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Possibility of Inverse Energy Cascade in Two-Dimensional Quantum Turbulence

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Abstract We numerically study two-dimensional quantum turbulence by using the Gross-Pitaevskii model and the Vortex-Point model. Two-dimensional classical turbulence has long been investigated as an ideal system of geophysical phenomena. The amazing character of this turbulence is inverse energy cascade which carries energy toward low wavenumbers and excites large-scale motion. We expect these phenomena in two-dimensional quantum turbulence because in three-dimensional turbulence we know classical and quantum analogue. However, we have not yet confirmed inverse cascade in two-dimensional quantum turbulence. In this work, we show numerical results and discuss why inverse cascade does not occur in two-dimensional quantum turbulence by referring to the mechanism of two-dimensional classical turbulence.

Keywords Quantum turbulence · Quantized vortices · BEC · Quantum fluids

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1 Introduction

Since 1955 when R. Feynmann predicts that turbulence consisting of quantized vortices occurs in superfluid ^{4}He, quantum turbulence has been actively studied in low-temperature physics. Especially, the analogue between classical and quantum turbulence has attracted much attention. The most remarkable one is that Kolmogorov's $-5/3$ law in the energy spectrum has been observed in both experiments [1] and numerical simulations [2–5] of isotropic homogeneous quantum turbulence. This analogue makes us expect the same statistical law in two-dimensional classical and quantum turbulence too.

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Two-dimensional turbulence in incompressible fluid is an ideal model of geophysical phenomena in the atmosphere, magnetosphere, and ocean [6, 7]. Particularly, it has been studied in order to reveal meso-scale meteorological phenomena. This turbulence has an surprising character, inverse energy cascade peculiar to two-dimensional system. This character comes from two inviscid quadratic constants, one is kinetic energy per unit mass $E = (1/2S)\int d^2\boldsymbol{r}\boldsymbol{v}^2(\boldsymbol{r},t) = \int dk E(k)$ and the other is enstrophy which is the integral of squared vorticity $\Omega = (1/2S)\int d^2\boldsymbol{r}\boldsymbol{\omega}^2(\boldsymbol{r},t) = \int dk\Omega(k)$, where $S = \int d^2\boldsymbol{r}$ is the size of the system. Here, $E(k)$ and $\Omega(k)$ denote energy spectrum and enstrophy spectrum respectively. These spectra are connected with the relation $k^2E(k) = \Omega(k)$. In energy cascade process, energy is carried by nonlinear wave interactions without being dissipated. Energy cascades toward both large and small wavenumbers so that the conservation of energy and enstrophy and the relation between spectra of these quantities may be satisfied in inertial ranges [6]. The cascade toward low wavenumbers generates large-scale motion and is called inverse cascade. Inverse cascade has been confirmed by both numerical simulations [8–10] and experiments [11]. Moreover, in direct numerical simulation of isotropic turbulence of surface gravity waves [12], second sound acoustic turbulence in He II [13], etc, inverse cascade has been confirmed in other two-dimensional systems. In [14], inverse cascade is predicted and studied through the weak turbulence approach to Kelvin turbulence.

We are very interested in this phenomenon and investigate if it also occurs in atomic Bose-Einstein condensates by numerical simulation and the film of superfluid helium by analytical calculation.

2 Gross-Pitaevskii Model

The GP model is intended for quantum turbulence in two-dimensional atomic Bose-Einstein condensates.

2.1 Numerical Analysis

We solve the nondimensionalized two-dimensional GP equation with small-scale dissipation normalized by the healing length ξ,

$$[i - \tilde{\gamma}(\boldsymbol{k})]\frac{\partial}{\partial t}\tilde{\Phi}(\boldsymbol{k},t) = [k^2 - \mu(t)]\tilde{\Phi}(\boldsymbol{k},t) + \tilde{h}(\boldsymbol{k},t). \tag{1}$$

