Normal modes calculation*

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I. NORMAL MODES CALCULATION

The force balance equation of two optically trapped spherical particles near to each other in a highly viscous medium (Reynold's number tends to zero) is,

$$-\gamma\dot{\mathbf{x}} - \mathbf{k}\mathbf{x} = \xi \tag{1}$$

Where γ is the many body friction tensor, $\mathbf{k}$ is the stiffness tensor for those two optical traps and ξ is the noise tensor of thermal origin which is correlated as $\left\langle \xi(t)\xi^{\mathbf{T}}(t')\right\rangle = 2K_BT\gamma\delta(t-t')$. The many body friction tensor is given by

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$$

where $\gamma_{12} = \gamma_{21}$ and for two spherical beads of same radius,$\gamma_{11} = \gamma_{22}$.The stiffness tensor

$$\mathbf{k} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

. If $\mathbf{S}$ be a similarity transformation matrix which diagonalizes γ then

$$\begin{aligned} -\mathbf{S}^{-1}\gamma_{\mathbf{D}}\mathbf{S}\dot{\mathbf{x}} - \mathbf{k}\mathbf{x} &= \xi \\ -\gamma_{\mathbf{D}}\mathbf{S}\dot{\mathbf{x}} - \mathbf{S}\mathbf{k}\mathbf{x} &= \mathbf{S}\xi \\ -\gamma_{\mathbf{D}}\dot{\mathbf{y}} - \mathbf{k}\mathbf{y} &= \mathbf{S}\xi \end{aligned} \tag{2}$$

where we used $\mathbf{S}\mathbf{x} = \mathbf{y}, \mathbf{S}^{-1}\gamma_{\mathbf{D}}\mathbf{S} = \gamma$ and $[\mathbf{S}, \mathbf{k}] = 0$. The diagonalized friction tensor is given by

$$\gamma_{\mathbf{D}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3}$$

$$\lambda_{1,2} = \left(\frac{\gamma_{11}+\gamma_{22}}{2}\right) \pm \frac{1}{2}\sqrt{(\gamma_{11}-\gamma_{22})^2 + 4\gamma_{12}\gamma_{21}}$$

and the similarity transformation matrix is

$$\begin{aligned} \mathbf{S} &= \begin{bmatrix} \frac{\lambda_1-\gamma_{22}}{\gamma_{21}} & \frac{\lambda_2-\gamma_{22}}{\gamma_{21}} \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Here we assumed $\gamma_{11} = \gamma_{22}$. So, from Eq 2 we get two equations of motions describing the normal modes of the movements of the two trapped beads as

$$-\lambda_1\dot{y_1} - k_1y_1 = (\xi_2 - \xi_1) \tag{4}$$
$$-\lambda_2\dot{y_2} - k_2y_2 = (\xi_1 + \xi_2) \tag{5}$$

From Eq 6 and 7 we get

$$y_1(\omega) = \frac{\xi_1(\omega) - \xi_2(\omega)}{i\omega\lambda_1 - k_1} \tag{6}$$

$$y_2(\omega) = \frac{\xi_1(\omega) + \xi_2(\omega)}{i\omega\lambda_2 - k_1} \tag{7}$$

Hence the corresponding power spectrums are given by

$$\begin{aligned} \langle y_1(\omega)y_1^*(\omega)\rangle &= \frac{\langle(\xi_1-\xi_2)(\xi_1-\xi_2)^*\rangle}{\omega^2\lambda_1^2 + k_1^2} \\ \langle y_1(\omega)y_1^*(\omega)\rangle &= \frac{4k_BT(\gamma_{11}-\gamma_{12})}{\omega^2\lambda_1^2 + k_1^2} \end{aligned} \tag{8}$$

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* A footnote to the article title
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Similarly,

$$\langle y_2(\omega) y_2^*(\omega)\rangle = \frac{4k_BT(\gamma_{11}+\gamma_{12})}{\omega^2\lambda_2^2+k_2^2} \tag{9}$$

