.

The SlickRod

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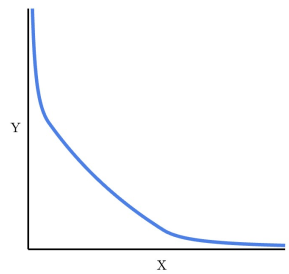


In this document, we introduce a new type of Automated Market Maker (AMM). This protocol, which we call the *SlickRod*, has two important features that allow us to think of it as a cross between Uniswap V3 and Stableswap. Specifically, the SlickRod bonding curve contains a broad flattened region of low slippage (similar to that of the Stableswap curve), but the location and width of this region depends dynamically on the wisdom of the market via LP choices (similar to Uniswap V3). This will be divided into five sections:

- **Section 1: The Idea -** Here we give an overview of the design for our bonding curve, providing a clear motivation for the steps presented in the following sections.
- **Section 2: Aggregating LP Positions -** In this section we give more details to the ideas presented in Section 1, providing quantitative statements and pictures for illustration.
- **Section 3: The Mathematics -** We derive the necessary mathematics hinted at (but not presented) in Section 2.
- **Section 4: A Convexity Condition -** In this section, we derive an additional condition that we choose to impose on the choice LP positions, in order to prevent something that we call *instantaneous impermanent gain.*
- **Section 5: Summary of Formulas -** Finally, we collect all the important results on one page for convenience.

Section 1: Idea

In this document we introduce a novel AMM design which we call the *SlickRod* (the name can be changed). One can roughly think of the SlickRod as being a blend between the Concentrated Liquidity of Uniswap V3 and the low slippage of the Stabeswap curve.

To understand how, we first construct a basic shape. In figure 1 below, we have three liquidity distributions; (1) equal liquidity throughout the entire range of prices, as it is in a constant product pool, (2) a limited band of liquidity provided in range $[p_a, p_b]$, as in concentrated liquidity, and (3) the superposition of the first two:

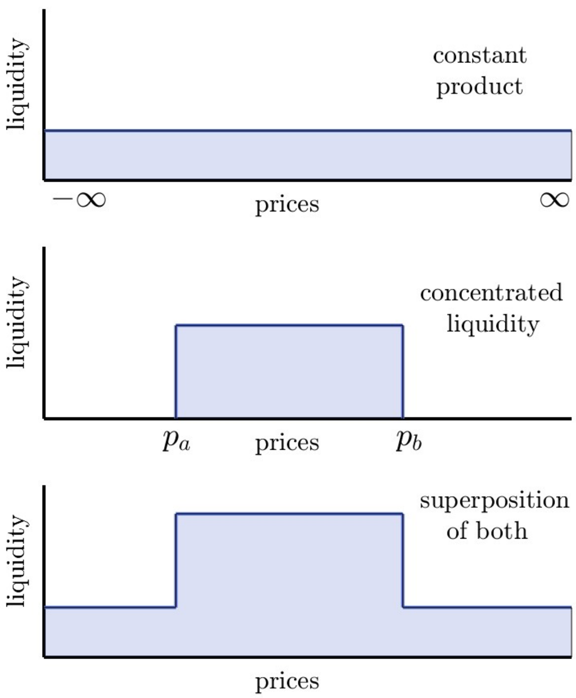


Figure 1. Three possible liquidity positions

We will refer to this third distribution as a *top-hat*. The bonding curve produced by a top-hat position will be a constant product hyperbola with a broad, flattened out region corresponding to the price range $[p_a, p_b]$. Such a curve is depicted in figure 2 below.

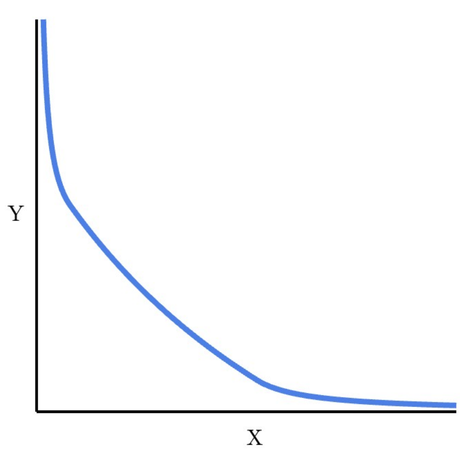


Figure 2. Bonding curve of a top-hat distribution.

The bonding curve shown in figure 2 is clearly very reminiscent of the famous Stableswap curve originally introduced in the Curve V1 whitepaper. In particular, we see that there exists a small range of prices for which transactions will experience significantly less slippage. It should be noted, however, that because our current curve is comprised of two simple hyperbolic arcs stitched together, the swap logic should actually be much simpler and less computationally costly than the analogous (iterative) logic required for a Stableswap transaction. In fact, the top-hat liquidity distribution is just a simple manifestation of concentrated liquidity, and thus requires the simplest case of the Uniswap v3 logic (or simpler, as we'll see in later sections).

Moreover, because the low slippage region is determined by the choice of the price range $[p_a, p_b]$, then it is a trivial matter to slide it around at will. For example, in figure 3 we can see two variations of our bonding curve resulting from top-hat distributions located over two different price ranges.

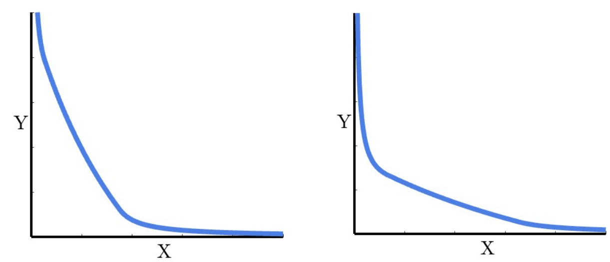


Figure 3. Variations of the top-hat bonding curve.

In the Curve V2 whitepaper, a design was presented to accomplish a similar objective, namely a bonding curve with a *moveable* low slippage region. However, their design differs from ours in two important respects:

- Their bonding curve still roughly consisted of the Stableswap invariant curve, but used a change of coordinates to accomplish the dynamic peg (the low slippage region). As we mentioned, the swap logic for this curve is computationally more costly than for the top-hat bonding curve.
- Their dynamic peg (low slippage region) was determined algorithmically, based on price oracles and profit/loss calculations. Alternatively (and more importantly), the low slippage region for *our* curve (the SlickRod AMM) will be determined by the market, and in particular, through the collective action of LPs.

This second point above justifies the initial claim that the SlickRod AMM can be thought of as being a kind of blend between Stableswap and Concentrated Liquidity; it has the low slippage region of a Stableswap curve, but its dynamic location is determined by the collective wisdom of the LPs, as in Uniswap V3. The precise way in which LP choices are synthesized into a top-hat distribution is non-trivial, as we will see in section 2.

Section 2: Aggregating LP Positions

As previously mentioned, the location of the low slippage region of our top-hat bonding curve should be determined dynamically by the collective set of LPs. In this section, we describe how this will be accomplished.

First, as we see from figure 1, a top-hat distribution can be decomposed into two pieces; a piece ranging over the entire price range (the *brim* of the top-hat), and a rectangular piece of concentrated liquidity (the *crown* of the top-hat). To describe these pieces quantitatively, define the variables L and H as in figure 4:

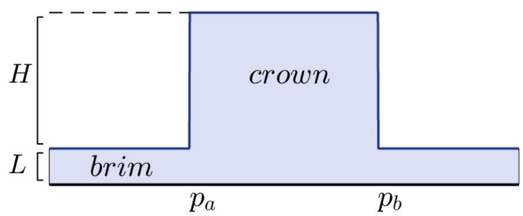


Figure 4. Variables defining a top-hat distribution.

Each LP chooses their own top-hat distribution, i.e. they choose their own set of parameters $\{L, H, p_a, p_b\}$. Our goal is to then synthesize these choices into one aggregate top-hat distribution. If, however, we were to simply take a superposition of these positions (as would be done in Unsiwap V3, for example), the resulting distribution would not itself be a top-hat, but rather a staggered collection of rectangles.

Our solution to this synthesis issue is the following. First, we will deal with the brim and crown separately. The brims actually *can* be added as a superposition (because the sum of arbitrary horizontal bands is still a horizontal band). The crowns, on the other hand, must be synthesized into one aggregate crown. The way we do this is illustrated in figure 5 (using four crowns).

