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Comparing Internal and External Fees For The Constant Product AMM

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In this article, we will compare two approaches to taking fees in the Constant Product AMM; either we store the fees as additional liquidity in the pool (internal fees), which is the current standard way to do it, or we deliver fees immediately to the LPs (external fees), as it is done for the concentrated liquidity pools. To compare these two fee structures, we will first derive two simple and abstract toy models for price movements, and then we will compare them to random simulations. Though we focus exclusively on the context of constant product AMMs, this article will lay the groundwork for future expansions on the topic. The tedious mathematical details will be left for the appendix at the end. This article will be divided into 5 sections:

- Section 1: Internal Fees v.s. External Fees
- Section 2: Two Simple Scenarios
- Section 3: Simulations
- Section 4: Discussion & Conclusions
- Section 5: Mathematical Appendix

Prerequisite: The reader should be familiar with the basics of the constant product protocol, as well as impermanent loss. For a refresher, see here.

— Section 1: Internal Fees v.s External Fees —

In a constant product trading pool, the fees that are taken from each trade are typically stored *internally in the pool*, as added liquidity. However, it is also possible to store the fees *externally* (outside of the pool), by delivering them directly to the LPs. Our goal will be to compare the advantages and disadvantages of *internal* and *external* fees structures, exclusively in this context of the constant product protocol.

First, let us articulate the difference between these two types of fee structures. Consider a constant product pool in some arbitrary state, and consider an amount Δx to be paid into the pool. A fraction γ splits this quantity into two pieces; $\gamma\Delta x$ and a fee (see Figure 1).

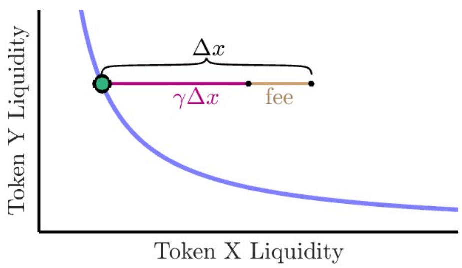

Figure 1. An incoming quantity.

A typical value for γ could be $\gamma = 0.997$, which means the fee is 0.3% of the incoming quantity. The quantity $\gamma\Delta x$ is used to determine the corresponding output quantity Δy via the constant product protocol, which is illustrated in Figure 2. With an external fee structure, the fee portion would then be delivered to the LPs.

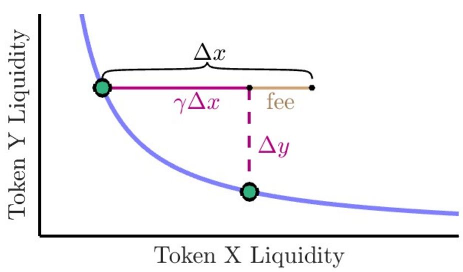

Figure 2. External Fee.

However, when storing fees internally, we insist on depositing the entire Δx into the pool (Figure 3).

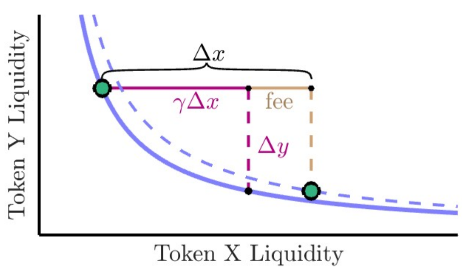

Figure 3. Internal Fee.

In this case, one easily sees that the new coordinates will land *above* the constant product curve $xy = k$. In effect, we graduate to a higher curve with each transaction.

There are pros and cons to each of these fee structures.

- Storing fees internally reduces gas costs (the fees do not need to be transferred after each transaction). Moreover, storing the fees internally allows them to *compound* - more liquidity in the pool tends to result in more fees earned.
- On the other hand, any fees stored externally are not subject to impermanent loss.

Impermanent loss is the dominant risk exposure that LPs tend to be concerned with, and so we will asses the differences between our two fee structures in the context of how they compensate for this loss.

——— Section 2: Two Simple Scenarios ———

In the battle between Impermanent Loss and Fees, the latter can surpass the former when there are many back and forth price movements that don't wander too far from the initial price. Conversely, if the price does indeed wander too far away, then the impermanent loss may overshadow the fees that are earned. Taking these two scenarios to the extreme, we begin by considering two simple toy models of price movement:

- (Best case scenario for the LPs) The market simply moves back and forth forever between two fixed prices $p_1 \leftrightarrow p_2$. We'll refer to this as the *Round Trip* model (Figure 4).

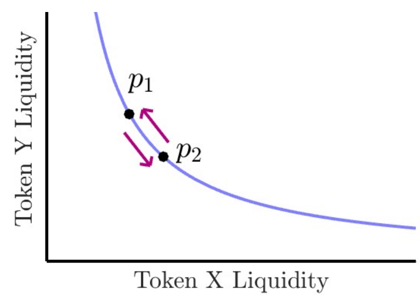

Figure 4 Round Trip model

- (Worst case scenario for LPs) The market moves indefinitely in one direction through a monotonic sequence of prices $p_0 \to p_1 \to p_2 \to ....$ We'll call this the *One Way* model (Figure 5).

