Giant gravitons and the supersymmetric states of $\mathcal{N} = 4$ Yang-Mills

Subhaneil Lahiri

Department of Physics,

Harvard University

based on `hep-th/0606087` with Indranil Biswas, Davide Gaiotto and Shiraz Minwalla

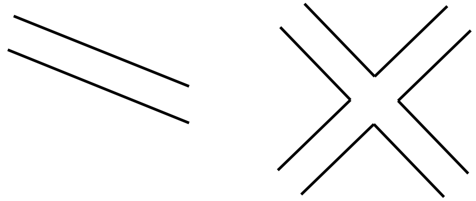
Outline

- Gauge theories and gravity
- Black holes and supersymmetric states
- Giant gravitons
- Quantisation
- Conclusions and future directions

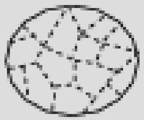
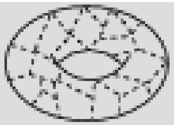
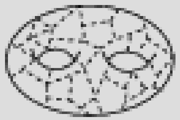
Gauge theories and gravity

If we take the 't Hooft limit of a U(N) gauge theory ($N \to \infty$, $\lambda = g^2 N$ fixed) the $\frac{1}{N}$ expansion looks like perturbative string theory.

If we draw propagators and vertices for matrix fields as:



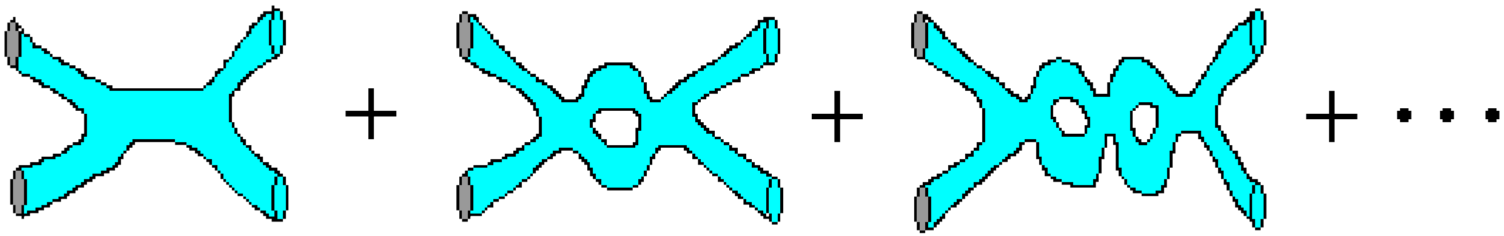
Feynman diagrams look like two dimensional surfaces.

$F = N^2$  $+ 1$  $+ \frac{1}{N^2}$  $+ \cdots = \sum_{g=0}^{\infty} \frac{1}{N^{2g-2}} \sum_{l=0}^{\infty} c_{g,l}\, \lambda^l$

Gauge theories and gravity

The power of $\frac{1}{N}$ is $(2 \times \# \, holes - 2)$.

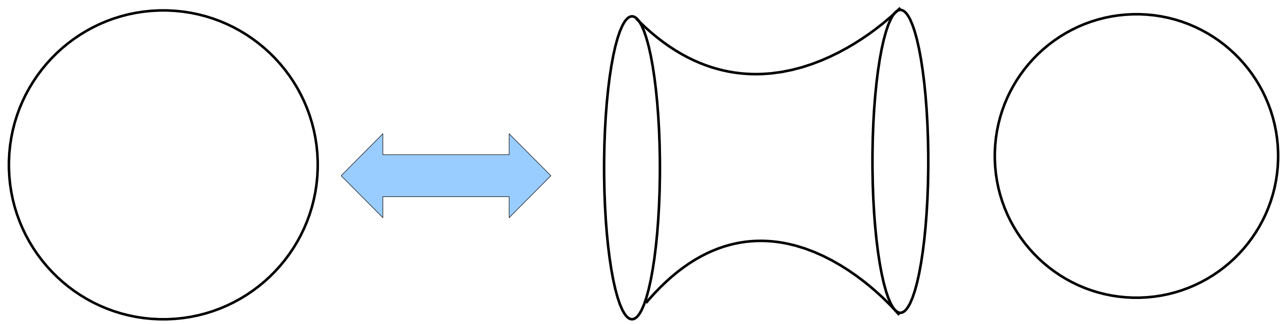
Just like g_s in string theory:



The AdS/CFT correspondence

The best known case is a correspondence between:

- U(N), $\mathcal{N} = 4$ super Yang-Mills on $S^3 \times$ Time;
- Type IIB string theory on $AdS_5 \times S^5$ with N units of five form flux;



Conformal invariance

$\mathcal{N} = 4$ super Yang-Mills theories happen to be scale invariant quantum mechanically as well as classically.

