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The Null Fee

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In this document, we propose a novel type of indicator called the *null fee* that may be a useful ingredient in the design of a dynamic fee algorithms for concentrated liquidity and constant product AMMs. This will be divided into five sections:

- **Section 1: The Null Fee Idea -** Here we give a brief overview of the motivation and basic idea of the *null fee* indicator.
- **Section 2: Deriving The Formula -** In this section we go through the complete mathematical derivation for the null fee formula (in the context of concentrated liquidity).
- **Section 3: How Do We Use This? -** We apply our null fee formula to some historical price data and make observations.
- **Section 4: Extending To Internal Fees -** In this section, we derive an alternative formula for the null fee in the context of constant product.
- **Section 5: Summary -** Finally, we give a brief summary of our formulas and results.

Section 1: The Null Fee Idea

In the world of AMMS, the notion of dynamically changing swap fees is not unheard of. Such fees would typically change in response to market conditions, in order to protect the liquidity providers (LPs). But how does one design a protocol for choosing these dynamic fees? We will propose a specific type of novel indicator called the *null fee*.

For the majority of this document, we will assume that our AMM is a Concentrated Liquidity protocol, as in Uniswap V3. In order to sensibly design a dynamic fee structure, we need to establish a clear motivating objective. With this in mind, we will take as our goal the mitigation of Impermanent Loss (IL), that is, the fractional difference of the value of the LP's assets at a current moment, compared to what it would be had they not provided liquidity at all.

It is well know that impermanent loss will always be negative. The size of this loss increases as the market price moves (in either direction) away from the price upon entry, and thus it can only be zero if the market price returns to wherever it was when the LP entered.

To compensate for this loss, fees are taken from every swap. When the fees are accounted for, we then denote the impermanent loss by ILF ('IL with fees'). Naturally, this quantity may be negative *or* positive. Roughly speaking, a relatively stationary market with many swaps but little net movement will be best for the LP (i,.e. positive ILF). Conversely, a market with notable drift (but little chop back and forth) is the worst case scenario for an LP (i.e. negative ILF). A dynamically changing fee should be lower in the former scenario, and higher in the latter.

This sounds simple enough, in principle, but it is not clear how to *quantify* this notion, and *use* it to generate a suitable dynamic fee. Some have suggested *volatility* could be a sufficient indicator, but it is doubtful that volatility can capture the nuances, the differences, and the interplay between *chop* and *drift*. Instead, we define a novel quantity called the *null fee* that is meant to make sense of all of this in a very natural way.

We proceed as follows. We consider a fixed window of time (on the order of several hours, for example).

- we consider a hypothetical LP that maintains a position only over this window
- based on the fixed fee size the LP may have a positive or negative ILF
- we compute the precise hypothetical fee size that *would have* resulted in ILF=0 for this window (i.e. the fee size such that LP breaks even)
- we call this value the *Null Fee* for this window

If we use the most recent market data as our running window, we then get a real time indicator that should guide us towards a natural choice for a dynamic fee. Rather than use this number directly, however, we may modify it slightly, as we will discuss in later sections.

Section 2: Deriving The Formula

First, we recall some important formulas pertaining to concentrated liquidity AMM. The state of liquidity that a particular LP can claim to own is described by an ordered pair (X, Y), corresponding to the amount of the two token types X and Y. These two values will be functions of the current price p. Specifically, for an LP providing in range $[p_a, p_b]$ with liquidity scale L, we have

$$X(p) := L\Big(1/\sqrt{[p]_{p_a}^{p_b}} - 1/\sqrt{p_b}\Big) \tag{1}$$

$$Y(p) := L\Big(\sqrt{[p]_{p_a}^{p_b}} - \sqrt{p_a}\Big) \tag{2}$$

where the box notation $[\cdot]$ is defined by

$$[t]_a^b = \left\{ \begin{array}{cc} a & t < a \\ t & a \le t \le b \\ b & b < t \end{array} \right\}. \tag{3}$$

The impermanent loss (without fees taken) over a sequence of price movements $\{p_n\}_{n=0}^N$ is given by

$$\begin{aligned} \text{IL} &= \frac{\begin{bmatrix}\text{LP current value} \\ \text{in pool}\end{bmatrix}}{\begin{bmatrix}\text{LP current value} \\ \textbf{if} \text{ stayed out of pool}\end{bmatrix}} - 1 \\ &= \frac{\Big[p_N X(p_N) + Y_N\Big]}{\Big[p_N X(p_0) + Y(p_0)\Big]} - 1 \end{aligned} \tag{4}$$

