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grupa 214

Lucrare scrisă la analiză matematică sesiunea iarnă 2021

1. Determinați mulțimea de convergență a seriei de puteri

$$\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)^n (x-2)^n,\ x\in\mathbb{R}$$

2. Determinați punctele critice și cele de extrem local (specificând tipul acestora) pentru funcția

$f:\mathbb{R}^3\to\mathbb{R}$ $f(x,y,z)=x^3-x+y^2+z^2$

3. Se dă funcția:

$$f(x,y)=\arcsin\frac{y}{\sqrt{x^2+y^2}}.$$

Det. $\alpha\in\mathbb{R}$ a.î.:

$$\frac{\partial^2 f}{\partial x^2}(x,y)=\alpha\frac{\partial^2 f}{\partial y^2}(x,y),\ \forall (x,y)\in(0,\infty)^2$$

4. Studiați natura integralei improprii în funcție de valorile lui $\alpha\in\mathbb{R}$.

$$\mathcal{I}(\alpha)=\int_0^1\frac{x^\alpha}{(e^{\sqrt{x}}-1)^2}\,dx$$

și calculați $\mathcal{I}(-\frac{1}{2})$.

1.

$$a_n = \left(\frac{n+1}{n+2}\right)^n \quad x_0 = 2$$

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\left(\frac{n+1}{n+2}\right)^n \cdot \left(\frac{n+3}{n+2}\right)^{n+1}\right|$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)(n+3)}{(n+2)^2}\right)^n \cdot \frac{n+3}{n+2} = \lim_{n\to\infty} \frac{n+3}{n+2} \cdot \lim_{n\to\infty} \left(\frac{n^2+4n+3}{(n+2)^2}\right)^n =$$

$$= (1^\infty) = 1 \cdot \lim_{n\to\infty} \left(1 + \frac{n^2+4n+3}{n^2+4n+4} - 1\right)^n =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{-1}{n^2+4n+4}\right)^{-(n^2+4n+4)}\right]^{-\frac{n}{n^2+4n+4}} =$$

$$= e^{\lim_{n\to\infty} -\frac{n}{n^2+4n+4}} = e^0 = 1$$

$$\cdot\ (x_0 - r, x_0 + r) \subseteq I \subseteq [x_0 - r, x_0 + r]$$

$$\Rightarrow (1, 3) \subseteq I \subseteq [1, 3]$$

pt. $x = 1$: $\sum_{n=0}^{\infty} \underbrace{\left(\frac{n+1}{n+2}\right)^n \cdot (-1)^n}_{x_n}$

$$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\left(\frac{n+1}{n+2}\right)^n \cdot (-1)^n}{\left(\frac{n+2}{n+3}\right)^{n+1} \cdot (-1)^{n+1}} =$$

$$= -\lim_{n\to\infty} \left(\frac{n+1}{n+2} \cdot \frac{n+3}{n+2}\right)^n \cdot \lim_{n\to\infty} \frac{n+3}{n+2} =$$

$= -e^0 \cdot 1 = -1 < 1 \Rightarrow$ serie divergenta pt. $x=3$: $\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)^n \cdot 1^n = \sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)^n$

$$D=\lim_{n\to\infty}\frac{x_n}{x_{n+1}}=\lim_{n\to\infty}\left(\frac{n+1}{n+2}\cdot\frac{n+3}{n+2}\right)^n\cdot\frac{n+3}{n+2}=1$$

- nu decide

$$\cancel{R=\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)=}$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n=\lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)^n=$$

$$=\lim_{n\to\infty}\left[\left(1-\frac{1}{n+2}\right)^{-(n+2)}\right]^{-\frac{n}{n+2}}=$$

$= e^{\lim_{n\to\infty} -\frac{n}{n+2}} = e^{-1} \neq 0 \Rightarrow$ seria este divergentă

$\Rightarrow \sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)^n\cdot(x-2)^n$ convergentă pentru $x \in (1,3)$

2. $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^3 - x + y^2 + z^2$

$$\frac{\partial f}{\partial x} f(x,y,z) = 3x^2 - 1$$

$$\frac{\partial f}{\partial y} f(x,y,z) = 2y$$

$$\frac{\partial f}{\partial z} f(x,y,z) = 2z$$

$$\nabla f(x,y,z) = (3x^2-1, 2y, 2z)$$

$$\nabla f(x,y,z) = 0_3$$

$$\Leftrightarrow \begin{cases} 3x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}} \\ 2y = 0 \Rightarrow y = 0 \\ 2z = 0 \Rightarrow z = 0 \end{cases}$$

