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Subject: Theory of Games and Statistical Decisions

1) a)

Let, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be the 2×2 matrix game.

To obtain the value of the game and an optimal mixed strategies for each player, we shall consider the following steps.

i) Check if $\max\limits_i \min\limits_j a_{ij} = \min\limits_j \max\limits_i a_{ij}$.

If they are equal, then ~~the~~ the common value is the value of the game. Also, both players then have an optimal strategy in terms of pure strategies in this case. The index $i \in \{1,2\}$ where the outer maxima happens is the optimal strategy for player I. Similarly the index $j' \in \{1,2\}$ where the outer minima happens is the optimal strategy for player II.

ii) If, $\max\limits_i \min\limits_j a_{ij} \neq \min\limits_j \max\limits_i a_{ij}$, then the matrix game has unique solution in mixed strategies, with $(x^*, 1-x^*)$ is a mixed strategy of player I and $(y^*, 1-y^*)$ is a mixed strategy of player II, where

$$x^* = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

$$y^* = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

and the value of the game is,

$$v(A) = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

1/b)

Let, $X = (x, 1-x)$ be a mixed strategy of player I and $Y = (y, 1-y)$ be a mixed strategy of player II, where $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Note that, the expected payoff of player I is :

$$H(x,y) = a_{11}xy + a_{12}x(1-y) + \cancel{a_{21}(1-y)x}$$
$$a_{21}(1-x)y + a_{22}(1-x)(1-y)$$

$$= a_{11}xy + a_{12}x - a_{12}xy + a_{21}y - a_{21}xy + a_{22} - a_{22}x$$
$$- a_{22}y + a_{22}xy.$$

$$= xy(a_{11} + a_{22} - a_{12} - a_{21}) + x(a_{12} - a_{22}) + y(a_{21} - a_{22}) + a_{22}$$

If (x^*, y^*) be an equilibrium situation, then it follows that

$$H(x^*, y) \geq H(x^*, y^*) \geq H(x, y^*), \quad \forall\, 0 \leq x, y \leq 1. \quad \text{——— (1)}$$

Claim: If any of the player has a pure optimal strategy, the other player also has a pure optimal strategy.

Proof: Let us assume without the loss of generality, player I has a pure optimal strategy, and it is strategy 1, i.e. $x^* = 1$.

Then eqn. (1) is given by,

$$H(1, y) \geq H(1, y^*) \geq H(x, y^*)$$

Since, $H(x, y^*)$ is linear in x,

and $H(1, y)$ is linear in y, their maximum and minimum value is attained at the endpoints i.e. $x, y = 0$ or 1.

So, the condition merely reduces to,

$$H(0, y^*) \leq H(1, y^*) \leq \min \{ H(1,0), H(1,1) \}$$

If, $a_{11} < a_{12}$, then we have, $a_{12} + y^*(a_{11} - a_{12}) \leq a_{11}$

$$\Rightarrow y^*(a_{11} - a_{12}) \leq (a_{11} - a_{12})$$

$$\Rightarrow y^* \geq 1. \quad (\text{since } (a_{11} - a_{12}) < 0)$$

$$\Rightarrow y^* = 1 \quad , \text{since } 0 \leq y^* \leq 1.$$

Therefore strategy 1 is optimal for player 2 and $v(A) = a_{11}$.

If, $a_{11} > a_{12}$, then $a_{12} + y^*(a_{11} - a_{12}) \leq a_{12}$

$$\Rightarrow y^*(a_{11} - a_{12}) \leq 0$$

$$\Rightarrow y^* \leq 0, \quad \text{since } (a_{11} - a_{12}) > 0$$

$$\Rightarrow y^* = 0$$

Therefore, strategy 2 is optimal for player 2 and $v(A) = a_{12}$.

Finally if, $a_{11} = a_{12}$, then $H(0, y^*) \leq H(1, y^*)$

$$\Rightarrow a_{21} y^* + a_{22}(1 - y^*) \leq a_{12}$$

$$\Rightarrow (a_{21} - a_{22}) y^* \leq a_{12} - a_{22}$$

If $a_{12} \geq a_{22}$, then $y^* = 0$ is a possible solution.