For transactions occurring within the range $[p_a, p_b]$, the changes in assets (going from p_1 to p_2) are given by

$$\Delta X := L\Big(1/\sqrt{p_2} - 1/\sqrt{p_1}\Big) \tag{5}$$

$$\Delta Y := L\Big(\sqrt{p_2} - \sqrt{p_1}\Big). \tag{6}$$

For every transaction, one of these quantities will be positive and one of them will be negative. The fee taken from each transaction is proportional to whichever one is positive, i.e. the incoming quantity. Specifically, for a fee size φ, only the fraction $1 - \varphi$ (often denoted γ) of the paid amount actually comes into the pool. If we call this amount Δ, then the original amount paid by the user must have been $\Delta/(1-\varphi)$. Therefore, the fee taken *from* this amount must have been $\varphi[\Delta/(1-\varphi)]$. We abbreviate this coefficient of Δ by $\widetilde{\varphi}$:

$$\widetilde{\varphi} := \varphi/(1-\varphi). \tag{7}$$

All together, the fees taken from either ΔX and ΔY, denoted f^X and f^Y, respectively, are given by

$$f^X = \max\Big(\widetilde{\varphi}\Delta X,\ 0\Big) =: \Big(\widetilde{\varphi}\Delta X\Big)^+ \tag{8}$$

$$f^Y = \max\Big(\widetilde{\varphi}\Delta Y,\ 0\Big) =: \Big(\widetilde{\varphi}\Delta Y\Big)^+. \tag{9}$$

For transactions that begin outside of $[p_a, p_b]$, no fees are awarded to the LP. For transactions that cross the boundary, the $(\Delta X,\ \Delta Y)$ quantities given in (5)-(6) must be modified slightly, but this is complicated and inconsequential as an edge case.

Now consider a window of time with a market trajectory consisting of the sequence of price movements $\{p_n\}_0^N$. Over such a window, the impermanent loss *including fees* (ILF) is given by the following:

$$
\begin{aligned}
\text{ILF} &= \frac{\begin{bmatrix}\text{LP current value}\\ \text{in pool + fees}\end{bmatrix}}{\begin{bmatrix}\text{LP current value}\\ \textbf{if}\text{ stayed out of pool}\end{bmatrix}} - 1 \\
&= \frac{\Big[p_N X(p_N) + Y_N\Big] + \Big[p_N \sum\limits_{n\in R} f_n^X + \sum\limits_{n\in R} f_n^Y\Big]}{\Big[p_N X(p_0) + Y(p_0)\Big]} - 1 \quad (10)
\end{aligned}
$$

where R is the set of indices corresponding to trades that begin in the range $[p_a, p_b]$ (otherwise the LP is not entitled to the fee). Moreover, the deltas implicit in the fees f_n^X and f_n^Y, are given by

$$\Delta X_n = L\Big(1/\sqrt{p_n} - 1/\sqrt{p_{n\text{-}1}}\Big) \quad (11)$$

$$\Delta Y_n = L\Big(\sqrt{p_n} - \sqrt{p_{n\text{-}1}}\Big). \quad (12)$$

We also note that we choose to represent every thing in the Y-denomination, which is why all X quantities are multiplied by the conversion factor p_N, i.e. the final market price. This is actually an immaterial choice, as the impermanent loss ends up being unitless (by the nature of the ratio). We define the *null fee* to be the value of φ that results in ILF=0. In other words, for a given set of transactions, the null fee is *the fee that would result in the LP breaking even.* If we set the ILF equal to zero, then equation (10) becomes

$$
\begin{aligned}
\Big[p_N \sum_{n\in R} f_n^X + \sum_{n\in R} f_n^Y\Big] &= \Big[p_N X(p_0) + Y(p_0)\Big] \\
&\quad - \Big[p_N X(p_N) + Y_N\Big] \quad (13)
\end{aligned}
$$

Using expressions (8)-(9), then (13) becomes

$$
\begin{aligned}
&\Big[p_N \sum_{n\in R} (\widetilde{\varphi}\Delta X_n)^+ + \sum_{n\in R} (\widetilde{\varphi}\Delta Y_n)^+\Big] \\
&\quad = \Big[p_N X(p_0) + Y(p_0)\Big] - \Big[p_N X(p_N) + Y_N\Big] \quad (14)
\end{aligned}
$$

