Indian Statistical Institute, Kolkata

Measure Theoretic Probability

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1 Problem 1

Problem. Let, $\mathcal{C} = \{A \subseteq \mathbb{R} : A \text{ is countable or } \mathbb{R}\backslash A \text{ is countable}\}$ be the countable-cocountable σ-field on $\mathbb{R}$. Let, $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that, $\forall B \in \mathcal{B}_{\mathbb{R}}$, $f^{-1}(B) \in \mathcal{C}$ iff there exists a countable set $A \subset \mathbb{R}$ and $c \in \mathbb{R}$ such that, $f(x) = c, \forall x \notin A$.

Solution: I shall first proceed with the if part and then the only if part.

If part:

Here, it is given that there exists a countable set $A \subset \mathbb{R}$ and $c \in \mathbb{R}$ such that, $f(x) = c$ for any $x \notin A$. Therefore, for any real $x \in \mathbb{R}$, either $f(x) = c$ or $x \in A$, i.e. $\mathbb{R} = A \cup f^{-1}(\{c\})$, i.e. $f(\mathbb{R}) = A^* \cup \{c\}$ where $A^* = f(A) = \{f(x) : x \in A\}$.

Now choose any set $B \in \mathcal{B}_{\mathbb{R}}$.

1. If $c \in B$, then $f^{-1}(B) \supseteq f^{-1}(\{c\}) = \mathbb{R}\backslash A$. Therefore, $\mathbb{R}\backslash f^{-1}(B) \subseteq A$, and as A is countable, $\mathbb{R}\backslash f^{-1}(B)$ is countable as well. This shows $f^{-1}(B) \in \mathcal{C}$ as it is cocountable.

2. If $c \notin B$, then
$$f^{-1}(B) = f^{-1}(B \cap A^*) \subseteq f^{-1}(A^*) = A$$
Hence, $f^{-1}(B)$ would be countable and thereby belongs to $\mathcal{C}$.

Only If:

Here, it is assumed that, for any $B \in \mathcal{B}_{\mathbb{R}}$, $f^{-1}(B) \in \mathcal{C}$, i.e. either $f^{-1}(B)$ or its complement is countable. Since, $f^{-1}(B^c) = (f^{-1}(B))^c$, hence either B or B^c must have its pre-image set as a countable set.

Claim 1. For all $B \in \mathcal{B}_{\mathbb{R}}$, either $f^{-1}(B)$ or $f^{-1}(B^c)$ is countable.

Let us consider a countable cover for the whole real line,

$$\mathbb{R} = \cup_{n=-\infty}^{\infty}(n, n+1]$$

Clearly, as f is a function, $f^{-1}(\mathbb{R})$ is uncountable. Since, $f^{-1}(\mathbb{R}) = \cup_{n=-\infty}^{\infty} f^{-1}((n, n+1])$, which is a countable union, atleast one of the set among the ones above must be uncountable in size. Without loss of generality, assume, $(0, 1]$ be one such set so that, $f^{-1}((0, 1])$ is uncountable.

Now, it will be shown that it is only such set whose pre-image is uncountable. By the claim 1 made earlier, it is obvious once we take $B = (0, 1]$ and $B^c = \cup_{n=-\infty n\neq 0}^{\infty}(n, n+1]$ whose preimage must be countable, and hence each of pre-image $f^{-1}((n, n+1]) \subseteq f^{-1}(B^c)$ is countable for any $n \in \mathbb{Z}\backslash \{0\}$.

Let, $a_1 = 0, b_1 = 1$. Then, the closed interval $[a_1, b_1]$ has its pre-image uncountable (as it contains $(0, 1]$). Now, we consider a division of the interval into two equal parts, $[0, 1/2]$ and $(1/2, 1]$, and since $f^{-1}([0, 1]) = f^{-1}([0, 1/2]) \cup f^{-1}((1/2, 1])$. Again by exactly same logic as before exactly one of these pre-images is uncountable. We call this $[a_2, b_2]$. (If the set is not closed then we add the endpoints to it so that it becomes closed and still retain the above property.) We apply this step inductively to obtain a sequence of nested closed intervals,

$I_1 \supset I_2 \supset I_3 \cdots \subset I_m \ldots$, where $I_m = [a_m, b_m]$. Also, since we are making these intervals of half length at each step, $(b_m - a_m) = 2^{-(m-1)} \to 0$, as $m \to \infty$.

Therefore, by Cantor's Intersection theorem[1], there exists exactly a single point $c \in R$ such that, $\cap_{m=1}^{\infty} I_m = \{c\}$. Due to the repeated application of claim 1 and the choice of the nested closed intervals I_m's, for the complementary sets we have, $f^{-1}(\mathbb{R}\backslash I_1), f^{-1}(I_1\backslash I_2), f^{-1}(I_2\backslash I_3), \ldots$ are countable sets. Since,

$$(\cap_{m=1}^{\infty} I_m)^c = \cap_{m=1}^{\infty} I_m^c = (\mathbb{R}\backslash I_1) \cup (I_1\backslash I_2) \cup (I_2\backslash I_3) \cup \ldots$$

Therefore,

$$f^{-1}(\{c\}^c) = f^{-1}((\cap_{m=1}^{\infty} I_m)^c) = f^{-1}(\mathbb{R}\backslash I_1) \cup \cup_{n=1}^{\infty} f^{-1}(I_n\backslash I_{n+1})$$

which is a countable union of countable sets, and hence is countable. We call this set A. Then clearly, for all $x \notin A$, $f(x) = c$, which was what we intended to show. ■