If, $a_{12} \geq a_{21}$, then $y^* = 1$ is a possible solution.

Otherwise if $a_{12} < a_{22}$ and $a_{12} < a_{21}$

and since $a_{11} = a_{12}$, this means that strategy 2 is dominant ~~for~~ over strategy 1 for player 1, hence this contradicts that strategy 1 is optimal for player 1.

This proves that if player I has an optimal strategy in pure strategies, player II also have an optimal pure strategy.

(Proof End)

Now, the worst that player I can gurantee by using strategy 'i' is $\min_j a_{ij}$. Therefore, he (she) would choose the pure strategy 'i' that maximizes this worst payoff, i.e. the ~~his~~ minimum payoff he can secure is $\max_i \min_j a_{ij}$. On the other hand, by using strategy 'j'. Player II allows a maximium loss of $\max_i a_{ij}$. Therefore, he would try to minimize the loss, hence would ensure a loss of at most $\min_j \max_i a_{ij}$.

Therefore, if, $\max_i \min_j a_{ij} = \min_j \max_i a_{ij}$, then both player I and player II would have optimal pure strategies.

However, if the equality does not hold, then Nash's theorem gurantees that there will be a solution in mixed strategies, such that,

$$H(x, y^*) \leqslant H(x^*, y^*) \leqslant H(x^*, y),$$

$$\forall 0 \leqslant x, y \leqslant 1$$

In other words, we require,

$$\max_{0 \leqslant x \leqslant 1} H(x, y^*) = H(x^*, y^*)$$

$$\min_{0 \leqslant y \leqslant 1} H(x^*, y) = H(x^*, y^*)$$

Since no solution exists in terms of pure strategies, both $0 < x^* < 1$, $0 < y^* < 1$, i.e. x^*, y^* are interior points.

Also, $H(x, y)$ is a continuous function of x and y, hence at the critical point (x^*, y^*) they must satisfy,

$$\left.\frac{\partial H(x, y^*)}{\partial x}\right|_{x = x^*} = 0 \quad \text{and} \quad \left.\frac{\partial H(x^*, y)}{\partial y}\right|_{y = y^*} = 0.$$

i.e. $\quad y^* (a_{11} + a_{22} - a_{12} - a_{21}) + (a_{12} - a_{22}) = 0.$

and $\quad x^* (a_{11} + a_{22} - a_{12} - a_{21}) + (a_{21} - a_{22}) = 0.$

If $a_{11} + a_{22} - a_{12} - a_{21} = 0$, then $(a_{12} - a_{22}) = (a_{22} - a_{21}) = 0.$

$\Rightarrow \quad a_{12} = a_{22} = a_{21} = a_{11}$

$\Rightarrow \quad A = \begin{pmatrix} a_{11} & a_{11} \\ a_{11} & a_{11} \end{pmatrix}$ is the matrix game hence

$(1,1)$ or $(2,2)$ becomes an equilibrium in pure strategies.

Hence if, $a_{11} + a_{22} - a_{12} - a_{21} \neq 0.$, then we have a unique solution,

$$x^* = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

and
$$y^* = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

Finally, the value of the game is,

$$H(x^*, y^*) = (a_{11} - a_{12} - a_{21} + a_{22})\, x^* y^* + x^* (a_{12} - a_{22}) + y^* (a_{21} - a_{22}) + a_{22}$$

$$= \frac{(a_{22} - a_{21})(a_{22} - a_{12})}{(a_{11} + a_{22} - a_{12} - a_{21})} + \frac{(a_{12} - a_{22})(a_{22} - a_{21})}{(a_{11} + a_{22} - a_{12} - a_{21})}$$

$$+ \frac{(a_{22} - a_{12})(a_{21} - a_{22})}{(a_{11} + a_{22} - a_{12} - a_{21})} + a_{22}$$

$$= \frac{a_{11} a_{22} - a_{21} a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

