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SS ZC416 Mathematical Foundations for Data Science
Linear Algebra

Recap

Recap

- Linear System of equation
  - Homogeneous and Non-Homogeneous
  - Consistent and Inconsistent Systems
- Gauss Method
  - REF
  - Elementary Row Operations & Pivots
- Rank of a Matric
- Conditions for System of linear Equations to have
  - Unique / Infinitely many / No Solutions
- Gauss Jordan Methods
  - RREF

Inverse of a Matrix

Inverse of Matrix

A is an nxn matrix – a square matrix

Suppose there is an A^{-1}, an n x n matrix A such that

$$\mathbf{A\,A^{-1} = A^{-1}\,A = I}$$

- Then A^{-1} is said to be the inverse of A
- f A has an inverse, then A is called a non singular matrix
- If A has no inverse A is called singular matrix
- If A has inverse , then it is inverse is unique

Gauss Jordan Elimination to find A^{-1}

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

Consider the Partitioned matrix

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Perform elementary row operations until A becomes RREF

A is non-singular $\Leftrightarrow$ RREF = $[\mathrm{I} \mid \mathrm{A}^{-1}]$

Compute A^{-1}

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Perform elementary row operations until A becomes RREF

A is non-singular $\Leftrightarrow$ RREF = $[I \mid A^{-1}]$

Compute A^{-1}

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -1 & 1 & 0 \\ 0 & -5 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -5 & 0 & -2 & 0 & 1 \\ 0 & -5 & -2 & -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0.4 & 0 & -0.2 \\ 0 & -5 & -2 & -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -0.2 & 0 & 0.6 \\ 0 & 1 & 0 & 0.4 & 0 & -0.2 \\ 0 & 0 & -2 & 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -0.2 & 0 & 0.6 \\ 0 & 1 & 0 & 0.4 & 0 & -0.2 \\ 0 & 0 & 1 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0.3 & 0.5 & 0.1 \\ 0 & 1 & 0 & 0.4 & 0 & -0.2 \\ 0 & 0 & 1 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0 & -0.2 \\ -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Gauss Jordan Elimination to find A^{-1}

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

Consider the Partitioned matrix

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Perform elementary row operations until A becomes RREF

A is non-singular ⇔ RREF = $[\mathrm{I} \mid A^{-1}]$

$$A^{-1} = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0 & -0.2 \\ -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Verify!

Determinants

Determinant of a Matrix

A **determinant of order** n is a scalar associated with an n × n (hence **square!**) matrix **A = [a_{jk}]**, and is denoted by

(1)
$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}.$$

For n = 1, this determinant is defined by

(2)
$$D = a_{11}.$$

The Formula

For $n \geq 2$ by

- $D = a_{j1}C_{j1} + a_{j2}C_{j2} + \dots + a_{jn}C_{jn}$ ($j = 1, 2, \dots$, or n)

or

- $D = a_{1k}C_{1k} + a_{2k}C_{2k} + \dots + a_{nk}C_{nk}$ ($k = 1, 2, \dots$, or n).

Here,

$$C_{jk} = (-1)^{j+k}M_{jk}$$

and M_{jk} is a determinant of order $n - 1$, namely, the determinant of the submatrix of **A** obtained from **A** by omitting the j-th row and k-th column of the entry a_{jk}.

- M_{jk} is called the **minor** of a_{jk} in D, and
- C_{jk} the **cofactor** of a_{jk} in D.

Example – Cofactors & Minors & Determinant

Find the determinant of $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$

Example – Cofactors & Minors & Determinant

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Find the determinant of $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} = 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12$$

This is the expansion by the first row. The expansion by the third column is

$$D = 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0 - 12 + 0 = -12.$$

Facts

Behavior under Elementary Row Operations

(a) Interchange of two rows multiplies the value of the determinant by −1.

(b) Addition of a multiple of a row to another row does not alter the determinant value

(c) Multiplication of a row by a nonzero constant c multiplies the determinant by c. (This holds also when c = 0, but no longer gives an elementary row operation.)

(a)–(c) in Theorem 1 hold also for columns.

Further Properties of nth-Order Determinants

(d) Transposition leaves the value of a determinant unaltered.

(e) A zero row or column renders the value of a determinant zero.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

(g) det (**AB**) = det (**BA**) = det **A** det **B**.

