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The Passive Price Formula

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In an ecosystem consisting of many interacting tokens, it may not always be clear how to attribute price movement for any particular token. If the price of a token goes up, is it because of its own intrinsic popularity? Or is it because it is primarily denominated with respect to some other token rising in popularity? In this article we define a novel type of performance metric called the *Passive Price* to help quantify this notion, along with a closely related metric called the *Active Price Score*. This article will consist of 6 sections:

- Section 1: The Setting / Relevant Variables
- Section 2: Motivation
- Section 3: The Passive Price Formula
- Section 4: Some Numerical Simulations
- Section 5: Discussion & Conclusion
- Section 6: Mathematical Appendix

Prerequisite: The reader should have a basic understanding of the constant product AMM. For a refresher, see here.

— Section 1: The Setting/Relevant Variables —

Throughout this document, our entire discussion will take place within a very simplified toy-model ecosystem consisting of a collection of constant product liquidity pools. We assume that there is a set of tokens that can be traded within our ecosystem, and our simple model assumes that there exists exactly one trading pool for any pair of tokens. We establish here some notational conventions that will be used throughout:

- P_n^m := the price of token n denominated with respect to token m. Note that for any intermediate token k, we will have the following product relationship:
$$P_m^n = P_k^n P_m^k \tag{1}$$
For example, if A is worth three of B, and if B is worth four of C, then A will be worth twelve of C. We will specifically write $P_n^\$$ for the price of token n in dollars with respect to some external liquid market. Additionally, we will usually write the price explicitly as a function of time, i.e. $P_n^m(t)$.

- L_n^m := the liquidity (number of tokens) of type n that are present in the (n, m) trading pool

- ℓ_n^m := the *fraction* that measures how much of token n is traded against token m:
$$\ell_n^m := \frac{L_n^m}{\sum_k L_n^k} \tag{2}$$
One should think of this as the following:
$$\ell_n^m = \frac{[\text{amount of token } n \text{ traded against token } m]}{[\text{total amount of token } n \text{ traded in ecosystem}]}$$
For example, if $\ell_5^7 = 0.3$, this means that 30% of all of token 5 is traded in a pool with token 7. We will refer to ℓ_n^m as the *liquidity fraction* of token n with respect to token m.

- $\hat{P}_n^\$$:= what we call the *passive price* of token n (to be defined in Sections 2 and 3), where the little 'hat' above P is specifically placed to distinguish the *passive price* from the *actual price*.

Section 2: Motivation

The focus of this article is to define the notion of the *passive price*. We will do this mathematically in the next section, but it will be easier to understand if in this section we first take a moment to give some motivation. To this end, let us begin with an extremely simple scenario. First, we fix a particular token, which we'll call token n. Next, we imagine that token n is *exclusively* traded against another specific token (let's say token m), in the sense that a pool exists between token n and m, but token n is not traded anywhere else. Token m, on the other hand, is widely traded elsewhere. Now, consider a time interval $[0, T]$. At time $t = 0$, tokens n and m each have a dollar price. Suppose that by time $t = T$ the dollar price of token m doubles. Consider the following possibilities:

1. The dollar price of token n does not double, meaning it did not keep up with its peer token m.
2. The dollar price of token n more than doubles, meaning the popularity of token n actually outpaced that of token m.
3. The dollar price of token n exactly doubles, meaning the two tokens kept pace with each other.

In possibility (3), given the assumption that token n is *only* traded against token m, it would stand to reason that token m was the asset that actually doubled in value in the eyes of society, whereas the doubling in dollar value of token n was simply *inherited* by its association with token m.

The passive price will be defined as an attempt to capture the following notion; *what would the price of token n be if its price is determined not by its intrinsic value to society, but simply by its trading adjacency to other tokens?*

In the simple scenario just described, it would be reasonable for the passive price of token n at time T, denoted $\hat{P}_n(T)$, to be computed by starting with the initial dollar value $P_n^\$(0)$ and scaling it up by the same factor by which token m increases or decreases:
$$\hat{P}_n^\$(T) = P_n^\$(0)\Big(P_m^\$(T)/P_m^\$(0)\Big). \tag{3}$$
However, if we now suppose that token n is traded against *many* other tokens, and not exclusively against token m, then we would no longer expect expression (3) to hold. Instead, we might expect that expression (3) should be replaced by a weighted sum over all the influences (other tokens) affecting token n. Specifically, we might expect the weights to be given by the liquidity fractions, and so we *could* (but *won't*) define the passive price by the following:
$$\hat{P}_n^\$(T) \stackrel{?}{=} P_n^\$(0)\left[\sum\nolimits_m \ell_n^m \Big(P_m^\$(T)/P_m^\$(0)\Big)\right]. \tag{4}$$
One can interpret (4) as though the price of token n is *trying* to evolve passively with respect to each other token according to (3), simultaneously, but they are each pushing and pulling the price in various directions, with a relative strength given by the liquidity fraction. Now, equation (4) will *not* be our formula for passive price, but it is not too far off. We derive the actual formula in the next section. It should be explicitly stated that the *actual* dollar price $P_n^\$(T)$ of token n at time T can be anything it damn well pleases. The *passive* price $\hat{P}_n^\$(T)$ is just a hypothetical value to which we can *compare* the actual price, as a kind of performance metric.

