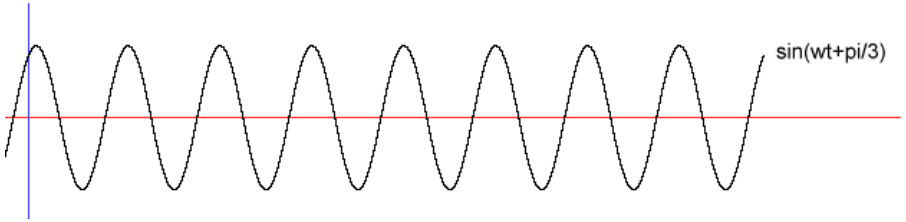
Image Enhancement:
Frequency representation

Dr. Tushar Sandhan

Introduction

- Signal decomposition



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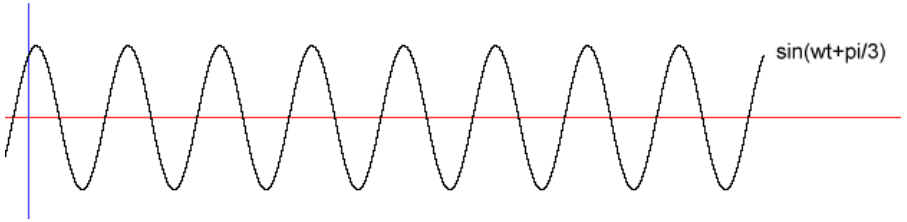

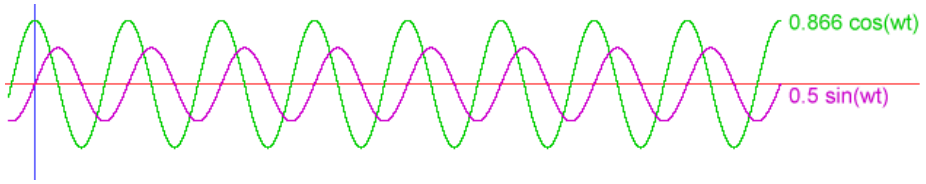

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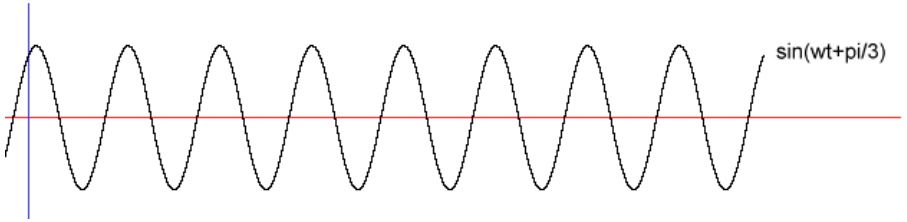

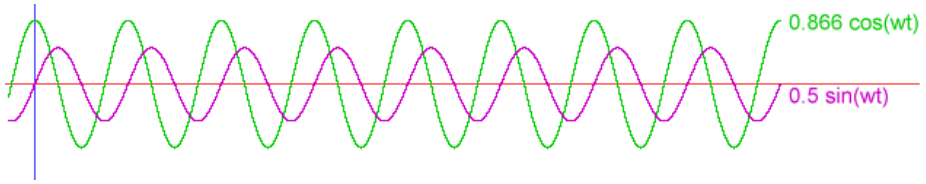

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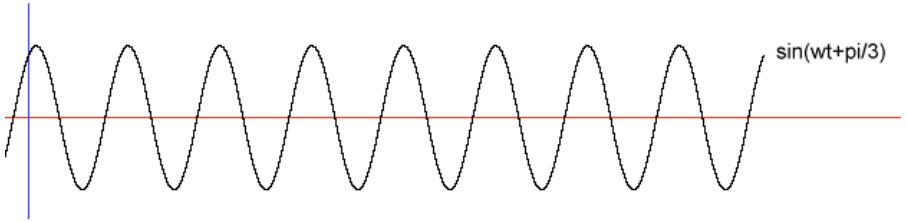

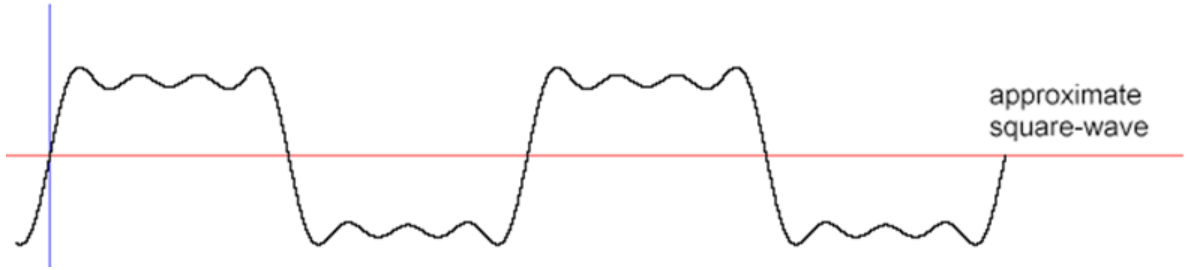

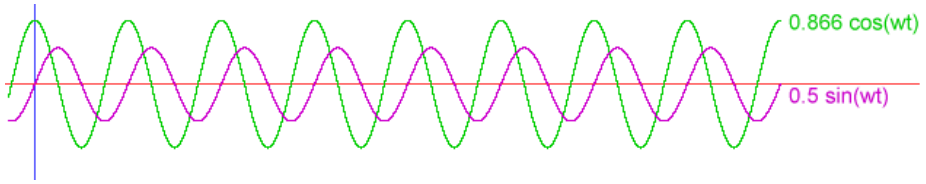

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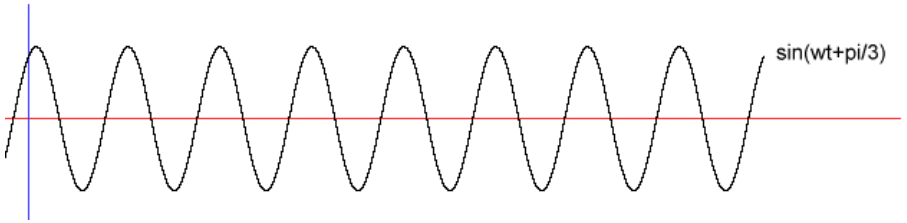

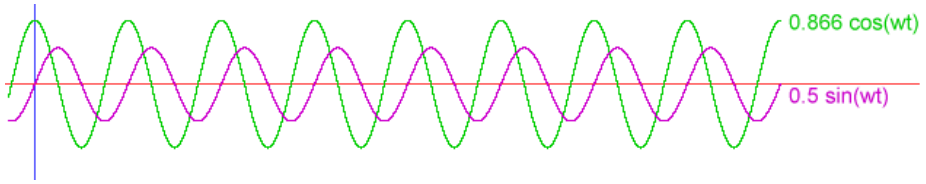

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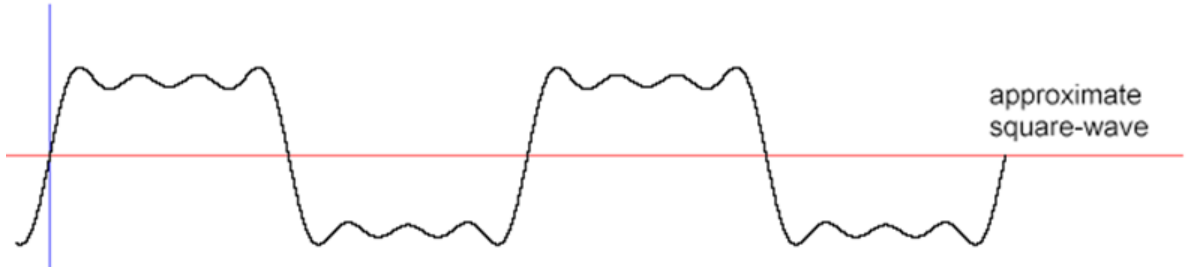

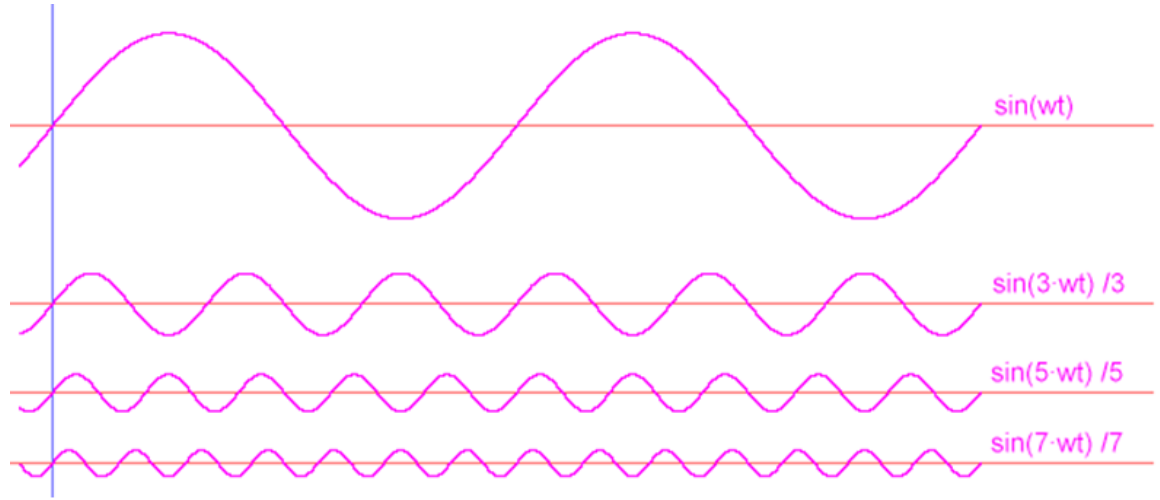

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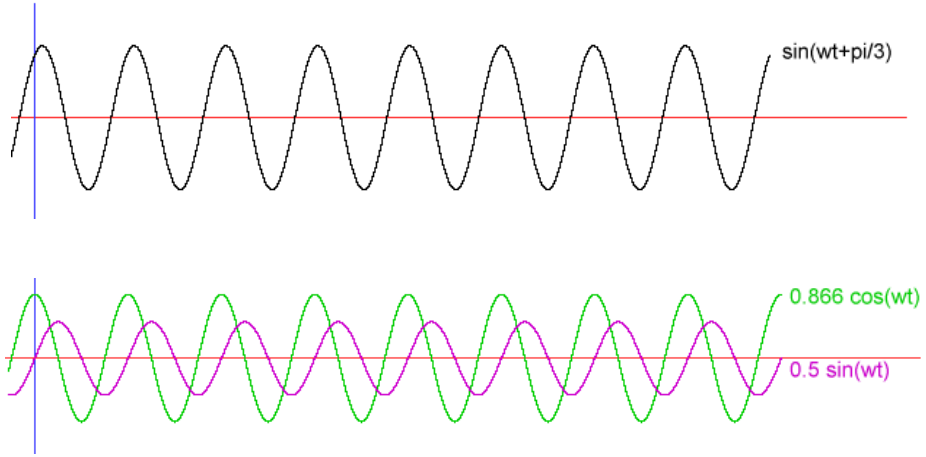

