Deriving Equation (10) in Toroidal–Poloidal Form

1 Governing momentum equation

Equation (10) of *Onset of Convection in Rotating Spherical Shells* gives the non-dimensional perturbation momentum balance

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\boldsymbol{\nabla} p' - 2\hat{\boldsymbol{z}} \times \boldsymbol{u}' + \frac{\mathrm{Ra}}{\mathrm{Pr}} \left(\frac{r}{r_o}\right)^2 \theta' \hat{\boldsymbol{r}} + E \boldsymbol{\nabla}^2 \boldsymbol{u}'. \tag{1.1}$$

Here $\boldsymbol{u}'$ is solenoidal, $\hat{\boldsymbol{r}}$ denotes the radial unit vector, and $\hat{\boldsymbol{z}}$ is the axis of rotation. Our goal is to rewrite (1.1) in terms of the toroidal and poloidal potentials introduced in Equations (13)–(15) of the paper.

2 Toroidal–poloidal decomposition

Following Equation (13), the divergence-free velocity field is decomposed into scalar potentials P (poloidal) and T (toroidal),

$$\boldsymbol{u}' = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (P\hat{\boldsymbol{r}}) + \boldsymbol{\nabla} \times (T\hat{\boldsymbol{r}}). \tag{2.1}$$

For compactness we define the surface gradient $\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \hat{\boldsymbol{r}}\,\partial_r$ and recall the vector spherical harmonics

$$\mathbf{Y}^{(r)}_{\ell m} = Y_{\ell m}\hat{\boldsymbol{r}}, \qquad \mathbf{Y}^{(p)}_{\ell m} = \boldsymbol{\nabla}_\perp Y_{\ell m}, \qquad \mathbf{Y}^{(t)}_{\ell m} = \hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_\perp Y_{\ell m}, \tag{2.2}$$

with $Y_{\ell m}$ the Schmidt semi-normalised spherical harmonics. Direct evaluation of (2.1) leads to the standard component form

$$\boldsymbol{u}'_P = \sum_{\ell=m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} P_{\ell m}\, \mathbf{Y}^{(r)}_{\ell m} + \frac{1}{r}\frac{\partial P_{\ell m}}{\partial r}\, \mathbf{Y}^{(p)}_{\ell m} \right], \tag{2.3}$$

$$\boldsymbol{u}'_T = \sum_{\ell=m}^{\infty} \left[\frac{1}{r} T_{\ell m}\, \mathbf{Y}^{(t)}_{\ell m} \right], \tag{2.4}$$

where the sums run over a fixed azimuthal order m and $P_{\ell m}(r,t)$, $T_{\ell m}(r,t)$ are the radial amplitudes of each degree ℓ.

Equation (14) expresses the angular dependence of the scalar potentials as

$$P(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty} P_{\ell m}(r,t) Y_{\ell m}(\theta,\phi), \qquad T(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty} T_{\ell m}(r,t) Y_{\ell m}(\theta,\phi), \tag{2.5}$$

while the radial profiles are expanded spectrally using Chebyshev polynomials as in Equation (15),

$$P_{\ell m}(r,t) = \sum_{n=0}^{N} P_{\ell m n}(t)\, C_n(r), \qquad T_{\ell m}(r,t) = \sum_{n=0}^{N} T_{\ell m n}(t)\, C_n(r). \tag{2.6}$$

The temperature perturbation is treated identically:

$$\theta'(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty} \theta_{\ell m}(r,t) Y_{\ell m}(\theta,\phi) = \sum_{\ell=m}^{\infty}\sum_{n=0}^{N} \Theta_{\ell mn}(t)\, C_n(r) Y_{\ell m}(\theta,\phi). \tag{2.7}$$

3 Operators acting on the toroidal–poloidal fields

To project (1.1) onto the basis (2.3)–(2.4) we evaluate each term separately.

3.1 Time derivative

Because the vector spherical harmonics are time-independent,

$$\frac{\partial \boldsymbol{u}'}{\partial t} = \sum_{\ell=m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} \frac{\partial P_{\ell m}}{\partial t} \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial P_{\ell m}}{\partial t}\right) \boldsymbol{Y}_{\ell m}^{(p)} + \frac{1}{r}\frac{\partial T_{\ell m}}{\partial t} \boldsymbol{Y}_{\ell m}^{(t)} \right]. \tag{3.1}$$

3.2 Pressure gradient

The pressure can be expanded as $p'(r,\theta,\phi,t) = \sum_{\ell m} p_{\ell m}(r,t) Y_{\ell m}(\theta,\phi)$. Its gradient decomposes into the same vector spherical harmonics,

$$\boldsymbol{\nabla} p' = \sum_{\ell=m}^{\infty} \left[\frac{\partial p_{\ell m}}{\partial r} \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r} p_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(p)} \right]. \tag{3.2}$$

When the curl or double curl of (1.1) is taken (as in the numerical formulation) the pressure term drops out; it is retained here only for completeness.

