Constraint Satisfaction

Russell & Norvig Ch. 6.1-6.4

Today's Class

- Constraint Processing / Constraint Satisfaction Problem (CSP) paradigm
- Algorithms for CSPs
  - Backtracking (systematic search)
  - Constraint propagation (k-consistency)
  - Variable and value ordering heuristics
  - Intelligent backtracking

Overview

- Constraint satisfaction offers a powerful problem-solving paradigm
  - View a problem as a **set of variables** to which we have to assign **values** that satisfy a number of **problem-specific constraints**.
  - Constraint programming, constraint satisfaction problems (CSPs), constraint logic programming…
- Algorithms for CSPs
  - Backtracking (systematic search)
  - Constraint propagation (k-consistency)
  - Variable and value ordering heuristics
  - Backjumping and dependency-directed backtracking

Informal Definition of CSP

- CSP = Constraint Satisfaction Problem
- Given

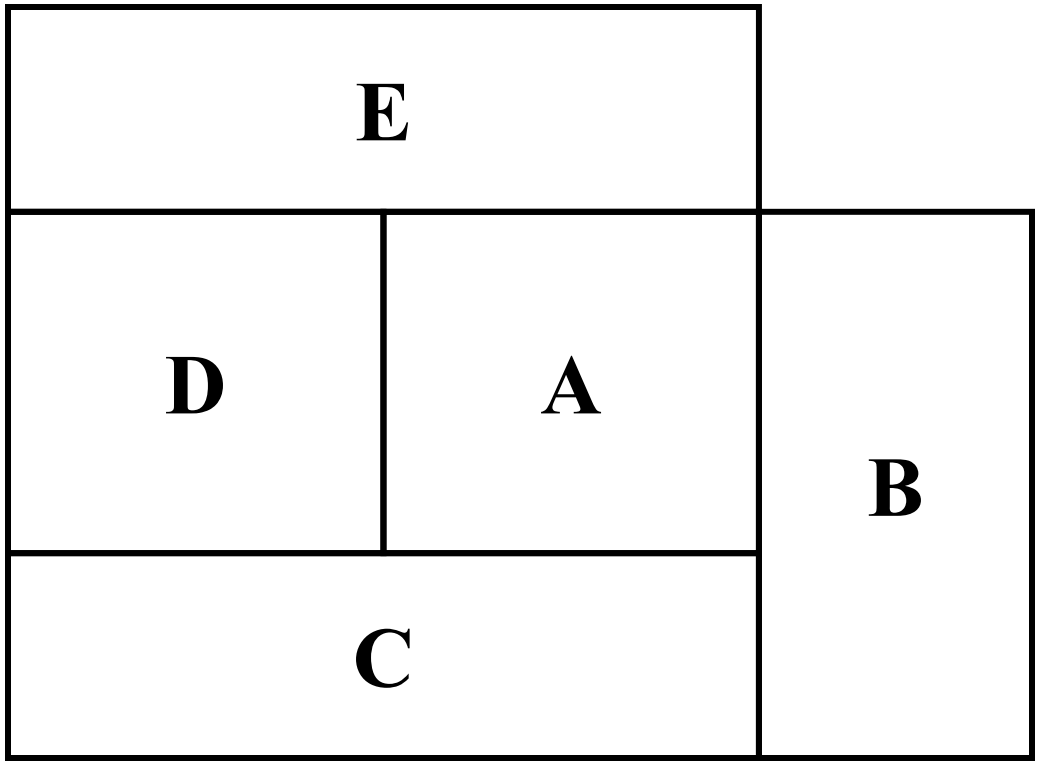
  (1) a finite set of variables

  (2) each with a domain of possible values (often finite)

  (3) a set of constraints that limit the values the variables can take on
- A **solution** is an assignment of a value to each variable such that the constraints are all satisfied.
- Tasks might be to decide if a solution exists, to find a solution, to find all solutions, or to find the "best solution" according to some metric (objective function).

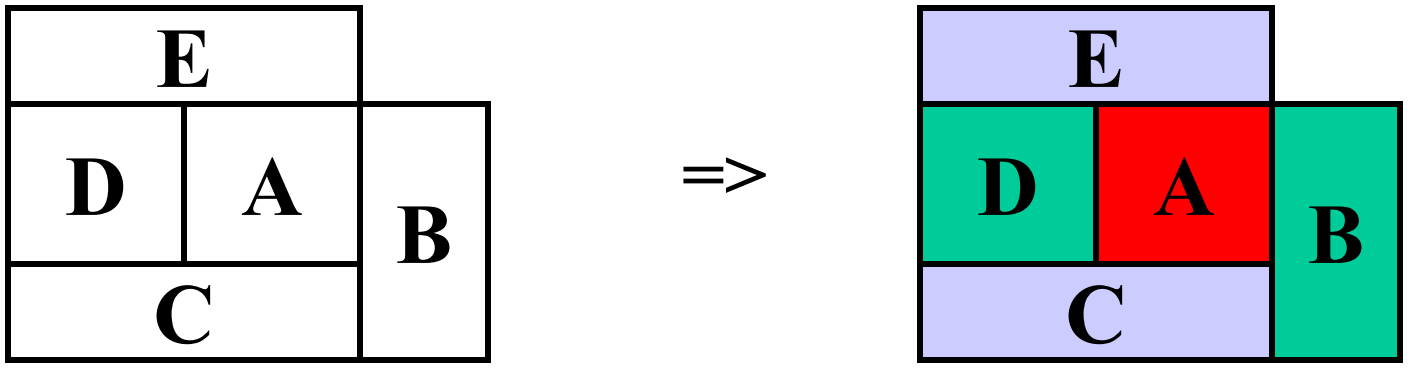
Informal Example: Map Coloring

- Color the following map using three colors (red, green, blue) such that no two adjacent regions have the same color.



Map Coloring II

- Variables: A, B, C, D, E all of domain RGB
- Domains: RGB = {red, green, blue}
- Constraints: A≠B, A≠C,A ≠ E, A ≠ D, B ≠ C, C ≠ D, D ≠ E
- One solution: A=red, B=green, C=blue, D=green, E=blue



Real-World Problems

- Scheduling
- Temporal reasoning
- Building design
- Planning
- Optimization/satisfaction
- Vision
- Graph layout
- Network management
- Natural language processing
- Molecular biology / genomics
- VLSI design

Formal Definition of a Constraint Network (CN)

A constraint network (CN) consists of

- a set of variables $X = \{x_1, x_2, \ldots x_n\}$
  - each with an associated domain of values $\{d_1, d_2, \ldots d_n\}$.
  - the domains are typically finite
- a set of constraints $\{c_1, c_2 \ldots c_m\}$ where
  - each constraint defines a predicate which is a relation over a particular subset of X.
  - e.g., C_i involves variables $\{X_{i1}, X_{i2}, \ldots X_{ik}\}$ and defines the relation $R_i \subseteq D_{i1} \times D_{i2} \times \ldots D_{ik}$
- **Unary** constraint: only involves one variable
- **Binary** constraint: only involves two variables

Formal Definition of a CN (cont.)

- Instantiations
  - An **instantiation** of a subset of variables S is an assignment of a value in its domain to each variable in S
  - An instantiation is **legal** iff it does not violate any constraints.
- A **solution** is an instantiation of all of the variables in the network.