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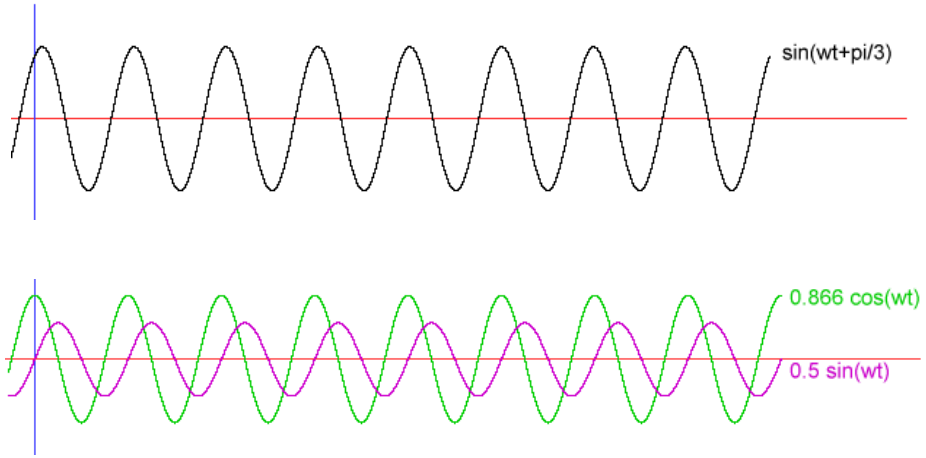

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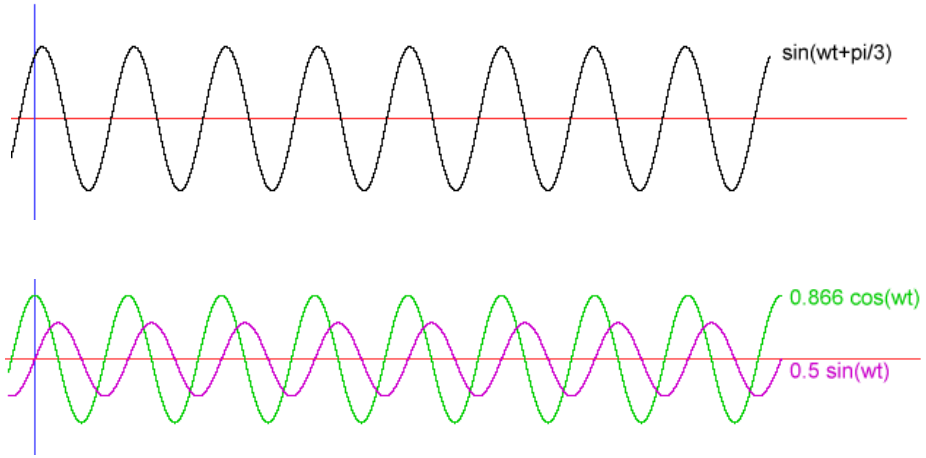

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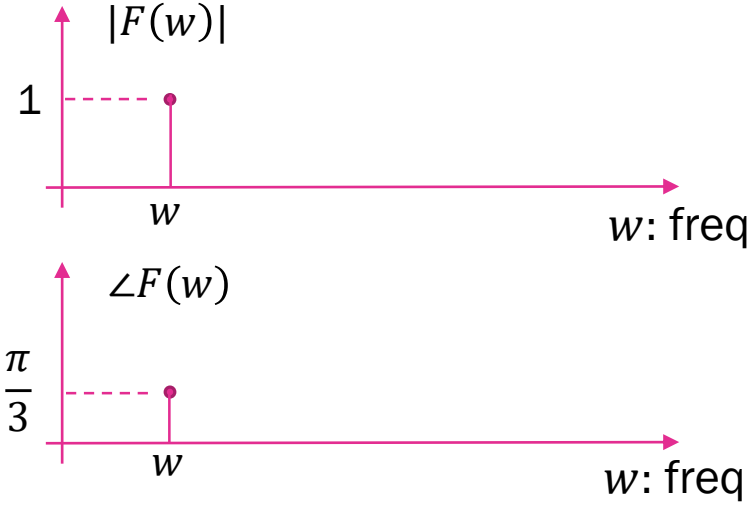

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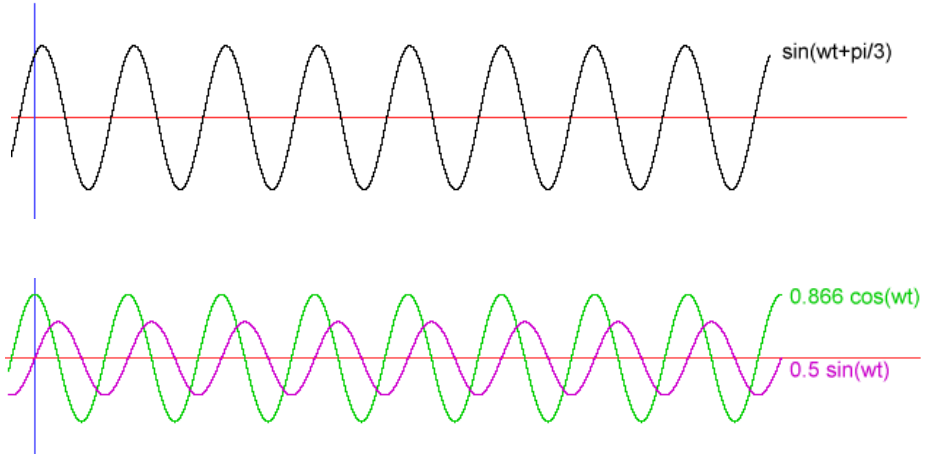

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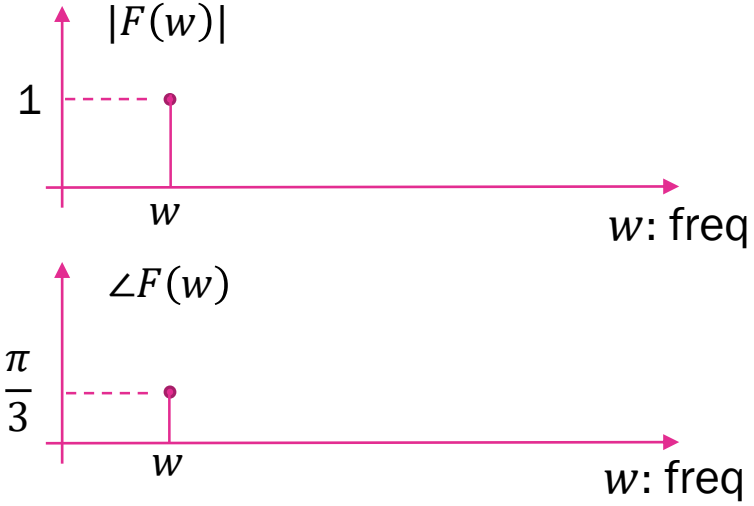

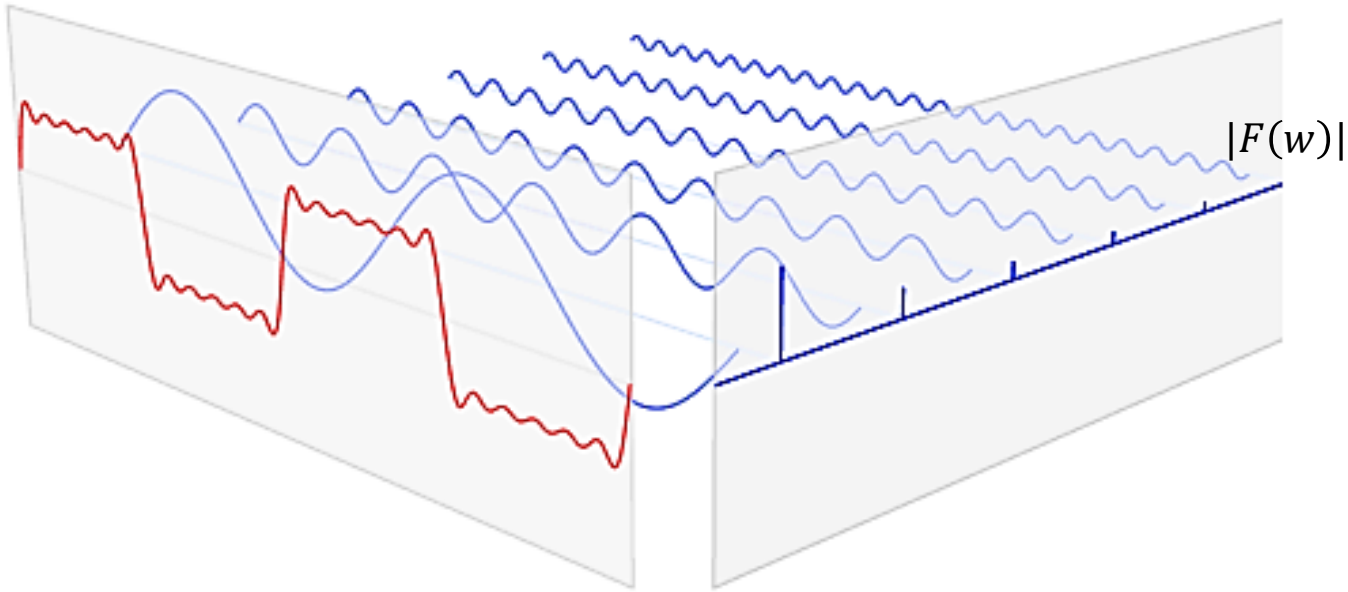

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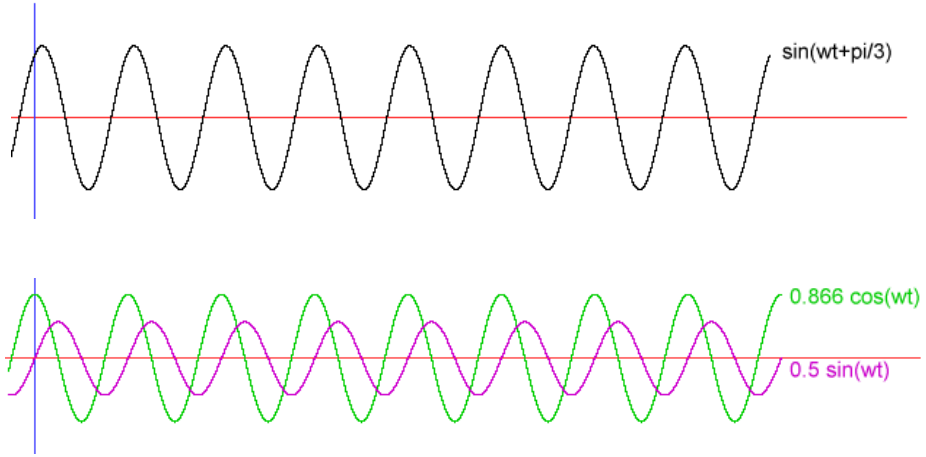

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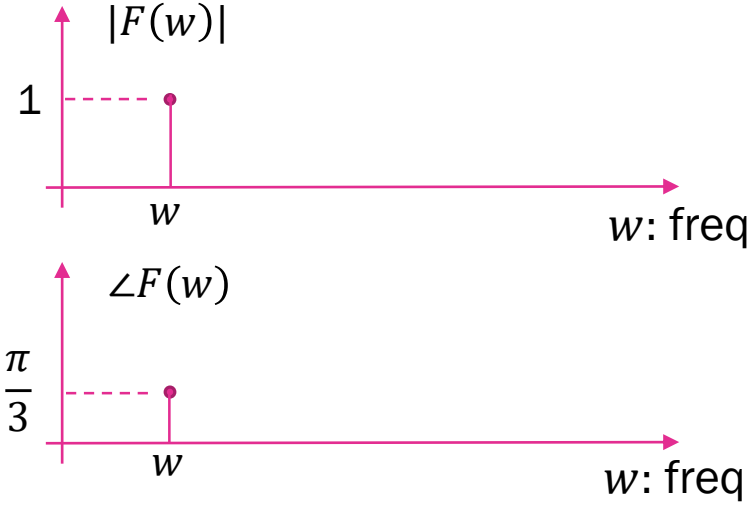

