Equilibrium configurations

Subhaneil Lahiri

Jefferson Physical Laboratory, Harvard University, Cambridge MA 02138, USA

Abstract

We look at equilibrium configurations of relativistic fluids.

1 Thermodynamics

sec:fltherm

A fluid satisfies

$$\mathrm{d}\mathcal{E} = \mathcal{T}\mathrm{d}\mathcal{S} - \mathcal{P}\mathrm{d}\mathcal{V} + \mathfrak{m}_i\,\mathrm{d}\mathcal{R}_i. \tag{1}$$ {firstlaw:eq}

Suppose we rescale the system by a factor $(1+\epsilon)$. Extensivity tells us that $\mathrm{d}\mathcal{E} = \epsilon\mathcal{E}$, $\mathrm{d}\mathcal{S} = \epsilon\mathcal{S}$, $\mathrm{d}\mathcal{V} = \epsilon\mathcal{V}$ and $\mathrm{d}\mathcal{R}_i = \epsilon\mathcal{R}_i$. Then (1) tells us that

$$\mathcal{E} = \mathcal{T}\mathcal{S} - \mathcal{P}\mathcal{V} + \mathfrak{m}_i\mathcal{R}_i.$$

Defining intensive quantities: density $\rho = \mathcal{E}/\mathcal{V}$, entropy density $s = \mathcal{S}/\mathcal{V}$ and charge density $\mathfrak{r}_i = \mathcal{R}/\mathcal{V}$, we have

$$\begin{aligned} \rho + \mathcal{P} &= s\mathcal{T} + \mathfrak{m}_i\mathfrak{r}_i, \\ \mathrm{d}\rho &= \mathcal{T}\mathrm{d}s + \mathfrak{m}_i\,\mathrm{d}\mathfrak{r}_i, \\ \mathrm{d}\mathcal{P} &= s\,\mathrm{d}\mathcal{T} + \mathfrak{r}_i\,\mathrm{d}\mathfrak{m}_i. \end{aligned} \tag{2}$$ {inttherm:eq}

Note that all intensive thermodynamic quantities can be written as functions of $(1+c)$ variables, which we will usually choose to be the temperature, $\mathcal{T}$ and chemical potentials $\mathfrak{m}_i$. Once we are given the pressure as a function of temperature and chemical potential, we can use (2) to determine the others.

2 Fluid mechanics

sec:stress

2.1 The equations

sec:basiceq

Provided all length scales are large compared to the thermalisation scale of the fluid (which we call l_{mfp}), each patch of the fluid is well described by equilibrium thermodynamics in its rest frame. The fluid is characterised by the velocity of these patches — described by a vector $u^\mu = \gamma(1, \vec{v})$ — and the intensive thermodynamic quantities in their rest frames — which can all be computed from the proper temperature $\mathcal{T}$ and $\mathfrak{m}_i$ using the equation of state and the first law of thermodynamics, as in §1.

The equations of fluid dynamics are simply a statement of the conservation of the stress tensor $T^{\mu\nu}$ and the charge currents J_i^μ.

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= 0\,, \\ \nabla_\mu J_i^\mu &= 0\,. \end{aligned} \tag{3}$$ {Epconsv:eq}

These provide $(d+c)$ equations for the evolution of for the $(d+c)$ quantities $\vec{v}$, $\mathcal{T}$ and μ_i.

2.2 Perfect fluid stress tensor

sec:perfstr

The dynamics of a fluid is completely specified once the stress tensor and charge currents are given as functions of $\mathcal{T}$, μ_i and u^μ. As we have explained in the introduction, fluid mechanics is an effective description at long distances (i.e, it is valid only when the fluid variables vary on distance scales that are large compared to the mean free path l_{mfp}). As a consequence it is natural to expand the stress tensor and charge current in powers of derivatives. In this subsection we briefly review the leading (i.e. zeroth) order terms in this expansion.

It is convenient to define a projection tensor

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{4}$$ {proj:eq}

$P^{\mu\nu}$ projects vectors onto the $(d-1)$ dimensional submanifold orthogonal to u^μ. In other words, $P^{\mu\nu}$ may be thought of as a projector onto spatial coordinates in the rest frame of the fluid. In a similar fashion, $-u^\mu u^\nu$ projects vectors onto the time direction in the fluid rest frame.

To zeroth order in the derivative expansion, Lorentz invariance and the correct static limit uniquely determine the stress tensor, charge and the entropy currents in terms of the thermodynamic variables. We have

$$\begin{aligned}
T^{\mu\nu}_{\text{perfect}} &= \rho u^\mu u^\nu + \mathcal{P} P^{\mu\nu}, \\
(J^\mu_i)_{\text{perfect}} &= \mathfrak{r}_i u^\mu, \\
(J^\mu_S)_{\text{perfect}} &= s u^\mu,
\end{aligned} \tag{5}$$

{currents:eq}

where all thermodynamic quantities are measured in the local rest frame of the fluid, so that they are Lorentz scalars. It is not difficult to verify that in this zero-derivative (or perfect fluid) approximation, the entropy current is conserved. Entropy production (associated with dissipation) occurs only at the first subleading order in the derivative expansion, as we will see in the next subsection.

2.3 Dissipation and diffusion

sec:visc

Now, we proceed to examine the first subleading order in the derivative expansion. In the first subleading order, Lorentz invariance and the physical requirement that entropy be non-decreasing determine the form of the stress tensor and the current to be (see, for example, §§14.1 of [1])

$$\begin{aligned}
T^{\mu\nu}_{\text{dissipative}} &= -\zeta\vartheta P^{\mu\nu} - 2\eta\sigma^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu, \\
(J^\mu_i)_{\text{dissipative}} &= j^\mu_i, \\
(J^\mu_S)_{\text{dissipative}} &= \frac{q^\mu - \mathfrak{m}_i j^\mu_i}{\mathcal{T}} \, .
\end{aligned} \tag{6}$$