3.3 Viscous term

Using the identity $\boldsymbol{\nabla}^2 \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{\nabla}^2 \boldsymbol{A}$ and the fact that $\boldsymbol{\nabla}^2(f\hat{\boldsymbol{r}}) = \left(\partial_r^2 f + \frac{2}{r}\partial_r f - \frac{\ell(\ell+1)}{r^2} f\right)\hat{\boldsymbol{r}}$ for each spherical harmonic degree, the Laplacian acts diagonally on the potentials:

$$\boldsymbol{\nabla}^2 \boldsymbol{u}'_P = \sum_{\ell=m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} \mathcal{L}_\ell P_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r}\frac{\partial}{\partial r}(\mathcal{L}_\ell P_{\ell m}) \boldsymbol{Y}_{\ell m}^{(p)} \right], \tag{3.3}$$

$$\boldsymbol{\nabla}^2 \boldsymbol{u}'_T = \sum_{\ell=m}^{\infty} \left[\frac{1}{r} \mathcal{L}_\ell T_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(t)} \right], \tag{3.4}$$

where the scalar radial operator $\mathcal{L}_\ell$ is

$$\mathcal{L}_\ell f \equiv \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} - \frac{\ell(\ell+1)}{r^2} f. \tag{3.5}$$

3.4 Buoyancy term

Because the buoyancy force is purely radial, only the $\boldsymbol{Y}_{\ell m}^{(r)}$ basis is involved:

$$\frac{\mathrm{Ra}}{\mathrm{Pr}} \left(\frac{r}{r_o}\right)^2 \theta' \hat{\boldsymbol{r}} = \sum_{\ell=m}^{\infty} \frac{\mathrm{Ra}}{\mathrm{Pr}} \left(\frac{r}{r_o}\right)^2 \theta_{\ell m}(r,t) \boldsymbol{Y}_{\ell m}^{(r)}. \tag{3.6}$$

3.5 Coriolis term

The Coriolis acceleration mixes neighbouring spherical-harmonic degrees. Writing $\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$ and using the angular momentum ladder relations

$$\cos\theta\, Y_{\ell m} = a^{+}_{\ell m} Y_{\ell+1,m} + a^{-}_{\ell m} Y_{\ell-1,m}, \tag{3.7}$$

$$\sin\theta\, \frac{\partial Y_{\ell m}}{\partial\theta} = \ell\, a^{-}_{\ell m} Y_{\ell-1,m} - (\ell+1) a^{+}_{\ell m} Y_{\ell+1,m}, \tag{3.8}$$

with coefficients

$$a^{+}_{\ell m} = \sqrt{\frac{(\ell+1)^2 - m^2}{(2\ell+1)(2\ell+3)}}, \qquad a^{-}_{\ell m} = \sqrt{\frac{\ell^2 - m^2}{(2\ell-1)(2\ell+1)}}, \tag{3.9}$$

one obtains

$$\begin{aligned} \hat{\boldsymbol{z}} \times \boldsymbol{u}'_T = &\sum_{\ell=m}^{\infty} \frac{1}{r}\left[(\ell-1)(\ell+1)a^{-}_{\ell m}\, T_{\ell-1,m} + \ell(\ell+2)a^{+}_{\ell m}\, T_{\ell+1,m}\right]\boldsymbol{Y}^{(p)}_{\ell m} \\ &- \sum_{\ell=m}^{\infty} \frac{\mathrm{i}m}{r}\left[a^{-}_{\ell m}\, T_{\ell-1,m} + a^{+}_{\ell m}\, T_{\ell+1,m}\right]\boldsymbol{Y}^{(r)}_{\ell m}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \hat{\boldsymbol{z}} \times \boldsymbol{u}'_P = &\sum_{\ell=m}^{\infty} \frac{1}{r}\left[(\ell-1)(\ell+1)a^{-}_{\ell m}\, \frac{\partial P_{\ell-1,m}}{\partial r} + \ell(\ell+2)a^{+}_{\ell m}\, \frac{\partial P_{\ell+1,m}}{\partial r}\right]\boldsymbol{Y}^{(t)}_{\ell m} \\ &+ \sum_{\ell=m}^{\infty} \frac{\mathrm{i}m}{r^2}\left[\ell(\ell-1)a^{-}_{\ell m}\, P_{\ell-1,m} + (\ell+1)(\ell+2)a^{+}_{\ell m}\, P_{\ell+1,m}\right]\boldsymbol{Y}^{(p)}_{\ell m}. \end{aligned} \tag{3.11}$$

Equations (3.10)–(3.11) summarise the toroidal–poloidal coupling produced by rotation; only $\ell \pm 1$ degrees interact at fixed m.