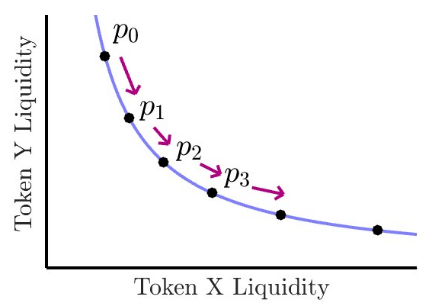

Figure 5 One Way model

We will derive relevant mathematical expressions for both scenarios.

This first step in quantifying these models is to define a set of prices $\{p_n\}$, for which we choose an initial price p_0, and a price increment factor α:

$$p_n := p_0 \alpha^n. \tag{1}$$

Thus, the prices are (geometrically) equispaced. Next, we recall the quantity ILF, which measures the impermanent loss (IL), but which takes into account the fees that are earned. Specifically, the ILF is computed in the same way as the IL, but we include the fees in the calculation of the current value:

$$\text{ILF} := \frac{\begin{pmatrix}\text{current value}\\ \text{held in pool}\end{pmatrix} + \text{Fees}\Big) - \begin{pmatrix}\text{current value had}\\ \text{they stayed out}\end{pmatrix}}{\begin{pmatrix}\text{current value had}\\ \text{they stayed out}\end{pmatrix}} \tag{2}$$

To be explicit, the '+Fees' term means two different things, depending on whether we are using an internal or external fee structure:

$$\left(\begin{matrix}\text{current value}\\ \text{held in pool}\end{matrix} + \text{Fees}\right) = \left\{\begin{matrix} \begin{pmatrix}\text{value in pool,}\\ \text{given rise in } k\end{pmatrix} & \begin{matrix}\text{if fees stored}\\ \text{internally}\end{matrix} \\ \\ \begin{pmatrix}\text{value in pool}\\ +\text{ external fees}\end{pmatrix} & \begin{matrix}\text{if fees stored}\\ \text{externally}\end{matrix}\end{matrix}\right\} \tag{3}$$

Now, given our choice of price definition in (1), we can (painstakingly) work out expressions for the ILF, for *both* of our toy models. The derivation of these will be presented in Section 5 (the mathematical appendix), but we will discuss the results presently. Beginning with the round trip model, we find that the ILF for n-many round trips is given by the following expressions:

$$\underset{\substack{\textit{Internal}\\ \text{(Round-Trip)}}}{\text{ILF}} = \left(\frac{1+\delta}{1+\gamma\delta}\right)^n - 1 \tag{4}$$

$$\underset{\substack{\textit{External}\\ \text{(Round-Trip)}}}{\text{ILF}} = n\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{1/\sqrt{\alpha}-\sqrt{\alpha}}{2}\right) \tag{5}$$

where δ is given by

$$\delta := \frac{(1+\gamma)}{2\gamma}\left(\sqrt{1+\frac{4\gamma(1/\alpha-1)}{(1+\gamma)^2}} - 1\right) \tag{6}$$

We note that in the round-trip case, there will be no impermanent loss, because the price always returns to where it started. In this case, the ILF will simply scale with the fees earned (and will thus be positive). To confirm this, we plot expressions (4)-(5), for two different choices of $\gamma = 0.99$ (1% fee) and $\gamma = 0.997$ (0.3% fee). We choose a value of $\alpha = 0.9$, which is a reasonable amount of movement over, say, a month. These are plotted in Figure 6.

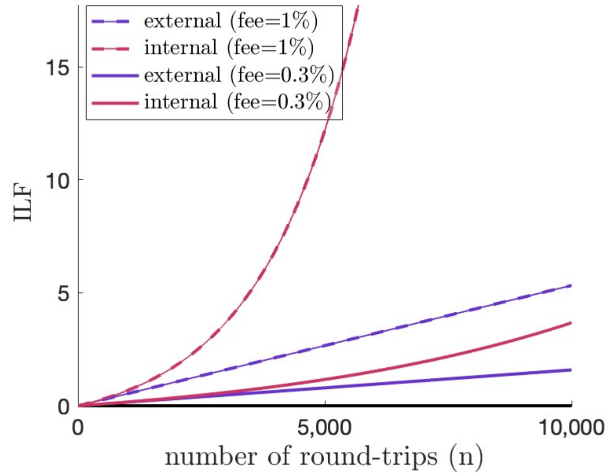

Figure 6. Round-Trip ($n \leq 10,000$), $\alpha = 0.9$

There are a few observations to make about the Round-Trip model at this point. First, the ILF growth rate (with respect to number of trades n) is *linear* with external fees, but *exponential* with internal fees. This is obvious from Figure 6, but it is also clear from inspecting formulas (4)-(5). Secondly, it actually takes some time for this exponential growth to dominate. With our choice of $\alpha = 0.9$ and a fee of 1%, the exponential growth of the internal fee structure only begins to dominate after 364 trades, though this is hardly noticible by eye. Indeed, one can check the limit of $\frac{d}{dn}(ILF)$ as $n \to 0$ for (4)-(5) and find that they are surprisingly close (but not equal).

