A universal tradeoff between power, precision and speed in physical communication

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Introduction

Physical devices and biological systems must perform their function

- precisely,
- in a reasonable time frame,
- without consuming too much energy.

Information theory and thermodynamics provide limits on the accuracy and energy efficiency of physical systems.

However, they assume infinite time / infinitesimal speed.

Can we extend this to systems operating at nonzero speed?

We show that power consumption bounds speed and precision of physical communication channels.

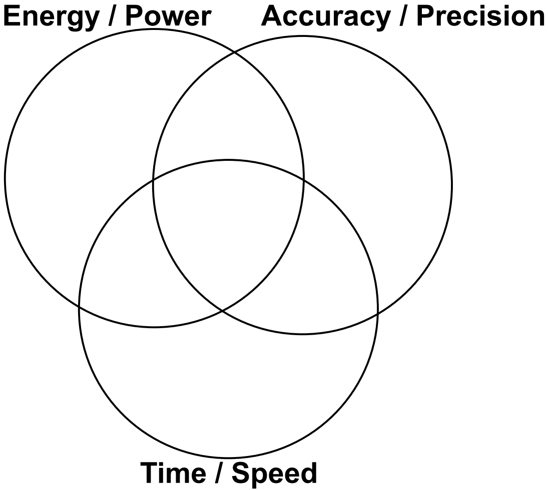
Outline

1. Background and motivation
2. General framework
3. Derivation of power-precision-speed tradeoff
4. Example systems
5. Conclusions

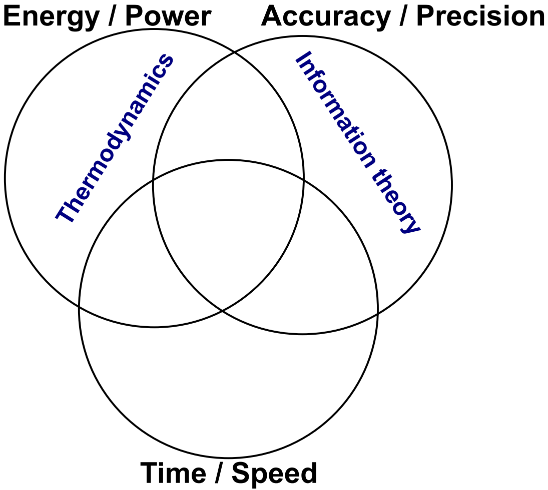
Section 1

Background and motivation

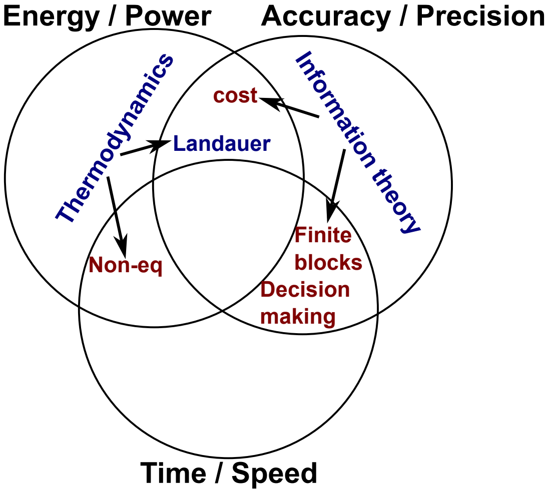
Tradeoffs between Power, precision and speed



Tradeoffs between Power, precision and speed



Tradeoffs between Power, precision and speed



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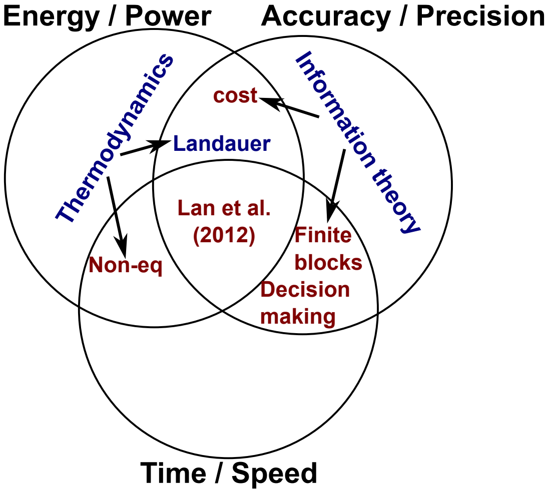


