Mathematics Talent Reward Programme

Question Paper for Class XI

14^{th} January, 2018

Total Marks: 100 Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

[***You should answer these questions in the first page according to the order given in the question paper***. *Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.*]

1. A coin is tossed 9 times. There are 2^9 possible outcomes. In how many of these outcomes does no two successive heads occur?

   (A) 55, **(B)** 34, **(C)** 89, **(D)** None of these.

2. $\lim_{x \to 0^+} \frac{[x]}{\tan(x)} =$

   (A) -1, **(B)** 1, **(C)** 0, **(D)** Does not exist.

3. Let F_n denote the Fibonacci sequence such that $F_1 = 0, F_2 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 3$. Then $\sum_{n=3}^{\infty} \frac{18+999F_n}{F_{n-1}F_{n+1}} =$

   (A) 2016, **(B)** 2017, **(C)** 2018, **(D)** None of these.

4. In ΔABC, O is an interior point such that $\angle BOC = 90°, \angle CAO = \angle ABO, \angle BAO = \angle BCO$. Then $\frac{AC}{OC} =$

   (A) $\sqrt{2}$, **(B)** 2, **(C)** $\sqrt{\frac{3}{2}}$, **(D)** None of these.

5. Let M and m denote the maximum value and the minimum value of the function $f(x) = \cos(x^{2018})\sin(x)$ in the interval $[-2\pi, 2\pi]$ respectively, then $m + M =$

   (A) $\frac{1}{2}$, **(B)** $-\frac{1}{\sqrt{3}}$, **(C)** $\frac{1}{2018}$, **(D)** None of these.

6. In a class of 80 students, 40 are male and 40 are female. Also, exactly 50 students wear glasses. Then which of the following is true?

   (A) Exactly 10 boys wear glasses, **(B)** At least 20 girls wear glasses,
   (C) At most 25 boys do not wear glasses, **(D)** At most 30 girls do not wear glasses.

7. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many functions $f : A \to A$ can be defined such that $f(1) < f(2) < f(3)$?

   (A) $\binom{8}{3}$, **(B)** $\binom{8}{3}5^8$, **(C)** $\binom{8}{3}8^5$, **(D)** $\frac{8!}{3!}$.

Short Answer Type Questions

[*Each question carries a total of 12 marks. Credits will be given to partially correct answers*]

1. If x, y, z are real numbers such that $x < y < z$, prove that

$$(x-y)^3 + (y-z)^3 + (z-x)^3 > 0$$

2. Let $P(x)$ be a polynomial with real coefficients such that $P(n)$ is an integer for any integer n. Prove that the coefficients of $P(x)$ must be rational.

3. Does there exist a continuous function f, such that $f(f(x)) = -x^{2019} \quad \forall x \in \mathbb{R}$?

4. Let S be a finite subset of $\mathbb{R}$. Let f be a function from S to S such that $\mid f(x_1) - f(x_2) \mid \leq \frac{1}{2} \mid x_1 - x_2 \mid$ $\forall x_1, x_2 \in S$. Prove that $f(x) = x$ for some $x \in S$.

5. (a) Prove that the sequence of remainders obtained when the Fibonacci numbers are divided by n is periodic, where $n \in \mathbb{N}$.

   (b) Prove that there does not exist a non-constant polynomial $P(x)$ with integer coefficients such that $P(F_n)$ is prime for all $n \in \mathbb{N}$, where F_n denotes the nth term of the Fibonacci sequence.

6. Let $d(n)$ be the number of divisors of n. Prove that we can colour the natural numbers using 2 colours such that if for an infinite increasing sequence $\{a_1, a_2, a_3, \cdots\}$, the sequence $\{d(a_1), d(a_2), \cdots\}$ is a non-constant geometric progression, then all the terms $\{a_1, a_2, a_3 \cdots\}$ cannot have the same colour. (You may use the fact that we can colour the natural numbers using 2 colours such that all the terms of any infinite increasing A.P cannot have the same colour.)

Use of calculators is not allowed. You may use a ruler and a compass for construction.

∼ Best of Luck ∼