Here, $\tilde{\Phi}(\boldsymbol{k},t)$ is the Fourier transformation of the wave function $\Phi(\boldsymbol{r},t)$, $\tilde{\gamma}(\boldsymbol{k})$ is the small-scale dissipation which works at length scales smaller than ξ, $\mu(t)$ is the chemical potential, $\tilde{h}(\boldsymbol{k},t)$ is also the Fourier transformation of $\{g|\Phi(\boldsymbol{r},t)|^2 + V(\boldsymbol{r},t)\}\Phi(\boldsymbol{r},t)$, where g is the coupling constant, and $V(\boldsymbol{r},t)$ is the external forcing potential. In this work, we define $V(\boldsymbol{r},t)$ in the Fourier space as

$$\tilde{V}(\boldsymbol{k},t) = \begin{cases} V_0 e^{i\theta(\boldsymbol{k},t)} & (k_f - 2\Delta k < k < k_f + 2\Delta k), \\ 0 & (\text{otherwise}), \end{cases} \tag{2}$$

where $\theta(\boldsymbol{k},t)$ is randomly taken $-\pi < \theta(\boldsymbol{k},t) < \pi$ and $k_f = 2\pi/2\xi$ in this simulation. We use 256^2 Fourier spectral method. Time step is $\Delta t = 1 \times 10^{-4}$, spatial grid is $\Delta x = 0.125$, and wavenumber grid is $\Delta k = 2\pi/(256\Delta x) \approx 0.196$.

We start from the uniform state and keep on pumping by moving $V(\boldsymbol{r},t)$. Then, we observe the time evolution of the total energy $E(t) = (1/N)\int d^2\boldsymbol{r}\Phi^*(\boldsymbol{r},t) \times [-\nabla^2 + (g/2)|\Phi(\boldsymbol{r},t)|^2]\Phi(\boldsymbol{r},t)$, the kinetic energy which comprises two components, $E_{kin}(t) = (1/N)\int d^2\boldsymbol{r}[\boldsymbol{p}(\boldsymbol{r},t)]^2 = E^c_{kin}(t) + E^i_{kin}(t)$ with $\boldsymbol{p}(\boldsymbol{r},t) = \mathrm{Im}\{\Phi^*(\boldsymbol{r},t)\nabla\Phi(\boldsymbol{r},t)\}/|\Phi(\boldsymbol{r},t)|$, where $N = \int d^2\boldsymbol{r}|\Phi(\boldsymbol{r},t)|^2$ denotes the total particle number [2]. $E^c_{kin}(t) = (1/N)\int d^2\boldsymbol{r}[\{\boldsymbol{p}(\boldsymbol{r},t)\}^c]^2$ is the compressible part of $E_{kin}(t)$, which comes from compressible elementary excitations and $\nabla \times \{\boldsymbol{p}(\boldsymbol{r},t)\}^c = \boldsymbol{0}$. $E^i_{kin}(t) = (1/N)\int d^2\boldsymbol{r}[\{\boldsymbol{p}(\boldsymbol{r},t)\}^i]^2$ is the incompressible part of $E_{kin}(t)$, which is related to the motion of quantized vortices and $\nabla \cdot \{\boldsymbol{p}(\boldsymbol{r},t)\}^i = 0$.

Figure 1 shows the time developments of E, E_{kin}, E^c_{kin}, and E^i_{kin}. At the early stage, as energy is injected, E, E_{kin}, E^c_{kin}, and E^i_{kin} increase in time. At the time $t \approx 20$, the system becomes statistical steady state, which means almost all energy cascades toward large wavenumbers. If energy cascades toward low wavenumbers, energy keeps on increasing because dissipation does not work at that wavenumber range.

Figure 2 shows the incompressible kinetic energy spectrum, being consistent with Fig. 1. In the early stage, we have verified in this numerical simulation that energy is mainly injected to the compressible component. As time goes by, quantized vortices

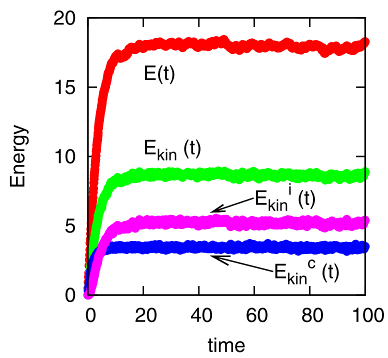

Fig. 1 (Color online) Time developments of $E(t)$, $E_{kin}(t)$, $E^c_{kin}(t)$, and $E^i_{kin}(t)$

