COMP 680
Statistics for Computing and Data Science

Week 9: Bayesian Inference

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Outline

Overview

- Bayesian v.s. Classical Statistics
  - Two major schools of statistics
  - Solve the same problem in two completely different ways
  - Somewhat philosophical

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- Classical Statistics
  - The unknown parameter is a fixed quantity
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  - Thus the name "Frequentist"
- Bayesian Statistics
  - The unknown parameter is a random variable follows some distribution
  - Has a prior belief about what that random distribution should be
  - Data comes in to update that prior belief to a posterior belief

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- The Frequentist way:
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- The Bayesian way:
  - prior distribution $\rightarrow$ data $\rightarrow$ posterior distribution
  - posterior distribution quantify uncertainty!

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- The "Original" Bayes paper: An essay towards solving a problem in the doctrine of chances
  - use Binomial data comprising r successes out of n attempts
  - learn about the underlying chance θ of each attempt succeeding
  - use a probability distribution to represent uncertainty about θ

Outline

1. The Big Picture
2. The Bayesian "Recipe"
3. Conjugate Family
4. Posterior Inference
5. Code Demo

The Bayes Theorem

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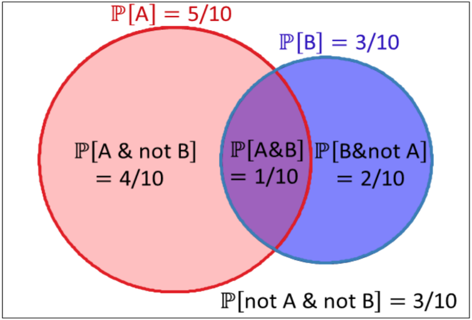

In this example;

- $\mathbb{P}[A|B] = \frac{1/10}{3/10} = 1/3$
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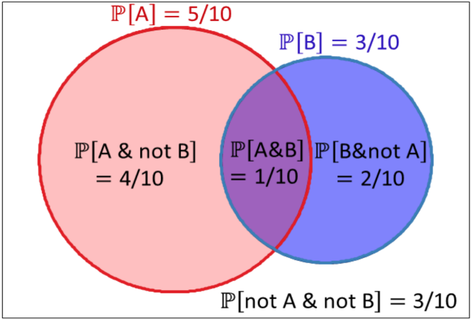

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- the conditional probability of A given B is the conditional probability of B given A scaled by the relative probability of A compared to B.

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- Marginal Likelihood: (aka the "normalizing constant")
  - $p(X) = \int p(X|\theta)\pi(\theta)d\theta$
  - $p(X)$ is a constant regarding the posterior $\pi(\theta|X)$

Bayesian Inference

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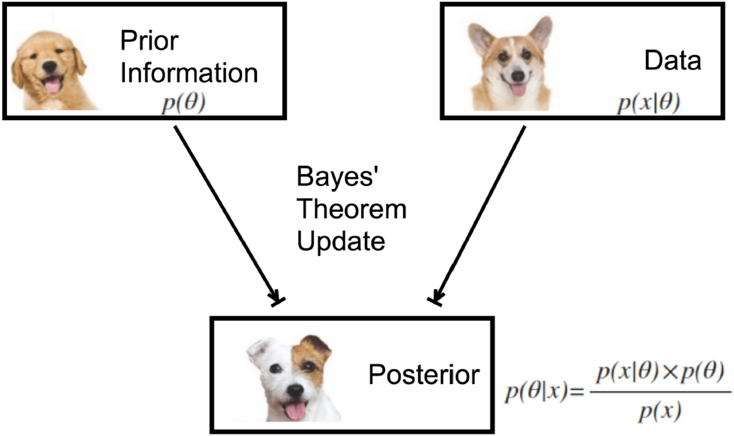
- So how to get a posterior distribution: starting what we know about θ, combining the prior with the data, Bayes Theorem used for inference tells us to multiply and scale ... and that is it! (essentially!)

