Seminar 8 (continuare)
- integrale improprii -

Calculați integrala $J = \int_{0+0}^{\frac{\pi}{2}} \ln(\sin x)\,dx \underset{\substack{x = \frac{\pi}{2} - y \\ dx = -dy \\ x \searrow 0 \Rightarrow y \nearrow \frac{\pi}{2} \\ x = \frac{\pi}{2} \Rightarrow y = 0}}{=}$

$$= \int_{\frac{\pi}{2}-0}^{0} \ln\left(\sin\left(\frac{\pi}{2} - y\right)\right) \cdot (-dy) = \int_{0}^{\frac{\pi}{2}-0} \ln(\cos y)\,dy$$

$$2J = J + J = \int_{0+0}^{\frac{\pi}{2}} \ln(\sin x)\,dx + \int_{0+0}^{\frac{\pi}{2}-0} \ln(\cos x)\,dx =$$

$$= \int_{0+0}^{\frac{\pi}{2}-0} \ln\frac{\sin 2x}{2}\,dx = \int_{0+0}^{\frac{\pi}{2}-0} \left(\ln(\sin 2x) - \ln 2\right)dx$$

$$= \underbrace{\int_{0+0}^{\frac{\pi}{2}-0} \ln(\sin 2x)\,dx}_{J} - x\ln 2 \Big|_{0}^{\frac{\pi}{2}} = J - \frac{\pi}{2}\ln 2$$

$$z = 2x$$

$$J = \frac{1}{2}\int_{0+0}^{\pi-0} \ln(\sin z)\,dz = \frac{1}{2}\left[\int_{0+0}^{\frac{\pi}{2}} \ln(\sin z)\,dz + \underbrace{\int_{\frac{\pi}{2}}^{\pi-0} \ln(\sin z)\,dz}_{z = \pi - u}\right] = \frac{1}{2}\left[J + \underbrace{\int_{\frac{\pi}{2}}^{0+0} \ln(\sin u)(-du)}_{J}\right] = J$$

$$= J$$

2 Studiati convergenta integralelor improprii

a) $\int_{0}^{3-0} \frac{x+1}{\sqrt{9-x^2}} dx$

a) $f:[0,3) \to [0,+\infty)$

$f(x) = \frac{x^3+1}{\sqrt{9-x^2}}$

$P_1 \quad \lambda = \lim_{x \nearrow 3} (3-x)^p \cdot f(x) = \lim_{x \nearrow 3} (3-x)^p \cdot \frac{x^3+1}{\sqrt{(3-x)(3+x)}} =$

$= \lim_{x \nearrow 3} (3-x)^{p-\frac{1}{2}} \cdot \frac{x^3+1}{\sqrt{3+x}}$

- $p < 1$, $\lambda < +\infty \Rightarrow$ int. conv.
- $p \geq 1$, $\lambda > 0 \Rightarrow$ int. div.

alegem $p = \frac{1}{2} < 1 \Rightarrow \lambda = \frac{28}{\sqrt{6}} < +\infty \Rightarrow$ int. conv.

b) $\int_{0+0}^{\infty} \frac{\operatorname{arctg} x}{x} dx = \underbrace{\int_{0+0}^{1} \frac{\operatorname{arctg} x}{x} dx}_{J_1} + \underbrace{\int_{1}^{\infty} \frac{\operatorname{arctg} x}{x} dx}_{J_2}$

pt. J_1: $f:(0,1] \to [0,+\infty)$, $f(x) = \frac{\operatorname{arctg} x}{x}$

$\lambda = \lim_{x \searrow 0} (x-0)^p \cdot f(x) = \lim_{x \searrow 0} x^p \cdot \frac{\operatorname{arctg} x}{x} =$

$= \lim_{x \searrow 0} x^p \cdot \underbrace{\lim_{x \searrow 0} \frac{\operatorname{arctg} x}{x}}_{=1}$

alegem $p = \frac{1}{2} < 1$, $\lambda = 0 < +\infty \Rightarrow J_1$ conv pt. J_2: [redacted] $f:[1,+\infty) \to [0,+\infty)$, $f(x)=\frac{\operatorname{arctg} x}{x}$

p_2: $\lambda = \lim_{x\to\infty} x^p \cdot f(x) = \lim_{x\to\infty} x^{p-1} \cdot \underbrace{\operatorname{arctg} x}_{\to \frac{\pi}{2}} = \frac{\pi}{2} > 0$ $\Rightarrow$