[1]**Statement:** If $C_1, \supseteq C_2 \supseteq \ldots C_n \subseteq \ldots$ be a sequence of nested closed bounded intervals in $\mathbb{R}$ such that $\lim_{n\to\infty} \mathrm{Diam}(C_n) \to 0$, then $\cap_{n=1}^{\infty} C_n$ contains exactly one point

2 Problem 2

Problem. Let, $a_0, a_n, b_n, n \geq 1$ be real numbers such that the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

converges absolutely on a set of positive Lebesgue measure. In other words, Lebesgue measure of $E = \{x \in [-\pi, \pi] : \sum_{n=1}^{\infty} |a_n||\cos nx| + \sum_{n=1}^{\infty} |b_n||\sin nx| < \infty\}$ is positive. Show that, $\sum_{n=1}^{\infty}(|a_n| + |b_n|) < \infty$.
Note: You would possibly need the following fact. If $f : [-\pi, \pi] \to \mathbb{R}$ be a bounded measurable function, then both $\int_{-\pi}^{\pi} f(x)\cos(nx)d\lambda(x)$ and $\int_{-\pi}^{\pi} f(x)\sin(nx)d\lambda(x)$ converge to zero as n goes to infinity.

Solution: We have, $E = \{x \in [-\pi, \pi] : \sum_{n=1}^{\infty} |a_n||\cos nx| + \sum_{n=1}^{\infty} |b_n||\sin nx| < \infty\}$ and it is given that $\lambda(E) > 0$, where λ denotes the usual Lebesgue measure.

Let, $F = \{x \in [-\pi, \pi] : \sum_{n=1}^{\infty} |a_n \cos nx + b_n \sin nx| < \infty\}$. Since, by triangle inequality;

$$|a_n \cos nx + b_n \sin nx| < |a_n||\sin nx| + |b_n||\cos nx|$$

Hence, $E \subseteq F$ and, therefore, $\lambda(F) \geq \lambda(E) > 0$, and obviously, $F \subseteq [-\pi, \pi]$ implies $\lambda(F) \leq \lambda([-\pi, \pi]) = 2\pi$. Now,

$$\begin{aligned} a_n \cos nx + b_n \cos nx &= r_n \left[\frac{a_n}{r_n}\cos nx + \frac{b_n}{r_n}\sin nx\right], \quad \text{where } r_n = \sqrt{(a_n^2 + b_n^2)} \\ &= r_n \left[\sin\theta_n \cos nx + \cos\theta_n \sin nx\right], \quad \text{where } \theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \\ &= r_n \sin(\theta_n + nx) \end{aligned}$$

Therefore,

$$F = \left\{x \in [-\pi, \pi] : \sum_{n=1}^{\infty} r_n|\sin(\theta_n + nx)| < \infty\right\}$$

Now, since $|\sin(\theta_n + nx)| \leq 1$, we try to bound the series so that, $\sum_{n=1}^{\infty} r_n$ can be bounded.

For this reason, we consider the sets $F_m = \{x \in F : \sum_{n=1}^{\infty} r_n|\sin(\theta_n + nx)| < m\}$. Clearly, the sets $F_m \uparrow F$, and hence the Lebesgue measure $\lambda(F_m) \uparrow \lambda(F)$, provided that F_m's and F are measurable, which we state as a claim.

Claim 2. The sets $F_m = \{x \in F : \sum_{n=1}^{\infty} r_n|\sin(\theta_n + nx)| < m\}$ are measurable for any $m \in \mathbb{N}$. Also, the set $F = \{x \in [-\pi, \pi] : \sum_{n=1}^{\infty} r_n|\sin(\theta_n + nx)| < \infty\}$ is measurable.

Proof. To prove the claim, let us consider a function $h : \mathbb{R} \to [0, \infty]$ given by;

$$h(x) = \sum_{n=1}^{\infty} r_n |\sin(\theta_n + nx)|$$

And let, $h_m(x)$ denotes the partial sums, namely $h_m(x) = \sum_{n=1}^{m} r_n |\sin(\theta_n + nx)|$. Note that, $\sin(\theta_n + nx)$ is measurable being a continuous function of x, and so is its absolute value. Being a finite sum of such terms, h_m is also a measurable function. Also, h_m is non-negative for each $m \geq 1$, h_m are non-decreasing (with respect to m) and $h_m \uparrow h$. Hence the function h is also measurable.

Finally, note that $F_m = F \cap h^{-1}((-\infty, m))$, which is measurable provided that F is measurable.