{extraTvisc:eq}

where

$$\begin{aligned}
a^\mu &= u^\nu \nabla_\nu u^\mu, \\
\vartheta &= \nabla_\mu u^\mu, \\
\sigma^{\mu\nu} &= \frac{1}{2}\left(P^{\mu\lambda}\nabla_\lambda u^\nu + P^{\nu\lambda}\nabla_\lambda u^\mu\right) - \frac{1}{d-1}\vartheta P^{\mu\nu}, \\
\omega^{\mu\nu} &= \frac{1}{2}\left(P^{\mu\lambda}\nabla_\lambda u^\nu - P^{\nu\lambda}\nabla_\lambda u^\mu\right) \\
q^\mu &= -\kappa P^{\mu\nu}(\partial_\nu \mathcal{T} + a_\nu \mathcal{T}) \, , \\
j^\mu_i &= -D_{ij} P^{\mu\nu} \partial_\nu\left(\frac{\mathfrak{m}_j}{\mathcal{T}}\right) ,
\end{aligned} \tag{7}$$

{fluidtensors:eq}

are the acceleration, expansion, shear tensor, vorticity heat flux and diffusion current respectively. We can write

$$\nabla_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{\vartheta}{d-1} P_{\mu\nu} - u_\mu a_\nu. \quad (8) \quad \text{\{velder:eq\}}$$

These equations define a set of new fluid dynamical parameters in addition to those of the previous subsection: ζ is the bulk viscosity, η is the shear viscosity, κ is the thermal conductivity and D_{ij} are the diffusion coefficients. Fourier's law of heat conduction $\vec{q} = -\kappa \vec{\nabla} \mathcal{T}$ has been relativistically modified to

$$q^\mu = -\kappa P^{\mu\nu} (\partial_\nu \mathcal{T} + a_\nu \mathcal{T}) \,, \quad (9) \quad \text{\{heatcond:eq\}}$$

with an extra term that is related to the redshifting of the temperature. The diffusive contribution to the charge current is the relativistic generalisation of Fick's law.

At this order in the derivative expansion, the entropy current is no longer conserved; however, it may be checked [1] that

$$\mathcal{T} \nabla_\mu J_S^\mu = \frac{q^\mu q_\mu}{\kappa \mathcal{T}} + \mathcal{T} (D^{-1})^{ij} j_i^\mu j_{j\mu} + \zeta \theta^2 + 2\eta \sigma_{\mu\nu} \sigma^{\mu\nu}. \quad (10) \quad \text{\{increase:eq\}}$$

As q^μ, j_i^μ and $\sigma^{\mu\nu}$ are all spacelike vectors and tensors, the RHS of (10) is positive provided η, ζ, κ and D are positive parameters, a condition we further assume. This establishes that (even locally) entropy can only be produced but never destroyed. In equilibrium, $\nabla_\mu J_S^\mu$ must vanish. It follows that, q^μ, j_i^μ, θ and $\sigma^{\mu\nu}$ each individually vanish in equilibrium.

For fluids with gravity duals, the shear viscosity is given by $\eta = \frac{s}{4\pi}$ [2]. We can estimate the thermalisation length of the fluid by comparing coefficients at different orders in the derivative expansion

$$l_{\text{mfp}} \sim \frac{\eta}{\rho} = \frac{s}{4\pi\rho}. \quad (11) \quad \text{\{mfp:eq\}}$$

This length scale may plausibly be identified with the thermalisation length scale of the fluid. This may be demonstrated within the kinetic theory, where l_{mfp} is simply the mean free path of colliding molecules, but is expected to apply to more generally to any fluid with short range interactions.

3 Surfaces

sec:surface

The plasma ball configurations we consider have a domain wall separating a bubble of the deconfined phase from the confined phase. As the density, pressure, etc. of the deconfined phase are a factor of N^2 larger than the confined phase, we can treat the confined phase as the vacuum and the domain wall as a surface bounding the deconfined fluid.

At surfaces, the density of the fluid changes too rapidly to be described by fluid mechanics. However, provided that we look at length scales much larger than the thickness of the surface, we can replace this region by a delta function localised piece of the stress tensor.

At these length scales, this stress tensor will depend on the direction of the surface, with dependance on its curvature being suppressed.

3.1 Thermodynamics

sec:surftherm

The surface energy, area, entropy and charge will satisfy

$$\mathrm{d}\mathcal{E}_{\mathcal{A}} = \mathcal{T}\,\mathrm{d}\mathcal{S}_{\mathcal{A}} + \sigma\,\mathrm{d}\mathcal{A} + \mathfrak{m}_i\,\mathrm{d}\mathcal{R}_{i,\mathcal{A}}\,. \tag{12}$$ {firstlawsurf:eq}

where σ is the surface tension and $\mathcal{A}$ is the surface area. Suppose we rescale the area by a factor $(1+\epsilon)$. Extensivity tells us that $\mathrm{d}\mathcal{E}_{\mathcal{A}} = \epsilon\mathcal{E}_{\mathcal{A}}$, $\mathrm{d}\mathcal{S}_{\mathcal{A}} = \epsilon\mathcal{S}_{\mathcal{A}}$, $\mathrm{d}\mathcal{A} = \epsilon\mathcal{A}$ and $\mathrm{d}\mathcal{R}_{i,\mathcal{A}}\epsilon\mathcal{R}_{i,\mathcal{A}}$. Then (12) tells us that

$$\mathcal{E}_{\mathcal{A}} = \mathcal{T}\mathcal{S}_{\mathcal{A}} + \sigma\mathcal{A} + \mathfrak{m}_i\mathcal{R}_{i,\mathcal{A}}\,.$$

Defining the intensive quantities: surface energy density $\sigma_E = \mathcal{E}_{\mathcal{A}}/\mathcal{A}$, surface entropy density $\sigma_S = \mathcal{S}_{\mathcal{A}}/\mathcal{A}$ and surface R-charge densities $\sigma_{R_i} = \mathcal{R}_{i,\mathcal{A}}/\mathcal{A}$, considerations similar to those leading to (2) lead to

$$\begin{aligned} \sigma_E &= \sigma + \mathcal{T}\sigma_S + \mathfrak{m}_i\,\sigma_{R_i}, \\ \mathrm{d}\sigma &= -\sigma_S\,\mathrm{d}\mathcal{T} - \sigma_{R_i}\mathrm{d}\mathfrak{m}_i. \end{aligned} \tag{13}$$ {surftherm:eq}

Note that using a temperature and chemical potential independent surface tension is equivalent to setting $\sigma_S = \sigma_{R_i} = 0$ and $\sigma_E = \sigma$.

