Faculty of Computer Science

School of Data Analysis and Artificial Intelligence

HOMEWORK REPORT

Practical homework №2

Subject: *Ordered Sets for Data Analysis*

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Moscow, 2024

Task. Given the following formal context, find all formal concepts, draw the concept lattice, and find all non-trivial implications.

	a	b	c	d	e
1	1		1		1
2	1		1	1	
3	1			1	1
4	1	1		1	
5	1	1			1

Solution.

1. *Reflexivity.* R is reflexive, since $R : \forall x \in A \ \ xRx$.

2. *Antireflexity.* R is not antireflexive, since $\exists x \in A : \neg x R^c x$. For example, $x = a$.

3. *Symmetricity.* R is not symmetric, since $\exists x, y \in A : xRy \implies \neg yRx$. For example, $x = a$, $y = b$.

4. *Asymmetricity.* R is not asymmetric, since $\exists x, y \in A : xRy \implies \neg yR^c x$. For example, $x = a = y$.

5. *Antisymmetricity.* R is not antisymmetric, since $aRb \,\&\, bRa \implies \neg a = b$.

6. *Transitivity.* R is not transitive, since $\exists x, y, z \in A : xRy \,\&\, yRz \implies \neg xRz$. For example, $x = a$, $y = c$, $z = b$.

7. *Linearity.* R is not linear, since $b \neq d \implies \neg aRb \vee bRa$.

QUESTION 2

Task. For the partial order given in the following table:

	1	2	3	4	5	6
1	1		1		1	
2		1	1	1	1	1
3			1		1	
4				1		1
5					1	
6						1

1. Draw its directed graph and its Hasse diagram.

2. Determine the dimension of the partial order.

3. Provide 3 different topological sortings for the graph of the partial order.

Solution. The partial order's graphical representation can be seen on fig. 1. Its Hasse diagram is shown on fig. 2. According to fig. 2, the dimension of the partial order is two. Fig. 3 shows three different topological sorting options: {2, 4, 6, 1, 3, 5}, {2, 1, 3, 4, 5, 6}, and {2, 1, 3, 4, 5, 6}.
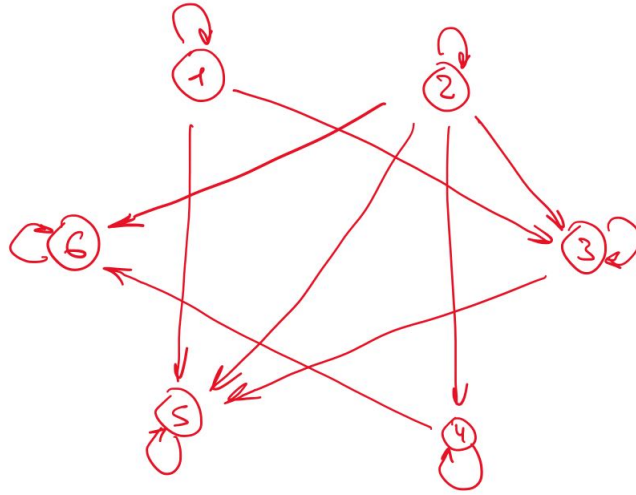
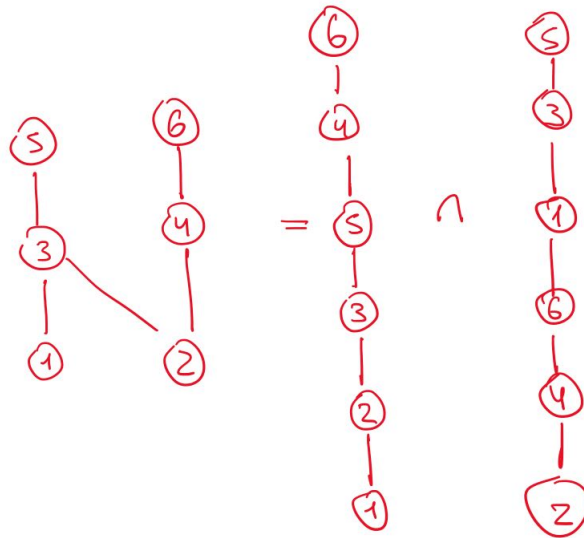**Figure 1.** Directed graph representing the partial order

