Sesiune 2021

1. Studiați convergența și absolut convergența seriei $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot \sin\frac{\pi}{\sqrt{n}}$ – serie alternantă

Convergența: $\frac{\pi}{\sqrt{n}}\in[0,\pi] \Rightarrow \sin\frac{\pi}{\sqrt{n}}\geq 0.$

dacă $\lim_{n\to\infty}\sin\frac{\pi}{\sqrt{n}}=0 \Rightarrow$ semi-convergentă

absolut convergența:

$\sum_{n=1}^{\infty}\underbrace{\sin\frac{\pi}{\sqrt{n}}}_{=x_n}$, dacă e convergentă $\Rightarrow$ e absolut convergentă.

Criteriul comparației $\quad y_n=\frac{\pi}{\sqrt{n}}$.

$=\lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{\sin\frac{\pi}{\sqrt{n}}}{\frac{\pi}{\sqrt{n}}}=1\in(0,\infty)\Rightarrow$ au aceeași natură.

$\sum_{n=1}^{\infty}\frac{\pi}{\sqrt{n}} > \sum_{n=1}^{\infty}\frac{1}{n}$ – divergentă.

e mai mare decât o serie divergentă. $\Rightarrow \sum_{n=1}^{\infty}\frac{\pi}{\sqrt{n}}$ – divergentă.

$=\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{\pi}{\sqrt{n}}$ nu e absolut convergentă

2. Calculați integrala improprie $\int_1^{\infty}\frac{1}{x^3+x}dx$

$$\int_1^{\infty}\frac{1}{x^3+x}dx=\lim_{v\to\infty}\int_1^{v}\frac{1}{x(x^2+1)}dx$$

$$\frac{1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1} \Leftrightarrow \frac{1}{x(x^2+1)}=\frac{Ax^2+A+Bx^2+Cx}{x^2+1}$$

$$\begin{cases}A+B=0\\C=0\\A=1\end{cases}\Leftrightarrow \begin{matrix}A=1\\B=-1\\C=0\end{matrix}$$

$$\Rightarrow \lim_{v\to\infty}\int_1^{v}\frac{1}{x}dx+\int_1^{v}\frac{-x}{x^2+1}dx=\lim_{v\to\infty}\ln x\Big|_1^{v}-\frac{1}{2}\ln(x^2+1)\Big|_1^{v}$$

$$=\lim_{v\to\infty}\ln v-\frac{1}{2}\ln(v^2+1)+\frac{1}{2}\ln 2=\frac{1}{2}\ln 2+\lim_{v\to\infty}\ln\frac{v}{(v^2+1)^{\frac{1}{2}}}$$

$$=\frac{1}{2}\ln 2+\lim_{v\to\infty}\underbrace{\ln\frac{\cancel{v}}{\cancel{v}\left(1+\frac{1}{v^2}\right)^{\frac{1}{2}}}}_{=\ln 1}=\frac{1}{2}\ln 2$$

3. Determinați punctele critice și punctele de extrem local. (specificând tipul acestora) pentru funcția:

$f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=(x+xy+y^2)\sqrt{e^x}$

$$\frac{\partial f}{\partial x}(x,y)=(1+y)\sqrt{e^x}+(x+xy+y^2)\frac{1}{2\sqrt{e^x}}\cdot e^x$$

$$=\sqrt{e^x}\left(1+y+\frac{x+xy+y^2}{2}\right)$$

$$\frac{\partial f}{\partial y}(x,y)=(x+2y)\sqrt{e^x}$$

$$\begin{cases}\sqrt{e^x}\left(1+y+\frac{x+xy+y^2}{2}\right)=0\\ \sqrt{e^x}(x+2y)=0\end{cases} \Leftrightarrow \begin{cases}1+y+\frac{x+xy+y^2}{2}=0\\ x+2y=0\end{cases} \Rightarrow x=-2y.$$

$$^{2/}1+\overset{2}{}y+\frac{x+xy+y^2}{2}=0 \Leftrightarrow 2+2y+x+xy+y^2=0.$$

$$2+\cancel{2y}-\cancel{2y}-2y^2+y^2=0$$

$$2-y^2=0 \Leftrightarrow y^2=2 \Rightarrow y=\pm\sqrt{2}.$$

pt $y=-\sqrt{2} \Rightarrow x=2\sqrt{2}$

pt $y=\sqrt{2} \Rightarrow x=-2\sqrt{2}$

$(2\sqrt{2},-\sqrt{2})$ și $(-2\sqrt{2},\sqrt{2})$ - puncte critice.

