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Application of Chebyshev Finite Difference Method (ChFDM) in Calculus of Variation

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ABSTRACT

The Chebyshev finite difference method (ChFDM) is presented for solving the ordinary differential equations which arise in problems of calculus of variations. This method is regarded as a non-uniform finite difference method for mathematics and fluid mechanics problems. The method has been implemented in three different mathematical problems to show the capability of this method in different areas. The performance and accuracy of the proposed method have been compared with the exact solution of each example. Obviously, we have observed that as number of components in series increase, the accuracy will increase and thought error decrease, successively.

Keywords: Chebyshev finite difference method (ChFDM), Calculus of variation, Chebyshev Gauss-Lobatto (ChGL) points

1. INTRODUCTION

In a lot of problems arising in fluid mechanics, engineering, mathematics which we are facing in analytical solution of the problems, it is necessary to determine the maximal and minimal of a certain functional. Since importance of this subject in science and engineering has been highlighted, considerable attention has been taken into account on this kind of problems. Such problems are called variational problems. Some popular methods for solving variational problems are direct methods.

Chebyshev polynomials are widely used as a numerical solution of various boundary value problems and boundary layer problems [1, 2]. One of the advantages of using Chebyshev polynomials $T_n(t)$ as a tool for expansion functions is the good representation of smooth functions by finite Chebyshev expansion provided that the function f(t) is infinitely differentiable. Chebyshev finite difference method (ChFD) has been proven to be successful in the numerical solution of various boundary value problems and in the solution of boundary layer equations. The accuracy of ChFDM has been proven to be much higher than conventional finite difference and finite element methods [3-8]. ChFD method can be regarded as a non-uniform finite difference scheme. In this method the derivatives of the function $f(t)$ at a grid point t_j is linear combination of the values of the function f at the Chebyshev Gauss-Lobatto (ChGL) points $t_k = \cos(k\pi/N),\ k = 0,1,2,....,N$. While the finite difference method produces a second order accurate derivative with the error decreasing as $1/m^2$ (m being the number of grid points), the error from the global method decreases exponentially. The coefficients in Chebyshev expansion, approach zero faster than any inverse power in n as n goes to infinity. Heinrichs [9] put forward a scheme for the adjoint of the linearized problem, and also presented a numerical investigation of the eigenvalues of the spectral differentiation operator. There are also recent articles by Ma and Sun [10] and by Shen [11] presenting spectral methods for third-order problems with boundary conditions.

Considerable attention has been received by many researches to investigate on arising problems in fluid mechanics as well as other controversial topics to determine the maximum and minimum of a certain functional. Chen and Hsiao [12] presented the Walsh series method to variational problems. Due to the nature of the Walsh functions, the solution obtained was piecewise constant. Tatari and Dehghan [13] implemented He's variational iteration method for solving of some problems in calculus of variations. They had demonstrated that using the sufficiently large number of iterations, He's method can obtain an accurate approximation of the exact solution. Dehghan and Tatari [14] found the solution of an ordinary differential equation which arises from the variational problems by the means of Adomian decomposition method. They have compared the results of the proposed method by the numerical ones which were in a good agreement.

This paper aims at introducing Chebyshev finite difference method (ChFDM) to find approximation solution of differential equations which arise from problems of calculus of variations. To go further, grid points are defined and ChFDM is applied to satisfy the differential equations and its boundary conditions at these grid points. The computational error has been calculated for each step which is shown to be smaller in large number of series, showing the accuracy of proposed method.

2. STATEMENT OF THE PROBLEM

The simplest form of a variational problem can be considered as finding the extremum of the functional is:

$$J\left[y(x)\right] = \int_{x_0}^{x_1} F(x, y(x), y'(x))dx. \tag{1}$$

To find the extreme value of *J*, the boundary points of the admissible curves are known in the following form:

$$y(x_0) = \alpha, \quad y(x_1) = \beta. \tag{2}$$

The necessary condition for *y(x)* to extremize *J [y(x)]* is that it should satisfy the Euler–Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0, \tag{3}$$

with boundary conditions given in (2). The boundary value problem (2) does not always have a solution and if the solution exists, it may not be unique. Note that in many variational problems the existence of a solution is obvious from the physical or geometrical meaning of the problem and if the solution of Euler's equation satisfies the boundary conditions, it is unique. Also this unique extremal will be the solution of the given variational problem. Thus another approach for solving the variational problem (1) is finding the solution of the ordinary differential equation (3) which satisfies the boundary conditions (2).

