BITS Pilani presentation

BITS Pilani
Pilani Campus

Shekhar Rajagopalan

innovate achieve lead

BITS Pilani

Pilani Campus

SS ZC416 Mathematical Foundations for Data Science

Introduction to Linear Programming

Introduction & Formulation

A Simple Maximization Problem

Product Mix Problem
We produce two products: A fuel additive and a solvent base.

3 raw materials are required:

- 1 Kg of fuel additive requires 1 Kg of material1, 2 Kg of material2 and 1 Kg of material 3
- 1 Kg of the solvent base requires 0 Kg of material1, 3 Kg of material2 and 1 Kg of material 3

There are 6 Kg of material1, 19 Kg of material2 and 8 Kg of material3.

Profit contributions are Rs5/Kg on Product 1 and Rs7/Kg on Product 2.

We want to maximize the profit

A Simple Maximization Problem

Product Mix Problem

We produce two products:
A fuel additive and a solvent base.

Maximize the profits when the profit contributions are Rs5/Kg on Product 1 and Rs7/Kg on Product 2.

3 raw materials are reqd.

- 1 Kg of fuel additive requires 1 Kg of material1, 2 Kg of material2 and 1 Kg of material 3.
- While 1 Kg of the solvent base requires 0, 3 and 1 Kg respectively.
- There are 6 Kg of material1, 19 Kg of material2 and 8 Kg of material3.

Formulation

Decision Variables

x_1: Amount (Kgs) of fuel additive produced
x_2: Amount (Kgs) of solvent base produced

Max $5x_1 + 7x_2$ Objective Function

Subject to

$x_1 \leq 6$ (Material 1)
$2x_1 + 3x_2 \leq 19$ (Material 2)
$x_1 + x_2 \leq 8$ (Material 3)

$x_1 \geq 0$ and $x_2 \geq 0$ (Non-Negativity)

Linear Programming (LP) Problem

If both the objective function and the constraints are linear, the problem is referred to as a linear programming problem.

Linear functions are functions in which each variable appears in a separate term raised to the first power and is multiplied by a constant (which could be 0).

Linear constraints are linear functions that are restricted to be "less than or equal to", "equal to", or "greater than or equal to" a constant.

Linear Programming

- Linear programming has nothing to do with computer programming.
- The use of the word "programming" here means "choosing a course of action."
- Linear programming involves choosing a course of action when the mathematical model of the problem contains only linear functions.

Linear Programming (LP) Problem

The maximization or minimization of some quantity is the objective in all linear programming problems.

All LP problems have constraints that limit the degree to which the objective can be pursued.

A feasible solution satisfies all the problem's constraints.

An optimal solution is a feasible solution that results in the largest possible objective function value when maximizing (or smallest when minimizing).

Problem Formulation

Problem formulation or modeling is the process of translating a verbal statement of a problem into a mathematical statement.

Formulating models is an art that can only be mastered with practice and experience.

Every LP problems has some unique features, but most problems also have common features.

Guidelines for Model Formulation

Understand the problem thoroughly.

Describe the objective.

Describe each constraint.

Define the decision variables.

Write the objective in terms of the decision variables.

Write the constraints in terms of the decision variables.

Graphical Solution

Graphical Solution

LPs with 2 decision variables can be solved in this manner:

1. Plot each constraint as an equation and then decide which side of the line is feasible (if it's an inequality).
2. Find the feasible region.
3. find the coordinates of the corner (extreme) points of the feasible region.
4. Substitute the corner point coordinates in the objective function
5. Choose the optimal solution

Example 1

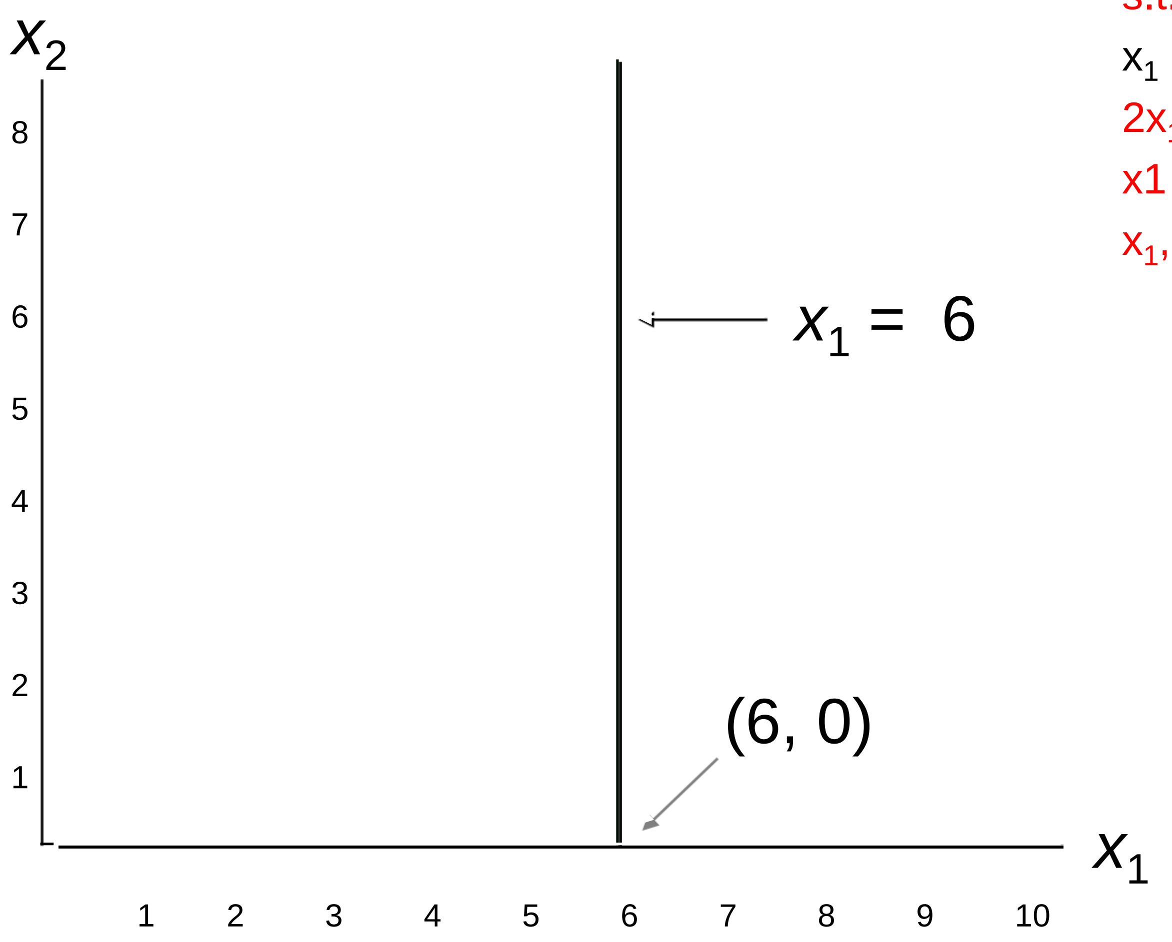
Example 1: A Maximization Problem

Max $\quad z = 5x_1 + 7x_2$

s.t.

$$\begin{aligned} x_1 \qquad\qquad &\leq 6 \\ 2x_1 + 3x_2 &\leq 19 \\ x_1 + \quad x_2 &\leq 8 \\ x_1, x_2 \qquad &\geq 0 \end{aligned}$$

1st Constraint Graphed



Max $z = 5x_1 + 7x_2$

s.t.

$x_1 \quad <= 6$

$2x_1 + 3x_2 <= 19$

$x1 + x2 <= 8$

$x_1, x_2 \geq 0$

2nd Constraint Graphed

Max $z = 5x_1 + 7x_2$

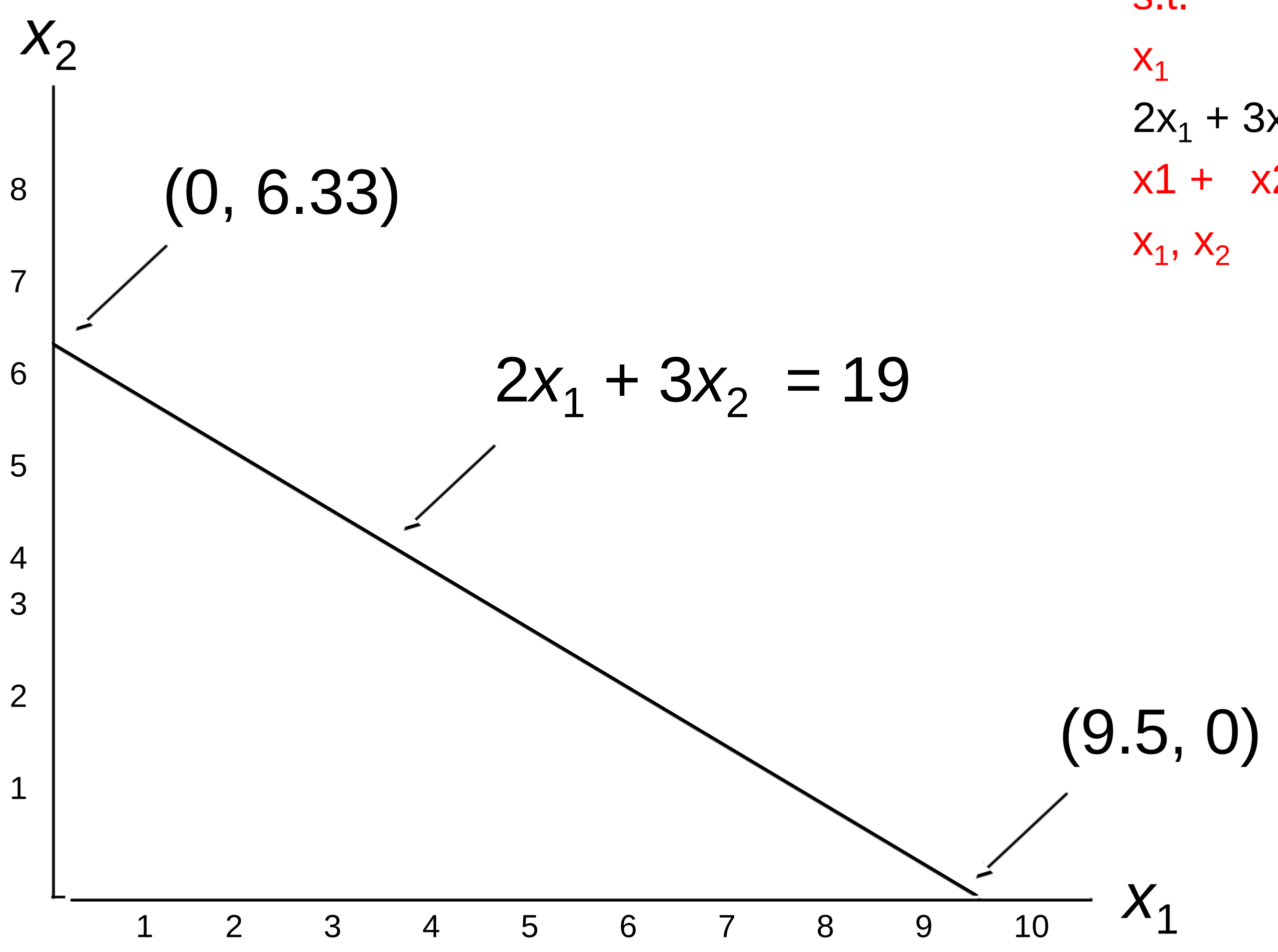
s.t.

