Seminar 6

① Calc. derivata de ordinul $n \in \mathbb{N}$ a funcțiilor de mai jos și precizați mulțimea pe care aceste funcții sunt în definit derivabile

a) $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$$f^{(n)}(x) = \begin{cases} \sin x, & n = 4k \\ \cos x, & n = 4k+1 \\ -\sin x, & n = 4k+2 \\ -\cos x, & n = 4k+3 \end{cases}, \quad k \in \mathbb{Z} = \begin{cases} (-1)^k \sin x, & n = 2k \\ (-1)^k \cos x, & n = 2k+1 \end{cases} \quad k \in \mathbb{Z}$$

f indef. derivabilă pe $\mathbb{R}$

b) $f(x) = \ln(x+1)$ CE: $x+1 > 0 \Rightarrow x > -1$

$f'(x) = \frac{1}{x+1} = (x+1)^{-1}$

$f''(x) = -\frac{1}{(x+1)^2} = -(x+1)^{-2}$

$f'''(x) = \frac{2}{(x+1)^3} = 2(x+1)^{-3}$

$f^{(4)}(x) = -2 \cdot 3 (x+1)^{-4}$

$f^{(5)}(x) = 2 \cdot 3 \cdot 4 (x+1)^{-5}$

$$f^{(n)}(x) = (-1)^{n+1} \cdot (n-1)! \cdot (x+1)^{-n}, \quad \forall n \geq 1$$

f indefinit derivabilă pe $(-1, +\infty)$

c) Formula lui Leibniz: $u = u(x)$, $v = v(x)$

$$(u \cdot v)^{(n)} = \sum_{k=0}^{n} C_n^k \cdot u^{(k)} \cdot v^{(n-k)} \quad (*)$$

$u(x) = x^2 - x$, $u'(x) = 2x - 1$, $u''(x) = 2$, $u'''(x) = 0$

$v(x) = e^x$, $v^{(k)}(x) = e^x$

$$\overset{(*)}{\Rightarrow} f^{(n)}(x) = C_n^0 \cdot (x^2 - x) \cdot e^x + \overbrace{C_n^1 \cdot (2x-1) \cdot e^x + C_n^2 \cdot 2 \cdot e^x}^{n \geq 2} + 0$$

$$= (x^2 - x) e^x + n(2x-1) \cdot e^x + \frac{n(n-1)}{2} \cdot 2e^x$$

$$= e^x \left(x^2 + x(2n-1) + n^2 - 2n \right), \ \forall n \in \mathbb{N}$$

se verifică pt. $n \in \{0, 1\}$

$f(x)$ indefinit derivabilă pe $\mathbb{R}$

d) $f(x) = \sqrt{1-x}$

$$f(x) = (1-x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} \cdot (1-x)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2} \cdot \frac{1}{2} (1-x)^{-\frac{3}{2}}$$

$$f'''(x) = -\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot (1-x)^{-\frac{5}{2}}$$

$$f^{(4)}(x) = -\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot (1-x)^{-\frac{7}{2}}$$

$$f^{(n)}(x) = -\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n} \cdot (1-x)^{-\frac{2n-1}{2}}$$

$$= -\frac{(2n-3)!!}{2^n} \cdot (1-x)^{-\frac{2n-1}{2}}, \ \forall n \geq 2$$

f indef. derivabilă pe $(-\infty, 1)$

② Pt. funcțiile de la exercițiul anterior, punctul $x_0 = 0$ și numărul $m \in \mathbb{N}$, determinați:

a) Polinomul lui Taylor de grad m asociat funcției f în punctul x_0.

b) Mulțimea de convergență a seriei Taylor corespunzătoare.

$$T_m(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \; ; \; \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m$$

1) $f(x) = \sin x$, $f^{(m)}(x) = \begin{cases} (-1)^k \sin x, & m = 2k \\ (-1)^k \cos x, & m = 2k+1 \end{cases}$,

$$f^{(m)}(0) = \begin{cases} 0, & m = 2k \\ (-1)^k, & m = 2k+1 \end{cases}$$

$$T_m(x) = f(0) + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} \cdot x^2 + \dots + \frac{f^{(m)}(0)}{m!} \cdot x^m =$$

$$= 0 + \frac{x}{1!} + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \dots + \frac{f^{(m)}(0)}{m!} \cdot x^m$$

Seria Taylor: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} \cdot x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1}$

Fie $x \in \mathbb{R}^*$ fixat și studiem absolut convergența:

$$\sum_{k=0}^{\infty} \underbrace{\frac{|x|^{2k+1}}{(2k+1)!}}_{x_k} \text{ s.t.p.}, \quad D = \lim_{k\to\infty} \frac{x_k}{x_{k+1}} = \lim_{k\to\infty} \frac{|x|^{2k+1}}{(2k+1)!} \cdot \frac{(2k+3)!}{|x|^{2k+3}}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+3)}{|x|^2} = +\infty > 1$$

$\Rightarrow$ serie conv.

