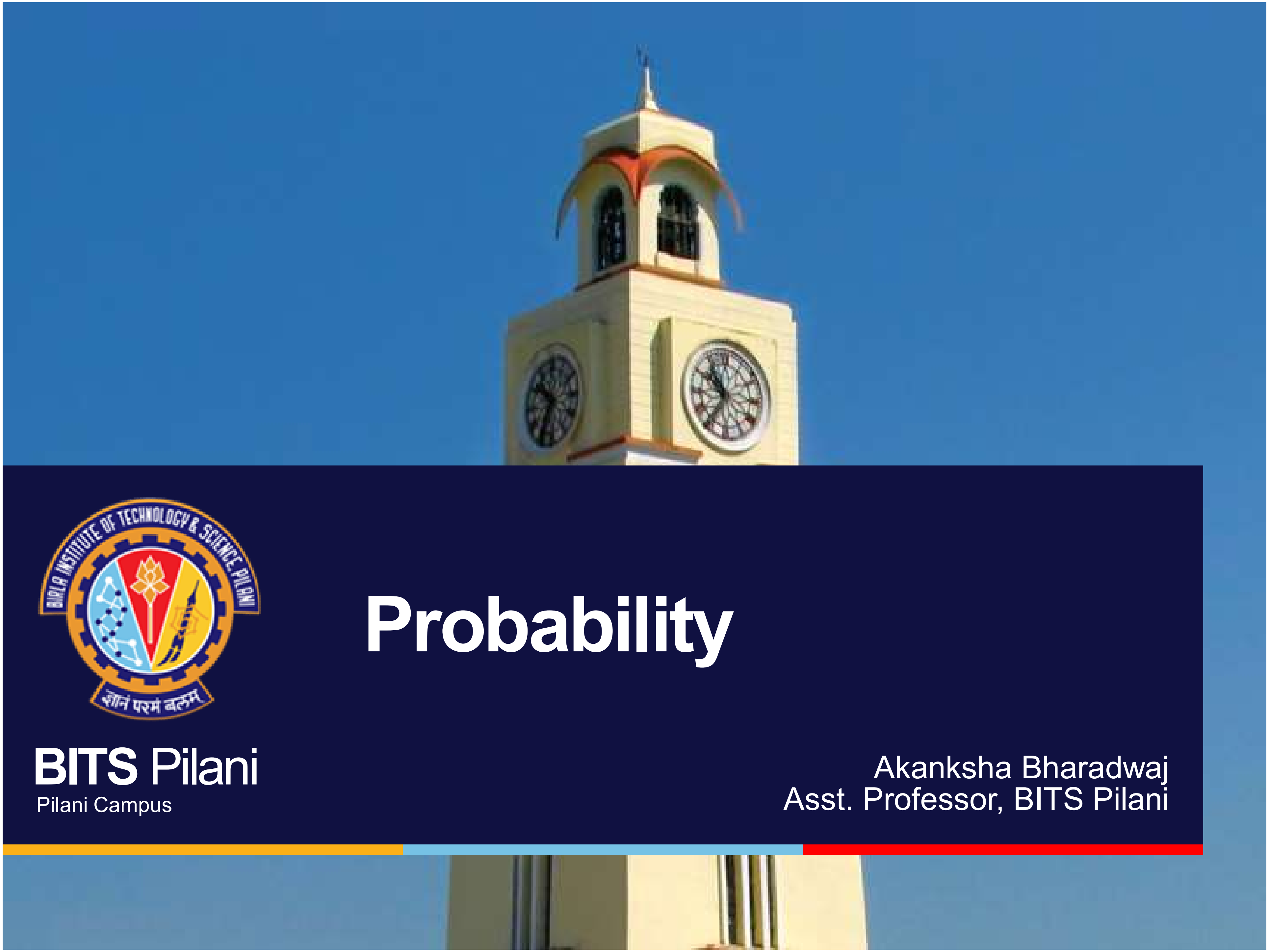

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Probability

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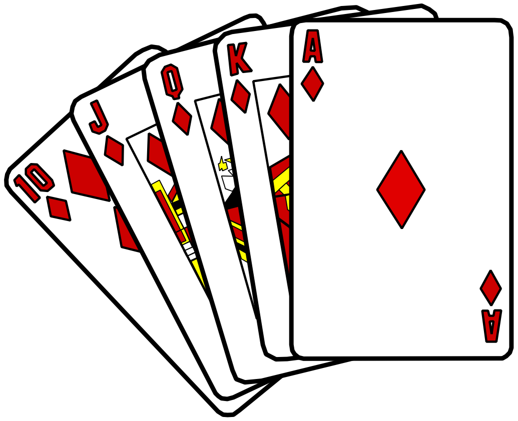
SS ZG536, ADV STAT TECHNIQUES FOR ANALYTICS
Contact Session 2

Agenda

- Experiments, Counting Rules, and Assigning Probabilities
- Events and Their Probability
- Some Basic Relationships of Probability
- Conditional Probability
- Bayes' Theorem



