Topics: Permutation tests, logistic regression Due on November 16, 2019

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## 1. Permutation test of independence [8]

Perform permutation tests of independence on Table 1 using the $\chi^2$ and the likelihood ratio test statistics. Plot the permutation null distributions of these statistics. Check how the results depend on the number of permutations used. Compare the results with the standard $\chi^2$ and likelihood ratio tests.

| Gender | Handedness | | | Total by Gender |
|---|---|---|---|---|
| | Right-handed | Left-handed | Ambidextrous | |
| Male | 14 | 3 | 3 | 20 |
| Female | 13 | 1 | 2 | 16 |
| Other | 6 | 1 | 1 | 8 |
| Total by Handedness | 33 | 5 | 6 | 44 |

Table 1: Gender and Handedness

**Solution.** While $\chi^2$ and Likelihood ratio test statistic can be computed by hand for the given table, I use the following R code to read the data into R and perform the test to circumvent unnecessary calculation.

```
data <- matrix(c(14, 3, 3, 13, 1, 2, 6, 1, 1), nrow = 3, byrow = T, dimnames =
   list("Gender" = c("Male","Female","Other"), "Handedness" = c("Right","Left","
   Both")))
print(data)
chisq.test(data)
```

```
              Handedness
Gender        Right    Left    Both
Male          14       3       3
Female        13       1       2
Other         6        1       1


Pearson's Chi-squared test

data:  data
X-squared = 0.81, df = 4, p-value = 0.9371
```

Since the p-value is 0.9371, which is extremely higher than the significance level of 0.05, it is clear that this data does not show any evidence that Gender and Handedness are not independent, thereby enabling us believe in favour of independence between Gender and Handedness.

```
DescTools::GTest(data)
```

```
    Log likelihood ratio (G-test) test of independence without correction

data:  data
G = 0.85984, X-squared df = 4, p-value = 0.9303
```

The above code snippet shows the output of Likelihood ratio test of independence. Clearly, the p-value obtained is quite similar to that of the usual $\chi^2$ test statistic, hence showing no evidence in rejecting in favour of the belief that Gender and Handedness are not independent.

To perform the permutation test, the dataset is first reshaped into a desired view.

```
meltdata <- reshape2::melt(data)
meltdata <- as.data.frame(lapply(meltdata, rep, meltdata$value))[, -3]
meltdata$Gender <- factor(meltdata$Gender)
meltdata$Handedness <- factor(meltdata$Handedness)
meltdata
```

```
     Gender  Handedness
  1    Male       Right
  2    Male       Right
  3    Male       Right
  ...
  14   Male       Right
  15 Female       Right
  16 Female       Right
  ...
  27 Female       Right
  28  Other       Right
  ...
  33  Other       Right
  34   Male        Left
  35   Male        Left
  36   Male        Left
  37 Female        Left
  38  Other        Left
  39   Male        Both
  40   Male        Both
  41   Male        Both
  42 Female        Both
  43 Female        Both
  44  Other        Both
```

Since there are lots of permutations ($\frac{44!}{20!16!8!} \approx 1.29 \times 10^{18}$) of the **Gender** variable, therefore, it may not be possible to completely enumerate all possible permutations of the **Gender** variables keeping the other variable fixed.

We begin by writing two customized function in **R**, by which we can compute the Pearson's chi-square test statistic and Likelihood Ratio test statistic according to the following equations.

$$\chi^2 = \sum_i \sum_j \frac{(n_{ij} - E_{ij})^2}{E_{ij}}$$

and

$$-2 \log \Lambda = 2 \sum_i \sum_j n_{ij} \log\left(\frac{n_{ij}}{E_{ij}}\right) \text{ where } E_{ij} = \frac{n_{i\cdot} n_{\cdot j}}{n}$$

```
chisq.stat <- function(tab) {
  n <- sum(tab)
  rowmars <- apply(tab, 1, sum)
  colmars <- apply(tab, 2, sum)
  exp.tab <- rowmars %*% t(colmars)/n

  return(sum((tab - exp.tab)^2 / exp.tab))
}

lrt.stat <- function(tab) {
  n <- sum(tab)
  rowmars <- apply(tab, 1, sum)
  colmars <- apply(tab, 2, sum)
  exp.tab <- rowmars %*% t(colmars)/n