Typical Tasks for CSP

- Solutions:
  - Does a solution exist?
  - Find one solution
  - Find all solutions
  - Given a partial instantiation, do any of the above
- Transform the CN into an equivalent CN that is easier to solve.

Binary CSP

- A **binary CSP** is a CSP in which all of the constraints are binary or unary.
- Any non-binary CSP can be converted into a binary CSP by introducing additional variables.
- Dual graph construction: one variable for each constraint; one binary constraint for each pair of original constraints that share variables
- A binary CSP can be represented as a **constraint graph**, which has a node for each variable and an arc between two nodes if and only there is a constraint involving the two variables.
  - Unary constraint appears as a self-referential arc

Example: Sudoku

	3		1
	1		4
3	4	1	2
		4	

Running Example: Sudoku

v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
3	4	1	2
v_{41}	v_{42}	4	v_{44}

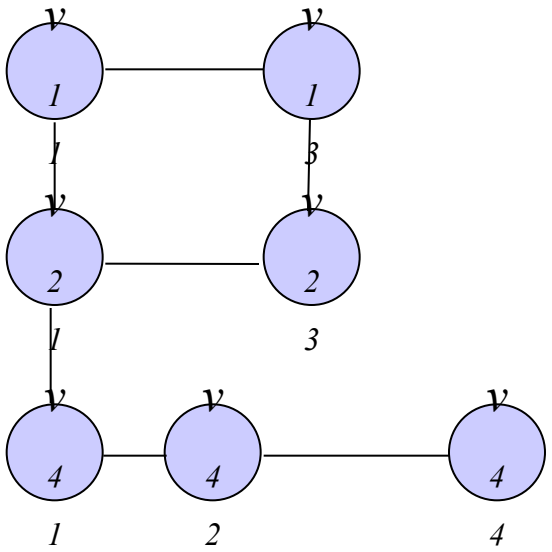
- Variables and their domains
  - v_{ij} is the value in the jth cell of the ith row
  - $D_{ij} = D = \{1, 2, 3, 4\}$
- Blocks:
  - $B_1 = \{11, 12, 21, 22\}$
  - ...
  - $B_4 = \{33, 34, 43, 44\}$
- Constraints (implicit/intensional)
  - $C^R : \forall i, \ \cup_j v_{ij} = D$ (every value appears in every row)
  - $C^C : \forall j, \ \cup_j v_{ij} = D$ (every value appears in every column)
  - $C^B : \forall k, \ \cup (v_{ij} \mid ij \in B_k) = D$ (every value appears in every block)
  - Alternative representation: pairwise inequality constraints:
    - $I^R : \forall i, j \neq j' : v_{ij} \neq v_{ij'}$ (no value appears twice in any row)
    - $I^C : \forall j, i \neq i' : v_{ij} \neq v_{i'j}$ (no value appears twice in any column)
    - $I^B : \forall k, ij \in B_k, i'j' \in B_k, ij \neq i'j' : v_{ij} \neq v_{i'j'}$ (no value appears twice in any block)
  - Advantage of the second representation: all binary constraints!

Sudoku Constraint Network

	3		1
	1		4
3	4	1	2
		4	

Sudoku Constraint Network

v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
3	4	1	2
v_{41}	v_{42}	4	v_{44}

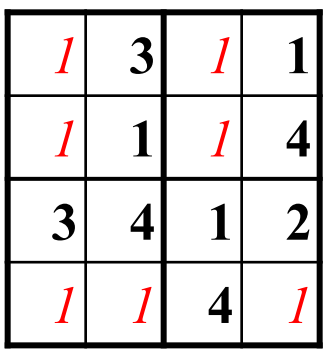


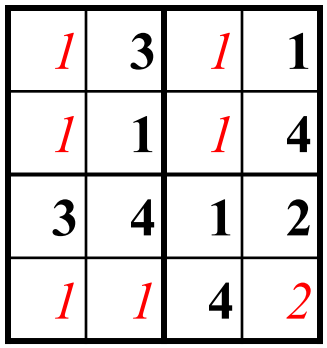
Solving Constraint Problems

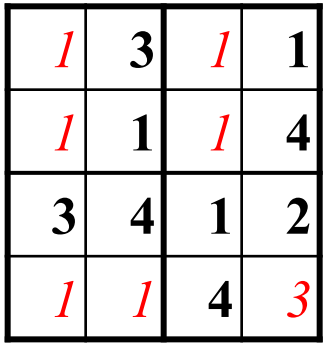
- Systematic search
  - Generate and test
  - Backtracking
- Constraint propagation (consistency)
- Variable ordering heuristics
- Value ordering heuristics
- Backjumping and dependency-directed backtracking

Generate and Test: Sudoku

- Try each possible combination until you find one that works:







.....

- Doesn't check constraints until all variables have been instantiated
- Very inefficient way to explore the space of possibilities (4^7 for this trivial Sudoku puzzle, most illegal)

Systematic Search: Backtracking

(a.k.a. depth-first search!)

- Consider the variables in some order
- Pick an unassigned variable and give it a provisional value such that it is consistent with all of the constraints
- If no such assignment can be made, we've reached a dead end and need to backtrack to the previous variable
- Continue this process until a solution is found or we backtrack to the initial variable and have exhausted all possible values

Backtracking: Sudoku

Let's try it…

v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
3	4	1	2
v_{41}	v_{42}	4	v_{44}