$x_1 \quad <= 6$

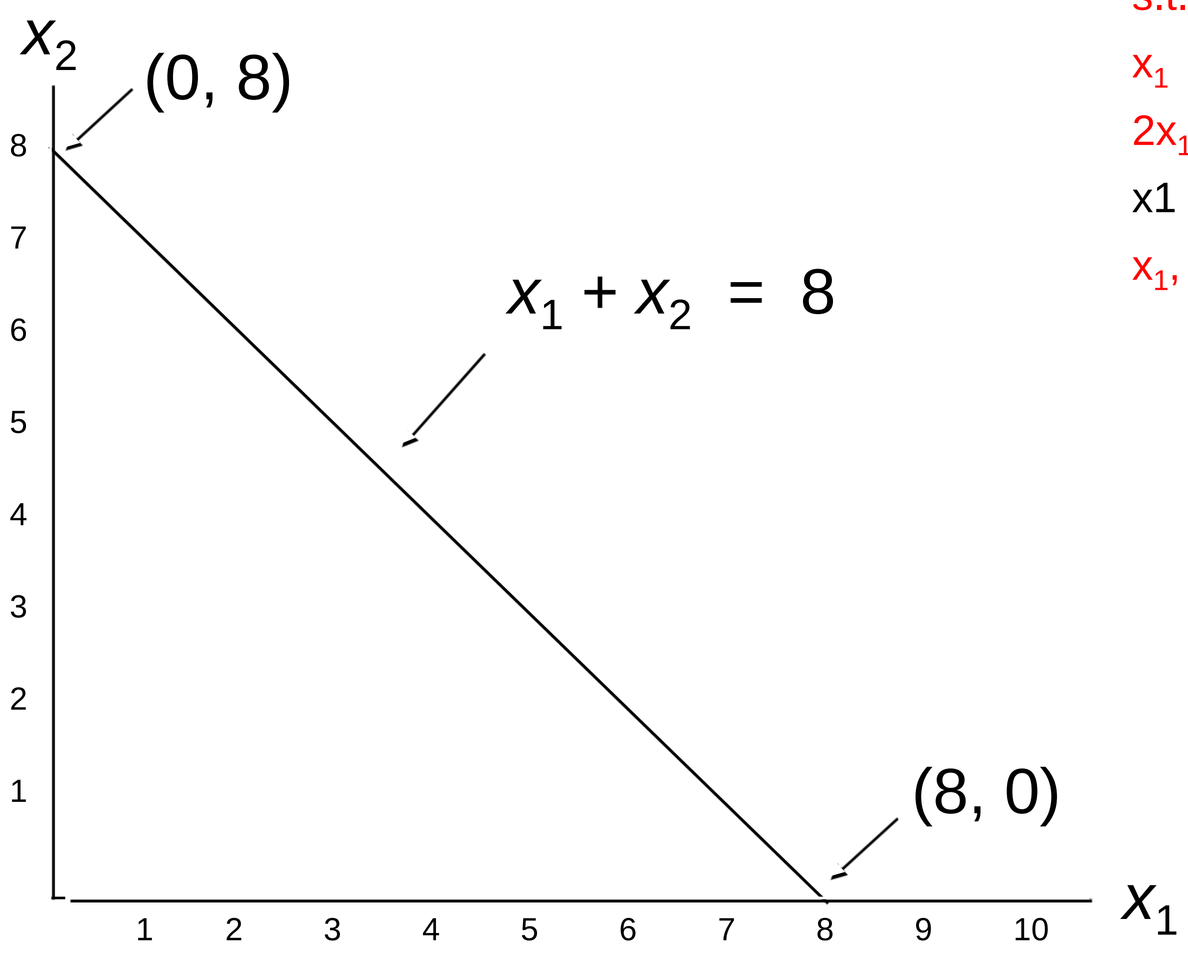
$2x_1 + 3x_2 <= 19$

$x1 + x2 <= 8$

$x_1, x_2 \geq 0$



3rd Constraint Graphed



Max $z = 5x_1 + 7x_2$

s.t.

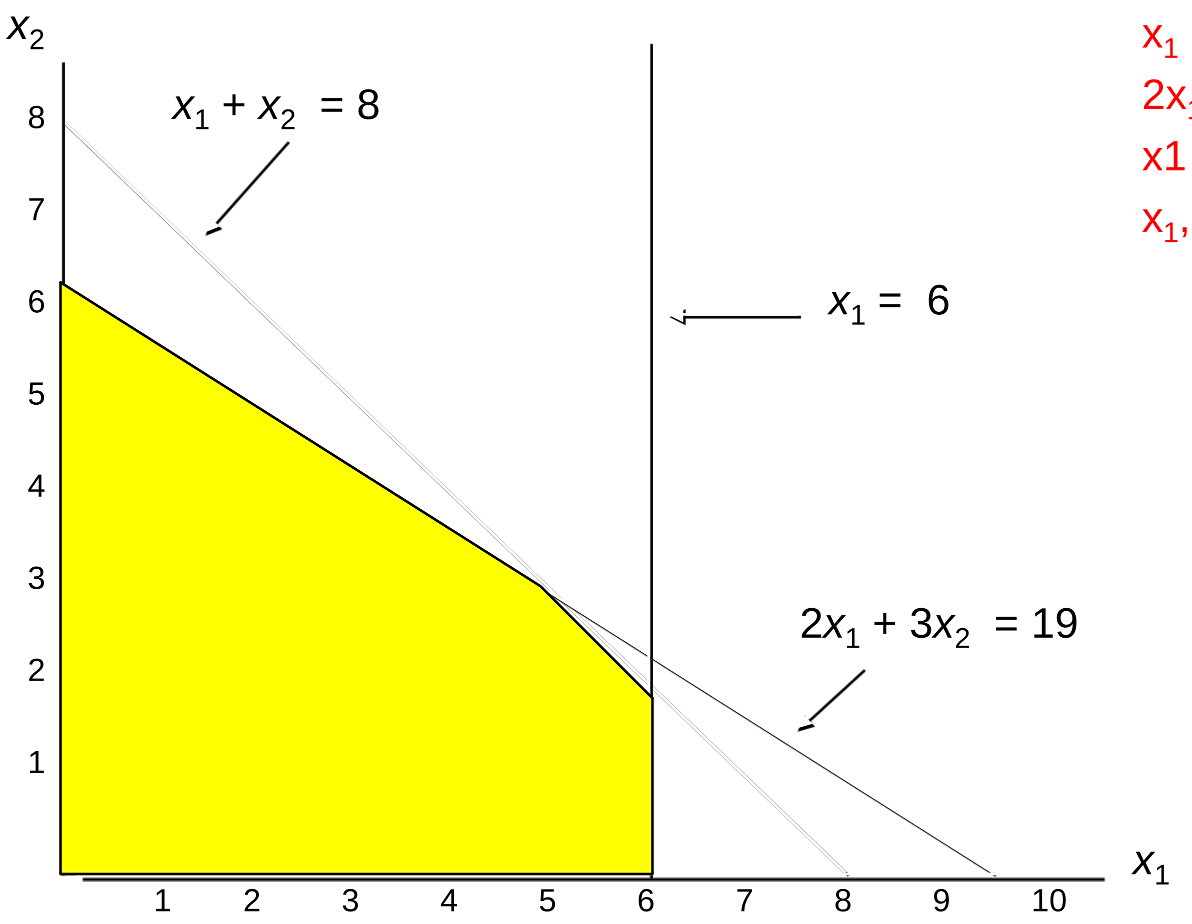
$x_1 \quad <= 6$

$2x_1 + 3x_2 <= 19$

x1 + x2 <= 8

$x_1, x_2 \geq 0$

Combined Constraint Graph



Max $z = 5x_1 + 7x_2$

s.t.

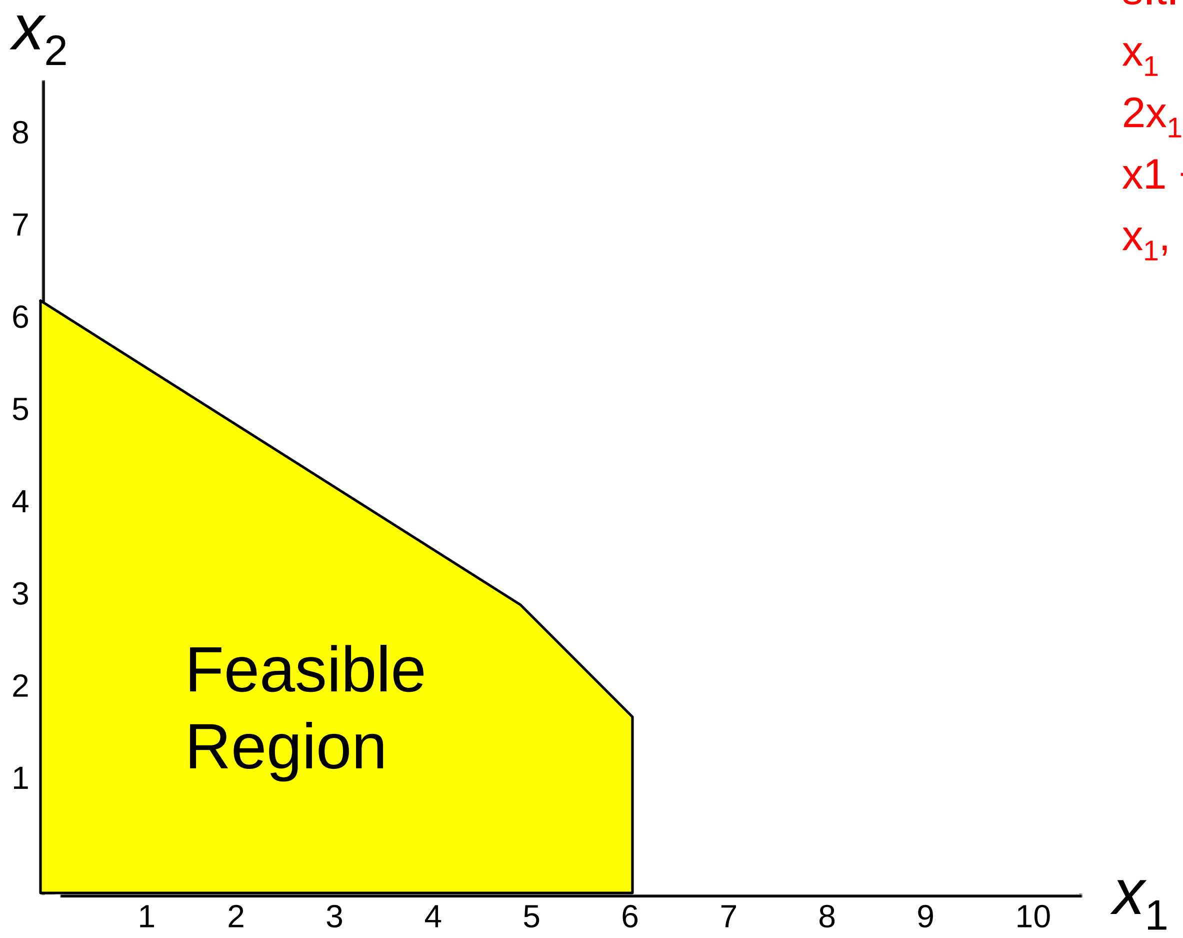
$x_1 \quad\quad <= 6$

$2x_1 + 3x_2 <= 19$

$x1 + x2 <= 8$

$x_1, x_2 \geq 0$

The Feasible Region



Max $\quad z = 5x_1 + 7x_2$

s.t.

