Practice assignment 2

Propositional Logic

Exercise 2–1

Angelo, Bruno and Carlo are three students that took the Logic exam. Let's consider a propositional language where

- A stands for Aldo passed the exam
- B stands for Bruno passed the exam
- C stands for Carlo passed the exam

Formalize the following sentences:

(a) Carlo is the only one passing the exam

(b) Aldo is the only one not passing the exam

(c) Only one, among Aldo, Bruno and Carlo, passed the exam

(d) At least one among Aldo, Bruno and Carlo passed the exam

(e) At least two among Aldo, Bruno and Carlo passed the exam

(f) At most two among Aldo, Bruno and Carlo passed the exam

(g) Exactly two, among Aldo, Bruno and Carlo passed the exam

Solution:

(a) Carlo is the only one passing the exam
$C \land \neg B \land \neg A$

(b) Aldo is the only one not passing the exam
$\neg A \land B \land C$

(c) Only one, among Aldo, Bruno and Carlo, passed the exam
$(A \land \neg B \land \neg C) \lor (\neg A \land B \land \neg C) \lor (\neg A \land \neg B \land C)$

(d) At least one among Aldo, Bruno and Carlo passed the exam
$A \lor B \lor C$

(e) At least two among Aldo, Bruno and Carlo passed the exam
$(A \land B) \lor (A \land c) \lor (B \land C)$

(f) At most two among Aldo, Bruno and Carlo passed the exam
$\neg A \lor \neg B \lor \neg C$

(g) Exactly two, among Aldo, Bruno and Carlo passed the exam
$(A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge c) \vee (\neg A \wedge B \wedge C)$

Exercise 2–2

(a) Let p be the proposition You have the flu, q be the proposition You miss the final examination and r be the proposition You pass the course. Express the following as an English sentence: $(p \to \neg r) \vee (q \to \neg r)$

(b) Let p be the proposition You get an A on the final exam, q be the proposition You do every exercise in the book and r be the proposition You get an A in this course Write down the following using p; q and r and logical connectives: Getting an A on the final and doing every exercise in the book is sufficient for getting an A in this course.

(c) Express the following statements in propositional logic:
1. If premises are either true or false, then arguments can only be either true or false.
2. If the set of sets that are not members of themselves is a member of itself, then it is not a member of itself.

Solution:

(a) If you have the flu or you miss the final examination then you will not pass the course.
(b)$(p \wedge q) \to r$
(c)
1. $(p \vee \neg p) \to (q \vee \neg q)$
2. $p \to \neg p$

Exercise 2–3

Translation
(a) If it is hot and humid, then it is raining.
(b) If it is humid, then it is hot.
(c) It is either humid or hot.
(d) It is never humid and cold.
(e) If it is raining and not humid then it is cold.

Translate the following statements into propositional logic formulas using the following propositions:
• h: it is hot.
• m: it is humid.
• r: it is raining.
• c: it is cold.

Solution:

(a) If it is hot and humid, then it is raining: $h \wedge m \to r$

(b) If it is humid, then it is hot: $m \to h$

(c) It is either humid or hot: $m \vee h$

(d) It is never humid and cold: $\neg(m \wedge c)$

(e) If it is raining and not humid then it is cold: $(r \wedge \neg h) \rightarrow c$

Exercise 2–4

"It is not the case that the car is red and the truck is blue". Let proposition p = "the car is red" and q = "the truck is blue". Which formula represents the above statement ?
i. $\neg p \wedge q$
ii. $\neg(p \vee q)$
iii. $\neg(p \wedge q)$
iv. $\neg p \wedge \neg q$

Solution:

iii. $\neg(p \wedge q)$

Exercise 2–5

Three boxes are presented to you. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are:

- Box 1 "The gold is not here"
- Box 2 "The gold is not here"
- Box 3 "The gold is in Box 2"

Only one message is true; the other two are false. Which box has the gold?

