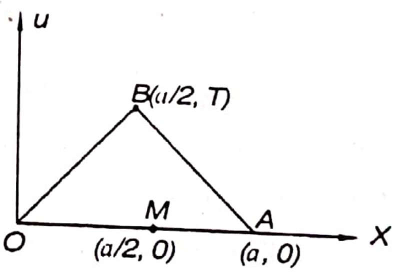
Ex. 2 (c) *Solve* $\partial u/\partial t = \partial^2 u/\partial x^2$, $0 < x < l$, $t > 0$ *given that* $u(0, t) = u(l, t) = 0$ *and* $u(x, 0) = x(l-x)$, $0 \le x \le l$. **[I.A.S. 2002]**

Sol. Refer solved Ex. 2(a). Here $k = 1$ and hence the solution reduces to

$$u(x,t) = \frac{8l^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin\frac{(2m-1)\pi x}{l} e^{-\{(2m-1)^2\pi^2 t\}/l^2}$$

Ex. 3. *A homogeneous rod of conducting material of length a has its ends kept at zero temperature. The temperature at the centre is T and falls uniformly to zero at the two ends. Find the temperature function u (x, t).*

Sol. We know that $u(x, t)$ is the solution of heat equation $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$. Here the boundary conditions are $u(0, t) = u(a, t) = 0$ for all $t \ge 0$. Let OA be the given rod and M be its middle point. Given that the temperature at the centre M is T and falls uniformly to zero at the two ends O and A of the rod. Hence, the temperature distribution at $t = 0$ is as given in the adjoining figure. The equations of straight lines OB and BA respectively are given by



$$u - 0 = \frac{T-0}{(a/2)-0}(x-0) \quad \text{and} \quad u - 0 = \frac{T-0}{(a/2)-a}(x-a) \qquad \ldots (i)$$

We can prove that the solution of heat equation $\qquad k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \ldots (1)$

subject to the boundary conditions $u(a, t) = u(a, t) = 0$, $\quad$ for all $t \qquad \ldots (2)$

and the initial condition $\qquad u(x, 0) = f(x), \quad 0 < x < a \qquad \ldots (3)$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)^{-C_n^2 t} \qquad \ldots (4)$$

where

$$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx, \quad n = 1, 2, 3, \qquad \ldots (5)$$

and

$$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \ldots (6)$$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = a$. Also, from (i), we have

$$u(x,0)=f(x)=\begin{cases}(2Tx)/a, \text{ where } 0\le x\le a/2\\ \{2T(a-x)\}/a, \text{ where } a/2\le x\le a\end{cases} \qquad \dots(7)$$

$$\therefore \quad (5)\Rightarrow \; E_n=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx=\frac{2}{a}\left[\int_0^{a/2} f(x)\sin\frac{n\pi x}{a}dx+\int_{a/2}^{a} f(x)\sin\frac{n\pi x}{a}dx\right]$$

$$=\frac{2}{a}\int_0^{a/2}\frac{2Tx}{a}\sin\frac{n\pi x}{a}dx+\frac{2}{a}\int_{a/2}^{a}\frac{2T(a-x)}{a}\sin\frac{n\pi x}{a}dx, \text{ using (7)}$$

$$=\int_0^{a/2}\frac{4Tx}{a^2}\sin\frac{n\pi x}{a}dx+\int_{a/2}^{a}\frac{4T(a-x)}{a^2}\sin\frac{n\pi x}{a}dx$$

$$=\left[\left(\frac{4Tx}{a^2}\right)\left(-\frac{\cos(n\pi x)/a}{(n\pi)/a}\right)-\left(\frac{4T}{a^2}\right)\left(-\frac{\sin(n\pi x)/a}{(n\pi)^2/a^2}\right)\right]_0^{a/2}$$

$$+\left[\left\{\frac{4T(a-x)}{a^2}\right\}\left(-\frac{\cos(n\pi x)/a}{(n\pi)/a}\right)-\left(-\frac{4T}{a^2}\right)\left(-\frac{\sin(n\pi x)/a}{(n\pi)^2/a^2}\right)\right]_{a/2}^{a}$$

$$=-(2T/n\pi)\cos(n\pi/2l)+(4T/n^2\pi^2)\sin(n\pi/2)+(2T/n\pi)\cos(n\pi/2)+(4T/n^2\pi^2)\sin(n\pi/2)$$

$$\therefore \quad E_n=\frac{8T}{n^2\pi^2}\sin\frac{n\pi}{2}=\begin{cases}0, \text{ if } n=2m \text{ and } m=1,2,3,\dots\\ \{8(-1)^{m+1}T\}/(2m-1)^2\pi^2, \text{ if } n=2m-1 \text{ and } m=1,2,3,\dots\end{cases}$$