$\Rightarrow J_2$ divergentă

$\left.\begin{matrix}\text{pt. } J_1 \\ \text{pt. } J_2\end{matrix}\right\} \Rightarrow J_1 + J_2$ divergentă

c) $\int_0^{\pi} x \ln(\sin x)\,dx = \int_{0+0}^{\pi-0} x \ln(\sin x)\,dx =$

$= -\int_{0+0}^{\frac{\pi}{2}} -x \ln(\sin x) dx + (-1)\int_{\frac{\pi}{2}}^{\pi-0} -x \ln(\sin x)\,dx$

pt. J_1: $f:(0,\frac{\pi}{2}] \to [0,+\infty)$, $f(x) = -x\ln(\sin x)$

$\lambda = \lim_{x\searrow 0} (x-0)^p \cdot (-x) \cdot \ln(\sin x) = -\lim_{x\searrow 0} x^{p+1} \cdot \ln(\sin x)$

alegem $p=0 \Rightarrow \lim_{x\searrow 0} x \ln(\sin x) = \lim_{x\searrow 0} \frac{\ln(\sin x)}{\frac{1}{x}}$

$\underset{-\frac{\infty}{\infty}}{\overset{\text{l'hopital}}{=}} \lim_{x\searrow 0} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x\searrow 0} -\frac{x^2 \cos x}{\sin x} =$

$= \lim_{x\searrow 0} -\underbrace{\frac{x}{\sin x}}_{\to 1} \cdot x\cos x = -1 \cdot 0 = 0$

$p<1$, $\lambda < +\infty \Rightarrow J_1$ conv.

pt. J_2 : $f : [\frac{\pi}{2}, \pi) \to [0, +\infty)$, $f(x) = -x \ln(\sin x)$

$$\lambda = \lim_{x \nearrow \pi} (\pi - x)^p \cdot (-x) \cdot \ln(\sin x) \underset{\pi - x = y}{=}$$

$$= \lim_{y \searrow 0} y^p (y - \pi) \ln(\sin(\pi - y)) \overset{\pi}{=} \lim_{y \searrow 0} y^p \cdot \ln(\sin y) = 0$$

$p = \frac{1}{2} < 1 \Rightarrow$ conv

$\Rightarrow J_1 + J_2$ conv

3. Studiați convergența integralei improprii

$$J(\alpha) = \int_0^1 \left(\frac{x}{1-x}\right)^\alpha dx, \quad \alpha \in \mathbb{R}$$

și calc. valoarea lui $J\left(\frac{1}{2}\right)$

caz I : $\alpha > 0 \Rightarrow J(\alpha) = \int_0^{1-0} \left(\frac{x}{1-x}\right)^\alpha dx$

$f : [0, 1) \to [0, +\infty)$, $f(x) = \left(\frac{x}{1-x}\right)^\alpha$

$$\lambda = \lim_{x \nearrow 1} (1-x)^p \cdot \frac{x^\alpha}{(1-x)^\alpha} = \lim_{x \nearrow 1} (1-x)^{p-\alpha} \cdot \underbrace{x^\alpha}_{\to 1}$$

$$= \lim_{x \nearrow 1} (1-x)^{p-\alpha}$$

alegem $p = \alpha \Rightarrow \lambda = 1 \Rightarrow \lambda > 0, \lambda < +\infty$

~~dacă $\alpha < 1$~~ dacă $p < 1$ ($\alpha < 1$) $\Rightarrow$ int. conv.

dacă $p \geq 1$ ($\alpha \geq 1$) $\Rightarrow$ int. div

caz II : $\alpha = 0 \Rightarrow J(0) = \int_0^1 dx = 1 \Rightarrow$ conv.

Cazul $\alpha < 0 \Rightarrow I(\alpha) = \int_0^1 \left(\frac{1-x}{x}\right)^{-\alpha} dx$

$f : (0,1] \to [0,+\infty),\ f(x) = \left(\frac{1-x}{x}\right)^{-\alpha}$

$$\lambda = \lim_{x \searrow 0} (x-0)^p \cdot f(x) = \lim_{x \searrow 0} x^p \cdot \frac{(1-x)^{-\alpha}}{x^{-\alpha}} =$$

$$= \lim_{x \searrow 0} x^{p+\alpha} \cdot \underbrace{(1-x)^{-\alpha}}_{\to 1} = \lim_{x \searrow 0} x^{p+\alpha}$$