However,

$$F = \left\{ x \in [-\pi, \pi] : \lim_{m \to \infty} h_m(x) < \infty \right\}$$

which is measurable due to Proposition 3.2.5 of the notes.[2] □

Now, due to claim 2, $\lambda(F_m) \uparrow \lambda(F)$, and since, $\lambda(F) > 0$, this means there exists M such that, $\forall m \geq M$, $\lambda(F_m) > 0$. Therefore,

$$\begin{aligned}
\int_{F_M} \sum_{n=1}^{\infty} r_n |\sin(\theta_n + nx)| d\lambda &= \int_{\mathbb{R}} \mathbf{1}_{F_M} \sum_{n=1}^{\infty} r_n |\sin(\theta_n + nx)| d\lambda, \\
&\qquad \text{where } \mathbf{1}_A \text{ is the indicator function of set } A \\
&\leq \int_{\mathbb{R}} \mathbf{1}_{F_M} M d\lambda, \quad \text{as } \forall x \in F_M, \text{ the series is bounded by } M \\
&\qquad \text{and the rest follows from monotonicity of integral} \\
&= M \int_{F_M} d\lambda \quad \text{by linearity of integral} \\
&= M\lambda(F_M)
\end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned}
|\sin(\theta_n + nx)| &\geq \sin^2(\theta_n + nx), \quad \text{since } |\sin(\theta_n + nx)| \leq 1 \\
&= \frac{1}{2} [1 - \cos(2\theta_n + 2nx)] \\
&= \frac{1}{2} - \frac{1}{2} \cos(2\theta_n) \cos(2nx) + \frac{1}{2} \sin(2\theta_n) \sin(2nx)
\end{aligned}$$

[2]**Statement of the Proposition:** Let $(\Omega, \mathcal{G})$ be a measurable space and $X_n : \Omega \to \mathbb{R}$ be sequence of real valued random variables. Then $A = \{w \in \Omega : \lim_n X_n(w) < \infty\}$ is a measurable set.

Now,

$$\int_{F_M} |\sin(\theta_n + nx)| d\lambda \geq \int_{F_M} \left[\frac{1}{2} - \frac{1}{2}\cos(2\theta_n)\cos(2nx) + \frac{1}{2}\sin(2\theta_n)\sin(2nx)\right] d\lambda$$

by monotonicity of integral

$$= \frac{\lambda(F_M)}{2} - \frac{1}{2}\cos(2\theta_n)\int_{F_M}\cos(2nx)d\lambda + \frac{1}{2}\sin(2\theta_n)\int_{F_M}\sin(2nx)d\lambda \tag{2}$$

Note that,

$$\int_{F_M} \cos(2nx)d\lambda(x) = \int_{-\pi}^{\pi} \mathbf{1}_{F_M}(x)\cos(2nx)d\lambda(x)$$

where, $\mathbf{1}_{F_M}(x) = 1$ if $x \in F_M$, 0 otherwise. Clearly, by the note given in the question, as $n \to \infty$, the above integral converges to zero, as $\mathbf{1}_{F_M}(x)$ is a bounded measurable function since F_M is a measurable set.

Similarly, $\int_{F_M} \sin(2nx)d\lambda(x) \to 0$ as $n \to \infty$. Also, since $\cos(2\theta_n)$ is a bounded sequence of real numbers, combining with above yields, $\frac{1}{2}\cos(2\theta_n)\int_{F_M}\cos(2nx)d\lambda \to 0$ as $n \to \infty$. By similar argument, we also have, $\frac{1}{2}\sin(2\theta_n)\int_{F_M}\sin(2nx)d\lambda \to 0$ as well.

Therefore, by applying the limit $n \to \infty$ on eq. (2), we obtain the existence of $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_{F_M} |\sin(\theta_n + nx)| d\lambda \geq \frac{\lambda(F_M)}{3}$$

as $\frac{\lambda(F_M)}{3} < \frac{\lambda(F_M)}{2}$ since $\lambda(F_M) > 0$. Therefore,

$$\int_{F_M} \sum_{n=n_0}^{\infty} r_n |\sin(\theta_n + nx)| d\lambda = \sum_{n=n_0}^{\infty} r_n \int_{F_M} |\sin(\theta_n + nx)| d\lambda$$

can interchange the sum and the integral since it is finite by eq. (1) (3)

$$\geq \sum_{n=n_0}^{\infty} r_n \frac{\lambda(F_M)}{3}$$

Now combining eq. (1) and eq. (3), we get that,

$$\frac{\lambda(F_M)}{3}\sum_{n=n_0}^{\infty} r_n \leq \int_{F_M}\sum_{n=n_0}^{\infty} r_n |\sin(\theta_n + nx)| d\lambda \leq M\lambda(F_M)$$

Since, $\lambda(F_M) > 0$, this yields;

$$\sum_{n=1}^{\infty} r_n \leq \sum_{n=1}^{(n_0-1)} r_n + M < \infty$$

Finally, an application of QM-AM inequality yields, $\sqrt{2}\sqrt{a_n^2+b_n^2} \geq (|a_n|+|b_n|)$. Thus,

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|) \leq \sqrt{2}\sum_{n=1}^{\infty} r_n < \infty$$

This completes the proof of the result.

■

3 Problem 3

Problem. Let, X be a random variable having distribution function F. Show that, $\mathbb{E}(F(X)) \geq 1/2$ with equality iff F is continuous.

Solution: We denote the measure space associated with the random variable X as $(\Omega, \mathcal{G}, \mu)$, where μ is the probability measure.

Let us assume existence of two random variables X_1 and X_2 such that both of these random variables have the same distribution function F as X.[3] Let, μ be the probability measure associated with X, hence associated with X_1 and X_2 as well.

