DEMOGRAPHY

ASSIGNMENT 4: LIFE TABLES, FERTILITY RATES

SUBHRAJYOTY ROY (BS - 1613)

PALLAB DAS (BS - 1614)

UDIT SURYA SAHA (BS - 1619)

SUPRAVAT SARKAR (BS - 1648)

Indian Statistical Institute, Kolkata

02/11/2018

Q1. The following table below gives the q-type age-specific death rates for male population in a developing country. Use it to calculate the life expectancies at birth for males in that country, assuming $l_0 = 10,000$. Assume that no man survives until his 100-th birthday, i.e. $l_{100} = 0$.

Answer:

From the q-type age-specific death rates ${}_nq_x$ we can figure out the p-type age-specific survival rates as, ${}_np_x = 1 - {}_nq_x$.

Since, the radix l_0 is given to be 10,000, hence we can figure out $l_{x+n} = l_x \times {}_np_x$ for each of the age groups.

Then we can obtain the number of person years lived between the ages x and $x+n$, using the formula;

$$\begin{aligned} {}_nL_x &= (n \times l_{x+n}) + ({}_nd_x \times {}_na_x) \\ &= (n \times l_{x+n}) + \{(l_x - l_{x+n}) \times {}_na_x\} \\ &= ({}_na_x \times l_x) + (n - {}_na_x)l_{x+n} \end{aligned}$$

Since the country is given to be a developing country, the mortality rates are high. In such a case, we use ${}_1a_0 = 0.330$ and ${}_4a_1 = 1.352$, obtained from Coale-Demeny family of West Model life tables, and for higher age groups with age difference of 5 years, ${}_na_x$ is taken as $0.5 \times 5 = 2.5$.

After this, we can compute the life expectancies at birth using the formula;

$$e_0 = \frac{T_0}{l_0} = \left(\sum_{\alpha=0}^{95} {}_nL_\alpha\right)/l_0$$

The calculation using the above mentioned formulas is summarized in the following table;

Age, x	**Width, n**	${}_nq_x$	${}_np_x$	l_x	${}_nL_x$
0	1	0.1402	0.8598	10000	9060.66
1	4	0.0714	0.9286	8598	32766.13
5	5	0.0207	0.9793	7984	39505
10	5	0.015	0.985	7818	38795
15	5	0.022	0.978	7700	38075
20	5	0.0313	0.9687	7530	37060
25	5	0.0343	0.9657	7294	35842.5
30	5	0.0393	0.9607	7043	34522.5
35	5	0.0466	0.9534	6766	33040
40	5	0.0578	0.9422	6450	31317.5
45	5	0.0715	0.9285	6077	29297.5
50	5	0.0947	0.9053	5642	26872.5
55	5	0.125	0.875	5107	23937.5
60	5	0.1751	0.8249	4468	20382.5
65	5	0.2412	0.7588	3685	16202.5
70	5	0.3362	0.6638	2796	11627.5
75	5	0.4549	0.5451	1855	7165
80	5	0.6126	0.3874	1011	3505

Age, x	**Width, n**	${}_n q_x$	${}_n p_x$	l_x	${}_n L_x$
85	5	0.7707	0.2293	391	1200
90	5	0.8984	0.1016	89	245
95	5	1	0	9	22.5
Total	-	-	-	-	470441.8

Therefore, the life expectancy at birth is given by; $e_0 = \frac{470441.8}{10,000} \approx 47.044$ years, i.e. about 47 years and 15 days.

Q2. Show that the crude birth rate in a stationary population corresponding to a life table is equal to $1/e_0$, where e_0 is the life expectation at birth.

Answer:

We know that;

$$e_0 = \frac{T_0}{l_0} = \frac{L_0 + L_1 + L_2 + \cdots}{l_0}$$

where each of the age groups considered are of length equal to 1 year. Now, observe that, $L_x = {}_1L_x$ is the number of person years lived by the population between age x to age $(x+1)$.

Since, we are considering that the population is stationary, i.e.

i. The age-specific death rates are constant over time. So, L_x is also constant over time.
ii. The flow of births is constant over time.
iii. Net migration is zero at all ages.

Therefore, the person year lived by the population between age x to age $(x+1)$ is uniformly spread over the age interval. Consider a single person aged y years, then as the population is stationary, so the person-year lived by that single person can be broken up into several parts, each representing the person-year lived by that person in each group, possibly in different time.

Therefore, $\text{The midyear population } = \sum_x \text{midyear population in age group } (x, x+1)$
$= \sum_x \text{total personyears lived by population aged } (x, x+1)$
$= \sum_x \sum_y \text{personyears lived by population currently aged } (x, x+1)$
$\text{during their age } (y, y+1) \text{ in past times}$

Now, observe that, because of stationary population, number of person year lived during the age $(y, y+1)$ in past times is constant over time, hence is equal to L_y. Hence, we obtain that,

$$L_0 + L_1 + \cdots = \text{mid year population}$$

Also, note that, l_0, the radix is the population in the age group 0 to 1, i.e. is the population of new live birth cohorts. As the flow of the birth is constant in the stationary population, the number of live births during the year is equal to the number of birth cohorts i.e. l_0.

