Seminar 10

① Studiați existența limitelor de funcții

ⓐ $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{1+xy}-1} \overset{xy=u}{=} \lim_{u\to 0} \frac{u}{\sqrt{1+u}-1} = \lim_{u\to 0} \frac{u(\sqrt{1+u}+1)}{u} = 2$$

ⓑ $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$; Fie $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$, alegem:

$$\begin{cases} a^n = \left(0, \frac{1}{n}\right) \to (0,0) ; \lim_{n\to\infty} f(a^n) = \frac{0-\frac{1}{n^2}}{0+\frac{1}{n^2}} = -1 \\ b^n = \left(\frac{1}{n} ; 0\right) \to (0,0) ; \lim_{n\to\infty} f(b^n) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1 \end{cases}$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

Limite iterate: $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 1 \neq -1 = \lim_{y\to 0}\lim_{x\to 0} f(x,y)$

ⓒ $$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2+y^2}{x^4+y^4}$$

$$\left|\frac{x^2+y^2}{x^4+y^4} - 0\right| = \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4} \leq \frac{x^2}{x^4} + \frac{y^2}{y^4} = \frac{1}{x^2} + \frac{1}{y^2} \to 0$$

ⓓ $$\lim_{(x,y)\to(0,0)} \frac{x\cdot\sin(x^2-y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2-y^2)}{x^2-y^2}\cdot\frac{x(x^2-y^2)}{x^2+y^2}$$

$$= \lim_{t\to 0} \frac{\sin t}{t} \cdot \lim_{(x,y)\to(0,0)} \frac{x(x^2-y^2)}{x^2+y^2}$$

$x^2-y^2=t$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x^2-y^2)}{x^2+y^2}$$

$$|g(x,y)-0| = \left|\frac{x^3}{x^2+y^2} - \frac{xy^2}{x^2+y^2}\right| \leq \frac{|x^3|+|-xy^2|}{x^2+y^2} =$$

$$= \frac{|x^3|}{x^2+y^2} + \frac{|-xy^2|}{x^2+y^2} = \frac{x^2|x|}{\underbrace{x^2+y^2}_{\leq 1}} + \frac{|-x|y^2}{\underbrace{x^2+y^2}_{\leq 1}} \leq 2|x| \to 0$$

$$\Rightarrow g(x,y) \to 0$$

e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy}$ $\qquad A = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0 \wedge y \neq 0$

$$a^n = \left(\frac{1}{n^2}, \frac{1}{n}\right) \to (0,0)$$

$$f(x,y) = \frac{x^3+y^3}{x \cdot y} ; \quad f(a^n) = \frac{\frac{1}{n^6}+\frac{1}{n^3}}{\frac{1}{n^3}} = n^3 \cdot \left(\frac{1}{n^6} + \frac{1}{n^3}\right)$$

$$= \frac{1}{n^3} + 1 \longrightarrow 1, \; n \to \infty$$

$$b^n = \left(\frac{1}{n} ; \frac{1}{n}\right) ; \quad f(b^n) = \frac{\frac{1}{n^3}+\frac{1}{n^3}}{\frac{1}{n^2}} = \frac{2}{n} \to 0, \; n \to \infty$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

f) $$\lim_{(x,y)\to(1,1)} \frac{(x-1)(y-1)}{xy-1} = \lim_{(u,v)\to(0,0)} \frac{u\,v}{(u+1)(v+1)-1} = \lim_{(u,v)\to(0,0)} \frac{uv}{uv+u+v}$$

$u = x-1$

$v = y-1$

$$a^n = \left(\frac{1}{n} ; \frac{1}{n}\right) \to (0,0)$$

$$g(a^n) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n}+\frac{1}{n}} = \frac{1}{2n+1} \xrightarrow[n\to\infty]{} 0$$

$$b^n = \left(\frac{1}{n} ; -\frac{1}{n}\right) \to (0,0)$$

$$g(b^n) = \frac{-\frac{1}{n^2}}{-\frac{1}{n^2}+\frac{1}{n}-\frac{1}{n}} = 1$$

$$\Rightarrow \nexists \lim_{(u,v)\to(0,0)} g(u,v)$$

(9) $$\lim_{(x,y,z)\to O_3} \underbrace{\frac{(x+y+z)^2}{x^2+y^2+z^2}}_{f(x,y,z)} = \lim_{(x,y,z)\to O_3} =$$

