Mathematics Talent Reward Programme

Question Paper for Class XI

18^{th} January, 2015

Total Marks: 150 Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

[***You should answer these questions in the first page according to the order given in the question paper***. *Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.*]

1. How many distinct arrangements are possible for wearing five different rings in the five fingers of the right hand? (We can wear multiple rings in one finger)

   (A) $\frac{10!}{5!}$, **(B)** 5^5, **(C)** $\frac{9!}{4!}$, **(D)** None of these.

2. Let $f_n(x) = \underbrace{xx\cdots x}_{n \text{ times}}$ that is,$f_n(x)$ is an n digit number with all digits x, where $x \in \{1, 2, \ldots, 9\}$. Then which of the following is $(f_n(3))^2 + f_n(2)$?

   (A) $f_n(5)$, **(B)** $f_{2n}(9)$, **(C)** $f_{2n}(1)$, **(D)** None of these.

3. If $A_i = \frac{x-a_i}{|x-a_i|}$, $i = 1, 2, \ldots n$ for n numbers $a_1 < a_2 < \cdots < a_m < \cdots < a_n$, then $\lim\limits_{x \to a_m} (A_1 A_2 \ldots A_n) =$?.

   (A) $(-1)^{m-1}$, **(B)** $(-1)^m$, **(C)** 1, **(D)** Does not exist.

4. Let n be an odd integer. Placing no more than one **X** in each cell of a $n \times n$ grid, what is the greatest number of **X**'s that can be put on the grid without getting n **X**'s together vertically, horizontally or diagonally?

   (A) $2\binom{n}{2}$, **(B)** $\binom{n}{2}$, **(C)** $2n$, **(D)** $2\binom{n}{2} - 1$.

5. How many integral solutions are there for the equation $x^5 - 31x + 2015 = 0$?

   (A) 2, **(B)** 4, **(C)** 1, **(D)** None of these.

6. Let AC and CE be perpendicular line segments, each of length 18. Suppose B and D are the midpoints of AC and CE respectively. If F be the point of intersection of EB and AD, then the area of $\triangle BDF$ is?

   (A) $27\sqrt{2}$, **(B)** $18\sqrt{2}$, **(C)** 13.5, **(D)** 18.

7. How many x are there such that x, $[x]$, $\{x\}$ are in harmonic progression (i.e, the reciprocals are in arithmetic progression)? (Here $[x]$ is the largest integer less than equal to x and $\{x\} = x - [x]$)

   (A) 0, **(B)** 1, **(C)** 2, **(D)** 3.

8. In $\triangle ABC$, $AB = AC$ and D is foot of the perpendicular from C to AB and E the foot of the perpendicular from B to AC, then

   (A) $BC^3 > BD^3 + BE^3$, **(B)** $BC^3 < BD^3 + BE^3$,
   (C) $BC^3 = BD^3 + BE^3$, **(D)** None of these.

9. How many 5×5 grids are possible such that each element is either 1 or 0 and each row sum and column sum is 4?

   (A) 64, **(B)** 32, **(C)** 120, **(D)** 96.

10. If $\sum\limits_{i=1}^{n} \cos^{-1}(\alpha_i) = 0$, then find $\sum\limits_{i=1}^{n} \alpha_i$.

    (A) $\frac{n}{2}$, **(B)** n , **(C)** $n\pi$, **(D)** $\frac{n\pi}{2}$.

11. $S = \{1, 2, \ldots, 6\}$. Then find out the number of unordered pairs of (A,B) such that $A, B \subseteq S$ and $A \cap B = \emptyset$.

    (A) 360, **(B)** 364, **(C)** 365, **(D)** 366.

12. The maximum value of $\sin^4\theta + \cos^6\theta$ will be?

    (A) $\frac{1}{2\sqrt{2}}$, **(B)** $\frac{1}{2}$, **(C)** $\frac{1}{\sqrt{2}}$, **(D)** 1.