We can then factor out $\widetilde{\varphi}$ from the left side of (14) and solve for it, giving us

$$\widetilde{\varphi} = \frac{\Big[p_N X(p_0) + Y(p_0)\Big] - \Big[p_N X(p_N) + Y_N\Big]}{\Big[p_N \sum\limits_{n\in R} (\Delta X_n)^+ + \sum\limits_{n\in R} (\Delta Y_n)^+\Big]} \quad (15)$$

We now have a function for $\widetilde{\varphi}$ (and thus φ). We can develop this further. Let us take expressions (1)-(2) and (11)-(12) and substitute them into (15), canceling a factor of L from every term:

$$\widetilde{\varphi} = \frac{\begin{array}{c}\Big[p_N\Big(1/\sqrt{[p_0]_{p_a}^{p_b}} - 1/\sqrt{p_b}\Big) + \Big(\sqrt{[p_0]_{p_a}^{p_b}} - \sqrt{p_a}\Big)\Big] \\ -\Big[p_N\Big(1/\sqrt{[p_N]_{p_a}^{p_b}} - 1/\sqrt{p_b}\Big) + \Big(\sqrt{[p_N]_{p_a}^{p_b}} - \sqrt{p_a}\Big)\Big]\end{array}}{\Big[p_N \sum\limits_{n\in R}\Big(1/\sqrt{p_n} - 1/\sqrt{p_{n\text{-}1}}\Big)^+ + \sum\limits_{n\in R}\Big(\sqrt{p_n} - \sqrt{p_{n\text{-}1}}\Big)^+\Big]} \quad (16)$$

Before we proceed with expression (16), we first make an observation. Recall that our goal is to find a useful expression for the null fee, and in particular so that we may use it as a real time indicator for designing a dynamic fee structure. As such, our expression should ideally be independent of any particular LP, and should therefore be independent of the LP choices $\{L, p_a, p_b\}$. Fortunately, the factor of L as already cancelled out, and one can easily see that the $\sqrt{p_a}$ and $\sqrt{p_b}$ terms in (16) will cancel out as well. The only place where these values persist is in the box function $[\cdot]$. Therefore, we will make the following assumption; we will only consider positions $[p_a, p_b]$ that completely contain the price trajectory $\{p_n\}$ in our window of time. In this case, every occurrence of the box notation can be replaced with its argument. Moreover, the sums may range over the index n without any constraints (such as $n \in R$). All that said, expression (16) now simplifies to

$$\widetilde{\varphi} = \frac{\Big[p_N\Big(1/\sqrt{p_0}\Big) + \Big(\sqrt{p_0}\Big)\Big] - \Big[p_N\Big(1/\sqrt{p_N}\Big) + \Big(\sqrt{p_N}\Big)\Big]}{\Big[p_N \sum\limits_{n}\Big(1/\sqrt{p_n} - 1/\sqrt{p_{n\text{-}1}}\Big)^+ + \sum\limits_{n}\Big(\sqrt{p_n} - \sqrt{p_{n\text{-}1}}\Big)^+\Big]} \quad (17)$$

Next, let's divide through by a factor of $\sqrt{p_N}$:

$$\widetilde{\varphi} = \frac{\Big[\Big(\sqrt{p_N}/\sqrt{p_0}\Big) + \Big(\sqrt{p_0}/\sqrt{p_N}\Big) - 2\Big]}{\begin{bmatrix}(\sqrt{p_N})\sum\limits_n\Big(1/\sqrt{p_n} - 1/\sqrt{p}_{n\text{-}1}\Big)^+ \\ +(1/\sqrt{p_N})\sum\limits_n\Big(\sqrt{p_n} - \sqrt{p}_{n\text{-}1}\Big)^+\end{bmatrix}} \quad (18)$$

This gives us $\widetilde{\varphi}$ in terms of the price history $\{p_n\}$, and then we can retrieve φ by inverting equation (7):

$$\varphi = \frac{\widetilde{\varphi}}{1 + \widetilde{\varphi}} \quad (19)$$

This gives us our formula for the null fee.

