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SS ZC416 Mathematical Foundations for Data Science
Revision

DISCLAIMER!

- Solving some “difficult” problems

NOTE:

- This slide pack does not cover the entire course
- This slide pack **does not even** cover the material covered after the mid term exam
- You will need to study the entire semester material for the comprehensive exam
- The comprehensive exam will focus on the post mid-sem material **BUT** there can always be stray questions from the mid-sem portion

1. Optimization: Newton's Method

Starting at x = 1, apply one iteration of the Newton's method to approximate the solution of Min f(x) $= 0.5 - xe^{-x^2}$

f'(x) = $(2x^2 - 1)e^{-x^2}$

f''(x) = $2\mathrm{x}(3 - 2x^2)e^{-x^2}$

And

$$\mathrm{x}_{n+1} = \mathrm{x}_n - \frac{f'(\mathrm{x}_n)}{f''(\mathrm{x}_n)} = \mathrm{x}_n - \frac{(2x^2-1)e^{-x^2}}{2\mathrm{x}(3-2x^2)e^{-x^2}}$$

$\mathrm{x}_0 = 1$

$$\mathrm{x}_1 = \mathrm{x}_0 - \frac{(2x_0^2-1)e^{-x_0^2}}{2x_0(3-2x_0^2)e^{-x_0^2}} = 1 - \frac{(2*1^2-1)e^{-1^2}}{2*1(3-2*1^2)e^{-1^2}} = 1 - \frac{(2-1)e^{-1}}{2(3-2)e^{-1}} = 1 - \frac{e^{-1}}{2e^{-1}} = 0.5$$

2. Optimization: Steepest Descent Method

Starting at x = 6 & y = 3, apply one iteration of the Steepest Descent Method to approximate the solution of Min $f(x, y) = x^2 + 3y^2$

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 6y \end{bmatrix}$$

$$z(t) = \begin{bmatrix} x \\ y \end{bmatrix} - t\nabla f(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} - t\begin{bmatrix} 2x \\ 6y \end{bmatrix} = \begin{bmatrix} (1-2t)x \\ (1-6t)y \end{bmatrix}$$

$$g(t) = f(z(t) = x^2(1-2t)^2 + 3y^2(1-6t)^2$$

$$g'(t) = -4x^2(1-2t) - 36y^2(1-6t)$$

$$g'(t) = 0 \Rightarrow t = \frac{4x^2+36y^2}{8x^2+216y^2}$$

1st Iteration

$$t = \frac{4*36+36*9}{8*36+216*9} = 0.2097, z(t) = \begin{bmatrix} (1-2t)x \\ (1-6t)y \end{bmatrix} = \begin{bmatrix} 3.4836 \\ -0.7746 \end{bmatrix} \Rightarrow x_1 = 3.4836 \& \, y_1 = -0.7746$$

3. Optimization: Lagrange Multipliers – 1 of 2

Max 5xy subject to 2x + y = 100

The augmented function: z = 5xy + λ (2x + y -100)

$$\frac{\delta z}{\delta x} = 5y + \lambda 2; \ \frac{\delta z}{\delta y} = 5x + \lambda; \ \frac{\delta z}{\delta \lambda} = 2x + y - 100$$

$$\frac{\delta z}{\delta x} = 5y + \lambda 2 = 0; \ \frac{\delta z}{\delta y} = 5x + \lambda = 0; \ \frac{\delta z}{\delta \lambda} = 2x + y - 100 = 0 \ \Rightarrow$$