```

```
  return(2*sum(tab * log(tab / exp.tab), na.rm = T))   # if one cell becomes 0, log is not defined, so we omit them in the sum

}
```

We then proceed to write a customized function in **R**, where we consider some permutations of the **Gender** variable. Under the null hypothesis that **Gender** and **Handedness** are independent of each other, it is clear that replacing the permuted row as the **Gender** variable, keeping the **Handedness** variable fixed, would give rise to a contingency table generated by the same mechanism as the original one. From that table, we can obtain the value of Pearson's chi square test statistic or an Likelihood Ratio statistic using above functions, and use that to infer about the p-value of the observed table.

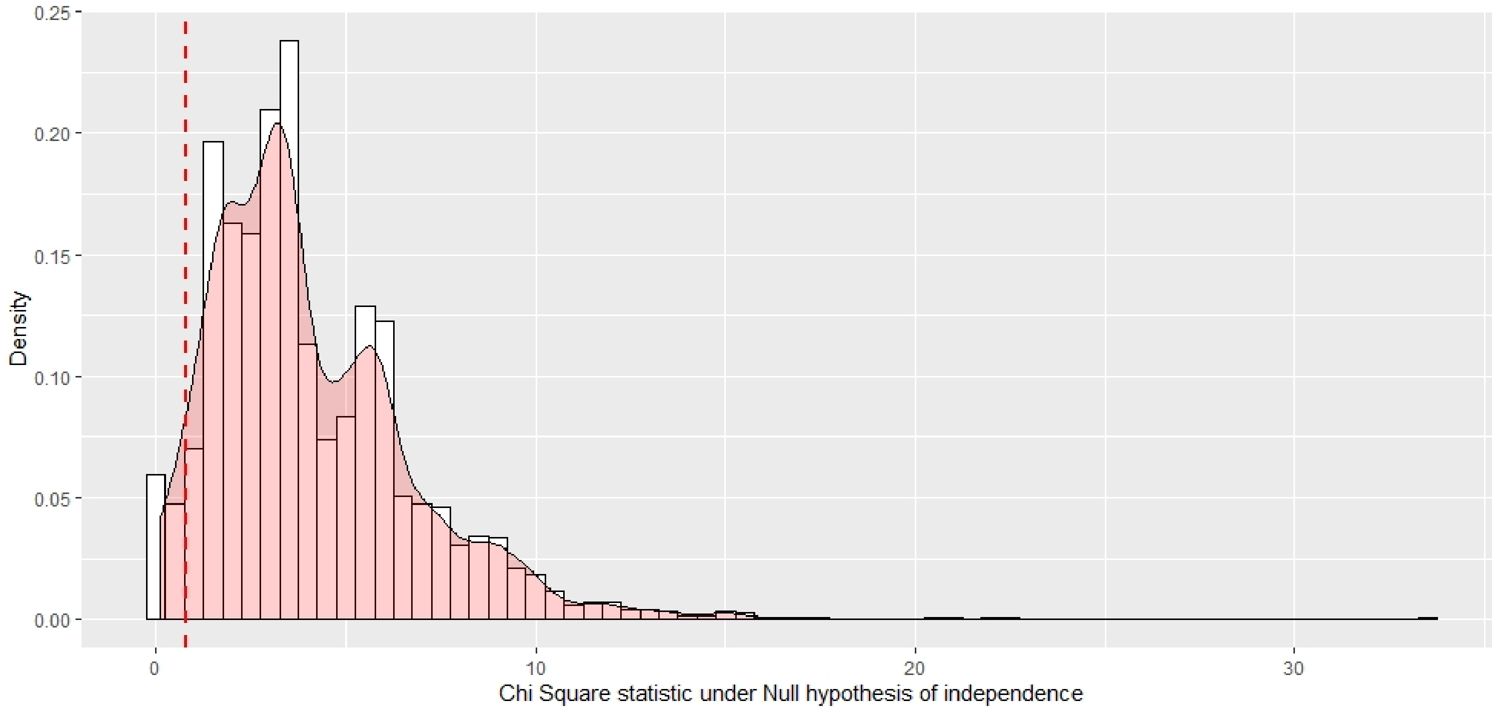
```
perm.stat <- function(data, stat, nperms = 10e4, seed = 1234) {

  if (! (stat %in% c("chisq","lrt"))) {
    stop("Not Implemented yet! Only 'chisq' and 'lrt' is available.")
  }
  else {
    set.seed(seed)
    x <- sapply((1:nperms), function(a){ sample(data[, 1], size = nrow(data)) })
    x <- t(x)
    x <- unique(x)

    if (stat == "chisq") {
      y <- apply(x, 1, function(a) {
        tab <- table(a, data[, 2])
        return(chisq.stat(tab))
      })
    }
    else if (stat == "lrt") {
      y <- apply(x, 1, function(a) {
        tab <- table(a, data[, 2])
        return(lrt.stat(tab))
      })
    }

    return(y)
  }
}
```

