1. Topological dimension

Topological dimension is denoted as d_T. It has following properties:

1. $d_T(\emptyset) = -1$, $d_T(\text{point}) = 0$, $d_T(\text{line}) = 1$
2. Border between A and B is a closed set Φ such that its complement is a union of such C and D that $C \cap D = \emptyset$, $A \subseteq C$ and $B \subseteq D$
3. Dimension of X is equal to dimension of the border increased by 1.

Consider a set A. Split it into subsets A_i, diam $A_i < \varepsilon$. Let

$$m(\varepsilon, p) = \inf_{\{A_i\}} \sum_i (\text{diam} A_i)^p,$$

$$d_M = \sup_p \left\{ p \mid \sup_{\varepsilon > 0} m(\varepsilon, p) > 0 \right\}.$$

Note that if $d_M > d_T$ A is a fractal.

Let $N(\varepsilon)$ be the number of non-empty cubes with diam $= \varepsilon$. Then, capacity is given by

$$D_0 = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

1. 1. Fractal dimension estimation

1. $\{x_1, ..., x_N\} \to \{y_1, ..., y_M\},\ y_i = \left[x_i, x_{i+\tau}, ..., x_{i+\tau \cdot (M-1)}\right]$. x_i — scalars, y_i — vectors, $y_i^{(k)}$ — k-th value of y_i
2. Normalization $\tilde{y}_i^{(k)} = \frac{y_i^{(k)} - \min_j y_j^{(k)}}{\max_j y_j^{(k)} - \min_j y_j^{(k)}}$.
3. $\varepsilon_l = \varepsilon_{\max} \cdot q^l$.
4. Calculate $N(\varepsilon) = \lfloor \frac{\tilde{y}_i}{\varepsilon_l} \rfloor$, $N(\varepsilon_l) = \text{unique}\{\varepsilon_i\}$.

Plotting $\ln N(\varepsilon)$ against $\ln \frac{1}{\varepsilon}$ we get that there is a line: $\ln N(\varepsilon) = \alpha + D_0 \ln\left(\frac{1}{\varepsilon}\right)$.

1. 2. Correlation dimension

$$D_2 = \lim_{r \to 0} \frac{\ln C(r)}{\ln r},$$

where $C(r)$ is correlation integral.

Consider a set fo points in m-dimensional phase space $\{y_i\}_{i=1}^{M}$, then:

$$C(r) = \frac{2}{M(M-1)} \sum_{i=1}^{M} \sum_{j=i+1}^{M} \theta\big(r - \|y_i - y_j\|\big),$$

where $\theta(x)$ is a Heaviside function. Generally,

$$C(r) = \int \mu(B(x,r))d\mu(x)$$

where $B(x,r)$ is ball of radius r with center at x and μ is a metric function.

1. Reconstruction $x_i \to y_i$.
2. Define a grid for r (usually as geometric progression).
3. $d_{ij} = \|y_i - y_j\|$
4. $C(r) = \frac{2}{M(M-1)} \sum_{i=1}^{M} \sum_{j=i+1}^{M} \theta\big(r - d_{ij}\big)$.
5. $C(r) <$ º))) $>< \gamma^{D_2} \Rightarrow \ln C(r) = \alpha + D_2 \ln r$ (use only part of data that creates the line).