1(c)

We have, $B = \begin{bmatrix} 17 & x+3 \\ x+2 & 20 \end{bmatrix}$. Putting x as the last digit of my roll number (MB1911) we obtain,

$$B = \begin{bmatrix} 17 & 4 \\ 3 & 20 \end{bmatrix}$$

Now,

$$\max_i \min_j b_{ij} = 4, \quad \text{and} \quad \min_j \max_i b_{ij} = 17$$

Since, $\max_i \min_j b_{ij} \neq \min_j \max_i b_{ij}$, ~~they~~ there is no equilibrium in ~~opti~~ pure strategies. Therefore, $(x^*, 1-x^*)$ is a mixed strategy of player I and $(y^*, 1-y^*)$ is a mixed strategy for player II where,

$$x^* = \frac{b_{22} - b_{21}}{b_{11} + b_{22} - b_{12} - b_{21}} = \frac{20-3}{20+17-4-3} = \frac{17}{30}$$

and $$y^* = \frac{b_{22} - b_{12}}{b_{11} + b_{22} - b_{12} - b_{21}} = \frac{20-4}{20+17-4-3} = \frac{16}{30} = \frac{8}{15}$$

and the value of the game is,

$$v(B) = \frac{17 \times 20 - 3 \times 4}{17+20-3-4} = \frac{328}{30} \simeq 10 \cdot 93$$

Therefore, the unique optimal strategy for player I is $\left(\frac{17}{30}, \frac{13}{30}\right)$ and the optimal strategy for player II is $\left(\frac{8}{15}, \frac{7}{15}\right)$.

This induces ~~a~~ the value of the game as $v(B) = \frac{328}{30} \simeq 10 \cdot 93$.

1) a)

Consider a matrix game A with two intelligent rational players. From the perspective of player 1, if he (she) selects a mixed strategy $\underset{\sim}{X}$, then ~~it~~ in the worst case, he (she) would receive a payoff of, $\min_{\underset{\sim}{Y}} \underset{\sim}{X} A \underset{\sim}{Y}^T$, where, $\underset{\sim}{Y}$ is chosen over ~~a~~ the sets of all mixed strategies of player 2. Since player 1 is rational, he (she) would try to choose $\underset{\sim}{X}$ to maximize his (her) worst case reward, effectively ~~allowing~~ ensuring to gain $\max_{\underset{\sim}{X}} \min_{\underset{\sim}{Y}} \underset{\sim}{X} A \underset{\sim}{Y}^T$. In other words, player 1 can ensure this amount, irrespective of player 2's strategy. On the other hand, if player 2 selects mixed strategy $\underset{\sim}{Y}$, then in the worst case, he (she) has to lose $\max_{\underset{\sim}{X}} \underset{\sim}{X} A \underset{\sim}{Y}^T$ where the maximization is over the choice of mixed strategies of player 1. Player 2, being rational, would try to minimize this loss, and choose such $\underset{\sim}{Y}$ such that the payoff is $\min_{\underset{\sim}{Y}} \max_{\underset{\sim}{X}} \underset{\sim}{X} A \underset{\sim}{Y}^T$ to player 1.

Also from the minimax theorem,

$$\max_{\underset{\sim}{X}} \min_{\underset{\sim}{Y}} \underset{\sim}{X} A \underset{\sim}{Y}^T = \min_{\underset{\sim}{Y}} \max_{\underset{\sim}{X}} \underset{\sim}{X} A \underset{\sim}{Y}^T \quad \text{——} \ (*)$$

Therefore, if both players player rationally, the payoff of the players will be pre-determined by the entries of A, and will be equal to the quantity above. Hence, in case of a game between ~~intelligente~~ intelligent players, any other value other than the common value in (*) would provide incentive for the players to deviate, and the game will be ~~settle~~ settled at the particular value as in (*). So, the playing of the game would simply be deterministic, where player 2 transfers (*) amount to player 1, ~~at~~ ultimately.

