Appendix. Important Graphs and Experiments

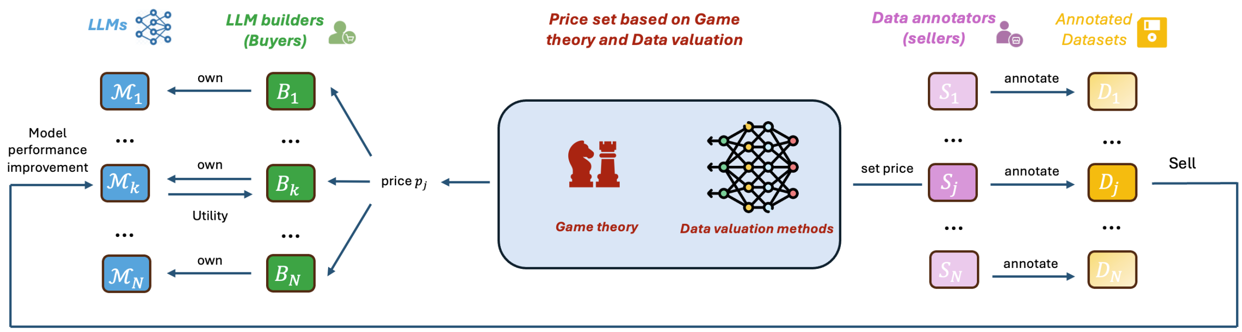
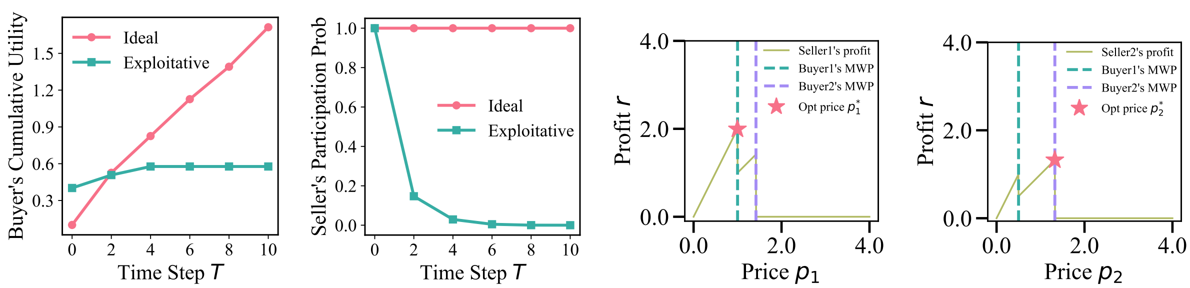
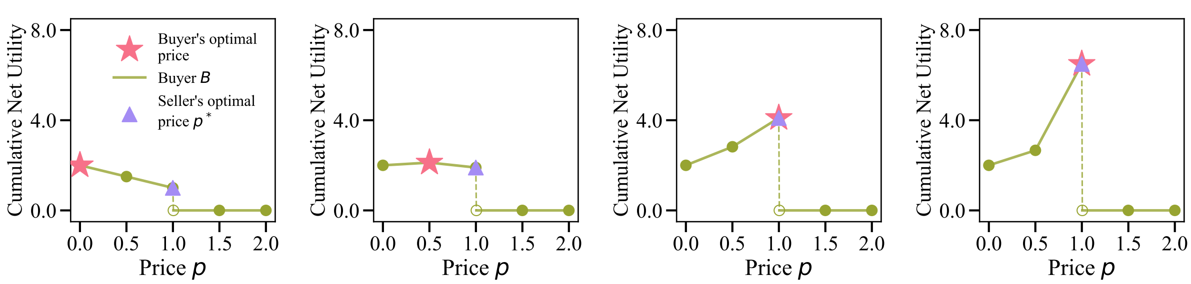


Figure 1 Illustration of the LLM data market, showing buyers (LLM builders) and sellers (annotators) interacting via a pricing mechanism based on game theory and data valuation.



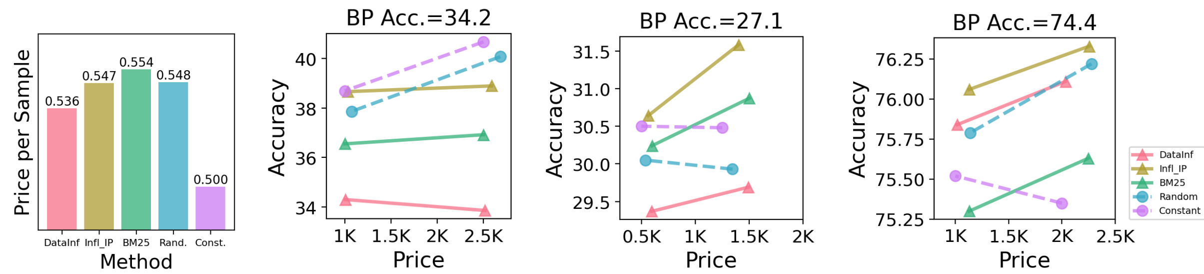
(a) Buyer's utility. (b) Seller's participation. (c) Seller S_1's profit. (d) Seller S_2's profit.

Figure 2 Buyer's cumulative utility and seller participation under ideal and exploitative pricing (Figures 2a and 2b); Profits of sellers S_1 and S_2 with buyer B_1's and B_2's MWP over price (Figures 2c and 2d).



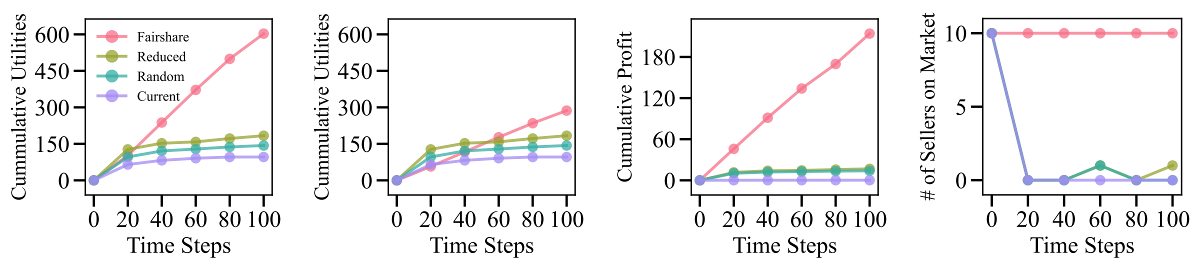
(a) $T = 1$ (b) $T = 2$ (c) $T = 5$ (d) $T = 10$

Figure 3 Analysis of the buyer's cumulative net utility as a function of the acquisition prices over time ($T = 1, 2, 5, 10$). The market has one buyer B and one seller S. Cumulative net utility over price is a discontinuous piecewise linear function that breaks at $p = 1$.



(a) Avg Price (Llama-1b). (b) MathQA (Llama-1b). (c) MedQA (Llama-1b). (d) PIQA (Llama-1b).

Figure 5 ***Left, middle-left, middle-right columns:*** **Buyers' model performance versus cost. Performance of models is shown *before purchasing* (BP) data, and after purchasing 2K and 4K data samples. *Right column:* Average price-per-sample cost of purchased data across math, medical, and physical reasoning data markets. Purchasing decisions were made using the constant, random, BM25, Infl_{IP}, and DataInf valuation methods. Additional analysis on the Pythia-1b/410m models is in Section 7.**



(a) High-budget buyer's utilities. (b) Low-budget buyer's utilities. (c) Sellers' profits. (d) Sellers' participation.

Figure 4 **Analysis of (1) buyer's cumulative utilities with high-budget buyer (Figure 4a) and low-budget buyer (Figure 4b), and (2) sellers' average cumulative profits (Figure 4c) and number of sellers in the market (Figure 4d) over time ($T = 100$). Model: Pythia-1b; Task: MedQA. Groups: (1) fairshare, (2) reduced, (3) random, and (4) current pricing.**

A. Ablation Study: Effect of Different Data Valuation Methods

This experiment assesses the impact of different *data valuation* strategies in the *fairshare* framework.

Setup. Using the models and datasets introduced earlier in Section 4.1, we run separate simulations for each market (math, medical, physical reasoning), testing different *data valuation* methods for the buyer. Each buyer is randomly assigned to use one of four valuation methods to select training data and fine-tune their model. The seller receives payments according to the data's assigned value. Valuation scores are normalized to $[0, 1]$ for pricing compatibility.

Data Valuation Methods. We consider the following methods, where each assigns a value to every data sample: (1) *Constant* baseline – assigns the same value, mimicking flat-rate pricing on platforms; (2) *Random* baseline – values drawn uniformly from $[0,1]$; (3) *Semantic* – uses BM25 (Trotman et al. 2014) to compute average similarity to the representative set; (4) *Influence-based:* returns a score which leverages learning gradients to estimate a data sample's avg. contribution to model learning of a representative dataset. Specifically, we use Infl_{IP} (Pruthi et al. 2020b, Xia et al. 2024) and DataInf (Kwon et al. 2023) (see Section 2). In addition, we provide experiments using data valuation through model re-training, which is considered an "oracle" for influence-based methods (Yu et al. 2024, Pruthi et al. 2020b) in Section 7.2. Further details on the experiment (e.g., model training) are described in Section 6.1.

Results. Figure 5 presents results across *data valuation* methods. Buyers using a valuation method (ie..g, BM25, Infl_{IP}, DataInf) in general achieved higher model performance across tasks. When considering the trade-off between cost and performance, Infl_{IP} offered the best balance – delivering strong model improvements at a lower cost than constant, random, and BM25 (see Figures 5a and 8a).

Our results highlight the benefits of learning-aware data valuation methods. By prioritizing high-impact data, they offer a better alternative for buyers, particularly those with limited budgets.

We also run the error analysis of *data valuation* methods. Following previous studies in approximating true data value (Jiao et al. 2024, Wang et al. 2024), we compute the correlation between oracle and Infl_{IP}, reporting spearman correlation of $0.54, 0.42$ for MathQA and PIQA (see Figure 7 in Section 7). As previously noted, our framework is method-agnostic and can directly benefit from future advances in data valuation accuracy and scalability.

