Extrinsic curvature for higher codimension

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Abstract

We define scalar measures of local extrinsic curvature and scale invariant measures of global extrinsic curvature for submanifolds of $\mathbb{R}^n$.

1   Fundamental forms (abstract)

Let $\mathcal{M}$ be a submanifold of $\mathcal{N}$ and let m_1, m_2 be points in $\mathcal{M}$. When $\mathcal{N} = \mathbb{R}^n$, the Grassmannian "distance" between these two points, $D_{III}(m_1, m_2)$, is (the sine of?) the largest principal angle between the tangent planes $T_{m_1}\mathcal{M}$ and $T_{m_2}\mathcal{M}$, regarded as subspaces of $T_{m_1}\mathcal{N} = T_{m_2}\mathcal{N} = \mathbb{R}^n$. This is a nonlocal measure of extrinsic curvature. When m_1, m_2 are infinitesimally near each other, it is related to a local measure of extrinsic curvature, the third fundamental form [1]:

$$D_{III}(m_1, m_2) = \int \sqrt{K(u(t), u(t))}\, \mathrm{d}t, \tag{1}$$

where $u(t)$ is the tangent vector to the "shortest" path $m(t)$ from m_1 to m_2.

The first fundamental form of $\mathcal{M}$ is the induced metric:

$$h(u, v) = \langle u, v \rangle_{\mathcal{M}} = \langle U, V \rangle_{\mathcal{N}}, \tag{2}$$

where u, v are tangent vectors to $\mathcal{M}$, $\langle \,,\, \rangle_{\mathcal{M}}$ is the inner product on $T_m\mathcal{M}$, U, V are the push-forwards of u, v to $\mathcal{N}$ and $\langle \,,\, \rangle_{\mathcal{N}}$ is the inner product on $T_m\mathcal{N}$. The first fundamental form maps two tangent vectors of $\mathcal{M}$ to a scalar.

The (vector-valued) second fundamental form and shape operators are defined as [2, §V.A]

$$S(u, v) = \widetilde{D}_U V - D_U V, \qquad \langle S(u), v \rangle_{\mathcal{M}} = S(u, v), \tag{3}$$

where $\widetilde{D}$ and D are the covariant derivatives on $\mathcal{N}$ and $\mathcal{M}$ respectively. We have used the same letter for the second fundamental form and the shape operators to conserve alphabetical resources, they can be distinguished by the number of arguments. The second fundamental form maps two tangent vectors of $\mathcal{M}$ to a tangent vector of $\mathcal{N}$. The shape

operator maps one tangent vectors of $\mathcal{M}$ to an object that is both a tangent vectors of $\mathcal{M}$ and a tangent vector of $\mathcal{N}$ (perpendicular to $T_m\mathcal{M}$).

The third fundamental form is defined by [3]

$$K(u,v) = \langle\langle S(u), S(v)\rangle_{\mathcal{M}}\rangle_{\mathcal{N}}. \tag{4}$$

It maps two tangent vectors of $\mathcal{M}$ to a scalar.

2   Constructing extrisic curvatures (concrete)

Let x^i, $i = 1\ldots n$, be cartesian coordinates on $\mathbb{R}^n$. Let σ^α , $\alpha = 1\ldots d$, be coordinates on $\mathcal{M}$.[1] As $\mathcal{M}$ is not necessarily Euclidean, we will have to distinguish covariant indices (subscripts) from contravariant indices (superscripts) for Greek letters. We need not make this distinction for roman letters, but we will use superscripts to keep them out of the way. We also use the Einstein summation convention, where there is implicit summation over indices that appear twice on the same side of any equation.

We parameterise $\mathcal{M}$ as

$$x^i = \phi^i(\sigma), \qquad \text{with} \qquad \phi^i_\alpha = \frac{\partial\phi^i}{\partial\sigma^\alpha},$$
$$\text{and} \qquad \phi^i_{\alpha\beta} = \frac{\partial^2\phi^i}{\partial\sigma^\alpha\partial\sigma^\beta}. \tag{5}$$

We can use ϕ^i_α to push-forward vectors and pull-back forms.

We denote the units of x and σ by $[x]$ and $[\sigma]$ respectively. Then

$$\left[\phi^i\right] = [x], \qquad \left[\phi^i_\alpha\right] = [x][\sigma]^{-1}, \qquad \left[\phi^i_{\alpha\beta}\right] = [x][\sigma]^{-2}. \tag{6}$$

This tells us how these quantities would scale if we were to rescale x or σ.

The induced metric and its matrix-inverse are:

$$h_{\alpha\beta} = \phi^i_\alpha\phi^i_\beta, \qquad\qquad h^{\alpha\beta} = \left(h_{\alpha\beta}\right)^{-1},$$
$$[h_{\alpha\beta}] = [x]^2[\sigma]^{-2}, \qquad \left[h^{\alpha\beta}\right] = [x]^{-2}[\sigma]^2. \tag{7}$$

The inverse metric can be pushed-forward to $\mathbb{R}^n$:

$$h^{ij} = \phi^i_\alpha\phi^j_\beta\, h^{\alpha\beta}, \qquad \tilde{h}^{ij} = \delta^{ij} - h^{ij}, \qquad \left[h^{ij}\right] = \left[\tilde{h}^{ij}\right] = 1. \tag{8}$$

These are the orthogonal projection operators parallel and perpendicular to the tangent plane of $\mathcal{M}$ respectively.

