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Theorem on Fractional Calculus of Variations and Solution to Fractional Variational Problems via Rayleigh - Ritz method

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Abstract

In this article, the authors present a new theorem and method for solution of fractional variational problems. We used Rayleigh – Ritz method to solve certain fractional variational problems (FVP).Illustrative examples are provided.

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1. INTRODUCTION

Calculus of variation and fractional calculus have played a significant role in various areas of sciences, engineering, and applied mathematics. In last three decades, there has been significant progress in this field. This paper is concerned with a method of solution to certain fractional variational calculus. We present some new results for fractional calculus of variation. In this paper it is assumed that the fractional derivatives have been defined in the sense of the Caputo fractional derivatives.
We list a number of review articles and monographs where the background and many applications of fractional derivatives could be found and focus on calculus of variations. Interested readers may find back ground and literature review in [1–8].

1.1. Caputo Fractional Derivative

Let $[a,b]$ be a finite interval on the real axis. The left Caputo fractional Derivative (LCFD) ${}_a^C D_x^\alpha y$ and the right Caputo fractional derivative (RCFD) ${}_x^C D_b^\alpha y$ of order

$\alpha > 0$ are defined by

$$ {}_a^C D_x^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}} \, dt $$

$$ {}_x^C D_b^\alpha y(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)}{(t-x)^{\alpha-n+1}} \, dt $$

where $n-1 \le \alpha < n$, $n \in N$.

The left Riemann – Liouville fractional derivative (LRLFD) and the right Riemann – Liouville fractional derivative (RRLFD) of order $\alpha > 0$ are defined by

$$ {}_a D_x^\alpha y(x) = (D)^n \, {}_a I_x^{n-\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x \frac{y(t)}{(x-t)^{\alpha-n+1}} \, dt $$

$$ {}_x D_b^\alpha y(x) = (-D)^n \, {}_x I_b^{n-\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx} \right)^n \int_x^b \frac{y(t)}{(t-x)^{\alpha-n+1}} \, dt $$

where $n-1 \le \alpha < n$, $n \in N$.

Let us recall some useful theorems that we need in this paper

Theorem 1.1. Let $I(y)$ be a functional of the form

$$ I(y(x)) = \int_a^b F(x, y, {}_a D_x^\alpha y, {}_x D_b^\beta y) \, dx $$

defined on the set of functions $y(x)$ which have continuous LRLFD of order $\alpha > 0$ and RRLFD of order $\beta > 0$ in $[a,b]$ and satisfy the boundary conditions $y(a) = y_a$ and $y(b) = y_b$. Then a necessary condition for $I(y)$ to have an extremum for a given function $y(x)$ is that $y(x)$ satisfy following Euler – Lagrange equation:

$$ \frac{\partial F}{\partial y} + {}_x D_b^\alpha \left(\frac{\partial F}{\partial \, {}_a D_x^\alpha y} \right) + {}_a D_x^\beta \left(\frac{\partial F}{\partial \, {}_x D_b^\beta y} \right) = 0 $$

Proof. See [1]

Theorem 1.2. Let $I(y)$ be a functional of the form

$$ I(y(x)) = \int_a^b F(x, y, {}_a^C D_x^\alpha y, {}_x^C D_b^\beta y) \, dx $$

defined on the set of functions $y(x)$ which have continuous LCFD of order $\alpha > 0$ and RCFD of order $\beta > 0$ in $[a,b]$ and satisfies the boundary conditions $y(a) = y_a$ and $y(b) = y_b$. Then a necessary condition for $I(y)$ to have an

Extremum for a given function $y(x)$ is that $y(x)$ satisfy the following Euler – Lagrange equation:

$$\frac{\partial F}{\partial y}+{}_xD_b^{\alpha}\left(\frac{\partial F}{\partial\, {}_a^CD_x^{\alpha}y}\right)+{}_aD_x^{\beta}\left(\frac{\partial F}{\partial\, {}_x^CD_b^{\beta}y}\right)=0$$

Proof. See [1].

