Hydrodynamic Synchronization of optically trapped micro-beads

I. INTRODUCTION

Hydrodynamic interaction plays an important role in the dynamics of microscopic particles. In a simple experimental setup using two optical traps, trapping a microbead each, we study the response of the second bead when first bead is driven with periodic force. We have further developed a theory to study and analyzed the experimental results.

The force balance equation of the optically trapped bead is,

$$\dot{\mathbf{R}}_i = \mathbf{v}_i \tag{1a}$$

$$m\dot{\mathbf{v}}_i = -\boldsymbol{\gamma}_{ij}\mathbf{v}_j - \boldsymbol{\nabla}_i U + \boldsymbol{\xi}_i \tag{1b}$$

In the above Eq. $\mathbf{R}_i$ and $\mathbf{v}_i$ are position and velocity of the i-th particle of mass m, $\boldsymbol{\gamma}_{ij}$ is manybody friction tensor and U is the potential and ξ is noise of thermal origin. In a optical trap, the potential is $U(t) = \frac{1}{2}\sum k_i(\mathbf{R}_i - \mathbf{R}_i^0)^2$ with $\mathbf{R}_i^0$ is the position of the potential minimum of the i-th optical trap. In the experimental set up the minimum of the optical trap is shifting with a periodic signal from outside and the response of the two beads have been studied. Assuming momentum to be rapidly relaxing on the time scale of the trap motion, we neglect inertia and average over the noise to get,

$$-\boldsymbol{\gamma}_{ij}\dot{\mathbf{R}}_j - \boldsymbol{\nabla}_i U = 0 \tag{2}$$

Eq.(2) can be inverted and presented in terms of mobility matrices $\boldsymbol{\mu}_{ij} = \boldsymbol{\gamma}_{ij}^{-1}$ as,

$$\begin{aligned}\dot{\mathbf{R}}_1 &= -\mu k_1\boldsymbol{\delta}(\mathbf{R}_1 - \mathbf{R}_1^0) - \boldsymbol{\mu}_{12}k_2(\mathbf{R}_2 - \mathbf{R}_2^0)\\ \dot{\mathbf{R}}_2 &= -\boldsymbol{\mu}_{21}k_1(\mathbf{R}_1 - \mathbf{R}_1^0) - \mu k_2\boldsymbol{\delta}(\mathbf{R}_2 - \mathbf{R}_2^0)\end{aligned} \tag{3}$$

Now we introduce new variable $\boldsymbol{\Delta}_i = \mathbf{R}_i - \mathbf{R}_i^0$, approximate mobility matrices with separation vector to be the average distance between two minimum of the optical traps. Thus we get,

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\Delta}_1\\ \boldsymbol{\Delta}_2\end{bmatrix} = -\begin{bmatrix}\mu k_1\boldsymbol{\delta} & \boldsymbol{\mu}_{12}k_2\\ \boldsymbol{\mu}_{21}k_1 & \mu k_2\boldsymbol{\delta}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Delta}_1\\ \boldsymbol{\Delta}_2\end{bmatrix} - \begin{bmatrix}\dot{\mathbf{R}}_1^0\\ \dot{\mathbf{R}}_2^0\end{bmatrix} \tag{4}$$

Steady state solution of the Eq.(4) can easily be calculated by taking Fourier transformation. Assuming $\mathbf{A} = \begin{bmatrix}\mu k_1\boldsymbol{\delta} & \boldsymbol{\mu}_{12}k_2\\ \boldsymbol{\mu}_{21}k_1 & \mu k_2\boldsymbol{\delta}\end{bmatrix}$, the solution in frequency is

$$\boldsymbol{\Delta}_i(\omega) = i\omega[-i\omega\boldsymbol{\delta} + \mathbf{A}]_{ij}^{-1}\mathbf{R}_j^0(\omega) = \boldsymbol{\chi}_{ij}(\omega)\mathbf{R}_j^0(\omega) \tag{5}$$

which defines the response for $\boldsymbol{\chi}$.

As the minimum of the optical trap is modulated by a sinusoidal wave with driving frequency Ω, $\mathbf{R}_j^0(\omega) = \frac{\mathbf{X}_i}{2}(\delta(\omega - \Omega) + \delta(\omega + \Omega))$. Further $\boldsymbol{\chi}$ is a block-diagonal matrix in cartesian indices. Given the experimental set up $\boldsymbol{\chi}$ can be decomposed in $\boldsymbol{\chi}_\parallel$ and $\boldsymbol{\chi}_\perp$ for motion along the trap separation and the motion perpendicular to it. Inserting this form of the signal into the Eq. above and taking the response equation back to time domain, we find

$$\Delta_{\parallel i}(t) = \chi'_{\parallel ij}(\Omega)\cos(\Omega t)X_j + \chi''_{\parallel ij}(\Omega)\sin(\Omega t)X_j$$