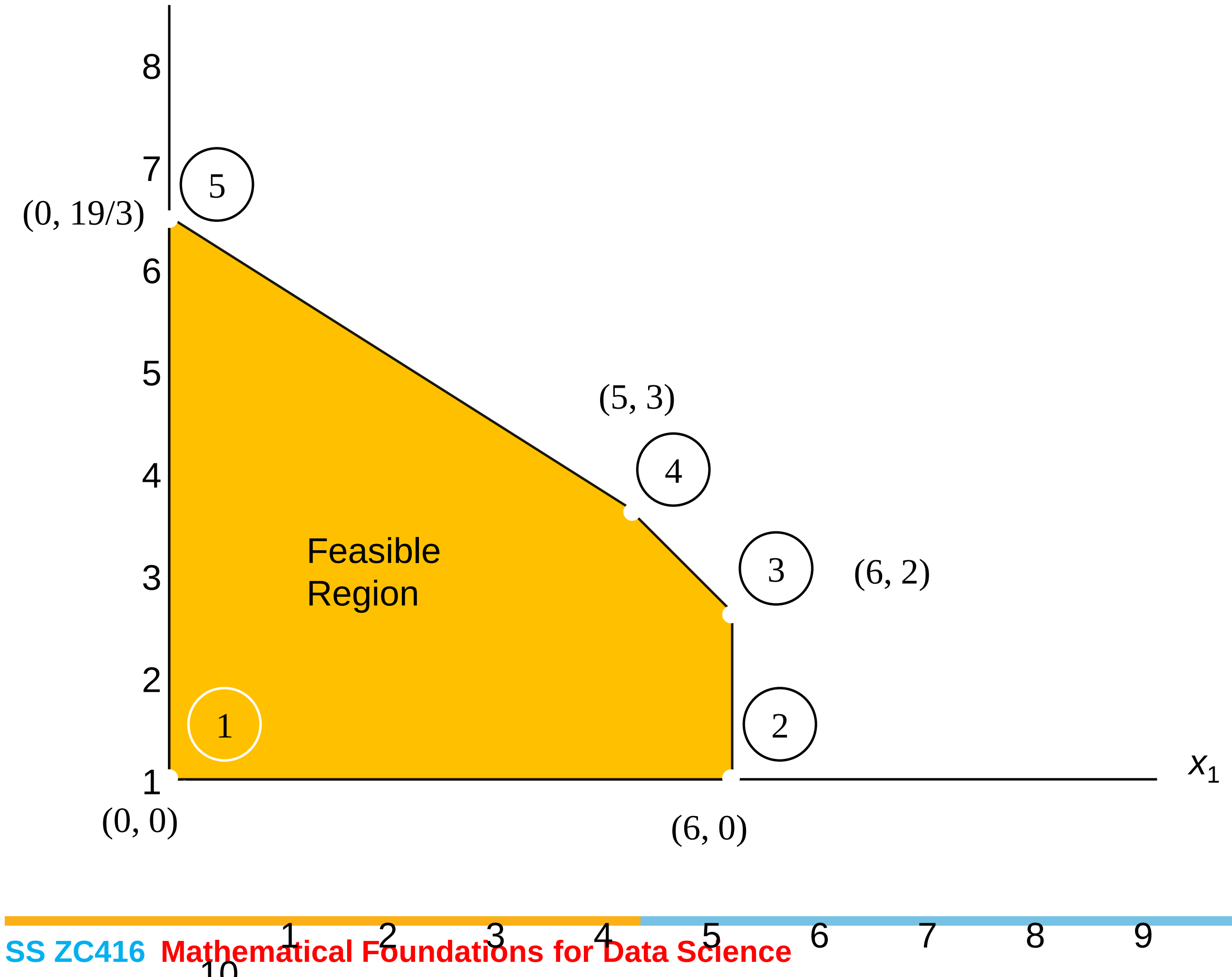
$x_1 \quad\quad <= 6$

$2x_1 + 3x_2 <= 19$

$x1 + \; x2 <= \; 8$

$x_1, x_2 \quad \geq 0$

The Five Extreme Points



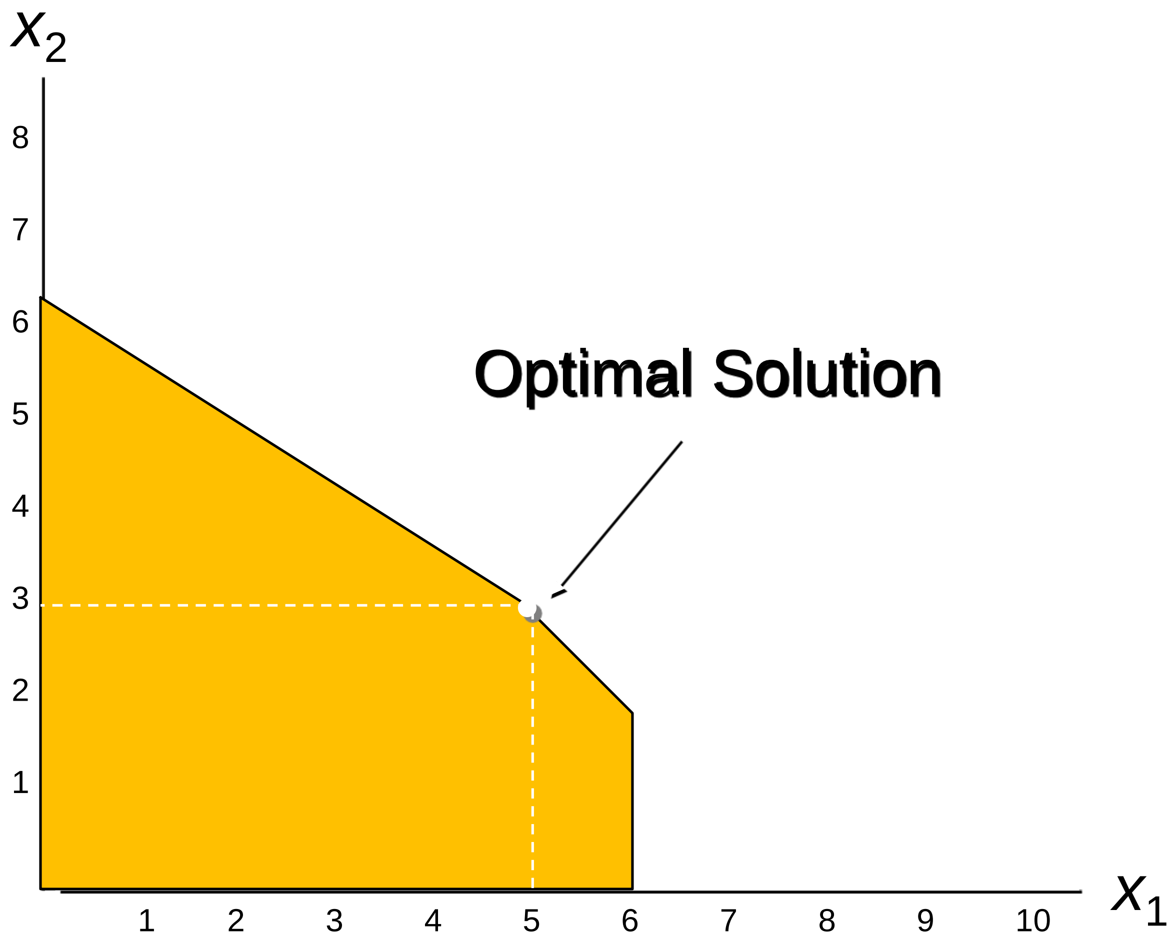
Identifying the Optimal Solution

Having identified the feasible region for the problem, we now search for the optimal solution, which will be the point in the feasible region with the largest (in case of maximization or the smallest (in case of minimization) of the objective function.

To find this optimal solution, we need to evaluate the objective function at each one of the corner points of the feasible region.

The Optimal Solution I

Point	Z
(0,0)	0
(6,0)	30
(6,2)	44
(5,3)	46
(0,19/3)	44.33



Max $z = 5x_1 + 7x_2$

s.t.

$x_1 \quad <= 6$

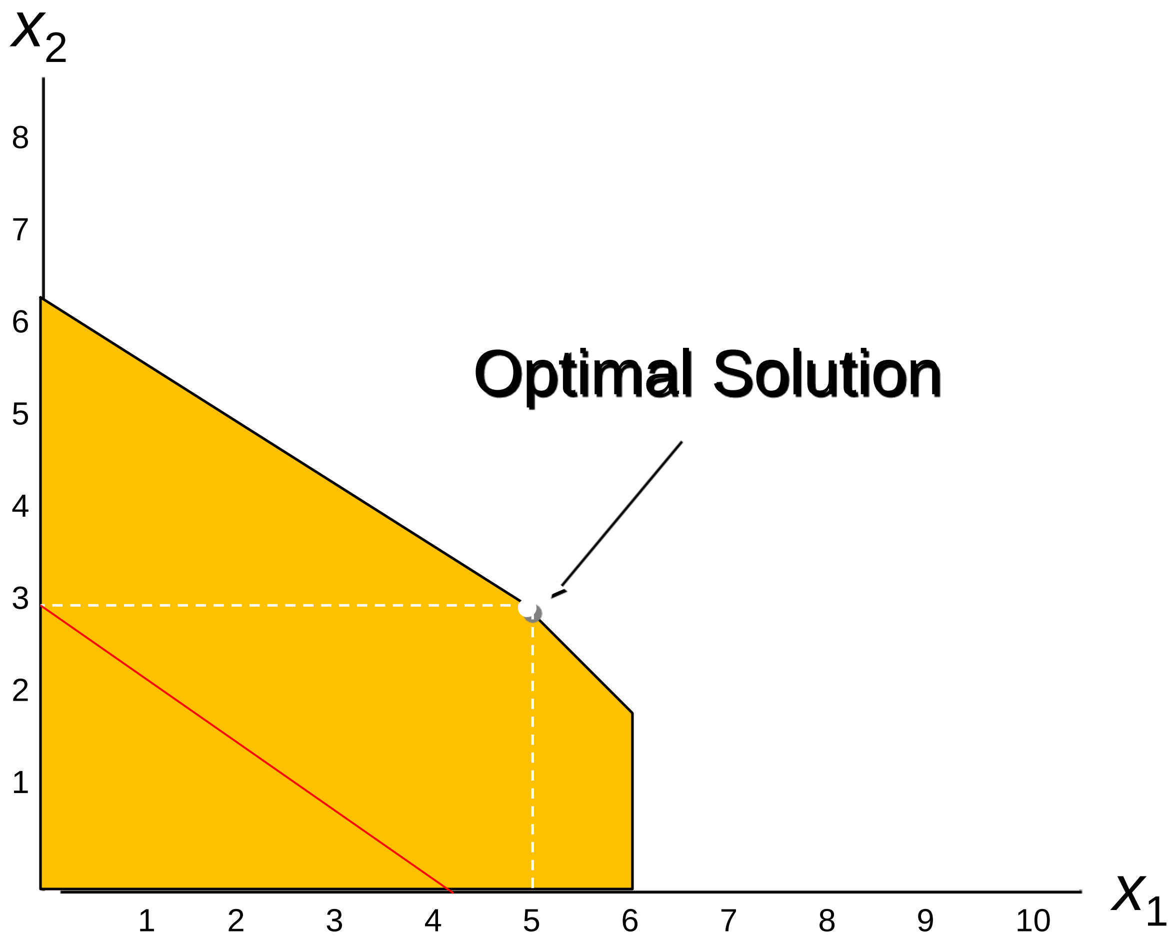
$2x_1 + 3x_2 <= 19$

$x1 + x2 <= 8$

$x_1, x_2 \geq 0$

The Optimal Solution II

Consider the contour line $z = 21$. That is $5x_1 + 7x_2 = 21$



Max $\quad z = 5x_1 + 7x_2$

s.t.

$x_1 \quad\quad\quad <= 6$

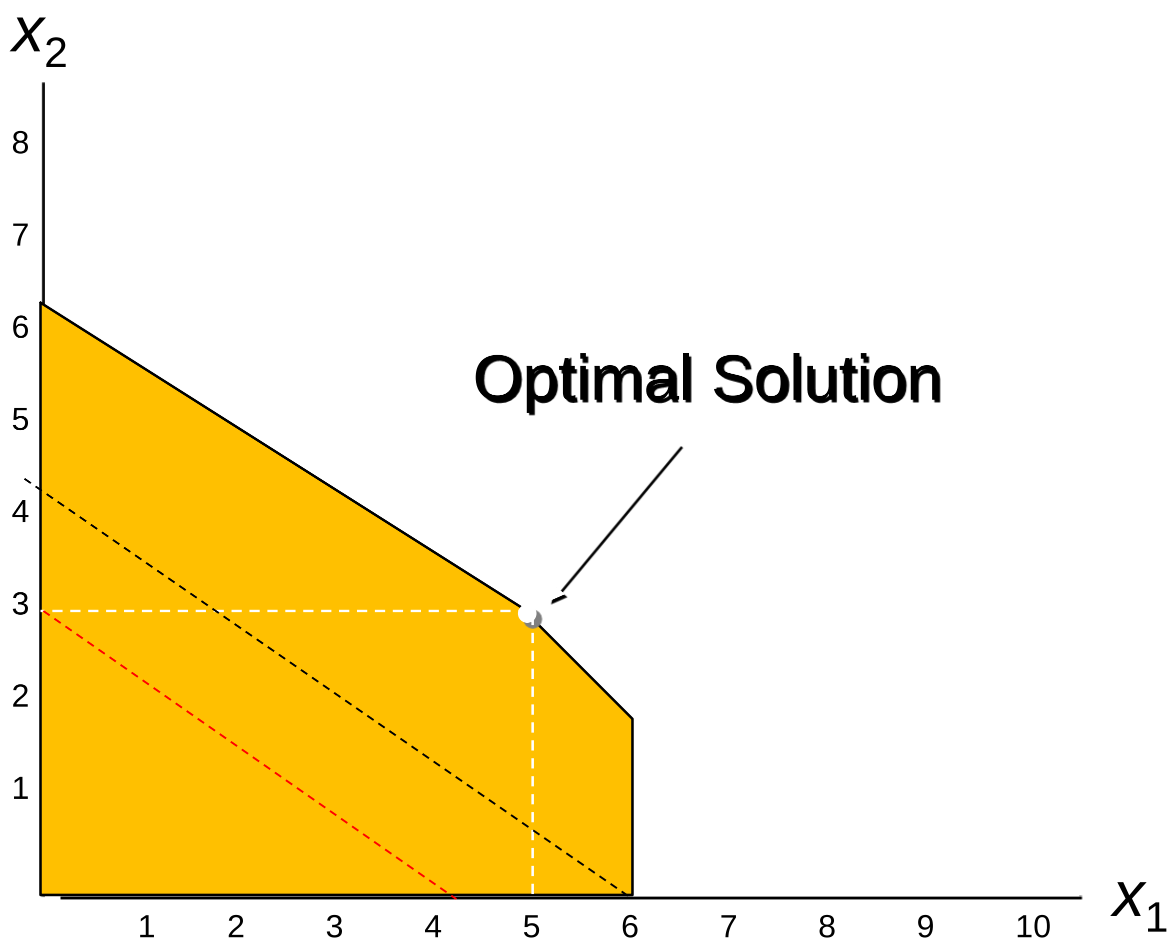
$2x_1 + 3x_2 <= 19$

$x1 + \ \ x2 \ <= \ \ 8$

$x_1, x_2 \quad\quad \geq 0$

The Optimal Solution II

Consider the contour line z = 30. That is $5x_1 + 7x_2 = 30$



Max $z = 5x_1 + 7x_2$

s.t.

