COMP 680
Statistics for Computing and Data Science
Week 1: Probability and Random Variable I

Su Chen, Assistant Teaching Professor,
Rice D2K Lab

Outline

Your Instructor

- Su Chen
- Email: Su.Chen@rice.edu
- Office: Duncan 2051
- About me

Administrative Stuff

You can find everything on the Canvas website for this course

- The syllabus
- Weekly slides, code demo and quiz
- Bi-weekly homework assignments
- Announcements and emails...

Prerequisites

- College level calculus: single & multivariate
- Basics of linear algebra: vectors and matrices
- Comfortable with mathematical reasoning...
- Basic programming skills (or not afraid to learn :)

Learning Objectives

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- Interpret inferences and modeling results in real-world contexts and communicate the findings effectively.

Course Work

- Weekly classes schedule: MW 2-3:15pm
  - Attendance
  - Participation
  - Feedback

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- Major assignments:
  - Bi-weekly homework assignments
  - Weekly quizzes
  - Midterm in-class exam
  - Final takehome exam

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- Assignment: 45%
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  - final: 30%
- Other: 5%
  - class participation and attendance: 3%
  - course exit survey: 2%

Etc.

- Disability accommodations.
- Tentative schedule.
- Syllabus is subject to change with reasonable advance notice by the instructor.

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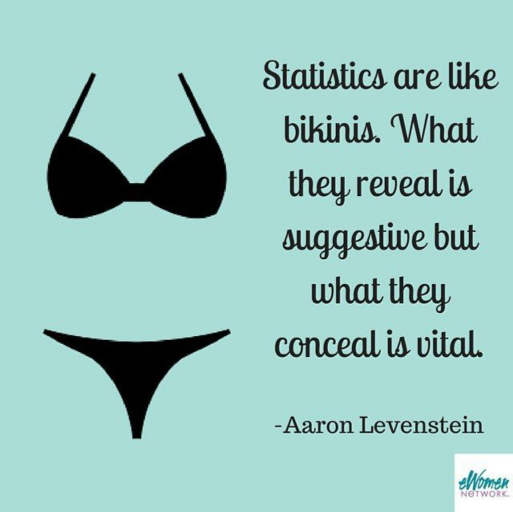
- A Brief History of Data Science
- Rooted in Statistics, powered by Computer Science, driven by application, interdisciplinary in nature
  - big data, cloud computing, data mining
  - machine learning/deep learning, artificial intelligence ...

What is Statistics?



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- Statistics as problem solving: harnessing the power of uncertainty
  - descriptive / exploratory
  - predictive / decision-making
  - inferential / causal

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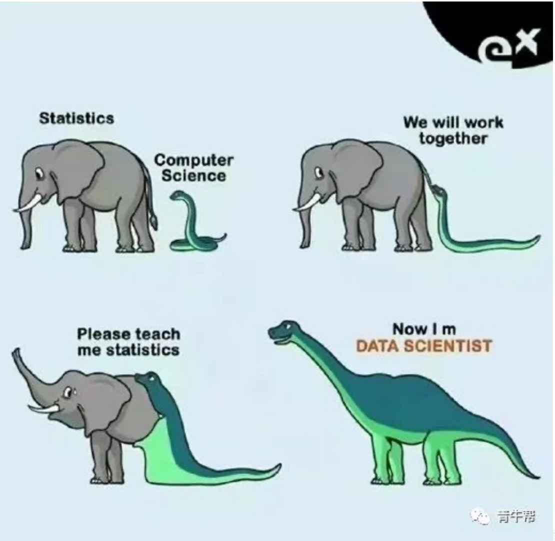
- "Lies, damn lies, and statistics."
  - statistics don't lie, people do.

Statistics vs. Data Science



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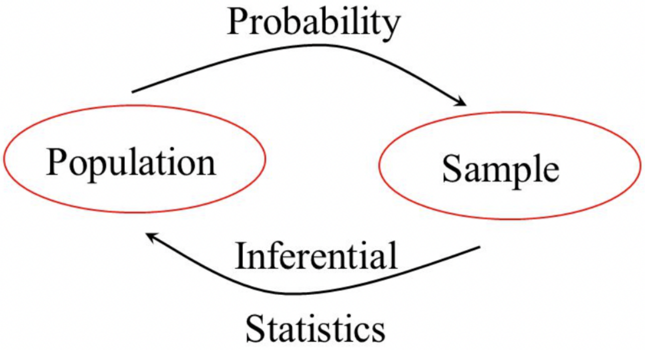
- " fair to consider applied statistics as a subset of data science":
  - harvesting, processing, storing and cleaning are more central to data science than hard core statistics

Outline

Probability and Statistics

"Probability is the vehicle of statistics."