Next, we turn to the One-Way model. Once again, one can compute ILF explicitly after n-many one-way movements, but the details are reserved for the appendix. The results are as follows:

$$\underset{\substack{\textit{Internal}\\ \text{(One-Way)}}}{\text{ILF}} = \frac{(1+\delta)^n(2-(1+\gamma\delta)^n)-1}{1+(1+\delta)^n(1+\gamma\delta)^n} \tag{7}$$

$$\underset{\substack{\textit{External}\\ \text{(One-Way)}}}{\text{ILF}} = \frac{(1-\alpha^{-n/2})+(1-\alpha^{n/2})/\gamma}{\alpha^{-n/2}+\alpha^{n/2}} \tag{8}$$

Expressions (7)-(8) are plotted in Figure 7:

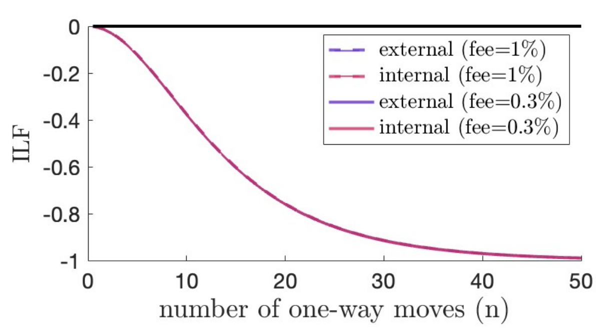

Figure 7. One Way ($n \leq 100$), $\alpha = 0.9$

For the One-Way model, it would appear that there is no appreciable difference between an internal or external fee structures (and regardless of fee size).

While this is indeed more or less the case, zooming in reveals for us slight differences (Figure 8):

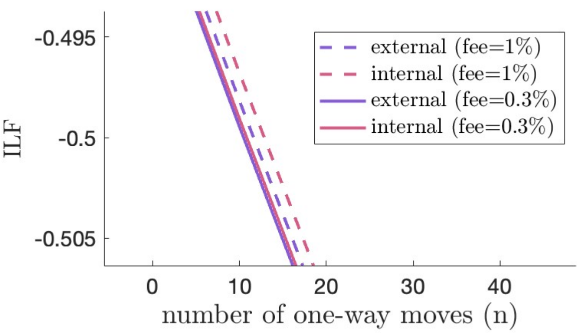

Figure 8. One Way (zoomed in)

Of course, the ILF is negative in this case, regardless of fee size and fee structure. This makes sense; if we are marching monotonically away in price, the impermanent loss is maximized, and the rate of fee accrual cannot keep pace.

So, what do we conclude from our two toy models? There are two basic observations:

- For back and forth price movements, the internal fees can be advantageous if the fee size is large enough, or if the movement lasts long enough. Otherwise, external fees may be more advantageous.

- For monotonic price movements, the internal and external fees are not appreciably different.

Section 3: Simulations

Our simplified models are useful for mathematical insight, but to truly compare internal and external fee structures, we must turn to random simulations. The structure of our simulations will be as follows:

- **Initialize** two distinct constant product pools; one with internal fees, one with external fees. The pools begin with identical liquidity and at the same initial price.

- **Iterate** over a random walk of price movements. For each new price $p(t)$, make an appropriate trade within each pool to move each pool to $p(t)$.

- **Calculate** the running ILF for each pool, as measured relative to the initial state at the start of the simulation.

For our random walk, we will simulate lognormal Brownian motion, with a standard deviation σ which we refer to as *volatility* (with an appropriate time scale):

$$p(t + \Delta t) = p(t)e^{W} \qquad \Big[W \sim N\big(0, \sigma\sqrt{\Delta t}\big)\Big]. \tag{9}$$

We choose a time step of $\Delta t = 1$ minute, and an initial volatility of $\sigma = 1.0$, or 100% (annually).

We begin with a fee size of 0.3%, executing trades once per minute over the course of a month. For each simulation, we plot the running ILF, as well as the underlying price time series $\{p(t)\}$ that is responsible. A typical run appears in Figure 10:

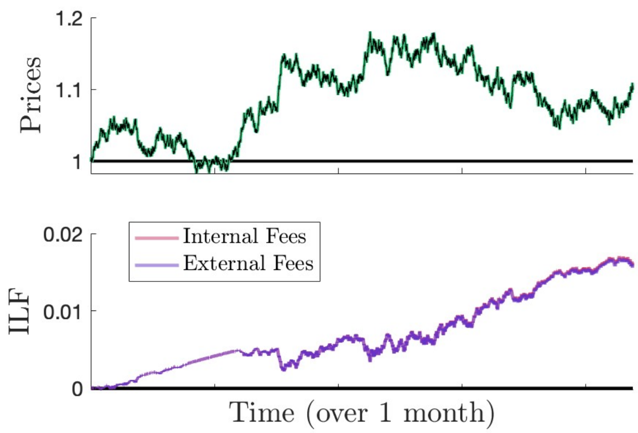

Figure 10. Fee=0.3%, volatility = 100%

We make a few observations.

- At times when the price deviates far from the initial price, the ILF makes a significant drop, as we'd expect. Additionally, when the price stabilizes again, the ILF steadily grows.

- With the current choice of parameters, the ILF for the two different fee structures move in lockstop and have no discernible difference.