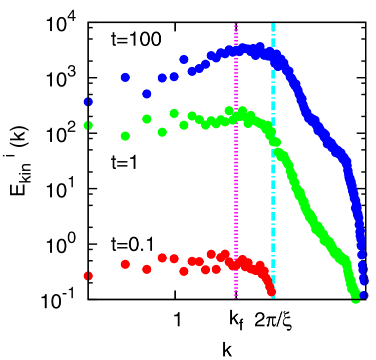

Fig. 2 (Color online) Incompressible kinetic energy spectrum $E^i_{kin}(t)$ at time 0.1, 5, 100

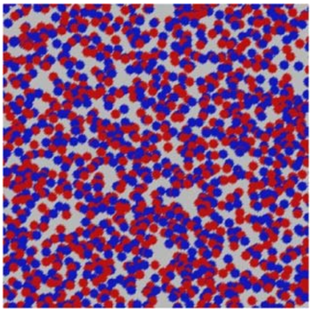

Fig. 3 (Color online) Distribution of quantized vortices. *Red circles* are clockwise vortices and *blue* vice versa

are generated as phase defects. Then, the incompressible component increase in both low and high wavenumbers.

Figure 3 shows the distribution of quantized vortices at time 100 in the statistical steady state. A lot of vortex pairs are excited by moving $V(\boldsymbol{r},t)$. We cannot see large scale motion in this figure such as vortices having the same circulation clump.

2.2 Discussion About the GP Model

We guess that there is an analogue between two-dimensional classical and quantum turbulence. Here, we should investigate whether the analogue is correct in two-dimensional systems or not.

In two-dimensional classical turbulence, energy cascade to two opposite directions stems from two inviscid constants, kinetic energy and enstrophy. Originally, in the GP model without dissipation term, the total particle number N and the total energy E are conserved. This combination generates inverse cascade in the total particle number spectrum $N(k)$, defined as $N=\int dkN(k)$ [15]. If we think two-dimensional quantum turbulence along the theory of two-dimensional classical turbulence [6], it is necessary to investigate enstrophy conservation in the GP model.

The GP equation with Madelung transformation $\Phi(\boldsymbol{r},t)=\sqrt{n_0(\boldsymbol{r},t)}e^{i\theta(\boldsymbol{r},t)}$ is rewrote to the Euler-like equation,

$$\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\boldsymbol{v}\times(\nabla\times\boldsymbol{v})=2\nabla\left(\frac{1}{\sqrt{n_0}}\nabla^2\sqrt{n_0}\right)-2\nabla(gn_0-\mu), \tag{3}$$

where $\boldsymbol{v}(\boldsymbol{r},t)=2\nabla\theta(\boldsymbol{r},t)$. In this scheme, vorticity is also defined as $\boldsymbol{\omega}=\nabla\times\boldsymbol{v}$. Equation (3) is transformed to the vorticity equation,

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\frac{1}{2}\nabla\times\left(\nabla\boldsymbol{v}^2\right)=\nabla\times\left\{2\nabla\left(\frac{1}{\sqrt{n_0}}\nabla^2\sqrt{n_0}\right)\right\}-\nabla\times2\nabla(gn_0-\mu). \tag{4}$$

The second term in the left hand side does not vanish when there are quantized vortices. This equation tells us whether enstrophy $\Omega=(1/2)\int d^2\boldsymbol{r}\boldsymbol{\omega}^2(\boldsymbol{r},t)$ is conserved or not by multiplying $\boldsymbol{\omega}$ both sides and integrating. However, since the second term in the left hand side and the first term in the right hand side are not simple when there are quantized vortices, it is difficult to handle this equation. Then, we think the problem phenomenologically.