The general form of the variational problem (1) is:

$$J[y_1, y_2, ..., y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, ..., y_n, y'_1, y'_2, ..., y'_n) dx, \quad (4)$$

with the given boundary conditions for all functions

$$y_1(x_0) = \alpha_1, \; y_2(x_0) = \alpha_2, \quad ..., \quad y_n(x_0) = \alpha_n, \quad (5)$$
$$y_1(x_1) = \beta_1, \; y_2(x_1) = \beta_2, \quad ..., \quad y_n(x_1) = \beta_n. \quad (6)$$

Here the necessary condition for the extremum of the functional (4) is to satisfy the following system of second-order differential equations

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y'_i} = 0, \; i = 1, 2, ..., n, \quad (7)$$

with boundary conditions given in (5) and (6). In the present work, we have found the solution of the variational problems by applying ChFD method on the Euler–Lagrange equations.

3. CHEBYSHEV FINITE DIFFERENCE METHOD (CHFDM)

The well-known Chebyshev polynomials of the first kind of degree n are defined on the interval[-1,1] as:

$$T_n(t) = \cos\left(n \cos^{-1} t\right). \quad (8)$$

Obviously, $T_0(t) = 1, T_1(t) = t$ and they satisfy the recurrence relations:

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t), \; n = 1, 2, ..., \quad (9)$$

we choose the grid (interpolation) points to be the extrema:

$$t_k = \cos\left(\frac{k\pi}{N}\right), \; k = 0, 1, 2, ..., N \quad (10)$$

of the N^{th} order Chebyshev polynomial T_N *(t)*. These grids $t_N = -1 < t_{N-1} < ... < t_1 < t_0 = 1$ are also viewed as zeros of $\left(1 - t^2\right)\dot{T}(t)$, where $\dot{T}(t) = dT(t)/dt$.

Clenshaw and Curtis introduced the following approximation of the function y (t),

$$y(t) = \sum_{n=0}^{N}{}'' a_n T_n(t), \quad a_n = \frac{2}{N}\sum_{j=0}^{N}{}'' y(t_j) T_n(t_j). \quad (11)$$

The summation symbol with double primes denotes a sum with both the first and last terms halved. The first and second derivatives of the function *y(t)* at the point t_k are given by:

$$y^{(n)}(t_k)=\sum_{j=0}^{N} d_{k,j}^{(n)} y(t_j), \quad d_{k,j}^{(1)}=\frac{4\theta_j}{N}\sum_{n=0}^{N}\sum_{\substack{l=0 \\ (n+l)odd}}^{n-1}\frac{n\theta_n}{c_l}T_n(t_j)T_l(t_k),\ n=1,2,\ k,j=0,1,\ldots,N \tag{12}$$

$$d_{k,j}^{(2)}=\frac{2\theta_j}{N}\sum_{n=0}^{N}\sum_{\substack{l=0 \\ (n+l)even}}^{n-2}\frac{n(n^2-l^2)\theta_n}{c_l}T_n(t_j)T_l(t_k), \quad k,j=0,1,\ldots,N \tag{13}$$

with $\theta_0=\theta_N=1/2,\ \theta_j=1\ for\ j=1,2,\ldots,N-1, and\ c_0=2, c_i=1, for\ i\geq 1.$

As we see from Eq. (12), the first and second derivatives of the function y(t) at any point from Gauss-Lobatto nodes are expanded as linear combination of the values of the function at these points.

To show the efficiency of the ChFD method described above, we present some examples. Appropriate codes have been developed in Matlab and Engineering Equation Solver (EES) which are believe to be in a good agreement with the previous works.

Example 1:

Consider the following vibrational problem:

$$\min J=\int_0^1\left[y(x)+y'(x)-4e^{3x}\right]^2 dx, \tag{14}$$

with the given boundary conditions:

$$y(0)=1, \quad y(1)=e^3$$

The corresponding Euler-Lagrange equation is:

$$f=\left[y(x)+y'(x)-4e^{3x}\right]^2,\ \frac{\partial f}{\partial y}=2\left[y(x)+y'(x)-4e^{3x}\right],\ \frac{\partial f}{\partial y'}=2\left[y(x)+y'(x)-4e^{3x}\right]$$