Scale invariant theories are usually invariant under conformal transformations: transformations that leave angles unchanged.

This group is SO(4,2).

It is convenient to study conformal theories on S^3 rather than $\mathbb{R}^3$ as there is a 1-1 correspondence between: States on S^3 and Local operators in $\mathbb{R}^4$.

Anti-de-Sitter space

Anti-de-Sitter space can be constructed as a hyperboloid in $\mathbb{R}^{4,2}$:

$$-(x^{-1})^2 - (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = -R^2$$

It has the metric:

$$\mathrm{d}s^2 = R^2(-\cosh^2\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\Omega^2)$$

This has the symmetry group SO(4,2)

The boundary is conformally $S^3\times$Time.

Super and R symmetries

The $\mathcal{N}=4$ theory has 16 supersymmetries, Q^i_α and $\overline{Q}_{i\,\dot\alpha}$, and their complex conjugates – the superconformal symmetries $S_{i\,\alpha}$ and $\overline{S}^i_{\dot\alpha}$.

Correspondingly, IIB in $AdS_5 \times S^5$ has 32 killing spinors.

The $\mathcal{N}=4$ theory also has an SU(4)=SO(6) R-symmetry that mixes the supersymmetries.

This is also the rotation group of S^5.

Counting states

We would like to classify the states of the theory using these symmetries.

i.e. count the number of states for each value of the Noether charges.

This can be summarised in a partition function:

$$Z = \mathrm{Tr}\, \mathrm{e}^{\mu_i Q_i}$$

Parameter matching

The parameters are related by:

$$g_s = \frac{\lambda}{N}$$

$$\frac{R^4}{(\alpha')^2} = 4\pi\lambda$$

This means:

$$\text{Low curvature} \quad \rightleftharpoons \quad \text{Strong coupling}$$

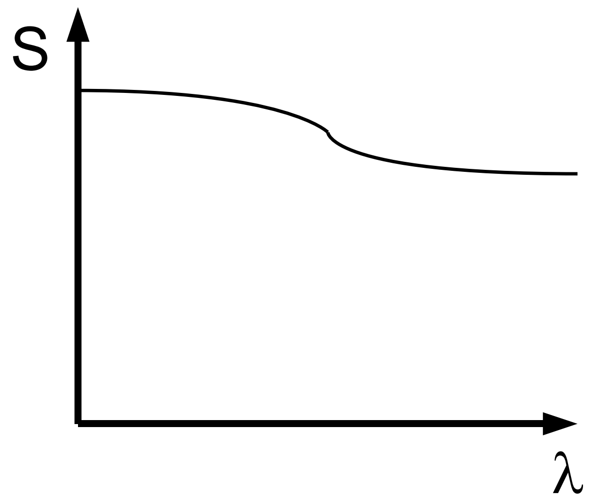
$$\text{High curvature} \quad \rightleftharpoons \quad \text{Weak coupling}$$

AdS-Schwarzschild black hole

The “deconfinement” phase transition in the gauge theory is dual to the formation of a black hole in the bulk.

Qualitative matching, but not quantitative – the entropy is off by $\frac{3}{4}$.

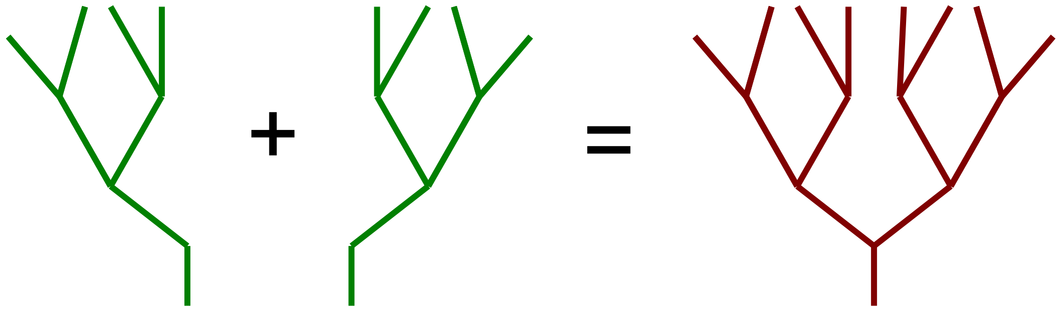
The spectrum of the theory varies with λ.