$x_1 \le 6$

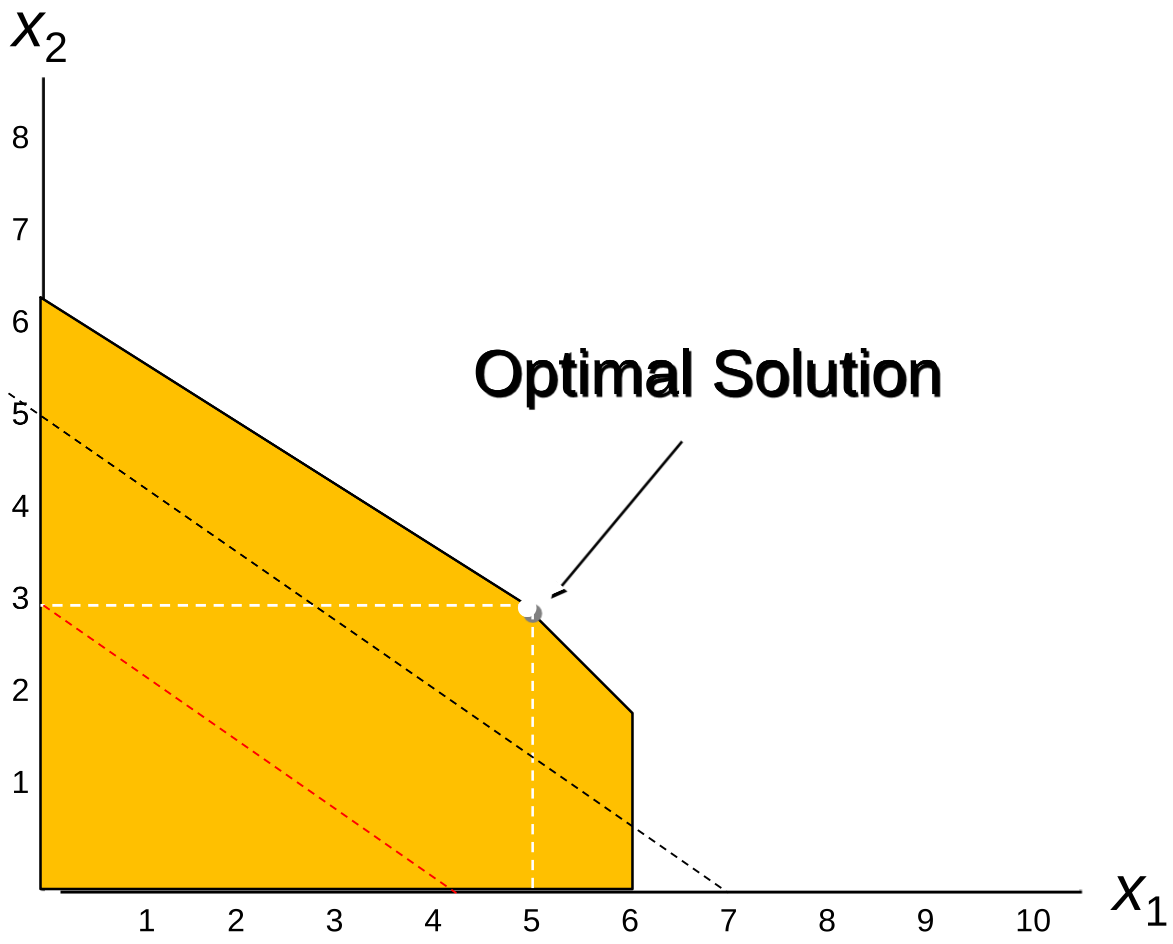
$2x_1 + 3x_2 \le 19$

$x1 + x2 \le 8$

$x_1, x_2 \ge 0$

The Optimal Solution II

Consider the contour line $z = 35$. That is $5x_1 + 7x_2 = 35$



Max $\quad z = 5x_1 + 7x_2$

s.t.

$x_1 \quad\quad \leq 6$

$2x_1 + 3x_2 \leq 19$

$x1 + x2 \leq 8$

$x_1, x_2 \geq 0$

The Optimal Solution II

Consider the contour line $z = 40$. That is $5x_1 + 7x_2 = 40$

Max $\quad z = 5x_1 + 7x_2$

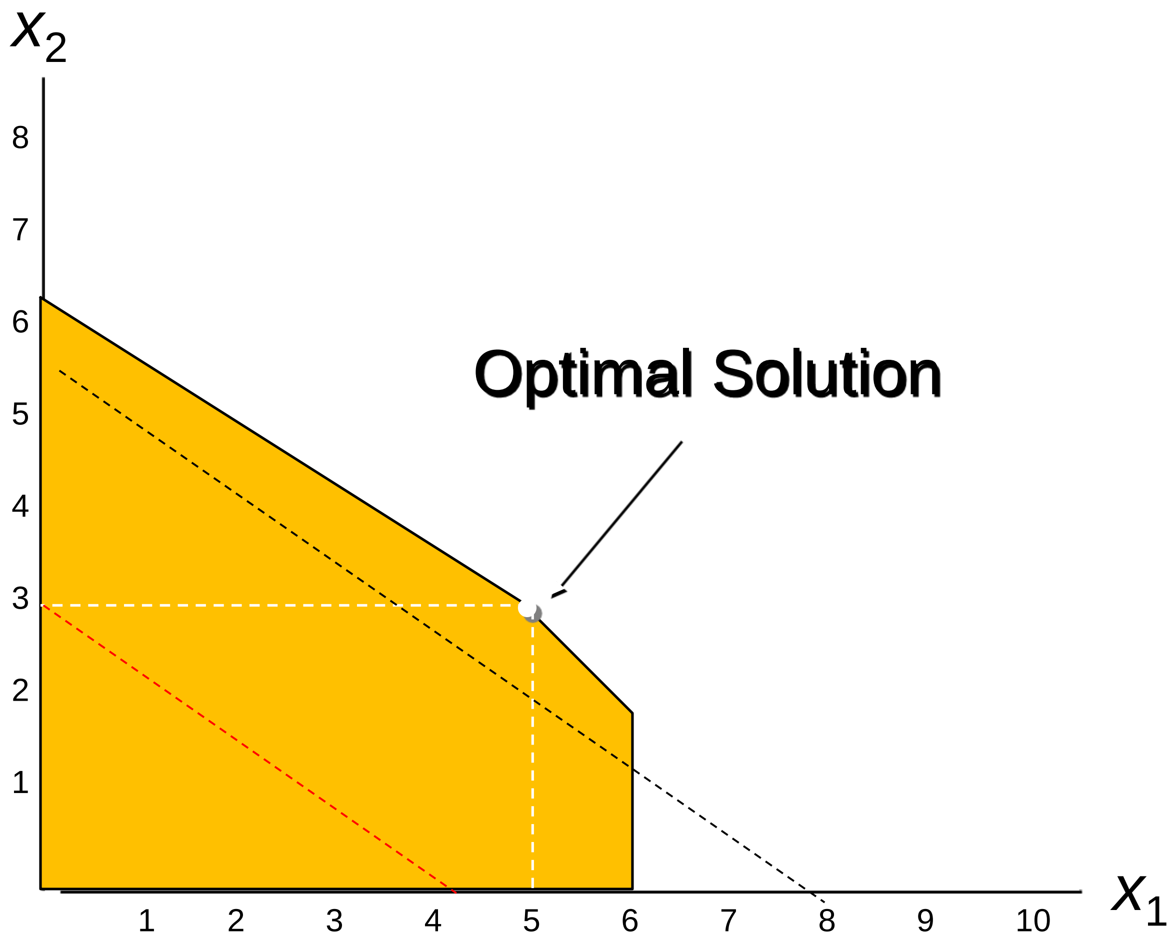
s.t.

$x_1 \quad\quad\quad <= 6$

$2x_1 + 3x_2 <= 19$

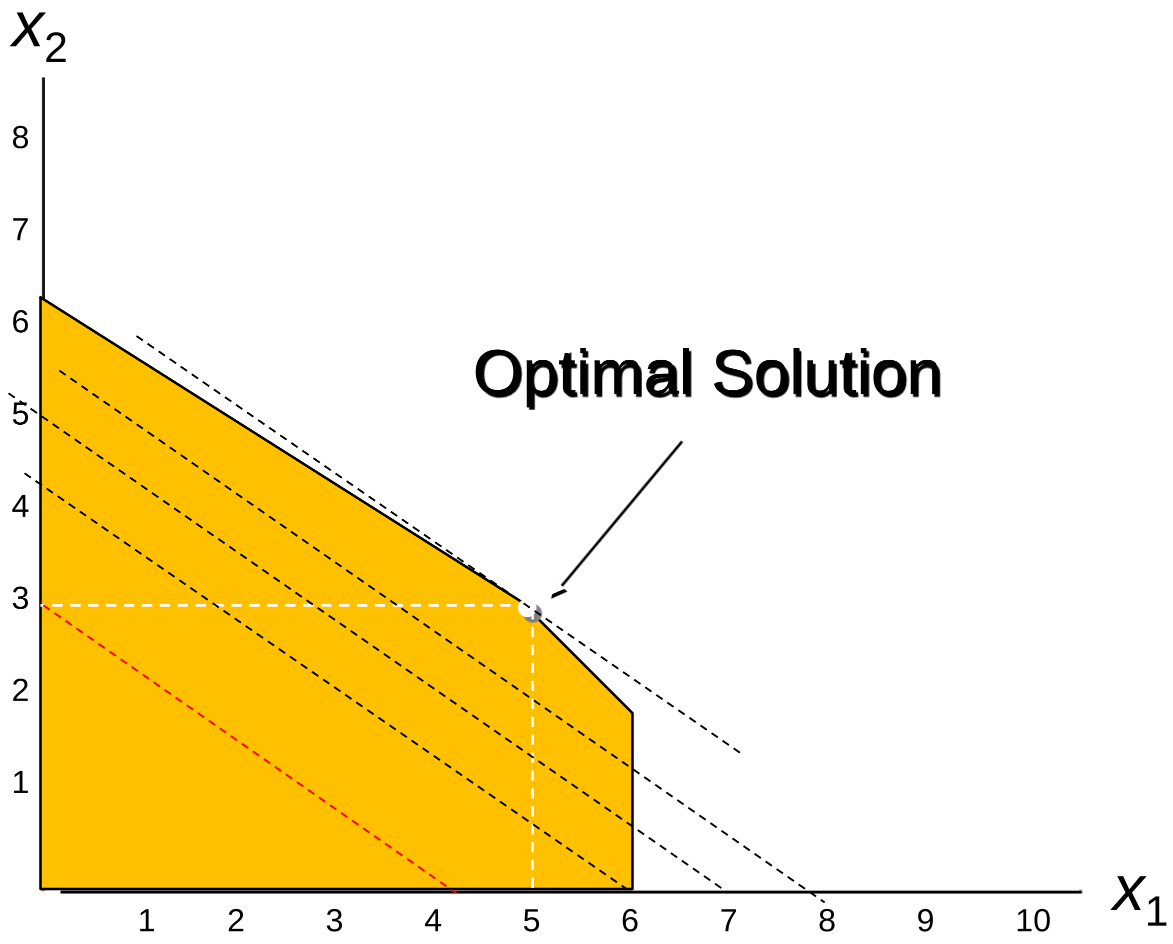
$x1 + \quad x2 \;<= \quad 8$

$x_1, x_2 \quad\quad \geq 0$



The Optimal Solution II

Consider the contour line z = 46. That is $5x_1 + 7x_2 = 46$



Max $z = 5x_1 + 7x_2$

s.t.

$x_1 \quad <= 6$

$2x_1 + 3x_2 <= 19$

$x1 + x2 <= 8$

$x_1, x_2 \geq 0$

Extreme Points and the Optimal Solution

The corners or vertices of the feasible region are referred to as the extreme points.

An optimal solution to an LP problem can be found at an extreme point of the feasible region.

When looking for the optimal solution, you do not have to evaluate all feasible solution points.

You have to consider only the extreme points of the feasible region.

Example 2

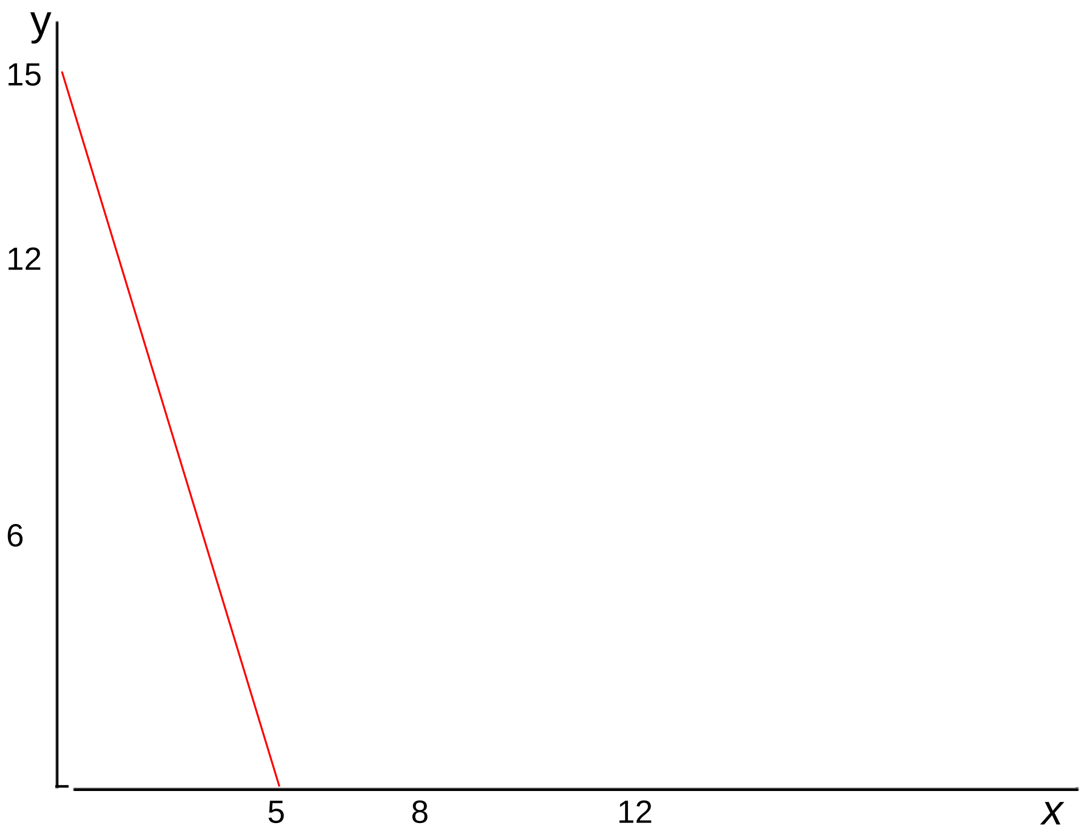
A Minimization Problem

$$\text{Min} \quad z = 5x + 6y$$

s.t.

$$\begin{aligned} 3x + y &\geq 15 \\ x + 2y &\geq 12 \\ 3x + 2y &\leq 24 \\ x \quad y &\geq 0 \end{aligned}$$

1st Constraint Graphed



Min $z = 5x + 6y$

s.t.