Previous work in biophysics

Sensory adaptation: three-way tradeoff between energy, speed and accuracy for a specific model. [Lan et al. (2012)]

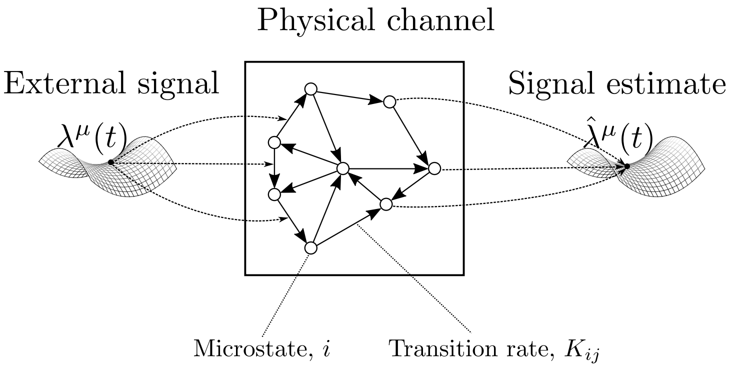
Kinetic proofreading: two-way tradeoff between energy and accuracy.
[Hopfield (1974), Freter and Savageau (1980), Ehrenberg and Blomberg (1980)]
[Savageau and Lapointe (1981), Qian (2006), Murugan et al. (2014)],
two-way tradeoff between speed and accuracy
[Murugan et al. (2012)]

Cellular chemosensation: two-way tradeoff between energy and precision.
[Endres and Wingreen (2009), Mehta and Schwab (2012), Lang et al. (2014)]
[Barato et al. (2014), Govern and Ten Wolde (2014), Sartori et al. (2014)]

Section 2

General framework

Model of physical signaling substrate

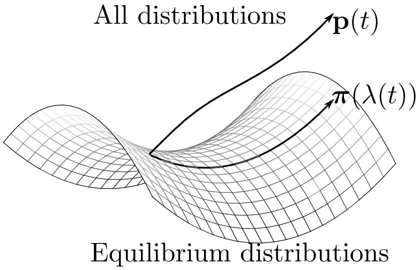


Channel dynamics: arbitrary Markov process.
Detailed balance: $\pi_i K_{ij} = \pi_j K_{ji}$.
Signal: control parameters for transition rates & energies.
Receiver: estimate λ^μ by observing state of channel.

noneq

Causes of energy dissipation

Move at nonzero speed $\rightarrow$ out of equilibrium.



$\mathbf{p}(t)$: current probability distribution,

$\boldsymbol{\pi}(\lambda(t))$: equilibrium distribution for current λ.

[Mandal and Jarzynski (2015)]

Move faster $\rightarrow$ further out of equilibrium $\rightarrow$ more energy.
More precision $\rightarrow$ more sensitivity to $\lambda \rightarrow$ more energy.

$\implies$ three-way tradeoff between energy, speed and precision.

Section 3

Derivation of power-precision-speed tradeoff

Dissipation and the friction tensor

When $\dot{\lambda}$ is small on channel dynamics' timescales, dissipation rate is:

[Sivak and Crooks (2012)]

$$\mathcal{P}_{\mathrm{ex}} = g_{\mu\nu}\,\dot{\lambda}^{\mu}\dot{\lambda}^{\nu},$$

$g_{\mu\nu}$ = friction tensor:

$$g_{\mu\nu} = k_{\mathrm{B}}T\int_0^{\infty}\mathrm{d}t'\,\left\langle\delta\phi_\mu(0)\delta\phi_\nu(t')\right\rangle_{\pi(\lambda(t))},$$

$$\phi_\mu = -\beta\frac{\partial E}{\partial\lambda^\mu}, \qquad \delta\phi_\mu = \phi_\mu - \langle\phi_\mu\rangle\,.$$

Riemannian metric on λ manifold $\rightarrow$ thermodynamic distance.
Optimal protocol = shortest path.

