Q1) $U_t = U_{xx}$ — ①, $0 \le x \le 1$, $t > 0$

$u(x,0) = x - x^2$, $u(0,t) = u(1,t) = 0$

Actual heat eq^n is $U_t = \alpha U_{xx}$ — ⓐ, $u(0,t) = u(l,t) = 0$, $u(x,0) = f(x)$

$0 \le x \le l$, $t > 0$

heat equation

(by Comparing)

eq^n ① & ⓐ, we get $\alpha = 1$

~~So~~ The solution of eq^n ⓐ is

$$U_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right) \cdot e^{-\alpha\lambda^2 t}$$

where $A_n = \dfrac{2}{l}\displaystyle\int_0^l f(x) \cdot \sin\left(\frac{n\pi}{l}x\right) dx.$

Now, sol^n of (1),

where $\alpha = 1$, lenght of rod is 1 i.e $l = 1$

$$U_n(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi)x \cdot e^{-\lambda^2 t} \qquad \text{where } \lambda = n\pi$$

So, $$U_n(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi)x \cdot e^{-(n^2\pi^2)t} \quad \text{— ③}$$

$n = 0, 1, 2, \ldots$

Now, find A_n, after finding A_n put in eq^n ③, which is the final solution.

Use, ~~[illegible]~~ $f(x) = x - x^2$

$$A_n = 2\int_0^1 (x - x^2) \cdot \sin(n\pi x)\, dx$$

then we get $A_n = \dfrac{4(1-(-1)^n)}{n^3\pi^3}$, $n = 1, 2, 3, \ldots$

If n is odd {means $n = 2m-1$} then $A_n = \dfrac{8}{(2m-1)^3\pi^3}$ for $m = 1, 2, 3, \ldots$

If n is even, then $A_n = 0$

So, solution is $$\boxed{U_n(x,t) = \frac{8}{\pi^3}\sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin(2m-1)\pi x\, e^{-(2m-1)^2\pi^2 t}}$$

Q2

$$u_t = u_{xx}, \quad u(0,t) = u(100,t) = 0$$

$$u(x,0) = \begin{cases} x ; & 0 \le x \le 50 \\ 100 - x ; & 50 \le x \le 100 \end{cases}$$

Now, $\alpha = 1, \ l = 100$

Sol^n is :-
$$u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{100}\right)x \, e^{-\frac{n^2\pi^2}{100^2}\cdot t}$$

where
$$A_n = \frac{2}{100}\int_0^{100} f(x) \sin\left(\frac{n\pi}{100}\right)x \, dx$$

$$A_n = \frac{2}{100}\left[\int_0^{50} x \sin\left(\frac{n\pi}{100}\right)x \, dx + \int_{50}^{100} (100-x) \sin\left(\frac{n\pi x}{100}\right) dx\right]$$

(using chain rule) Integrate A_n

$$= \frac{2}{100}\left[\left\{ x\left(\frac{-\cos\frac{n\pi}{100}x}{(n\pi/100)}\right) + \frac{\sin\left(\frac{n\pi}{100}\right)x}{n^2\pi^2/100^2}\right\}_0^{50} + \left\{(100-x)\left(\frac{-\cos\frac{n\pi}{100}x}{(n\pi/100)}\right) - \left(\frac{\sin\frac{n\pi}{100}x}{\frac{n^2\pi^2}{100^2}}\right)\right\}_{50}^{100}\right]$$

to make it easy let $l = 100$, then $50 = \frac{l}{2}$

$$= \frac{2}{l}\left[\left\{ x\left(\frac{-\cos\left(\frac{n\pi}{l}x\right)}{\left(\frac{n\pi}{l}\right)}\right) + \frac{\sin\left(\frac{n\pi}{l}\right)}{\frac{(n\pi)^2}{l^2}}\right\}_0^{l/2} + \left\{(l-x)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - \frac{\sin\left(\frac{n\pi}{l}\right)x}{\frac{(n\pi)^2}{l^2}}\right\}_{l/2}^{l}\right]$$

$$\therefore A_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4l}{(2m-1)^2\pi^2}, & n = 2m-1, \ m = 1,2,3\ldots \end{cases}$$

$$\boxed{u_n(x,t) = \frac{4l}{\pi^2}\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \sin\frac{(2m-1)\pi x}{l} \, e^{-\frac{(2m-1)^2\pi^2 t}{l^2}}, \text{ where } l = 100}$$

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Q3) $u_t = k\, u_{xx}$, $u(x,0) = 6\sin\left(\frac{\pi x}{l}\right)$ — ⓐ

$0 \le x \le l,\ t > 0,\quad u(0,t) = u(l,t) = 0$

Sol^n

$$u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-k\left(\frac{n^2\pi^2}{l^2}\right) t} \quad \text{— (1)}$$

$$A_n = \frac{2}{l}\int_0^l 6\sin\left(\frac{\pi x}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

, solve by $2\sin A \sin B$ formula, & integrate, then put $A_n = ?$ in 1.