From the form of the gravity solution, we would expect the thickness of the surface to be approximately

$$\xi \sim \frac{\sigma_E}{\rho} \sim \frac{\sigma_S}{s} \sim \frac{\sigma}{\mathcal{P}} \sim \frac{\sigma_{R_i}}{\mathfrak{r}_i}\,. \tag{14}$$ {thick:eq}

In general, it will be of order N^0 and is similar to l_{mfp} (if 8π can be considered similar to 1).

For the domain wall of [3], the thickness and surface tension are $6 \times \frac{1}{2\pi\mathcal{T}_c}$ and $\sigma = 2.0 \times \frac{\pi^2 N^2 \mathcal{T}_c{}^2}{2}$ respectively. If we use the estimate $\xi = \frac{\sigma}{\rho_c} = \frac{2.0}{\mathcal{T}_c}$, which is pretty close to the thickness.

sec:surfstr

3.2 Stress tensor and currents

Let's describe the location of the surface by a function $f(x)$ that is positive inside the fluid and has a first order zero on the surface.

$$T^{\mu\nu} = \theta(f) T^{\mu\nu}_{\mathrm{fluid}} + \delta(f) T^{\mu\nu}_{\mathrm{surface}}. \tag{15}$$ {fluidsurf:eq}

At large length scales, as mentioned above, $T^{\mu\nu}_{\mathrm{surface}}$ will only depend on the first derivative of f and no higher derivatives.

If we demand invariance under reparameterisations of the function $f(x) \to g(x)f(x)$, where $g(x) > 0$, and that the surface moves at the velocity of the fluid

$$u^\mu \partial_\mu f \delta(f) = 0, \tag{16}$$ {surfvel:eq}

the most general surface stress tensor we can have is (see §2.3 of [4])

$$T^{\mu\nu}_{\mathrm{surface}} \delta(f) = \left[A n^\mu n^\nu + B u^\mu u^\nu + C \left(u^\mu n^\nu + n^\mu u^\nu \right) + D g^{\mu\nu} \right] \sqrt{\partial f \cdot \partial f} \delta(f) \,.$$

where $n_\mu = -\partial_\mu f / \sqrt{\partial f \cdot \partial f}$ is the outward pointing unit normal to the surface.

We can fix A, B, C, D by looking at a fluid at rest, $u^\mu = (1, 0, 0, \ldots)$, with a surface $f(x) = x$

$$T^{\mu\nu}_{\mathrm{surface}} = \begin{pmatrix} B - D & -C & 0 \\ -C & A + D & 0 \\ 0 & 0 & D \end{pmatrix} \delta(x) = \begin{pmatrix} \sigma_E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sigma \end{pmatrix} \delta(x).$$

This gives

$$T^{\mu\nu}_{\mathrm{surface}} = \sqrt{\partial f \cdot \partial f} \left[\sigma_E u^\mu u^\nu - \sigma (h^{\mu\nu} + u^\mu u^\nu) \right], \tag{17}$$ {surfstressgen:eq]

where $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the induced metric of the surface.

Similar reasoning leads to

$$\begin{aligned}
J_i^\mu &= \theta(f) J^\mu_{i,\text{fluid}} + \delta(f) J^\mu_{i,\text{ surface}}\,, \\
J^\mu_{i,\text{ surface}} &= \sqrt{\partial f\!\cdot\!\partial f}\,\sigma_{R_i} u^\mu. \\
J_S^\mu &= \theta(f) J^\mu_{S,\text{fluid}} + \delta(f) J^\mu_{S,\text{ surface}}\,, \\
J^\mu_{S,\text{ surface}} &= \sqrt{\partial f\!\cdot\!\partial f}\,\sigma_S u^\mu.
\end{aligned} \tag{18}$$

{surfcurr:eq}

The factor of $\sqrt{\partial f\!\cdot\!\partial f}$ also has a simple interpretation: suppose we use a coordinate system where f is one of the coordinates. Then

$$\sqrt{\partial f\!\cdot\!\partial f} = \sqrt{g^{ff}} = \sqrt{\frac{\det h}{\det g}}\,, \tag{19}$$

{surfmeasure:eq}

which provides the correct change of integration measure for localisation to the surface. If we used some other coordinates, there'd be an extra Jacobian factor.

3.3 Equations of motion

sec:surfeom

We have

$$\nabla_\mu T^{\mu\nu} = \theta(f)\nabla_\mu T^{\mu\nu}_{\text{fluid}} + \delta(f)(\partial_\mu f) T^{\mu\nu}_{\text{fluid}} + \delta(f)\nabla_\mu T^{\mu\nu}_{\text{surface}} + \delta'(f)(\partial_\mu f) T^{\mu\nu}_{\text{surface}}. \tag{20}$$

{surfeom:eq}

Taking the derivative of (16) and contracting with $\nabla_\nu f$ gives

$$(\nabla^\nu u^\mu)(\nabla_\mu f)(\nabla_\nu f)\delta(f) + u^\mu(\nabla^\nu f)(\nabla_\nu\nabla_\mu f)\delta(f) + u^\mu(\nabla_\mu f)(\partial f\!\cdot\!\partial f)\delta'(f) = 0.$$

This can be used to eliminate the last term of (20)

$$\begin{aligned}
&\nabla_\mu T^{\mu\nu} = \theta(f)\nabla_\mu T^{\mu\nu}_{\text{fluid}} + \\
&\delta(f)\sqrt{\partial f\!\cdot\!\partial f}\left[-n_\mu T^{\mu\nu}_{\text{fluid}} + \frac{\nabla_\mu T^{\mu\nu}_{\text{surface}}}{\sqrt{\partial f\!\cdot\!\partial f}} + (\sigma_E - \sigma)u^\nu\left(\frac{u^\mu n^\lambda \nabla_\mu\nabla_\lambda f}{\sqrt{\partial f\!\cdot\!\partial f}} - n_\mu n_\lambda \nabla^\mu u^\lambda\right)\right].
\end{aligned} \tag{21}$$

{surfeom2:

So, in addition to the equation of motion (3), we also have the boundary conditions coming from the term in square brackets [1]

$$-\mathcal{P} n^\mu + u^\mu u^\nu \partial_\nu(\sigma_E - \sigma) + (\sigma_E - \sigma)\left(\frac{d-2}{d-1}\vartheta u^\mu + a^\mu - u^\mu n_\lambda n_\nu \sigma^{\lambda\nu}\right) - h^{\mu\nu}\partial_\nu\sigma + \sigma\Theta n^\mu\Big|_{f=0} = 0 \tag{22}$$

{surfbc:eq}

[1] As we are only keeping terms to leading order in the derivative expansion for the surface, we will do the same for the fluid here.

where Θ is the trace of the extrinsic curvature of the surface, as seen from outside the fluid (see §A).