$$H(f)(x,y)=\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{pmatrix}.$$

orice funcție fundamentală de clasă C^2 are $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$

$$\frac{\partial^2 f}{\partial x^2}=\frac{1}{2\sqrt{e^x}}\cdot e^x\left(1+y+\frac{x+xy+y^2}{2}\right)+\sqrt{e^x}\left(\frac{1}{2}(1+y)\right)$$

$$=\frac{1}{2}\sqrt{e^x}\left(1+y+\frac{x+xy+y^2}{2}+1+y\right)=\frac{1}{2}\sqrt{e^x}\left(2+2y+\frac{x+xy+y^2}{2}\right).$$

$$\frac{\partial^2 f}{\partial y^2}=2\sqrt{e^x}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\left(\sqrt{e^x}(x+2y)\right)'_x=\frac{1}{2\sqrt{e^x}}\cdot e^x(x+2y)+\sqrt{e^x}$$

$$=\frac{1}{2}\sqrt{e^x}\left(\frac{x+2y+1}{2}\right)$$

$(-2\sqrt{2}, \sqrt{2})$

$$\frac{\partial^2 f}{\partial x^2}(-2\sqrt{2}, \sqrt{2}) = \frac{1}{2}\cdot\sqrt{e^{-2\sqrt{2}}}\left(2+2\sqrt{2}+\frac{-2\sqrt{2}-4+2}{2}\right)$$

$$= \frac{1}{2}\cdot e^{-\sqrt{2}}\left(2+2\sqrt{2}+\frac{-\cancel{2}(\sqrt{2}+1)}{\cancel{2}}\right)$$

$$= \frac{1}{2}e^{-\sqrt{2}}(2+2\sqrt{2}-\sqrt{2}-1) = \frac{1}{2}e^{-\sqrt{2}}(1+\sqrt{2})$$

$$\frac{\partial^2 f}{\partial y^2}(-2\sqrt{2}, \sqrt{2}) = 2\cdot e^{-\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x\partial y}(-2\sqrt{2}, \sqrt{2}) = \sqrt{e^{-2\sqrt{2}}}\left(\frac{-2\sqrt{2}+2\sqrt{2}}{2}+1\right) = e^{-\sqrt{2}}$$

$$H(f)(-2\sqrt{2}, \sqrt{2}) = \begin{pmatrix} \frac{1}{2}e^{-\sqrt{2}}(1+\sqrt{2}) & e^{-\sqrt{2}} \\ e^{-\sqrt{2}} & 2\cdot e^{-\sqrt{2}} \end{pmatrix}$$

$\Delta_1 > 0.$

$$\Delta_2 = e^{-2\sqrt{2}}(1+\sqrt{2}) - e^{-2\sqrt{2}} = e^{-2\sqrt{2}}(1+\sqrt{2}-1) = e^{-2\sqrt{2}}\cdot\sqrt{2} > 0.$$

$$\left.\begin{matrix}\Delta_2 > 0 \\ \Delta_1 > 0\end{matrix}\right\} \Rightarrow d^2 f(-2\sqrt{2}, \sqrt{2}) \text{ - e pozitiv definită.}$$

$\Rightarrow (-2\sqrt{2}, \sqrt{2})$ - pct. de minim

$(2\sqrt{2}, -\sqrt{2})$

$$\frac{\partial^2 f}{\partial x^2}(2\sqrt{2}, -\sqrt{2}) = \frac{1}{2}\sqrt{e^{2\sqrt{2}}}\left(2-2\sqrt{2}+\frac{2\sqrt{2}-4+2}{2}\right)$$

$$= \frac{1}{2}e^{\sqrt{2}}(2-2\sqrt{2}+\sqrt{2}-1)$$

$$= \frac{1}{2}e^{\sqrt{2}}(1-\sqrt{2})$$

$$\frac{\partial^2 f}{\partial y^2}(2\sqrt{2}, -\sqrt{2}) = 2\cdot\sqrt{e^{2\sqrt{2}}} = 2e^{\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x\partial y}(2\sqrt{2}, -\sqrt{2}) = \sqrt{e^{2\sqrt{2}}}\left(\frac{2\sqrt{2}-2\sqrt{2}}{2}+1\right) = e^{\sqrt{2}}$$

$$H(f)(2\sqrt{2}, -\sqrt{2}) = \begin{pmatrix} \frac{1}{2}e^{\sqrt{2}}(1-\sqrt{2}) & e^{\sqrt{2}} \\ e^{\sqrt{2}} & 2e^{\sqrt{2}} \end{pmatrix}$$

$\Delta_1 < 0$

$$\Delta_2 = (1-\sqrt{2})e^{2\sqrt{2}} - e^{2\sqrt{2}} = e^{2\sqrt{2}}(1-\sqrt{2}-1) = -\sqrt{2}e^{2\sqrt{2}} < 0.$$

$\left.\begin{array}{l}\Delta_1 < 0 \\ \Delta_2 < 0\end{array}\right\}$ $d^2f(2\sqrt{2}, -\sqrt{2})$ - nu e nici pozitiv definită, nici negativ definită

Verificăm dacă e indefinită ; dacă e $\Rightarrow$ e punct şa cu definiţia diferenţialei:

$d^2(2\sqrt{2}, -\sqrt{2}) = \frac{1}{2}e^{\sqrt{2}}(1-\sqrt{2})u_1^2 + (-2)e^{\sqrt{2}} \cdot u_2^2 + 2u_1u_2 \cdot e^{\sqrt{2}}$

