Generalized Alpha-Beta Divergence, its Properties and Associated Entropy

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Statistical Divergence

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You can extend for **sub-densities**

$$\mathcal{F}^* := \left\{ f : f \geq 0, \int f d\mu \leq 1 \right\}.$$

Minimum divergence estimation

Given a statistical model $\{f_\theta : \theta \in \Theta\}$ and a divergence measure $d(\cdot, \cdot)$ between distributions, the **minimum divergence estimator** (MDE) of θ minimizes the divergence between a "proxy" data density g and the model density f_θ:

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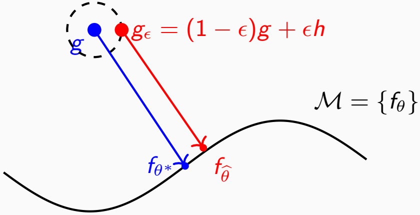

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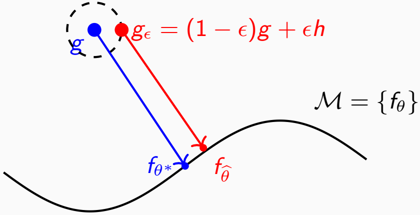

The choice of divergence $d(\cdot, \cdot)$ controls the curvature of these geodesic lines.

Useful divergences for Robust Estimation

Several divergences have been proposed to balance **efficiency** and **robustness**:

- **Density Power Divergence (DPD)** [1]:

$$d_{DPD}^{\alpha}(g,f) = \int f^{1+\alpha} - \left(1+\frac{1}{\alpha}\right)\int g f^{\alpha} + \frac{1}{\alpha}\int g^{1+\alpha}.$$

- **Log-Density Power Divergence (LDPD)** [2]:

$$d_{\mathrm{LDPD}}^{\alpha}(g,f) = \log\int f^{1+\alpha} - \left(1+\frac{1}{\alpha}\right)\log\int g f^{\alpha} + \frac{1}{\alpha}\log\int g^{1+\alpha}.$$

- **Bridge Divergence** [3]

$$\begin{aligned} d_{\mathrm{BD}}^{\alpha,c_1,c_2}(g,f) = \log\left(c_1 + c_2\int f^{1+\alpha}\right) - \left(1+\frac{1}{\alpha}\right)\log\left(c_1 + c_2\int g f^{\alpha}\right) \\ + \frac{1}{\alpha}\log\left(c_1 + c_2\int g^{1+\alpha}\right). \end{aligned}$$

Useful divergences for Robust Estimation

- **S-divergence family** [4]:

$$d_{SD}^{\alpha,\lambda}(g,f) = \frac{1}{A}\int f^{1+\alpha} - \frac{1+\alpha}{AB}\int f^B g^A + \frac{1}{B}\int g^{1+\alpha},$$

where $A = 1 + \lambda(1-\alpha)$, $B = 1 + \alpha - \lambda(1-\alpha)$.

- **Logarithmic S-divergence family** [5]:

$$d_{LSD}^{\alpha,\lambda}(g,f) = \frac{1}{A}\log\left(\int f^{1+\alpha}\right) - \frac{1+\alpha}{AB}\log\left(\int f^B g^A\right) + \frac{1}{B}\log\left(\int g^{1+\alpha}\right),$$

where $A = 1 + \lambda(1-\alpha)$, $B = 1 + \alpha - \lambda(1-\alpha)$.

Generalized Alpha-Beta Divergence

We define the generalised alpha-beta divergence (GABD) as

$$d_{GAB}^{(\alpha,\beta),\psi}(f,g) = \frac{1}{\beta(\alpha+\beta)}\psi\left(\int f^{\alpha+\beta}\right) - \frac{1}{\alpha\beta}\psi\left(\int f^{\alpha}g^{\beta}\right) + \frac{1}{\alpha(\alpha+\beta)}\psi\left(\int g^{\alpha+\beta}\right),$$

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- $\psi(x) = x$, gives you super divergence.
- $\psi(x) = x, \beta = 1$ gives you density power divergence.
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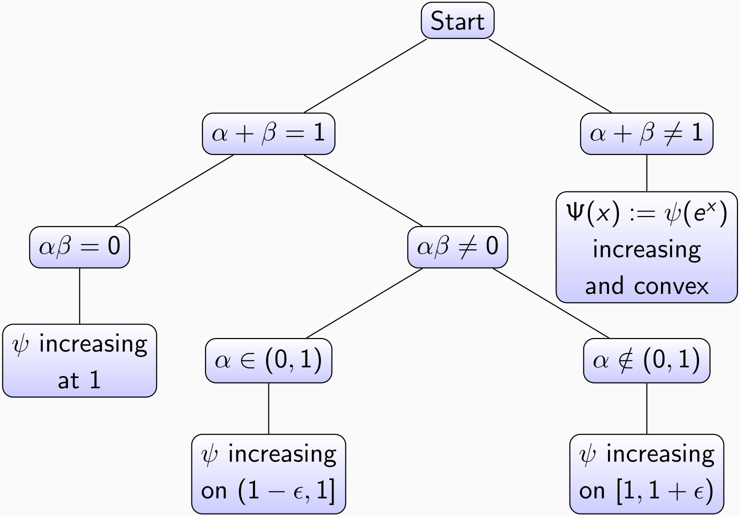
Does every $\psi(\cdot)$ make it a divergence?