Supersymmetric spectrum

Supersymmetric states lie in short representations

SUSY $\rightarrow$ non-SUSY requires short reps joining to form a long rep.

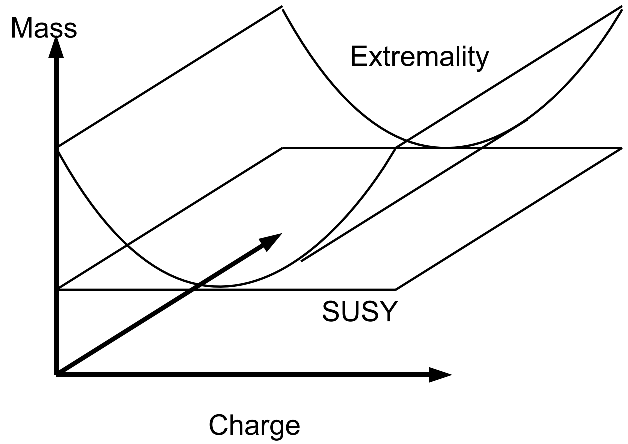


AdS$_5$ black holes

A general black hole has six parameters $(\Delta, J, \bar{J}, R_1, R_2, R_3)$.

For each value of $(J, \bar{J}, R_1, R_2, R_3)$, there is an extremal black hole.

If these five charges satisfy an additional relation, this black hole preserves 1/16 of the supersymmetries.



1/16 BPS states

At zero coupling:

- Qualitative but not quantitative matching
- No sign of relation between charges

At weak coupling:

- Haven't found the spectrum
- Computing an index doesn't work

1/8 BPS states

These are invariant under both components of Q^1_α and their complex conjugates $S_{1\,\alpha}$.

They are in 1-1 correspondence with cohomology classes of Q^1_α, i.e.

$$Q\,|\psi\rangle = 0$$

$$|\psi\rangle \sim |\psi\rangle + Q\,|\phi\rangle$$

Each cohomology class contains one state that is also annihilated by S.

1/8 BPS states

The supercharge acts (in $\mathcal{N} = 1$ language) as:

$$Q_\alpha \bar{\phi}^m = 0\,, \qquad Q_\alpha \phi_m = \psi_{m\alpha}\,,$$

$$Q_\alpha \psi_{m\beta} = g\epsilon_{\alpha\beta}\epsilon_{mkl}[\bar{\phi}^k, \bar{\phi}^l]\,, \qquad Q_\alpha \lambda_\beta = f_{\alpha\beta} + g\epsilon_{\alpha\beta}[\phi_m, \bar{\phi}^m]\,,$$

$$Q_\alpha \bar{\psi}^m_{\dot\beta} = D_{\alpha\dot\beta}\bar{\phi}^m\,, \qquad Q_\alpha \bar{\lambda}_{\dot\beta} = 0\,,$$

$$Q_\alpha A_{\beta\dot\gamma} = \epsilon_{\alpha\beta}\bar{\lambda}_{\dot\gamma}\,. \qquad \Longrightarrow Q_\alpha D_{\beta\dot\gamma} = g\epsilon_{\alpha\beta}[\bar{\lambda}_{\dot\gamma},\ \ ]\,.$$

The Q-closed letters are $\bar{\phi}^m$ and $\bar{\lambda}_{\dot\beta}$. Their commutators are Q-exact. They are simultaneously diagonalisable in cohomology.

They can be counted in terms of eigenvalues.

1/8 BPS states

The states built out of the scalars can be thought of as N bosons moving in a three dimensional harmonic oscillator.

$(\bar{\phi}_a^1)^{n_1}(\bar{\phi}_a^2)^{n_2}(\bar{\phi}_a^3)^{n_3}$ maps onto boson number a in the state $|n_1, n_2, n_3\rangle$.