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}=0\Rightarrow 2\left[y(x)+\cancel{y'(x)}-4e^{3x}\right]-2\left[\cancel{y'(x)}+y''(x)-12e^{3x}\right]=0$$

$$\Rightarrow \boxed{y''(x)-y(x)-8e^{3x}=0} \tag{15}$$

For exact solution of this problem we can have:

$$y_h(x)=Ae^{x}+Be^{-x},\ y_p(x)=Ce^{3x}.$$

After substituting the particular solution on the general equation and implementing of boundary conditions in overall solution, one can obtain:

$$A = B = 0, \ \ C = 1$$

So,

$$y_{exact}(x) = e^{3x}$$

In order to solve the Eq. (15) by ChFD method, since the Gauss-Lobatto nodes lie in the computational interval [-1, 1], in the first step of this method, the transformation $t = 2x - 1$ is used to change Eq. (15) and boundary conditions to the following form:

$$y' = \dot{y}t_x = 2\dot{y}, \ y'' = (2\dot{y})_x = 2\ddot{y}t_x = 4\ddot{y} \ \Rightarrow \ 4\ddot{y}(x) - y(x) - 8e^{3\left(\frac{t+1}{2}\right)} = 0,$$
$$y(-1) = 1, \ y(1) = e^3. \tag{16}$$

Substitute Eq. (12) into (16) and evaluate the result at the Gauss-Lobatto nodes t_k *for* $k = 1,...,N-1$. This gives:

$$4\sum_{j=0}^{N} d_{k,j}^{(2)} y(t_j) - y(t_k) - 8e^{3Z_k} = 0, \ k = 1,...,N-1, \tag{17}$$

where $d_{k,j}^{(2)}$ is given in Eq. (13) and $Z_k = \frac{t_k + 1}{2}$. For *k=0* and *k=N* by using the boundary conditions in (16) we have:

$$y(t_0) = e^3, \ y(t_N) = 1 \tag{18}$$

Eq. (16) gives N-1 nonlinear algebraic equations which can be solved for the unknown coefficients $y(t_1), y(t_2), ..., y(t_{N-1})$ by using Newton's method. Consequently, y(t) given in Eq. (11) can be calculated as follow:

Now, define the maximum errors for $y_N(t)$ as:

$$E_N = \|y_N(t) - y_{exact}(t)\|_\infty = \max\left\{|y_N(t) - y_{exact}(t)|\right\}, \ -1 \le t \le 1$$

where y_N is the computed results with N and y_{exact}(t) is the exact solution. The written code has been attached in appendix A based on the different *N*. In Table **1** we give the errors E_N for different values of *N*. From Table **1** we see the errors decrease rapidly as *N* increases.

Table 1. The maximum error of E_N for different values of N for Example 1 in present work.

N	4	6	8	10	12	14
E_N	1.99×10^{-2}	2.150×10^{-4}	1.227×10^{-6}	4.843×10^{-9}	1.411×10^{-11}	2.766×10^{-13}

Example 2:

Consider the following brachistochrone problem:

$$\min J=\int_0^1 \frac{1+y^2(x)}{y'^2(x)}dx\,,y(0)=0,\;\; y(1)=0.5 \tag{19}$$

The corresponding Euler-Lagrange equation is:

$$\begin{aligned}
& f=\frac{1+y^2(x)}{y'^2(x)},\ \frac{\partial f}{\partial y}=2\frac{y(x)}{y'^2(x)},\ \frac{\partial f}{\partial y'}=-2\frac{1+y^2(x)}{y'^3(x)}\ ,\ \frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}=0 \Rightarrow \\
& 2\frac{y(x)}{y'^2(x)}+2\left[\frac{2y(x)y'^2(x)-3y''(x)(1+y^2(x))}{y'^4(x)}\right]=0 \\
& \Rightarrow \boxed{y''(x)+y^2(x)y''(x)-y(x)y'^2(x)=0}
\end{aligned} \tag{20}$$

The exact solution of this problem is:

$$y_{exact}(x)=\sinh(0.481211825x) \tag{21}$$

Similar to example 1, the transformation *t=2x-1* is used to change Eq. (20) and boundary conditions to the following form:

$$\begin{aligned}
& y'=\dot{y}t_x=2\dot{y},\ y''=(2\dot{y})_x=2\ddot{y}t_x=4\ddot{y}\ \Rightarrow\ \ddot{y}(x)+y^2(x)\ddot{y}(x)-y(x)\dot{y}^2(x)=0, \\
& y(-1)=0,\ y(1)=0.5
\end{aligned} \tag{22}$$

By substituting Eqs. (12) and (13) into (22) and evaluating the result at the Gauss-Lobatto nodes $t_k\ for\ k=1,\ldots,N-1$, we have:

$$\left(1+y^2(t_k)\right)\sum_{j=0}^{N}d_{k,j}^{(2)}y(t_j)-y(t_k)\left(\sum_{j=0}^{N}d_{k,j}^{(1)}y(t_j)\right)^2=0,\ k=1,\ldots,N-1 \tag{23}$$

Using the boundary conditions (22) we have:

$$y(t_0)=0.5,\ y(t_N)=0 \tag{24}$$

Thus, by solving *N-1* nonlinear algebraic equations (23) the approximation solution can be found. The appropriate developed codes have been written in Matlab and Engineering Equation Solver (EES). In Table **2** we have reported the errors E_N for different values of *N*.