Geometric Properties of Determinants

Given the following matrices and vectors

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \& B = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Compute

- Det(A), Det(B)
- Av_1, Av_2, Bv_1, Bv_2
- Sketch all the vectors
- Compare the unit square with the resulting parallelograms

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Compute

- Det(A), Det(B)
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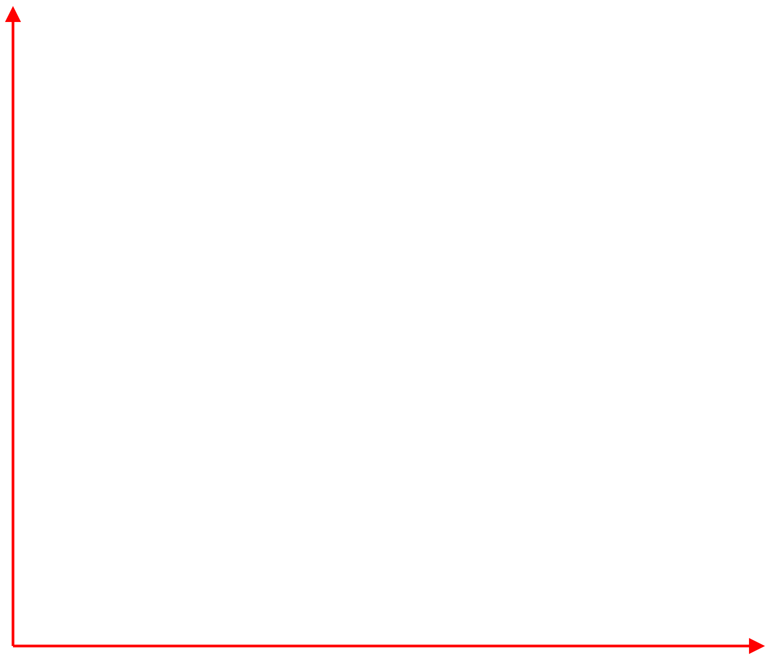
- Sketch all the vectors
- Compare the unit square with the resulting parallelograms

Det(A) = 4, Det(B) = -4

$[Av_1]^T$= [2 1], $[Av_2]^T$= [2 3],

$[Bv_1]^T$= [1 2], $[Bv_2]^T$= [3 2]]



Geometric Properties of Determinants

Given the following matrices and vectors

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Compute

- Det(A), Det(B)
- Av_1, Av_2, Bv_1, Bv_2
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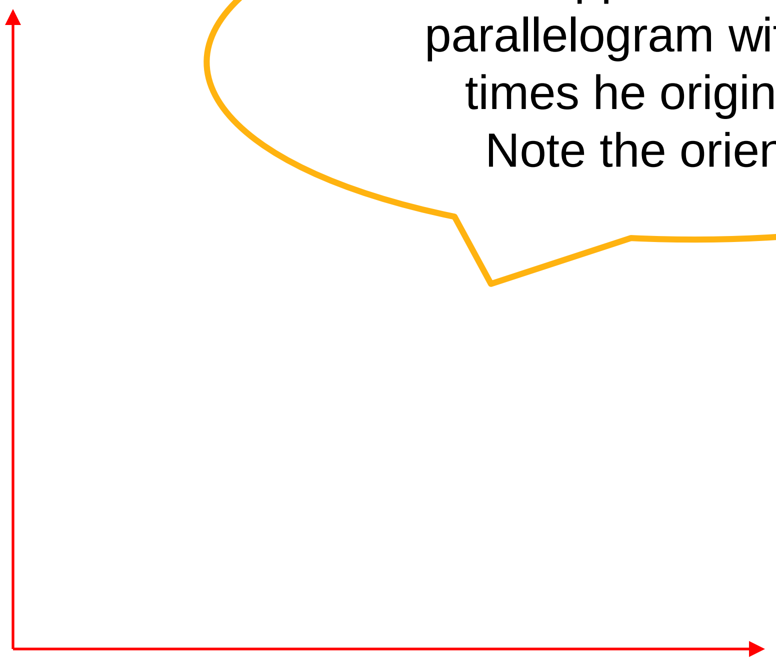
- Compare the unit square with the resulting parallelograms

Det(A) = 4, Det(B) = -4

$[Av_1]^T$= [2 1], $[Av_2]^T$= [2 3],

$[Bv_1]^T$= [1 2], $[Bv_2]^T$= [3 2]]

A has mapped the square to a parallelogram with area 4 times he original area
Note the orientation



Theorem: Rank in Terms of Determinants

Consider an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$:

1. A has rank $r \geq 1$ if and only if
- **A** has an $r \times r$ submatrix with a nonzero determinant
- The determinant of any square submatrix with more than **r** rows, contained in **A** (if such a matrix exists!) has a value equal to zero.