$f: \mathbb{R}^3 \setminus \{O_3\} \to \mathbb{R}$; $a^n = \left(0, 0, \frac{1}{n}\right) \to O_3$, $n \to \infty$

$$b^n = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right)$$

$$f(b^n) = 3$$

$$\Rightarrow \nexists \lim_{(x,y,z)\to O_3} f(x,y,z)$$

(2) $f: \mathbb{R}^2 \to \mathbb{R}$; $f(x,y) = \begin{cases} x \cdot \cos\frac{1}{y^2} + y \cdot \cos\frac{1}{x^2} ; xy \neq 0 \\ 0 ; xy = 0 \end{cases}$

f continuă în $(0,0)$? Dar în $(1,0)$

f cont în $(0,0) \Leftrightarrow \exists \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

$\lim_{x,y\to(0,0)} x \cdot \cos\frac{1}{y^2} + y \cdot \cos\frac{1}{x^2} = 0 \Rightarrow f$ este continuă în $(0,0)$

f cont în $(1,0) \Leftrightarrow \exists \lim_{(x,y)\to(1,0)} f(x,y) = f(1,0) = 0$

$$\lim_{(x,y)\to(1,0)} x \cos\frac{1}{y^2} + y \cdot \cos\frac{1}{x^2} = \underbrace{\lim_{(x,y)\to(1,0)} x \cdot \cos\frac{1}{y^2}}_{\nexists} + \underbrace{\lim_{(x,y)\to(1,0)} y \cdot \cos\frac{1}{x^2}}_{=0}$$

$a^n = \left(1, \frac{1}{\sqrt{2n\pi}}\right) \to (1,0)$; $g(x,y) = x \cdot \cos\frac{1}{y^2}$

$\Rightarrow g(a^n) = 1$

$$b^n = \left(1, \frac{1}{\sqrt{(2n+1)\pi}}\right);\ g(b^n) = -1 \Rightarrow \nexists \lim_{(x,y)\to(1,0)} g(x,y)$$

③ Verificați dacă funcțiile următ. își ating valorile extreme.

ⓐ $f : (0;+\infty)^2 \to \mathbb{R},\ f(x,y) = \frac{x}{y} + \frac{y}{x}$

$A = (0;+\infty)^2 \to$ nu este compactă

$f(x,y) \geq 2 \Leftrightarrow \frac{x}{y} + \frac{y}{x} \geq 2 \Rightarrow x^2+y^2 \geq 2xy$

$\Leftrightarrow (x-y)^2 \geq 0$

$\Rightarrow \inf f(A) = 2 \wedge$ se atinge

$f : A \to \mathbb{R}$

$\inf f(A), \sup f(A)$

se ating dacă

$\exists x,y \in A$ a.î.:

$f(x) = \inf A$

$f(y) = \sup A$

$a^n = (1, n) \in A,\ f(a^n) = \frac{1}{n} + n \longrightarrow \infty$

$\Rightarrow \sup f(A) = +\infty \wedge$ nu se atinge

ⓑ $B(O_2, 1) = \forall x \in \mathbb{R}^2, \{(x,y) \in \mathbb{R}^2 \mid \underline{x^2+y^2<1}\}$

$f : B(O_2, 1) \to \mathbb{R};\ f(x,y) = \frac{1}{1+x^2+y^2}$

nu este compactă

$0 \leq x^2+y^2 < 1 \mid +1 \Rightarrow 1 \leq x^2+y^2+1 < 2$

$\frac{1}{2} < \frac{1}{x^2+y^2+1} \leq 1$

$\Rightarrow \inf f(A) = \frac{1}{2} \wedge$ nu se atinge

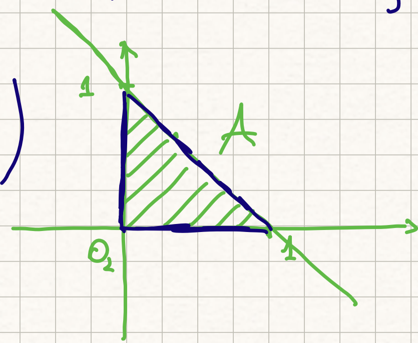
$\sup f(A) = 1 \wedge$ se atinge

$f(0,0) = 1$

ⓒ $f : A \to \mathbb{R}$, $f(x,y) = xy(1-x-y)$

$$A = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1 \}$$

$$\Leftarrow \begin{cases} A \text{ - mărginită} \\ A \text{ - închisă } (\text{fr}\, A \subseteq A) \end{cases}$$



$\Rightarrow A$ - compactă

$\xRightarrow{T.W.}$ f își atinge extremele

$$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \\ 1-x-y \geq 0 \end{array}\right| \; f(x,y) \geq 0 \Rightarrow \inf f(A) = f(0,0)$$

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \Rightarrow \frac{1}{3} \geq \sqrt[3]{f(x,y)} \;\Big|^{3}$$

$$\frac{1}{27} \geq f(x,y) \Rightarrow \sup f(A) = \frac{1}{27} = f\left(\frac{1}{3}, \frac{1}{3}\right)$$