Probability Spaces

Definition: A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of:

- A sample space Ω, which contains all possible outcomes.
- A set of events $\mathcal{F}$, which must be a σ-algebra defined on Ω. $\mathcal{F}$ is a collection of subsets of Ω such that:

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  - $\Omega \in \mathcal{F}$
  - if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
  - if $A_1, A_2, \cdots A_i, \cdots \in \mathcal{F}$, then $\cup_i A_i \in \mathcal{F}$
- A probability measure $\mathbb{P}$, which can assign probabilities to the events in $\mathcal{F}$.

Probability Spaces Example

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  - sample space Ω:
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- Generalize to any finite sample space:

Probability Measure

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3. If $A_1, A_2, \cdots A_n \ldots$ are disjoint then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

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Based on 1 - 3, can you show $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$?

Conditional Probability

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$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \times \mathbb{P}(B) = \mathbb{P}(B|A) \times \mathbb{P}(A)$$

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- **Law of total probability**: $A_1, A_2, \cdots$ be a partition of Ω

$$\mathbb{P}(S) = \sum_{i=1} \mathbb{P}(S \cap A_i) = \sum_{i=1} \mathbb{P}(S|A_i)\mathbb{P}(A_i)$$

Bayes Theorem

- Apply Bayes Theorem and (simple case) of Law of total probability:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)}$$

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- Example: Suppose 1% of population has certain cancer. A medical test can detect this cancer with 100% accuracy, but also has 1% false positive rate. If a person randomly selected tested positive, what is the probability this person actually has the cancer?

Independence

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Equivalently, if A and B are independent then:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Conditional Independence

Definition: Two events $A, B \in \mathcal{F}$ are independent conditioned on another event C **if and only if**:

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- A box contains two coins: a regular coin and one fake two-headed coin. I choose a coin at random and toss it twice. Define the following events:
    - $A=$ 1st coin toss is a head.
    - $B=$ 2nd coin toss is a head.
    - $C=$ the regular coin has been selected.

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  - interpretation of probability: long-term (relative) frequency
- Run a computer program to repeatedly simulate a random process!
  - Monte Carlo simulation
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- An approximation of the true probability
  - good enough approximation for “sufficient” repetition

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Notation:

- capital letter X for the random variable
- lower case x for values the random variable can take

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- Important discrete random variables:
  - Uniform
  - Bernoulli
  - Binomial
  - Poisson
  - See more here

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- A probability density function (pdf) $f_X(x)$ is defined as:

$$f_X(x) = F_X'(x)$$

for all points x at which $F_X(x)$ is differentiable.

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- $\int_{-\infty}^{\infty} f_X(x) = 1$

Continuous Random Variables

- Important continuous random variables:
  - Uniform
  - Normal (Gaussian)
  - Student-t
  - Cauchy
  - Exponential
  - Gamma
  - Beta
  - χ^2
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- If g is a strictly increasing function, then

$$F_Y(y) = \int_{-\infty}^{g^{-1}(y)} f_X(x)dx = F_X(g^{-1}(y))$$

$$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$$

Generating Random Variables

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- Let X be a random variable with a continuous CDF $F_X(x)$ and define the random variable $Y = F_X(x)$. Can you show $Y \sim U[0, 1]$?
- This is how most computer programs are generating random variables of common distributions:
    - sample from Uniform [0, 1] $u_1, u_2, \cdots u_n$
    - get $x_i = F_X^{-1}(u_i)$ for a specific distribution with cdf $F_X(\cdot)$

Outline

Joint Distributions

A joint probability mass function:

	Y=0	Y=1	
X=0	1/10	2/10	3/10
X=1	3/10	4/10	7/10
	4/10	6/10	1

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- Conditional distribution of $X|Y=0$ and $Y|X=1$?

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A joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} x + y & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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Independence and IID Samples

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- If $X_1, X_2, \cdots X_n$ are independent and each has the same marginal distribution of pdf $f_X(\cdot)$, we say that $X_1, X_2, \cdots X_n$ are IID (independent and identically distributed) and has joint pdf:

$$f_{X_1,X_2,\cdots X_n}(x_1, x_2, \cdots x_n) = \prod_{i=1}^{n} f_X(x_i)$$

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Python Jupyter Notebook

- Birthday problem.
- Monty Hall problem.
- Su's favorite probability brain teaser.

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- Assignment this week:
  - Install Python 3 and Jupyter Notebook through Anaconda
    - install tutorial for mac and window
  - Run the demo notebook and make sure nothing breaks.
  - Homework 1 posted, due in 2 weeks, please start early!