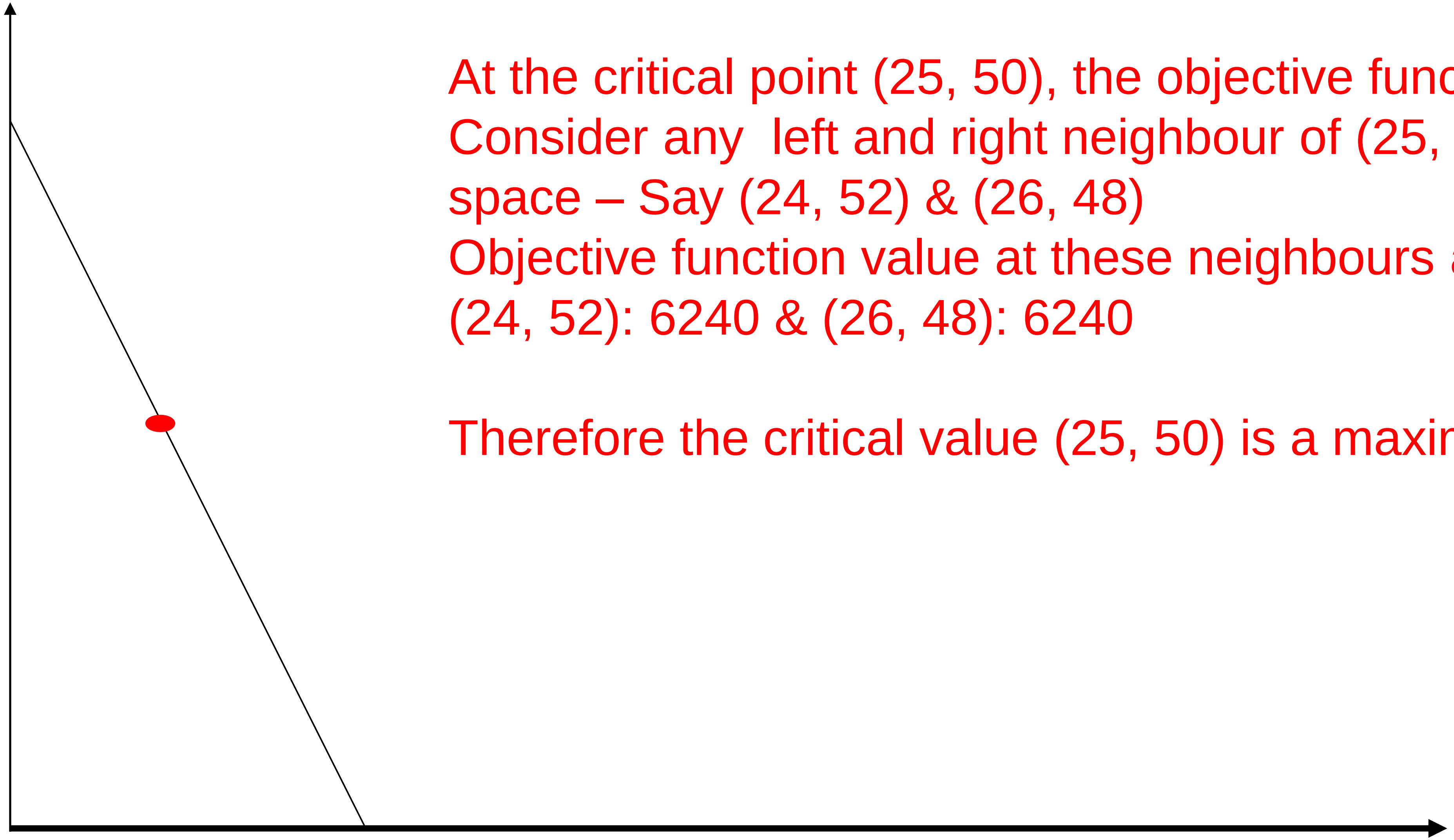
y = -2λ/5, x = -λ/5 & 2x + y -100 = -2λ/5 - 2λ/5 -100 = 0 → λ = 500 / 4 = 125

That is: x = 25 and y = 50 & 5xy = 6250

Is x = 25 and y = 50 a maxima?

3. Optimization: Lagrange Multipliers – 2 of 2

Max 5xy subject to 2x + y = 100



At the critical point (25, 50), the objective function value is 6250
Consider any left and right neighbour of (25, 50) on the feasible space – Say (24, 52) & (26, 48)
Objective function value at these neighbours are:
(24, 52): 6240 & (26, 48): 6240

Therefore the critical value (25, 50) is a maxima

4. Truth Table

Construct the Truth Table for F(x, y, z) = (xz + y)’

x	y	z	xz	xz+y	(xz + y)’
0	0	0	0	0	1
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	0
1	0	0	0	0	1
1	0	1	1	1	0
1	1	0	0	1	0
1	1	1	1	1	0

5. Maxima & Minima of f(x, y)

Find the maxima, minima and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$

$f_x = 4x^3 - 4y = 0 \Rightarrow y = x^3$. Hence $f_y = 4y^3 - 4x = 0 \Rightarrow x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$
$\Rightarrow$ The three real roots are: $x = 0, 1, -1$

Now $f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$ & $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$

- $D(0, 0) = -16 < 0 \Rightarrow$ The origin is a saddle point.
- $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0 \Rightarrow (1, 1)$ is a local minimum
- $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0 \Rightarrow (-1, -1)$ is also a local minimum

6. Maxima & Minima of f(x)

Minimize / Maximize $y = 2\sqrt{x} + \sqrt{1-x}$ over the interval $[0,1]$

$y' = \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{1-x}}$

$y' = 0 \Rightarrow \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{1-x}} \Rightarrow 4(1-x) = x \Rightarrow x = \frac{4}{5}$ is the critical point

y = 1 at x = 0, y = $\sqrt{5}$ at x = 4/5 & y = 2 at x = 1.

