Seminar 11

1. Calculati derivatele partiale de ordinul 1, gradientul ∇f si diferentiala $\mathrm{d}f$ pentru functiile

a) $f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = x^2y^3 + y\sin x - 2z$

b) $f : (0,\infty)^2 \to \mathbb{R}, \quad f(x,y) = \operatorname{arctg}\frac{x-y}{x+y}$

c) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = x\sqrt{x^2+y^2}$

2. Aratati ca functia $f(x,y) = y\ln(x^2 - y^2)$ verifica relatia

$$\frac{1}{x}\frac{\partial f}{\partial x} + \frac{1}{y}\frac{\partial f}{\partial y} = \frac{f}{y^2}, \quad \forall x > y > 0$$

3. Studiati existenta derivatelor partiale in origine si a derivatelor dupa directie in origine pentru

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

4. Calculati derivatele partiale ale functiei compuse $g \circ f$, unde $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (xe^y + xe^{-y}, xe^y - xe^{-y})$ si $g = g(u,v) : \mathbb{R}^2 \to \mathbb{R}$ este o functie oarecare de clasa C^1 pe $\mathbb{R}^2$.

5. Exprimati ecuatia

$$u\frac{\partial g}{\partial u}(u,v) + v\frac{\partial g}{\partial v}(u,v) = \sqrt{u^2+v^2}, \quad \forall (u,v) \in (0,\infty)^2$$

in variabilele $(x,y) \in (0,\infty) \times (0,\pi/2)$, efectuand transformarea $u = x\cos y$, $v = x\sin y$. Determinati apoi o functie g de clasa C^1 ce verifica relatia respectiva.

6. Calculati derivatele partiale de ordinul 2 ale functiilor

a) $f : (1,\infty) \times \mathbb{R} \to \mathbb{R}, \quad f(x,y) = \ln(x + y^2 - 1)$

b) $f : \mathbb{R} \times (0,\infty) \to \mathbb{R}, \quad f(x,y) = xy\,\mathrm{e}^{\frac{x}{y}}$

Exercitii suplimentare

1. Calculati derivatele partiale de ordinul 1, gradientul ∇f si diferentiala $\mathrm{d}f$ pentru functiile
a) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \sin^2(x^3 + y)$
b) $f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = (x+y+z)\mathrm{e}^{x^2+y^2+z^2}$

2. Calculati matricea Jacobi $J(f)$ in punctul $(1,1)$ pentru urmatoarele functii vectoriale
a) $f : \mathbb{R}^2 \to \mathbb{R}^3, \quad f(x,y) = (x^2 - y, 3x - 2y, 2xy + y^2)$
b) $f : (0,\infty)^2 \to \mathbb{R}^2, \quad f(x,y) = (\frac{1}{xy}, \operatorname{arctg} \frac{y}{x})$

Functii omogene. Fie $p \in \mathbb{R}$. O functie $f : (0,\infty)^m \to \mathbb{R}$ se numeste omogena (de grad p) daca $f(tx) = t^p f(x)$, $\forall x \in (0,\infty)^m$ si $\forall t > 0$.

3. Aratati ca functia $f : (0,\infty)^3 \to \mathbb{R}$, $f(x,y,z) = \frac{1}{x+y+z}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$ este omogena (de un anumit grad) si justificati egalitatea

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = -2f, \quad \forall (x,y,z) \in (0,\infty)^3$$

4. Aratati ca functia de mai jos nu este continua in $(0,0)$, dar admite derivate dupa orice directie in acest punct

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} \frac{xy}{x+y} & , x+y \neq 0 \\ 0 & , x+y = 0 \end{cases}$$

5. Calculati derivatele partiale ale functiei compuse $g \circ f$, unde $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x^2 - y, 3x - 2y, 2xy + y^2)$ si $g = g(u,v,w) : \mathbb{R}^3 \to \mathbb{R}$ este o functie oarecare de clasa C^1 pe $\mathbb{R}^3$.

6. Exprimati ecuatia

$$v\frac{\partial g}{\partial u}(u,v) - u\frac{\partial g}{\partial v}(u,v) = 1, \quad \forall (u,v) \in (0,\infty)^2$$

in variabilele $(x,y) \in (0,\infty) \times (0,\pi/2)$, efectuand transformarea $u = x\cos y$, $v = x\sin y$. Determinati apoi o functie g de clasa C^1 ce verifica relatia respectiva.

7. Aratati ca functia $f(x,y) = (x^2 + y^2)\operatorname{arctg} \frac{y}{x}$ verifica relatia

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = 2f, \quad \forall (x,y) \in (0,\infty)^2$$