Deriving Equation (10) in Toroidal–Poloidal Form

1 Governing momentum equation

Equation (10) of *Onset of Convection in Rotating Spherical Shells* gives the non-dimensional perturbation momentum balance

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\boldsymbol{\nabla} p' - 2\hat{\boldsymbol{z}} \times \boldsymbol{u}' + \frac{\mathrm{Ra}}{\mathrm{Pr}}\left(\frac{r}{r_o}\right)^2 \theta' \hat{\boldsymbol{r}} + E\boldsymbol{\nabla}^2 \boldsymbol{u}'. \tag{1.1}$$

Here $\boldsymbol{u}'$ is solenoidal, $\hat{\boldsymbol{r}}$ denotes the radial unit vector, and $\hat{\boldsymbol{z}}$ is the axis of rotation. Our goal is to rewrite (1.1) in terms of the toroidal and poloidal potentials introduced in Equations (13)–(15) of the paper.

2 Toroidal–poloidal decomposition

Following Equation (13), the divergence-free velocity field is decomposed into scalar potentials P (poloidal) and T (toroidal),

$$\boldsymbol{u}' = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (P\hat{\boldsymbol{r}}) + \boldsymbol{\nabla} \times (T\hat{\boldsymbol{r}}). \tag{2.1}$$

For compactness we define the surface gradient $\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \hat{\boldsymbol{r}}\,\partial_r$ and recall the vector spherical harmonics

$$\boldsymbol{Y}_{\ell m}^{(r)} = Y_{\ell m}\hat{\boldsymbol{r}}, \qquad \boldsymbol{Y}_{\ell m}^{(p)} = \boldsymbol{\nabla}_\perp Y_{\ell m}, \qquad \boldsymbol{Y}_{\ell m}^{(t)} = \hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_\perp Y_{\ell m}, \tag{2.2}$$

with $Y_{\ell m}$ the Schmidt semi-normalised spherical harmonics. Direct evaluation of (2.1) leads to the standard component form

$$\boldsymbol{u}'_P = \sum_{\ell=m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} P_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r}\frac{\partial P_{\ell m}}{\partial r}\, \boldsymbol{Y}_{\ell m}^{(p)} \right], \tag{2.3}$$

$$\boldsymbol{u}'_T = \sum_{\ell=m}^{\infty} \left[\frac{1}{r} T_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(t)} \right], \tag{2.4}$$

where the sums run over a fixed azimuthal order m and $P_{\ell m}(r,t)$, $T_{\ell m}(r,t)$ are the radial amplitudes of each degree ℓ.

Equation (14) expresses the angular dependence of the scalar potentials as

$$P(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty} P_{\ell m}(r,t)Y_{\ell m}(\theta,\phi), \qquad T(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty} T_{\ell m}(r,t)Y_{\ell m}(\theta,\phi), \tag{2.5}$$

while the radial profiles are expanded spectrally using Chebyshev polynomials as in Equation (15),

$$P_{\ell m}(r,t) = \sum_{n=0}^{N} P_{\ell mn}(t)\, C_n(r), \qquad T_{\ell m}(r,t) = \sum_{n=0}^{N} T_{\ell mn}(t)\, C_n(r). \tag{2.6}$$

The temperature perturbation is treated identically:

$$\theta'(r,\theta,\phi,t) = \sum_{\ell=m}^{\infty} \theta_{\ell m}(r,t) Y_{\ell m}(\theta,\phi) = \sum_{\ell=m}^{\infty} \sum_{n=0}^{N} \Theta_{\ell m n}(t)\, C_n(r) Y_{\ell m}(\theta,\phi). \tag{2.7}$$

3   Operators acting on the toroidal–poloidal fields

To project (1.1) onto the basis (2.3)–(2.4) we evaluate each term separately.