Similar reasoning leads to

$$\nabla_\mu J_i^\mu = \theta(f)\nabla_\mu J_{i,\text{fluid}}^\mu + \delta(f)\sqrt{\partial f\cdot\partial f}\left(\frac{d-2}{d-1}\sigma_{R_i}\vartheta + u^\mu\partial_\mu\sigma_{R_i} - \sigma_{R_i}n_\mu n_\lambda\sigma^{\mu\nu}\right), \tag{23}$$

{surfchbc:eq}

which leads to an additional boundary condition to go with (3).

Also

$$\nabla_\mu J_S^\mu = \theta(f)\nabla_\mu J_{S,\text{fluid}}^\mu,$$

so there is no entropy production at the surface to leading order in the derivative expansion.

4 Rigid rotation

sec:rigidrot

sec:rotint

4.1 Solutions for the interior

We want to find solutions of (3) that are independent of time, which means we need to set (10) to zero. This means we need velocity configurations that have zero expansion and shear. As argued in [5], combined with the equations of fluid mechanics, this implies that the velocity must be proportional to a Killing vector. In general, this would be a combination of a uniform boost and rigid rotation. We can always boost to a frame where the centre of rotation is static and the rotation lies in the Cartan directions of the rotation group. This gives

$$u = \gamma(\partial_t + \Omega_a l_a), \tag{24}$$

{rigidrot:eq}

where Ω_a are the angular velocities and l_a are a set of commuting rotational Killing vectors. The important feature is that the velocity is a normalisation factor times a Killing vector:

$$u^\mu = \gamma K^\mu, \qquad \gamma^2 K^\mu K_\mu = -1, \qquad \nabla_{(\mu}K_{\nu)} = 0. \tag{25}$$

{eqvel:eq}

One can deduce that

$$\theta = \sigma^{\mu\nu} = 0, \qquad u^\mu\partial_\mu\gamma = 0, \qquad a_\mu = -\frac{\partial_\mu\gamma}{\gamma}. \tag{26}$$

{rotvelder:eq}

Which leads to

$$q^\mu = -\kappa\gamma P^{\mu\nu}\partial_\nu\left[\frac{\mathcal{T}}{\gamma}\right], \qquad j_i^\mu = -D_{ij}P^{\mu\nu}\partial_\nu\left[\frac{\mathfrak{m}_j}{\mathcal{T}}\right].$$

One can also show that

$$\begin{aligned}\nabla_\mu T^{\mu\nu}_{\text{perfect}} =&\gamma \left(sP^{\nu\mu} + \left\{ \mathcal{T}\left(\frac{\partial s}{\partial \mathcal{T}}\right) + \mathfrak{m}_i \left(\frac{\partial \mathfrak{r}_i}{\partial \mathcal{T}}\right)\right\} u^\nu u^\mu \right) \partial_\mu \left[\frac{\mathcal{T}}{\gamma}\right] \\ &+ \gamma \left(\mathfrak{r}_i P^{\nu\mu} + \left\{ \mathcal{T}\left(\frac{\partial s}{\partial \mathfrak{m}_i}\right) + \mathfrak{m}_j \left(\frac{\partial \mathfrak{r}_j}{\partial \mathfrak{m}_i}\right)\right\} u^\nu u^\mu \right) \partial_\mu \left[\frac{\mathfrak{m}_i}{\gamma}\right], \\ \nabla_\mu J^\mu_{i,\text{perfect}} =&\gamma \left(\frac{\partial \mathfrak{r}_i}{\partial \mathcal{T}}\right) u^\mu \partial_\mu \left[\frac{\mathcal{T}}{\gamma}\right] + \gamma \left(\frac{\partial \mathfrak{r}_i}{\partial \mathfrak{m}_j}\right) u^\mu \partial_\mu \left[\frac{\mathfrak{m}_j}{\gamma}\right]\end{aligned}$$

So the velocity configuration (24) will be an equilibrium solution to the equations of motion provided that

$$\frac{\mathcal{T}}{\gamma} = T = \text{constant}, \qquad \frac{\mathfrak{m}_i}{\gamma} = \mu_i = \text{constant}, \qquad \frac{\mathfrak{m}_i}{\mathcal{T}} = \nu_i = \frac{\mu_i}{T} = \text{constant}. \tag{27}$$ {rotsol:eq}

Using the equation of state and (2), this determines all of the intensive thermodynamic quantities in the fluid.

4.2 Solutions for surfaces

sec:rotsurf

The fluid configurations described in the previous subsection have $\sigma^{\mu\nu} = \vartheta = u^\mu\partial_\mu\gamma = 0$. In addition, the quantities σ, σ_E and σ_{R_i} are functions of $\mathcal{T}$ and $\mathfrak{m}_i$, which in turn are proportional to γ. Therefore, using (13) and (26),

$$u^\mu\partial_\mu\sigma = u^\mu\partial_\mu\sigma_E = u^\mu\partial_\mu\sigma_{R_i} = 0\,, \qquad \partial_\mu\sigma = (\sigma_E - \sigma)a_\mu\,. \tag{28}$$ {rotsigmagrad:eq}

This means that (22) and (23) reduce to

$$\mathcal{P}|_{f=0} = \sigma\Theta + (\sigma_E - \sigma) n\cdot a = \nabla\cdot(n\sigma), \tag{29}$$ {rotsurfbc:eq}

where, for tha last expression, we defined n off the surface such that $n\cdot\nabla n = 0$, as in §A. As the pressure is determined by (27), this provides a differential equation that determines allowed positions of surfaces. Demanding that the surface has no conical singularities turns out to provide enough boundary conditions to determine the position of the surface completely (up to discrete choices) in terms of the parameters Ω_a, T and μ_i.