Probability

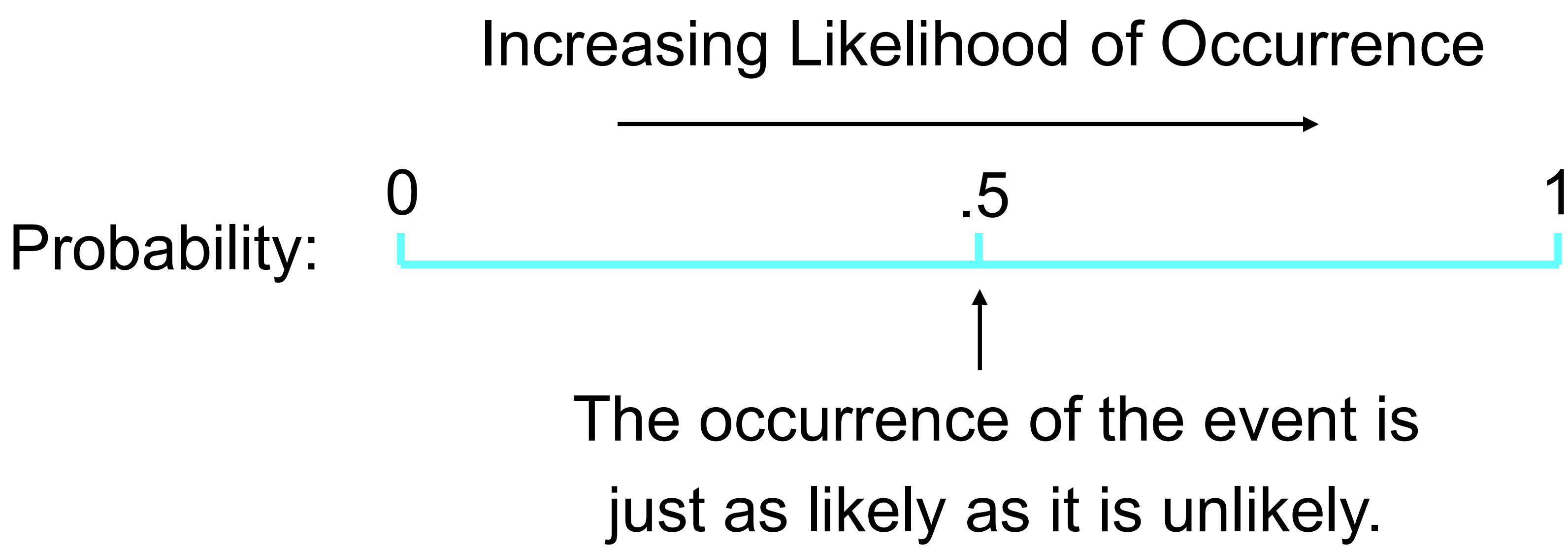
- The term probability refers to the study of randomness and uncertainty

- One way to think of probability is that it is the **likelihood** that something will occur.

- **Notation**: P(A) is the probability that event A will occur

$$P(A) = \frac{\text{no. of times event occurred}}{\text{total no. of events.}}$$

Probability as a Numerical Measure of the Likelihood of Occurrence



Sample Space

- The **sample space** of an experiment, denoted by S, is the set of all possible outcomes of that experiment

- Example: If we examine three fuses in sequence and note the result of each examination, then an outcome for the entire experiment is any sequence of N's and D's of length 3, where N represents not defective, D represents defective

- $S = \{NNN, NND, NDN, NDD, DNN, DND, DDN, DDD\}$

for 2 coins :-

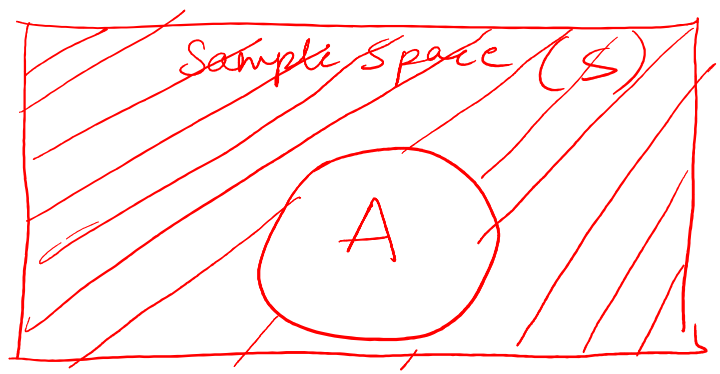
$S = \{HH, HT, TH, TT\}$

Event

- An event is a set of outcomes of the experiment. This includes the *null* (empty) set of outcomes and the set of *all* outcomes.