Variational Problems of Fractional Order with Two unknown Functions

Theorem 1.3. A necessary condition for the curve

$y_i=y_i(x)\ (i=1,2)$

which satisfies the boundary conditions

$y_i(a)=y_{ia}\ ,\ y_i(b)=y_{ib}\ \ (i=1,2)$

to be an extremal of the functional

$$I(y_1(x),y_2(x))=\int_a^b F(x,\,y_1,\,y_2,\,{}_aD_x^{\alpha}y_1,\,{}_aD_x^{\alpha}y_2)\,dx \qquad (0<\alpha\le 1)$$

Is that the functions $y_i(x)\ (i=1,2)$ satisfy the following Euler – Lagrange equation

$$\frac{\partial F}{\partial y_i}+{}_xD_b^{\alpha}\left(\frac{\partial F}{\partial\, {}_aD_x^{\alpha}y_i}\right)=0 \qquad (i=1,2)$$

Proof. See [1]

Theorem 1.4. A necessary condition for the curve

$y_i=y_i(x)\ (i=1,2)$

which satisfies the boundary conditions $y_i(a)=y_{ia}\ ,\ y_i(b)=y_{ib}\ \ (i=1,2)$ to be an extremal of the functional

$$I(y_1(x),y_2(x))=\int_a^b F(x,\,y_1,y_2,{}_a^CD_x^{\alpha}y_1,\,{}_a^CD_x^{\alpha}y_2)\,dx, \qquad (0<\alpha\le 1)$$

is that the functions $y_i(x)\ (i=1,2)$ satisfy the following Euler – Lagrange equation

$$\frac{\partial F}{\partial y_i}+{}_xD_b^{\alpha}\left(\frac{\partial F}{\partial\, {}_a^CD_x^{\alpha}y_i}\right)=0 \qquad (i=1,2)$$

Proof. See [1]

Lemma 1.5. In the discussion to follow, we will also need formulae for fractional integration by parts. These formulae are given as

$$\int_a^b f\ {}_a^CD_x^{\alpha}\eta\,dx=\int_a^b \eta\ {}_xD_b^{\alpha} f\ dx+\sum_{j=0}^{n-1}{}_xD_b^{\alpha+j-n} f\ \eta^{(n-1-j)}(x)\Bigg|_{x=a}^{x=b}$$

$$\int_a^b f\ {}_x^CD_b^{\alpha}\eta\,dx=\int_a^b \eta\ {}_aD_x^{\alpha} f\ dx+\sum_{j=0}^{n-1}(-1)^{n+j}\ {}_aD_x^{\alpha+j-n} f\ \eta^{(n-1-j)}(x)\Bigg|_{x=a}^{x=b}.$$

$n-1<\alpha\le n\ \ (n\in N)$ these formulae can be proven easily, where .

Example 1.6. Consider the following FVP

$$I(y(x), z(x)) = \int_0^1 {}_0D_x^{\frac{1}{2}} y \exp\left(f(x)\, {}_0D_x^{\frac{1}{2}} z \right) dx$$

where

$$f(x) = \begin{cases} 1 & : 0 \le x \le \frac{1}{2} \\ -1 & : \frac{1}{2} < x \le 1 \end{cases}$$

and boundary conditions are $y(0)=0$, $z(0)=0$, $y(1) = \frac{1}{\sqrt{\pi}}(-2\sqrt{2}+2)$, $z(1) = -\frac{2\ln 2}{\sqrt{\pi}}$.

The Euler – Lagrange equations are as following

$${}_xD_1^{\frac{1}{2}}\left(\exp\left(f(x)\, {}_0D_x^{\frac{1}{2}} z \right)\right) = 0,$$

$${}_xD_1^{\frac{1}{2}}\left(f(x)\, {}_0D_x^{\frac{1}{2}} y \exp\left(f(x)\, {}_0D_x^{\frac{1}{2}} z \right)\right) = 0,$$

i.applying the operator ${}_xI_1^{\alpha}$ on both sides of the above equations, we obtain

$$\exp\left(f(x)\, {}_0D_x^{\frac{1}{2}} z \right) = c(1-x)^{-\frac{1}{2}}$$

$$f(x)\, {}_0D_x^{\frac{1}{2}} y \exp\left(f(x)\, {}_0D_x^{\frac{1}{2}} z \right) = k(1-x)^{-\frac{1}{2}},$$

hence

$${}_0D_x^{\frac{1}{2}} z = -\frac{1}{2}\frac{\ln(1-x)}{f(x)} + \frac{c_0}{f(x)} \quad , \quad {}_0D_x^{\frac{1}{2}} y = \frac{k_0}{f(x)},$$

once again, applying the operator ${}_0I_x^{\alpha}$ on both sides of the above equations, we get

$$z(x) = C_1 x^{-\frac{1}{2}} + \frac{c_0}{\sqrt{\pi}} \int_0^x \frac{(x-t)^{-\frac{1}{2}}}{f(t)}\, dt - \frac{1}{2\sqrt{\pi}} \int_0^x \frac{(x-t)^{-\frac{1}{2}} \ln(1-t)}{f(t)}\, dt$$

