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The ASC

(The Annual Stability Cost)

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In this document, we consider the costs (implicit and explicit) that the users of stablecoins must endure to keep the coins stable. To quantify this notion, we define a simple metric called the ASC, or the *Annual Stability Cost*, and we will apply to five case studies: {DAI, UST, USDe, RAI, USDC}. This document will be arranged into four sections:

- Section 1: Introduction to the ASC
- Section 2: The Five Case Studies
- Section 3: Summary
- Section 4: Mathematical Appendix

Section 1: Introduction to the ASC

"The tree of liberty must be refreshed from time to time with the blood of patriots and tyrants." Likewise, the stability of Stablecoins must be managed over time with a similarly dramatic and equally noble sacrifice; irretrievable capital inflow. If price stability were free, it would happen on its own. But the fact that it doesn't tells us that there is a hidden (or not so hidden) cost.

In this document we aim to quantify this sacrifice (or at least put together rough estimates) for a variety of well known Stablecoins. We will call this quantity the *Annual Stability Cost*, or the *ASC* (which, based on its definition, should obviously be pronounced as the *ask*). The raw amount of capital required to keep a project stable would obviously scale with the size of the project itself, and so for our definition, we will require that the lost amount is normalized by the overall project size in some reasonable way. We thus define the following:

$$\text{ASC} = \frac{\begin{bmatrix}\text{amount of irretrievable capital} \\ \text{required to keep project stable} \\ \text{over the course of a year}\end{bmatrix}}{\begin{bmatrix}\text{some measure of the overall} \\ \text{size of the project (in value)}\end{bmatrix}} \tag{1}$$

There are a few important observations to make based on this definition:

- We emphasize the word *irretrievable* appearing in the definition above. Unlike collateral (which may be retrieved),we want to look at the value that is truly spent and lost in the name of stability.
- As a ratio of two values, the ASC will be *unitless*, and we will often quote its value as a percentage. For example, a project with \$100M TVL that eats up \$3M per year to maintain stability would have an ASC value of 3%.
- The ASC is measured over one year, which is an arbitrary time period (so it is not *totally* unitless). But this feels natural, and we are ok with it.
- We are looking to quantify the capital required to *keep* the project stable. In other words, we will be considering best case scenario stable periods, not periods of instability and/or collapse.
- The most important observation is the following; the ASC is **not** entirely **well defined**. The types of factors that we will identify for the ASC may include broad market forces such as interest rates, demand pressures, and so on. With these types of things, it can be ambiguous as to what extent they are 'maintaining stability'. The analyses to follow are meant to be rough toy models, at best, and we invite criticism and/or suggested improvements.

To make this notion concrete, we consider five different Stablecoins, and model the ASC for each as a separate case study, using historical data. Notably, we will find that this requires many different approaches.

We will consider the following five Stablecoins (with their respective project names) for our case studies:

$$\left\{\begin{array}{l}\text{DAI (Maker)} \\ \text{UST (Terra Luna)} \\ \text{USDe (Ethena)} \\ \text{RAI (Reflexer)} \\ \text{USDC (Circle)}\end{array}\right\}$$

A unifying feature for all of these Stablecoins is that they require some collateral to be provided when they are minted. In a real sense, this collateral is the primary mechanism that is responsible for maintaining stability. Collateral, however, is *retrievable* in normal conditions, and hence it should not be a component of the ASC. In effect, determining the ASC will involve identifying everything besides (or auxilary to) the collateral itself.

Additionally, the definition given in expression (1) is somewhat ambiguous about how to measure the project size over the course of a year, as the project size will almost certainly fluctuate throughout the year in which we are measuring the ASC. To deal with this, we may want to compute a value for the ASC at specific times throughout the year and sum these into one aggregate quantity. For example, if there are identifiable events during the year in which major stability maintenance is required, we may define the ASC as the following:

$$\text{ASC} = \sum_{\text{events}} \frac{[\text{capital inflow during event}]}{[\text{project size during event}]} \tag{2}$$

On the other hand, if we use a continuous model for our Stablecoin ecosystem, then the sum appearing above would be more appropriately written as an integral:

$$\text{ASC} = \int_{\text{year}} \frac{[\text{momentary rate of capital inflow}]}{[\text{momentary project size}]} \tag{3}$$

We will see examples of both (2)-(3) in our case studies.

Section 2: The Five Case Studies

With each Stablecoin, we must do a varying amount of detective work, and we will save the easiest for last. Our first two case studies (DAI & UST) will require the most attention, and because the mathematical details are a bit too involved to sit comfortably in the text of this section, we save the full treatment for the appendix.

Section 2.1. DAI

The Maker platform (now known as 'Sky') allows its users to mint DAI tokens into existence in the form of collateral backed loans. The (over) collateralization requirements ensure that the circulating supply of DAI is fully backed, with emergency liquidizations acting as a backstop. As we described, the collateral that backs these loans should *not* count towards the ASC, as it is (in principle) retrievable. However, there is an interest that accrues on the debt, called the *Stability Fee*, which *does* represent an irretrievable loss on the part of the users, and it is this interest that we identify as the ASC.

