Point Separation and Obstacle Removal by Finding and Hitting Odd Cycles

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joint work with

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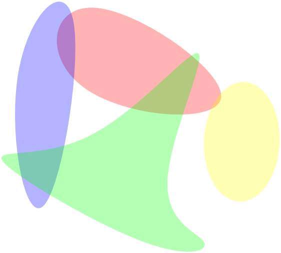
Background

- **Obstacles: connected geometric objects in the plane**



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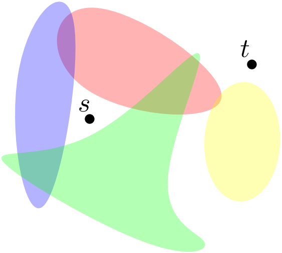

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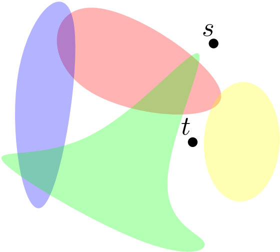

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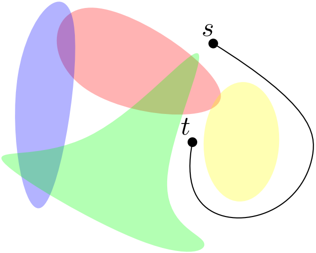

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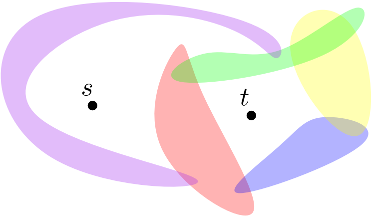
Problems

- **Point separation**

  Input: $s, t \in \mathbb{R}^2$ and a set $\mathcal{O}$ of obstacles in $\mathbb{R}^2$
  Output: a minimum subset $\mathcal{O}' \subseteq \mathcal{O}$ that separates s and t



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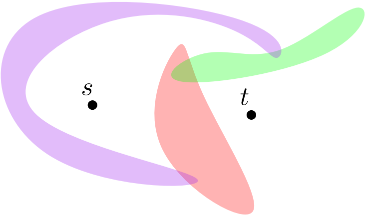
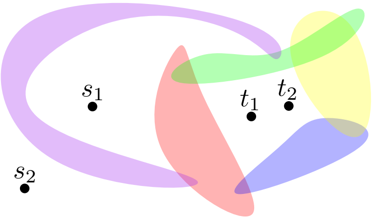


Problems

- **A generalization of point separation**

  Given p point pairs $(s_1, t_1), \ldots, (s_p, t_p)$ in $\mathbb{R}^2$, want a minimum subset $\mathcal{O}' \subseteq \mathcal{O}$ that separates every pair (s_i, t_i).



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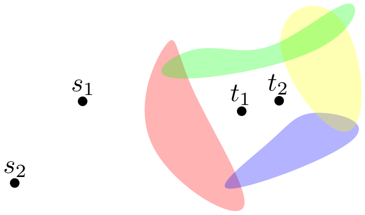

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  Input: $s, t \in \mathbb{R}^2$ and a set $\mathcal{O}$ of obstacles in $\mathbb{R}^2$
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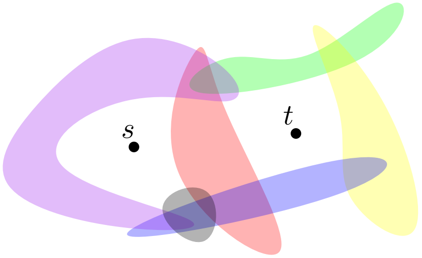

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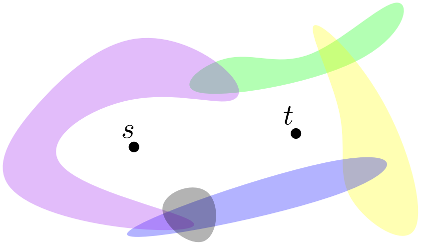

Previous work

- **Cabello and Giannopoulos [SoCG'13]**
  Point separation for one pair is polynomial-time solvable.

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- **Kumar et al. [SODA'21]**
  Obstacle removal admits a (complicated) constant approximation.

Main results of our paper

- **Point separation (with multiple pairs)**
  - Can be solved in $n^{O(p)}$ time for p pairs of points
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  - For pseudo-disk obstacles:
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- **Obstacle removal**
  - Improved FPT algorithm: $2.3146^k \cdot n^{O(1)}$ time
  - Simpler constant-approximation algorithm

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Theorem (one-pair version)

Given two points $s, t \in \mathbb{R}^2$ and a set $\mathcal{O}$ of obstacles in $\mathbb{R}^2$, there exists a (multi-)graph G with vertex set $\mathcal{O}$ and edges labeled from $\{0, 1\}$ such that a subset $\mathcal{O}' \subseteq \mathcal{O}$ of obstacles separates s and t iff the induced subgraph $G[\mathcal{O}']$ contains a cycle whose sum of labels is odd.

The graph G can be built in $n^{O(1)}$ time given s, t and $\mathcal{O}$ where $n = |\mathcal{O}|$.

A structural characterization

- More pairs to be separated?

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Theorem (multi-pair version)

Given p point pairs $(s_1, t_1), \ldots, (s_p, t_p)$ in $\mathbb{R}^2$ and a set $\mathcal{O}$ of obstacles in $\mathbb{R}^2$, there exists a multi-graph G with vertex set $\mathcal{O}$ and edges labeled from $\{0, 1\}^p$ such that a subset $\mathcal{O}' \subseteq \mathcal{O}$ separates all p point pairs iff for every $i \in \{1, \ldots, p\}$ the induced subgraph $G[\mathcal{O}']$ contains a cycle whose sum of labels has an odd i-th entry.