3.1   Time derivative

Because the vector spherical harmonics are time-independent,

$$\frac{\partial \boldsymbol{u}'}{\partial t} = \sum_{\ell=m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} \frac{\partial P_{\ell m}}{\partial t}\, \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial P_{\ell m}}{\partial t}\right) \boldsymbol{Y}_{\ell m}^{(p)} + \frac{1}{r}\frac{\partial T_{\ell m}}{\partial t}\, \boldsymbol{Y}_{\ell m}^{(t)} \right]. \tag{3.1}$$

3.2   Pressure gradient

The pressure can be expanded as $p'(r,\theta,\phi,t) = \sum_{\ell m} p_{\ell m}(r,t) Y_{\ell m}(\theta,\phi)$. Its gradient decomposes into the same vector spherical harmonics,

$$\boldsymbol{\nabla} p' = \sum_{\ell=m}^{\infty} \left[\frac{\partial p_{\ell m}}{\partial r} \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r} p_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(p)} \right]. \tag{3.2}$$

When the curl or double curl of (1.1) is taken (as in the numerical formulation) the pressure term drops out; it is retained here only for completeness.

3.3   Viscous term

Using the identity $\boldsymbol{\nabla}^2 \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{\nabla}^2 \boldsymbol{A}$ and the fact that $\boldsymbol{\nabla}^2(f\hat{\boldsymbol{r}}) = \left(\partial_r^2 f + \frac{2}{r}\partial_r f - \frac{\ell(\ell+1)}{r^2} f \right)\hat{\boldsymbol{r}}$ for each spherical harmonic degree, the Laplacian acts diagonally on the potentials:

$$\boldsymbol{\nabla}^2 \boldsymbol{u}_P' = \sum_{\ell=m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} \mathcal{L}_\ell P_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(r)} + \frac{1}{r}\frac{\partial}{\partial r}(\mathcal{L}_\ell P_{\ell m}) \boldsymbol{Y}_{\ell m}^{(p)} \right], \tag{3.3}$$

$$\boldsymbol{\nabla}^2 \boldsymbol{u}_T' = \sum_{\ell=m}^{\infty} \left[\frac{1}{r} \mathcal{L}_\ell T_{\ell m}\, \boldsymbol{Y}_{\ell m}^{(t)} \right], \tag{3.4}$$

where the scalar radial operator $\mathcal{L}_\ell$ is

$$\mathcal{L}_\ell f \equiv \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} - \frac{\ell(\ell+1)}{r^2} f. \tag{3.5}$$

3.4   Buoyancy term

Because the buoyancy force is purely radial, only the $\boldsymbol{Y}_{\ell m}^{(r)}$ basis is involved:

$$\frac{\mathrm{Ra}}{\mathrm{Pr}}\left(\frac{r}{r_o}\right)^2 \theta'\hat{\boldsymbol{r}} = \sum_{\ell=m}^{\infty} \frac{\mathrm{Ra}}{\mathrm{Pr}}\left(\frac{r}{r_o}\right)^2 \theta_{\ell m}(r,t)\boldsymbol{Y}_{\ell m}^{(r)}. \tag{3.6}$$

3.5 Coriolis term

The Coriolis acceleration mixes neighbouring spherical-harmonic degrees. Writing $\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$ and using the angular momentum ladder relations

$$\cos\theta\, Y_{\ell m} = a_{\ell m}^+ Y_{\ell+1,m} + a_{\ell m}^- Y_{\ell-1,m}, \tag{3.7}$$

$$\sin\theta\, \frac{\partial Y_{\ell m}}{\partial\theta} = \ell\, a_{\ell m}^- Y_{\ell-1,m} - (\ell+1)a_{\ell m}^+ Y_{\ell+1,m}, \tag{3.8}$$

with coefficients

$$a_{\ell m}^+ = \sqrt{\frac{(\ell+1)^2 - m^2}{(2\ell+1)(2\ell+3)}}, \qquad a_{\ell m}^- = \sqrt{\frac{\ell^2 - m^2}{(2\ell-1)(2\ell+1)}}, \tag{3.9}$$

one obtains

$$\hat{\boldsymbol{z}} \times \boldsymbol{u}'_T = \sum_{\ell=m}^{\infty} \frac{1}{r}\Big[(\ell-1)(\ell+1)a_{\ell m}^- T_{\ell-1,m} + \ell(\ell+2)a_{\ell m}^+ T_{\ell+1,m}\Big]\boldsymbol{Y}_{\ell m}^{(p)}$$