As an additional mechanism for maintaining stability, the interest rate itself can evolve over time in response to market conditions (decided by governance). The rate can change quickly at any second, and for this reason we construct a toy model of the DAI ecosystem that is *continuous* in time. Specifically, we define the following:

$$r(t) := \text{second by second interest rate at time } t \tag{4}$$

$$D(t) := \text{total outstanding } \textbf{debt} \text{ at time } t \tag{5}$$

$$F(t) := \text{total accumulated } \textbf{fees} \text{ at time } t \tag{6}$$

In the appendix, we will build a detailed quantitative model for this dynamic system, but even without going through all of the details, one can see that rate at which fees are accruing (i.e. the derivative $F'(t)$) is simply given by the 'instantaneous' interest growing on the debt,

$$F'(t) = r(t)D(t). \tag{7}$$

Moreover, the total supply of DAI in existence is simply equal to the total outstanding debt itself, $D(t)$. Thus, the integral expression for the ASC in equation (3) can be written more explicitly as

$$\text{ASC} = \int_{\text{year}} \frac{[F'(t)]}{[D(t)]} dt = \int_{\text{year}} \frac{[r(t)D(t)]}{[D(t)]} dt = \int_{\text{year}} r(t)\, dt. \tag{8}$$

Expression (8) is not quite complete, however, as there is one more ingredient to consider. Holders of DAI have the option to lock up their DAI for some time in order to receive a fraction of the fees collected. This yield is determined by another variable interest rate $s(t)$, called the *Dai Savings Rate*, or DSR. Analogous to definitions (4)-(6), let us define the following:

$$s(t) := \text{the DSR interest rate at time } t \tag{9}$$

$$L(t) := \text{total amount of DAI } \textbf{locked} \text{ up at time } t \tag{10}$$

$$Y(t) := \text{total accumulated } \textbf{yield} \text{ at time } t \tag{11}$$

Just as in expression (7), one can see that the rate of growth in the yield is simply the interest rate times the amount locked up (details in the appendix):

$$Y'(t) = s(t)L(t). \tag{12}$$

As these are fees being returned to the users, we must subtract them from the fees collected when calculating the ASC. Thus, expression (8) needs to be refined:

$$\begin{aligned} \text{ASC} &= \int_{\text{year}} \frac{[F'(t) - Y'(t)]}{[D(t)]} dt \\ &= \int_{\text{year}} \frac{r(t)D(t) - s(t)L(t)}{D(t)} dt \\ &= \int_{\text{year}} \Big[r(t) - \lambda(t)s(t) \Big] dt \qquad (13) \end{aligned}$$

where $\lambda(t)$ will be defined as the fraction of the DAI supply that is currently locked up in the DSR:

$$\lambda(t) := L(t)/D(t) \tag{14}$$

With equation (13), we now have our ASC for DAI.

With a formula in hand, we can use historical data to obtain a concrete value for the ASC involved for DAI. The historical rates for the Stability Fee and the DSR are shown in figure 1 below, with two separate year-long intervals highlighted in red (09/2022 - 09/2023) and in green (09/2023 - 09/2024):

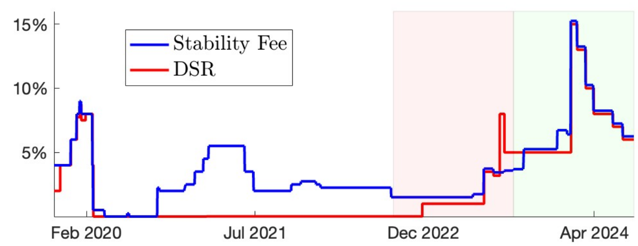


Figure 1 The Stability Fee and DSR for Maker

Obtaining a time series for $\lambda(t)$ (i.e. the fraction of DAI that is locked up in the DSR at any given time) proved to be more difficult, but we find that a reasonable order of magnitude estimate is around $\lambda = 0.25$ (that is, 25% of DAI is locked up). Using this value as a reasonable stand-in for $\lambda(t)$ and integrating over our interest rate time series $r(t)$ and $s(t)$, expression (13) results in an ASC of **1.6%** for the red year and **5.4%** for the green year. In other words, during this past year, users paid an amount equivalent to 5.4% of the total system size in the name of keeping the system stable.

Section 2.2. UST

Terra UST actually involved two tokens; a stablecoin called UST (pegged to the U.S. dollar), and an auxiliary coin LUNA, which is meant to absorb the volatility that UST would otherwise experience. These two coins could be traded independently, but one UST needed to always be redeemable for \$1 worth of LUNA.

In an event where the dollar price of UST depegs to the downside (dips below 1 U.S. dollar), the stabilizing mechanism works as follows:

(a) LUNA is minted and used to purchase UST from individuals on the open market.