2. An $n \times n$ square matrix **A** has rank n if and only if det $\mathbf{A} \neq 0$.

3. If Det(A) = 0, the inverse does not exist

4. If Det(A) ≠ 0, the inverse exists

Cramer's Rule

(a) If a linear system of n equations in the same number of unknowns $x_1, \ldots, x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

has a nonzero coefficient determinant D = det **A**, the system has precisely one solution.

Example

Solve

$$x + y + z = 6$$
$$3x + 3y + 4z = 20$$
$$2x + y + 3z = 13$$

Last class we solved using Gauss Jordan Elimination Method,

$x = 3,\ y = 1,\ z = 2$

We can write this system of equations as Av = b, that is, $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$

Det(A) = 1 ≠ 0 and $A^{-1} = \begin{bmatrix} 5 & -2 & 1 \\ -1 & 1 & -1 \\ -3 & 1 & 0 \end{bmatrix}$ & $A^{-1}b = \begin{bmatrix} 5 & -2 & 1 \\ -1 & 1 & -1 \\ -3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Matrix Inverse Using Determinants

Unique Inverse

Recall

The inverse of an $n \times n$ matrix $A = [a_{jk}]$ is denoted by A^{-1} and is an $n \times n$ matrix

$$AA^{-1} = A^{-1}A = I$$

where I is the $n \times n$ unit matrix

If A is invertible then A is called a nonsingular matrix.

If A has no inverse, then A is called a singular matrix.

If A has an inverse, the inverse is unique.

- Suppose B and C are inverses of A
- Then $AB = I$ and $CA = I$
- Then $B = IB = (CA)B = C(AB) = CI = C$.

When does an Inverse of a Matrix Exist?

The inverse A^{-1} of an $n \times n$ matrix A exists if and only if rank $A = n$
The inverse A^{-1} of an $n \times n$ matrix A exists if and only if $\det A \neq 0$.

Hence A is nonsingular if rank $A = n$ and is singular if rank $A < n$.

Inverse of a Matrix by Determinants

The inverse of a nonsingular n × n matrix **A** = [a_{jk}] *is given by*

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\left[C_{jk}\right]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{21} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix},$$

where C_{jk} *is the cofactor of* a_{jk} *in* det **A**

Remarks

CAUTION

- **Note** that in A^{-1}, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in A

In particular, the inverse of

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad is \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Example

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

Example

Find the inverse of $\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$

Det **A** = −1(−7) − 1 · 13 + 2 · 8 = 10, and

$$\begin{bmatrix} +M_{11} & -M_{21} & +M_{31} \\ -M_{12} & +M_{22} & -M_{32} \\ +M_{13} & -M_{23} & +M_{33} \end{bmatrix}$$

Example

Find the inverse of $\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$.

Det **A** = −1(−7) − 1 · 13 + 2 · 8 = 10, and

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \quad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2, \quad C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \quad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} = 7,$$

$$C_{13} = \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} = 8, \quad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2, \quad C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

Example – Final Solution

Det **A** = 10, and

C11 = -7, C21 = 2, C31 = 3

C12 = -13, C22 = -2, C32 = 7

C13 = 8, C23 = 2, C33 = -2

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Fact

- $(AC)^{-1} = C^{-1}A^{-1}$.
- $(AC \ldots PQ)^{-1} = Q^{-1}P^{-1} \ldots C^{-1}A^{-1}$.
- $\left(A^{T}\right)^{-1} = (A^{-1})^{T}$

Unusual Properties of Matrix Multiplication

- Matrix multiplication is not commutative. In general, **AB ≠ BA**.
- **AB** = **0** does not generally imply **A** = **0** or **B** = **0**

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- **AC** = **AD** does not generally imply **C** = **D** (even when **A** ≠ **0**).

Verify

$C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ & $D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ & $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

$C \times D, D \times C, A \times C, A \times D$

Cancellation Laws

Let A, B, C be $n \times n$ matrices. Then:

- If rank $A = n$ and $AB = AC$, then $B = C$.
- If rank $A = n$, then $AB = 0$ implies $B = 0$.

  Hence if $AB = 0$, but $A \neq 0$ as well as $B \neq 0$, then rank $A < n$ and rank $B < n$.
- If A is singular, so are BA and AB.

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Thank you!!