Subject: Time Series Analysis.

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1) Let, $m_t = \alpha + \beta t$, be a linear trend.

Since the filter allows to pass linear trends undistorted, we want,

$$\left(a_{-1} B^{-1} + a_0 + a_1 B\right) m_t = m_t \quad , \quad \forall\, \alpha, \beta \in \mathbb{R}.$$

$$\Rightarrow a_{-1}\left(\alpha + \beta(t+1)\right) + a_0\left(\alpha + \beta t\right) + a_1\left(\alpha + \beta(t-1)\right) = \alpha + \beta t$$

Comparing coefficients of α and β yields,

$$a_{-1} + a_0 + a_1 = 1. \quad \text{--- } (1)$$

and, $$\left(a_{-1} - a_1\right) = 0. \quad \text{--- } (2).$$

Let, $s_t = [p, q, -(p+q)]$ be a seasonal component of order 3. Since the filter absorbs a seasonal component of order 3, we must have,

$$a_{-1} p + a_0 q - a_1 (p+q) = 0. \quad , \quad \forall\, p, q \in \mathbb{R}.$$

$$\Rightarrow a_{-1} = a_1 \text{ and, } \quad a_0 = a_1 \quad \text{--- } (3).$$

From (2) and (3), we get, $a_{-1} = a_0 = a_1$

Hence from (1), we get, $3a_{-1} = 1 \Rightarrow a_{-1} = \frac{1}{3} = a_0 = a_1$.

Therefore, the required linear filter is, $\left(\frac{1}{3} B^{-1} + \frac{1}{3} + \frac{1}{3} B\right)$.

(Answer)

2)

We have the equation, $X_t = \alpha X_{t-2} + Z_t$., where $\alpha \in \mathbb{R}$, $|\alpha| < 1$, and $Z_t \sim WN(0, c^2)$., $c^2 > 0$.

Now,

$$X_t = \alpha X_{t-2} + Z_t.$$

$$= \alpha \cdot (\alpha X_{t-4} + Z_{t-2}) + Z_t.$$

$$= \alpha^2 \cdot X_{t-4} + \alpha Z_{t-2} + Z_t.$$

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Therefore by induction, we have, $X_t = \alpha^{n+1} X_{t-2(n+1)} + \sum_{k=0}^{n} \alpha^k Z_{t-2k}$

In this sense, we consider the process, $X_t = \sum_{k=0}^{\infty} \alpha^k Z_{t-2k}$ — (1)

The process is well-defined since $|\alpha| < 1$,

hence, $$E|X_t|^2 = \sum_{k=0}^{\infty} (\alpha^k)^2 Var(Z_{t-2k})$$

$$= c^2 \cdot \frac{1}{1-\alpha^2} < \infty, \quad \text{as } |\alpha| < 1.$$

Finally, $$X_t = \sum_{k=0}^{\infty} \alpha^k Z_{t-2k} = Z_t + \sum_{k=1}^{\infty} \alpha^k Z_{t-2k}$$

$$= Z_t + \alpha \cdot \sum_{k'=(k-1)=0}^{\infty} \alpha^{k'} \cdot Z_{(t-2)-2k'}$$

$$= Z_t + \alpha \cdot X_{t-2}$$

Hence, X_t as defined by (1) satisfy the given equation.

Clearly, X_t as defined by (1) is a stationary process as it is a linear filter of the white noise process.

To show the almost sure convergence, note that,

$$X_t = \alpha^{n+1} X_{t-2(n+1)} + \sum_{k=0}^{n} \alpha^k Z_{t-2k}$$

$$\Rightarrow \left(X_t - \sum_{k=0}^{n} \alpha^k Z_{t-2k}\right) = \alpha^{n+1} X_{t-2(n+1)}$$

$$\Rightarrow E\left(X_t - \sum_{k=0}^{n} \alpha^k Z_{t-2k}\right)^2 = \alpha^{(n+1)2} E\left(X_{t-2(n+1)}^2\right)$$

$$\rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

since $|\alpha| < 1$,

and X_t is stationary

(as described by 1)

Therefore, $X_t \xrightarrow{L_2} \sum_{k=0}^{n} \alpha^k Z_{t-2k}$, as $n \rightarrow \infty$. (mean-square convergence)

Since $E\left(\left|\sum_{k=0}^{\infty} \alpha^k Z_{t-2k}\right|\right) \leq \left(\sum_{k=0}^{\infty} \alpha^k\right) \cdot c = \frac{c}{1-\alpha} < \infty$

as $|\alpha| < 1$

$$\Rightarrow \sum_{k=0}^{\infty} \alpha^k Z_{t-2k} < \infty \text{ almost surely,}$$

hence, $X_t = \sum_{k=0}^{\infty} \alpha^k Z_{t-2k}$, almost surely.