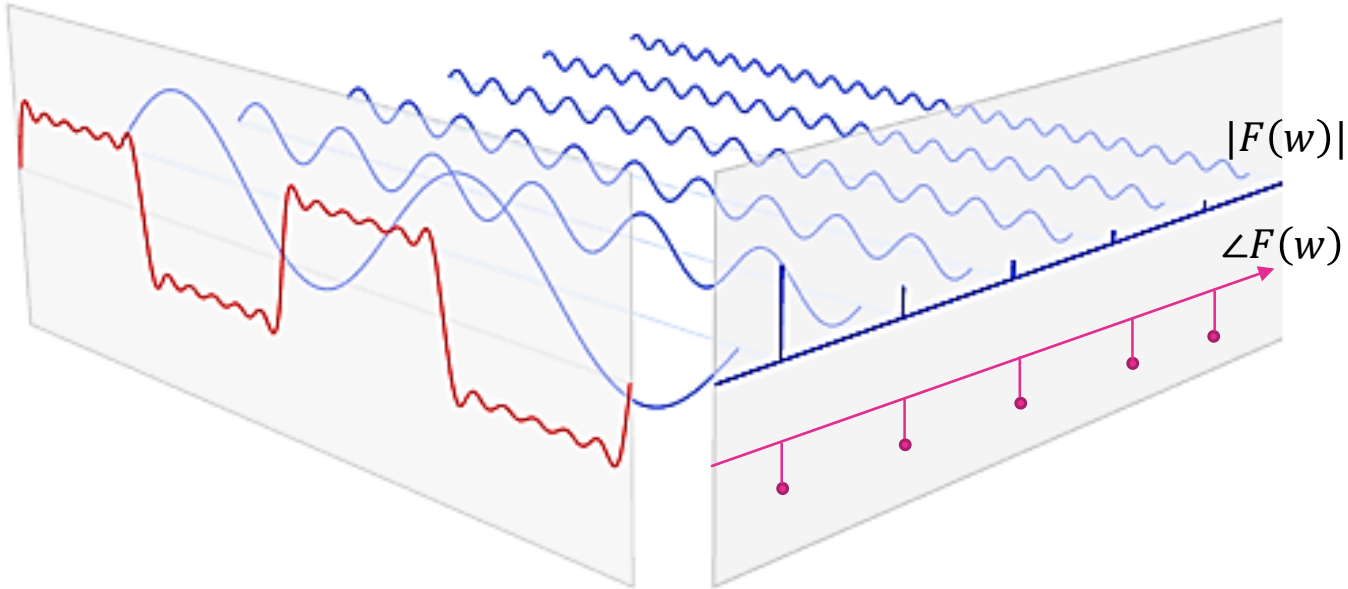

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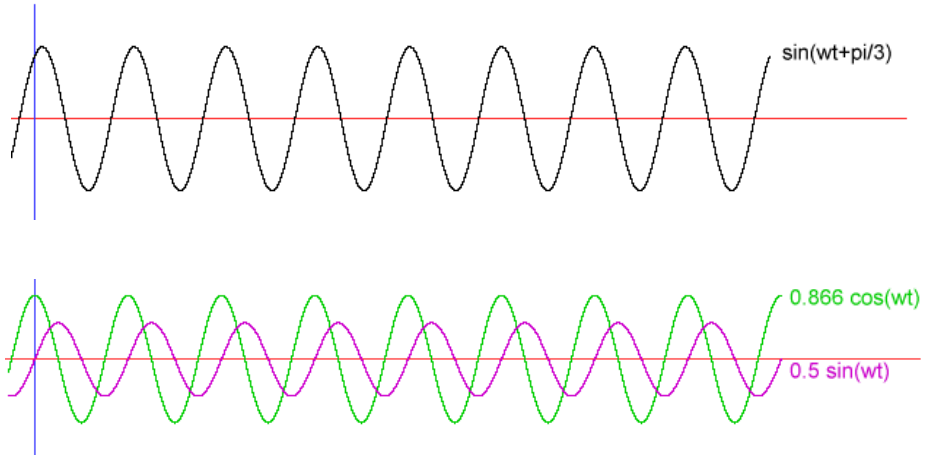

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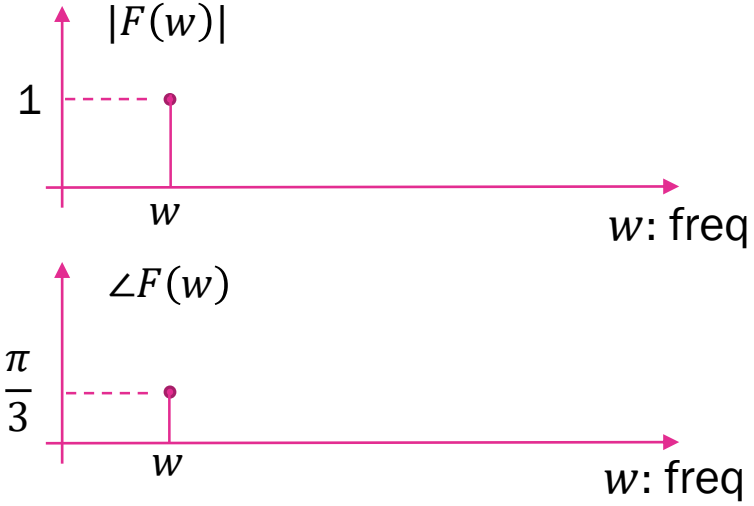

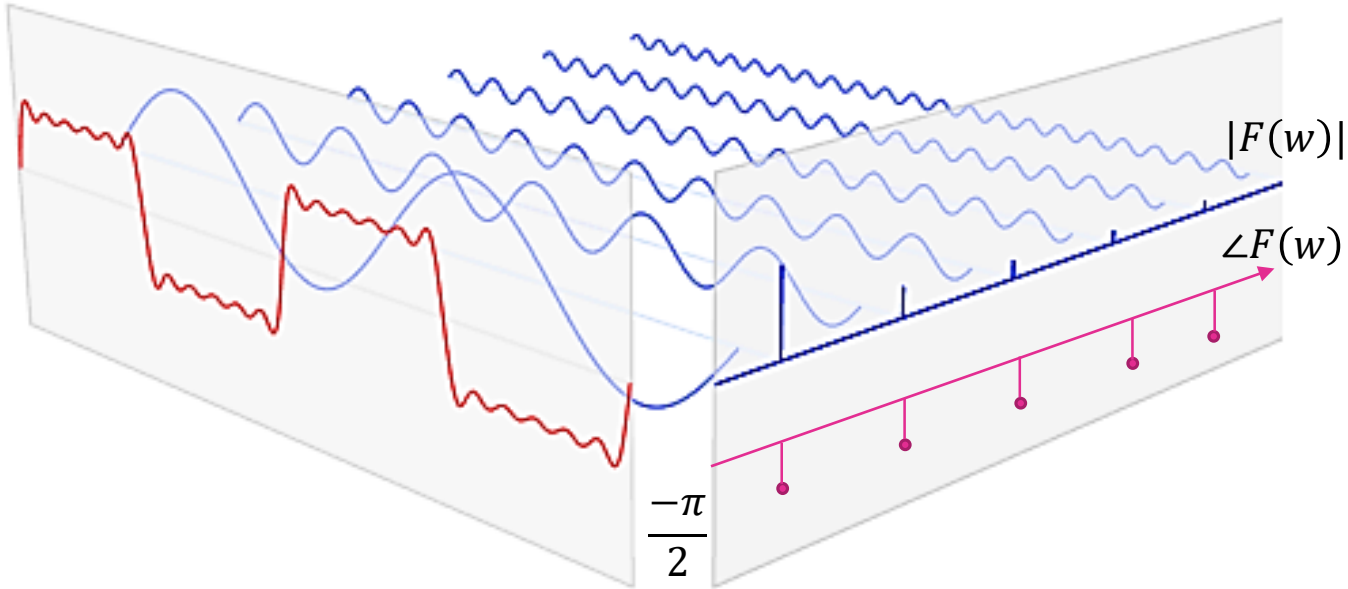

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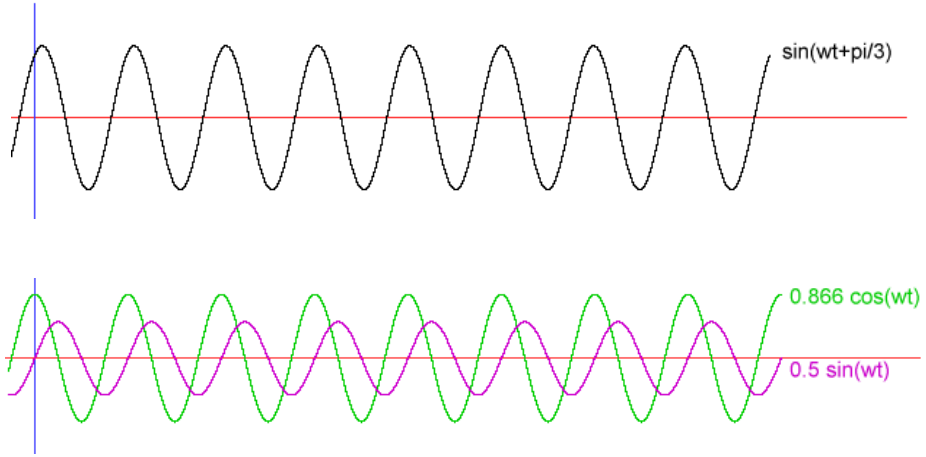

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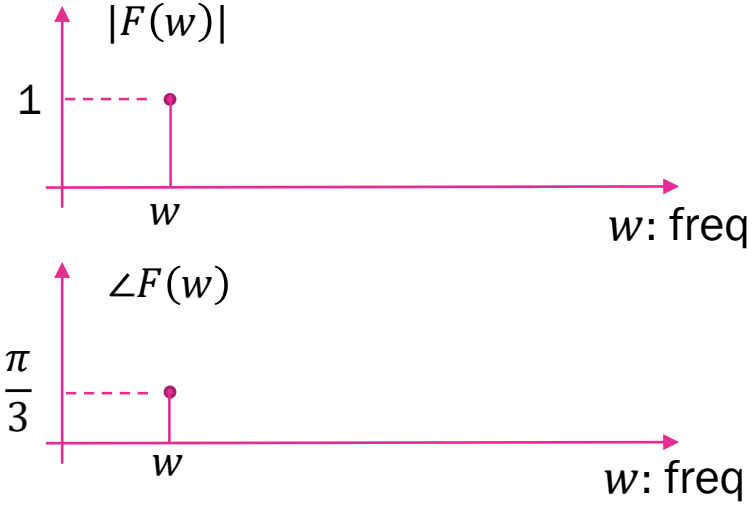
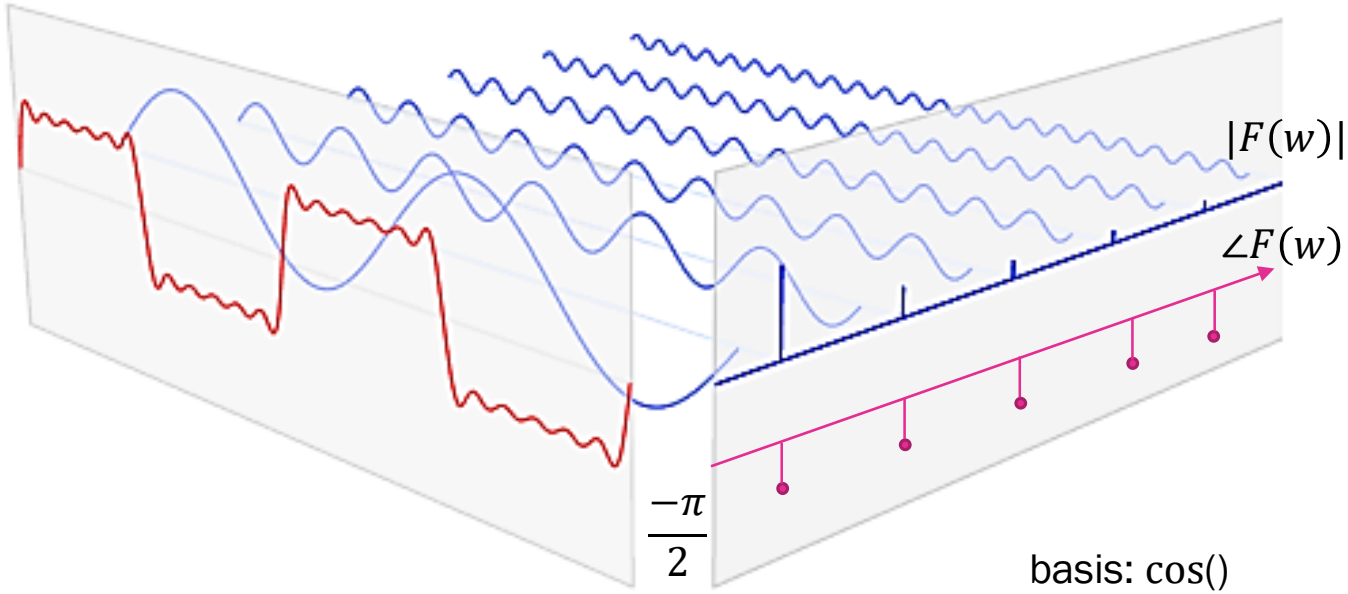