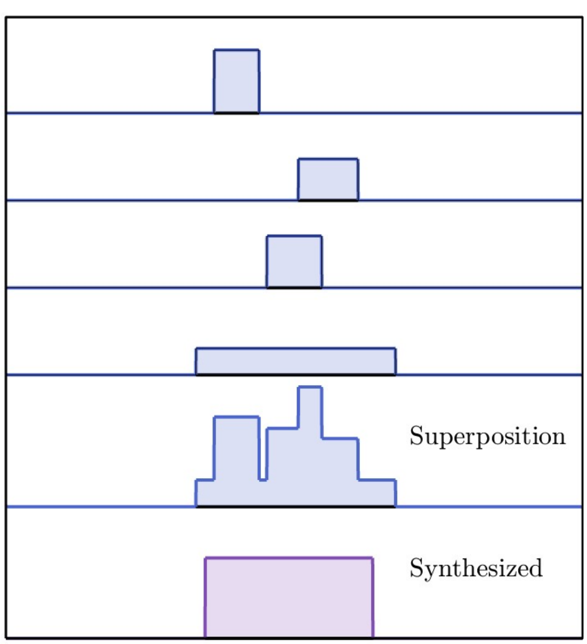


Figure 5. Four crowns, superimposed and synthesized

More precisely, we imagine the simple superposition of crowns as a starting point. From there, we can then define the *synthesized* crown by to be a *new* crown, but one that shares some important quantitative features of the superposition of crowns. Specifically,

> the synthesized crown will be the rectangle with the same *area*, *mean* and *standard deviation* as the superimposed LP crowns.

The mathematical details are left for the next section, but figure 6 shows the high level flow of calculations:

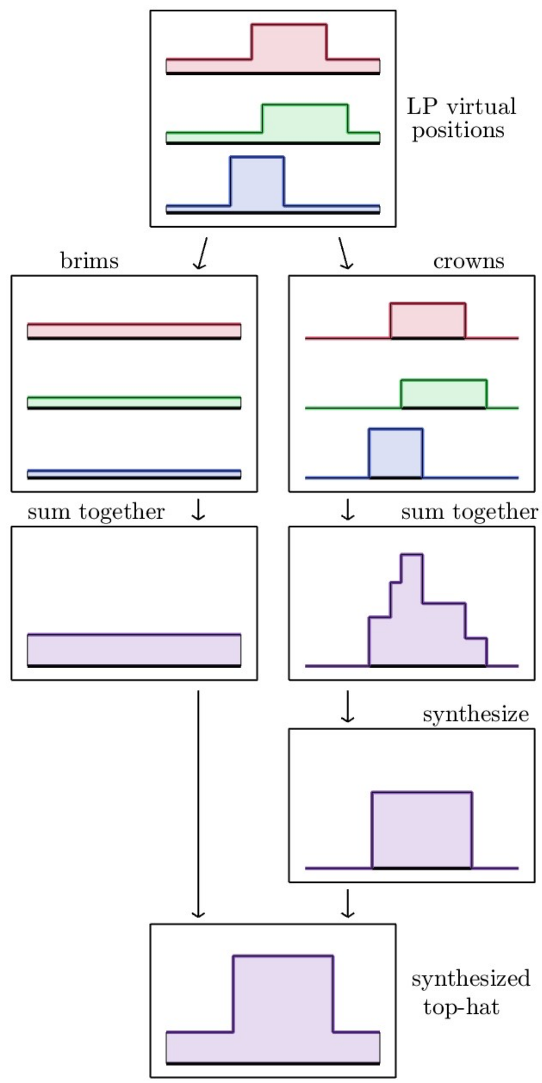


Figure 6. Synthesizing the top-hat

To summarize, we have the following:

- LPs choose their own virtual top-hot positions, and each top-hat is split into a brim and a crown.
- The brims are summed into an aggregate brim.
- The crowns are summed as well, and we calculate the the area, mean, and standard deviation of the resulting distribution. We then generate a simple crown with the same parameters.
- Together, this aggregate brim and crown give us a synthesized top-hat.

Section 3: The Mathematics

Now we can begin to derive all the necessary mathematics. This section is divided into four separate subsections.

Section 3.1. Synthesizing the Top-Hat

First, we will work with the logarithm of prices, which is the more natural and symmetric coordinate to work with. Moreover, because of the presence of square roots in the constant product mathematics, we define the following modified price variable ω by:

$$\omega := \log\left(\sqrt{p}\right). \tag{1}$$

We continue to define the parameters L and H as we did in figure 4, but now we depict them relative to our new ω variable,

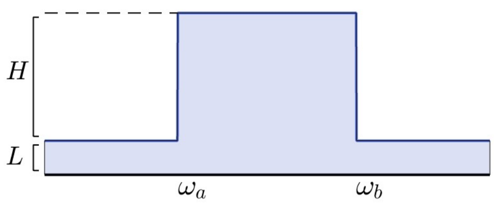


Figure 7. Variables defining a top-hat distribution.

where the values of $[\omega_a, \omega_b]$ are computed from the choice of price range $[p_a, p_b]$ using definition (1).

We will use the index n to denote the n^{th} LP. Thus, as the LPs choose their virtual top-hat positions, the collection of parameter choices can be denoted by $\{L_n, H_n, p_{an}, p_{bn}\}$. As we mentioned, in our effort to derive a synthesized top-hat distribution $\{L, H, p_a, p_b\}$, we treat the brim and the crown separately. For our synthesized brim, we simply take the superposition over all LPs, which only requires summing all the L_n values:

$$L = \sum\nolimits_n L_n. \tag{2}$$

To synthesize the crown, however, we must do a bit more work. To start, we note that although the crown parameters $\{H, p_a, p_b\}$ may be intuitive variables from the user perspective, it will be more convenient for the underlying mathematics if we convert these three degrees of freedom into three alternative quantities; the area A, the mean μ, and the standard deviation σ. One can easily check that the mean and standard deviation of the rectangular distribution are given by

$$\mu = (\omega_a + \omega_b)/2, \tag{3}$$
$$\sigma = (\omega_b - \omega_a)/2\sqrt{3}, \tag{4}$$

as illustrated in figure 8:

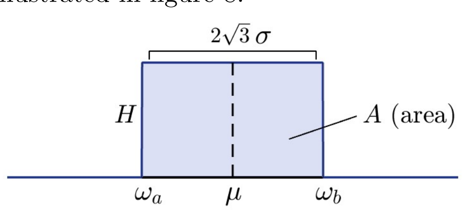


Figure 8. The crown and its distributional parameters.

Moreover, the area is clearly then given by:

$$A = (\omega_b - \omega_a)H = 2\sqrt{3}\,\sigma H \tag{5}$$

However, rather than us working with the standard deviation directly, it will be much cleaner if we will work with the *second moment*.

For any given distribution $f(\omega)$, the *zeroth moment* A (the area), the first moment μ (the mean), and the *second moment* τ (related to variance) are given by:

$$A = \int f(\omega)d\omega \tag{6}$$
$$\mu = \int \omega\big[f(\omega)/A\big]d\omega \tag{7}$$
$$\tau = \int \omega^2\big[f(\omega)/A\big]d\omega. \tag{8}$$

The quantity τ is related to the standard deviation by

$$\tau = \sigma^2 + \mu^2. \tag{9}$$

If we substitute (3)-(4) into (9) and simplify, we get the expression $\tau = (\omega_a^2 + \omega_b^2 + \omega_a\omega_b)/3$. Altogether then, we can convert from the quantities $\{H, \omega_a, \omega_b\}$ to the quantities $\{A, \mu, \tau\}$ according to

$$A = (\omega_b - \omega_a)H \tag{10}$$
$$\mu = (\omega_a + \omega_b)/2 \tag{11}$$
$$\tau = (\omega_a^2 + \omega_b^2 + \omega_a\omega_b)/3 \tag{12}$$

With a little algebra, we can also invert these equations:

$$\omega_a = \mu - \sqrt{3\tau - 3\mu^2} \tag{13}$$
$$\omega_b = \mu + \sqrt{3\tau - 3\mu^2} \tag{14}$$
$$H = A/(2\sqrt{3\tau - 3\mu^2}) \tag{15}$$

Now, for the n^{th} LP, let $\{A_n, \mu_n, \tau_n\}$ be their chosen crown parameters, generating the crown distribution $f_n(\omega)$. Then the *superposition* $f(\omega)$ of all LP crowns is given by

$$f(\omega) := \sum\nolimits_n f_n(\omega) \tag{16}$$

First, we find the area of f:

$$A = \int f(\omega)d\omega = \sum_n \int f_n(\omega)d\omega = \sum_n A_n. \tag{17}$$

Meanwhile, the mean of f is given by

$$\mu = \int \omega\left[\frac{f(\omega)}{A}\right]d\omega = \sum_n \frac{A_n}{A}\int \omega\left[\frac{f_n(\omega)}{A_n}\right]d\omega = \sum_n \lambda_n \mu_n, \tag{18}$$

where we have defined

$$\lambda_n := A_n/A \tag{19}$$

A nearly identical calculation for the second moment gives us

$$\tau = \sum_n \lambda_n \tau_n \tag{20}$$

With (17)-(20), we now have a definite prescription for computing the synthesized crown. For clarity, we will now summarize the synthesis process:

