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Taking the Continuum Limit of Concentrated Liquidity

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In a Concentrated Liquidity AMM, the space of possible prices $[0, \infty)$ is divided into a collection of discrete *ticks* $\{p_i\}$ which allows LPs to concentrate liquidity into specific price ranges. In this article, we look at what happens when we let the tick spacing go to zero, i.e. the continuum limit, and we find a general framework for arbitrary liquidity distributions. While this will primarily be a mathematical exercise, there is potential for actual applications. The document will be divided into 4 sections:

- Section 1: Background & Motivation
- Section 2: Taking the Continuum Limit
- Section 3: The Mechanics
- Section 4: Discussion

In this section, we first review the basic structure of the Constant Product and Concentrated liquidity AMM. Not only will this serve as motivation for what is to follow, but we will also establish some foundational mathematical relationship that we will use as a starting point in the next section.

For a liquidity pool consisting of two asset types, we represent the state of the pool as a point on the x-y plane, where the x and y coordinates represent the amounts of the two assets present in the pool. The *constant product* AMM operates off of the basic rule that any swap must keep the state of the pool on the curve $xy = k$, for some constant k. The hyperbolic curve $xy = k$ has the special feature that the slope p of the tangent line (and therefore the exchange rate for small movements along the curve) is equal to the ratio of coordinates $p = y/x$ (Figure 1).

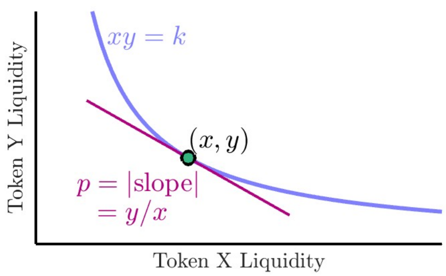

Figure 1. The constant product hyperbola

The equations $xy = k$ and $p = y/x$ can be solved for x and y in terms of p and k, although it is more common to work with the constant $L := \sqrt{k}$. One finds the following:

$$x = L/\sqrt{p} \tag{1}$$
$$y = L\sqrt{p} \tag{2}$$

For a *concentrated liquidity* AMM, a liquidity provider chooses a price range $[p_a, p_b]$ on which to provide liquidity. Each point on the curve $xy = k$ corresponds to a unique price, and thus the price range $[p_a, p_b]$ corresponds to a curve *segment*. One can imagine such a segment being shifted down towards the coordinate axes, as in Figure 2. The advantage of doing so is that now trades may occur along this segment, in exactly the same way they would on the original hyperbola, but with less overall liquidity required to facilitate it.

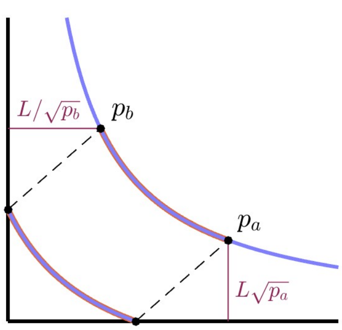

Figure 2. Shifting a segment to the coordinate axes

For a point along the shifted curve segment corresponding to some price $p \in [p_a, p_b]$, one easily sees from the figure that the coordinates will be given by formulas (1)-(2), but with a shift:

$$x = L/\sqrt{p} - L/\sqrt{p_b} \tag{3}$$
$$y = L\sqrt{p} - L\sqrt{p_a}. \tag{4}$$

Of course, this particular curve segment can *only* facilitate trades that occur within the range $[p_a, p_b]$.

To facilitate trades on a much wider price range, we next imagine a collection of LPs, each one of them making an individual choice of a price range $[p_a, p_b]$ in which to provide liquidity, along with an individual choice of a liquidity factor L. The complete collection of these prices $\{(p_a, p_b)\}$ can be listed in ascending order and given an index (say, i), thus partitioning our price axis along a set of prices $\{p_i\}$, called *ticks*. Between each pair of consecutive ticks, we have a range $[p_i, p_{i+1}]$. On this range, the total liquidity that is available to facilitate trades is that which comes from any LP whose chosen price range $[p_a, p_b]$ overlaps with $[p_i, p_{i+1}]$. Thus the liquidity factor L_i corresponding to the range $[p_i, p_{i+1}]$ will just be the *sum* of the L factors from each relevant LP. In Figure 3, we illustrate this notion. A set of simple positions $\{[p_a, p_b], L\}$ are represented as rectangular regions along the price axis, where height corresponds to the factor L. The net result (plotted at bottom) is a sequence of ticks $\{p_i\}$ and a distribution of resultant liquidity factors contained within each pair of adjacent ticks.