Complement of an Event

- The <u>complement</u> of event *A* is defined to be the event consisting of all sample points that are not in *A.*
- The complement of *A* is denoted by A^c or A’
- <u>Venn diagram</u> can illustrate the concept of a complement



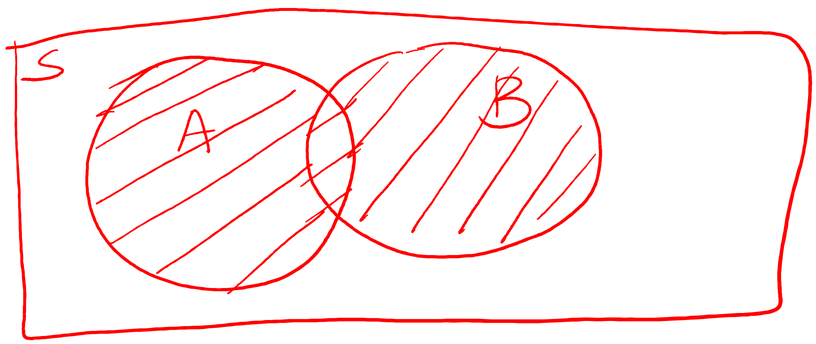
not A?

↳ Shaded part

↳ S − A

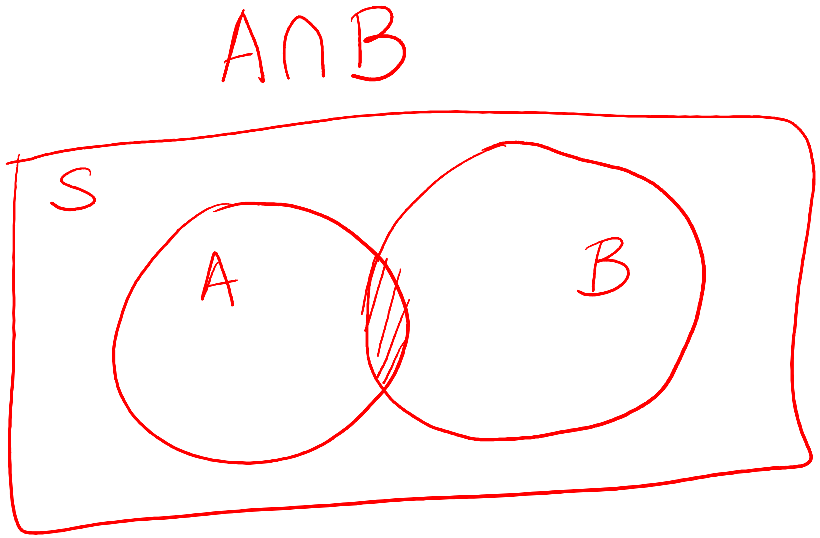
Union of Two Events

- The <u>union</u> of events *A* and *B* is the event containing all sample points that are in *A* or *B* or both.
- The union is denoted by *A* U *B*
- The union of *A* and *B* is:



Intersection of Two Events

- The <u>intersection</u> of events *A* and *B* is the set of all sample points that are in both *A* and *B*.
- The intersection of *A* and *B* is the area of overlap



Set Theory Example

- For the experiment in which the number of pumps in use at a single six-pump gas station is observed, let A={0,1,2,3,4}, B={3,4,5,6} , and C={1,3,5}.

Then, S= {0,1,2,3,4,5,6}

A'= $S-A = \{5,6\}$

$A \cap C = \{1,3\}$

A U B= $\{0,1,2,3,4,5,6\}$

A U C= $\{0,1,2,3,4,5\}$

A ∏ B= $\{3,4\}$

$A \cap B$

Disjoint events

- Two events that cannot occur at the same time are called **disjoint** or **mutually exclusive.**
- The idea of **disjoint events** is about whether or not it is possible for the events to occur at the same time

eg:- flipping of coin

event A = Head

event B = tail

$A \cap B = ?$

A & B are disjoint

$A \cap B = 0$

Non-disjoint events

- Two events can occur at the same time

- For non-disjoint events A and B

$A \cap B \neq 0$ / NULL

Independent events

- The idea of **independent events** is about whether or not the events affect each other in the sense that the occurrence of one event affects the probability of the occurrence of the other

- The event of getting a head on the first toss of a coin is independent of getting a head on the second toss.

Dependent events

- The occurrence of one event gives information about the occurrence of the other

- Suppose we have 5 blue marbles and 5 red marbles in a bag. We pull out one marble, which may be blue or red. Now there are 9 marbles left in the bag.

- The probability that the second marble will be red depends on first outcome

Basic properties of probability

- For any event *A*, P(A) >= 0

- P(S)= 1

A_1 A_2 A_3 A_4 ....

- If $A_{1,}$ $A_{2,}$ A_3......... is an infinite collection of disjoint events, then $P(A_1 \cup A_2 \cup A_3 \ldots\ldots\ldots) = \sum_{i=1}^{\infty} P(A_i)$

- Complement rule: P(not A) = 1 - P(A);

Determining Probability

These three represent distinct conceptual approaches to the study of probability theory.

1. Classical approach
2. Relative frequency approach
3. Subjective probability

Classical method

- **Classical** methods are used for games of chance, such as flipping coins, rolling dice, spinning spinners, roulette wheels, or lotteries.

- When probabilities are assigned based on laws and rules, the method is referred to as the classical method of assigning probabilities.