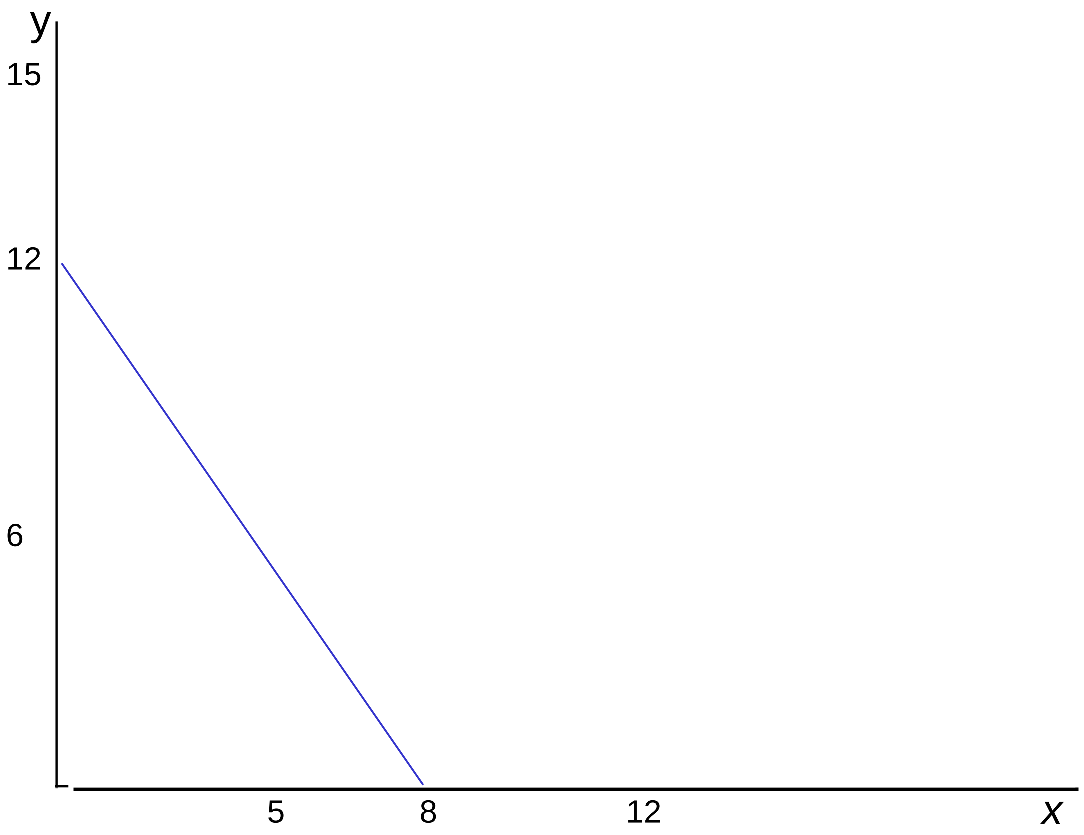
$3x + y \geq 15$

$x + 2y \geq 12$

$3x + 2y \leq 24$

$x, y > 0$

2^{nd} Constraint Graphed



Min $z = 5x + 6y$

s.t.

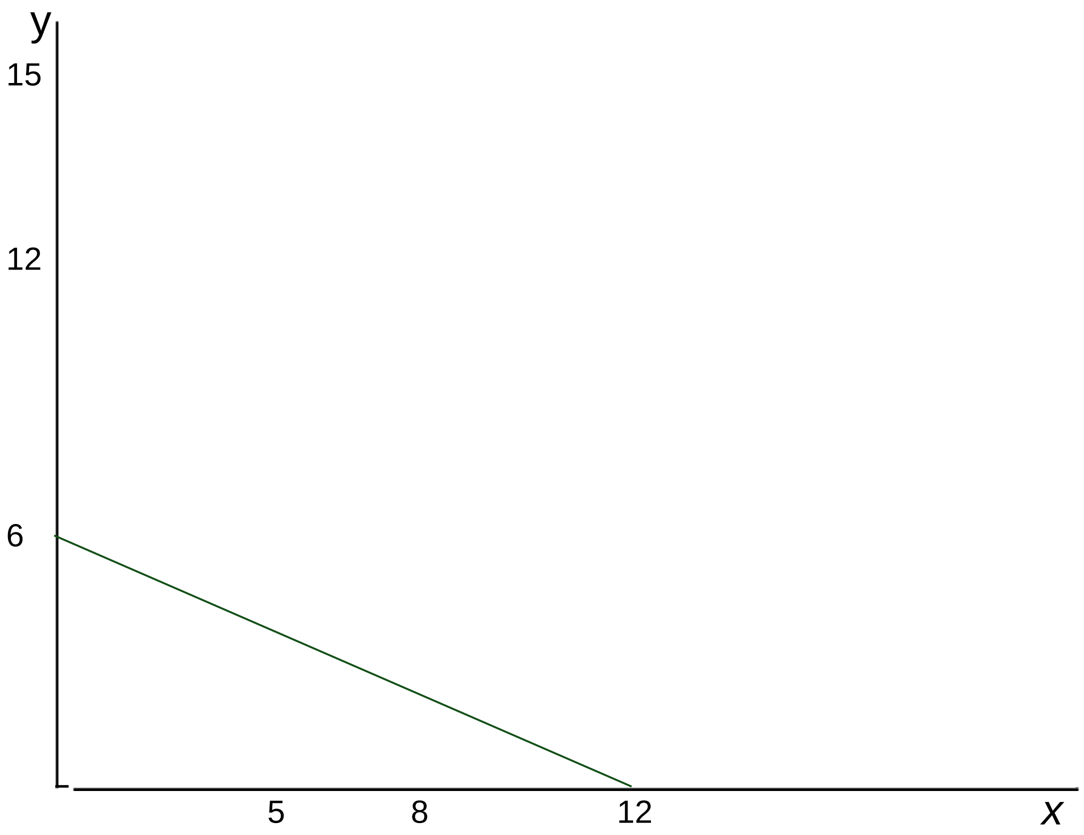
$3x + y \geq 15$

$x + 2y \geq 12$

$3x + 2y \leq 24$

X y > 0

3rd Constraint Graphed



Min z = 5x + 6y

s.t.

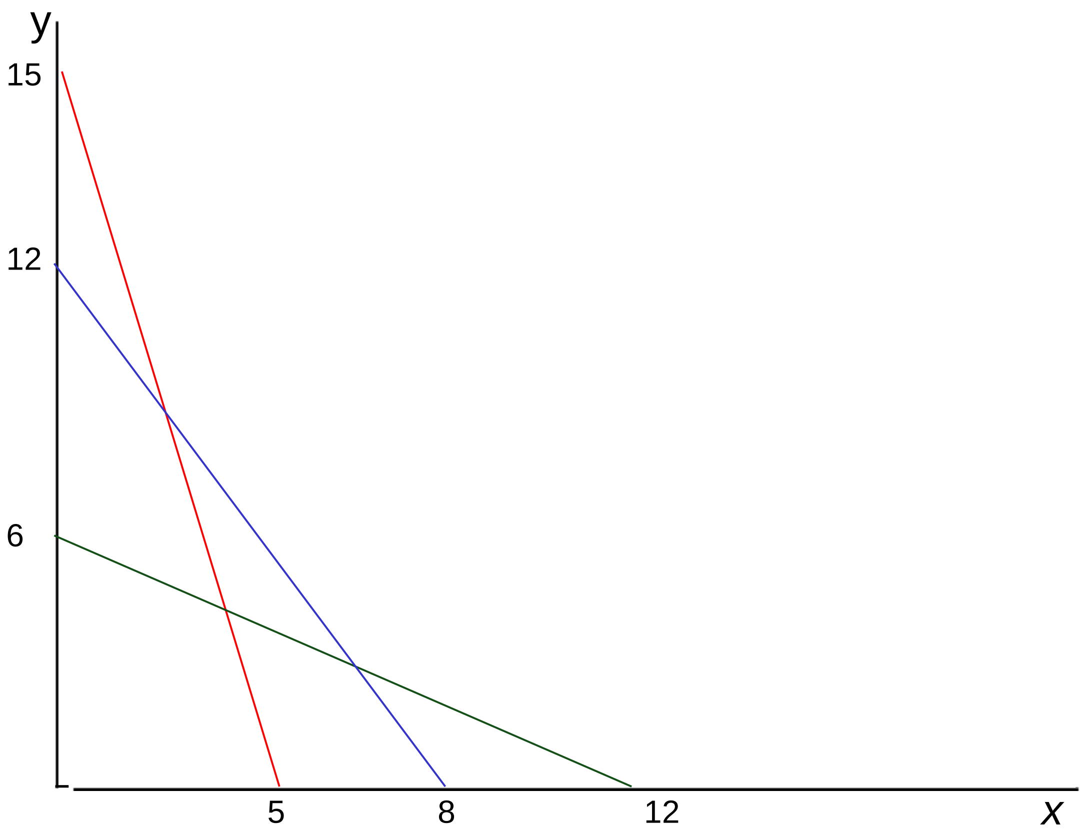
3x + y ≥ 15

x + 2y ≥ 12

3x + 2y ≤ 24

X y > 0

Combined Constraint Graphed



Min $z = 5x + 6y$

s.t.

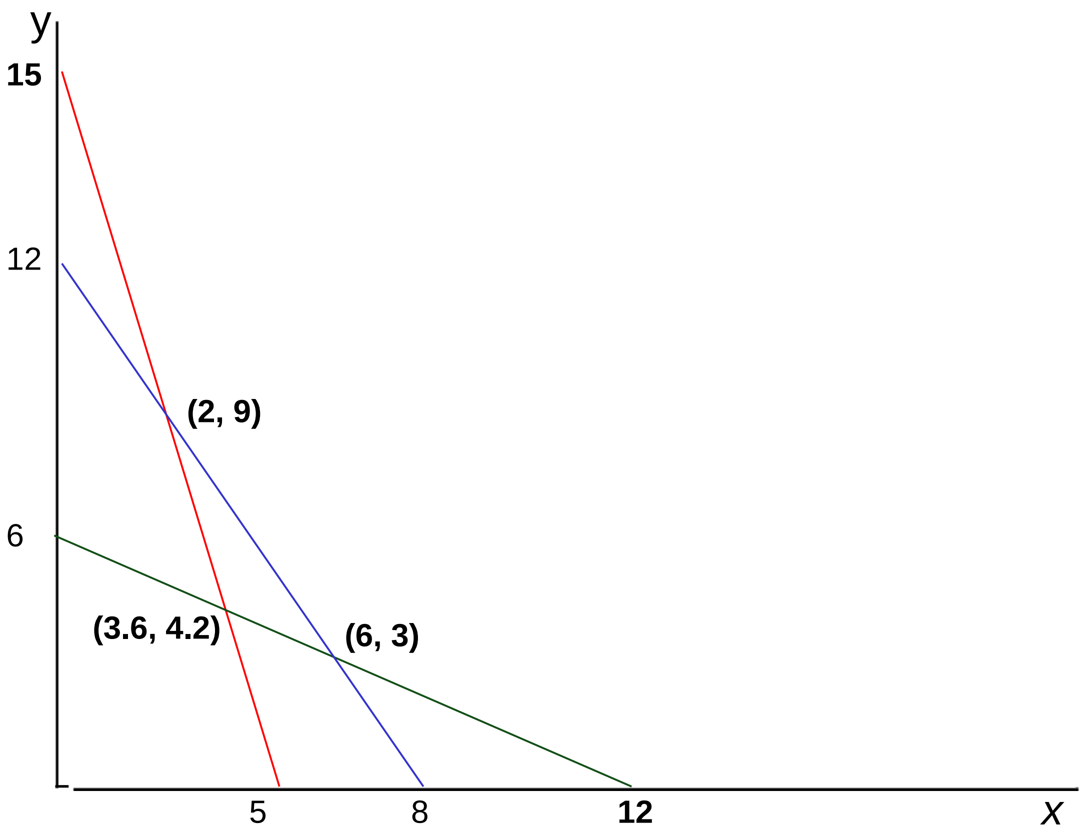
$3x + y \geq 15$

$x + 2y \geq 12$

$3x + 2y \leq 24$

$X \quad y > 0$

The Feasible Region with the Extreme Points



Min $z = 5x + 6y$

s.t.

