Euler's Formula - History, Derivation, and Applications. An Exploration of Complex Analysis.

Abstract:

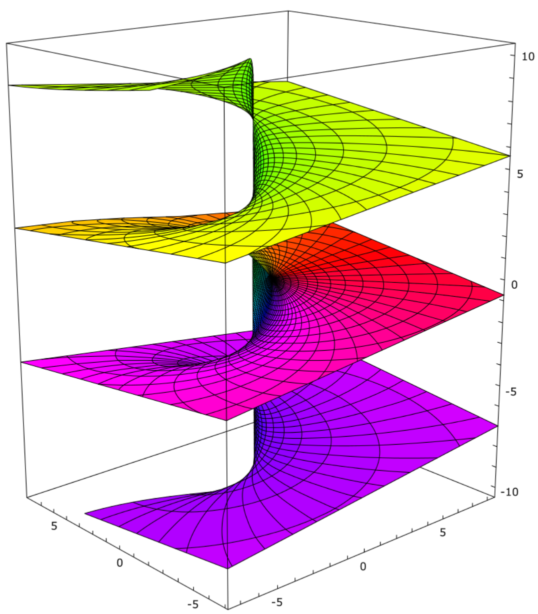
Euler's Formula, given by $e^{i\theta} = cos(\theta) + isin(\theta)$ has an interesting history and many applications and theorems that follow from it. Leonard Euler formally published the formula we know today in 1748. There have been many derivations since the initial publishing. The mathematical implications of this formula are far reaching, forever changing our understanding of complex analysis, as well as the field of trigonometry by allowing complex relations that allow neat and simple derivations of many previously tedious formulas. Very difficult integrals are now able to be solved in efficient and beautiful ways. In application, much of electrical engineering, specifically the study of alternating current would be neigh impossible if not for our new understanding of complex sinusoidals being equivalent to complex exponentials.

History:

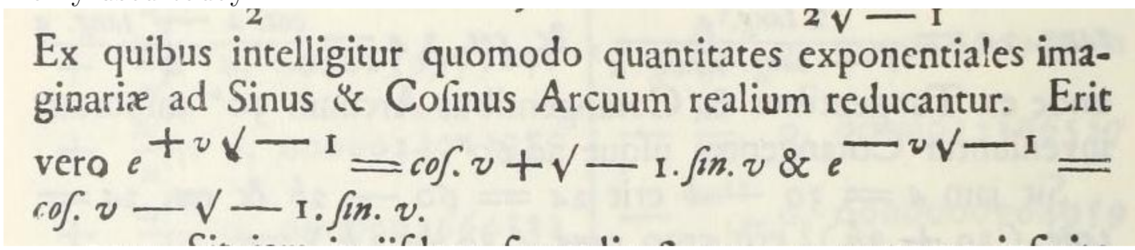
Euler's formula is the product of the work of a few mathematicians. The first to contribute to this was Roger Cotes. In 1714, in a treatise titled *Harmonia Mensurarum* (harmony of measures), a logarithmic version of the formula was written as $i\theta = ln(cos(\theta) + isin(\theta))$. (5)
Cotes died at age 34, before he could expand on the theorem. His intent was to study logarithm and inverse tangent functions and how they related to hyperbolas and circles. He nearly made the important connection between complex functions and circles.

There is an issue with this statement. Due to the nature of complex logarithms, for any input in $\mathbb{C}$, we will have an infinite number of solutions, each with a difference of $2\pi i$. This leads to the statement not being a function.

Below is a graph of the exponential logarithm (2). We can see the problem graphically. At this time in history however, there was not yet a visual representation of the complex plane.



There are remedies to make this formula more useful, such as restricting the output. Ultimately however, in the 1748 work *Inductio in Analysin Infinitorum* Leonard Euler published a large exploration of exponential functions, their properties and gave us the complex exponential formula that is commonly used today.



Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Coſinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = co\!f.\, v + \sqrt{-1}.\, \mathit{ſin}.\, v$ & $e^{-v\sqrt{-1}} = co\!f.\, v - \sqrt{-1}.\, \mathit{ſin}.\, v$.