A.1. Important Lemmas and Proofs

Lemma 1. With Assumptions 1 to 2, any exploitative pricing (i.e., $p_t < p_t^*, \forall t$) will only maximize cumulative utility within a finite horizon – after which it is strictly suboptimal.

Proof. Lemma 1 equivalently states that with Assumptions 1 to 2, the optimal pricing strategy for the buyer is also the ideal price p_t^*. Thus, we show that when the buyer pays the ideal price p_t^*, its total value is the largest, i.e.,

$$\mathbb{E}\left[u_t - p_t^* + \delta \mathbb{E}\left[r(p_t^*, p_t^*)G \mid u_t, b_t\right]\right] \geq \mathbb{E}\left[u_t - p_t + \delta \mathbb{E}\left[\pi(p_t, p_t^*)G \mid u_t, b_t\right]\right] \tag{11}$$

for all $p_t \in [0, p_t^*)$.

The seller will not set a price above its ideal price p_t^* as it will decreases its profit, since the ideal price p_t^* should be its profit-maximizing price. Any $p_t > p_t^*$ results lower profits.

Also, the buyer will not accept a price $p_t > p_t^*$, since it decreases the net *utility* gain $u_t - p_t$ without increasing seller's participation probability $\prod_{t=0}^{T-1} \pi(p_t, p_t^*)$.

Through some linear transformation, this is equivalent to show that

$$\mathbb{E}\left[p_t - p_t^* + \delta \mathbb{E}\left[G(\pi(p_t^*, p_t^*) - \pi(p_t, p_t^*)) \mid u_t, b_t\right]\right] \geq 0. \tag{12}$$

We first find the lower bound of G. We see that p_t^* gives a payoff of

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t \left(u_t - p_t^*\right)\right] \geq \frac{\min_{t\in[0,\infty)} \mathbb{E}[u_t - p_t^*]}{1-\delta}. \tag{13}$$

Therefore, we must have $G \geq \frac{\min_{t\in[0,\infty)} \mathbb{E}[u_t - p_t^*]}{1-\delta}$. Along with Assumptions 1 to 2, this gives us, for a given u_t and b_t,

$$\frac{\delta G\left(\pi(p_t^*, p_t^*) - \pi(p_t, p_t^*)\right)}{p_t^* - p_t} \geq \delta GL \geq 1, \tag{14}$$

implying that

$$\mathbb{E}\left[p_t - p_t^* + \delta \mathbb{E}\left[G(\pi(p_t^*, p_t^*) - \pi(p_t, p_t^*)) \mid u_t, b_t\right]\right] \geq 0. \tag{15}$$

Lemma 3 The optimal price for the seller S under our framework is

$$p_t^* = \min\{u_t, b_t\}, \forall t. \tag{16}$$

With assumptions 1 to 2, p_t^* gives the buyer the maximum cumulative *net utility* over infinite horizon.

Proof. In a single buyer and seller setting, we could trivially see that the optimal price for the seller $p_t^* = \min\{u_t, b_t\}$: if $p_t > p_t^*$, then the buyer would not purchase this dataset since the *net utility* would be negative; if $p_t < p_t^*$, then the seller's profit is not maximized.

Further, we refer to the proof of Lemma 1 for the second part.

Lemma 4 (*The Trade-Off Threshold Is Increasing as δ Decreases*)**.** *The threshold time period where the fairshare pricing obtains higher cumulative utility than any class of exploitative pricing is:*

$$t^* := \sup_{p_t < p_t^*, \forall t} \left\{ T \in \mathbb{N} : \mathbb{E}\left[\sum_{t=0}^{T} \delta^t \left((u_t - p_t^*) - \prod_{i=0}^{t-1} \pi(p_i, p_i^*)\,(u_i - p_i) \right)\right] \leq 0 \right\}. \tag{17}$$

And t^* is increasing as δ increases.

Proof. For any given class of exploitative pricing strategy, when δ increases, the part: $\sum_{t=0}^{T} \delta^t \left((u_t - p_t^*) - \prod_{i=0}^{t-1} \pi(p_i, p_i^*)\,(u_i - p_i)\right)$ increases. Therefore, for all class of exploitative pricing strategy, i.e., $p_t < p_t^*, \forall t$, then t^* also decreases.

A.2. Extra Appendices

Please refer to the supplementary appendix, which includes additional references, figures, and proofs, available at the following anonymous link: `https://submissionpaper1234.github.io/CIST-2025/appendix.pdf`.

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2. Influence-based Data Valuation

In Section 4, we introduced a gradient-based data attribution method, denoted as Infl_{IP}. In this section, we provide additional information on Infl_{IP}, which has been shown to be effective in training data selection in previous works (Pruthi et al. 2020b, Xia et al. 2024). Suppose we have a LLM parameterized by θ, and a train set D and a test set $\mathscr{D}'$. For a training sample $d \in D$, we wish to estimate its training impact on a test sample $d' \in D$. That is, we want to measure the impact of d on the model's loss on d' (i.e., $\mathscr{L}(d';\theta)$). As simple method of achieving this is to take training step – that is, a gradient descent step – on d and obtain:

$$\hat{\theta} = \theta - \eta \nabla \mathscr{L}(d;\theta) \tag{18}$$

where η is the learning rate. Then, in order to measure the influence of d towards d', we wish to find the change in loss on d':

$$\mathscr{L}(d';\theta) - \mathscr{L}(d';\hat{\theta}) \tag{19}$$

Instead of taking a single training step to measure the influence of $d \in D$ on d', we can instead approximate Equation (19) with using the following:

Lemma 5. *Suppose we have a LLM with parameters θ. We perform a gradient descent step with training sample d with learning rate η such that $\hat{\theta} = \theta - \eta \nabla \mathscr{L}(d;\theta)$. Then,*

$$\mathscr{L}(d';\theta) - \mathscr{L}(d';\hat{\theta}) \approx \nabla \mathscr{L}(d';\theta) \cdot \nabla \mathscr{L}(d;\theta)$$

See Section 5 for the proof.

Then, we set Infl_{IP} to be:

$$\text{Infl}_{\text{IP}} = \nabla \mathscr{L}(d';\theta) \cdot \nabla \mathscr{L}(d;\theta) \tag{20}$$

which is the dot-product between the learning gradients of d' and d.

3. Royalty model

So far, we have shown the case of *flat rate* (see Section 3.1), which is well-suited resource-rich buyers, such as leading tech companies whose LLMs generate significant economic value due to their wide-ranging impact and scalability. In this section, we introduce the *royalty model*, a contract framework that differs from the flat rate by offering a subscription-like structure. Under the royalty model, the price paid for training data is proportional to the future economic value generated by the LLM, providing a flexible and performance-based approach to data valuation. This scenario incorporates buyers in a less dominant position – those who are (1) uncertain about the prospective model outcome or (2) do not own a sufficient cash flow for purchasing data with full prices. We present updated decision-making models for buyers and sellers as follows.

Buyers. Unlike the flat pricing setting, the buyer B_k would alternatively pay with a *fractional price*. Suppose each dataset D_j is priced with an individual rate $\alpha_j \in [0,1)$ (as we denote $\alpha = (\alpha_1, \cdots, \alpha_N)$), then the price of an arbitrary data collection $u_k(x)$ is a fraction of its future marginal gain, i.e., $x^T p = f(\alpha, x) u_k(x)$, where the overall rate function $f : [0,1)^{|\alpha|} \times \{0,1\}^{|x|} \to [0,1)$ depends on specific contexts. We assume that f is a monotonically non-decreasing function of α. In this sense, the buyer B_k reduces the risk of losing b_k from its cash flow while the seller is betting on the potential value of the LLM M_k. Then we obtain an updated objective function for B_k:

$$g_{k,N,\text{frac}}(x) := (1 - f(\alpha, x)) u_k(x), \tag{21}$$

On the other hand, similar to the budget constraint (see Equation (2)), here each buyer B_k has a maximum rate $\overline{\alpha}_k$ that it is willing to pay. Then the buyer's purchasing problem is given as

$$\tilde{x}^{k,N,\text{frac}} := \underset{x \in \mathscr{X}_{k,N,\text{frac}}}{\arg\max} \; g_{k,N,\text{frac}}(x), \quad \text{s.t.} \tag{22}$$

$$\mathscr{X}_{k,N,\text{frac}} := \{x \mid g_{k,N,\text{frac}}(x) \geq 0, f(\alpha, x) \leq \overline{\alpha}_k\}, \tag{23}$$

And $\tilde{x}^{k,N,\text{frac}}$ is the optimal solution to $\max_{x \in \mathscr{X}_{k,N,\text{frac}}} g_{k,N,\text{frac}}(x)$ with a given rate vector α.

Sellers. In the fractional pricing setting, since the buyer B_k pays for the entire data collection, there should exists a fair and transparent allocation mechanism that distributes a portion of the total price charged to each individual dataset D_j. That is, $x_j p_j = \sum_{k=1}^{M} f_j(\alpha, \tilde{x}^{k,N,\text{frac}}) u_k(\tilde{x}^{k,N,\text{frac}})$, where $f(\cdot) = \sum_{j=1}^{N} f_j(\cdot)$. And we assume that for all $j \in [N]$, $f_j(\cdot)$ is monotonically non-decreasing over α. Therefore, we have an updated profit function for Se_j:

$$r_{\text{frac}}(\alpha_j) := \sum_{k=1}^{M} f_j(\alpha, \tilde{x}^{k,N,\text{frac}}) u_k(\tilde{x}^{k,N,\text{frac}}) - c_j. \tag{24}$$

which gives the following problem:

$$\alpha_j^* := \underset{\alpha_j \in \mathscr{A}_{j,M,\text{frac}}}{\arg\max} \; r_{\text{frac}}(\alpha_j), \quad \text{s.t.} \tag{25}$$

$$\mathscr{A}_{j,M} := \{\alpha_j \in [0,1) \mid r_{\text{frac}}(\alpha_j) \geq 0\}, \tag{26}$$

From this point onward, the market dynamics stays the same as in the previous section. It is noted that, compared to $\boxed{\max_{p_j \in \mathscr{P}_{j,M}} r(p_j)}$ where optimal flat rate is indirectly connected to the *utility*, the optimal rate of $\boxed{\max_{\alpha_j \in \mathscr{A}_{j,M,\text{frac}}} r_{\text{frac}}(\alpha_j)}$ offers a more direct representation of the *utility*.