Punctele critice sunt: $\left(\frac{1}{\sqrt{3}}, 0, 0\right)$, $\left(-\frac{1}{\sqrt{3}}, 0, 0\right)$

$$H(f)(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x^2} f(x,y,z) = (3x^2-1)_x = 6x$$

$$\frac{\partial^2 f}{\partial y^2} f(x,y,z) = (2y)_y = 2$$

$$\frac{\partial^2 f}{\partial z^2} f(x,y,z) = (2z)_z = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} f(x,y,z) = (2y)_x = 0$$

$$\frac{\partial^2 f}{\partial x \partial z} f(x,y,z) = (2z)_x = 0$$

$$\frac{\partial^2 f}{\partial y \partial z} f(x,y,z) = (2z)_y = 0$$

$$H(f)(x,y,z) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$H(f)\left(\frac{1}{\sqrt{3}}, 0, 0\right) = \begin{pmatrix} \frac{6}{\sqrt{3}} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = \left|\frac{6}{\sqrt{3}}\right| = \frac{6}{\sqrt{3}} > 0 \qquad \Delta_2 = \begin{vmatrix} \frac{6}{\sqrt{3}} & 0 \\ 0 & 2 \end{vmatrix} = \frac{12}{\sqrt{3}} > 0$$

$$\Delta_3 = \begin{vmatrix} \frac{6}{\sqrt{3}} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \frac{24}{\sqrt{3}} > 0$$

$\Rightarrow d^2 f\left(\frac{1}{\sqrt{3}}, 0, 0\right)$ pozitiv definită

$\Rightarrow \left(\frac{1}{\sqrt{3}}, 0, 0\right)$ punct de minim local

$$Hf\left(-\frac{1}{\sqrt{3}},0,0\right)=\begin{pmatrix}-\frac{6}{\sqrt{3}} & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$\Delta_1=-\frac{6}{\sqrt{3}}<0$$

$$\Delta_2=\begin{vmatrix}-\frac{6}{\sqrt{3}} & 0\\ 0 & 2\end{vmatrix}=-\frac{12}{\sqrt{3}}<0$$

$$d^2f\left(-\frac{1}{\sqrt{3}},0,0\right)(u_1,u_2,u_3)=-\frac{6}{\sqrt{3}}u_1^2+2u_2^2+2u_3^2$$

$$d^2f\left(-\frac{1}{\sqrt{3}},0,0\right)(0,1,1)=4>0$$

$$d^2f\left(-\frac{1}{\sqrt{3}},0,0\right)(1,0,0)=-\frac{6}{\sqrt{3}}<0$$

$\Rightarrow d^2f\left(-\frac{1}{\sqrt{3}},0,0\right)$ indefinită $\Rightarrow$

$\Rightarrow \left(-\frac{1}{\sqrt{3}},0,0\right)$ punct şa

3. $f(x,y)=\arcsin\frac{y}{\sqrt{x^2+y^2}}$

$$\frac{\partial^2 f}{\partial x^2}(x,y)=\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$\frac{\partial f}{\partial x}(x,y)=\frac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}}\cdot\left(\frac{y}{\sqrt{x^2+y^2}}\right)_x=$$

$$=\frac{1}{\sqrt{\frac{x^2}{x^2+y^2}}}\cdot y\cdot\left(-\frac{1}{x^2+y^2}\right)\cdot\left(\sqrt{x^2+y^2}\right)_x=$$

$$= \sqrt{\frac{x^2+y^2}{x^2}} \cdot y \cdot \left(-\frac{1}{x^2+y^2}\right) \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x =$$

$$= \frac{\sqrt{x^2+y^2}}{|x|} \cdot y \cdot \left(-\frac{1}{x^2+y^2}\right) \cdot \frac{x}{\sqrt{x^2+y^2}} =$$

$$\overset{x>0}{=} \frac{\sqrt{x^2+y^2}}{x} \cdot y \cdot \left(-\frac{1}{x^2+y^2}\right) \cdot \frac{x}{\sqrt{x^2+y^2}} =$$

$$= -\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \left(-\frac{y}{x^2+y^2}\right)_x = -y \cdot \left(-\frac{1}{(x^2+y^2)^2} \cdot 2x\right) =$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{y}{\sqrt{x^2+y^2}}\right)_y \cdot \frac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}} =$$

$$= \sqrt{\frac{x^2+y^2}{x^2}} \cdot \frac{\sqrt{x^2+y^2} - y \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}y}{x^2+y^2} =$$

$$= \sqrt{\frac{x^2+y^2}{x^2}} \cdot \frac{x^2+y^2-y^2}{(x^2+y^2)\sqrt{x^2+y^2}} =$$

$$= \frac{\sqrt{x^2+y^2}}{x} \cdot \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \left(\frac{x}{x^2+y^2}\right)_y = x \cdot \left(-\frac{1}{(x^2+y^2)^2}\right) \cdot 2y =$$

$$= -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \alpha \cdot \frac{\partial^2 f}{\partial y^2}$$

$$\Leftrightarrow \frac{2xy}{(x^2+y^2)^2} = \alpha \cdot \left(-\frac{2xy}{(x^2+y^2)^2}\right)$$

$$\Leftrightarrow \alpha = -1 \in \mathbb{R}$$

4. $$\mathcal{J}(\alpha) = \int_{0+0}^{1} \frac{x^\alpha}{(e^{\sqrt{x}}-1)^2}\, dx$$