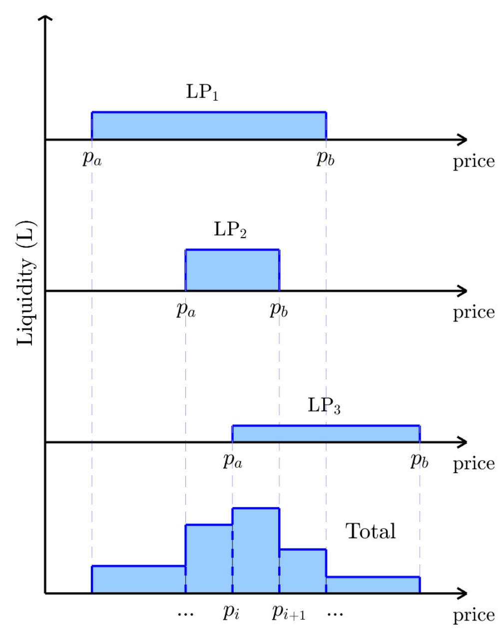

Figure 3. Superposition of LP positions

Our goal for the remainder of this document will be to work out a framework in which we consider infinitesimal tick spacing (taking the continuum limit) that allow for *smooth* liquidity distributions to replace the staggered step functions seen above.

We are now ready to begin our journey towards the continuum limit and to discover a framework for general liquidity distributions. As equations (1)-(2) and (3)-(4) illustrate, the values of the coordinates x and y will naturally be functions of the current price p. With this in mind, our ultimate goal will be to find a general way of defining a pair of functions $x(p)$ and $y(p)$ for *arbitrary* liquidity distributions, and more importantly, to articulate the corresponding rules for making swaps.

Because the concentrated liquidity AMM will be our foundation, we will start by adding a bit more information to formulas (3)-(4). Specifically, because these formulas only apply for prices within the range $[p_a, p_b]$, then we make the following observations:

- When $p = p_a$, the formulas give
$$x = L\big(1/\sqrt{p_a} - 1/\sqrt{p_b}\big), \quad y = 0, \tag{5}$$
and thus all assets are converted into the x type.

- When $p = p_b$, we have
$$x = 0, \quad y = L\big(\sqrt{p_b} - \sqrt{p_a}\big) \tag{6}$$
and all assets are converted into the y type.

- When the market price p is outside of $[p_a, p_b]$, trades are no longer possible on the curve segment, and so one of (5) or (6) will simply hold statically until the market price comes back in to the range.

We can articulate these observations by introducing the *box* notation (for any real numbers t, a, b):

$$[t]_a^b := \left\{ \begin{array}{ll} a & \text{for } t < a \\ t & \text{for } a \le t \le b \\ b & \text{for } b < t \end{array} \right\}. \tag{7}$$

Then all of the above can be neatly summarized as

$$x(p) = L\Big(1/\sqrt{[p]_{p_a}^{p_b}} - 1/\sqrt{p_b}\Big) \tag{8}$$
$$y(p) = L\Big(\sqrt{[p]_{p_a}^{p_b}} - \sqrt{p_a}\Big), \tag{9}$$

which is valid for any $p > 0$.

When our liquidity pool consists of a collection of positions $\{(p_a, p_b), L\}$, they give rise to a sequence of ticks $\{p_i\}$ (as in Figure 3), where each subrange $[p_i, p_{i+1}]$ has a corresponding liquidity factor L_i. If we denote the asset coordinates for each subrange by $x_i(p)$ and $y_i(p)$, then these will be given by expressions identical to (8)-(9):

$$x_i(p) = L_i\Big(1/\sqrt{[p]_{p_i}^{p_{i+1}}} - 1/\sqrt{p_{i+1}}\Big) \tag{10}$$
$$y_i(p) = L_i\Big(\sqrt{[p]_{p_i}^{p_{i+1}}} - \sqrt{p_i}\Big). \tag{11}$$

Note these are valid for any $p > 0$, as well.

The maximum possible values of x_i and y_i (analogous to expressions (5) and (6)), will be denoted by Δx_i, Δy_i:

$$\Delta x_i := L_i\Big(1/\sqrt{p_i} - 1/\sqrt{p_{i+1}}\Big) \tag{12}$$
$$\Delta y_i := L_i\Big(\sqrt{p_{i+1}} - \sqrt{p_i}\Big) \tag{13}$$

We can use these expressions to write down the *total* amount of liquidity in the pool. Suppose the current price p is in interval with index i_c, that is $[p_{i_c}, p_{i_c+1}]$. There will be assets of both the x and y type in this subrange. However, from our previous observations, we see that all subranges *above* this range will contain only the x type, while subranges *below* this range will contain only the y type. Thus, the *total* liquidity in the pool, which we denote $x(p)$ and $y(p)$, will be given by summing:

$$x(p) = x_{i_c}(p) + \sum_{i > i_c} \Delta x_i, \tag{14}$$
$$y(p) = y_{i_c}(p) + \sum_{i < i_c} \Delta y_i. \tag{15}$$

Expressions (12)-(13) and (14)-(15) are begging for a continuum limit.