Key Result: Necessary and Sufficient Conditions



Recipe for new divergences

1. Start with any nonnegative function f.
2. Define, $F(x) = \int_{-\infty}^{x} f(t)dt$. If f is density, F is the cdf.
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Some examples are:

- LSD when $f = \delta(0)$.
- (ϕ, γ)-divergence ($x \leq 1$) when f is the density of $\exp(\gamma)$.
- Bridge divergence when f is $\text{Logistic}(\log(c_1/c_2), 1)$.

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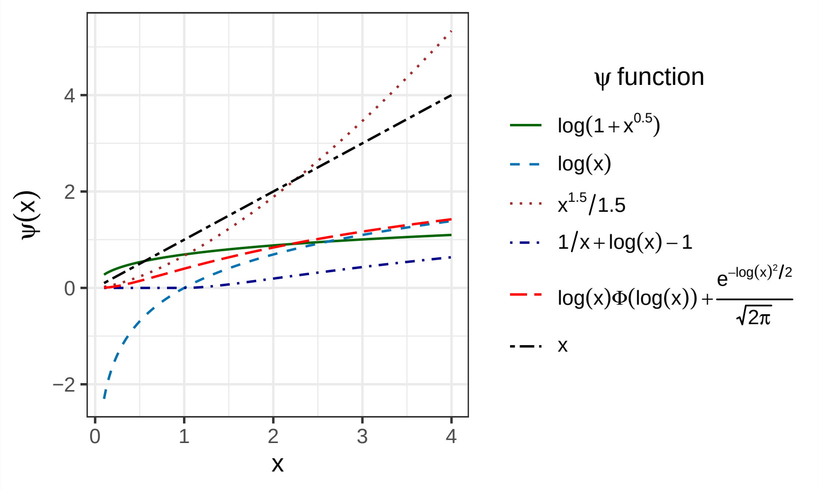

Special cases

- In location models, all MGABDE are equivalent, i.e., $d_{GAB}^{(\alpha,\beta),\psi_1}(f,g) = h(d_{GAB}^{(\alpha,\beta),\psi_2}(f,g))$ for some monotonic function h.

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- For a density f, let $f^{[\alpha]}(x) = f^\alpha(x)/\int f^\alpha(u)du$. When $\beta = 0, \alpha \neq 0$,

$$d_{GAB}^{(\alpha,0),\psi}(f,g) = \frac{\psi'(\int f^\alpha)\int f^\alpha}{\alpha^2}\left(d_{KL}(f^{[\alpha]}, g^{[\alpha]}) + \ln\left(\frac{\int f^\alpha}{\int g^\alpha}\right)\right) \\ - \psi\left(\int f^\alpha\right) + \psi\left(\int g^\alpha\right)$$

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- When $\alpha + \beta = 1, \alpha \notin \{0, 1\}$,

$$d_{GAB}^{(\alpha,1-\alpha),\psi} = \frac{1}{\alpha(1-\alpha)}\left[\psi(1) - \psi\left(d_{PD,\alpha}(f,g) + \text{constant}\right)\right]$$

Properties

- **Duality**: $d_{GAB}^{(\alpha,\beta),\psi}(f,g) = d_{GAB}^{(\beta,\alpha),\psi}(g,f)$.
- **Scaling**: $d_{GAB}^{(\alpha,\beta),\psi}(cf,cg) = d_{GAB}^{(\alpha,\beta),\psi(c^{\alpha+\beta}\cdot)}(f,g)$.
- **Zooming**: $d_{GAB}^{(\alpha,\beta),\psi}(f^{\tau},g^{\tau}) = \tau^2 d_{GAB}^{(\tau\alpha,\tau\beta),\psi}(f,g)$.

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Theorem (Approximate Pythagorean Identity)

1. *Let ψ be continuously differentiable.*
2. $\alpha\beta(\alpha+\beta) \neq 0$.
3. $g_\epsilon^\alpha = (1-\epsilon)g^\alpha + \epsilon\delta^\alpha$.

