Seminar 5 – ANALIZĂ

1. Justificați rasp.

i) $\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}, \ \forall n \in \mathbb{N}^*$

ii) șirul $c_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$ e convergent

i) fie $f(x) = \ln x, \ \forall x > 0$ și intervalul $[n, n+1], n \in \mathbb{N}^*$

Aplicăm Teorema de medie a lui Lagrange $\Rightarrow$

$\Rightarrow \exists\, c \in (n, n+1)$

$$f'(c) = \frac{f(n+1) - f(n)}{n+1-n} = \ln(n+1) - \ln(n)$$

$$f'(c) = \frac{1}{c}, \quad n < c < n+1 \ \Big|\ (\ )^{-1}$$

$$\frac{1}{n} > \underbrace{\frac{1}{c}}_{\shortparallel} > \frac{1}{n+1}$$

$$\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}$$

~~ii) arătăm~~ c_n

ii) $c_{n+1} - c_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \frac{1}{n+1} - \ln(n+1) -$

$\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n\right) = \frac{1}{n+1} - \ln(n+1) + \ln n$

< 0

$\Rightarrow (c_n)$ descrescător

~~ii) $\ln 2 - \ln 1 < 1$~~

~~$\ln 3 - \ln 2 < \frac{1}{2}$~~

~~$\ln 3 - \ln$~~

$\overset{i)}{\Rightarrow}$ $\ln 2 - \ln 1 < 1$

$\ln 3 - \ln 2 < \frac{1}{2}$

$\ln(n+1) - \ln n < \frac{1}{n}$

$(+)$

$\ln(n+1) - \ln 1 < 1 + \frac{1}{2} + \dots + \frac{1}{n} \quad \big| - \ln n$

$\Rightarrow 0 < \ln(n+1) - \ln n < c_n \ , \ \forall n \geq 1$

$\Rightarrow (c_n)$ mărginit inf

$\Rightarrow (c_n)$ convergent

$\lim\limits_{n\to\infty} c_n = \gamma \approx 0,57$ (constanta lui Euler)

ex: Natura seriei $\sum\limits_{n=1}^{\infty} a^{1+\frac{1}{2}+\dots+\frac{1}{n}}$, $a > 0$

pt. $a = \frac{1}{e}$ ✓

$\sum\limits_{n=1}^{\infty} \left(\frac{1}{e}\right)^{1+\frac{1}{2}+\dots+\frac{1}{n}} \sim \sum\limits_{n=1}^{\infty} \frac{1}{n}$, c.d.

$\lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{e}\right)^{1+\frac{1}{2}+\dots+\frac{1}{n}}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{e}\right)^{1+\frac{1}{2}+\dots+\frac{1}{n}}}{e^{\ln \frac{1}{n}}} =$

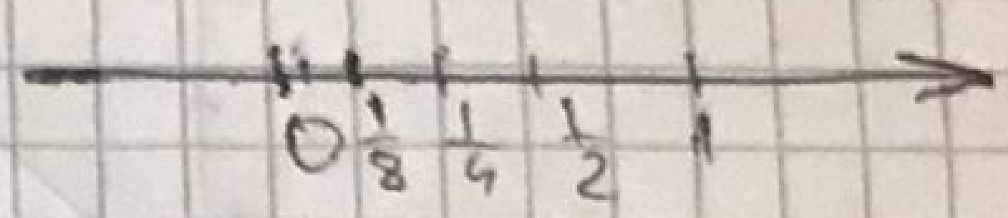
$= \lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{e}\right)^{1+\frac{1}{2}+\dots+\frac{1}{n}}}{\left(\frac{1}{e}\right)^{\ln n}}$

② Det. mult. pct. de acumulare A pt.

a) $A = \{ \frac{1}{2^n} \mid n \in \mathbb{N} \}$.

b) $A = \mathbb{Q}$

a) $0 = \lim\limits_{n \to \infty} \frac{1}{2^n} \Rightarrow 0 \in A'$



? b) $\forall r \in \overline{\mathbb{R}}, \exists (q_n) \subseteq \mathbb{Q}$ şir cu prop. - - -

~~lim~~.

③ Verificaţi dacă funcţiile următoare îşi ating valorile extreme şi det. aceste valori:

a) $f : (-1,1) \to \mathbb{R}$

$f(x) = \ln \frac{1-x}{1+x}$

$f : A \to \mathbb{R}$.