$${}_0D_x^{\frac{1}{2}} y = k_1 x^{-\frac{1}{2}} + \frac{k_0}{\sqrt{\pi}} \int_0^x \frac{(x-t)^{-\frac{1}{2}}}{f(t)}\, dt$$

using the boundary conditions , leads to

$$z(x) = -\frac{1}{2\sqrt{\pi}} \int_0^x \frac{(x-t)^{-\frac{1}{2}} \ln(1-t)}{f(t)}\, dt$$

$${}_0D_x^{\frac{1}{2}} y = \frac{1}{\sqrt{\pi}} \int_0^x \frac{(x-t)^{-\frac{1}{2}}}{f(t)}\, dt.$$

Theorem 1.7 (Main theorem) Consider the functional [8]

$$I(y(x), z(x)) = \int_{x_1}^{x_2} F(x, y, z, {}_{x_1}^{C}D_x^{\alpha} y, {}_{x_1}^{C}D_x^{\alpha} z)\, dx \qquad (1.1)$$

where $0 < \alpha \le 1$. Let the point $A(x_1, y_1, z_1)$ corresponding to the lower limit in the above integral be fixed, and let the other point $B(x_2, y_2, z_2)$ moves along an arbitrary manner, along a given curve or surface. We want to derive transversality conditions in each case.

Proof. The extremum can be attained by Euler – Lagrange equations:

$$\frac{\partial F}{\partial y} + {}_xD_{x_2}^{\alpha}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} y}\right) = 0, \quad \frac{\partial F}{\partial z} + {}_xD_{x_2}^{\alpha}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} z}\right) = 0.$$

The general solution of these equations contains four arbitrary constants. Since the boundary point $A(x_1, y_1, z_1)$ is fixed, it is possible to eliminate two arbitrary constants. The other two constants have to be determined from the necessary condition $\delta I = 0$ for extremum, where δI is the variation of I, hence

$$\delta I = \int_{x_1}^{x_2+\delta x_2} F(x, y+\delta y, z+\delta z, {}_{x_1}^{C}D_x^{\alpha} y + {}_{x_1}^{C}D_x^{\alpha} \delta y, {}_{x_1}^{C}D_x^{\alpha} z + {}_{x_1}^{C}D_x^{\alpha} \delta z)\, dx$$
$$- \int_{x_1}^{x_2} F(x, y, z, {}_{x_1}^{C}D_x^{\alpha} y, {}_{x_1}^{C}D_x^{\alpha} z)\, dx$$
$$= \int_{x_1}^{x_2} \Big(F(x, y+\delta y, z+\delta z, {}_{x_1}^{C}D_x^{\alpha} y + {}_{x_1}^{C}D_x^{\alpha} \delta y, {}_{x_1}^{C}D_x^{\alpha} z + {}_{x_1}^{C}D_x^{\alpha} \delta z) - F \Big)\, dx$$
$$+ \int_{x_2}^{x_2+\delta x_2} F(x, y+\delta y, z+\delta z, {}_{x_1}^{C}D_x^{\alpha} y + {}_{x_1}^{C}D_x^{\alpha} \delta y, {}_{x_1}^{C}D_x^{\alpha} z + {}_{x_1}^{C}D_x^{\alpha} \delta z)\, dx = 0.$$

(1.2)

Using the mean value theorem, the second term on the right – hand side of the above expression can be written as

$$\int_{x_2}^{x_2+\delta x_2} F(x, y+\delta y, z+\delta z, {}_{x_1}^{C}D_x^{\alpha} y + {}_{x_1}^{C}D_x^{\alpha} \delta y, {}_{x_1}^{C}D_x^{\alpha} z + {}_{x_1}^{C}D_x^{\alpha} \delta z)\, dx$$
$$= F\Big|_{x_2+\theta\delta x_2} \delta x_2.$$

