INDIAN STATISTICAL INSTITUTE, KOLKATA
FINAL EXAMINATION: SECOND SEMESTER 2019 - '20
M.STAT I YEAR

Subject	:	**Metric Topology and Complex Analysis**
Time	:	7 days
Maximum score	:	50

This is a take-home exam. You are expected to follow the honour code and answer the questions on your own. Attempt all the problems. Please use separate pages /reverse side of same page for separate answers and send the answers in order. Justify every step in order to get full credit of your answers but be concise. All arguments should be clearly mentioned on the answer script. The onus is on you to provide clearly readable answerscripts. No marks will be awarded to the portion of the answer that is illegible.

(1) Where does the function $f(z) = zRe(z) + \overline{z}Im(z) + \overline{z}$ have a complex derivative? Find the derivative wherever it exists.

[3 + 3 = 6 marks]

(2) (i) Show that a Möbius transformation has 0 and ∞ as its only fixed points iff it is a dilation.
(ii) Find the image of the set $\{z : |z| < 1\} \cap \{z : |z-1| < 1\}$ under the map $Tz = \frac{z-1}{z+1}$.

[3 + 5 = 8 marks]

(3) $f : \Omega \to \mathbb{C}$ is an analytic function where Ω is an open connected set containing the closure of the unit disc $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$. Show that if f is real on the boundary of D, then f must be constant.

[8 marks]

(4) Evaluate the following integrals using residue formula:-
(i) $\int_{|z|=2} \frac{e^z}{\cos z} dz$
(ii) $\int_0^{\pi} \frac{1}{5+3\cos(\theta)} d\theta$

[5 + 5 = 10 marks]

(5) Let f be an analytic function in the region $\{z : |z| > 1\}$ and suppose that $\lim_{z\to\infty} f(z) = 0$. Show that if $|z| > 2$, then

$$\frac{1}{2\pi i} \int_{|\zeta|=2} \frac{f(\zeta)}{\zeta - z} d\zeta = -f(z)$$

[8 marks]

(6) Use contour integration and the residue method to evaluate the integral

$$\int_0^{\infty} \frac{\cos x}{(1+x^2)^2} dx$$

[10 marks]