$$- \sum_{\ell=m}^{\infty} \frac{\mathrm{i}m}{r}\Big[a_{\ell m}^- T_{\ell-1,m} + a_{\ell m}^+ T_{\ell+1,m}\Big]\boldsymbol{Y}_{\ell m}^{(r)}, \tag{3.10}$$

$$\hat{\boldsymbol{z}} \times \boldsymbol{u}'_P = \sum_{\ell=m}^{\infty} \frac{1}{r}\left[(\ell-1)(\ell+1)a_{\ell m}^- \frac{\partial P_{\ell-1,m}}{\partial r} + \ell(\ell+2)a_{\ell m}^+ \frac{\partial P_{\ell+1,m}}{\partial r}\right]\boldsymbol{Y}_{\ell m}^{(t)}$$

$$+ \sum_{\ell=m}^{\infty} \frac{\mathrm{i}m}{r^2}\Big[\ell(\ell-1)a_{\ell m}^- P_{\ell-1,m} + (\ell+1)(\ell+2)a_{\ell m}^+ P_{\ell+1,m}\Big]\boldsymbol{Y}_{\ell m}^{(p)}. \tag{3.11}$$

Equations (3.10)–(3.11) summarise the toroidal–poloidal coupling produced by rotation; only $\ell \pm 1$ degrees interact at fixed m.

4 Projected evolution equations

Projecting Equation (1.1) onto the vector spherical harmonics and using the orthogonality relations $\int \boldsymbol{Y}_{\ell m}^{(r)}\cdot\boldsymbol{Y}_{\ell'm'}^{(r)}\,\mathrm{d}\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}$, $\int \boldsymbol{Y}_{\ell m}^{(p)}\cdot\boldsymbol{Y}_{\ell'm'}^{(p)}\,\mathrm{d}\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}$, and similarly for the toroidal branch, yields a coupled set of radial equations for each (ℓ,m):

$$\left(\frac{\partial}{\partial t} - E\mathcal{L}_\ell\right)\mathcal{L}_\ell P_{\ell m} - 2\,\mathcal{C}_{\ell m}[T] = \frac{\mathrm{Ra}}{\mathrm{Pr}}\frac{\ell(\ell+1)}{r^2}\left(\frac{r}{r_o}\right)^2\theta_{\ell m}, \tag{4.1}$$

$$\left(\frac{\partial}{\partial t} - E\mathcal{L}_\ell\right)T_{\ell m} + 2\,\mathcal{D}_{\ell m}[P] = 0, \tag{4.2}$$

with coupling operators

$$\mathcal{C}_{\ell m}[T] = \frac{\mathrm{i}m}{r^2}\Big[\ell(\ell-1)a_{\ell m}^- T_{\ell-1,m} + (\ell+1)(\ell+2)a_{\ell m}^+ T_{\ell+1,m}\Big], \tag{4.3}$$

$$\mathcal{D}_{\ell m}[P] = \frac{1}{r}\left[(\ell-1)(\ell+1)a_{\ell m}^- \frac{\partial P_{\ell-1,m}}{\partial r} + \ell(\ell+2)a_{\ell m}^+ \frac{\partial P_{\ell+1,m}}{\partial r}\right]. \tag{4.4}$$

Equations (4.1)–(4.4) show Equation (10) rewritten entirely in the toroidal–poloidal basis, together with the buoyancy forcing and viscous diffusion acting through $\mathcal{L}_\ell$.

5   Chebyshev representation

Substituting the radial expansions (2.6) into (4.1)–(4.2) produces a linear system for the time-dependent Chebyshev coefficients. Define the modal vectors

$$\boldsymbol{P}_{\ell m}(t) = (P_{\ell m 0}(t), \ldots, P_{\ell m N}(t))^\top, \quad \boldsymbol{T}_{\ell m}(t) = (T_{\ell m 0}(t), \ldots, T_{\ell m N}(t))^\top, \quad \boldsymbol{\Theta}_{\ell m}(t) = (\Theta_{\ell m 0}(t), \ldots, \Theta_{\ell m N}(t))^\top.$$
(5.1)