basis: sin()



basis: cos()

Frequency

- 2D harmonics
  - $v_o = 0$?

$$f(x, y) = A \cos(u_0 x + v_0 y + \varphi)$$

Frequency

- 2D harmonics
  - $v_o = 0$?
  - identical sinusoids with
    - amplitude A
    - spatial period $P_x = \frac{2\pi}{u_0}$
    - vertical stripes in the image

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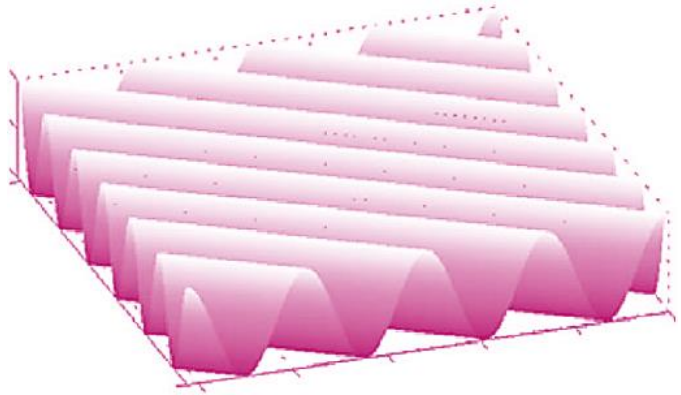

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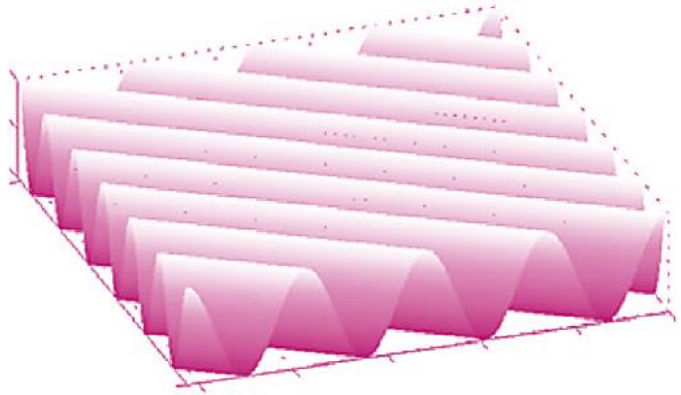

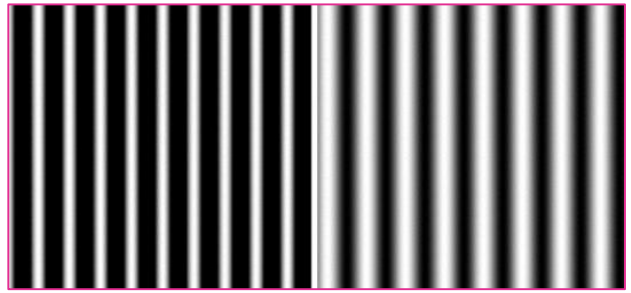

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  - $v_o = 0$?
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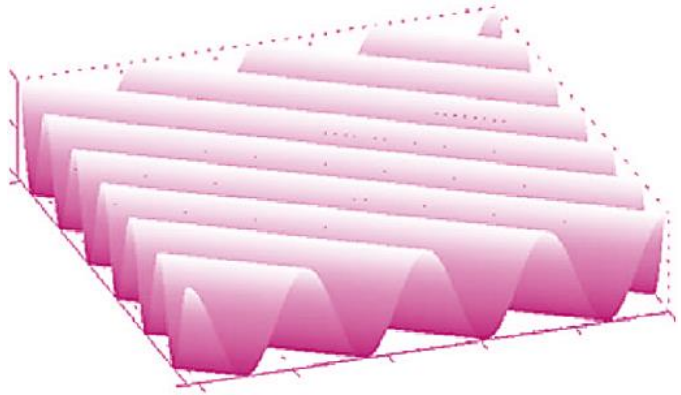
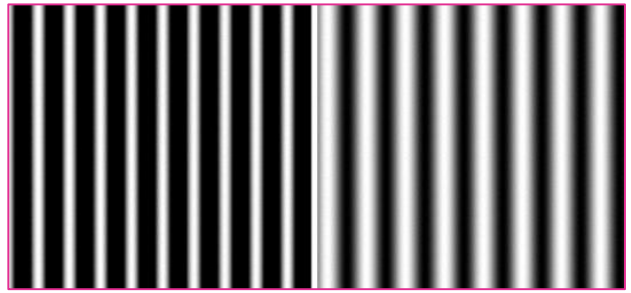
    - amplitude A
    - spatial period $P_x = \frac{2\pi}{u_0}$
    - vertical stripes in the image
  - space frequency
    - in both x, y directions
    - x direction: $\frac{u_0}{2\pi}(m^{-1})$
    - number of stripes per meter in x direction

$$f(x, y) = A\cos(u_0 x + v_0 y + \varphi)$$





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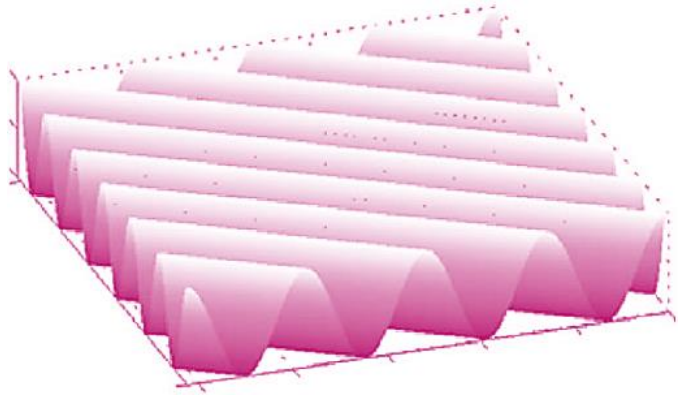
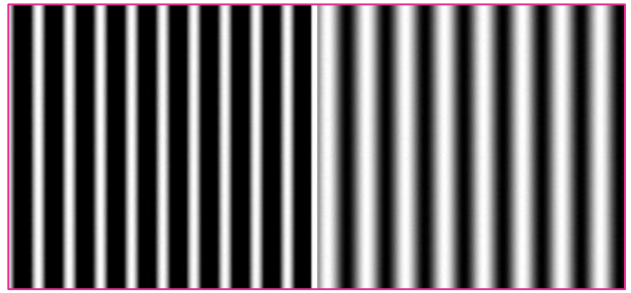


  - space frequency
    - in both x, y directions
    - x direction: $\frac{u_0}{2\pi}(m^{-1})$
    - number of stripes per meter in x direction
  - phase
    - determines shift d_x of ridges from origin of coordinates
    - d_x is from origin of coordinates as a fraction of harmonic's period
    - $d_x = P_x \frac{\phi}{2\pi} = \frac{\phi}{u_0}$



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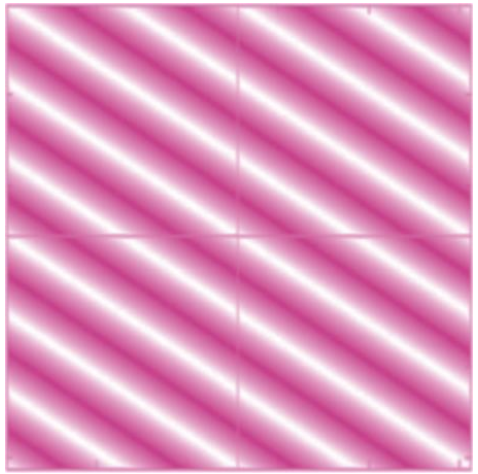

Frequency

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  - $u_0 \neq 0 \;\&\; v_o \neq 0$?
  - stripes become oblique
    - They make angel ϑ with x axis
    - $\vartheta = \arctan(\frac{v_0}{u_0})$
    - ratio of both frequencies determines orientation
    - ridges of stripes are characterized by $u_0 x + v_0 y + \varphi = \pm k2\pi$

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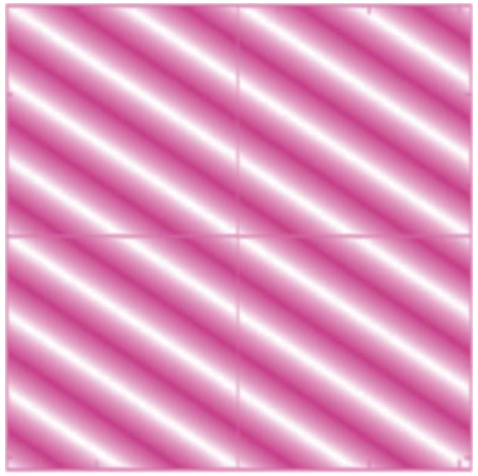

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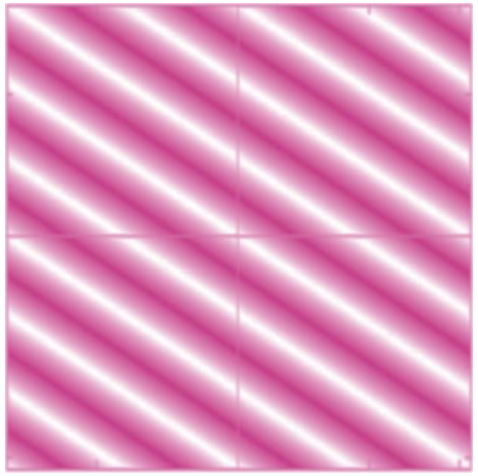
    - family of linear equations
    - oblique parallel lines
    - distance between them = period P of harmonics
    - $P = \frac{2\pi}{\sqrt{u_0^2+v_0^2}},\; w_0 = \sqrt{u_0^2 + v_0^2}$
    - shifting of strides by distance d wrt origin & $\perp$ to ridges
    - $d = \frac{P\varphi}{2\pi} = \frac{\varphi}{\omega}$

$$f(x,y) = A\cos(u_0 x + v_0 y + \varphi)$$



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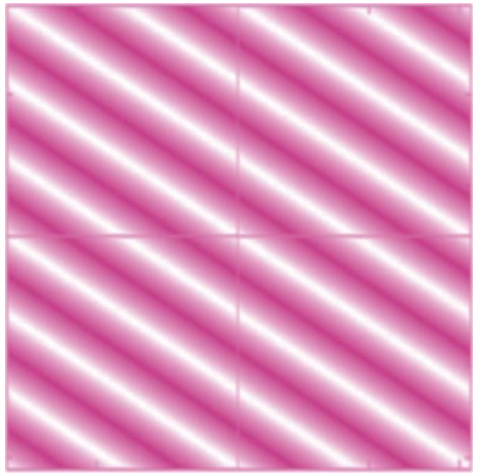