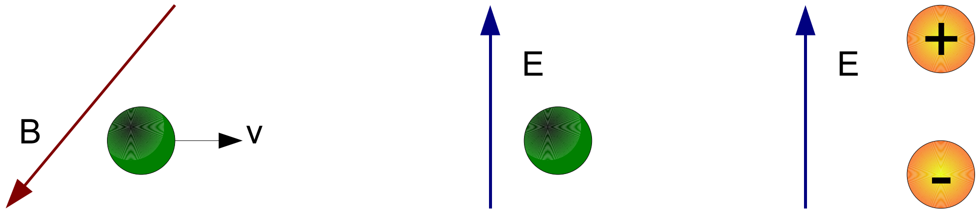
Bosons because permutations are part of the gauge invariance.

SO(6) charges are the total excitation $\#$'s of each oscillator.

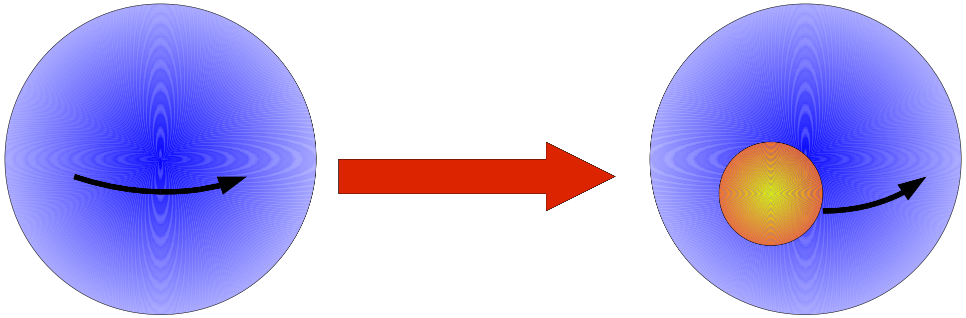
Energies $<< N$ described by multi-gravitons.
Energies $\sim N$ described by giant gravitons.

Giant gravitons

Dubious analogy: neutral particle moving in a magnetic field.



Graviton moving fast around S^5



Likes to puff out into a D3 brane.

Quantising these produces finite N, 1/2 BPS spectrum.

1/8 BPS giant gravitons

Mikhailov's construction:

- Embed S^5 in $\mathbb{C}^3$

$$|x|^2 + |y|^2 + |z|^2 = 1$$

- Pick a holomorphic function $f(x, y, z)$
- The D3-brane wraps the intersection of the surface $f(\mathrm{e}^{-\mathrm{i}t}x, \mathrm{e}^{-\mathrm{i}t}y, \mathrm{e}^{-\mathrm{i}t}z) = 0$ with the S^5.

It can be shown that this preserves 1/8 of the supersymmetries.

Doesn't include worldvolume gauge fields and fermions.

Quantisation

First we need to describe the system in the Hamiltonian formalism.

We need a phase space and a Poisson bracket:

$$\{x^i, x^j\} = \omega^{ij}\,, \qquad \{f, g\} = \omega^{ij}(\partial_i f)(\partial_j g)\,,$$

or, equivalently, a symplectic form: $\omega_{ij} = \left[\omega^{ij}\right]^{-1}$.

This must be closed and non-degenerate:

$$\mathrm{d}\omega = 0\,, \qquad \det \omega \neq 0\,.$$

Then we can use the standard procedure of Geometric Quantisation.

Crnkovic-Witten-Zuckerman formalism

We can identify the phase space with the space of solutions to the equations of motion.

We then find ω by plugging the solutions into:

$$\omega = \int \mathrm{d}x \, \delta p_i \wedge \delta\phi^i$$

where ϕ^i are the dynamical fields and p_i are their conjugate momenta:

$$p_i = \frac{\partial L}{\partial \dot{\phi}^i}$$

Symplectic form

Using the Born-Infeld + Wess-Zumino action:

$$S = S_{\mathrm{BI}} + S_{\mathrm{WZ}}$$
$$= \frac{1}{(2\pi)^3(\alpha')^2 g_s}\int \mathrm{d}^4\sigma\,\sqrt{-\tilde{g}} + \int \mathrm{d}t\,\mathrm{d}^3\sigma\; A_{\mu_0\mu_1\mu_2\mu_3}\,\dot{x}^{\mu_0}\,\frac{\partial x^{\mu_1}}{\partial\sigma^1}\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3}\,,$$

we can find the symplectic form:

$$\omega_{\mathrm{full}} = \omega_{\mathrm{BI}} + \omega_{\mathrm{WZ}} = \frac{N}{2\pi^2}\int_\Sigma \mathrm{d}^3\sigma\,\delta\left(\sqrt{-g}g^{0\alpha}\frac{\partial x^\mu}{\partial\sigma^\alpha}G_{\mu\nu}\right)\wedge\delta x^\nu$$
$$+\frac{2N}{\pi^2}\int_\Sigma \mathrm{d}^3\sigma\,\frac{\delta x^\lambda\wedge\delta x^\mu}{2}\left(\frac{\partial x^\nu}{\partial\sigma^1}\frac{\partial x^\rho}{\partial\sigma^2}\frac{\partial x^\sigma}{\partial\sigma^3}\right)\epsilon_{\lambda\mu\nu\rho\sigma}\,.$$

Mikhailov's phase space

Solutions parameterised by one holomorphic function.

This is an infinite dimensional space. We regulate it by restricting to polynomials made from a finite number of monomials:

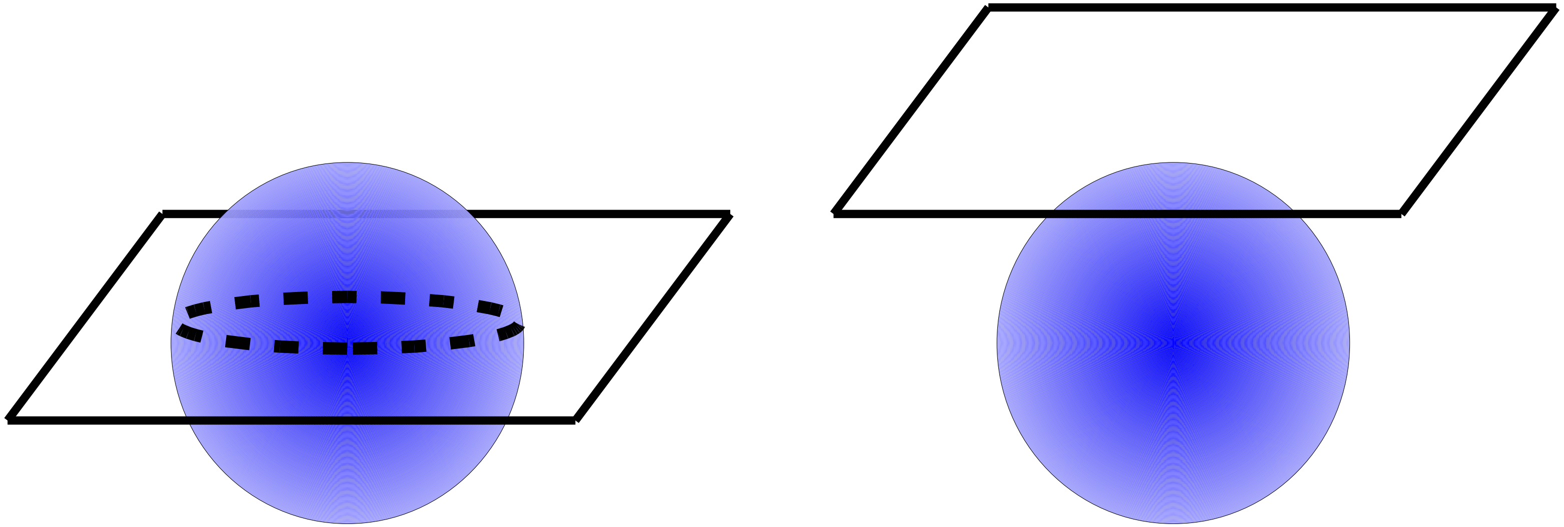
$$f(z_1, z_2, z_3) = \sum_{\vec{n} \in C} c_{\vec{n}} \, (z^1)^{n_1} (z^2)^{n_2} (z^3)^{n_3} \, .$$

$c_{\vec{n}}$ and $\lambda c_{\vec{n}}$ describe the same surface.

It looks like $\mathbb{CP}^{n_C - 1}$.

But unfortunately . . .

. . . not all surfaces touch the sphere:



Eats holes out of phase space.

Geometric quantisation of $\mathbb{CP}^n$

$\mathbb{CP}^n$ has a canonical two-form (Fubini-Study):

$$\omega_{\rm FS} = \frac{1}{4\pi \mathrm{i}} \frac{1}{|z|^2} \left[\mathrm{d}\bar{z}^i - \frac{\bar{z}^i z^j}{|z|^2} \mathrm{d}\bar{z}^j \right] \wedge \left[\mathrm{d}z^i - \frac{z^i \bar{z}^j}{|z|^2} \mathrm{d}z^j \right] .$$

Suppose that our symplectic form is in the cohomology class $(2\pi N)[\omega_{\rm FS}]$.