Now, we perform the Permutation test with chi-square test statistic with 5000 permutations, and obtain a p-value of 0.9298.

```
y <- perm.stat(meltdata, stat = "chisq", nperms = 5000, seed = 1911)
head(y)
paste("Approximated p-Value is", sum(y > chisq.stat(data))/5000)
```

```
[1] 4.748333 5.348333 5.785000 2.451667 4.748333 2.785000
[1] "Approximated p-Value is 0.9298"
```

The plot for the null distribution looks as follows.

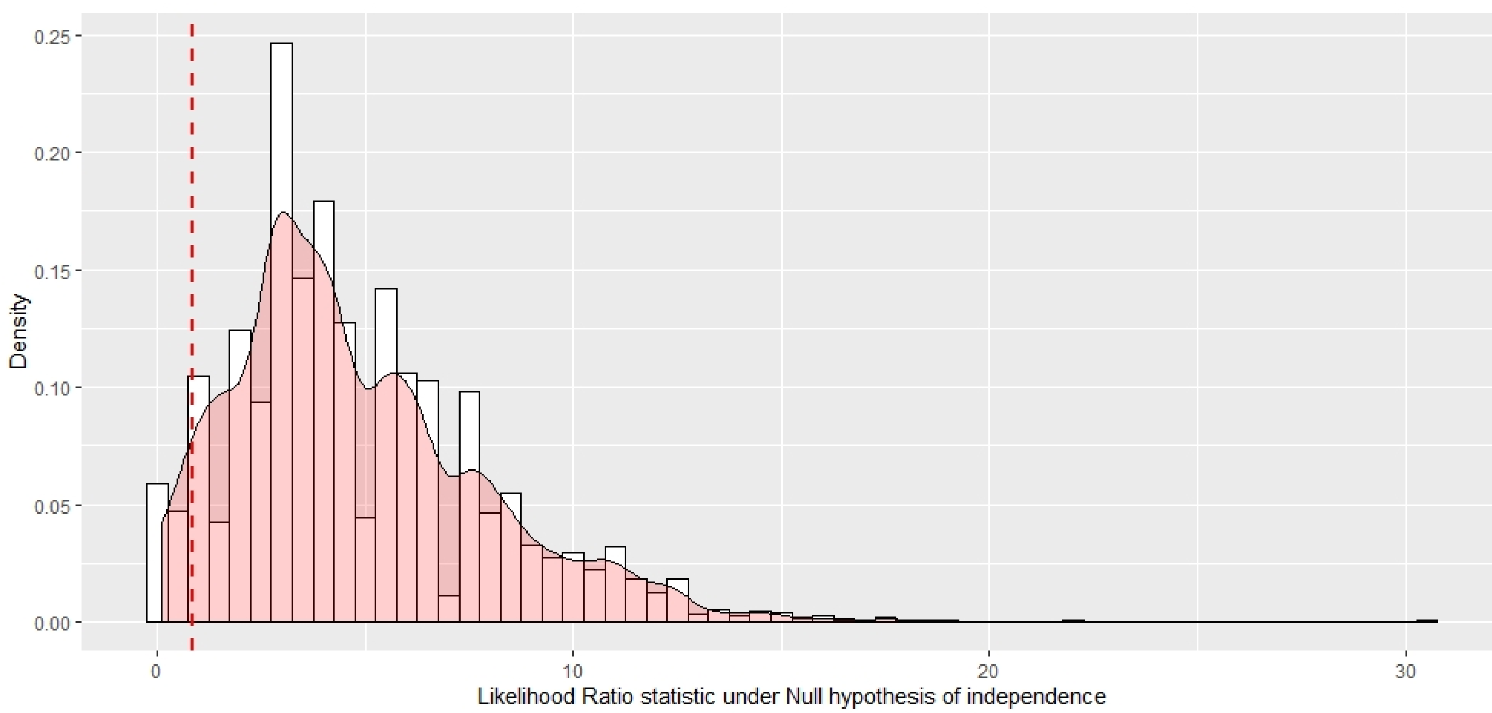



With same permutations, the p-value with Likelihood Ratio statistic turns to be exactly same, 0.9298, which should be true, as both the statistic measure the same kind of deviations from independence setup.

```
y <- perm.stat(meltdata, stat = "lrt", nperms = 5000, seed = 1911)
head(y)
paste("Approximated p-Value is", sum(y > lrt.stat(data))/5000)
```

```
[1] 5.655498 6.280661 5.001904 2.184919 4.975902 3.566309
[1] "Approximated p-Value is 0.9298"
```

and the null distribution looks as follows;



From the figures above, we find that the distribution of chi-square statistic is less dispersed than the null distribution of Likelihood ratio statistic. However, both the distributions are positively skewed, as expected and resembles the structure of a chi-square distribution.

Now, since both these p-values are 0.9298 which is extremely higher than the significance level of 0.05, hence we cannot reject the null hypothesis of independence between **Gender** and **Handedness**. As far as this data is concerned, it does not show any evidence that these two variables are associated in some way (i.e. not independent).

Also note that, the exact p-values obtained are extremely close to the approximate p-value based on asymptotic distribution of Pearson's chi-square and Likelihood Ratio test statistic. Clearly, it indicates that the asymptotic distribution well approximates the behavior of exact distribution in the light of the data.