3) Let, $(a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n)$ be a linear predictor of X_{n+1}. To find the best linear unbiased predictor, we need to minimize the MSE,

$$E = E\left(\left(X_{n+1} - a_0 - a_1 X_1 - \cdots - a_n X_n\right)^2\right)$$

Since, the above is differentiable with respect to the parameters $a_0, a_1, \ldots, a_n$, we can differentiate it and set it equal to 0.

Therefore, we obtain the normal equations,

$$\frac{\partial E}{\partial a_0} = E\left(\left(X_{n+1} - a_0 - a_1 X_1 - \cdots - a_n X_n\right)\right) = 0$$

$$\Rightarrow \quad E(X_{n+1}) = a_0 + a_1 E(X_1) + \cdots + a_n E(X_n).$$

Since $\{X_n\}$ is stationary, let its mean ~~function~~ be μ and autocovariance ~~be~~ function be $\gamma(h)$, then, we have,

$$\mu = a_0 + (a_1 + \cdots + a_n)\mu$$

$$\Rightarrow \quad a_0 = \mu(1 - (a_1 + \cdots + a_n)) \quad \text{—— } ①$$

This condition ensures that the predictor is unbiased

Also, we have, for $i = 1, 2, \ldots, n$;

$$\frac{\partial E}{\partial a_i} = E\left(\left(X_{n+1} - a_0 - a_1 X_1 - \cdots - a_n X_n\right)(-X_i)\right) = 0.$$

$$\Rightarrow E(X_{n+1} X_i) = a_0 E(X_i) + a_1 E(X_1 X_i) + \cdots + a_n E(X_n X_i)$$

$$\Rightarrow E(X_{n+1} X_i) = \mu^2 (1 - (a_1 + \cdots + a_n)) + \sum_{k=1}^{n} a_k E(X_k X_i)$$

$$\Rightarrow \gamma(n+1-i) = \sum_{k=1}^{n} a_k \gamma(k-i). \qquad \text{(From ①)}$$

This suggests the form,

$$\Gamma_n \underset{\sim}{a_n} = \underset{\sim}{\gamma_n}, \quad \text{where } \Gamma_n = ((\gamma(i-j)))_{i,j=1}^{n}$$

$$\underset{\sim}{a_n}' = (a_1, a_2, \ldots, a_n)$$

$$\text{and } \underset{\sim}{\gamma_n} = (\gamma(n), \gamma(n-1), \ldots, \gamma(1))'$$

Thus, a solution to the above normal equations yields ~~a~~ the best linear predictor of X_{n+1} in terms of $1, X_1, \ldots, X_n$.

This completes the existence.

To show that is predictor is unique, let,

$P_1(X_{n+1}) = a_0^{(1)} + a_1^{(1)} X_1 + \cdots + a_n^{(1)} X_n$.

and $P_2(X_{n+1}) = a_0^{(2)} + a_1^{(2)} X_1 + \cdots + a_n^{(2)} X_n$., be two different predictors.

Now, $E\left[\left(P_1(X_{n+1}) - P_2(X_{n+1})\right)^2\right]$

$$= E\left[\left(X_{n+1} - P_2(X_{n+1})\right) - \left(X_{n+1} - P_1(X_{n+1})\right)\right]^2$$

$$= E\left[(\epsilon_2 - \epsilon_1)\left(P_1(X_{n+1}) - P_2(X_{n+1})\right)\right], \quad \text{where, } \underline{\epsilon_i = (X_{n+1} - P_i(X_{n+1}))}$$

$$= E\left[(\epsilon_2 - \epsilon_1)\left(\sum_{k=1}^{n}(a_k^{(1)} - a_k^{(2)})X_k + a_0^{(1)} - a_0^{(2)}\right)\right]$$

$$= (a_0^{(1)} - a_0^{(2)}) E(\epsilon_2 - \epsilon_1) + \sum_{k=1}^{n} (a_k^{(1)} - a_k^{(2)}) E((\epsilon_2 - \epsilon_1)X_k)$$

$= 0$, because of the normal equations,

as, $E(\epsilon_i X_k) = 0, \quad \forall\ k = 1, 2, \ldots, n$
$\forall\ i = 1, 2.$

This shows that the predictor is unique.

In general, the solution to minimum square error prediction of X_{n+1} based on $X_1, \ldots, X_n$ is given by, the conditional expectation $E(X_{n+1} | X_1, \ldots, X_n)$. The computation of this conditional expectation is extremely difficult, and requires knowledge of the dependence structure of the joint distribution $(X_{n+1}, X_n, \ldots, X_2, X_1)$, and possibly higher order moments.

4)

We are given an MA process of order 2 as,

$$X_t = Z_t + Z_{t-1} - 0.5 Z_{t-2}, \quad \text{where } Z_t \overset{iid}{\sim} N(0,1).$$

We have $Y_t = X_t + \mu$, and $Y_1, Y_2, \ldots, Y_n$ are observed.

