COMP 680
Statistics for Computing and Data Science
Week 12: Generalized Linear Models

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Outline

Generalized Linear Models

- A large class of models!
  - where the error distribution is no longer restricted to be Gaussian

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  - exponential family: Gaussian, Binomial, Poisson, Gamma, Beta ...

Generalized Linear Models

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  - where the error distribution is no longer restricted to be Gaussian
- The conditional distribution of $Y|X$ follows exponential family
  - linear regression is one type of GLM
  - exponential family: Gaussian, Binomial, Poisson, Gamma, Beta ...
- Model the conditional mean of $Y|X$
  - $\mathbb{E}[Y|X] = X\beta$ for LM
  - $g(\mathbb{E}[Y|X]) = X\beta$ for GLM
  - different link functions g for specific conditional distributions

LM as GLM

A **generalized linear model** to model conditional mean $\mu = \mathbb{E}[Y|X]$:

$$g(\mu) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

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- a link function $g(\mu)$
  - describes how the transformed mean depends on the linear predictor
- a variance function $\phi V(\mu)$
  - how the variance of Y depends on the mean μ where ϕ is the dispersion parameter

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LM (a special case)

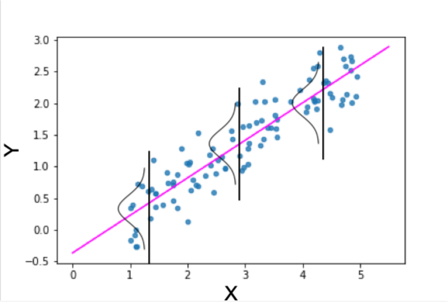
- $g(\mu) = \mu$
  - link function for LM is the identity function, i.e. no transformation needed
- $V(\mu) = 1$
  - for Gaussian distribution, variance does not depend on mean, and $\phi = \sigma^2$

Link Functions

Response Y	Conditional Distribution	Link Function
Continuous	Gaussian	Identity function
Continuous	Gamma	Negative inverse function
Continuous	Inverse Gaussian	Inverse squared function
Binary	Binomial	Logit function
Categorical	Multinomial	(generalized) Logit function
Ordinal	Multinomial	Logit or Probit function
Count	Poisson	Log function
Count	Negative Binomial	Log function

Illustration

- Linear regression



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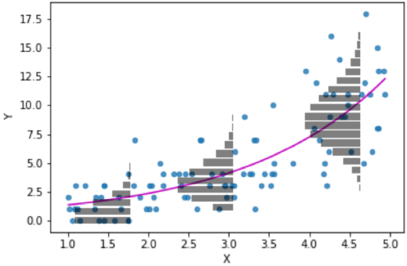
- Poisson regression



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Binary Outcome

- Examples of binary responses
  - spam filter
  - fraud detection
  - cancer screening
  - ...

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  - spam filter
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  - ...
- In Machine Learning: Binary Classification
  - a type of supervised learning
  - Logistic Regression is one classifier
  - many others: KNN, NaiveBayes, DecisionTrees, SVM, ...

Bernoulli and Binomial Distributions

- A Bernoulli random variable Y is a discrete random variable that can only take values 0 or 1:
  - a single parameter p: "success" probability
  - $\mathbb{P}(Y = 1) = p$: "success"
  - $\mathbb{P}(Y = 0) = 1 - p$: "failure"
  - $\mathbb{E}[Y] = p$, $\text{var}(Y) = p(1 - p)$

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- A Binomial random variable Y is the sum of n independent Bernoulli's:
  - two parameters: n and p
  - Y is a discrete random variable and can take values from 0 to n
  - $\mathbb{P}(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k}$: k "success" out of n trials
  - $\mathbb{E}[Y] = np$, $\text{var}(Y) = np(1 - p)$