Following is table containing the p-values for these permutation tests, based on the number of permutations used. All were produced with a seed of 1234 in the previous functions.

| Number of Permutations | p-value with Chi-Square statistic | p-value with LRT |
|---|---|---|
| 5 | 1 | 1 |
| 10 | 1 | 1 |
| 50 | 0.92 | 0.92 |
| 100 | 0.93 | 0.93 |
| 500 | 0.932 | 0.932 |
| 1000 | 0.932 | 0.932 |
| 5000 | 0.9324 | 0.9324 |
| 10,000 | 0.9336 | 0.9336 |
| 100,000 | 0.93652 | 0.93652 |

Interestingly, the p-values for both type of test statistic for permutation test turns out to be same.

**2. Exact logistic regression** [12]

In a standard binomial logistic regression set-up

$$\mathrm{logit}(\mathbb{P}(Y_i = 1)) = \beta_0 + \beta^\top X_i, \qquad i = 1, \dots, n,$$

write down sufficient statitsics $T_j$ for the parameters $\beta_j$, $j = 0, \dots, p$. Show that the distribution of $T_p$ conditional on $T_0, \dots, T_{p-1}$ depends only on $\beta_p$. Thus, using this conditional distribution, one can estimate and perform inference on $\beta_p$. Does a conditional MLE always exist?

Write down the conditional distribution under $H_0 : \beta_p = 0$. Describe how you would do an exact test of this hypothesis.

Use the **R** packages `logistiX` and `elrm` to perform exact logistic regression on the data in Table 2 with "White-collar job" as response, and "Gender" and "College education" as explanatory variables. Also, perform a standard logistic regression on the same data and compare the results.

| Gender | College education | White-collar job | Number of cases |
|---|---|---|---|
| M | No | 1 | 8 |
| F | No | 1 | 6 |
| M | Yes | 7 | 10 |
| F | Yes | 6 | 6 |

Table 2: Another fictitious dataset.