Let $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ denote the first and second radial differentiation matrices associated with the Chebyshev basis. Evaluating $\mathcal{L}_\ell$ at the collocation points and projecting with the appropriate quadrature weights yields the discrete operator matrices used in the numerical code. Symbolically,

$$\left(\frac{\partial}{\partial t} \boldsymbol{L}_\ell - E \boldsymbol{L}_\ell^2 \right) \boldsymbol{P}_{\ell m} - 2 \boldsymbol{C}_{\ell m} \boldsymbol{T}_{\ell m} = \frac{\mathrm{Ra}}{\mathrm{Pr}} \boldsymbol{B}_\ell \boldsymbol{\Theta}_{\ell m},$$
(5.2)

$$\left(\frac{\partial}{\partial t} - E \boldsymbol{L}_\ell \right) \boldsymbol{T}_{\ell m} + 2 \boldsymbol{D}_{\ell m} \boldsymbol{P}_{\ell m} = \boldsymbol{0},$$
(5.3)

where $\boldsymbol{L}_\ell$, $\boldsymbol{C}_{\ell m}$, $\boldsymbol{D}_{\ell m}$, and $\boldsymbol{B}_\ell$ follow directly from the continuous operators (3.5), (4.3), and (4.4). This is precisely the algebraic form discretised in the software package CROSS.JL.

6   Temperature equation in spectral form

Equation (11) of the manuscript—referred to hereafter for consistency, even though the user prompt mentions Equation (12)—gives the non-dimensional temperature perturbation dynamics,

$$\frac{\partial \theta'}{\partial t} = -u_r' \frac{\mathrm{d}\bar{\theta}}{\mathrm{d}r} + \frac{E}{\mathrm{Pr}} \boldsymbol{\nabla}^2 \theta'.$$
(6.1)

The modal ansatz of Equation (12) and the angular/radial expansions of Equations (14)–(15) lead to

$$\theta'(r, \theta, \phi, t) = \sum_{\ell=m}^{\infty} \theta_{\ell m}(r, t) Y_{\ell m}(\theta, \phi) = \sum_{\ell=m}^{\infty} \sum_{n=0}^{N} \Theta_{\ell m n}(t) C_n(r) Y_{\ell m}(\theta, \phi),$$
(6.2)

with the radial velocity supplied by the poloidal potential,

$$u_r'(r, \theta, \phi, t) = \sum_{\ell=m}^{\infty} \frac{\ell(\ell+1)}{r^2} P_{\ell m}(r, t) Y_{\ell m}(\theta, \phi).$$
(6.3)

The spherical Laplacian acting on θ' is diagonal in ℓ and m:

$$\boldsymbol{\nabla}^2 \theta' = \sum_{\ell=m}^{\infty} \mathcal{S}_\ell[\theta_{\ell m}] Y_{\ell m}(\theta, \phi), \qquad \mathcal{S}_\ell[f] \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) - \frac{\ell(\ell+1)}{r^2} f.$$
(6.4)

Substituting (6.2)–(6.4) into (6.1) and projecting onto $Y_{\ell m}$ delivers the scalar evolution equation for each (ℓ, m):

$$\left(\frac{\partial}{\partial t} - \frac{E}{\mathrm{Pr}} \mathcal{S}_\ell \right) \theta_{\ell m}(r, t) = -\frac{\ell(\ell+1)}{r^2} \frac{\mathrm{d}\bar{\theta}}{\mathrm{d}r} P_{\ell m}(r, t).$$
(6.5)

Finally, applying the Chebyshev expansion (6.2) and evaluating at the radial collocation nodes provides the discrete system for the coefficient vectors $\boldsymbol{\Theta}_{\ell m}(t)$,

$$\left(\frac{\partial}{\partial t} \boldsymbol{I} - \frac{E}{\mathrm{Pr}} \boldsymbol{S}_\ell \right) \boldsymbol{\Theta}_{\ell m} = -\boldsymbol{G} \boldsymbol{Q}_\ell \boldsymbol{P}_{\ell m},$$
(6.6)

where $\boldsymbol{S}_\ell$ represents $\mathcal{S}_\ell$ in the Chebyshev basis, $\boldsymbol{G}$ is the diagonal matrix containing the conduction gradient $d\bar{\theta}/dr$ at each collocation point, and $\boldsymbol{Q}_\ell$ multiplies by $\ell(\ell+1)/r^2$. These matrices already appear in the numerical implementation of CROSS.JL, aligning the temperature subsystem with the toroidal–poloidal formulation derived above.