$3x + y \geq 15$

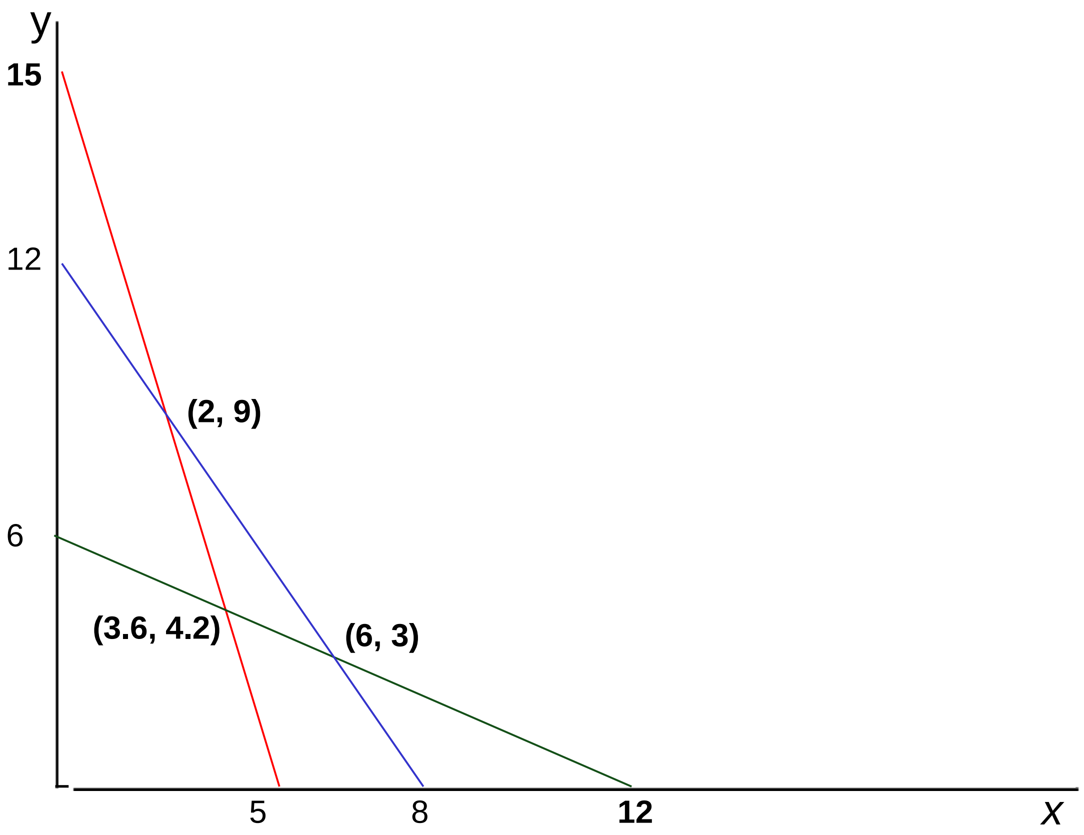
$x + 2y \geq 12$

$3x + 2y \leq 24$

$X \quad y > 0$

The Optimal Solution

Point	Z
(2,9)	64
(6,3)	48
(3.6, 4.2)	43.2



Min $z = 5x + 6y$

s.t.

$3x + y \geq 15$

$x + 2y \geq 12$

$3x + 2y \leq 24$

$X \quad y > 0$

Additional Remarks

Feasible Region

The feasible region for a two-variable linear programming problem can be nonexistent, a single point, a line, a polygon, or an unbounded area.

Any linear program falls in one of three categories:

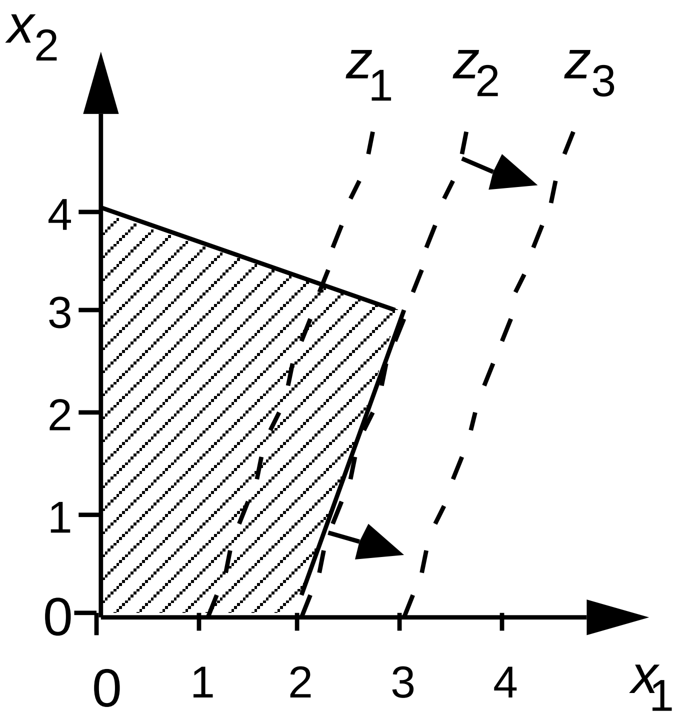
- is infeasible if there is no feasible space
- has a unique optimal solution or alternate optimal solutions
- has an objective function that can be increased without bound

A feasible region may be unbounded and yet there may be optimal solutions. This is common in minimization problems and is possible in maximization problems.

Feasible Region

The feasible region is a Convex Space

Multiple Optimal Solutions



$$\text{Maximize} \quad z = 3x_1 - x_2$$

$$\begin{aligned} \text{subject to} \quad & 15x_1 - 5x_2 \leq 30 \\ & 10x_1 + 30x_2 \leq 120 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

Example: Infeasible Problem

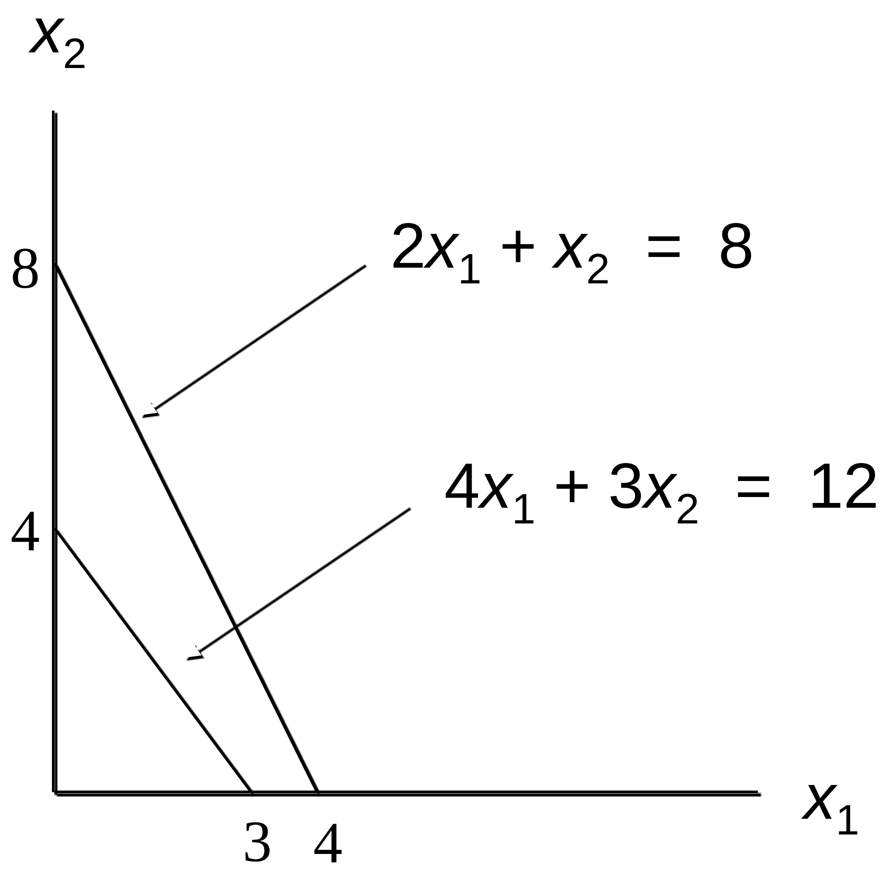
Solve graphically

Max $z = 2x_1 + 6x_2$

s.t. $4x_1 + 3x_2 \leq 12$

$2x_1 + x_2 \geq 8$

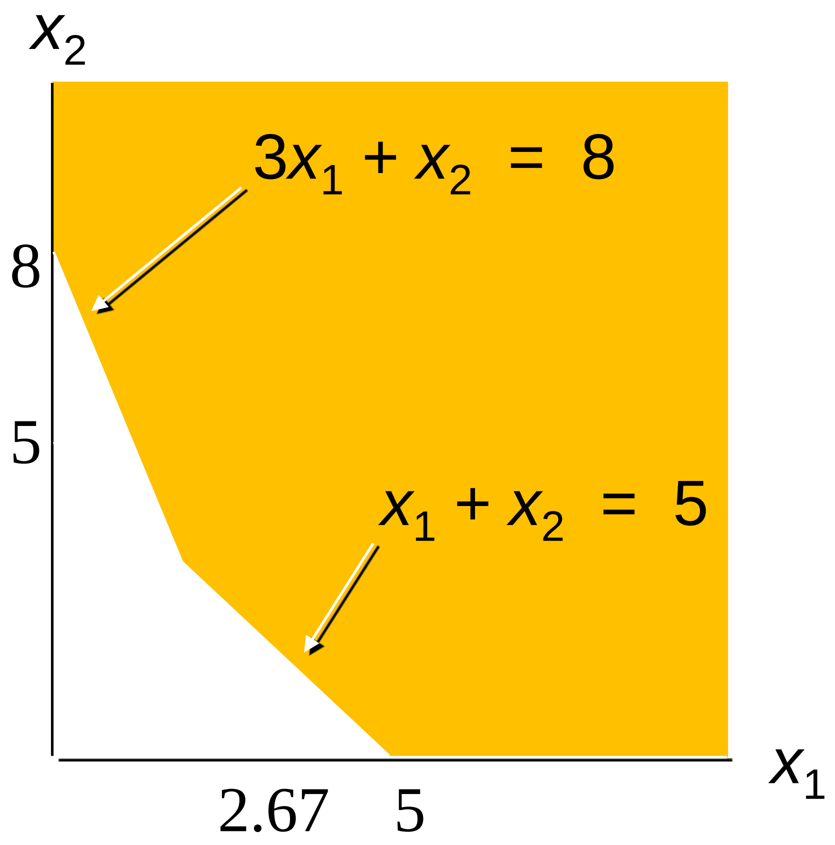
$x_1, x_2 \geq 0$



Example: Unbounded Problem

Solve graphically

$$\begin{array}{lrl} \text{Max} & z = & 3x_1 + 4x_2 \\ \text{s.t.} & x_1 + x_2 & \geq 5 \\ & 3x_1 + x_2 & \geq 8 \\ & x_1, x_2 & \geq 0 \end{array}$$



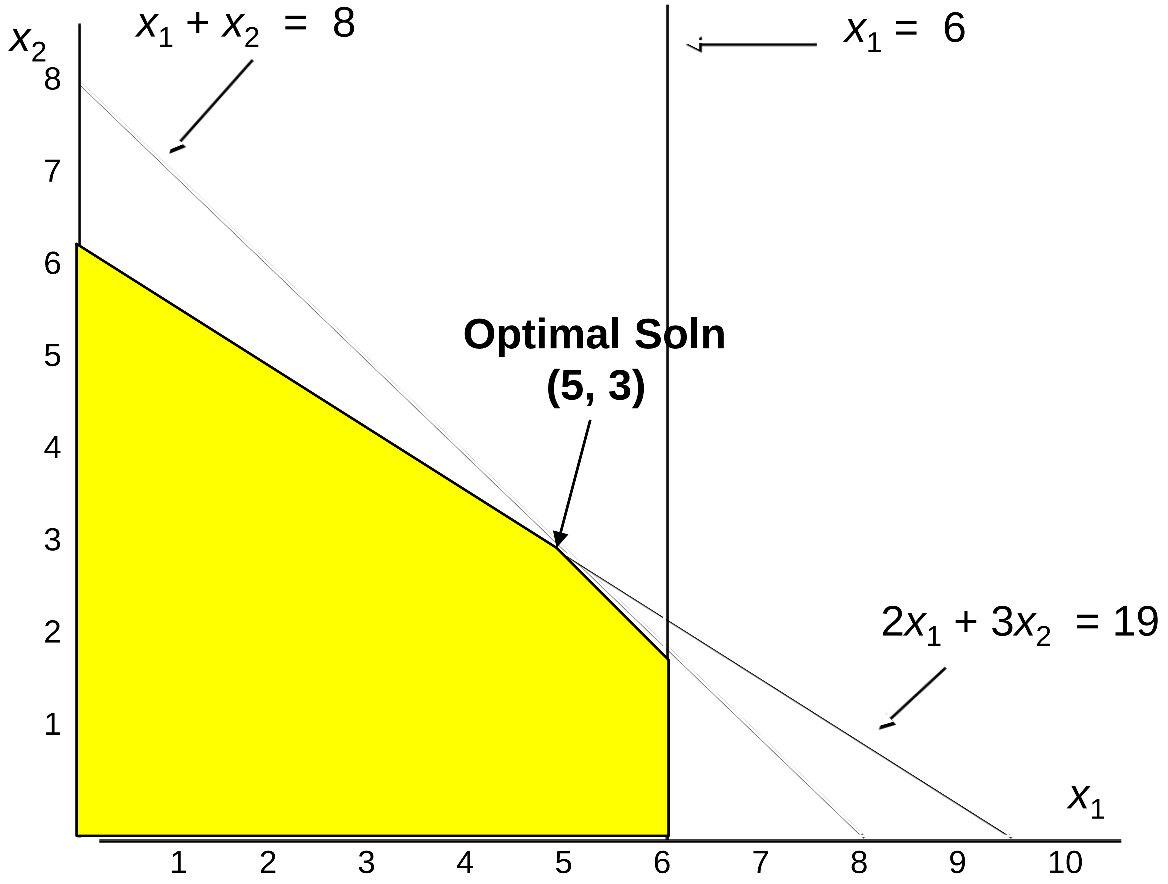
Binding & Non-Binding Constraints

A binding constraint is one where the slack or surplus variable equals 0.