- Each LP chooses a virtual top-hat distribution. At the interface level, this is done via the four parameters $\{L_n, H_n, p_{an}, p_{bn}\}$ (for n^{th} LP)
- We convert the price variable p to the logarithmic price ω by $\omega = \log(\sqrt{p})$. We define the change of variables $\{L, H, \omega_a, \omega_b\} \leftrightarrow \{L, A, \mu, \tau\}$ by

$$L = L \tag{21}$$
$$A = (\omega_b - \omega_a)H \tag{22}$$
$$\mu = (\omega_a + \omega_b)/2 \tag{23}$$
$$\tau = (\omega_a^2 + \omega_b^2 + \omega_a\omega_b)/3 \tag{24}$$

and the inverse transformation

$$L = L \tag{25}$$
$$\omega_a = \mu - \sqrt{3\tau - 3\mu^2} \tag{26}$$
$$\omega_b = \mu + \sqrt{3\tau - 3\mu^2} \tag{27}$$
$$H = A/(2\sqrt{3\tau - 3\mu^2}) \tag{28}$$

- The LP parameters $\{L_n, H_n, \omega_{an}, \omega_{bn}\}$ can then be converted into the quantities $\{L_n, A_n, \mu_n, \tau_n\}$.
- We define synthesized parameters $\{L, A, \mu, \sigma\}$ by

$$L := \sum_n L_n \tag{29}$$
$$A := \sum_n A_n \tag{30}$$
$$\lambda_n := A_n/A \tag{31}$$
$$\mu := \sum_n \lambda_n \mu_n \tag{32}$$
$$\tau := \sum_n \lambda_n \tau_n \tag{33}$$

We should also note that these synthesized parameters need not be recomputed from scratch each time an LP joins or leaves. Instead, one finds that the equations above inductively imply a simple update formula when a single LP joins/leaves. Specifically, let the current synthesized top-hat position be given by $\{L, A, \mu, \sigma\}$, and consider a new LP with parameters $\{L^*, A^*, \mu^*, \sigma^*\}$ joining the pool. Then the *new* synthesized parameters $\{\widetilde{L}, \widetilde{A}, \widetilde{\mu}, \widetilde{\sigma}\}$ will be given by

$$\widetilde{L} = L + L^* \tag{34}$$
$$\widetilde{A} = A + A^* \tag{35}$$
$$\lambda = A/\widetilde{A} \tag{36}$$
$$\lambda^* = A^*/\widetilde{A} \tag{37}$$
$$\widetilde{\mu} = \lambda A + \lambda^* A^* \tag{38}$$
$$\widetilde{\tau} = \lambda \tau + \lambda^* \tau^* \tag{39}$$

Similarly, if the LP is *leaving* the pool, the update formulas (34)-(39) need only be changed by switching the plus signs to minus signs.

Section 3.2. Token Reserves and Swaps

For a constant product AMM given by $xy = L^2$, the token reserves at current price p are given by

$$x = L/\sqrt{p} \tag{40}$$
$$y = L\sqrt{p} \tag{41}$$

For an LP in a concentrated liquidity pool with liquidity scale L and price range $[p_a, p_b]$, we instead have

$$x = L\Big(1/\sqrt{[p]_{p_a}^{p_b}} - 1/\sqrt{p_b}\Big) \tag{42}$$
$$y = L\Big(\sqrt{[p]_{p_a}^{p_b}} - \sqrt{p_a}\Big) \tag{43}$$

where the 'box' notation $[\cdot]$ is defined by

$$[p]_{p_a}^{p_b} := \left\{ \begin{array}{cc} p_a & (p < p_a) \\ p & (p_a \le p \le p_b) \\ p_b & (p_b < p) \end{array} \right\}, \tag{44}$$

and we will sometimes abbreviate the expression in (44) as simply $[p]$, if the range $[p_a, p_b]$ is understood from the context. For the top-hat distribution with parameters $\{L, H, p_a, p_b\}$, the token reserves can be expressed by superimposing expressions (40)-(41) (the brim) with expressions (42)-(43) (the crown). This then give us

$$x = L/\sqrt{p} + H\left(1/\sqrt{[p]} - 1/\sqrt{p_b}\right) \tag{45}$$
$$y = L\sqrt{p} + H\left(\sqrt{[p]} - \sqrt{p_a}\right). \tag{46}$$

It will be necessary for us to invert these expressions in the sense that, for given parameters $\{p_a, p_b, H, L\}$, and a given token amount x or y, we should be able to extract the price p. To this end, we write (45)-(46) as

$$\left(x + H/\sqrt{p_b}\right) = L/\sqrt{p} + H/\sqrt{[p]} \tag{47}$$
$$\left(y + H\sqrt{p_a}\right) = L\sqrt{p} + H\sqrt{[p]}. \tag{48}$$

and we note that (47) and (48) are both of the form

$$C = Lt + H[t]_{t_a}^{t_b}, \tag{49}$$

for a given $\{C, L, H\}$ and unknown t standing in for the quantity $\sqrt{p}^{\pm 1}$ (keeping in mind that the box function commutes with square roots). In figure 9 below, we plot the graph of $s = Lt + H[t]_{t_a}^{t_b}$. By setting $s = C$, it is easy to work out the case-by-case solutions to (49).

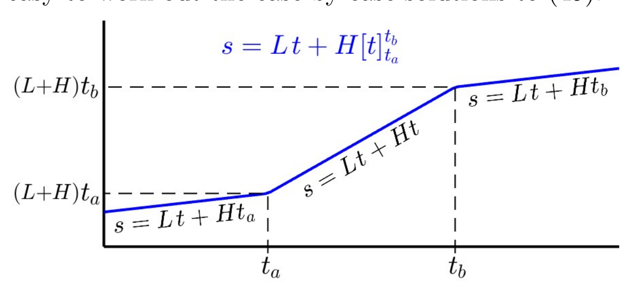


Figure 9. A useful plot of expression (49).

From Figure 9, we find the solutions of (49) to be:

$$\left\{ \begin{array}{ll} t = (C\text{-}Ht_a)/L & \text{if } C<t_a(L\text{+}H) \\ t = C/(L+H) & \text{if } t_a(L\text{+}H)<C<t_b(L\text{+}H)] \\ t = (C\text{-}Ht_b)/L & \text{if } t_b(L\text{+}H)<C \end{array} \right\} \tag{50}$$

If we define $k := C/(L+H)$, then (50) is equivalent to

$$\left\{\begin{array}{ll} t = k + \frac{H}{L}(k - t_a) & \text{if } k < t_a \\ t = k & \text{if } k \in [t_a, t_b] \\ t = k + \frac{H}{L}(k - t_b) & \text{if } k > t_b \end{array}\right\} \tag{51}$$

Expressions (51) can be succinctly stated as

$$t = k + \frac{H}{L}\Big(\min\{0, k\text{-}t_a\} + \max\{0, k\text{-}t_b\}\Big). \tag{52}$$

We recall that C stands in for the left hand sides of either (47) or (48), while the unknown t stands for the quantities $\sqrt{p}$ or $1/\sqrt{p}$. Thus, for example, if we want to solve (48) for $\sqrt{p}$, we would compute the following:

$$k = \big(y + H\sqrt{p}_a\big)/(L+H) \tag{53}$$

$$\sqrt{p} = k + \frac{H}{L}\Big(\min\{0, k\text{-}\sqrt{p}_a\} + \max\{0, k\text{-}\sqrt{p}_b\}\Big) \tag{54}$$

On the other hand, if we want to solve (47) for $1/\sqrt{p}$, we first note that because t now stands for $1/\sqrt{p}$, our upper and lower bounds are no longer $[\sqrt{p}_a, \sqrt{p}_b]$ but rather $[1/\sqrt{p}_b, 1/\sqrt{p}_a]$. Thus, expression (52) becomes

$$k = \big(x + H/\sqrt{p}_b\big)/(L+H) \tag{55}$$

$$1/\sqrt{p} = k + \frac{H}{L}\Big(\min\{0, k\text{-}1/\sqrt{p}_b\} + \max\{0, k\text{-}1/\sqrt{p}_a\}\Big) \tag{56}$$

With these results, we can easily compute the swaps. Admittedly, we *could* just employ the Uniswap V3 logic, because a top-hat distribution is, after all, just a very simple case of concentrated liquidity. However, we can actually articulate simpler, more compact formulas.