27/04

Two players game where each of the player has a finite set of strategies to choose from is called a "Bi-matrix" game, since the payoffs of the two players can be expressed as two matrices.

Let, there are two player I and player II. Player I has set of strategies $S_I = \{1, 2, \ldots, m\}$ and player II has set of strategies $S_{II} = \{1, 2, \ldots, n\}$. If player I plays $i \in S_I$ and player II plays $j \in S_{II}$, then the payoff of player I is a_{ij} and payoff of player II is b_{ij}.

Set of mixed strategies for player I is $\Sigma_I = \{x = (x_1, \ldots, x_m) : x_i \geq 0,$ and $\sum_{i=1}^{m} x_i = 1\}$

and for player II is, $\Sigma_{II} = \{y = (y_1, \ldots, y_n) : y_i \geq 0, \sum_{i=1}^{n} y_i = 1\}$

In such mixed strategies, the expected payoff of player I becomes, $H_I(X, Y) = XAY^T$, and, payoff of player II is $H_{II}(X, Y) = XBY^T$, where, $A = ((a_{ij}))$ and $B = ((b_{ij}))$.

This new game ~~with~~ $\Gamma = \{ I = \{I, II\}, \{\Sigma_I, \Sigma_{II}\}, \{H_I, H_{II}\}\}$

such that,
$$\Sigma_I = \{x : (x_1, \ldots, x_m) : x_i \geq 0, \sum_{i=1}^{m} x_i = 1\}$$
$$\Sigma_{II} = \{y : (y_1, \ldots, y_n) : y_i \geq 0, \sum_{i=1}^{n} y_i = 1\}$$

$$H_I, H_{II} : \Sigma_I \times \Sigma_{II} \to \mathbb{R} \quad \text{s.t.} \quad H_I(X, Y) = XAY^T$$
$$H_{II}(X, Y) = XBY^T$$

is called a mixed extension of the Bi-matrix game $\Gamma' = \{I = \{I, II\}, \{S_I, S_{II}\}, \{A, B\}\}$.

2) b)

Let, $X = (x, 1-x)$ and $Y = (y, 1-y)$ be a mixed strategy for player I and player II respectively. This situation (X, Y) can be represented by the point $(x, y) \in [0,1] \times [0,1]$

The payoff of player I is $H_I(x, y) = XCY^T$

and of player II is $H_{II}(x, y) = XDY^T$

where,
$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

- **Admissible strategies for player I:**

Let, $\mathcal{C} = c_{11} + c_{22} - c_{12} - c_{21}$

$c = c_{22} - c_{12}$

i) $\mathcal{C} = 0$, $c = 0$, then solution $(x, y) \in [0,1] \times [0,1]$

ii) $\mathcal{C} = 0$, $c > 0$, then the solution $(x, y) \in \{0\} \times [0,1]$

iii) $\mathcal{C} = 0$, $c < 0$, then the solution $(x, y) \in \{1\} \times [0,1]$

iv) $\mathcal{C} > 0$, then let $\alpha = \frac{c}{\mathcal{C}}$.

The solution is either $(x, y) \in \{0\} \times \left((-\infty, \alpha] \cap [0,1]\right)$

or $(x, y) \in \{1\} \times \left([\alpha, \infty) \cap [0,1]\right)$

or $(x, y) \in (0,1) \times \{\alpha\}$.

v) $\mathcal{C} < 0$, then letting $\alpha = \frac{c}{\mathcal{C}}$,

The solution is either $(x, y) \in \{0\} \times \left([\alpha, \infty) \cap [0,1]\right)$

or $(x, y) \in \{1\} \times \left((-\infty, \alpha] \cap [0,1]\right)$

or $(x, y) \in (0,1) \times \{\alpha\}$.

- **Admissible strategies for player II :**

Let, $\mathcal{D} = d_{11} + d_{22} - d_{12} - d_{21}$

$d = d_{22} - d_{21}$.