13. Define $f(x) = \max\{\sin x, \cos x\}$. Find at how many points in $(-2\pi, 2\pi)$, $f(x)$ is not differentiable?

    (A) 0, **(B)** 2, **(C)** 4, **(D)** ∞.

14. $z = x + iy$ where x and y are two real numbers. Find the locus of the point (x, y) in the plane, for which $\dfrac{z+i}{z-i}$ is purely imaginary (that is, it is of the form ib where b is a real number). [Here, $i = \sqrt{-1}$]

    (A) A straight line, **(B)** A circle, **(C)** A parabola, **(D)** None of these.

15. Find out the number of real solutions of $x^2 e^{\sin x} = 1$

(A) 0, **(B)** 1, **(C)** 2, **(D)** 3.

Short Answer Type Questions

[Each question carries a total of 15 marks. Credits will be given to partially correct answers]

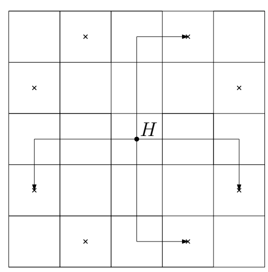
1. In a room there is a series of bulbs on a wall and corresponding switches on the opposite wall. If you put on the n-th switch the n-th bulb will light up. There is a group of men who are operating the switches according to the following rule: they go in one by one and starts flipping the switches starting from the first switch until he has to turn on a bulb; as soon as he turns a bulb on, he leaves the room. For example the first person goes in, turns the first switch on and leaves. Then the second man goes in, seeing that the first switch on turns it off and then lights the second bulb. Then the third person goes in, finds the first switch off and turns it on and leaves the room. Then the fourth person enters and switches off the first and second bulbs and switches on the third. The process continues in this way. Finally we find out that first 10 bulbs are off and the 11-th bulb is on. Then how many people were involved in the entire process?

2. Let x, y be numbers in the interval (0, 1) such that for some $a > 0$, $a \neq 1$

$$\log_x a + \log_y a = 4 \log_{xy} a.$$

Prove that $x = y$.

3. Show that, in a chessboard, it is possible to traverse to any given square from another given square using a knight. (A knight can move in a chessboard by going two steps in one direction and one step in a perpendicular direction as shown in the given figure)

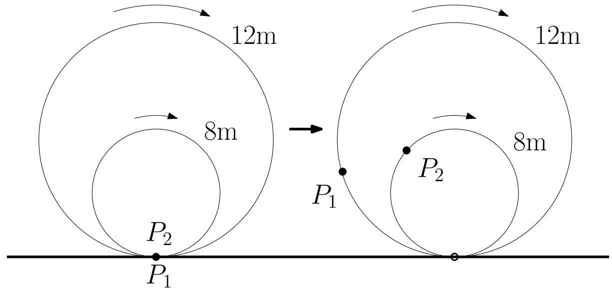


4. Find all real numbers $x_1, x_2, \ldots, x_n$ satisfying,

$$\sqrt{x_1 - 1^2} + 2\sqrt{x_2 - 2^2} + \cdots + n\sqrt{x_n - n^2} = \frac{1}{2}(x_1 + x_2 + \cdots + x_n).$$

5. Let a be the smallest and A the largest of n distinct positive integers. Prove that the least common multiple of these numbers is greater than or equal to na and that the greatest common divisor is less than or equal to $\frac{A}{n}$.

6. In the following figure, the bigger wheel has circumference 12 m and the inscribed wheel has circumference 8 m. P_1 denotes a point on the bigger wheel and P_2 denotes a point on the smaller wheel. Initially P_1 and P_2 coincide as in the figure. Now we roll the wheels on a smooth surface and the smaller wheel also rolls in the bigger wheel smoothly. What distance does the bigger wheel have to roll so that the points will be together again?



Use of calculators is not allowed. You may use a ruler and a compass for construction.

~ Best of Luck ~