Table 2. The maximum error of E_N for different values of N for Example 2 for present work.

N	2	3	4	5	6
E_N	1.34×10^{-4}	7.607×10^{-6}	4.368×10^{-7}	1.34×10^{-9}	4.399×10^{-10}

Example 3:

Consider the problem of finding the extremals of the functional:

$$J(y(x),z(x))=\int_0^{\frac{\pi}{2}}\left[y'^2(x)+z'^2(x)+2y(x)z(x)\right]dx,\ y(0)=0,\ y(\frac{\pi}{2})=1,\ z(0)=0,\ z(\frac{\pi}{2})=-1, \tag{25}$$

the system of Euler's differential equations is of the form:

$$\begin{aligned}&f=y'^2(x)+z'^2(x)+2y(x)z(x),\ \frac{\partial f}{\partial y}=2z(x),\ \frac{\partial f}{\partial y'}=2y'(x),\frac{\partial f}{\partial z}=2y(x),\ \frac{\partial f}{\partial z'}=2z'(x)\\ &\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}=0\Rightarrow 2z(x)-2y''(x)=0\Rightarrow \boxed{z(x)-y''(x)=0}\\ &\frac{\partial f}{\partial z}-\frac{d}{dx}\frac{\partial f}{\partial z'}=0\Rightarrow 2y(x)-2z''(x)=0\Rightarrow \boxed{y(x)-z''(x)=0}\end{aligned} \tag{26}$$

The exact solutions of the problem are $y = sin(x)$ and $z = - sin(x)$. For this problem the transformation $t=\frac{4}{\pi}x-1$ is used to change Eqs. in (26) to the following form:

$$\begin{aligned}&z(t)-\frac{16}{\pi^2}y''(t)=0,\\ &y(t)-\frac{16}{\pi^2}z''(t)=0,\end{aligned} \tag{27}$$

We use Eqs. (12) and (13) to approximate $y(t)$ and $z(t)$. Now substituting $y(t)$ and $z(t)$ into Eq. (27) and evaluating the result at the Gauss–Lobatto nodes t_k we obtain:

$$\begin{cases}\frac{16}{\pi^2}\sum_{j=0}^{N}d_{k,j}^{(2)}y(t_j)-z(t_k)=0\\ \frac{16}{\pi^2}\sum_{j=0}^{N}d_{k,j}^{(2)}z(t_j)-y(t_k)=0\end{cases},\ k=1,\dots,N-1 \tag{28}$$

Also by using the boundary conditions in (25) we then have:

$$y(t_0)=1,\ y(t_N)=0 \qquad z(t_0)=-1,\ z(t_N)=0 \tag{29}$$

Again, by solving 2N-2 non-linear algebraic equations (28) for obtaining y(t) and z(t), the approximation solution can be found. Suppose E_N *and* E'_N be the maximum errors for $y_N(t)$ *and* $z_N(t)$, respectively. Table **3** shows E_N *and* E'_N for different values of N based on the present work.

Table 3. The maximum error of E_N and E'_N for different values of N for Example 3 for present work.

N	4	6	8	10	12
E_N	1.313×10^{-4}	3.116×10^{-7}	5.725×10^{-10}	4.962×10^{-12}	4.887×10^{-14}
E'_N	1.313×10^{-4}	3.116×10^{-7}	5.725×10^{-10}	4.962×10^{-12}	4.887×10^{-14}

As we can see for large value of N our errors decrease.

4. CONCLUSIONS

This article presented an efficient method for finding the minimum of a functional over the specified domain. The main objective was to find the solution of an ordinary differential equation which arises from the variational problem. This paper introduced Chebyshev finite difference method (ChFDM) to find approximation solution of ordinary differential equations which arise from problems of calculus of variations. Properties of the ChFD method are utilized to reduce the computation of this problem to some algebraic equations. The computational error had been calculated for each step which was so small, illustrating the accuracy of proposed method.

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