Deriving Equation (10) in Toroidal–Poloidal Form

1 Governing momentum equation

Equation (10) of *Onset of Convection in Rotating Spherical Shells* gives the non-dimensional perturbation momentum balance

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\boldsymbol{\nabla} p' - 2\hat{\boldsymbol{z}} \times \boldsymbol{u}' + \frac{\mathrm{Ra}}{\mathrm{Pr}}\left(\frac{r}{r_o}\right)^2 \theta' \hat{\boldsymbol{r}} + E\boldsymbol{\nabla}^2 \boldsymbol{u}'. \tag{1.1}$$

Here $\boldsymbol{u}'$ is solenoidal, $\hat{\boldsymbol{r}}$ denotes the radial unit vector, and $\hat{\boldsymbol{z}}$ is the axis of rotation. Our goal is to rewrite (1.1) in terms of the toroidal and poloidal potentials introduced in Equations (13)–(15) of the paper.

2 Toroidal–poloidal decomposition

Following Equation (13), the divergence-free velocity field is decomposed into scalar potentials P (poloidal) and T (toroidal),

$$\boldsymbol{u}' = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (P\hat{\boldsymbol{r}}) + \boldsymbol{\nabla} \times (T\hat{\boldsymbol{r}}). \tag{2.1}$$

For compactness we define the surface gradient $\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \hat{\boldsymbol{r}}\,\partial_r$ and recall the vector spherical harmonics

$$\mathbf{Y}^{(r)}_{\ell m} = Y_{\ell m}\hat{\boldsymbol{r}}, \qquad \mathbf{Y}^{(p)}_{\ell m} = \boldsymbol{\nabla}_\perp Y_{\ell m}, \qquad \mathbf{Y}^{(t)}_{\ell m} = \hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_\perp Y_{\ell m}, \tag{2.2}$$

with $Y_{\ell m}$ the Schmidt semi-normalised spherical harmonics. Direct evaluation of (2.1) leads to the standard component form

$$\boldsymbol{u}'_P = \sum_{\ell=m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} P_{\ell m}\,\mathbf{Y}^{(r)}_{\ell m} + \frac{1}{r}\frac{\partial P_{\ell m}}{\partial r}\,\mathbf{Y}^{(p)}_{\ell m}\right], \tag{2.3}$$

$$\boldsymbol{u}'_T = \sum_{\ell=m}^{\infty} \left[\frac{1}{r} T_{\ell m}\,\mathbf{Y}^{(t)}_{\ell m}\right], \tag{2.4}$$

where the sums run over a fixed azimuthal order m and $P_{\ell m}(r,t)$, $T_{\ell m}(r,t)$ are the radial amplitudes of each degree ℓ.

Equation (14) expresses the angular dependence of the scalar potentials as

$$P(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty} P_{\ell m}(r,t) Y_{\ell m}(\theta,\phi), \qquad T(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty} T_{\ell m}(r,t) Y_{\ell m}(\theta,\phi), \tag{2.5}$$

while the radial profiles are expanded spectrally using Chebyshev polynomials as in Equation (15),

$$P_{\ell m}(r,t) = \sum_{n=0}^{N} P_{\ell m n}(t)\, C_n(r), \qquad T_{\ell m}(r,t) = \sum_{n=0}^{N} T_{\ell m n}(t)\, C_n(r). \tag{2.6}$$

The temperature perturbation is treated identically:

$$\theta'(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty} \theta_{\ell m}(r,t) Y_{\ell m}(\theta,\phi) = \sum_{\ell=m}^{\infty} \sum_{n=0}^{N} \Theta_{\ell m n}(t)\, C_n(r) Y_{\ell m}(\theta,\phi). \tag{2.7}$$

3 Operators acting on the toroidal–poloidal fields

To project (1.1) onto the basis (2.3)–(2.4) we evaluate each term separately.