CLASSICAL METHOD OF ASSIGNING PROBABILITIES

$$P(E) = \frac{n_e}{N}$$

where

N = total possible number of outcomes of an experiment

n_e = the number of outcomes in which the event occurs out of N outcomes

Example

- Example: Each traditional (cube-shaped) die has six sides, marked in dots with the numbers 1 through 6.
- On a "fair" die, these numbers are equally likely to end up face-up when the die is rolled.
- Thus, P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = ?

$1/6$

Limitation of Classical definition

The classical definition of probability has certain drawbacks and fails at times in different situations as described below:

- This emphasizes that the events must be equally likely. Thus, it fails when various outcomes of a trail are not equally likely.

- For example if a die is biased that gives numbers greater than 3 more often than the numbers less than 3, then the occurrence of numbers on the die is not equally probable.

Limitation of Classical definition

- It does not consider those situations that are unlikely but that could conceivably happen.

- Like the occurrence of a coin landing on its edge or our room burning down while watching TV etc., which are extremely unlikely but not impossible.

Random experiment

- Term "**random experiment**" is used to describe any action whose outcome is not known in advance.

Relative frequency

- To estimate the probability of event A, written P(A), we may repeat the random experiment many times and count the number of times event A occurs.

PROBABILITY BY RELATIVE FREQUENCY OF OCCURRENCE

$$\frac{\text{Number of Times an Event Occurred}}{\text{Total Number of Opportunities for the Event to Occur}}$$

- This is also called the **relative frequency of event A**.

- Relative frequency of occurrence is not based on rules or laws but on what has occurred in the past.

Exercise

- The random experiment is rolling a fair die once.
- The sample space of all possible outcomes in this case this is S = {1, 2, 3, 4, 5, 6}.
- If we had following events in past 6,5,1,4,2,4,6,3,1,1,2,4
- We are interested in a particular type of outcome, which is represented by event E—getting an even number. What is the probability for this based on the past outcomes?

$P(4) = 3/12$

$P(2) = 2/12$

$P(6) = 2/12$

$P(even) = P(4) + P(2) + P(6)$

$= 7/12$

Limitations of relative frequency

1. The condition of an experiment may not remain the same in long series of trials

2. The relative frequency may not attain a unique value in spite of a large number of trials

Subjective Probability

- It is an estimate that reflects a person’s opinion, or best guess about whether an outcome will occur.

- These are values (between 0 and 1 or 0 and 100%) assigned by individuals based on how likely they think events are to occur.

- Example: The probability of candidate winning in an election is based on opinion poll is 60%.

Exercise

- On the "Information for the Patient" label of a certain diabetes medicine it is claimed when taking this medication
- there is a 10% chance of experiencing sleeping problems (denote this event by S) $P(S) = 0.1$
- there is a 29% chance of experiencing headaches (denote this event by H), and $P(H) = 0.29$
- - there is a 35% chance of experiencing at least one of these two side effects (denote this event by A) $0.35 = P(A)$

Ques. What is the probability that a patient taking this drug will not experience insomnia?

Ques. The probability of "the patient will experience neither of the two side effects"?

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P(Sleep problem)=0.1 $(P(S))$

P(Headache)=0.29 / $P(H)$

P(Atleast 1)=0.35 / $P(A)$

Q1. $P(\text{not } S) = 1 - 0.1 = 0.9$

Q2. $P(\text{neither of 2 problems}) = 1 - 0.35$

$= 0.65$

Exercise

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- The sales manager of an e-commerce company says that 80% of those who visit their website for the first time do not buy any mobile. If a new customer visits the website first-time, what is the probability that the customer would buy mobile

$$P(\text{visits \& buy}) = 1 - 0.8$$
$$= 0.2$$

Exercise

- A woman's pocket contains two dollars and two pennies. She randomly extracts one of the coins and, after looking at it, replaces it before picking a second coin.
- Let Q1 be the event that the first coin is a dollar and Q2 be the event that the second coin is a dollar.
- **Are Q1 and Q2 independent events? Why?**

Yes because of replacement of coin

Exercise

- A woman's pocket contains two dollars and two pennies. She randomly extracts one of the coins, and **without placing** it back into her pocket, she picks a second coin. As before, let Q1 be the event that the first coin is a dollar, and Q2 be the event that the second coin is a dollar.
- Are Q1 and Q2 independent events? Why?

No

Not replacing

Exercise

- 40% people have blood group as B, 30% have blood group as A, 25% have blood group as O and 5% have blood group as AB. From a **large population** three people are randomly selected. Let,
- P(B1)=probability that first person is of blood group B
- P(B2)=probability that second person is of blood group B
- P(B3)=probability that third person is of blood group B

- Are P(B1), P(B2) and P(B3) independent?

Yes

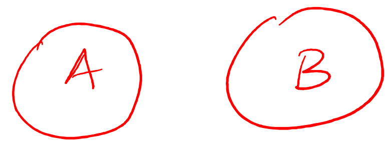
large population & random selection

P(A or B)

$P(A \cup B)$

- For disjoint events

  P(A and B)= 0

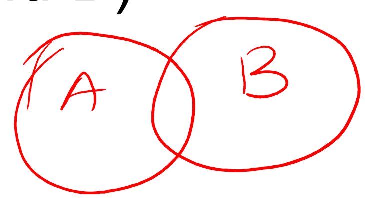
  P(A or B) = P(event A occurs or event B)

  i.e, P(A or B) = P(A) + P(B) (**addition rule**)

- For non-disjoint events

  P(A or B) = P(A) + P(B) - P(A and B)

  (**General Addition Rule)**

Exercise

- In a city it was observed that 80% of the families owns a two wheeler and 43% owns a car. Those who owns both are 38%. If a family is selected at random what is the probability that they own either a two wheeler or a car.