Linear vs. Logistic Regression

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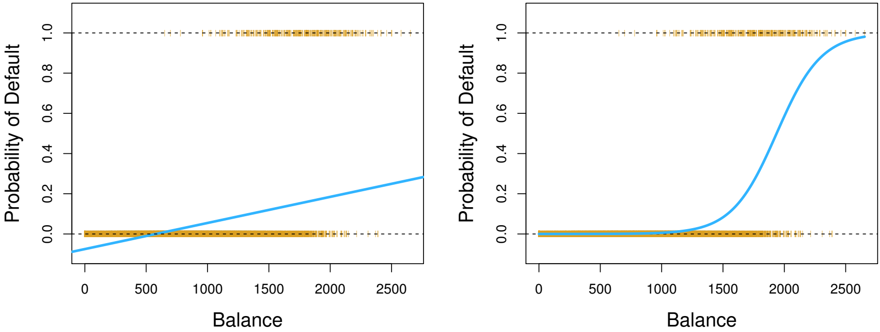

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Logit and Logistic

The logit function:

- $\text{logit}(x) = \ln\left(\frac{x}{1-x}\right)$
- log odds ratio

The logistic function:

- $f(x) = \frac{e^x}{1+e^x}$
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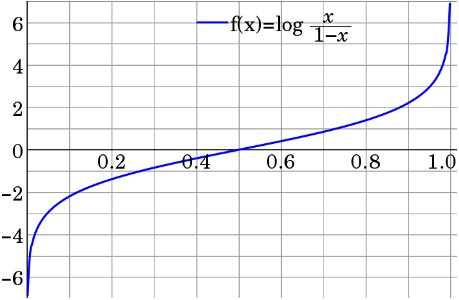

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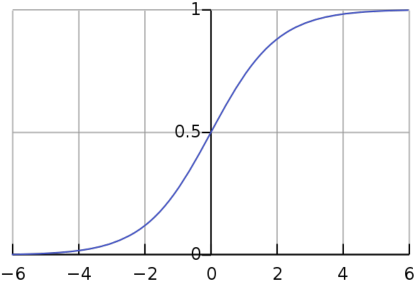

Simple Logistic Model

$$\mathbb{P}(Y = 1|X) = p_x = \frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}} \Leftrightarrow \text{logit}(p_x) = \ln\left(\frac{p_x}{1-p_x}\right) = \beta_0 + \beta_1 x$$

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- remember $\mu = \mathbb{E}[Y|X] = p_x$ for binary Y
- the link function $g(\mu)$ is the logit function
- log odds ratio is a linear function of X!

Model Fitting and Coefficients

- How to estimate model parameters: Maximum Likelihood Estimate
    - likelihood function?

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  - can make inference
    - asymptotic normality of MLE
  - can make prediction
    - predict log odds $\rightarrow$ probability $\rightarrow$ label

Model Interpretation

Linear Regression:

- $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$
- $\hat{\beta}_0$: average value of Y when $X = 0$
- $\hat{\beta}_1$: average change in Y when X increase by 1 unit
- $\hat{\beta}_1 > 0$ and significant: X is positively correlated with Y

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Logistic Regression

- $\ln\left(\frac{\hat{p}}{1-\hat{p}}\right) = \hat{\beta}_0 + \hat{\beta}_1 x$
- $\hat{\beta}_0$: log odds of $Y = 1$ when $X = 0$
- $\hat{\beta}_1$: change in log odds of $Y = 1$ when X increase by 1 unit
- $\hat{\beta}_1 > 0$ and significant: X is positively correlated with log odds of $Y = 1$

Multiple Covariates

- simple linear regression $\rightarrow$ multiple regression
  - interpretation of single slope means controlling others

Multiple Covariates

- simple linear regression $\rightarrow$ multiple regression
  - interpretation of single slope means controlling others
- simple logistic regression $\rightarrow$ multiple logistic regression
  - A set of p covariates $X_1, X_2, \cdots X_p$

$$\mathbb{P}(Y = 1|X) = p_x = \frac{e^{\beta_0+\beta_1 X_1+\cdots+\beta_p X_p}}{1 + e^{\beta_0+\beta_1 X_1+\cdots+\beta_p X_p}} \Leftrightarrow$$

$$\text{logit}(p_x) = \ln\left(\frac{p_x}{1-p_x}\right) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

  - interpretation of single slope means controlling others (**holding others constant**), exactly the same!