The second fundamental form is given by:

$$S^i_{\alpha\beta} = \phi^i_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}\,\phi^i_\gamma = \tilde{h}^{ij}\,\phi^j_{\alpha\beta}, \qquad \left[S^i_{\alpha\beta}\right] = [x][\sigma]^{-2}, \tag{9}$$

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[1]For us, σ parameterise translations of the image and x could represent pixel values or neuron outputs.

where $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols for the covariant derivative on $\mathcal{M}$.

The third fundamental form is given by:

$$K_{\alpha\beta} = h^{\gamma\delta}\, S^i_{\alpha\gamma} S^i_{\beta\delta} = h^{\gamma\delta}\, \tilde{h}^{ij}\, \phi^i_{\alpha\gamma}\phi^j_{\beta\delta} \qquad [K_{\alpha\beta}] = [\sigma]^{-2}. \tag{10}$$

If we use the special coordinate system of [1, Appendix B], we get

$$\sigma^\alpha = x^\alpha, \quad \alpha = 1\ldots d, \qquad \implies \qquad K_{\alpha\beta} = \sum_{i=d+1}^{n}\sum_{\gamma=1}^{d} \phi^i_{\alpha\gamma}\phi^i_{\beta\gamma}, \tag{11}$$

in agreement with the expression therein. The Grassmanian distance between two infinitesimally near points, i.e. (the sine of?) the largest principal angle between the tangent planes to $\mathcal{M}$ at σ_1 and σ_2, is given by

$$D_{III}(\sigma_1, \sigma_2) = \int \sqrt{K_{\alpha\beta}\, \dot\sigma^\alpha(t)\dot\sigma^\beta(t)}\, dt \qquad [D_{III}] = 1, \tag{12}$$

where $\sigma(t)$ is the "shortest" path from σ_1 to σ_2.

If we think of $K_{\alpha\beta}$ as a linear operator, it maps vectors to covectors. Therefore, it does not make sense to talk of its eigenvalues as they would be coordinate dependent quantities. Instead, consider

$$K^\alpha{}_\beta = h^{\alpha\gamma} K_{\gamma\beta}, \qquad K = K^\alpha{}_\alpha,$$
$$K^\alpha{}_\beta\, u^\beta_{(a)} = \lambda^{(a)}_K\, u^\alpha_{(a)}, \qquad [K^\alpha{}_\beta] = [K] = \left[\lambda^{(a)}_K\right] = [x]^{-2}. \tag{13}$$

Under a coordinate change, $K^\alpha{}_\beta$ undergoes a similarity transform. Therefore, the eigenvalues, $\lambda^{(a)}_K$, are scalars, as is K, their sum. In the $d = 2$, $n = 3$ case, the Gaussian curvature is equal to

$$\bar{K} = \sqrt{\frac{\det K_{\alpha\beta}}{\det h_{\alpha\beta}}} = \sqrt{\det K^\alpha{}_\beta} = \sqrt{\prod_a \lambda^{(a)}_K}, \qquad [\bar{K}] = [x]^{-d}. \tag{14}$$

This is also a scalar, as the numerator and denominator both transform with the square of the jacobian determinant under a coordinate change.

We can use the square roots of the determinants of the first and third fundamental forms to construct invariant measures for integration over $\mathcal{M}$. They have units:

$$\left[\sqrt{\det h_{\alpha\beta}}\, d^d\sigma\right] = [x]^d, \qquad \left[\sqrt{\det K_{\alpha\beta}}\, d^d\sigma\right] = 1. \tag{15}$$

We can then construct the following global, coordinate-invariant, dimensionless measures of extrinsic curvature:

$$K_{(a)} = \int_{\mathcal{M}} \left(\lambda^{(a)}_K\right)^{\frac{d}{2}} \sqrt{\det h_{\alpha\beta}}\, d^d\sigma,$$

$$K_{\mathrm{tr}} = \int_{\mathcal{M}} K^{\frac{d}{2}} \sqrt{\det h_{\alpha\beta}}\, d^d\sigma = \int_{\mathcal{M}} \left(\sum_a \lambda^{(a)}_K\right)^{\frac{d}{2}} \sqrt{\det h_{\alpha\beta}}\, d^d\sigma, \tag{16}$$

$$K_{\mathrm{det}} = \int_{\mathcal{M}} \sqrt{\det K_{\alpha\beta}}\, d^d\sigma = \int_{\mathcal{M}} \left(\prod_a \lambda^{(a)}_K\right)^{\frac{1}{2}} \sqrt{\det h_{\alpha\beta}}\, d^d\sigma.$$

For our case of $d = 2$, the second line is the sum of the first line over a. For higher dimensional manifolds, one could come up with many other ways of combining eigenvalues in the integral. The third line can be thought of as the Grassmannian volume of the submanifold. The fact that these quantities are dimensionless means that they are invariant under rescaling of x and σ.

References

[1] K. L. Clarkson, "Tighter bounds for random projections of manifolds," in *Proceedings of the twenty-fourth annual symposium on Computational geometry*, pp. 39–48, ACM. 2008.

[2] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, vol. 3. Springer, 1990.

[3] J.-H. Eschenburg, B. Kruglikov, V. S. Matveev, and R. Tribuzy, "Compatibility of Gauss maps with metrics," *Differential Geometry and its Applications* **28** (2010) no. 2, 228 – 235, arXiv:0903.3225 [math.DG].