Then,

$$\begin{aligned}
\mathbb{E}(F(X)) &= \mathbb{E}(F(X_1)) \\
&= \int_{-\infty}^{\infty} F(x_1)\mu(dx_1) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} \mu(dx_2)\mu(dx_1) \qquad \text{, since } F \text{ is distribution function of } X_2 \text{ also} \\
&= \int_{\mathbb{R}^2} \mathbf{1}_{\{(x_1,x_2):x_2\leq x_1\}}(x_1,x_2)\mu(dx_2)\mu(dx_1) \qquad \text{where } \mathbf{1}_A \text{ is the indicator function of } A
\end{aligned}$$

Now in order to get a product probability measure in the product space, we extend these individual measures to a transition probability measure, simply as follows:

Define, $\mu_{12} : \Omega \times \mathcal{G} \to [0, \infty]$ as $\mu_{12}(w, B) = \mu(B) \quad \forall B \in \mathcal{G}$. Then clearly,

1. For any $w \in \Omega$, $\mu_{12}(w, \cdot)$ is σ-finite as $\mu_{12}(w, \Omega) = \mu(\Omega) = 1 < \infty$.
2. For any $B \in \mathcal{G}$, $\mu_{12}(\cdot, B) : \Omega \to [0, \infty]$ is measurable, as $\mu_{12}(\cdot, B) = \mu(B)$, the constant function which is trivially measurable.

Now that we have μ_{12} as a transition measure, it is also easy to note that it is uniformly σ-finite. This is because, $\sup_{w\in\Omega} \mu_{12}(w, B) = \mu(B) \leq \mu(\Omega) = 1 < \infty$.

Hence, letting $S = \{(x_1, x_2) : x_2 \leq x_1\}$, an application of Fubini's theorem yields;

$$\mathbb{E}(F(X)) = \int_{\mathbb{R}^2} \mathbf{1}_S(x_1,x_2)\mu(dx_2)\mu(dx_1) = \int_{\mathbb{R}^2} \mathbf{1}_S(x_1,x_2)\mu_{12}(x_1,dx_2)\mu(dx_1) = \lambda(S)$$

where $\lambda(\cdot)$ is a σ-finite measure on the product space $\mathcal{G} \otimes \mathcal{G}$. Consequently, reversing the role of x_1 and x_2 in the order to integration, we could consider,

$$\lambda(S) = \int_{\mathbb{R}^2} \mathbf{1}_S(x_1,x_2)\mu_{12}(x_2,dx_1)\mu(dx_2) = \int_{\mathbb{R}^2} \mathbf{1}_S(x_1,x_2)\mu(dx_1)\mu(dx_2)$$

Therefore, finally we have;

[3] Note that, independence of X_1 and X_2 are not assumed.

$$\mathbb{E}(F(X)) = \int_{\mathbb{R}^2} \mathbf{1}_{\{(x_1,x_2):x_2 \le x_1\}}(x_1, x_2)\mu(dx_1)\mu(dx_2) \tag{4}$$

However, for all $(x_1, x_2) \in \mathbb{R}^2$ we have,

$$\mathbf{1}_{\{(x_1,x_2):x_2 \le x_1\}}(x_1, x_2) + \mathbf{1}_{\{(x_1,x_2):x_2 \ge x_1\}}(x_1, x_2) - \mathbf{1}_{\{(x_1,x_2):x_2 = x_1\}}(x_1, x_2) = 1 \tag{5}$$

Let us call the sets $\{(x_1, x_2) : x_2 \ge x_1\}$ and $\{(x_1, x_2) : x_2 = x_1\}$ as T and V respectively. Combining eq. (4) and eq. (5) together, it yields that;

$$\begin{aligned}
\mathbb{E}(F(X)) &= \int_{\mathbb{R}^2} (1 - \mathbf{1}_T(x_1, x_2) + \mathbf{1}_V(x_1, x_2))\, \mu(dx_1)\mu(dx_2) \\
&= \int_{\mathbb{R}^2} \mu(dx_1)\mu(dx_2) - \int_{R^2} \mathbf{1}_T(x_1, x_2)\mu(dx_1)\mu(dx_2) + \int_{\mathbb{R}^2} \mathbf{1}_V(x_1, x_2)\mu(dx_1)\mu(dx_2) \\
&= \int_{-\infty}^{\infty} \mu(\Omega)\mu(dx_2) - \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x_2} \mu(dx_1) \right) \mu(dx_2) + \lambda(V) \\
&= \int_{-\infty}^{\infty} \mu(dx_2) - \int_{-\infty}^{\infty} F(x_2)\mu(dx_2) + \lambda(V) \\
&\qquad \text{since, } \mu(\Omega) = 1 \text{ and } \int_{-\infty}^{x_2} \mu(dx_1) = F(x_2) \\
&= \mu(\Omega) - \mathbb{E}(F(X_2)) + \lambda(V) \\
&= 1 + \lambda(V) - \mathbb{E}(F(X)) \quad \text{, as } X \text{ and } X_2 \text{ are identically distributed}
\end{aligned}$$

Hence,

$$2\mathbb{E}(F(X)) = 1 + \lambda(V) \ge 1$$

as $\lambda(V)$ is a well defined measure (as noted earlier due to Fubini's theorem). Hence, $\mathbb{E}(F(X)) \ge \frac{1}{2}$.

Note that, the equality holds if and only if $\lambda(V) = 0$. So, let us prove this concerning the equality case as two seperate parts.

- **If Part:** Suppose, F is continuous. Then,

  Claim 3. F is uniformly continuous, since it is bounded between 0 and 1.