Then, by (6), $C_n^2=\{(2m-1)^2\pi^2k\}/a^2$ and so from (4), the required solution is given by

$$u(x,t)=\frac{8T}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{a}e^{-\{(2m-1)^2\pi^2kt\}/a^2}$$

Ex. 4. *Solve the one-dimensional diffusion equation* $\partial^2u/\partial x^2=(1/k)(\partial u/\partial t)$ *in the range* $0\le x\le 2\pi, t\ge 0$ *subject to the boundary conditions :* $u(x,0)=\sin^3 x$ *for* $0\le x\le 2\pi$ *and* $u(0,t)=u(2\pi,t)=0$ *for* $t\ge 0$ **[Meerut 2008; Delhi Maths (H), 2004, 06, 09]**

Sol. Proceed upto equation (18) as in Art. 2.3 A by taking $a=2\pi$ and $f(x)=\sin^3 x$. Then, equation (18) for the present problem reduces to

$$u(x,t)=\sum_{n=1}^{\infty}E_n\sin\left(\frac{nx}{2}\right)e^{-(n^2kt)/4}, \quad n=1,2,3,\dots \qquad \dots(i)$$

Putting $t=0$ in (*i*) and using given condition $u(x,0)=\sin^3 x$, we get

$$\sum_{n=1}^{\infty}E_n\sin(nx/2)=\sin^3 x=(3/4)\times\sin x-(1/4)\times\sin 3x$$

$$[\because \sin 3x=3\sin x-4\sin^3 x\Rightarrow \sin^3 x=(1/4)\times(3\sin x-\sin 3x)]$$

or $$E_1\sin(x/2)+E_2\sin x+E_3\sin(3x/2)+E_4\sin 2x+E_5\sin(5x/2)$$
$$+E_6\sin 3x+E_7\sin(7x/2)+\dots=(3/4)\times\sin x-(1/4)\times\sin 3x. \qquad \dots(ii)$$

Equating the coefficients of the like terms on both sides of (*ii*), we get

$E_2 = 3/4$, $E_6 = -1/4$ and $E_n = 0$ when $n \neq 2$ or $n \neq 6$. Substituting these values in (i), the required solution is

$$u(x, t) = E_2 \sin x\, e^{-kt} + E_6 \sin 3x\, e^{-9kt} = (1/4) \times [3 \sin x\, e^{-kt} - \sin 3x\, e^{-9kt}].$$

Ex. 5. *Determine u such that* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *and satisfy the conditions (i)* $u \to 0$ *as* $t \to \infty$ *(ii)* $u = \sum_n c_n \cos nx$ *for* $t = 0$. **[Delhi Maths (H) 2004]**

Sol. Given $$\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t). \quad \text{... (1)}$$

Also given that $$u \to 0 \quad \text{as} \quad t \to \infty. \quad \text{... (2)}$$

and $$u(x, 0) = \sum_n c_n \cos nx. \quad \text{... (3)}$$

Let a solution of (1) be $$u(x, t) = X(x)\, T(t), \quad \text{...(4)}$$

where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone.

Substituting (4) in (1), we get $$(1/X)X'' = (1/kT)T' \quad \text{... (5)}$$

Since the L.H.S of (5) is a function of x alone and the R.H.S. of (6) is a function of t alone, hence the two sides of (6) can be equal only if each side is equal to a constant, λ say. In view of condition (2), we choose $\lambda = -n^2$, where n is a non-zero constant. Then (6) gives

$$(1/X)X'' = -n^2 \quad \text{so that} \quad (D^2 + n^2)X = 0, \quad \text{where} \quad D \equiv d/dx.$$

and $$(1/kT)T' = -n^2 \quad \text{so that} \quad (1/t)dT = -n^2 k dt.$$

Solving these $\quad X_n(x) = a_n \cos nx + b_n \sin nx \quad$ and $\quad T_n(t) = e_n e^{-n^2 kt}$

Keeping (3) and (4) in view, the most general solution of (1) may be written as

$$u(x,t) = \sum_n u_n(x,t) = \sum_n X_n(x) T_n(t) \quad \text{or} \quad u(x,t) = \sum_n (c_n \cos nx + d_n \sin nx) e^{-n^2 kt}. \quad \text{... (1)}$$

where $c_n (= a_n e_n)$ and $d_n (= b_n e_n)$ are new arbitrary constants.

