A universal tradeoff between energy, speed and precision in neural communication

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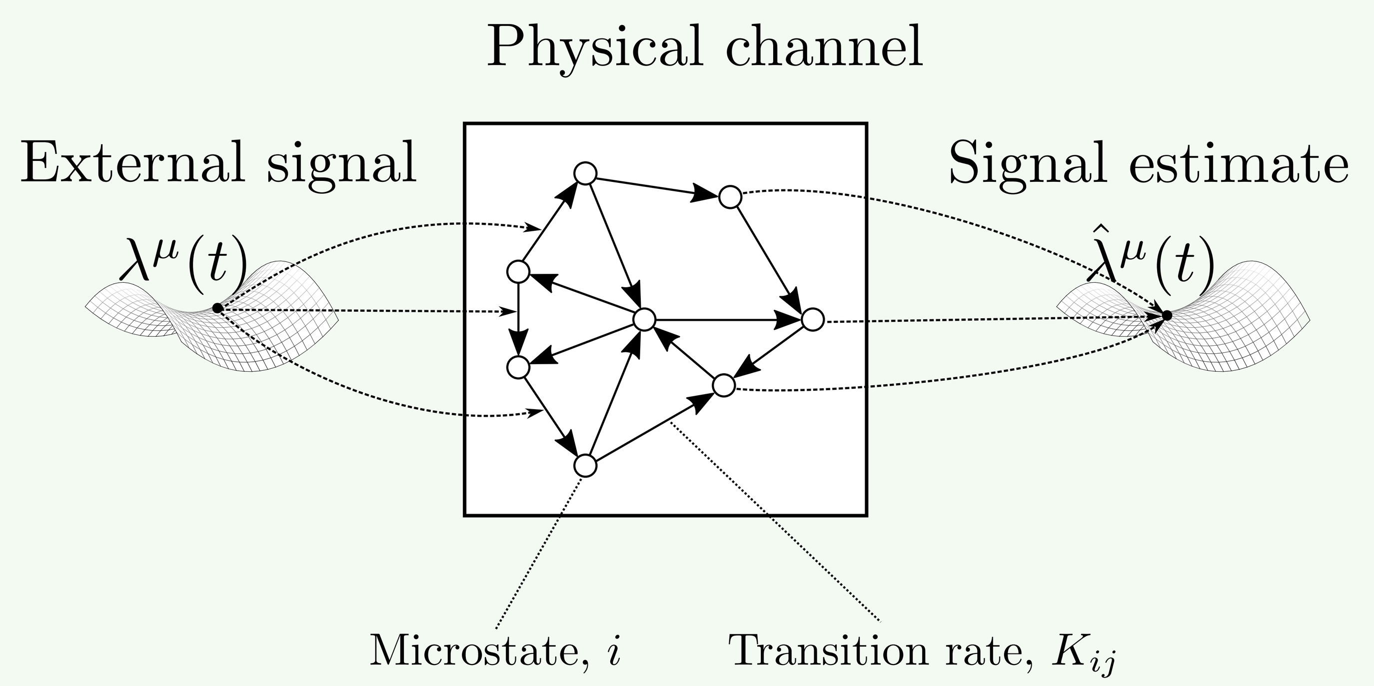
Background

Information theory and thermodynamics provide limits on the accuracy and energy efficiency of physical systems.

However, they assume infinite time / infinitesimal speed.

Can we extend this to systems operating at nonzero speed?

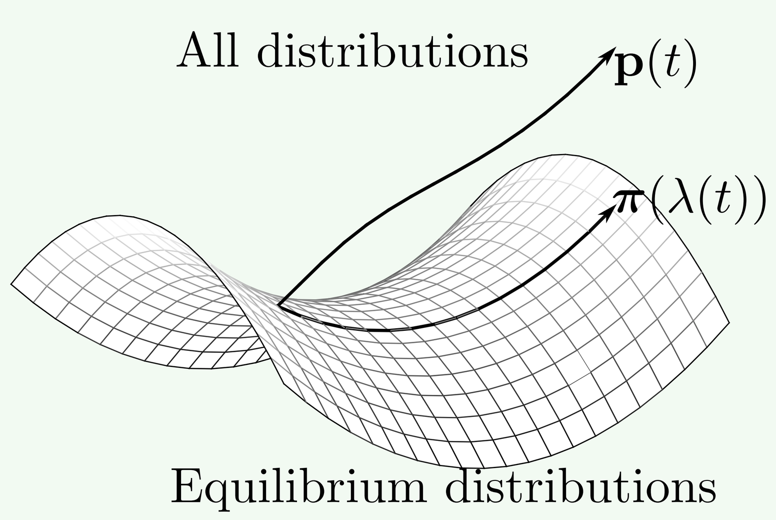
Model of physical signaling substrate



Channel dynamics: arbitrary Markov process.
Signal: control parameters for transition rates & energies.
Receiver: estimate λ^μ by observing state of channel.