Where $0 < \theta < 1$. But, by virtue of the continuity of F, we may write

$$F\Big|_{x_2+\theta\delta x_2} = F\Big|_{x_2} + \varepsilon,$$

where ε is an infinitesimal such that $\varepsilon \to 0$ as $\delta x_2 \to 0$, $\delta y_2 \to 0$ and $\delta z_2 \to 0$. Thus

$$\int_{x_2}^{x_2+\delta x_2} F(x, y+\delta y, z+\delta z, {}_{x_1}^{C}D_x^{\alpha} y + {}_{x_1}^{C}D_x^{\alpha}\delta y, {}_{x_1}^{C}D_x^{\alpha} z + {}_{x_1}^{C}D_x^{\alpha}\delta z)\, dx$$

$$= F\Big|_{x_2} \delta x_2 + \varepsilon \delta x_2$$

using Taylor's theorem, we now transform the first term on the right side of (1.2) as relation

$$\int_{x_1}^{x_2} \Big(F(x, y+\delta y, z+\delta z, {}_{x_1}^{C}D_x^{\alpha} y + {}_{x_1}^{C}D_x^{\alpha}\delta y, {}_{x_1}^{C}D_x^{\alpha} z + {}_{x_1}^{C}D_x^{\alpha}\delta z) - F \Big) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} y}\, {}_{x_1}^{C}D_x^{\alpha}\delta y + \frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} z}\, {}_{x_1}^{C}D_x^{\alpha}\delta z \right) dx + R$$

where R is an infinitesimal of the order higher than that of δy, δz, ${}_{x_1}^{C}D_x^{\alpha}\delta y$ or ${}_{x_1}^{C}D_x^{\alpha}\delta z$. Therefore

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} y}\, {}_{x_1}^{C}D_x^{\alpha}\delta y + \frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} z}\, {}_{x_1}^{C}D_x^{\alpha}\delta z \right) dx$$

$$+ F\Big|_{x_2} \delta x_2 + \varepsilon \delta x_2 + R = 0.$$

Further, after fractional integration by parts and applying the previous lemma, we get

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial z}\delta z\, dx$$

$$+ \int_{x_1}^{x_2} {}_{x}D_{x_2}^{\alpha}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} y} \right)\delta y\, dx + {}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} y} \right)\delta y \Big|_{x_1}^{x_2}$$

$$+ \int_{x_1}^{x_2} {}_{x}D_{x_2}^{\alpha}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} z} \right)\delta z\, dx + {}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} z} \right)\delta z \Big|_{x_1}^{x_2} + F\Big|_{x_2}\delta x_2$$

$$= \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} + {}_{x}D_{x_2}^{\alpha}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} y} \right) \right\} \delta y\, dx + {}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} y} \right)\delta y \Big|_{x_1}^{x_2}$$

$$+ \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial z} + {}_{x}D_{x_2}^{\alpha}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} z} \right) \right\} \delta z\, dx + {}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha} z} \right)\delta z \Big|_{x_1}^{x_2}$$

$$+ F\Big|_{x_2}\delta x_2 = 0.$$

Since the value of the functional I are taken only on the extremals, it follows that

$$\frac{\partial F}{\partial y} + {}_x D_{x_2}^{\alpha}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, y} \right) = 0 \quad , \frac{\partial F}{\partial z} + {}_x D_{x_2}^{\alpha}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, z} \right) = 0$$

Note that $\delta y \Big|_{x_1} = \delta z \Big|_{x_1} = 0$. Hence

$$\delta I = F\Big|_{x_2} \delta x_2 + {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, y} \right)\delta y\Big|_{x_2} + {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, z} \right)\delta z\Big|_{x_2} = 0.$$

We can write

$$\delta y\Big|_{x_2} = \delta y_2 - y'(x_2)\delta x_2 \quad , \delta z\Big|_{x_2} = \delta z_2 - z'(x_2)\delta x_2$$

$\delta I = 0$ gives

$$\delta I = \left\{ F - y'\, {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, y} \right) - z'\, {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, z} \right) \right\}\Bigg|_{x_2} \delta x_2$$

$$+ {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, y} \right)\Bigg|_{x_2} \delta y_2 + {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, z} \right)\Bigg|_{x_2} \delta z_2 = 0 \qquad (1.3)$$

For an extremum, If the variations δx_2, δy_2 and δz_2 are independent, then (1.3) gives

$$F - y'\, {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, y} \right) - z'\, {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, z} \right)\Bigg|_{x_2} = 0$$

$${}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, y} \right)\Bigg|_{x_2} = 0$$

$${}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, z} \right)\Bigg|_{x_2} = 0.$$

If the boundary point $B(x_2, y_2, z_2)$ moves along some curve $y_2 = \phi(x_2)$, $z_2 = \psi(x_2)$ then $\delta y_2 = \phi'(x_2)\delta x_2$ and $\delta z_2 = \psi'(x_2)\delta x_2$. Thus from (1.3) we have

$$\left\{ F + (\phi' - y')\, {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, y} \right) + (\psi' - z')\, {}_x D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\, {}_{x_1}^{C}D_x^{\alpha}\, z} \right)\Bigg|_{x_2} \right\} \delta x_2 = 0.$$