Figure 2. Hasse diagram of the partial order and it's decomposition into an intersection of linear orders
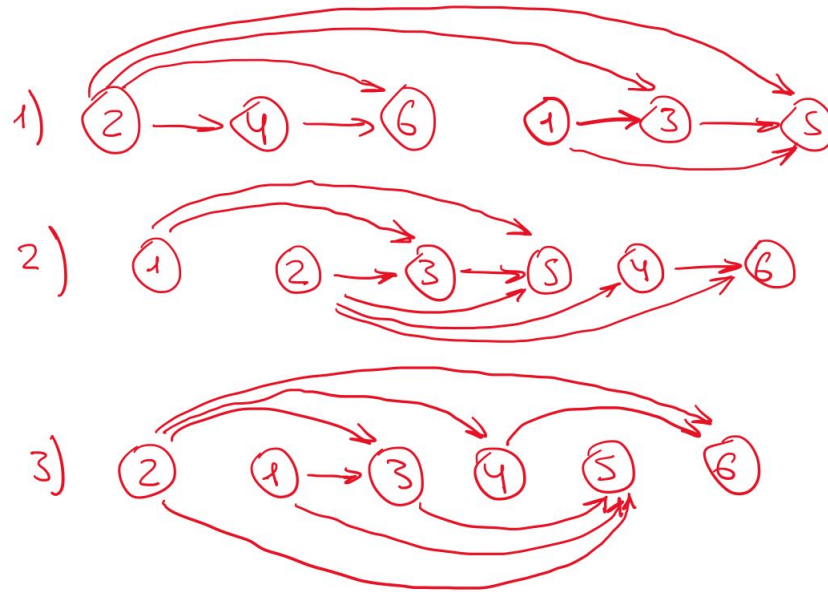
Figure 3. Three different topological sorts for the graph on fig. 1

QUESTION 3

Task. Prove that in any arbitrary undirected graph, the number of vertices with odd degree is even.

◀ Let $G = (V, E)$ be an undirected graph, where V is the set of its vertexes and E — the set of its edges. It is a well-known fact that

$$\sum_{v \in V} \deg(v) = 2|E|, \tag{1}$$

where $\deg(v)$ is degree of a vertex $v \in V$ and $|E|$ is the number of edges in G. Let us suppose that the number of vertices with odd degree is odd. Then $\sum_{v \in V} \deg(v)$ must also be an odd number. This contradicts (1), hence the number of vertices with odd degree must be even. ▶

QUESTION 4

Theorem 1. Let $(P, \leq_p)$ and $(Q, \leq_q)$ be finite posets with covering relations $\prec_p$ and $\prec_q$, and let $\varphi : P \to Q$ be a bijection. Then the following two statements are equivalent:

1. Bijection φ is an order isomorphism, i.e., $x \leq_p y$ iff $\varphi(x) \leq_q \varphi(y)$.

2. $x \prec_p y$ iff $\varphi(x) \prec_q \varphi(y)$.

Task. Prove the set of all positive integer divisors of the number 30 with the relation *"to be a divisor"* as an order relation is isomorphic to the set of all subsets $\{x, y, z\}$ ordered by inclusion.

◀ Let $S = \{x, y, z\}$ and

$$\mathbb{P}(S) = \{\{\varnothing\}, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$$

be the powerset(set of all subsets) over it. The set of all positive integer divisors of the number 30 is

$$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}.$$

Partial order on poset $(D_{30}, |)$ is defined by a following table:

	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2		1			1	1		1
3			1		1		1	1
5				1		1	1	1
6					1			1
10						1		1
15							1	1
30								1

6

Let us consider the table that defines partial order on poset $(\mathbb{P}(S), \subseteq)$ as well:

	$\{\varnothing\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$	$\{x, y, z\}$
$\{\varnothing\}$	1	1	1	1	1	1	1	1
$\{x\}$		1			1	1		1
$\{y\}$			1		1		1	1
$\{z\}$				1		1	1	1
$\{x, y\}$					1			1
$\{x, z\}$						1		1
$\{y, z\}$							1	1
$\{x, y, z\}$								1

From those tables Hasse diagrams(fig. 4) can be constructed.

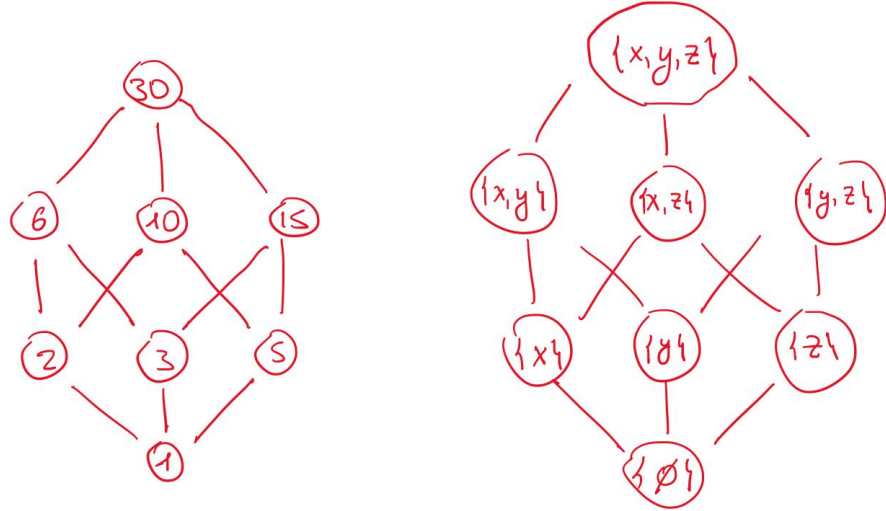**Figure 4.** Hasse diagrams of $(D_{30}, |)$ (left) and $(\mathbb{P}(S), \subseteq)$ (right)

If we define a relation $\psi : D_{30} \to \mathbb{P}(S)$ so that

$$\psi(1) = \{\varnothing\}, \psi(2) = \{x\}, \psi(3) = \{y\}, \psi(5) = \{z\}, \psi(6) = \{x, y\},$$
$$\psi(10) = \{x, z\}, \psi(15) = \{y, z\}, \psi(30) = \{x, y, z\},$$

then $(D_{30}, |)$ and $(\mathbb{P}(S), \subseteq)$ along with ψ satisfy the conditions for theorem 1, thus they are isomorphic. ▶

QUESTION 5

Task. For the binary relation $(\{1, 2, 3, 4, 6, 24, 36, 72\}, |)$:

1. Prove that it is a partial order.

2. Draw its diagram.

3. Using the diagram, find the following:

   (a) Maximal elements.