$P(A)$ = owns two wheeler = 0.8

$P(B)$ = owns a car = 0.43

$P(A \cap B) = 0.38$ non-disjoint events

$P(A \text{ or } B) / P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.8 + 0.43 - 0.38$

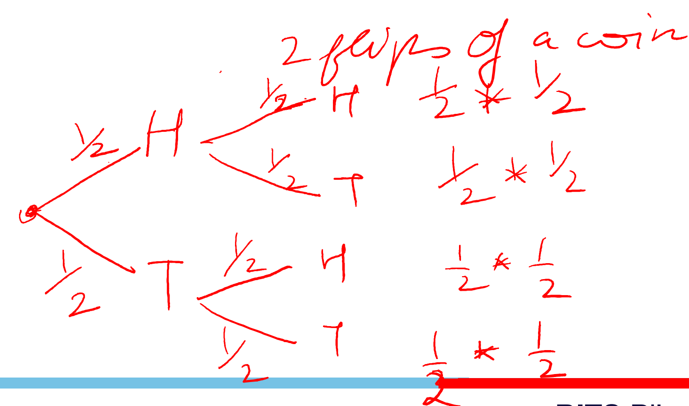
$= 0.85$

P(A and B)

- P(A and B), the probability that both events A and B occur.
- P(A and B) = P(event A occurs and event B occurs)
- So, if events **A and B are disjoint**, then (by definition)

  P(A and B)= 0
- For **non-disjoint** events, P(A and B) **is not equal to 0**
- For **independent** events

  P(A and B) = P(A) * P(B)



Exercise

- A fair coin is tossed 10 times. Which of the following two outcomes is more likely?

$\frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} \ldots \ldots$ 10 times

independent events? yes

(a) HHHHHHHHHH $\left(\frac{1}{2}\right)^{10}$

(b) THTTTTHTTT $\left(\frac{1}{2}\right)^{10}$

Exercise (HW)

- A quiz consists of 10 multiple-choice questions, each with 4 possible answers, only one of which is correct. A student an independent random guess to answer each of the 10 questions. What is the probability that the student gets at least one question right?

$P(\text{correct}) = 0.25$

$P(\text{not correct}) = 0.75$

The mn Rule

- If an experiment is performed in two stages, with ***m*** ways to accomplish the first stage and ***n*** ways to accomplish the second stage, then there are ***mn*** ways to accomplish the experiment.

- This rule is easily extended to ***k*** stages, with the number of ways equal to $n_1 n_2 n_3 \ldots n_k$

- **Example:** Toss two coins. The total number of simple events is:
- **2 X 2 = 4**

Sampling from a Population with Replacement

- Suppose in a lottery six numbers are drawn from the digits 0 through 9, with replacement
- (digits can be reused). How many different groupings of six numbers can be drawn?
- N is the population of 10 numbers (0 through 9) and n is the sample size, six numbers.

$$N^n = (10)^6$$

- That is, a million six-digit numbers are available!
- Here N is the population size and n is the sample size

Permutation

- The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

where $n! = n(n-1)(n-2)...(2)(1)$ and $0! \equiv 1$.

$4! = 4 \times 3 \times 2 \times 1 \times 0!$
$= 24$

Example: How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?

The order of the choice is important!

123, 321, 132 ...

$$^{4}P_{3} = \frac{4!}{1!} = 4(3)(2) = 24$$

Permutation

- **Example:** A lock consists of five parts and can be assembled in any order. A quality control engineer wants to test each order for efficiency of assembly. How many orders are there?

The order of the choice is important!

$$^{5}P_{5} = \frac{5!}{0!} = 5(4)(3)(2)(1) = 120$$

Combination: Sampling from a Population Without Replacement

- The number of distinct combinations of ***n*** distinct objects that can be formed, taking them ***r*** at a time is

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Example: Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

The order of the choice is not important!

$$^{5}C_{3} = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

$P_1P_2P_3 = P_2P_1P_3 = P_3P_2P_1 \cdots$

Combination

- A box contains six balls, four red and two green. A child selects two balls at random. What is the probability that exactly one is red?

The order of the choice is not important!

$$^{6}C_{2} = \frac{6!}{2!4!} = \frac{6(5)}{2(1)} = 15$$

ways to choose 2 balls.

$$^{4}C_{1} = \frac{4!}{1!3!} = 4$$

ways to choose 1 red ball.

$$^{2}C_{1} = \frac{2!}{1!1!} = 2$$

ways to choose 1 green ball.

4×2 =8 ways to choose 1 red and 1 green ball.