Putting $t = 0$ in (1) and using (3), we have $\quad \sum_n c_n \cos nx = \sum_n (c_n \cos nx + d_n \sin nx)$,

showing that for the present problem $d_n = 0$. Then, from (1) the required solution is

$$u(x, t) = \sum_n c_n \cos nx\, e^{-n^2 kt}$$

Ex. 6. *A uniform rod 20 cm in length is insulated over its sides. Its ends are kept at 0°C. Its initial temperature is* $\sin(\pi x/20)$ *at a distance x from an end. Find temperature u(x, t) at time t. Given that* $\partial u/\partial t = a^2(\partial^2 u/\partial x^2)$. **[Nagpur 1996]**

Sol. Given $$\partial u/\partial t = a^2(\partial^2 u/\partial x^2). \quad \text{... (1)}$$

Boundary Conditions (B.C.) : $$u(0, t) = u(20, t) = 0 \text{ for all } t \quad \text{... (2)}$$

Initial Conditions (I.C.) : $$u(x, 0) = \sin(\pi x/20),\ 0 \leq x \leq 20 \quad \text{... (3)}$$

Let a solution of (1) be $$u(x, t) = X(x)\, T(t). \quad \text{... (4)}$$

Substituting this value of u in (1), we have

$$XT' = a^2 X'' T \quad \text{or} \quad X''/X = T'/a^2 T. \quad \text{... (5)}$$

$$u(x,t) = (E\cos\lambda x + F\sin\lambda x)e^{-\lambda^2 kt}, \quad \text{... (14)}$$

where $A = a_1 a_3$, $B = a_2 a_3$, $C = b_1 b_3$, $D = b_2 b_3$, $E = c_1 c_3$ and $F = c_2 c_3$

Now, the condition (2) demands that u should remain finite as $t \to \infty$. We therefore reject solution (13).

Next, in view of B.C. (3), solution (12) gives $0 = A\cdot 0 + B$ and $0 = A\cdot\pi + B$. These give $A = B = 0$ and hence from (12), $u = 0$ for all t. This is a trivial solution. Since we are looking for a non-trivial solution, we reject the solution (12) also. Thus, the only possible solution satisfying the condition (2) is given by (14).

Putting $x = 0$ in (4) and using B.C. $u(0, t) = 0$ given by (3), we obtain $E = 0$. Then, (14) is simplified as

$$u(x,t) = F\sin\lambda x\, e^{-\lambda^2 kt} \quad \text{... (15)}$$

Putting $x = \pi$ in (15) and using the BC $u(\pi,t) = 0$ given by (3), we obtain

$$0 = F\sin\lambda\pi\, e^{-\lambda^2 kt} \quad \text{giving} \quad F\sin\lambda\pi = 0 \quad \text{... (16)}$$

Now, in view of $E = 0$ we must take $F \neq 0$ in order to obtain a non-trivial solution of (1). Accordingly (16) yields

$\sin\lambda\pi = 0$ or $\lambda\pi = n\pi$ so that $\lambda = n$, $n = 1, 2, 3, \ldots$

Thus, from (15), we arrive at a solution of the form

$$u_n(x,t) = F_n \sin nx\, e^{-n^2 kt}, \; n = 1, 2, 3, \ldots$$

Noting that the diffusion equation (1) is linear, its most general solution is obtained by applying the principle of superposition. Thus, the general solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} F_n \sin nx\, e^{-n^2 kt} \quad \text{... (17)}$$

Putting t = 0 in (17), we have

$$u(x,0) = \sum_{n=1}^{\infty} F_n \sin nx, \quad \text{... (18)}$$

which is a half-range Fourier-sine series and, hence

$$F_n = \frac{2}{\pi}\int_0^{\pi} u(x,0)\sin nx\, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} u(x,0)\sin nx\, dx + \int_{\pi/2}^{\pi} u(x,0)\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx, \text{ using (5)}$$

$$= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

[Using the chain rule of Integrating by parts]