It is a standard result that the Hilbert space is the space of degree N homogeneous polynomials in the z^i.

3D harmonic oscillator

We can map this to the 3D harmonic oscillator as follows:

$c_{\vec{n}} \to a^\dagger_{\vec{n}}$: the creation operator for a particle in the state $|n_1, n_2, n_3\rangle$.

A monomial of degree N acting on the vacuum produces a N-particle state.

These states transform the same way under U(3) as our 3D harmonic oscillator.

Problems

- Some surfaces do not touch the sphere, e.g.

$$c_i z^i - 1 = 0 \quad \text{for} \quad |c|^2 < 1\,.$$

- Singularities when the function factorises - the surface degenerates, e.g.

$$x^2 + y^2 + \epsilon z^2 = 0 \quad \text{as} \quad \epsilon \to 0\,.$$

- Factorised functions where one factor doesn't touch the sphere.
- Finding the cohomology class of ω.

Example: Linear functions

Let's look at the space of functions $f(z^i) = c_i z^i - 1$.

The symplectic form is:

$$\omega = 2N\left[\left(\frac{1}{|c|^2} - \frac{1}{|c|^4}\right)\frac{\mathrm{d}\bar{c}^i \wedge \mathrm{d}c_i}{2\mathrm{i}} - \left(\frac{1}{|c|^2} - \frac{2}{|c|^4}\right)\frac{\bar{c}^i c_j}{|c|^2}\frac{\mathrm{d}\bar{c}^j \wedge \mathrm{d}c_i}{2\mathrm{i}}\right].$$

It is zero inside $|c|^2 < 1$ and has four null directions on the boundary.

While it is not zero at the boundary, its restriction to the boundary is.

Contracting the hole

Coordinate change:

$$w_i = c_i \sqrt{\frac{|c|^2 - 1}{|c|^2}}$$

Shrinks sphere $|c|^2 = 1$ to the point $|w|^2 = 0$. We get:

$$\omega = \frac{2N}{1+|w|^2}\left(\frac{\mathrm{d}\overline{w}^i \wedge \mathrm{d}w_i}{2\mathrm{i}} - \frac{w_i \overline{w}^j}{1+|w|^2}\frac{\mathrm{d}\overline{w}^i \wedge \mathrm{d}w_j}{2\mathrm{i}}\right)$$

This is precisely $(2\pi N)\omega_{\mathrm{FS}}$ on $\mathbb{CP}^3$!

General case

Define the distance function: $\rho(c, \bar{c})$ - the minimum distance from the origin of $\mathbb{C}^3$ to the surface.

The boundary of the hole is $\rho = 1$

Under $c_{\vec{n}} \to \lambda^{-(n_1+n_2+n_3)} c_{\vec{n}}$, we have $\rho \to \lambda\rho$.

Each ray $c_{\vec{n}}(\lambda) = \lambda^{-(n_1+n_2+n_3)} c_{\vec{n}}^{(0)}$ intersects the boundary $\rho = 1$ once.

This means that the hole has the topology of a ball – can be contracted, e.g. with $\lambda = (1 - \rho^2)^{-1/2}$.

Factorised submanifolds

There are submanifolds of phase space where the function factorises.

These submanifolds also have holes. These holes, and their intersections with each other, are also balls.

This means we can contract all of the holes without changing the topology.

The phase space is $\mathbb{CP}^{n_C-1}$ after all!

As ρ is U(3) invariant, this doesn't change the charges of the coordinates.

Geometric description of ω

Wess-Zumino contribution:

$$\omega_{\mathrm{WZ}} = \frac{2N}{\pi^2} \int_\Sigma \mathrm{d}^3\sigma \, \frac{\delta x^\lambda \wedge \delta x^\mu}{2} \left(\frac{\partial x^\nu}{\partial \sigma^1} \frac{\partial x^\rho}{\partial \sigma^2} \frac{\partial x^\sigma}{\partial \sigma^3} \right) \epsilon_{\lambda\mu\nu\rho\sigma} \, .$$

For two deformations of the surface, this is $\frac{2N}{\pi^2}$ times the volume swept out.