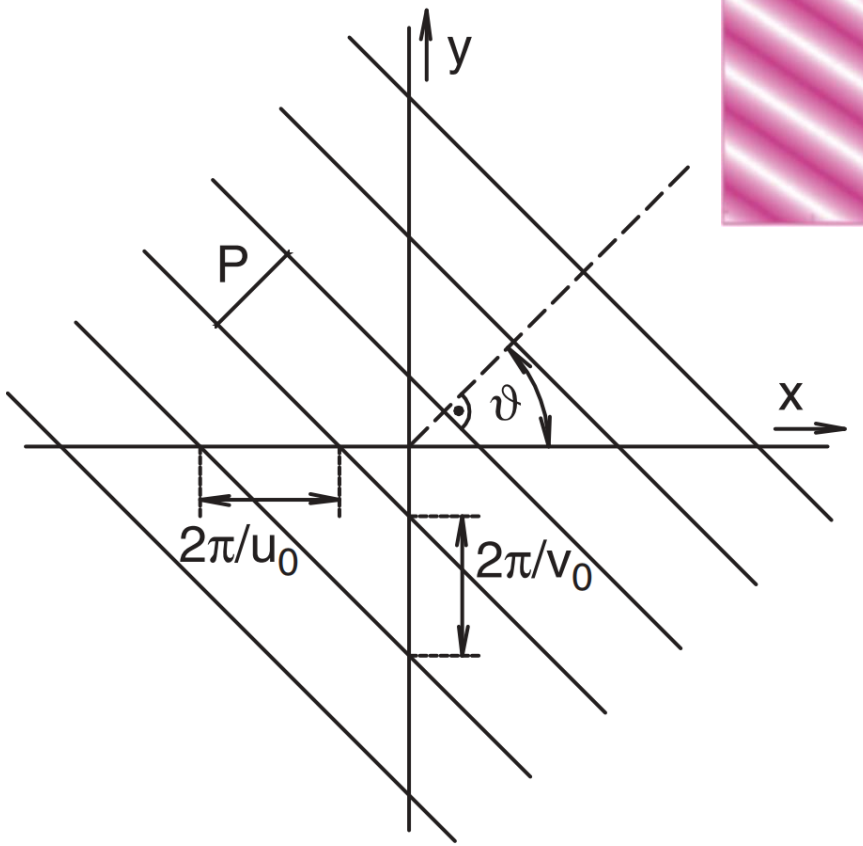

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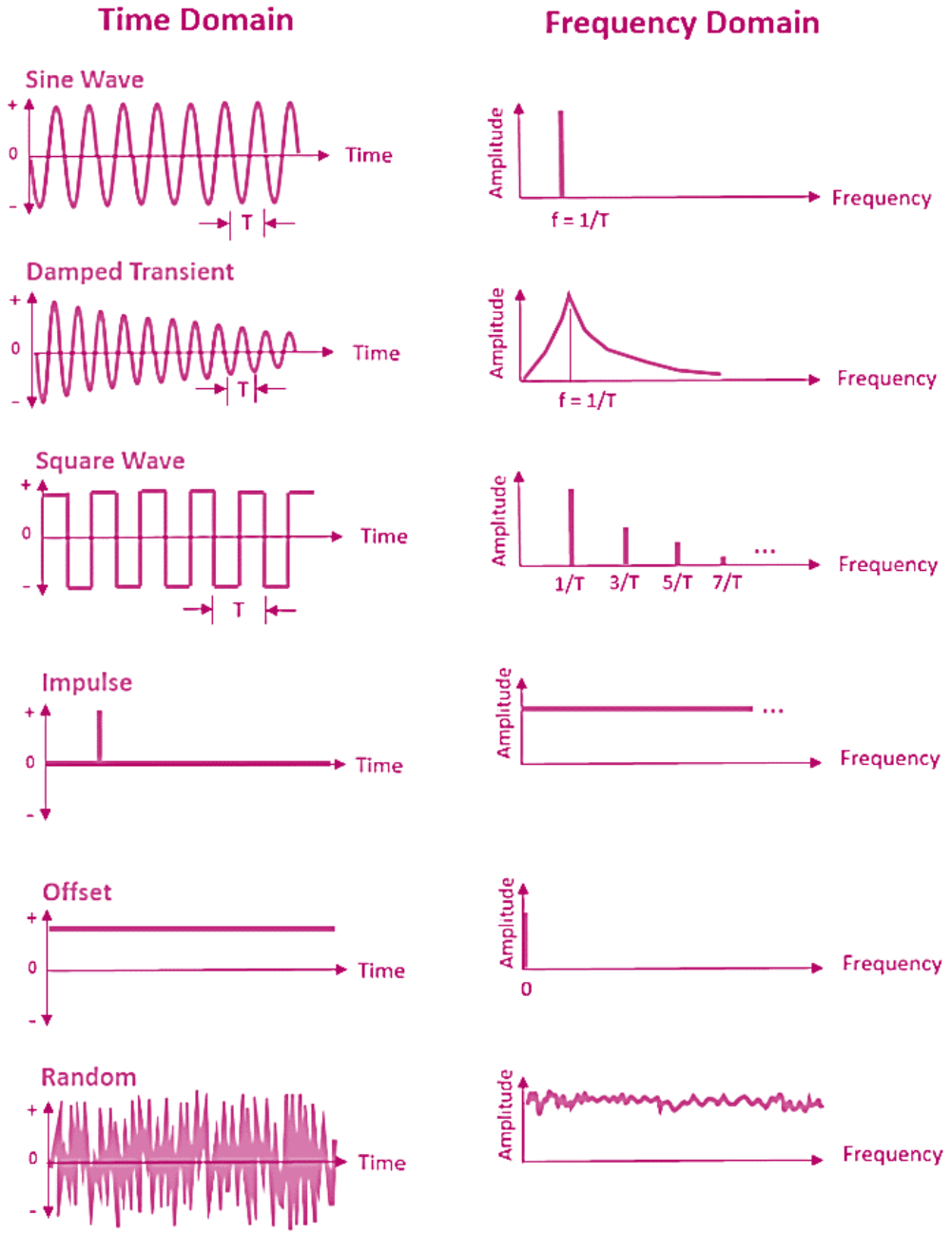
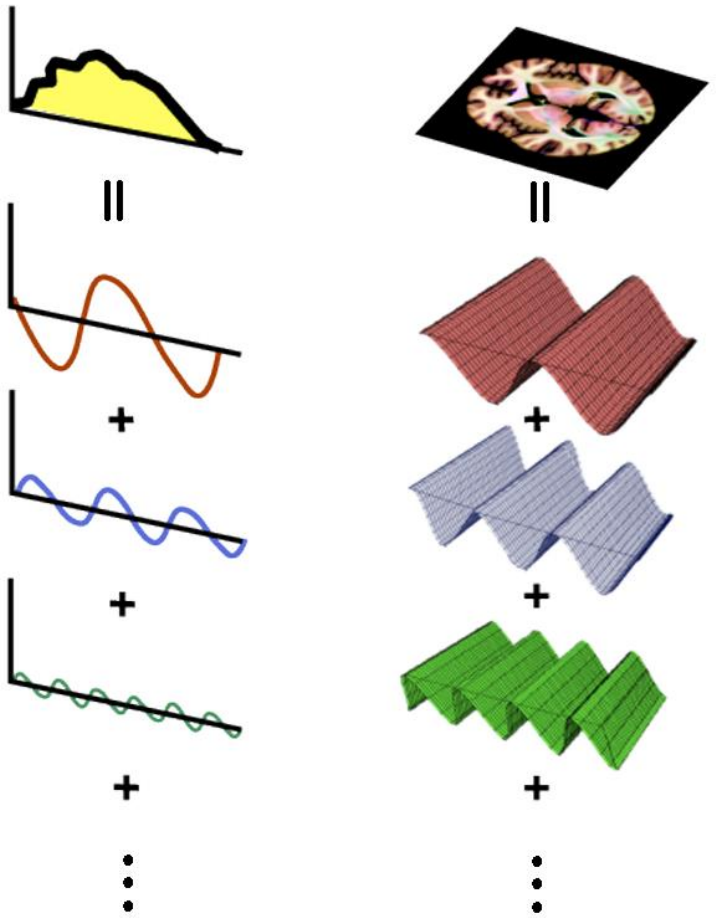
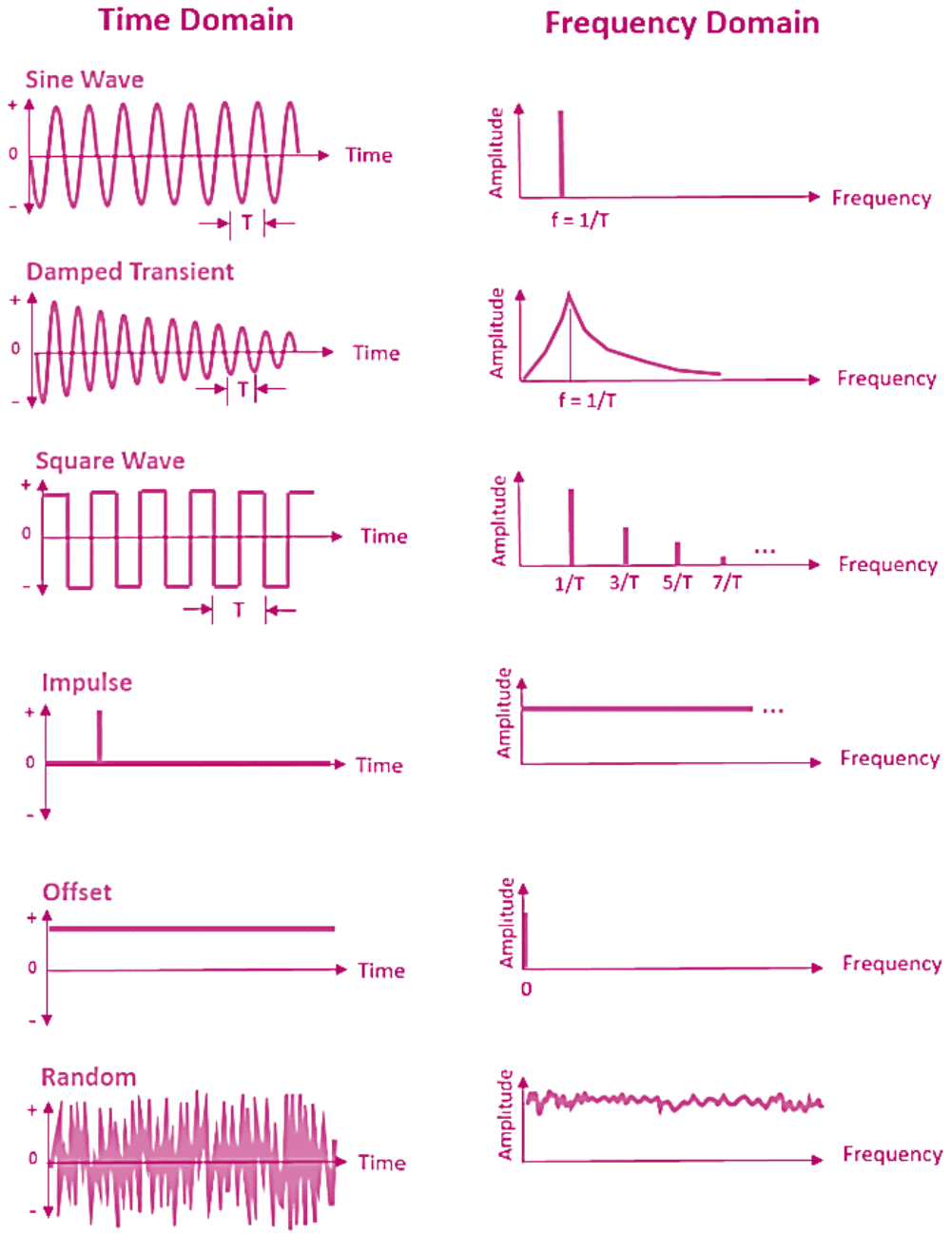