Above is the famous statement itself from book 1, chapter 8, page 104. (3)

This theorem allowed for the formal connection of complex functions and complex trigonometry. Further, this allowed the connection to be made between circles and complex functions. Further, it led to the derivation of important formulas not explicitly used in complex analysis. (4)

Today, the formula is key in our general understanding of how to work with complex numbers and functions. Fields in which this formula is commonly used are electrical engineering, namely signal processing, control theory, a field important to engineering in general, the physics of electromagnetism, fluid dynamics, and quantum mechanics, vibration analysis, and cartography .

Derivations: (7)

In the *modern* form, the formula, where $\theta \in \mathbb{R}$, and i is the imaginary number, the formula looks like:

$e^{i\theta} = cos(\theta) + isin(\theta)$.

The formula has multiple derivations.

- Series expansion (this is the method Euler used):

  Proof. Note $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

  Let $x \in \mathbb{R}$

  $e^{xi} = \sum_{n=0}^{\infty} \frac{xi^n}{n!}$

  $e^{xi} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!}...$

  $e^{xi} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + ...$

  $e^{xi} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...)$

  Note $sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...$

  and $cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...$

  $\therefore e^{ix} = cos(x) + isin(x)$ $\square$

- Integration:
  Note: the usage of differentials as mathematical objects unto themselves is not accepted as valid, and thus this proof is debatably not mathematically sound, however is an interesting proof to consider.

  Proof.
  Let $y = cos(x) + isin(x)$

$dy = (-cos(x) + icos(x))dx$

$dy = (icos(x) - sin(x))dx$

$dy = (iy)dx$

$\frac{dy}{y} = (i)dx$

$\int \frac{dy}{y} = \int (i)dx$

$ln(y) = ix$

$y = e^{ix}$

$\therefore cos(x) + isin(x) = e^{ix}$ $\square$

- Differentiation:

  Proof.
  Consider the complex function $f(\theta)$where $\theta \in \mathbb{R}$

  $f(\theta) = \frac{cos\theta + isin\theta}{e^{i\theta}}$

  $f(\theta) = e^{-i\theta}(cos\theta + isin\theta)$

  Now differentiate using the product rule

  $f'(\theta) = e^{-i\theta}(icos\theta - sin\theta) - ie^{-i\theta}(cos\theta + isin\theta)$

  which simplifies to 0.

  the rate of change is zero meaning the function is equal to a constant

  To find what this constant is equal to, we need to plug in any value. For ease, plug in $\theta = 0$.

  $f(0) = 1$

$\Rightarrow 1 = e^{-i\theta}(cos\theta + isin\theta)$

By algebra, we get $e^{i\theta} = cos\theta + isin\theta$ $\square$

- Polar coordinates

Complex numbers can be written in polar coordinates, and thus we can write the formula

$e^{ix} = r(cos\theta + isin\theta)$
Where $x \in \mathbb{R}$, $0 \leq \theta < \pi$, $r > 0$.

Proof. Consider $e^{ix} = r(cos\theta + isin\theta)$

Set $x = 0$ which implies initial conditions $r(0) = 1, \theta(0) = 0$.
Take the derivative with respect to x
$ie^{ix} = \frac{dr}{dx}(cos\theta + isin\theta) + r(-sin\theta + icos\theta)\frac{d\theta}{dx}$

Substitute our initial equation to remove exponential.
$ir(cos\theta + isin\theta) = \frac{dr}{dx}(cos\theta + isin\theta) + r(-sin\theta + icos\theta)\frac{d\theta}{dx}$

$r(icos\theta - sin\theta) = \frac{dr}{dx}(cos\theta + isin\theta) + r(-sin\theta + icos\theta)\frac{d\theta}{dx}$

Real and imaginary parts are equal.
$\Rightarrow ircos\theta = isin\theta\frac{dr}{dx} + ircos\theta\frac{d\theta}{dx}$ (1)

$\Rightarrow -rsin\theta = cos\theta\frac{dr}{dx} - rsin\theta\frac{d\theta}{dx}$ (2)

Hence there is a system of equations. Let:
$\frac{dr}{dx} = \alpha$

$\frac{d\theta}{dx} = \beta$

Multiply (1) by $cos\theta$ and (2) by $sin\theta$.