**Solution.** We write the likelihood of the binomial logistic regression setup, where $(X_i, Y_i)$ be the set of datapoints with $Y_i$ taking values 0 and 1.

$$
\begin{aligned}
\mathsf{L} &= \prod_{i=1}^{n} \left(\mathbb{P}(Y_i = 1)\right)^{Y_i} \left(\mathbb{P}(Y_i = 0)\right)^{1-Y_i} \\
&= \prod_{i=1}^{n} \left(\frac{e^{\beta_0+\beta^\intercal X_i}}{1+e^{\beta_0+\beta^\intercal X_i}}\right)^{Y_i} \left(\frac{1}{1+e^{\beta_0+\beta^\intercal X_i}}\right)^{1-Y_i} \\
&= \prod_{i=1}^{n} \left(\frac{1}{1+e^{\beta_0+\beta^\intercal X_i}}\right) \times e^{\beta_0 \sum_{i=1}^{n} Y_i + \sum_{j=1}^{p} \beta_j \sum_{i=1}^{n} X_{ij} Y_i}
\end{aligned}
$$

where $X_{ij}$ is the $i$-th observation corresponding to $j$-th variable (or predictor). Note that, the first term in the multiplication does not depend on the random observation $Y_i$, and hence is a fixed function of the parameters $\beta$, (as $X_i$'s are treated as constants). Therefore, by use of Neymann Factorization theorem, we obtain that; $T = (T_0, T_1, \dots T_p)$ with $T_0 = \sum_{i=1}^{n} Y_i$ and $T_j = \sum_{i=1}^{n} X_{ij} Y_i$ is a sufficient statistic for $(\beta_0, \beta_1, \dots \beta_p)$.

Note that, the joint density of $(T_0, T_1, \dots T_p)$ is; $f(t_0, t_1, \dots t_p) \propto \exp\left[\sum_{j=0}^{p} \beta_j t_j\right]$. Therefore,

$$
\begin{aligned}
f(t_p | t_0, t_1, \dots t_{p-1}) &= \frac{f(t_0, t_1, \dots t_p)}{f(t_0, t_1, \dots t_{p-1})} \\
&= \frac{f(t_0, t_1, \dots t_p)}{\sum_{t_p} f(t_0, t_1, \dots t_p)} \\
&= \frac{c(t_0, \dots t_p) \exp\left[\sum_{j=0}^{p} \beta_j t_j\right]}{\sum_{t_p} c(t_0, \dots t_p) \exp\left[\sum_{j=0}^{p} \beta_j t_j\right]}, \text{ where } c(t_0, \dots t_p) \text{ is the proportionality constant} \\
&= \frac{c(t_{(p-1)}, t_p) \exp\left[\beta_p t_p\right]}{\sum_{t_p} c(t_{(p-1)}, t_p) \exp\left[\beta_p t_p\right]} \text{ where } t_{(p-1)} = (t_0, t_1, \dots t_{(p-1)})
\end{aligned}
$$

which does only depend on $\beta_p$. Also note that, the conditional MLE does not always exist. This can be seen as follows. Since $X_{ip}$'s are fixed quantities, and $Y_i$ is 0 or 1, it is clear that, the sum in the denominator runs through all elements of the set $\mathbb{T}_p = \{\sum_{i=1}^{n} Y_i X_{ip} : Y_i \in \{0, 1\}\}$. Let, the observed $t_p^*$ that we get is $\max \mathbb{T}_p$, which exists since the set $\mathbb{T}_p$ is finite for a fixed set of $X_i$'s. Then;

$$
\begin{aligned}
f(t_p^* | t_0, t_1, \dots t_{p-1}) &= \frac{c(t_{(p-1)}, t_p^*) \exp\left[\beta_p t_p^*\right]}{\sum_{t_p \in \mathbb{T}_p} c(t_{(p-1)}, t_p) \exp\left[\beta_p t_p\right]} \\
&= \frac{1}{1 + \sum_{t_p \in \mathbb{T}_p - \{t_p^*\}} \frac{c(t_{(p-1)}, t_p^*)}{c(t_{(p-1)}, t_p)} \exp\left[\beta_p (t_p - t_p^*)\right]} \\
&\to 1 \quad \text{as } \beta_p \to \infty, \text{ since } (t_p - t_p^*) < 0
\end{aligned}
$$

