Quantization of odd dimensional symplectic manifolds

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Goals

- quantum dynamics = parallel transport?
- quantization = flat connection? [1, 2]

- beyond formality?
- quantum Darboux? [3]

G. Herczeg and A. Waldron, "Contact geometry and quantum mechanics," *Physics Letters B*, vol. 781, pp. 312–315, 2018.

R. Casals, G. Herczeg, and A. Waldron, "Dynamical quantization of contact structures," 2021.

O. Corradini, E. Latini, and A. Waldron, "Quantum Darboux theorem," *Phys. Rev. D*, vol. 103, no. 10, p. 105021, 2021.

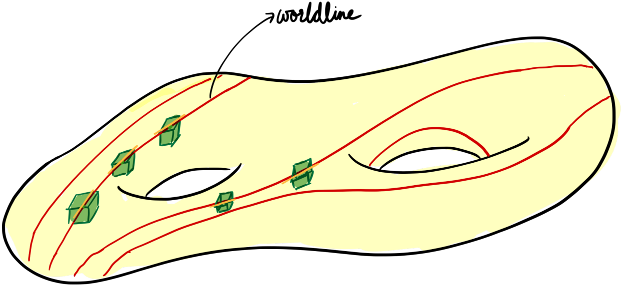
Motivation

Why odd dim symplectic manifold?

- $(\mathcal{M}^{2n+1}, \omega \in \Omega^2\mathcal{M})$ where,

$$\omega^n \neq 0, \qquad d\omega = 0$$

- covariant "phasespacetime"



Examples

- **Hamiltonian dynamics**: $\omega = (dp - \partial_q H\, dt) \wedge (dq + \partial_p H\, dt)$

$$\ker \omega = \langle \partial_q H\, \partial_p - \partial_p H\, \partial_q + \partial_t \rangle$$

- **Relativistic particle**: $\mathcal{M} = ST^* M_g$

$$\omega = dp_\mu \wedge dx^\mu \big|_{p^2 = -m^2}$$

e.g. $\mathcal{M} = S_m T^* \mathbb{R}^{1,1}$

$$\omega = dp_+ \wedge dx^+ + dp_- \wedge dx^- \Big|_{p_+ p_- = \frac{m^2}{2}}$$

$$\ker \omega = \langle p_- \partial_{x^+} + p_+ \partial_{x^-} \rangle \Big|_{p_+ p_- = \frac{m^2}{2}} \xrightarrow{m=0} \langle \partial_{x^\pm} \rangle$$

- **Reeb dynamics**: Contact manifold

$$(\mathcal{M}^{2n+1}, \alpha \in \Omega^1 \mathcal{M}), \quad \omega = d\alpha, \quad \alpha \wedge \omega^n \neq 0 \quad \textit{max non-integrable}$$

$$\exists! \quad R \in \ker \omega \quad \text{with} \quad \alpha(R) = 1 \qquad \text{Reeb vector field}$$

Why quantization?

- Covariance & Geometry:

  Independent of reference frames

- Correspondence Principle:

  Classical observables $\leftrightarrow$ Quantum observables

- Classical Lab:

  Measurement devices with dials, knobs, metersticks and clocks

Quantization Techniques

- **Canonical/Dirac:** *Heisenberg*

$$q \to \hat{q}, \quad p \to \hat{p}, \quad f(p,q) \to \hat{f} = f(\hat{p},\hat{q}), \quad \{\ ,\ \} \to \frac{1}{i\hbar}[\ ,\ ]$$

