Seminar 7 - integrala Riemann

① Evaluați integralele

a) $\int_0^1 \frac{e^x}{\sqrt{e^{2x}+1}}\,dx = J$ b) $\int_0^2 \max\{x,x^2\}dx$ c) $\int_1^{\sqrt3} \frac{arctg\,x}{x^2}dx$

a) $\int \frac{u'(x)}{\sqrt{u^2(x)+a^2}}\,dx = \ln\left(u(x)+\sqrt{u^2(x)+a^2}\right)+c$

$u(x)=e^x \Rightarrow J = \ln\left(e^x+\sqrt{e^{2x}+1}\right)\Big|_0^1 = \ln(e+\sqrt{e^2+1})-\ln(1+\sqrt2)$

b) $\int_0^1 \max\{x,x^2\}dx + \int_1^2 \max\{x,x^2\}dx = \int_0^1 x\,dx + \int_1^2 x^2dx$

$\max\{x,x^2\} = \begin{cases} x, & x\in[0,1] \\ x^2, & x\in[1,2]\end{cases}$ $= \frac{x^2}{2}\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 = \frac12 - 0 + \frac83 - \frac13$

$= \frac12 + \frac73$

c) $\int_1^{\sqrt3} \left(-\frac1x\right)' arctg\,x\,dx = -\frac{arctg\,x}{x}\Big|_1^{\sqrt3} + \underbrace{\int_1^{\sqrt3} \frac1x\cdot\frac{1}{1+x^2}dx}_{J}$

$= -\frac{arctg\sqrt3}{\sqrt3} + \frac{arctg\,1}{1} + J = -\frac{\pi}{3\sqrt3} + \frac{\pi}{4} + J$

$J = \int_1^{\sqrt3}\left(\frac1x - \frac{x}{1+x^2}\right)dx = \ln x\Big|_1^{\sqrt3} - \frac12\ln(1+x^2)\Big|_1^{\sqrt3}$

$= \ln\sqrt3 - \frac12\ln(1+3) + \frac12\ln 2$

d) $\int_{-1}^1 \sqrt{1-x^2}\,dx$

Substituții trigonometrice pt. integrale algebrice:

Fie $R(U,V)$ o funcție rațională și $a>0$,

i) $\int R(x,\sqrt{a^2-x^2})\,dx$, $x=a\sin t$ sau $x=a\cos t$

ii) $\int R(x,\sqrt{a^2+x^2})\,dx$, $x=a\,tg\,t$ sau $x=a\,ctg\,t$

iii) $\int R(x,\sqrt{x^2-a^2})\,dx$, $x=\frac{a}{\sin t}$ sau $x=\frac{a}{\cos t}$

d) $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$\ulcorner x = \varphi(t)$
$dx = \varphi'(t)\,dt \lrcorner$

// folosim i)

$x = \sin t$
$dx = \cos t\,dt$

$x = 1 \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$
$x = -1 \Rightarrow \sin t = -1 \Rightarrow t = -\frac{\pi}{2}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cdot \cos t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos t|\cdot \cos t\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt = \qquad \cos 2t = 2\cos^2 t - 1$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt = \left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{\pi}{4} + 0$$

e) $\int_{2}^{4} \frac{\sqrt{x^2-4}}{x}\,dx = \mathcal{I}$

$$x = \frac{2}{\sin t},\quad dx = \left(\frac{2}{\sin t}\right)' dt = -\frac{2\cos t}{\sin^2 t}\,dt$$

$x = 2 \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$
$x = 4 \Rightarrow \sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$

$$\frac{\sqrt{x^2-4}}{x} = \frac{\sqrt{\frac{4}{\sin^2 t} - 4}}{\frac{2}{\sin t}} = \frac{\sin t}{2}\cdot\sqrt{\frac{4\cdot\cos^2 t}{\sin^2 t}}$$

$$= \frac{\sin t}{2}\cdot 2\left|\frac{\cos t}{\sin t}\right| = \cos t$$

$$\mathcal{I} = \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \cos t\cdot\left(-\frac{2\cos t}{\sin^2 t}\right)dt = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\overbrace{\cos^2 t}^{=1-\sin^2 t}}{\sin^2 t}\,dt$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{\sin^2 t} - 1\right)dt = 2(-\operatorname{ctg} t - t)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 2\left(0 + \sqrt{3} - \frac{\pi}{2} + \frac{\pi}{6}\right)$$

Seminar 8

- integrale improprii

a) $\int_0^{\infty} \frac{\operatorname{arctg} x}{1+x^2}\,dx =$ b) $\int_{-1}^{1} \frac{x+1}{\sqrt{1-x^2}}\,dx$

a) $= \lim_{v\to\infty} \int_0^v \frac{\operatorname{arctg} x}{1+x^2}\,dx$

$= \lim_{v\to\infty} \int_0^v \operatorname{arctg} x \cdot (\operatorname{arctg} x)'\,dx = \lim_{v\to\infty} \frac{\operatorname{arctg}^2 x}{2}\Big|_0^v$

$= \lim_{v\to\infty} \frac{\operatorname{arctg}^2 v}{2} = \frac{1}{2}\cdot\left(\frac{\pi}{2}\right)^2$

b) $\int_{-1+0}^{1-0} \frac{x+1}{\sqrt{1-x^2}}\,dx = \int_{-1+0}^{0} \frac{x+1}{\sqrt{1-x^2}}\,dx + \int_0^{1-0} \frac{x+1}{\sqrt{1-x^2}}\,dx =$

$= \lim_{u\searrow -1} \int_u^0 \frac{x+1}{\sqrt{1-x^2}}\,dx + \lim_{v\nearrow 1} \int_0^v \frac{x+1}{\sqrt{1-x^2}}\,dx = \lim_{u\searrow -1} \left(-\sqrt{1-x^2} + \arcsin x\right)\Big|_u^0 +$ ⊛

