Quantitative Finance Problem Solving Assignment

As a part of the Master of Statistics (M. Stat.) Second year curriculum

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Contents

Problem 1. Pilska, Chapter 1, Problem 9: Suppose $K = 2, N = 1$, and the interest rate is a scalar parameter $r \geq 0$. Also, suppose $S_0 = 1, S_1(\omega_1) = u$ (up), and $S_1(\omega_2) = d$ (down), where the parameters satisfy $u > d > 0$. For what values of u, d and r, does there exist a risk neutral probability measure? Say what this measure is. For the complementary values of these parameters, say what all the arbitrage opportunities are.

Solution.

Problem 2. Pilska, Chapter 2, Problem 2: Suppose $u(w) = \log(w)$. Show that the inverse function $I(i) = i^{-1}$, the Lagrange multiplier is $\lambda = v^{-1}$, the optimal attainable wealth is $W = vL^{-1}B_1$, and the optimal objective value is $\ln(v) - \mathbb{E}\left[\ln(L/B_1)\right]$. Compute these expressions and solve for the optimal trading strategy in the case where $N = 1, K = 2, r = 1/9, S_0 = 5, S_1(\omega_1) = 20/3, S_1(\omega_2) = 40/9$ and $P(\omega_1) = 3/5$.

Solution.

Problem 3. Pilska, Chapter 4, Problem 9: Consider the binomial stoch price model with $T = 4, S_0 = 20, u = 1.2214$, and $d = 0.8187 = u^{-1}$. The interest rate is $r = 3.82\%$. What is the time 0 price on an American put that has exercise price $e = 18$? Is it optimal to exercise early? If so, when?

Solution.

Problem 4. Hoel Port Stone, Chapter 4, Problem 3: Let $X(t), -\infty < t < \infty$ be a second order stationary process and set $Y(t) = X(t+1) - X(t), -\infty < t < \infty$. Show that the $Y(t)$ process is a second order stationary process having zero means and covariance function

$$r_Y(t) = 2r_X(t) - r_X(t-1) - r_X(t+1)$$

Solution.

Problem 5. Hoel Port Stone, Chapter 4, Problem 11: Let, $R_1, \dots R_n$, and $\Theta_1, \dots \Theta_n$ be independent random variables such that Θ's are uniformly distributed on $[0, 2\pi)$ and R_k has the density

$$f_{R_k}(r) = \begin{cases} \dfrac{r}{\sigma_k^2} e^{-r^2/2\sigma_k^2}, & 0 < r < \infty \\ 0, & r \leq 0 \end{cases}$$

where $\sigma_1, \dots \sigma_n$ are positive constants. Let, $\lambda_1, \dots \lambda_n$ be positive constants and set

$$X(t) = \sum_{k=1}^{n} R_k \cos(\lambda_k t + \Theta_k)$$

Show that $X(t)$ is a Gaussian process.

Solution.

Problem 6. Hoel Port Stone, Chapter 4, Problem 19: Let $W(t)$ denotes the Weiner process. Define,

$$X(t) = e^{-\alpha t} W(e^{2\alpha t}), \qquad -\infty < t < \infty$$

where α is a positive constant. Show that $X(t)$ process is a stationary Gaussian process having the covariance function

$$r_X(t) = \sigma^2 e^{-\alpha |t|}, \qquad -\infty < t < \infty$$

Solution.

Problem 7. Hoel Port Stone, Chapter 5, Problem 2: Find the correlation between $W(t)$ and

$$\int_0^1 W(s) ds$$

for $0 \leq t \leq 1$.

Solution.

Problem 8. Hoel Port Stone, Chapter 5, Problem 15: Find the mean and the variance of

$$X = \int_0^1 t dW(t) \qquad \text{and} \qquad Y = \int_0^1 t^2 dW(t)$$

and find the correlation between these two random variables.

Solution.

Problem 9. Hoel Port Stone, Chapter 6, Problem 7: Show that the left side of the stochastic differential equation

$$a_0 X"(t) + a_1 X'(t) + a_2 X(t) = W'(t)$$

is stable if and only if the coefficients a_0, a_1 and a_2 are either all positive or all negative.

Solution.

Problem 10. Oksendal B, Chapter 3, Problem 6: Prove that $N_t = B_t^3 - 3tB_t$ is a martingale, where B_t denotes a Brownian motion.

Solution.

Problem 11. Oksendal B, Chapter 4, Problem 4: Consider the vector $\theta(t,\omega) = (\theta_1(t,\omega), \ldots \theta_n(t,\omega)) \in \mathbb{R}^n$ with $\theta_k(t,\omega) \in \mathcal{V}[0,T]$ for $k = 1, 2, \ldots n$, where $T \leq \infty$. Define,

$$Z_t = \exp\left\{ \int_0^t \theta(s,\omega) dB(s) - \frac{1}{2} \int_0^t \theta^2(s,\omega) ds \right\}, \qquad 0 \leq t \leq T$$

where $B(s) \in \mathbb{R}^n$ and $\theta^2 = \theta \cdot \theta$ (dot product).

(a) Use Itô's formula to prove that

$$dZ_t = Z_t \theta(t,\omega) dB(t)$$

(b) Deduce that Z_t is a martingale for $t \leq T$, provided that $Z_t \theta_k(t,\omega) \in \mathcal{V}[0,T]$ for $1 \leq k \leq n$.

Solution.