4.3 Thermodynamics of solutions

sec:rottherm

In this section. we will use the symbol d to indicate the change in a quantity due to a small change in the thermodynamic state of the system, i.e. it is the exterior derivative on the space of thermodynamic states and not the space-time exterior derivative. The only exception will be the integration measures over the region occupied by the fluid, Γ, and the surface, $\partial\Gamma$:

$$\int_\Gamma \mathrm{d}^{d-1}x \, [\ldots] \,, \qquad \int_{\partial\Gamma} \mathrm{d}^{d-2}x \, [\ldots] \,.$$

We compute the extensive thermodynamic properties of these solutions by integrating the time components of the corresponding currents (noting that the current associated with a Killing vector ζ^μ is $J^\mu_\zeta = T^{\mu\nu}\zeta_\nu$):

$$Q_X = \int_\Gamma \mathrm{d}^{d-1}x J^0_X, \tag{30}$$

{noetherch:eq}

In particular, also noting that for equilibrium configurations $\partial^0 f = 0$,

$$Q_\zeta = \int_\Gamma \mathrm{d}^{d-1}x \left[(\rho+\mathcal{P})\gamma^2 K^0 K\cdot\zeta + \mathcal{P}\zeta^0\right] + \int_{\partial\Gamma} \mathrm{d}^{d-2}x \left[(\sigma_E-\sigma)\gamma^2 K^0 K\cdot\zeta - \sigma\zeta^0\right] . \tag{31}$$

{killingcharge:eq}

Noting that $K^0 = (\partial_t)^0 = 1$ and $l^0_a = 0$, this gives

$$\begin{aligned}
E = -Q_{\partial_t} &= -\int_\Gamma \mathrm{d}^{d-1}x \left[(\rho+\mathcal{P})\gamma^2 K\cdot\partial_t + \mathcal{P}\right] - \int_{\partial\Gamma} \mathrm{d}^{d-2}x \left[(\sigma_E-\sigma)\gamma^2 K\cdot\partial_t - \sigma\right] , \\
L_a = Q_{l_a} &= \int_\Gamma \mathrm{d}^{d-1}x \left[(\rho+\mathcal{P})\gamma^2 K\cdot l_a\right] + \int_{\partial\Gamma} \mathrm{d}^{d-2}x \left[(\sigma_E-\sigma)\gamma^2 K\cdot l_a\right] , \\
S = Q_S &= \int_\Gamma \mathrm{d}^{d-1}x \, [\gamma s] + \int_{\partial\Gamma} \mathrm{d}^{d-2}x \, [\gamma\sigma_S] \,, \\
R_i = Q_{R_i} &= \int_\Gamma \mathrm{d}^{d-1}x \, [\gamma\mathfrak{r}_i] + \int_{\partial\Gamma} \mathrm{d}^{d-2}x \, [\gamma\sigma_{R_i}] \,.
\end{aligned} \tag{32}$$

{thermcharge:eq}

From these quantities, we can compute overall angular velocities Ω_a, temperature T and chemical potentials μ_i thermodynamically

$$\mathrm{d}E = \Omega_a \, \mathrm{d}L_a + T \, \mathrm{d}S + \mu_i \, \mathrm{d}R_i. \tag{33}$$

{chpotdef:eq}

A priori, it may not seems that these quantities have to be the same as Ω_a, T and μ_i from (24) and (27). However, we can show that they are the same by checking that (33) holds with Ω_a, T and μ_i taken from (24) and (27). In practice, it is easier to verify the equivalent statement

$$\mathrm{d}(E - \Omega_a L_a - TS - \mu_i R_i) = -L_a \, \mathrm{d}\Omega_a - S \, \mathrm{d}T - R_i \, \mathrm{d}\mu_i. \tag{34}$$

{chpotcheck:eq}

First, making use of (2) and (13), we see that

$$\Phi \equiv E - \Omega_a L_a - TS - \mu_i R_i = -Q_K - TQ_S - \mu_i Q_{R_i} = -\int_\Gamma \mathrm{d}^{d-1}x\,\mathcal{P} + \int_{\partial\Gamma} \mathrm{d}^{d-2}x\,\sigma. \tag{35}$$

{thermpot:eq}

Let's split $\mathrm{d}\Phi$ into two pieces, the contribution from the change in $(\mathcal{P},\sigma)$ and the contribution from the change in the region occupied by the fluid:

$$\mathrm{d}\Phi = \mathrm{d}\Phi_{\mathcal{P},\sigma} + \mathrm{d}\Phi_f$$

Consider an infinitesimal change of Ω_a, T and μ_i. We have

$$\begin{aligned}
\mathrm{d}\mathcal{P} &= s\,\mathrm{d}(\gamma T) + r_i\,\mathrm{d}(\gamma\mu_i) = \frac{\rho+\mathcal{P}}{\gamma}\,\mathrm{d}\gamma + \gamma s\,\mathrm{d}T + \gamma\mathfrak{r}_i\,\mathrm{d}\mu_i,\\
\mathrm{d}\sigma &= -\sigma_S\,\mathrm{d}(\gamma T) - \sigma_{R_i}\,\mathrm{d}(\gamma\mu_i) = -\frac{\sigma_E-\sigma}{\gamma}\,\mathrm{d}\gamma - \gamma\sigma_S\,\mathrm{d}T - \gamma\sigma_{R_i}\,\mathrm{d}\mu_i,\\
\gamma^{-3}\,\mathrm{d}\gamma &= K\cdot\mathrm{d}K = K\cdot l_a\,\mathrm{d}\Omega_a.
\end{aligned}$$

From this, we see that

$$\mathrm{d}\Phi_{\mathcal{P},\sigma} = -L_a\,\mathrm{d}\Omega_a - S\,\mathrm{d}T - R_i\,\mathrm{d}\mu_i.$$

This means that we need $\mathrm{d}\Phi_f = 0$ if (34) is to hold.

