Large Sample Statistical Methods

Assignment

1. Let F be a distribution function on $\mathbb{R}$ and $0<p<1$. Show that a real number c is the unique quantile of order p of F if and only if
$$F(c-\varepsilon) < p < F(c+\varepsilon) \quad \forall\, \varepsilon > 0.$$

2. Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution symmetric about zero and having finite 8th moment. Find the asymptotic distribution of the following measures of skewness and kurtosis:
$$(a)\ g_{1n} = \frac{m_{3n}}{m_{2n}^{3/2}} \quad \text{and} \quad (b)\ g_{2n} = \frac{m_{4n}}{m_{2n}^{2}} - 3$$
where m_{rn} denotes the sample central moments of order r. Firt find the result in the general case, and then use it to derive for the case where the underlying distribution is $N(\mu, \sigma^2)$.

3. Let $X_1, \dots, X_n$ be i.i.d. $N(\theta, 1)$ variables where it is known that $|\theta| \leq 1$. Find the MLE of θ and also find its asymptotic distribution under any $\theta_0 \in (-1, 1)$.

4. Let $X_1, \dots, X_n$ be a random sample from a distribution with a density $f(x|\theta)$, $\theta \in \Theta$, an open interval in $\mathbb{R}$. Assume suitable regularity conditions on the densities so that there is a consistent solution $\hat{\theta}_n$ of the likelihood equation for which $\sqrt{n}(\hat{\theta}_n - \theta)$ is $AN(0, I^{-1}(\theta))$ (under θ). Fix $\theta_0 \in \Theta$ and set
$$T_n = \begin{cases} \hat{\theta}_n, & \text{if } |\hat{\theta}_n - \theta_0| > n^{-1/4} \\ \theta_0, & \text{if } |\hat{\theta}_n - \theta_0| \leq n^{-1/4}. \end{cases}$$
Find the asymptotic distribution of $\sqrt{n}(T_n - \theta)$ (under θ).