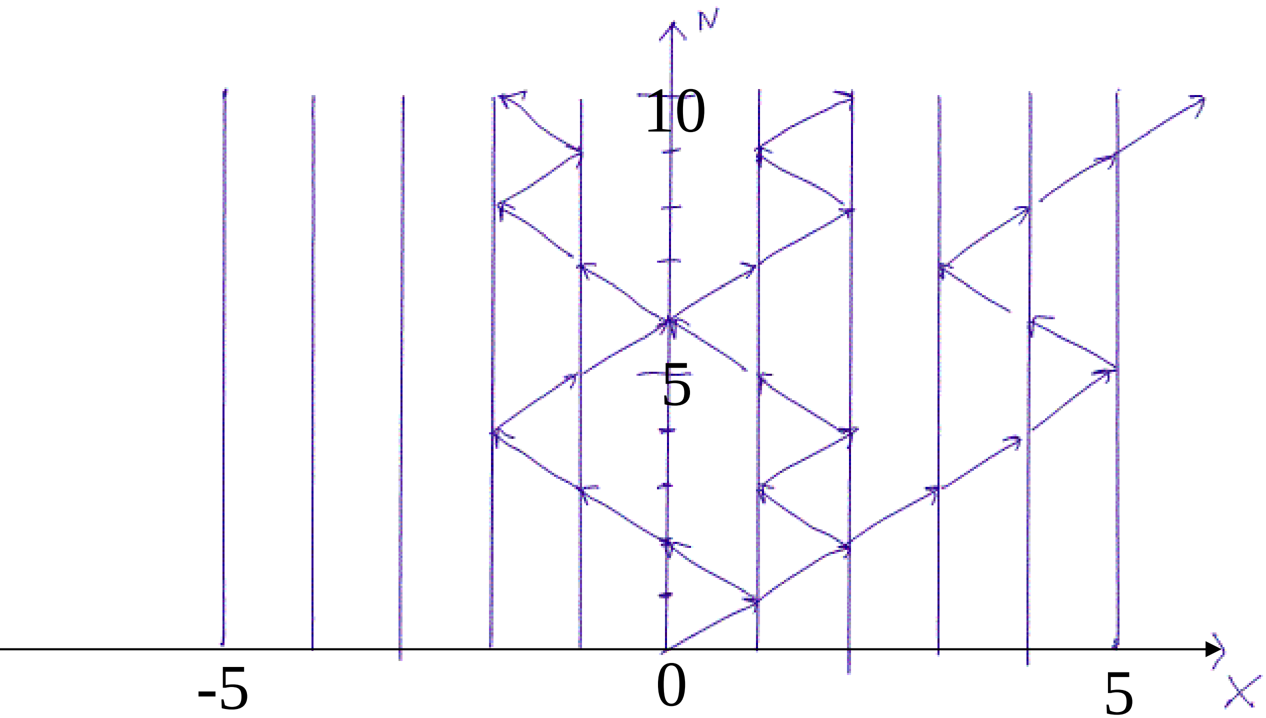
Random Walk in 1-D

Random walks appear in many contexts:
• diffusion is a random walk process
• understanding buffering, waiting times, queuing (more generally the theory of stochastic processes)
• gambling (choosing the best strategies for specific desired outcomes)

A random walk is basically looking at the time sequence in a binomial process. We will only consider the simplest random walks here …

Random Walk in 1-D

Consider an asymmetric random walk along the x-axis, beginning at the origin. There is a probability p that the step will be +1, and q=1-p that the step will be –1. We could additionally define the probability to stay put.



3 possible paths
x vs n

For our random walk, we assume the probabilities p,q do not depend on time (n) - stationary

Random Walk in 1-D

1-D Random Walk:

First question: what is the probability to be at x after n steps ?

Let n_+ represent the number of steps in the +x direction and n_- the number of steps in the -x direction. Then □

$$n_+ - n_- = x \qquad n_+ + n_- = n$$

so

$$n_+ = (n+x)/2 \qquad n_- = (n-x)/2$$

Note that this is only possible if n+x is even !