To take this limit, we first rearrange (12)-(13):

$$\Delta x_i = -L_i\left[\frac{1/\sqrt{p_{i+1}} - 1/\sqrt{p_i}}{p_{i+1} - p_i}\right](p_{i+1} - p_i) \tag{16}$$
$$\Delta y_i = L_i\left[\frac{\sqrt{p_{i+1}} - \sqrt{p_i}}{p_{i+1} - p_i}\right](p_{i+1} - p_i) \tag{17}$$

Using the two derivatives

$$\frac{d}{dp}\Big(1/\sqrt{p}\Big) = -\frac{1}{2p\sqrt{p}}, \qquad \frac{d}{dp}\Big(\sqrt{p}\Big) = \frac{1}{2\sqrt{p}}, \tag{18}$$

we take the limit as the intervals shrink in size so that $|p_{i+1} - p_i| \to dp$. The difference quotients in (16)-(17) thus become

$$dx(p) = L(p)\left(\frac{1}{2p\sqrt{p}}\right)dp \tag{19}$$
$$dy(p) = L(p)\left(\frac{1}{2\sqrt{p}}\right)dp \tag{20}$$

We note that in (19)-(20), we replaced the dependence on the index i with the continuous variable p, because each interval $[p_i, p_{i+1}]$ has has been shrunk down to an instantaneous spot price. In this context, we should think of $L(p)$ as being a liquidity *density*. Moreover, we note that (19)-(20) can be put into a slightly more symmetric form:

$$dx(p) = \Big[L(p)/\sqrt{p}\Big]\,\frac{dp}{2p} \tag{21}$$
$$dy(p) = \Big[L(p)\sqrt{p}\Big]\,\frac{dp}{2p} \tag{22}$$

Expressions (21)-(22) are encouraging, considering their similarity to our original expressions in (1)-(2).

Meanwhile, the expressions in (14)-(15) are clearly meant to become integrals. We note that the isolated i_c term should be dropped altogether, as it vanishes in the limit (any individual subrange becomes infinitesimally thin - they only count if they are aggregated in a sum). Moreover, the ranges $i < i_c$ and $i_c < i$ should be replaced with $(0, p)$ and (p, ∞), respectively. Lastly, because we are using p to denote the current price, we will use the variable ρ for integration. Thus, (14)-(15) become

$$x(p) = \int_p^\infty dx(\rho) = \int_p^\infty \Big[L(\rho)/\sqrt{\rho}\Big] \frac{d\rho}{2\rho} \tag{23}$$

$$y(p) = \int_0^p dy(\rho) = \int_0^p \Big[L(\rho)\sqrt{\rho}\Big] \frac{d\rho}{2\rho} \tag{24}$$

One checks that, assuming $L(\rho) \geq 0$ for all ρ, then we will have $x(p)$ monotonically decreasing with p, and $y(p)$ monotonically increasing with p. This makes sense; as the price increases, we should *gain* y tokens and *lose* x tokens.

We also note that there is an (optional) change of variables that, in some sense, may be more natural. In particular, the appearance of the $(dp/2p)$ term in our integrals suggest that we make a change of variables defined by

$$w := \log\big(\sqrt{p}\big), \tag{25}$$

for then we have

$$dw = dp/(2p). \tag{26}$$

If we define the corresponding liquidity density

$$\ell(w) := L\Big(p(w)\Big), \tag{27}$$

and note that (25) implies that $\sqrt{p} = e^w$, then our expressions become

$$x(p) = \int_{w(p)}^\infty dx(\omega) = \int_{w(p)}^\infty \Big[\ell(\omega)e^{-\omega}\Big]\, d\omega \tag{28}$$

$$y(p) = \int_{-\infty}^{w(p)} dy(\omega) = \int_{-\infty}^{w(p)} \Big[\ell(\omega)e^{\omega}\Big]\, d\omega \tag{29}$$