4 Projected evolution equations

Projecting Equation (1.1) onto the vector spherical harmonics and using the orthogonality relations $\int \boldsymbol{Y}^{(r)}_{\ell m} \cdot \boldsymbol{Y}^{(r)}_{\ell' m'}\, \mathrm{d}\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}$, $\int \boldsymbol{Y}^{(p)}_{\ell m} \cdot \boldsymbol{Y}^{(p)}_{\ell' m'}\, \mathrm{d}\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}$, and similarly for the toroidal branch, yields a coupled set of radial equations for each (ℓ, m):

$$\left(\frac{\partial}{\partial t} - E\mathcal{L}_\ell\right)\mathcal{L}_\ell P_{\ell m} - 2\,\mathcal{C}_{\ell m}[T] = \frac{\mathrm{Ra}}{\mathrm{Pr}}\frac{\ell(\ell+1)}{r^2}\left(\frac{r}{r_o}\right)^2 \theta_{\ell m}, \tag{4.1}$$

$$\left(\frac{\partial}{\partial t} - E\mathcal{L}_\ell\right) T_{\ell m} + 2\,\mathcal{D}_{\ell m}[P] = 0, \tag{4.2}$$

with coupling operators

$$\mathcal{C}_{\ell m}[T] = \frac{\mathrm{i}m}{r^2}\left[\ell(\ell-1)a^{-}_{\ell m}\, T_{\ell-1,m} + (\ell+1)(\ell+2)a^{+}_{\ell m}\, T_{\ell+1,m}\right], \tag{4.3}$$

$$\mathcal{D}_{\ell m}[P] = \frac{1}{r}\left[(\ell-1)(\ell+1)a^{-}_{\ell m}\, \frac{\partial P_{\ell-1,m}}{\partial r} + \ell(\ell+2)a^{+}_{\ell m}\, \frac{\partial P_{\ell+1,m}}{\partial r}\right]. \tag{4.4}$$

Equations (4.1)–(4.4) show Equation (10) rewritten entirely in the toroidal–poloidal basis, together with the buoyancy forcing and viscous diffusion acting through $\mathcal{L}_\ell$.

5 Chebyshev representation

Substituting the radial expansions (2.6) into (4.1)–(4.2) produces a linear system for the time-dependent Chebyshev coefficients. Define the modal vectors

$$\boldsymbol{P}_{\ell m}(t) = (P_{\ell m0}(t), \dots, P_{\ell mN}(t))^\top, \quad \boldsymbol{T}_{\ell m}(t) = (T_{\ell m0}(t), \dots, T_{\ell mN}(t))^\top, \quad \boldsymbol{\Theta}_{\ell m}(t) = (\Theta_{\ell m0}(t), \dots, \Theta_{\ell mN}(t))^\top. \tag{5.1}$$

Let $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ denote the first and second radial differentiation matrices associated with the Chebyshev basis. Evaluating $\mathcal{L}_\ell$ at the collocation points and projecting with the appropriate quadrature weights yields the discrete operator matrices used in the numerical code. Symbolically,

$$\left(\frac{\partial}{\partial t}\boldsymbol{L}_\ell - E\boldsymbol{L}_\ell^2\right)\boldsymbol{P}_{\ell m} - 2\boldsymbol{C}_{\ell m}\boldsymbol{T}_{\ell m} = \frac{\mathrm{Ra}}{\mathrm{Pr}}\boldsymbol{B}_\ell\boldsymbol{\Theta}_{\ell m}, \tag{5.2}$$

$$\left(\frac{\partial}{\partial t} - E\boldsymbol{L}_\ell\right)\boldsymbol{T}_{\ell m} + 2\boldsymbol{D}_{\ell m}\boldsymbol{P}_{\ell m} = \boldsymbol{0}, \tag{5.3}$$

where $\boldsymbol{L}_\ell$, $\boldsymbol{C}_{\ell m}$, $\boldsymbol{D}_{\ell m}$, and $\boldsymbol{B}_\ell$ follow directly from the continuous operators (3.5), (4.3), and (4.4). This is precisely the algebraic form discretised in the software package Cross.jl.