In practice, however, it is more common to have access to the fees acquired (per unit liquidity) over a window of time. Thus, if φ_0 was the *actual* fee size from the preceding window, and we define $\widetilde{\varphi}_0 := \varphi_0/(1-\varphi_0)$, we can multiply the top and bottom of (18) by $\widetilde{\varphi}_0$:

$$\widetilde{\varphi} = \frac{\widetilde{\varphi}_0\Big[\Big(\sqrt{p_N}/\sqrt{p_0}\Big) + \Big(\sqrt{p_0}/\sqrt{p_N}\Big) - 2\Big]}{\begin{bmatrix}(\sqrt{p_N})\sum\limits_{n\in R}\widetilde{\varphi}_0\Big(1/\sqrt{p_n} - 1/\sqrt{p_{n\text{-}1}}\Big)^+ \\ +(1/\sqrt{p_N})\sum\limits_{n\in R}\widetilde{\varphi}_0\Big(\sqrt{p_n} - \sqrt{p_{n\text{-}1}}\Big)^+\end{bmatrix}} \quad (20)$$

We can now recognize the quantities in the denominator as the fees taken per unit liquidity in the time window. Thus, we can express our formula very simply in words as the following:

$$\widetilde{\varphi} = \frac{\begin{bmatrix}\text{fee}\\ \text{size}\end{bmatrix}\Bigg[\sqrt{\dfrac{\text{final price}}{\text{initial price}}} + \sqrt{\dfrac{\text{initial price}}{\text{final price}}} - 2\Bigg]}{\Bigg[\sqrt{\begin{matrix}\text{final}\\ \text{price}\end{matrix}}\Bigg]\begin{bmatrix}\text{fees collected}\\ \text{in } X \text{ token}\\ \begin{pmatrix}\text{per unit}\\ \text{liquidity}\end{pmatrix}\end{bmatrix} + \Bigg[1/\sqrt{\begin{matrix}\text{final}\\ \text{price}\end{matrix}}\Bigg]\begin{bmatrix}\text{fees collected}\\ \text{in } Y \text{ token}\\ \begin{pmatrix}\text{per unit}\\ \text{liquidity}\end{pmatrix}\end{bmatrix}} \quad (21)$$

Section 3: How Do We Use This?

We would now like to test our null fee formula (21) with a set of historical prices. However, before we do this, there is an observation we need to discuss. Namely, the null fee measured over small time windows will almost certainly be much higher than what we actually have in mind for a dynamic fee. To see why this is, consider the fact that the aggregated fees can only ever increase over time, whereas price drift can be (possibly) reversed over time. Indeed, when when we naively calculate formula (21) as is, we find comically high fee sizes. For example, we consider the price movement for ETH between Apr/14/23 and Jul/07/23 (about 8,000 hours). These prices and null fees are shown below:

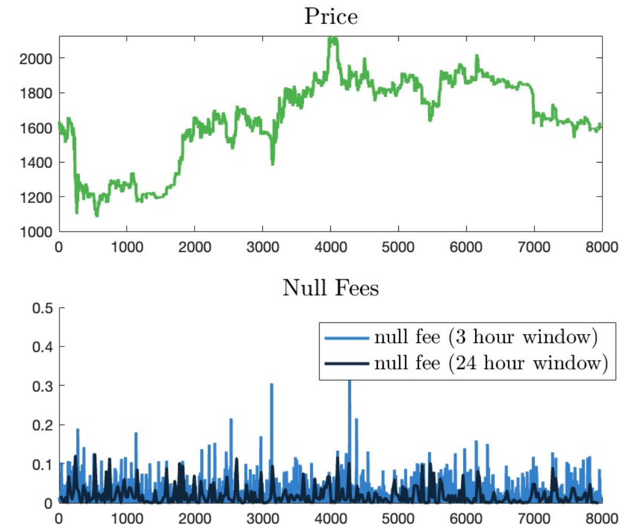

Figure 1. Prices and corresponding null fees

Over a 3 hour window, the null fee values reach up to 0.3 (30% fee!). Extending the time window alleviates the problem slightly, but even over a 24 hour window the formula is giving us a null fee on the order of 5%. For this reason, we will renormalize the null fee. To do this, we will take the output of (21) and put it through a quadratic function. We choose quadratic because there are three degrees of freedom (which we will think of as minimum, maximum, and some central value). One easily checks that the following quadratic takes the values of (x_1, x_2, x_3) to the values (y_1, y_2, y_3):

$$f(x) = \frac{(x\text{-}x_2)(x\text{-}x_3)}{(x_1\text{-}x_2)(x_1\text{-}x_3)}y_1 + \frac{(x\text{-}x_1)(x\text{-}x_3)}{(x_2\text{-}x_1)(x_2\text{-}x_3)}y_2 + \frac{(x\text{-}x_1)(x\text{-}x_2)}{(x_3\text{-}x_1)(x_3\text{-}x_2)}y_3 \quad (22)$$