Supposing we have current token reserves (x, y), a swap is initiated when a user pays in an incoming quantity, Δx or Δy. Depending on which is provided, we will know the new reserve amount, either $(x+\Delta x)$ or $(y+\Delta y)$. With this modification to either (53) or (55), we can then compute the new price p'. From there we compute the *other* reserve amount, using either (45) or (46), from which we can compute the outgoing amount:

- if Δx is given
  - compute $k := \Big(x + \Delta x + H/\sqrt{p_b}\Big)/(L+H)$
  - compute p' by
    $$1/\sqrt{p'} = k + \frac{H}{L}\left(\begin{array}{c} \max\{0, k - 1/\sqrt{p_a}\} \\ + \min\{0, k - 1/\sqrt{p_b}\} \end{array}\right)$$
  - compute $y' = L\sqrt{p'} + H\left(\sqrt{[p']} - \sqrt{p_a}\right)$
  - compute $\Delta y = y' - y$
- if Δy is given
  - compute $k := \Big(y + \Delta y + H\sqrt{p_a}\Big)/(L+H)$
  - compute p' by
    $$\sqrt{p'} = k + \frac{H}{L}\left(\begin{array}{c} \max\{0, k - \sqrt{p_b}\} \\ + \min\{0, k - \sqrt{p_a}\} \end{array}\right)$$
  - compute $x' = L/\sqrt{p'} + H\left(1/\sqrt{[p']} - 1/\sqrt{p_b}\right)$
  - compute $\Delta x = x' - x$

Now, we notice that in the preceding prescription, the formulas are essentially the same regardless of whether it is Δx or Δy that is given; the only difference is that certain quantities must be replaced with others. If we introduce a more general notation, we can actually express both of these cases at once (this notation will be particularly useful in the next section when we discuss how fees are taken). To this end, we begin by thinking of the token reserves as *functions* of the current price, and we rewrite (45)-(46) as explicit functions of p:

$$x(p) = L/\sqrt{p} + H\left(1/\sqrt{[p]} - 1/\sqrt{p}_b\right) \tag{57}$$

$$y(p) = L\sqrt{p} + H\left(\sqrt{[p]} - \sqrt{p_a}\right). \tag{58}$$

Now, we consider a swap that moves the market from a current price p to new price p'. This swap will require an *incoming* token amount and an *outgoing* token amount, which we will denote by Δt_i and Δt_o. These quantities will be equal to either Δx or Δy, depending on whether the price is increasing or decreasing. We will also denote the corresponding token reserves (at the beginning of the swap) by the expressions t_i and t_o. Moreover, as we saw in the previous development, the roles of the price bounds p_a and p_b appear to alternate between *upper* and *lower* bounds, and we will denote these generally as p_u and p_ℓ. Finally, there occurs a plus/minus sign that depends on the trade direction, as well. Altogether, this notation can be summarized in the following table:

	if $(p' < p)$	if $(p' > p)$
$t_i :=$	$x(p)$	$y(p)$
$t_o :=$	$y(p)$	$x(p)$
$\Delta t_i :=$	Δx (given)	Δy (given)
$p_\ell :=$	p_b	p_a
$p_u :=$	p_a	p_b
$s :=$	-1	$+1$

Then we can succinctly summarize the swap logic as the following. Given (t_i, t_o) and Δt_i, we first compute

$$k := \Big(t_i + \Delta t_i + H\sqrt{p_\ell}^{\,s}\Big)/(L+H), \tag{59}$$

from which we can compute the new price p',

$$\sqrt{p'}^{\,s} = k + \frac{H}{L}\left(\begin{array}{c} \max\{0, k - \sqrt{p_u}^{\,s}\} \\ + \min\{0, k - \sqrt{p_\ell}^{\,s}\} \end{array}\right). \tag{60}$$

Then the new t_o value (denoted t'_o) is given by

$$t'_o = L\sqrt{p'}^{\,(\text{-}s)} + H\Big(\sqrt{[p']}^{\,(\text{-}s)} - \sqrt{p}_u^{\,(\text{-}s)}\Big) \tag{61}$$

Finally then, we can compute the outgoing quantity:

$$\Delta t_o = |t'_o - t_o| \tag{62}$$

(We will take Δt_o to be positive for later convenience.)

Section 3.3. Fees

The implementation of fees for the SlickRod can be somewhat nuanced. We *could* store the fees externally from the pool (as in Uniswap V3), but it is unclear how to then fairly distribute them to LPs (because of the fact that the individual LP positions are *virtual* and do not exactly equate to liquidity provided). Instead, we will store the fees internally as extra pool liquidity (as in Uniswap V2). There are several ways of doing this, but we will use the GIFS method (discussed in depth in this paper). We briefly summarize the ideas and relevant formulas here.

The GIFS method for implementing fees can be interpreted in two (equivalent) ways:

- We first imagine executing the swap as if there are *no fees taken*. Then, we scale up the liquidity parameters $\{H, L\}$ by a scale factor η, leaving the spot price unchanged. This is shown in the figure below, for an incoming Δx and outgoing Δy:

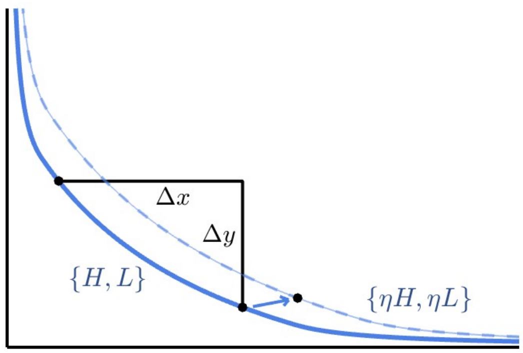


Figure 10. GIFS method (interpretation 1)

- Alternatively, we can think of it as though we are taking our fee factor φ and dynamically splitting it into two components φ_i and φ_o to be applied to the *incoming* and *outgoing* quantities. These satisfy $\varphi = \varphi_i + \varphi_o$, so for example, a fee size of 0.3% might be split into an incoming fee of 0.2% and an outgoing fee of 0.1%. This is shown below:

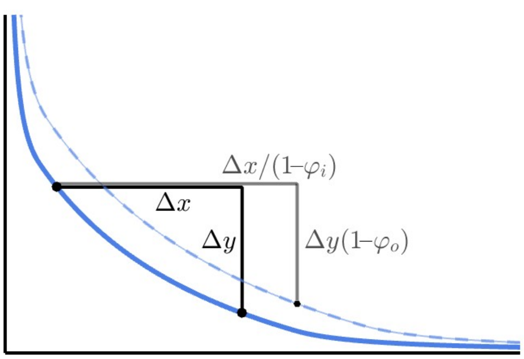


Figure 11. GIFS method (interpretation 2)

The beauty of the GIFS method is that the formulas are crafted in such a way so that these two interpretations are mathematically equivalent (*almost* exactly).

The GIFS formulas depend on whether the price is moving up or down, but using the notation established in the previous section (for t_i, t_o, etc.), we can express both cases at once. We start with some given token reserves (t_i, t_o) and we suppose that a user specifies an incoming quantity Δt_i. The corresponding outgoing quantity Δt_o can be computed via formulas (59)-(62). No fees have been taken yet, and so we now invoke the GIFS formulas. First, we compute three intermediate quantities:

$$a := (t_i)(\Delta t_o) \tag{63}$$

$$b := (t_o)(\Delta t_i) \tag{64}$$

$$c := (\Delta t_i)(\Delta t_o) \tag{65}$$

Then the scaling factor η can be computed as

$$\eta = 1 + \frac{c(a+b)\,\varphi}{(a+b)^2 - (a+c)^2\varphi} \tag{66}$$

where φ represents the fee size (for example $\varphi \sim 0.003$). Then we can compute the *actual* deltas $(\underline{\Delta} t_i, \underline{\Delta} t_o)$ by

$$\underline{\Delta} t_i = \eta \Delta t_i + (\eta\text{-}1) t_i \tag{67}$$

$$\underline{\Delta} t_o = \eta \Delta t_o - (\eta\text{-}1) t_o \tag{68}$$

Thus, the token reserves are then updated as

$$t_i \to t_i + \underline{\Delta} t_i \tag{69}$$

$$t_o \to t_o - \underline{\Delta} t_o \tag{70}$$

Finally, we scale up the liquidity parameters $\{H, L\}$:

$$H \to \eta H \tag{71}$$

$$L \to \eta L \tag{72}$$

As a practical matter, we observe that updating the liquidity parameters $\{H, L\}$ could be somewhat costly, at least in principle. As we saw in Section 3.1, the values of $\{H, L\}$ are synthesized from the LP parameters, and so updating them would require an update over *all* LPs. However, rather than updating $\{H_n, L_n\}$ for *every* LP during *every* swap, we could instead keep an accumulator variable η_{acc} that is the accumulating product of the scale factors. For swap computations, we use the modified parameters $\{H\eta_{acc}, L\eta_{acc}\}$. When an LP wishes to open/close a position, we multiply all LP liquidity parameters by η_{acc}, and we reset $\eta_{acc} = 1$.

