ASSIGNMENT 1: SQC & OR

Formulation of Linear Programming Problems

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Question 1

This problem is known in the literature as ***Diet Problem***. Dieticians tell us that a balanced diet must contain quantities of nutrients such as fats, vitamins, minerals etc. The medical experts and dieticians tell us that it is necessary for an adult to consume at least 75 g of proteins, 85 g of fats, and 300 g of carbohydrate daily. Table 1 below gives the food items (which are readily available in the market), analysis and their respective cost.

Food Type	**Food Value (g) per 100 g**			**Cost per kg (Rs)**
	Proteins	**Fats**	**Carbohydrates**	
1	8.0	1.5	35.0	1.00
2	18.0	15.0	-	3.00
3	16.0	4.0	7.0	4.00
4	4.0	20.0	2.5	2.00
5	5.0	8.0	40.0	1.50
6	2.5	-	25.0	3.00
Minimum daily requirements	75	85	300	

Find out the food that should be recommended from a large number of alternative sources of these nutrients so that the total cost of food satisfying the minimum requirements of balanced diet is the lowest.

Answer – Formulation of LP Problem

Let x_j denote the amount of food intake of food type j, where $j = 1,2, \ldots 6$, calculated in the units of per 100 g.

As per given by the table above, the cost for consuming x_1 amount (in units of 100 g) of food of type 1 would cost rupees $(1.00 \times x_1/10) = (0.1 \times x_1)$, since the cost of per kg of food of type 1 is 1.00 rupee. In a similar way, $(x_2 \times 100)$ grams of food of type 2 would cost $(0.3 \times x_2)$ rupees, $(x_3 \times 100)$ grams of food of type 3 would cost $(0.4 \times x_3)$ rupees, $(x_4 \times 100)$ grams of food of type 4 would cost $(0.2 \times x_4)$ rupees, $(x_5 \times 100)$ grams of food of type 5 would cost $(0.15 \times x_5)$ rupees and $(x_6 \times 100)$ grams of food of type 6 would cost $(0.3 \times x_6)$ rupees.

Therefore, the total cost of consuming the food would be given by;

$$x_0 = 0.1x_1 + 0.3x_2 + 0.4x_3 + 0.2x_4 + 0.15x_5 + 0.3x_6$$

Now, according to the above table, we get to know that consuming 100 grams of food of type 1 would give 8 grams of proteins as nutrients. Hence, consuming $(100 \times x_1)$ grams of food of type 1 would give $8x_1$ grams of proteins as nutrients. Similarly, consuming $\left(100x_j\right)$ grams of food of type j, would give $18x_2, 16x_3, 4x_4, 5x_5$ and $2.5x_6$ grams of proteins as nutrients respectively, where j is equal to 2, 3, 4, 5 and 6.

Therefore, the total amount (in gram) of protein gained from the combination of food is;

$$8x_1 + 18x_2 + 16x_3 + 4x_4 + 5x_5 + 2.5x_6$$

According to dieticians, this amount of protein intake should be at least 75 grams daily. It introduces the constraint,

$$8x_1 + 18x_2 + 16x_3 + 4x_4 + 5x_5 + 2.5x_6 \geq 75$$

Similarly, considering the minimum daily requirements of Fats and Carbohydrates, we also obtain another two constraints;

$1.5x_1 + 15x_2 + 4x_3 + 20x_4 + 8x_5 \geq 85$, the constraint for minimum requirement of Fats and,

$35x_1 + 7x_3 + 2.5x_4 + 40x_5 + 25x_6 \geq 300$, the constraint for minimum requirement of Carbohydrate.

Also, since x_j's denote the amount of food intake of type j, it clearly must be non-negative.

The objective of the problem is to minimize the total cost of the food combination, i.e. minimizing the quantity x_0. Therefore, the Linear Programming Problem can be formulated as follows;

Minimize $\boldsymbol{x_0 = 0.1x_1 + 0.3x_2 + 0.4x_3 + 0.2x_4 + 0.15x_5 + 0.3x_6}$

Subject to the constraints;

$$\begin{aligned} \boldsymbol{8x_1 + 18x_2 + 16x_3 + 4x_4 + 5x_5 + 2.5x_6} &\geq \boldsymbol{75} \\ \boldsymbol{1.5x_1 + 15x_2 + 4x_3 + 20x_4 + 8x_5} &\geq \boldsymbol{85} \\ \boldsymbol{35x_1 + 7x_3 + 2.5x_4 + 40x_5 + 25x_6} &\geq \boldsymbol{300} \end{aligned}$$

And $\boldsymbol{x_1, x_2, x_3, x_4, x_5, x_6 \geq 0}$

Question 2

Two alloys, *A* and *B* are made from four different metals, I, II, III and IV, according to the following specifications:

Alloy	Specifications	Selling price ($)/ton
A	At most 80% of I	200
	At least 30% of II	
	At least 50% of IV	
B	Between 40% & 60% of II	300
	At least 30% of III	
	At most 70% of IV	

The four metals, in turn, are extracted from three different ores with the following data;

Ore	Max. Quantity (tons)	Constituents (%)					Purchase Price ($)/ton
		I	II	III	IV	others	
1	1000	20	10	30	30	10	30
2	2000	10	20	30	30	10	40
3	3000	5	5	70	20	0	50

How much of each alloy should be produced to maximize the profit. Formulate the problem as a LP model.