However, if we run the simulation over a longer period of time, say over 6 months, then the differences begin to emerge. A typical run appears in Figure 11:

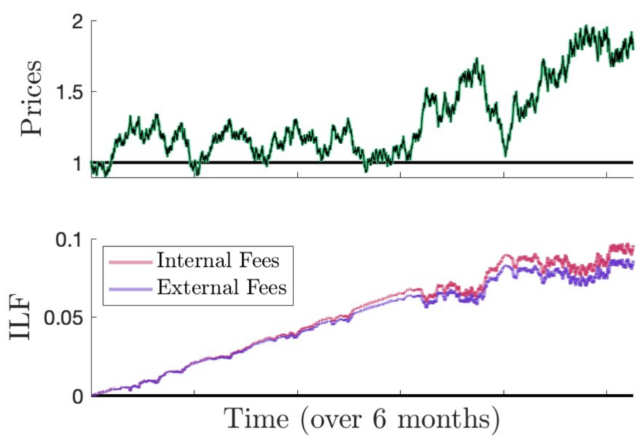

Figure 11. Fee=0.3%, volatility = 100%

This agrees with our observations from the Round-Trip model; the advantage of using internal fees requires some time to emerge. By eye, the difference appears to emerge roughly two months into the simulation. However, *unlike* our Round-Trip model, the internal fee structure *does not* underperform before this point. Rather, upon zooming in, they seem to be indiscernible.

If we increase the fee size to 1%, then the difference in ILF is clear and present (Figure 12):

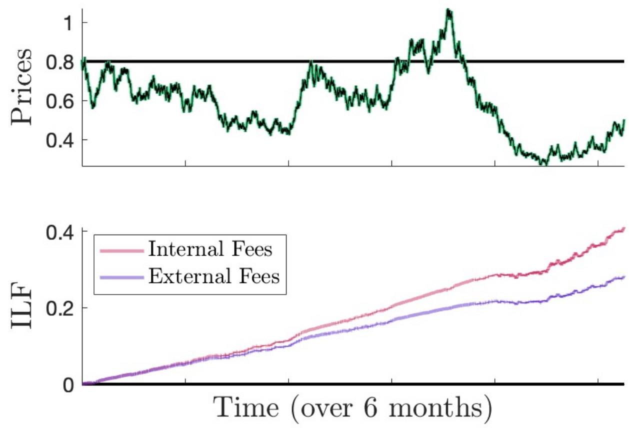

Figure 12. Fee=1%, volatility = 100%

This also confirms our observations from before that larger fee sizes amplifies the advantage of internal fee structures (see Figure 6 again).

So far, we have confirmed observations that were mostly drawn from the Round-Trip model. To approach the One-Way model, we consider raising the volatility. Let us increase it to 200% (i.e. $\sigma = 2.0$), and return to a fee size of 0.3%. A typical run is pictured in Figure 13:

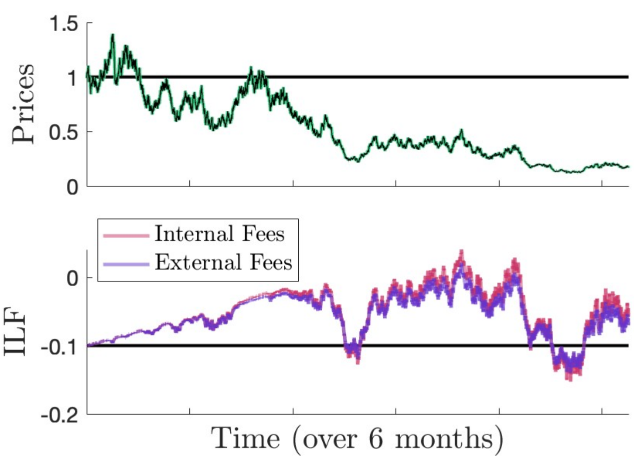

Figure 13. Fee=0.3%, volatility = 200%

Two interesting things emerge:

- Unlike our previous simulations, the ILF dips into the negative. In other words, fees cannot be trusted to always offset impermanent loss (as any real LP already knows).
- The internal fee structure does not necessarily have higher ILF, but rather *more extreme* ILF. More specifically, when the ILF goes up overall, it goes even higher for internal fees, but when the ILF goes lower overall, it goes even lower for internal fees.

This behavior, however, can be mitigated be raising the fee size. Bringing it back up to a fee of 1%, we return to strictly positive ILF, with a clearly dominant internal fee structure (Figure 14).

