rSVDdpd

A Robust Singular Value Decomposition Technique

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Background Modelling Problem

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Applications ranging security, defence, object tracking, motion detection, video filters, etc.

Background Modelling as Low Rank Decomposition

$h \times w \times t$

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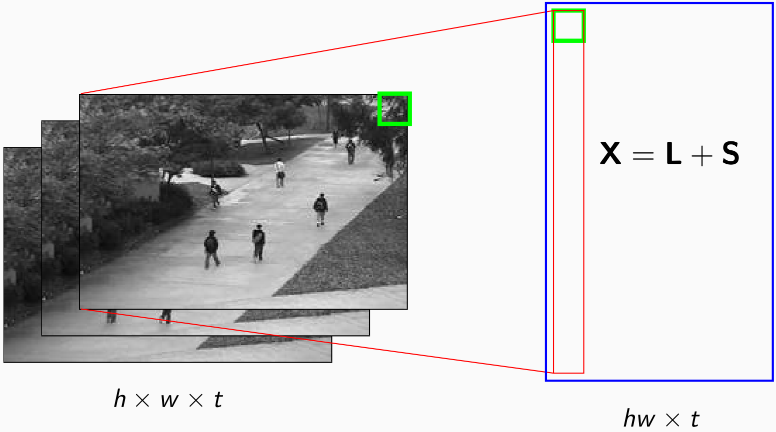

Singular Value Decomposition (SVD)

Definition
For a $n \times p$ matrix $\mathbf{M}$, its singular value decomposition is defined by the factorization

$$\mathbf{M} = \mathbf{U}_{n \times r} \mathbf{D}_{r \times r} (\mathbf{V}_{p \times r})^{\intercal}$$

where,

1. $\mathbf{U}, \mathbf{V}$ are unitary matrices (left and right singular vectors).
2. $\mathbf{D}$ is a diagonal matrix comprising singular values, in decreasing order of magnitude.
3. r is the approximate rank.

Robust Background Modelling

- Singular Value Decomposition (SVD) can be used to estimate **L**.
- Advantages compared to Deep learning.
    1. Unsupervised. Need no training data.
    2. Low hardware requirements.
    3. Generalizable.
    4. Better theoretical guarantees.

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  3. Camera tampering

SVD as alternating regression

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Fixing j yields,

$$
\begin{bmatrix} X_{1j} \\ X_{2j} \\ \vdots \\ X_{nj} \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{r1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{rn} \end{bmatrix}_{n \times r} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix} + \begin{bmatrix} \epsilon_{1j} \\ \epsilon_{2j} \\ \vdots \\ \epsilon_{nj} \end{bmatrix}
$$

Density Power Divergence

Definition
Given two density functions f and g, Basu et al. (1998) defines the density power divergence between them as

$$d_\alpha(g, f) = \begin{cases} \int f^{1+\alpha} - \left(1 + \frac{1}{\alpha}\right) \int f^\alpha g + \frac{1}{\alpha} \int g^{1+\alpha}, & \alpha > 0 \\ \int f \log(f/g) & \alpha = 0 \end{cases}$$

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Definition
Given an independent sample of observations $X_1, \ldots X_n$ modelled by non-homogeneous family of distributions $\{f_{i,\theta} : \theta \in \Theta\}$, the MDPDE (Ghosh and Basu) is defined as

$$\widehat{\theta}_\alpha = \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} \left[\int f_{i,\theta}^{1+\alpha} - \left(1 + \alpha^{-1}\right) f_{i,\theta}^{\alpha}(X_i) \right]$$

Robust SVD using DPD

For background modelling problem, usually $r = 1$, so we have

$$X_{ij} = a_i b_j + \epsilon_{ij}$$

So we choose $f_{ij,\theta} \equiv N(a_i b_j, \sigma^2)$.

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b_j^{(t+1)} &= \sum_i (x_{ij}/a_i^{(t+1)})(a_i^{(t+1)})^2 w_{ij}^{(t)} / \sum_i (a_i^{(t+1)})^2 w_{ij}^{(t)},\ j = 1, \dots p; \\
(\sigma^{(t+1)})^2 &= \sum_i \sum_j (x_{ij} - a_i^{(t+1)} b_j^{(t+1)})^2 w_{ij}^{(t)} / \left(\sum_i \sum_j w_{ij}^{(t)} - \frac{\alpha}{(1+\alpha)^{3/2}}\right).
\end{aligned}$$

Features and Theoretical Results (Roy, Basu and Ghosh, 2021)

1. Massively parallelizable iteration rules.
2. Converges to a global optimum of the restricted DPD objective function.
3. Estimator has equivariance property under general g.
4. Asymptotically consistent under reasonable assumptions:
    4.1 Thrice differentiability of density g and exchange of derivative under integral sign.
    4.2 Lindeberg-like condition on third derivative of density function g.
    4.3 Both $n = hw, p = t \to \infty$ with $n/p \to c$ for some $0 < c < \infty$.
    4.4 $\sigma = O((np)^{-1/2})$, i.e., $\sigma^2 \to 0$ as $n, p \to \infty$.
5. When $p = t \to \infty$, asymptotic breakdown point is $\alpha/(1+\alpha)$, free of $n = hw$. Very robust and attractive for large resolution images (Roy, Ghosh and Basu, 2023).

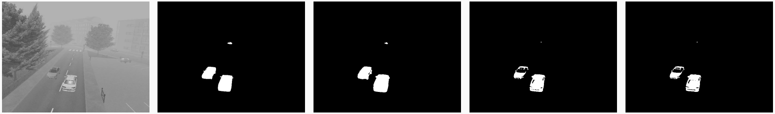
Benchmarks

- We consider competing algorithms like GoDec, GRASTA, Variational Bayes, Outlier Pursuit, PCP, Sparse PCP, ADMM, ALM, etc.
- rSVDdpd is about 50x faster than PCP, and 5-10x faster than all other algorithms. Following are comparisons on some videos in **Background Model Challenger** Dataset.

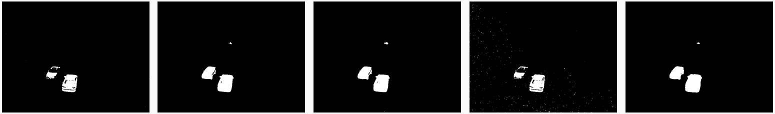
Dataset	Metric	rSVDdpd	GoDec	GRASTA	VB	PCP
BMC (street)	F1	0.754	0.672	0.528	0.728	0.739
BMC (rotary)	F1	0.757	0.723	0.624	0.614	0.726

- rSVDdpd has applications beyonds background modelling, can be used as a general purpose robust SVD.

Background Model Challenger Dataset



(a) (b) (c) (d) (e)



(f) (g) (h) (i) (j)

One frame of the street video (a), ground truth (b) and estimated foreground mask from exact RPCA (c), inexact ALM (d), SRPCP (e), Variational Bayes (f), Outlier Pursuit (g), GoDec (h), GRASTA (i) and rSVDdpd (j) algorithms. Enlarged better quality images are available online at https://subroy13.github.io/rsvddpd-home/.

University of Houston Camera Tampering Dataset

Truth OP ADMM GoDec rSVDdpd

References

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- Roy, Subhrajyoty, et al. "Breakdown Point Analysis of the Minimum S-Divergence Estimator." arXiv preprint arXiv:2304.07466 (2023).
- R package `rsvddpd` `https://cran.r-project.org/web/packages/rsvddpd`.

Questions?