Using a 12 hour window, we find an average null fee of around 0.016. Thus, as our min, max, and central value, we will choose $(x_1, x_2, x_3) = (0, 0.016, 1)$. We then map these to the values of $(y_1, y_2, y_3) = (0.003, 0.0035, 1)$ so that our transformed null fee will have a lower bound of 0.003 (0.3% fee) and most value will fall somewhere just above that. Finally, we will take an exponential running average to smoothen things out. We choose an exponential weight of 0.9, so that the most recent values have relative weights of 0.9, 0.9^2,0.9^3, etc.

For comparison, we will also compute the running volatility (standard deviation in price). Over the same sequence of 12 hour windows, we find the following:

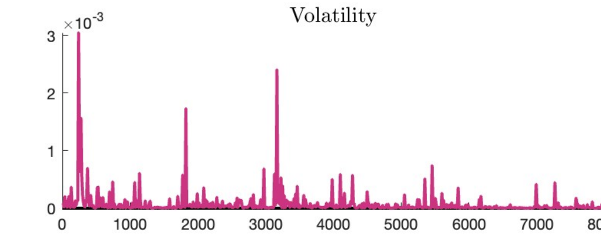

Figure 2. Volatilities (12 hour windows)

We will use the same quadratic expression from (22) to convert these values into sensible fee sizes. We take the values $(x_1, x_2, x_3) = (0, 0.0001, 0.01)$ and map them to $(y_1, y_2, y_3) = (0.003, 0.0035, 0.9)$. These choices are arbitrary, and we can always increase the payoff to LPs by choosing larger fee sizes, but these choices are made primarily so that the two indicators have *similar* effects on the ILF. Moreover, we take an exponential moving average with factor 0.9, as well.

Let's look at the results. For the given price series, we plot the (renormalized) indicators of the null fee and the volatility. Additionally, we show the raw ILF as it would be with a constant fee size of 0.3%, as well as the hypothetical ILF resulting from the dynamic fees.

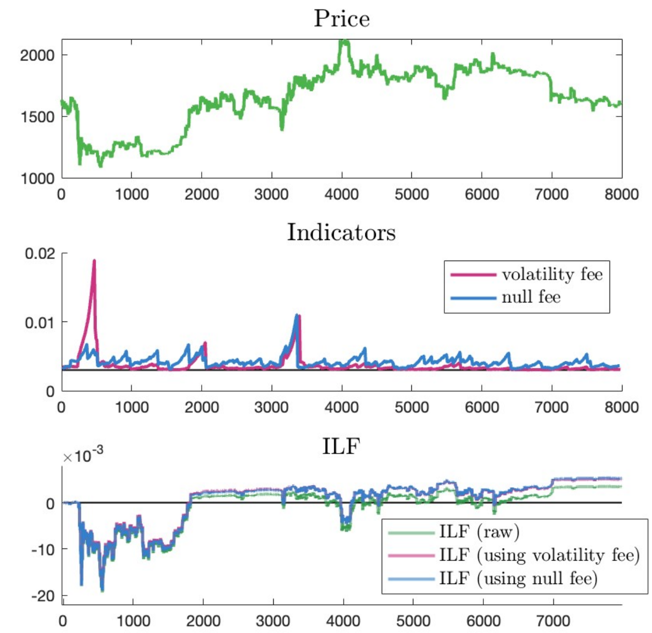

Figure 3. Indicators and resulting ILF

As promised, the ILF resulting from the two different dynamic fee algorithms ends up being roughly similar (and manifestly better than raw ILF, as we'd expect). What is more interesting is to compare the indicators themselves. To get roughly the same net results for ILF, we see from figure 3 that the volatility indicator tends to temporarily spike with extremely high fees (almost 2%), while the null fee tends to be more evenly spaced out in time, spiking occasionally, but not as dramatically. It is not necessarily obvious which approach is favorable, but it is clear that they are distinct.