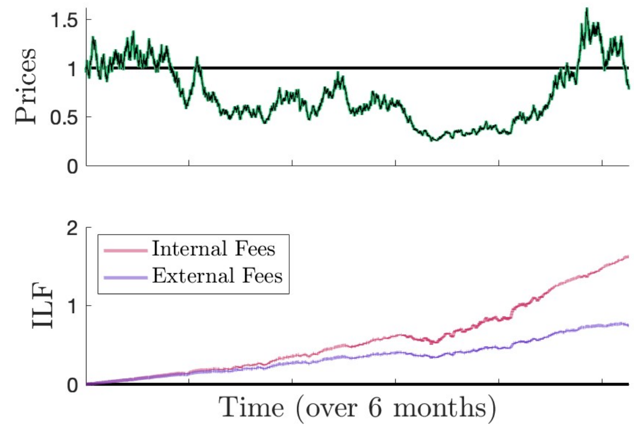

Figure 14. Fee=0.3%, volatility = 200%

So far we have only displayed individual simulations, but any single run is nothing but an anecdote. In order to gather some statistics, we next run 1000 simulations for each set of parameters explored above, and for each simulation we record the value of the *final* ILF (after 6 months). We can then view the distribution of outcomes. For example, in Figure 15, we do this with fee of 0.3% and volatility of 100%:

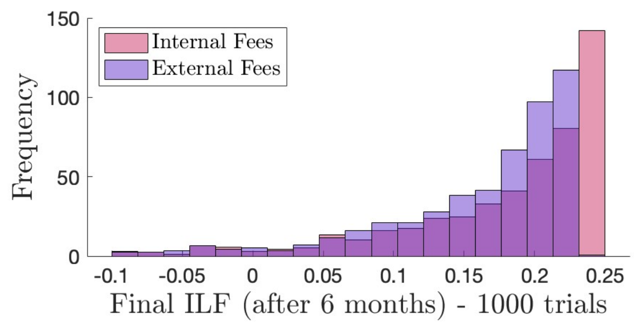

Figure 15. Fee=1%, volatility = 100%

In Figure 15 we find that internal fees marginally out perform external fees, being over represented in the upper extreme end of the final ILF distribution. More interestingly, if we look at the final ILF difference *per trial*, we see that internal fees dominate in the vast majority of trials (about 95% of them) in Figure 16:

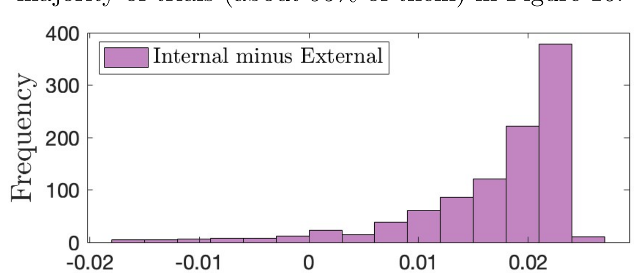

Figure 16. Fee=0.3%, volatility = 100%

When do this again (1,000 trials) but with a fee size of 1%, then the distributions clearly separate, shown below in Figure 17 (capturing the phenomena seen observed in Figure 12):

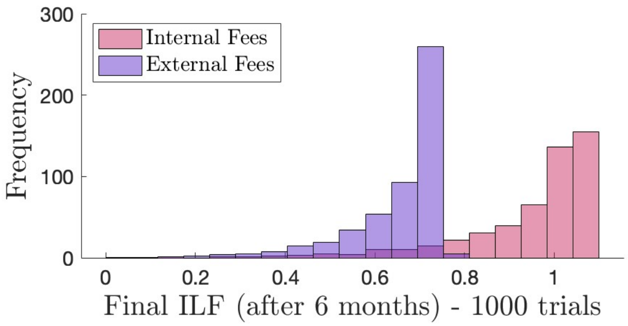

Figure 17. Fee=1%, volatility = 100%

This time, the difference in final ILF values *per trial* shows internal fees being desirable in 100% of the trials:

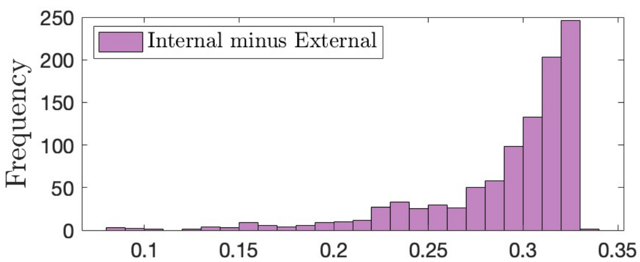

Figure 18. Fee=1%, volatility = 100%

Lastly, we bring the volatility back up to 200% for Figures 19 through 22. We basically find exaggerated versions of the previous results:

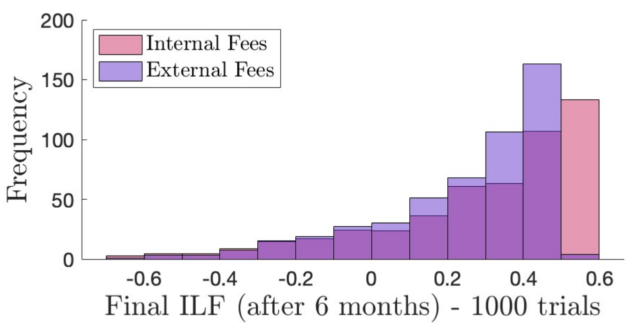

Figure 19. Fee=0.3%, volatility = 200%

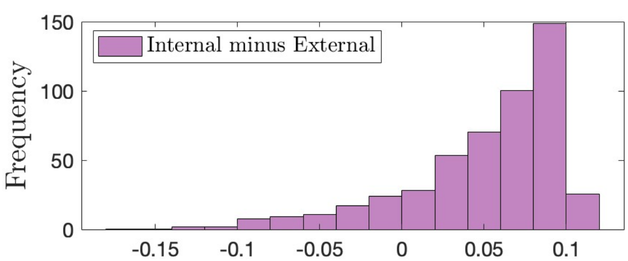

Figure 20. Fee=0,3%, volatility = 200%

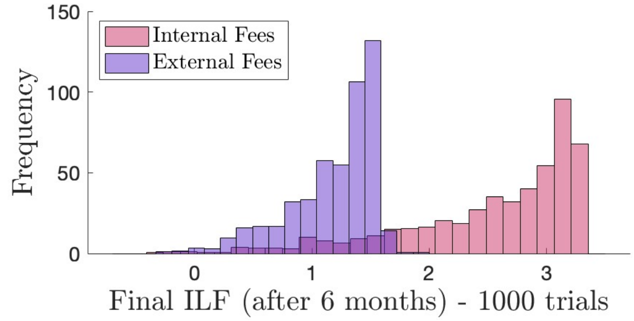

Figure 21. Fee=1%, volatility = 200%

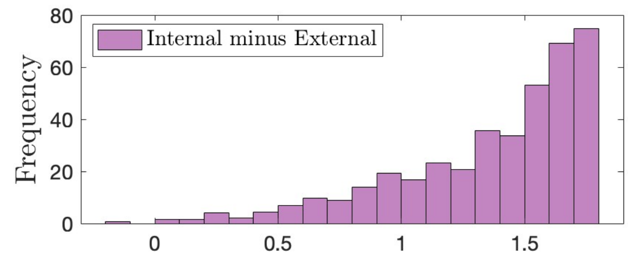

Figure 22. Fee=1%, volatility = 200%

—— Section 4: Discussion & Conclusions ——

Now, let us summarize what we have observe. Through mathematical toy models, and direct simulation, we hypothesize the following claim (for a constant product trading pool):

- Using an internal fee structure can be preferable to an external fee structure. In particular, the internal fee is advantageous, provided the following:
  - The LP period is long (*the more trades, the higher the advantage*)
  - The fee size is large (*the higher the fee size, the higher the advantage*)
  - The volatility is high (*the higher the volatility, the higher the advantage - although this principle only really applies if the fee size is already large*).
- For smaller fee sizes, and higher volatilities, the internal fee structure can have more *extreme* ILF values (both higher and lower), and could therefore be less preferable, at least to the risk-adverse.

Finally, we note that the advantages articulated in the first two observations above are due primarily to the fact that internal fees can *compound*. Specifically, more liquidity in the pool means that greater fees can be taken (assuming similar price movements), and this has a feedback effect.

In this appendix, we derive expressions (4)-(8) for our two toy models. We begin by collecting a few results, but we won't put them all together until the next page. First, we recall that in the constant product protocol, the constant product formula $xy = k$ and the price $p = y/x$ can be solved for x and y:

$$x = \sqrt{k/p}, \qquad y = \sqrt{kp} \tag{10}$$

Now consider a transaction that moves the market from price p_1 to p_2, with $p_1 > p_2$. Because the price is going down, it is $\Delta x = \sqrt{k/p_2} - \sqrt{k/p_1}$ that is paid into the pool. If we are using external fees, then this means an amount $\left(\sqrt{k/p_2} - \sqrt{k/p_1}\right)/\gamma$ must have been originally paid, and hence the fee taken from this transaction would be $(1-\gamma)\left(\sqrt{k/p_2} - \sqrt{k/p_1}\right)/\gamma$. An analogous statement can be made when we move from p_2 to p_1, paying into the y direction. Thus, we have

$$\underset{[p_1 \leftrightarrow p_2]}{\text{Fee}} = \begin{cases} \left(\sqrt{k/p_2} - \sqrt{k/p_1}\right)(1\text{-}\gamma)/\gamma & p_1 \to p_2 \\ \left(\sqrt{kp_1} - \sqrt{kp_2}\right)(1\text{-}\gamma)/\gamma & p_2 \to p_1 \end{cases} \tag{11}$$

for *external fees.*

Now, for internal fees, there *are no* external fees, but we must instead consider the changes in liquidity. This is a little more subtle. First, the constant product relation $(x + \Delta x)(y + \Delta y) = xy$ allows us to convert an incoming quantity into an outgoing quantity. For example, if we input $\Delta x > 0$ into the pool, then the equation can be solved for the outgoing Δy:

$$\Delta y = \frac{-y\Delta x}{x + \Delta x} \tag{12}$$

When fees are taken (internal *or* external), we calculate the outgoing amount by using a fraction γ of the incoming amount. So, for example, (12) becomes

$$\Delta y = \frac{-y(\gamma\Delta x)}{x + (\gamma\Delta x)} \tag{13}$$

Suppose again that we are moving from p_1 to p_2. If the price at p_1 is given by $p_1 = y/x$, then the price at p_2 is given by $p_2 = (y + \Delta y)/(x + \Delta x)$. By substituting our expression (13) for Δy, we get

$$p_2 = \frac{y + \frac{-y\gamma\Delta x}{x+\gamma\Delta x}}{x + \Delta x} = \frac{xy}{(x + \Delta x)(x + \gamma\Delta x)} \tag{14}$$