3.1. Solving for optimal price of the royalty model

Similar to *flat rate*, in the case of *royalty model*, we need to solve buyer's problem $\boxed{\max_{x \in \mathscr{X}_{k,N-1,\text{frac}}} g_{k,N-1,\text{frac}}(x)}$ (before the arrival of S_j) and $\boxed{\max_{x \in \mathscr{X}_{k,N,\text{frac}}} g_{k,N,\text{frac}}(x)}$ (after the arrival of S_j) for all $k \in [M]$ and seller's problem $\boxed{\max_{\alpha_j \in \mathscr{A}_{j,M,\text{frac}}} r_{\text{frac}}(\alpha_j)}$.

Solve buyer's problems. For each feasible collection of datasets $x \in \mathscr{X}_{k,N-1,\text{frac}}$ (before the arrival of S_j), denotes it union with dataset D_j as x^{new}. Then we run the check: (1) if the *net utility* of x^{new} is larger than the one of $\tilde{x}^{k,N-1,\text{frac}}$, i.e., $g_{k,N,\text{frac}}(x^{\text{new}}) > g_{k,N,\text{frac}}(\tilde{x}^{k,N-1,\text{frac}})$, and (2) if the rate for purchasing x^{new} is still under the budget $\overline{\alpha}_k$, i.e., $f(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x^{\text{new}}) \leq \overline{\alpha}_k$, where $\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}$ denotes concatenating α_j to α. If the answer is positive to bother tests, then we can determine that the buyer B_k will change its decision and purchase S_j under the rate α_j.

Solve seller's problem. First, we consider when $\alpha_j = 0$. We could first find all $x \in \mathscr{X}_{k,N-1,\text{frac}}$ such that $g_{k,N,\text{frac}}(x^{\text{new}}) > g_{k,N,\text{frac}}(\tilde{x}^{k,N-1,\text{frac}})$. And we denote the set that contains such x as $\mathscr{X}^1_{k,N-1,\text{frac}}$. If $\mathscr{X}^1_{k,N-1,\text{frac}}$ is empty, then $\mathbb{1}_{\{B_k,D_j,p_j\}} = 0$ (indicating whether B_k will purchase D_j at price p_j or not), as S_j cannot bring positive value to B_k; else, then for all $x \in \mathscr{X}^1_{k,N-1,\text{frac}}$, thanks to the monotonicity of f_j over α_j, we could gradually increase α_j until the either of the two criterion are met first: (1) we find the largest α_j such that $g_{k,N,\text{frac}}(x^{\text{new}}) > g_{k,N,\text{frac}}(\tilde{x}^{k,N-1,\text{frac}})$, and (2) $f_j(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x^{\text{new}}) \leq \overline{\alpha}_k$. Then we have the following property about the optimal rate α_j^* for Equation (25):

Lemma 6 (Characterization of α_j^* under *royalty model*)**.** *Define α_j^x as*

$$\min\left\{\sup_{\alpha_j\in[0,1)}\left\{\alpha_j : f_j(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x^{new}) < 1-(1-f_j(\alpha,\tilde{x}^{k,N-1,frac}))\frac{u_k(\tilde{x}^{k,N-1,frac})}{u_k(x^{new})}\right\},\right.$$
$$\left.\sup_{\alpha_j\in[0,1)}\left\{\alpha_j : f_j(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x^{new}) \leq \overline{\alpha}_k\right\}\right\}. \tag{27}$$

For every $x \in \mathscr{X}^1_{k,N-1,frac}$ and all $k \in [M]$, we obtain α_j^x and their union $\cup_{k=1}^M \cup_{x\in\mathscr{X}^1_{k,N-1,frac}} \{\alpha_j^x\}$. Then we have $\alpha_j^* \in \cup_{k=1}^M \cup_{x\in\mathscr{X}^1_{k,N-1,frac}} \{\alpha_j^x\}$.

Remark 3.1 (Similarities between *flat rate* and *royalty model*)**.** *Observing from Lemmas 2 and 6, we see that the both the optimal price p_j^* and the optimal rate α_j^* are closely tied to B_k's maximum willingness to pay. That is, compared to the market prior to the arrival of S_j, the optimal values are characterized by the minimum of two factors: (1) marginal utility that S_j provides to B_k and (2) B_k's budget surplus. It is also noted that, under royalty model, the rate function f also plays an important role as it determines the how the single rate α_j affects the total rate that B_k pays.*

4. Applications for Real-Life Scenarios

In real-life settings, the relationship between the data valuation of a training sample and the buyer's utility u_k (i.e., the economical value, which may be expressed in dollar amounts) can have different mappings, as mentioned in Section 3.1. Suppose the data valuation function is denoted as $v_k : D \to \mathbb{R}$

for a dataset D. Then, a buyer may expect a linear relationship between v_k and u_k, where the utility increases as the data valuation score increases. Alternatively, a buyer may prefer to only purchase data beyond a certain threshold for v_k. In this section, we present three types mappings between v_k and u_k to reflect these scenarios: *linear*, *discrete*, and *zero-one* mappings. We show that these mappings can be easily adapted to our proposed framework in Section 3. We only present the updated buyer's purchasing problem (Equation (2)) since the seller's pricing problem (Equation (4)) stays the same.

4.1. Linear Outcome

In practice, there are many applications where u_k is an affine function of v_k. As previously mentioned, training LLMs on data with higher valuation scores v_k can result in better economic value towards downstream model performance, as shown in previous works (Xia et al. 2024, Yu et al. 2024). In this outcome setting, in addition to considering u_k to be a affine function of v_k, we also include a bias variable β to account for other potential other factors that are independent of v_k. Therefore, we can set $u_k = \gamma v_k(x) + \beta$ into Equation (1), where $\gamma \in \mathbb{R}_+$ is a known coefficient, and obtain buyer B_k's net utility function for the linear outcome:

$$g_{k,N}(x) = \gamma v_k(x) + \beta - x^T p, \tag{28}$$

To obtain optimal price p^*, we can directly refer to same procedure described in Section 3 using set values for γ and β.

4.2. Discrete Outcome

There are also many applications where u_k is discrete. For instance, if the data buyers are participating in an LLM benchmark challenge, such as MMLU (Hendrycks et al. 2021), then training on data that falls within various ranges v_k may lead to drastically different model performance, and hence leaderboard rankings.

To mirror this, consider u_k to be a category variable. We denote $\{c_h\}_{h=1}^H$ as a strictly increasing set of numbers such that when $v_k \in [c_h, c_{h+1})$, the buyer will receive reward $u_{k,h}$. We also assume that $u_{k,h+1} > u_{k,h}$ since higher data valuation scores may lead to a larger reward. Therefore, we could set $u_k = \sum_{h=1}^H \mathbb{1}_{\{v_k(x)\in[c_h,c_{h+1})\}} u_{k,h}(x)$ and rewrite buyer B_k's net utility function as

$$g_{k,N}(x) = \sum_{h=1}^{H} \mathbb{1}_{\{v_k(x)\in[c_h,c_{h+1})\}} u_{k,h}(x) - x^T p. \tag{29}$$

We again apply the same procedure in Section 3 to solve for the optimal pricing.

4.3. Zero-One Outcome

There are scenarios where the data buyers are risk-adverse and focus on the effects of rare events. In these cases, suppose that v_k is normalized between [0, 1]. Then buyers may wish to purchase training data with higher values of v_k, assuming that purchasing data with lower v_k may result in severe adverse effects. For instance, data buyers who are building AI for healthcare should not purchase data with incorrect medical information, and even a small amount of contaminated data can result in severe real-life consequences such as mis-diagnosis (Jin et al. 2021, Zhou et al. 2023) or unsuitable medical protocols in emergency situations (Sun et al. 2024). Therefore, in this context, we consider u_k as a generalized Bernoulli distribution. The downstream outcome has a small positive reward $\underline{u}$ with probability v_k (normal events) and a massive negative reward $\overline{u}$ with probability $1 - v_k$ (undesirable rare events). And we assume that $\mathbb{E}(u_k) > 0$. Therefore, we can plug in and obtain buyer B_k's net utility function:

$$g_{k,N}(x) = \mathbb{E}[u_k(x)] - x^T p \tag{30}$$

$$= v_k(x)(\underline{u} - \overline{u}) + \overline{u} - x^T p, \tag{31}$$

which is an affine function of v_k. Therefore, we again apply same procedure in Section 3 to solve for the optimal pricing.

4.4. Multiple tasks

In practice, many LLMs are evaluated over multiple tasks (Hendrycks et al. 2021). To this end, we consider the context where buyer B_k wishes their model $\mathscr{M}_k$ to perform well across multiple tasks, denoted as Q. Each data valuation score for a task is denoted by $v_1^k, \cdots, v_Q^k$ and the vector of all task valuations is denoted as $v_k = (v_{k,1} \cdots v_{k,Q})$. Then we consider that the utility u_k is an affine function of the utility in each task, denoted by $u_k = (u_{k,1} \cdots u_{k,Q})$ that is, $u_k = \theta^T u_k + \varepsilon$, where $\theta \in \mathbb{R}^Q$ is a coefficient vector and $\varepsilon \in \mathbb{R}$ denotes other factors independent from u_k. We also assume that the each task is one of three categories mentioned in the last section. Therefore, we can rewrite u_k as a function of v_k, which gives $u_k = \theta^T u_k(v_k) + \varepsilon$. Therefore, the buyer's net utility function becomes

$$g_{k,N}(x) = \theta^T u_k(v_k(x)) + \varepsilon - x^T p. \tag{32}$$

whose solution could adopt the same procedure as described in Section 3 to solve for the optimal pricing.

5. Extra Lemmas and Proofs

Lemma 7 (*Participation Loss Is Lower-Bounded*)**.** *With Assumption 1, let* $P := \lim_{T\to\infty} P_T$ *and* $S := \sum_{i=0}^{\infty}(1 - \pi(p_i, p_i^*))$. *Then for the class of all exploitative pricing strategies where* $p_t \leq p_t^*, \forall t$, *we have (1)* P_T *is strictly decreasing and (2) The limit of reduced participation is lower bounded, i.e.,* $1 - P \geq 1 - e^{-S} > 0$.