Section 3.4. Impermanent Loss

We will now compute the impermanent loss (IL) experienced by an individual LP. To do this, however, we must first articulate how much of the token reserves the LP can claim as their own. Using the notation from the end of Section 3.1, suppose the following:

- there are background parameters $\{L, H, p_a, p_b\}$ with current token reserves (x, y) at price p
- a new LP chooses parameters are $\{L^*, H^*, p_a^*, p_b^*\}$
- we get new synthesized parameters $\{\widetilde{L}, \widetilde{H}, \widetilde{p}_a, \widetilde{p}_b\}$, which requires the token reserves $(\widetilde{x}, \widetilde{y})$ at price p

The token quantities that the LP can claim, denoted (x^*, y^*) would therefore just be the differential that the LP must pay into the pool in order to bring the amounts (x, y) up to the amounts $(\widetilde{x}, \widetilde{y})$. In other words,

$$x^* = \widetilde{x} - x \tag{73}$$

$$y^* = \widetilde{y} - y \tag{74}$$

Using the definitions in (45)-(46), we can write (73)-(74) more explicitly. For x^*, we have

$$\begin{aligned} x^* &= \widetilde{x} - x \\ &= \frac{\widetilde{L}}{\sqrt{p}} + \widetilde{H}\left(\frac{1}{\sqrt{\widetilde{[p]}}} - \frac{1}{\sqrt{\widetilde{p}_b}}\right) - \frac{L}{\sqrt{p}} - H\left(\frac{1}{\sqrt{[p]}} - \frac{1}{\sqrt{p_b}}\right) \\ &= \frac{L^*}{\sqrt{p}} + \widetilde{H}\left(\frac{1}{\sqrt{\widetilde{[p]}}} - \frac{1}{\sqrt{\widetilde{p}_b}}\right) - H\left(\frac{1}{\sqrt{[p]}} - \frac{1}{\sqrt{p_b}}\right), \end{aligned} \tag{75}$$

where we used the fact that $\widetilde{L} - L = L^*$ to simplify slightly. For y^*, we find

$$\begin{aligned} y^* &= \widetilde{y} - y \\ &= \widetilde{L}\sqrt{p} + \widetilde{H}\left(\sqrt{\widetilde{[p]}} - \sqrt{\widetilde{p}_a}\right) - L\sqrt{p} - H\left(\sqrt{[p]} - \sqrt{p_b}\right) \\ &= L^*\sqrt{p} + \widetilde{H}\left(\sqrt{\widetilde{[p]}} - \sqrt{\widetilde{p}_a}\right) - H\left(\sqrt{[p]} - \sqrt{p_a}\right), \end{aligned} \tag{76}$$

where we are using the abbreviations

$$[p] := [p]_{p_a}^{p_b} \tag{77}$$

$$\widetilde{[p]} := [p]_{\widetilde{p}_a}^{\widetilde{p}_b} \tag{78}$$

to help with overall readability.

Now, to compute IL, we need to consider two points in time - an initial point, and some arbitrary later point. And although the specific LP position $\{L^*, H^*, p_a^*, p_b^*\}$ won't change over that time, the background parameters might. We will use a subscript of 0 for all quantities measured at the initial time (whether price or parameters). Beginning with the definition of IL, we find the following:

$$\begin{aligned} IL &= \frac{px^* + y^*}{px_0^* + y_0^*} - 1 \\ &= \frac{\left[\begin{array}{c} p\left[\frac{L^*}{\sqrt{p}} + \widetilde{H}\left(\frac{1}{\sqrt{\widetilde{[p]}}} - \frac{1}{\sqrt{\widetilde{p}_b}}\right) - H\left(\frac{1}{\sqrt{[p]}} - \frac{1}{\sqrt{p_b}}\right)\right] \\ + \left[L^*\sqrt{p} + \widetilde{H}\left(\sqrt{\widetilde{[p]}} - \sqrt{\widetilde{p}_a}\right) - H\left(\sqrt{[p]} - \sqrt{p_a}\right)\right] \end{array}\right]}{\left[\begin{array}{c} p\left[\frac{L^*}{\sqrt{p_0}} + \widetilde{H}_0\left(\frac{1}{\sqrt{\widetilde{[p_0]}_0}} - \frac{1}{\sqrt{\widetilde{p}_{b0}}}\right) - H_0\left(\frac{1}{\sqrt{[p_0]_0}} - \frac{1}{\sqrt{p_{b0}}}\right)\right] \\ + \left[L^*\sqrt{p_0} + \widetilde{H}_0\left(\sqrt{\widetilde{[p_0]}_0} - \sqrt{\widetilde{p}_{a0}}\right) - H_0\left(\sqrt{[p_0]_0} - \sqrt{p_{a0}}\right)\right] \end{array}\right]} - 1 \\ &:= \frac{\left[\begin{array}{c} 2L^*\sqrt{p} \\ +p\left[\widetilde{H}\left(\frac{1}{\sqrt{\widetilde{[p]}}} - \frac{1}{\sqrt{\widetilde{p}_b}}\right) - H\left(\frac{1}{\sqrt{[p]}} - \frac{1}{\sqrt{p_b}}\right)\right] \\ + \left[\widetilde{H}\left(\sqrt{\widetilde{[p]}} - \sqrt{\widetilde{p}_a}\right) - H\left(\sqrt{[p]} - \sqrt{p_a}\right)\right] \end{array}\right]}{\left[\begin{array}{c} L^*\left(\frac{p}{\sqrt{p_0}} + \sqrt{p_0}\right) \\ +p\left[\widetilde{H}_0\left(\frac{1}{\sqrt{\widetilde{[p_0]}_0}} - \frac{1}{\sqrt{\widetilde{p}_{b0}}}\right) - H_0\left(\frac{1}{\sqrt{[p_0]_0}} - \frac{1}{\sqrt{p_{b0}}}\right)\right] \\ + \left[\widetilde{H}_0\left(\sqrt{\widetilde{[p_0]}_0} - \sqrt{\widetilde{p}_{a0}}\right) - H_0\left(\sqrt{[p_0]_0} - \sqrt{p_{a0}}\right)\right] \end{array}\right]} - 1. \end{aligned} \tag{79}$$

We note that the expression $[p_0]_0$ has a subscript of zero in two places because we are inputing the price p_0, and the box function is being evaluated using the initial parameters $[p_{a0}, p_{b0}]$ (similarly for the expression $\widetilde{[p_0]}_0$).

As is the convention, we would like to express the IL in terms of the price change ratio

$$r := p/p_0. \tag{80}$$

If we divide the top and bottom of (79) by $\sqrt{p_0}$, all quantities can be resolved in terms of this variable r as:

$$IL = \frac{\left[\begin{array}{c} 2L^*\sqrt{r} \\ +r\left[\widetilde{H}\left(\frac{1}{\sqrt{\widetilde{[r]}}} - \frac{1}{\sqrt{\widetilde{r}_b}}\right) - H\left(\frac{1}{\sqrt{[r]}} - \frac{1}{\sqrt{r_b}}\right)\right] \\ + \left[\widetilde{H}\left(\sqrt{\widetilde{[r]}} - \sqrt{\widetilde{r}_a}\right) - H\left(\sqrt{[r]} - \sqrt{r_a}\right)\right] \end{array}\right]}{\left[\begin{array}{c} L^*(r+1) \\ +r\left[\widetilde{H}_0\left(\frac{1}{\sqrt{\widetilde{[r_0]}_0}} - \frac{1}{\sqrt{\widetilde{r}_{b0}}}\right) - H_0\left(\frac{1}{\sqrt{[r_0]_0}} - \frac{1}{\sqrt{r_{b0}}}\right)\right] \\ + \left[\widetilde{H}_0\left(\sqrt{\widetilde{[r_0]}_0} - \sqrt{\widetilde{r}_{a0}}\right) - H_0\left(\sqrt{[r_0]_0} - \sqrt{r_{a0}}\right)\right] \end{array}\right]} - 1 \tag{81}$$

where expressions like $[r]$ are simply defined by

$$[r] = [p]/p_0. \tag{82}$$

Expression (81) is certainly unsightly, but it is at least encouraging to check that in the case of constant product, all the H terms vanish, and expression (81) reduces to $IL = 2\sqrt{r}/(1+r) - 1$, i.e. the classic result.