(b) The purchased UST is burned to decrease supply and therefore raise the price back up to the peg.

Of course, the newly minted LUNA ought to cause the market price of LUNA to drop appreciably. However, if the project is to remain stable, there must be some organic backstop preventing the price from falling too fast, thus avoiding a death spiral (which is indeed what ultimately happened). Thus, under normal conditions, we assume there exists some amount of capital inflow to maintain buying pressure on LUNA. Importantly, we observe that, from the point of view of the market, this capital flow can be seen as *irretrievable loss*, in the sense that their capital is spent, but their LUNA does not appreciate beyond its initial value. Hence, because this irretrievable loss is required to maintain the stability of the project, we will identify this as the ASC.

The various mechanisms involved here are not as well defined as they were in our previous case, and thus our analysis will be somewhat looser. First, we make the observation that for assets trading on the open market, there is *no deterministic* way in which supply influences price. Naturally, one expects that increasing the supply should depress the price, and vice versa, but there are many immaterial factors that can be difficult to model. Thus, we will make a few assumptions along the way.

Let us first define some variables that we will use. The supplies of UST and LUNA will be denoted by

$$T := \text{supply of UST} \tag{15}$$
$$L := \text{supply of LUNA} \tag{16}$$

Their dollar prices will be denoted by

$$p_T := \text{price of UST (in dollars)} \tag{17}$$
$$p_L := \text{price of LUNA (in dollars)} \tag{18}$$

For a depegging/recovery event as previously described, we let the supply and price of UST *before* and *after* the event be denoted by (T, p_T) and $(T,' p'_T)$, respectively. Our first simplifying assumption will be that as we burn some UST, the price scales inversely with the supply.

Next, this amount of burned UST must have been given up in exchange for some newly minted LUNA, which we denote by ΔL. From an initial price of p_L, we suppose that the increase in LUNA supply brings the price down to p'_L. However, due to organic market demand (capital inflow, denoted ΔC), the drop in price is not as severe as it otherwise could have been. We consider the hypothetical *intermediate* price p''_L, that *would have been* the result of the increased supply, in the absence of capital inflow (illustrated in figure 2):

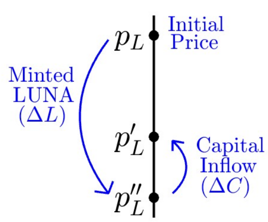


Figure 2. A toy model for LUNA price movement

Once again, there is no well defined deterministic way in which these prices $\{p'_L, p''_L\}$ can be computed from the incoming/outgoing quantities ΔL and ΔC. Here we make our second assumption; we choose to invoke a constant product framework to relate these quantities.

The details are left for the appendix, but the result is the following:

$$\text{ASC} = \left(1 - p_T/p'_T\right)\left(\frac{\Delta L\sqrt{p_L\, p'_L} - \Delta T}{\Delta L\, p_L - \Delta T}\right) \tag{19}$$

Expression (19) is for one recovery event. Consequently, we must sum it over the course of a full year:

$$\text{ASC} = \sum_{\substack{\text{recovery}\\ \text{events}}} \left(1 - p_T/p'_T\right)\left(\frac{\Delta L\sqrt{p_L\, p'_L} - \Delta T}{\Delta L\, p_L - \Delta T}\right) \tag{20}$$

where each recovery event is characterized by the price change $(p_T, p_L) \to (p'_T, p'_L)$ and supply change $(\Delta T, \Delta L)$.

Our expression for the ASC in (19) is pretty opaque (again, see the appendix for details), but we can probe it by considering some edge cases. In particular, there are three extreme cases for us to consider:

- If $\boxed{p'_T = p_T}$, then there would not actually exist any recovery at all because the price of UST does not even rise. Indeed, one can easily verify from expression (19) that $\boxed{\text{ASC=0}}$ in this case, and this makes sense as there can be no capital expended to maintain stability if stability is not maintained.
- If $\boxed{p'_L = p''_L}$, then this means that LUNA had no price recovery from the minting event. According to our toy model, this means there was no capital inflow, and thus there is no ASC to speak of. This one is harder to check, but in the appendix we can verify that we will indeed have $\boxed{\text{ASC=0}}$.
- If $\boxed{p'_L = p_L}$, then the capital inflow allows for a complete price recovery for LUNA. In this case, one easily checks from (19) that the ASC then simplifies to $\boxed{\chi = (1 - p_T/p'_T)}$. In other words, if we can assume total recovery in LUNA, then the only factor determining the ASC is the measure of the severity of the UST price journey $p_T \to p'_T$ which needs to be funded. Moreover, if we make the additional assumption $\boxed{p_T\text{=0}}$, we find $\boxed{\text{ASC=1}}$ (i.e. 100%). In this case, the project has lost all of its value (p_T=0), and so bringing it back up to any non-zero price requires an infusion of capital equal to 100% of the total system's value.