Solution:

using truth table:

- Box 1 "The gold is not here" means $\neg B1$
- Box 2 "The gold is not here" means $\neg B2$
- Box 3 "The gold is in Box 2" means $B2$

We can formalize the statements of the problem as follows:

- One box contains gold, the other two are empty. $\rightarrow$ $Eqn(1) : (B1 \wedge \neg B2 \wedge \neg B3) \vee (\neg B1 \wedge B2 \wedge \neg B3) \vee (\neg B1 \wedge \neg B2 \wedge B3)$
- Only one message is true; the other two are false. $\rightarrow$ $Eqn(2) : (\neg B1 \wedge \neg\neg B2 \wedge \neg B2) \vee (\neg\neg B1 \wedge \neg B2 \wedge \neg B2) \vee (\neg\neg B1 \wedge \neg\neg B2 \wedge B2)$

B1	B2	B3	Eqn(1)	Eqn(2)
t	t	t	f	t
t	t	f	f	t
t	f	t	f	t
t	f	f	t	t
f	t	t	f	f
f	t	f	t	f
f	f	t	t	f
f	f	f	f	f

The only message that verifies both Eqn(1) and Eqn(2) is the one with I(B1) =T and I(B2) = I(B3) = F, which implies that the gold is in the first box.

Exercise 2–6

Kyle, Neal, and Grant find themselves trapped in a dark and cold dungeon (how they arrived there is another story). After a quick search the boys find three doors, the first one red, the second one blue, and the third one green. Behind one of the doors is a path to freedom. Behind the other two doors, however, is an evil fire-breathing dragon. Opening a door to the dragon means almost certain death. On each door there is an inscription:

- Red Door: Freedom is behind this door.
- Blue Door: Freedom is not behind this door.
- Green Door: Freedom is not behind the blue door.

Given the fact that at LEAST ONE of the three statements on the three doors is true and at LEAST ONE of them is false, which door would lead the boys to safety?

Solution:

• r: "freedom is behind the red door"
• b: "freedom is behind the blue door"
• g: "freedom is behind the green door"

We can formalize the statements of the problem as follows:

- "behind one of the door is a path to freedom, behind the other two doors is an evil dragon"
  $(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$
- "at least one of the three statements is true"
  $(r \vee \neg b)$
- " at least one of the three statements is false"
  $(\neg r \vee b)$

r	b	g	Eqn(2)	Eqn(3)	Eqn($2 \wedge 3$)
t	f	f	t	f	f
f	t	f	f	t	f
f	f	t	t	t	t

Freedom is behind the green door.

Exercise 2–7

Let $F = ((b \vee c) \rightarrow ((\neg a \vee b) \leftrightarrow (c \rightarrow a)))$ be a formula and $\mathcal{A} = \{(a \mapsto f), (b \mapsto f), (c \mapsto t)\}$ be a valuation for F. Find the interpretation of F.

Solution:

$I(F) = f$

Exercise 2–8

Show that the following formulas can be replaced by equivalent formulas that only use $\neg$ and $\vee$ as connectives.

(a) $\perp$

(b) $\top$

(c) $a \wedge b$

(d) $a \rightarrow b$

(e) $a \leftrightarrow b$

Solution:

(a) $\perp \equiv \neg(a \vee \neg a)$

(b) $\top \equiv (a \vee \neg a)$

(c) $a \wedge b \equiv \neg(\neg a \vee \neg b)$

(d) $a \rightarrow b \equiv \neg a \vee b$

(e) $a \leftrightarrow b \equiv \neg(\neg(\neg a \vee b) \vee \neg(\neg b \vee a))$

Exercise 2–9

Show that the following formulas can be replaced by equivalent formulas that only use $\rightarrow$ and $\perp$ as connectives.

(a) $\neg a$

(b) $\top$

(c) $a \wedge b$

(d) $a \vee b$

(e) $a \leftrightarrow b$

Solution:

(a) $\neg a \equiv a \rightarrow \perp$

(b) $\top \equiv a \to a$

(c) $a \wedge b \equiv (a \to (b \to \bot)) \to \bot$

(d) $a \vee b \equiv ((a \to \bot) \to b)$

(e) $a \leftrightarrow b \equiv ((a \to b) \to ((b \to a) \to \bot)) \to \bot$

Exercise 2–10

For each of the following propositions, decide whether the proposition is valid, satisfiable , unsatisfiable or falsifiable.