- **Geometric:** *Schrödinger*

$$(M,\omega) \longrightarrow \left(\begin{array}{c} \mathbb{C} \hookrightarrow (L,\nabla) \\ \downarrow \\ (M,\omega,P) \end{array} \right)$$

$$f \to \hat{f} = -i\hbar\nabla_{X_f} + f \qquad \text{if polarization P preserving}$$

$$\mathcal{H} = \{ s \in \Gamma(L) \big| \, \nabla|_P \, s = 0 \}$$

- **Deformation:** *Wigner-Weyl-Moyal*

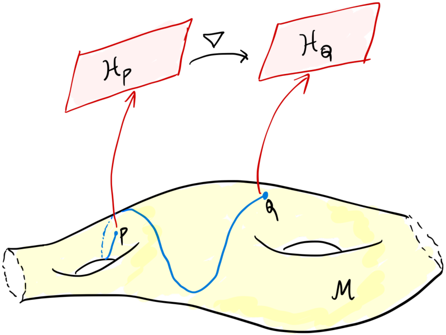
$$(C^\infty(M), ., \{,\}_{PB}) \xrightarrow{Q} (\mathbb{R}[T^*M][[\hbar]], *, \{,\}_{MB})$$

$$Q_{\{f,g\}_{PB}} = \{Q_f, Q_g\}_{MB}$$

Quantization

Quantum Dynamical System

$$\begin{tikzcd} & \mathcal{U}(\mathcal{H}) \arrow[d, phantom, "\circlearrowright"] \\ \mathcal{H} \arrow[r, hook] & (\mathscr{H}\mathcal{M}, \nabla) \arrow[d, "\pi"] \\ & \mathcal{M} \end{tikzcd}$$



- Hilbert space $\mathcal{H}$
- Hilbert bundle

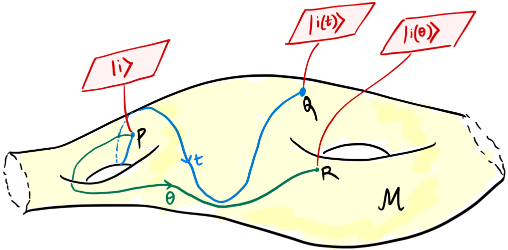
$$\mathscr{H}\mathcal{M} = {}^{\mathcal{H}\mathcal{M}}/_{\mathrm{Hol}(\nabla)}$$

- flat connection ∇,

$$\nabla^2 = 0$$

- structure group $\mathcal{U}(\mathcal{H})$

Quantum dynamics = Parallel transport along experiments



$$U_\gamma = Pe^{\int_\gamma \hat{A}} = \begin{cases} Pe^{-\frac{i}{\hbar}\int \hat{H}(t)dt} & \textit{time evolution} \\ e^{-\frac{i}{\hbar}\theta B \hat{J}} & \textit{rotation due to magnetic field} \\ \vdots & \end{cases}$$

geometry	**physics**
$\mathcal{M}$	classical lab
paths	experiments
line operators	unitary evolution
non-contractible loops	quanta

Example : Anharmonic Oscillator

$$\mathcal{M} = S^1 \times \mathbb{R}^2 \ni (\theta, p, t), \quad \mathcal{H} = L^2(S^1)$$

One-parameter family of flat connections:

$$\nabla^\hbar = d + \frac{d\theta}{i\hbar}\left(p + \frac{\hbar}{i}\partial_\Theta\right) - \frac{dp}{i\hbar}\Theta - \frac{dt}{i\hbar}\left(\frac{1}{2}\left(p + \frac{\hbar}{i}\partial_\Theta\right)^2 + V(\theta + \Theta)\right)$$

Prolonged Schrödinger equation $\nabla^\hbar \Psi = 0$:

$$\begin{cases} d\theta : \left(\partial_\theta + \frac{1}{i\hbar}\left(p + \frac{\hbar}{i}\partial_\Theta\right)\right)\Psi = 0 \\ dp : (\partial_p - \frac{\Theta}{i\hbar})\Psi = 0 \\ dt : \left(\partial_t - \frac{1}{i\hbar}\hat{H}_{AHO}\right)\Psi = 0 \end{cases} \implies i\hbar\dot{\psi} = \hat{H}_{AHO}\psi$$