$\Rightarrow rcos^2\theta = (sin\theta cos\theta)\alpha + (rcos^2\theta)\beta$ (3)

$$\Rightarrow -rsin^2\theta = (sin\theta cos\theta)\alpha - (rsin^2\theta)\beta \tag{4}$$

Subtract: $(3) - (4)$.

$r(cos^2\theta + sin^2\theta) = r(cos^2\theta + sin^2\theta)\beta$

Note: $r = r\beta, r > 0$
$\Rightarrow \beta = \frac{d\theta}{dx} = 1.$

Plug back into (1) and (2)

$0 = (sin\theta)\alpha$
$0 = (cos\theta)\alpha$

$\Rightarrow \alpha = \frac{dr}{dx} = 0.$

$r = C.$
$\theta = x + C.$

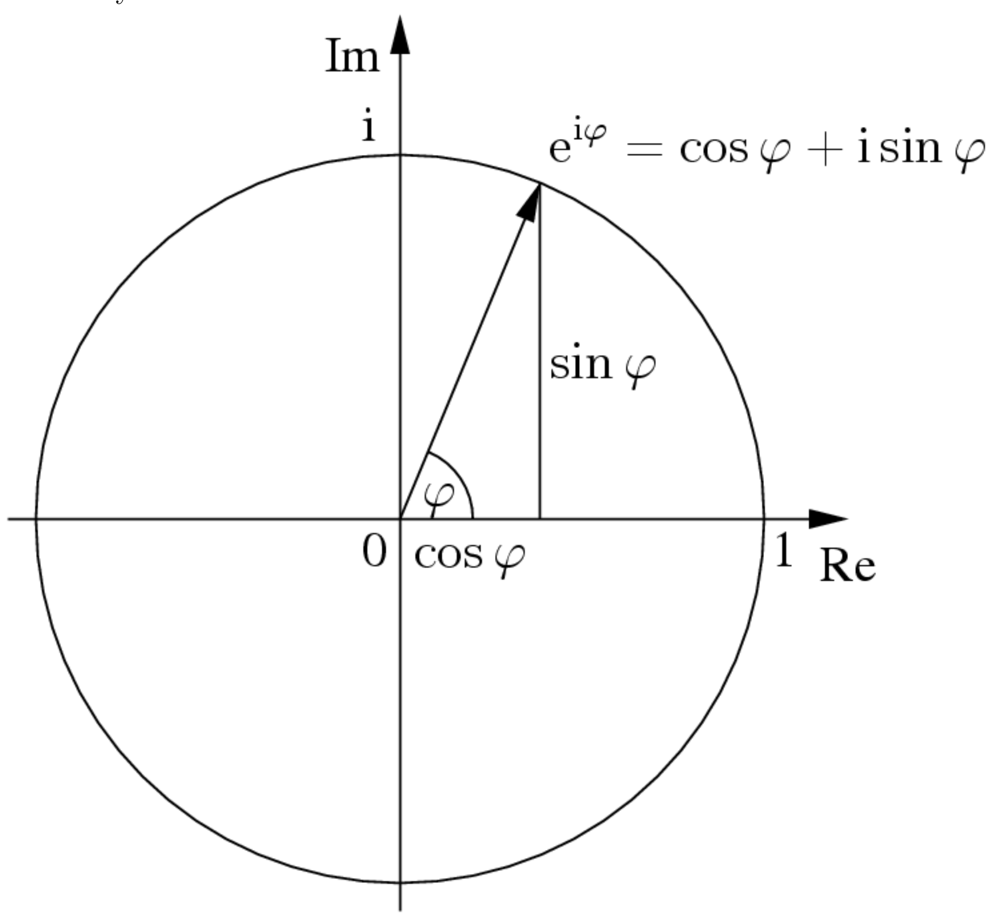
Initial conditions imply $r(0) = 1, r = 1.$
$\theta(0) = 0, C = 0.$
$\theta = x.$

$\therefore e^{ix} = cos(x) + isin(x).$ $\square$

Applications:

There are numerous applications for Euler's formula, both in the domain of pure math, through complex analysis and trigonometry, as well as in scientific fields that use mathematics.