Two time scales from two traps can be calculated to be $\tau_i = 1/\mu k_i$.

$$\boldsymbol{\chi}_\parallel(\Omega) = i\Omega\begin{bmatrix}(\mu k_1 - i\Omega) & \mu_{12}k_2\\ \mu_{21}k_1 & (\mu k_2 - i\Omega)\end{bmatrix}^{-1} = \frac{i\Omega}{Det\,A_\parallel - \Omega^2 - i\Omega Tr\,A_\parallel}\begin{bmatrix}(\mu k_2 - i\Omega) & -\mu_{21}k_1\\ -\mu_{12}k_2 & (\mu k_1 - i\Omega)\end{bmatrix}$$

$$\boldsymbol{\chi}'_\parallel(\Omega) = \frac{\Omega^2}{(Det\,A_\parallel - \Omega^2)^2 + \Omega^2(Tr\,A_\parallel)^2}\begin{bmatrix}Det\,A_\parallel - \Omega^2 - \mu k_2 Tr\,A_\parallel & \mu_{21}k_1 Tr\,A_\parallel\\ \mu_{12}k_2 Tr\,A_\parallel & Det\,A_\parallel - \Omega^2 - \mu k_1 Tr\,A_\parallel\end{bmatrix}$$

$$\boldsymbol{\chi}''_\parallel(\Omega) = \frac{\Omega}{(Det\,A_\parallel - \Omega^2)^2 + \Omega^2(Tr\,A_\parallel)^2}\begin{bmatrix}\mu k_2(Det\,A_\parallel - \Omega^2) + \Omega^2 Tr\,A_\parallel & -\mu_{21}k_1(Det\,A_\parallel - \Omega^2)\\ -\mu_{12}k_2(Det\,A_\parallel - \Omega^2) & \mu k_1(Det\,A_\parallel - \Omega^2) + \Omega^2 Tr\,A_\parallel\end{bmatrix}$$

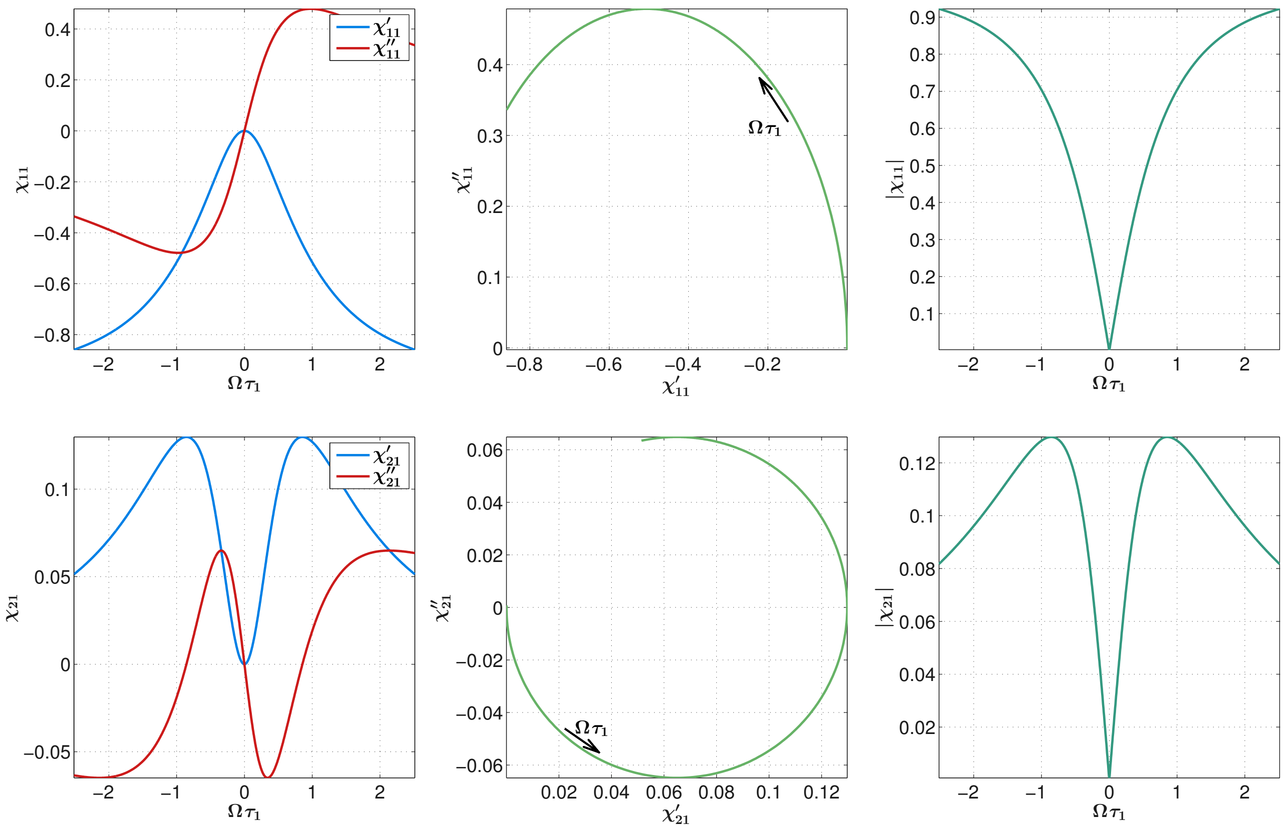

Figure 1. Real and imaginary part of the response function(a). Cole-Cole plot (b) and the amplitude of the response function.