$\begin{cases} \text{Pozitiv definită: } d^2f(x)\cdot u > 0 & \forall u \neq 0_m \\ \text{Negativ definită: } d^2f(x)\cdot u < 0 & \forall u \neq 0_m \\ \text{Indefinită: } \exists a, b \in \mathbb{R}^m \text{ a.î. } d^2f(x)\, a > 0 \\ \qquad\qquad\qquad\qquad\qquad\quad d^2f(x)\, b < 0. \end{cases}$

$u_1 = 0 \Rightarrow d^2f = (-2)e^{\sqrt{2}} \cdot u_2^2 \Rightarrow u_2 = 1.$

pt $u_1 = 0$, $u_2 = 1 \Rightarrow d^2f(x) > 0.$

$u_2 = 0 \Rightarrow d^2f = \frac{1}{2}e^{\sqrt{2}}(1-\sqrt{2})u_1^2 \Rightarrow u_1 = 1$

pt $u_1 = 1$, $u_2 = 0 \Rightarrow d^2f(x) < 0.$

$\Rightarrow$ funcţia e indefinită $\Rightarrow (2\sqrt{2}, -\sqrt{2})$ - punct şa.

4. Fie $g: (0,\infty)^2 \to \mathbb{R}$ o funcţie de clasă C^1. Exprimaţi relaţia $-u\frac{\partial g}{\partial u}(u,v) + \frac{v}{1+2u}\frac{\partial g}{\partial v}(u,v) = 1 \quad \forall (u,v) \in (0,\infty)^2$ în variabilele $(x,y) \in (0,\infty)^2$, efectuând transformarea $u = \frac{y}{x}$, $v = x+2y$. Determinaţi apoi o funcţie g cu proprietăţile de mai sus. Verificare.

Transformăm:

$-\frac{y}{x}\frac{\partial g}{\partial u}\left(\frac{y}{x}, x+2y\right) + \frac{x+2y}{\cancel{x}+2\frac{y}{x}}\frac{\partial g}{\partial v}\left(\frac{y}{x}, x+2y\right) = 1.$

$-\frac{y}{x}\underbrace{\frac{\partial g}{\partial u}\left(\frac{y}{x}, x+2y\right)}_{A} + x\underbrace{\frac{\partial g}{\partial v}\left(\frac{y}{x}, x+2y\right)}_{B} = 1$

$f(x,y) = \left(\frac{y}{x}, x+2y\right)$

Folosim $\nabla(g\circ f)(x^0) = \nabla g(f(x^0)) \cdot J(f)(x^0)$

$g\circ f = F(a,b)$

$\nabla g(f(x^0)) = \frac{\partial g}{\partial u}(f)(x^0), \frac{\partial g}{\partial v}(f)(x^0)$

$= \frac{\partial g}{\partial u}\left(\frac{y}{x}, x+2y\right), \frac{\partial g}{\partial v}\left(\frac{y}{x}, x+2y\right) = (A, B)$

Scriem matricea Jacobi.

$$J(f)(x,y) = \begin{pmatrix} \left(\frac{y}{x}\right)'_x & \left(\frac{y}{x}\right)'_y \\ (x+2y)'_x & (x+2y)'_y \end{pmatrix} = \begin{pmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 1 & 2 \end{pmatrix}$$

$$\nabla F(a,b) = (A, B) \cdot \begin{pmatrix} -\frac{b}{a^2} & \frac{1}{b} \\ 1 & 2 \end{pmatrix} = \left(A\left(-\frac{b}{a^2}\right) + B, \frac{A}{b} + 2B\right)$$

$-\frac{b}{a}A + aB = 1 \quad /: a$

$-\frac{b}{a^2}A + B = \frac{1}{a} \qquad \frac{\partial f}{\partial a}(a,b) = \frac{1}{a} \Rightarrow F(a,b) = \ln a + C(b)$

$\Rightarrow F(a,b) = \ln a$

$g\circ f(a,b) = \ln a \Rightarrow g(f(a,b)) = \ln a$

$g(\underbrace{\frac{b}{a}}_{u}, \underbrace{a+2b}_{v}) = \ln a$

$\frac{b}{a} = u \Rightarrow b = a\cdot u \Rightarrow b = \frac{a\cdot v}{1+2u}$

$a+2b = v \Rightarrow a+2au = v$

$a(1+2u) = v \Rightarrow a = \frac{v}{1+2u}$

Verificare: $g(u,v) = \ln\frac{v}{1+2u} = \ln v - \ln(1+2u)$

$\frac{\partial f}{\partial u} = \frac{1}{1+2u}\cdot(-2) = \frac{-2}{1+2u}$

$\frac{\partial f}{\partial v} = \frac{1}{v}$

$+u\cdot\frac{2}{1+2u} + \frac{\cancel{v}}{1+2u}\cdot\frac{1}{\cancel{v}} = \frac{\cancel{2u+1}}{\cancel{1+2u}} = 1.$

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