Proof. The first part is trivial to see as we assume that $\pi(p_i, p_i^*) < 1$ for all $p_i < p_i^*$ from Assumption 1.

For the second part, we first denote $P := \lim_{t\to\infty} P_t$. Then we take the following transformation:

$$\log P = \log\left(\prod_{i=0}^{\infty} \pi(p_i, p_i^*)\right) = \sum_{i=0}^{\infty} \log\left(\pi(p_i, p_i^*)\right). \tag{33}$$

Since we have $0 < \pi(p_i, p_i^*) < 1$, then $\log\left(\pi(p_i, p_i^*)\right) \leq \pi(p_i, p_i^*) - 1$, indicating that

$$\log P = \sum_{i=0}^{\infty} \log\left(\pi(p_i, p_i^*)\right) \leq \sum_{i=0}^{\infty} \left(\pi(p_i, p_i^*) - 1\right) = -S, \tag{34}$$

which gives $P \leq e^{-S}$ by exponentiating both sides.

Lemma 2. Seller S_j's optimal price for D_j is characterized as one of buyers' MWP:

$$p_j^* \in \cup_{k=1}^M \mathrm{MWP}_k. \tag{35}$$

Proof. Recall that without dataset D_j, each buyer B_k has already solved $\max_{x \in \mathscr{X}_{k,N-1}} g_{k,N-1}(x)$ according to our market definition in Section 3, where $\mathscr{X}_{k,N-1}$ is the set of all feasible purchase decisions. Next, after seller S_j (with D_j) has arrived on the market, we analyze the conditions in which B_k will purchase D_j at a potential price p_j. For each feasible purchase decision (i.e., a collection of datasets), represented by $x \in \mathscr{X}_{k,N-1}$, let $x + e_j$ denote its union with D_j, where e_j is the unit vector indicating D_j is selected. For buyer B_k to change their previous decision to purchase D_j, there are two requirements that need to be satisfied. First, we must have:

$$g_{k,N}(x + e_j) > g_{k,N-1}(\tilde{x}^{k,N-1}). \tag{36}$$

That is, the *net utility* $g_{k,N}(x + e_j)$ of purchasing decisions $x + e_j$, must be larger than the *net utility* $g_{k,N-1}(\tilde{x}^{k,N-1})$ of a previous optimal purchasing decision $\tilde{x}^{k,N-1}$. It is also noted that $g_{k,N}(\tilde{x}^{k,N-1}) = g_{k,N-1}(\tilde{x}^{k,N-1})$. Second, for buyer B_k to purchase $x + e_j$ at price p_j, we must fulfill the budget constraint:

$$p_j \leq b_k - \left(\tilde{x}^{k,N-1}\right)^T p = \Delta b_k(\tilde{x}^{k,N-1}), \tag{37}$$

which ensures that purchasing D_j does not exceed the buyer's budget b_k. If both requirements are satisfied, then the buyer B_k will change their previous purchasing decision in order to purchase D_j under the price p_j. This procedure is presented in detail in Algorithm 1 in Section 7.1.

Next, given the conditions for the buyer B_k to purchase D_j, the seller must solve $\max_{p_j \in \mathscr{P}_{j,M}} r(p_j)$ to find the optimal price p_j^*. First, we consider an edge case where the price of dataset D_j is set as $p_j = 0$. For a buyer B_k, we denote $\mathscr{X}_{k,N}^1$ as the set of all purchasing decisions where including D_j in the purchase improves the buyer's previous net utility $g_{k,N}(\tilde{x}^{k,N-1})$. That is, for every $x+e_j \in \mathscr{X}^1_{k,N}$, we have $g_{k,N}(x+e_j) > g_{k,N}(\tilde{x}^{k,N-1})$. If $\mathscr{X}^1_{k,N}$ is empty, then B_k will not purchase D_j at any price, since D_j cannot bring positive improved *net utility* to B_k. Then, when p_j gradually increases and exceeds $\max_{x\in\mathscr{X}_{k,N-1}}\{\min\{\Delta u_k(x+e_j), \Delta b_k(\tilde{x}^{k,N-1})\}\}$, then B_k will decide not to purchase D_j, causing the value of $\sum_{k=1}^{M} \tilde{x}_j^{k,N}(p)$ to drop by one. Since the profit function $r(p_j)$ is a piecewise linear function, the its optimal point must be one of its breakpoints.

Lemma 5. Suppose we have a LLM with parameters θ. We perform a gradient descent step with training sample d with learning rate η such that $\hat{\theta} = \theta - \eta\nabla\mathscr{L}(d;\theta)$. Then,

$$\mathscr{L}(d';\theta) - \mathscr{L}(d';\hat{\theta}) \approx \nabla\mathscr{L}(d';\theta)\cdot\nabla\mathscr{L}(d;\theta)$$

Proof. First, we consider the change in loss of z' using a first-order approximation:

$$\mathscr{L}(d';\hat{\theta}) = \mathscr{L}(d';\theta) + \nabla\mathscr{L}(d';\theta)\dot{(}\hat{\theta}-\theta) + \mathscr{O}(||\hat{\theta}-\theta||^2) \tag{38}$$

$$\mathscr{L}(d';\theta) - \mathscr{L}(d';\hat{\theta}) = -\nabla\mathscr{L}(d';\theta)\dot{(}\hat{\theta}-\theta) + \mathscr{O}(||\hat{\theta}-\theta||^2) \tag{39}$$

Next, suppose a gradient descent step is taken on training sample d, and the model parameters are updated as: $\hat{\theta} = \theta - \eta\nabla\mathscr{L}(d;\theta)$. Thus, we have $\hat{\theta} - \theta = -\eta\nabla\mathscr{L}(d;\theta)$, and the change in loss can be written as

$$\mathscr{L}(d';\theta) - \mathscr{L}(d';\hat{\theta}) \approx \eta\nabla\mathscr{L}(d';\theta)\cdot\nabla\mathscr{L}(z;\theta) \propto \nabla\mathscr{L}(d';\theta)\cdot\nabla\mathscr{L}(d;\theta) \tag{40}$$

Given that η is a constant.

Lemma 6 Define α_j^x as

$$\min\left\{\sup_{\alpha_j\in[0,1)}\left\{\alpha_j : f_j(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x^{\text{new}}) < 1-(1-f_j(\alpha,\tilde{x}^{k,N-1,\text{frac}}))\frac{u_k(\tilde{x}^{k,N-1,\text{frac}})}{u_k(x^{\text{new}})}\right\},\right.$$
$$\left.\sup_{\alpha_j\in[0,1)}\left\{\alpha_j : f_j(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x^{\text{new}}) \le \overline{\alpha}_k\right\}\right\}. \tag{41}$$

For every $x \in \mathscr{X}^1_{k,N-1,\text{frac}}$ and all $k \in [M]$, we obtain α_j^x and their union $\cup_{k=1}^M \cup_{x\in\mathscr{X}^1_{k,N-1,\text{frac}}} \{\alpha_j^x\}$. Then we have $\alpha_j^* \in \cup_{k=1}^M \cup_{x\in\mathscr{X}^1_{k,N-1,\text{frac}}} \{\alpha_j^x\}$.

Proof. We show that, for every $x \in \mathscr{X}^1_{k,N-1,\text{frac}}$, α_j^x gives the largest revenue of $x+e_j$ for S_j (note that $x+e_j = x^{\text{new}}$). Recall that in the main text, we need to increase α_j from zero until we find the largest α_j such that either of:

1. $g_{k,N,\text{frac}}(x+e_j) > g_{k,N,\text{frac}}(\tilde{x}^{k,N-1,\text{frac}})$,
2. $f_j(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x+e_j) = \overline{\alpha}_k$.

If we rewrite the first condition, we are essentially looking for α_j such that

$$\sup_{\alpha_j\in[0,1)} \left\{\alpha_j : f_j(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x+e_j) < 1-(1-f_j(\alpha, \tilde{x}^{k,N-1,\text{frac}}))\frac{u_k(\tilde{x}^{k,N-1,\text{frac}})}{u_k(x+e_j)}\right\} \tag{42}$$

Then we see that the revenue that for each $x \in \mathscr{X}^1_{k,N-1,\text{frac}}$, seller S_j can make from buyer B_k is

$$f_j(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x+e_j)u_k(x_j(\alpha)), \tag{43}$$

where $f_j(\begin{bmatrix}\alpha^T & \alpha_j\end{bmatrix}, x+e_j)$ is a non-decreasing function over α_j while other terms stays fixed. It indicates that α_j^x is the largest α_j that the seller S_j could set for buyer B_k to purchase S_j. Therefore, the optimal rate α_j^* is one of the rates $\cup_{k=1}^M \cup_{x\in\mathscr{X}^1_{k,N-1,\text{frac}}} \{\alpha_j^x\}$.

6. Additional Experimental Details

6.1. Data Valuation Experiments

For each dataset, we randomly sample 200 demonstrations from the validation set to form a representative dataset [4]. Each data sample in the market is then scored based on its similarity to this representative set.

Model Training: After obtaining purchasing decisions for all data samples, the buyers train their models using the purchased data. In order to conduct a fair comparison across buyers, we sample a set number of data from the buyers' purchases (shown in Figure 5). We train each model (i.e.,

[4] Note: For PIQA we take 200 samples from the training set since the validation set is commonly reserved for testing.

buyer) on these samples separately using LoRA Hu et al. (2021) for 3 epochs, with a learning rate of 2e-5 and batch size 32. All models are trained on A6000 GPUs on single GPU settings and take less than 1 hour.