If we zoom in a little bit, we can see that the ILF from the two dynamic fees are not entirely equal - sometimes one is higher than the other, and vice versa.

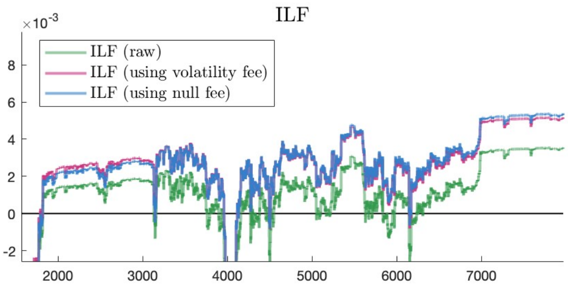

Figure 4. Zoom in from figure 3.

At the end of the day, we now have another useful tool in our arsenal for choosing dynamic fees. Notably, the null fee may be *cheaper* to calculate than the running volatility. The accumulated fees (per unit liquidity) are already tracked in Uniswap V3, and so formula (21) only involves a couple additions and divisions. Compare that to the standard deviation, which requires storing many price values, squaring them, etc.

—— Section 4: Extending To Internal Fees ——

Our formula (21) assumed the use of external fees. If we are dealing with a constant product pool (like Uniswap V2), the fees will be stored internally, and this changes the calculation significantly. We begin with the basic equations of constant product:

$$X(p) = L/\sqrt{p}, \quad Y(p) = L\sqrt{p}. \tag{23}$$

The impermanent loss (with fees) is computed by:

$$\text{ILF} = \frac{p_N X(p_N) + Y(p_N)}{p_N X(p_0) + Y(p_0)} - 1. \tag{24}$$

Where are the fees located in (24)? The liquidity factor L will have evolved from its initial value L_0 to its final value L_N:

$$\text{ILF} = \frac{p_N L_N/\sqrt{p_N} + L_N\sqrt{p_N}}{p_N L_0/\sqrt{p_0} + L_0\sqrt{p_0}} - 1. \tag{25}$$

Setting ILF$= 0$, we find (25) becoming

$$2L_N\sqrt{p_N} = p_N L_0/\sqrt{p_0} + L_0\sqrt{p_0} \tag{26}$$

Dividing both sides by $2\sqrt{p_N}L_0$ gives us the following:

$$\frac{L_N}{L_0} = \frac{1}{2}\Big(\sqrt{p_N/p_0} + \sqrt{p_0/p_N}\Big) \tag{27}$$

We next subtract 1 from both sides:

$$\frac{L_N}{L_0} - 1 = \frac{1}{2}\Big(\sqrt{p_N/p_0} + \sqrt{p_0/p_N} - 2\Big), \tag{28}$$

which will be useful momentarily. Now, let's consider a swap starting at a point with price p_1 and liquidity L_1, and thus has coordinates

$$X_1 = L_1/\sqrt{p_1}, \tag{29}$$
$$Y_1 = L_1\sqrt{p_1}. \tag{30}$$

Let us suppose specifically that a positive Δx is paid *into* the pool. Then, for a fee parameter $\gamma = 1 - \varphi$, the outgoing amount $\Delta y = Y_1\gamma\Delta x/(X_1 + \gamma\Delta x)$ will lead us to the new coordinates given by

$$X_2 = X_1 + \Delta x \tag{31}$$
$$Y_2 = Y_1 - \frac{Y_1\gamma\Delta x}{(X_1 + \gamma\Delta x)} = \frac{X_1 Y_1}{X_1 + \gamma\Delta x} \tag{32}$$

We rewrite these in terms of liquidities and prices:

$$L_2/\sqrt{p_2} = L_1/\sqrt{p_1} + \Delta x \tag{33}$$
$$L_2\sqrt{p_2} = \frac{L_1^2}{L_1/\sqrt{p_1} + \gamma\Delta x} \tag{34}$$

We will solve (33) for Δx and substitute it into (34):

$$\begin{aligned} L_2\sqrt{p_2} &= \frac{L_1^2}{L_1/\sqrt{p_1} + \gamma\big(L_2/\sqrt{p_2} - L_1/\sqrt{p_1}\big)} \\ &= \frac{L_1^2}{(1\text{-}\gamma)L_1/\sqrt{p_1} + \gamma L_2/\sqrt{p_2}} \end{aligned} \tag{35}$$

