6.434/16.391 Statistics for Engineers and Scientists

Lecture 13 11/05/2012

Laboratory for Information and Decision Systems
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RAO-BLACKWELL THEOREM

Rao-Blackwell Theorem

- Theorem: Let $S(\mathbf{X})$ be an estimator of $g(\theta)$. Suppose $T(\mathbf{X})$ is sufficient statistic for θ, and $\mathbb{E}\{|S(\mathbf{X})|\} < \infty$, for all $\theta \in \Theta$. Let

  $$T^*(\mathbf{X}) = \mathbb{E}\{S(\mathbf{X})|T(\mathbf{X})\}$$

  Then $\forall\theta \in \Theta$, $T^*(\mathbf{X})$ is also an estimator for $g(\theta)$ with

  $$\mathbb{E}\left\{\left[T^*(\mathbf{X}) - g(\theta)\right]^2\right\} \leq \mathbb{E}\left\{\left[S(\mathbf{X}) - g(\theta)\right]^2\right\}$$

  If $\mathbb{V}\{S\} < \infty$, $\forall\theta \in \Theta$, then the inequality is strict, unless

  $$T^*(\mathbf{X}) = S(\mathbf{X})$$

Remark

- Remark:
$$\begin{aligned}\mathbb{E}\{T^*(\mathbf{X})\} &= \mathbb{E}\{\mathbb{E}\{S(\mathbf{X})|T(\mathbf{X})\}\} \\ &= \mathbb{E}\{S(\mathbf{X})\}\end{aligned}$$

  If $S(\mathbf{X})$ is unbiased, then $T^*(\mathbf{X})$ is also unbiased, and the conclusion becomes
$$\mathbb{V}\{T^*(\mathbf{X})\} \leq \mathbb{V}\{S(\mathbf{X})\}, \ \forall\theta \in \Theta$$

- Interpretation: Suppose $S(\mathbf{X})$ is an estimator for θ and we have a sufficient statistic $T(\mathbf{X})$ for θ. We can always construct a uniformly better (no worse) estimator by conditioning on $T(\mathbf{X})$

- Thus, in our search for best estimator, we only need to consider statistic that are functions of a sufficient statistic.

Double expectation formula

- Before the proof of Rao-Blackwell Theorem, we provide the following three facts
  - Double expectation formula
  - Conditioning reduces variance
  - Mean squared error property

- Consider two random variables X and Y. We have

$$\mathbb{E}\{\mathbb{E}\{X|Y\}\} = \mathbb{E}\{X\}$$

Conditioning reduces variance

- Consider two random variables S and T. We have

$$\mathbb{V}\{\mathbb{E}\{S|T\}\} = \mathbb{V}\{S\} - \mathbb{E}\{\mathbb{V}\{S|T\}\}$$

- Proof:
- Step 1. Conditional variance:

$$\mathbb{V}\{S|T\} = \mathbb{E}\{S^2|T\} - \mathbb{E}^2\{S|T\}$$

Then we have

$$\mathbb{E}\{\mathbb{V}\{S|T\}\} = \underbrace{\mathbb{E}\{\mathbb{E}\{S^2|T\}\}}_{=\mathbb{E}\{S^2\}} - \mathbb{E}\{\mathbb{E}^2\{S|T\}\} \qquad (1)$$

Conditioning reduces variance

- Step 2. Variance of $\mathbb{E}\{S|T\}$

$$\mathbb{V}\{\underbrace{\mathbb{E}\{S|T\}}_{Z}\} = \mathbb{E}\{\underbrace{\mathbb{E}^2\{S|T\}}_{Z^2}\} - \mathbb{E}^2\{\underbrace{\mathbb{E}\{S|T\}}_{Z}\}$$

$$= \mathbb{E}\{\mathbb{E}^2\{S|T\}\} - \mathbb{E}^2\{S\} \qquad (2)$$

Adding (1) and (2), and move $\mathbb{E}\{\mathbb{V}\{S|T\}\}$ to the right hand side, we obtain

$$\mathbb{V}\{\mathbb{E}\{S|T\}\} = \mathbb{V}\{S\} - \mathbb{E}\{\mathbb{V}\{S|T\}\}$$

