Substitution Method (cont.)

Idea:
1. Propose (guess) an order of time complexity for $T(n)$.
2. Find constants (by induction) that support your proposed solution.

Example. Prove that $T(n) = 2T(\frac{n}{2}) + n$ is $O(n\log(n))$

Substitution Method (cont.)

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1. Propose (guess) an order of time complexity for *T*(*n*).
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Example. Prove that $T(n) = 2T(\frac{n}{2}) + n$ is $O(n \log(n))$

(1). $T(n) \leq c\, n \log(n) \;\; \exists c > 0$ `// definition of Big-O`

(2). $T(m) \leq c\, m \log(m) \;\; \forall m \leq n$ `// inductive step (assumption)`

Substitution Method (cont.)

Idea:
1. Propose (guess) an order of time complexity for *T*(*n*).
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Example. Prove that $T(n)=2T(\frac{n}{2})+n$ is $O(n\log(n))$

(1). $T(n) \leq c\,n\log(n) \;\; \exists c>0$

(2). $T(m) \leq c\,m\log(n) \;\; \forall m<n$

$Let\, m=\frac{n}{2}$

(3). $T(\frac{n}{2}) \leq c\frac{n}{2}\log(\frac{n}{2})$ `// placing m = n/2 in (2)`

Substitution Method (cont.)

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1. Propose (guess) an order of time complexity for *T*(*n*).
2. Find constants (by induction) that support your proposed solution.

Example. Prove that $T(n)=2\boxed{T\left(\frac{n}{2}\right)}+n$ is $O(n\log(n))$

(1). $T(n) \le c\,n\log(n) \;\; \exists c>0$

(2). $T(m) \le c\,m\log(n) \;\; \forall m<n$

$Let\, m=\frac{n}{2}$

(3). $\boxed{T\left(\frac{n}{2}\right)} \le c\frac{n}{2}\log\left(\frac{n}{2}\right)$

(4). $T(n) \le 2c\frac{n}{2}\log\left(\frac{n}{2}\right) + n$ `// substitution of T(n/2) in definition of T(n)`

Substitution Method (cont.)

Idea:
1. Propose (guess) an order of time complexity for *T*(*n*).
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Example. Prove that $T(n) = 2T(\frac{n}{2}) + n$ is $O(n\log(n))$

(1). $T(n) \le c\,n\log(n) \quad \exists c > 0$

(2). $T(m) \le c\,m\log(n) \quad \forall m < n$

$Let\, m = \frac{n}{2}$

(3). $T(\frac{n}{2}) \le c\frac{n}{2}\log(\frac{n}{2})$

```
// log(a/b) = log(a) - log(b)
```

(4). $T(n) \le \cancel{2}c\frac{n}{\cancel{2}}\log(\frac{n}{2}) + n = c\,n\log(\frac{n}{2}) + n = c\,n(\log(n) - \log(\cancelto{1}{2})) + n$

Substitution Method (cont.)

Idea:
1. Propose (guess) an order of time complexity for *T*(*n*).
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Example. Prove that $T(n)=2T(\frac{n}{2})+n$ is $O(n\log(n))$

(1). $T(n) \leq cn\log(n) \;\; \exists c>0$

(2). $T(m) \leq cm\log(n) \;\; \forall m<n$

$Let\, m=\frac{n}{2}$

(3). $T(\frac{n}{2}) \leq c\frac{n}{2}\log(\frac{n}{2})$

(4). $T(n) \leq \cancel{2}c\frac{n}{\cancel{2}}\log(\frac{n}{2}) + n = cn\log(\frac{n}{2}) + n = cn(\log(n)-\log\overset{1}{\cancel{(2)}}) + n$

$T(n) \leq cn\log(n)-cn + n$

$T(n) \leq cn\log(n)-(c-1)n$

Substitution Method (cont.)

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1. Propose (guess) an order of time complexity for *T*(*n*).
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Example. Prove that $T(n)=2T(\frac{n}{2})+n$ is $O(n\log(n))$

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$T(n) \le c\,n\log(n)-cn + n$

$\boxed{T(n) \le c\,n\log(n)-(c-1)n}$

(5). $c\,n\log(n)-(c-1)n \le c\,n\log(n)$ `// for which values of c?`

Substitution Method (cont.)

Idea:
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Example. Prove that $T(n) = 2T(\frac{n}{2}) + n$ is $O(n\log(n))$

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(3). $T(\frac{n}{2}) \leq c\frac{n}{2}\log(\frac{n}{2})$

(4). $T(n) \leq \cancel{2}c\frac{n}{\cancel{2}}\log(\frac{n}{2}) + n = cn\log(\frac{n}{2}) + n = cn(\log(n) - \overset{1}{\cancel{\log(2)}}) + n$

$T(n) \leq cn\log(n) - cn + n$

$T(n) \leq cn\log(n) - (c-1)n$

(5). $cn\log(n) - (c-1)n \leq cn\log(n)$ `// for which values of c?`

$n\log(n) \leq n\log(n)$ if $c = 1$

$\frac{1}{2}n\log(n) + \frac{1}{2}\mathrm{n} \leq \frac{1}{2}n\log(n)$ if $c = \frac{1}{2}$

Substitution Method (cont.)

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(3). $T(\frac{n}{2}) \leq c\frac{n}{2}\log(\frac{n}{2})$

(4). $T(n) \leq \cancel{2}c\frac{n}{\cancel{2}}\log(\frac{n}{2}) + n = cn\log(\frac{n}{2}) + n = cn(\log(n)-\log(\cancelto{1}{2})) + n$

$T(n) \leq cn\log(n)-cn + n$

$T(n) \leq cn\log(n)-(c-1)n$

(5). $cn\log(n)-(c-1)n \leq cn\log(n)$

Solution: $T(n) \leq cn\log(n) \;\; \forall c\geq 1$