The first part of $\mathrm{d}\Phi_f = 0$ is

$$-\mathrm{d}\left[\int_\Gamma \mathrm{d}^{d-1}x\,\mathcal{P}\right]_f = -\int_{\Delta\Gamma} \mathrm{d}^{d-1}x\,\mathcal{P},$$

where $\Delta\Gamma$ is the region between the new and the old surfaces (with appropriate signs).

The change in the surface area can be written as

$$\mathrm{d}\left[\int_{\partial\Gamma} \mathrm{d}^{d-2}x\,\sigma\right]_f = -\int_{\partial\Delta\Gamma} \mathrm{d}^{d-2}x\,\sigma\hat{n}\cdot\tilde{n},$$

where $\hat{n}$ is a unit normal vector pointing into the initial fluid and out of the final fluid and $\tilde{n}$ is some vector field that is equal to the outward pointing normal at both the initial and final surfaces. By Gauss' theorem, this can be written as

$$\mathrm{d}\left[\int_{\partial\Gamma} \mathrm{d}^{d-2}x\,\sigma\right]_f = \int_{\Delta\Gamma} \mathrm{d}^{d-1}x, \nabla\cdot(\sigma\tilde{n}),$$

As the region of integration is already infinitesimal, we can replace $\tilde{n}$ with the vector field n described in (50), as the difference would be infinitesimal, i.e. $\nabla\cdot(\sigma\tilde{n}) \to \sigma\Theta + n\cdot\nabla\sigma$. Making use of (28)

$$\mathrm{d}\Phi_f = \int_{\Delta\Gamma} \mathrm{d}^{d-1}x \left[\sigma\Theta + (\sigma_E - \sigma) n\cdot a - \mathcal{P}\right].$$

which vanishes due to (29).

The thermodynamics of the solution can be summarised by defining a grand partition function

$$\mathcal{Z}_{\mathrm{gc}} = \operatorname{Tr}\exp\left(-\frac{E - \Omega_a L_a - \mu_i R_i}{T}\right). \tag{36}$$

{gpf:eq}

In the thermodynamic limit,

$$\begin{aligned} -T\ln\mathcal{Z}_{\mathrm{gc}} = \Phi &= E - \Omega_a L_a - TS - \mu_i R_i, \\ \mathrm{d}(T\ln\mathcal{Z}_{\mathrm{gc}}) &= L_a\,\mathrm{d}\Omega_a + S\,\mathrm{d}T + R_i\,\mathrm{d}\mu_i. \end{aligned} \tag{37}$$

{gpftherm:eq}

We have seen that

$$T\ln\mathcal{Z}_{\mathrm{gc}} = \int_{f>0} \mathrm{d}V\,\mathcal{P} - \int_{f=0} \mathrm{d}A\,\sigma \tag{38}$$

{gpfrot:eq}

and the Ω_a, T and μ_i are the same as those given by (24) and (27).

5 Conformal fluids

sec:Conformal

Determining the velocity above depended on the following arguments:

- From (10) we see that we require $\sigma_{\mu\nu} = \vartheta = 0$.
- From [5] we see that $\sigma_{\mu\nu} = \vartheta = 0$ combined with $\nabla_\mu T^{\mu\nu}_{\mathrm{perfect}} = 0$ imply that a_μ is an exact form: $a = \mathrm{d}\alpha$ for some scalar field α.
- From [5] we see that $\sigma_{\mu\nu} = \vartheta = 0$ and $a = \mathrm{d}\alpha$ for some α imply that $\mathrm{e}^\alpha u$ is a Killing vector: $\nabla_{(\mu}\left(\mathrm{e}^\alpha u_{\nu)}\right) = \mathrm{e}^\alpha(\partial_{(\mu}\alpha - a_\mu)u_{\nu)} = 0$.

However, for a conformal fluid the bulk viscosity $\zeta = 0$. Therefore (10) does not imply that $\vartheta = 0$.

Instead we can use the following series of arguments:

- From (10) we see that we require $\sigma_{\mu\nu} = 0$.

- We will see that $\sigma_{\mu\nu} = 0$ combined with $\nabla_\mu T^{\mu\nu}_{\text{perfect}} = 0$ and a conformal equation of state, $\rho = (d-1)\mathcal{P}$, imply that Loganayagam's Weyl connection from [6], $A_\mu = a_\mu - \frac{\vartheta}{d-1} u_\mu$, is an exact form: $A = d\alpha$ for some scalar field α.

- We will see that $\sigma_{\mu\nu} = 0$ and $A = \mathrm{d}\alpha$ for some α imply that $\mathrm{e}^\alpha u$ is a *conformal* Killing vector. $\nabla_{(\mu}\left(\mathrm{e}^\alpha u_{\nu)}\right) - \frac{g_{\mu\nu}}{d}\nabla_\lambda\left(\mathrm{e}^\alpha u^\lambda\right) = 0$.

Then performing a conformal transformation to bring this vector into the Cartan subalgebra of the conformal group leaves us with the same space of possible fluid velocities studied above.

To prove the second and third propositions, begin by noting that $\sigma_{\mu\nu} = 0$ means that (8) becomes

$$\nabla_\mu u_\mu = \omega_{\mu\nu} + \frac{P_{\mu\nu}}{d-1}\vartheta - u_\mu a_\nu. \tag{39}$$ {eq:confvelder}

To prove the second proposition, we now use the conformal equation of state to write $T^{\mu\nu}_{\text{perfect}} = \mathcal{P}\left(d\, u^\mu u^\nu + g^{\mu\nu}\right)$. Taking the divergence and using (39) we find

$$\nabla_\mu T^{\mu\nu}_{\text{perfect}} = \left(d\, u^\nu u^\mu + g^{\nu\mu}\right)\partial_\mu \mathcal{P} + d\,\mathcal{P}\left(\vartheta u^\nu + a^\nu\right) = 0. \tag{40}$$ {eq:confdiv}

If we project (40) onto the velocity, we find

$$\vartheta = -\frac{(d-1)u^\mu \partial_\mu \mathcal{P}}{d\,\mathcal{P}},$$

and if we project (40) onto the rest frame, we find

$$a^\mu = -\frac{P^{\mu\nu}\partial_\nu \mathcal{P}}{d\,\mathcal{P}},$$

and therefore

$$A_\mu = a_\mu - \frac{\vartheta}{d-1}u_\mu = \left(\frac{u_\mu u^\nu - P_\mu{}^\nu}{d\,\mathcal{P}}\right)\partial_\nu \mathcal{P} = \partial_\mu\left(-\frac{\log \mathcal{P}}{d}\right). \tag{41}$$ {eq:confweyl}

This is the second proposition with $\alpha = -\frac{\log \mathcal{P}}{d}$.