Since δx_2 is arbitrary, it leads to

$$F+(\phi'-y')\,{}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\,{}_{x_1}^{C}D_x^{\alpha}y}\right)+(\psi'-z')\,{}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\,{}_{x_1}^{C}D_x^{\alpha}z}\right)\Bigg|_{x_2}=0. \qquad (1.4)$$

This is the transversality condition in the problem of extremum of (1.1), along with the equations $y_2=\phi(x_2)$, $z_2=\psi(x_2)$, the condition (1.4) gives the equations for determining the two arbitrary constants in the general solution of Euler's equations.
On the other hand, the boundary point B (x_2, y_2, z_2) moves along a given surface $z_2=\phi(x_2, y_2)$, then
$\delta z_2=\phi_{x_2}\,\delta x_2+\phi_{y_2}\,\delta y_2.$
Such that the variations δx_2 and δy_2 are arbitrary. In this case (3) reduced to

$$F-y'\,{}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\,{}_{x_1}^{C}D_x^{\alpha}y}\right)+(\phi_x-z')\,{}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\,{}_{x_1}^{C}D_x^{\alpha}z}\right)\Bigg|_{x_2}=0 \qquad (1.5)$$

$${}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\,{}_{x_1}^{C}D_x^{\alpha}y}\right)+\phi_y\,{}_{x}D_{x_2}^{\alpha-1}\left(\frac{\partial F}{\partial\,{}_{x_1}^{C}D_x^{\alpha}z}\right)\Bigg|_{x_2}=0. \qquad (1.6)$$

Example 1.8. Consider the following fractional variational problem
Minimize $\quad I(y(x), z(x))=2\int_0^b\left(\sqrt{{}_0^{C}D_x^{\alpha}y}+\sqrt{{}_0^{C}D_x^{\alpha}z}\right)dx.$
Where $0<\alpha\le 1$ with the boundary conditions $y(0)=0$, $z(0)=0$ and the boundary point (b, y_b, z_b) moves on the surface $\phi(x,y)=x+\dfrac{y}{2}$.
Solution. For this problem, the Euler's equations are as following

$${}_{x}D_{b}^{\alpha}\left(\frac{1}{\sqrt{{}_0^{C}D_x^{\alpha}y}}\right)=0,$$

$${}_{x}D_{b}^{\alpha}\left(\frac{1}{\sqrt{{}_0^{C}D_x^{\alpha}z}}\right)=0.$$

Hence

$$\frac{1}{\sqrt{{}_0^{C}D_x^{\alpha}y}}=d_1(b-x)^{\alpha-1}$$

$$\frac{1}{\sqrt{{}_0^{C}D_x^{\alpha}z}}=d_2(b-x)^{\alpha-1}.$$

It leads to

$$y(x)=\frac{1}{d_1^2\,\Gamma(\alpha)}\int_0^x(b-t)^{2-2\alpha}(x-t)^{\alpha-1}\,dt$$

$$z(x)=\frac{1}{d_2^2\,\Gamma(\alpha)}\int_0^x (b-t)^{2-2\alpha}(x-t)^{\alpha-1}\,dt.$$

By (1.5) and (1.6) the transversality conditions are as follows

$$2\left(\frac{(b-x)^{1-\alpha}}{d_1}+\frac{(b-x)^{1-\alpha}}{d_2}\right)-y'\,{}_xD_b^{\alpha-1}\left(d_1(b-x)^{\alpha-1}\right)$$

$$+(1-z')\,{}_xD_b^{\alpha-1}\left(d_2(b-x)^{\alpha-1}\right)\Big|_{x=b}=0$$

$${}_xD_b^{\alpha-1}\left(d_1(b-x)^{\alpha-1}\right)+\frac{1}{2}\,{}_xD_b^{\alpha-1}\left(d_2(b-x)^{\alpha-1}\right)\Big|_{x=b}=0.$$