Time Domain
Frequency Domain
Sine Wave
Time
T
Amplitude
Frequency
f = 1/T
Damped Transient
Time
T
Amplitude
Frequency
f = 1/T
Square Wave
Time
T
Amplitude
Frequency
1/T
3/T
5/T
7/T
Impulse
Time
Amplitude
Frequency
Offset
Time
Amplitude
Frequency
0
Random
Time
Amplitude
Frequency

Frequency

- 1D
- 2D





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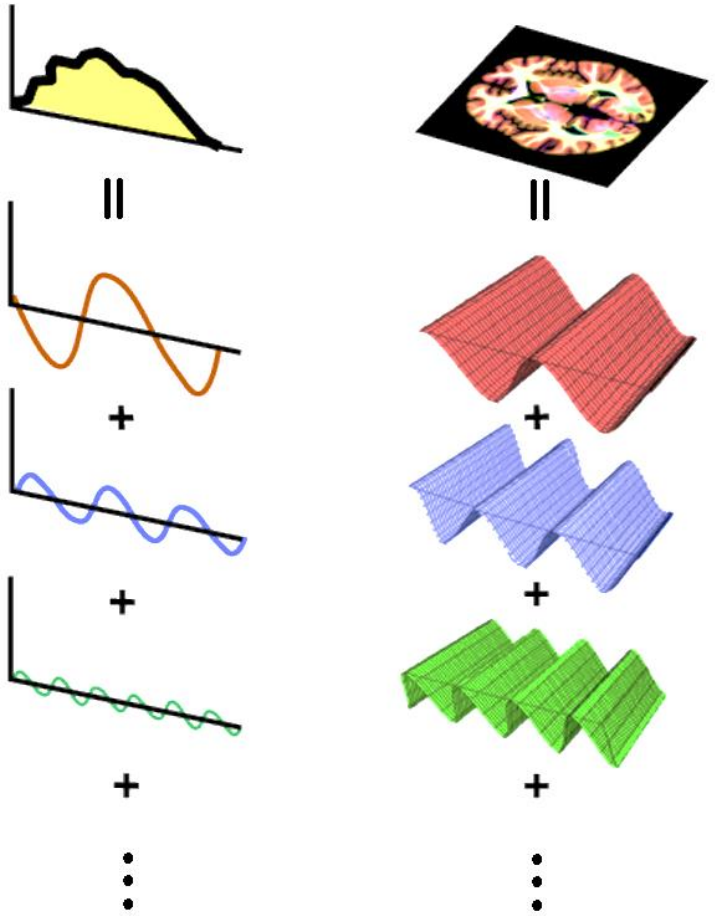

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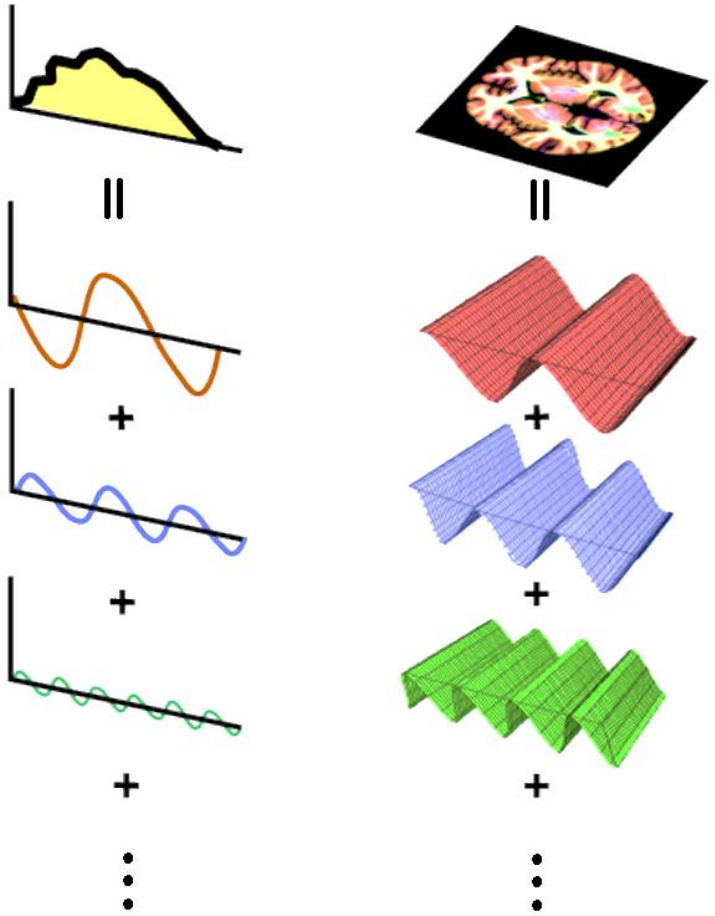

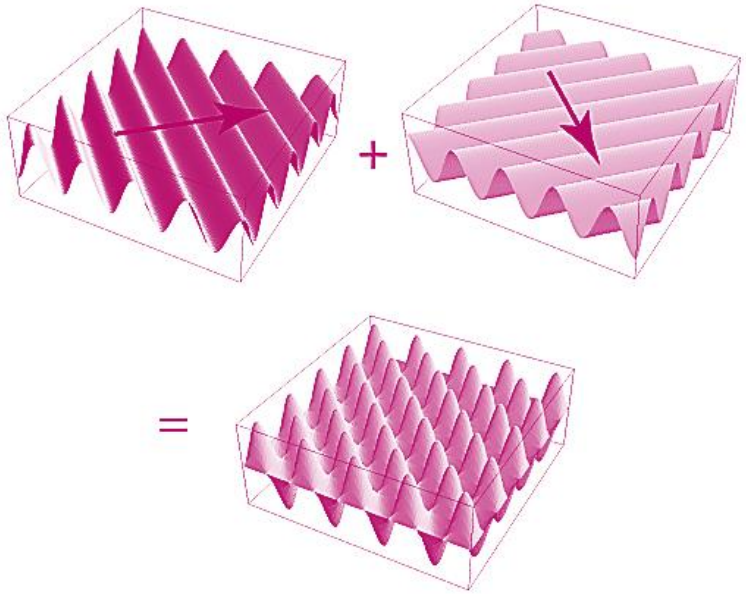

Frequency

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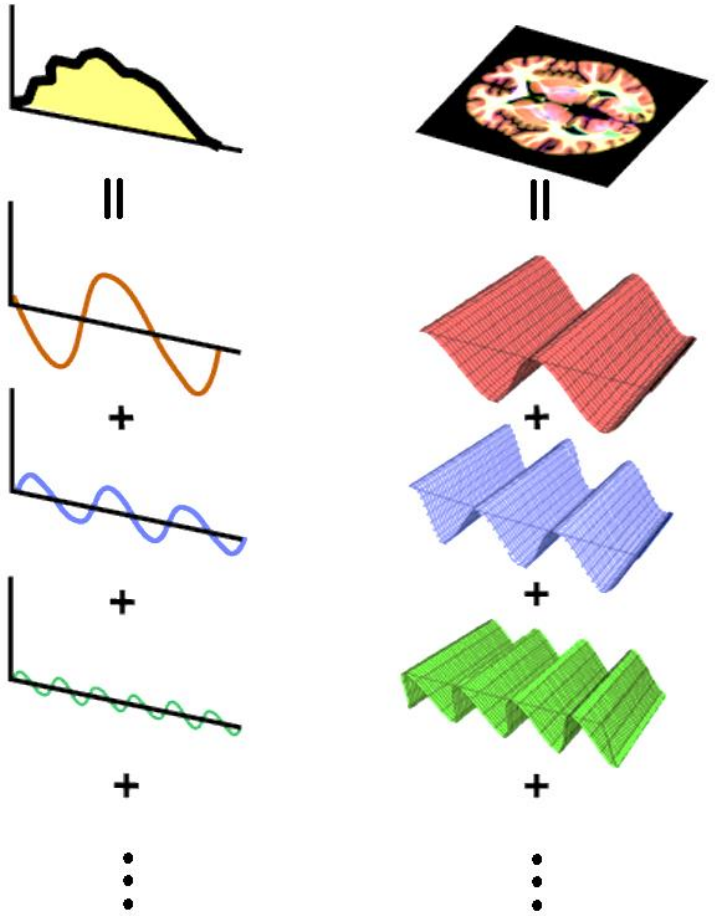
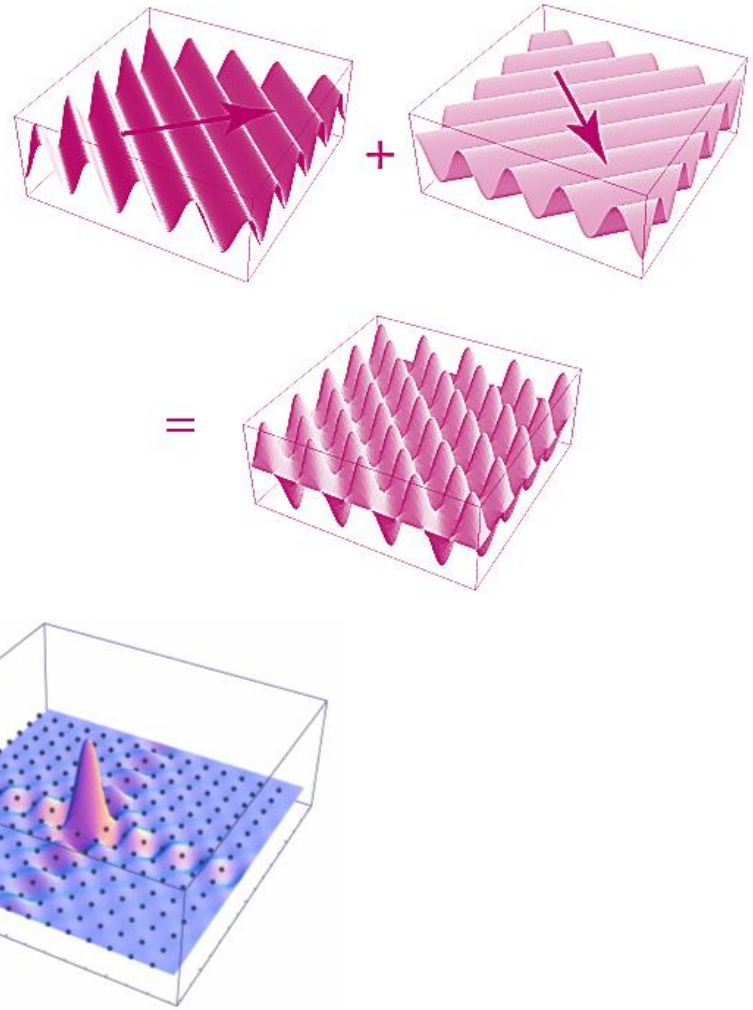
- 2D