Section 3: The Passive Price Formula

In this section, we will define the actual passive price formula, to replace formula (4). First, for the trading pool between tokens n and m, consider the following helpful visual aid:

$$\begin{array}{|c|}\hline L_m^n \\ \hline L_n^m \\ \hline\end{array}$$

This figure is meant to represent the (n, m) pool that (by definition) contains L_m^n many tokens of type m, and L_n^m many tokens of type n. Now, as before, fix a particular token index n, and imagine that there are $m = 1, 2, ..., M$ other tokens out there to be traded against. The state of the ecosystem (at least, the portion interacting with token n) can be visually represented by the following:

$$\begin{array}{|c|}\hline L_1^n \\ \hline L_n^1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline L_2^n \\ \hline L_n^2 \\ \hline\end{array} \quad \cdots \quad \begin{array}{|c|}\hline L_M^n \\ \hline L_n^M \\ \hline\end{array} \tag{5}$$

At time $t = 0$, our token n will have a dollar price $P_n^\$(0)$, as will the other M tokens:

$$\left\{P_1^\$(0), P_2^\$(0), ..., P_M^\$(0)\right\} \tag{6}$$

We assume that, through arbitrage, the prices *between* tokens will reflect their *relative* dollar prices. Now, at some time later $t = T$, the world moves on and the dollar prices are given by a new set of values:

$$\left\{P_1^\$(T), P_2^\$(T), ..., P_M^\$(T)\right\} \tag{7}$$

Consequently, the prices between tokens in the pools must adjust to reflect these new dollar prices. The liquidity amounts present in each pool will therefore be altered and we will denote these changes by ΔL. Thus, we may represent the new state of the pools by the following figure:

$$\begin{array}{|c|}\hline L_1^n + \Delta L_1^n \\ \hline L_n^1 + \Delta L_n^1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline L_2^n + \Delta L_2^n \\ \hline L_n^2 + \Delta L_n^2 \\ \hline\end{array} \quad \cdots \quad \begin{array}{|c|}\hline L_M^n + \Delta L_M^n \\ \hline L_n^M + \Delta L_n^M \\ \hline\end{array} \tag{8}$$

Now we state the interesting (and critical) observation that allows us to define the passive price:

- It can be shown (in Section 6) that given the initial state of the ecosystem (5) and (6), and given the new set of prices (7), it turns out that the resulting changes in liquidity (the ΔL's) shown in (8) are mathematically **not uniquely determined**.

- In fact, there will be *one* mathematical degree of freedom left over. We can, however, generate a uniquely determined set of ΔL's by imposing **one additional constraint**. The additional constraint is chosen to capture the notion that token n evolves *passively*, and so we propose the following choice for our constraint: *no new n tokens are created or destroyed in this system.* Mathematically,

$$\sum\nolimits_m \Delta L_n^m = 0. \tag{9}$$

In other words, constraint (9) demands that any amount of token n that leaves one pool just ends up in another. In a sense, this constraint suggests that token n is just acting as a lubricant for the other tokens, but not gaining or losing popularity on its own.

It will be shown in Section 6 precisely how constraint (9) results in a set of ΔL's that are uniquely determined. More importantly, with this new uniquely determined state of the trading pools, our token n will inherit a resulting dollar price. It will be shown that this dollar price is (miraculously) given by the following formula:

$$\boxed{\hat{P}_n^\$(T) = P_n^\$(0)\left(\sum\nolimits_m \ell_n^m \sqrt{P_m^\$(T)/P_m^\$(0)}\right)^2} \tag{10}$$

This, then, will be our official definition of the *passive price* of token n. Compare this to our guess in formula (4) - the similarity is actually quite miraculous, but this result (with its squares and square roots) is totally unintuitive. The real utility of formula (10) is seen when we compare the hypothetical passive price to the actual price:

(Given at $t = 0$)

$$\begin{array}{|r|}\hline \text{State of pools: } \{\ell_n^m\} \\ \text{Price of token } n\text{: } \{P_n^\$(0)\} \\ \text{Other prices: } \{P_1^\$(0), ..., P_M^\$(0)\} \\ \hline\end{array}$$

$$\Big\downarrow \ (\text{at } t = T)$$

$$\underset{\text{(New prices)}}{\left\{P_1^\$(T), ..., P_M^\$(T)\right\}} \longrightarrow \begin{array}{|c|}\hline \text{Actual price } P_n^\$(T) \\ \text{v.s.} \\ \text{Passive price } \hat{P}_n^\$(T) \\ \hline\end{array}$$

Moreover, we can then define a simple way of comparing the two prices (actual v.s passive) to yield a performance metric. This metric, which will call the *active price score*, will be defined as the logarithm of their ratio:

$$\text{active price score} := \log\left(P_n^\$(t)/\hat{P}_n^\$(t)\right) \tag{11}$$

Let us briefly interpret the meaning of this score. If the performance of the token is *better* than a passive performance, then the ratio of $P/\hat{P}$ will be greater than one, and so the logarithm will be positive. Conversely, if the performance is *worse* than passive, then the ratio is less than one and so the logarithm will be negative. In short,

$$\begin{array}{c}\text{active price}\\ \text{score}\end{array} = \left\{\begin{array}{cl} \text{negative} & \begin{array}{l}\text{if token performs}\\ \text{worse than passively}\end{array} \\ \text{zero} & \begin{array}{l}\text{if token performs}\\ \text{passively}\end{array} \\ \text{positive} & \begin{array}{l}\text{if token performs}\\ \text{better than passively}\end{array} \end{array}\right\}$$

In the next section, we run some numerical simulations, and the active price score will be used ubiquitously.

