SIT B

Part I Ans:

Partial pivoting is needed to eliminate the 'division by zero' error in gaussian elimination, when in case the pivot element becomes zero. ~~To avoid~~

Part II Ans:

A system of linear equations can be written in the form:-

$[A][X] = [C]$; where A = matrix of coefficients, X = matrix of variables, C = matrix of constants.

We are asked to find $[A]^{-1}$.

So, converting the system of linear equations in matrix form gives us:-

$$\underbrace{\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}}_{[A]} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{[X]} = \underbrace{\begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix}}_{[C]}$$

In LU decomposition, we divide the matrix A into L & U matrices such that

$$[A] = [L]\cdot[U]$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Step 1: Find the ~~[L]~~ [U] matrix from the "forward elimination" step of Gaussian elimination on [A]

$$[A] = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$$

1st step of forward elimination:

pivot element $= a_{11} = 7$

$$R_3' = R_3 - \frac{-3}{7} \times R_1$$

$$a_{31}' = a_{31} - \frac{a_{31}}{a_{11}} \times a_{11} = 0 \;;\; l_{31} = \frac{-3}{7} = -0.42857$$

$$a_{32}' = a_{32} - \frac{a_{31}}{a_{11}} \times a_{12} = 4 - \frac{-3}{7} \times 2 = 4.8571$$

$$a_{33}' = a_{33} - \frac{a_{31}}{a_{11}} \times a_{13} = -2 - \frac{-3}{7} \times 1 = -1.5715$$

$$[A] = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 4.8571 & -1.5715 \end{bmatrix}$$

2nd step of forward elimination:

pivot element $= a_{22} = 3$

$$R_3' = R_3 - \frac{4.8571}{3} \times R_2$$

$$a_{32}' = a_{32} - \frac{a_{32}}{a_{22}} \times a_{22} = 0 \;;\; l_{32} = \frac{a_{32}}{a_{22}} = \frac{4.8571}{3} = 1.6190$$

$$a_{33}' = a_{33} - \frac{a_{32}}{a_{22}} \times a_{23} = -1.5715 - \frac{4.8571}{3} \times (-1)$$
$$= 0.047533$$

$$[A] = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & .047533 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = [U]$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -.42857 & 1.6190 & 1 \end{bmatrix}$$

Step 2 :

Since, $[A][A]^{-1} = [I]$

We are to find $[A]^{-1}$, let $[A]^{-1} = [B]$

Then, $[A][B] = [I]$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $[L][Z] = [I]$

& $[U][B] = [Z]$ for finding each column of the inverse matrix, i.e. the $[B]$ matrix

For first column,

$$[A] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

[NOTE : Shrinking the no. of columns in [B] matrix doesn't effect the product value of the [I] matrix at the corresponding column]

To find $\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$;

first, $[L][Z] = [I]$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.42857 & 1.6190 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Using $\boxed{\text{Forward substitution}}$, we get

$z_1 = 1$;

$z_2 = 0$;

$$-0.42857 z_1 + 1.6190 z_2 + z_3 = 0$$

$$\Rightarrow z_3 = 0.42857 z_1 - 1.6190 z_2$$

$$= 0.42857 \times 1 - 0$$

$$= 0.42857$$

$$= 0.42857$$

Next, $[U][B] = [Z]$

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0.047533 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.42857 \end{bmatrix}$$

Using $\boxed{\text{Back substitution}}$, we get,

$$0.047533\, b_{31} = 0.42857$$

$$\Rightarrow b_{31} = 9.01626;$$

$3b_{21} - b_{31} = 0 \therefore b_{21} = 3.00542$; $7b_{11} + 2b_{21} + b_{31} = 1$; $\therefore b_{11} = -2.1467$

for second column,

$$[A] \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

To find $\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix}$;

First, $[L][Z] = [I]$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.42857 & 1.6190 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Using forward substitution,

$z_1 = 0$

$z_2 = 1$

$-0.42857 z_1 + 1.6190 z_2 + z_3 = 0$

$-1.6190 z_2 = -1.6190$

$z_3 = .42857 z_1 - 1.6190$

~~Usin~~ Then, $[U][B] = [Z]$

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0.047533 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} > \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1.6190 \end{bmatrix}$$

Using back Substitution,

$0.047533 b_{32} = -1.6190$

$\Rightarrow b_{32} = -34.0605$

~~$3b_{22}$~~ $3b_{22} - b_{32} = 1$

$\Rightarrow b_{22} = \dfrac{b_{32} + 1}{3} = -11.0202$

$7b_{12} + 2b_{22} + b_{32} = 0$

$\Rightarrow b_{12} = 8.0144$

Last column:

$$[A]\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To find $\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix}$,

First, $[L][Z] = [I]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -.42857 & 1.6190 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

forward substitution:

$z_1 = 0$

$z_2 = 0$

$-.42857 z_1 + 1.6190 z_2 + z_3 = 1$

$\Rightarrow z_3 = 1 + .42857 \times 0 - 1.6190 \times 0$

$\Rightarrow z_3 = 1$

~~Back substitution to find~~

Then, $[U][B] = [Z]$

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0.047532 \end{bmatrix}\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Back substitution:

$b_{33} = 21.0384$

$3b_{23} - b_{33} = 0 \quad \therefore b_{23} = 7.0128$

$7b_{13} + 2b_{23} + b_{33} = 1 \quad \therefore b_{13} = -4.8663$

So, the inverse matrix is :-

$$[B] = [A]^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -2.1467 & 8.0144 & -4.8663 \\ 3.00542 & -11.0202 & 7.0128 \\ 0.01626 & -34.0605 & 21.0384 \end{bmatrix}$$

[check in your calculator & match your answer]

To solve the system of linear equations given above;

$$[A][x] = [C] \quad \ldots\ldots (1)$$

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix}$$

we can do, $[x] = [A]^{-1}[C]$ from eqn (1), then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2.1467 & 8.0144 & -4.8663 \\ 3.00542 & -11.0202 & 7.0128 \\ 0.01626 & -34.0605 & 21.0384 \end{bmatrix} \begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix}$$

Using calculator, we get, $x = -0.142$, $y = 1$, $z = -2$