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Riemannian metric on λ manifold $\to$ thermodynamic distance.
Optimal protocol = shortest path.

Friction tensor and Fisher information

We can show that

proof

$$\mathbf{g} = k_\mathrm{B}\,T \sum_a \tau_a\,\mathbf{F}^a,$$

$\tau_a =$ time constant of eigenmode a,
$\mathbf{F}^a =$ Fisher information from eigenmode a,

$$\boldsymbol{\eta}^a\,\mathbf{K} = -\tau_a^{-1}\boldsymbol{\eta}^a, \qquad F^a_{\mu\nu} = (\boldsymbol{\eta}^a\cdot\delta\boldsymbol{\phi}_\mu)(\boldsymbol{\eta}^a\cdot\delta\boldsymbol{\phi}_\nu).$$

Then

$$\mathbf{g} \geq k_\mathrm{B}\,T\,\tau_\mathrm{min}\,\mathbf{F},$$

$\tau_\mathrm{min} = \min_a \tau_a$ (only over eigenmodes with $\mathbf{F}^a \not\approx 0$).

noneq

Power-precision-speed bound

Define precision $\Psi = 1/\text{std. error}^2$ of unbiased estimator $\hat{\lambda}$.
Cramér-Rao bound:

$$\Psi \leq F.$$

Define $V = \dot{\lambda}^2$,

$$\Psi V \leq \frac{\mathcal{P}_{\text{ex}}}{k_{\text{B}} T \, \tau_{\text{min}}}.$$

This bound is tightest when

- λ couples to a narrow range of timescales,
- estimator $\hat{\lambda}$ is efficient.

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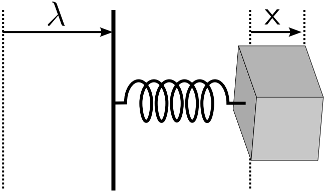
Section 4

Example systems

Heavily damped harmonic oscillator



Obeys Langevin equation:

$$\zeta \dot{x} = \kappa(\lambda(t) - x) + \sqrt{2\zeta k_B T}\xi(t).$$

Let $\tau = \frac{\zeta}{\kappa}$ and $\sigma = \sqrt{\frac{k_B T}{\kappa}}$.

Solution is Gaussian with:

$$\begin{aligned}
\langle x(t)\rangle &= \int_0^\infty \frac{dt'}{\tau}\, e^{-t'/\tau}\lambda(t-t') = \sum_n \left[-\tau\frac{\mathrm{d}}{\mathrm{d}t}\right]^n \lambda(t) \\
&\approx \lambda(t), \\
\langle \delta x(t)\delta x(t')\rangle &= \sigma^2 e^{-|t-t'|/\tau}.
\end{aligned}$$

Heavily damped harmonic oscillator (continued)

Excess power:

$$\mathcal{P}_{\text{ex}} = \zeta\dot{\lambda}(t)\int_0^\infty \frac{\mathrm{d}t'}{\tau}\,\mathrm{e}^{-t'/\tau}\dot{\lambda}(t-t') \approx \zeta\dot{\lambda}(t)^2.$$

Eigenmodes: $\tau_n = \frac{\tau}{n}$.
But: $\boldsymbol{\eta}^n\cdot\delta\boldsymbol{\phi} = \frac{\delta_{n,1}}{\sigma}$.

Combining all these results:

$$\frac{\Psi\,V}{\mathcal{P}_{\text{ex}}} = \frac{[\sigma^{-2}][\dot{\lambda}^2]}{[\zeta\dot{\lambda}^2]} = \frac{1}{k_{\text{B}}\,T\,\tau_{\text{min}}}.$$

Saturates bound!