Causes of energy dissipation

Move at nonzero speed → out of equilibrium.



$\mathbf{p}(t)$: current probability distribution,

$\boldsymbol{\pi}(\lambda(t))$: equilibrium distribution for current λ.

Move faster → further out of equilibrium → more energy.
More precision → more sensitivity to λ → more energy.

$\implies$ three-way tradeoff between energy, speed and precision.

Previous work

Sensory adaptation: three-way tradeoff between energy, speed and accuracy for a specific model [1].
Kinetic proofreading: two-way tradeoff between energy and accuracy [2], or speed and accuracy [3].
Chemosensation: two-way tradeoff between energy and precision [4].

Proof of bound

Dissipation and the friction tensor

When $\dot{\lambda}$ is small on channel dynamics' timescales, dissipation rate is [5]:

$$\mathcal{P}_{\text{ex}} = g_{\mu\nu}\,\dot{\lambda}^\mu\dot{\lambda}^\nu,$$

$g_{\mu\nu}$ = friction tensor:

$$g_{\mu\nu} = k_{\text{B}}T\int_0^\infty \mathrm{d}t'\,\langle\delta\phi_\mu(0)\delta\phi_\nu(t')\rangle\,,$$

$$\phi_\mu = -\beta\frac{\partial E}{\partial\lambda^\mu}, \qquad \delta\phi_\mu = \phi_\mu - \langle\phi_\mu\rangle\,.$$

Riemannian metric on λ manifold → thermodynamic distance.
Optimal protocol = shortest path.

Friction tensor and Fisher information

We can show that

$$\mathbf{g} = k_{\text{B}}T\sum_a \tau_a\mathbf{F}^a,$$

τ_a = time constant of eigenmode a,
$\mathbf{F}^a$ = Fisher information projected onto eigenmode a,

$$\boldsymbol{\eta}^a\mathbf{K} = -\tau_a^{-1}\boldsymbol{\eta}^a, \qquad F^a_{\mu\nu} = (\boldsymbol{\eta}^a\cdot\delta\phi_\mu)(\boldsymbol{\eta}^a\cdot\delta\phi_\nu).$$

Then

$$\mathbf{g} \geq k_{\text{B}}T\,\tau_{\text{min}}\,\mathbf{F},$$

$\tau_{\text{min}} = \min_a\tau_a$ (only over eigenmodes with $\mathbf{F}^a \not\approx 0$).

Energy-speed-precision bound

Define precision $\Phi = 1/\text{std. error}^2$ of unbiased estimator $\hat{\lambda}$.
Cramér-Rao bound:

$$\Phi \leq F.$$

Define $V = \dot{\lambda}^2$,

$$\Phi V \leq \frac{\mathcal{P}_{\text{ex}}}{k_{\text{B}}T\,\tau_{\text{min}}}.$$

This bound is tightest when

- λ couples to a narrow range of timescales,
- estimator $\hat{\lambda}$ is efficient.

Dual coordinates for exponential families

We're dealing with Boltzmann distributions: $\pi_i = \frac{\mathrm{e}^{-\beta E_i}}{Z}$.
Exponential coordinates: $E = \sum_\mu \lambda^\mu\,\mathcal{O}_\mu$.
Dual coordinates: $\tilde{\lambda}^\mu = \langle\mathcal{O}_\mu\rangle\,, \qquad \hat{\tilde{\lambda}}^\mu = \mathcal{O}_\mu$.
These are the only coordinates that saturate the Cramér-Rao bound [6].

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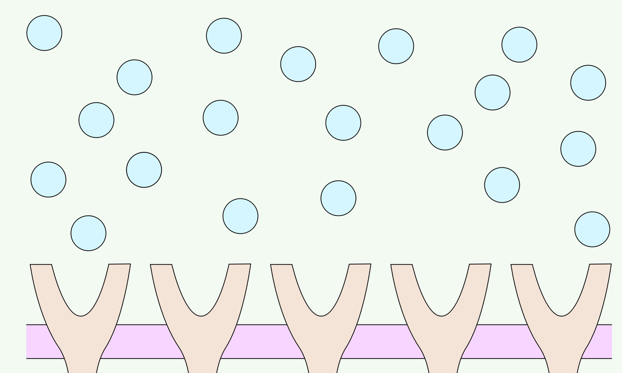
Examples