$f(A)$ - imaginea funcţiei

$\left.\begin{array}{l} \inf f(A) \\ \sup f(A) \end{array}\right\}$ valori extreme

se ating, dacă $\exists x_1, x_2 \in A$ a.î.

$f(x_1) = \inf f(A) = \min f(A)$

$f(x_2) = \sup f(A) = \max f(A)$

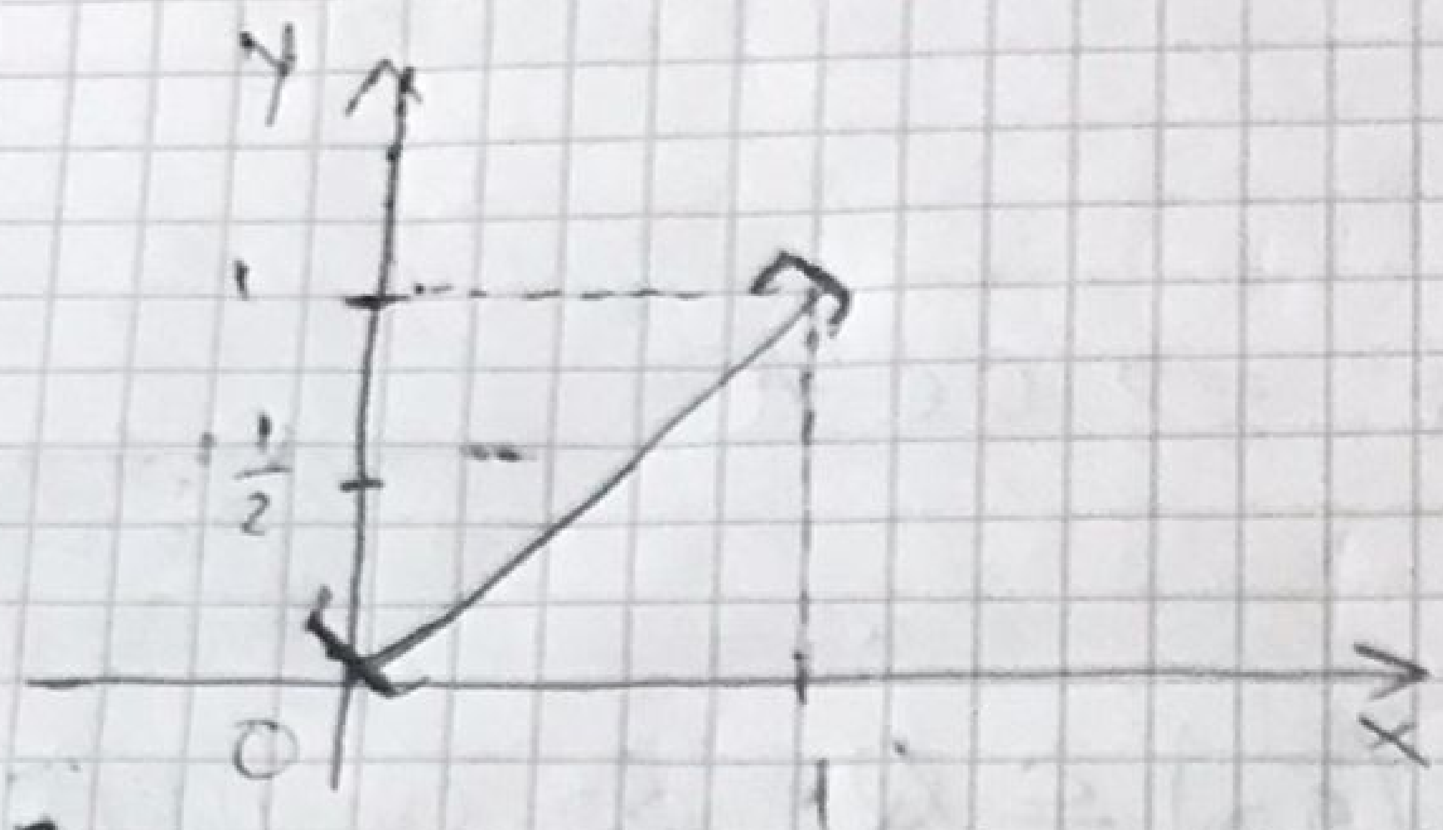
$A = (-1,1)$

$\left.\begin{array}{l} \lim\limits_{x \searrow -1} f(x) = \ln \frac{2}{0_+} = \infty \\ \lim\limits_{x \nearrow 1} f(x) = \ln \frac{0_+}{2} = -\infty \end{array}\right\} \Rightarrow f(A) = \mathbb{R}$, $\inf f(A) = -\infty$, $\sup f(A) = \infty$