Ising model

$\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow$ $\qquad E = -h\sum_n \sigma_n - J\sum_n \sigma_n\,\sigma_{n+1}.$

Estimate $\tilde{\lambda} = \mathrm{e}^{2\beta J}\tanh\beta h$ with $\hat{\tilde{\lambda}} = \frac{\sum_n \sigma_n}{N}$.

At the instant we pass through $h = 0$, we find:

$$\Psi = N\mathrm{e}^{-2\beta J}, \quad \mathcal{P}_{\text{ex}} = \frac{N\,k_{\text{B}}T\;V\cosh 2\beta J}{\alpha}, \quad \tau_{\text{min}} = \frac{\mathrm{e}^{2\beta J}\cosh 2\beta J}{\alpha},$$
$$\implies \frac{\Psi V}{\mathcal{P}_{\text{ex}}} = \frac{\alpha}{k_{\text{B}}T\,\mathrm{e}^{2\beta J}\cosh 2\beta J}$$

Saturates bound!

Ising model

$\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow$

$$E = -h \sum_n \sigma_n - J \sum_n \sigma_n \sigma_{n+1}.$$

Also estimate $\tilde{\lambda}^2 = \tanh \beta J$ with $\hat{\tilde{\lambda}}^2 = \frac{\sum_n \sigma_n \sigma_{n+1}}{N}$.

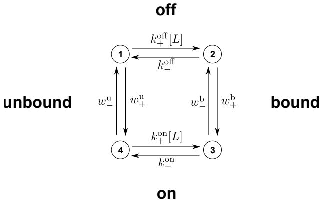
At the instant we pass through $h = 0$, we find:

$$\Psi_{11} = N \mathrm{e}^{-2\beta J}, \qquad \mathcal{P}_{\text{ex}} = \cosh 2\beta J \left[(\dot{\tilde{\lambda}}^1)^2 + \cosh^2 \beta J (\dot{\tilde{\lambda}}^2)^2 \right],$$

$$\Psi_{22} = N \cosh^2 \beta J, \qquad \tau_{\text{min}} = \frac{\mathrm{e}^{-2\beta |J|} \cosh 2\beta J}{2\alpha},$$

$$\implies \frac{\operatorname{tr}(\boldsymbol{\Psi}\mathbf{V})}{\mathcal{P}_{\text{ex}}} \leq \frac{\frac{1}{2} \mathrm{e}^{-2\beta [J]_+}}{(k_{\text{B}} T)\, \tau_{\text{min}}}.$$

Nonequilibrium receptor



Define: [Skoge et al. (2013)]

$$\kappa = \ln \frac{k_-^{\text{off}}}{k_-^{\text{on}}}, \qquad \gamma = \ln \frac{k_-^{\text{off}} w_-^{\text{b}} k_+^{\text{on}} w_+^{\text{u}}}{k_+^{\text{off}} w_+^{\text{b}} k_-^{\text{on}} w_-^{\text{u}}},$$

$$\lambda = \ln \frac{k_+[L]}{k_-}, \qquad \alpha = w_+^{\text{u/b}} + w_-^{\text{u/b}},$$

$$k_- = \sqrt{k_-^{\text{off}} k_-^{\text{on}}}.$$

Assume $k_+^{\text{on}} = k_+^{\text{off}} = k_+$ and $\alpha \ll k_-$.

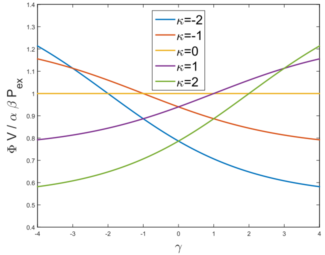
Estimate $\tilde{\lambda}$ with #active − #inactive.

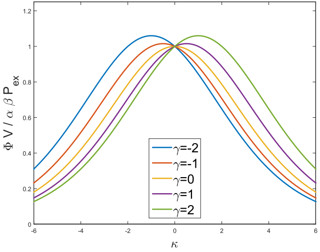
Estimate $\tilde{\lambda}$ with #active − #inactive.