P(exactly one red) = 8/15

Exercise

- Suppose that there were 120 students in the classroom, and that they could be classified as follows. Calculate P(A U B).

non-disjoint events

A B

P (A and B)

$P(A \cap B) = ? = 30/120$

	Brown hair	Not Brown hair
Male	20	40
Female	30	30

A: brown hair

P(A) = 50/120

B: female

P(B) = 60/120

$P(A \cup B) = ?$

P(A or B) / $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{50}{120} + \frac{60}{120} - \frac{30}{120} = 80/120$

Exercise

- Suppose that there were 120 students in the classroom, and that they could be classified as follows. Calculate P(A U B).

Disjoint events

	Brown hair	Not Brown hair
Male	20	40
Female	30	30

A: male with brown hair
$P(A) = 20/120$
B: female with brown hair
$P(B) = 30/120$

$P(A \cup B) = ?$

$P(A \cup B) = P(A) + P(B) = 50/120$

Probability table

- Complete the following table using P(D) = 0.95, P(B) = 0.40, D and B are independent → P(D and B) = 0.38

$P(\text{not } B) = ?$
$= 1 - 0.4$
$= 0.6$

	B	not B	Total
D	0.38	x	0.95
not D	0.02	y	0.05
Total	0.4	0.6	1

$P(\text{not } D) = ?$
$= 1 - 0.95$
$= 0.05$

$P(B \text{ and not } D) = ? = 0.4 - 0.38 = 0.02$

$P(D \text{ and not } B) = x = ? = 0.95 - 0.38$

$y = ? = 0.6 - x$ or $0.05 - 0.02$
$= 0.03$

Conditional probability

- Conditional probability is denoted P(E1 | E2)
- This expression is read: the probability that *E1* will occur given that *E2* is known to have occurred.
- Conditional probabilities involve knowledge of some prior information.
- The information that is known or given is written to the right of the vertical line in the probability statement.

Example

- An example of conditional probability is the probability that a person owns a Chevrolet given that she owns a Ford.
- This conditional probability is only a measure of the proportion of Ford owners who have a Chevrolet—not the proportion of total car owners who own a Chevrolet.

Conditional Probability

If A and B are any two events in S, and P(B) ≠ 0, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{provided } P(B) \neq 0$$

Similarly, if A and B are any two events in S, and P(A) ≠ 0, the conditional probability of B given A is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{provided } P(A) \neq 0$$

Conditional Probability

$$P(A|B)=\frac{P(A\cap B)}{P(B)} \Rightarrow \underline{P(A\cap B)=P(B)P(A|B)} \quad \ldots (1)$$

and $P(B|A)=\frac{P(A\cap B)}{P(A)} \Rightarrow P(A\cap B)=P(A)P(B|A)$

$\ldots (2)$

dependent events / independent event $\longrightarrow P(A\cap B) = P(A) * P(B)$

P(A and B)

$$\therefore \underline{P(A\cap B)=P(B)P(A|B)=P(A)P(B|A)} \quad \ldots (3)$$

Exercise

- A manufacturer of airplane parts knows from the past experience that the probability is 0.80 that an order will be ready for shipment on time, and it is 0.72 that an order will be ready for shipment and will also be delivered on time. What is the probability that such an order will be delivered on time given that it was ready for shipment on time?

$P(R) = 0.8$ $\quad$ $P(R \cap D) = 0.72$

$$P(D|R) = \frac{P(R \cap D)}{P(R)} = \frac{0.72}{0.8}$$

$$= 0.9$$

Exercise: Marginal Probabilities

Consider the example on Sex wise blood group distribution

Blood group	Male	Female	Total
O	20	20	40
A	17	18	35
B	8	7	15
AB	5	5	10
Total	50	50	100

What is the probability of a person selected randomly will have blood group A? $P(A) = 35/100$

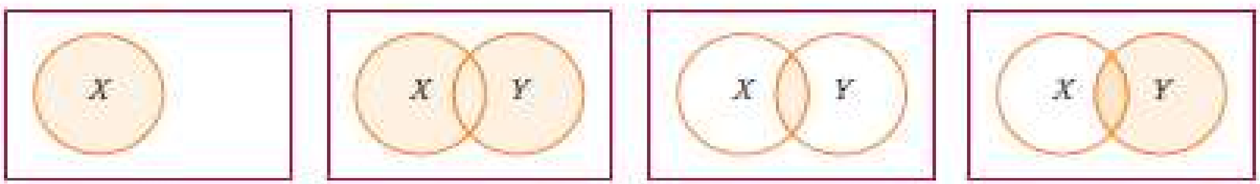
Solution

Marginal probabilities appear on the “margins” of a probability table. It is probability of single outcome

Blood group	Male	Female	Total	Row probabilities
O	20	20	40	40/100
A	17	18	35	35/100
B	8	7	15	15/100
AB	5	5	10	10/100
Total	50	50	100	1
Column probabilities	$\frac{50}{100}$	$\frac{50}{100}$	1	

Marginal, Union, Joint, and Conditional Probabilities

Marginal	Union	Joint	Conditional
$P(X)$	$P(X \cup Y)$	$P(X \cap Y)$	$P(X \mid Y)$
The probability of X occurring	The probability of X or Y occurring	The probability of X and Y occurring	The probability of X occurring given that Y has occurred
Uses total possible outcomes in denominator	Uses total possible outcomes in denominator	Uses total possible outcomes in denominator	Uses subtotal of the possible outcomes in denominator



Example

What is the probability that a person selected has blood group B given that he is male?