To prove the third proposition, we can use (40) to expand the following

$$\begin{aligned}\nabla_{(\mu}\left(\mathrm{e}^{\alpha}u_{\nu)}\right) - \frac{g_{\mu\nu}}{d}\nabla_{\lambda}\left(\mathrm{e}^{\alpha}u^{\lambda}\right) &= \mathrm{e}^{\alpha}\left(\partial_{(\mu}\alpha - A_{\mu}\right)u_{\nu)} - \frac{\mathrm{e}^{\alpha}g_{\mu\nu}}{d}\left(u^{\lambda}\partial_{\lambda}\alpha - \tfrac{\vartheta}{d-1}\right)\\ &= \mathrm{e}^{\alpha}\left(u_{(\mu}\delta^{\lambda}_{\nu)} - \frac{g_{\mu\nu}}{d}u^{\lambda}\right)\left(\partial_{\lambda}\alpha - A_{\lambda}\right).\end{aligned} \tag{42}$$

{eq:confckv}

Therefore, if we choose α to be the scalar field that appears in (41), $\mathrm{e}^{\alpha}u$ must be a conformal Killing vector.

6 Determining surface tension from gravity duals

sec:fromgravity

In this section we will describe how we could determine the surface tension of a fluid with a gravity dual from finite-sized plasmaball solutions, if we had them, similar to how we determine the equation of state of the fluid from black brane solutions.

From (13), we see that we only need to find the function $\sigma(\mathcal{T},\mathfrak{m}_i)$. We can then determine everything else by differentiating this function. Suppose we had gravitational solutions for static, spherical plasmaballs. Then we would know how the radius of the plasmaball depends on the temperature and chemical potential (maybe only at $\mathfrak{m}_i = 0$ for now), $r(\mathcal{T},\mathfrak{m}_i)$, with negative values corresponding to inverted plasmaballs (plasmaholes?). For these static, spherical plasmaballs, the boundary condition (29) reduces to

$$\mathcal{P} = \frac{(d-2)\sigma}{r}. \tag{43}$$

{eq:staticbc}

Since we already know the functions $\mathcal{P}(\mathcal{T},\mathfrak{m}_i)$ and $r(\mathcal{T},\mathfrak{m}_i)$, this determines $\sigma(\mathcal{T},\mathfrak{m}_i)$.

This would be most useful if we could handwave our way to the general functional form, with a few constants to be determined from gravity, much like we do for the equation of state for the AdS-soliton plasma. We can determine two of these constants from what we already know.

At the phase transition, the pressure vanishes (the free energy of the two phases are equal, but the confined phase is $\mathcal{O}(N^{-2})$ compared to the deconfined phase). From (2), near the transition temperature, $\mathcal{T}_{\mathrm{c}}$,

$$\mathcal{P} \sim s_{\mathrm{c}}(\mathcal{T}-\mathcal{T}_{\mathrm{c}}) = \frac{\rho_{\mathrm{c}}}{\mathcal{T}_{\mathrm{c}}}(\mathcal{T}-\mathcal{T}_{\mathrm{c}}). \tag{44}$$

{eq:neartc}

Then we find that the radius behaves as

$$r \sim \left(\frac{\sigma_c \mathcal{T}_c}{\rho_c}\right) \frac{d-2}{\mathcal{T} - \mathcal{T}_c}, \tag{45}$$

{eqradiustc}

where the combination $\frac{\sigma_c \mathcal{T}_c}{\rho_c}$ is precisely what was determined in [3]. Thus we know both the location of a pole in the radius function and its residue.

Appendices

A Extrinsic curvature

sec:extrinsic

Suppose we have a timelike surface with unit normal vector n pointing toward us (spacelike surfaces will require some sign differences). The induced metric on the surface is

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu. \tag{46}$$

{indmet:eq}

The extrinsic curvature is given by [7]

$$\Theta_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu} = \nabla_\mu n_\nu. \tag{47}$$

{extrdef:eq}

We have to be a little careful with the last expression. It agrees with the first expression when projected tangent to the surface. The first expression has vanishing components normal to the surface. The normal components of the second expression depend on how we extend n off the surface.

The conventional choice for extending n is as follows: at each point on the surface, construct the geodesic that passes through that point tangent to n and parallel transport n along it. In other words

$$n^\mu \nabla_\mu n^\nu = 0. \tag{48}$$

{geodesic:eq}

This ensures that the second expression in (47) has vanishing components normal to the surface. The other normal component, $n^\nu \nabla_\mu n_\nu$, vanishes due to the normalisation of n.

For the surfaces given by $f(x) = 0$, considered in §3, the unit normal on the surface is given by

$$n_\mu = -\frac{\partial_\mu f}{\sqrt{\partial f \cdot \partial f}}. \tag{49}$$

{normonsurf:eq}

However, if we used this vector away from the surface, it would not satisfy (48). We could still use either expression in (47) with this vector — we would just have to project the second one tangent to the surface. Alternatively, we can use

$$n_\mu = -\frac{\partial_\mu f}{(\partial f \cdot \partial f)^{1/2}} + \left[\frac{\partial^\nu f\, \nabla_\nu \partial_\mu f}{(\partial f \cdot \partial f)^{3/2}} - \frac{\partial_\mu f\, \partial^\lambda f\, \partial^\nu f\, \nabla_\lambda \partial_\nu f}{(\partial f \cdot \partial f)^{5/2}}\right] f + \mathcal{O}(f^2). \quad (50)$$

{normoffsurf:eq}

The $\mathcal{O}(f^2)$ terms don't contribute to (47) or (48) on the surface. The contribution of the $\mathcal{O}(f)$ terms on the surface to (47) are normal to the surface and ensure that n satisfies (48).