Note that $E(Y_t) = E(X_t) + \mu = \mu$, as $E(X_t) = 0$

Since X_t is stationary, Y_t is also stationary.

$$\mathrm{cov}(Y_t, Y_{t-h})$$

$$= \mathrm{cov}(X_t, X_{t-h})$$

$$= \mathrm{cov}(Z_t + Z_{t-1} - 0.5 Z_{t-2}, Z_{t-h} + Z_{t-h-1} - 0.5 Z_{t-h-2})$$

$$= \begin{cases} (1 + 1 + \frac{1}{4}) & \text{if } h = 0. \\ (1 - \frac{1}{2}) & \text{if } |h| = 1, \\ (-\frac{1}{2}) & \text{if } |h| = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 9/4, & \text{if } h = 0 \\ 1/2 & \text{if } |h| = 1 \\ -1/2 & \text{if } |h| = 2 \\ 0, & \text{otherwise.} \end{cases}$$

We know that the sample mean of a stationary process, Y_t, has the asymptotic distribution,

$$\sqrt{n}(\bar{Y}_n - \mu) \to N(0, v),$$

where $\mu = E(Y_t)$, and $v = \sum_{|h| < \infty} \gamma(h)$, where $\gamma(h)$ is the covariance function of Y_t

Here, $v = \sum_{|h|<\infty} \gamma(h) = \frac{9}{4} + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{2} - \frac{1}{2}\right) = \frac{9}{4}$.

Therefore, a 95% Confidence interval of μ would be given by,

$$\left(\bar{Y}_n - \frac{1.96}{\sqrt{n}} \sqrt{v} \, , \; \bar{Y}_n + \frac{1.96}{\sqrt{n}} \times \sqrt{v} \right)$$

which for the given setup with $\bar{Y}_n = 2.1$, and $v = \frac{9}{4}$ becomes,

$$\left(2.1 - \frac{1.96}{\sqrt{100}} \sqrt{\frac{9}{4}} \, , \; 2.1 + \frac{1.96}{\sqrt{100}} \sqrt{\frac{9}{4}} \right)$$

$$\equiv \left(2.1 - \frac{1.96}{10} \times \frac{3}{2} \, , \; 2.1 + \frac{1.96}{10} \times \frac{3}{2} \right)$$

$$\equiv (1.806, \; 2.394).$$

To test the null hypothesis $H_0 : \mu = 1$, at significance level $\alpha = 0.05$, we notice that the ~~null~~ 95% confidence interval $(1.809, 2.394)$ does not contain the null value $\mu = 1$. Therefore, the obtained data shows strong evidence against the null hypothesis and the null hypothesis should be rejected.

5) Let, X_t be a stationary time series with auto-covariance function $\gamma(h)$, $h = 0, \pm 1, \pm 2, \cdots$

and, $g(k) = \sum_{|h|<\infty} e^{ihk} \gamma(h)$.

Since the definition of $g(k)$ requires the specification of only the ~~covariance~~ auto-covariance function, it is independent of the mean $E(X_t)$. Hence, without any loss of generality, we may assume $E(X_t) = 0$.

Define for any $N \in \mathbb{N}$,

$$g_N(k) = \frac{1}{N} E\left[\left|\sum_{r=1}^{N} e^{irk} X_r\right|^2\right]$$

$$= \frac{1}{N} E\left[\left(\sum_{r=1}^{N} e^{irk} X_r\right)\left(\sum_{s=1}^{N} e^{-isk} X_s\right)\right]$$

$$= \frac{1}{N} \sum_{r=1}^{N}\sum_{s=1}^{N} e^{i(r-s)k} E(X_r X_s)$$

$$= \frac{1}{N} \sum_{r=1}^{N}\sum_{s=1}^{N} e^{i(r-s)k} \gamma(r-s)$$

$$= \sum_{|h|<N} \left(1 - \frac{|h|}{N}\right) \gamma(h) e^{ihk}$$

By definition of $g_N(k)$, $g_N(k) \geq 0$, $\forall N \geq 0$
(since $\left|\sum_{r=1}^{N} e^{irk} X_r\right|^2 \geq 0$)

Also note that,

$$|g_N(k)| \leq \left|\sum_{|h|<N} \gamma(h) e^{ihk}\right| \leq \left|\sum_{|h|<\infty} \gamma(h) e^{ihk}\right| = |g(k)|,$$

which is finite and well defined.

Hence, by Dominated Convergence Theorem (DCT), it follows that,

$$g_N(k) \to g(k), \quad \text{as } N \to \infty, \quad \forall k \in \mathbb{R}.$$

Since, $g_N(k) \geq 0$, $\forall N \geq 0$, $\forall k \in \mathbb{R}$.

it must be that, $g(k) \geq 0$, for any $k \in \mathbb{R}$.