With these edge cases verified, we can now have some confidence in the validity of our ASC model in (19).

Let us now look at some historical data. In figure 3 below, we have the time series for the price of UST and LUNA over period beginning in January 2021 lasting until the collapse in mid 2022. First, we notice that we can see a clear slip-and-recovery event in May of 2021.

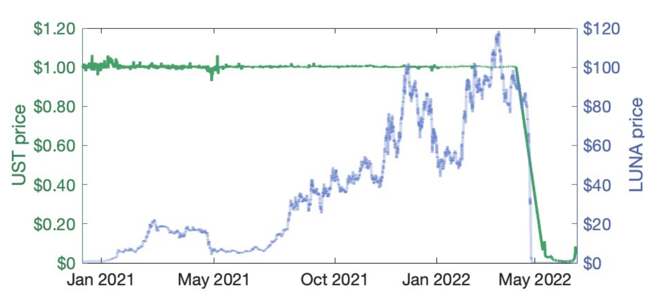


Figure 3. Time Series of UST & LUNA prices

During this event, the price of UST fell to \$0.93 before it was brought back up to the \$1 peg. About 0.27B LUNA was minted in order to burn about 0.33B UST. The price of LUNA consequently dropped from \$15 to \$6. Plugging these numbers into formula (19) gives us an ASC of **4.2%**, for that event alone. Admittedly, an *event* is not necessarily well defined, but there are two more big LUNA minting events that we find occurring in the year 2021. Carrying out the same analysis, we sum together to find a total ASC of about **10%**.

Section 2.3. USDe

The Ethena project offers users the opportunity to steak collateral in order to mint USDe tokens, much like the other lending protocols. However, this collateral is then used to open a perpetual futures short position as a hedge against the volatility in the price of the collateral itself. The collateral that is required is commensurate with the USDe tokens minted, and may be redeemed at any time by burning USDe. Thus, the collateral does not involve any irretrievable loss.

The perpetual short position requires a continuous premium payment, called the *funding amount*, which is paid from the long position holders to the short position holders, or vice versa, depending on market conditions. In principle, these premium payments *could* constitute an irretrievable loss on the part of the users (that is, the short position holders), but in practice it does not. Historically, the funding amounts tend to be *positive*, meaning they to go *from* the long positions *to* the short positions, and a good portion of this income is stored by Ethena in an emergency fund. Moreover, some of the collateral assets allow for liquid staking, and some of the yield generated from therein also contributes to the emergency fund. Thus, even when the funding amounts are *negative* (*from* the short position holders *to* the long position holders), these payments can be made from the emergency fund, and do *not* represent an additional cost to those who are posting the collateral.

Additionally, the type of perpetual futures contract that Ethena employs is typically an unleveraged inverse perpetual (and occasionally linear perpetuals, as well). These types of positions always guarantee a constant dollar amount of the collateral asset, by construction, and thus liquidations will be nearly impossible. Indeed, liquidations can only occur when the funding amounts are not met and the collateral asset must be sold off to meet the funding requirements. As stated, this is not expected to happen, based on historical precedent.

Of course, precedent is only precedent until it is not. If sentiment in the perpetuals market were to change dramatically due to world events, it is entirely possible that the emergency fund could be deplenished and the collateral assets liquidated. To their credit, the Ethena documentation addresses these prospects thoroughly.

Thus, we have an interesting result for our purposes here; the ASC is essentially **0%**, but the mechanism for stability is possibly subject to catastrophic collapse, depending on market conditions. Of course, the latter claim could be made about any stablecoin project.

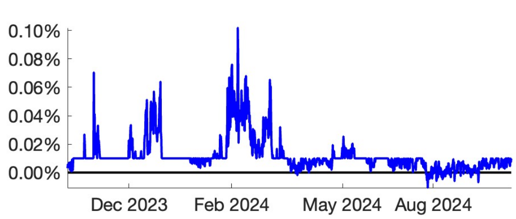


Figure 4 Funding amount (as percentage of position size)

Section 2.4. RAI

Reflexer is an offshoot of the Maker project, which we already analyzed in depth for Section 2.1. However, the major difference now is that, rather than using an adjustable Stability Fee, Reflexer employs an adjustable *redemption price*, which is the exchange rate between dollars and RAI. Though this may be counter-intuitive at first, it allows for incentive structures that bring the price of RAI back towards the one dollar peg. Though the mechanics here are quite interesting, they do not introduce any irretrievable loss on the part of the users, and are thus irrelevant for this current paper. The only irretrievable loss is a minor *constant* Stability Fee of 0.5% per year, and thus we can easily conclude that the ASC for RAI is **0.5%**.