The auto correlations can be calculated by performing inverse Fourier transformations on the corresponding power spectrums. We get

$$\langle y_1(t) y_1(0)\rangle = \frac{2k_BT}{k_1} e^{-\frac{k_1}{\lambda_1}t} \tag{10}$$

$$\langle y_2(t) y_2(0)\rangle = \frac{2k_BT}{k_2} e^{-\frac{k_2}{\lambda_2}t} \tag{11}$$

Here, $\lambda_1 = (\gamma_{11} - \gamma_{12})$ and $\lambda_2 = (\gamma_{11} + \gamma_{12})$. Now, we can write

$$x_1 = a_1 y_1 + a_2 y_2 \tag{12}$$

$$x_2 = a_1 y_1 - a_2 y_2 \tag{13}$$

Where a_1, a_2 are constants.

$$\begin{aligned}\langle x_1(\omega) x_1^*(\omega)\rangle &= \left\langle a_1^2 y_1(\omega) y_1^*(\omega)\right\rangle + \langle a_1 a_2 y_1(\omega) y_2^*(\omega)\rangle \\ &+ \langle a_2 a_1 y_2(\omega) y_1^*(\omega)\rangle + \left\langle a_2^2 y_2(\omega) y_2^*(\omega)\right\rangle\end{aligned}$$

Now,$\langle y_1(\omega) y_2^*(\omega)\rangle = 0$, if $\gamma_{11} = \gamma_{22}$ and $\gamma_{12} = \gamma_{21}$. So,

$$\begin{aligned}\langle x_1(\omega) x_1^*(\omega)\rangle &= \left\langle a_1^2 y_1(\omega) y_1^*(\omega)\right\rangle + \left\langle a_2^2 y_2(\omega) y_2^*(\omega)\right\rangle \\ \langle x_1(0) x_1^*(t)\rangle &= \left\langle a_1^2 y_1(0) y_1^*(t)\right\rangle + \left\langle a_2^2 y_2(0) y_2^*(t)\right\rangle\end{aligned}$$

Hence,

$$\begin{aligned}\langle x_1(\omega) x_1^*(\omega)\rangle &= \left\langle a_1^2 y_1(\omega) y_1^*(\omega)\right\rangle + \left\langle a_2^2 y_2(\omega) y_2^*(\omega)\right\rangle \\ \langle x_1(0) x_1^*(t)\rangle &= 2k_BT\left[\frac{a_1^2}{k_1} e^{-\frac{k_1}{\lambda_1}t} + \frac{a_2^2}{k_2} e^{-\frac{k_2}{\lambda_2}t}\right]\end{aligned}$$

Similarly,

$$\langle x_2(0) x_2^*(t)\rangle = 2k_BT\left[\frac{a_1^2}{k_1} e^{-\frac{k_1}{\lambda_1}t} + \frac{a_2^2}{k_2} e^{-\frac{k_2}{\lambda_2}t}\right]$$

II. EXPERIMENT

We know k_1, k_2 experimentally and also know that $\lambda_1 = \gamma_{11} - \gamma_{12}, \lambda_1 = \gamma_{11} + \gamma_{12}$.If

$$\gamma = \mu^{-1}$$

then,

$$\begin{bmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{bmatrix} = \frac{1}{\mu_{11}^2 - \mu_{12}^2}\begin{bmatrix}\mu_{22} & -\mu_{12}\\ -\mu_{21} & \mu_{11}\end{bmatrix} \tag{14}$$

$$\lambda_1 = \frac{\mu_{11}+\mu_{12}}{\mu_{11}^2-\mu_{12}^2} = \frac{1}{\mu_{11}-\mu_{12}} \tag{15}$$

$$\lambda_1 = \frac{\mu_{11}-\mu_{12}}{\mu_{11}^2-\mu_{12}^2} = \frac{1}{\mu_{11}+\mu_{12}} \tag{16}$$

So, the first time constant should match with $\frac{k_1}{\lambda_1}$ and the second time constant should match with $\frac{k_2}{\lambda_2}$.