Internal Fees For Concentrated Liquidity

Introduction

In the present document, we will outline our approach for implementing a novel fee structure for concentrated liquidity pools, as well as present some simulation results illustrating the utility of such a fee structure.

It is well known that the fee structure for Uniswap V2 was significantly changed upon transitioning to V3. In the former, the elegant simplicity of the constant product protocol allows us to take fees and deposit them directly back into the pool. In the latter, the more complicated nature of concentrated liquidity necessitates an alternative approach where fees are taken, kept out of the pool, and subsequently delivered directly to LPs. We will refer to these two approaches to fee taking as *internal* and *external*, respectively.

There are advantages and disadvantages for the two structures. Storing the fees *internally* in the pool is desirable because

- it can reduce the gas costs,
- it allows fees to compound.

On the other hand, storing the fees *externally* has the benefit that

- fees are not subject to impermanent loss.

The tradeoff here is interesting. In particular, when restricted to a constant product AMM, simulations suggest that internal fees are preferable (in terms of the payoff to LPs) when either

- volatility is small,
- or fee size is large.

However, when volatility is high, we find that

- internal fees have more *extreme* payoffs,

in the sense that the payoff for LPs will have both higher highs *and* lower lows. The reader is invited to find the results here. We will present similar simulation results for concentrated liquidity pools later in this document.

Finally, it is worth pointing out that regardless of the advantages and disadvantages of internal fee structures described above, a primary reason that Uniswap V3 uses *external* fees may be due to the fact that an *internal* fee structure (in this context) is not entirely mathematically trivial. Having found an elegant mathematical framework for doing so, we hope to build and develop an internal fee structure that can be deployed in real world concentrated liquidity pools.

Method

In the context of concentrated liquidity, the conventional way to take fees is to remove a small fee from the incoming quantity and awarded to the LPs (pro rata). This requires us to store fee variables that scale with the number of active ticks. We would like to eliminate these extra fee variables by putting the fees back in the pool. However, if we were to naively import that internal fee method from V2, this would not agree with the structure of the LP positions for concentrated liquidity.

Specifically, each price range between adjacent active ticks has a liquidity scale factor L that is the *sum* of all LP liquidities that overlap with the current range. For a trade that moves the price by Δp, there is a strict algebra relating $(L, \Delta p)$ to the input/output amounts $(\Delta x, \Delta y)$. By depositing a single fee into the pool, the relations would be broken, and a complicated rewriting of all the chosen LP positions would be necessary to maintain mathematical cohesion.

Instead, we present an alternative approach. First, we calculate the appropriate $(\Delta x, \Delta y)$ as if *no fees were taken*, and then we simply scale up the liquidity L by some factor η. Uniform scaling of the liquidity results in a commensurate scaling of the coordinates (x, y), but leaves the current price p unchanged (see the Figure 1).

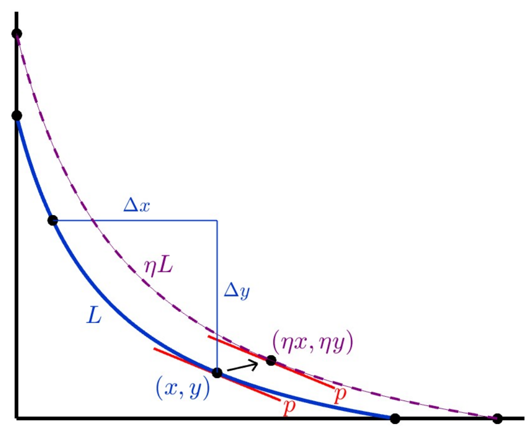

Figure 1. Scaling the liquidity

One can show (and indeed we have a paper to do so) that this approach is completely consistent with the mathematical framework of concentrated liquidity. The only question is *how to choose* this factor η. We have an elegant answer to this question, and the mathematical derivation will appear in the official paper.

Computing η and updating $L \to \eta L$ requires additional arithmetic. Thus, we have a tradeoff between eliminating the need to store fee variables (which scales with the number of LPs) and replacing it with extra arithmetic (which *also* scales with the number of LPs).