$\Rightarrow$ y is maximized at $(\frac{4}{5}, \sqrt{5})$ and minimized at $(0,1)$.

7. Transitive Closure: Warshall's Algorithm

Find the transitive closure of R, where R ={(1,2), (2,3), (3,4)} is a relation on X = {1,2,3, 4}

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

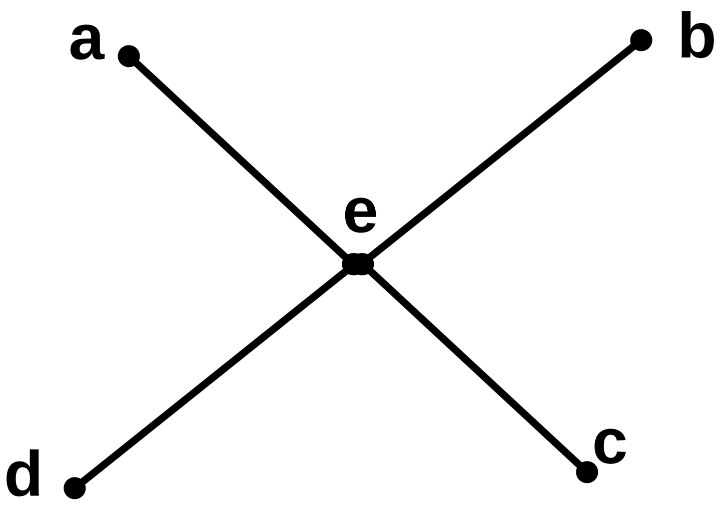
The transitive closure: $R^* = R \vee R^2 \vee R^3 \vee, R^4 =$ ={(1,2), (2,3), (3,4), (1,3), (2, 4), (1,4)}

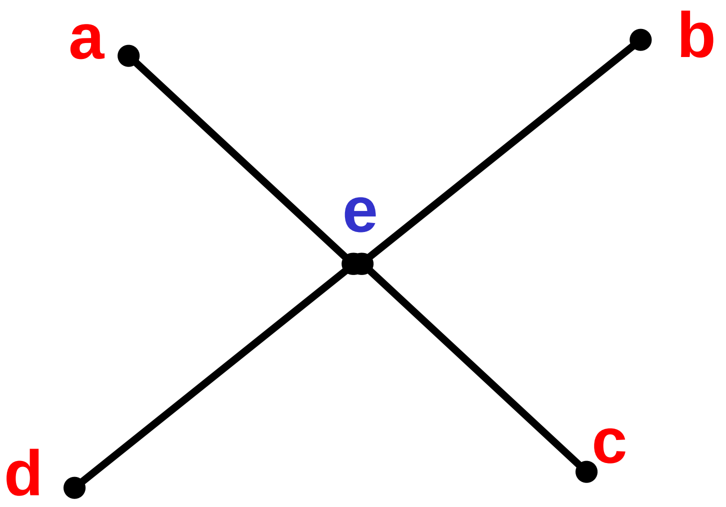
The Boolean product (example)