Therefore,

$$CBR = \frac{\text{number of live births during the year}}{\text{mid year population}}$$
$$= \frac{l_0}{L_0 + L_1 + L_2 + \cdots} = \frac{1}{T_0/l_0} = \frac{1}{e_0}$$

Q3. In a life table, $\boldsymbol{e_0}$ is equal to $\boldsymbol{e_1}$, and

$$\boldsymbol{l_x = \left(\frac{l_1}{l_0}\right)^x l_0, \qquad \text{for } 0 \leq x < 1}$$

Show that, $\boldsymbol{p_0 = exp(-1/e_1)}$.

Answer:

Observe that, p_0 is the probability that a person survives till age 1 given that he/she is born, i.e. of age 0, hence clearly,

$$p_0 = \frac{l_1}{l_0}$$

Therefore, the given formula for l_x can be rewritten as $l_x = (p_0)^x \times l_0$, where x lies between 0 and 1.

Now, observe that, ${}_nL_x$ is the number of person years lived by the cohort between ages x to $(x+n)$. Clearly, ${}_nL_x$ should be the infinitesimal sums of number of surviving cohorts at every instant between age x and $(x+n)$. This reduces to the formula;

$${}_nL_x = \int_x^{x+n} l_x \, \mathrm{d}x$$

Using the above formula for ${}_nL_x$ and the given expression for l_x, we can calculate;

$${}_1L_0 = \int_0^1 l_x \, \mathrm{d}x = \int_0^1 (p_0)^x l_0 \, \mathrm{d}x = l_0 \times \frac{(p_0 - 1)}{\ln(p_0)}$$

Therefore, T_x i.e. the number of person years lived by the cohort after age x, can be expressed as;

$$T_x = \sum_{\alpha=x}^{\infty} {}_nL_\alpha = \int_x^\infty l_x \, \mathrm{d}x$$

Now, from the given information that $e_0 = e_1$, and $e_x = T_x/l_x$, we obtain;

$$\begin{aligned}
&e_0 = e_1 \\
&\Rightarrow \frac{T_0}{l_0} = \frac{T_1}{l_1}, \qquad \text{since } e_x = \frac{T_x}{l_x} \\
&\Rightarrow \frac{l_1}{l_0} = \frac{T_1}{T_0} \\
&\Rightarrow p_0 = \frac{T_1}{T_0}, \qquad \text{since} \frac{l_1}{l_0} = p_0 \\
&\Rightarrow (1 - p_0) = \frac{T_0 - T_1}{T_0} = \frac{{}_1L_0}{T_0}, \qquad \text{since } T_0 = {}_1L_0 + T_1 \\
&\Rightarrow (1 - p_0) = \frac{l_0}{T_0} \times \frac{(p_0 - 1)}{\ln(p_0)}, \ \text{ from the formula of } {}_1L_0 \\
&\Rightarrow \frac{T_0}{l_0} = e_0 = -\frac{1}{\ln(p_0)} \\
&\Rightarrow p_0 = \exp\left(-\frac{1}{e_0}\right) = \exp\left(-\frac{1}{e_1}\right), \ \text{ since } e_0 = e_1
\end{aligned}$$

This proves the result.

Q4. The table below gives estimates of the life expectation e_x at various ages x for females in Nicaragua, 1990-95 and the United States, 1989. Use them to estimate q_0, ${}_4q_1$, ${}_5q_5$ and ${}_5q_{10}$ for these two countries.

Answer:

We know that, the life expectation at age x, i.e. $e_x = T_x/l_x$. Therefore, we can rewrite it as;

$$e_x l_x = T_x = \sum_{\alpha=x}^{\infty} {}_nL_\alpha$$

Therefore, we can consider the successive difference of the above to obtain;

$$e_x l_x - e_{x+n} l_{x+n} = \sum_{\alpha=x}^{\infty} {}_nL_\alpha - \sum_{\alpha=x+n}^{\infty} {}_nL_\alpha = {}_nL_x$$

However, as show in the answer to question 1, we can write ${}_nL_x$ as a weighted sum of the l_x's where the weights depend only on ${}_na_x$. Therefore,

