Seminar 3 - Analiză

x_n convergent $\Rightarrow x_n \to 0$

$x_n \to 0 \not\Rightarrow$ convergentă

Criteriul Stolz - Cesaro

$\boxed{\text{T}}$ Fie $(a_n)_{n\in\mathbb{N}}$ un șir oarecare de nr. reale și $(b_n)_{n\in\mathbb{N}}$ un șir monoton și divergent

dacă $\exists \lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} = l \in \overline{\mathbb{R}}$, atunci $\exists \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = l$

1. $(x_n)_{n\in\mathbb{N}}$ șir cu termeni strict pozitivi. S.s.a.că dacă $\exists \lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = l$ atunci $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = l$. Reciproca e adev?

$$\sqrt[n]{x_n} = x_n^{\frac{1}{n}} \overset{\ln}{\Rightarrow} \frac{1}{n}\ln x_n = \frac{\ln x_n}{n} \quad (x_n > 0)$$

În S.C. luăm $a_n = \ln x_n$
$b_n = n$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\ln x_{n+1} - \ln x_n}{\cancel{n}+1-\cancel{n}} = \lim_{n\to\infty} \ln \frac{x_{n+1}}{x_n} =$$

$$= \ln\left(\lim_{n\to\infty} \frac{x_{n+1}}{x_n}\right) = \ln l \quad \left(\begin{array}{l} l=0 \to \ln 0 = -\infty \\ l = +\infty \to \ln \infty = \infty \end{array}\right)$$

$$\overset{S.C.}{\Rightarrow} \lim_{n\to\infty} \frac{\ln x_n}{n} = \ln l \Rightarrow \lim_{n\to\infty} \ln x_n^{\frac{1}{n}} = \ln l$$

$$\ln\left(\lim_{n\to\infty} \sqrt[n]{x_n}\right) = \ln l \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = l$$

✱ Reciproca nu e adevărată

$$x_n = e^{(-1)^n}$$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \left[e^{(-1)^n} \right]^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{(-1)^n}{n}} = e^0 = 1$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{e^{(-1)^{n+1}}}{e^{(-1)^n}} = \lim_{n\to\infty} e^{(-1)^{n+1} - (-1)^n} =$$

$$= \begin{cases} e^{-2}, & n \text{ par} \\ e^{2}, & n \text{ impar} \end{cases} \Rightarrow \nexists \lim$$

2. Calc limita sirurilor

a) $y_n = \dfrac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\ln n}$

b) $y_n = \sqrt[n]{n!}$

c) $y_n = \dfrac{\sqrt[n]{n!}}{n}$

d) $y_n = \sqrt[n]{n}$

a) $a_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

$b_n = \ln n$ (crescator, $\lim = \infty$)

$$\overset{S.C}{\Rightarrow} \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln \frac{n+1}{n}} =$$

$$= \lim_{n\to\infty} \frac{1}{(n+1)\ln \frac{n+1}{n}} = \lim_{n\to\infty} \frac{1}{(n+1)\left(\ln\left(1+\frac{1}{n}\right)\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\ln \underbrace{\left(1+\frac{1}{n}\right)^{n+1}}_{\to e}} = \frac{1}{\ln e} = 1 \Rightarrow y_n \to 1$$

b) Fie $x_n = n! \Rightarrow \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n+1 = \infty$

$$\overset{ex.1)}{\Rightarrow} \lim_{n\to\infty} \sqrt[n]{x_n} = \infty$$

d) ~~$\lim_{n\to\infty}$~~

c) $y_n = \frac{\sqrt[n]{n!}}{n} = \sqrt[n]{\frac{n!}{n^n}}$

$x_n = \frac{n!}{n^n}$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)}{(n+1)^{n+1}} \cdot n^n$

$\lim_{n\to\infty} = \frac{n^n}{(n+1)^n}$

$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$ ~~$= \frac{1}{e}$~~

$= \lim \frac{1}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}} = \frac{1}{e}$

$\Rightarrow \lim_{n\to\infty} y_n = \frac{1}{e}$

3. Pt șirurile de mai jos:

$a_n = \sum_{k=1}^{n} \frac{1+(-1)^k}{2}, \quad b_n = n, \quad \forall n \in \mathbb{N}^*$

Calc $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ și $\lim_{n\to\infty} \frac{a_n}{b_n}$

$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} \frac{1+(-1)^k}{2} - \sum_{k=1}^{n} \frac{1+(-1)^k}{2}}{1} =$