Simulations

As we have just discussed, implementing internal fees in concentrated liquidity comes with a computational tradeoff; more arithmetic for less storage. There is reason to believe that this may result in a net reduction of gas costs, but this remains to be seen. In the meantime, however, there is a more important feature of our fee structure that we should discuss; the payoff for LPs.

For this, we will run some market simulations. Our primary metric will be the impermanent loss (IL). The typical mathematical treatment of IL does not take fees into account. We will of course factor in the the fees, and we denote this quantity by ILF ('Impermanent Loss with Fees'). The structure of our simulations is as follows:

- **Initialize** identical LP positions in two distinct concentrated liquidity pools; one with internal fees and one with external fees. Each pool starts at the same initial price.
- **Iterate** over a random walk of price movements. For each new price $p(t)$, make an appropriate trade within each pool to move each pool to $p(t)$.
- **Calculate** the running ILF for each pool, as measured relative to the initial state.

For our random walk, we will simulate lognormal Brownian motion, with a standard deviation σ which we refer to as *volatility* (with an appropriate time scale):

$$p(t + \Delta t) = p(t)e^W \quad \left(W \sim N(0, \sigma\sqrt{\Delta t})\right). \qquad (1)$$

We choose a time step of $\Delta t = 1$ minute, and an initial volatility of $\sigma = 1.0$, or 100% (annually). For each simulation, we plot the running ILF over a total period of 3 months, as well as the time series of prices driving the simulation (see Figure 2 for our first example).

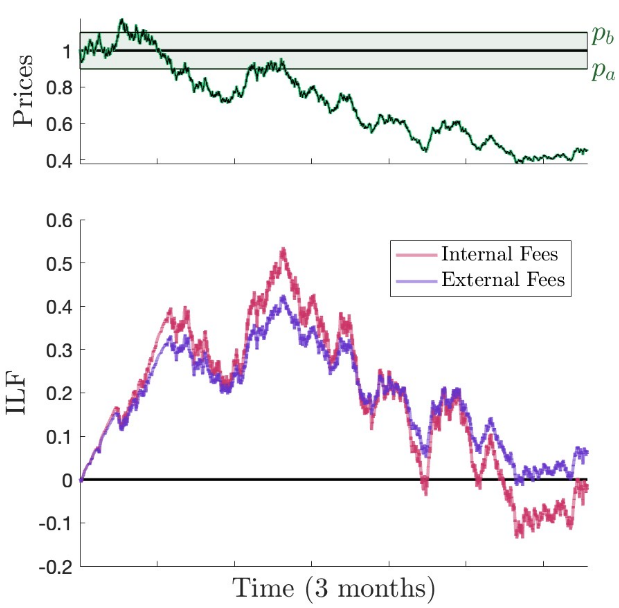

Figure 2. A single simulation.
Fee=0.3%, vol=100%, $[p_a, p_b] = [0.9, 1.1]$

In Figure 2, we begin with a fee size of 0.3%, an initial price of $p_0 = 1$, and an LP position of $[p_a, p_b] = [0.9, 1.1]$ We see that when the ILF is growing, it tends to grow *more* for internal fees, but when it is falling, it tends to fall more as well. We can also see how these movements roughly correspond to times when the market price is *within* our LP position.

In fact, in Figure 3 we have a particular run where the price stays within our position most of the time, and the internal fee structure performs dramatically well.