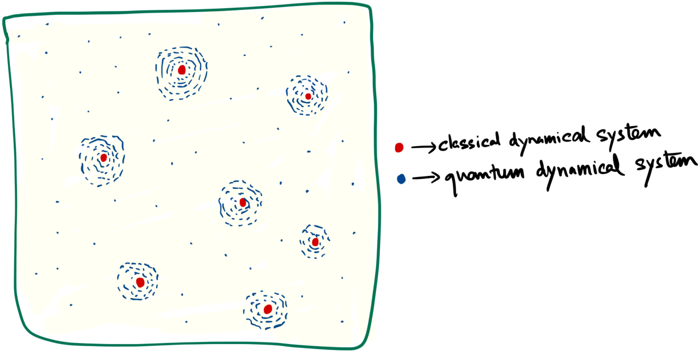
Classical Limit:

$$\nabla^\hbar \xrightarrow{\hbar\to 0} \frac{p d\theta - \left(\frac{1}{2}p^2 + V(\theta)\right)dt}{i\hbar} \quad \textit{U(1) connection 1-form}$$

Quantum dynamics with classical limit

- $\mathbf{M}_{\mathcal{M}}(\mathcal{HZ}) := \check{\mathrm{H}}^1(\mathcal{M},\mathcal{U}(\mathcal{H}))$ moduli space of flat connections
  $\cong \mathrm{Hom}(\pi_1(\mathcal{M}),\mathcal{U}(\mathcal{H}))/\mathcal{U}(\mathcal{H})$
- Classical limit

$$\lim_{\hbar\to 0} i\hbar\nabla^{\hbar} = \alpha \in \Omega^1\mathcal{M}, \qquad \omega = d\alpha$$



BRST/Fedosov

- Constraints and gauge symmetries

$$S = \int \alpha = \int \alpha_\mu(z)\, \dot{z}^\mu d\tau, \quad \omega = d\alpha$$

$$C_\mu = p_\mu - \alpha_\mu, \quad \{C_\mu, C_\nu\} = \omega_{\mu\nu}$$

2n second class

1 first class

- Formal BRST charge

$$Q_{BRST} = \underbrace{\frac{\alpha\, \mathrm{Id}}{i\hbar}}_{\text{abelian}} + \underbrace{\frac{e^a \hat{s}_a}{i\sqrt{\hbar}}}_{\text{symplectic spinors}} + d + \underbrace{\frac{1}{2i}\, \Omega^{ab} \hat{s}_a \hat{s}_b}_{\text{symplectic connection}} + \mathcal{O}(\hbar^{1/2})$$

$$\omega = \frac{1}{2} J_{ab}\, e^a \wedge e^b, \qquad \textit{symplectic vielbein}$$

$$[\hat{s}_a, \hat{s}_b] = -iJ_{ab}, \qquad \textit{Stone-von Neumann}$$

$$de^a + \Omega^{ab} \wedge e_b = 0 \qquad \textit{torsion-free connection}$$

Future Directions

Exact quantization of a 3-torus

$$\mathcal{M} = T^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3 \ni (x, y, \theta)$$

- Maurer-Cartan coframe

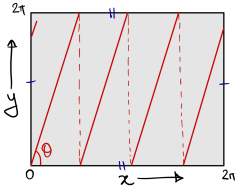
$$\alpha = \cos\theta dx + \sin\theta dy$$
$$\beta = d\theta$$
$$\gamma = -\sin\theta dx + \cos\theta dy$$
$$d\alpha = \beta \wedge \gamma, \quad d\beta = 0, \quad d\gamma = \alpha \wedge \beta$$

- Quantum connection

$$\nabla^\hbar = d + \frac{\alpha \cos\left(s\sqrt{\hbar}\right)}{i\hbar} - \frac{\beta}{\sqrt{\hbar}}\frac{\partial}{\partial s} + \frac{\gamma \sin\left(s\sqrt{\hbar}\right)}{i\hbar} \quad \textit{exact!}$$

- Hilbert space is finite dimensional

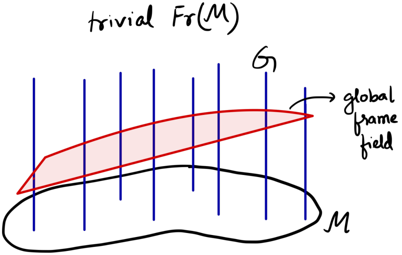
$$\psi_{n_1,n_2} = f(\Theta) e^{\frac{i}{\hbar}(x\cos\Theta + y\sin\Theta)}$$
$$\sim e^{i(n_1 x + n_2 y)}, \quad n_1, n_2 \in \mathbb{Z}, \quad n_1^2 + n_2^2 = \frac{1}{\hbar^2}$$