3.1 Time derivative

Because the vector spherical harmonics are time-independent,

$$\frac{\partial \boldsymbol{u}'}{\partial t} = \sum_{\ell=m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} \frac{\partial P_{\ell m}}{\partial t} \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial P_{\ell m}}{\partial t} \right) \boldsymbol{Y}_{\ell m}^{(p)} + \frac{1}{r} \frac{\partial T_{\ell m}}{\partial t} \boldsymbol{Y}_{\ell m}^{(t)} \right]. \tag{3.1}$$

3.2 Pressure gradient

The pressure can be expanded as $p'(r,\theta,\phi,t) = \sum_{\ell m} p_{\ell m}(r,t) Y_{\ell m}(\theta,\phi)$. Its gradient decomposes into the same vector spherical harmonics,

$$\boldsymbol{\nabla} p' = \sum_{\ell=m}^{\infty} \left[\frac{\partial p_{\ell m}}{\partial r} \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r} p_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(p)} \right]. \tag{3.2}$$

When the curl or double curl of (1.1) is taken (as in the numerical formulation) the pressure term drops out; it is retained here only for completeness.

3.3 Viscous term

Using the identity $\boldsymbol{\nabla}^2 \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{\nabla}^2 \boldsymbol{A}$ and the fact that $\boldsymbol{\nabla}^2 (f \hat{\boldsymbol{r}}) = \left(\partial_r^2 f + \frac{2}{r} \partial_r f - \frac{\ell(\ell+1)}{r^2} f \right) \hat{\boldsymbol{r}}$ for each spherical harmonic degree, the Laplacian acts diagonally on the potentials:

$$\boldsymbol{\nabla}^2 \boldsymbol{u}'_P = \sum_{\ell=m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} \mathcal{L}_\ell P_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r} \frac{\partial}{\partial r} (\mathcal{L}_\ell P_{\ell m}) \boldsymbol{Y}_{\ell m}^{(p)} \right], \tag{3.3}$$

$$\boldsymbol{\nabla}^2 \boldsymbol{u}'_T = \sum_{\ell=m}^{\infty} \left[\frac{1}{r} \mathcal{L}_\ell T_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(t)} \right], \tag{3.4}$$

where the scalar radial operator $\mathcal{L}_\ell$ is

$$\mathcal{L}_\ell f \equiv \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} - \frac{\ell(\ell+1)}{r^2} f. \tag{3.5}$$

3.4 Buoyancy term

Because the buoyancy force is purely radial, only the $\boldsymbol{Y}_{\ell m}^{(r)}$ basis is involved:

$$\frac{\mathrm{Ra}}{\mathrm{Pr}} \left(\frac{r}{r_o} \right)^2 \theta' \hat{\boldsymbol{r}} = \sum_{\ell=m}^{\infty} \frac{\mathrm{Ra}}{\mathrm{Pr}} \left(\frac{r}{r_o} \right)^2 \theta_{\ell m}(r,t) \boldsymbol{Y}_{\ell m}^{(r)}. \tag{3.6}$$

3.5 Coriolis term

The Coriolis acceleration mixes neighbouring spherical-harmonic degrees. Writing $\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$ and using the angular momentum ladder relations

$$\cos\theta\, Y_{\ell m} = a^{+}_{\ell m} Y_{\ell+1,m} + a^{-}_{\ell m} Y_{\ell-1,m}, \tag{3.7}$$

$$\sin\theta\, \frac{\partial Y_{\ell m}}{\partial\theta} = \ell\, a^{-}_{\ell m} Y_{\ell-1,m} - (\ell+1) a^{+}_{\ell m} Y_{\ell+1,m}, \tag{3.8}$$

with coefficients

$$a^{+}_{\ell m} = \sqrt{\frac{(\ell+1)^2 - m^2}{(2\ell+1)(2\ell+3)}}, \qquad a^{-}_{\ell m} = \sqrt{\frac{\ell^2 - m^2}{(2\ell-1)(2\ell+1)}}, \tag{3.9}$$

