INDIAN STATISTICAL INSTITUTE

Semestral Assignment : Second Semester 2019-20

Course : B Stat

Subject : Measure Theoretic Probability

Date : 2nd July 2020 | Marks :60 | Duration :Submit by 11 a.m 15th July 2020

Before using any result clearly state that.

1. Let $\mathcal{C} = \{A \subseteq \mathbb{R} : A$ is countable or $\mathbb{R} \setminus A$ is countable$\}$ be the countable-cocountable σ-field on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that $\forall B \in \mathbb{R}, f^{-1}(B) \in \mathcal{C}$ iff there exists a countable set $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ such that $f(x) = c, \forall x \notin A$. **[10]**

2. Let $a_0, a_n, b_n, n \geq 1$ be real numbers such that the series
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$
converges absolutely on a set of positive Lebesgue measure. In other words Lebesgue measure of $E = \{x \in [-\pi, \pi] : \sum_{n=1}^{\infty} |a_n||\cos nx| + \sum_{n=1}^{\infty} |b_n||\sin nx| < \infty\}$ is positive. Show that $\sum_{n=1}^{\infty}(|a_n| + |b_n|) < \infty$. (You would need the following fact: If $f : [-\pi, \pi] \to \mathbb{R}$ is a bounded measurable function then both $\int_{-\pi}^{\pi} f(x) \cos nx d\lambda(x)$ and $\int_{-\pi}^{\pi} f(x) \sin nx d\lambda(x)$ converge to zero as n goes to infinity. **[15]**

3. Let X be a random variable having distribution function F. Show that $\mathbb{E}(F(X) \geq 1/2$ with equality iff F is continuous. **[6+2+2=10]**

4. Let X be a random variable such that $\mathbb{E}(X^2) < \infty$. Show that the characteristic function of X is twice differentiable. **[10]**

5. Let S_n be the group of permutations of n symbols and σ_n be a randomly chosen element. This means all elements of S_n are equally likely. Consider random variables $X_{j,n}$ for $j = 1, \ldots, n$ defined as $X_{j,n} = \#\{i : 1 \leq i < j : \sigma_n(i) > \sigma_n(j)\}$ and $L_n = \sum_{j=1}^{n} X_{j,n}$. Show that

   (a) $X_{1,n}, \ldots, X_{n,n}$ are independent.

   (b) $\mathbb{E}(X_{j,n}) = \frac{j-1}{2}, \mathbb{V}(X_{j,n}) = \frac{j^2-1}{12}$.

   (c) $\frac{L_n - n^2/4}{n^{3/2}/6}$ converges in distribution to $N(0, 1)$. **[5+5+5=15]**