- **Development of complex analysis**
  Complex analysis is a term referring to the study of complex numbers and functions. This is a large field of mathematics, however without the Euler formula, we would not have the connection to complex numbers through the lens of trigonometry. The groundbreaking discovery of this connection allowed us to visualize the complex plane in a more clear way.



  Beyond this simple yet important connection, this allowed further development of complex analysis.

Complex Algebra

This Euler formula allows for complex numbers to be rewritten into exponential form, and then trigonometric form. Effectively, we now know how to connect any complex number to polar coordinates. This can turn tedious and difficult problems into very manageable ones.

Given any complex number $z = x + iy$ we can find the distance of this number from the origin, calling this distance r, find the angle, θ between 0 and our point on the complex plane, z, and then write $z = re^{i\theta}$ (10)

Consider a case we are asked to solve

$(1 + \sqrt{3}i)^{-10}$

This is a tedious and time consuming calculation, but we can translate this.

$(1 + \sqrt{3}i) = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2})$

$\Rightarrow 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = re^{i\theta}$

$\theta = \frac{\pi}{3}$

$r = 2$

$(1 + \sqrt{3}i)^{-10} = (2(e^{i\frac{\pi}{3}}))^{-10}$

$(1 + \sqrt{3}i)^{-10} = (2)^{-10}(e^{i\frac{-10\pi}{3}}))$

$(1 + \sqrt{3}i)^{-10} = (2)^{-10}(e^{i\frac{-2\pi}{3}}))$

$(1 + \sqrt{3}i)^{-10} = 2^{-10}(-\frac{1}{2} + i\frac{\sqrt{3}}{2})$

$(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + i\sqrt{3})$

- **Trigonometric Identites**
  Trigonometry has been studied for most of recorded history by the Egyptians and Babylonians up until the present. However before Euler formalized his theorem, many theorems that are commonly used today were either not yet derived, proved, or their proofs were extremely tedious.
  The advent of Euler's formla allowed for nearly trivial proofs for some extremely key trigonometric identities used in nearly all branches of science, engineering, and math in some form.

  A few of these identities are:

  Theorem - De Moine's Formula:
  $(cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta)$

  Now that all complex trig functions can be mapped to exponentials, De Moine's Formula now has a very simple proof. Prior to the Euler's Formula, it was proved through mathematical induction and a number of trig identities. The proof below is the one that results from Euler's formula.

  Proof. $(cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta)$

  The proof goes as follows:

  $(cos\theta + isin\theta)^n = (e^{i\theta})^n$

  $(cos\theta + isin\theta)^n = (e^{i(n\theta)})$

  $\therefore (cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta)$ $\square$

Theorem - Double Angle:
$cos2\theta = cos^2\theta - sin^2\theta$
$sin2\theta = 2sin\theta cos\theta$

From De Moine's Formula, double angle identities become simple to derive.
Proof. $(cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta)$

Let n = 2.

$(cos\theta + isin\theta)^2 = cos2\theta + icos2\theta$

$= cos^2\theta - sin^2\theta + i2sin\theta cos\theta$

Real and Imaginary parts are equal.
$\therefore$
$cos2\theta = cos^2\theta - sin^2\theta$
$sin2\theta = 2sin\theta cos\theta$ $\square$

Note that this process can be used for higher values of n. If we use the binomial expansion formula, this can be a very efficient even for relatively large values.