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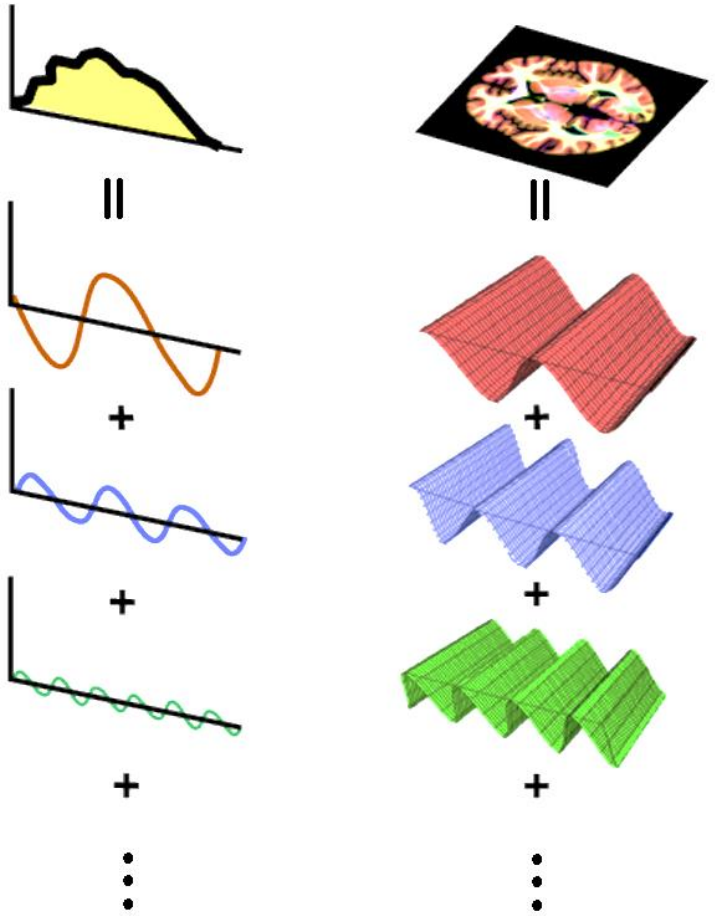

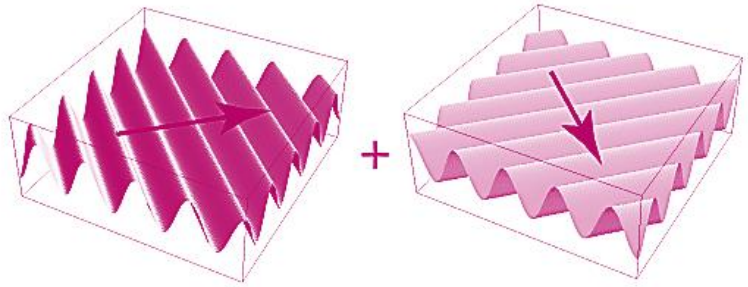

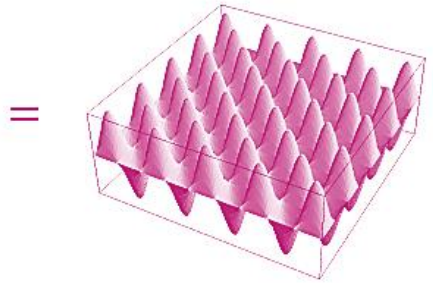

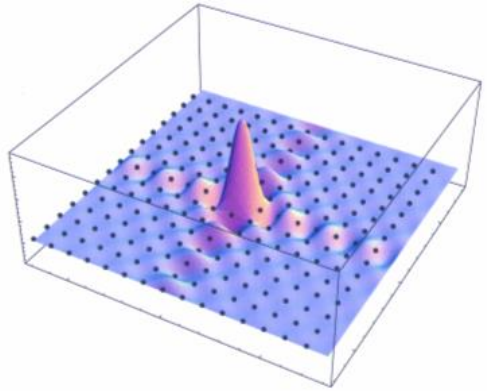

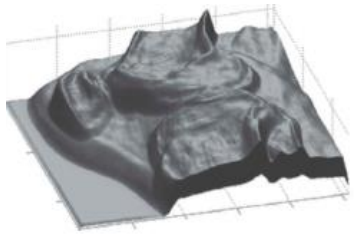

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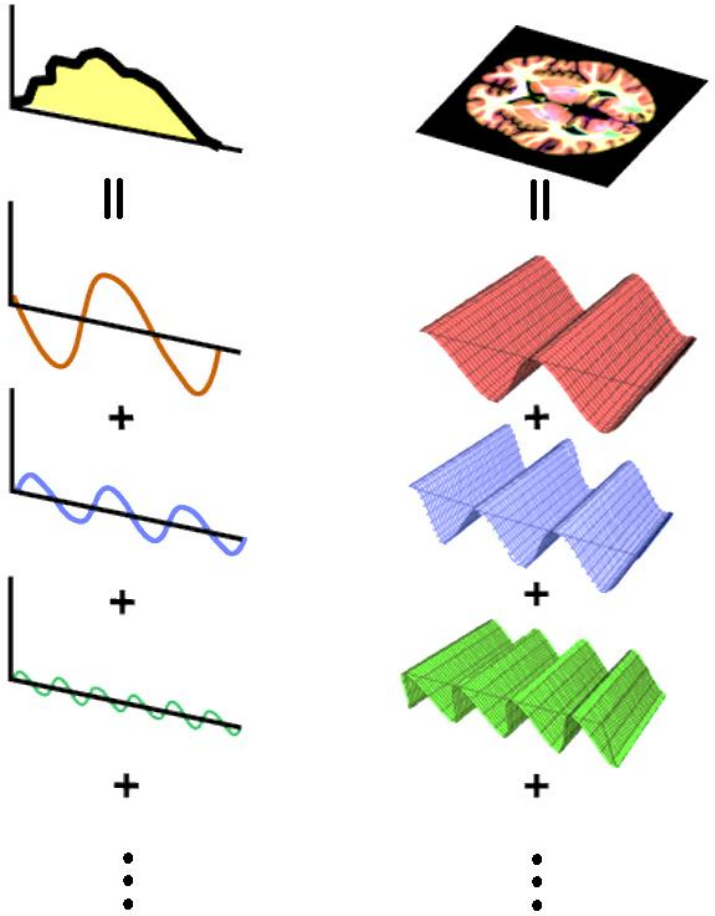

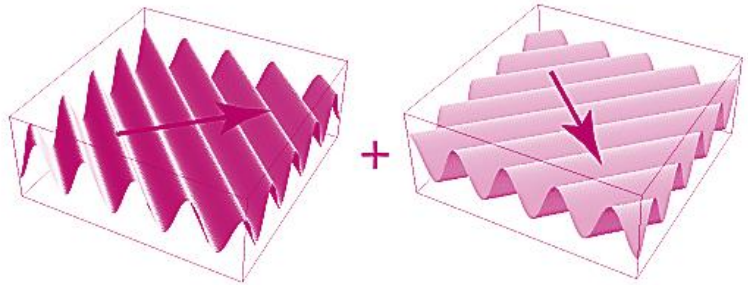

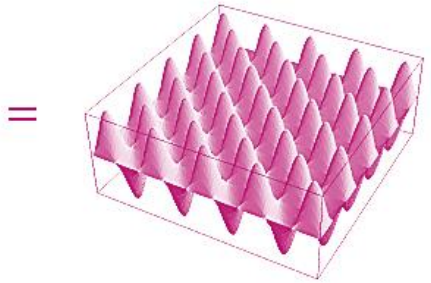

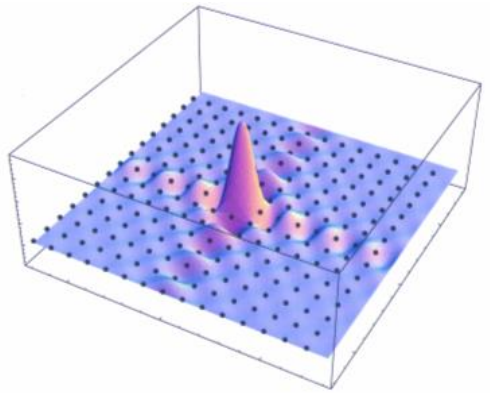

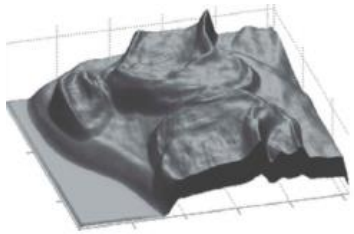

Fourier

- Fourier series
  - any periodic function can be approximated with series of harmonics
  - sinusoids are considered as harmonics
  - $f(x)$ with period L

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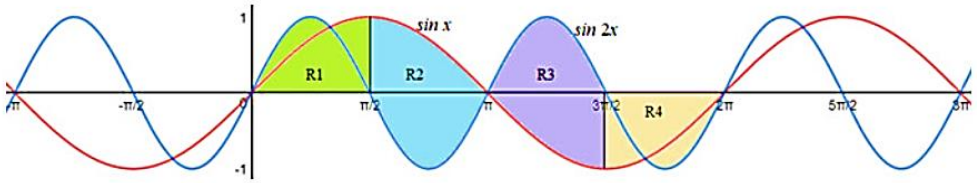

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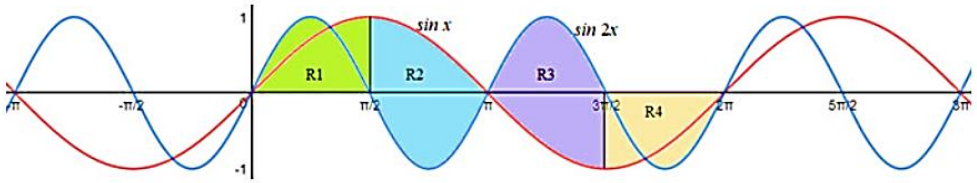

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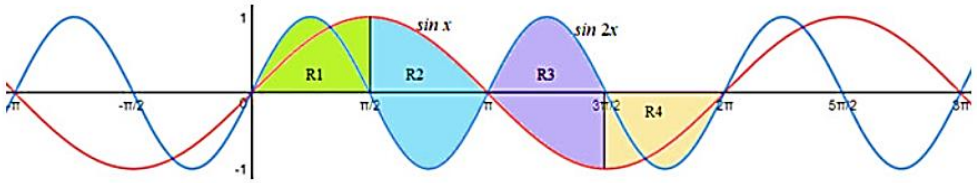

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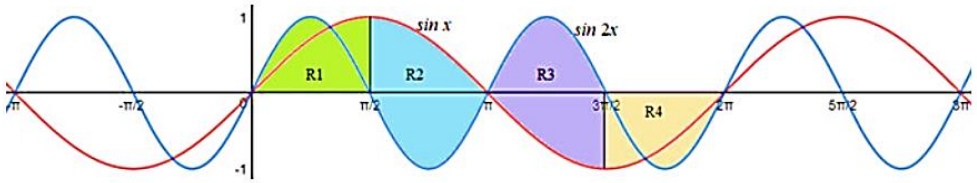