one obtains

$$\begin{aligned}\hat{\boldsymbol{z}} \times \boldsymbol{u}'_T = &\sum_{\ell=m}^{\infty} \frac{1}{r}\big[(\ell-1)(\ell+1)a^{-}_{\ell m}\, T_{\ell-1,m} + \ell(\ell+2)a^{+}_{\ell m}\, T_{\ell+1,m}\big]\boldsymbol{Y}^{(p)}_{\ell m} \\ &- \sum_{\ell=m}^{\infty} \frac{\mathrm{i}m}{r}\big[a^{-}_{\ell m}\, T_{\ell-1,m} + a^{+}_{\ell m}\, T_{\ell+1,m}\big]\boldsymbol{Y}^{(r)}_{\ell m},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\hat{\boldsymbol{z}} \times \boldsymbol{u}'_P = &\sum_{\ell=m}^{\infty} \frac{1}{r}\left[(\ell-1)(\ell+1)a^{-}_{\ell m}\, \frac{\partial P_{\ell-1,m}}{\partial r} + \ell(\ell+2)a^{+}_{\ell m}\, \frac{\partial P_{\ell+1,m}}{\partial r}\right]\boldsymbol{Y}^{(t)}_{\ell m} \\ &+ \sum_{\ell=m}^{\infty} \frac{\mathrm{i}m}{r^2}\big[\ell(\ell-1)a^{-}_{\ell m}\, P_{\ell-1,m} + (\ell+1)(\ell+2)a^{+}_{\ell m}\, P_{\ell+1,m}\big]\boldsymbol{Y}^{(p)}_{\ell m}.\end{aligned} \tag{3.11}$$

Equations (3.10)–(3.11) summarise the toroidal–poloidal coupling produced by rotation; only $\ell \pm 1$ degrees interact at fixed m.

4 Projected evolution equations

Projecting Equation (1.1) onto the vector spherical harmonics and using the orthogonality relations $\int \boldsymbol{Y}^{(r)}_{\ell m} \cdot \boldsymbol{Y}^{(r)}_{\ell' m'}\, \mathrm{d}\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}$, $\int \boldsymbol{Y}^{(p)}_{\ell m} \cdot \boldsymbol{Y}^{(p)}_{\ell' m'}\, \mathrm{d}\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}$, and similarly for the toroidal branch, yields a coupled set of radial equations for each (ℓ, m):

$$\left(\frac{\partial}{\partial t} - E\mathcal{L}_\ell\right)\mathcal{L}_\ell P_{\ell m} - 2\,\mathcal{C}_{\ell m}[T] = \frac{\mathrm{Ra}}{\mathrm{Pr}}\frac{\ell(\ell+1)}{r^2}\left(\frac{r}{r_o}\right)^2 \theta_{\ell m}, \tag{4.1}$$

$$\left(\frac{\partial}{\partial t} - E\mathcal{L}_\ell\right)T_{\ell m} + 2\,\mathcal{D}_{\ell m}[P] = 0, \tag{4.2}$$

with coupling operators

$$\mathcal{C}_{\ell m}[T] = \frac{\mathrm{i}m}{r^2}\big[\ell(\ell-1)a^{-}_{\ell m}\, T_{\ell-1,m} + (\ell+1)(\ell+2)a^{+}_{\ell m}\, T_{\ell+1,m}\big], \tag{4.3}$$

$$\mathcal{D}_{\ell m}[P] = \frac{1}{r}\left[(\ell-1)(\ell+1)a^{-}_{\ell m}\, \frac{\partial P_{\ell-1,m}}{\partial r} + \ell(\ell+2)a^{+}_{\ell m}\, \frac{\partial P_{\ell+1,m}}{\partial r}\right]. \tag{4.4}$$

Equations (4.1)–(4.4) show Equation (10) rewritten entirely in the toroidal–poloidal basis, together with the buoyancy forcing and viscous diffusion acting through $\mathcal{L}_\ell$.