Alternative Solution :

apply ⓐ condition in (1) $\quad u(x,0) = 6\sin\left(\frac{\pi x}{l}\right)$, $t=0$

in (1) $\quad u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \cdot 1$

$$6\sin\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

Expand it by putting $n=1$, then $n=2$, & so on.

$$\underbrace{6\sin\left(\frac{\pi x}{l}\right)}_{f(x)} = \underbrace{A_1 \sin\left(\frac{\pi x}{l}\right)}_{I^{st}\text{ term}} + \underbrace{A_2 \sin\left(\frac{2\pi x}{l}\right)}_{II^{nd}\text{ term}} + \underbrace{A_3 \sin\left(\frac{3\pi x}{l}\right)}_{III^{rd}\text{ term}} + \cdots$$

Now, $f(x)$ & I^{st} term are equal i.e $\sin\left(\frac{\pi x}{l}\right)$,

So by comparing coefficients of $\sin\left(\frac{\pi x}{l}\right)$, we get $\boxed{A_1 = 6}$,

& on L.H.S there are no terms of $\sin\left(\frac{2\pi x}{l}\right), \sin\left(\frac{3\pi x}{l}\right), \ldots$

$\therefore\quad A_2 = A_3 = A_4 = \cdots = A_n = 0$

Now, put $\boxed{A_1 = 6}$ in (1), where $n = 1$

$$\boxed{u(x,t) = 6\sin\left(\frac{\pi x}{l}\right) \cdot e^{-k\frac{\pi^2}{l^2}t}} \quad \text{Sol}^n$$

Q 4) $u_t = k u_{xx}, \quad 0 \le x \le l, \ t > 0$

$u(0,t) = u(l,t) = 0, \quad u(x,0) = 12 \sin\left(\frac{9\pi x}{l}\right) - 7\sin\left(\frac{4\pi x}{l}\right).$

Sol^n is $$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{-k\frac{n^2\pi^2}{l^2}t} \quad \text{—} (1)$$

Use $u(x,0) = 12\sin\left(\frac{9\pi x}{l}\right) - 7\sin\left(\frac{4\pi x}{l}\right)$ in 1, by comparing coefficients.

Now in (1), ~~Expand it~~ use $u(x,0)$,

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \cdot 1$$

Expand it

$$12\sin\left(\frac{9\pi x}{l}\right) - 7\sin\left(\frac{4\pi x}{l}\right) = A_1 \sin\left(\frac{\pi x}{l}\right) + A_2 \sin\left(\frac{2\pi x}{l}\right) + A_3 \sin\left(\frac{3\pi x}{l}\right) + \dots - - - - + A_n\left(\sin\left(\frac{n\pi x}{l}\right)\right)$$

↓

here we have only $\sin\left(\frac{9\pi x}{l}\right)$ & $\sin\left(\frac{4\pi x}{l}\right)$ terms, so after comparing it with R.H.S

only $n = 9$ & $n = 4$ remains non-zero, otherwise all are zero $A_1 = A_2 = A_3 = A_5 = A_6 = A_7 = A_8 = A_{10} = \dots = A_n = 0$

So, by comparing coefficients of

$$\boxed{12\sin\left(\frac{9\pi x}{l}\right)} + \boxed{-7\sin\left(\frac{4\pi x}{l}\right)} = A_1 \sin\left(\frac{\pi x}{l}\right) + \dots + \boxed{A_4 \sin\left(\frac{4\pi x}{l}\right)} + A_5\left(\sin\left(\frac{5\pi x}{l}\right)\right) + \dots - + \boxed{A_9 \sin\left(\frac{9\pi x}{l}\right)} + \dots + A_n \sin\left(\frac{n\pi x}{l}\right)$$

compare

Compare

we get $\underbrace{A_4}_{n=4} = -7$, $\underbrace{A_9}_{n=9} = 12$, now put A_4 & A_9 in (1), we get

Solution

$$\boxed{u(x,t) = -7\sin\left(\frac{4\pi x}{l}\right) e^{-k\frac{4^2\pi^2}{l^2}t} + 12\sin\left(\frac{9\pi x}{l}\right) \cdot e^{-k\frac{9^2\pi^2}{l^2}t}}$$