6 Temperature equation in spectral form

Equation (11) of the manuscript—referred to hereafter for consistency, even though the user prompt mentions Equation (12)—gives the non-dimensional temperature perturbation dynamics,

$$\frac{\partial\theta'}{\partial t} = -u_r'\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}r} + \frac{E}{\mathrm{Pr}}\boldsymbol{\nabla}^2\theta'. \tag{6.1}$$

The modal ansatz of Equation (12) and the angular/radial expansions of Equations (14)–(15) lead to

$$\theta'(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty}\theta_{\ell m}(r,t)Y_{\ell m}(\theta,\phi) = \sum_{\ell=m}^{\infty}\sum_{n=0}^{N}\Theta_{\ell mn}(t)C_n(r)Y_{\ell m}(\theta,\phi), \tag{6.2}$$

with the radial velocity supplied by the poloidal potential,

$$u_r'(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty}\frac{\ell(\ell+1)}{r^2}P_{\ell m}(r,t)Y_{\ell m}(\theta,\phi). \tag{6.3}$$

The spherical Laplacian acting on θ' is diagonal in ℓ and m:

$$\boldsymbol{\nabla}^2\theta' = \sum_{\ell=m}^{\infty}\mathcal{S}_\ell[\theta_{\ell m}]\,Y_{\ell m}(\theta,\phi), \qquad \mathcal{S}_\ell[f] \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2}f. \tag{6.4}$$

Substituting (6.2)–(6.4) into (6.1) and projecting onto $Y_{\ell m}$ delivers the scalar evolution equation for each (ℓ, m):

$$\left(\frac{\partial}{\partial t} - \frac{E}{\mathrm{Pr}}\mathcal{S}_\ell\right)\theta_{\ell m}(r,t) = -\frac{\ell(\ell+1)}{r^2}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}r}\,P_{\ell m}(r,t). \tag{6.5}$$

Finally, applying the Chebyshev expansion (6.2) and evaluating at the radial collocation nodes provides the discrete system for the coefficient vectors $\boldsymbol{\Theta}_{\ell m}(t)$,

$$\left(\frac{\partial}{\partial t}\boldsymbol{I} - \frac{E}{\mathrm{Pr}}\boldsymbol{S}_\ell\right)\boldsymbol{\Theta}_{\ell m} = -\boldsymbol{G}\boldsymbol{Q}_\ell\boldsymbol{P}_{\ell m}, \tag{6.6}$$

where $\boldsymbol{S}_\ell$ represents $\mathcal{S}_\ell$ in the Chebyshev basis, $\boldsymbol{G}$ is the diagonal matrix containing the conduction gradient $d\bar{\theta}/dr$ at each collocation point, and $\boldsymbol{Q}_\ell$ multiplies by $\ell(\ell+1)/r^2$. These matrices already appear in the numerical implementation of Cross.jl, aligning the temperature subsystem with the toroidal–poloidal formulation derived above.

7 Boundary conditions

Boundary conditions appear as algebraic constraints on the Chebyshev coefficients at the inner ($r = r_i$) and outer ($r = r_o$) radii. They are imposed by replacing rows of the discrete operators with the appropriate evaluation or derivative stencils. The manuscript lists the no-slip, fixed-temperature conditions in Equations (16)–(18); here we summarise those and add their stress-free and fixed-flux counterparts.

7.1 Velocity boundary conditions

Regardless of the mechanical condition, impermeability requires $u_r' = 0$ at both boundaries, so from (6.3)

$$P_{\ell m}(r_b, t) = 0, \qquad r_b \in \{r_i, r_o\}. \tag{7.1}$$

No-slip. Vanishing tangential velocities enforce

$$\frac{\partial P_{\ell m}}{\partial r} = 0, \qquad T_{\ell m}(r_b, t) = 0, \tag{7.2}$$

for every degree $\ell \geq m$. In the Chebyshev discretisation these are simply point evaluations of $\boldsymbol{P}_{\ell m}$, $\boldsymbol{D}_1\boldsymbol{P}_{\ell m}$, and $\boldsymbol{T}_{\ell m}$ at $r = r_i, r_o$, reproducing Equations (16)–(17) of the paper.

Stress-free. Free-slip boundaries require the tangential shear stresses to vanish. Using the component forms (2.3)–(2.4), the conditions $\partial_r(u_\theta/r) = \partial_r(u_\phi/r) = 0$ translate into

$$r_b \frac{\partial^2 P_{\ell m}}{\partial r^2} - 2\frac{\partial P_{\ell m}}{\partial r} = 0, \tag{7.3}$$

$$r_b \frac{\partial T_{\ell m}}{\partial r} - 2T_{\ell m}(r_b, t) = 0. \tag{7.4}$$

Equations (7.1), (7.3), and (7.4) therefore provide the two independent constraints per potential required for stress-free boundaries.