Prediction

- From predicted probability $\hat{p} = \mathbb{P}(\hat{Y} = 1)$ to predicted outcome $\hat{Y}$
  - a natural threshold 0.5
  - $\hat{p} = \mathbb{P}(\hat{Y} = 1) \geq 0.5 \Rightarrow \hat{Y} = 1$
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$$\frac{y_i - \hat{p}_i}{\sqrt{\hat{p}_i(1 - \hat{p}_i)}}$$

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$$\frac{y_i - \hat{p}_i}{\sqrt{\hat{p}_i(1 - \hat{p}_i)}}$$

- Can vary threshold to minimize error measure
  - sensitivity: true positive rate
  - specificity: true negative rate

Outline

Categorical Outcome

- More than 2 outcomes
  - multi-class classification

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- More than 2 outcomes
  - multi-class classification
- Multinomial regression
  - extension on logistic regression
  - multi-class logistic regression

Multinomial Distribution

- A Multinomial random variable (vector) $Y = (Y_1, \cdots Y_K)$
  - K parameters: n and probabilities $(p_1, p_2, \cdots p_K)$
  - $\sum_{k=1}^{K} p_k = 1$
  - Y represents number of k different outcomes

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  - $\sum_{k=1}^{K} p_k = 1$
  - Y represents number of k different outcomes
- A Binomial is a special case of Multinomial where $K = 2$
  - $(p_1 = p, p_2 = 1 - p)$

Generalized Logit

- You have total of K categories, pick a category to be baseline

$$\mathbb{P}(Y = j|X) = p_x^j = \frac{e^{\beta_0^j + \beta_1^j X_1 + \cdots + \beta_p^j X_p}}{1 + \sum_{k=1}^{K-1} e^{\beta_0^k + \beta_1^k X_1 + \cdots + \beta_p^k X_p}}, \quad j = 1, \cdots K - 1$$

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  - Softmax function

  - which one to be your baseline DOES NOT matter
  - interpretation of coefficients: odds of category j vs. baseline

- Binary Logistic is a special case of $K=2$ and $Y=0$ as baseline

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  - less efficient, $(K-1)(p+1)$ parameters
- Use ordinal information
  - more efficient, $K+p-1$ parameters
- Intuition
  - unlike multinomial case, now the categories are ordered
  - the difference between category $k-1$ and k should be comparable to the difference between category k and $k+1$

Ordinal Logistic Regression

- The Proportional-Odds Cumulative Logit Model

$$\log\left(\frac{\mathbb{P}(Y \leq k|X)}{1-\mathbb{P}(Y \leq k|X)}\right) = \beta_0^k + \beta_1 X_1 + \cdots \beta_p X_p, \quad k = 1, 2, \cdots K-1$$

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  - estimate $K-1$ intercept and p slopes
- Interpretation of coefficients:
  - β_0^k is the log odds of falling into or below category k when all $X = 0$
  - β_j is increase in log odds of falling into or below any category associated with a one unit increase in X_j while controlling all other X

Outline

Poisson Distribution

- A Poisson random variable X is discrete and can take any integer values with single parameter λ:

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

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- $\mathbb{E}[Y] = \mathrm{Var}(Y) = \lambda$

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- Interpretation of slopes:
  - one unit increase of X_j impacts average value of Y by multiplication of e^{β_j} while holding all other X constant

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- the term $\log(t)$ is referred to as an offset, an adjustment term
- the mean count is proportional to t

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- In practice, the apparent variance of data often exceeds the mean, reflecting overdispersion in the model parameters.
  - cause covariates to be "more significant"
  - narrower confidence intervals than warranted by the data
- Possible solutions
  - quasi-Poisson or negative binomial generalized linear model
  - more parameter and no longer force mean = variance

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Exponential Family

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- $\eta(\theta)$ is called the natural parameterization and is the link function!
  - each exponential distribution corresponds to a GLM
  - logistic and Poisson are most common

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- Poisson

$$\mathbb{P}(X = x \mid \lambda) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{x!} \exp[x \ln \lambda - \lambda]$$

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- Software output usually includes:
  - the null deviance shows how well the response variable is predicted by a model that includes only the intercept (grand mean).
  - the residual deviance can be compared to the null deviance:
- Akaike Information Criterion (AIC):
  - based on deviance, but penalize on model size
  - more useful in model comparison

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- In general, GLM are estimated by MLE
  - Likelihood ratio test (χ^2 test)
  - significant results means larger model provide significant improvement

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