5. Suppose that we observe a group of p-dimensional independent and identically distributed samples $\{X_i\}_{i=1}^N$, where $X_i \sim N_p(\mu, \Sigma)$ with μ and Σ unknown. We are interested in testing $H_0 : \mu = \mathbf{0}$. Suppose we use Hotelling's T^2-test

$$T^2 = N\bar{X}_N^\top S_N^{-1}\bar{X}_N$$

where $\bar{X}_N$ and S_N are the sample mean and sample covariance matrix.

(a) Show that the likelihood ratio test takes the form

$$LRT = \left\{\frac{1}{1+T^2/(N-1)}\right\}^{N/2}$$

and therefore is equivalent to Hotelling's T^2-test.

(b) When p is fixed, show that the test statistic T^2 has a limiting χ^2-distribution under the null hypothesis as $N \to \infty$.

(c) Under the setting with $N = 100$ and $p = 3$, perform a simulation study to check if the limiting χ^2-distribution in (b) controls the type I error well (Note that the simulation study would need at least a few hundreds replications to estimate the type I error well).

(d) Under the setting of (c) but increasing p to 10, 40, and 80, does the limiting χ^2-distribution still control the type I error well?

(e) When p is assumed to increase with N such that $p/N \to \gamma \in (0, 1)$ as $N \to \infty$, show that the test statistic T^2 has a limiting normal distribution (in the sense that there exist a_N and b_N such that $a_N(T^2 - b_N) \to N(0, 1)$ as $N, p \to \infty$).

(f) If $p > N$, can we still use Hotelling's T^2-test? Explain why. If not, please propose an alternative testing procedure and conduct a simulation study to verify (This is an open-ended question and theoretical results are not needed for your new procedure).

$\bar{X} = \frac{1}{N}\sum_{i=1}^N X_i$

$S_N = \frac{1}{N-1}\sum_{i=1}^N (X_i-\bar{X})(X_i-\bar{X})^T$

$H_0: \mu = 0 \quad H_1: \mu \neq 0$

(a) LRT: $\Lambda = \frac{\sup_{\mu=0,\Sigma} L(\mu,\Sigma)}{\sup_{\mu,\Sigma} L(\mu,\Sigma)} \quad \frac{H_0}{H_1}$

$f(x;\mu,\Sigma) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right) \Rightarrow L(\mu,\Sigma) = \frac{1}{(2\pi)^{\frac{Np}{2}}|\Sigma|^{\frac{N}{2}}}\exp\left(-\frac{1}{2}\sum_{i=1}^N(x_i-\mu)^T\Sigma^{-1}(x_i-\mu)\right)$

$\Rightarrow H_0: \hat{\mu}_0 = 0,\ \hat{\Sigma}_0 = \frac{1}{N}\sum_{i=1}^N X_iX_i^T \quad H_1: \hat{\mu}_1 = \bar{X},\ \hat{\Sigma}_1 = \frac{1}{N}\sum_{i=1}^N(X_i-\bar{X})(X_i-\bar{X})^T$

denote $A := \sum_{i=1}^N (X_i-\bar{X})(X_i-\bar{X})^T \Rightarrow \hat{\Sigma}_1 = \frac{1}{N}A$, besides, $\sum_{i=1}^N X_iX_i^T = \sum_{i=1}^N (X_i-\bar{X})(X_i-\bar{X})^T + N\bar{X}\bar{X}^T \Rightarrow \hat{\Sigma}_0 = \hat{\Sigma}_1 + \bar{X}\bar{X}^T$

$\Rightarrow \Lambda = \left(\frac{|\hat{\Sigma}_1|}{|\hat{\Sigma}_0|}\right)^{\frac{N}{2}} = \left(\frac{|\hat{\Sigma}_1|}{|\hat{\Sigma}_1+\bar{X}\bar{X}^T|}\right)^{\frac{N}{2}}$, here $|\hat{\Sigma}_1 + \bar{X}\bar{X}^T| = |\hat{\Sigma}_1|(1+\bar{X}^T\hat{\Sigma}_1^{-1}\bar{X}) \Rightarrow \Lambda = \left(\frac{1}{1+\bar{X}^T\hat{\Sigma}_1^{-1}\bar{X}}\right)^{\frac{N}{2}}$