Therefore, it implies that

$$y'(b)=\frac{d_2}{d_1}(1-z'(b)) \qquad (1.7)$$

$$d_2=-2\,d_1 \qquad (1.8)$$

but we have

$$y'(b)=-\frac{b^{1-\alpha}}{d_1^2\,\Gamma(\alpha)}$$

$$z'(b)=-\frac{b^{1-\alpha}}{d_2^2\,\Gamma(\alpha)}$$

by replacing these expressions in (1.7) we get

$$d_2^2\,\Gamma(\alpha)=-\left(\frac{d_2}{d_1}+1\right)b^{1-\alpha}.$$

Now by taking into account (1.8) we will have

$$d_1^2\,\Gamma(\alpha)=\frac{1}{4}\,b^{1-\alpha}$$

$$d_2^2\,\Gamma(\alpha)=b^{1-\alpha}$$

then

$$y(x)=4\,b^{\alpha-1}\int_0^x (b-t)^{2-2\alpha}(x-t)^{\alpha-1}\,dt$$

$$z(x)=b^{\alpha-1}\int_0^x (b-t)^{2-2\alpha}(x-t)^{\alpha-1}\,dt$$

minimize I

Example1.9. Let us consider the following FVP and boundary conditions $y(0)=0$, $y(1)=1$;

$$I(y(x))=\int_0^1 g(x)\sqrt{f^2(x)+\left({}_0^CD_x^\alpha y\right)^2}\,dx \qquad (0<\alpha\le 1)$$

Solution. Using the theorem 1.2, we have the Euler – Lagrange equation as

$$ {}_xD_1^{\alpha}\left(\frac{g(x)\,{}_0^CD_x^{\alpha}y(x)}{\sqrt{f^2(x)+\left({}_0^CD_x^{\alpha}y(x)\right)^2}}\right)=0. $$

ii.Applying the operator ${}_xI_1^{\alpha}$ on both sides of the above equation, we obtain

$$ \frac{g\,{}_0^CD_x^{\alpha}y}{\sqrt{f^2+\left({}_0^CD_x^{\alpha}y\right)^2}}=C(1-x)^{\alpha-1}, $$

this implies that

$$ {}_0^CD_x^{\alpha}y(x)=\frac{C(1-x)^{\alpha-1}f(x)}{\sqrt{g^2(x)-C^2(1-x)^{2\alpha-2}}}\ . $$

Applying the operator ${}_0I_x^{\alpha}$ on both sides of the above equation, we get

$$ y(x)=C\,{}_0I_x^{\alpha}\left(\frac{(1-x)^{\alpha-1}f(x)}{\sqrt{g^2(x)-C^2(1-x)^{2\alpha-2}}}\right) $$

hence

$$ y(x)=\frac{C}{\Gamma(\alpha)}\int_0^x\frac{f(t)(1-t)^{\alpha-1}(x-t)^{\alpha-1}}{\sqrt{g^2(t)-C^2(1-t)^{2\alpha-2}}}\,dt. $$

At this point, using the boundary condition $y(1)=1$, C will be obtained.

Example 1.10. Let us solve the FVP with boundary conditions $y(0)=0$, $y(1)=1$.

$$ I(y(x))=\int_0^1\left(\frac{g(x)+{}_0^CD_x^{\alpha}y(x)}{f(x)+{}_0^CD_x^{\alpha}y(x)}-(1-x)^{-\alpha}y\right)dx \qquad (0<\alpha\le 1) $$

Solution. The Euler – Lagrange equation can be written as the following,

$$ {}_xD_1^{\alpha}\left(\frac{f(x)-g(x)}{\left(f(x)+{}_0^CD_x^{\alpha}y(x)\right)^2}\right)=(1-x)^{-\alpha} $$

by applying the operator ${}_xI_1^{\alpha}$ on both sides of the above relation, one gets

$$ \frac{f(x)-g(x)}{\left(f(x)+{}_0^CD_x^{\alpha}y(x)\right)^2}=C(1-x)^{\alpha-1}+{}_xI_1^{\alpha}(1-x)^{-\alpha}=C(1-x)^{\alpha-1}+\Gamma(1-\alpha) $$

then

$${}_0^C D_x^{\alpha}\, y(x) = \sqrt{\frac{f(x)-g(x)}{C(1-x)^{\alpha-1}+\Gamma(1-\alpha)}} - f(x),$$

now by applying the operator ${}_0 I_x^{\alpha}$ on both sides of the above equation we get;

$$y(x) = {}_0 I_x^{\alpha}\left(\sqrt{\frac{f(x)-g(x)}{C(1-x)^{\alpha-1}+\Gamma(1-\alpha)}} - f(x) \right)$$

then

$$y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\sqrt{\frac{f(t)-g(t)}{C(1-t)^{\alpha-1}+\Gamma(1-\alpha)}} - f(t) \right)(x-t)^{\alpha-1}\, dt\,,$$

by the boundary condition $y(1)=1$, C will be obtained.

2. MAIN RESULT

In this section, the authors used Rayleigh – Ritz method in order to establish approximate

solution to certain fractional variational problem.