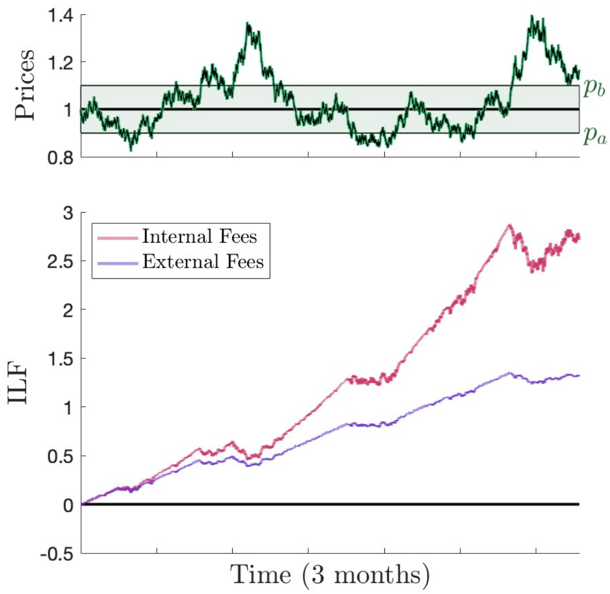

Figure 3. A single simulation
Fee=0.3%, vol=100%, $[p_a, p_b] = [0.9, 1.1]$

If we widen the LP position to $[p_a, p_b] = [0.5, 2.0]$, then we can almost guarantee that the market price stays within our position. However, this comes at a cost; the relative advantage of internal fees tends to diminish (compare the scale on the vertical axes between Figures 3 and 4).

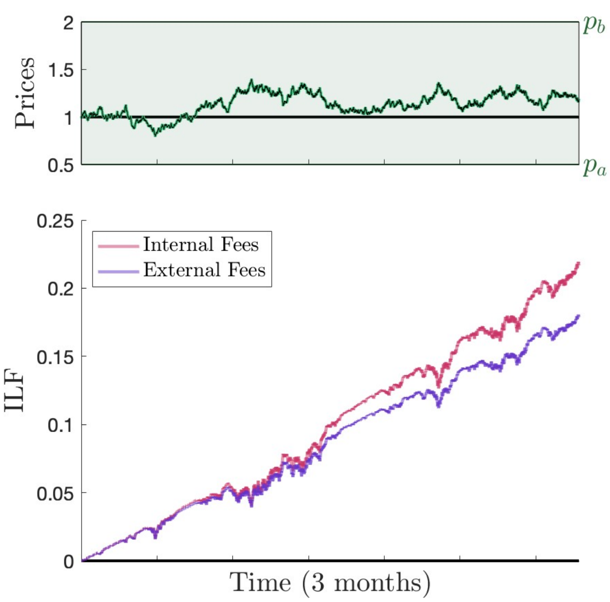

Figure 4. A single simulation.
Fee=0.3%, vol=100%, $[p_a, p_b] = [0.5, 2.0]$

While these individual simulation runs are interesting, we should really collect results over *many* trials. Once again beginning with a fee size of 0.3%, volatility at 100%, and an LP position of $[p_a, p_b] = [0.9, 1.1]$, we run 1,000 trials and collect the *final* ILF value over the 3 month period for each trial. The results are plotted in Figure 5.

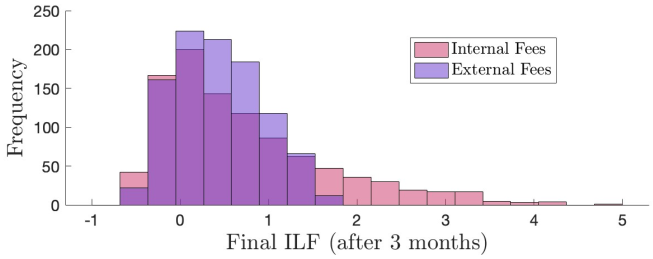

Figure 5. Distribution of final ILF for 1000 trials. Fee=0.3%, vol=100%, $[p_a, p_b] = [0.9, 1.1]$

These distributions seem to confirm our previous anecdotal observation that internal fees can produce *more extreme* ILF values. In particular, we see in Figure 5 that the the final ILF values for internal fees are over represented in both the positive and negative end of the distribution, while the distribution for external fees is bulkier in the middle.

(We should note that despite the apparent skewness of the distributions seen here, the situation is actually quite symmetric in the sense that negative ILF values live in the range $[-1, 0]$, while positive ILF values live in the range $[0, \infty)$. In some sense, the ILF should be judged from a *logarithmic* perspective.)

More tangibly, we can look at the difference in final ILF value *per trial*. We compute the difference as

$$\begin{bmatrix} \text{per trial} \\ \text{difference} \end{bmatrix} = \begin{bmatrix} \text{Final ILF} \\ \text{(internal fees)} \end{bmatrix} - \begin{bmatrix} \text{Final ILF} \\ \text{(external fees)} \end{bmatrix}$$

Note that the internal fee structure is preferable when this difference is *positive*.