Random Walk in 1-D

The distribution for n_+ is a Binomial distribution:

$$P(n_+) = \binom{n}{n_+} p^{n_+} q^{n-n_+} = \frac{n!}{n_+!(n-n_+)!} p^{n_+} q^{n-n_+}$$

so

$$<n_+> = np \qquad <n_+^2> = npq + n^2 p^2 \qquad \sigma_{n_+}^2 = npq$$

Now for x

$$<x> = <2n_+ - n> = 2 <n_+> - n = n(2p-1)$$

$$<x^2> = <(2n_+ - n)^2> = 4 <n_+^2> - 4 <n_+> n + n^2 = 4npq + n^2(1-4pq)$$

$$\sigma_x^2 = 4npq$$

Random Walk

Example: symmetric random walk p=q=1/2. Then

$$\sigma_x^2 = \frac{npq}{4} = n$$

Package

We now look at the probability of returning to the origin. Use the symbol P(x,n) to represent the probability of being at x after n steps.

$$P(x,n) = \frac{n!}{((n+x)/2)!((n-x)/2)!} p^{(n+x)/2} q^{(n-x)/2}$$

Random Walk

Take n even and x=0.

Change variables to simplify the math somewhat:

$$2\mathrm{m} = \mathrm{n}$$

$$P(0{,}2m) = \frac{(2m)!}{m!m!} p^m q^m$$

For large n, use Stirling's approximation:

$$m! \approx \; m^{m+1/2} e^{-m} \sqrt{2\pi}$$

$$P(0{,}2m) \approx \frac{(2m)^{2m+1/2} e^{-2m}}{m^{2m+1} e^{-2m} \sqrt{2\pi}} p^m q^m = \frac{2^{2m} (pq)^m}{\sqrt{m\pi}} = \frac{(4pq)^m}{\sqrt{m\pi}}$$

Random Walk

$$P(0,2m) \approx \frac{(4pq)^m}{\sqrt{m\pi}}$$

Note that pq≤1/4, so that $P(0,2m) \to 0$ for $m \to \infty$. The probability to be at the origin goes to 0. However, the number of returns to the origin after N steps

$$R(N) = \sum_{\mathrm{m}=0}^{\mathrm{N}/2} \mathrm{P}(0,2\mathrm{m}) = \sum_{\mathrm{m}=0}^{\mathrm{N}/2} \frac{(2\mathrm{m})!}{\mathrm{m}!\mathrm{m}!} p^m q^m$$

Taking p=q=1/2,

$$R(N) = \sum_{\mathrm{m}=0}^{\mathrm{N}/2} \frac{(2\mathrm{m})!}{\mathrm{m}!\mathrm{m}!} (1/2)^{2m} = \frac{(N+1)!}{2^N (\frac{N}{2}!)^2} \xrightarrow[N \to \infty]{} \sqrt{\frac{2N}{\pi}}$$

Stirling's approx.

The state is said to be recurrent - there is probability one of eventually returning to the origin. Only true of p=q=1/2.

Random Walk in 1-D

Let *f(m)* be the probability that *x=m* is ***ever*** reached (*m>0*).

Then:

$$f(m+1) = f(m)\,f(1)$$

The probability that we get to x=m+1 (at some time) is just the probability that we got to x=m (at some earlier time) times the probability that we ever get to +1 from there.

This implies: $$f(m) = f(1)^m$$

What is *f(1)* ? Starting at the origin, we can either reach m=1 on the first step, or we have to each it starting from x=-1. I.e.,

$$f(1) = p + qf(2) = p + qf(1)^2$$

Random Walk in 1-D

We can solve for *f(1):*

$$f(1)=\frac{1\pm\sqrt{1-4pq}}{2q}=\frac{1\pm\sqrt{1-4p(1-p)}}{2(1-p)}=\frac{1\pm(1-2p)}{2(1-p)}$$

For $p \leq 1/2$, we take the negative sign:

$$f(1)=\frac{p}{1-p}=\frac{p}{q}$$

For $p>1/2$, we take the negative sign

$$f(1)=1$$

The chance to reach $x=m$ at some time is 1 for $p>1/2$, $m>0$, and $(p/q)^m$ for $p<1/2$, $m>0$.