The Bayesian “Recipe”



Puppies borrowed by Kruschke J., Doing Bayesian Data Analysis, A tutorial with R, JAGS and STAN, Academic Press

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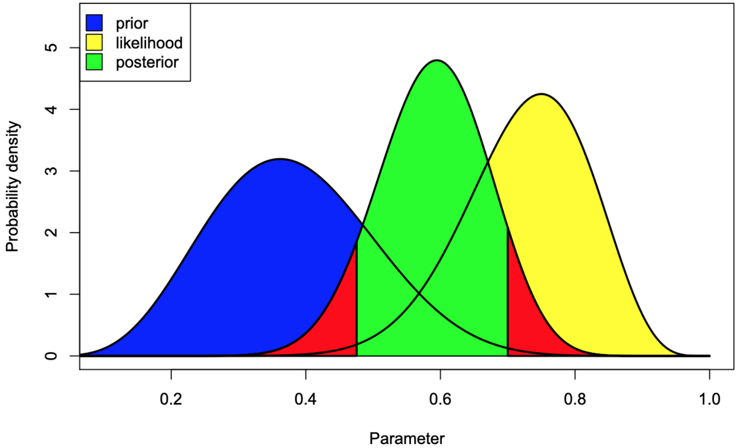
- The "prior" distribution $\pi(\theta)$ is combined with "likelihood" $p(X|\theta)$ to provide a "posterior" distribution $\pi(\theta|X)$.
- The likelihood is derived from an sampling model $p(X|\theta)$ but considered as function of θ for fixed X.

Illustration

$$\pi(\theta|X) \propto p(X|\theta)\pi(\theta) \quad \Longrightarrow \quad \text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$



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- How to choose the prior for the unknown parameter θ?
- How to calculate the posterior, in particular, the normalizing constant $p(X) = \int p(X|\theta)d\theta$ where you have to do an integral (possibly high dimension integral)!

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- Priors come from all data external to the current study i.e. everything else.
- "Boil down" to what subject-matter experts know/think is known as eliciting a prior.
- It is not easy!

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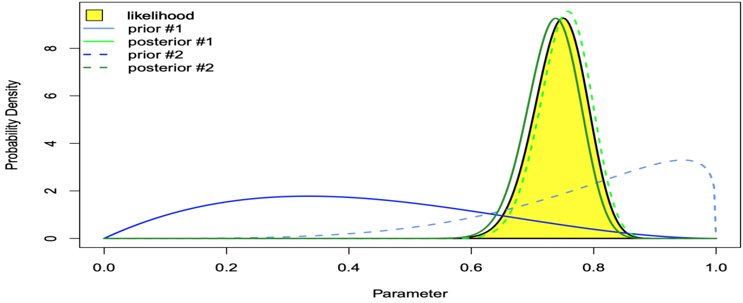

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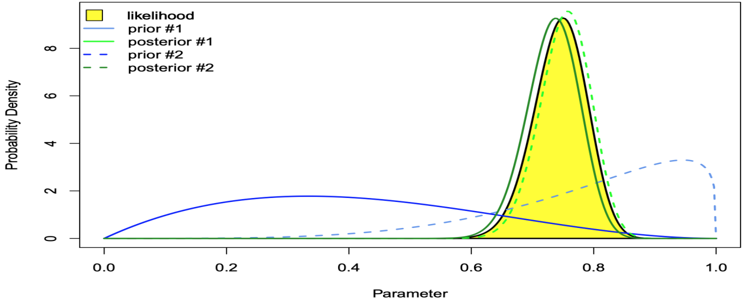

These two priors (and many more) are dominated by the likelihood, and they give very similar posteriors - i.e. everyone agrees. (Phew!)

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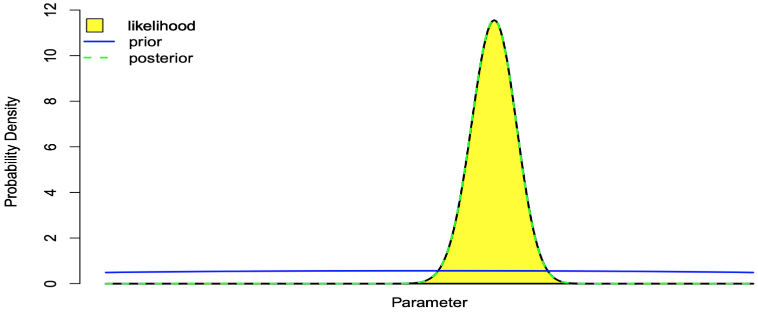