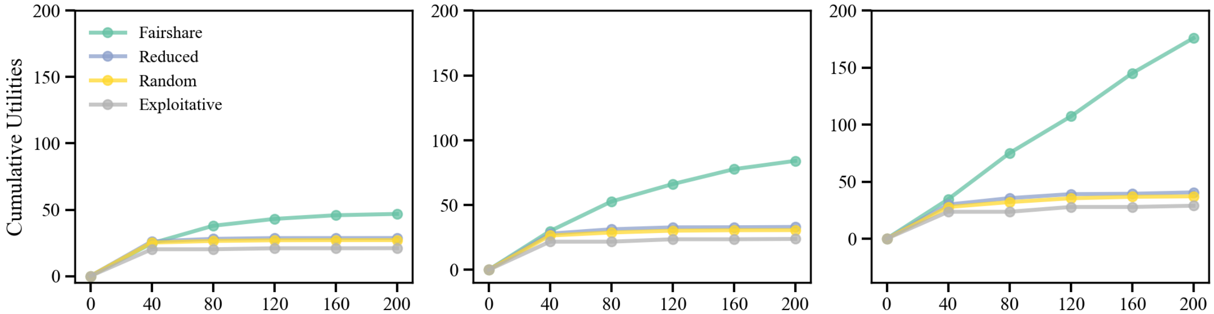
Model Evaluation: For evaluation, we use the test splits of the previously mentioned datasets. In particular, we use 5-shot evaluation on the MathQA test set, and 4-shot evaluation in on the MedQA test. Table 2 in Section 7 shows the demonstrations used for 5-shot and 4-shot evaluation.

Market/Pricing Setup: We reserve 1% of the samples from each dataset's training split to represent the existing data in their respective markets. Each data sample was randomly priced between $(0,1]$. Next, for each remaining data sample in the training set, we determine whether each buyer will purchase the data sample at potential price points $[0.5, 0.625, 0.75, 0.875, 1.0]$ by solving Equation (2). The seller then sets their prices according to Equation (4). We price data separately for each data valuation method. This assesses the method's ability to discern whether a new data sample is worth purchasing for each buyer given the existing market data, as noted in our analysis in Section 3.

6.2. Data Pricing Experiments

Experiment Setups: We simulate two buyer budgets at each time step t. The first buyer (high budget) has a budget uniformly randomly generated between 95% and 100% of the total utilities of all 10 datasets listed in the market. The second buyer (low budget) has a budget uniformly randomly generated between 90% and 95% of the total utilities of all 10 datasets listed in the market. At each time period, seller's arriving orders are randomly shuffled. And they prices their own datasets sequentially.

Robustness Check. As discussed in Lemma 4, the threshold t^* when *fairshare* becomes optimal for the buyer) increases as the discount factor δ decreases. To run a robustness check, for the high-budget buyer in Figure 4a, setting $\delta = 0.999, 0.99, 0.98$ yields $t^* = 32, 38, 44$ respectively, which is consistent with our theoretical analysis. In below, we compare the change of buyer's accumulative utilities over different values of δ.



(a) $\delta = 0.980$

(b) $\delta = 0.990$

(c) $\delta = 0.999$

Figure 6 **Buyer's accumulative utilities over time periods for** $\delta = 0.980, 0.990, 0.999$.

7. Additional Experiments and Figures

7.1. Algorithms

Algorithm 1 Determine if buyer B_k will purchase dataset D_j at price p_j

Inputs: prices p, previous optimal $\tilde{x}^{k,j-1}$, feasible set $\mathscr{X}_{k,j-1}$, price p_j, budget b_k

Output: indicator $\mathbb{1}_{\{B_k,D_j,p_j\}}$

Initialize $\mathbb{1}_{\{B_k,D_j,p_j\}} \leftarrow 0$

for $x \in \mathscr{X}_{k,j-1}$ **do**

  $x^{\text{new}} \leftarrow x + e_j$

  if $g_{k,j}(x^{\text{new}}) > g_{k,j-1}(\tilde{x}^{k,j-1})$ **and** $x^\top p + p_j \leq b_k$ **then**

    $\mathbb{1}_{\{B_k,D_j,p_j\}} \leftarrow 1$

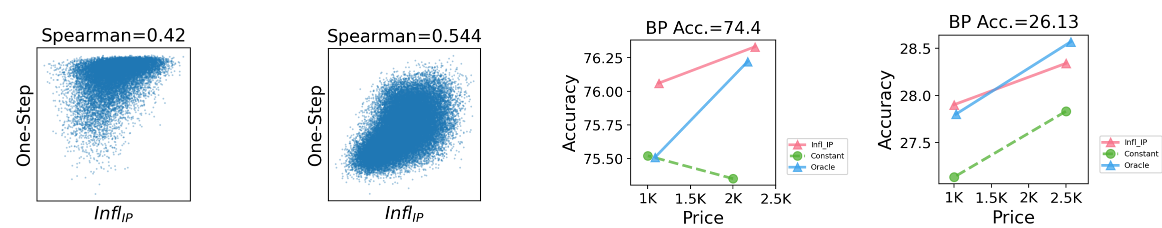
    break

  end if

end for

return $\mathbb{1}_{\{B_k,D_j,p_j\}}$

7.2. Data Valuation Oracle Experiments



(a) PIQA (Llama-1b). (b) MathQA (Pythia-1b). (c) PIQA (Llama-1b). (d) MathQA (Pythia-1b).

Figure 7 ***Left, middle-left columns:*** **Correlation analysis between oracle and Infl_{IP} valuation.** ***Right, middle-right columns:*** **Performance analysis between oracle and Infl_{IP} valuation.**

Influence-based methods, such as Infl_{IP} approximate the influence of a sample d on a d' for a model parameterized by θ by estimating the effects of training or "upweighting" (alternatively,

Algorithm 2 Market Dynamic Procedure

Inputs: Buyers $\{B_k\}_{k=1}^{M}$, Sellers $\{S_j\}_{j=1}^{N}$
Initialization: Buyers $\{B_k\}_{k=1}^{M}$ enter the market
for $j = 1$ to N **do**
  Seller S_j enters with potential prices $\mathscr{P}_{j,M}$ for dataset D_j
  for all $p_j \in \mathscr{P}_{j,M}$ **do**
    for $k = 1$ to M **do**
      Buyer B_k solves:

$$\tilde{x}^{k,j-1} = \arg\max_{x \in \mathscr{X}_{k,j}} g_{k,j}(x)$$

      to decide whether to purchase D_j at price p_j ▷ See Eqn. 2
    end for
    Seller computes net profit $r(p_j)$ assuming price p_j ▷ See Eqn. 3
  end for
  Seller selects:

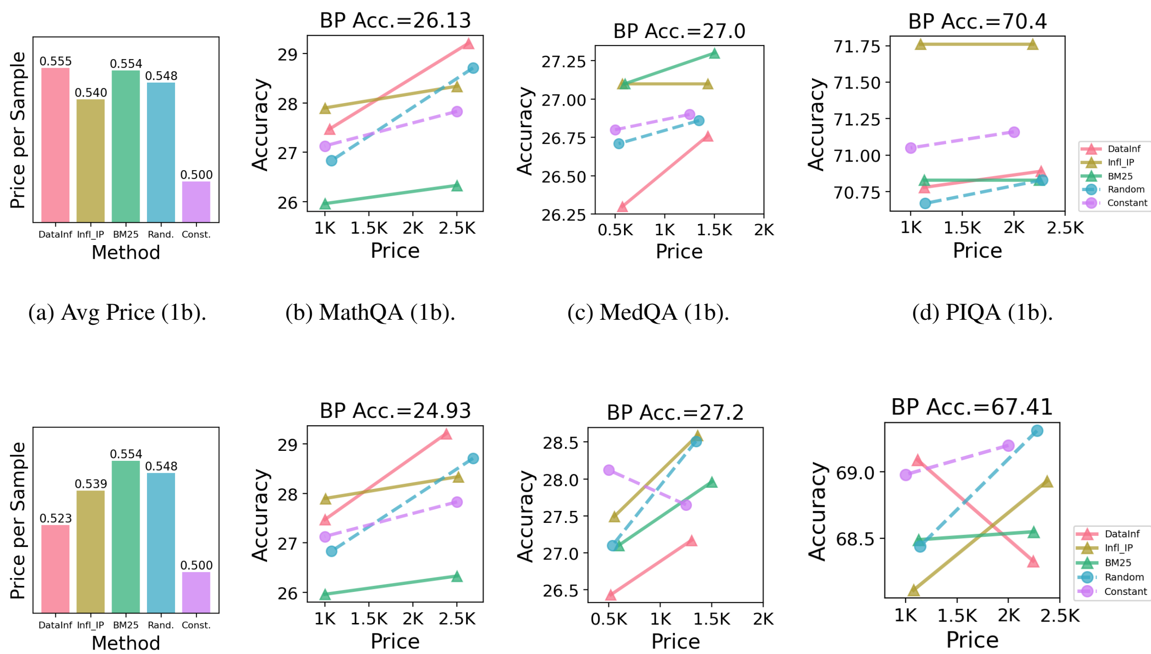
$$p_j^* = \arg\max_{p_j \in \mathscr{P}_{j,M}} r(p_j)$$

  and sets p_j^* as the fixed price for D_j ▷ See Eqn. 4
end for

removing/"downweighting") on d (see Section 2). In past literature, $\mathrm{Infl}_{\mathrm{IP}}$ is validation through an "Oracle" (One-Step Training) score, which we denote as $\mathrm{Oracle}(d,d') = \mathscr{L}(d';\theta) - \mathscr{L}(d';\hat{\theta})$, where $\hat{\theta} = \theta - \eta\nabla\mathscr{L}(d;\theta)$ and η is the learning rate Pruthi et al. (2020b), Jiao et al. (2024), Yu et al. (2024).

To compare the difference between $\mathrm{Infl}_{\mathrm{IP}}$ valuation versus its oracle valuation, we conduct the same experiments described in Section A. Figure 7 shows that $\mathrm{Infl}_{\mathrm{IP}}$ and Oracle have decent correlation in their agreement in their valuation of the sellers' data, which supports findings in previous works Jiao et al. (2024). We note that in the case where correlation is decent, such as in Figure 7b, the final model performance between these methods is close, as seen in Figure 7d. In the case where correlation is lower, such as in Figure 7a, the final model performance between these methods initially have a gap, but become more similar as the amount of data purchased increases, as seen in Figure 7c. This suggests that in practice, even in cases when the agreement between $\mathrm{Infl}_{\mathrm{IP}}$ and Oracle may not be very high, final model performance resulting from these two methods can still be similar.