Therefore, conditional MLE need not always exist in the above case. Also, note that the proportionality constants are positive. On the other hand, if the observed $t_p^{**}$ that we obtain is actually $\min \mathbb{T}_p$, then;

$$
\begin{aligned}
f(t_p^{**} | t_0, t_1, \dots t_{p-1}) &= \frac{c(t_{(p-1)}, t_p^{**}) \exp\left[\beta_p t_p^{**}\right]}{\sum_{t_p \in \mathbb{T}_p} c(t_{(p-1)}, t_p) \exp\left[\beta_p t_p\right]} \\
&= \frac{1}{1 + \sum_{t_p \in \mathbb{T}_p - \{t_p^{**}\}} \frac{c(t_{(p-1)}, t_p^{**})}{c(t_{(p-1)}, t_p)} \exp\left[\beta_p (t_p - t_p^{**})\right]} \\
&\to 1 \quad \text{as } \beta_p \to -\infty, \text{ since } (t_p - t_p^{**}) > 0
\end{aligned}
$$

Hence, in this case also, the conditional MLE need not exist.

Under the null hypothesis $H_0 : \beta_p = 0$, the conditional distribution reduces to; $f_{H_0}(t_p|t_0, \dots t_{p-1}) = \frac{c(t_{(p-1)}, t_p)}{\sum_{u \in \mathbb{T}_p} c(t_{(p-1)}, u)}$.

Now, to perform a hypothesis testing for $\beta = 0$, note that; $f_{T_p}(t; \beta_p)/f_{T_p}(t; \beta_p = 0) \propto \exp[\beta_p t]$. As the likelihood ratio is monotonically increasing as a function of $t$, hence, we obtain the p-value for the observed value of the statistic $T_p = \sum_{i=1}^{n} Y_i X_{ip}$ as;

$$p_+ = \sum_{u \geq t} f_{T_p}(u; \beta_p = 0) = \frac{\sum_{u \geq t} c(t_{(p-1)}, u)}{\sum_{u \in \mathbb{T}_p} c(t_{(p-1)}, u)}$$

for testing against the alternative $\beta > 0$. Similarly, for testing against the alternative that $\beta < 0$, we could obtain the p-value as;

$$p_- = \sum_{u \leq t} f_{T_p}(u; \beta_p = 0) = \frac{\sum_{u \leq t} c(t_{(p-1)}, u)}{\sum_{u \in \mathbb{T}_p} c(t_{(p-1)}, u)}$$

And for testing against the two sided alternative; we can use the p-value as $2 \min\{p_+, p_-\}$. Then, we use this p-values and compare them with the nominal significance level $\alpha$, to perform the hypothesis test (i.e. reject $H_0$ when the p-value is less than the significance level, and accept otherwise).

