Seminar 9

1. Fie $x = (1, 0, -1)$, $y = (3, -1, 1) \in \mathbb{R}^3$. Calculati $x + y$, $x \cdot y$, $||x||$, $||-2y||$ si $||x - y||$.

2. Fie $x, y \in \mathbb{R}^m$ si notam $a = x \cdot y$, $b = ||x||$ si $c = ||y||$. Exprimati urmatoarele marimi in functie de a, b si c
   a) $(x + y) \cdot y$
   b) $x \cdot (2x - y)$
   c) $||x - y||$

3. Fie $x, y \in \mathbb{R}^m$. Demonstrati **identitatea paralelogramului**

$$||x + y||^2 + ||x - y||^2 = 2\left(||x||^2 + ||y||^2\right)$$

4. Determinati int$A$, fr$A$, precum si daca A este multime deschisa, respectiv multime inchisa.
   a) $A = B(O_2, 1) \subseteq \mathbb{R}^2$
   b) $A = [2, \infty) \times (2, \infty) \subseteq \mathbb{R}^2$
   c) $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$
   d) $A = \mathbb{R} \setminus \mathbb{Z} \subseteq \mathbb{R}$

5. $\forall A \subseteq \mathbb{R}^m$ multime nevida, au loc afirmatiile
   a) $\mathrm{int}A \subseteq A$
   b) $\mathrm{int}A \cap \mathrm{fr}A = \emptyset$
   c) $A \subseteq \mathrm{int}A \cup \mathrm{fr}A$ (cu "=" daca multimea A este inchisa)
   d) $\mathrm{int}A \cup \mathrm{fr}A \cup \mathrm{int}\left(\mathbb{R}^m \setminus A\right) = \mathbb{R}^m$

6. Fie $A, B \subseteq \mathbb{R}^m$ multimi nevide. Numarul real

$$d(A, B) = \inf\{||x - y|| \,\big|\, x \in A, y \in B\}$$

se numeste **distanta dintre multimile** A si B.
   a) Determinati distanta dintre multimile $A = [1, 2]^2$ si $B = B(O_2, 1)$
   b) Dati exemplu de doua multimi nevide $A, B \subseteq \mathbb{R}^2$ cu $A \cap B = \emptyset$ si $d(A, B) = 0$.

Exercitii suplimentare

1. Fie $x = (-1, 2, 3)$, $y = (-2, 1, -3) \in \mathbb{R}^3$.
   a) Determinati valorile lui $r > 0$ astfel incat $y \notin B(x, r)$
   b) Determinati valorile lui $t \in \mathbb{R}$ astfel incat vectorul $(1, -1, t)$ sa apartina bilei $\overline{B}(x, 5)$.

2. Fie $x, y \in \mathbb{R}^m$. Demonstrati ca
   a) $x \cdot y = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right)$
   b) $\big|||x|| - ||y||\big| \leq ||x - y||$

3. Doi vectori $x, y \in \mathbb{R}^m$ se numesc **ortogonali** daca $x \cdot y = 0$. Justificati afirmatia

$$x, y \in \mathbb{R}^m \text{ sunt ortogonali } \iff ||x - y||^2 = ||x||^2 + ||y||^2$$

4. Determinati $\text{int}A$, $\text{fr}A$, precum si daca A este multime deschisa, respectiv multime inchisa.
   a) $A = \overline{B}(O_2, 1) \setminus \{O_2\} \subseteq \mathbb{R}^2$
   b) $A = [0, 1] \times (0, 1) \subseteq \mathbb{R}^2$
   c) $A = \mathbb{Q} \times \mathbb{Q} \subseteq \mathbb{R}^2$
   d) $A = \left\{\left(1 + \frac{1}{n}\right)^n \,\middle|\, n \in \mathbb{N}^\star\right\} \subseteq \mathbb{R}$

5. $\forall A \subseteq \mathbb{R}^m$ multime nevida, au loc afirmatiile
   a) $A' \subseteq A \cup \text{fr}A$
   b) $\text{int}A \cap \text{int}\,(\mathbb{R}^m \setminus A) = \emptyset$
   c) $\text{fr}A = \text{fr}\,(\mathbb{R}^m \setminus A)$
   d) $\text{int}A = A \setminus \text{fr}A$

6. Fie $A \subseteq \mathbb{R}^m$ multime nevida. Au loc afirmatiile
   a) Daca A este multime deschisa atunci $A \subseteq A'$
   b) Daca A este multime inchisa atunci $A' \subseteq A$

Reciprocele afirmatiilor sunt adevarate?

7. Fie $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. Numarul real pozitiv

$$||x||_M \overset{\text{not}}{=} |x_1| + |x_2| + \ldots + |x_m|$$

se numeste **norma Minkowski** a vectorului x. Aratati ca aceasta verifica proprietatile normei euclidiene.