Fig. 4 (Color online) Image of vorticity distribution. *Circles* express quantized vortices

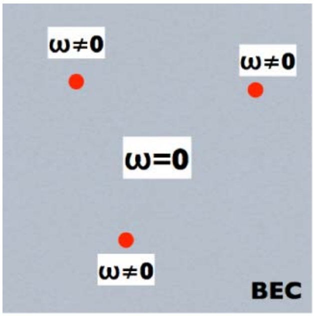

Figure 4 shows that vorticity is 0 all over the condensate except the cores of quantized vortices because the velocity field is described as potential flow. Quantized vortices are generated as the phase defects. Vorticity is **0** on all over the condensate except the cores of quantized vortices because the velocity field is described as potential flow. Therefore, quantized vortices are the source of enstrophy and it depends on the number of quantized vortices. We could reach the relation,

$$\Omega \propto N_{qv}, \tag{5}$$

where N_{qv} is the number of quantized vortices. Thus, we think whether N_{qv} is conserved or not in arbitrary wave function. Generally, N_{qv} is not conserved. For example, if initial configuration of phase of wavefunction is randomly taken, creation and annihilation of quantized vortices occur repeatedly and N_{qv} fluctuates in time. Consequently, enstrophy does not conserve exactly in the GP equation without dissipation term. It is compressibility of atomic Bose-Einstein condensates that leads to this effect because condensates are nonviscous fluids. We conclude that inverse cascade does not occur from the combination of energy and enstrophy in the GP model.

3 Vortex-Point Model

This model corresponds to superfluid turbulence on the thin film. Since this is an exactly incompressible model, we can expect the analogue with classical Navier-Stokes model. We will show this expectation analytically.

Although there is no reality in classical fluid, this model has long been studied as an important and interesting system. This is a Hamiltonian system,

$$H = -\frac{1}{4\pi}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\Gamma_i\Gamma_j \log|\boldsymbol{s}_i - \boldsymbol{s}_j|, \tag{6}$$

with canonical conjugates $\Gamma_i x_i$ and y_i.

On the other hand, this is a realistic model for superfluid helium because all quantized vortices have atomic-scale cores so that we can handle quantized vortices as points and the circulation takes two values, $\pm\kappa$ with circulation quantum κ. The equation of motion is

$$\frac{d\boldsymbol{s}_i(\boldsymbol{r},t)}{dt} = \sum_{j\neq i, j=1}^{N} \boldsymbol{v}_{j\to i}, \tag{7}$$

$$\boldsymbol{v}_{j\to i} = \frac{\kappa n_j}{2\pi}\frac{\hat{\boldsymbol{z}}\times(\boldsymbol{s}_i-\boldsymbol{s}_j)}{|\boldsymbol{s}_i-\boldsymbol{s}_j|^2}, \tag{8}$$

where $\boldsymbol{s}_i(\boldsymbol{r},t)$ is the position vector of ith quantized vortex, N is the total vortex number, $\hat{z}$ is the unit vector perpendicular to the field, and $n_j = \pm 1$. This equation of motion means that quantized vortices are moved by the velocity field of all the other vortices. Nonlinear effect stems from $\boldsymbol{v}_{j\to i}$. This term makes energy cascade, wavenumber interaction is not clear though.

In this model, there are several constants, such as energy, moment of inertia, and angular momentum. Then, we will show enstrophy does not depend on time and therefore it is also one constant. Vorticity field is expressed as the sum of delta functions localized distributed around quantized vortex cores,

$$\boldsymbol{\omega}(\boldsymbol{r},t) = \kappa\sum_{i=1}^{N} n_i \hat{\boldsymbol{z}}\delta^2(\boldsymbol{r}-\boldsymbol{s}_i). \tag{9}$$

Enstrophy is

$$\begin{aligned}\Omega &= \frac{1}{2}\int d^2\boldsymbol{r}\boldsymbol{\omega}^2(\boldsymbol{r},t) = \frac{1}{2}\int d^2\boldsymbol{r}\kappa^2\sum_{i,j} n_i n_j \delta^2(\boldsymbol{r}-\boldsymbol{s}_i)\delta^2(\boldsymbol{r}-\boldsymbol{s}_j) \\ &= \frac{1}{2}\kappa^2\sum_{i,j} n_i n_j \delta^2(\boldsymbol{s}_i-\boldsymbol{s}_j) = \frac{1}{2}N\kappa^2\delta^2(\boldsymbol{0}) \propto N.\end{aligned} \tag{10}$$