nu se ating

b) $f:[0,1]\to\mathbb{R}$

$$f(x)=\begin{cases}\frac{1}{2}, & x=0\\ x, & x\in(0,1]\end{cases}$$



$A=[0,1]$

$f(A)=(0,1]$

$\inf f(A)=0$

$\sup f(A)=1=f(1)$

T. Weierstrass nu se aplică deoarece ~~trebuie să~~ nu avem interval închis ($f(A)=(0,1]$)

c) $A=[-1,1]$, f cont. pe $A \overset{T.W}{\Longrightarrow} f$ mărginită și își atinge extremele

$f:[-1,1]\to\mathbb{R}$

$f(x)=x\sqrt{1-x^2}$

$x\in[-1,1]$, fie $x=\sin t$, $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$f(x)=f(\sin t)=\sin t\sqrt{1-\sin^2 t}=\sin t\cdot|\cos t|$

$=\sin t\cdot\cos t=\frac{1}{2}\sin(2t)\in\left[-\frac{1}{2},\frac{1}{2}\right]$

$\Rightarrow \inf f(A)=-\frac{1}{2}=f\left(-\frac{\sqrt{2}}{2}\right)$

$\sup f(A)=\frac{1}{2}=f\left(\frac{\sqrt{2}}{2}\right)$

$\frac{1}{2}\sin(2t)=\frac{1}{2}\Rightarrow\sin 2t=1\Rightarrow 2t=\frac{\pi}{2}\Rightarrow$

$t=\frac{\pi}{4}\Rightarrow x=\frac{\sqrt{2}}{2}$

④ ~~Demo~~

a) $\forall x_1, x_2 \in (a,b)$, $x_1 < x_2$ să avem $f(x_1) \leq f(x_2)$
(reciproca lui c))
atunci este crescătoare pe (a,b)

„$\Rightarrow$" $f'(x) = \lim\limits_{y \to x} \underbrace{\frac{f(y)-f(x)}{y-x}}_{\geq 0}$

„$\Leftarrow$" fie $x, y \in (a,b)$, $x<y$, din T. de medie a lui Lagrange pe $[x,y] \Rightarrow \exists\, c \in (x,y)$:

$f'(c) = \frac{f(y)-f(x)}{y-x} \Rightarrow f(y) - f(x) \geq 0 \Rightarrow f$ cresc.

b) analog ; c), d) analog
(reciproca lui d))

Contraexemplu: f s. $\nearrow$, dar $f' \not> 0$

$f(x) = x^3$ s. $\nearrow$ pe $\mathbb{R}$

$f'(x) = 3x^2$, $f'(0) = 0 \Rightarrow f' \not> 0$ pe $\mathbb{R}$

⑤ Determinați punctele de extrem local ale fct. de la ex. 3

a) $f:(-1,1) \to \mathbb{R}$

$f(x) = \ln \frac{1-x}{1+x} = \ln(1-x) - \ln(1+x)$

~~$f'(x) = \frac{1+x}{1-x}$~~ . $f'(x) = \frac{-1}{1-x} - \frac{1}{1+x}$

$= \frac{-1-\cancel{x}-1+\cancel{x}}{(1-x)(1+x)} = \frac{-2}{(1-x)(1+x)}$

$= \frac{-2}{1-x^2} \neq 0$,

$\forall x \in (-1,1) \Rightarrow f$ nu are pct. de extrem local

c) $f:[-1,1] \to \mathbb{R}$

$f(x) = x\sqrt{1-x^2}$

f derivabilă pe $(-1,1)$; $f'(x) = \sqrt{1-x^2} + x \cdot \dfrac{-2x}{2\sqrt{1-x^2}}$

$= \sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}} = \dfrac{1-2x^2}{\sqrt{1-x^2}}$.

$f'(x) = 0 \Leftrightarrow 1-2x^2 = 0 \Leftrightarrow x^2 = \frac{1}{2} \Leftrightarrow x = \pm\sqrt{\frac{1}{2}} \in (-1,1)$

x	-1		$-\sqrt{\frac{1}{2}}$		$+\sqrt{\frac{1}{2}}$		1
f'	$-$	$- - -$	0	$+ + +$	0	$- - -$	
f							