7.2 Thermal boundary conditions

Two standard thermal conditions are considered. For an isothermal boundary (fixed temperature) one imposes

$$\theta_{\ell m}(r_b, t) = 0, \tag{7.5}$$

as in Equation (18) of the paper. For a fixed heat flux boundary the radial gradient is specified, typically as zero for insulating walls:

$$\frac{\partial \theta_{\ell m}}{\partial r} = 0. \tag{7.6}$$

In the Chebyshev representation these reduce to evaluating $\boldsymbol{\Theta}_{\ell m}$ or $\boldsymbol{D}_1\boldsymbol{\Theta}_{\ell m}$ at the boundary nodes. Mixed boundary conditions (e.g. fixed temperature at r_i and fixed flux at r_o) are handled by applying (7.5) and (7.6) at the respective radii.

8 Growth-rate eigenproblem

Substituting the normal-mode ansatz $e^{\lambda t}$ into Equations (4.1)–(6.5) yields a generalized eigenvalue problem for the complex growth rate $\lambda = \sigma + i\omega$:

$$\big(\lambda\mathcal{L}_\ell - E\mathcal{L}_\ell^2\big)P_{\ell m} - 2\,\mathcal{C}_{\ell m}[T] = \frac{\mathrm{Ra}}{\mathrm{Pr}}\frac{\ell(\ell+1)}{r^2}\left(\frac{r}{r_o}\right)^2\theta_{\ell m}, \tag{8.1}$$

$$\big(\lambda - E\mathcal{L}_\ell\big)T_{\ell m} + 2\,\mathcal{D}_{\ell m}[P] = 0, \tag{8.2}$$

$$\big(\lambda - \tfrac{E}{\mathrm{Pr}}\mathcal{S}_\ell\big)\theta_{\ell m} = -\frac{\ell(\ell+1)}{r^2}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}r}P_{\ell m}. \tag{8.3}$$

After discretising in radius with the Chebyshev basis, these relations can be written compactly as

$$\boldsymbol{A}_{\ell m}\boldsymbol{x}_{\ell m} = \lambda\boldsymbol{B}\boldsymbol{x}_{\ell m}, \tag{8.4}$$

where $\boldsymbol{x}_{\ell m} = (\boldsymbol{P}_{\ell m}, \boldsymbol{T}_{\ell m}, \boldsymbol{\Theta}_{\ell m})^\top$, the matrix $\boldsymbol{A}_{\ell m}$ contains diffusion, Coriolis, and buoyancy operators (with the prescribed Rayleigh number appearing explicitly in (8.1)), and $\boldsymbol{B}$ is block-diagonal with identity submatrices on the velocity and temperature blocks and zeros elsewhere. Solving this eigenproblem at fixed Ra yields the growth rate σ and drift frequency ω for each azimuthal order m.

9 Implementation in Cross.jl

The Julia package Cross.jl implements the toroidal–poloidal framework described above using Chebyshev collocation in radius and provides both field reconstruction tools and a complete linear stability eigenvalue solver for computing critical parameters.

9.1 Linear stability analysis

The onset of convection problem is solved via the generalized eigenvalue formulation (8.1)–(8.3) in file `src/linear_stability.jl`:

- `OnsetParams` encapsulates all physical and numerical parameters: Ekman number E, Prandtl number Pr, Rayleigh number Ra, radius ratio χ, azimuthal wavenumber m, spherical harmonic truncation $\ell_{\max}$, radial resolution N_r, and boundary condition types.

- `LinearStabilityOperator` assembles the discrete operators corresponding to Equations (4.1)–(6.5). The radial operator $\mathcal{L}_\ell$ (3.5) and scalar Laplacian $\mathcal{S}_\ell$ (6.4) are constructed for each spherical harmonic degree ℓ using Chebyshev differentiation. The Coriolis coupling operators $\mathcal{C}_{\ell m}$ (4.3) and $\mathcal{D}_{\ell m}$ (4.4) implement the ladder relations (3.9), coupling adjacent $\ell \pm 1$ modes.