Section 4: Some Numerical Simulations

We will now demonstrate the utility of the passive price formula via numerical simulations. For this we choose $M = 10$ background tokens for our primary token to trade against. Over 500 time steps, we evolve the dollar prices as lognormal Brownian motion (with a suitably chosen variance). All token prices are evolved independently (both primary and background tokens). Because the prices can vary across orders of magnitudes, prices will normalized by their initial value when plotted (i.e. we plot $P^{\$}_m(t)/P^{\$}_m(0)$). Also, the liquidity fractions for each background token (ℓ^m_n) are indicated by the thickness and shading of the time series. An example of these price time series are plotted below in Figure 1:

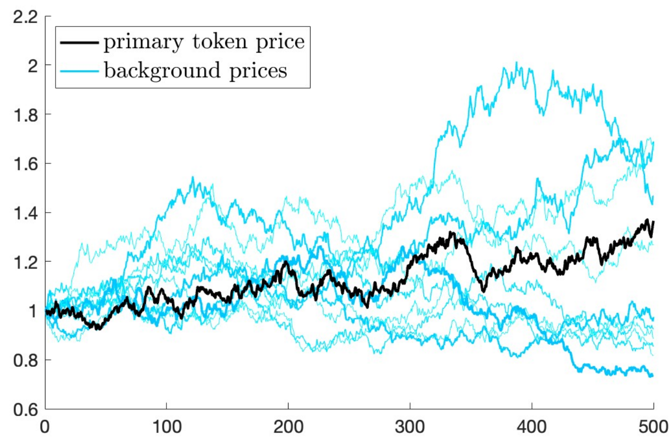

Figure 1. Token dollar price movements.

At each point in time, we compute the passive price, and superimpose this time series as well (see Figure 2). We also plot the corresponding active price score, in parallel:

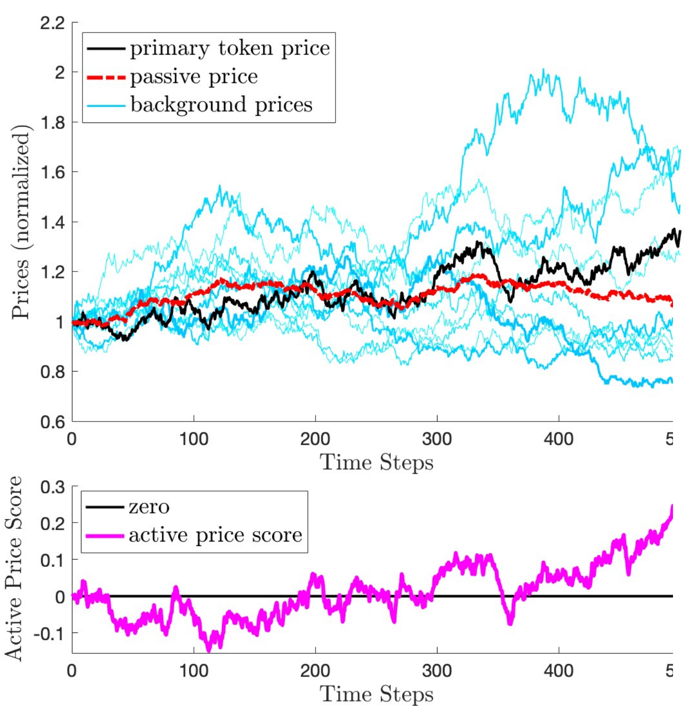

Figure 2. Time series from Figure 1, with passive price.

In some cases, it is pretty easy to judge the performance of the primary token by eye. In Figure 3, it is pretty clear when the primary token is over performing (relative to the background tokens) and when it is under performing, and it is questionable whether the active price score reveals anything interesting.