   (b) Largest element (Maximum).

   (c) Minimal elements.

   (d) Least element (Minimum).

   (e) $\downarrow \{4, 6\}$

   (f) $\uparrow \{4, 6\}$

   (g) $\downarrow \{3, 4\}$

   (h) $\uparrow \{3, 4\}$

   (i) $\downarrow \{36, 24\}$

   (j) $\uparrow \{36, 24\}$

Solution. Let us denote $\{1, 2, 3, 4, 6, 24, 36, 72\}$ as A. Consider the relation table for $(A, |)$.

	1	2	3	4	6	24	36	72
1	1	1	1	1	1	1	1	1
2		1		1	1	1	1	1
3			1		1	1	1	1
4				1		1	1	1
6					1	1	1	1
24						1		1
36							1	1
72								1

For a binary relation to be a partial order, it must be reflexive, antisymmetric and transitive. Since the table consists of upper triangular matrix, $\forall x \in A \; x|x$ and $\forall x, y \in A \; x|y \implies \neg y|x$, therefore the relation is reflexive and antisymmetric. Let us consider matrix

$$M = (m_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where zeros correspond to empty cells in the relation table. Its matrix product with itself is

$$V = (v_{ij}) = M^2 = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 6 & 6 & 8 \\ 0 & 1 & 0 & 2 & 2 & 4 & 4 & 6 \\ 0 & 0 & 1 & 0 & 2 & 3 & 3 & 5 \\ 0 & 0 & 0 & 1 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here, according to the definition of matrix product, $v_{ij} = \sum_{k=1}^{n} m_{ik}m_{kj}$, where n is size of the matrix M. Therefore, $v_{ij} > 0$ iff there is a two-step path between i and j and the value of v_{ij} is the number of two-step paths between i and j.

The relation is transitive, if there no pair of incomparable elements with a two-way path between them:

$$\nexists i, j: \; m_{ij} = 0 \land v_{ij} > 0.$$

Since there are no elements that do not satisfy this condition, the relation $(A, |)$ is also transitive, therefore it is a partial order.

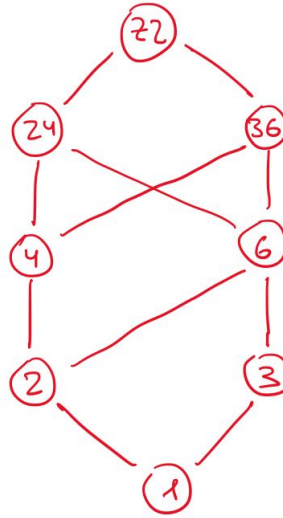Let us consider its Hasse diagram:

Figure 5. Hasse diagram of the partial order $(A, |)$

using it, we can determine that:

1. The set of maximal elements consists only of 72.

2. The maximum is 72.

3. The set of minimal elements consists only of 1.

4. The minimum is 1.

5. $\downarrow \{4, 6\} = \{4, 6, 2, 3, 1\}$

6. $\uparrow \{4, 6\} = \{4, 6, 24, 36, 72\}$

7. $\downarrow \{3, 4\} = \{3, 4, 2, 1\}$

8. $\uparrow \{3, 4\} = \{3, 4, 6, 24, 36, 72\}$

9. $\downarrow \{36, 24\} = \{36, 24, 4, 6, 2, 3, 1\}$

10. $\uparrow \{36, 24\} = \{36, 24, 72\}$

QUESTION 6

Task. Prove that the incomparability relation for a strict order is a tolerance relation.

◄ Let incomparability relation be denoted as $\not\sim$ and the strict order to be defined on some set S. A strict order is a antireflexive relation, thus

$$\forall x \in S \ \neg x < x \implies x \not\sim x.$$

so incomparability relation is reflexive. Suppose that incomparability relation is not symmetrical. Then there would be such pair $x, y \in S$, $x \neq y^{1}$ that $x \not\sim y$ and $\neg y \not\sim x$. Since elements x and y are called incomparable if

$$x \neq y \implies \neg(x < y \vee y > x),$$

this would imply that both $\neg(x < y \vee y > x)$ and $y < x \vee x > y$, which a contradictory statement. Therefore, incomparability relation is symmetrical, hence it is a tolerance relation. ▶

QUESTION 7

Q. Which of the three options have you picked for the big project?
A. Neural FCA.

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[1]   We are not considering the case when x and y are the same element, since it was already shown that incomparability relation is reflexive.