- `apply_operator` evaluates the right-hand side of the spatial eigenvalue problem:

$$\boldsymbol{A}\boldsymbol{x} = \left(-E\boldsymbol{L}_\ell^2\boldsymbol{P} - 2\boldsymbol{C}_{\ell m}\boldsymbol{T} + \frac{\mathrm{Ra}}{\mathrm{Pr}}\boldsymbol{B}_\ell\boldsymbol{\Theta},\ E\boldsymbol{L}_\ell\boldsymbol{T} + 2\boldsymbol{D}_{\ell m}\boldsymbol{P},\ \frac{E}{\mathrm{Pr}}\boldsymbol{S}_\ell\boldsymbol{\Theta} - \boldsymbol{G}\boldsymbol{Q}_\ell\boldsymbol{P}\right)$$

  including viscous diffusion, Coriolis coupling, buoyancy forcing, and advection of the background temperature gradient.

- `apply_mass` forms the mass matrix $\boldsymbol{B}\boldsymbol{x}$ that multiplies the time-derivative block in the generalized eigenvalue problem $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{B}\boldsymbol{x}$.

- Boundary conditions (7.1)–(7.6) are enforced by replacing the appropriate operator rows at $r = r_i$ and $r = r_o$ with constraints for impermeability ($P = 0$), no-slip ($\partial P/\partial r = 0$, $T = 0$) or stress-free (7.3)–(7.4), and fixed-temperature ($\Theta = 0$) or fixed-flux ($\partial\Theta/\partial r = 0$).
- `solve_eigenvalue_problem` uses the KrylovKit library with shift-and-invert around $\lambda = 0$ to compute eigenvalues near marginal stability, directly yielding the complex growth rate $\lambda = \sigma + i\omega$.
- `find_critical_rayleigh` automates the search for the critical Rayleigh number Ra_c where $\sigma = 0$ via Brent's root-finding method, returning the critical parameters $(\mathrm{Ra}_c, \omega_c)$ along with the associated eigenmode.

The implementation has been benchmarked against Table 5 of Dormy et al. (2004) for $\chi = 0.35$, $\mathrm{Pr} = 1$, no-slip and fixed-temperature boundary conditions, reproducing critical Rayleigh numbers and drift frequencies to better than 0.1% accuracy (see `test/test_onset_benchmark.jl`).

9.2 Field reconstruction

Post-processing utilities reside in `src/get_velocity.jl`:

- `velocity_fields_from_poloidal_toroidal` synthesizes (u_r, u_θ, u_ϕ) from spectral coefficients using Equations (2.3)–(2.4) and SHTnsKit spherical harmonic transforms.
- `temperature_field_from_coefficients` reconstructs $\theta'(r, \theta, \phi)$ from the expansion (2.7).
- `fields_from_coefficients` combines both reconstructions.

9.3 Chebyshev radial discretization

`ChebyshevDiffn` (`src/Chebyshev.jl`) constructs spectral differentiation matrices up to fourth order on the Chebyshev–Gauss–Lobatto grid with automatic domain transformation from $[-1, 1]$ to $[r_i, r_o]$, ensuring that $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, etc. correspond to physical radial derivatives.

10 Extension: Basic State with Meridional Variations

The preceding sections analyzed linear stability around a quiescent conduction state with purely radial temperature variation and zero velocity, $\bar{\boldsymbol{u}} = \boldsymbol{0}$ and $\bar{\theta}(r)$. This section extends the formulation to study onset of convection on top of a *basic state* that includes:

- Meridionally-varying temperature: $\bar{\theta}(r, \theta)$
- Axisymmetric zonal flow: $\bar{u}_\phi(r, \theta)$ from thermal wind balance
- No meridional circulation: $\bar{u}_r = \bar{u}_\theta = 0$

10.1 Physical Motivation

In rapidly rotating systems, meridional (latitudinal) temperature gradients drive zonal (east–west) flows through thermal wind balance. This occurs when:

- Differential heating creates latitudinal temperature contrasts

- Geostrophic balance between Coriolis force and pressure gradient holds
- Rotation constrains motion to be quasi-two-dimensional

Such configurations arise in planetary atmospheres, stellar convection zones, and laboratory experiments with imposed thermal patterns. Understanding how background differential rotation affects convective onset is crucial for interpreting observations and numerical simulations.

