Homework I
Due date: October 19, 11 pm

Instruction: You have to submit your answer to iCampus. Hand-writing is not accepted. You don't have to put your code, but please describe briefly how you proceeded your simulation and empirical studies. For example, in the simulation exercise, please make it self-explanatory by introducing the model and parameter values, and by explaining how you simulated. Your report does not look like an academic paper, but please make it as if I were your boss at your internship.

1. Let X_t and Y_t be mutually uncorrelated covariance stationary processes, i.e, $cov(X_s, Y_t) = 0$ for all s, t. Show that $Z_t = X_t + Y_t$ is stationary with autocovariance function equal to the sum of the autocovariance function of X_t and Y_t.

2. Let Z_t be weakly stationary with mean zero. Consider

$$X_t = Z_t + aZ_{t-1}, \quad Y_t = Z_t + bZ_{t-1},$$

where $|a| < 1$ and $b = 1/a$.

   (a) Express the autocovariance functions of X_t and Y_t in terms of the autocovariance function of Z_t.

   (b) Show that X_t and Y_t have the same autocorrelation functions.

3. Consider an AR(2) process

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2).$$

   (a) Show that the AR(2) process becomes weakly stationary if

$$\alpha_1 + \alpha_2 < 1, \quad -\alpha_1 + \alpha_2 < 1, \quad -1 < \alpha_2 < 1.$$

   (b) Under stationarity, obtain unconditional mean and variance of X_t.

   (c) Let $\rho(k)$ be the autocorrelation function of X_t. Show that we have

$$\rho(k) - \alpha_1 \rho(k-1) - \alpha_2 \rho(k-2) = 0 \quad \text{for} \quad k \geq 2.$$

   (d) Let $\bar{X} = T^{-1} \sum_{t=1}^{T} X_t$. Obtain the asymptotic distribution of $\sqrt{T}\bar{X}$.

   (e) Construct a 95% confidence interval of $\mathbb{E}[X_t]$ (Use Part (a)).

4. Let X_t be an MA(1) process

$$X_t = \varepsilon_t + \beta \varepsilon_{t-1}, \quad |\beta| < 1, \quad \varepsilon_t \sim WN(0, \sigma^2).$$

   (a) Let $\bar{X} = T^{-1} \sum_{t=1}^{T} X_t$. Obtain the asymptotic distribution of $\sqrt{T}\bar{X}$.

   (b) Construct a 95% confidence interval of $\mathbb{E}[X_t]$ (Use Part (a)).

5. Let X_t be a weakly stationary ARMA(1,1) process

$$X_t = \phi X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1},$$

where $\varepsilon_t \sim iidN(0, \sigma^2)$, $|\phi| < 1$ and $|\theta| < 1$. Let $\mathbb{E}[X_t] = \mu$ and $var(X_t) = \sigma_x^2$

   (a) Obtain μ and σ_x^2.

   (b) Determine the coefficients $\{\pi_i\}$ in

$$X_t = \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}.$$

   (c) Show that the autocorrelation functions of X_t is given by

$$\rho(1) = \frac{(1 + \phi\theta)(\phi + \theta)}{1 + \theta^2 + 2\phi\theta}$$

   and for $k \geq 2$ we have $\rho(k) = \phi^{k-1}\rho(1)$.

   (d) Let $(\phi, \theta, \sigma^2) = (0.3, 0.3, 1)$. Simulate the ARMA(1,1) model with $T = 200$ and $S = 10$ by drawing $X_1 \sim N(\mu, \sigma_x^2)$. Report a graph of $S = 10$ simulated time series (x-axis: time (1-T), y-axis: X_t) in one plot.

   (e) Simulate the same ARMA(1,1) model in Part (d) with $T = 200$ and $S = 10$ by drawing $X_1 = 10$. Report a graph of $S = 10$ simulated time series (x-axis: time (1-T), y-axis: X_t) in one plot. If you want to simulate a stationary process, how many samples would you recommend to throw away?

6. Download *Gross Private Domestic Investment* (GPDI) from the following website

   - https://fred.stlouisfed.org/series/GPDI

   Consider the quarterly data from 1970/01/01 to 2017/12/31, and let $Y_t = \log(GPDI_t) - \log(GPDI_{t-1})$.

   (a) Compute the Newey-West HAC estimator with a truncation parameter that is set according to the rule of thumb, $L = 0.75T^{1/3}$ (round this number to get an integer).

   (b) Using the HAC estimator obtained above, compute the 95% confidence interval for $\mathbb{E}[Y_t]$.

   (c) For each time series, draw ACF and PACF.

   (d) Using Ljung-Box test statistic, test the following null hypothesis

$$H_0 : \rho(1) = \rho(2) = \rho(3) = 0.$$

   What are the value of the test statistic, p-value and the test result at 5% significance level?