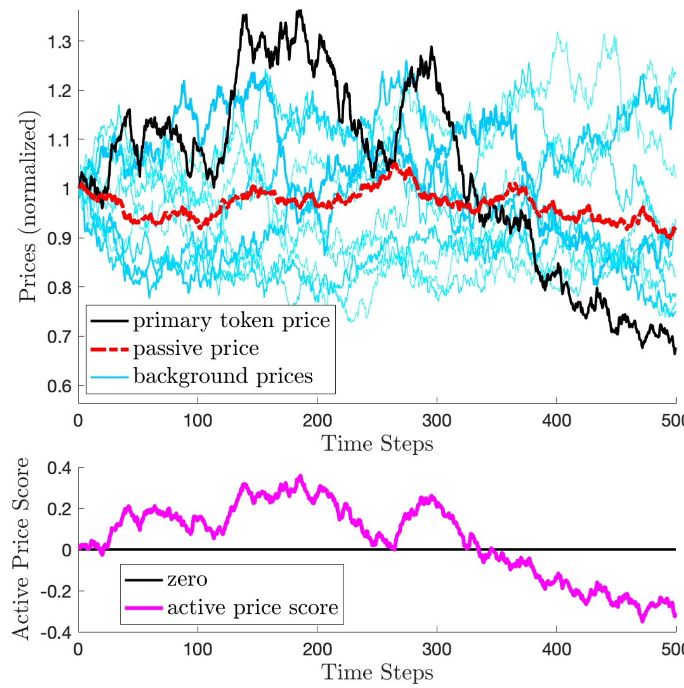

Figure 3. An unambiguous performance.

On the other hand, there are cases where it is less clear. Consider the time series of token prices given below in Figure 4. While there do exist intervals in which our primary token is clearly doing better than the pack (for example, when $t \sim [100, 300]$), there are also regions in which it is hard to say. For example, consider the interval $t \sim [350, 500]$. On one hand, it appears that a greater weight of the liquidity fractions are *below* the primary token. On the other hand, the liquidity fractions above the primary token are *far* above. How do these two factors wash out? For this, we can use the passive price.

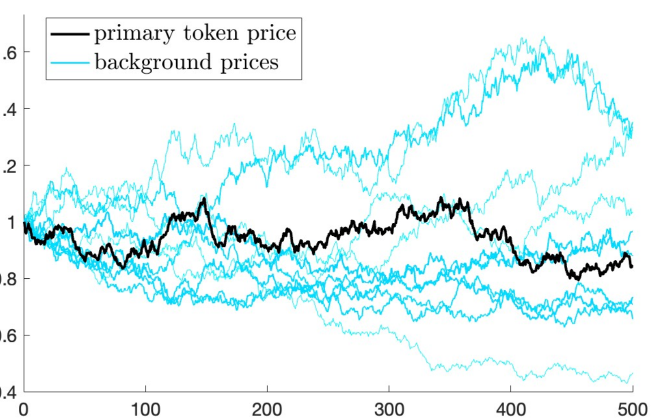

Figure 4. A more ambiguous performance.

In Figure 5, we plot the same time series from Figure 4, but as before, we superimpose the passive price and we plot the corresponding active price score below it.

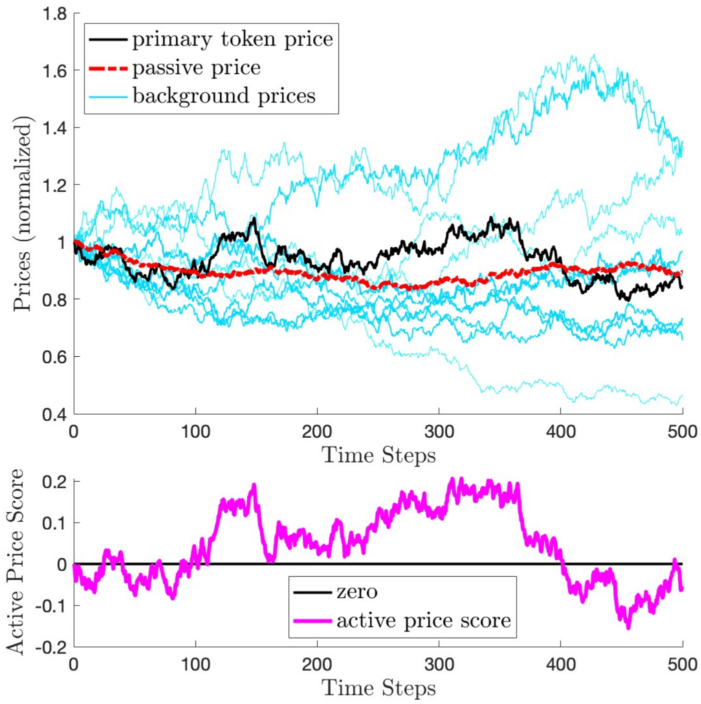

Figure 5. A performance disambiguated.

Notice that we can now say precisely where the primary token begins to underperform or overperform.

Section 5: Discussion & Conclusions

The *passive price formula*, and the associated *active price score*, appear to be interesting and potentially useful performance metrics for assessing how a token does relative to its (constant product style) ecosystem. There are a couple observations we can make at this point.

- To calculate the passive price, we need to know the dollar prices of our tokens (easily found) as well as the liquidity fractions (not so easily found). The latter may not be readily available because they may not even be well defined, as our model assumed only one pool per token pair, whereas real life is much more complicated. In practice, these quantities must be approximated.
- We used the dollar as a universal background denomination, but obviously we could have used anything else. For example, one might choose to measure background prices relative to ETH.

