Ministry of Science and Higher Education of the Russian Federation

National Research University Higher School of Economics

Faculty of Computer Science

School of Data Analysis and Artificial Intelligence

HOMEWORK REPORT

GAUSSIAN PROCESSES

Subject: *Machine Learning*

Student *M.D. Kirdin*

Teaching Assistant *M.A. Shirokikh*

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Task 19.

Suppose that $\boldsymbol{x} = (x_0,\, x_1)^T \sim N(\mu,\, \Sigma)$. Find the distribution of $(a,\, b)^T$ of line $y = ax + b$ passing through points $(0,\, x_0)$ and $(1,\, x_1)$. Derive the distribution and its parameters, implement it and check if it works.

Solution.

Let us consider a line $y = f(x)$ going from $(0,\, x_0)$ to $(1,\, x_1)$. It is defined by a following expression:

$$y = \frac{x_1 - x_0}{1 - 0} x + x_0.$$

Therefore, coefficients a and b are equal to $x_1 - x_0$ and x_0 respectively. Note that vector $\boldsymbol{k} = (a,\, b)^T$ is, in fact, a linear transformation of $\boldsymbol{x}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 - x_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = A\boldsymbol{x}.$$

Hence, we can easily deduce the distribution of $\boldsymbol{k}$ using characteristic functions (CFs). Since CF for multivariate normal distribution is

$$\varphi_X(t) = E[e^{it^T X}] = \exp\left(it^T\mu - \frac{1}{2}t^T\Sigma t\right),$$

for a linear transform defined by matrix A we get that CF is

$$\varphi_{AX}(t) = E[e^{it^T AX}] = \varphi_X(A^T t) = \exp\left(it^T A\mu - \frac{1}{2}t^T A\Sigma A^T t\right).$$

Therefore $\boldsymbol{k} \sim N(A\mu,\, A\Sigma A^T)$. ▶

Task 20.

Same, but for $\boldsymbol{x} = (x_0,\, x_1,\, x_2)^T$ and coefficients of quadratic polynomials passing through all three points $(0,\, x_1), (1,\, x_1), (2,\, x_2)$.

Solution.

For a second degree polynomial $f(x) = ax^2 + bx + c$ its coefficients can be derived from elements of $\boldsymbol{x}$ with a linear transform since:

$$\begin{cases} x_0 = c, \\ x_1 = a + b + c, \\ x_2 = 4a + 2b + c. \end{cases}$$

Or, in matrix form:

$$\boldsymbol{x} = B^{-1}\boldsymbol{k},$$

where $B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$ and $\boldsymbol{k} = (a,\, b,\, c)^T$. Hence, analogously to the previous task, we get that $\boldsymbol{k} \sim N(B\mu,\, B\Sigma B^T)$, where $B = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$. ▶

Task 21.

Same, but for $\boldsymbol{x} = (x_0,\, x_1,\, \ldots,\, x_n)^T$ and coefficients of polynomials with degree n passing through all $n+1$ points $(i,\, x_i)$, $i = 0, \ldots, n$.

Solution.

It can be proven that the vector of coefficients $\boldsymbol{k} = (k_0,\, k_1,\, \ldots,\, k_n)^T$ for a n-th degree polynomial $f(x) = k_0 x^n + k_1 x^{n-1} + \cdots + k_n$, which passes through points $(i,\, x_i)$, $i = 0, \ldots, n$, is a linear transformation of vector $\boldsymbol{x}$.

Since it goes through each point in said set, it holds that:

$$\begin{cases} x_0 = k_n, \\ x_1 = k_0 + k_1 + \cdots + k_{n-1} + k_n, \\ \quad\vdots \\ x_n = k_0 n^n + k_1 n^{n-1} + \cdots + k_{n-1} n + k_n. \end{cases}$$

Or in matrix form:

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & 1 \\ \vdots & & & & \vdots \\ n^n & n^{n-1} & \ldots & n^1 & n^0 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_n \end{pmatrix}.$$

If we designate the second term as C^{-1}, we get that $\boldsymbol{x} = C^{-1}\boldsymbol{k}$. It indeed follows that $\boldsymbol{k} = C\boldsymbol{x}$, where C in an inverse of matrix C^{-1}. This is a linear transformation, hence, $\boldsymbol{k} \sim N(C\mu,\, C\Sigma C^T)$. ▶