Section 2.5. USDC

Circle's USDC is the easiest case of ours to consider. Just as it is with any stablecoin that we've considered, users exchange existing assets in order to mint USDC, and they may burn USDC to retrieve said assets. Cirlce requires no fees for these transactions, and there are no other hidden costs that could possibly count towards the ASC. In this case, the ASC is trivially equal to zero. Of course, the reason this is so simple to manage is that USDC is minted and burned by a large, trusted, centralized authority. From an idealogical point of view, this could possibly be considered the most severe possible cost. Nevertheless, the ASC is **0%**.

Section 3: Summary

Now we summarize our case studies. Recalling that we have defined the ASC according to the following,

$$\text{ASC} = \frac{\left[\begin{array}{c}\text{amount of irretrievable capital}\\ \text{required to keep project stable}\\ \text{over the course of a year}\end{array}\right]}{\left[\begin{array}{c}\text{some measure of the overall}\\ \text{size of the project (in value)}\end{array}\right]}$$

we now aggregate the values we found, as well as some accompanying observations:

Coin	ASC	notes
DAI	**∼5%**	safe and reliable medium sized ASC
UST	**∼10%**	already collapsed and had excessive ASC
USDe	**0%**	zero ASC, but vulnerable to derivatives market
RAI	**0.5%**	minimal ASC, and apparently robust
USDC	**0%**	zero ASC, but highly centralized

Ultimately, the definition and calculation of the ASC is admittedly loose, and this is only meant to start the conversation. Further development of the ideas and/or further analysis on other coins is certainly welcome.

Section 4: Mathematical Appendix

Here we provide some of the mathematic details that were omitted from Sections 2.1 and 2.2, regarding the ASC for DAI and UST.

Section 4.1. DAI

For the Maker DAI ecosystem, there exists a global interest rate that is variable in time. The interest rate has a value for each second in time, and we denote this sequence of interest rates by $\{r_{t_k}\}$, where t_k represents the sequence of seconds, and each r_{t_k} gives the rate of return over one second (it is actually only updated in 12 second intervals, but this is immaterial for us here).

The protocol keeps track of a running cumulative interest rate R defined by

$$R(t) = \prod_{t_k \le t} (1 + r_{t_k}) \tag{21}$$

The main utility of this quantity R is that, regardless of when this cumulative product begins, the interest factor *between* two times t_a and t_b is simply given by the ratio of R at those two times:

$$\begin{bmatrix} \text{interest factor} \\ \text{between} \\ t_a \text{ and } t_b \end{bmatrix} = \prod_{t_a < t_k \le t_b} (1 + r_{t_k}) = \frac{R(t_b)}{R(t_a)} \tag{22}$$

Let us take the logarithm on both sides of (21),

$$\log\big[R(t)\big] = \sum_{t_k \le t} \log(1 + r_{t_k}) \tag{23}$$

and consider the difference quotient over one second Δt,

$$\frac{\log[R(t+\Delta t)] - \log[R(t)]}{\Delta t} = \frac{\sum\limits_{t_k \le t+\Delta t} \log(1+r_{t_k}) - \sum\limits_{t_k \le t} \log(1+r_{t_k})}{\Delta t}$$
$$= \log(1 + r_{t_k'}) \tag{24}$$

where t_k' is the most recent update time $t < t_k' \le t + \Delta t$ and we use the fact that $\Delta t = 1$ on the right hand side. Now, because the rate has a value for each second, we may approximate it as a continuous function. In this case, the difference quotient in (24) becomes the logarithmic derivative $R'(t)/R(t)$, and (24) becomes

$$R'(t)/R(t) \approx \log\big[1 + r(t)\big] \tag{25}$$

The values of $r(t)$ will be extremely small (being returns over one second), and by the fact that $\log(1+\epsilon) \approx \epsilon$, we rewrite this once more:

$$R'(t)/R(t) \approx r(t) \tag{26}$$

or, more simply

$$R'(t) = r(t)R(t) \tag{27}$$

For our model, we will take (27) as the *definition* of R. In particular, we assume a continuous instantaneous rate of return $r(t)$, and we define the cumulative rate $R(t)$ to be the solution to (27) with $R(0) = 1$, which is easily found to be

$$R(t) = \exp\left(\int_0^t r(s)ds\right) \tag{28}$$

We begin by considering the vault of one fixed user. We suppose there is a discrete sequence of withdrawals from the vault in which DAI must be *minted*. We will denote these amounts by $\{m_1, m_2, ...\}$ and they occur at times $\{t_{m_1}, t_{m_2}, ...\}$. Similarly, we suppose there are a sequence of repayments into the vault in which the DAI is *burned*. We denote these amounts by $\{b_1, b_2, ...\}$ and they occur at times $\{t_{b_1}, t_{b_2}, ...\}$.