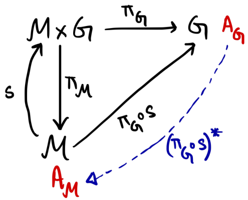


How to go beyond formality?

$$\nabla = d + \hat{A}, \qquad d\hat{A} + \frac{1}{2}[\hat{A}, \hat{A}] = 0, \quad \hat{A} \in \Omega^1(\mathcal{M}, W_n(\mathbb{C}))$$

- bonafide quantum connection = Maurer-Cartan form on $\mathcal{M}$
- When can we hope to get global Maurer-Cartan?
  - For a Lie group, we have Maurer-Cartan frames with Lie algebra structure (structure constants).
  - For a parallelizable manifold (e.g. S^7)





Darboux Theorems

- Local rigidity

$$\omega_{std} = dp_\mu \wedge dx^\mu$$

- Locally formally there exists a quantum Darboux theorem

$$\begin{tikzcd}
(M|_U, \omega) \arrow[r, "\text{symplectomorphism}"] \arrow[d, "\text{quantize}"'] & (M|_U, \omega_{std}) \arrow[d, "\text{quantize}"] \\
(\mathcal{H}M|_U, \nabla) \arrow[r, "\text{gauge transformation}"] & (\mathcal{H}M|_U, \nabla_{std})
\end{tikzcd}$$

$$\boxed{\begin{aligned} \omega_{std} &= dp \wedge dx \\ \nabla_{std} &= d + \frac{p\,dx - dt}{i\hbar} + \frac{dx\, s + dp\left(-i\frac{\partial}{\partial s}\right)}{i\sqrt{\hbar}} \end{aligned}}$$

Commutative ?

- For pure Grassmann systems, there exists global Darboux (closed $\implies$ exact $\implies$ linear)!
- For Grassmann+time, can setup a Moser-like flow problem using Hamilton-Jacobi
- Global Darboux for pure fermionic suggests more efficient algebraic techniques to understand Grassmann/mixed dynamics

Thank you

Fantasies

- Classical limit of fermionic/spin systems = Discrete probabilistic system? Quantization of arbitrary supermanifolds
- Quantum dynamics in higher cohomology
- Fermionic quantum Darboux - Fermionic HJ, super Moser
- Conditions for exact quantization - parallelizability, G-structure
- Structure of moduli space of flat connections on Hilbert bundle - classification of quantum systems

Induced contact Hilbert Bundles

$$
\begin{array}{c}
\mathsf{Sp}(2n) \\
\circlearrowright \\
(D, \alpha) \\
\downarrow \\
(\mathcal{M}, \omega)
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
\mathsf{Sp}(2n) & \hookrightarrow & \mathsf{Fr}(D) \\
 & & \downarrow \\
 & & (\mathcal{M}, \omega)
\end{array}
\quad \rightarrow
$$

$$
\begin{array}{c}
\mathsf{Mp}(2n) \\
\circlearrowright \\
{\color{red}\mathcal{H}'\mathcal{M}} = \mathsf{Fr}(D) \times_{\mathsf{Mp}(2n)} \mathcal{H} \\
\downarrow \\
(\mathcal{M}, \omega)
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
\mathcal{U}(\mathcal{H}) & \hookrightarrow & \mathsf{Fr}(\mathcal{H}'\mathcal{M}) \\
 & & \downarrow \\
 & & (\mathcal{M}, \omega)
\end{array}
\quad \rightarrow
$$

$$
\begin{array}{c}
\mathcal{U}(\mathcal{H}) \\
\circlearrowright \\
{\color{red}\mathcal{H}\mathcal{M}} = \mathsf{Fr}(\mathcal{H}'\mathcal{M}) \times_{Id} \mathcal{H} \\
\downarrow \\
(\mathcal{M}, \omega)
\end{array}
$$