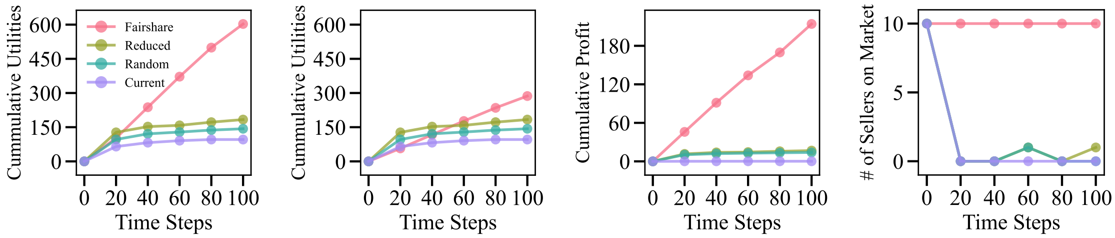
7.3. Additional Experimental Results



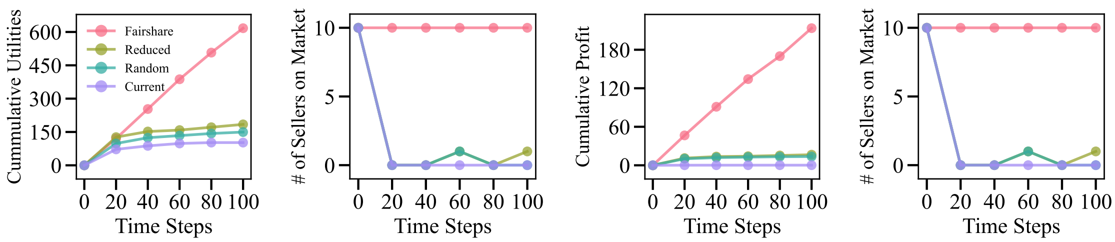
(a) Avg Price (1b). (b) MathQA (1b). (c) MedQA (1b). (d) PIQA (1b).

(e) Avg Price (410m). (f) MathQA (410m). (g) MedQA (410m). (h) PIQA (410m).

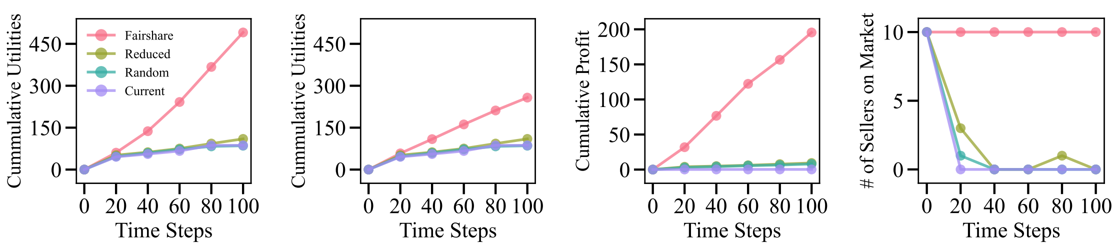
Figure 8 Buyers' model performance and costs across math, medical, and physical reasoning data markets. Top row: Pythia-1b; bottom row: Pythia-410m. Purchasing decisions were made using the constant, random, BM25, and $\mathrm{Infl}_{\mathrm{IP}}$ data valuation methods (see Section A for details).



(a) High-budget buyers' utilities (Pythia-1b). (b) Low-budget buyers' utilities (Pythia-1b). (c) Sellers' profits (Pythia-1b). (d) Sellers' participation (Pythia-1b).

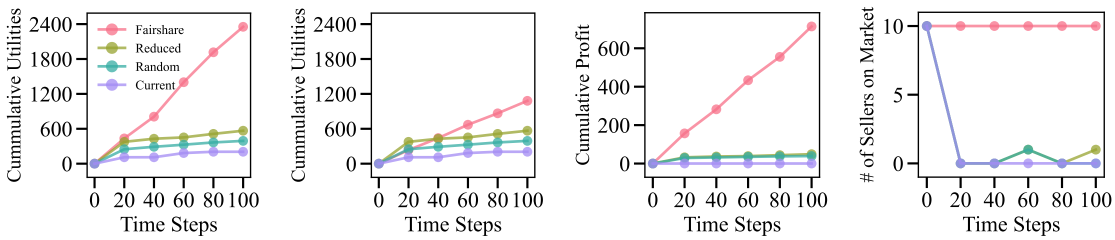


(e) High-budget buyers' utilities (Pythia-410m). (f) Low-budget buyers' utilities (Pythia-410m). (g) Sellers' profits (Pythia-410m). (h) Sellers' participation (Pythia-410m).



(i) High-budget buyers' utilities (Llama-3.2-Inst.-1b). (j) Low-budget buyers' utilities (Llama-3.2-Inst.-1b). (k) Sellers' profits (Llama-3.2-Inst.-1b). (l) Sellers' participation (Llama-3.2-Inst.-1b).

Figure 9 Analysis of (1) buyer's cumulative utilities with high-budget buyer (Figures 9a, 9e and 9i) and low-budget buyer (Figures 9b, 9f and 9j), and (2) sellers' average cumulative profits (Figures 9c, 9g and 9k) and number of sellers in the market (Figures 9d, 9h and 9l) over time ($T = 100$). Model: Pythia-1b, Pythia-410m, and Llama-3.2-Inst.-1b; Task: MedQA. Experimental groups: (1) fairshare, (2) reduced, (3) random, and (4) current pricing.

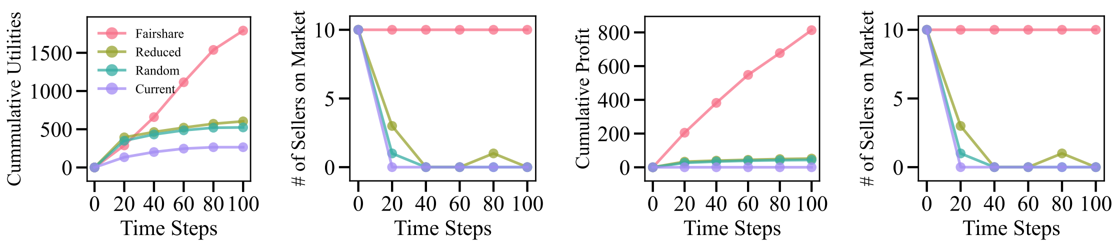


(a) High-budget buyers' utilities (Pythia-1b).

(b) Low-budget buyers' utilities (Pythia-1b).

(c) Sellers' profits (Pythia-1b).

(d) Sellers' participation (Pythia-1b).

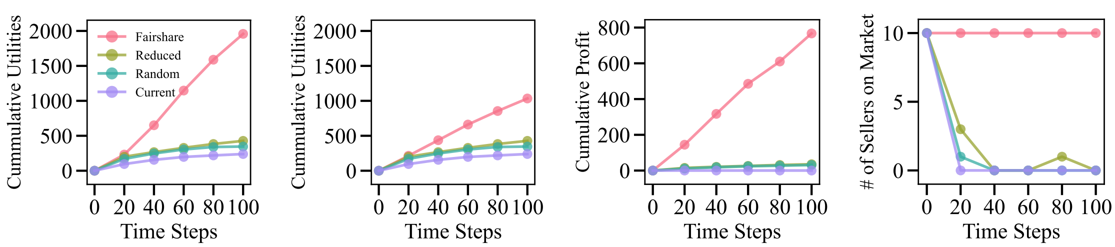


(e) High-budget buyers' utilities (Pythia-410m).

(f) Low-budget buyers' utilities (Pythia-410m).

(g) Sellers' profits (Pythia-410m).

(h) Sellers' participation (Pythia-410m).



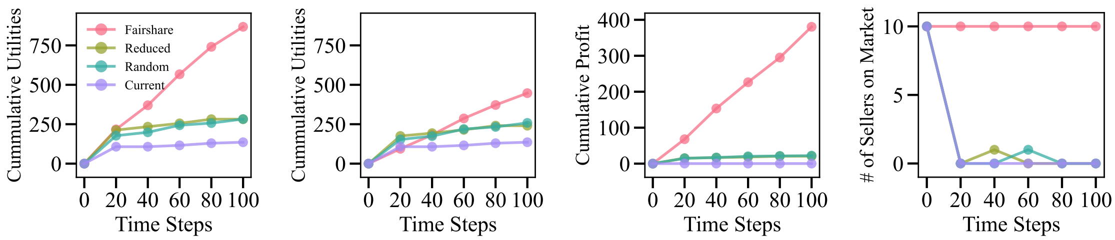
(i) High-budget buyers' utilities (Llama-3.2-Inst.-1b).

(j) Low-budget buyers' utilities (Llama-3.2-Inst.-1b).

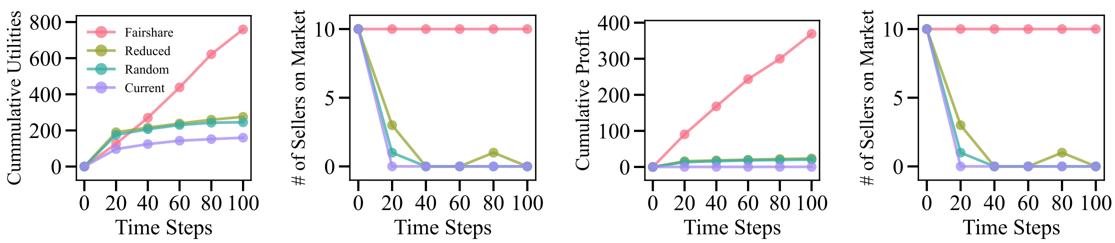
(k) Sellers' profits (Llama-3.2-Inst.-1b).

(l) Sellers' participation (Llama-3.2-Inst.-1b).