At time t, the total amount of minted DAI $M(t)$ is given by summing all the withdrawals up to time t,

$$M(t) = \sum_{i \le t} m_i \tag{29}$$

where the notation $i \le t$ is shorthand for $\{i \,|\, t_{m_i} \le t\}$. Similarly, the total amount of burned DAI, $B(t)$, is

$$B(t) = \sum_{i \le t} b_i \tag{30}$$

Now, when an amount m_i is withdrawn from the vault, it begins accruing interest (the stability fee). Using equation (22), we can say that the outstanding debt at time t from a withdrawal m_i at time t_{m_i} is given by

$$\begin{bmatrix} \text{outstanding debt} \\ \text{on } m_i \text{ at time } t \end{bmatrix} = m_i \frac{R(t)}{R(t_{m_i})} \tag{31}$$

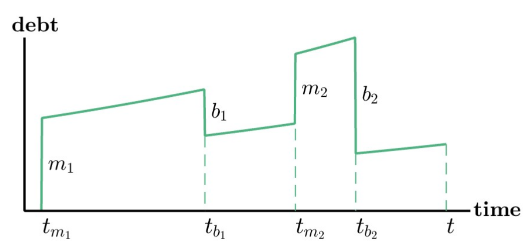
The total outstanding debt in the vault would then be given by summing over all withdrawals up to time t. However, the repayments must also be accounted for. In particular, they are not simply subtracted at one point in time, but rather they undo all the *future* interest that *would* have been accrued. In other words, the total outstanding debt at time t, denoted by $D(t)$, will be

$$D(t) = \sum_{i,j \le t} \left[m_i \frac{R(t)}{R(t_{m_i})} - b_j \frac{R(t)}{R(t_{b_j})} \right] \tag{32}$$

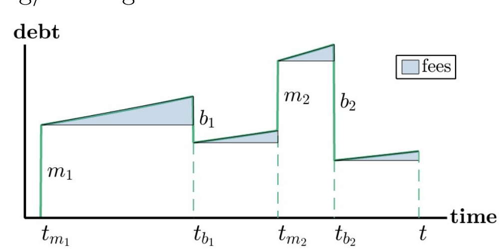
or alternatively written,

$$D(t) = R(t) \sum_{i,j \le t} \Big[m_i / R(t_{m_i}) - b_j / R(t_{b_j}) \Big] \tag{33}$$

We can visualize these quantities in the following figure:



The interest accruing in this vault (the fees) is precisely equal to the climb in debt we see occurring between each minting/burning event. We visualize this below:



To compute cumulative fees, we could work out a complicated calculation by summing the interest over each stretch of time between minting/burning events. However, we could instead simply observe that the debt at any given time t is simply the sum of every minting event plus the total interest accrued, minus every burning event. In other words, we have

$$D(t) = M(t) - B(t) + \begin{bmatrix} \text{cumulative fees} \\ \text{at time } t \end{bmatrix} \tag{34}$$

If we use $F(t)$ to denote these cumulative fees taken (interest accrued) at time t, then expression (34) gives us the following simple formula:

$$F(t) = D(t) - M(t) + B(t) \tag{35}$$

If we wish to write this more explicitly, we substitute definitions (29), (30) and (32) into equation (35) directly which simplifies into the following:

$$F(t) = \sum_{i,j \le t} \left[m_i \left(\frac{R(t)}{R(t_{m_i})} - 1 \right) - b_j \left(\frac{R(t)}{R(t_{b_j})} - 1 \right) \right] \tag{36}$$

Because it will be useful later on, let us investigate the rate of change (derivative) of $F(t)$:

$$F'(t) = D'(t) - M'(t) + B'(t) \tag{37}$$

The notion of the derivative is loose here, because the functions $M(t)$ and $B(t)$ have jump discontinuities at the points $t \in \{t_{m_1}, t_{m_2}, ..., t_{b_1}, t_{b_2}, ...\}$. However, each jump in $M(t)$ or $B(t)$ is canceled out by a corresponding jump in $D(t)$, as one can check by the signs in (37), ensuring that $F(t)$ is at least continuous. Nevertheless, let us consider a point $t \notin \{t_{m_1}, t_{m_2}, .., t_{b_1}, t_{b_2}, ..\}$. Then we know $M'(t) = B'(t) = 0$, and so, using the definition (33) of $D(t)$, we have

$$\begin{aligned} F'(t) &= D'(t) \\ &= R'(t) \sum_{i,j \le t} \Big[m_i / R(t_{m_i}) - b_j / R(t_{b_j}) \Big] \\ &= \Big[R'(t)/R(t) \Big] R(t) \sum_{i,j \le t} \Big[m_i / R(t_{m_i}) - b_j / R(t_{b_j}) \Big] \\ &= r(t) D(t) \end{aligned} \tag{38}$$

where we used (27) in the last line.

Now that we have a description of the stability fee (the interest rate), let us consider the DAI Savings Rate (DSR). Specifically, a user has the option to lock up some of their DAI for a fixed period of time and receive interest on it at the end of the period. Let us denote the length of this time period by T, and suppose the user locks up the amounts $\{\ell_1, \ell_2, ...\}$ at times $\{t_{\ell_1}, t_{\ell_2}, ...\}$. We denote the instantaneous rate of return by $p(t)$, and the cumulative rate $P(t)$ as the solution to

$$P'(t) = p(t) P(t) \tag{39}$$

completely analogously to $r(t)$ and $R(t)$.