Next, we divide through by $L_1\sqrt{p_2}$ to obtain

$$\begin{aligned} \frac{L_2}{L_1} &= \frac{1}{\sqrt{p_2}}\frac{L_1}{(1\text{-}\gamma)L_1/\sqrt{p_1} + \gamma L_2/\sqrt{p_2}} \\ &= \frac{1}{(1\text{-}\gamma)\sqrt{p_2/p_1} + \gamma(L_2/L_1)} \end{aligned} \tag{36}$$

We will define two quantities to simplify things:

$$r := \sqrt{p_2/p_1} \tag{37}$$
$$\eta := L_2/L_1 \tag{38}$$

Then (37) can be rewritten using (36) and ($\varphi = 1\text{-}\gamma$):

$$\eta = \frac{1}{\varphi r + (1\text{-}\varphi)\eta} \tag{39}$$

This gives a quadratic equation $(1\text{-}\varphi)\eta + \varphi r\eta - 1 = 0$, which has the solutions

$$\eta = -\left[\frac{r\varphi}{2(1\text{-}\varphi)}\right] \pm \sqrt{\left[\frac{r\varphi}{2(1\text{-}\varphi)}\right]^2 + \frac{1}{(1\text{-}\varphi)}} \tag{40}$$

The quantity η is the (necessarily greater than 1) liquidity scale factor L after a swap. Clearly, the only way to get a positive value out of this is to take the 'plus' solution in (40).

Now, expression (40) is somewhat intractable to extract a null fee formula. For this reason, we will need to make an approximation. To this end, we note that in (40), the first term in the radical is ≈ 0, while the second term will be ≈ 1. We might consider dropping the first term altogether, but we also note that $r \approx 1$ (except for extremely large outlier trades). Thus, we will replace $r \to 1$, and because the overall term is ≈ 0, it shouldn't matter too much. Altogether, our new approximation for η will be

$$\eta \approx -\left[\frac{r\varphi}{2(1\text{-}\varphi)}\right] + \sqrt{\left[\frac{\varphi}{2(1\text{-}\varphi)}\right]^2 + \frac{1}{(1\text{-}\varphi)}} \tag{41}$$

By defining the constants b and c such that

$$b := \frac{\varphi}{2(1\text{-}\varphi)} \tag{42}$$

$$c := \sqrt{\left[\frac{\varphi}{2(1\text{-}\varphi)}\right]^2 + \frac{1}{(1\text{-}\varphi)}} \tag{43}$$

then we see that we have a linear expression for eta ($\eta = -br + c$). Though it is not obvious, we actual have the relation $c = 1 + b$, as we can see from the following; we start with the observation that $c = \sqrt{b^2 + 1/(1\text{-}\varphi)}$. Then we do a calculation starting with c^2:

$$c^2 = b^2 + \frac{1}{1\text{-}\varphi} = b^2 + \frac{\varphi}{1\text{-}\varphi} + 1 = b^2 + 2b + 1 = (b+1)^2 \tag{44}$$

From this it then follows that $c = 1 + b$. Now we can write our linear expression $\eta = -br + c$ as

$$\eta = 1 + b(1 - r) \tag{45}$$

Now, our derivation so far only applied to a trade moving from p_1 to p_2, with $p_1 > p_2$ (because of the fact that Δx was coming *into* the pool). We could repeat this whole derivation with a trade with $p_1 < p_2$, but we can appeal to symmetry arguments instead. Specifically, all calculations are symmetric with respect to X and Y, with the exception that the role of p becomes $1/p$. Consequently, we define the following variation of r (which will now be defined over a sequence of trades with index n):

$$\tilde{r}_n := \left\{ \begin{array}{ll} \sqrt{p_n/p_{n\text{-}1}} & \text{if } p_n < p_{n\text{-}1} \\ \sqrt{p_{n\text{-}1}/p_n} & \text{if } p_{n\text{-}1} < p_n \end{array} \right\} \tag{46}$$