  Proof. A simple proof of this uniform continuity can be established by noting that, since $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$, for any given $\epsilon > 0$, there exists M large enough such that, $F(x) < \epsilon/2$ for any $x \le (-M)$ and $(1 - F(x)) > \epsilon/2$, for any $x \ge M$. However, within the interval $[-M, M]$, F restricted to this interval is a continuous function defined on a compact set, hence is uniformly continuous. Together, we have some δ such that, $\forall x, y \in \mathbb{R}$ such that, $|x - y| < \delta$;

1. If x, y both are either in $(-\infty, -M)$ or in (M, ∞), $|F(x) - F(y)| < \epsilon/2 + \epsilon/2 = \epsilon$.
2. If x, y are both in $[-M, M]$, then $|F(x) - F(y)| < \epsilon$ by uniform continuity of F in the interval $[-M, M]$.
3. If $x \in (-\infty, -M)$ and $y \in [-M, M]$ for example, then,

$$|F(x) - F(y)| \leq |F(x)| + |F(y) - F(-M)| + |F(-M)| < 2\epsilon$$

which completes the argument for establishing the claim. □

Now, fix any $\epsilon > 0$. Since, F is established to be uniformly continuous, we have some $\delta > 0$ such that for any $x \in \mathbb{R}$,

$$F(x + \delta) - F(x - \delta) < \epsilon \tag{6}$$

Therefore,

$$\begin{aligned}
\lambda(V) &= \int_{\mathbb{R}^2} \mathbf{1}_V(x_1, x_2)\mu(dx_1)\mu(dx_2) \\
&\leq \int_{\mathbb{R}^2} \mathbf{1}_{\{(x_1,x_2):x_1 \in (x_2-\delta, x_2+\delta)\}}(x_1, x_2)\mu(dx_1)\mu(dx_2) \\
&\qquad\qquad \text{since, } \{(x_1, x_2) : x_1 \in (x_2 - \delta, x_2 + \delta)\} \supseteq V \\
&= \int_{-\infty}^{\infty} (F(x_2 + \delta) - F(x_2 - \delta))\,\mu(dx_2) \\
&\leq \epsilon \int_{-\infty}^{\infty} \mu(dx_2) \qquad \text{because of eq. (6)} \\
&= \epsilon \qquad \text{since, } \mu(\Omega) = 1
\end{aligned}$$

Since, ϵ is arbitrary, $\lambda(V) = 0$, and consequently the equality holds, i.e. $\mathbb{E}(F(X)) = \dfrac{1}{2}$.

- **Only If part:** Suppose, $\lambda(V) = 0$. We need to show that F is continuous.

  For the sake of contradiction, assume that F is not continuous. Since by definition of distribution function, F is right continuous everywhere, hence there must exist $x_0 \in \mathbb{R}$ such that, $\lim_{x \to x_0 -} F(x) < F(x_0)$. Hence,

$$
\begin{aligned}
\lambda(V) &= \int_{\mathbb{R}^2} \mathbf{1}_V(x_1, x_2)\mu(dx_1)\mu(dx_2) \\
&\geq \int_{\mathbb{R}^2} \mathbf{1}_{\{(x_1,x_2):x_1=x_2=x_0\}}(x_1, x_2)\mu(dx_1)\mu(dx_2) \\
&\qquad\qquad \text{since, } V \subseteq \{(x_1, x_2) : x_1 = x_2 = x_0\} \\
&= \int_{\mathbb{R}^2} \mathbf{1}_{\{(x_1,x_2):x_1=x_0\}}(x_1, x_2)\mathbf{1}_{\{(x_1,x_2):x_2=x_0\}}(x_1, x_2)\mu(dx_1)\mu(dx_2) \\
&= \int_{-\infty}^{\infty} \mu(x_0)\mathbf{1}_{\{(x_1,x_2):x_2=x_0\}}(x_1, x_2)\mu(dx_2) \\
&= \mu(x_0)\int_{-\infty}^{\infty} \mathbf{1}_{\{(x_1,x_2):x_2=x_0\}}(x_1, x_2)\mu(dx_2) \\
&= (\mu(x_0))^2 \\
&= \left(F(x_0) - \lim_{x \to x_0-} F(x)\right)^2 > 0
\end{aligned}
$$

contradicting the fact that we assumed $\lambda(V) = 0$. Hence, it must be the case that F is continuous.

This completes the proof for the equality case.

■

4 Problem 4

Problem. Let, X be a random variable such that $\mathbb{E}(X^2) < \infty$. Show that the characteristic function of X is twice differentiable.

Solution:

Let, $\phi(t) = \mathbb{E}(e^{itX})$ be the characteristic function of X, where i is the complex number such that $(i^2 + 1) = 0$. Before proceeding with the main proof, we start by establishing two claims.

Claim 4.

$$\lim_{h \to 0} \frac{e^{ihx} - 1}{h} = ix$$

Proof. The proof of this claim follows simply from the Taylor series expansion of e^z for any complex number z. Hence,

$$\begin{aligned}
\lim_{h \to 0} \frac{e^{ihx} - 1}{h} &= \lim_{h \to 0} \frac{\sum_{k=0}^{\infty} \frac{(ihx)^k}{k!} - 1}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \sum_{k=1}^{\infty} \frac{(ihx)^k}{k!} \\
&= ix + \lim_{h \to 0} \sum_{k=2}^{\infty} \frac{(ix)^k h^{(k-1)}}{k!} \\
&= ix
\end{aligned}$$

since, the later power series is continuous at $h = 0$ and takes value 0 at $h = 0$. □

Claim 5. If $x \in \mathbb{R}$, then $|e^{ix} - 1| \leq |x|$.