While expression (81) is certainly the most general representation of IL for the SlickRod, there are some important specific cases we should consider while we are on the topic. First, we consider the case of *no change in background parameters*. This is equivalent to erasing all of the 0 subscripts on all parameters (because we do not need to distinguish them from the current values. This gives us the following:

$$IL = \frac{\left[\begin{array}{c} 2L^*\sqrt{r} \\ +r\left[\widetilde{H}\left(\frac{1}{\sqrt{\widetilde{[r]}}} - \frac{1}{\sqrt{\widetilde{r}_b}}\right) - H\left(\frac{1}{\sqrt{[r]}} - \frac{1}{\sqrt{r_b}}\right)\right] \\ + \left[\widetilde{H}\left(\sqrt{\widetilde{[r]}} - \sqrt{\widetilde{r}_a}\right) - H\left(\sqrt{[r]} - \sqrt{r_a}\right)\right] \end{array}\right]}{\left[\begin{array}{c} L^*(r+1) \\ +r\left[\widetilde{H}\left(\frac{1}{\sqrt{\widetilde{[r_0]}}} - \frac{1}{\sqrt{\widetilde{r}_b}}\right) - H\left(\frac{1}{\sqrt{[r_0]}} - \frac{1}{\sqrt{r_b}}\right)\right] \\ + \left[\widetilde{H}\left(\sqrt{\widetilde{[r_0]}} - \sqrt{\widetilde{r}_a}\right) - H\left(\sqrt{[r_0]} - \sqrt{r_a}\right)\right] \end{array}\right]} - 1 \tag{83}$$

Furthermore, we consider the case where all prices are *within* the crown range, that is $p_0, p \in [p_a, p_b], [\widetilde{p}_a, \widetilde{p}_b]$, and so we can simplify $[r] = \widetilde{[r]} = r$ and $[r_0] = \widetilde{[r_0]} = 1$. Then after getting a common denominator with the minus one, (83) simplifies dramatically to the following:

$$IL = \frac{-\left[L^* + (\widetilde{H}\text{-}H)\right](1-\sqrt{r})^2}{\left[\begin{array}{c} \left(\left[L^* + (\widetilde{H}\text{-}H)\right] + \left[\widetilde{H}/\sqrt{\widetilde{r}_b} - H/\sqrt{r_b}\right]\right)r \\ + \left(\left[L^* + (\widetilde{H}\text{-}H)\right] + \left[\widetilde{H}\sqrt{\widetilde{r}_a} - H\sqrt{r_a}\right]\right) \end{array}\right]} \tag{84}$$

Expression (84) will be very useful in the next section.

Section 4: A Convexity Condition

In this section we discover one additional constraint over the space of possible top-hat distributions that an LP can choose. This condition will come from a surprising feature of the Impermanent Loss, which we will discover numerically. Towards this end, we will now numerically illustrate various scenarios by plotting the following relevant features:

- an *initial background* top hat distribution
- a *new LP* top hat distribution
- the *background* distribution that exists *later* (at the time when we measure the IL)
- the graph of IL at this later time

For example, in figure 12 below, we have an initial price around 1750, and background distribution with a crown spanning the range [1500,2000]. The new LP chooses a virtual crown that spans the range [1620,2120]. At some time later, the background distribution has shifted slightly up to a crown in the range [1550,2100].

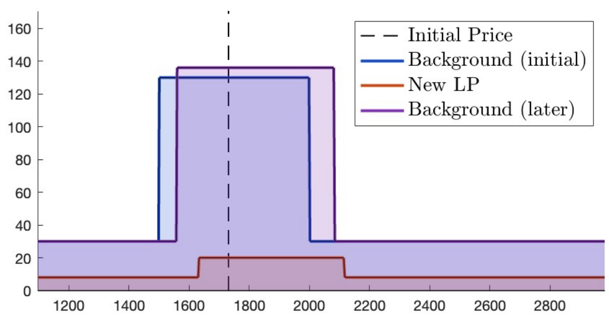


Figure 12. An example scenario.

The corresponding graph of IL versus r is shown in figure 13 (plotted on the $\log(r)$ axis, for symmetry). For comparison, we simultaneously plot the IL that one would experience for a standard constant product pool (Uniswap V2), as well as the IL for a concentrated liquidity position (Uniswap V3) that corresponds to the choice of the LP crown position.

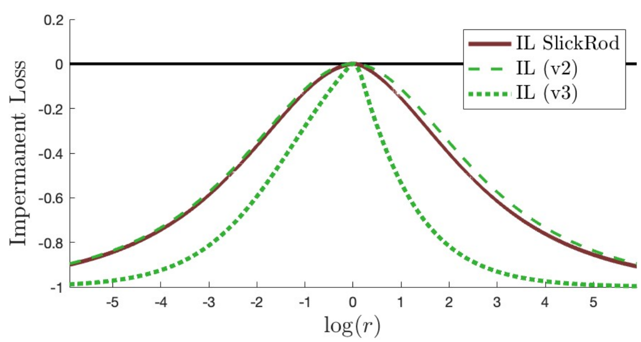


Figure 13. The IL corresponding to Figure 12.

What we see in this case is fairly interesting; the IL experienced by the LP in this case is similar to that in a constant product pool (V2).

This is particularly encouraging because we can say that at least the IL is not as bad as it is would be in the context of concentrated liquidity, this being a particular concern for LPs in recent years.

For our next example, we have the same initial background parameters as we did before, but now the background parameters drift slightly down in price. Meanwhile, the LP crown parameters are chosen slightly higher here, and most importantly, to a range that exists *above* the initial price (figure 14):

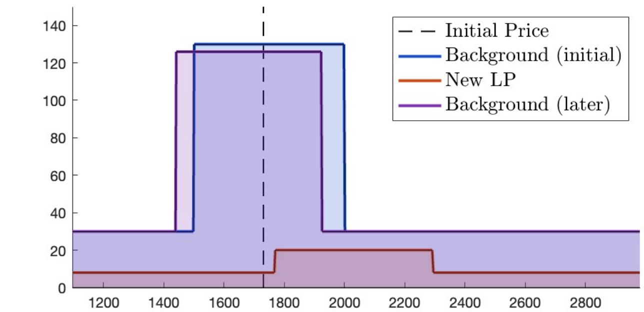


Figure 14. Another example scenario.

Once again, we see that the IL for the LP is closer to that of constant product (V2). It is worth noting that in this scenario (where an LP selects a range that is *above* the initial price), the IL in V3 is actually zero for any downward price movement. Our current LP in the SlickRod does not enjoy this protection, as their selected price range is just a hypothetical vote, not an actual position.

More interestingly, if we zoom into the $r = 1$ region ($\log(r) = 0$) corresponding to small price changes, we see that there is actually some *positive* IL! This should more aptly be called *Impermanent Gain* (figure 15):

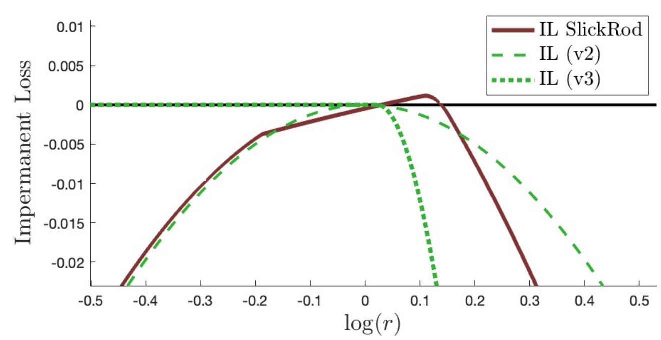


Figure 15. An interesting IL profile.

This potential for impermanent gain is actually concerning. In fact, there exists many scenarios with *guaranteed* gain for small price movements. Consider, for example, the scenario in figure 16 in which the initial and final background parameters are identical and the LP selects a position *above* the initial price.

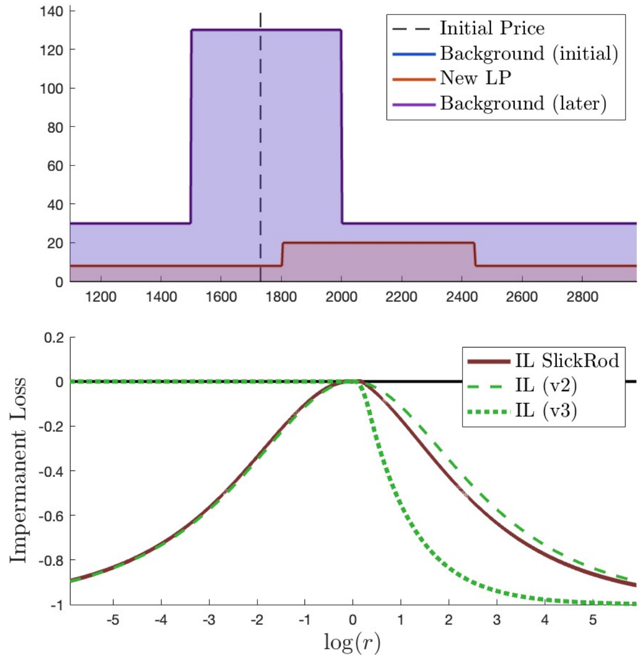


Figure 16. Another example scenario.

