6.434/16.391 Statistics for Engineers and Scientists

Lecture 14 11/07/2012

Laboratory for Information and Decision Systems
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FURTHER COMMENTS ON RAO-BLACKWELL THEOREM

Necessity of sufficiency

- In our proof of Rao-blackwell theorem, we did not use that $T(\mathbf{X})$ is sufficient statistic
- It may first seem that conditioning on anything will result in improvement
- The next example shows that $T(\mathbf{X})$ needs to be sufficient statistic

Example 7.3.18

- Conditioning on an insufficient statistic
- Let X_1 and X_2 be iid $\mathcal{N}(\mu, 1)$. The statistic $\bar{X} = \frac{1}{2}(X_1 + X_2)$ has

$$\mathbb{E}_\mu\{\bar{X}\} = \mu \text{ and } \mathbb{V}_\theta\{\bar{X}\} = \frac{1}{2}$$

Consider conditioning on X_1, which is not sufficient. Let

$$\phi(X_1) = \mathbb{E}_\theta\{\bar{X}|X_1\}$$

It can be verified that $\mathbb{E}_\theta\{\,\phi(X_1)\} = \mu$ and

$$\mathbb{V}_\theta\{\phi(X_1)\} \leq \mathbb{V}_\theta\,\{\bar{X}\}$$

so $\phi(X_1)$ is better than $\bar{X}$

Example 7.3.18

- However,

$$\begin{aligned}\phi(X_1) &= \mathbb{E}_\theta\left\{\bar{X}|X_1\right\} \\ &= \frac{1}{2}\mathbb{E}_\theta\left\{X_1|X_1\right\} + \frac{1}{2}\mathbb{E}_\theta\left\{X_2|X_1\right\} \\ &= \frac{1}{2}X_1 + \frac{1}{2}\theta\end{aligned}$$

  which is not an estimator

- We now know that, in looking for a best unbiased estimator, we need consider only estimators based on a sufficient statistic

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LEHMANN SCHEFFÉ THEOREM

Lehmann Scheffé Theorem

- Theorem: If $T(\mathbf{X})$ is complete sufficient statistic and $S(\mathbf{X})$ is an unbiased estimator for $g(\theta)$, then

$$T^*(\mathbf{X}) = \mathbb{E}\{S(\mathbf{X})|T(\mathbf{X})\}$$

is a uniformly minimum variance unbiased (UMVU) estimate for $g(\theta)$. If $\mathbb{V}\{T^*(\mathbf{X})\} < \infty$, $\forall\theta$, then $T^*(\mathbf{X})$ is <u>the</u> UMVU estimator for $g(\theta)$

Proof (1 of 5)

- 1) Unbiased: We have

$$\begin{aligned}\mathbb{E}\left\{T^*(\mathbb{X})\right\} &= \mathbb{E}\left\{\mathbb{E}\left\{S(\mathbf{X})|T(\mathbf{X})\right\}\right\} \\ &= \mathbb{E}\left\{S(\mathbf{X})\right\} \\ &= g(\theta)\end{aligned}$$

- 2) From Rao-Blackwell Theorem,

$$\mathbb{V}\left\{T^*(\mathbf{X})\right\} \leq \mathbb{V}\left\{S(\mathbf{X})\right\}, \quad \forall\theta$$

Proof (2 of 5)

- 3) Minimum variance: Suppose U is another unbiased estimate for $g(\theta)$. Let

  $$\widetilde{T}(\mathbf{X}) = \mathbb{E}\{U(\mathbf{X})|T(\mathbf{X})\}$$

  then

  $$\mathbb{V}\left\{\widetilde{T}(\mathbf{X})\right\} \leq \mathbb{V}\left\{U(\mathbf{X})\right\}$$

  Also

  $$T^*(\mathbf{X}) = g_1\left(T(\mathbf{X})\right)$$
  $$\widetilde{T}(\mathbf{X}) = g_2\left(T(\mathbf{X})\right)$$

  i.e., they are both functions of the complete sufficient statistic. Since T^* and $\widetilde{T}$ are unbiased,

  $$\mathbb{E}\left\{g_1\left(T\right) - g_2\left(T\right)\right\} = g(\theta) - g(\theta) = 0$$

Proof (3 of 5)

- Further, by completeness,

$$g_1(t) - g_2(t) = 0$$

  for all possible outcomes t of $T(\mathbf{X})$. Hence,

$$\widetilde{T}(\mathbf{X}) = T^*(\mathbf{X})$$

  and $T^*(\mathbf{X})$ does not depend on $S(\mathbf{X})$

- Note: When $\mathbb{V}\left\{T^*(\mathbf{X})\right\}$ is finite then one can achieve strict inequality unless $T^*(\mathbf{X}) = S(\mathbf{X})$. That is why <u>the</u> comes from

Proof (4 of 5)