$\varnothing$

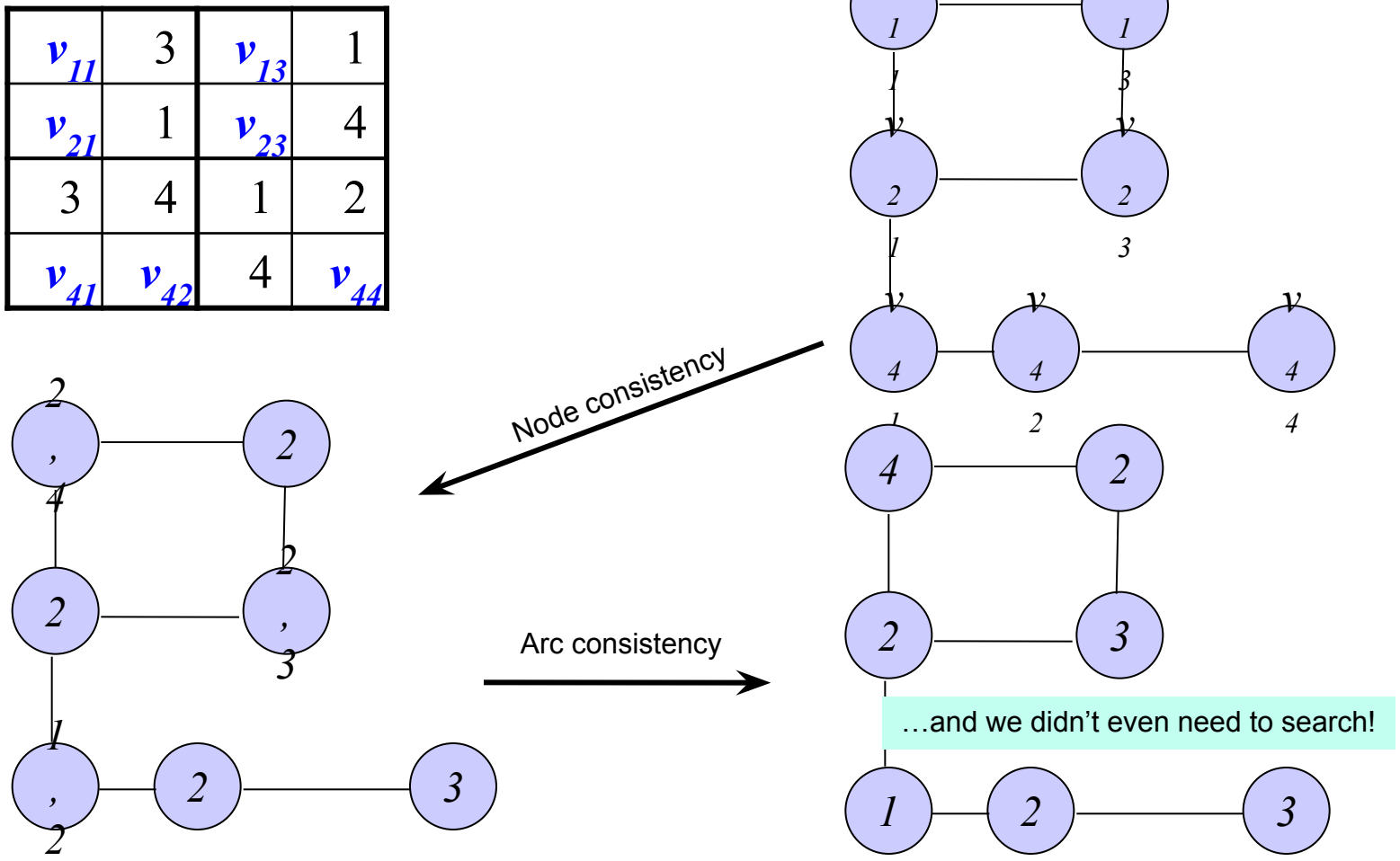
Problems with Backtracking

- Thrashing: keep repeating the same failed variable assignments
  - Consistency checking can help
  - Intelligent backtracking schemes can also help
- Inefficiency: can explore areas of the search space that aren't likely to succeed
  - Variable ordering can help

Consistency

- Node consistency
  - A node X is **node-consistent** if every value in the domain of X is consistent with X's unary constraints
  - A graph is node-consistent if all nodes are node-consistent
- Arc consistency
  - An arc (X, Y) is **arc-consistent** if, for every value x of X, there is a value y for Y that satisfies the constraint represented by the arc.
  - A graph is arc-consistent if all arcs are arc-consistent.
- To create arc consistency, we perform **constraint propagation**: that is, we repeatedly reduce the domain of each variable to be consistent with its arcs

Constraint Propagation: Sudoku



Constraint satisfaction problems (CSPs)

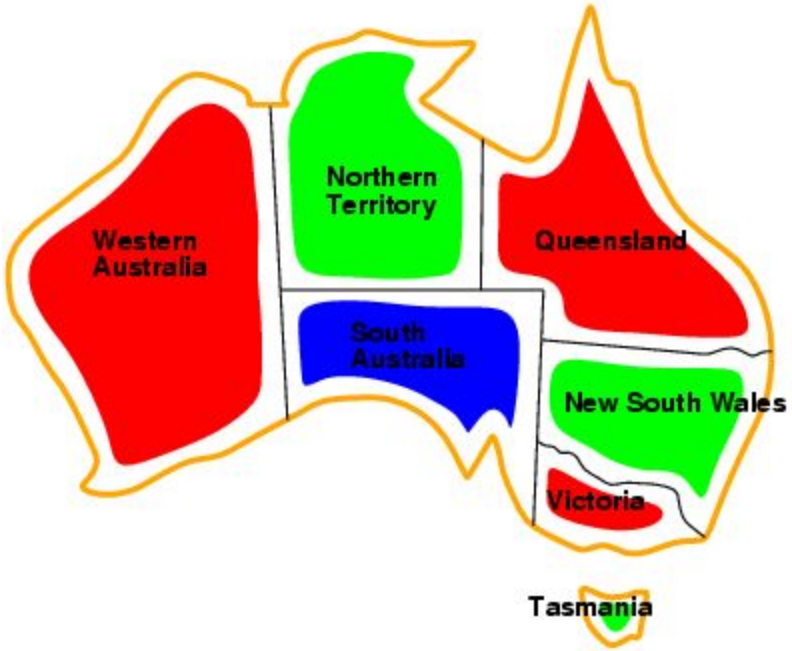
- Standard search problem: state is a "black box“ – any data structure that supports successor function and goal test
  CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables

- Simple example of a formal representation language
- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map-Coloring

- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors
  e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red), (green,blue),(blue,red),(blue,green)}

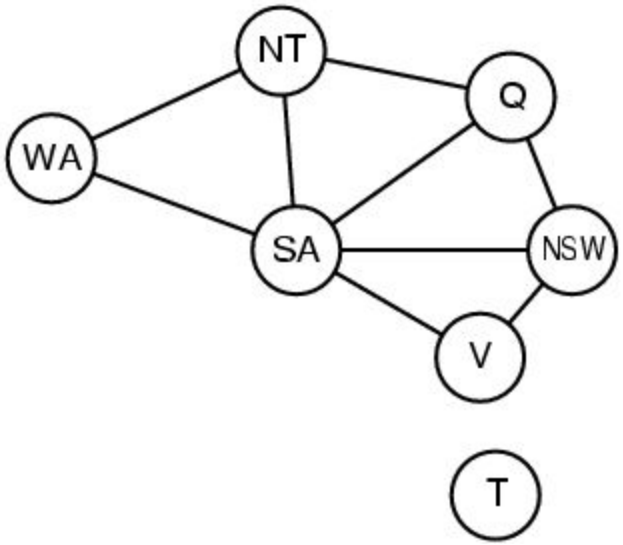
Example: Map-Coloring



- Solutions are complete and consistent assignments
- e.g., WA = red, NT = green, Q = red, NSW = green,V = red,SA = blue,T = green

Constraint graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints



Varieties of CSPs

- Discrete variables
  - finite domains:
    - n variables, domain size d □ $O(d^n)$ complete assignments
    - e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$

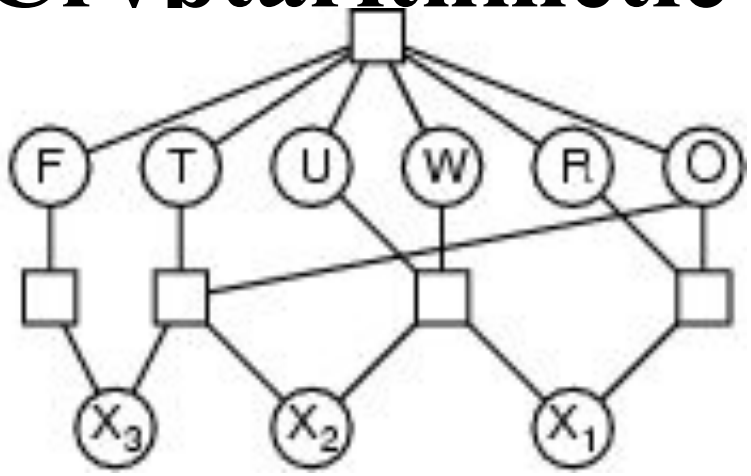
- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by LP

Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables,
  - e.g., cryptarithmetic column constraints

Example: Cryptarithmetic

$$\begin{array}{r} \text{T W O} \\ +\ \text{T W O} \\ \hline \text{F O U R} \end{array}$$



- Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
- Domains: $\{0,1,2,3,4,5,6,7,8,9\}$
- Constraints: *Alldiff (F,T,U,W,R,O)*
  - $O + O = R + 10 \cdot X_1$
  - $X_1 + W + W = U + 10 \cdot X_2$
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F,\ T \neq 0,\ F \neq 0$

Standard search formulation (incremental)

Let's start with the straightforward approach, then fix it

States are defined by the values assigned so far

Initial state: the empty assignment { }

- Successor function: assign a value to an unassigned variable that does not conflict with current assignment
  - fail if no legal assignments

- Goal test: the current assignment is complete

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables
   - use depth-first search
3. Path is irrelevant, so can also use complete-state formulation
4. $b = (n - \ell)d$ at depth ℓ, hence $n! \cdot d^n$ leaves

Backtracking search

- Variable assignments are commutative, i.e.,

  [WA = red then NT = green] same as [NT = green then WA = red]

- => Only need to consider assignments to a single variable at each node

- Depth-first search for CSPs with single-variable assignments is called backtracking search

- Can solve n-queens for $n \approx 25$

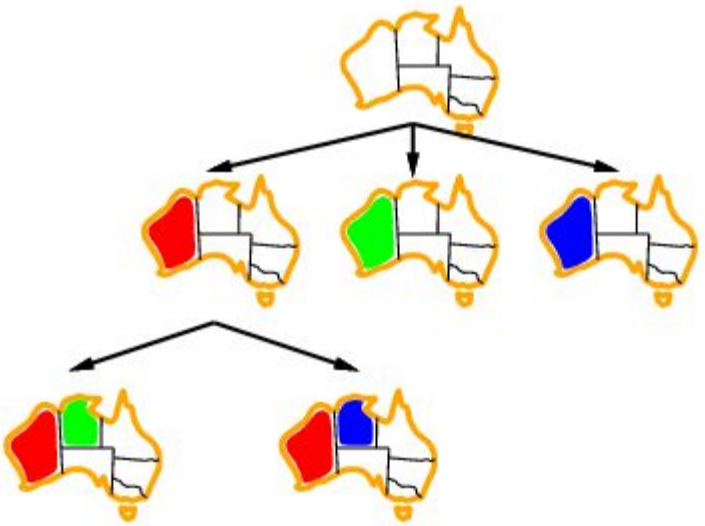
Backtracking search

```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
    return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment according to Constraints[csp] then
            add { var = value } to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failue then return result
            remove { var = value } from assignment
    return failure
```

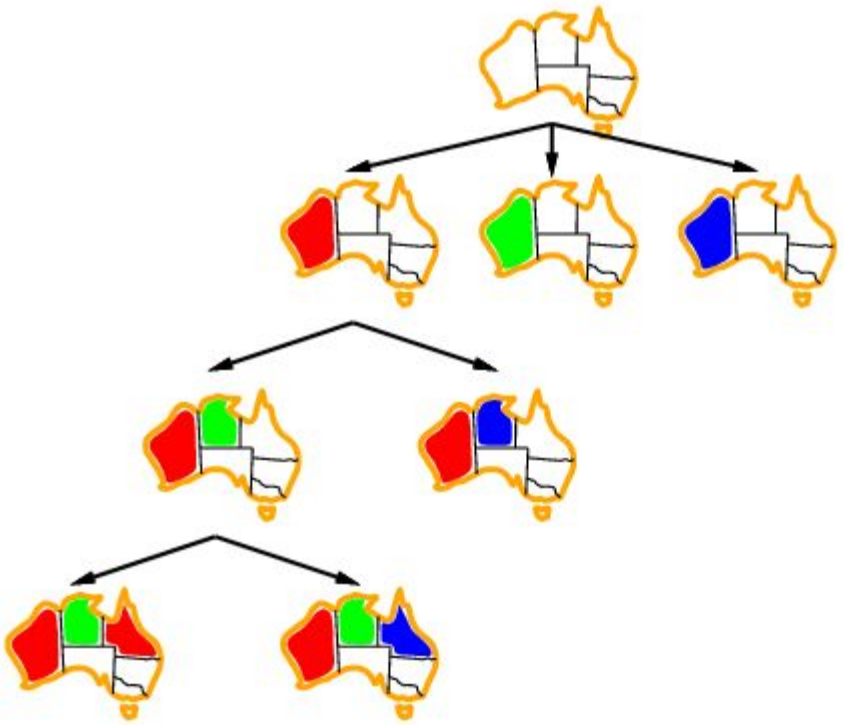
Backtracking example

Backtracking example

Backtracking example



Backtracking example



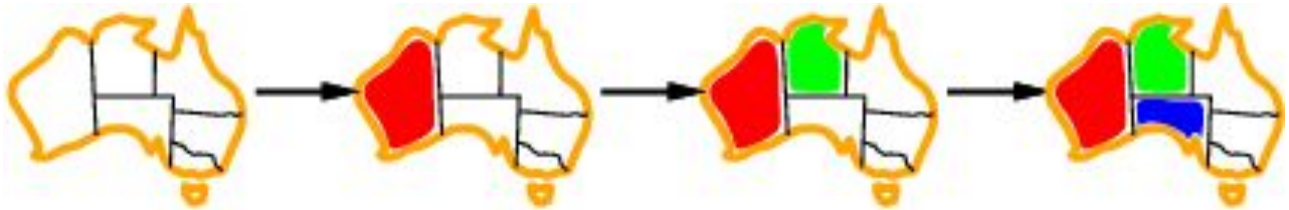
Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Most constrained variable

- Most constrained variable:

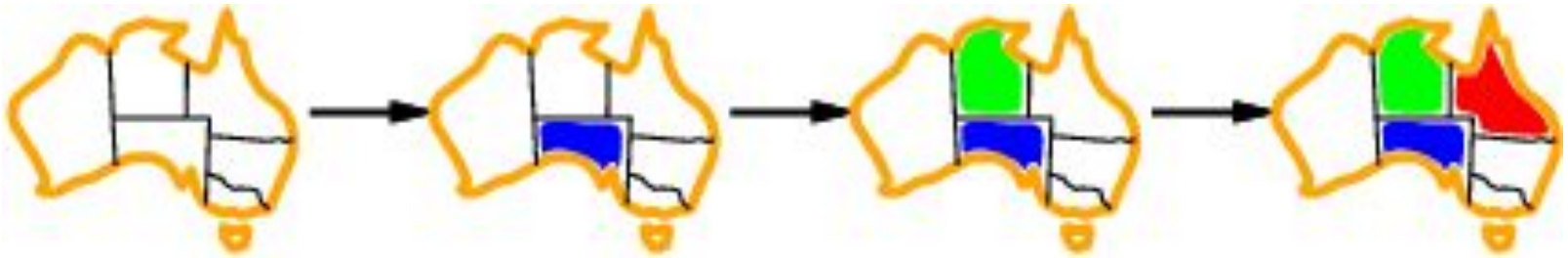
  choose the variable with the fewest legal values