Proof. If we consider the function $f(x) = e^{ix}$, then due to the above claim 4, we have;

$$\lim_{h \to 0} \frac{e^{i(x+h)} - e^{ix}}{h} = e^{ix} \lim_{h \to 0} \frac{e^{ih} - 1}{h} = ie^{ix}$$

Therefore, for $x \geq 0$, we have;

$$|e^{ix} - 1| = \left| \int_0^x ie^{iu} du \right| \leq \int_0^x |ie^{iu}| du = \int_0^x du = x$$

For $x < 0$, $|e^{ix} - 1| = |e^{ix}||1 - e^{-ix}| = |e^{-ix} - 1| \leq (-x)$
Together, we have $|e^{ix} - 1| \leq |x|$. □

Note that,

$$\begin{aligned}
\frac{\phi(t+h)-\phi(t)}{h} &= \frac{\mathbb{E}(e^{i(t+h)X})-\mathbb{E}(e^{itX})}{h} \\
&= \int \frac{(e^{i(t+h)x}-e^{itx})}{h} dP \\
&= \int e^{itx}\frac{(e^{ihx}-1)}{h} dP \\
&= \mathbb{E}\left(e^{itX}\frac{(e^{ihX}-1)}{h}\right)
\end{aligned}$$

Now, due to claim 5,

$$\left| e^{itX}\frac{(e^{ihX}-1)}{h}\right| = |e^{itX}| \left|\frac{(e^{ihX}-1)}{h}\right| \leq |hX| \leq |X|$$

if $|h| \leq 1$. Since, $h \to 0$, this is possible to ensure at the limiting stage. Turning to the integral, we see that $e^{itx}\frac{(e^{ihx}-1)}{h} \to ixe^{itx}$ as $h \to 0$, the integrand is bounded by $|X|$ for small h, and since $\mathbb{E}(X^2) < \infty$, $\mathbb{E}(|X|) < \infty$ as well, since by Holder's inequality, $\mathbb{E}(|X|) < \mathbb{E}(X^2)^{1/2} < \infty$. Therefore, by Dominated Convergence Theorem,

$$\phi'(t) = \lim_{h\to 0}\frac{\phi(t+h)-\phi(t)}{h} = \mathbb{E}(iXe^{itX})$$

Now, proceeding similarly with this $\phi'(t)$ as starting point, we note that,

$$\begin{aligned}
\frac{\phi'(t+h)-\phi'(t)}{h} &= \frac{\mathbb{E}(iXe^{i(t+h)X})-\mathbb{E}(iXe^{itX})}{h} \\
&= \int \frac{ix(e^{i(t+h)x}-e^{itx})}{h} dP \\
&= \int ixe^{itx}\frac{(e^{ihx}-1)}{h} dP \\
&= \mathbb{E}\left(iXe^{itX}\frac{(e^{ihX}-1)}{h}\right)
\end{aligned}$$

Again, from claim 5, we get;

$$\left| iXe^{itX}\frac{(e^{ihX}-1)}{h}\right| = |X| \left| e^{itX}\frac{(e^{ihX}-1)}{h}\right| \leq |X^2|$$

and $\mathbb{E}(X^2) < \infty$ as given in the question. Also, the limit of the integrad as $h \to 0$, yields $(ix)^2e^{itx}$. Again, by using Dominated Convergence Theorem, we get;

$$\phi''(t+h) = \lim_{h\to 0}\frac{\phi'(t+h)-\phi'(t)}{h} = \mathbb{E}\left((iX)^2e^{itX}\right)$$

This shows that the characteristic function $\phi(\cdot)$ is twice differentiable. ■

5 Problem 5

Problem. Let S_n be the group of permutations of n symbols and σ_n be a randomly chosen element. This means, all elements of S_n are equally likely. Consider random variables $X_{j,n}$ for $j = 1, 2, \dots n$ defined as,

$$X_{j,n} = \#\{i : 1 \le i < j : \sigma_n(i) > \sigma_n(j)\}$$

and $L_n = \sum_{j=1}^n X_{j,n}$. Show that,

(a) $X_{1,n}, X_{2,n}, \dots X_{n,n}$ are independent.

(b) $\mathbb{E}(X_{j,n}) = \dfrac{j-1}{2}$ and $\mathbb{V}\text{ar}(X_{j,n}) = \dfrac{j^2-1}{12}$.

(c) $\dfrac{L_n - n^2/4}{n^{3/2}/6}$ converges in distribution to $N(0,1)$.

Solution:

(a) Note that, if $X_{j,n} = k$, then it means that there are k symbols $i_1, i_2, \dots i_k$ such that their image under the permutation goes somewhere after $\sigma_n(j)$. Hence, the image of the rest of the symbols $\{1, 2, \dots (j-1)\} \setminus \{i_1, i_2, \dots i_k\}$ goes somewhere before $\sigma_n(j)$. Therefore, knowing $X_{j,n} = k$ tells us that among the symbols $\sigma_n(1), \sigma_n(2), \dots \sigma_n(j)$, the symbol $\sigma_n(j)$ appears at $(j-k)$-th position if there were arranged in increasing order.