(i) $\mathcal{D} = 0$, $d = 0$, then solution $(x, y) \in [0,1] \times [0,1]$

(ii) $\mathcal{D} = 0$, $d > 0$, then solution $(x, y) \in [0,1] \times \{0\}$.

(iii) $\mathcal{D} = 0$, $d < 0$, then solution $(x, y) \in [0,1] \times \{1\}$

(iv) $\mathcal{D} > 0$, then letting $\beta = \frac{d}{\mathcal{D}}$ we have,

The solution is either $(x, y) \in [0, \beta] \times \{0\}$

or $(x, y) \in [\beta, 1] \times \{1\}$

or $(x, y) \in \{\beta\} \times (0,1)$

(v) $\mathcal{D} < 0$, then letting $\beta = \frac{d}{\mathcal{D}}$, we get

The solution is either $(x, y) \in [\beta, 1] \times \{0\}$

or $(x, y) \in [0, \beta] \times \{1\}$

or $(x, y) \in \{\beta\} \times (0,1)$

2> c>

We consider the Bi-matrix game (C, D) where,

$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ and $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$. Let, H_1 and H_2 be the payoff function of player I and player II respectively

Consider, $\alpha = (c_{11} - c_{12}) > 0$, (according to the question)

Consider the new pay off functions as,

$$\widetilde{H}_1(i,j) = \frac{1}{\alpha}\left(H_1(i,j) - c_{12}\right)$$

$$\text{and } \widetilde{H}_2(i,j) = \frac{1}{\alpha}\left(H_2(i,j) - d_{12}\right)$$

$\left(\frac{1}{\alpha} > 0, \frac{c_{12}}{\alpha}, \frac{d_{12}}{\alpha} \in \mathbb{R}\right)$

Then,

$$\widetilde{H}_1(1,1) = \frac{1}{\alpha}\left(H_1(1,1) - c_{12}\right) = \frac{1}{\alpha}(c_{11} - c_{12}) = 1.$$

$$\widetilde{H}_1(1,2) = \frac{1}{\alpha}\left(H_1(1,2) - c_{12}\right) = \frac{1}{\alpha}(c_{12} - c_{12}) = 0.$$

$$\widetilde{H}_1(2,1) = \frac{1}{\alpha}\left(H_1(2,1) - c_{12}\right) = \frac{c_{21} - c_{12}}{\alpha} = p_{21} \text{ (say)}$$

$$\widetilde{H}_1(2,2) = \frac{1}{\alpha}\left(H_1(2,2) - c_{12}\right) = \frac{c_{22} - c_{12}}{\alpha} = p_{22} \text{ (say)}$$

and $\widetilde{H}_2(1,1) = \frac{1}{\alpha}\left(H_2(1,1) - d_{12}\right) = \frac{1}{\alpha}(d_{11} - d_{12}) < 0$

since $c_{11} > c_{12} \Rightarrow d_{11} < d_{12}$ as it is almost antagonistic

So, we let $\widetilde{H}_2(1,1) = (-k)$, where $k > 0$.

$$\widetilde{H}_2(1,2) = \frac{1}{\alpha}\left(H_2(1,2) - d_{12}\right) = \frac{1}{\alpha}(d_{12} - d_{12}) = 0,$$

and, $\widetilde{H}_2(2,1) = \dfrac{d_{21} - d_{12}}{\alpha} = q_{21}$ (say)

and, $\widetilde{H}_2(2,2) = \dfrac{d_{22} - d_{12}}{\alpha} = q_{22}$ (say)

This shows that the game $\Gamma = \{\{I, II\}, \{S_I, S_{II}\}, \{C, D\}\}$ is strategically equivalent to, $\Gamma' = \{\{I, II\}, \{S_I, S_{II}\}, \{\widetilde{H}_I, \widetilde{H}_{II}\}\}$ and, Γ' is a bi-matrix game (P, Q) with

$$P = \begin{bmatrix} 1 & 0 \\ p_{21} & p_{22} \end{bmatrix} \text{ and } Q = \begin{bmatrix} -k & 0 \\ q_{21} & q_{22} \end{bmatrix}$$

where $k > 0$, and $p_{21}, p_{22}, q_{21}, q_{22} \in \mathbb{R}$.