Then,

$$d_{GAB}^{(\alpha,\beta),\psi}(g_\epsilon,f) = d_{GAB}^{(\alpha,\beta),\psi}(g_\epsilon,g) + d_{GAB}^{(\alpha,\beta),\psi}(g,f) + O_\psi(\epsilon) + O_\psi(\ln(1-\epsilon))$$

Generalized Alpha-Beta Entropy

Definition (GAB Entropy)

$$\varepsilon_{GAB}^{(\alpha,\beta),\psi}(f) = -\frac{1}{\beta}\left[\frac{\psi(\int f^{\alpha+\beta})}{\alpha+\beta} - \frac{\psi(\int f^{\alpha})}{\alpha}\right].$$

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1. $\psi(x) = \log(x)$ yields constant $\times$ logarithmic norm entropy.
2. If $\alpha \neq 0, \beta = 0$, $\varepsilon_{GAB}^{(\alpha,\beta),\psi}(f) = c_1 + c_2 H_\alpha(f)$ (Affine transformation of Réyni entropy).

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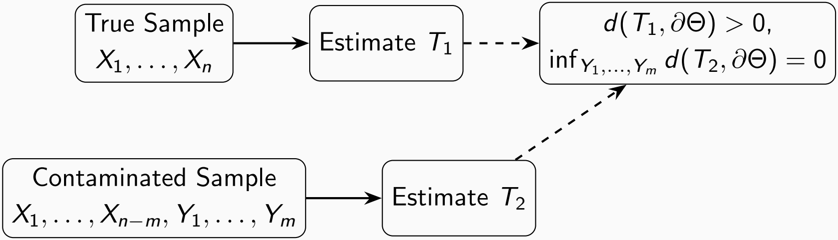
Theorem (Concavity)

$\varepsilon_{GAB}^{(\alpha,\beta),\psi}(f)$ is concave if any of the following holds:

1. *$f \mapsto \ln(\int f^\alpha)$ is convex, and either $\beta > 0, \alpha \in (-\beta, 0)$ or $\alpha > 0, \beta < -\alpha$.*
2. *$\psi(\cdot)$ is convex, and either $\alpha < 0, \beta > (1-\alpha)$ or $\alpha > 1, \beta < -\alpha$.*

Asymptotic Breakdown Point

Breakdown point of an estimator T is the maximum amount of outliers it can tolerate before giving an egregiously bad estimate.



Robustness of Generalized Alpha-Beta Divergence

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$$T_{MGABD}(G) = \arg\min_{\theta \in \Theta} d_{GAB}^{(\alpha,\beta),\psi}(f_\theta, g)$$

2. Asymptotic BP of MGABD functional at location model is $1/2$.

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2. Asymptotic BP of MGABD functional at location model is $1/2$.
3. Under suitable assumptions, asymptotic breakdown point is atleast

$$\min\left\{\liminf_{d(\theta_m,\partial\Theta)\to 0}\left[\frac{\psi^{-1}\left(\frac{\alpha}{\alpha+\beta}\psi(\int f_{\theta_m}^{\alpha+\beta})\right)}{\int f_{\theta_m}^{\alpha+\beta}}\right]^{1/\beta}, 1-\left[\frac{\psi^{-1}\left(\frac{\alpha}{\alpha+\beta}\psi(\int f_{\theta g}^{\alpha+\beta})\right)}{\int f_{\theta g}^{\alpha} g^{\beta}}\right]^{1/\beta}\right\}$$

In many cases with $\psi(x) = x$, this is $\left(\frac{\alpha}{\alpha+\beta}\right)^{1/\beta}$.

Asymptotic Distribution of MGABDE

1. When $\beta = 1$, the MGABDE is an M-estimator with data-dependent $\psi_M(\cdot)$ or $\rho_M(\cdot)$ functions.
2. As a result, typical consistency and asymptotic normality holds as

$$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{d} N(0, V)$$

$$V = \frac{N_{1+2\alpha} - M_{1+\alpha}^2}{\left| N_{1+\alpha} + M_{1+\alpha}^2 \frac{\psi''(L_{1+\alpha})}{\psi'(L_{1+\alpha})} \right|^2}$$

   where

$$L_{1+\alpha} = \int f_{\theta^g}^{1+\alpha},\ M_{1+\alpha} = \int f_{\theta^g}^{1+\alpha} u_{\theta^g},\ N_{1+\alpha} = \int f_{\theta^g}^{1+\alpha} u_{\theta^g} u_{\theta^g}^\intercal$$

3. This means, given a model family f_θ, possible to find $\psi(\cdot)$ function that achieves optimal efficiency.

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Thank you!

Questions?