A non-binding constraint has a non-zero slack / surplus variable

$$
\begin{array}{lll}
\text{Max} & 5x + 3y & \\
\text{s.t.} & x & \leq 6 \\
 & 2x + 3y & \leq 19 \\
 & x + y & \leq 8 \\
 & x \geq 0 \text{ and } y \geq 0 &
\end{array}
$$

Identify the binding constraints!



Appendix

Slack and Surplus Variables

Standard Form: An LPP where all the variables are non-negative and all the constraints are equalities.

1. Standard form is attained by adding slack variables to "≤" constraints, and by subtracting surplus variables from "≥" constraints.
2. These additional variables represent the difference between the left and right sides of the constraints.
3. These variables are inserted in the objective function with coefficients equal to 0.

$$
\begin{array}{ll}
\text{Max} & 2x + 4y \\
\text{s.t.} & x \le 6 \\
& 2x + 3y \le 20 \\
& x + y \ge 8 \\
& x, y \ge 0
\end{array}
$$

$$
\begin{array}{ll}
\text{Max} & 2x + 4y + 0s1 + 0s2 + 0s3 \\
\text{s.t.} & x + s1 = 6 \\
& 2x + 5y + s2 = 20 \\
& x + y - s3 = 8 \\
& x, y, s1, s2, s3 \ge 0
\end{array}
$$

Non Linear Optimization

Introduction

- Many business processes behave in a nonlinear manner.
  - The price of a bond is a nonlinear function of interest rates.
  - Stock option prices are nonlinear function of the price of the underlying stock.
  - The marginal cost of production often decreases with the quantity produced.
  - The quantity demanded for a product is often a nonlinear function of the price.
- A nonlinear optimization problem is any optimization problem in which at least one term in the objective function or a constraint is nonlinear.
- The nonlinear optimization problems presented on the upcoming slides can be solved using computer software such as *LINGO* and *Excel Solver*.

Example

ABC Bike Co. produces two new lightweight bicycle frames, the Flyer and the Razor, that are made from special aluminum and steel alloys. The cost to produce a Flyer frame is $100, and the cost to produce a Razor frame is $120.

We cannot assume that ABC will sell all the frames it can produce.

- As the selling price increases, the demand goes down

Assume that the demands for Flyer frames F and Razor frames R are given by:

- $F = 750 – 5P_F$ & $R = 400 – 2P_R$
  where P_F = the price of a Flyer frame & P_R = the price of a Razor frame.

The profit contributions (revenue – cost) are:

- Flyer frames $P_F F - 100F = (150 – (1/5)F)F – 100 F = 50 F – (1/5) F^2$
- Razor frames $P_R R - 120R = (200 – (1/2)R)R – 120 R = 80 R – (1/2)R^2$

Unconstrained Profit Contribution

Total Profit Contribution = $50F - 1/5\,F^2 + 80R - 1/2\,R^2$

This objective function is a quadratic function.

Using Calculus, we can show Maximum at (125, 80)

Excel Solver generates the same solution

$D_F = 50 - (2/5)F = 0 \Rightarrow F = 125$ & $D_R = 80 - R \Rightarrow R = 80$
$D_{FF} * D_{RR} - (D_{FR})^2 = (2/5) - 0 > 0$ & Both D_{FF} & D_{RR} are negative
Therefore maximum at (125, 80)

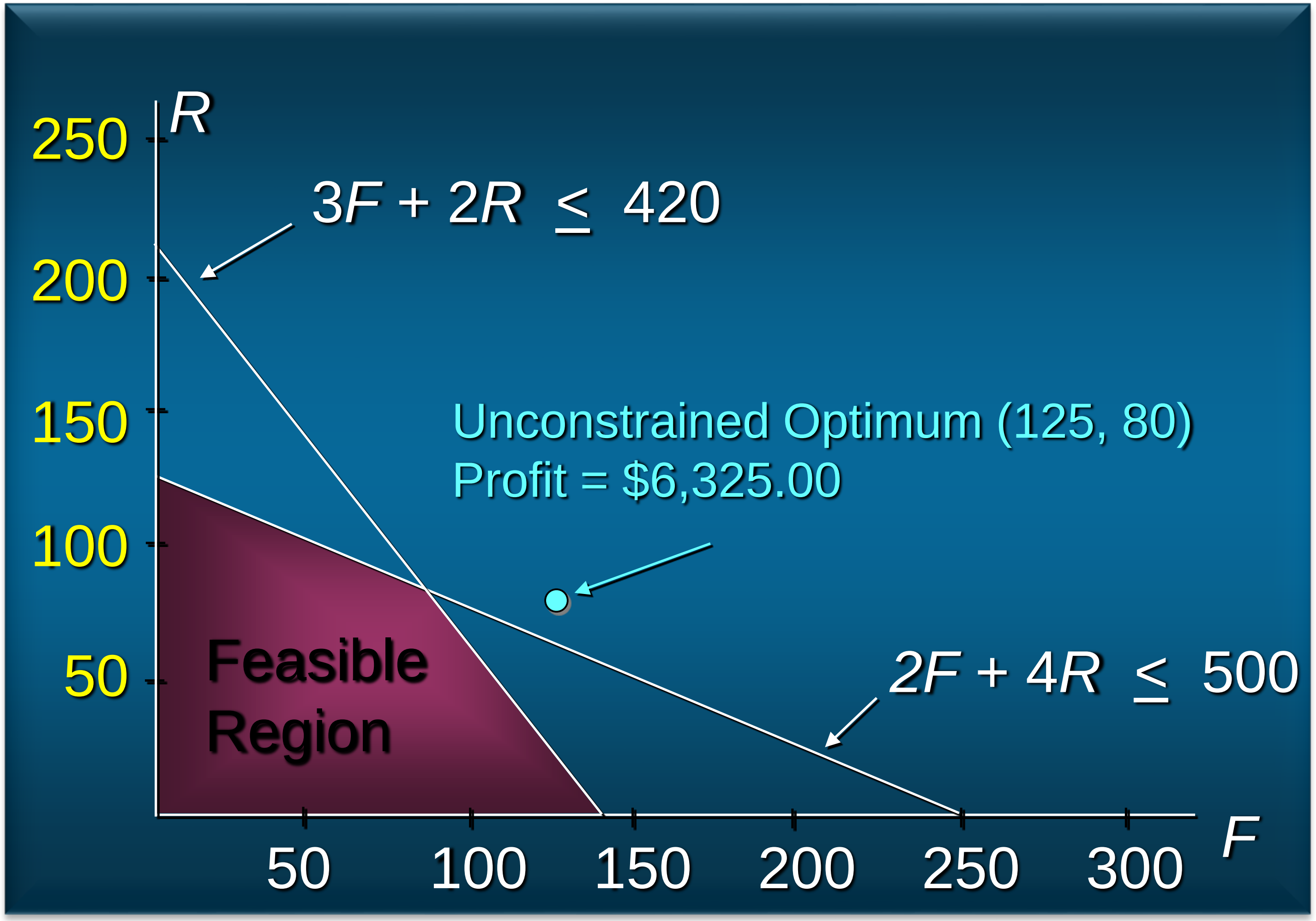
Constraints & Formulation

A supplier can deliver a maximum of 500 pounds of the aluminum alloy and 420 pounds of the steel alloy weekly. The number of pounds of each alloy needed per frame is summarized below. How many Flyer and Razor frames should ABC produce each week?

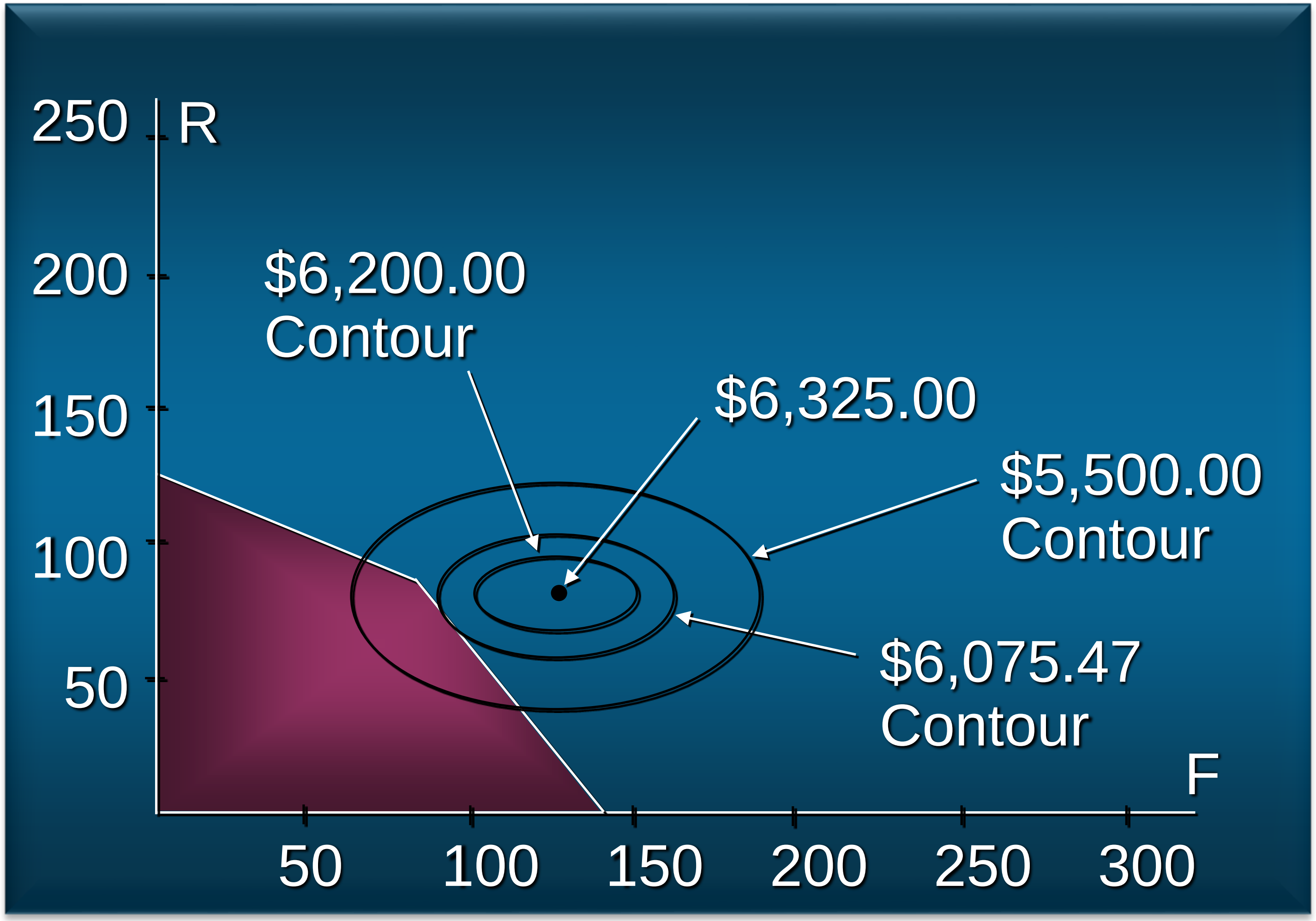
	Aluminum Alloy	Steel Alloy
Flyer	2	3
Razor	4	2

$$
\begin{array}{llr}
\text{Max} & 50F - 1/5\,F^2 + 80R - 1/2\,R^2 & \text{(Total Weekly Profit)} \\
\text{s.t.} & 2F + 4R \le 500 & \text{(Aluminum Available)} \\
 & 3F + 2R \le 420 & \text{(Steel Available)} \\
 & F, R \ge 0 & \text{(Non-negativity)}
\end{array}
$$