Ans

Let us consider that there are 2 companies manufacturing the same product. These companies are the 2 players, company A is a smaller manufacturer and company B is a much bigger manufacturer. Also assume that there are two markets to sell the items / products, namely M_1 and M_2 where M_1 is a bigger market than M_2. Each of the players (companies) has the strategy to sell the items to either in market M_1 or in market M_2, but not both.

If both choose market M_1 (bigger), then player A get a big loss of (-10), and player B (by eliminating player A) gains $(+5)$. If player A choose M_1 and player B choose M_2, then player A gains $(+2)$ whereas player B (bigger company) being in smaller market gets (-2). On the other hand, in the smaller market M_2, player A if unchallenged gets less profit $(+1)$ and if challenged ~~ta~~ incurs a loss of (-1). Player B, ~~at~~ on the other hand gets the reverse in the cases as (-1) and $(+1)$ respectively.

So, we can describe this situation as the bi-matrix game (C, D) where,

$$C = \begin{bmatrix} -10 & 2 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & -2 \\ -1 & 1 \end{bmatrix}$$

where, C and D respectively denotes the payoff matrix of player A and B. Clearly, this is an almost antagonistic game.

Here, $\mathcal{C} = -10 - 1 - 2 - 1 = (-14)$

$c = -1 - 2 = (-3)$,

and, $\mathcal{D} = 5 + 1 + 1 + 2 = 9$

$d = 1 - (-1) = 2$.

We have $\alpha = \frac{c}{\mathcal{C}} = \frac{3}{14}$, and $\beta = \frac{d}{\mathcal{D}} = \frac{2}{9}$.

Since $\mathcal{C} < 0$, the admissible situations for player A are simply given by,

$$\text{either } (x, y) \in \{0\} \times \left[\tfrac{3}{14}, 1\right]$$

$$\text{or } (x, y) \in \{1\} \times [0, 3/14]$$

$$\text{or } (x, y) \in (0, 1) \times \left\{\tfrac{3}{14}\right\}$$

Here, (x, y) denotes the situation (X, Y) where player A uses the mixed strategy $X = (x, 1-x)$, and player B uses the mixed strategy $Y = (y, 1-y)$

On the other hand, as $\mathcal{D} > 0$, the admissible situations for player B are,

$$\text{either } (x, y) \in [0, 2/9] \times \{0\}$$

$$\text{or } (x, y) \in [2/9, 1] \times \{1\}$$

$$\text{or } (x, y) \in \{2/9\} \times (0, 1)$$

Since equilibrium situation is admissible for both players, it turns out that the unique equilibrium situation is, $\left(\frac{2}{9}, \frac{3}{14}\right)$, i.e. player A chooses market M_1 with probability $\frac{2}{9}$ and chooses market M_2 with probability $\frac{7}{9}$. Player B chooses market M_1 and M_2 with probabilities $\frac{3}{14}$ and $\frac{11}{14}$ respectively.

Therefore, in equilibrium, the expected payoff of player A is,

$$v(C) = \frac{2}{9} \times \frac{3}{14} \times (-10) + \frac{2}{9} \times \frac{11}{14} \times 2 + \frac{7}{9} \times \frac{3}{14} \times 1 + \frac{7}{9} \times \frac{11}{14} \times (-1)$$

$$= -\frac{72}{9 \times 14} = -\frac{4}{7}$$

and the player B's payoff in expectation would be,

$$v(D) = \frac{2}{9} \times \frac{3}{14} \times 5 + \frac{2}{9} \times \frac{11}{14} \times (-2) + \frac{7}{9} \times \frac{3}{14} \times (-1) + \frac{7}{9} \times \frac{11}{14} \times 1$$

$$= \frac{42}{9 \times 14} = \frac{1}{3}.$$