$\Rightarrow$ mulțimea de conv. este $\mathbb{R}$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}, \quad \forall x \in \mathbb{R}}$$

2) $f(x) = \ln(x+1)$ indef deriv pe $(-1, +\infty)$

$$f^{(n)}(0) = (-1)^{n+1} \cdot (n-1)!, \quad \forall n \geq 1;$$

$$T_n(x) = 0 + \sum_{k=1}^{n} \frac{(-1)^{k+1} \cdot (k-1)!}{k!} \cdot x^k$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \cdot x^k =$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1}}{n} \cdot x^n$$

Seria Taylor $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}}_{a_n} \cdot x^n$ este s.p. raza de conv. $r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{n+1}{n} = 1$

$x_0 = 0$

$\Rightarrow (-1, 1) \subseteq I \subseteq [-1, 1]$

pt. $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ conv. (serie armonică) $\Rightarrow 1 \in I$,

$x = -1 \Rightarrow \sum_{n=1}^{\infty} -\frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ divergentă $\Rightarrow$

$\Rightarrow -1 \notin I \Rightarrow I = (-1, 1] \Rightarrow \ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot x^n,$

$\forall x \in (-1, 1]$

c) Temă

d) $f(x) = \sqrt{1-x}$ indef. derivabilă pe $(-\infty, 1)$

$$f^{(n)}(0) = -\frac{(2n-3)!!}{2^n} \cdot 1^{-\frac{2n-1}{2}}$$

$$f^{(n)}(0) = -\frac{(2n-3)!!}{2^n}, \forall n \geq 2$$

$f(0) = 1$

$f'(0) = -\frac{1}{2}$

$$T_n(x) = 1 - \frac{1}{2}x + \sum_{k=2}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k$$

$$= 1 - \frac{x}{2} + \sum_{k=2}^{n} -\frac{(2k-3)!!}{2^k - k!} \cdot x_k$$

Seria Taylor : $1 - \frac{x}{2} - \underbrace{\sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!} \cdot x^n}_{\text{mulțimea de conv}}$

$$a_n = \frac{(2n-3)!!}{2^n \cdot n!}, \quad r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(2n-3)!!}{2^n \cdot n!} \cdot \frac{2^{n+1} \cdot (n+1)!}{(2n-1)!!}$$

$$= \lim_{n\to\infty} \frac{2(n+1)}{2n-1} = 1 \Rightarrow (-1,1) \subseteq I \subseteq [-1,1]$$

pt. $x = 1 \Rightarrow \sum_{n=2}^{\infty} \frac{2n-3}{2^n \cdot n!}$ s.t.p.

$$D = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{2(n+1)}{2n-1} = 1 \text{ – nu decide}$$

$$R = \lim_{n\to\infty} n \cdot \left(\frac{a_n}{a_{n+1}} - 1 \right) = \lim_{n\to\infty} n \cdot \left[\frac{2n+2}{2n-1} - 1 \right] = \lim_{n\to\infty} \frac{3n}{2n-1}$$

$$= \frac{3}{2} > 1 \Rightarrow \text{conv}$$

pt. $x=-1 \Rightarrow \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!} \cdot (-1)^n$ este abs. conv. deci conv $\Rightarrow -1 \in I$

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③ Utilizând operații cu serii de puteri, justificați egalitățile:

a) $\sum_{n=0}^{\infty} (-1)^n (n+1) \cdot x^n = \frac{1}{(1+x)^2}, \forall x \in (-1,1)$

b) $1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot x^n = \frac{1}{\sqrt{1-x}}, \forall x \in [-1,1)$

a) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \forall x \in (-1,1)$

$\sum_{n=0}^{\infty} (-1)^n \cdot x^n = \frac{1}{1+x}, \forall x \in (-1,1)$

$\Rightarrow \sum_{n=0}^{\infty} \left[(-1)^n \cdot x^n\right]' = \frac{-1}{(1+x)^2}$

$n-1=m \Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot n \cdot x^{n-1} = -\frac{1}{(1+x)^2}$

$\Rightarrow \sum_{m=0}^{\infty} (-1)^m \cdot (m+1) \cdot x^m = +\frac{1}{(1+x)^2}, \forall x \in (-1,1)$

are aceeași rază de conv. $r=1$