At the instant we pass through $\lambda = 0$:

$$\Psi = N, \quad \mathcal{P}_{\text{ex}} = \frac{N\, k_{\text{B}} T\; V \cosh\left(\frac{\kappa-\gamma}{4}\right) \cosh\left(\frac{\kappa}{4}\right)}{\alpha \cosh\left(\frac{\gamma}{4}\right)}, \quad \tau_{\min} = \frac{1}{k_{-}(1+\mathrm{e}^{|\kappa|/2})}.$$

Satisfies bound, but a long way from saturating it, as $\tau_3 \sim \mathcal{O}(1/\alpha)$.





Section 5

Conclusions

Summary and future directions

We derive general relations between friction and information
$\Longrightarrow$ speed $\times$ precision bounded by power consumption.

We did this for an extremely *general* class of physical channels.

How close does biology get to this bound?
Need to *simultaneously* measure power, precision and speed.

Observations extended in time?
Beyond the slow signal limit?

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Section 6

Bonus slides

Autocorrelation of forces

$$
\begin{aligned}
\left\langle \delta\phi_\mu(0)\delta\phi_\nu(t')\right\rangle &= \sum_{ij} p_{ij}(0,t')\,\delta\phi^i_\mu\delta\phi^j_\nu \\
&= \sum_{ij} \pi_i \left[\exp(\mathbf{K}t')\right]_{ij}\delta\phi^i_\mu\delta\phi^j_\nu \\
&= \sum_{ij}\sum_a \pi_i u^a_i \,\mathrm{e}^{-t'/\tau_a}\eta^a_j\,\delta\phi^i_\mu\delta\phi^j_\nu \\
&= \sum_{ij}\sum_a \eta^a_i\,\mathrm{e}^{-t'/\tau_a}\,\eta^a_j\delta\phi^i_\mu\delta\phi^j_\nu \\
&= \sum_a \mathrm{e}^{-t'/\tau_a}\left(\boldsymbol{\eta}^a\cdot\delta\boldsymbol{\phi}_\mu\right)\left(\boldsymbol{\eta}^a\cdot\delta\boldsymbol{\phi}_\nu\right).
\end{aligned}
$$

$\tau_a =$ time constant of eigenmode a,
$\boldsymbol{\eta}^a/\mathbf{u}^a =$ left/right eigenvectors.

Beyond detailed balance

In general:

$$\zeta_{\mu\nu} = -\sum_{ij} \pi_i K^{\mathrm{D}}_{ij}\, \delta\phi^i_\mu \delta\phi^j_\nu, \qquad g_{\mu\nu} = \frac{1}{2}\left(\zeta_{\mu\nu} + \zeta_{\nu\mu}\right).$$

[Mandal and Jarzynski (2015)]

But: $\quad \mu \leftrightarrow \nu \quad \Longleftrightarrow \quad \mathbf{K}^{\mathrm{D}} \leftrightarrow \mathbf{K}^{\mathrm{D}\dagger} = \mathbf{K}^{\dagger\mathrm{D}}.$

$\Longrightarrow$ use $\widehat{\mathbf{K}}^{\mathrm{D}} = \frac{1}{2}\left(\mathbf{K}^{\mathrm{D}} + \mathbf{K}^{\mathrm{D}\dagger}\right).$

Satisfies detailed balance $\to$ use its eigenmodes.

back

Dual coordinates for exponential families

We're dealing with Boltzmann distributions:

$$\pi_i = \frac{e^{-\beta E_i}}{\mathcal{Z}}.$$

Exponential coordinates:

$$E = \sum_\mu \lambda^\mu \, \mathcal{O}_\mu.$$

Dual coordinates:

$$\tilde{\lambda}^\mu = \langle \mathcal{O}_\mu \rangle \,, \qquad \hat{\tilde{\lambda}}^\mu = \mathcal{O}_\mu.$$

These are the only coordinates that saturate the Cramér-Rao bound [Amari and Nagaoka (2007)].

back

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