fie $f : (0,1] \to \mathbb{R}$ $\quad f(x) = \dfrac{x^\alpha}{(e^{\sqrt{x}}-1)^2}$

$$\lambda = \lim_{x \to 0} (x-0)^p \cdot f(x) =$$

$$= \lim_{x \to 0} x^p \cdot \frac{x^\alpha}{(e^{\sqrt{x}}-1)^2} = \lim_{x \to 0} \frac{x^{p+\alpha}}{(e^{\sqrt{x}}-1)^2} =$$

$$= \lim_{x \to 0} \frac{x^{p+\alpha}}{\left(\frac{e^{\sqrt{x}}-1}{\sqrt{x}} \cdot \sqrt{x}\right)^2} = \lim_{x \to 0} \frac{x^{p+\alpha}}{(\ln e \cdot \sqrt{x})^2} =$$

$$= \lim_{x \to 0} \frac{x^{p+\alpha}}{x} = \lim_{x \to 0} x^{p+\alpha-1}$$

alegem $p = -\alpha + 1$

$\Rightarrow \lambda = 1$

pt. $p < 1 \Rightarrow -\alpha + 1 < 1 \Rightarrow \alpha > 0$

$\lambda < \infty \Rightarrow \mathcal{I}(\alpha)$ convergentă pt. $\alpha > 0$

pt. $p \geq 1 \Rightarrow -\alpha + 1 \geq 1 \Rightarrow \alpha \leq 0$

$\lambda > 0 \Rightarrow \mathcal{I}(\alpha)$ divergentă pt. $\alpha \leq 0$

$$\mathcal{I}\left(-\frac{1}{2}\right) = \int_0^1 \frac{x^{-\frac{1}{2}}}{(e^{\sqrt{x}}-1)^2} dx = \int_{0+0}^1 \frac{dx}{\sqrt{x}\,(e^{\sqrt{x}}-1)^2}$$

fie $t = e^{\sqrt{x}} - 1 \Rightarrow e^{\sqrt{x}} = t + 1 \Rightarrow$

$\Rightarrow \sqrt{x} = \ln(t+1) \Rightarrow x = \ln^2(t+1)$

$\Leftrightarrow dx = 2\ln(t+1) \cdot \frac{1}{t+1} dt$

pt. $x = 1 \Rightarrow t = e - 1$

pt. $x \to 0 \Rightarrow t \to \lim_{x \to 0} (e^{\sqrt{x}} - 1) = 1 - 1 = 0$

$$\mathcal{I}\left(-\frac{1}{2}\right) = \lim_{v \to 0} \int_v^1 \frac{1}{\ln(t+1) \cdot t^2} \cdot 2\ln(t+1) \cdot \frac{1}{t+1} dt$$

$$= 2 \lim_{v \to 0} \int_v^1 \frac{1}{t^2(t+1)} dt =$$

$$= 2 \lim_{v \to 0} \int_v^1 \frac{1+t-t}{t^2(t+1)} dt =$$

$$= 2 \lim_{v \to 0} \left(\int_v^1 \frac{1}{t^2} dt - \int_v^1 \frac{1}{t(t+1)} dt \right) =$$

$$= 2 \lim_{v \to 0} \left(\int_v^1 \frac{1}{t+1} dt - \int_v^1 \frac{1}{t} dt + \int_v^1 \frac{1}{t^2} dt \right)$$

$$= 2 \lim_{v \to 0} \left(\ln(t+1) \Big|_v^1 - \ln t \Big|_v^1 - \frac{1}{t} \Big|_v^1 \right)$$

$$= 2 \lim_{v \to 0} (\ln 2 - \ln(v+1) - \ln 1 + \ln v -$$

$$- 1 + \frac{1}{v}) =$$

$$= 2 \lim_{v \to 0} \left(\ln v - \ln(v+1) \right) + 2 \ln 2 -$$

$$- 2 \ln 1 + \lim_{v \to 0} \frac{1}{v} - 2 =$$

$$= 2 \lim_{v \to 0} \left(\ln v + \frac{1}{v} \right) - 2 \ln 2 - 2 =$$

$$= 2 \lim_{v \to 0} \frac{v \cdot \ln v + 1}{v^2} + 2 \ln 2 - 2 = 2 \lim_{v \to 0} \left(\frac{1}{v} \cdot \frac{\ln v}{v} + \frac{1}{v^2} \right)$$

$$+ 2 \ln 2 - 2 = 2 \lim_{v \to 0} \left(\frac{1}{v} \cdot \frac{\ln v}{v} \right) + 2 \lim_{v \to 0} \frac{1}{v^2} +$$

$$+ 2 \ln 2 - 2 = 2 \cdot \frac{1}{0+} \cdot 1 + 2 \cdot \frac{1}{0+} + 2 \ln 2 - 2 = + \infty$$