Mathematics Talent Reward Programme

Model Solutions for Class XI

Multiple Choice Questions

[*Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.*]

1. **(C)**	2. **(C)**	3. **(D)**	4. **(B)**	5. **(D)**
6. **(C)**	7. **(B)**	8. **(A)**	9. **(C)**	10. **(B)**
11. **(C)**	12. **(D)**	13. **(C)**	14. **(B)**	15. **(B)**

Short Answer Type Questions

[*Each question carries a total of 15 marks. Credit will be given to partially correct answers*]

1. When the n-th person leaves the room, the bulbs can be represented as a binary number, say A_n whose k-th digit from right is 1 if k-th bulb is ON and 0 if k-th bulb is OFF. Denote by $(B)_{10}$ to be the decimal representation of a binary number B. Then we will show that $(A_n)_{10} = n$.

   Clearly when the 1st person leaves, only the first switch is turned on. Therefore $A_1 = 1 \implies (A_1)_{10} = 1$. Now we use induction. Suppose it $(A_n)_{10} = n$ holds for all $n \le m$. We will show $(A_{m+1})_{10} = m+1$. If the 1st switch is turned off when the $(m+1)$-th person enters, then $(A_{m+1})_{10} - (A_m)_{10} = 1 \implies (A_{m+1})_{10} = m+1$. If the first r switches are turned on and $(r+1)$-th switch is turned off when the $(m+1)$-th person enters, then

$$(A_{m+1})_{10} - (A_m)_{10} = (\dots 1\underbrace{00\dots0}_{r\text{times}})_{10} - (\dots 0\underbrace{11\dots1}_{r\text{times}})_{10} = 2^r - \sum_{k=0}^{r-1} 2^k = 1$$

   Hence by induction the claim is true. We use this method to compute the number of involved people. Now note that our configuration gives A_n to be a binary number with 1 followed by 10 0's. So, n, the number of persons is

$$(1\underbrace{00\dots0}_{10\text{times}})_{10} = 2^{10} = 1024$$

2. From the given equation we have

$$\begin{aligned}
&\log_x a + \log_y a = 4\log_{xy} a \\
\implies &\frac{\log a}{\log x} + \frac{\log a}{\log y} = 4\frac{\log a}{\log xy} \\
\implies &\log a\left(\frac{1}{\log x} + \frac{1}{\log y}\right) = \frac{4\log a}{\log x + \log y}
\end{aligned}$$

   Since $a > 0$ and $a \neq 1$, this ensures $\log a \neq 0$. Hence we can cancel $\log a$ both sides to get

$$\begin{aligned}
&\frac{1}{\log x} + \frac{1}{\log y} = \frac{4}{\log x + \log y} \\
\implies &2 + \frac{\log y}{\log x} + \frac{\log x}{\log y} = 4 \\
\implies &\left(\sqrt{\log_x y} - \sqrt{\log_y x}\right)^2 = 0
\end{aligned}$$

   Now note that if $t \in (0,1)$, then $\log t < 0$. Therefore as $x, y \in (0,1)$, $\log_y x = \frac{\log x}{\log y} > 0$. Thus taking square roots is justified. Therefore

$$\log_x y - \log_y x = 0 \implies (\log x)^2 - (\log y)^2 = 0 \implies \log x - \log y = 0 \implies x = y$$

   Since $\log x, \log y$ are both negative $\log x + \log y = 0$ was ruled out.

3. Define $a_n = \sum_{k=2}^{2015}(k^{1/n} - 1)$. Observe that

$$\lim_{n\to\infty} a_n = \sum_{k=2}^{2015}\left[\left(\lim_{n\to\infty} k^{1/n}\right) - 1\right] = 0$$

With this observation, the limit in the problem can be computed as follows

$$\begin{aligned}
&\lim_{n\to\infty} (1^{1/n} + 2^{1/n} + \cdots + 2015^{1/n} - 2014)^n \\
=&\lim_{n\to\infty}\left(1 + \sum_{k=2}^{2015}(k^{1/n} - 1)\right)^n \\
=&\lim_{n\to\infty} (1 + a_n)^n \\
=&\lim_{n\to\infty}\left[(1 + a_n)^{1/a_n}\right]^{na_n} \\
=&\left[\lim_{x\to 0}(1 + x)^{1/x}\right]^{\lim\limits_{n\to\infty} na_n} \qquad [\because \lim_{n\to\infty} a_n = 0\,] \\
=&e^{\lim\limits_{n\to\infty} na_n}
\end{aligned}$$

Now let us calculate the limit in the power separately.

$$\begin{aligned}
\lim_{n\to\infty} na_n &= \sum_{k=2}^{2015}\lim_{n\to\infty} n(k^{1/n} - 1) \\
&= \sum_{k=2}^{2015}\lim_{x\to 0}\frac{k^x - 1}{x} \\
&= \sum_{k=2}^{2015}\log k \\
&= \log(2015)!
\end{aligned}$$

Thus the required limit turns out to be $e^{\log(2015)!} = (2015)!$

4. From the given equation we have

$$\begin{aligned}
&\sum_{k=1}^{n} k\sqrt{x_k - k^2} = \frac{1}{2}\sum_{k=1}^{n} x_k \\
\implies &\sum_{k=1}^{n}(x_k - 2k\sqrt{x_k - k^2}) = 0 \\
\implies &\sum_{k=1}^{n}(\sqrt{x_k - k^2} - k)^2 = 0 \\
\implies &x_k - k^2 = k^2 \ \forall\ k = 1, \ldots, n \\
\implies &x_k = 2k^2 \ \forall\ k = 1, \ldots, n
\end{aligned}$$

5. Let $a = a_1 < a_2 < \cdots < a_n = A$ be the n distinct positive integers. Suppose d and l are the gcd and lcm of these numbers respectively. Then $\dfrac{a_i}{d}$ are all positive integers. Also

$$1 \le \frac{a_1}{d} < \frac{a_2}{d} < \cdots < \frac{a_n}{d}$$

Since $\frac{a_i}{d}$ is an integer strictly greater than $\frac{a_{i-1}}{d}$, we have $\frac{a_i}{d} \ge 1 + \frac{a_{i-1}}{d}$. Therefore

$$\frac{a_n}{d} \ge 1 + \frac{a_{n-1}}{d} \ge 1 + 1 + \frac{a_{n-2}}{d} \ge \cdots \ge n - 1 + \frac{a_1}{d} \ge n \implies d \le \frac{a_n}{n} = \frac{A}{n}.$$

Similarly, we have that $\dfrac{l}{a_i}$ are all integers and

$$1 \le \frac{l}{a_n} < \frac{l}{a_{n-1}} < \cdots < \frac{l}{a_1}$$

Since $\frac{l}{a_i}$ is an integer strictly greater than $\frac{l}{a_{i+1}}$, we have $\frac{l}{a_i} \ge 1 + \frac{l}{a_{i+1}}$. Therefore

$$\frac{l}{a_1} \ge 1 + \frac{l}{a_2} \ge 1 + 1 + \frac{l}{a_3} \ge \cdots \ge n - 1 + \frac{l}{a_n} \ge n \implies d \ge na_1 = na.$$

6. First observe that the points P_1 and P_2 can meet together only when both of them are on the ground surface. Suppose the wheels starts initially at point A on the surface and they meet again at point B on the surface t m away. Suppose the bigger wheel revolves k times to reach B. Therefore $12k = t$. So, 12 divides t. Similarly 8 must also divide t. Thus, t must be divisible by 24. Since the points meet for the first time after A at B, value of t must be the minimum possible multiple of 24, that is the required distance is 24 m.