Dem 5 (4) $\ell = \lim_{n\to\infty} \frac{x_n}{y_n}$

* dacă $\ell < \infty$ și $\sum y_n$ Conv $\Rightarrow \sum x_n$ Conv
* dacă $\ell > 0$ și $\sum y_n$ Diverg $\Rightarrow \sum x_n$ Diverg.

dacă $\exists \lim_{n\to\infty} \frac{x_n}{y_n} \in (0, \infty)$ atunci $\sum x_n$ și $\sum y_n$ au aceeași natură

* seria armonică generalizată

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv $p > 1$ $\qquad \sum_{n=1}^{\infty} \frac{1}{n}$ divergentă

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{x_n \to 0} \frac{\ln(1 + x_n)}{x_n} = 1 \in (0, \infty)$$

$\Rightarrow \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$ are aceeași natură cu $\sum_{n=1}^{\infty} \frac{1}{n^2}$ e convergentă

② Consecințe ale criteriului lui Kummer

P1) Criteriul lui d'Alambert

$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}}$, dacă $D > 1 \Rightarrow \sum x_n$ conv.

$D < 1 \Rightarrow \sum x_n$ diverg.

$D = 1 \Rightarrow$ trecem la pasul 2

P2) Criteriul Raabe-Duhamel

$$R = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n(D - 1)$$

dacă
- $R > 1 \Rightarrow \sum x_n$ conv.
- $R < 1 \Rightarrow \sum x_n$ diverg.
- $R = 1 \Rightarrow$ trecem la pasul 3

3) Criteriul lui Bertrand

$$B = \lim_{n \to \infty} \ln n \left[n\left(\frac{x_n}{x_{n+1}} - 1 \right) - 1 \right] = \lim_{n \to \infty} \ln n \cdot (R - 1)$$

dacă
- $B > 1 \Rightarrow \sum x_n$ conv.
- $B < 1 \Rightarrow \sum x_n$ diverg.
- $B = 1 \Rightarrow$ nu se poate stabili natura

a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$$x_n = \frac{2^n}{n!}$$

$$D = \lim_{n \to \infty} \frac{x_n}{x_{n+1}} = \lim_{n \to \infty} \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}} =$$

$$= \lim_{n \to \infty} \frac{n+1}{2} = +\infty > 1 \Rightarrow \text{conv.}$$

b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{\sqrt{n}}$

$$x_n = \left(\frac{1}{2}\right)^{\sqrt{n}}$$

$$D = \lim_{n \to \infty} \frac{\left(\frac{1}{2}\right)^{\sqrt{n}}}{\left(\frac{1}{2}\right)^{\sqrt{n+1}}} = \lim_{n \to \infty} \left(\frac{1}{2}\right)^{\frac{\sqrt{n} - \sqrt{n+1}}{1}} {}^{(\sqrt{n}+\sqrt{n+1})}$$

$$= \lim_{n \to \infty} \left(\frac{1}{2}\right)^{\frac{n - n - 1}{\sqrt{n} + \sqrt{n+1}}}$$

$$= \lim_{n \to \infty} \left(\frac{1}{2}\right)^{-\frac{1}{\sqrt{n}+\sqrt{n+1}}} = \lim_{n \to \infty} 2^{\frac{1}{\sqrt{n}+\sqrt{n+1}}} = 2^0 = 1 \Rightarrow \text{nu decide}$$

$$R = \lim_{n \to \infty} n(D - 1) = \lim_{n \to \infty} n \cdot \left(2^{\frac{1}{\sqrt{n}+\sqrt{n+1}}} - 1 \right)$$

$$= \lim_{n\to\infty} \frac{2^{\frac{1}{\sqrt{n}+\sqrt{n+1}}} - 1}{\frac{1}{\sqrt{n}+\sqrt{n+1}}} \cdot \frac{n}{\sqrt{n}+\sqrt{n+1}} =$$

$$= \ln 2 \cdot \lim_{n\to\infty} \frac{n}{\sqrt{n}+\sqrt{n+1}} = +\infty > 1 \Rightarrow \text{converg.}$$