7 Generalised eigenvalue problem for the Rayleigh number

At the onset of convection the growth rate vanishes, so the exponential ansatz of Equation (12) gives $\partial/\partial t \to 0$ in the discrete system. Combining (4.1)–(6.6) with this marginal-stability condition produces

$$-E\boldsymbol{L}_\ell^2 \boldsymbol{P}_{\ell m} - 2\boldsymbol{C}_{\ell m}\boldsymbol{T}_{\ell m} = \frac{\mathrm{Ra}}{\mathrm{Pr}}\boldsymbol{B}_\ell \boldsymbol{\Theta}_{\ell m}, \tag{7.1}$$

$$-E\boldsymbol{L}_\ell \boldsymbol{T}_{\ell m} + 2\boldsymbol{D}_{\ell m}\boldsymbol{P}_{\ell m} = \boldsymbol{0}, \tag{7.2}$$

$$-\frac{E}{\mathrm{Pr}}\boldsymbol{S}_\ell \boldsymbol{\Theta}_{\ell m} + \boldsymbol{G}\boldsymbol{Q}_\ell \boldsymbol{P}_{\ell m} = \boldsymbol{0}. \tag{7.3}$$

Introducing the block state vector

$$\boldsymbol{x}_{\ell m} = \begin{bmatrix} \boldsymbol{P}_{\ell m} \\ \boldsymbol{T}_{\ell m} \\ \boldsymbol{\Theta}_{\ell m} \end{bmatrix}, \tag{7.4}$$

Equations (7.1)–(7.3) can be written compactly as the Rayleigh-number eigenproblem

$$\boldsymbol{A}_{\ell m}\,\boldsymbol{x}_{\ell m} = \mathrm{Ra}\,\boldsymbol{B}_\ell\,\boldsymbol{x}_{\ell m}, \tag{7.5}$$

with

$$\boldsymbol{A}_{\ell m} = \begin{bmatrix} -E\boldsymbol{L}_\ell^2 & -2\boldsymbol{C}_{\ell m} & 0 \\ 2\boldsymbol{D}_{\ell m} & -E\boldsymbol{L}_\ell & 0 \\ \boldsymbol{G}\boldsymbol{Q}_\ell & 0 & -\frac{E}{\mathrm{Pr}}\boldsymbol{S}_\ell \end{bmatrix}, \qquad \boldsymbol{B}_\ell = \begin{bmatrix} \frac{1}{\mathrm{Pr}}\boldsymbol{B}_\ell & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.6}$$

The matrix $\boldsymbol{B}_\ell$ is singular because the Rayleigh number enters only through the buoyancy forcing; this structure matches the generalised eigenvalue problems solved numerically in CROSS.JL. One may optionally eliminate $\boldsymbol{\Theta}_{\ell m}$ using (7.3) to reduce (7.5) to a smaller system involving only the poloidal and toroidal coefficients, but the block form (7.6) makes the dependence on Ra explicit.

8 Summary

Starting from the non-dimensional momentum balance (1.1), we introduced the toroidal and poloidal potentials of Equations (13)–(15), expanded them in spherical harmonics and radial Chebyshev polynomials, and projected each term onto the vector spherical harmonic basis. The resulting Equations (4.1) and (4.2) express the dynamics entirely in terms of the scalar potentials, providing the foundation of the matrix eigenvalue problem solved in the accompanying code base.

In parallel, Equation (6.5) (and its discrete counterpart (6.6)) expresses the temperature perturbation equation in the same spectral framework, while the marginal-stability system (7.5)–(7.6) casts the full problem as a generalised eigenvalue problem with the Rayleigh number as eigenvalue, completing the derivation requested.