5 Chebyshev representation

Substituting the radial expansions (2.6) into (4.1)–(4.2) produces a linear system for the time-dependent Chebyshev coefficients. Define the modal vectors

$$\boldsymbol{P}_{\ell m}(t) = (P_{\ell m0}(t), \dots, P_{\ell mN}(t))^\top, \quad \boldsymbol{T}_{\ell m}(t) = (T_{\ell m0}(t), \dots, T_{\ell mN}(t))^\top, \quad \boldsymbol{\Theta}_{\ell m}(t) = (\Theta_{\ell m0}(t), \dots, \Theta_{\ell mN}(t))^\top. \tag{5.1}$$

Let $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ denote the first and second radial differentiation matrices associated with the Chebyshev basis. Evaluating $\mathcal{L}_\ell$ at the collocation points and projecting with the appropriate quadrature weights yields the discrete operator matrices used in the numerical code. Symbolically,

$$\left(\frac{\partial}{\partial t}\boldsymbol{L}_\ell - E\boldsymbol{L}_\ell^2\right)\boldsymbol{P}_{\ell m} - 2\boldsymbol{C}_{\ell m}\boldsymbol{T}_{\ell m} = \frac{\mathrm{Ra}}{\mathrm{Pr}}\boldsymbol{B}_\ell\boldsymbol{\Theta}_{\ell m}, \tag{5.2}$$

$$\left(\frac{\partial}{\partial t} - E\boldsymbol{L}_\ell\right)\boldsymbol{T}_{\ell m} + 2\boldsymbol{D}_{\ell m}\boldsymbol{P}_{\ell m} = \boldsymbol{0}, \tag{5.3}$$

where $\boldsymbol{L}_\ell$, $\boldsymbol{C}_{\ell m}$, $\boldsymbol{D}_{\ell m}$, and $\boldsymbol{B}_\ell$ follow directly from the continuous operators (3.5), (4.3), and (4.4). This is precisely the algebraic form discretised in the software package Cross.jl.

6 Temperature equation in spectral form

Equation (11) of the manuscript—referred to hereafter for consistency, even though the user prompt mentions Equation (12)—gives the non-dimensional temperature perturbation dynamics,

$$\frac{\partial \theta'}{\partial t} = -u_r' \frac{\mathrm{d}\bar{\theta}}{\mathrm{d}r} + \frac{E}{\mathrm{Pr}}\boldsymbol{\nabla}^2\theta'. \tag{6.1}$$

The modal ansatz of Equation (12) and the angular/radial expansions of Equations (14)–(15) lead to

$$\theta'(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty}\theta_{\ell m}(r,t)Y_{\ell m}(\theta,\phi) = \sum_{\ell=m}^{\infty}\sum_{n=0}^{N}\Theta_{\ell mn}(t)C_n(r)Y_{\ell m}(\theta,\phi), \tag{6.2}$$

with the radial velocity supplied by the poloidal potential,

$$u_r'(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty}\frac{\ell(\ell+1)}{r^2}P_{\ell m}(r,t)Y_{\ell m}(\theta,\phi). \tag{6.3}$$

The spherical Laplacian acting on θ' is diagonal in ℓ and m:

$$\boldsymbol{\nabla}^2\theta' = \sum_{\ell=m}^{\infty}\mathcal{S}_\ell[\theta_{\ell m}]\,Y_{\ell m}(\theta,\phi), \qquad \mathcal{S}_\ell[f] \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2}f. \tag{6.4}$$

Substituting (6.2)–(6.4) into (6.1) and projecting onto $Y_{\ell m}$ delivers the scalar evolution equation for each (ℓ, m):

$$\left(\frac{\partial}{\partial t} - \frac{E}{\mathrm{Pr}}\mathcal{S}_\ell\right)\theta_{\ell m}(r,t) = -\frac{\ell(\ell+1)}{r^2}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}r}\,P_{\ell m}(r,t). \tag{6.5}$$