c) $\displaystyle\sum_{n=1}^{\infty} \left[\frac{(2n)!!}{(2n+1)!!}\right]^2$

$(2n)!! = 2\cdot 4\cdot 6\cdot \ldots \cdot 2n$

$(2n+1)!! = 1\cdot 3\cdot 5 \cdot \ldots \cdot (2n+1)$

$$\boxed{(*)\ \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \ln \sqrt[n]{n} = \ln 1 = 0}$$

$$x_n = \left[\frac{(2n)!!}{(2n+1)!!}\right]^2$$

$$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left[\frac{(2n)!!}{(2n+1)!!}\right]^2 \cdot \left[\frac{(2n+3)!!}{(2n+2)!!}\right]^2$$

$$= \lim_{n\to\infty} \left(\frac{2n+3}{2n+2}\right)^2 = 1 \Rightarrow \text{nu decide}$$

$$R = \lim_{n\to\infty} n(D-1) = \lim_{n\to\infty} n\left[\left(\frac{2n+3}{2n+2}\right)^2 - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{4n^2+12n+9}{4n^2+8n+4} - 1\right] = \cancel{\lim_{n\to\infty} n\left[4n^2-4n^2-8n-\right.}$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{4n^2}+12n+9-\cancel{4n^2}-8n-4}{4n^2+8n+4}\right] = \lim_{n\to\infty} n \cdot \frac{4n+5}{4n^2+8n+4}$$

$$= \lim_{n\to\infty} \frac{4n^2+5n}{4n^2+8n+4} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(4+\frac{5}{n}\right)}{\cancel{n^2}\left(4+\frac{8}{n}+\frac{4}{n^2}\right)} = \frac{4}{4} = 1 \text{ nu decide}$$

$$B = \lim_{n\to\infty} \ln n\,(R-1) = \lim_{n\to\infty} \ln n \cdot \left(\frac{4n^2+5n}{4n^2+8n+4} - 1\right)$$

$$= \lim_{n\to\infty} \ln n \cdot \frac{\cancel{4n^2}+5n-\cancel{4n^2}-8n-4}{4n^2+8n+4} = \lim_{n\to\infty} \ln n \cdot \frac{-3n-4}{4n^2+8n+4}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n} \cdot \frac{-3n^2-4n}{4n^2+8n+4} \overset{(*)}{=} 0 \cdot \left(-\frac{3}{4}\right) = 0 < 1 \Rightarrow \text{divergentă}$$

Criteriul radicalului

(x_n) șir cu termeni strict pozitivi
și $\exists\ c = \lim\limits_{n\to\infty} \sqrt[n]{x_n}$, $c \in \overline{\mathbb{R}}$

* dacă $c < 1 \Rightarrow \sum x_n$ convergentă
* dacă $c > 1 \Rightarrow \sum x_n$ divergentă

$$\sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

$$x_n = \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{\left(2+\frac{1}{n}\right)^n}} = \lim_{n\to\infty} \frac{n^{\frac{2}{n}}}{2+\frac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^2}{2+\frac{1}{n}} = \frac{1}{2} < 1 \Rightarrow \text{convergentă}$$

Criteriul condensării

Fie (x_n) un șir descrescător de numere pozitive, atunci $\sum x_n$ și $\sum 2^n \cdot x_{2^n}$ au aceeași natură

$$S = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}, \quad p > 0$$

$$x_n = \frac{1}{n(\ln n)^p} \Rightarrow x_{2^n} = \frac{1}{2^n(\ln 2^n)^p}$$

$$\sum_{n=2}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n}(\ln 2^n)^p} = \sum_{n=2}^{\infty} \frac{1}{(\ln 2^n)^p} = \sum_{n=2}^{\infty} \frac{1}{(n \ln 2)^p}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^p} \cdot \frac{1}{(\ln 2)^p} = \frac{1}{(\ln 2)^p} \cdot \underbrace{\sum_{n=2}^{\infty} \frac{1}{n^p}}_{\text{conv} \Leftrightarrow p > 1}$$

Caz part. $p = 1 \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n}$ div $\Rightarrow S =$ conv, $p > 1$; div, $p = 1$