We are interested in the resonance in amplitude of the second bead due to the forcing of the first bead, that is given by the maximising modulus of $\chi_{\parallel 21}$ with respect to Ω.

$$
\begin{aligned}
|\chi_{21}| &= \left|\frac{-i\Omega\mu_{12}k_2}{Det\, A_{\parallel} - \Omega^2 - i\Omega Tr\, A_{\parallel}}\right| \\
&= \frac{\Omega\mu_{12}k_2}{\sqrt{\left(Det\, A_{\parallel} - \Omega^2\right)^2 + \Omega^2 (Tr\, A_{\parallel})^2}}
\end{aligned}
$$

Clearly the resonance frequency in dimensionless unit is $\tau_1\Omega_{res} = \sqrt{\frac{Det A_{\parallel}}{\mu^2 k_1^2}} = \sqrt{\frac{k_2}{k_1}\left(1 - \frac{\mu_{\parallel 12}^2}{\mu^2}\right)}$ when $\mu_{12} \neq 0$.

We can further calculate auto-correlation and cross-correlation functions. To do that, we have kept two separate particles in two different optical traps and let them jiggle due to the thermal motion. The equation of motion for such motion is described by,

$$
\frac{d}{dt}\begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} = -\begin{bmatrix} \mu k_1 \boldsymbol{\delta} & \boldsymbol{\mu}_{12} k_2 \\ \boldsymbol{\mu}_{21} k_1 & \mu k_2 \boldsymbol{\delta} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} \tag{6}
$$

Writing this equation in index form we get,

$$
\dot{\Delta}_i = -A_{ij}\Delta_j + B_{ij} w_j
$$

where noise is correlated Wiener processes with variances goes as $\mathbf{B}\mathbf{B}^T = k_b T \mathbf{A}$ and white noise w satisfy $\langle w \rangle = 0$ $\langle w_i(t) w_j(t') \rangle = \delta_{ij}\delta(t-t')$. The solution of this multivariate Stochatic differential equation is

$$
\boldsymbol{\Delta}(t) = \boldsymbol{\Delta}(0) e^{-\mathbf{A}t} + \int_0^t ds\, e^{-\mathbf{A}(t-s)} \mathbf{B}\mathbf{w}
$$

Keeping the particles in the trap for long time will let them equilibriate and all transient effect will die out and hence will not have any consequences. The correlation matrix can be calculated which will only depend on contribution from noise.

$$\mathbf{\Delta}(t)\mathbf{\Delta}^T(t') = \int_0^{t'} \int_0^{t} ds' ds\, e^{-\mathbf{A}(t-s)} \mathbf{B}\mathbf{w}\mathbf{w}^T \mathbf{B}^T e^{-\mathbf{A}^T(t'-s')}.$$

For stationary case average of the correlator become,

$$\langle \mathbf{\Delta}(t)\mathbf{\Delta}^T(t')\rangle_{eq} = \int_{-\infty}^{t'} \int_{-\infty}^{t} ds' ds\, e^{-\mathbf{A}(t-s)} \mathbf{B}\langle\mathbf{w}\mathbf{w}^T\rangle_{eq} \mathbf{B}^T e^{-\mathbf{A}^T(t'-s')}.$$
$$\langle \mathbf{\Delta}(t)\mathbf{\Delta}^T(t')\rangle_{eq} = e^{-\mathbf{A}(t-t')} \int_{-\infty}^{t} ds\, e^{-\mathbf{A}(t'-s)} \mathbf{B}\mathbf{B}^T e^{-\mathbf{A}^T(t'-s)} = e^{-\mathbf{A}(t-t')}\mathbf{C}$$

where $\mathbf{C}$ is the stationary covariance matrix,

For a two dimensional system C can easily be calculated \cite{gardinar}, the solution is

$$\mathbf{C} = \frac{(Det\,\mathbf{A})\mathbf{B}\mathbf{B}^T + [\mathbf{A} - Tr\,\mathbf{A}]\mathbf{B}\mathbf{B}^T[\mathbf{A} - Tr\,\mathbf{A}]}{2Tr\,\mathbf{A}\,Det\,\mathbf{A}}$$