Finally, applying the Chebyshev expansion (6.2) and evaluating at the radial collocation nodes provides the discrete system for the coefficient vectors $\boldsymbol{\Theta}_{\ell m}(t)$,

$$\left(\frac{\partial}{\partial t}\boldsymbol{I} - \frac{E}{\mathrm{Pr}}\boldsymbol{S}_\ell\right)\boldsymbol{\Theta}_{\ell m} = -\boldsymbol{G}\boldsymbol{Q}_\ell\boldsymbol{P}_{\ell m}, \tag{6.6}$$

where $\boldsymbol{S}_\ell$ represents $\mathcal{S}_\ell$ in the Chebyshev basis, $\boldsymbol{G}$ is the diagonal matrix containing the conduction gradient $d\bar{\theta}/dr$ at each collocation point, and $\boldsymbol{Q}_\ell$ multiplies by $\ell(\ell+1)/r^2$. These matrices already appear in the numerical implementation of Cross.jl, aligning the temperature subsystem with the toroidal–poloidal formulation derived above.

7 Boundary conditions

Boundary conditions appear as algebraic constraints on the Chebyshev coefficients at the inner ($r = r_i$) and outer ($r = r_o$) radii. They are imposed by replacing rows of the discrete operators with the appropriate evaluation or derivative stencils. The manuscript lists the no-slip, fixed-temperature conditions in Equations (16)–(18); here we summarise those and add their stress-free and fixed-flux counterparts.

7.1 Velocity boundary conditions

Regardless of the mechanical condition, impermeability requires $u_r' = 0$ at both boundaries, so from (6.3)

$$P_{\ell m}(r_b, t) = 0, \qquad r_b \in \{r_i, r_o\}. \tag{7.1}$$

No-slip. Vanishing tangential velocities enforce

$$\frac{\partial P_{\ell m}}{\partial r} = 0, \qquad T_{\ell m}(r_b, t) = 0, \tag{7.2}$$

for every degree $\ell \geq m$. In the Chebyshev discretisation these are simply point evaluations of $\boldsymbol{P}_{\ell m}$, $\boldsymbol{D}_1\boldsymbol{P}_{\ell m}$, and $\boldsymbol{T}_{\ell m}$ at $r = r_i, r_o$, reproducing Equations (16)–(17) of the paper.

Stress-free. Free-slip boundaries require the tangential shear stresses to vanish. Using the component forms (2.3)–(2.4), the conditions $\partial_r(u_\theta/r) = \partial_r(u_\phi/r) = 0$ translate into

$$r_b \frac{\partial^2 P_{\ell m}}{\partial r^2} - 2\frac{\partial P_{\ell m}}{\partial r} = 0, \tag{7.3}$$

$$r_b \frac{\partial T_{\ell m}}{\partial r} - 2T_{\ell m}(r_b, t) = 0. \tag{7.4}$$

Equations (7.1), (7.3), and (7.4) therefore provide the two independent constraints per potential required for stress-free boundaries.

7.2 Thermal boundary conditions

Two standard thermal conditions are considered. For an isothermal boundary (fixed temperature) one imposes

$$\theta_{\ell m}(r_b, t) = 0, \tag{7.5}$$

as in Equation (18) of the paper. For a fixed heat flux boundary the radial gradient is specified, typically as zero for insulating walls:

$$\frac{\partial \theta_{\ell m}}{\partial r} = 0. \tag{7.6}$$

In the Chebyshev representation these reduce to evaluating $\boldsymbol{\Theta}_{\ell m}$ or $\boldsymbol{D}_1\boldsymbol{\Theta}_{\ell m}$ at the boundary nodes. Mixed boundary conditions (e.g. fixed temperature at r_i and fixed flux at r_o) are handled by applying (7.5) and (7.6) at the respective radii.