$e_x l_x - e_{x+n} l_{x+n} = ({}_na_x \times l_x) + (n - {}_na_x) l_{x+n}$

$\Rightarrow (e_x - {}_na_x) l_x = (n - {}_na_x + e_{x+n}) l_{x+n}$

$\Rightarrow l_{x+n}/l_x = (e_x - {}_na_x)/(n - {}_na_x + e_{x+n})$

Since, ${}_nq_x = 1 - (l_{x+n}/l_x)$, we obtain the final formula as;

$${}_nq_x = \frac{(n + e_{x+n} - e_x)}{(n + e_{x+n} - {}_na_x)}$$

To obtain the values of ${}_na_x$, we again use the Coale-Demeny family of West model life tables, and for female population, we have ${}_1a_0 = 0.350$ and ${}_4a_1 = 1.361$ and 2.5 for the rest of the age groups. The necessary calculations are summarized in the following table;

Age, x	**Nicaragua, 1990-95**		**United States, 1989**	
	e_x	**${}_nq_x$**	**e_x**	**${}_nq_x$**
0	67.7	0.046709	78.6	0.008866
1	70	0.024169	78.3	0.001298
5	67.7	0.00761	74.4	0.001389
10	63.2	0.004918	69.5	0.00149
15	58.5	-	64.6	-

Q5. If the crude birth rate in a country remains constant over a number of years, but the general fertility rate increases steadily, what does this tell you about the country's population?

Answer:

We know that,

$$CBR = \frac{\text{number of live births during the year}}{\text{mid year population}} \times 1{,}000$$

And

$$GFR = \frac{\text{number of live births during the year}}{\text{mid year female population in reproductive age group}} \times 1{,}000$$

Now, if the Crude Birth Rate (CBR) remains constant over a number of years, that means the number of live births as a proportion of the whole population is not changing over the years. However, as General Fertility Rate (GFR) steadily increases, it indicates that the number of live births as proportion of the women population of reproductive age group is increasing. This indicates that the denominator is the GFR formula is decreasing over the years as proportion of the whole population, i.e. the number of women in reproductive age group as proportion of the whole population.

The above claim is also indicated by the following formula which can be obtained by taking the ratio of CBR to GFR, i.e.

$$\frac{CBR}{GFR} = \frac{\text{mid year women population in reproductive age group}}{\text{mid year population}}$$

As the numerator is held constant over time, but the denominator is increasing, this indicates the ratio being declined over time, which shows a steady decrement of the female share in the age distribution.

Q6. The data below relate to fertility in a country in 1976 and 1993.

Age Group	1976		1993	
	No. of births ('000)	Mid-year female population ('000)	No. of births ('000)	Mid-year female population ('000)
15-19	57.9	1809	45.1	1455
20-24	182.2	1672	152.0	1831
25-29	220.7	1855	236.0	2070
30-34	90.8	1593	171.1	1967
35-39	26.1	1374	58.8	1729
40-44	6.5	1300	10.5	1750

a) **Calculate the general fertility rate for 1976 and 1993.**
b) **Calculate age specific fertility rate for these two years.**
c) **Using the 1976 population as standard calculate the standardized fertility rate for 1993.**

Answer:

To calculate the General Fertility Rate (GFR), we use the following formula;

$$GFR = \frac{\text{number of live births during the year}}{\text{mid year female population in reproductive age group}} \times 1{,}000$$

The total number of live births during the year 1976 is = (57.9 + 182.2 + 220.7 + 90.8 + 26.1 + 6.5) = 584.2 thousands. And, the mid-year female population in reproductive period during the year 1976 is = (1809 + 1672 + 1855 + 1593 + 1374 + 1300) = 9603 thousands.

Therefore, $GFR_{1976} = \frac{584.2}{9603} \times 1{,}000 = 60.83$ children per thousand females.

The total number of live births during the year 1993 is = (45.1 + 152.0 + 236.0 + 171.1 + 58.8 + 10.5) = 673.5 thousands. And, the mid-year female population in reproductive period during the year 1993 is = (1455 + 1831 + 2070 + 1967 + 1729 + 1750) = 10802 thousands.

Therefore, $GFR_{1993} = \frac{673.5}{10802} \times 1{,}000 = 62.35$ children per thousand females.

Hence, General Fertility Rate for the year 1976 is 60.83 children per thousand females and for the year 1993 is 62.35 children per thousand females.