For the same data set giving us Figure 5, we plot these per-trial-differences in Figure 6 below.

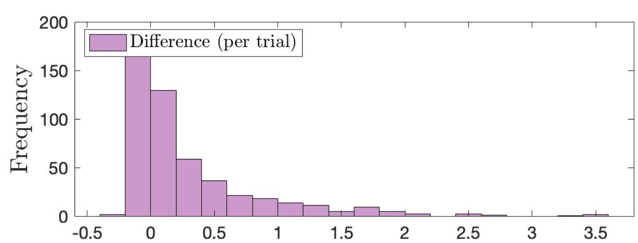

Figure 6. Difference, *per trial*, for the trials in Figure 4. Fee=0.3%, vol=100%, $[p_a, p_b] = [0.9, 1.1]$

We see from the figure that the distribution is split is pretty even between positive/negative values, but there does exist a slight advantage for using internal fees. Specifically, internal fees resulted in a better ILF value in 63% of the trials.

Next, we widen the LP range to $[p_a, p_b] = [0.5, 2.0]$. Again, we run 1,000 trials, and the aggregated results are shown in Figure 7.

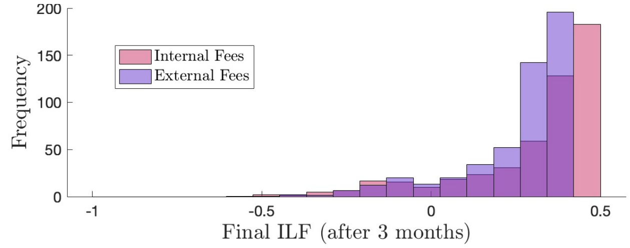

Figure 7. Distribution of final ILF for 1000 trials. Fee=0.3%, vol=100%, $[p_a, p_b] = [0.5, 2.0]$

This agrees with what we would expect in that the wider LP range will bring the distribution inward (less extreme ILF values). However, it is worth noting that in this case, using internal fees was preferable in 86% of the trials. Also, we note that the shape of the distribution has changed notably.

Returning to our initial LP position of $[0.9, 1.1]$, we next increase the volatility to 200%. The results of 1,000 trials are shown in Figure 8. As we'd expect, the ILF values are more extreme overall, but now the internal fees were only preferable in 46% of the trials (a minority!).

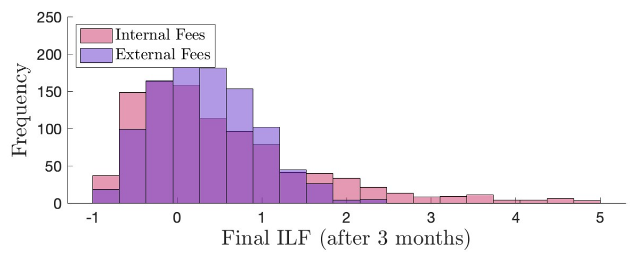

Figure 8. Distribution of final ILF for 1000 trials. Fee=0.3%, vol=200%, $[p_a, p_b] = [0.9, 1.1]$

Lastly, we raise the fee size to 1%. In this case, the ILF is able to reach comically high values in some trials, and overall, internal fees won out in 88% of the trials.

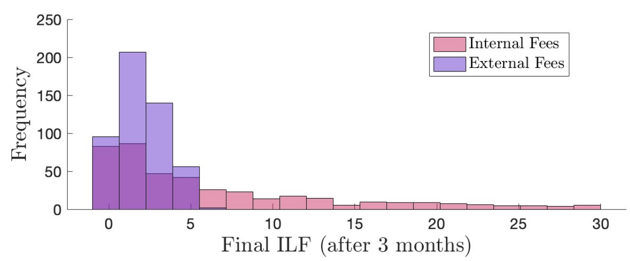

Figure 9. Distribution of final ILF for 1000 trials. Fee=1%, vol=100%, $[p_a, p_b] = [0.9, 1.1]$

Even if we bring the volatility back up to 200% and run 1,000 more trials, the internal fees are still preferable 73% of the time (compared to the 46% when the fee size was at 0.3%).

Based on these simulations, we can conjecture that using internal fees will be generally preferable to the LP whenever volatility is low, and/or the fee size is large.