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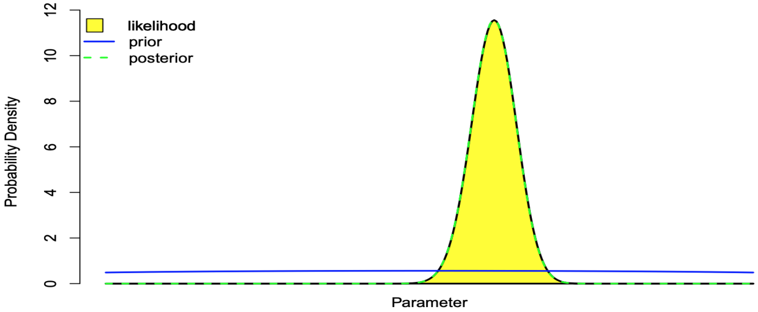

"Objective Bayes": use flat (non-informative) priors.

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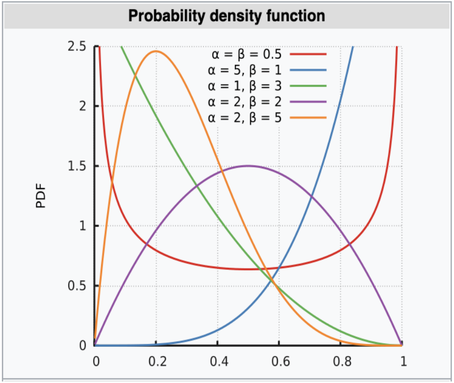
- We almost always choose a conjugate prior if there is one!

Beta Distribution



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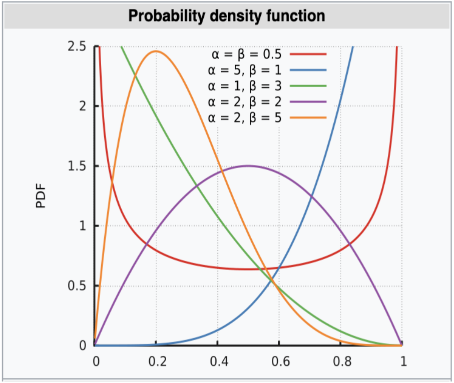

- A continuous random variable defined on $[0, 1]$ with PDF:

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$$

  - $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
  - $\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$
  - $\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

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Gamma Distribution

- Exponential(β): a continuous, non-negative random variable
- PDF:

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- Gamma(α, β): sum of α i.i.d. Exponential(β)
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  - Posterior mean is a weighted average of prior mean and sample mean
  - $\mu_n = \frac{1/\tau_0^2}{1/\tau_0^2 + n/\sigma^2}\mu_0 + \frac{n/\sigma^2}{1/\tau_0^2 + n/\sigma^2}\bar{X} = \omega_n\mu_0 + (1-\omega_n)\bar{X}$

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- Uncertainty quantification of the unknown parameter:
  - standard error: posterior standard deviation
  - confidence interval: the "middle chunk" of posterior distribution
  - in Bayesian this is called credible interval $[\theta_l, \theta_u]$:

$$\mathbb{P}(\theta \in [\theta_l, \theta_u]) = 95\%$$

the Monte Carlo Method

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- Samples from the posterior can be generated in several ways, without exact knowledge of $p(\theta|X)$.

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- Developing practical algorithms to approximate posterior distributions for complex problems remains an active area of research.

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- Likelihood: $p(y_1, y_2, \cdots y_n|\mu, \tau) = \prod_{i=1}^{n} \mathbf{N}(y_i|\mu, \tau)$
- Prior: $p(\mu, \tau) =???$
- Posterior: $\pi(\mu, \tau|y_1, y_2, \cdots y_n) \propto p(y_1, y_2, \cdots y_n|\mu, \tau)p(\mu, \tau)$