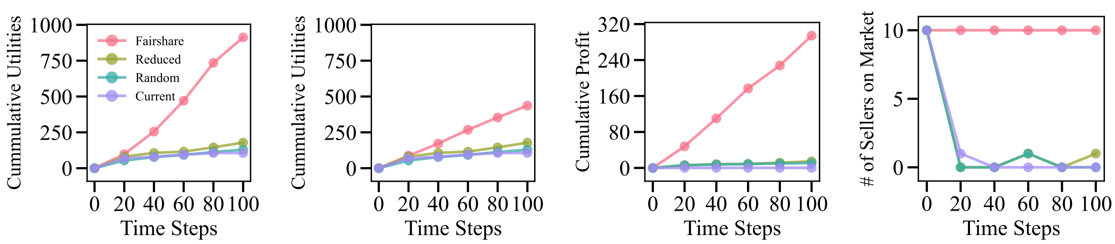
Figure 10 **Analysis of (1) buyer's cumulative utilities with high-budget buyer (Figures 10a, 10e and 10i) and low-budget buyer (Figures 10b, 10f and 10j), and (2) sellers' average cumulative profits (Figures 10c, 10g and 10k) and number of sellers in the market (Figures 10d, 10h and 10l) over time ($T = 100$). Model: Pythia-1b, Pythia-410m, and Llama-3.2-Inst.-1b; Task: MathQA. Experimental groups: (1) fairshare, (2) reduced, (3) random, and (4) current pricing.**



(a) High-budget buyers' utilities (Pythia-1b).
(b) Low-budget buyers' utilities (Pythia-1b).
(c) Sellers' profits (Pythia-1b).
(d) Sellers' participation (Pythia-1b).

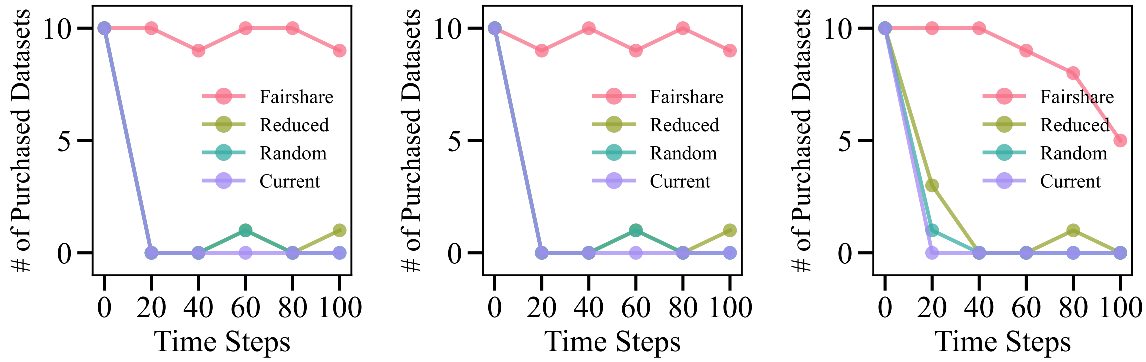


(e) High-budget buyers' utilities (Pythia-410m).
(f) Low-budget buyers' utilities (Pythia-410m).
(g) Sellers' profits (Pythia-410m).
(h) Sellers' participation (Pythia-410m).

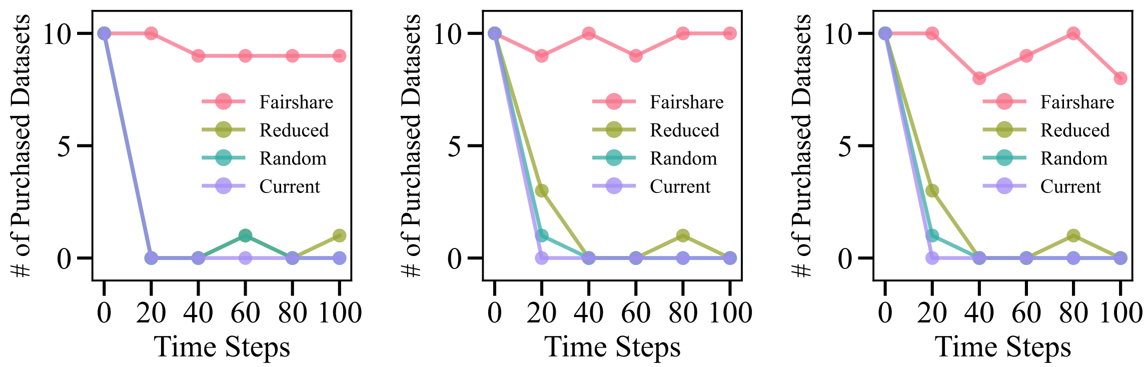


(i) High-budget buyers' utilities (Llama-3.2-Inst.-1b).
(j) Low-budget buyers' utilities (Llama-3.2-Inst.-1b).
(k) Sellers' profits (Llama-3.2-Inst.-1b).
(l) Sellers' participation (Llama-3.2-Inst.-1b).

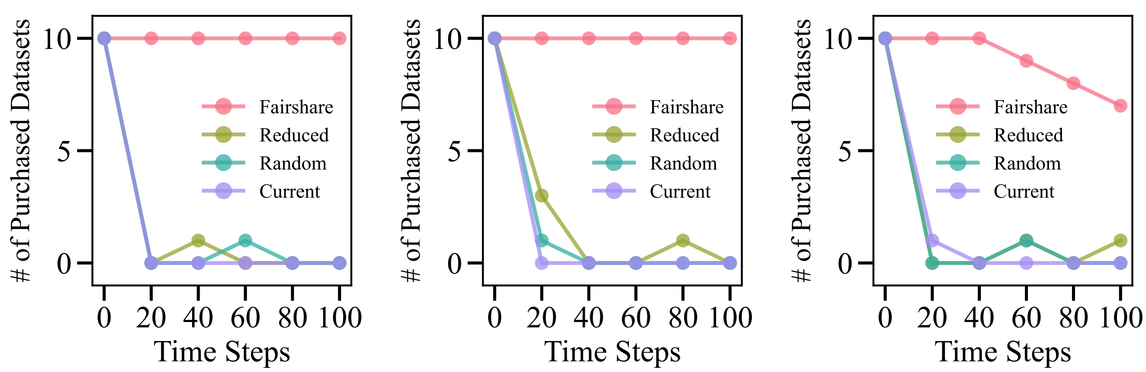
Figure 11 **Analysis of (1) buyer's cumulative utilities with high-budget buyer (Figures 11a, 11e and 11i) and low-budget buyer (Figures 11b, 11f and 11j), and (2) sellers' average cumulative profits (Figures 11c, 11g and 11k) and number of sellers in the market (Figures 11d, 11h and 11l) over time ($T = 100$). Model: Pythia-1b, Pythia-410m, and Llama-3.2-Inst.-1b; Task: PIQA. Experimental groups: (1) fairshare, (2) reduced, (3) random, and (4) current pricing.**



(a) High-budget buyer (MedQA, Pythia-1b). (b) High-budget buyer (MedQA, Pythia-410m). (c) High-budget buyer (MedQA, Llama-3.2-Inst.-1b).

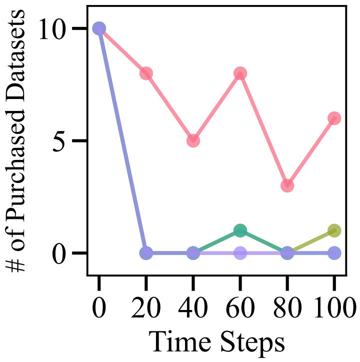


(d) High-budget buyer (MathQA, Pythia-1b). (e) High-budget buyer (MathQA, Pythia-410m). (f) High-budget buyer (MathQA, Llama-3.2-Inst.-1b).

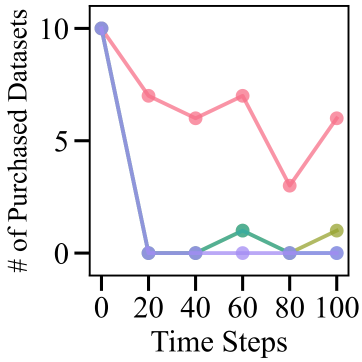


(g) High-budget buyer (PiQA, Pythia-1b). (h) High-budget buyer (PiQA, Pythia-410m). (i) High-budget buyer (PiQA, Llama-3.2-Inst.-1b).

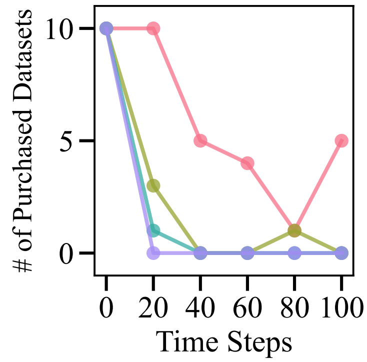
Figure 12 **Number of purchased datasets for the buyer with high budget over time periods ($T = 100$). Models: Pythia-1b, Pythia-410m, and Llama-3.2-Inst.-1b. Tasks: MedQA, MathQA, and PiQA. Experimental groups: (1) fairshare, (2) reduced, (3) random, and (4) current pricing.**



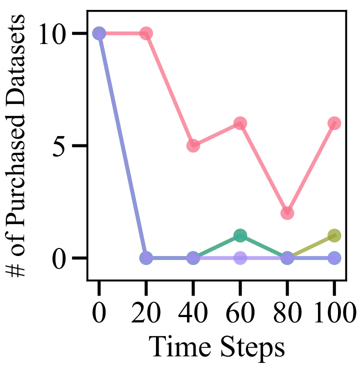
(a) Low-budget buyer (MedQA, Pythia-1b).



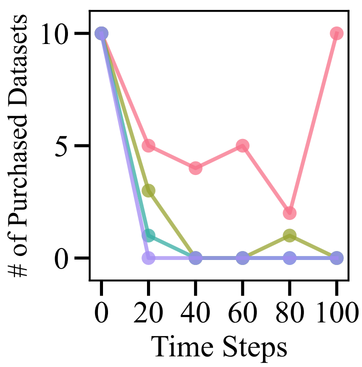
(b) Low-budget buyer (MedQA, Pythia-410m).



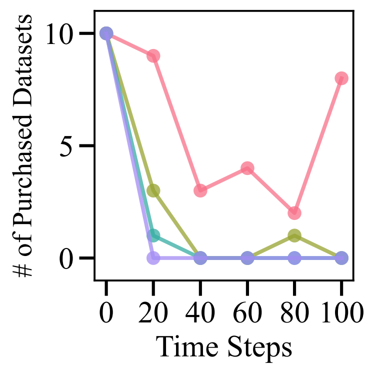
(c) Low-budget buyer (MedQA, Llama-3.2-Inst.-1b).