Next, we use package `elrm` to perform exact logistic regression on the given data.

```r
df <- data.frame(Gender = c("M","F","M","F"), CollegeEd = c("No", "No","Yes","Yes"), WhiteJob = c(1, 1, 7, 6), Total = c(8, 6, 10, 6))

library(elrm)

# CI for Gender variable
fit <- elrm(WhiteJob / Total ~ Gender + CollegeEd, interest = ~ Gender, dataset = df, iter = 10e5, burnIn = 100)
summary(fit)

# CI for College Education Variable
fit <- elrm(WhiteJob / Total ~ Gender + CollegeEd, interest = ~ CollegeEd, dataset = df,
iter = 10e5, burnIn = 100)
summary(fit)
```

```
Progress: 100%  Progress:  95%  Progress:  90%  Progress:  85%  Progress:  80%
Generation of the Markov Chain required 35 secs
Conducting inference ...
Inference required 5 secs

Call:

elrm(formula = WhiteJob/Total ~ Gender + CollegeEd, interest = ~Gender, iter = 1e+06, dataset = df, burnIn
    = 100)


Results:

        estimate   p-value   p-value_se   mc_size
GenderM -1.35482   0.33699   0.00094      999900


95% Confidence Intervals for Parameters

           lower       upper
GenderM    -5.342778   1.128712
```

```
Call:

elrm(formula = WhiteJob/Total ~ Gender + CollegeEd, interest = ~CollegeEd, iter = 1e+06, dataset = df,
    burnIn = 100)


Results:

               estimate   p-value   p-value_se   mc_size
CollegeEdYes   3.6278     0.00031   2e-05        999900


95% Confidence Intervals for Parameters

               lower      upper
CollegeEdYes   1.107514   7.948637
```

From the above confidence interval, we note that the for the Gender variable, the 95% confidence interval does contain the value 0, and hence we are unable to reject that null hypothesis. So, conclude that effect of Gender on probability of getting White Collar Job is insignificant based on the data. However, for the College Education variable, the 95% exact confidence interval contains the value 0, and hence we can reject the null hypothesis that coefficient for College Education in logistic regression model is 0, and conclude that effect of College Education on probability of getting White Collar Job is significant at a 5% level of significance.

Using the package `logistiX`, almost similar confidence intervals are obtained;

```r
library(logistiX)
Gender <- factor(c(rep("M", 8), rep("F", 6), rep("M", 10), rep("F", 6)))
CollegeEd <- factor(c(rep("No", 14), rep("Yes", 16)))
WhiteJob <- c(1, rep(0, 7), 1, rep(0, 5), rep(1, 7), rep(0, 3), rep(1, 6))
longdf <- data.frame(Gender = Gender, CollegeEd = CollegeEd, WhiteJob = WhiteJob
  )

x <- model.matrix(WhiteJob ~ Gender + CollegeEd, longdf)[, -1]
fit <- logistiX(x, y = longdf$WhiteJob)
summary(fit)
```

```
Exact logistic regression

Call:
logistiX(x = x, y = longdf$WhiteJob)

Estimation method:         LX
CI method:                 exact
Test method:               TST

Summary of estimates, confidence intervals and parameter hypotheses tests:

estimates     2.5 %    97.5 % statistic      pvalue cardinality
1 -1.360474 -5.367983 1.128986         8 0.4556514914           6
2  3.338699  1.101657 7.265954        13 0.0005817118          14
```

It seems that the `elrm` approximately obtains the same values of upper and lower bounds for confidence interval. Also, based on whether the confidence interval contains 0 or not, or checking whether the p-value is less than $\alpha = 0.05$ or not, we obtain same decision as before. The contribution of **Gender** variable is not significant in deciding the probability of getting a **White Collar Job**, but the contribution of **College Education** is significant.

We also perform the standard logistic regression on the data.

```r
fit <- glm(cbind(WhiteJob, Total - WhiteJob) ~ Gender + CollegeEd, data = df,
  family = "binomial")
summary(fit)
```

```
Call:
glm(formula = cbind(WhiteJob, Total - WhiteJob) ~ Gender + CollegeEd,
family = "binomial", data = df)

Deviance Residuals:
1         2         3         4
0.5628   -0.4437   -0.3179    0.9629

Coefficients:
                Estimate   Std. Error  z value    Pr(>|z|)
(Intercept)     -1.1471    0.8763      -1.309     0.19054
GenderM         -1.4515    1.2038      -1.206     0.22790
CollegeEdYes     3.6687    1.1904       3.082     0.00206 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' '   1

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 17.9365  on 3  degrees of freedom
Residual deviance:  1.5418  on 1  degrees of freedom
AIC: 13.877

Number of Fisher Scoring iterations: 4
```

We note that, using Wald's asymptotic test for standard logistic regression, we also find that Gender does not have a significant effect, whereas College Education has one on predicting the log odds of having White Collar job.