Here $T^2 = N\bar{X}^TS_N^{-1}\bar{X}$, $\hat{\Sigma}_1 = \frac{N-1}{N}S_N \Rightarrow S_N^{-1} = \frac{N-1}{N}\hat{\Sigma}_1^{-1} \Rightarrow T^2 = (N-1)\bar{X}^T\hat{\Sigma}^{-1}\bar{X} \Rightarrow \Lambda = \left(\frac{1}{1+\frac{T^2}{N-1}}\right)^{\frac{N}{2}}$

(b) $H_0: \mu=0$: Under CLT: $\sqrt{N}\bar{X} \sim N_p(0,\Sigma)$, Under LLN: $S_N \sim \Sigma \Rightarrow S_N^{-1} \sim \Sigma^{-1}$

$T^2 = N\bar{X}^TS_N^{-1}\bar{X} = (\sqrt{N}\bar{X})^TS_N^{-1}(\sqrt{N}\bar{X})$, according to Slutsky: $T^2 = (\sqrt{N}\bar{X})^T\Sigma^{-1}(\sqrt{N}\bar{X})$

Let $Y = \Sigma^{-\frac{1}{2}}\sqrt{N}\bar{X}$, then $Y \sim N_p(0, I_p) \Rightarrow T^2 = Y^TY = \sum_{j=1}^p y_j^2 \sim \chi^2_p$

(e) denote $Z := \sqrt{N}\bar{X}$, $Z \sim N_p(0,\Sigma)$ $T^2 = Z^TS^{-1}Z$; $W = \sum_{i=1}^N(X_i-\bar{X})(X_i-\bar{X})^T \sim W_p(N-1,\Sigma)$; $\bar{X} \perp S$

According to theorem: $Z \sim N_p(0,\Sigma)$, $W \sim W_p(m,\Sigma)$, $Z \perp W$, $\Rightarrow \frac{m-p+1}{p}Z^TW^{-1}Z \sim F_{p,m-p+1}$

here $T^2 \sim \frac{p(N-1)}{N-p}F_{p,N-p} = \frac{p(N-1)}{N-p}\cdot\frac{U/p}{V/(N-p)}$, $U \sim \chi^2_p$, $V \sim \chi^2_{(N-p)}$ $\Rightarrow T^2 \sim (N-1)\frac{U}{V}$

when $N,p \to \infty$, $\frac{p}{N} \to \gamma \in (0,1)$, $\chi^2_p \approx N(p,2p)$, $\chi^2_{N-p} \approx (N-p, 2(N-p)) \Rightarrow U \approx p + \sqrt{2p}Z_1$, $V \approx (N-p) + \sqrt{2(N-p)}Z_2$

$T^2 \sim (N-1)\frac{U}{V} = g(U,V)$, accordind to CLT, g is approximately Gaussian

, Taylor's formula in $g(m)$: $g(U,V) \approx g(m) + \nabla g(m)^T\begin{pmatrix}U-m_U\\V-m_V\end{pmatrix}$ (denote $m = \begin{pmatrix}m_U\\m_V\end{pmatrix} = \begin{pmatrix}E[U]\\E[V]\end{pmatrix}$)

So $b_N = g(p,N-p) = (N-1)\frac{p}{N-p}$ $(E[U]=p,\ E[V]=N-p)$

$\frac{\partial g}{\partial U} = (N-1)\frac{1}{V}$, $\frac{\partial g}{\partial V} = (N-1)\cdot\left(-\frac{U}{V^2}\right) \Rightarrow \nabla g(p,N-p) = \left(\frac{N-1}{N-p}, -\frac{(N-1)p}{(N-p)^2}\right)$,

$Var\begin{pmatrix}U\\V\end{pmatrix} = \begin{pmatrix}2p & 0\\0 & 2(N-p)\end{pmatrix}$, $Var(g(U,V)) \approx \nabla g(m)^T Var(U,V)\nabla g(m) \Rightarrow Var(T^2) \approx \left(\frac{N-1}{N-p}\right)^2(2p) + \left(\frac{(N-1)p}{(N-p)^2}\right)^2(2(N-p))$,

To let $a_N(T^2-b_N) \sim N(0,1)$, $a_N = \frac{1}{\sqrt{Var(T^2)}}$, $b_N = (N-1)\frac{p}{N-p}$

(f) $p>N$, $rank(S_N) \leq N-1 < p \Rightarrow S_N$ is singular, so T^2 cannot work

alternative option: Use the norm of mean vector, test how it's close to zero

the rest is in Notebook