alegem $p = -\alpha \Rightarrow \lambda = 1$

dacă $p < 1$ $(\alpha > -1) \Rightarrow$ int conv

dacă $p \geq 1$ $(\alpha \leq -1) \Rightarrow$ int div

În concluzie: $I(\alpha)$ conv $\Leftrightarrow \alpha \in (-1,1)$

$I(\alpha)$ div $\Leftrightarrow \alpha \in \mathbb{R} \setminus (-1,1)$

$$I\left(\frac{1}{2}\right) = \int_0^1 \left(\frac{x}{1-x}\right)^{\frac{1}{2}} dx = \int_0^{1-0} \sqrt{\frac{x}{1-x}}\, dx$$

$$\sqrt{\frac{x}{1-x}} = t \geq 0 \Rightarrow \frac{x}{1-x} = t^2 \Rightarrow x = t^2 - t^2 x$$

$$\Rightarrow x = \frac{t^2}{1+t^2}$$

$\cancel{t^2 + t^2 x - x = 0}$

$\cancel{t^2(1-x) - x = 0}$

$\cancel{x = -t^2(1-x)}$

$$dx = \frac{2t(1+t^2) - (t^2 \cdot 2t)}{(1+t^2)^2} dt = \frac{2t}{(1+t^2)^2} dt$$

$x = 0 \Rightarrow t = 0$

$x \nearrow 1 \Rightarrow \lim_{x \nearrow 1} \sqrt{\frac{x}{1-x}} = +\infty \Rightarrow t \to +\infty$

$$\Rightarrow I\left(\frac{1}{2}\right) = \int_0^{+\infty} t \cdot \frac{2t}{(1+t^2)^2} dt$$

$$\left(\frac{1}{1+t^2}\right)' = \frac{-2t}{(1+t^2)^2} \Rightarrow \int_0^\infty t\cdot\frac{2t}{(1+t^2)^2}\,dt = \int_0^\infty t\cdot\left(-\frac{1}{1+t^2}\right)'\,dt$$

$$= \left(\frac{1}{1+t^2}\right)' = \frac{-2t}{(1+t^2)^2} \quad\Bigg|\; = -\frac{t}{1+t^2}\Big|_0^\infty + \underbrace{\int_0^\infty \frac{1}{1+t^2}\,dt}_{\operatorname{arctg} t|_0^\infty}$$

$$= \lim_{v\to\infty}\left(-\frac{t}{1+t^2}\right)\Big|_0^v + \lim_{v\to\infty}\operatorname{arctg} t\Big|_0^v$$

$$= \lim_{v\to\infty}\frac{-v}{1+v^2} + \lim_{v\to\infty}\operatorname{arctg} v = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

④ (funcția Γ) Considerăm integrala improprie

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\cdot e^{-x}\,dx, \quad \alpha\in\mathbb{R}$$

Dem. urm. prop.:

a) $\Gamma(\alpha)$ este convergentă, $\forall \alpha>0$

b) $\Gamma(n+1) = n!$, $\forall n\in\mathbb{N}$

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c) $\Gamma(\alpha+1) = \alpha\cdot\Gamma(\alpha)$, $\forall\alpha>0$

d) $\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\cdot\Gamma(\frac{1}{2})$, $\forall n\in\mathbb{N}^*$

$\Gamma(n+1) = \int_0^\infty x^n\cdot e^{-x}\,dx = n!$, $\forall n\in\mathbb{N}$

a) $\alpha>0$

$$\Gamma(\alpha) = \int_{0+0}^1 x^{\alpha-1}\cdot e^{-x}\,dx + \int_1^{+\infty} x^{\alpha-1}\cdot e^{-x}\,dx$$

pt. J_1: $0<x\le 1 \Rightarrow \frac{x^{\alpha-1}}{e^x} < x^{\alpha-1}$

$$\int_{0+0}^1 x^{\alpha-1}\,dx = \frac{x^\alpha}{\alpha}\Big|_{0}^1 = \frac{1}{\alpha} < +\infty$$

$\xRightarrow{\text{Criteriul comparației}} J_1$ conv

pt. J_2: $f:[1,+\infty)\to[0,+\infty)$, $f(x) = x^{\alpha-1}\cdot e^{-x}$

$$\lambda = \lim_{x\to\infty} x^p\cdot f(x) = \lim_{x\to\infty} x^{p+\alpha-1}\cdot e^{-x} \underset{p=2>1}{=} \lim_{x\to\infty} x^{\alpha+1}\cdot e^{-x} = 0 < +\infty \Rightarrow$$

$\Rightarrow \Gamma(\alpha)$ conv pt. $\alpha>0$