The graph G can be built in $2^{O(p)} \cdot n^{O(1)}$ time given the pairs and $\mathcal{O}$.

Applications

- **Obstacle removal**
- Using our characterization theorem, we construct the multi-graph G.
- We want a minimum subset $\mathcal{X} \subseteq \mathcal{O}$ such that $\mathcal{O}' := \mathcal{O} \backslash \mathcal{X}$ does not separate s and t, i.e., $G[\mathcal{O}']$ does not contain an odd labeled cycle.

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- (Minimum) Odd Cycle Transversal
Given a graph G, find a minimum subset X of vertices of G that hit all odd cycles in G (or equivalently, $G - X$ is bipartite).
- (Parameterized) Odd Cycle Transversal
Given a graph G and a parameter $k \geq 0$, find a subset X of at most k vertices of G that hit all odd cycles in G.

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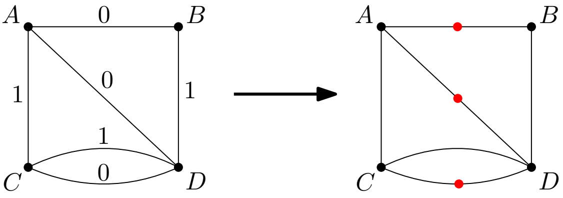

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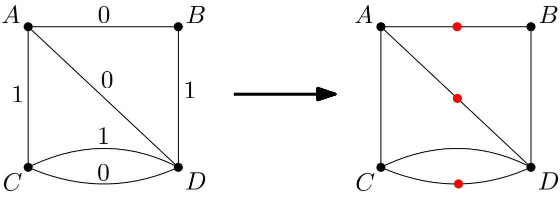

- **FPT algorithm:** Directly apply the $2.3146^k \cdot n^{O(1)}$-time algorithm for (Parameterized) Odd Cycle Transversal.
- **Constant approximation:** LP rounding using the technique of Lee (similar to the basic idea of **Kumar et al. [SODA'21]**).

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- We want a minimum subset $\mathcal{O}' \subseteq \mathcal{O}$ such that for any $i \in \{1, \ldots, p\}$, $G[\mathcal{O}']$ contains a cycle with odd i-th entry.
- **Key observation:**
  There is a subgraph H of $G[\mathcal{O}']$ that consists of $O(p)$ special vertices and $O(p)$ paths connecting these special vertices such that for every $i \in \{1, \ldots, p\}$, H contains a cycle with odd i-th entry.

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  1. Guess the structure of H and the label parities of the paths in H.
  2. Guess the special vertices of H.
  3. For each path in H connecting X and Y with label parity $\ell \in \{0, 1\}^p$, find the shortest path in G between X and Y with label parity ℓ.

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  2. Guess the special vertices of H.
  3. For each path in H connecting X and Y with label parity $\ell \in \{0,1\}^p$, find the shortest path in G between X and Y with label parity ℓ.
- In this way, we can compute H and solve the problem in $n^{O(p)}$ time.

Proof sketch of the characterization theorem

- Recall that we need a multi-graph G whose vertex set is $\mathcal{O}$ and edges are labeled from $\{0, 1\}$ such that

  (*) A subset $\mathcal{O}' \subseteq \mathcal{O}$ of obstacles separates s and t iff the induced subgraph $G[\mathcal{O}']$ contains a cycle whose sum of labels is odd.

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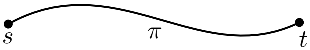

Fix a curve π connecting s and t in the plane.

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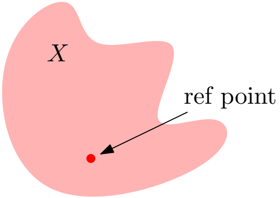

For each obstacle X, fix a reference point inside X.

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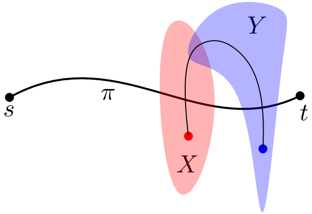

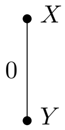

If there exists a curve connecting the ref points of X and Y in $X \cup Y$ that crosses π an **even** number of times, add an edge (X, Y) to G with label 0.

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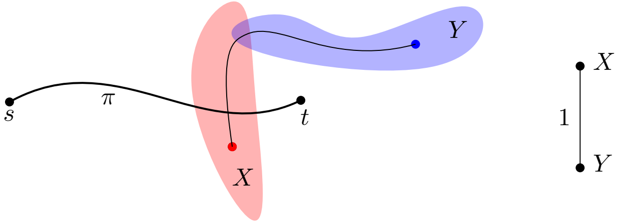

If there exists a curve connecting the ref points of X and Y in $X \cup Y$ that crosses π an **odd** number of times, add an edge (X, Y) to G with label 1.

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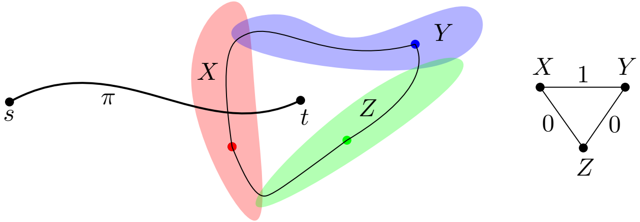

An odd labeled cycle in the graph corresponds to a closed curve in the union of the obstacles that crosses π an odd number of times.

Open questions

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- Approximation for point separation?

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- Higher dimensions?

Thank you!

Q & A