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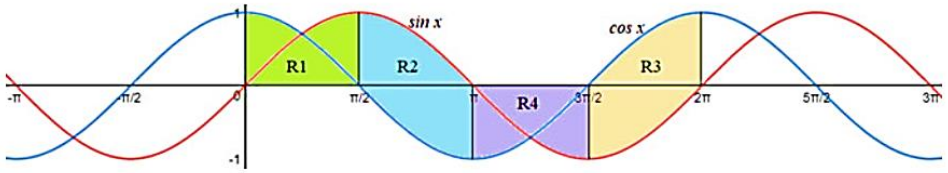

Fourier

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Fourier

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  - Complex
  - $e^{i\theta} = ?$

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  - any signal
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  - $\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t < \infty$

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$$L\,c_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-j\frac{2\pi nx}{L}} dx \longrightarrow F(u) = \int_{-\infty}^{\infty} f(x) \exp(-j2\pi ux)\, dx$$

$L \to \infty$ Frequency n/L becomes continuous

$$n/L \to u$$

Fourier

- Fourier Transform
  - any signal
  - finite discontinuities in finite interval
  - $\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t < \infty$

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Fourier

- Discrete Fourier Transform
  - any digital signal
  - N samples in [0, L]
  - Δx sample step in x direction
  - $L = ?$

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$$c_n = \frac{\Delta x}{N\Delta x}\sum_{k=0}^{N-1} f(k)e^{-j\frac{2\pi nk\Delta x}{N\Delta x}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} f(k)e^{-j\frac{2\pi nk}{N}} n = 0, 1, 2, \ldots, N-1$$

Fourier

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Fourier

- 2D DFT
  - MxN samples

$$0 \leq x < M$$
$$0 \leq y < N$$

  - image $f(x, y)$

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$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

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Fourier

- 2D DFT
  - separability

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$$F(u,v) = \sum_{x=0}^{M-1} \left[\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi vy/N} \right] e^{-j2\pi ux/M}$$

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$$F(u,v) = FT_x \left\{ FT_y \left[f(x,y) \right] \right\}$$

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FT_x and FT_y are the 1D FTs on row and column, respectively.

Fourier

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  - image
  - use grayscale image for DFT

Fourier

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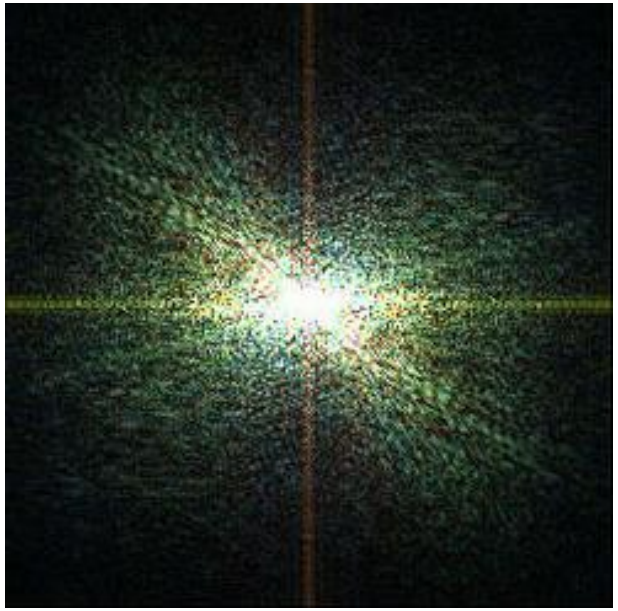

Fourier

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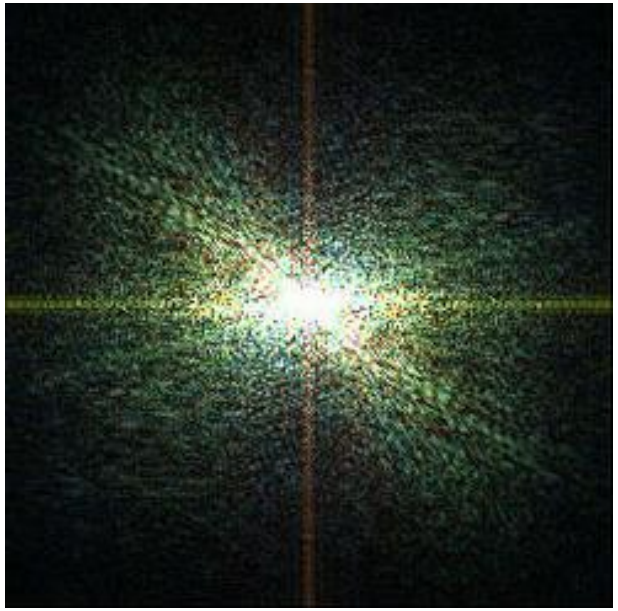
$f(x, y)$



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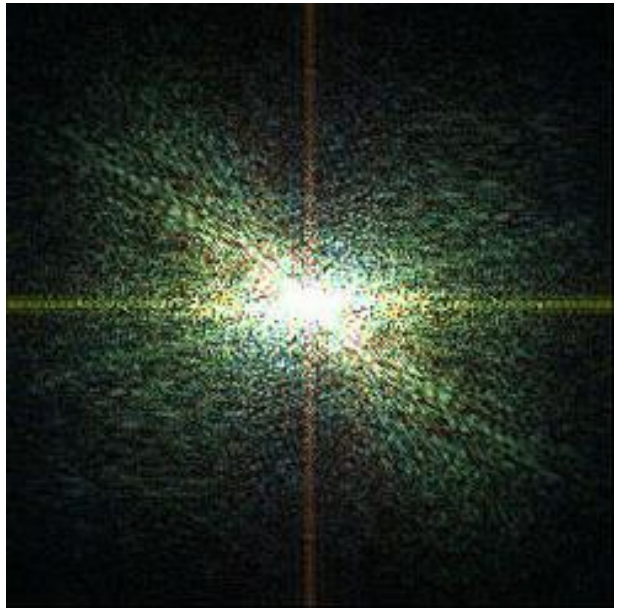
Enhanced($|F(u, v)|$)



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$$f(x, y)$$

Fourier

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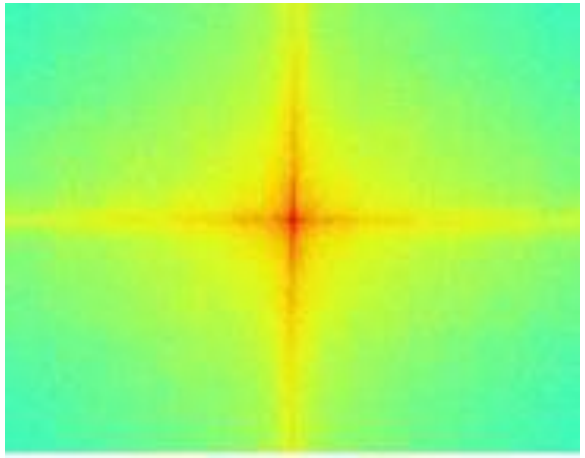
$$f(x, y)$$



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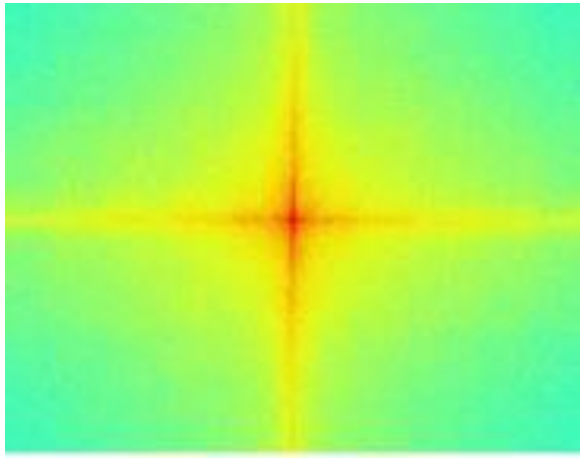
$|F(u, v)|$



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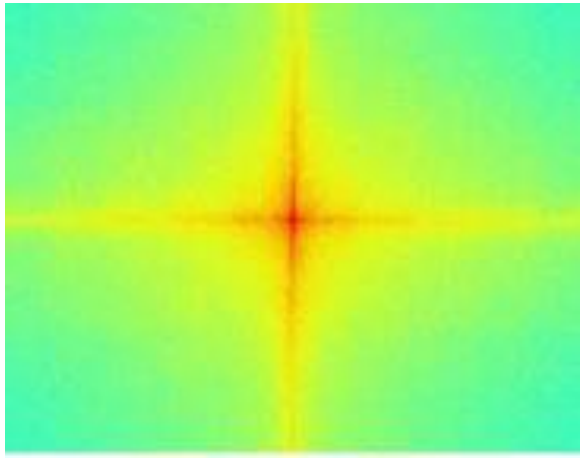

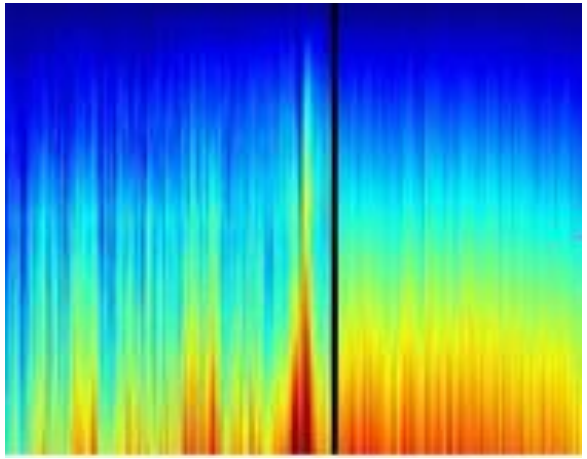

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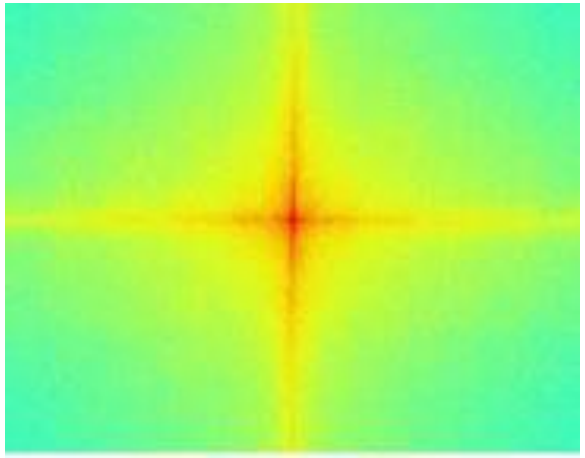
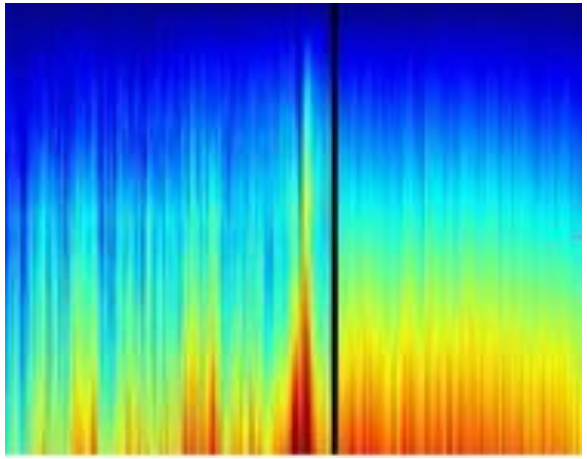


$\angle F(u,v)$



Conclusion

- Frequency representation
- Fourier
  - Series
  - Transform
  - 2D DFT

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"Four years at IIT transform $\left(f^*_F T\right)$ life phenomenally"

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Lagrange, Laplace, Monge