If we zoom in on the $r = 1$ region of this figure, we find a convex piece of the IL graph (figure 17):

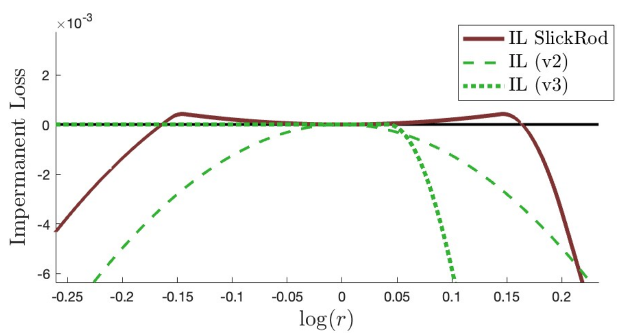


Figure 17. Convex Impermanent Gain

This is a problem - LPs in such a scenario will be motivated to withdraw their assets as soon as the price moves in either direction! To deal with this issue, we would like to impose a condition that forbids such a choice of parameters by the LP. However, to calculate IL, one needs to know what the background parameters at a later time will be, and a new LP will obviously not have this information. The only reasonable constraint therefore, is the following:

> *A new LP cannot select a position that, under the assumption of* **static** *background parameters, has a* **convex** *impermanent loss function near* $r = 1$.

Fortunately, we have our expression (84) which expresses the IL under the scenario of static background parameters. Additionally, (84) assumes $[r] = \widetilde{[r]} = r$, but this seems to be a reasonable assumption for our current analysis (assume the LP is joining a set of background parameters *containg* the current price). Our convexity condition that we would like to enforce is the following:

$$\mathrm{IL}(1) = 0 \tag{85}$$
$$\mathrm{IL}'(1) = 0 \tag{86}$$
$$\mathrm{IL}''(1) < 0. \tag{87}$$

To begin, let us define the following helpful quantities:

$$A := \widetilde{H}\sqrt{\widetilde{r}_a} - H\sqrt{r_a} \tag{88}$$
$$B := \widetilde{H}/\sqrt{\widetilde{r}_b} - H/\sqrt{r_b} \tag{89}$$
$$D := L^* + (\widetilde{H}\text{-}H) \tag{90}$$

Then (84) becomes

$$\begin{aligned}\mathrm{IL}(r) &= \frac{-D(1-\sqrt{r})^2}{\big[D+B\big]r + \big[D+A\big]} \\ &= \frac{Br + 2D\sqrt{r} + A}{\big[D+B\big]r + \big[D+A\big]} - 1 \end{aligned} \tag{91}$$

Clearly we have $\mathrm{IL}(1) = 0$. Then, taking the derivative,

$$\mathrm{IL}'(r) = \frac{\begin{bmatrix} \Big[-D(D+B)\Big]\sqrt{r} \\ +\Big[D(B-A)\Big] \\ +\Big[D(D+A)\Big]1/\sqrt{r} \end{bmatrix}}{\Big(\big[D+B\big]r + \big[D+A\big]\Big)^2} \tag{92}$$

we plug in $r = 1$, and find

$$\mathrm{IL}'(1) = \frac{\begin{bmatrix} -\Big[D(D+B)\Big] \\ +\Big[D(B-A)\Big] \\ +\Big[D(D+A)\Big] \end{bmatrix}}{\Big(\big[D+B\big] + \big[D+A\big]\Big)^2} = 0 \tag{93}$$

Thus, we find $\mathrm{IL}'(1) = 0$ as well. This is going well so far. All we have left is condition (87). Taking the second derivative, we find

$$\mathrm{IL}''(r) = \frac{\begin{bmatrix} \Big[\frac{3}{2}D(D{+}B)^3\Big]r^{3/2} \\ +\Big[2D(D{+}B)^2(A\text{-}B)\Big]r \\ +\Big[\frac{-3}{2}D(D{+}A)(D{+}B)^2\Big]\sqrt{r} \\ +\Big[2D(D{+}A)(D{+}B)(A\text{-}B)\Big] \\ +\Big[\frac{-7}{2}D(D{+}A)^2(D{+}B)\Big]/\sqrt{r} \\ +\Big[\frac{-1}{2}D(D{+}A)^3\Big]/r^{3/2} \end{bmatrix}}{\Big(\big[D+B\big]r + \big[D+A\big]\Big)^4} \tag{94}$$

where we've expanded and sorted the numerator by like terms. This is a little less exciting.

When we plug $r = 1$ into (94) we find

$$\mathrm{IL}''(r) = \frac{\left[\begin{array}{c} \left[\frac{3}{2}D(D{+}B)^3\right] \\ +\left[2D(D{+}B)^2(A\text{-}B)\right] \\ +\left[\frac{-3}{2}D(D{+}A)(D{+}B)^2\right] \\ +\left[2D(D{+}A)(D{+}B)(A\text{-}B)\right] \\ +\left[\frac{-7}{2}D(D{+}A)^2(D{+}B)\right] \\ +\left[\frac{-1}{2}D(D{+}A)^3\right] \end{array}\right]}{\Big(\big[D+B\big]+\big[D+A\big]\Big)^4} \tag{95}$$

This expression can be shockingly simplified. Defining $u := D + A$ and $v := D + B$, and separating the terms with a fractional coefficient, then (95) becomes

$$\begin{aligned} IL''(r) &= D\frac{\left[\begin{array}{c} \frac{1}{2}\Big(3v^3 - 3uv^2 - 7u^2v - u^3\Big) \\ +2(v^2+uv)(u-v) \end{array}\right]}{(u+v)^4} \\ &= D\frac{\left[\begin{array}{c} \frac{1}{2}(u+v)\Big(3v^2 - 6uv - u^2\Big) \\ +\frac{4}{2}(u+v)v(u-v) \end{array}\right]}{(u+v)^4} \\ &= \frac{D}{2}\frac{\left[\begin{array}{c} \Big(3v^2 - 6uv - u^2\Big) \\ +4v(u-v) \end{array}\right]}{(u+v)^3} \\ &= \frac{D}{2}\frac{\left[\ -v^2 - 2uv - u^2\ \right]}{(u+v)^3} \\ &= -\frac{D}{2}(u+v) \end{aligned} \tag{96}$$

and thus we have the remarkable simplification of (95):

$$\mathrm{IL}''(1) = -\frac{D/2}{2D + A + B}. \tag{97}$$

Recall that our goal is to find conditions that guarantee $\mathrm{IL}''(r) < 0$. In other words, we want to satisfy the inequality $-(D/2)/(2D + A + B) < 0$. But this will still hold if we invert: $-(2D + A + B)/(D/2) < 0$. We can divide by -2 to find $(2D + A + B)/D > 0$, which we can then rearrange into $(A + B)/D > -2$. Thus, we have a rephrasing of our condition:

$$\mathrm{IL}''(r) < 0 \quad \Leftrightarrow \quad (A + B)/D > -2 \tag{98}$$

Going back to our original parameters, (98) becomes:

$$\frac{\Big(\widetilde{H}\sqrt{\widetilde{r}_a} - H\sqrt{r}_a\Big) + \Big(\widetilde{H}/\sqrt{\widetilde{r}_b} - H/\sqrt{r}_b\Big)}{L^*{+}(\widetilde{H}\text{-}H)} > -2 \tag{99}$$

Now let's make another change of variables to help with calculation:

$$K := \Big(\widetilde{H}\sqrt{\widetilde{r}_a} - H\sqrt{r}_a\Big) + \Big(\widetilde{H}/\sqrt{\widetilde{r}_b} - H/\sqrt{r}_b\Big) \tag{100}$$

$$J := -(\widetilde{H}\text{-}H) \tag{101}$$

Then our condition in (99) now becomes:

$$K/(L^* - J) > -2. \tag{102}$$

Notice that the quantities K and J depend on all of the *crown* parameters (for both the background and the LP). The beauty of (102) is that the *brim* parameter L^* has been isolated. This suggests a strategy; given the background parameters, the LP can choose whatever crown they want, but their brim parameter L^* (which they shall choose last) must then satisfy (102). All that is left is to put (102) in a more convenient form. This boils down to some cases.