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- Notice here we still don't know how to calculate the joint posterior or sample from it directly. However, conditional conjugacy allows us to sample μ and τ iteratively from the conditional posterior:
  - Randomly set initial value $\mu^{(0)}$ and $\tau^{(0)}$, for $m = 1, 2, 3, \cdots M$.
  - Sample $\mu^{(m)}$ from $\pi(\mu|\tau, y_1, y_2, \cdots y_n) = \mathbf{N}\left(\mu|\frac{\bar{y}n\tau}{n\tau+1/v^2}, \frac{1}{n\tau+1/v^2}\right)$ where we use $\tau = \tau^{(m-1)}$.
  - Sample $\tau^{(m)}$ from $\pi(\tau|\mu, y_1, y_2, \cdots y_n) = \mathbf{Gamma}\left(\tau|\alpha + \frac{n}{2}, \beta + \frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2}\right)$ where we use $\mu = \mu^{(m)}$.

Posterior Inference for Parameters

- For M large enough, we assume the chain has converged, and $(\mu^{(m)}, \tau^{(m)})$ can be treated as samples from the joint posterior $\pi(\mu, \tau | y_1, y_2, \cdots y_n)$.

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- Use Monte Carlo approximation to get: point estimator, uncertainty measure, credit interval, any function of μ and τ etc.
- In practice, usually discard the first $x\%$ samples (burn-in) from the chain $(\mu^{(m)}, \tau^{(m)})$ and use the rest for posterior inference.

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  - to predict the next data point.
  - in practice, can sample X^{pred} using samples of θ from Gibbs sampling, then use Monte Carlo approximation again.
  - after all, we can view the predictive density as a function of θ.

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  - Converge in what sense?
  - How fast is the convergence rate?
  - How does the Bayesian estimate compare to MLE?

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- To make inferences on a huge joint probability model where there are possibly thousands of observations and parameters.
- To use Bayesian ideas to quantify uncertainty of parameters.
- The "updating" inherent in the Bayesian approach is suitable in machine-learning.

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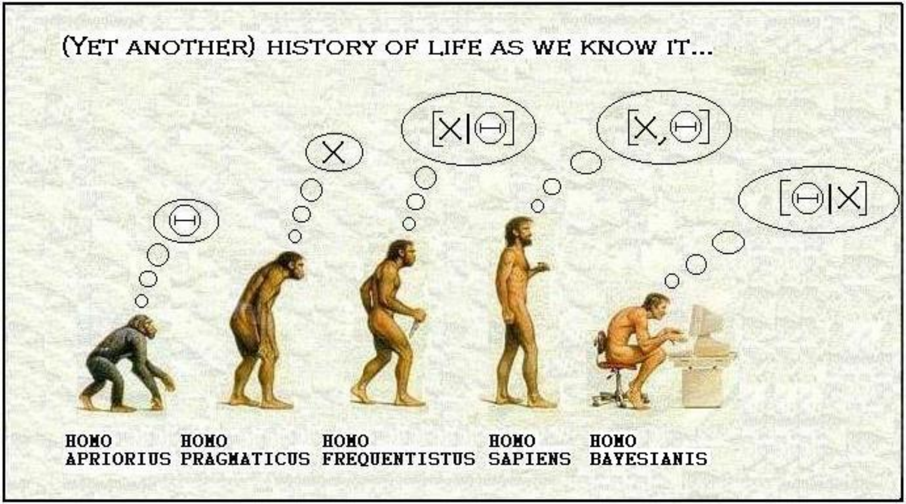
- There is deep philosophical differences between Bayesian and frequentist inference.
- Bayesian make statements about the relative evidence for parameter values given a dataset, while frequentists compare the relative chance of datasets given a parameter value.
- Bayesian statistics is getting more and more popular due to its advantages and increase of computation power.

I'm a homo Bayesian



Outline

1. The Big Picture
2. The Bayesian "Recipe"
3. Conjugate Family
4. Posterior Inference
5. Code Demo

Recommended References

- Textbook: **A first course in Bayesian Statistic method, by Peter Hoff.**
- Lecture notes: **Introduction to Bayesian Statistics, by Brendon Brewer.**
- Talk slides: **Bayesian Statistics, a very brief introduction, by Ken Rice.**
- Websites:
  - **Scholarpedia entry on Bayesian statistics.**
  - **Bayesian statistics for beginners in simple English.**