Blood group	Male	Female	Total
O	20	20	40
A	17	18	35
B	8	7	15
AB	5	5	10
Total	50	50	100

Probability to be computed

Given information

$$P(B \mid Male) = \frac{P(B \cap Male)}{P(Male)} = \frac{8/100}{50/100}$$

8/50 = 0.16 ???

How ???

Example

What is the probability that a person selected is male given that his blood group is B?

Blood group	Male	Female	Total
O	20	20	40
A	17	18	35
B	8	7	15
AB	5	5	10
Total	50	50	100

Given information

$P(Male \mid B) = \frac{P(Male \cap B)}{P(B)}$

8/15 = 0.53 ???

Probability to be computed

How ???

Rule of total Probability

Theorem (Rule of total probability):

If B_1, B_2, …, B_n are mutually exclusive events of which one must occur, and A is a common event among them then

$P(A|B) = \frac{P(A\cap B)}{P(B)}$ $\Rightarrow P(A\cap B) = P(B) * P(A|B)$

conditional prob.

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A|B_i)$$

$= (A\cap B_1) + (A\cap B_2) + (A\cap B_3) \ldots\ldots$

Example

- Suppose for instance, that an assembly plant receives its voltage regulators from three suppliers,
- 60% from supplier B1, 30% from supplier B2, and 10% from B3.
- If 95% of voltage regulators from B1, 80% from B2 and 65% from B3 perform according to specification,
- **what we would like to know is the probability that any one voltage regulator received by the plant will perform according to specifications**

$P(A)$ = performing according to specification

Solution

- If A denotes the event that a voltage regulator received by the plant performs according to specifications and B1, B2, and B3 are the events that it comes from the respective suppliers, we can write

  $A = A \cap [B_1 \cup B_2 \cup B_3]$

  $\quad = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$

  $P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$

- since $A \cap B_1$, $A \cap B_2$, $A \cap B_3$ are mutually exclusive. By using general multiplication rule we get

  $P(A) = P(B_1).P(A|B_1)+P(B_2).P(A|B_2) +P(B_3).P(A|B_3)$

$P(A) = (0.60)+(0.95) + (0.30)(0.80) + (0.10)(0.65)$

$\quad = 0.875$

Bayes' Theorem

- Bayes' rule is a formula that extends the use of the law of conditional probabilities to allow revision of original probabilities with new information.

$P(A|B_i)$

$$\frac{P(X_i \cap Y)}{P(Y)} = P(X_i|Y) = \frac{P(X_i)\cdot P(Y|X_i)}{P(X_1)\cdot P(Y|X_1) + P(X_2)\cdot P(Y|X_2) + \cdots + P(X_n)\cdot P(Y|X_n)}$$

- The numerators of Bayes' rule and the law of conditional probability are the same
- This denominator is sometimes referred to as the "total probability formula."

Example

Example 1:

In a certain assembly plant, three machines, B1, B2, and B3 make 30%, 45%, and 25% respectively of the products. It is known from the past experience that 2%, 3% and 2% of the products made by each machine respectively, are defective.

(i) Suppose a finished product is randomly selected, what is the probability that it is defective? P(Defective)

(ii) If a product chosen randomly is found defective, what is the probability that it was made by machine B3?

P (B3| defective)

Solution

Let, A: event that the product is defective

B1: an event that product made by machine B1

B2: an event that product made by machine B2

B3: an event that product made by machine B3

P(B1)= 0.3, P(B2)= 0.45, P(B3)= 0.25

P(A|B1)= 0.02, P(A|B2)= 0.03, P(A|B3)= 0.02

P(A and B1)= P(B1) X P(A|B1)= 0.3 X 0.02= 0.006

P(A and B2)= 0.45 X 0.03= 0.0135

P(A and B3)= 0.25 X 0.02= 0.005

P(A)= 0.006+0.0135+0.005= 0.0245 (Rule of total probability)

P(B3|A)= P(A and B3)/P(A)= 0.005/0.0245= 0.204

(Bayes' Theorem)

Example

Example 2:

Four technicians regularly make repairs when breakdowns occur on an automated production line. Janet, who services **20**% of the breakdowns, makes an incomplete repair 1 time in 20; Tom, who services **60**% of the breakdowns makes an incomplete repair 1 time in 10; Georgia, who services **15**% of the breakdowns, makes an incomplete repair 1 time in 10; and Peter, who services **5**% of the breakdowns, makes an incomplete repair 1 time in 20.

For the next problem with the production line diagnosed as being due to an initial repair that was incomplete, what is the probability that this initial repair was made by Janet?.