Geometric description of ω

Born-Infeld contribution:

For Mikhailov's solutions, we can write:

$$\omega_{\mathrm{BI}} = \mathrm{d}\theta_{\mathrm{BI}}$$

$$\theta_{\mathrm{BI}} = \frac{N}{\pi^2} \int_S \mathrm{d}^4\sigma \, \epsilon_{\mu_1 \cdots \mu_6} \left[\frac{\partial x^{\mu_1}}{\partial \sigma^1} \cdots \frac{\partial x^{\mu_4}}{\partial \sigma^4} \right] e_{\perp}^{\mu_5} \, \delta x^{\mu_6} \, \delta\big(|z^i|^2 - 1\big) \ ,$$

i.e. compute $\frac{N}{2\pi^2}$ times the volume, inside a ball of radius r, swept out by a deformation of S and the unit radial vector $e_\perp$. Differentiate it with respect to r and set $r = 1$.

Singularities of ω

These can be formally rewritten as fibre integrals of closed forms.

This means ω is a current – i.e. $\int \beta \wedge \omega$ is finite, even if ω isn't.

We can also see that ω vanishes when restricted to the boundary, so no singularities from contracting it.

This is enough for geometric quantisation

Cohomology class of ω

Every closed two form in $\mathbb{CP}^{n_C-1}$ can be written as:

$$M\omega_{\mathrm{FS}} + \mathrm{d}\beta\,.$$

We want to determine M.

Consider two sets of monomials $\widetilde{C} \subset C$. Restricting the the phase space from C to $\widetilde{C}$ maps:

$$\mathbb{CP}^{n_C-1} \to \mathbb{CP}^{n_{\widetilde{C}}-1}\,,$$

$$\omega_{\mathrm{FS}} \to \omega_{\mathrm{FS}}\,.$$

This means $M_C = M_{\widetilde{C}}$

Cohomology class of ω

Looking at a tree like:

$$\begin{tikzcd} & C_1 \cup C_2 \arrow[dl, "M_{C_1\cup C_2} = M_{C_1}"'] \arrow[dr, "M_{C_1\cup C_2} = M_{C_2}"] & \\ C_1 & & C_2 \end{tikzcd}$$

We see that $M_{C_1} = M_{C_2}$.

We have already found M for linear functions, so $M = 2\pi N$ for all C.

Summary

- The Phase space is $\mathbb{CP}^{n_C-1}$
- The symplectic form is cohomologically $(2\pi N)\omega_{\mathrm{FS}}$
- The coordinates have U(3) charges (n_1, n_2, n_3)
- This is isomorphic to the 3D harmonic oscillator

This gives the partition function

$$\sum_N \zeta^N Z_N(\mu_1, \mu_2, \mu_3) = \prod_{\vec{n}} \frac{1}{1 - \zeta \mathrm{e}^{-\mu_i n_i}}$$

Conclusions and future directions

We get exact (finite N) matching between the gauge theory and giant gravitons.

We are getting ordinary gravitons by quantising D-branes.

Giant/goliath duality still works.

Should be extended to include worldvolume gauge fields and fermions.

Can be extended to other AdS/CFT duals.

Classical 1/16 BPS giants are known, but much harder to quantise.

Dual giants

These are spherically symmetric in AdS_5, point-like in S^5 and move as: $(x, y, z) = (x_0 \mathrm{e}^{\mathrm{i}t}, y_0 \mathrm{e}^{\mathrm{i}t}, z_0 \mathrm{e}^{\mathrm{i}t})$.

The phase space is given by the radius of the giant and an initial position on S^5: $\mathbb{R}^+ \times S^5 = \mathbb{C}^3$.

This has the standard symplectic form and Noether charges $\frac{1}{2}N(x^2, y^2, z^2)$.

This *is* the 3D harmonic oscillator.

1/16 BPS giants

As well as embedding S^5 in $\mathbb{C}^3$, we can embed AdS_5 in $\mathbb{C}^{2,1}$ as: $|w|^2 - |u|^2 - |v|^2 = 1$.

Pick three holomorphic functions, $f_i(w_j, x_k)$ that satisfy a homogeneity condition

$$f_i\left(\frac{w_j}{\lambda}, \lambda x_k\right) = \lambda^n f_i(w_j, x_k)\,.$$

The worldvolume is given by the the intersection of the surface $f_i = 0$ with $AdS_5 \times S^5$.

This is much harder to quantise.