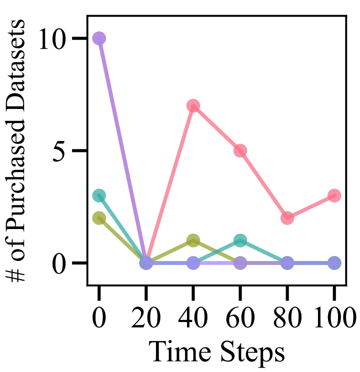
(d) Low-budget buyer (MathQA, Pythia-1b).



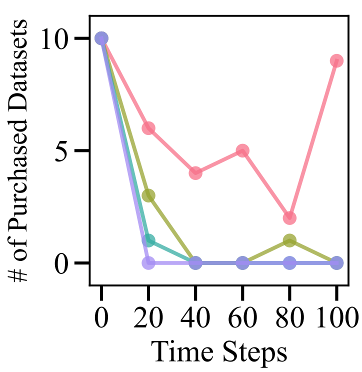
(e) Low-budget buyer (MathQA, Pythia-410m).



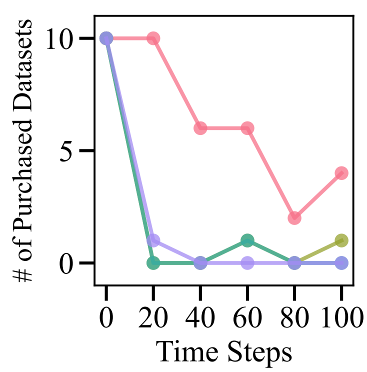
(f) Low-budget buyer (MathQA, Llama-3.2-Inst.-1b).



(g) Low-budget buyer (PIQA, Pythia-1b).



(h) Low-budget buyer (PIQA, Pythia-410m).



(i) Low-budget buyer (PIQA, Llama-3.2-Inst.-1b).

Figure 13 **Number of purchased datasets for the buyer with low budget over time periods ($T = 100$). Models: Pythia-1b, Pythia-410m, and Llama-3.2-Inst.-1b. Tasks: MedQA, MathQA, and PIQA. Experimental groups: (1) fairshare, (2) reduced, (3) random, and (4) current pricing.**

7.4. Datasets

Dataset	# of Train/Valid/Test	Example
MathQA	29837/4475/2985	**Question:** A train running at the speed of 48 km / hr crosses a pole in 9 seconds . what is the length of the train? a) 140 , b) 130 , c) 120 , d) 170 , e) 160 **Answer:** C
GSM8K	7473/1319	**Question:** Natalia sold clips to 48 of her friends in April, and then she sold half as many clips in May. How many clips did Natalia sell altogether in April and May? **Answer:** 72
MedQA	10178/1272/1273	**Question:** A 27-year-old man presents to the emergency room with persistent fever, nausea, and vomiting for the past 3 days. While waiting to be seen, he quickly becomes disoriented and agitated. Upon examination, he has visible signs of difficulty breathing with copious oral secretions and generalized muscle twitching. The patient's temperature is 104°F (40°C), blood pressure is 90/64 mmHg, pulse is 88/min, and respirations are 18/min with an oxygen saturation of 90% on room air. When the nurse tries to place a nasal cannula, the patient becomes fearful and combative. The patient is sedated and placed on mechanical ventilation. Which of the following is a risk factor for the patient's most likely diagnosis? a) Contaminated beef b) Epiglottic cyst c) Mosquito bite d) Spelunking **Answer:** D
PIQA	16000/2000	**Question:** How do I ready a guinea pig cage for it's new occupants? a) Provide the guinea pig with a cage full of a few inches of bedding made of ripped paper strips, you will also need to supply it with a water bottle and a food dish. b) Provide the guinea pig with a cage full of a few inches of bedding made of ripped jeans material, you will also need to supply it with a water bottle and a food dish. **Answer:** A

Table 1 Dataset splits and demonstrations from the MathQA, GSM8K, MedQA, and PIQA datasets

Dataset	Prompts
MathQA	**Question:** the banker ' s gain of a certain sum due 3 years hence at 10 % per annum is rs . 36 . what is the present worth ? a) rs . 400 , b) rs . 300 , c) rs . 500 , d) rs . 350 , e) none of these **Answer:** A **Question:** average age of students of an adult school is 40 years . 120 new students whose average age is 32 years joined the school . as a result the average age is decreased by 4 years . find the number of students of the school after joining of the new students . a) 1200 , b) 120 , c) 360 , d) 240 , e) none of these **Answer:** D **Question:** sophia finished 2 / 3 of a book . she calculated that she finished 90 more pages than she has yet to read . how long is her book ? a) 229 , b) 270 , c) 877 , d) 266 , e) 281 **Answer:** B **Question:** 120 is what percent of 50 ? na) 5 % , b) 240 % , c) 50% , d) 2 % , e) 500 **Answer: B** **Question:** there are 10 girls and 20 boys in a classroom . what is the ratio of girls to boys ? a) 1 / 2 , b) 1 / 3 , c) 1 / 5 , d) 10 / 30 , e) 2 / 5 **Answer:** A
MedQA	**Question:** A mother brings her 3-week-old infant to the pediatrician's office because she is concerned about his feeding habits. He was born without complications and has not had any medical problems up until this time. However, for the past 4 days, he has been fussy, is regurgitating all of his feeds, and his vomit is yellow in color. On physical exam, the child's abdomen is minimally distended but no other abnormalities are appreciated. Which of the following embryologic errors could account for this presentation? a) Abnormal migration of ventral pancreatic bud b) Complete failure of proximal duodenum to recanalize c) Abnormal hypertrophy of the pylorus d) Failure of lateral body folds to move ventrally and fuse in the midline **Answer:** A **Question:** A 53-year-old man comes to the emergency department because of severe right-sided flank pain for 3 hours. The pain is colicky, radiates towards his right groin, and he describes it as 8/10 in intensity. He has vomited once. He has no history of similar episodes in the past. Last year, he was treated with naproxen for swelling and pain of his right toe. He has a history of hypertension. He drinks one to two beers on the weekends. Current medications include amlodipine. He appears uncomfortable. His temperature is 37.1ŏ0b0C (99.3ŏ0b0F), pulse is 101/min, and blood pressure is 130/90 mm Hg. Examination shows a soft, nontender abdomen and right costovertebral angle tenderness. An upright x-ray of the abdomen shows no abnormalities. A CT scan of the abdomen and pelvis shows a 7-mm stone in the proximal ureter and grade I hydronephrosis on the right. Which of the following is most likely to be seen on urinalysis? a) Urinary pH: 7.3 b) Urinary pH: 4.7 c) Positive nitrites test d) Largely positive urinary protein **Answer:** B **Question:** A 48-year-old woman comes to the emergency department because of a photosensitive blistering rash on her hands, forearms, and face for 3 weeks. The lesions are not itchy. She has also noticed that her urine has been dark brown in color recently. Twenty years ago, she was successfully treated for Coats disease of the retina via retinal sclerotherapy. She is currently on hormonal replacement therapy for perimenopausal symptoms. Her aunt and sister have a history of a similar skin lesions. Examination shows multiple fluid-filled blisters and oozing erosions on the forearms, dorsal side of both hands, and forehead. There is hyperpigmented scarring and patches of bald skin along the sides of the blisters. Laboratory studies show a normal serum ferritin concentration. Which of the following is the most appropriate next step in management to induce remission in this patient? a) Pursue liver transplantation b) Begin oral thalidomide therapy c) Begin phlebotomy therapy d) Begin oral hydroxychloroquine therapy **Answer:** C **Question:** A 23-year-old pregnant woman at 22 weeks gestation presents with burning upon urination. She states it started 1 day ago and has been worsening despite drinking more water and taking cranberry extract. She otherwise feels well and is followed by a doctor for her pregnancy. Her temperature is 97.7ŏ0b0F (36.5ŏ0b0C), blood pressure is 122/77 mmHg, pulse is 80/min, respirations are 19/min, and oxygen saturation is 98% on room air. Physical exam is notable for an absence of costovertebral angle tenderness and a gravid uterus. Which of the following is the best treatment for this patient? a) Ampicillin b) Ceftriaxone c) Doxycycline d) Nitrofurantoin **Answer:** D

Table 2 **Demonstrations included for 5-shot evaluation on the MathQA dataset and for 4-shot evaluation on the MedQA dataset. Demostrations were randomly selected from their respective dataset's training sets.**

8. Limitations and Impact

This paper addresses the critical issue of fairshare pricing in the data market for large language models (LLMs) by proposing a framework and methodologies for fair compensation of datasets from LLM developers to data annotators. Our work directly tackles the ethical and societal challenges in the current data market, where many data annotators are underpaid and receive compensation significantly disconnected from the true economic value their contributions bring to LLMs.

8.1. Limitations

Since our work proposes a novel fairshare framework, there are several lines of future research that can investigate future adjustments to this framework, which lie beyond the scope of our paper. For instance, a large-scale simulation of this market with a wider range of datasets and models is one possibility. In addition, running the simulation with human buyers/sellers is another avenue. Finally, there are several other diverse market dynamics (e.g., incomplete information between buyers/sellers) that can be explored with our proposed framework.

8.2. Impact Statement

From ethical and societal perspectives, our framework prioritizes the welfare of both data annotators and LLM developers. Our methodology ensures that data annotators are fairly compensated for their labor, promoting equity and fairness in the data ecosystem. This contributes to mitigating the exploitation of vulnerable annotators in the data market and aligns the incentives of stakeholders toward a more ethical and sustainable practice. In addition, our framework also benefits LLM developers, by demonstrating that our framework maximizes their utilities and welfare in the long term. Fair compensation encourages ongoing participation of data annotators in the market, ensuring a steady supply of diverse, high-quality datasets essential for LLM development. By addressing existing inequities, our work lays the foundation for a more sustainable, equitable, and mutually beneficial ecosystem for all stakeholders in the LLM data market.