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left(\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right)$$

$$\therefore \quad F_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 4(-1)^{m+1}/\pi(2m-1)^2, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

Substituting the above value of F_n in (17), the required solution is

$$u(x,t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin(2m-1)x\, e^{-(2m-1)^2 kt}$$

Ex. 9. *Make use of the method of separating variables to solve*

$$\partial u/\partial t = c^2(\partial^2 u/\partial x^2),\ t > 0,\ 0 \le x \le 1 \quad \ldots (1)$$

$$u(0,t) = 2, \qquad u(1,t) = 3 \quad \ldots (2)$$

$$u(x,0) = x(1-x) \quad \ldots (3)$$

Sol. Let

$$g(x) = c_1 x + c_2 \quad \ldots (4)$$

that

$$g(0) = 2 \quad \text{and} \quad g(1) = 3. \quad \ldots (5)$$

Using (5), (4) reduces to $2 = c_2$ and $3 = c_1 + c_2$, so that $c_1 = 1$ and $c_2 = 2$, Then (4) becomes

$$g(x) = x + 2 \quad \ldots (6)$$

Let

$$v(x,t) = v(x,t) - g(x) = u(x,t) - x - 2. \quad \ldots (7)$$

Using (7), (1), (2) and (3) respectively may be re-written as

$$\partial v/\partial t = c^2(\partial^2 v/\partial x^2) \quad \ldots (8)$$

$$v(0,t) = 0, \qquad v(1,t) = 0 \quad \ldots (9)$$

$$v(x,0) = x(1-x) - x - 2 = -(x^2+2). \quad \ldots (10)$$

Proceed as Art. 2.3B taking $u = v$, $a = 1$, $k = c^2$, $C_n^2 = n^2\pi^2c^2$ and $f(x) = -(x^2+2)$

$$\therefore \qquad v(x,t) = \sum_{n=1}^{\infty} E_n \sin n\pi x\, e^{-n^2\pi^2c^2 t} \quad \ldots (11)$$

$$E_n = -2\int_0^1 (x^2+2)\sin n\pi x\, dx = 2\left[(x^2+2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

[Using the chain rule of integration by parts].

$$= -2\left[-\frac{3(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} + \frac{2}{n\pi} - \frac{2}{n^3\pi^3}\right] \quad \ldots (12)$$

Using (11), (7) gives the required solution $\qquad u(x,t) = v(x,t) + x + 2$

$$u(x,t) = x + 2 + \sum_{n=1}^{\infty} E_n \sin n\pi x\, e^{-n^2\pi^2c^2 t}, \qquad \text{where} \quad E_n \text{ is given by (12)}$$

4A. General solution of heat equation when both the ends of a bar are insulated and the initial temperature is prescribed. **(Meerut 2002)**

If both the ends of a bar of length a are insulated and the initial temperature f(x) is prescribed, then to find the temperature at a subsequent time t.

Sol. Here the temperature $u(x, t)$ in the given bar is governed by one dimensional heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \ldots (1)$$

Physical experiments show that the rate of heat flow is proportional to the gradient $\partial u/\partial x$ of the temperature $u(x, t)$. Hence if the ends $x = 0$ and $x = a$ of the bar are insulated, so that no heat can flow through the ends, we have the boundary conditions:

$$u_x(0,t) = u_x(a,t) = 0 \text{ for all } t \quad \ldots (2)$$

2.4 C. Solved examples based on Art. 2.4 A and Art 2.4 B

Ex. 1 **(*a*)** *Solve* $k(\partial^2 u/\partial x^2) = \partial u/\partial t$ *for* $0 < x < \pi, t > a$, if $u_x(0,t) = u_x(\pi,t) = 0$ and $u(x, 0) = \sin x$. [I.A.S. 2002]

(*b*) *Find the temperature in a laterally insulated bar of length a whose ends are insulated assuming that the initial temperature is* $f(x) = \begin{cases} x, \text{ if } a < x < a/2 \\ a - x, \text{ if } a/2 < x < a \end{cases}$

Sol. We can prove that (prove is examination for complete solution) that the solution of heat equation.