Mean squared error

- The expression of mean squared error is given by

$$\mathbb{E}\left\{\left[T(\mathbf{X}) - g(\theta)\right]^2\right\} = \mathbb{V}\left\{T(\mathbf{X})\right\} + \mathbb{B}^2\left(\theta,\, T(\mathbf{X})\right)$$

where $\mathbb{B}\left(\theta,\, T(\mathbf{X})\right) = \mathbb{E}\left\{T(\mathbf{X})\right\} - g(\theta)$

- Proof:

$$\begin{aligned}\text{L.H.S} &= \mathbb{E}\left\{\left[T(\mathbf{X}) - \mathbb{E}\left\{T(\mathbf{X})\right\} + \mathbb{E}\left\{T(\mathbf{X})\right\} - g(\theta)\right]^2\right\} \\ &= \mathbb{V}\left\{T(\mathbf{X})\right\} + \left[\mathbb{E}\left\{T(\mathbf{X})\right\} - g(\theta)\right]^2\end{aligned}$$

where the second equality is due to

$$\mathbb{E}\left\{T(\mathbf{X}) - \mathbb{E}\left\{T(\mathbf{X})\right\}\right\} = 0$$

Proof of Rao-Blackwell Theorem (1 of 2)

- Proof: First, according to the relationship between MSE and bias (lecture note 11, slide 11), we have

$$\mathbb{E}\left\{[\,T^*(\mathbf{X}) - g(\theta)\,]^2\right\} = \mathbb{V}\left\{T^*(\mathbf{X})\right\} + \mathbb{B}^2\left\{\theta,\, T^*(\mathbf{X})\right\}$$
$$\mathbb{E}\left\{[\,S(\mathbf{X}) - g(\theta)\,]^2\right\} = \mathbb{V}\left\{S(\mathbf{X})\right\} + \mathbb{B}^2\left\{\theta,\, S(\mathbf{X})\right\}$$

where

$$\mathbb{B}\left\{\theta,\, T^*(\mathbf{X})\right\} = \mathbb{E}\left\{T^*(\mathbf{X})\right\} - g(\theta)$$
$$\mathbb{B}\left\{\theta,\, S(\mathbf{X})\right\} = \mathbb{E}\left\{S(\mathbf{X})\right\} - g(\theta)$$

Recall that

$$\begin{aligned}\mathbb{E}\{T^*(\mathbf{X})\} &= \mathbb{E}\{\mathbb{E}\left\{S(\mathbf{X})|T(\mathbf{X})\right\}\} \\ &= \mathbb{E}\left\{S(\mathbf{X})\right\}\end{aligned}$$

Thus

$$\mathbb{B}\left\{\theta,\, T^*(\mathbf{X})\right\} = \mathbb{B}\left\{\theta,\, S(\mathbf{X})\right\}$$

Proof of Rao-Blackwell Theorem (2 of 2)

- Since conditioning reduces variance, we have

$$\mathbb{V}\left\{T^*(\mathbf{X})\right\} \leq \mathbb{V}\left\{S(\mathbf{X})\right\}$$

- Therefore,

$$\mathbb{E}\left\{\left[T^*(\mathbf{X}) - g(\theta)\right]^2\right\} \leq \mathbb{E}\left\{\left[S(\mathbf{X}) - g(\theta)\right]^2\right\}$$

with equality only if $\mathbb{E}\{\mathbb{V}\{S|T\}\} = 0$. Thus, $\mathbb{V}\{S|T\} = 0$, i.e.,

$$\mathbb{E}\left\{\left[S - \mathbb{E}\{S|T\}\right]^2 \middle| T\right\} = 0$$

which implies

$$S = \mathbb{E}\{S|T\} = T^*$$