The amount of DAI locked up at a time t, denoted $L(t)$, is given by

$$L(t) = \sum_{t-T < i \le t} \ell_i \frac{P(t)}{P(t_{\ell_i})} \tag{40}$$

where the condition $t - T < i \le t$ ensures that only the DAI that is *currently locked* is counted. The total cumulative amount of yield earned by locked DAI up to a time t will be denoted by $Y(t)$ and is given by

$$Y(t) = \sum_{i \le t} \ell_i \left(\frac{P(\lceil t \rceil_{t_{\ell_i}})}{P(t_{\ell_i})} - 1 \right) \tag{41}$$

where the minus 1 ensures that we are only considering the returns of the DAI Savings Rate, rather than the principle locked amounts themselves, and the argument appearing in the numerator $\lceil t \rceil_{t_{\ell_i}}$ is defined by

$$\lceil t \rceil_{t_{\ell_i}} = \min\{t, t_{\ell_i} + T\}, \tag{42}$$

which is necessary because the locked amount ℓ_i only earns interest over a time period T, and not thereafter. One can verify that $Y(t)$ is continuous, and moreover differentiable for $t \notin \{t_{\ell_1}, t_{\ell_2}, ...\}$. Let us compute $Y'(t)$:

$$\begin{aligned} Y'(t) &= \frac{d}{dt} \left[\sum_{i \le t} \ell_i \left(\frac{P(\lceil t \rceil_{t_{\ell_i}})}{P(t_{\ell_i})} - 1 \right) \right] \\ &= \frac{d}{dt} \left[\sum_{i \le t-T} \ell_i \frac{P(t_{\ell_i} + T)}{P(t_{\ell_i})} + \sum_{t-T < i \le t} \ell_i \frac{P(t)}{P(t_{\ell_i})} \right] \\ &= \sum_{t-T < i \le t} \ell_i \frac{P'(t)}{P(t_{\ell_i})} \\ &= \left[\frac{P'(t)}{P(t)} \right] \sum_{t-T < i \le t} \ell_i \frac{P(t)}{P(t_{\ell_i})} \\ &= p(t) L(t) \end{aligned} \tag{43}$$

where we used (39) and (40) to simplify the last line.

So far, we have only been considering the vault for a single user, but all of the relevant quantities (such as minted/burned tokens, total debt, etc.) sum together linearly over all vaults without changing our formulas.

Finally, our ASC will be given by the integral form of expression (3). Specifically, the capital inflow is the Stability Fee, minus the DSR. Moreover, because the current outstanding debt must be repaid eventually (through repayments or liquidations), then this is a good measure for the total project size. Thus, we have

$$\text{ASC} = \int_{\text{year}} \frac{[F'(t) \text{-} Y'(t)]}{[D(t)]} dt = \int_{\text{year}} \frac{r(t)D(t) \text{-} s(t)L(t)}{D(t)} dt \tag{44}$$

If we define $\lambda(t)$ to be the fraction of DAI locked up in the DSR, $\lambda(t) := L(t)/D(t)$, then we have

$$\text{ASC} = \int_{\text{year}} \Big[r(t) - \lambda(t) s(t) \Big] dt \tag{45}$$

which is the expression presented in (13).

——— *Section 4.1. UST* ———

Let us first define some variables that we will use. The supplies of UST and LUNA will be denoted by

$$T := \text{supply of UST} \tag{46}$$
$$L := \text{supply of LUNA} \tag{47}$$

Their dollar prices will be denoted by

$$p_T := \text{price of UST (in dollars)} \tag{48}$$
$$p_L := \text{price of LUNA (in dollars)} \tag{49}$$

First, we make the simplifying assumption that the price of UST scales inversely with the supply. In other words, at a current supply T and price p_T, we imagine that bringing the supply to a new value T' should result in a new price p'_T given by

$$p'_T = p_T\big(T/T'\big) \tag{50}$$

So, for example, doubling supply cuts the price in half.

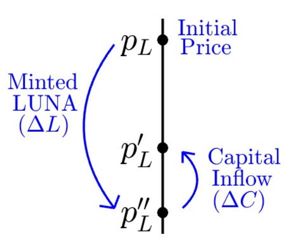
Now, we imagine that the price of UST has slipped to some price p_T below the peg, and by burning an amount ΔT, the price rises to p'_T. Then (50) becomes

$$p'_T = p_T\Big(T/(T\text{-}\Delta T)\Big). \tag{51}$$

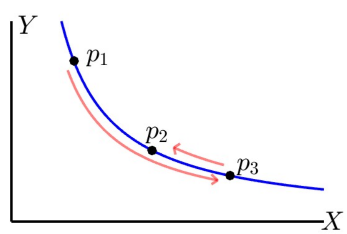
Equation (51) can be rearranged to give us

$$\Delta T = T\big(1 - p_T/p'_T\big). \tag{52}$$

Next, the amount of burned UST must have been given up in exchange for some newly minted LUNA, which we denote by ΔL. From an initial price of p_L, we suppose that the increase in LUNA supply brings the price down to p'_L. However, due to organic market demand (capital inflow ΔC), the drop in price is not as severe as it otherwise could have been. We consider the hypothetical *intermediate* price p''_L, that *would have been* the result of the increased supply, in the absence of capital inflow. This is illustrated below:



We will model these quantities on a constant product curve, with a price journey $p_1 \to p_3 \to p_2$, shown below:



The constant product formulas $(X, Y) = (\sqrt{K/p}, \sqrt{Kp})$ allow us to compute a few results that will be useful in a moment. First, we have the following:

$$\underset{[p_1\to p_3]}{\frac{|\Delta Y|}{|\Delta X|}} = \frac{\sqrt{Kp_1} - \sqrt{Kp_3}}{\sqrt{K/p_3} - \sqrt{K/p_1}} = \sqrt{p_1 p_3} \tag{53}$$

Next, we compute

$$\begin{aligned}
\underset{[p_3\to p_2]}{|\Delta Y|} &= \sqrt{Kp_2} - \sqrt{Kp_3} \\
&= \left(\sqrt{K/p_3} - \sqrt{K/p_1}\right)\left(\frac{\sqrt{p_2} - \sqrt{p_3}}{1/\sqrt{p_3} - 1/\sqrt{p_1}}\right) \\
&= \underset{[p_1\to p_3]}{|\Delta X|}\left(\frac{\sqrt{p_2} - \sqrt{p_3}}{1/\sqrt{p_3} - 1/\sqrt{p_1}}\right) \\
&= \underset{[p_1\to p_3]}{|\Delta X|}\sqrt{p_1 p_3}\left(\frac{\sqrt{p_2 p_1}/\sqrt{p_1 p_3} - 1}{p_1/\sqrt{p_1 p_3} - 1}\right)
\end{aligned} \tag{54}$$

Equations (53)-(54) may appear needlessly tedious, but they can be made useful by thoughtfully mapping them onto our current problem. For example, we have an obvious correspondence between the prices:

$$p_1 \sim p_L, \qquad p_2 \sim p'_L, \qquad p_3 \sim p''_L \tag{55}$$

When we reference the price p as the ratio $\Delta Y/\Delta X$, we are treating X as the risk asset and Y as the numeraire. Thus, on our journey $p_1 \to p_3$, corresponding to price movement $p_L \to p''_L$, we can translate expression (53) as the exchange between minted LUNA and burned UST:

$$\underset{[p_1\to p_3]}{|\Delta Y|/|\Delta X|} \sim \Delta T/\Delta L \tag{56}$$

Moreover, on the price journey $p_3 \to p_2$, the quantity ΔY that we computed in (54) corresponds to the capital inflow ΔC over the movement $p''_L \to p_L$:

$$\underset{[p_3\to p_2]}{|\Delta Y|} \sim \Delta C \tag{57}$$

Now we can fully translate everything. First, expression (53) can be translated as the following:

$$\Delta T/\Delta L = \sqrt{p_L\, p''_L} \tag{58}$$

Moreover, (54) becomes

$$\Delta C = \Delta L\sqrt{p_L\, p''_L}\left(\frac{\sqrt{p'_L\, p_L}/\sqrt{p_L\, p''_L} - 1}{p_L/\sqrt{p_L\, p''_L} - 1}\right) \tag{59}$$

Substituting (58) into (59), we find

$$\Delta C = \Delta L\left(\frac{\Delta T}{\Delta L}\right)\left(\frac{\sqrt{p'_L\, p_L}(\Delta L/\Delta T) - 1}{p_L(\Delta L/\Delta T) - 1}\right) \tag{60}$$

which simplifies as

$$\Delta C = \Delta T\left(\frac{\Delta L\sqrt{p_L\, p'_L} - \Delta T}{\Delta L\, p_L - \Delta T}\right) \tag{61}$$

Finally then, to compute the ASC, we need to view the capital cost ΔC as a fraction of the total project size, i.e. the supply of UST. Thus, we divide by T:

$$\text{ASC} = \frac{\Delta T}{T}\left(\frac{\Delta L\sqrt{p_L\, p'_L} - \Delta T}{\Delta L\, p_L - \Delta T}\right) \tag{62}$$

But the ratio of $\Delta T/T$ is given to us by (52), and so we arrive at our final expression:

$$\text{ASC} = \Big(1 - p_T/p'_T\Big)\left(\frac{\Delta L\sqrt{p_L\, p'_L} - \Delta T}{\Delta L\, p_L - \Delta T}\right) \tag{63}$$

This is the expression presented in (19), as promised.