$$R^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8. Bipartite Graph

Determine whether the following is bipartite

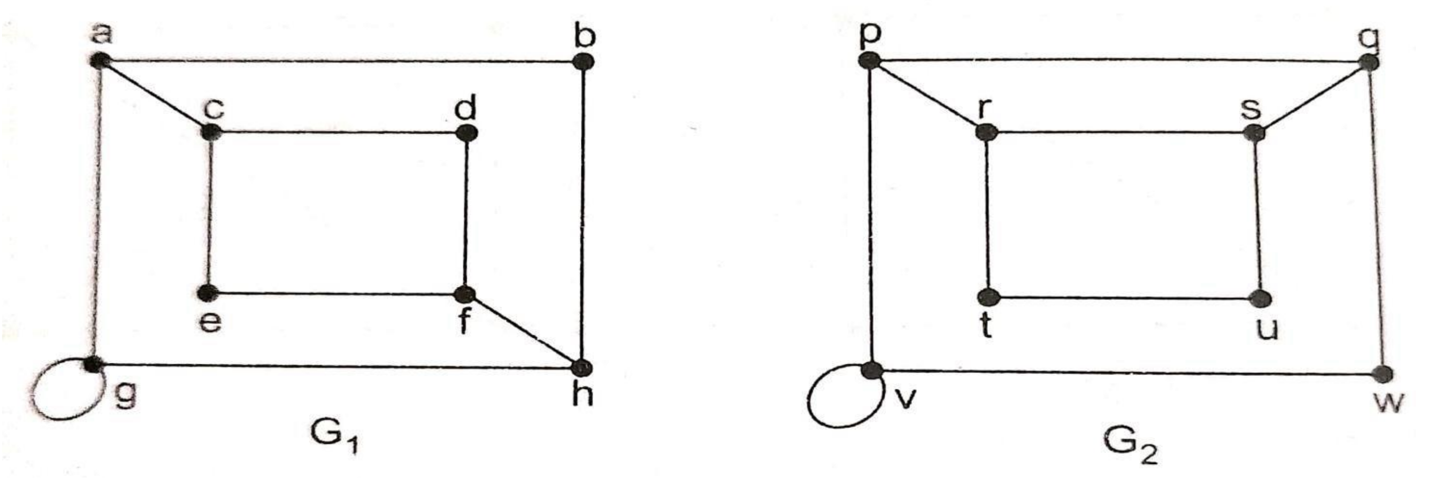




Since no adjacent vertices have the same colour, the graph is bipartite.

9. Isomorphic Graphs

Determine whether the following graphs are isomorphic.



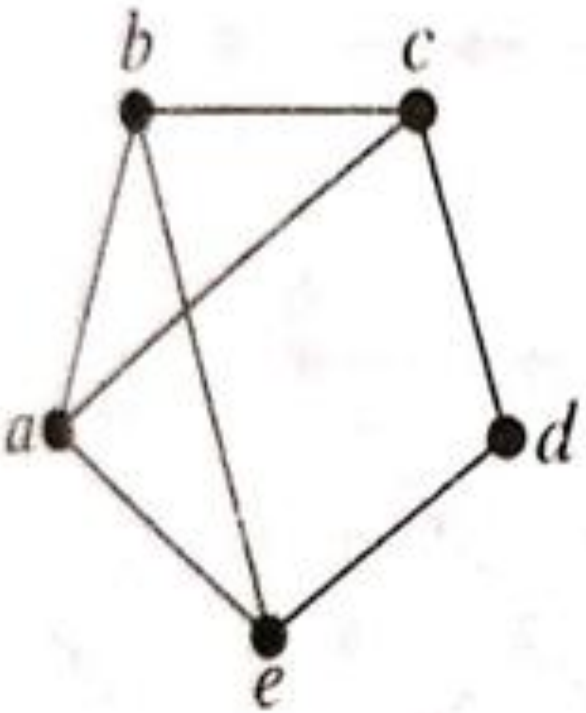
The degree sequence of G_1: 3, 3, 3, 3, 3, 2, 2

The degree sequence of G_2: 3, 3, 3, 3,2, 2, 2

Therefore the graphs are not isomorphic

10. Euler Path / Circuit

Determine whether the graph has an Euler Path or Circuit. If yes, construct it.



Each vertex has even degree? No ⇒ no Euler circuit

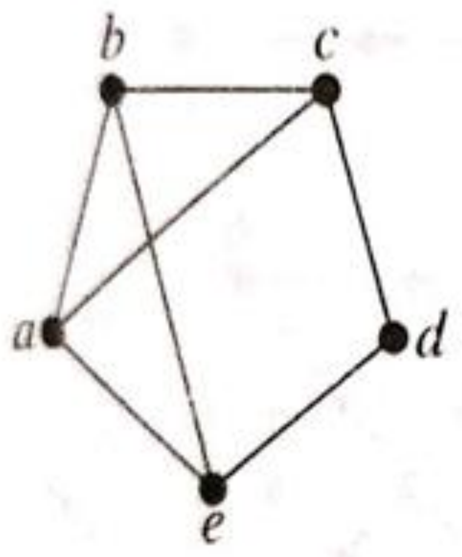
- For example, vertex a has degree = 3

Exactly two vertices of odd degree? No ⇒ no Euler path

- Vertices a, b, c, & e have odd degrees

11. Hamilton Path / Circuit

Does the graph have a Hamilton path or Hamilton circuit? If so, find such a path



n = 5

- Dirac's Theorem fails since deg(d) = 2 < ceiling (n/2) = 3
- Ore's theorem is satisfied: since the degree of any vertex >= 2 and therefore deg(u)+ deg (v) ≥ n for every pair of non adjacent vertices