Either way, on the surface, we get

$$\Theta_{\mu\nu} = -\frac{\nabla_\mu \partial_\nu f}{(\partial f \cdot \partial f)^{1/2}} + \frac{\partial_\mu f\, \partial^\lambda f\, \nabla_\lambda \partial_\nu f + \partial_\nu f\, \partial^\lambda f\, \nabla_\lambda \partial_\mu f}{(\partial f \cdot \partial f)^{3/2}} - \frac{\partial_\mu f\, \partial_\nu f\, \partial^\lambda f\, \partial^\sigma f\, \nabla_\lambda \partial_\sigma f}{(\partial f \cdot \partial f)^{5/2}}. \quad (51)$$

{extrsurf:eq}

As this is perpendicular to n, it doesn't matter if we contract its indices with the full metric $g_{\mu\nu}$ or the induced metric $h_{\mu\nu}$. We get

$$\Theta = \Theta^\mu_\mu = -\frac{\Box f}{(\partial f \cdot \partial f)^{1/2}} + \frac{\partial^\mu f\, \partial^\nu f\, \nabla_\mu \partial_\nu f}{(\partial f \cdot \partial f)^{3/2}}. \quad (52)$$

{trextrsurf:eq}

B Notation

app:notation

We work in the $(-+++)$ signature. μ, ν denote space-time indices, $i, j = 1 \ldots c$ label the c different R-charges and $a, b = 1 \ldots n$ label the n different angular momenta. The dimensions of the AdS space is denoted by D whereas the spacetime dimensions of its boundary is denoted by $d = D - 1$. In this paper we consider fluids on $S^{D-2} \times \mathbb{R}$ or equivalently $S^{d-1} \times \mathbb{R}$. Here we present some relations which are useful in converting between n, D and d:

$$\begin{aligned} D &= d + 1 = 2n + 2 - (D \mod 2) \\ d &= D - 1 = 2n + (d \mod 2) \\ n &= \left[\frac{D-1}{2}\right] = \left[\frac{d}{2}\right] \end{aligned}$$

where $[x]$ represents the integer part of a real number x.

A summary of the variables used in this paper appears in table 1.

Symbol	Definition	Symbol	Definition
D	Dimension of bulk	d	$D-1$, Dimension of boundary
G_D	Newton Constant in AdS_D	n	$[d/2]$, no. of commuting angular momenta
V_d	Volume of S^{d-1}, $\frac{2\pi^{d/2}}{\Gamma(d/2)}$		
R_{AdS}	AdS radius (taken to be unity)	c	no. of commuting R-charges
R_H, r_+	Horizon radius	l_{mfp}	Mean free path, η/ρ
$\mathcal{E}$	Fluid energy	ρ	Proper density
$\mathcal{S}$	Fluid entropy	s	Proper entropy density
$\mathcal{T}$	Fluid temperature	$\mathcal{P}$	Pressure
$\mathcal{R}_i$	Fluid R-charge	$\mathfrak{r}_i$	Proper R-charge density
$\mathfrak{m}_i$	Fluid chemical potential	ν_i	$\mathfrak{m}_i/\mathcal{T}$
$\mathcal{V}$	Volume	$\mathcal{A}$	Area
$T^{\mu\nu}$	Stress tensor	J_S^μ	Entropy current
J_i^μ	R-charge current	u^μ	$\mathrm{d}x^\mu/\mathrm{d}\tau = \gamma(1,\vec{v})$, fluid velocity
Ω_a	Angular velocities	γ	$(1-v^2)^{-1/2}$
v^2	$\sum_a g_{\phi_a\phi_a}\Omega_a^2$	$P^{\mu\nu}$	Projection tensor, $g^{\mu\nu}+u^\mu u^\nu$
$a^\mu, \vartheta, \sigma^{\mu\nu}$	see (7)	ζ, η	Bulk, shear viscosity
q^μ	Heat flux, see (9)	κ	Thermal conductivity
j_i^μ	Diffusion current, see (7)	D_{ij}	Diffusion coefficients
E	Total energy (32)	S	Total entropy (32)
L_a	Angular momenta (32)	R_i	Total R-charges (32)
Ω_a	Angular velocities (33)	T	Overall temperature (33)
μ_i	Overall chemical potentials (33)	$\mathcal{Z}_{\mathrm{gc}}$	Partition function (36)
Φ	Grand thermodynamic potential (35)	Γ	region occupied by fluid
$\Theta^{\mu\nu}$	Extrinsic curvature (47)	Θ	Θ^μ_μ
n^μ	Unit normal to surface	$h^{\mu\nu}$	Induced metric of surface
σ	Surface tension	$f(x)$	Surface at $f(x)=0$
σ_E	Surface energy density	σ_S	Surface entropy density
σ_{R_i}	Surface charge density		

tab:notation

Table 1: Summary of variables used. Numbers in parentheses refer to the equation where it is defined

References

Andersson:2006nr [1] N. Andersson and G. L. Comer, "Relativistic fluid dynamics: Physics for many different scales," `arXiv:gr-qc/0605010`.

Son:2007vk [2] D. T. Son and A. O. Starinets, "Viscosity, Black Holes, and Quantum Field Theory," *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 95–118, `arXiv:0704.0240 [hep-th]`.

Aharony:2005bm [3] O. Aharony, S. Minwalla, and T. Wiseman, "Plasma-balls in large N gauge theories and localized black holes," *Class. Quant. Grav.* **23** (2006) 2171–2210, `arXiv:hep-th/0507219`.

Lahiri:2007ae [4] S. Lahiri and S. Minwalla, "Plasmarings as dual black rings," *JHEP* **05** (2008) 001, `arXiv:0705.3404 [hep-th]`.

Caldarelli:2008mv [5] M. M. Caldarelli, O. J. C. Dias, R. Emparan, and D. Klemm, "Black Holes as Lumps of Fluid," `arXiv:0811.2381 [hep-th]`.

oganayagam:2008is [6] R. Loganayagam, "Entropy Current in Conformal Hydrodynamics," *JHEP* **05** (2008) 087, `arXiv:0801.3701 [hep-th]`.

Wald-GeneRela:84 [7] R. M. Wald, *General Relativity.* The University of Chicago Press, Chicago 60637, 1984.