Random walk in 1-D

Now let *g(m,n)* be the probability of reaching the point *x=+m* ***before*** *x=-n*. How do we find this quantity (very important if have absorbing boundaries). Start with what we know:

$$f(m) = g(m,n) + [1 - g(m,n)]\, f(m+n)$$

first reached x=-n first

Got there by first reaching x=m

Prob to eventually get to x=m

$$f(1)^m = g(m,n) + [1 - g(m,n)]\, f(1)^{m+n}$$

Solving for *g(m,n)* gives

This is for p<1/2.

$$g(m,n) = \left. \left(\frac{p}{q}\right)^m - \left(\frac{p}{q}\right)^{m+n} \middle/ 1 - \left(\frac{p}{q}\right)^{m+n} \right.$$

Random Walk in 1-D

Example: A physicist with some special powers is able to guess correctly the flip of a coin with 60% probability. He starts with two marbles, and plays a game of guessing the toss with someone who has an infinite number of marbles. Calculate the probability that the physicist will ultimately lose both marbles.

Our formula works for p<1/2, so we take (1-p) and calculate the probability to get to x=2 before reaching x=-∞

$$g(2,\infty) = \frac{f(1)^2 - f(1)^\infty}{1 - f(1)^\infty} = \frac{\left(\frac{0.4}{0.6}\right)^2}{1} = \frac{4}{9} \quad \text{since } f(1)^\infty = 0$$

I.e., better than 50% chance of winning all the marbles !

Random Walk in 1-D

Another example: suppose you have 100 Euros and play roulette. You bet on black every time, so have 18/38 chance of being correct on any given spin. What is the probability of making 100 Euros before losing 100 ? What is the best strategy ?

Suppose you make 1 Euro bets:

$$\mathrm{p} = 0.474,\ \mathrm{q} = 0.526,\ \mathrm{p/q} = 0.9,\ \ \mathrm{m} = 100,\ \mathrm{n} = 100$$

$$g(m,n) = \frac{(0.9)^{100} - (0.9)^{200}}{1-(0.9)^{200}} \quad = 2.66 \cdot 10^{-5}$$

A better strategy would be to make large bets ! Try 50 Euro bets.

$$g(m,n) = \frac{(0.9)^{2} - (0.9)^{4}}{1-(0.9)^{4}} \quad = 0.45$$

Random Walk

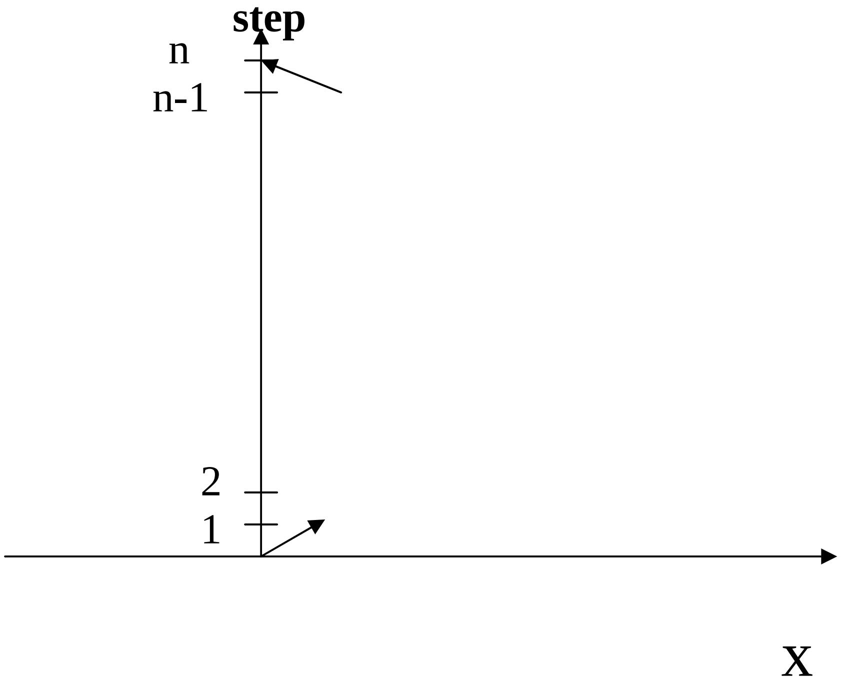
Fun fact: the symmetric random walk in 3 dimensions has a non-zero probability of never returning to the starting point.