Answer – Formulation of LP Problem

Let y_A and y_B denote the amount (in ton) of produced alloys for alloy A and B respectively. Let x_1, x_2 and x_3 denote the amount (in ton) of the ores used of type 1, 2 and 3 respectively. Let z_{iA} denotes the amount (in ton) of i-th Metal used to produce alloy A. Similarly, let z_{iB} denotes the amount (in ton) of the i-th Metal used to produce alloy B. Here, i can be any of $\{1, 2, 3, 4\}$.

It is assumed that the metals are extracted from the ores, and then they are mixed to form the alloys, rather than blending the ores together to form alloys which would cause the alloys to be comprised of other impurities in the ore. With this assumption in mind, the LP problem is formulated.

Selling y_A tons of alloy A would yield a earning of $200y_A$ rupees, while selling y_B tons of alloy B would yield a earning of $300y_B$ rupees. Also, using x_1 tons of ore of first type would rise to a cost of $30x_1$ rupees and similar cost would be incurred for usage of other ores. Therefore, the profit is given by;

$$x_0 = 200y_A + 300y_B - 30x_1 - 40x_2 - 50x_3$$

Now, based on the availability of the ores in tons as given in the table, we have the constraints; $x_1 \leq 1000, x_2 \leq 2000$ and $x_3 \leq 3000$.

Observe that, the total amount of metal I used to produce the alloys is $(z_{1A} + z_{1B})$. On the other hand, the total amount of metal I extracted from the ores is $(0.2x_1 + 0.1x_2 + 0.05x_3)$, since we get 20% of x_1 i.e. $0.2x_1$ tons of metal I from ore of first type and so on. Obviously, the total amount of metal I extracted must be more than the total amount of metal I used to produce the alloys. Therefore, for metal I we have the constraint that;

$$z_{1A} + z_{1B} \leq 0.2x_1 + 0.1x_2 + 0.05x_3$$

We get three similar constraints for the other metals as well;

$$\begin{aligned} z_{2A} + z_{2B} &\leq 0.1x_1 + 0.2x_2 + 0.05x_3 \\ z_{3A} + z_{3B} &\leq 0.3x_1 + 0.3x_2 + 0.7x_3 \\ z_{4A} + z_{4B} &\leq 0.3x_1 + 0.3x_2 + 0.2x_3 \end{aligned}$$

Also, the alloys should meet the required specifications. Therefore, for alloy A, the amount of metal 1 used to produce alloy A, i.e. z_{1A} to be less than or equal to 80% of the total amount of alloy A, i.e. y_A. This give us the constraint; $z_{1A} \leq 0.8y_A$. Having at most 30% of alloy A being metal II, we get $z_{2A} \leq 0.3y_B$ and at least 50% being metal IV gives $z_{4A} \geq 0.5y_A$. In addition to that, we must have the total amount of metal used to build up an alloy equal to the total amount of alloy produced. Therefore,

$$y_A = z_{1A} + z_{2A} + z_{3A} + z_{4A}$$

We can put this equality back into the objective function and the constraints to get rid of the variable y_A. However, using y_A simplifies our notation. Similar inequality and equality constraint comes up when dealing with specification requirements of alloy B.

The goal is to maximize the total amount of profit or x_0. The Linear Programming problem can be formulated as follows;

Maximize $x_0 = 200y_A + 300y_B - 30x_1 - 40x_2 - 50x_3$

Subject to;

$$\begin{aligned} z_{1A} + z_{1B} &\leq 0.2x_1 + 0.1x_2 + 0.05x_3 \\ z_{2A} + z_{2B} &\leq 0.1x_1 + 0.2x_2 + 0.05x_3 \\ z_{3A} + z_{3B} &\leq 0.3x_1 + 0.3x_2 + 0.7x_3 \\ z_{4A} + z_{4B} &\leq 0.3x_1 + 0.3x_2 + 0.2x_3 \end{aligned}$$

$$\begin{aligned} y_A &= z_{1A} + z_{2A} + z_{3A} + z_{4A} \\ y_B &= z_{1B} + z_{2B} + z_{3B} + z_{4B} \end{aligned}$$

$$z_{1A} \leq 0.8y_A,\ z_{2A} \leq 0.3y_A,\ z_{4A} \leq 0.5y_A$$
$$z_{1B} \geq 0.4y_B,\ z_{1B} \leq 0.6y_B,\ z_{3B} \geq 0.3y_B,\ z_{4B} \leq 0.7y_B$$

$x_1, x_2, x_3, z_{1A}, z_{2A}, z_{3A}, z_{4A}, z_{1B}, z_{2B}, z_{3B}, z_{4B} \geq 0$ and $x_1 \leq 1000, x_2 \leq 2000, x_3 \leq 3000$

Note: $y_A, y_B \geq 0$ is implied from the equality constraints and the non-negativity of z_{iA}'s and z_{iB}'s. Hence, they are excluded.