(a) $p \to (\neg q \vee r)$

Solution:

p	q	r	$p \to (\neg q \vee r)$
f	f	f	t
f	f	t	t
f	t	f	t
f	t	t	t
t	f	f	t
t	f	t	t
t	t	f	f
t	t	t	t

$p \to (\neg q \vee r)$ is satisfiable and falsifiable.

(b) $\neg p \to (q \to r)$

Solution:

p	q	r	$\neg p \to (q \to r)$
f	f	f	t
f	f	t	t
f	t	f	f
f	t	t	t
t	f	f	t
t	f	t	t
t	t	f	t
t	t	t	t

$\neg p \to (q \to r)$ is satisfiable and falsifiable.

(c) $(p \to q) \vee (\neg p \to r)$

Solution:

p	q	r	$(p \to q) \vee (\neg p \to r)$
f	f	f	t
f	f	t	t
f	t	f	t
f	t	t	t
t	f	f	t
t	f	t	t
t	t	f	t
t	t	t	t

$(p \to q) \vee (\neg p \to r)$ is valid and satisfiable.

(d) $(p \to q) \wedge (\neg p \to r)$

Solution:

p	q	r	$(p \to q) \wedge (\neg p \to r)$
f	f	f	f
f	f	t	t
f	t	f	f
f	t	t	t
t	f	f	f
t	f	t	f
t	t	f	t
t	t	t	t

$(p \to q) \wedge (\neg p \to r)$ is satisfiable and falsifiable.

(e) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

Solution:

p	q	r	$(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$
f	f	f	t
f	f	t	t
f	t	f	t
f	t	t	f
t	f	f	f
t	f	t	t
t	t	f	t
t	t	t	t

$(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$ is satisfiable and falsifiable.

(f) $(\neg p \wedge \neg q) \wedge (p \vee q)$

Solution:

p	q	$(\neg p \wedge \neg q) \wedge (p \vee q)$
f	f	f
f	t	f
t	f	f
t	t	f

$(\neg p \wedge \neg q) \wedge (p \vee q)$ is unsatifsiable and falsifiable.

Exercise 2–11

Determine which of the following are equivalent to each other:

(a) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

(b) $\neg P \vee Q$

(c) $(P \vee \neg Q) \wedge (Q \vee \neg P)$

(d) $\neg(P \vee Q)$

(e) $(Q \wedge P) \vee \neg P$

Solution:

p	q	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$	$\neg P \vee Q$	$(P \vee \neg Q) \wedge (Q \vee \neg P)$	$\neg(P \vee Q)$	$(Q \wedge P) \vee \neg P$
f	f	t	t	t	t	t
f	t	f	t	f	f	t
t	f	f	f	f	f	f
t	t	t	t	t	f	t

$(P \wedge Q) \vee (\neg P \wedge \neg Q) \equiv (P \vee \neg Q) \wedge (Q \vee \neg P)$
$(\neg P \vee Q) \equiv ((Q \wedge P) \vee \neg P)$

Exercise 2–12

A formula F is said to be unstable if for every valuation $\mathcal{A}$, changing the truth assignment to a single variable, changes the interpretation of F. Find an unstable formula containing the atoms $\{a, b, c\}$.

Solution:

a	b	c	F
f	f	f	t
f	f	t	f
f	t	f	f
f	t	t	t
t	f	f	f
t	f	t	t
t	t	f	t
t	t	t	f

$F = (\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$

Useful Links

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https://www.geeksforgeeks.org/propositional-and-first-order-logic-qq/
http://www.logicinaction.org/docs/ch2.pdf
http://turner.faculty.swau.edu/mathematics/materialslibrary/truth/
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