Seminar 11

1. Calc. derivatele parțiale de ordinul 1, gradientul ∇f și diferențiala df pentru:

a) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2y^3 + y\sin x - 2z$

$$\frac{\partial f}{\partial x} = 2xy^3 + y\cos x \quad \text{(derivata pt. x)}$$

(derond f / derond x)

$$\frac{\partial f}{\partial y} = 3x^2y^2 + \sin x \quad \text{(derivata pt. y)}$$

$$\frac{\partial f}{\partial z} = -2 \quad \text{(derivata pt. z)}$$

$$df(x,y,z): \mathbb{R}^3 \to \mathbb{R}$$

$$df(x,y,z)(u_1,u_2,u_3) = \frac{\partial f}{\partial x}(x,y,z)\cdot u_1 + \frac{\partial f}{\partial y}(x,y,z)\cdot u_2 + \frac{\partial f}{\partial z}(x,y,z)\cdot u_3$$

b) $$\frac{\partial f}{\partial x}(x,y) = \frac{1}{\left(\frac{x-y}{x+y}\right)^2+1}\cdot\left(\frac{x-y}{x+y}\right)'_x = \frac{1}{\frac{(x-y)^2+(x+y)^2}{(x+y)^2}}\cdot\frac{x+y-(x-y)}{(x+y)^2}$$

$$= \frac{\cancel{(x+y)^2}}{(x-y)^2+(x+y)^2}\cdot\frac{2y}{\cancel{(x+y)^2}} = \frac{2y}{2x^2+2y^2} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{(x+y)^2}{(x-y)^2+(x+y)^2}\cdot\left(\frac{x-y}{x+y}\right)' = \frac{(x+y)^2}{(x-y)^2+(x+y)^2}\cdot\frac{-(x+y)-(x-y)}{(x+y)^2}$$

$$= \frac{-2x}{2x^2+2y^2} = \frac{-x}{x^2+y^2}$$

c) $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = x\sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x}(x,y) = \left(x\sqrt{x^2+y^2}\right)'_x = \sqrt{x^2+y^2} + x \cdot \frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\left(\sqrt{x^2+1}\right)' = \frac{x}{\sqrt{x^2+1}}$$

$$\frac{\partial f}{\partial y}(x,y) = \left(x\sqrt{x^2+y^2}\right)'_y = x \frac{y}{\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{x\sqrt{x^2}-0}{x} =$$

$$= \lim_{x\to 0} |x| = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{0}{y} = 0$$

2. Arătați că funcția $f(x,y) = y\ln(x^2-y^2)$ verifică relația

$$\frac{1}{x}\cdot\frac{\partial f}{\partial x}(x,y) + \frac{1}{y}\cdot\frac{\partial f}{\partial y}(x,y) = \frac{1}{y^2} f(x,y),\ \forall x>y>0$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{2xy}{x^2-y^2}$$

$$\Rightarrow \frac{1}{x}\cdot\frac{2xy}{x^2-y^2} + \frac{1}{y}\left[\ln(x^2-y^2) - \frac{2y^2}{x^2-y^2}\right]$$

$$\frac{\partial f}{\partial y}(x,y) = \ln(x^2-y^2) + y\cdot\frac{-2y}{x^2-y^2}$$

3. Studiați existența derivatelor parțiale în origine și a derivatelor după direcție în origine pentru

$$f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=\begin{cases}\dfrac{x^2\cdot y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{\frac{0}{y^2}-0}{x}=0$$

analog pt y $\lim \ldots = 0$

fie $v=(v_1,v_2)\in\mathbb{R}^2$, $f_v'(0,0)=\lim\limits_{t\to 0}\frac{1}{t}\cdot[f((0,0)+t(v_1,v_2))-f(0,0)]$

$$=\lim_{t\to 0}\frac{1}{t}[f(tv_1,tv_2)-f(0,0)]=\lim_{t\to 0}\frac{1}{t}\left[\frac{t^2v_1^2\cdot tv_2}{t^2v_1^2+t^2v_2^2}-0\right]=$$

$$=\lim_{t\to 0}\frac{v_1^2v_2}{t^2v_1^4+v_2^2}=\begin{cases}\dfrac{v_1^2}{v_2}, & v_2\neq 0\\ 0, & v_2=0\end{cases}$$

$\Rightarrow$ f derivabilă după direcția lui v în $(0,0)$

În part, dacă $v=e^1=(1,0)\Rightarrow f'_{e_1}(0,0)=0$ și $v=e^2=(0,1)\Rightarrow$

$\Rightarrow f'_{e^2}(0,0)=0$

4. Calc. derivatele partiale ale functii compuse $g \circ f$, unde $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (xe^y + xe^{-y}, xe^y - xe^{-y})$