Solution

Let, A: be the event that the initial repair was incomplete,

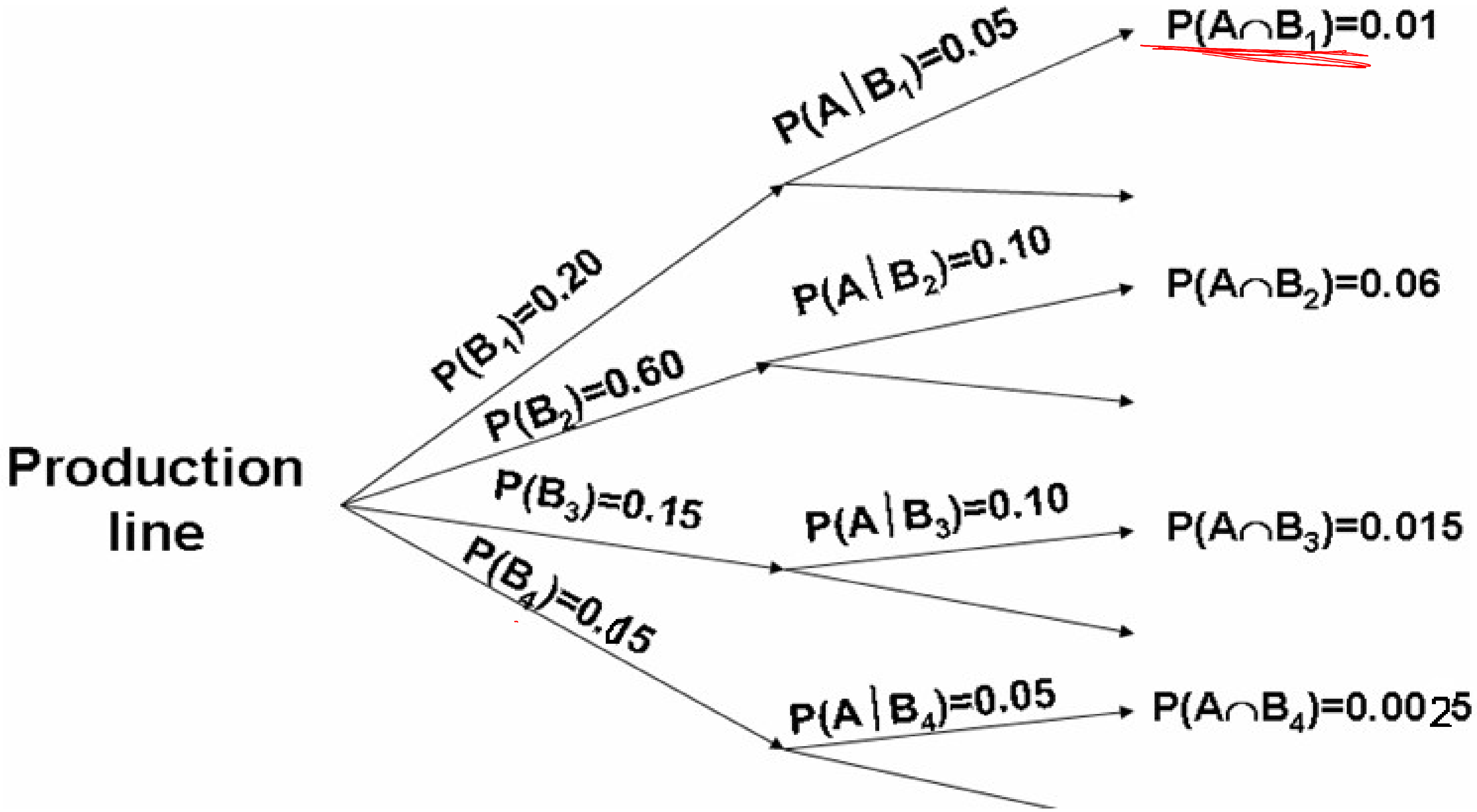
B1: an event that the initial repair was made by Janet,

B2: an event that the initial repair was made by Tom,

B3: an event that the initial repair was made by Georgia,

B4: an event that the initial repair was made by Peter.

Solution



P(A)= 0.01+0.06+0.015+0.0025= 0.0875 (Rule of total probability)
P(B1|A)= P(A and B1)/P(A)= 0.01/0.0875= 0.1142 (Bayes' Theorem)

Exercise (HW)

- Using different rules of probability how can we determine if events A and B are indepedent?

- Hint: use conditional probability and multiplication rule

Solution

A and B are independent if

- $P(A)=P(A|B)$
- $P(B)=P(B|A)$
- $P(A|B)=P(A|\text{ not } B)$
- $P(A \text{ and } B)= P(A) * P(B)$

Exercise (HW)

- An individual has 3 different mail accounts. Most of her messages, in fact 70% come into account #1,whereas 20% come into account #2 and the remaining 10% into account #3.
- Of the messages into account #1, only 1% are spam where as the corresponding for accounts # 2 and # 3 are 2% and 5% respectively.

- What is the probability that a randomly selected message is spam?

References

- Probability and Statistics for Engineering and Sciences,8th Edition, Jay L Devore, Cengage Learning
- Applied Business Statistics, Ken Black