$\int \frac{x+1}{\sqrt{1-x^2}}\,dx = \int \underbrace{\frac{x}{\sqrt{1-x^2}}}_{\left(-\sqrt{1-x^2}\right)'}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + \arcsin x + c$

⊛ $+ \lim_{v\nearrow 1} \left(-\sqrt{1-x^2} + \arcsin x\right)\Big|_0^v = \lim_{u\searrow -1} \left(-1 + \sqrt{1-u^2} - \arcsin u\right) +$

$+ \lim_{v\nearrow 1} \left(-\sqrt{1-v^2} + \arcsin v + 1\right) = -1 - \left(-\frac{\pi}{2}\right) + \left(\frac{\pi}{2} + 1\right) = \cancel{-1} + \frac{\pi}{2} + \frac{\pi}{2} \cancel{+1}$

$= \pi$

c) $\int_0^{\infty} x^n \cdot e^{-x}\,dx = I_n, \quad n \in \mathbb{N}$

$I_n = \int_0^{\infty} x^n \cdot (-e^{-x})'\,dx = x^n \cdot (-e^{-x})\Big|_0^{\infty} + \int_0^{\infty} n \cdot x^{n-1} \cdot e^{-x}\,dx$

$= \lim_{v\to\infty} x^n (-e^{-x})\Big|_0^v + n \cdot I_{n-1} = \lim_{v\to\infty} \left[-v^n \cdot e^{-v} + 0\right] + n I_{n-1}$

$\lim_{v\to\infty} \frac{-v^n}{e^v} \underset{\text{se aplică l'Hôpital de n ori}}{\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=}} 0$ ✓

$= n\, I_{n-1}, \quad \forall n \geq 1$

$$J_0 = \int_0^{\infty} e^{-x}\,dx = -e^{-x}\Big|_0^{\infty} = \lim_{v\to\infty}(-e^{-x})\Big|_0^{v}$$

$$= \lim_{v\to\infty}\left(-e^{-v}-(-e^{0})\right) = \lim_{v\to\infty}\left(-e^{-v}+1\right) = 1$$

$J_1 = 1\cdot J_0 = 1$
$J_2 = 2\cdot J_1 = 2$
$J_3 = 3\cdot J_2 = 6$

$J_n = n!,\ \forall n\in\mathbb{N}$ (se dem. prin inducție)

d) $$\int_1^{2-0} \frac{1}{\sqrt{x(2-x)}}\,dx$$

$$= \int_1^{2-0} \frac{1}{\sqrt{2x-x^2}} = \int_1^{2-0} \frac{1}{\sqrt{2x-x^2-1+1}}\,dx = \int_1^{2-0} \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$t = x-1$

$$\int_0^{1-0} \frac{1}{\sqrt{1-t^2}}\,dt = \arcsin t\Big|_0^{1-0} = \arcsin t\Big|_0^{1} = \frac{\pi}{2}$$

$t = x-1$
$dt = dx$

e) $$J = \int_0^{\infty} e^{-x}\cdot\cos x\,dx = \int_0^{\infty} (-e^{-x})'\cdot\cos x\,dx$$

$$= (-e^{-x})\cdot\cos x\Big|_0^{\infty} - \int_0^{\infty}(-e^{-x})\cdot(-\sin x)\,dx$$

$$= \lim_{v\to\infty}(-e^{-x}\cdot\cos x)\Big|_0^{v} - \int_0^{\infty}(-e^{-x})'\cdot\sin x\,dx$$

$$= \lim_{v\to\infty}\left(\underbrace{-e^{-v}\cdot\cos v}_{0}+1\right) - \left((-e^{-x})\sin x\Big|_0^{\infty} + \int_0^{\infty}(e^{-x})\cdot\cos x\,dx\right)$$

$$-\frac{1}{e^v}\le\frac{\cos v}{e^v}\le\frac{1}{e^v}$$
$\searrow 0 \swarrow$

$$= 1 - \left(\lim_{v\to\infty}\left((-e^{-x})\cdot\sin x\right)\Big|_0^{v} + J\right) =$$

$$= 1 - \lim_{v\to\infty}\left(-e^{-v}\cdot\sin v\right) - J = 1-0-J = 1-J \Rightarrow 2\cdot J = 1 \Rightarrow J = \frac{1}{2}$$

f) $\int_{0+0}^{1-0} \frac{\ln x}{\sqrt{1-x}}\, dx = -2\sqrt{1-x}\ln x \Big|_{0+0}^{1-0} + 2\int_{0+0}^{1-0} \frac{\sqrt{1-x}}{x}\, dx$

$\circledast\ \int \frac{\ln x}{\sqrt{1-x}}\, dx = \int (-2\sqrt{1-x})' \cdot \ln x = -2\sqrt{1-x}\cdot \ln x + 2\int \frac{\sqrt{1-x}}{x} dx$

$\left(\lim_{u\downarrow 0} -2\sqrt{1-x}\ln x \Big|_u^1 = \lim_{u\downarrow 0} 2\sqrt{1-u}\cdot \ln u = -\infty \right)$

$= -2\sqrt{1-x}\ln x + 2\int \frac{u}{1-u^2}\cdot(-2u)\, du =$

$1-x = u^2,\ u \geq 0$

$2\int \frac{u}{1-u^2}\cdot(-2u)du = 2\int \frac{u^2-1+1}{u^2-1}\, du = 2\left(\int \left(1+\frac{1}{u^2-1}\right) du \right) =$

$= 2\left(u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| \right) = 2\left(\sqrt{1-x} + \frac{1}{2}\ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| \right)$