Theorem - Angle Addition:
$cos(\theta_1 + \theta_2) = cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2$
$sin(\theta_1 + \theta_2) = sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2$

Proof. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

$\Rightarrow (cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2) = cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)$

$= cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2 + i(sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2)$

$\therefore$
$\Rightarrow cos(\theta_1 + \theta_2) = cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2$
$\Rightarrow sin(\theta_1 + \theta_2) = sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2$ $\square$

Note also that using very simple algebra we can write sine and cosine in a complex exponential very similar to the formulas of real exponentials for hyperbolic sine and hyperbolic cosine.

Theorem - Exponential form of Sine and Cosine:
$cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$
$sin(\theta) = \frac{e^{i\theta}-e^{i\theta}}{2i}$

Proof.
$e^{ix} = cos(x) + isin(x)$

$e^{-ix} = cos(-x) + isin(-x) = cos(x) - isin(x)$

$\Rightarrow cos(x) = e^{ix} - isin(x) = e^{-ix} + isin(x)$

$2cos(x) = e^{ix} - isin(x) + e^{-ix} + isin(x)$

$\therefore cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ $\square$

The same process can also yield a complex exponential version of sine:
$sin(x) = \frac{e^{ix}-e^{ix}}{2i}$

Theorem - Lagrange Trig Identity:
$1 + cos\theta + cos2\theta + ... + cosn\theta = \frac{1}{2} + \frac{sin\frac{(2n+1)\theta}{2}}{2sin\frac{\theta}{2}}$

Proof. Let $S = 1 + z + z^2 + ... + z^n$
$\Rightarrow zS = z + z^2 + z^3 + ... + z^{n+1}$
$S - zS = 1 + z + z^2 + ... + z^n - (z + z^2 + ... + z^n + z^{n+1})$
$S - zS = 1 - z^{n+1}$
$S(1 - z) = 1 - z^{n+1}$
$\Rightarrow S = \frac{1-z^{n+1}}{1-z}$

Let $z = e^{i\theta}$
$1 + e^{i\theta} + e^{2i\theta} + ... + e^{ni\theta} = \frac{1-e^{(n+1)\theta}}{1-e^{i\theta}}$
$= \frac{1-e^{(n+1)\theta}}{-e^{i\theta/2}(e^{i\theta/2}-e^{-i\theta/2})}$
$= \frac{-e^{-i\theta/2}(1-e^{(n+1)\theta})}{2isin(\theta/2)}$

$= \frac{i(e^{-i\theta/2} - e^{(n+\frac{1}{2})i\theta})}{2sin(\theta/2)}$
$= \frac{1}{2} + \frac{sin((n+\frac{1}{2})\theta)}{2sin(\theta/2)} + i\frac{cos(\theta/2) - cos((n+\frac{1}{2})\theta)}{2sin(\theta/2)}$
by equating real and imaginary parts:
$\therefore$
$1 + cos\theta + cos2\theta + ... + cosn\theta = \frac{1}{2} + \frac{sin\frac{(2n+1)\theta}{2}}{2sin\frac{\theta}{2}}$ $\square$

- **Integration**
  Using the ability to write sine and cosine in complex exponential form, we are now more capable to solve extremely difficult integrals with relative ease.

  Consider the integral $\int cos^2(x)dx$.
  The techniques required to solve this using only the knowledge of integration taught in typical college math courses are not sufficient, though it can alternatively be solved using trig identities.
  However using the Euler Formula, the problem is made very simple.

  $\int cos^2(x)dx = \int (\frac{e^{ix}+e^{-ix}}{2})^2 dx$

  $\int cos^2(x)dx = \frac{1}{4}\int (e^{2ix} + 2 + e^{-2ix})dx$

  $\int cos^2(x)dx = \frac{1}{4}(\frac{e^{2ix}}{2i} + 2x - \frac{e^{-2ix}}{2i}) + C$

  $\int cos^2(x)dx = \frac{1}{4}(\frac{e^{2ix}}{2i} - \frac{e^{-2ix}}{2i} + 2x) + C$