Now, it is obvious that $X_{1,n} = 0$, by definition. Hence, $X_{1,n}$ and $X_{2,n}$ are independent. Now, knowledge of $X_{2,n}$ tells one about the relative position of $\sigma_n(2)$ among $\{\sigma_n(1), \sigma_n(2)\}$, hence once you get the position for $\sigma_n(2)$, the relative position of $\sigma_1(n)$ is automatically obtained.

In general, if one assume the knowledge of the random variables $X_{1,n}, X_{2,n}, \dots X_{j,n}$, then starting with the knowledge of $X_{j,n}$, it tells us about the relative position of $\sigma_n(j)$ among $\{\sigma_n(1), \sigma_n(2), \dots \sigma_n(j)\}$. Once we know the relative position of $\sigma_n(j)$, knowledge of $X_{j-1,n}$ tells us about the relative position of $\sigma_n(j-1)$ among the rest and so on. Therefore, knowing $(X_{1,n}, X_{2,n}, \dots X_{j,n})$ is essentially same as knowing only the relative ordering of $A = \{\sigma_n(1), \sigma_n(2), \dots \sigma_n(j)\}$.

Now, to talk about independence, we consider the sample space $\Omega = S_n$, the σ algebra $\mathcal{G} = \mathcal{P}(\Omega)$ and P as the probability measure proportional to the counting measure on Ω. Also note that, $X_{j,n} : \Omega \to \{0, 1, 2, \dots (j-1)\}$ for all $j = 1, 2, \dots n$.

Therefore,

$$
\begin{aligned}
& P(X_{1,n} = x_1, X_{2,n} = x_2, \dots X_{j,n} = x_j) \\
= \; & P\left(\{\sigma_n : \sigma_n \text{ maps the relative ordering of } 1, 2, \dots j \text{ to the one specified by } X_{\cdot,n}\}\right) \\
= \; & \frac{\#\left\{\sigma_n : \sigma_n \text{ maps } 1, 2, \dots j \text{ to a specific relative ordering}\right\}}{|S_n|} \\
= \; & \frac{n!/j!}{n!} = \frac{1}{j!}
\end{aligned}
$$

provided that, $x_1 = 0, x_2 \in \{0, 1\}, x_3 \in \{0, 1, 2\}, \dots x_j \in x_2 \in \{0, 1, \dots (j-1)\}$. Otherwise, the probability measure of the event is 0.

Hence,

$$
\begin{aligned}
P(X_{j,n} = x_j) &= \sum_{\substack{x_1 = 0, \\ 0 \le x_2 < 2, \\ 0 \le x_3 < 3, \\ \dots \\ 0 \le x_{j-1} < (j-1)}} P(X_{1,n} = x_1, X_{2,n} = x_2, \dots X_{j,n} = x_j) \\
&= \sum_{\substack{x_1 = 0, \\ 0 \le x_2 < 2, \\ 0 \le x_3 < 3, \\ \dots \\ 0 \le x_{j-1} < (j-1)}} \frac{1}{j!} \\
&= \frac{1 \times 2 \times 3 \times \dots (j-1)}{j!} \\
&= \frac{(j-1)!}{j!} = \frac{1}{j}
\end{aligned}
$$

Note that, this holds for any $j = 1, 2, \dots n$ and any $x_j \in \{0, 1, 2, \dots (j-1)\}$.

Therefore,

$$
P(X_{1,n} = x_1, \dots X_{n,n} = x_n) = \frac{1}{n!} = \prod_{j=1}^{n} \frac{1}{j} = \prod_{j=1}^{n} P(X_{j,n} = x_j) \tag{7}
$$

assuming $0 \le x_j < j$ for all $j = 1, 2, \dots n$, otherwise, both side of the above equality equals 0. Therefore, eq. (7) holds regardless of any choice of x_j's. Now, considering the whole sigma algebra as the sub-sigma algebras for each $X_{j,n}$'s, i.e. with $\mathcal{G}_j = \mathcal{G}$, and by the fact that P is simply probability measure corresponding to the counting measure, the independence of $X_{1,n}, X_{2,n}, \dots X_{n,n}$ follows from eq. (7).

(b) From part (a) above, we have obtained that,

$$
P(X_{j,n} = x_j) = \begin{cases} \frac{1}{j} & \text{if } x_j \in \{0, 1, \dots (j-1)\} \\ 0 & \text{otherwise} \end{cases}
$$

As, $X_{j,n}$ is a simple random variable, hence,

$$\begin{aligned}
\mathbb{E}(X_{j,n}) &= \int X_{j,n} dP \\
&= \sum_{k=0}^{(j-1)} kP(X_{j,n} = k) \\
&= \sum_{k=0}^{(j-1)} \frac{k}{j} \\
&= \frac{j(j-1)}{2j} = \frac{(j-1)}{2}
\end{aligned}$$

And,

$$\begin{aligned}
\mathbb{E}(X_{j,n}^2) &= \int X_{j,n}^2 dP \\
&= \sum_{k=0}^{(j-1)} k^2 P(X_{j,n} = k) \\
&= \sum_{k=0}^{(j-1)} \frac{k^2}{j} \\
&= \frac{(j-1)j(2j-1)}{6j} = \frac{(j-1)(2j-1)}{6}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{V}\text{ar}(X_{j,n}) &= \mathbb{E}(X_{j,n}^2) - \mathbb{E}(X_{j,n})^2 \\
&= \frac{(j-1)(2j-1)}{6} - \left(\frac{(j-1)}{2}\right)^2 \\
&= \frac{(j-1)(j+1)}{12} = \frac{j^2-1}{12}
\end{aligned}$$

This completes part (b).