- a.k.a. minimum remaining values (MRV) neuristic

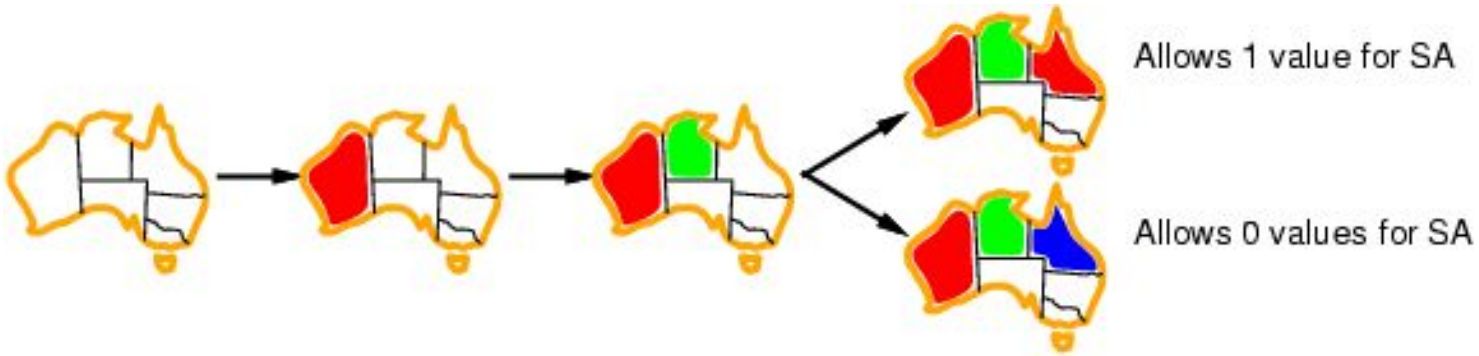
Most constraining variable

- A good idea is to use it as a tie-breaker among most constrained variables
- Most constraining variable:
  - choose the variable with the most constraints on remaining variables



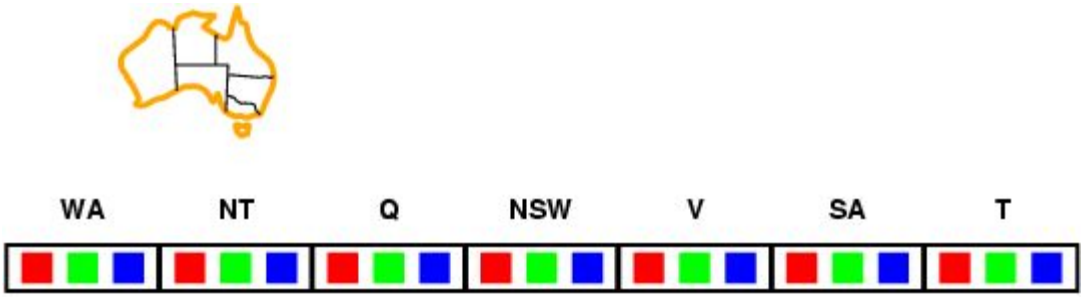
Least constraining value

- Given a variable to assign, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



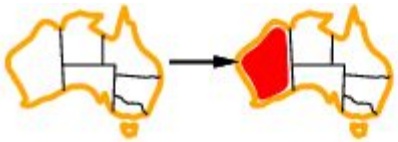
Forward checking

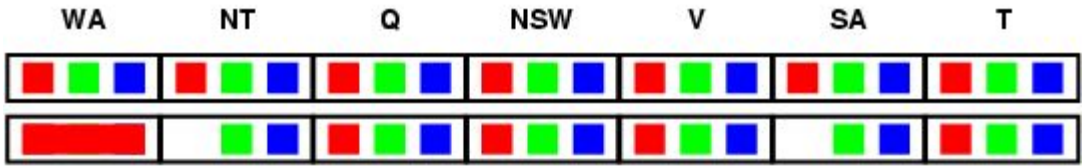
- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



Forward checking

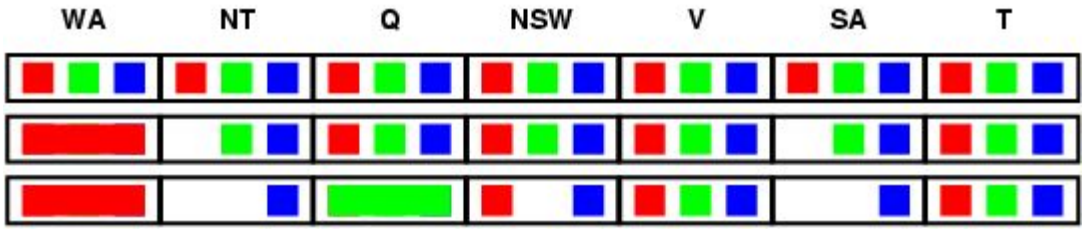
- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values





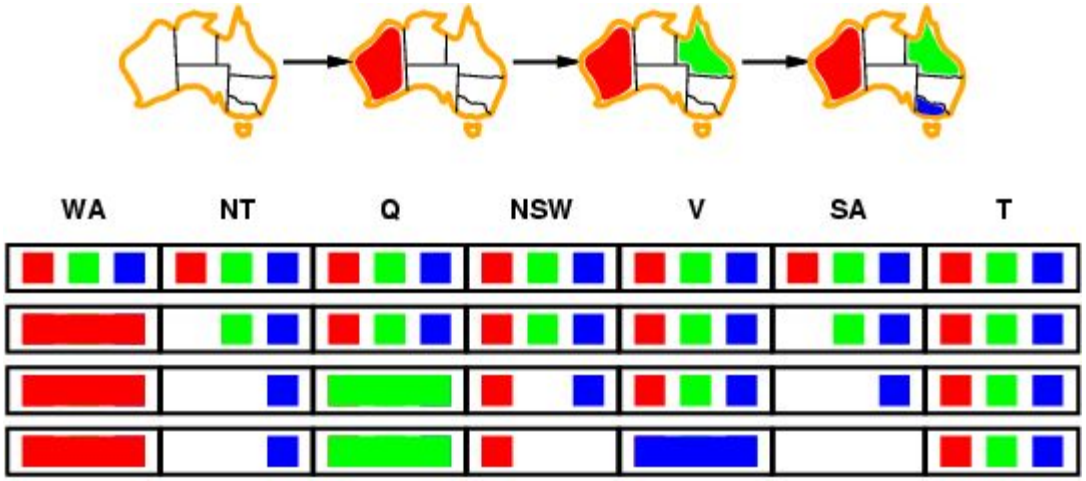
Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