This model does not include vortex pair annihilation process so that enstrophy does not depend on time. Energy spectrum can be calculated analytically using Bessel function J_0 [16],

$$\begin{aligned}E &= \frac{1}{2}\int d^2\boldsymbol{r}|\boldsymbol{v}(\boldsymbol{r},t)|^2 = \frac{1}{2}(2\pi)^2\int d^2\boldsymbol{k}|\hat{\boldsymbol{v}}(\boldsymbol{k},t)|^2 = \frac{1}{2}(2\pi)^2\int d^2\boldsymbol{k}\frac{|\hat{\boldsymbol{\omega}}(\boldsymbol{k},t)|^2}{|\boldsymbol{k}|^2} \\ &= \frac{1}{2}(2\pi)^2\int d^2\boldsymbol{k}\frac{\kappa^2}{(2\pi)^4|\boldsymbol{k}|^2}\sum_{j,l} n_j n_l e^{i\boldsymbol{k}\cdot(s_j-s_l)},\end{aligned} \tag{11}$$

$$E(k) = \frac{\kappa^2}{2(2\pi)^2 k}\sum_{j,l} n_j n_l \int_0^{2\pi} d\theta e^{i\boldsymbol{k}\cdot(s_j-s_l)} = \frac{\kappa^2}{4\pi k}\sum_{j,l} n_j n_l J_0(k|\boldsymbol{s}_j-\boldsymbol{s}_l|).$$

Enstrophy spectrum can be similarly calculated,

$$\Omega(k) = \frac{\kappa^2 k}{4\pi}\sum_{j,l} n_j n_l J_0(k|\boldsymbol{s}_j-\boldsymbol{s}_l|), \tag{12}$$

$$k^2 E(k) = \Omega(k). \tag{13}$$

This model has two constants, energy and enstrophy. Moreover, their spectra are connected with the relation (13), same as that of Navier-Stokes model. So, if we add proper forcing term to (7), this model may occur inverse cascade.

4 Conclusion

We have investigated the possibility of inverse cascade in quantum turbulence in two models by numerical simulation, phenomenology, and analysis.

In the GP model, we conclude that compressibility of condensates prevent energy from cascading toward low wavenumbers. However, things may change in large system because the velocity of sound wave $c = \sqrt{2g}|\Phi(\boldsymbol{r}, t)|$ becomes fast as $|\Phi(\boldsymbol{r}, t)|$ is getting large. We are now trying the large simulation. In Vortex-Point model, we obtain some conditions that may guarantees to occur inverse cascade. We are also trying the numerical simulation adding proper forcing term.

References

1. J. Maurer, P. Tabeling, Europhys. Lett. **43**(1), 29 (1998)
2. C. Nore, M. Abid, M.E. Brachet, Phys. Rev. Lett. **78**, 3896 (1997)
3. T. Araki, M. Tsubota, S.K. Nemirovskii, J. Low Temp. Phys. **126**, 303 (2002)
4. M. Kobayashi, M. Tsubota, Phys. Rev. Lett. **94**, 065302 (2005)
5. M. Kobayashi, M. Tsubota, J. Phys. Soc. Jpn. **78**, 3248 (2005)
6. R.H. Kraichnan, Phys. Fluids **10**, 1417 (1967)
7. R.H. Kraichnan, D. Montgomery, Rep. Prog. Phys. **43**, 547 (1980)
8. M. Chertkov, C. Connaughton, I. Kolokolov, V. Levedev, Phys. Rev. Lett. **99**, 084501 (2007)
9. L.M. Smith, V. Yakhot, Phys. Rev. Lett. **71**, 352 (1993)
10. U. Frisch, P.L. Sulem, Phys. Fluids. **27**, 1921 (1984)
11. M.G. Shats, H. Xia, H. Punzmann, Phys. Rev. Lett. **99**, 164502 (2007)
12. A.O. Korotkevich, Phys. Rev. Lett. **101**, 074504 (2008)
13. A.N. Ganshin, V.B. Efimov, G.V. Kolmakov, L.P. Mezhov-Deglin, P.V.E. McClintock, Phys. Rev. Lett. **101**, 065303 (2008)
14. S. Nazaranko, JETP Lett. **83**, 198 (2006)
15. A. Dyachenko, G. Falkovich, Phys. Rev. E **54**, 5095 (1996)
16. E.A. Novikov, Sov. Phys. JETP **41**, 937 (1976)