- 4) Uniqueness:
  Suppose $T^*(\mathbf{X})$ is not unique $\Rightarrow\ \exists$ UMVUE $V(\mathbf{X}) \neq T^*(\mathbf{X})$
  Let $\breve{T}(\mathbf{X}) = \mathbb{E}\{V(\mathbf{X})|T(\mathbf{X})\}$.
  By Rao-Blackwell Theorem,

$$\mathbb{V}\{\breve{T}(\mathbf{X})\} \leq \mathbb{V}\{V(\mathbf{X})\} \qquad (A)$$

  and the inequality is strict unless $\breve{T}(\mathbf{X}) = V(\mathbf{X})$
  Since $V(\mathbf{X})$ is a UMVUE,

$$\mathbb{V}\{V(\mathbf{X})\} \leq \mathbb{V}\{\breve{T}(\mathbf{X})\} \qquad (B)$$

  (A) and (B) gives $\mathbb{V}\{V(\mathbf{X})\} = \mathbb{V}\{\breve{T}(\mathbf{X})\}$. Thus

$$V(\mathbf{X}) = \breve{T}(\mathbf{X}) \qquad (C)$$

Proof (5 of 5)

From 3) we know that

$$\breve{T}(\mathbf{X}) = T^*(\mathbf{X}) \qquad (D)$$

(C) and (D) gives $V(\mathbf{X}) = T^*(\mathbf{X})$. Contradiction!
Thus $T^*(\mathbf{X})$ is unique.

Recipe to find UMVUE

- Suppose that we have a complete sufficient statistic $T(\mathbf{X})$

- If we can find $h\left(T(\mathbf{X})\right)$ such that it is an unbiased estimate of $g(\theta)$, then $h\left(T(\mathbf{X})\right)$ is a UMVUE. This follows since

$$h\left(T(\mathbf{X})\right) = \mathbb{E}\left\{h\left(T(\mathbf{X})\right)|T(\mathbf{X})\right\}$$

- If we can find <u>any</u> unbiased estimate $S(\mathbf{X})$ of $g(\theta)$, then

$$\mathbb{E}\left\{S(\mathbf{X})|T(\mathbf{X})\right\}$$

  is UMVME

Best unbiased estimator is unique

- If W is best unbiased estimator for $g(\theta)$, then W is unique
- Proof: Suppose that W' is another best unbiased estimator, i.e., $\mathbb{V}\{W\} = \mathbb{V}\{W'\}$. Let

$$W^* = \frac{1}{2}(W + W')$$

which is unbiased. The variance of W^* is

$$\begin{aligned}
\mathbb{V}\{W^*\} &= \frac{1}{4}\mathbb{V}\{W\} + \frac{1}{4}\mathbb{V}\{W'\} + \frac{1}{2}\mathbb{C}\{W, W'\} \\
&\leq \frac{1}{4}\mathbb{V}\{W\} + \frac{1}{4}\mathbb{V}\{W'\} + \frac{1}{2}\sqrt{\mathbb{V}\{W\}\,\mathbb{V}\{W'\}} \\
&= \frac{1}{4}\left(\sqrt{\mathbb{V}\{W\}} + \sqrt{\mathbb{V}\{W'\}}\right)^2 \\
&= \mathbb{V}\{W\}
\end{aligned}$$

where the inequality is due to Cauchy-Schwarz inequality

Best unbiased estimator is unique

- Since W is best, equality must hold, which holds for Cauchy-Schwarz iff $W = a(\theta)W' + b(\theta)$. Since $\mathbb{V}\{W\} = \mathbb{V}\{W'\}$, $a(\theta) = 1$. Further, since W and W' are unbiased, $b(\theta) = 0$. Thus,

$$W = W'$$

Best unbiased estimator

- Theorem 7.3.20: If $\mathbb{E}_\theta\{W\} = \tau(\theta)$, W is the best unbiased estimator of $\tau(\theta)$ iff W is uncorrelated with all unbiased estimators of θ
- Proof: Let $\phi_a = W + aU$, where $\mathbb{E}_\theta\{U\} = 0$, $\forall\theta$, i.e., U is an unbiased estimator of θ. The estimator ϕ_a satisfies $\mathbb{E}_\theta\{\phi_a\} = \tau(\theta)$ and hence is also an unbiased estimator of $\tau(\theta)$. Further,

  $$\mathbb{V}_\theta\{\phi_a\} = \mathbb{V}_\theta\{W + aU\} = \mathbb{V}\{W\} + 2a\mathbb{C}_\theta\{W, U\} + a^2\mathbb{V}_\theta\{U\}$$

  It can be verified that, if $\mathbb{C}_\theta(W, U) \neq 0$, appropriate choice of a will give $\mathbb{V}_\theta\{\phi_a\} < \mathbb{V}_\theta\{W\}$.
- Therefore, $\mathbb{C}_\theta\{W, U\} = 0$