There is, however, one big shortcoming that we must address. Many tokens are traded primarily with respect to one major token, typically ETH. In such a case, the liquidity fractions may be close to zero for all other tokens, and so the passive price formula will simply be dominated by the price of ETH.

As an example, we track the price of price of UNI between January and July of 2023. The majority of UNI is traded against ETH, and as a result, the passive price of UNI simply follows the price of ETH, move for move without telling us anything interesting (Figure 6):

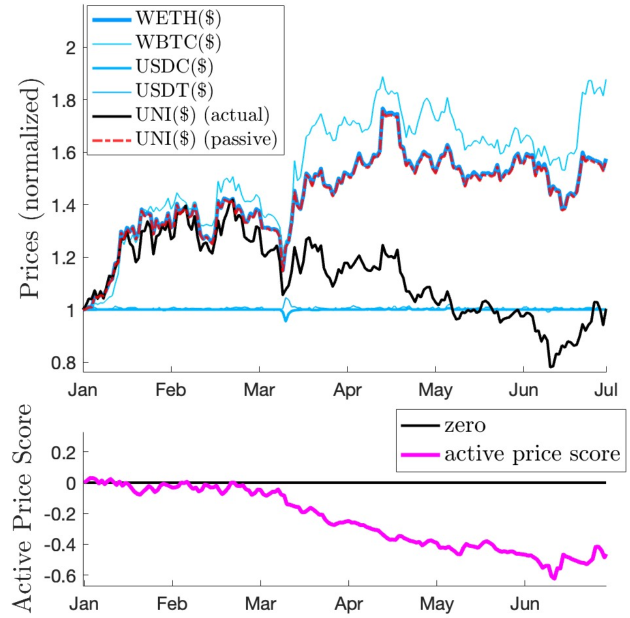

Figure 6. Passive price for UNI token.

However, there exists another arena where the passive price formula may be relevant; foreign exchange markets. Here we have numerous different assets all being inter-traded amongst themselves. In Figure 7 below, we compute the passive price for the New Zealand Dollar (NZD) over the first half of 2022 (the liquidity fractions were estimated from one single day and are probably not representative of the general state of things, unfortunately).

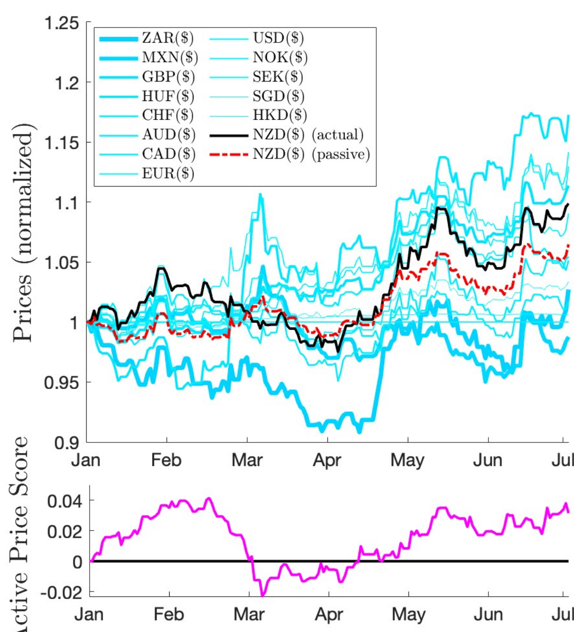

Figure 7. The Passive Price for the NZD.

In this context, we are abandoning the constant product protocol, an essential assumption in our derivation for the passive price formula. However, we speculate that the formula may be more universal than it seems.

Section 6: Mathematical Appendix

In this section we will finally derive the passive price formula given in (10). For constant product pools, there are two fundamental equations that we need. Specifically, for two tokens in a pool with liquidities x and y, their ratio is equal to the price between them, while their product is equal to some constant k:

$$k = xy \tag{12}$$
$$p = y/x. \tag{13}$$

Now, as in Section 3, fix a primary token n and consider the other $m = 1, 2, ..., M$ tokens that token n can be traded against. At time $t = 0$, we represent the state of the pool liquidities by the following:

$$\begin{array}{|c|}\hline L_1^n \\ \hline L_n^1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline L_2^n \\ \hline L_n^2 \\ \hline\end{array} \quad \cdots \quad \begin{array}{|c|}\hline L_M^n \\ \hline L_n^M \\ \hline\end{array} \tag{14}$$

Being constant product pools, we can write down versions of (12) and (13). First, their products will be equal to some constant, which we'll denote by the letter K:

$$K_m := L_n^m L_m^n \qquad (\text{for } m = 1, ..., M) \tag{15}$$

Second, the price in each pool is given by the ratio of the coordinates, and thus, the initial prices in each pool are given by

$$P_n^m(0) = L_m^n / L_n^m \qquad (\text{for } m = 1, ..., M) \tag{16}$$

(One may verify from the definitions of these quantities that the ratio is taken in the correct order to represent the price P_n^m). At this time, each token has a dollar price determined by the world at large:

$$P_n^{\$}(0), \quad \text{and} \quad \left\{P_1^{\$}(0), P_2^{\$}(0), ..., P_M^{\$}(0)\right\} \tag{17}$$