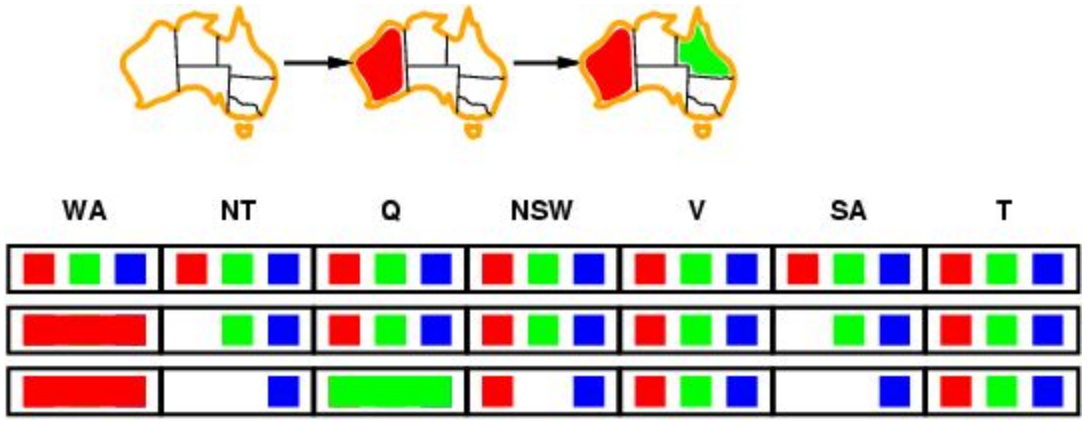
Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

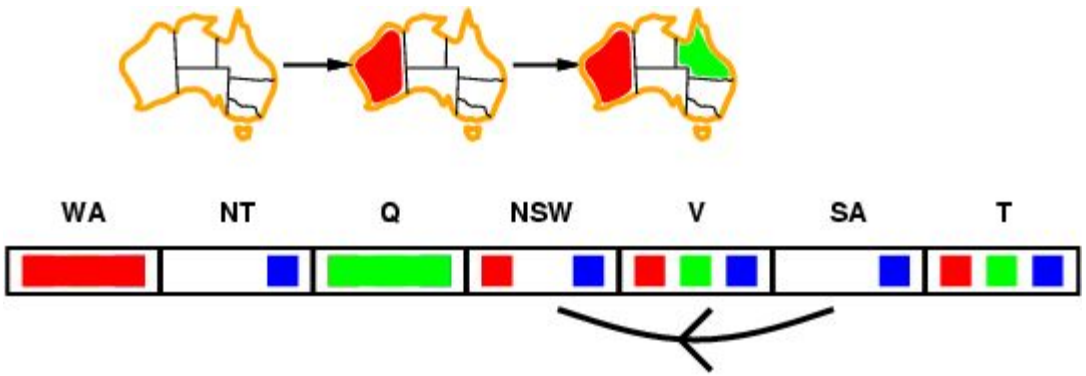


- NT and SA cannot both be blue!
- Constraint propagation algorithms repeatedly enforce constraints locally…

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

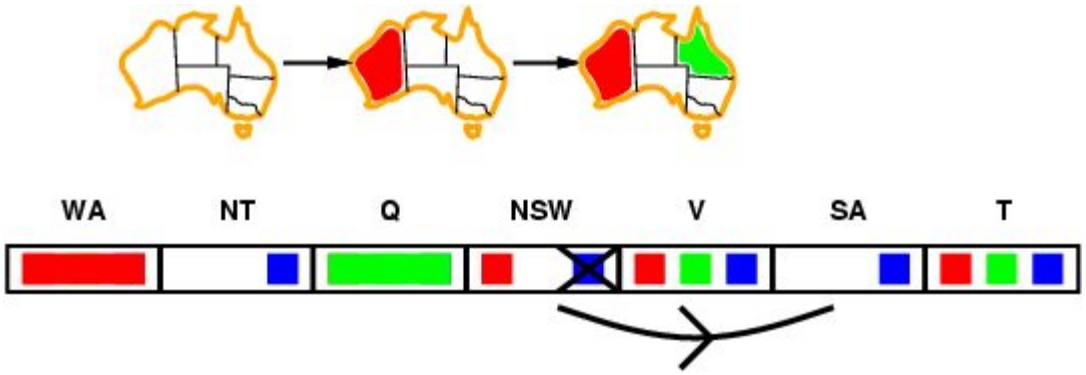
  for every value x of X there is some allowed y



Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

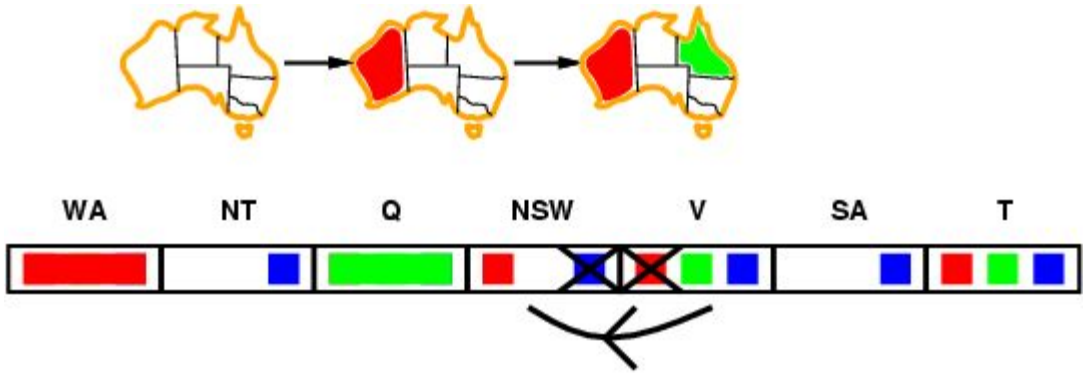
  for every value x of X there is some allowed y



Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \square Y$ is consistent iff

  for every value x of X there is some allowed y

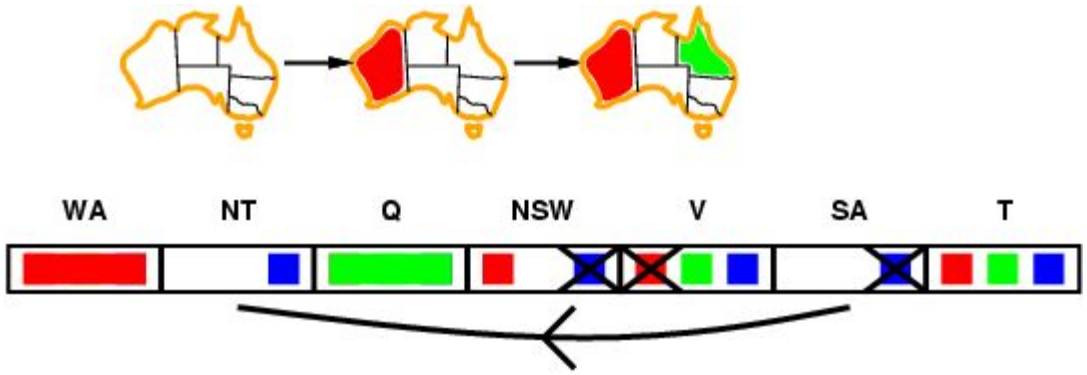


- If X loses a value, neighbors of X need to be rechecked

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \square Y$ is consistent iff

for every value x of X there is some allowed y



- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc consistency algorithm AC-3