(c) We start by considering the following random variable:

$$Y_{nj} = X_{j,n} - \mathbb{E}(X_{j,n}) = X_{j,n} - \frac{(j-1)}{2}$$

We start by noting that $\{Y_{nj}\}_{\substack{j=1 \\ n\in\mathbb{N}}}^{n}$ is a triangular array, which is independent as $X_{1,n}, X_{2,n}, \ldots X_{n,n}$ are independent random variables as shown in part (a). The row sums are denoted as $S_n = \sum_{j=1}^{n} Y_{nj}$. Note that,

$$S_n == \sum_{j=1}^{n} Y_{nj} = L_n - \sum_{j=1}^{n} \frac{(j-1)}{2} = L_n - \frac{n(n-1)}{4}$$

Clearly, due to part (b), $\mathbb{E}(Y_{nj}) = 0$ and $\mathbb{V}\mathrm{ar}(Y_{nj}) = \mathbb{V}\mathrm{ar}(X_{nj}) = \dfrac{(j^2-1)}{12}$. Then,

$$s_n^2 = \sum_{j=1}^{n} \mathbb{V}\mathrm{ar}(Y_{nj}) = \frac{1}{12}\left[\frac{n(n+1)(2n+1)}{6} - n\right] = \frac{n(2n^2+3n-5)}{72}$$

We start by showing that $\{Y_{nj}\}_{\substack{j=1 \\ n\in\mathbb{N}}}^{n}$ satisfies Lindeberg's condition.

Claim 6. The triangular array $\{Y_{nj}\}_{\substack{j=1 \\ n\in\mathbb{N}}}^{n}$ satisfies Lindeberg's condition.

Proof. First note that, for any $n \in \mathbb{N}$ and any $1 \leq j \leq n$;

$$\mathbb{E}(|Y_{nj}|^3) = \frac{1}{j}\sum_{k=0}^{(j-1)} k^3 = \frac{1}{j}\left[\frac{j(j-1)}{2}\right]^2 < \infty$$

Also, it satisfies Lyapunov's condition for $\delta = 1$.

$$\begin{aligned}
\lim_{n\to 0} \frac{1}{s_n^3} \sum_{j=1}^{n} \mathbb{E}(|Y_{nj}^3|) &= \lim_{n\to 0} \frac{1}{\left(\dfrac{n(2n^2+3n-5)}{72}\right)^{3/2}} \sum_{j=1}^{n} \frac{1}{j}\left[\frac{j(j-1)}{2}\right]^2 \\
&\leq \lim_{n\to 0} \frac{1}{\left(\dfrac{2n^3+3n^2-5n}{72}\right)^{3/2}} \times \frac{n^4}{4} \\
&= \text{constant} \times \lim_{n\to 0} \left(\frac{n^{8/3}}{2n^3+3n^2-5n}\right)^{3/2} \\
&= 0 \qquad \text{as, } \frac{8}{3} < 3
\end{aligned}$$

Therefore, by proposition, 6.4.6 of the notes, the triangular array satisfies Lindeberg's condition. $\square$

Therefore, by Lindeberg's Central Limit Theorem, $\dfrac{S_n}{s_n}$ converges in distribution to a standard normal random variable.

Now note that,

$$\frac{L_n - n^2/4}{n^{3/2}/6} = \frac{L_n - \dfrac{n(n-1)}{4}}{s_n} \times \frac{s_n}{n^{3/2}/6} + \frac{n(n-1)-n^2}{4s_n} = a_n \left[\frac{L_n - \dfrac{n(n-1)}{4}}{s_n} \right] + b_n$$

where a_n and b_n are the quantities it is replacing. Now note that, $a_n \to 1$ and $b_n \to 0$, as a sequence of real numbers, as $n \to \infty$.

Claim 7. The random variables degenerate at a_n, i.e. $\delta_{a_n} \xrightarrow{P} \delta_1$ and similarly, $\delta_{b_n} \xrightarrow{P} \delta_0$.

Proof. Assume, $\lambda(\cdot)$ is the usual Lebesgue measure and $\delta_x : (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

$$\begin{aligned}
P(|\delta_{a_n} - \delta_1| > \epsilon) &\leq P(\delta_{a_n} \neq \delta_1) \\
&= \int \mathbf{1}_{\{x:\delta_{a_n}(x) \neq \delta_1(x)\}}(x) dP \\
&= \lambda(|a_n - 1|) \qquad \text{as } \delta_{a_n}(x) \neq \delta_1(x) \text{ iff } x \text{ lies between } a_n \text{ and } 1 \\
&\to 0 \qquad \text{as, } a_n \to 1
\end{aligned}$$

□

Therefore, by claim 7, it follows that $\delta_{a_n} \xrightarrow{P} \delta_1$ and $\delta_{b_n} \xrightarrow{P} \delta_0$. Hence, by applying Slutsky's theorem, we note that $\dfrac{L_n - n^2/4}{n^{3/2}/6}$ converges in distribution to the same asymptotic distribution of $\dfrac{S_n}{s_n} = \dfrac{L_n - \dfrac{n(n-1)}{4}}{s_n}$ which due to Central Limit theorem is already establishing as a standard normal random variable.

■

Thank you