$= \lim_{n\to\infty} \frac{1+(-1)^{n+1}}{2}$; n par $\Rightarrow \lim_{n\to\infty} 0 = 0$; $\neq \Rightarrow \nexists \lim$

n impar $\Rightarrow \lim_{n\to\infty} 1 = 1$

* n par

$$a_n = 0+1+0+1+\dots+0+1 = \frac{n}{2}$$

* n impar

$$a_n = \underbrace{0+1+0+1+\dots 0+1}_{n-1 \text{ termeni}} + 0 = \frac{n-1}{2}$$

$n = 2K$

$$\lim_{K\to\infty} \frac{a_{2K}}{b_{2K}} = \lim_{K\to\infty} \frac{\frac{2K}{2}}{2K} = \frac{1}{2}$$

$n = 2K+1$

$$\lim_{K\to\infty} \frac{a_{2K+1}}{b_{2K+1}} = \lim_{K\to\infty} \frac{\frac{2K+1-1}{2}}{2K+1} = \frac{1}{2}$$

deci $\exists \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$

Reciproca lui Stolz-Cesaro e falsă

$$\exists \lim \frac{a_n}{b_n} \not\Rightarrow \exists \lim \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

4. Scrieți urm sume cu ajutorul Σ

a) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots = \sum_{K=0}^{\infty} \frac{1}{2K+1}$

b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{K=0}^{\infty} \frac{1}{2^K}$

c) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} = \sum_{K=1}^{\infty} (-1)^{K-1} \cdot \frac{1}{K^2}$

5. a) $\sum_{n=0}^{\infty} \frac{1}{n!} = e$

b) $\sum_{n=1}^{\infty} \frac{1}{5^n}$

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}}$

d) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

e) $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$

f) $\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{(n+2)!}$

b) $S_k = \frac{1}{5} + \frac{1}{5^2} + \dots + \frac{1}{5^k} = \frac{1}{5} \cdot \frac{1-\left(\frac{1}{5}\right)^k}{1-\frac{1}{5}}$

$$= \frac{1}{5} \cdot \frac{\left(1-\frac{1}{5}\right)^k}{\frac{4}{5}} = \frac{1}{4} \cdot \left(1-\left(\frac{1}{5}\right)^k\right)$$

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \dots = \frac{1}{1-a}, \quad a \in (-1,1)$$

c) $S_k = \frac{\overset{\sqrt{1}-\sqrt{0}}{1}}{\sqrt{1}+\sqrt{0}} + \frac{\overset{\sqrt{2}-\sqrt{1}}{1}}{\sqrt{2}+\sqrt{1}} + \frac{\overset{\sqrt{3}-\sqrt{2}}{1}}{\sqrt{3}+\sqrt{2}} + \dots + \frac{\overset{\sqrt{k}-\sqrt{k-1}}{1}}{\sqrt{k}+\sqrt{k-1}}$

$$S_k = \frac{\cancel{\sqrt{1}}-\sqrt{0}+\cancel{\sqrt{2}}-\cancel{\sqrt{1}}+\cancel{\sqrt{3}}-\cancel{\sqrt{2}}+\dots+\sqrt{k}-\cancel{\sqrt{k-1}}}{1} = \sqrt{k}$$

$$S = \lim_{k\to\infty} S_k = \infty$$

d) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

$$S_k = \sum_{k=1}^{n} \frac{1}{4k^2-1} = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \sum_{k=1}^{n} \frac{1}{2} \cdot \frac{2}{(2k-1)(2k+1)}$$

$$= \frac{1}{2} \sum_{k=1}^{n} \frac{2}{(2k-1)(2k+1)} = \frac{1}{2} \sum_{k=1}^{n} \frac{(2k+1)-(2k-1)}{(2k-1)(2k+1)} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{2k-1} - \frac{1}{2k+1}$$

$$= \frac{1}{2} \left(\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k+1} \right)$$

$$= \frac{1}{2} \left(\frac{1}{1} + \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{2n-1}} - \cancel{\frac{1}{3}} - \dots - \frac{1}{2n+1} \right)$$

$$= \frac{1}{2} \left(1 - \frac{1}{2n+1} \right)$$

$$\lim_{n \to \infty} \frac{1}{2} \cdot \frac{2n+1-1}{2n+1} = \cancel{\frac{1}{2} \lim_{n \to \infty} \frac{2n}{2n+1}}$$

$$= \frac{1}{2} \lim_{n \to \infty} 1 - \frac{1}{2n+1} = \frac{1}{2}$$