Hamiltonian circuit: a – b – c – d – e – a

12. LPP

Max $z = 5x_1 + 7x_2$

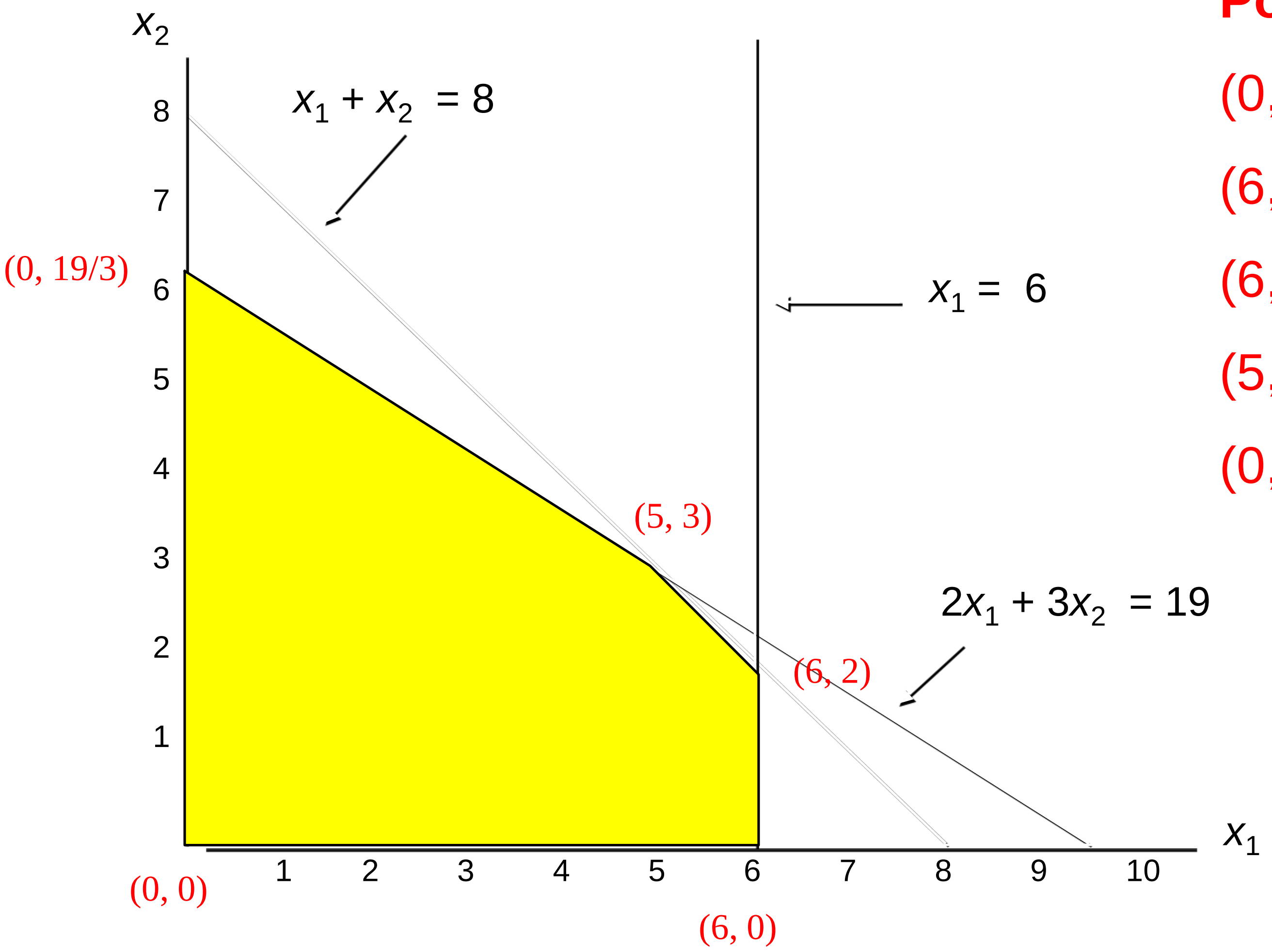
s.t.

$x_1 \quad <= 6$

$2x_1 + 3x_2 \quad <= 19$

$x1 + x2 \quad <= 8$

$x_1, x_2 \geq 0$



Point	Z
(0,0)	0
(6,0)	30
(6,2)	44
(5,3)	46
(0,19/3)	44.33

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Thank you!!