If we clear the denominator in (14), this rearranges into a quadratic equation for Δx:

$$0 = \Big[\gamma\Big]\Delta x^2 + \Big[x(1+\gamma)\Big]\Delta x + x^2\Big[1 - p_1/p_2\Big] \tag{15}$$

The solution (upon simplification) is given by

$$\Delta x = x\left[\frac{(1+\gamma)}{2\gamma}\left(\sqrt{1 + \frac{4\gamma(p_1/p_2 - 1)}{(1+\gamma)^2}} - 1\right)\right] \tag{16}$$

This coefficient of x in (16) will be a repeatedly relevant quantity, so we give it the name δ. Moreover, we use $\alpha < 1$ to denote the factor $p_2 = \alpha p_1$ between prices, and we express δ in terms of this:

$$\delta := \frac{(1+\gamma)}{2\gamma}\left(\sqrt{1 + \frac{4\gamma(1/\alpha - 1)}{(1+\gamma)^2}} - 1\right) \tag{17}$$

If we were to repeat this analysis but in the opposite direction (from p_2 to p_1), then we would have an expression analogous to (16) but for Δy. The only effected term would be the ratio p_1/p_2. But this is actually invariant, because switching the roles of p_1 and p_2 is accomplished by switching the roles of x and y and *inverting* the prices, and this leaves p_1/p_2 unchanged. Thus, the ratio would remain the same, and hence

$$\left\{\begin{array}{ll} \Delta x = x\,\delta & \text{if } p_1 \to p_2 \\ \Delta y = y\,\delta & \text{if } p_2 \to p_1 \end{array}\right\} \tag{18}$$

Now, if we are taking internal fees on our One Way sequence of prices $p_n = p_0\alpha^n$, then we can use (18) to define the increment $\Delta x_n = \delta x_{n-1}$, and so we have

$$\begin{aligned} x_n &= x_{n-1} + \Delta x_n \\ &= x_{n-1} + \delta x_{n-1} \\ &= x_{n-1}(1+\delta) \end{aligned} \tag{19}$$

Inductively, we conclude that $x_n = x_0(1+\delta)^n$. Moreover, using this result together with (13), we can calculate y_n:

$$\begin{aligned} y_n &= y_{n-1} + \Delta y_n \\ &= y_{n-1} - \frac{y_{n-1}(\gamma\Delta x_n)}{x_{n-1} + (\gamma\Delta x_n)} \\ &= y_{n-1}\left(1 - \frac{\gamma\,\delta\,x_0(1+\delta)^{n-1}}{x_0(1+\delta)^{n-1} + \gamma\,\delta\,x_0(1+\delta)^{n-1}}\right) \\ &= y_{n-1}\left(\frac{1}{1+\gamma\delta}\right) \end{aligned} \tag{20}$$

and again we inductively conclude $y_n = y_0/(1+\gamma\delta)^n$. To summarize, the values of (x_n, y_n) for this situation are

$$\left\{\begin{array}{l} x_n = x_0(1+\delta)^n \\ y_n = y_0\left(\frac{1}{1+\gamma\delta}\right)^n \end{array}\right\} \text{ (Internal Fees, One Way)} \tag{21}$$

Importantly, we note that after n movements, the liquidity product $k_n = x_n y_n$ will be given by

$$\begin{aligned} k_n &= x_n y_n \\ &= x_0(1+\delta)^n y_0\left(\frac{1}{1+\gamma\delta}\right)^n \\ &= k_0\left(\frac{1+\delta}{1+\gamma\delta}\right)^n \end{aligned} \tag{22}$$

Moreover, because of the fact that δ is invariant whether the price increases or decreases, then (22) is valid regardless of the direction of the transaction. This is useful for when we consider the values (x_n, y_n) after n-many Round Trips (which terminate at price p_1):

$$\begin{aligned} x_n &= \sqrt{k_{2n}/p_1} \\ &= \sqrt{k_0\left(\frac{1+\delta}{1+\gamma\delta}\right)^{2n}/p_1} \\ &= x_0\left(\frac{1+\delta}{1+\gamma\delta}\right)^{n} \end{aligned} \tag{23}$$

Note we have 2 factors of (22), for *one* round trip. An identical calculation can be done for y_n, resulting in

$$\left\{\begin{array}{l} x_n = x_0\left(\frac{1+\delta}{1+\gamma\delta}\right)^n \\ y_n = y_0\left(\frac{1+\delta}{1+\gamma\delta}\right)^n \end{array}\right\} \quad \text{(Internal Fees, Round Trip)} \tag{24}$$

With this, we have all the formulas we need to finally analyze the ILF for our two models. We will write everything with respect to the y denomination (an arbitrary and immaterial convention), so that a pair of assets (x, y) has a total value of $px + y$ (the factor of p converts x tokens into y denomination).