Section 3: The Mechanics

For this framework to be consistent, a trade that moves the price from p_1 to p_2 must have corresponding incoming and outgoing quantities given by

$$\Delta x = x(p_2) - x(p_1), \tag{30}$$

$$\Delta y = y(p_2) - y(p_1). \tag{31}$$

Using the definitions of $x(p)$ and $y(p)$ in (23)-(24), and the basic algebra of integrals, this is equivalent to

$$\Delta x = -\int_{p_1}^{p_2} \Big[L(\rho)/\sqrt{\rho}\Big] \frac{d\rho}{2\rho}, \tag{32}$$

$$\Delta y = \int_{p_1}^{p_2} \Big[L(\rho)\sqrt{\rho}\Big] \frac{d\rho}{2\rho}. \tag{33}$$

Note that the negative sign (32) arises because the integral in (23) has the variable p in its *lower* limit. This makes practical sense; because $L(p) \geq 0$, then we can see from expressions (32)-(33) that we have the following:

if: $p_1 < p_2$	then: $\Delta x < 0$, $\Delta y > 0$
if: $p_1 > p_2$	then: $\Delta x > 0$, $\Delta y < 0$

This is actually quite natural:

- If $p_1 < p_2$, then we are moving to a higher price, and so the pool loses some x and gains some y.
- If $p_1 > p_2$, then we are moving to a lower price, and so the pool gains some x and lose some y.

Thus, we now have naturally *signed* deltas.

For taking fees in this framework, there are two options that come to mind:

1. We may take a fee from the incoming quantity (as it is typically done in a concentrated liquidity AMM).
2. We may store the fees internally as additional pool liquidity. The mathematics of this is not entirely trivial, but the *GIFS* method (found here) will work just fine.

We won't explore the fee structure any further in this document, but the reader is encouraged to read the paper cited above.

Finally, we note that while the preceding section was not terribly rigorous, it should serve only as a motivation for exploration. At this point, we can take expressions (23)-(24) *as our definition* of the pool dynamics. More explicitly, the logic is as follows:

- An LP chooses a liquidity density $L(p)$
- At initial price p_0, the LP provides x and y tokens in the amount of $x(p_0)$ and $y(p_0)$, given by (23)-(24).
- For any trade moving the market price from p_1 to p_2, we have formulas (32)-(33) that give us the required incoming and outgoing quantities.

From the basic rules of integrals, these steps are mutually consistent, and thus the framework is well defined.

Lastly, we note that liquidity providing *superimposes*. In other words, for a collection of LPs, each with their own liquidity densities $L_n(p)$, the aggregate liquidity density for the whole pool will be given by $L = \sum_n L_n(p)$.

One may justifiably consider the preceding development to be largely just a mathematical exercise. Indeed, it is not immediately clear whether or not it is of any use to put this framework into practice. A few considerations:

- For the vast majority of choices for the liquidity density functions $L(p)$, actually *doing* the integrals in (32)-(33) will be impossible to do in any efficient manner. For most density functions, a closed form expression for the integrals will be unattainable. There will be approximations available, of course, but these may be computationally expensive.
- On the other hand, for trades that would otherwise cross *many* ticks in a concentrated liquidity pool, that *sequence* of calculations can now be replaced with a *single* calculation (the integral).

It is our opinion that, in practice, the most likely utility of this development will be to serve more as a conceptual foundation for further exploration.

As a final note, we take an arbitrary liquidity density function (Figure 4) with several notable bumps, and plot the resulting curve in the x/y plane (Figure 5). The flat regions on the curve in Figure 5 correspond with the spikes in concentrated liquidity in Figure 4.

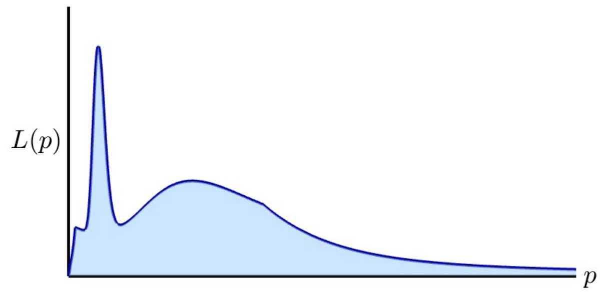

Figure 4. A possible liquidity distribution $L(p)$

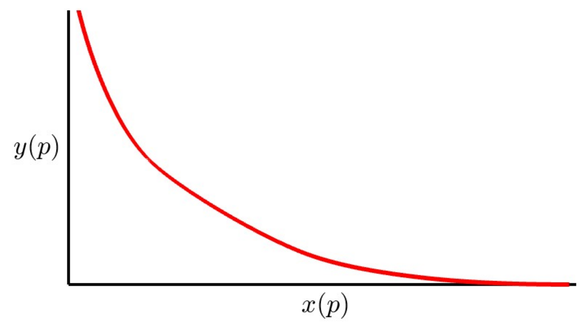

Figure 5. The corresponding curve $x(p)$ v.s. $y(p)$