To calculate the age-specific fertility rates for different years, we use the following formula;

$$_{n}ASFR_{x} = \frac{\text{number of live births to females in age group } (x, x+n)}{\text{midyear female population in the age group } (x, x+n)} \times 1{,}000$$

The necessary results for Age-specific fertility rates are shown in the following table;

Age group	**Age-specific Fertility Rate in the year 1976 (per '000 females)**	**Age-specific Fertility Rate in the year 1993 (per '000 females)**
15-19	32.00663	30.99656
20-24	108.9713	83.01475
25-29	118.9757	114.0097
30-34	56.99937	86.98526
35-39	18.99563	34.0081
40-44	5	6

If we use the year 1976 as the standard, then we can calculate the expected number of live births to females in each age group, from the fertility distribution in different age groups in the year 1993, and using the age distribution of the female population in 1976 as the standard. We use the following formula for each age group to calculate the expected number of live births;

$$\text{Expected number of live births to females in age group } (x, x+n)$$
$$= \frac{\text{Age specific fertility rate in 1993} \times \text{age specific female population in 1976}}{1000}$$

The necessary calculation is shown in the table below;

Age Group	$\boldsymbol{ASFR_{1993}}$	**Mid-year female population ('000)**	**Expected number of live births ('000)**
15-19	30.99656	1809	56.07
20-24	83.01475	1672	138.8
25-29	114.0097	1855	211.49
30-34	86.98526	1593	138.57
35-39	34.0081	1374	46.73
40-44	6	1300	7.8
Total	-	9603	599.46

Therefore, the standardized fertility rate for the year 1993 is given as; $= \frac{599.46}{9603} \times 1{,}000 =$ 62.42 children per thousand females.

Clearly, this also indicates that the fertility has been increased in 1993 compared to 1976 in the country.

Q7. The table below gives the parity progression ratios for a number of recent birth cohorts in a country. Assuming that no woman in any of these birth cohorts had a fifth child, calculate;

- **a) The proportion of women in each birth cohort who had exactly 0, 1, 2, 3 and 4 children.**
- **b) The total fertility rate for women in each birth cohort.**

Calendar years of birth	Parity Progression Ratio (PPR)			
	0-1	1-2	2-3	3-4
1931-33	0.861	0.804	0.555	0.518
1934-36	0.885	0.828	0.555	0.489
1937-39	0.886	0.847	0.543	0.455
1940-42	0.890	0.857	0.516	0.416
1943-45	0.892	0.854	0.458	0.378
1946-48	0.885	0.849	0.418	0.333

Answer:

Suppose, N is the total female population in the reproductive period, and W_x to be the number of females in the reproductive age group with parity x. Hence,

$$W_0 + W_1 + \cdots + W_4 = N$$

Also, let us denote the parity progression ratio from parity x to $(x+1)$ by a_x. Clearly, a_x is the probability that a women will have another child, i.e. will increase in parity given that she already is at parity x.

Therefore, assuming no women has parity 5 or more,

$$a_x = \frac{W_{x+1} + \cdots + W_4}{W_x + W_{x+1} + \cdots + W_4}$$

Hence,

$$(1 - a_x) = \frac{W_x}{W_x + W_{x+1} + \cdots + W_4}$$

Hence, observe that;

$$a_0 . a_1 \dots a_{x-1}(1 - a_x) = \frac{W_x}{W_x + \cdots + W_4} . \frac{W_x + \cdots + W_4}{W_{x-1} + \cdots + W_4} \cdots \frac{W_1 + \cdots W_4}{N} = \frac{W_x}{N}$$

which is the proportion of female population having parity exactly equal to x.

Therefore, we use the following formula to calculate the proportions of female having parity x,

$$\text{Proportion of women of parity exactly } 0 = (1 - a_0)$$
$$\text{Proportion of women of parity exactly } 1 = a_0(1 - a_1)$$
$$\text{Proportion of women of parity exactly } 2 = a_0 . a_1(1 - a_2)$$
$$\text{Proportion of women of parity exactly } 3 = a_0 . a_1 . a_2(1 - a_3)$$
$$\text{Proportion of women of parity exactly } 4 = a_0 . a_1 . a_2 . a_3$$

The answers are given in the following tabular form.

Calendar years of birth	Proportion of Women having exactly a specified parity				
	0	1	2	3	4
1931-33	0.139	0.168756	0.308049	0.185182	0.199013
1934-36	0.115	0.15222	0.326087	0.20782	0.198873
1937-39	0.114	0.135558	0.342952	0.222082	0.185408
1940-42	0.11	0.12727	0.369161	0.229844	0.163725
1943-45	0.108	0.130232	0.412878	0.217009	0.13188
1946-48	0.115	0.133635	0.437294	0.209485	0.104585

Now, to obtain the Total Fertility Rate (TFR) for women in each birth cohort, we compute the Completed Fertility Rate (CFR) for each birth cohort which is known to be equivalent of Total Fertility Rate (TFR). The formula for computing CFR is as follows;

$$CFR = a_0 + a_0 a_1 + a_0 a_1 a_2 + a_0 a_1 a_2 a_3$$

since, $a_x = 0$ for $x \geq 4$.

The answers are summarized in the following table;

Calendar years of birth	**Completed Fertility Rate (CFR) or Total Fertility Rate (TFR) for birth cohorts ('000)**
1931-33	2.136453
1934-36	2.223346
1937-39	2.22934
1940-42	2.210023
1943-45	2.134538
1946-48	2.055021

THANK YOU