If we have $L^* > J$, then (102) implies $K > \text{-}2(L^*\text{-}J)$, which then rearranges to $L^* > J\text{-}K/2$. Combined with the assumption that $L^* > J$, then we have:

$$L^* > \max\{J, J - K/2\}. \tag{103}$$

Alternatively, if $L^* < J$, then (102) gives $K < \text{-}2(L^*\text{-}J)$. This rearranges to $L^* < J\text{-}K/2$. Again, combined with our assumption $L^* < J$, then we have

$$L^* < \min\{J, J - K/2\}. \tag{104}$$

We should point out that the conditions in (103)-(104) are *both* acceptable. We just need *either one* of them to hold in order for $\mathrm{IL}''(r) < 0$. We can be more concrete, however, based on the sign of K. Specifically, (103)-(104) become:

$$\text{if } K > 0 \quad \Rightarrow \quad (L^* {>} J) \text{ or } (L^* {<} J\text{-}K/2) \tag{105}$$

$$\text{if } K < 0 \quad \Rightarrow \quad (L^* {>} J\text{-}K/2) \text{ or } (L^* {<} J) \tag{106}$$

These expression (105)-(106) can be visualized below:

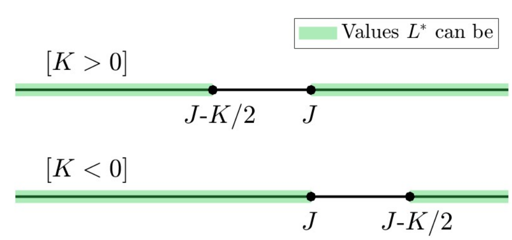


Figure 18. Visualizing the convexity condition.

This gives us the following simple prescription:

$$\boxed{L^* \notin \mathrm{sort}(J, J\text{-}K/2)} \tag{107}$$

Not surprisingly, if we take L^* to be on the boundary of (107), then we find nearly flat regions of IL near $r = 1$:

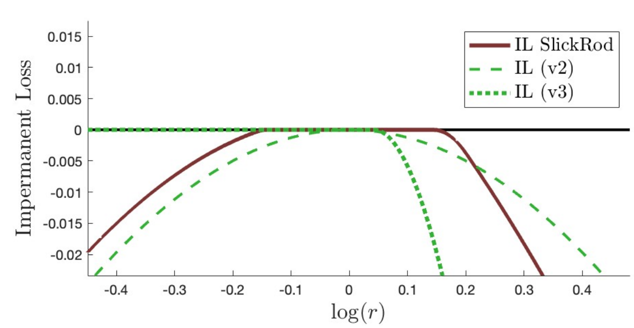
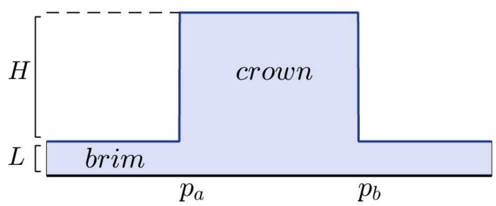


Figure 19. A flat IL region.

Section 5: Summary of Formulas

For clarity, let us now summarize the synthesis process:

- Each LP chooses a virtual top-hat distribution. At the interface level, this is done via the four parameters $\{L_n, H_n, p_{an}, p_{bn}\}$ (for n^{th} LP)



- We convert the price variable p to the logarithmic price ω by $\omega = \log(\sqrt{p})$. We define the change of variables $\{L, H, \omega_a, \omega_b\} \leftrightarrow \{L, A, \mu, \tau\}$ by

$$\begin{bmatrix} L = L & \mu = (\omega_a + \omega_b)/2 \\ A = H(\omega_b\text{-}\omega_a) & \tau = (\omega_a^2 + \omega_b^2 + \omega_a\omega_b)/3 \end{bmatrix}$$

  and the inverse transformation

$$\begin{bmatrix} L = L & \omega_a = \mu - \sqrt{3\tau\text{-}3\mu^2} \\ H = A/(2\sqrt{3\tau\text{-}3\mu^2}) & \omega_b = \mu + \sqrt{3\tau\text{-}3\mu^2} \end{bmatrix}$$

- The LP parameters $\{L_n, H_n, \omega_{an}, \omega_{bn}\}$ can then be converted into the quantities $\{L_n, A_n, \mu_n, \tau_n\}$.

- We define synthesized parameters $\{L, A, \mu, \sigma\}$ by

$$\begin{bmatrix} L := \sum_n L_n & \mu := \sum_n \lambda_n \mu_n \\ A := \sum_n A_n & \tau := \sum_n \lambda_n \tau_n \\ \lambda_n := A_n/A & \end{bmatrix}$$

The synthesized parameters need not be recomputed from scratch each time an LP chooses to join/leave the pool. If, for example, we have a *current* synthesized distribution given by $\{L, A, \mu, \sigma\}$, and a new LP joining the pool with parameters $\{L^*, A^*, \mu^*, \sigma^*\}$, then the *new* synthesized parameters $\{\widetilde{L}, \widetilde{A}, \widetilde{\mu}, \widetilde{\sigma}\}$ will be given by

$$\begin{bmatrix} \widetilde{L} = L + L^* & \lambda = A/\widetilde{A} & \widetilde{\mu} = \lambda A + \lambda^* A^* \\ \widetilde{A} = A + A^* & \lambda^* = A^*/\widetilde{A} & \widetilde{\tau} = \lambda\tau + \lambda^*\tau^* \end{bmatrix}$$

Similarly, if the LP is *leaving* the pool, we simply switch all plus signs to minus signs.

For a current synthesized top-hat distribution with parameters $\{L, H, p_a, p_b\}$, a new user is allowed to freely choose their crown position $\{H^*, p_a^*, p_b^*\}$. From this, the synthesized crown $\{\widetilde{H}, \widetilde{p}_a, \widetilde{p}_b\}$ can be computed . Then we define the following quantities:

$$K := \left(\widetilde{H}\sqrt{\widetilde{r}_a} - H\sqrt{r_a}\right) + \left(\widetilde{H}/\sqrt{\widetilde{r}_b} - H/\sqrt{r_b}\right)$$

$$J := -(\widetilde{H}\text{-}H)$$

The user is allowed to choose their brim parameter L^* subject to the following constraint:

$$L^* \notin \mathrm{sort}(J, J\text{-}K/2)$$

This prevents instantaneous impermanent gain.

The token reserves are given by the formulas

$$x(p) = L/\sqrt{p} + H\left(1/\sqrt{[p]} - 1/\sqrt{p_b}\right)$$

$$y(p) = L\sqrt{p} + H\left(\sqrt{[p]} - \sqrt{p_a}\right).$$

with the box $[\cdot]$ defined by $[p] = \max\{p_a, \min\{p_b, p\}\}$. To articulate the mechanics of swaps/fees, we consider a swap going from price p to p', and we define the following notation to accommodate the calculation of trades in *either* direction:

	if $(p' < p)$	if $(p' > p)$
$t_i :=$	$x(p)$	$y(p)$
$t_o :=$	$y(p)$	$x(p)$
$\Delta t_i :=$	Δx (given)	Δy (given)
$p_\ell :=$	p_b	p_a
$p_u :=$	p_a	p_b
$s :=$	-1	$+1$

Given the current amount of token reserves (t_i, t_o), the user initiates a swap with an incoming quantity Δt_i. We can then compute the outgoing quantity Δt_o by:

$$k := \left(t_i + \Delta t_i + H\sqrt{p_\ell}^{\,s}\right)/(L + H)$$

$$\sqrt{p'}^{\,s} = k + \frac{H}{L}\begin{pmatrix} \max\{0, k - \sqrt{p_u}^{\,s}\} \\ + \min\{0, k - \sqrt{p_\ell}^{\,s}\} \end{pmatrix}$$

$$t_o' = L\sqrt{p'}^{(\text{-}s)} + H\left(\sqrt{[p']}^{(\text{-}s)} - \sqrt{p_u}^{(\text{-}s)}\right)$$

$$\Delta t_o = |t_o' - t_o|$$

To take fees, we use the GIFS method. After computing Δt_o above, we then compute the following quantities:

$$a := (t_i)(\Delta t_o)$$
$$b := (t_o)(\Delta t_i)$$
$$c := (\Delta t_i)(\Delta t_o)$$

Then we compute a scaling factor η by

$$\eta = 1 + \frac{c(a+b)\varphi}{(a+b)^2 - (a+c)^2\varphi}$$

where φ is the fee size. The actual deltas $(\underline{\Delta} t_i, \underline{\Delta} t_o)$ are

$$\underline{\Delta} t_i = \eta\Delta t_i + (\eta\text{-}1)t_i$$
$$\underline{\Delta} t_o = \eta\Delta t_o - (\eta\text{-}1)t_o$$

The token reserves are then updated as

$$t_i \to t_i + \underline{\Delta} t_i$$
$$t_o \to t_o - \underline{\Delta} t_o$$

Finally, we scale up the liquidity parameters $\{H, L\}$:

$$H \to \eta H$$
$$L \to \eta L$$

This scaling of liquidity parameters implies that the fees are stored *internally* as extra liquidity in the pool.