Nonlinear Optimization: Feasible Space



Objective Function Contour Lines & Solution

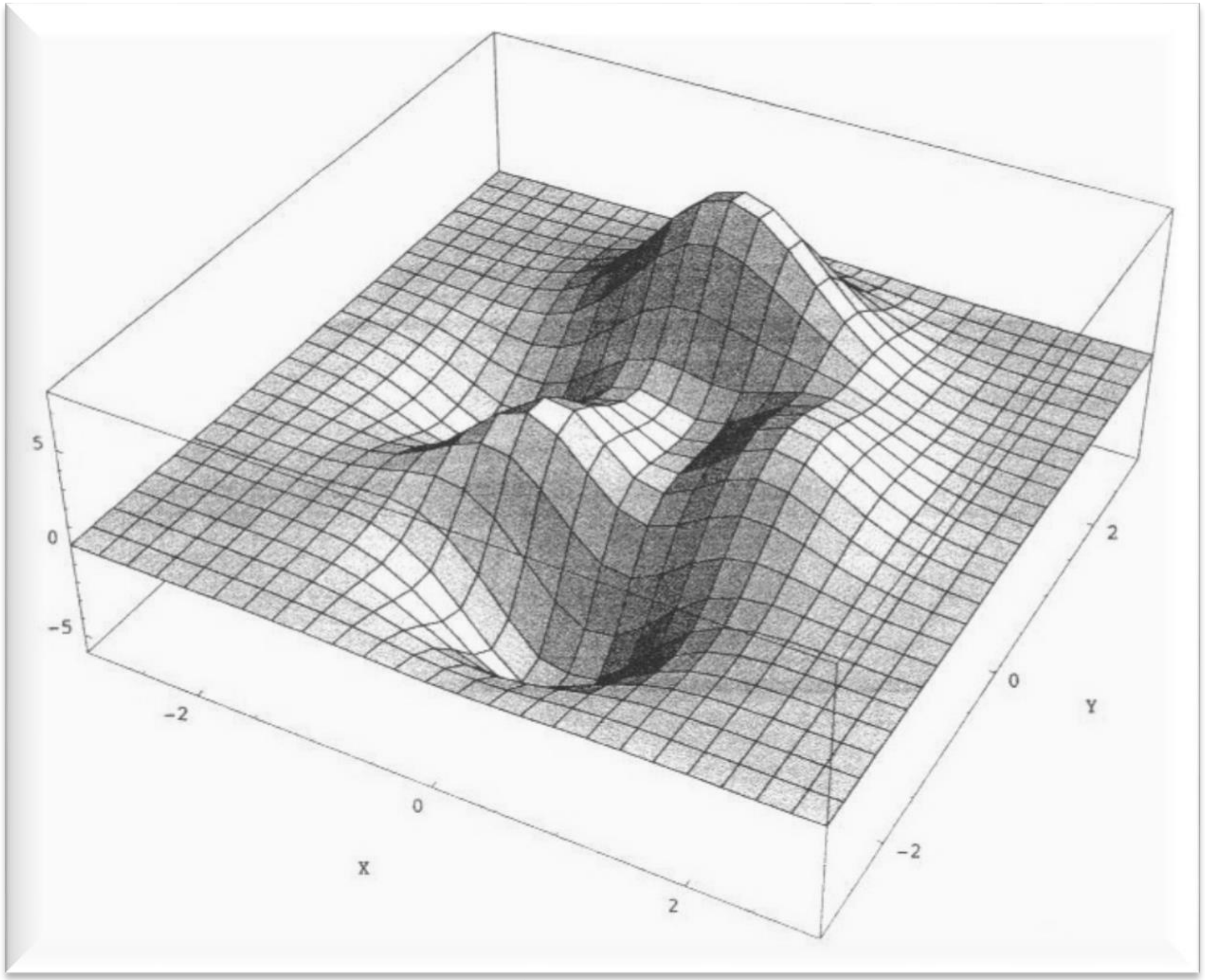


- Produce 92.45 Flyer frames per week.
- Produce 71.32 Razor frames per week.
- Profit per week is $6,075.47.
- Use 470.2 pounds of aluminum alloy per week (of the 500 pounds available per week).
- Use the entire 420 pounds of steel alloy available per week.

Issues in Non Linear Optimization

Multiple Local Optima

- Consider the function
- $f(X, Y) = 3(1-X)^2 e^{-X^2-(Y+1)^2} - 10\left(\frac{X}{5} - X^3 - Y^5\right) e^{-X^2-Y^2} - e^{--(X+1)^2-Y^2}/3$
- The hills and valleys in the graph show that this function has several local maximums and local minimums.
- There are two local minimums, one of which is the global minimum.
- There are three local maximums, one of which is the global maximum.



Local and Global Optima

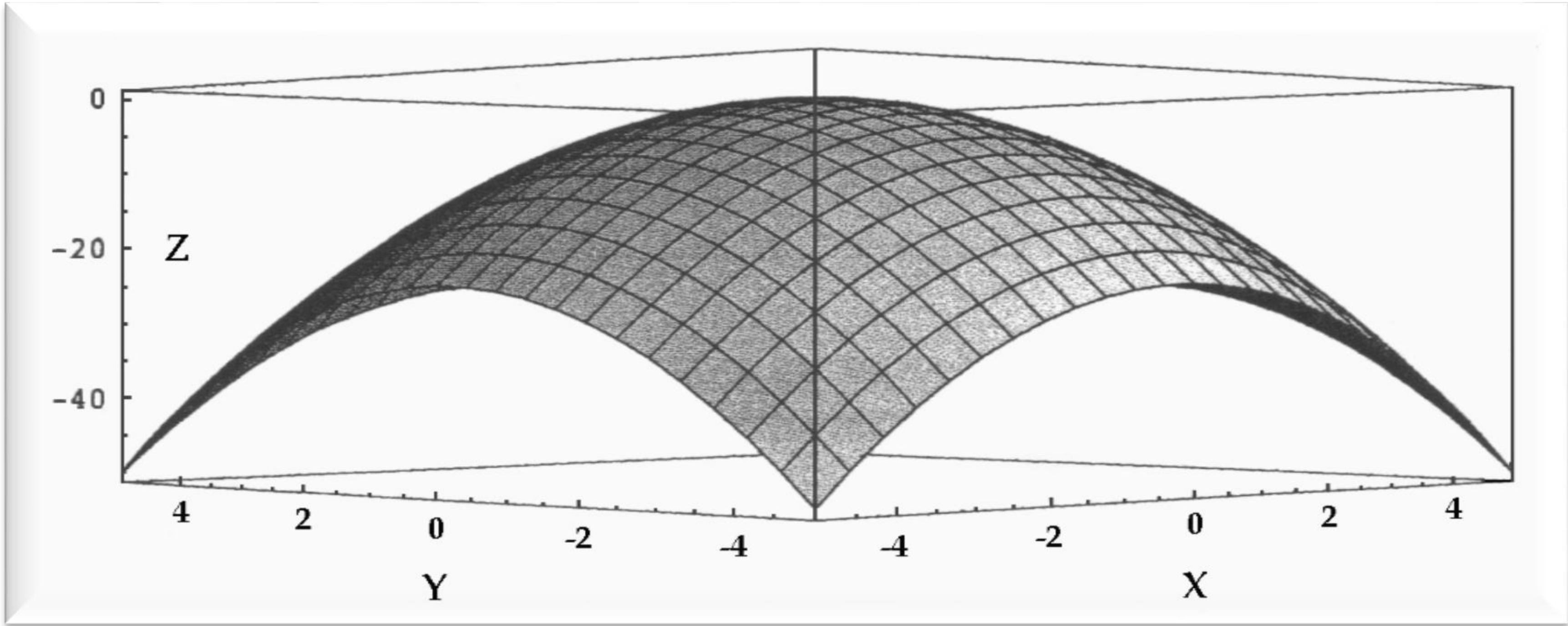
- A feasible solution is a local optimum if there are no other feasible solutions with a better objective function value in the immediate neighborhood.
    - For a maximization problem the local optimum corresponds to a local maximum.
    - For a minimization problem the local optimum corresponds to a local minimum.
- A feasible solution is a global optimum if there are no other feasible points with a better objective function value in the feasible region.
- Obviously, a global optimum is also a local optimum.

Multiple Local Optima

- Nonlinear optimization problems can have multiple local optimal solutions, in which case we want to find the best local optimum.
- Nonlinear problems with multiple local optima are difficult to solve and pose a serious challenge for optimization software.
- In these cases, the software can get “stuck” and terminate at a local optimum.
- There can be a severe penalty for finding a local optimum that is not a global optimum.
- Developing algorithms capable of finding the global optimum is currently a very active research area.
- If the Objective Function is Convex and the Feasible Region is Convex then every local optimal solution is a global optimal solution

Concave Functions & Global Maxima

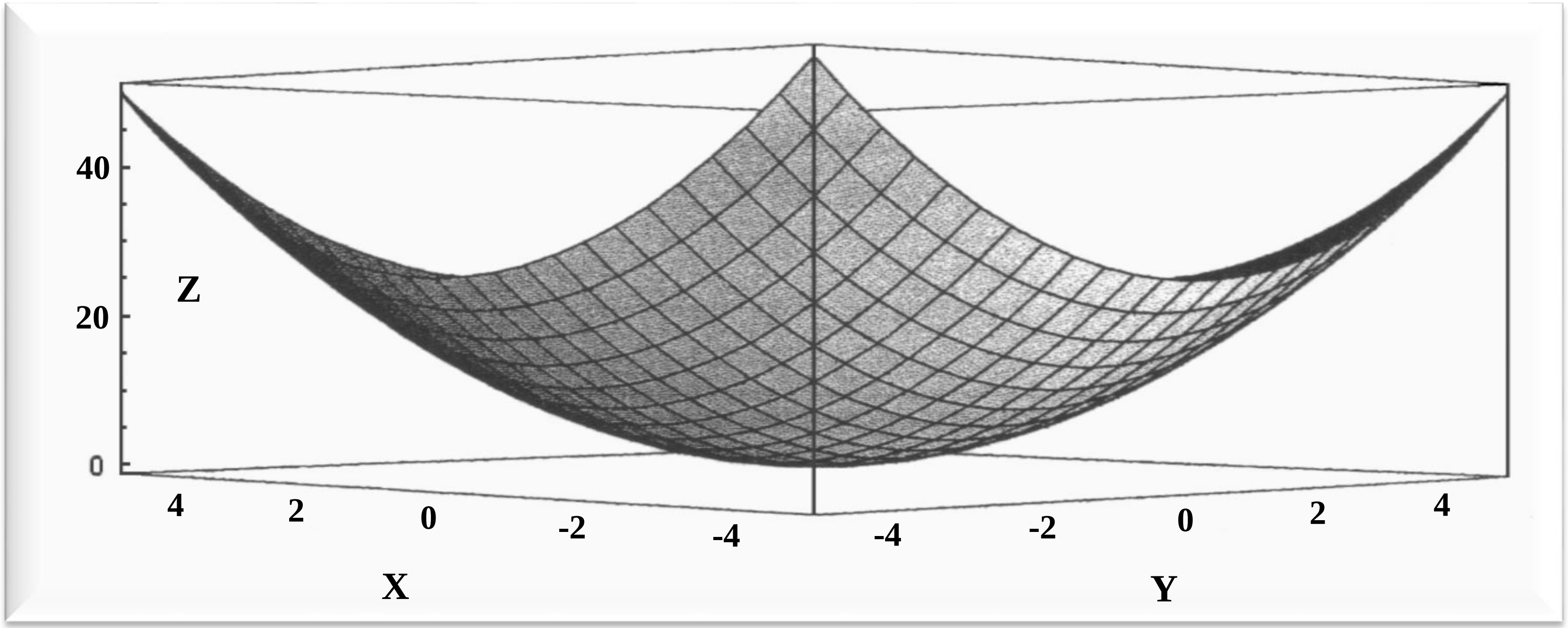
- Consider the function $f(x, y) = -x^2 - y^2$
- A function that is bowl-shaped down is called a concave function.
- The maximum value for f(x, y) is 0 and the point (0, 0) gives the optimal value of 0.
- Functions such as f(x, y) have a single local maximum that is also a global maximum
- This type of nonlinear problem is relatively easy to maximize.



Convex Functions & Global Minima

Consider the objective function $f(x, y) = x^2 + y^2$

- A function that is bowl-shaped up is called a convex function.
- The minimum value for f(x, y) is 0 and the point (0, 0) gives the optimal value of 0.
- Functions such as f(x, y) have a single local minimum that is also a global minimum.
- This type of nonlinear problem is relatively easy to minimize.



innovate achieve lead

BITS Pilani
Pilani Campus

Thank you!!