Cooperative receptors



Cooperative receptors can be modeled as an Ising chain [7]

$$E = -h\sum_n\sigma_n - J\sum_n\sigma_n\sigma_{n+1}.$$

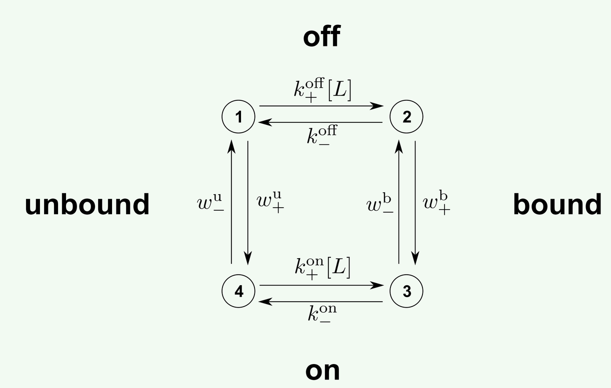
h = function of ligand concentration,
J = receptor cooperativity.

Estimate $\tilde{\lambda} = \mathrm{e}^{2\beta J}\tanh\beta h$ with $\hat{\tilde{\lambda}} = \frac{\sum_n\sigma_n}{N}$.
At the instant we pass through $h = 0$, we find:

$$\Phi = N\mathrm{e}^{-2\beta J}, \quad \mathcal{P}_{\text{ex}} = \frac{N\,k_{\text{B}}T\,V\cosh 2\beta J}{\alpha}, \quad \tau_{\text{min}} = \frac{\mathrm{e}^{2\beta J}\cosh 2\beta J}{\alpha}.$$

Saturates bound!

Nonequilibrium receptor



Following [8], define:

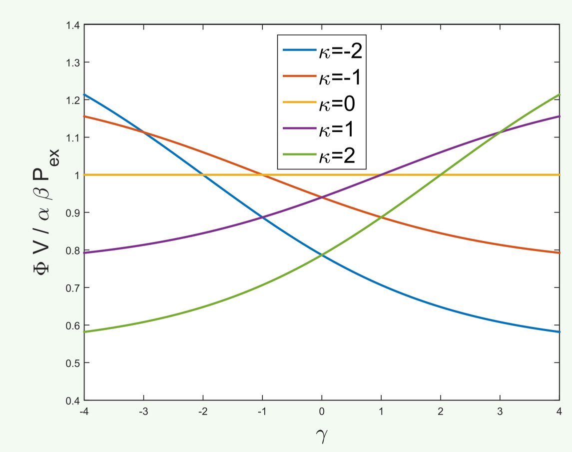
$$\kappa = \ln\frac{k_-^{\text{off}}}{k_-^{\text{on}}}, \qquad \gamma = \ln\frac{k_-^{\text{off}}w_-^{\text{b}}k_+^{\text{on}}w_+^{\text{u}}}{k_+^{\text{off}}w_+^{\text{b}}k_-^{\text{on}}w_-^{\text{u}}}$$

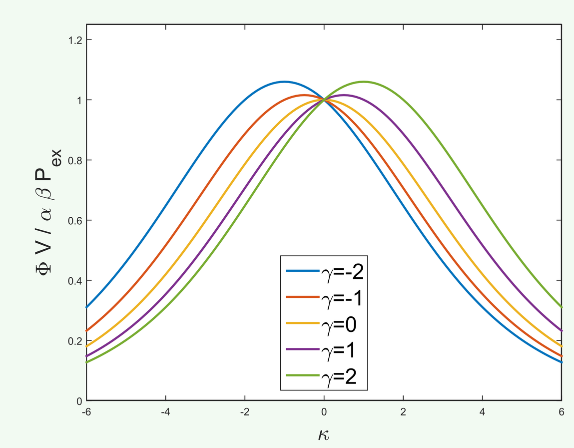
$$\lambda = \ln\frac{k_+[L]}{(k_-^{\text{off}}k_-^{\text{on}})^{1/2}}, \qquad \alpha = w_+^{\text{u/b}} + w_-^{\text{u/b}},$$

Estimate $\tilde{\lambda}$ with #active − #inactive.
At the instant we pass through $\lambda = 0$, assuming $\alpha \ll k_- = (k_-^{\text{off}}k_-^{\text{on}})^{1/2}$,

$$\Phi = N, \quad \mathcal{P}_{\text{ex}} = \frac{N\,k_{\text{B}}T\,V\cosh\left(\frac{\kappa-\gamma}{4}\right)\cosh\left(\frac{\kappa}{4}\right)}{\alpha\cosh\left(\frac{\gamma}{4}\right)}, \quad \tau_{\text{min}} = \frac{1}{k_-(1+\mathrm{e}^{|\kappa|/2})}.$$

Satisfies bound, but a long way from saturating it.





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