10.2 Basic State Equations

The basic state is *axisymmetric* ($\partial/\partial\phi = 0$) and *steady* ($\partial/\partial t = 0$). The temperature field is expanded in axisymmetric spherical harmonics:

$$\bar{\theta}(r,\theta) = \sum_{\ell=0}^{\ell_{\max}^{\mathrm{bs}}} \bar{\theta}_{\ell 0}(r) Y_{\ell 0}(\theta), \tag{10.1}$$

where typically $\ell_{\max}^{\mathrm{bs}} \ll \ell_{\max}$ (the basic state uses fewer modes than the perturbation). Similarly, the zonal velocity is

$$\bar{u}_\phi(r,\theta) = \sum_{\ell=0}^{\ell_{\max}^{\mathrm{bs}}} \bar{u}_{\phi,\ell 0}(r) Y_{\ell 0}(\theta). \tag{10.2}$$

10.3 Thermal Wind Balance

The zonal flow is determined by thermal wind balance. In geostrophic equilibrium, the meridional and radial momentum equations read:

$$2\Omega \sin\theta \, \bar{u}_\phi = \frac{1}{\bar{\rho}} \frac{1}{r} \frac{\partial \bar{p}}{\partial \theta}, \tag{10.3}$$

$$2\Omega \cos\theta \, \bar{u}_\phi - \alpha g_0 \left(\frac{r}{r_o}\right) \bar{\theta} = \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial r}. \tag{10.4}$$

Eliminating pressure by taking $\partial/\partial r$ of (**??**) and $(1/r)\,\partial/\partial\theta$ of (**??**) yields the *thermal wind equation*:

$$\frac{\partial \bar{u}_\phi}{\partial r} + \frac{\bar{u}_\phi}{r} = -\frac{\alpha g_0}{2\Omega^2 r^2 \sin\theta} \left(\frac{r}{r_o}\right) \frac{\partial \bar{\theta}}{\partial \theta}. \tag{10.5}$$

Using the same non-dimensionalization as in Section 1 (length L, time $1/\Omega$, temperature ΔT), and recognizing that

$$\frac{\alpha g_0 \Delta T L}{2\Omega^2 L^2} = \frac{\mathrm{Ra}\, E^2}{2\,\mathrm{Pr}},$$

the non-dimensional thermal wind balance becomes:

$$\frac{\partial \bar{u}_\phi}{\partial r} + \frac{\bar{u}_\phi}{r} = -\frac{\mathrm{Ra}}{2\,\mathrm{Pr}} \frac{1}{r\sin\theta} \left(\frac{r}{r_o}\right) \frac{\partial \bar{\theta}}{\partial \theta}. \tag{10.6}$$

Equivalently, writing in terms of $r\bar{u}_\phi$:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}_\phi) = -\frac{\mathrm{Ra}}{2\,\mathrm{Pr}} \frac{r}{r_o \sin\theta} \frac{\partial \bar{\theta}}{\partial \theta}. \tag{10.7}$$

For a given basic state temperature $\bar{\theta}(r,\theta)$, Equation (**??**) is a first-order ODE in r (for each latitude θ) that can be integrated subject to boundary conditions $\bar{u}_\phi(r_i,\theta) = \bar{u}_\phi(r_o,\theta) = 0$ (no-slip).

10.4 Modified Perturbation Equations

With a non-zero basic state, the linearized perturbation equations acquire advection terms. The momentum equation becomes:

$$\frac{\partial \boldsymbol{u}'}{\partial t} + (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\boldsymbol{u}' + (\boldsymbol{u}' \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}} = -\boldsymbol{\nabla} p' - 2\hat{\boldsymbol{z}} \times \boldsymbol{u}' + \frac{\mathrm{Ra}}{\mathrm{Pr}}\left(\frac{r}{r_o}\right)^2 \theta' \hat{\boldsymbol{r}} + E\boldsymbol{\nabla}^2 \boldsymbol{u}', \tag{10.8}$$

where the new terms are:

- $(\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\boldsymbol{u}' = \bar{u}_\phi \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{u}'}{\partial \phi}$ – advection of perturbation by basic state flow
- $(\boldsymbol{u}' \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}} = u_r' \frac{\partial \bar{u}_\phi}{\partial r}\hat{\boldsymbol{\phi}} + \frac{u_\theta'}{r}\frac{\partial \bar{u}_\phi}{\partial \theta}\hat{\boldsymbol{\phi}}$ – perturbation advection of basic state shear

The temperature equation becomes:

$$\frac{\partial \theta'}{\partial t} + (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\theta' + (\boldsymbol{u}' \cdot \boldsymbol{\nabla})\bar{\theta} = \frac{E}{\mathrm{Pr}}\boldsymbol{\nabla}^2\theta', \tag{10.9}$$

where:

- $(\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\theta' = \bar{u}_\phi \frac{1}{r\sin\theta}\frac{\partial \theta'}{\partial \phi}$ – basic state advection of perturbation temperature
- $(\boldsymbol{u}' \cdot \boldsymbol{\nabla})\bar{\theta} = u_r' \frac{\partial \bar{\theta}}{\partial r} + \frac{u_\theta'}{r}\frac{\partial \bar{\theta}}{\partial \theta}$ – *perturbation advection of meridional temperature gradient*

The last term is crucial: the meridional gradient $\partial\bar{\theta}/\partial\theta$ introduces coupling between different spherical harmonic degrees, modifying the onset conditions.