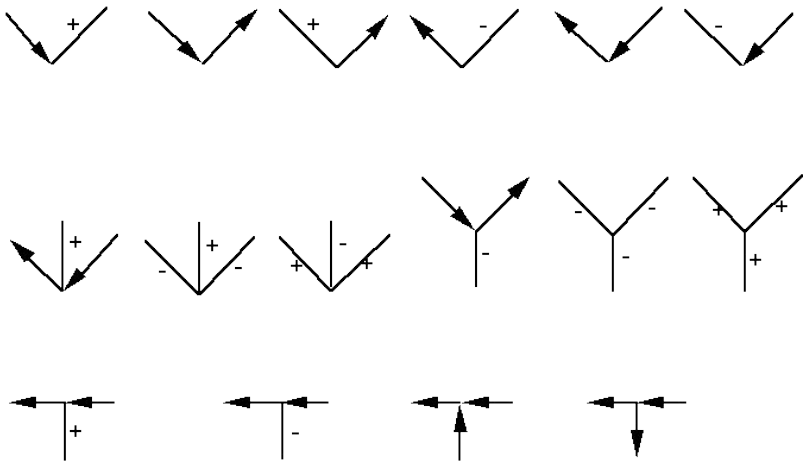
```
function AC-3( csp) returns the CSP, possibly with reduced domains
  inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
  local variables: queue, a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if RM-INCONSISTENT-VALUES(X_i, X_j) then
      for each X_k in NEIGHBORS[X_i] do
        add (X_k, X_i) to queue

function RM-INCONSISTENT-VALUES( X_i, X_j) returns true iff remove a value
  removed ← false
  for each x in DOMAIN[X_i] do
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy constraint(X_i, X_j)
      then delete x from DOMAIN[X_i]; removed ← true
  return removed
```

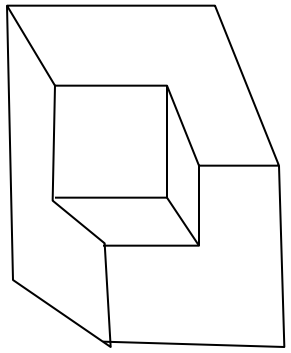
- Time complexity: O(#constraints $|\text{domain}|^3$)

Checking consistency of an arc is O($|\text{domain}|^2$)

A Famous Example: Labelling Line Drawings

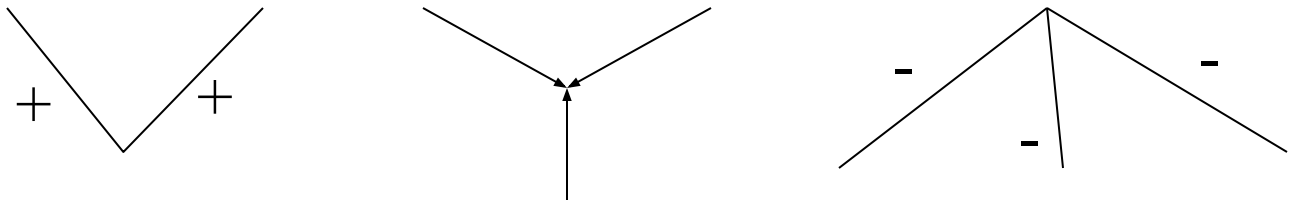
- Waltz labelling algorithm – one of the earliest CSP applications
  - Convex interior lines are labelled as +
  - Concave interior lines are labeled as –
  - Boundary lines are labeled as ⟶
- There are 208 labellings (most of which are impossible)
- Here are the 18 legal labellings:





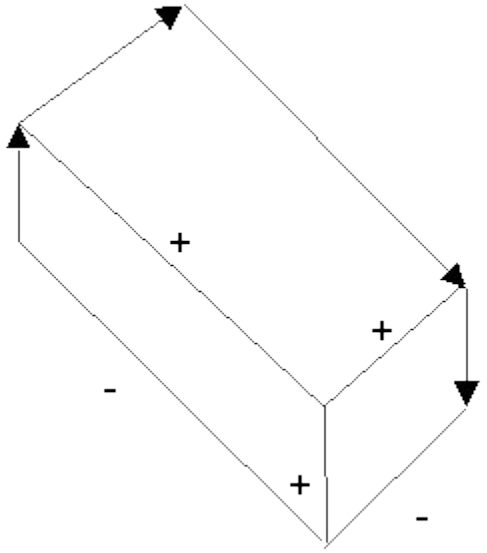
Labelling Line Drawings II

- Here are some illegal labelings:

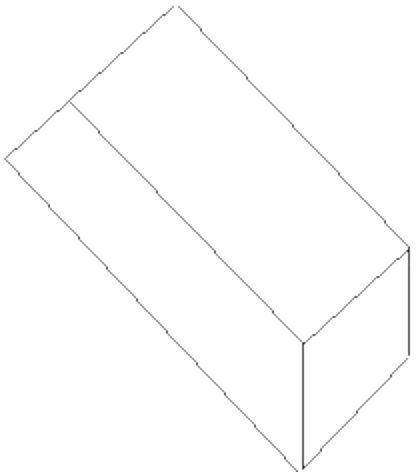


Labelling Line Drawings (cont.)

- Waltz labelling algorithm: Propagate constraints repeatedly until a solution is found



A solution for one labelling problem



A labelling problem with no solution

K-consistency

- K- consistency generalizes the notion of arc consistency to sets of more than two variables.
  - A graph is K-consistent if, for legal values of any K-1 variables in the graph, and for any Kth variable V_k, there is a legal value for V_k
- Strong K-consistency = J-consistency for all J<=K
- **Node consistency = strong 1-consistency**
- **Arc consistency = strong 2-consistency**
- Path consistency = strong 3-consistency

Why Do We Care?

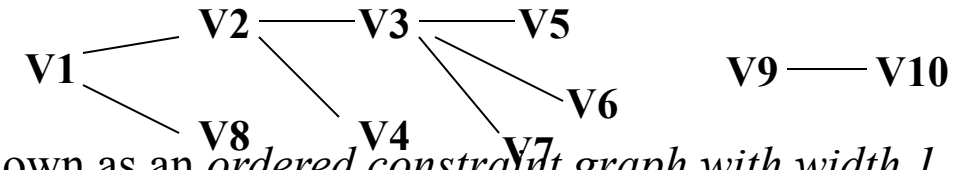
1. If we have a CSP with N variables that is known to be **strongly N-consistent**, we can solve it **without backtracking**
2. For any CSP that is **strongly K-consistent**, if we find an **appropriate variable ordering** (one with "small enough" branching factor), we can solve the CSP **without backtracking**

Ordered Constraint Graphs

- Select a variable ordering, $V_1, \ldots, V_n$
- **Width of a node** in this OCG is the number of arcs leading to earlier variables:
  - $w(V_i) = \text{Count}\ (\ (V_i, V_k) \mid k < i)$
- **Width of the OCG** is the maximum width of any node:
  - $w(G) = \text{Max}\ (w\ (V_i)),\ 1 <= i <= N$
- **Width of an unordered CG** is the minimum width of all orderings of that graph ("best you can do")

Tree-Structured Constraint Graph

- A **constraint tree** rooted at V_1 satisfies the following property:
  - There exists an ordering V1, …, Vn such that every node has zero or one parents (i.e., each node only has constraints with at most one "earlier" node in the ordering)



  - Also known as an *ordered constraint graph with width 1*
- If this constraint tree is also node- and arc-consistent (a.k.a. *strongly 2-consistent*), then it can be solved without backtracking
- (More generally, if the ordered graph is strongly k-consistent, and has width w < k, then it can be solved without backtracking.)