Because of arbitrage, the prices in the pools given by (16) must also reflect the current dollar prices. To see how, we note that for any two tokens m and m', using our relation (1), we can write

$$P_n^{\$}(0) = P_n^m(0) P_m^{\$}(0) = P_n^{m'}(0) P_{m'}^{\$}(0) \tag{18}$$

Thus, using equation (16), we may write the following:

$$\frac{L_m^n / L_n^m}{L_{m'}^n / L_n^{m'}} = \frac{P_{m'}^{\$}(0)}{P_m^{\$}(0)} \qquad (\text{for any pair } (m, m')) \tag{19}$$

Now, at some time later $t = T$, the external dollar prices have possibly changed:

$$\left\{P_1^{\$}(T), P_2^{\$}(T), ..., P_M^{\$}(T)\right\} \tag{20}$$

Through arbitrage, these will induce a new set of liquidities present in the trading pools:

$$\begin{array}{|c|}\hline L_1^n + \Delta L_1^n \\ \hline L_n^1 + \Delta L_n^1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline L_2^n + \Delta L_2^n \\ \hline L_n^2 + \Delta L_n^2 \\ \hline\end{array} \quad \cdots \quad \begin{array}{|c|}\hline L_M^n + \Delta L_M^n \\ \hline L_n^M + \Delta L_n^M \\ \hline\end{array} \tag{21}$$

At this time, we can write down two equations analogous to (15) and (19). First, the liquidity product for each pool must be maintained:

$$K_m = (L_n^m + \Delta L_n^m)(L_m^n + \Delta L_m^n) \tag{22}$$
$$(\text{for } m = 1, ..., M)$$

Second, the pool prices must again reflect the external dollar prices:

$$\frac{(L_m^n + \Delta L_m^n)/(L_n^m + \Delta L_n^m)}{(L_{m'}^n + \Delta L_{m'}^n)/(L_n^{m'} + \Delta L_n^{m'})} = \frac{P_{m'}^{\$}(T)}{P_m^{\$}(T)} \tag{23}$$
$$(\text{for any pair } (m, m'))$$

However, we can use (22) to eliminate both the ΔL_m^n and $\Delta L_{m'}^n$ terms in (23):

$$\frac{L_n^{m'} + \Delta L_n^{m'}}{L_n^m + \Delta L_n^m} = \sqrt{\frac{K_{m'} P_{m'}^{\$}(T)}{K_m P_m^{\$}(T)}} \tag{24}$$
$$(\text{for any pair } (m, m'))$$

Expression (24) represents a set of equations (for any pair m and m') which all must independently hold, and as such it represents a system of equations for the collection of unknowns $\{\Delta L_n^1, \Delta L_n^2, ..., \Delta L_n^M\}$ (recall that we assume that the liquidity constants K_m are given, as well as the dollar prices $P_m^{\$}(T)$). The number of equations we have in this system is $(M-1)M/2$ (being the number of ways to choose a pair m and m'), while the number of unknowns is M. Hence, it would appear that our system is *overdetermined*. However, many of the equations in (24) are redundant. To see why, fix an arbitrary token index m_0. Then we make the following claim; the system of equations in (24) is equivalent to the following system of equations:

$$\frac{L_n^m + \Delta L_n^m}{L_n^{m_0} + \Delta L_n^{m_0}} = \sqrt{\frac{K_m P_m^{\$}(T)}{K_{m_0} P_{m_0}^{\$}(T)}} \qquad (\text{for any } m) \tag{25}$$

To justify this claim, suppose we have a solution set for the unknowns $\{\Delta L_n^m\}$ to the equations in (25). Then equation (25) is satisfied when we plug in any m. In particular, pick an arbitrary pair (m_1, m_2). Next, take equation (25) with $m = m_2$, and divide it by equation (25) with $m = m_1$. The result will simply be equation (24) with $(m, m') = (m_1, m_2)$. Since m_1 and m_2 were arbitrary, we see that (25) being satisfied implies that (24) is also satisfied for every possible choice of (m, m'). Showing that (24) implies (25) is trivial.

Most importantly, we see that equation (25) really only represents $M-1$ distinct equations (because the case $m = m_0$ is tautologically satisfied). Therefore, our system actually consists of $M-1$ equations with M unknowns, and so it is actually *underdetermined*, with one degree of freedom. In other words, for a given set of initial conditions, and a given set of new updated dollar prices, there are still infinitely many ways the liquidity can exist in the pools.