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \text{... (1)}$$

subject to the boundary conditions $u_x(0, t) = u_x(a, t) = 0$ for all t ... (2)

and the initial condition $u(x, t) = f(x), \ 0 < x < a$ ... (3)

is given by

$$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{a} e^{-C_n^2 t} \quad \text{... (4)}$$

where $$E_0 = \frac{2}{a}\int_0^a f(x)dx, \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx, \ n = 1, 2, 3, \ldots \quad \text{... (5)}$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \quad \text{... (6)}$$

(*a*) Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = \pi$ and $f(x) = \sin x$. Hence, from (5), we get

$$E_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\, dx = \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}\times 2 = \frac{4}{\pi} \quad \text{... (7)}$$

and $$E_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi}\{\sin(n+1) - \sin(n-1)x\}dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

Thus, $$E_n = \frac{1}{\pi}\left[\frac{1-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right] \quad \text{... (8)}$$

If $n = 2m - 1$ (odd) with $m = 1, 2, 3, \ldots$, then $E_n = 0$, by (8)

If $n = 2m$ (even) with $m = 1, 2, 3, \ldots$, then from (8), we get

$$E_{2m} = \frac{2}{m}\left(\frac{1}{2m+1} - \frac{1}{2m-1}\right) = -\frac{4}{\pi(4m^2-1)}, \ m = 1, 2, 3 \quad \text{... (9)}$$

Also, from (6), $$C_{2m}^2 = \{(2m)^2 \times \pi^2 \times k\}/\pi^2 = 4m^2\pi^2$$

Substituting the values of E_0 and E_{2m} given by (7) and (9) in (4), the required solution is

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum \frac{\cos 2mx}{4m^2-1} e^{-4m^2 k t}$$

(*b*) Comparing the given boundary value problem with boundary value problem given by (1), (2) and (3), we have $k = k$, $a = a$ and

$$u(x,0)=f(x)=\begin{cases} x, & \text{if } 0<x<a/2 \\ a-x, & \text{if } a/2<x<a \end{cases} \qquad \text{... (7)}$$

From (5) and (7), we have

$$E_0=\frac{2}{a}\int_0^a f(x)dx=\frac{2}{a}\left[\int_0^{a/2} f(x)dx+\int_{a/2}^a f(x)\,dx\right]$$

$$=\frac{2}{a}\left[\int_0^{a/2} x\,dx+\int_{a/2}^a (a-x)dx\right]=\frac{2}{a}\left\{\left[\frac{x^2}{2}\right]_0^{a/2}+\left[ax-\frac{x^2}{2}\right]_{a/2}^a\right\}$$

$$=(2/a)\times\left\{a^2/8+(a^2/2-3a^2/8\right\}=(2/a)\times(a^2/4)=a/2$$

$$E_n=\frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx=\frac{2}{a}\left[\int_0^{a/2} f(x)\cos\frac{n\pi x}{a}dx+\int_{a/2}^a f(x)\cos\frac{n\pi x}{a}dx\right]$$

$$=\frac{2}{a}\int_0^{a/2} x\cos\frac{n\pi x}{a}dx+\frac{2}{a}\int_{a/2}^a (a-x)\cos\frac{n\pi x}{a}dx, \text{ using (7)}$$

$$=\frac{2}{a}\left[(x)\frac{\sin(n\pi x/a)}{(n\pi/a)}-(1)\left(-\frac{\cos(n\pi x/a)}{(n^2\pi^2/a^2)}\right)\right]_0^{a/2}+\frac{2}{a}\left[(a-x)\frac{\sin(n\pi x/a)}{(n\pi/a)}-(-1)\left(-\frac{\cos(n\pi x/a)}{(n^2\pi^2/a^2)}\right)\right]_{a/2}^a$$

[Using chain rule of integrating by parts]

$$=\frac{2}{a}\left[\frac{a}{2}\times\frac{a}{n\pi}\sin\frac{n\pi}{2}+\frac{a^2}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{a^2}{n^2\pi^2}\right]+\frac{2}{a}\left[-\frac{a^2}{n^2\pi^2}\cos n\pi-\frac{a}{2}\times\frac{a}{n\pi}\sin\frac{n\pi}{2}+\frac{a^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$=\frac{2}{a}\left[\frac{2a^2}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{a^2}{n^2\pi^2}-\frac{a^2}{n^2\pi^2}\cos n\pi\right]=\frac{2a}{n^2\pi^2}\left(2\cos\frac{n\pi}{2}-\cos n\pi-1\right)$$