Then, recalling that η is just the increment scaling of the liquidity, we can extend (45) to an arbitrary trade $\eta_n = L_n/L_{n\text{-}1}$. Combined with (46), we now can write

$$\frac{L_n}{L_{n\text{-}1}} = 1 + b(1 - \tilde{r}_n) \tag{47}$$

We'll temporarily define the quantity $q_n = 1 - \tilde{r}_n$:

$$\frac{L_n}{L_{n\text{-}1}} = 1 + bq_n \tag{48}$$

Looking back at equation (28), we are looking for an expression for L_N/L_0. We can write this quantity as:

$$\begin{aligned} \frac{L_N}{L_0} &= \frac{L_1}{L_0}\frac{L_2}{L_1}\frac{L_3}{L_2}\cdots\frac{L_N}{L_{N-1}} \\ &= \Big(1 + bq_1\Big)\Big(1 + bq_2\Big)\Big(1 + bq_3\Big)\ldots\Big(1 + bq_N\Big) \\ &= 1 + b\Big(q_1 + q_2 + q_3 + \ldots + q_N\Big) + O\Big(b^2\Big) \end{aligned} \tag{49}$$

Because the quantity b is small (scaling with φ), then we will make another approximation and drop the higher order terms $O(b^2)$ and just write

$$\begin{aligned} \frac{L_N}{L_0} &\approx 1 + b\Big(q_1 + q_2 + q_3 + \ldots + q_N\Big) \\ &= 1 + b\Big((1\text{-}\tilde{r}_1) + (1\text{-}\tilde{r}_2) + (1\text{-}\tilde{r}_3) + \ldots + (1\text{-}\tilde{r}_N)\Big) \\ &= 1 + b\Big(N - \textstyle\sum_n \tilde{r}_n\Big) \end{aligned} \tag{50}$$

From (50), we have an expression for $L_N/L_0 - 1$, as we need for (28). Substituting this gives the following:

$$b\Big(N - \textstyle\sum_n \tilde{r}_n\Big) = \frac{1}{2}\Big(\sqrt{p_N/p_0} + \sqrt{p_0/p_N} - 2\Big) \tag{51}$$

If we restore the value of b, and rearrange some things, expression (51) becomes

$$\frac{\varphi}{1\text{-}\varphi} = \frac{\Big(\sqrt{p_N/p_0} + \sqrt{p_0/p_N} - 2\Big)}{\Big(N - \sum_n \tilde{r}_n\Big)} \tag{52}$$

Note that the numerator on the right hand side of (52) is identical to the numerator in (21). We even see the natural emergence of the expression $\tilde{\varphi}$ on the left hand side of (52). For readability, we can put (52) into words:

$$\tilde{\varphi} = \frac{\left[\sqrt{\frac{\text{final price}}{\text{initial price}}} + \sqrt{\frac{\text{initial price}}{\text{final price}}} - 2\right]}{\left(\left[\begin{array}{c}\text{total number}\\ \text{of trades}\end{array}\right] - \sum\limits_{\text{trades}} \sqrt{\left[\begin{array}{c}\text{ratio of prices}\\ \text{before/after trade,}\\ \textbf{smaller}\text{ one on }\textbf{top}\end{array}\right]}\right)} \tag{53}$$

Alternatively, we could stop short of (50), and use the fact that $1 - a/b = (b - a)/b$ to rewrite the denominator, giving us the following:

$$\tilde{\varphi} = \frac{\left[\sqrt{\frac{\text{final price}}{\text{initial price}}} + \sqrt{\frac{\text{initial price}}{\text{final price}}} - 2\right]}{\left[\sum\limits_{\text{trades}} \frac{\left|\Delta\sqrt{\text{price}}\right|}{\max\left\{\sqrt{\text{price before}}, \sqrt{\text{price after}}\right\}}\right]} \tag{54}$$

Section 5: Summary

We introduced a novel type of indicator that can be used in the potential design of a dynamic fee structure for AMM trading pools (both concentrated liquidity and constant product). This indicator is called the *null fee* and it is defined to be the hypothetical fee size that, for a given window of time, would have resulted in the LP breaking even (i.e. experiencing zero impermanent loss). Due to the nature of the interplay between price drift and transaction activity over short time windows, this quantity needs to be renormalized in some way. We propose a quadratic change of variables. We also take an exponentially weighted average of the running values of the indicator.

Depending on the type of AMM, the formula for the null fee φ is given by

- using formulas (21) and (19), in the context of a concentrated liquidity AMM
- using formulas (54) and (19), in the context of a constant product AMM

The quadratic renormalization function we use is given by formula (22), with parameter choices depending on things like time window size, desired fee range, etc.

Applied to historical datasets, the null fee indicator can provide a reasonable recipe for dynamic fees, and offers a qualitatively different alternative from the more common indicator choice of volatility.