  $\int cos^2(x)dx = \frac{1}{4}(sin2x + 2x) + C.$

  Consider a more challenging example: $\int (sin^2xcos4x)dx$

  $\int (sin^2xcos4x)dx = \int (\frac{e^{ix}-e^{-ix}}{2i})^2(\frac{e^{4ix}+e^{-4ix}}{2})dx$

  $\int (sin^2xcos4x)dx = -\frac{1}{8}\int (e^{2ix} - 2 + e^{-2ix})(e^{4ix} + e^{-4ix})dx$

  $\int (sin^2xcos4x)dx = -\frac{1}{8}\int (e^{6ix} - 2e^{4ix} + e^{2ix} + e^{-2ix} - 2e^{-4ix} + e^{-6ix})dx$

  $\int (sin^2xcos4x)dx = -\frac{1}{8}(\int 2cos(6x) - 4cos(4x) + 2cos(2x))$

  $\int (sin^2xcos4x)dx = -\frac{1}{24}sin(6x) + \frac{1}{8}sin(4x) - \frac{1}{8}sin(2x) + C.$

  An example illustrates the use of matching real parts of numbers.

  $\int e^x cos(x)dx = \mathbf{Re}\int e^x e^{ix} dx$

$$\Rightarrow \int e^x e^{ix} = \int e^{(i+1)x} dx$$

$$\int e^x e^{ix} = \frac{e^{(1+i)x}}{1+i} + C.$$

$$\Rightarrow \int e^x cos(x) dx = \mathbf{Re}(\frac{e^{(1+i)x}}{1+i} + C)$$

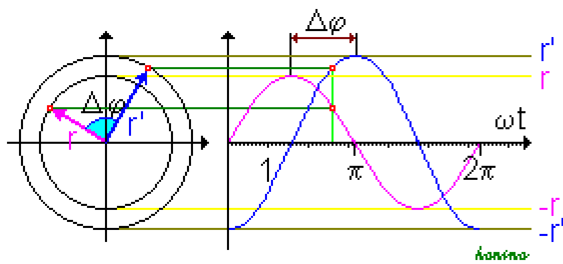
$$\int e^x cos(x) dx = e^x \mathbf{Re}(\frac{e^{ix}}{1+i}) + C$$

$$\int e^x cos(x) dx = e^x \mathbf{Re}(\frac{e^{ix}(1-i)}{2}) + C$$

$$\int e^x cos(x) dx = e^x \frac{cos(x) + sin(x)}{2} + C$$

- **Phasor**

A phasor is a way of representing a sinusoidal function as a complex number (8). The origin of phasor measurements, according to the Institute of Electrical and Electronics Engineers originated in "computer relaying to present applications in power system operation, protection, and control." (9)



This sinusoidal function will have time dependent amplitude, initial phase, and angular frequency.

The mathematics behind phasors is as follows:

Allowing A to be the amplitude, ω be the angular frequency, and θ to be the phase.

The real component of the phasor is modeled as

$Acos(\omega t + \theta)$

and the imaginary component is modeled as

$iAcos(\omega t + \theta)$

By the Euler formula, we write
$Acos(\omega t + \theta) + iAsin(\omega t + \theta) = Ae^{i(\omega t+\theta)}$

Two voltages being added can be thought of as adding as adding their vertical direction, followed by their horizontal direction, corresponding to to real and imaginary components. (6)

In the study of electrical engineering, particularly, signal processing, the representation of signals as complex functions is called the analytic signal, and the primary usage of this is that it allows for computations that are magnitudes simpler than using trig in the real number line.

The main application of analytic signals is in the computation for alternating current voltages.

- **"The Most Beautiful Theorem in Mathematics"**

  A trivial result from Euler's Formula is the Euler Identity. This is an equation that relates five of the most important constants in math (the additive identity, the unity, the ratio of a circumference to diameter, base of natural log, and unity of imaginary number line) in an extremely simple and neat manor. (7)

  Let $\theta = \pi$ within the Euler Formula.
  $e^{i\pi} = cos(\pi) + isin\pi$

  $e^{i\pi} = -1 + 0$

  $e^{i\pi} + 1 = 0$

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