10.5 Spectral Formulation

In Fourier space (azimuthal mode m), the advection terms simplify. For the temperature equation (**??**), the azimuthal advection becomes:

$$\bar{u}_\phi \frac{1}{r\sin\theta}\frac{\partial \theta'}{\partial \phi} \to \frac{im}{r\sin\theta}\bar{u}_\phi \theta' \quad \text{(in Fourier space)}. \tag{10.10}$$

The perturbation advection of the basic state gradient involves:

- **Radial advection**: $u_r' \partial\bar{\theta}/\partial r$ couples to all ℓ' components of $\bar{\theta}_{\ell' 0}(r)$
- **Meridional advection**: $(u_\theta'/r)\partial\bar{\theta}/\partial\theta$ requires evaluating $\partial Y_{\ell' 0}/\partial\theta$ and mode-coupling integrals

The full treatment requires computing Gaunt coefficients for integrals of the form:

$$\int Y_{\ell m} \frac{\partial Y_{\ell' 0}}{\partial \theta} Y_{\ell'' m}\, d\Omega, \tag{10.11}$$

which couple different (ℓ, ℓ', ℓ'') triads.

10.6 Implementation Notes

The basic state capability is implemented in `src/basic_state.jl` with the following structure:

- `BasicState` type stores the spectral coefficients $\bar{\theta}_{\ell 0}(r)$, $\bar{u}_{\phi,\ell 0}(r)$, and their radial derivatives.
- `conduction_basic_state` creates the standard conduction profile (only $\ell = 0$ component, $\bar{u}_\phi = 0$), ensuring backward compatibility.
- `meridional_basic_state` constructs a basic state with a prescribed meridional temperature variation (e.g., adding a Y_{20} component) and solves the thermal wind equation (**??**) to obtain the corresponding zonal flow.
- `solve_thermal_wind_balance!` integrates Equation (**??**) for each spectral mode using the radial Chebyshev grid.

The `OnsetParams` structure accepts an optional `basic_state` argument. When provided, the operator `apply_operator` in `src/linear_stability.jl` includes the advection terms from Equations (**??**) and (**??**).

Current Status: The implementation includes:

- Thermal wind balance solver for the $\ell = 2$ mode (simplified)
- Radial advection $u_r' \partial\bar{\theta}/\partial r$ for $\ell' = 0, 2$ components
- Framework for meridional and azimuthal advection (marked as TODO)

Full mode coupling through Gaunt coefficients and complete advection terms remain as future enhancements. See `README_BASIC_STATE.md` for details.

10.7 Physical Implications

The presence of a meridional temperature gradient and associated zonal flow can:

- **Modify** Ra_c: The basic state shear can stabilize or destabilize convection depending on the sign of $\partial\bar{\theta}/\partial\theta$ and the interaction with the Coriolis force.
- **Alter** ω_c: Basic state zonal flow advects perturbations azimuthally, shifting the drift frequency. Prograde (retrograde) basic state flows can enhance (reduce) the prograde drift of thermal Rossby waves.
- **Change** m_c: The interaction between the azimuthal structure of perturbations and the basic state can favor different critical wavenumbers.

For small amplitude meridional perturbations ($\|\bar{\theta} - \bar{\theta}_{\mathrm{cond}}\| \ll 1$), effects scale linearly with amplitude. Larger perturbations require the full nonlinear coupling.

11 Summary

Starting from the non-dimensional momentum balance (1.1), we introduced the toroidal and poloidal potentials of Equations (13)–(15), expanded them in spherical harmonics and radial Chebyshev polynomials, and projected each term onto the vector spherical harmonic basis. The resulting Equations (4.1) and (4.2) express the dynamics entirely in terms of the scalar potentials, providing the foundation of the matrix eigenvalue problem solved in the accompanying code base.

In parallel, Equation (6.5) (and its discrete counterpart (6.6)) expresses the temperature perturbation equation in the same spectral framework, while the growth-rate system (8.1)–(8.3) casts the full problem as a generalized eigenvalue problem with the complex growth rate as eigenvalue.

Section **??** extends this formulation to study onset on top of a basic state with meridional temperature variations and thermal wind-balanced zonal flows. The thermal wind equation (**??**) relates the zonal flow to the meridional temperature gradient through geostrophic balance. Modified perturbation equations (**??**)–(**??**) include advection by the basic state, introducing mode coupling that alters the critical parameters for onset.