Proof Sketch for Constraint Trees

- Perform backtracking search in the order that satisfies the constraint tree condition
- Every node, when instantiated, is constrained only by at most one previous node
- Arc consistency tells us that there must be at least one legal instantiation in this case
  - (If there are no legal solutions, the arc consistency procedure will collapse the graph – some node will have no legal instantiations)
- Keep doing this for all *n* nodes, and you have a legal solution – without backtracking!

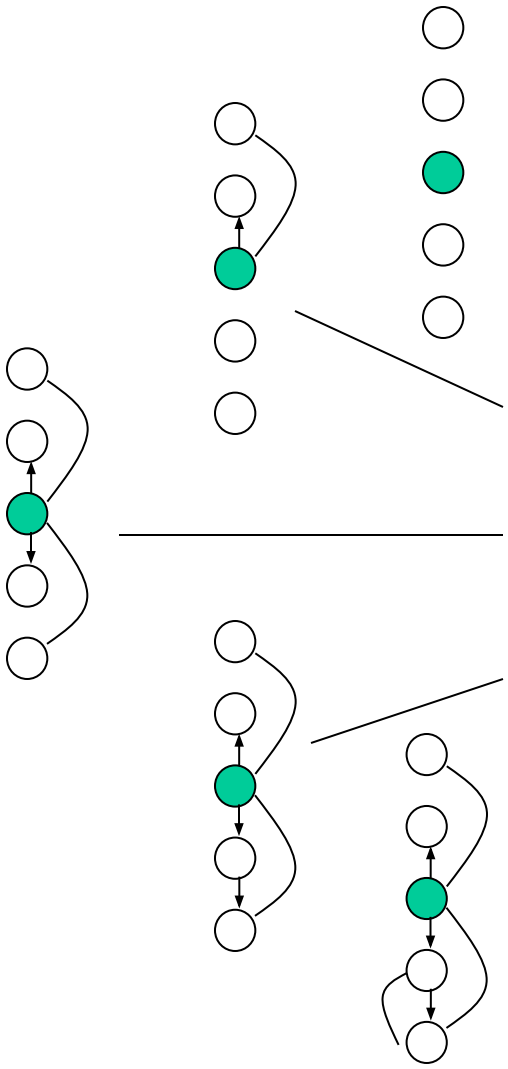
Backtrack-Free CSPs: Proof Sketch

- Given a strongly k-consistent OCG, G, with width w < k:
  - Instantiate variables in order, choosing values that are consistent with the constraints between V_i and its parents
  - Each variable has at most w parents, and k-consistency tells us we can find a legal value consistent with the values of those w parents
- *Unfortunately*, achieving k-consistency is hard (and can increase the width of the graph in the process!)
- *Fortunately*, 2-consistency is relatively easy to achieve, so constraint trees are easy to solve
- *Unfortunately*, many CGs have width greater than one (that is, no equivalent tree), so we still need to improve search

So What If We Don't Have a Tree?

- Answer #1: Try **interleaving** constraint propagation and backtracking
- Answer #2: Try using **variable-ordering** heuristics to improve search
- Answer #3: Try using **value-ordering** heuristics during variable instantiation
- Answer #4: See if **iterative repair** works better
- Answer #5: Try using **intelligent backtracking** methods

Interleaving Constraint Propagation and Search



Generate and Test	No constraint propagation: assign all variable values, **then** test constraints
Simple Backtracking	Check constraints only for variables "up the tree"
Forward Checking	Check constraints for immediate neighbors "down the tree"
Partial Lookahead	Propagate constraints forward "down the tree"
Full Lookahead	Ensure complete arc consistency after each instantiation (AC-3)

Variable Ordering

- **Intuition**: choose variables that are highly constrained early in the search process; leave easy ones for later
- **Minimum width ordering** (MWO): identify OCG with minimum width
- **Maximum cardinality ordering**: approximation of MWO that's cheaper to compute: order variables by decreasing cardinality (a.k.a. **degree heuristic**)
- **Fail first principle** (FFP): choose variable with the fewest values (a.k.a. **minimum remaining values** (MRV))
  - **Static** FFP: use domain size of variables
  - **Dynamic** FFP (**search rearrangement method**): At each point in the search, select the variable with the fewest remaining values

Variable Ordering II

- **Maximal stable set**: find largest set of variables with no constraints between them and save these for last
- **Cycle-cutset tree creation**: Find a set of variables that, once instantiated, leave a tree of uninstantiated variables; solve these, then solve the tree without backtracking
- **Tree decomposition**: Construct a tree-structured set of connected subproblems

Value Ordering

- **Intuition**: Choose values that are the least constrained early on, leaving the most legal values in later variables
- **Maximal options method** (a.k.a. **least-constraining-value** heuristic): Choose the value that leaves the most legal values in uninstantiated variables
- **Min-conflicts**: Used in iterative repair search (see below)

- Symmetry: Introduce **symmetry-breaking constraints** to constrain search space to "useful" solutions (don't examine more than one symmetric/isomorphic solution)

Iterative Repair

- Start with an initial complete (but invalid) assignment
- Hill climbing, simulated annealing
- Min-conflicts: Select new values that minimally conflict with the other variables
  - Use in conjunction with hill climbing or simulated annealing or…
- Local maxima strategies
  - Random restart
  - Random walk
  - Tabu search: don't try recently attempted values

Min-Conflicts Heuristic

- Iterative repair method
  1. Find some "reasonably good" initial solution
     - E.g., in N-queens problem, use greedy search through rows, putting each queen where it conflicts with the smallest number of previously placed queens, breaking ties *randomly*
  2. Find a variable in conflict (randomly)
  3. Select a new value that minimizes the number of constraint violations
     - O(N) time and space
  4. Repeat steps 2 and 3 until done
- Performance depends on quality and informativeness of initial assignment; inversely related to distance to solution

Intelligent Backtracking

- **Backjumping**: if V_j fails, jump back to the variable V_i with greatest i such that the constraint (V_i, V_j) fails (i.e., most recently instantiated variable in conflict with V_i)
- **Backchecking**: keep track of incompatible value assignments computed during backjumping
- **Backmarking**: keep track of which variables led to the incompatible variable assignments for improved backchecking

Some Challenges for Constraint Reasoning

- What if not all constraints can be satisfied?
  - Hard vs. soft constraints
  - Degree of constraint satisfaction
  - Cost of violating constraints
- What if constraints are of different forms?
  - Symbolic constraints
  - Numerical constraints [constraint solving]
  - Temporal constraints
  - Mixed constraints

Some More Challenges for Constraint Reasoning

- What if constraints are represented intensionally?
  - Cost of evaluating constraints (time, memory, resources)
- What if constraints, variables, and/or values change over time?
  - Dynamic constraint networks
  - Temporal constraint networks
  - Constraint repair
- What if you have multiple agents or systems involved in constraint satisfaction?
  - Distributed CSPs
  - Localization techniques