Studiați convergența și absolut convergența

Fie $\sum x_n$ o serie cu termeni oarecare. Seria este absolut convergentă dacă $\sum |x_n|$ e convergentă

absolut conv $\Rightarrow$ conv

$\not\Leftarrow$

a) $S=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{2n+1}{3^n}$

$S'=\sum_{n=0}^{\infty}\left|(-1)^n\cdot\frac{2n+1}{3^n}\right|=\sum_{n=0}^{\infty}\frac{2n+1}{3^n}$

$D=\lim_{n\to\infty}\frac{2n+1}{3^n}\cdot\frac{3^{n+1}}{2n+3}=\lim_{n\to\infty}\frac{6n+3}{2n+3}=\lim_{n\to\infty}\frac{n(6+\frac{3}{n})}{n(2+\frac{3}{n})}$

$=3>1\Rightarrow S'$ conv.

$\Rightarrow S$ absolut convergentă $\Rightarrow S$ convergentă

b) $S=\sum_{n=1}^{\infty}\frac{\sin n}{2^n}$

$S'=\sum_{n=1}^{\infty}\left|\frac{\sin n}{2^n}\right|$ $\sum_{n=1}^{\infty}\frac{1}{2^n}\Rightarrow\left|\frac{\sin n}{2^n}\right|<\frac{1}{2^n}$

$\sum_{n=1}^{\infty}\frac{1}{2^n}$ convergentă $\Rightarrow\sum_{n=1}^{\infty}\left|\frac{\sin n}{2^n}\right|$ convergentă

③ Criteriul raportului pentru șiruri

Fie $(x_n)_{n\in\mathbb{N}}$ un șir cu termeni strict pozitivi pt. care $\exists\lim_{n\to\infty}\frac{x_n}{x_{n+1}}=l$, atunci:

a) dacă $l>1\Rightarrow\lim_{n\to\infty}x_n=0$

b) dacă $l<1\Rightarrow\lim_{n\to\infty}x_n=+\infty$

a) fie $\sum_{n=0}^{\infty} x_n$ STP

$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = l > 1 \overset{\text{D'Alambert}}{\Rightarrow} \sum x_n$ conv $\Rightarrow \lim_{n\to\infty} x_n = 0$

b) fie $\sum_{n=0}^{\infty} \frac{1}{x_n}$ STP

$D = \lim_{n\to\infty} \frac{\frac{1}{x_n}}{\frac{1}{x_{n+1}}} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{l} > 1$ (pt că $l < 1$)

$\sum \frac{1}{x_n}$ conv $\Rightarrow \lim_{n\to\infty} \frac{1}{x_n} = 0 \Rightarrow \lim_{n\to\infty} x_n = +\infty$

(4) fie $\sum_{n=1}^{\infty} x_n$ STP. Arătați că:

$\sum_{n=1}^{\infty} x_n \overset{\text{(au aceeași natură)}}{\sim} \sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$

$x_n > \frac{x_n}{1+x_n}$, $\forall n \in \mathbb{N}$

dacă $\sum x_n$ conv $\overset{C.C.}{\Rightarrow} \sum \frac{x_n}{1+x_n}$ conv.

dacă $\sum \frac{x_n}{1+x_n}$ divergentă $\Rightarrow \sum x_n$ divergentă

dacă $\sum \frac{x_n}{1+x_n}$ convergentă $\Rightarrow \lim_{n\to\infty} \frac{x_n}{1+x_n} = 0$

$\Rightarrow \lim_{n\to\infty} \frac{1}{1+\frac{1}{x_n}} = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{x_n} = +\infty \Rightarrow \lim_{n\to\infty} x_n = 0$

$\lim_{n\to\infty} \frac{x_n}{\frac{x_n}{1+x_n}} = \lim_{n\to\infty} 1 + x_n = 1 + 0 = 1 \in (0, +\infty) \Rightarrow \sum x_n \sim \sum \frac{x_n}{1+x_n}$

$\Rightarrow \sum x_n$ conv

dacă $\sum x_n$ divergentă

$$\frac{x_n}{\frac{x_n}{1+x_n}} = 1 + x_n$$

1) $\lim_{n \to \infty} x_n = \infty \Rightarrow \lim_{n \to \infty} \frac{x_n}{1+x_n} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}} = 1 > 0 \Rightarrow$

$\Rightarrow \sum \frac{x_n}{1+x_n}$ div

$\cancel{2) \lim_{n \to \infty}}$