Also, from (6), we have $\qquad C_n^2=(n^2\pi^2k)/a^2$

Substituting the above values of E_0, E_n and C_n^2 in (4), the required solution is given by

$$u(x,t)=\frac{a}{4}+\frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\cos\frac{n\pi}{2}-\cos n\pi-1\right)\cos\frac{n\pi x}{a}e^{-(n^2\pi^2kt)/a^2}$$

or

$$u(x,t)=\frac{a}{4}-\frac{8a}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{a}e^{-(4\pi^2kt)/a^2}+\frac{1}{6^2}\cos\frac{6\pi x}{a}e^{-(36\pi kt/a^2}+\ldots\right)$$

Ex. 2. *Find the solution of the one-dimensional diffusion equation* $k(\partial^2u/\partial x^2)=\partial u/\partial t$ *satisfying the following boundary conditions : (i) u is bounded as* $t\to\infty$ *(ii)* $u_x\,(0,t)=0$, $u_x(a,t)=0$ *for all t (iii)* $u(x,0)=x\,(a-x)$, $0<x<a$. **(Himanchal 2007)**

Sol. We know that the bounded solution the diffusion equation

$$k(\partial^2u/\partial x^2)=\partial u/\partial t \qquad \text{... (1)}$$

subject to the boundary conditions $\qquad u_x\,(0,t)=u_x\,(a,t)=0$ for all t $\qquad$ ... (2)

and the initial condition $$u(x, 0) = f(x),\ 0 < x < a \qquad \dots(3)$$

is given by $$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{a} e^{-C_n^2 t} \qquad \dots(4)$$

where $$E_0 = \frac{2}{a}\int_0^a f(x)dx, \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx,\ n = 1, 2, 3, \dots \qquad \dots(5)$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \dots(6)$$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = a$ and $f(x) = ax - x^2$. So from (5), we have

$$E_0 = \frac{2}{a}\int_0^a (ax - x^2)dx = \frac{2}{a}\left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{a^2}{3}$$

$$E_n = \frac{2}{a}\int_0^a (ax - x^2)\cos\frac{n\pi x}{a}dx$$

$$= \frac{2}{a}\left[(ax - x^2)\frac{\sin(n\pi x/a)}{(n\pi/a)} - (a - 2x)\left(-\frac{\cos(n\pi x/a)}{n^2\pi^2/a^2}\right) + (-2)\left(-\frac{\sin(n\pi x/a)}{n^3\pi^3/a^3}\right)\right]_0^a$$

[Using chain rule of integrating by parts]

$$= \frac{2}{a}\left[-ax\frac{a^2}{n^2\pi^2}(-1)^n - ax\frac{a^2}{n^2\pi^2}\right] = -\frac{2a^2}{n^2\pi^2}\{1 + (-1)^n\}$$

Hence, if $n = 2m$ (even), then $$E_n = E_{2m} = -(a^2/m^2\pi^2)$$

and if $n = 2m-1$ (odd), then $$E_n = E_{2m-1} = 0.$$

Also, from (6), $$C_n^2 = (n^2\pi^2 k)/a^2 = (4m^2\pi^2 k)/a^2, \text{ if } n = 2m$$

Substituting the above values of E_0, E_n and C_n^2 in (4), the required solution is given by

$$u(x,t) = \frac{a^2}{6} + \sum_{m=1}^{\infty}\left(-\frac{a^2}{m^2\pi^2}\right)\cos\frac{2m\pi x}{a}e^{-(4m^2\pi^2 kt)/a^2}$$

or $$u(x,t) = \frac{a^2}{6} - \frac{a^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\frac{2m\pi x}{a}e^{-(4m^2\pi^2 kt)/a^2}$$

2.5. Solution of heat equation when one end is insulated

While finding the solution of heat equation when one end is insulated while the other end is kept at a constant temperature, we proceed as explained in the following example.

Example: *Obtain temperature distribution $y(x, t)$ in a uniform bar of unit length whose one end is kept at $10°$ C and the other end is insulated. Further it is given that $y(x, 0) = 1 - x$, $0 < x < 1$.*

Sol. Suppose the bar be placed along the x-axis with its one end (which is at $10°C$) at origin and the other end at $x = 1$ (which is insulated so that flux $-K(\partial y/\partial x)$ is zero there, K being the thermal conductivity). Then we are to solve heat equation

$$\partial y/\partial t = k(\partial^2 y/\partial x^2) \qquad \dots(1)$$

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