8 Growth-rate eigenproblem

Substituting the normal-mode ansatz $e^{\lambda t}$ into Equations (4.1)–(6.5) yields a generalized eigenvalue problem for the complex growth rate $\lambda = \sigma + i\omega$:

$$\big(\lambda\mathcal{L}_\ell - E\mathcal{L}_\ell^2\big)P_{\ell m} - 2\,\mathcal{C}_{\ell m}[T] = \frac{\mathrm{Ra}}{\mathrm{Pr}}\frac{\ell(\ell+1)}{r^2}\left(\frac{r}{r_o}\right)^2\theta_{\ell m}, \tag{8.1}$$

$$\big(\lambda - E\mathcal{L}_\ell\big)T_{\ell m} + 2\,\mathcal{D}_{\ell m}[P] = 0, \tag{8.2}$$

$$\big(\lambda - \tfrac{E}{\mathrm{Pr}}\mathcal{S}_\ell\big)\theta_{\ell m} = -\frac{\ell(\ell+1)}{r^2}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}r}P_{\ell m}. \tag{8.3}$$

After discretising in radius with the Chebyshev basis, these relations can be written compactly as

$$\boldsymbol{A}_{\ell m}\boldsymbol{x}_{\ell m} = \lambda\boldsymbol{B}\boldsymbol{x}_{\ell m}, \tag{8.4}$$

where $\boldsymbol{x}_{\ell m} = (\boldsymbol{P}_{\ell m}, \boldsymbol{T}_{\ell m}, \boldsymbol{\Theta}_{\ell m})^\top$, the matrix $\boldsymbol{A}_{\ell m}$ contains diffusion, Coriolis, and buoyancy operators (with the prescribed Rayleigh number appearing explicitly in (8.1)), and $\boldsymbol{B}$ is block-diagonal with identity submatrices on the velocity and temperature blocks and zeros elsewhere. Solving this eigenproblem at fixed Ra yields the growth rate σ and drift frequency ω for each azimuthal order m.

9 Implementation in Cross.jl

The Julia package Cross.jl evaluates the radial–meridional operator using SHTnsKit for the angular transforms together with Chebyshev collocation in radius. The normal-mode system (8.1)–(8.3) is encoded in two linear maps:

- `apply_operator` (file `src/linear_stability.jl:244`) evaluates the right-hand sides of (8.1)–(8.3), including the buoyancy term proportional to the specified Rayleigh number:

$$\left(-\nabla p - 2\hat{\boldsymbol{z}}\times\boldsymbol{u} + \frac{\mathrm{Ra}}{\mathrm{Pr}}\left(\frac{r}{r_o}\right)^2\theta\hat{\boldsymbol{r}} + E\nabla^2\boldsymbol{u},\ -u_r\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}r} + \frac{E}{\mathrm{Pr}}\nabla^2\theta,\ \nabla\cdot\boldsymbol{u}\right)$$

  with the requested mechanical and thermal boundary conditions enforced by `enforce_mechanical_boundary!` and `apply_thermal_boundary!`.

- `apply_mass` (file `src/linear_stability.jl:321`) forms the inertial block $\boldsymbol{B}\boldsymbol{x}$ by returning the velocity and temperature fields (with zeros in the constraint rows), so that the generalized eigenproblem solved by `eigsolve` yields the complex growth rate λ directly.

The eigenproblem constructed from these maps therefore reproduces (8.1)–(8.3), delivering the growth rate λ for the chosen Rayleigh number. Users can call `leading_modes` to obtain the most unstable eigenvalues for a prescribed azimuthal wavenumber while choosing between no-slip or stress-free velocity boundaries and fixed-temperature or fixed-flux thermal boundaries via the keyword arguments of `setup_operator`.

10 Summary

Starting from the non-dimensional momentum balance (1.1), we introduced the toroidal and poloidal potentials of Equations (13)–(15), expanded them in spherical harmonics and radial Chebyshev polynomials, and projected each term onto the vector spherical harmonic basis. The resulting Equations (4.1) and (4.2) express the dynamics entirely in terms of the scalar potentials, providing the foundation of the matrix eigenvalue problem solved in the accompanying code base.

In parallel, Equation (6.5) (and its discrete counterpart (6.6)) expresses the temperature perturbation equation in the same spectral framework, while the growth-rate system (8.1)–(8.3) casts the full problem as a generalized eigenvalue problem with the complex growth rate as eigenvalue, completing the derivation requested.