Sesiune 2020

1. Studiați natura seriei cu termeni pozitivi în funcție de valorile parametrului $a \in \mathbb{R}$. $\sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^a$

I Criteriul raportului al lui d'Alembert

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left(\frac{\ln n}{n}\right)^a \cdot \left(\frac{n+1}{\ln(n+1)}\right)^a = \lim_{n\to\infty} \left(\frac{(n+1)\ln n}{n\ln(n+1)}\right)^a = 1, \text{ nu decide}$$

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{\ln n}{\ln n(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{\ln n}{\ln n + \ln(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{\cancel{\ln n}}{\cancel{\ln n}\left(1+\frac{\ln(1+\frac{1}{n})}{\ln n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{\ln(1+\frac{1}{n})}{\ln n}} = 1.$$

II Criteriul lui Raabe-Duhamel.

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\left(\frac{n+1}{n}\frac{\ln n}{n+1}\right)^a - 1\right]$$

$$= \lim_{n\to\infty} n \frac{\left(\frac{n+1}{n}\frac{\ln n}{n+1}\right)^a - 1}{\frac{n+1}{n}\frac{\ln n}{n+1} - 1} \cdot \left(\frac{n+1}{n}\cdot\frac{\ln n}{\ln(n+1)} - 1\right)$$

$$= a \lim_{n\to\infty} n\left(\frac{n+1}{n}\frac{\ln n}{\ln n+1} - 1\right) = a\lim_{n\to\infty} \frac{(n+1)\ln n}{\ln(n+1)} - \frac{\ln(n+1)}{n}$$

$$= a\lim_{n\to\infty} \frac{(n+1)\ln n - n\ln(n+1)}{\ln(n+1)} = a\lim_{n\to\infty} \frac{n\ln n + \ln n - n\ln(n+1)}{\ln(n+1)}$$

$$= a\lim_{n\to\infty} \underbrace{\frac{\ln n}{\ln n+1}}_{=1} + a\lim_{n\to\infty} \frac{n\ln\frac{n}{n+1}}{\ln(n+1)}$$

$$= a + a\lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)^n}{\ln(n+1)} = a + a\lim_{n\to\infty} \frac{\ln\left(1+\frac{-1}{n+1}\right)^n}{\ln(n+1)}$$

$$= a + a\lim_{n\to\infty} \underbrace{\frac{e^{-1}}{\ln(n+1)}}_{=0} = a + a\cdot 0 = a.$$

I $a > 1 \Rightarrow$ șirul e convergent

II $a < 1 \Rightarrow$ șirul e divergent

III $a = 1$ $\quad x_n = \frac{\ln n}{n}$ $\quad y_n = \frac{1}{n}$

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} > \underbrace{\sum_{n=2}^{\infty} \frac{1}{n}}_{=\infty} \text{ - e divergent}$$

$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\ln n}{\cancel{n}}\cdot \cancel{n} = \infty > 0$, $\sum y_n = \frac{1}{n}$ - diverg. $\Rightarrow \sum \frac{\ln n}{n}$ - divergentă pt $a = 1$.

2. Determinați constanta $\alpha>0$ pentru care funcția $f:(1,\infty)^2 \to \mathbb{R}$, $f(x,y)=\sqrt{(1+x)(1+y)^\alpha}$ verifică egalitatea

$$\frac{\partial^2 f}{\partial x^2}(0,0)+\frac{\partial^2 f}{\partial y^2}(0,0)=2\frac{\partial^2 f}{\partial x \partial y}(0,0)$$

$$\frac{\partial f}{\partial x}=\left(\sqrt{1+x}\cdot\sqrt{1+y^\alpha}\right)'_x=\sqrt{1+y^\alpha}\cdot\frac{1}{2\sqrt{1+x}}$$

$$\frac{\partial f}{\partial y}=\left(\sqrt{1+x}\cdot(1+y)^{\frac{\alpha}{2}}\right)'=\sqrt{1+x}\cdot\frac{\alpha}{2}(1+y)^{\frac{\alpha}{2}-1}$$

$$\frac{\partial^2 f}{\partial x^2}=\sqrt{1+y^\alpha}\cdot\frac{1}{2}\cdot\left((1+x)^{-\frac{1}{2}}\right)'=\sqrt{1+y^2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot(1+x)^{-\frac{3}{2}}=-\frac{\sqrt{1+y^2}}{4}\cdot(1+x)^{\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial y^2}=\sqrt{1+x}\cdot\frac{\alpha}{2}\cdot\left(\frac{\alpha}{2}-1\right)(1+y)^{\frac{\alpha}{2}-2}=\frac{\alpha}{2}\sqrt{1+x}\cdot\frac{\alpha-2}{2}(1+x)^{\frac{\alpha-4}{2}}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\left(\sqrt{1+x}\cdot\frac{\alpha}{2}\cdot(1+y)^{\frac{\alpha-2}{2}}\right)'_x=\frac{\alpha}{2}(1+y)^{\frac{\alpha-2}{2}}\cdot\frac{1}{2\sqrt{1+x}}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0)+\frac{\partial^2 f}{\partial y^2}(0,0)=-\frac{1}{4}+\frac{\alpha}{2}\cdot\frac{\alpha-2}{2}=\frac{\alpha^2-2\alpha-1}{4}$$

$$2\frac{\partial^2 f}{\partial x\partial y}(0,0)=\cancel{2}\cdot\frac{\alpha}{\cancel{2}}\cdot 1\cdot\frac{1}{2}=\frac{\alpha}{2}$$

$$\frac{\alpha^2-2\alpha-1}{4}=\frac{\alpha}{2}\quad /\cdot 2$$

$$\frac{\alpha^2-2\alpha-1}{2}=\alpha \iff \alpha^2-2\alpha-1=2\alpha \iff \alpha^2-4\alpha-1=0.$$

$\Delta=16+4=20$ $\qquad \alpha_1=\frac{4-\sqrt{20}}{2}<0 \Rightarrow$ nu aparține. $\qquad \alpha_2=\frac{4+\sqrt{20}}{2}>0$

$\alpha>0 \Rightarrow \alpha=\frac{4+\sqrt{20}}{2}.$

4. a) Definiți noțiunea de rază de convergență a unei serii de puteri.

Raza de convergență e un nr pt care seria de puteri (centrată în 0) este absolut convergentă pe (x_0-r, x_0+r) și divergentă pe $(-\infty, x_0-r)\cup(x_0+r,\infty)$.

pt. o serie $\sum_{n=1}^{\infty} a_n(x-x_0)^n \Rightarrow r=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

b) Ex. de serie de puteri cu raza de convergență $r=0$.

$a_n=n^n$ $\qquad \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=0$ $\qquad \sum_{n=1}^{\infty} a_n\cdot x^n$

$$\lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\right|=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}=\lim_{n\to\infty}\frac{e^{-1}}{n+1}=0.$$

$\Rightarrow$ pt. $a_n=n^n$ $\quad r=0$

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