Hydrodynamic Synchronization of optically trapped micro-beads

I. INTRODUCTION

Hydrodynamic interaction plays an important role in the dynamics of microscopic particles. In a simple experimental setup using two optical traps, trapping a microbead each, we study the response of the second bead when first bead is driven with periodic force. We have further develop a theory to study and analyse the experimental results.

The force balance equation of the optically trapped bead is,

$$\dot{\mathbf{R}}_i = \mathbf{v}_i \tag{1a}$$

$$m\dot{\mathbf{v}}_i = -\boldsymbol{\gamma}_{ij}\mathbf{v}_j - \boldsymbol{\nabla}_i U + \boldsymbol{\eta}_i \tag{1b}$$

In the above Eq. $\mathbf{R}_i$ and $\mathbf{v}_i$ are position and velocity of the i-th particle of mass m, $\boldsymbol{\gamma}_{ij}$ is manybody friction tensor and U is the potential and η is noise of thermal origin. In a optical trap, the potential is $U(t) = \sum k_i(\mathbf{R}_i - \mathbf{R}_i^0)^2$ with $\mathbf{R}_i^0$ is the position of the potential minimum of the i-th optical trap. In the experimental set up the minimum of the optical trap is shifting with a periodic signal from outside and the response of the two beads have been studied. If the adiabatic approximation, the velocity reaches to the equilibrium faster. So in that time scale, the particle will move steadily without acceleration. So the velocity equation can be presented in terms of mobility matrix as,

$$\mathbf{v}_i = \boldsymbol{\mu}_{ij}\left[-\boldsymbol{\nabla}_j U + \boldsymbol{\eta}_j\right] \tag{2}$$

This Eq.(2) can be written as,

$$\dot{\mathbf{R}}_1 = -\mu k_1 \boldsymbol{\delta}(\mathbf{R}_1 - \mathbf{R}_1^0) - \boldsymbol{\mu}_{12} k_2(\mathbf{R}_2 - \mathbf{R}_2^0) + \boldsymbol{\eta}_1$$
$$\dot{\mathbf{R}}_2 = -\boldsymbol{\mu}_{21} k_1(\mathbf{R}_1 - \mathbf{R}_1^0) - \mu k_2 \boldsymbol{\delta}(\mathbf{R}_2 - \mathbf{R}_2^0) + \boldsymbol{\eta}_2$$

Now we introduce new variable $\boldsymbol{\Delta}_i = \mathbf{R}_i - \mathbf{R}_i^0$, approximate mobility matrices with separation vector to be the average distance between two minimum of the optical traps and we average the Eq over many realizations and thus the variables are become ensemble averaged. Thus we get,

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\Delta}_1\\ \boldsymbol{\Delta}_2\end{bmatrix} = -\begin{bmatrix}\mu k_1\boldsymbol{\delta} & \boldsymbol{\mu}_{12}k_2\\ \boldsymbol{\mu}_{21}k_1 & \mu k_2\boldsymbol{\delta}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Delta}_1\\ \boldsymbol{\Delta}_2\end{bmatrix} - \begin{bmatrix}\dot{\mathbf{R}}_1^0\\ \dot{\mathbf{R}}_2^0\end{bmatrix} \tag{3}$$

Steady state solution of the Eq.(4) can easily be calculated by taking Fourier transformation. Assuming $\mathbf{A} = \begin{bmatrix}\mu k_1\boldsymbol{\delta} & \boldsymbol{\mu}_{12}k_2\\ \boldsymbol{\mu}_{21}k_1 & \mu k_2\boldsymbol{\delta}\end{bmatrix}$, the solution in frequency is

$$\boldsymbol{\Delta}_i(\omega) = -[i\omega\boldsymbol{\delta} + \mathbf{A}]_{ij}^{-1}\dot{\mathbf{R}}_j^0(\omega) = \boldsymbol{\chi}_{ij}(\omega)\dot{\mathbf{R}}_j^0(\omega)$$

As the minium of the optical trap is modulated by a sinusoidal wave with driving frequency Ω, $\dot{\mathbf{R}}_j^0(\omega) = \frac{\mathbf{X}_i}{2}(\delta(\omega-\Omega)+\delta(\omega+\Omega))$. Further $\boldsymbol{\chi}$ is a block-diagonal matrix in cartesian indices. Given the experimental set up $\boldsymbol{\chi}$ can be decomposed in $\boldsymbol{\chi}_{\parallel}$and $\boldsymbol{\chi}_{\perp}$. Inserting this form of the signal into the Eq. above and taking the response equation back to time domain, we find

$$\Delta_{\parallel i}(t) = \chi'_{\parallel ij}(\Omega)\cos(\Omega t)X_j + \chi''_{\parallel ij}(\Omega)\sin(\Omega t)X_j$$

where $\boldsymbol{\chi}_{\parallel} = \boldsymbol{\chi}'_{\parallel} + i\boldsymbol{\chi}''_{\parallel}$, and $\mathbf{X}_j$'s are the amplitude of the signal.

$$\boldsymbol{\chi}_{\parallel}(\Omega) = -\begin{bmatrix} (\mu k_1 + i\Omega) & \mu_{12}k_2 \\ \mu_{21}k_1 & (\mu k_2 + i\Omega) \end{bmatrix}^{-1} = -\frac{1}{Det\,A_{\parallel} - \Omega^2 + i\Omega Tr\,A_{\parallel}} \begin{bmatrix} (\mu k_2 + i\Omega) & -\mu_{21}k_1 \\ -\mu_{12}k_2 & (\mu k_1 + i\Omega) \end{bmatrix}$$

$$\boldsymbol{\chi}_{\parallel}^{'}(\Omega) = -\frac{1}{(Det\,A_{\parallel} - \Omega^2)^2 + \Omega^2(Tr\,A_{\parallel})^2} \begin{bmatrix} \mu k_2(Det\,A_{\parallel} - \Omega^2) + \Omega^2 Tr\,A_{\parallel} & -\mu_{21}k_1(Det\,A_{\parallel} - \Omega^2) \\ -\mu_{12}k_2(Det\,A_{\parallel} - \Omega^2) & \mu k_1(Det\,A_{\parallel} - \Omega^2) + \Omega^2 Tr\,A_{\parallel} \end{bmatrix}$$

$$\boldsymbol{\chi}_{\parallel}^{''}(\Omega) = -\frac{\Omega}{(Det\,A_{\parallel} - \Omega^2)^2 + \Omega^2(Tr\,A_{\parallel})^2} \begin{bmatrix} Det\,A_{\parallel} - \Omega^2 - \mu k_2 Tr\,A_{\parallel} & \mu_{21}k_1 Tr\,A_{\parallel} \\ \mu_{12}k_2 Tr\,A_{\parallel} & Det\,A_{\parallel} - \Omega^2 - \mu k_1 Tr\,A_{\parallel} \end{bmatrix}$$