Round Trip: For the external fees, the value in the pool is unchanged when we make a round trip journey. Therefore, the only thing appearing in the numerator of (2) will be the fees collected after n round trips, for which we use (11), converting the 'x' fees with a factor of p_1. Moreover, the denominator of (2) is simply the entire value held in the pool at p_1, which is given by $p_1x_1 + y_1$:

$$\begin{aligned} \underset{\substack{External \\ \text{(Round-Trip)}}}{\text{ILF}} &= n\left(\frac{1\text{-}\gamma}{\gamma}\right)\frac{p_1\left(\sqrt{k/p_2}-\sqrt{k/p_1}\right)+\left(\sqrt{kp_1}-\sqrt{kp_2}\right)}{x_1p_1+y_1} \\ &= n\left(\frac{1\text{-}\gamma}{\gamma}\right)\frac{p_1\left(1/\sqrt{p_2}-1/\sqrt{p_1}\right)+\left(\sqrt{p_1}-\sqrt{p_2}\right)}{2\sqrt{p_1}} \\ &= n\left(\frac{1\text{-}\gamma}{\gamma}\right)\left(\frac{1/\sqrt{\alpha}-\sqrt{\alpha}}{2}\right) \end{aligned} \tag{25}$$

Meanwhile, putting (24) into (2), and remembering that the initial and final prices are equal after round trips, we compute the ILF for internal fees:

$$\begin{aligned} \underset{\substack{Internal \\ \text{(Round-Trip)}}}{\text{ILF}} &= \frac{[(y_n/x_n)x_n + y_n] + 0}{[(y_n/x_n)x_0 + y_0]} - 1 \\ &= \frac{2y_0\left(\frac{1+\delta}{1+\gamma\delta}\right)^n}{(y_0/x_0)x_0 + y_0} - 1 \\ &= \left(\frac{1+\delta}{1+\gamma\delta}\right)^n - 1 \end{aligned} \tag{26}$$

Thus, (25)-(26) confirm the expressions in (4)-(5).

One Way: For the external fees, the constant product factor k is unchanged, so for each price $p_n = p_0\alpha^n$, we know the values of x_n and y_n:

$$\left\{\begin{array}{l} x_n = \sqrt{k/(p_0\alpha^n)} = x_0\alpha^{-n/2} \\ y_n = \sqrt{k(p_0\alpha^n)} = y_0\alpha^{n/2} \end{array}\right\} \quad \text{(External Fees, One Way)} \tag{27}$$

The fees taken on the n^{th} transaction ΔF_n is given by the generalization of (11):

$$\begin{aligned} \Delta F_n &= \frac{(1-\gamma)}{\gamma}\left(\sqrt{k/p_n}-\sqrt{k/p_{n-1}}\right) \\ &= \frac{(1-\gamma)}{\gamma}\sqrt{k/p_0}\left(\sqrt{1/\alpha^n}-\sqrt{1/\alpha^{(n-1)}}\right) \\ &= \frac{(1-\gamma)}{\gamma}x_0\left(\alpha^{-n/2}-\alpha^{-(n-1)/2}\right) \end{aligned} \tag{28}$$

The total amount of fees aggregated by the n^{th} transaction F_n is given by a telescoping sum:

$$\begin{aligned} F_n &= \sum_i^n \Delta F_i \\ &= \frac{(1-\gamma)}{\gamma}x_0\left[\left(\alpha^{-1/2}-\alpha^{-0/2}\right)+\left(\alpha^{-2/2}-\alpha^{-1/2}\right)+ \right. \\ &\qquad \left. ... + \left(\alpha^{-n/2}-\alpha^{-(n-1)/2}\right)\right] \\ &= \frac{(1-\gamma)}{\gamma}x_0\left[\alpha^{-n/2}-1\right] \end{aligned} \tag{29}$$

We can plug (27) and (29) into (2) to compute the ILF:

$$\begin{aligned} \underset{\substack{External \\ \text{(One-Way)}}}{\text{ILF}} &= \frac{[p_nx_n + y_n] + p_nF_n}{[p_nx_0 + y_0]} - 1 \\ &= \frac{[2y_n + p_nF_n]}{[p_nx_0 + y_0]} - 1 \\ &= \frac{2y_0\alpha^{n/2} + (p_0\alpha^n)\frac{(1-\gamma)}{\gamma}x_0\left(\alpha^{-n/2}-1\right)}{x_0p_0\alpha^n + y_0} - 1 \\ &= \frac{2 + \frac{(1-\gamma)}{\gamma}\left(1-\alpha^{n/2}\right)}{\alpha^{n/2}+\alpha^{-n/2}} - 1 \end{aligned} \tag{30}$$

where we used the fact that $p_0 = y_0/x_0$. Finally, to compute the ILF for internal fees on the One Way trip, we use (2) and (22), and we find

$$\begin{aligned} \underset{\substack{Internal \\ \text{(One-Way)}}}{\text{ILF}} &= \frac{2y_n + p_nF_n}{x_0p_n + y_0} - 1 \\ &= \frac{2y_0\left(\frac{1}{1+\gamma\delta}\right)^n + 0}{x_0\left[\frac{y_0\left(\frac{1}{1+\gamma\delta}\right)^n}{x_0(1+\delta)^n}\right] + y_0} - 1 \\ &= \frac{2(1+\delta)^n}{1+(1+\delta)^n(1+\gamma\delta)^n} - 1 \end{aligned} \tag{31}$$

Expressions (30)-(31) confirm the formulas in (6)-(7), as promised.