Random Walk in 1-D

Now a somewhat trickier problem: What is the probability that the walk returns to the origin at the n^{th} step for the ***first*** time ?

Clearly, we need n to be even. Let us first look at the case where the first step is to the right

So now we have a random walk starting at x=1 and ending up at x=1 and there are n-2 steps.



The probability for ending at x after n steps is

$$P(x,n) = \frac{n!}{\left(n+x/2\right)!\left(n-x/2\right)!} p^{n+x/2} q^{n-x/2} = \frac{n!}{\left(n+x/2\right)!\left(n-x/2\right)!} \left(\frac{1}{2}\right)^n$$

We need to subtract all paths which reach *x=0 (in our shifted Coordinate system)*..

To count these, we use a symmetry argument. Note that once we are at *x=0*, then we have equal probability to end up at *x=-1* and *x=+1*. Therefore, the probability to hit *x=0* and end up at *x=+1* is the same as the probability to end up at *x=-1* starting from *x=+1*. I.e

$$P(0,n-2)\,|_{\text{reach x=-1}} = P(-2,n-2)$$

The probability not to reach x=0 in our n-2 steps is therefore

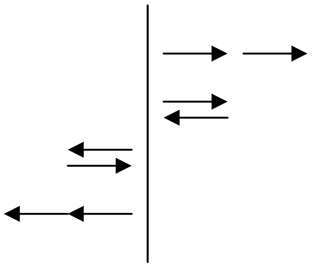
$$P(0,n-2) - P(-2,n-2)$$

We now put the pieces together. We have two possibilities: start left or start right:

$$P_F(0,n) = P(0,n)\,|_{\text{first time to return}} = 2\cdot(1/2)^2\cdot\left[P(0,n-2) - P(-2,n-2)\right]$$

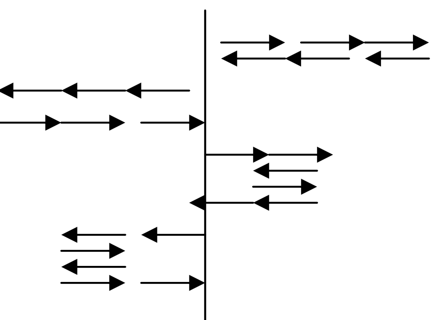
Let's try it for a few cases

$$n = 2 \qquad P_F(0,n) = 2\cdot(1/4)\cdot\left[1-0\right] = 1/2$$



By construction, we see ½ the paths return to zero for the first time at n=2

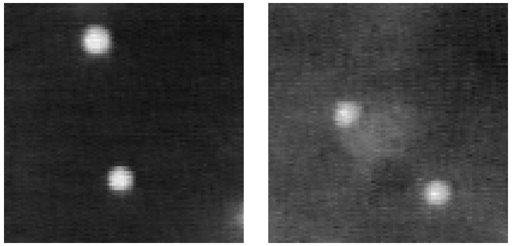
$$n = 6 \qquad P_F(0,6) = 2\cdot(1/4)\cdot\left[3/8-1/4\right] = 1/16$$



By construction, we see 4 paths return to zero for the first time at n=6, from a total of $(1/2)^6$ paths, or 4/64=1/16.