To generate a system with a uniquely determined solution, we must impose one additional constraint. As we described in Section 3, we impose the *passive price* condition that no new n tokens are created or destroyed in total:

$$\sum_m \Delta L_n^m = 0 \tag{26}$$

Thus, we seek a (uniquely determined) solution $\{\Delta L_n^m\}$ to the following $M \times M$ system of equations:

$$\left\{ \begin{array}{l} \frac{L_n^m + \Delta L_n^m}{L_n^{m_0} + \Delta L_n^{m_0}} = \sqrt{\frac{K_m P_m^\$(T)}{K_{m_0} P_{m_0}^\$(T)}} \quad (m \neq m_0) \\ \sum_m \Delta L_n^m = 0 \end{array} \right\} \tag{27}$$

Though it may not immediately appear to be, system (27) is actually *linear*. To see this, define the simplifying quantity $Q_{m_0}^m$ by the following:

$$Q_{m_0}^m := \sqrt{\frac{K_m P_m^\$(T)}{K_{m_0} P_{m_0}^\$(T)}}, \tag{28}$$

Then we can write system (27) as

$$\left\{ \begin{array}{c} \Delta L_n^m - \left[Q_{m_0}^m\right] \Delta L_n^{m_0} = \left[Q_{m_0}^m L_n^{m_0} - L_n^m\right] \quad (m \neq m_0) \\ \sum_m \Delta L_n^m = 0 \end{array} \right\} \tag{29}$$

which is clearly a $M \times M$ linear system when we write it as follows:

$$\begin{pmatrix} 1 & 0 & \dots & \text{-}Q_{m_0}^1 & \dots & 0 \\ 0 & 1 & \dots & \text{-}Q_{m_0}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \text{-}Q_{m_0}^M & \dots & 1 \end{pmatrix} \begin{pmatrix} \Delta L_n^1 \\ \Delta L_n^2 \\ \vdots \\ \Delta L_n^{m_0} \\ \vdots \\ \Delta L_n^M \end{pmatrix} = \begin{pmatrix} Q_{m_0}^1 L_n^{m_0} \text{-} L_n^1 \\ Q_{m_0}^2 L_n^{m_0} \text{-} L_n^2 \\ \vdots \\ 0 \\ \vdots \\ Q_{m_0}^M L_n^{m_0} \text{-} L_n^M \end{pmatrix} \tag{30}$$

Note that we can fit the passive constraint $\sum \Delta L = 0$ into the m_0^{th} row because that is precisely where we don't already have an equation.

Now, finding the solution to a linear system of equations is trivial in any programming language, but not necessarily insightful. We *could* explicitly write down the solutions $\{\Delta L_n^1, \Delta L_n^2, ..., \Delta L_n^M\}$ in terms of obnoxious formulas with matrix determinants and the like. Then we *could* use these and equation (22) to find the other (complementary) set of M many deltas, and then we *could* write down the new prices $\{P_n^1(T), P_n^2(T), ..., P_n^M(T)\}$ in terms of the new liquidity ratios, and then we *could* use these prices to finally write the new dollar price $P_n^\$(T)$, which due to the fact that we solve system (27), would actually be the passive price $\hat{P}_n^\$(T)$ (remember that this was our goal in the first place).

Doing all of this would be a **complete mess**. Fortunately, however, there is simple formula for the resulting price $\hat{P}_n^\$(T)$ that allows us to bypass all of this mess. We will derive this next.

To begin, we know that for any m, we must have

$$P_n^\$(T) = P_n^m(T) P_m^\$(T) \tag{31}$$

Now let's confirm that for solutions $\{\Delta L_n^m\}$ to (27), and the resulting values of $P_n^m(T)$, the expression $P_n^m(T)P_m^\$(T)$ will not depend on the index m. To do this, fix any two distinct m and m'. Using the definitions of P_n^m and K_m, we have

$$P_n^m(T) = \frac{L_m^n + \Delta L_m^n}{L_n^m + \Delta L_n^m} = \frac{K_m}{(L_n^m + \Delta L_n^m)^2} \tag{32}$$

with a similar expression for m'. Using this, let's consider the following ratio:

$$\begin{aligned} \frac{P_n^{m'}(T)P_{m'}^\$(T)}{P_n^m(T)P_m^\$(T)} &= \frac{K_{m'}/(L_n^{m'} + \Delta L_n^{m'})^2 P_{m'}^\$(T)}{K_m/(L_n^m + \Delta L_n^m)^2 P_m^\$(T)} \\ &= \left[\sqrt{\frac{P_{m'}^\$(T)K_{m'}}{P_m^\$(T)K_m}} \left(\frac{L_n^m + \Delta L_n^m}{L_n^{m'} + \Delta L_n^{m'}}\right)\right]^2 \\ &= 1 \end{aligned} \tag{33}$$

where the final equality is due to the fact that our solution satisfies equation (24) for any m and m'. Since this was true for any choice of m and m', then we see that $P_n^m P_m^\$ = P_n^{m'} P_{m'}^\$$ for all m and m', and thus expression (31) is well defined, regardless of a choice of m.