$g = g(u,v) : \mathbb{R}^2 \to \mathbb{R}$ este o fctie oarecare de clasă C_1 pe $\mathbb{R}^2$

$$\underset{\text{(gradientul)}}{\nabla (g \circ f)}(x,y) = \nabla g(f(x,y)) \cdot J(f)(x,y)$$

$$\left(\frac{\partial}{\partial x}(g \circ f)(x,y), \frac{\partial}{\partial y}(g \circ f)(x,y)\right) = \left(\frac{\partial g}{\partial u}(f(x,y)), \frac{\partial g}{\partial v}(f(x,y))\right) \cdot \begin{pmatrix} e^y + e^{-y} & x \cdot e^y - x \cdot e^{-y} \\ e^y - e^{-y} & x \cdot e^y + x \cdot e^{-y} \end{pmatrix} \Rightarrow$$

$$J(f)(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} e^y + e^{-y} & x \cdot e^y - x \cdot e^{-y} \\ e^y - e^{-y} & x \cdot e^y + x \cdot e^{-y} \end{pmatrix}$$

$$\Rightarrow \frac{\partial (g \circ f)}{\partial x}(x,y) = \frac{\partial g}{\partial u}(f(x,y)) \cdot (e^y + e^{-y}) + \frac{\partial g}{\partial v}(f(x,y)) \cdot (e^y - e^{-y})$$

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5. Exprimati ecuatia:

$$u \cdot \frac{\partial g}{\partial u}(u,v) + v \cdot \frac{\partial g}{\partial v}(u,v) = \sqrt{u^2 + v^2}, \forall (u,v) \in (0, +\infty)^2$$

în variabilele $(x,y) \in (0, +\infty) \times (0, \frac{\pi}{2})$, efectuând transformarea $u = x \cos y$, $v = x \sin y$. Determinati apoi o functie g de clasă C_1 ce verifică relatia respectivă

$$\Rightarrow x \cdot \cos y \cdot \frac{\partial g}{\partial u}(x\cos y, x\sin y) + x\sin y \cdot \frac{\partial g}{\partial v}(x\cos y, x\sin y) =$$

$$= x$$

$$\sqrt{u^2+v^2} = \sqrt{x^2(\cos^2 y+\sin^2 y)} = \sqrt{x^2} = x \quad (x>0)$$

fie $f(x,y) = (x\cos y, x\sin y)$, $G = g \circ f$

$$\frac{\partial G}{\partial x}(x,y) = \frac{\partial g}{\partial u}(f(x,y)) \cdot \frac{\partial f_1}{\partial x}(x,y) + \frac{\partial g}{\partial v}(f(x,y)) \cdot \frac{\partial f_2}{\partial x}(x,y)$$

$$= \frac{\partial g}{\partial u}(x\cos y, x\sin y) \cdot \cos y + \frac{\partial g}{\partial v}(x\cos y, x\sin y) \cdot \sin y$$

$$= 1 \quad \Rightarrow \frac{\partial G}{\partial x}(x,y) = 1, \quad \forall (x,y) \in A$$

$$\Rightarrow G(x,y) = \int 1\,dx = x + C(y)$$

$$\Rightarrow g(f(x,y)) = x + C(y)$$

$$g(x\cos y, x\sin y) = x + C(y)$$

$$\begin{cases} u = x\cos y \\ v = x\sin y \end{cases}$$

$$\Rightarrow g(u,v) = \sqrt{u^2+v^2} + C\left(\operatorname{arctg} \frac{v}{u}\right)$$

6. Calc. derivatele de ord. 2 ale fcțiilor

a) $\frac{\partial f}{\partial x} = \frac{1}{x+y^2-1}$

$$\frac{\partial f}{\partial y} = \frac{2y}{x+y^2-1}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \left(\frac{1}{x+y^2-1}\right)'_x = \frac{-1}{(x+y^2-1)^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \left(\frac{2y}{x+y^2-1}\right)' = \frac{2(x+y^2-1)-2y\cdot 2y}{(x+y^2-1)^2} = \frac{2x-2y^2-2}{(x+y^2-1)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \left(\frac{2y}{x+y^2-1}\right)'_x = \frac{-2y}{(x+y^2-1)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \left(\frac{1}{x+y^2-1}\right)'_y = \frac{-1}{(x+y^2-1)^2}\cdot(2y) = \frac{-2y}{(x+y^2-1)^2}$$

b) $\frac{\partial f}{\partial x} = y\cdot e^{\frac{x}{y}} + x\cdot y\cdot e^{\frac{x}{y}}\cdot\frac{1}{y}$

$$= e^{\frac{x}{y}}(x+y)$$

$$\frac{\partial f}{\partial y} = x\cdot e^{\frac{x}{y}} + x\cdot y\cdot e^{\frac{x}{y}}\cdot\left(-\frac{x}{y^2}\right) = x\cdot e^{\frac{x}{y}}\cdot\left(1-\frac{x}{y}\right)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = \left[e^{\frac{x}{y}}(x+y)\right]'_x =$$

$$= e^{\frac{x}{y}}\cdot\frac{1}{y}(x+y) + e^{\frac{x}{y}}\cdot 1 = e^{\frac{x}{y}}\left[\frac{x}{y}+2\right]$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = \left[x\cdot e^{\frac{x}{y}}\cdot\left(1-\frac{x}{y}\right)\right]'_y =$$

$$= x\cdot e^{\frac{x}{y}}\cdot\frac{x}{y^2} + x\cdot e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)\left(1-\frac{x}{y}\right) = \frac{x}{y^2}\cdot e^{\frac{x}{y}}\cdot\frac{x^2}{y}$$

$$= \frac{x}{y^3}\cdot e^{\frac{x}{y}}$$

Temă