Brownian Motion

We consider a physics example of a random walk – Brownian motion. Discovered in 1827 by the English botanist Brown, who observed that small particles immersed in a liquid exhibit irregular motion. Mathematical description from the laws of physics by Einstein in 1905, who started with the assumption that the motion was caused by repeated collisions of the molecules with the medium. Subject of intense interest since.



http://www.deas.harvard.edu/projects/weitzlab/research/brownian.html

Brownian Motion

Formulation: Let *X(t)* denote the displacement from the starting point projected into a fixed axis at time *t*. The displacement $X(t_2)-X(t_1)$ over the time interval t_2-t_1 can be views as a large number of small displacements. We postulate that the distribution of $X(t_2)-X(t_1)$ does not depend on the values of *t*, but only on the interval t_2-t_1. We can then apply the central limit theorem and expect that $X(t_2)-X(t_1)$ follows a Gaussian distribution.

Let x_0 be the *x* component at time t_0, I.e., $X(t_0)=x_0$. The probability of finding the particle at a distance *x* at a later time $t+t_0$ is $p(x|t,x_0)$. The normalization condition is satisfied:

$$\int_{-\infty}^{\infty} p(x \mid t, x_0)\, dx = 1$$

And we assume that $\lim_{t\to 0} p(x \mid t, x_0) = 0 \qquad x \neq x_0$

Brownian Motion-cont.

Einstein showed that:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}$$

The conditional probability for the position of the particle follows a diffusion equation, with diffusion coefficient D given by

$$D = \frac{2RT}{N_0 f}$$

Where R is the gas constant, T the temperature, N_0 Avogadro's number and f the coefficient of friction. By judicious choice of units, we can set D=1/2. The solution to the diffusion equation is then

$$p(x \mid t, x_0) = \frac{1}{\sqrt{2\pi t}} e^{-\left(\frac{(x-x_0)^2}{2t}\right)}$$

Brownian Motion-cont.

Check Gaussian is a solution to the diffusion equation:

$$\frac{\partial p}{\partial t} = -\frac{1}{2}\left(\frac{1}{2\pi t}\right)^{-3/2} 2\pi\, e^{-(x-x_0)^2/2t} + \frac{1}{\sqrt{2\pi t}}\left(\frac{(x-x_0)^2}{2t^2}\right) e^{-(x-x_0)^2/2t}$$

$$\frac{\partial p}{\partial x} = \frac{1}{\sqrt{2\pi t}}\left(-\frac{2(x-x_0)}{2t} e^{-(x-x_0)^2/2t}\right)$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\sqrt{2\pi t}}\left(-\frac{1}{t} e^{-(x-x_0)^2/2t} + \frac{(x-x_0)^2}{t^2} e^{-(x-x_0)^2/2t}\right)$$

Factoring out $\dfrac{1}{\sqrt{2\pi t}} e^{-(x-x_0)^2/2t}$, we get

$$\frac{\partial p}{\partial t} = \frac{1}{\sqrt{2\pi t}} e^{-(x-x_0)^2/2t}\left[-\frac{1}{2t} + \frac{(x-x_0)^2}{2t^2}\right]$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\sqrt{2\pi t}} e^{-(x-x_0)^2/2t}\left[-\frac{1}{t} + \frac{(x-x_0)^2}{t^2}\right]$$

so

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}$$

Brownian Motion–cont.

Another approach – start from symmetric random walk:

To get to position x at step n+1, we have to be at either x-1 or x+1 at step n:

$$P(x,n+1)=\frac{1}{2}P(x-1,n)+\frac{1}{2}P(x+1,n)$$

Now rewrite by subtracting *P(x,n)* from each side

$$P(x,n+1)-P(x,n)=\frac{1}{2}[P(x+1,n)-2P(x,n)+P(x-1,n)]$$

Notice that this looks like a first derivative in time on the LHS of the equation - remember that n is a time variable - and a 2nd derivative of space of the RHS (n fixed). So, we recover the diffusion equation from the random walk. Need physics input to get the units.