2.1. Rayleigh – Ritz Method for estimation of exact solution of FVP

Let us consider the following functional

$$I(y(x)) = \int_a^b F(x, y, {}_a D_x^{\alpha} y, {}_x D_b^{\beta} y)\, dx = extremum$$

where $y(a) = y_a$, $y(b) = y_b$. This method gives an approximate solution of the above

problem. This approximate solution is presented by linear combination of linearly

independent functions, namely

$$\bar{y}(x) = \alpha_0 b_0(x) + \alpha_1 b_1(x) + \cdots + \alpha_n b_n(x),$$

such that $\bar{y}(x)$ is satisfied in boundary conditions and α_i 's are unknown coefficients.

By setting $\bar{y}(x)$ into I , we have

$$I(\bar{y}) = \int_a^b F(x, \bar{y}, {}_a D_x^{\alpha} \bar{y}, {}_x D_b^{\beta} \bar{y})\, dx = extremum$$

where $\bar{y}(a)=y_a$, $\bar{y}(b)=y_b$. In order to obtain the desired extremum, we must have

$$\frac{\partial I(\bar{y})}{\partial \alpha_i}=0 \qquad (i=0,1,\ldots,n)$$

Problem 2.1. Let us consider the following FVP

$$I(y(x))=\frac{1}{2}\int_0^1 \left\{ \left({}_0D_x^{\frac{1}{2}} y(x) \right)^2 - \frac{y(x)}{\sqrt{1-x}} \right\} dx \text{ ,}$$

such that $y(0)=0,\ y(1)=1$.

Then, one gets the following approximate solution (third approximation).

$$u(x)=2.97x-5.2x^2+5.04x^3-1.81x^4.$$

Solution. First we obtain the exact analytical solution of this problem.

The Euler – Lagrange equation of this problem is written as

$${}_xD_1^{\frac{1}{2}} \left({}_0D_x^{\frac{1}{2}} y(x) \right) - \frac{1}{2\sqrt{1-x}}=0,$$

so that

$${}_0D_x^{\frac{1}{2}} y(x)=C_1(1-x)^{-\frac{1}{2}}+\frac{1}{2}\,{}_xI_1^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}=C_1(1-x)^{-\frac{1}{2}}+\frac{\sqrt{\pi}}{2},$$

since $y(0)=0$, we have $C_0=0$. But $y(1)=1$, hence $C_1=0$. It follows that $y(x)=\sqrt{x}$.

At this point, we approximate the solution by Rayleigh – Ritz method. For the first approximation, let us assume that

$$z(x)=x+C\,x(1-x)=(C+1)x-Cx^2,$$

such that $z(0)=0,\ z(1)=1$. Therefore

$${}_0D_x^{\frac{1}{2}} z(x)=(C+1)\frac{x^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}-C\frac{\Gamma(3)\,x^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}=\frac{2(C+1)}{\sqrt{\pi}}\sqrt{x}-\frac{8C}{3\sqrt{\pi}}\sqrt{x^3},$$

this implies that

$$I(z(x)) == \frac{1}{\pi}(C+1)^2 + \frac{8}{9\pi}C^2 - \frac{16}{9\pi}(C^2+C) - \frac{2}{3}(C+1) - \frac{8}{15}C$$

by setting

$$I(C) = \frac{1}{\pi}(C+1)^2 + \frac{8}{9\pi}C^2 - \frac{16}{9\pi}(C^2+C) - \frac{2}{3}(C+1) - \frac{8}{15}C$$

In order to calculate the extremum, we must have

$$\frac{dI(C)}{dC} = 0$$

therefore

$$\frac{dI(C)}{dC} = \frac{2}{\pi}(C+1) + \frac{16}{9\pi}C - \frac{16}{9\pi}(2C+1) - \frac{2}{3} + \frac{8}{15} = 0$$

from the above equation, we obtain $C \approx 0/885$. Therefore, we get

$$z(x) = 1.885x - 0.885x^2$$

in order to obtain a better approximation, we write the second approximation as

following;

$$t(x) = x + \alpha x(1-x) + \beta x^2(1-x) = (\alpha+1)x + (\beta-\alpha)x^2 - \beta x^3$$

this leads to

$${}_0D_x^{\frac{1}{2}}\, t(x) = \frac{2(\alpha+1)}{\sqrt{\pi}}\sqrt{x} + \frac{8(\beta-\alpha)}{3\sqrt{\pi}}\sqrt{x^3} - \frac{16\beta}{5\sqrt{\pi}}\sqrt{x^5}$$