Now we are ready to show that solutions to (27) will result in a token price $P_n^\$(T)$ that satisfies a simple formula. First, using the constraint $\sum_m \Delta L_n^m = 0$, we write down the statement

$$\sum_{m'} L_n^{m'} = \sum_m (L_n^m + \Delta L_n^m) \tag{34}$$

Now, we make the simple observation that solutions to (27) will satisfy

$$L_n^m + \Delta L_n^m = \sqrt{\frac{K_m P_m^\$(T)}{K_{m_0} P_{m_0}^\$(T)}} \left(L_n^{m_0} + \Delta L_n^{m_0}\right) \tag{35}$$

by definition. Then we substitute this into expression (34) and begin to compute:

$$\begin{aligned} \sum_{m'} L_n^{m'} &= \sum_m (L_n^m + \Delta L_n^m) \\ &= \sum_m \sqrt{\frac{K_m P_m^\$(T)}{K_{m_0} P_{m_0}^\$(T)}} \left(L_n^{m_0} + \Delta L_n^{m_0}\right) \\ &= \frac{\left(L_n^{m_0} + \Delta L_n^{m_0}\right)}{\sqrt{K_{m_0} P_{m_0}^\$}} \sum_m \sqrt{K_m P_m^\$} \end{aligned} \tag{36}$$

Next, we cross multiply expression (36) and square both sides:

$$\frac{K_{m_0} P_{m_0}^\$}{\left(L_n^{m_0} + \Delta L_n^{m_0}\right)^2} = \left(\frac{\sum_m \sqrt{K_m P_m^\$}}{\sum_{m'} L_n^{m'}}\right)^2 \tag{37}$$

Now, from equation (32), we see that the left hand side of (37) can be written as $P_n^{m_0}(T)P_{m_0}^{\$}(T)$. But, as we have just showed, regardless of the index (in this case m_0), this quantity is equal to $P_n^{\$}(T)$. Thus, we can rewrite equation (37) as the following:

$$P_n^{\$}(T) = \left(\frac{\sum_m \sqrt{K_m P_m^{\$}(T)}}{\sum_{m'} L_n^{m'}}\right)^2 \tag{38}$$

Finally, we use the definition of $K_m = L_n^m L_m^n$ one last time:

$$\begin{aligned}
P_n^{\$}(T) &= \left(\frac{\sum_m \sqrt{L_m^n L_n^m P_m^{\$}(T)}}{\sum_{m'} L_n^{m'}}\right)^2 \\
&= \left(\frac{\sum_m L_n^m \sqrt{(L_m^n/L_n^m) P_m^{\$}(T)}}{\sum_{m'} L_n^{m'}}\right)^2 \\
&= \left(\frac{\sum_m L_n^m \sqrt{P_n^m(0) P_m^{\$}(T)}}{\sum_{m'} L_n^{m'}}\right)^2 \\
&= \left(\sum\nolimits_m \left(\frac{L_n^m}{\sum_{m'} L_n^{m'}}\right) \sqrt{P_n^m(0) P_m^{\$}(T)}\right)^2 \\
&= \left(\sum\nolimits_m \ell_n^m(0) \sqrt{P_n^m(0) P_m^{\$}(T)}\right)^2
\end{aligned} \tag{39}$$

Thus, we have shown that for solutions to (27), the dollar price of the token n at time T is given by

$$P_n^{\$}(T) = \left(\sum\nolimits_m \ell_n^m(0) \sqrt{P_n^m(0) P_m^{\$}(T)}\right)^2 \tag{40}$$

Now, we note that, as we expect from equation (1), the price between two tokens must reflect their respective prices against the dollar:

$$P_n^m = P_n^{\$}/P_m^{\$} \tag{41}$$

Carrying on from (40), we have

$$\begin{aligned}
P_n^{\$}(T) &= \left(\sum\nolimits_m \ell_n^m(0) \sqrt{P_n^m(0) P_m^{\$}(T)}\right)^2 \\
&= \left(\sum\nolimits_m \ell_n^m(0) \sqrt{\left(P_n^{\$}(0)/P_m^{\$}(0)\right) P_m^{\$}(T)}\right)^2 \\
&= P_n^{\$}(0) \left(\sum\nolimits_m \ell_n^m(0) \sqrt{P_m^{\$}(T)/P_m^{\$}(0)}\right)^2
\end{aligned} \tag{42}$$

Finally, we can summarize: for the solutions $\{\Delta L_n^m\}$ for the system of equations (27), the resulting dollar price for token n will be equal to expression (42). Moreover, because the system in (27) invokes the passive constraint ($\sum \Delta L = 0$), then we will take (42) as the *definition* of the passive price, denoted by $\hat{P}_n^{\$}$:

$$\boxed{\hat{P}_n^{\$}(T) = P_n^{\$}(0) \left(\sum_m \ell_n^m(0) \sqrt{P_m^{\$}(T)/P_m^{\$}(0)}\right)^2} \tag{43}$$

As a final note, we also point out that (43) rearranges to

$$\sqrt{\hat{P}_n^{\$}(T)/P_n^{\$}(0)} = \sum_m \ell_n^m(0) \sqrt{P_m^{\$}(T)/P_m^{\$}(0)} \tag{44}$$

and so it is natural to give a name π_m to the square root or the price growth factor,

$$\pi_m[0,T] := \sqrt{P_m^{\$}(T)/P_m^{\$}(0)}. \tag{45}$$

With this notation, we car rewrite our passive price formula in a simpler form:

$$\hat{\pi}_n[0,T] = \sum\nolimits_m \ell_n^m(0)\ \pi_m[0,T] \tag{46}$$

where this *linear* weighted sum is reminiscent of our guess in expression (4).