putting $t(x)$ into I, we get

$$I(\alpha,\beta) = \frac{1}{\pi}(\alpha+1)^2 + \frac{8}{9\pi}(\beta-\alpha)^2 + \frac{64}{75\pi}(\alpha+1)(\beta-\alpha)$$

$$-\frac{8}{5\pi}\beta(\alpha+1) - \frac{128}{75\pi}\beta(\beta-\alpha) - \frac{2}{3}(\alpha+1) - \frac{8}{15}(\beta-\alpha) + \frac{16}{35}\beta,$$

we must have

$$\begin{cases} \dfrac{\partial I(\alpha,\beta)}{\partial \alpha} = 0 \\ \dfrac{\partial I(\alpha,\beta)}{\partial \beta} = 0 \end{cases},$$

the equation $\dfrac{\partial I(\alpha,\beta)}{\partial \alpha} = 0$ implies that

$$\frac{2}{9\pi}\alpha + \frac{8}{75\pi}\beta = \frac{6\pi-10}{45\pi}$$

then

$$0.11\alpha + 0.053\beta = 0.1$$

moreover, by setting $\frac{\partial I(\alpha,\beta)}{\partial\beta}=0$ one has the following

$$\frac{8}{75\pi}\alpha+\frac{16}{225\pi}\beta=\frac{56-24\pi}{315}$$

equivalently,

$$0.034\alpha+0.023\beta=0.02$$

then, one gets $\alpha\approx 1.70$, $\beta\approx -1.64$. So that

$$t(x)=2.7x-3.34x^2+1.64x^3,$$

now, the third approximation can be written as

$$\begin{aligned}u(x)&=x+\alpha x(1-x)+\beta x^2(1-x)+\gamma x^3(1-x)\\&=(\alpha+1)x+(\beta-\alpha)x^2+(\gamma-\beta)x^3-\gamma x^4\end{aligned}$$

we have

$${}_0D_x^{\frac{1}{2}}u(x)=\frac{2(\alpha+1)}{\sqrt{\pi}}\sqrt{x}+\frac{8(\beta-\alpha)}{3\sqrt{\pi}}\sqrt{x^3}+\frac{48(\gamma-\beta)}{15\sqrt{\pi}}\sqrt{x^5}-\frac{384\gamma}{105\sqrt{\pi}}\sqrt{x^7}$$

by setting $u(x)$ into I, we get the following;

$$\begin{aligned}I(\alpha,\beta,\gamma)=&\frac{1}{\pi}(\alpha+1)^2+\frac{8}{9\pi}(\beta-\alpha)^2+\frac{192}{225\pi}(\gamma-\beta)^2+\frac{9216}{11025\pi}\gamma^2\\&+\frac{16}{9\pi}(\alpha+1)(\beta-\alpha)+\frac{24}{15\pi}(\alpha+1)(\gamma-\beta)-\frac{768}{525\pi}\gamma(\alpha+1)\\&+\frac{384}{225\pi}(\beta-\alpha)(\gamma-\beta)-\frac{512}{315\pi}\gamma(\beta-\alpha)-\frac{18432}{11025\pi}\gamma(\gamma-\beta)\\&-\frac{2}{3}(\alpha+1)-\frac{8}{15}(\beta-\alpha)-\frac{16}{35}(\gamma-\beta)+\frac{128}{315}\gamma,\end{aligned}$$

on the other hand

$$\begin{cases}\dfrac{\partial I(\alpha,\beta,\gamma)}{\partial\alpha}=0\\[2mm]\dfrac{\partial I(\alpha,\beta,\gamma)}{\partial\beta}=0\\[2mm]\dfrac{\partial I(\alpha,\beta,\gamma)}{\partial\gamma}=0\end{cases}$$

it follows that

$$\begin{cases} 0.071\alpha + 0.034\beta + 0.018\gamma = 0.063 \\ 0.034\alpha + 0.023\beta + 0.015\gamma = 0.02 \\ 0.018\alpha + 0.015\beta + 0.011\gamma = 0.007 \end{cases},$$

the above system of equations has the following solution
$\alpha \approx 1.97$, $\beta \approx -3.23$, $\gamma \approx 1.81$,
therefore, we get the following solution

$$u(x) = 2.97x - 5.2x^2 + 5.04x^3 - 1.81x^4.$$

Note. The third approximation is very close to exact solution.

3. CONCLUSION

In this article, the Rayleigh – Ritz method was implemented to solve some variational problems. It may be concluded that the Rayleigh – Ritz method is powerful tool in finding approximate solutions for ordinary and partial fractional differential equations. This method could lead to a promising approach for many applications in applied sciences.

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