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SS ZC416 Mathematical Foundations for Data Science
Calculus Refresher

Objectives

Calluses Refresher

– Concept of Limits
– Simple differentiations
– Newton’s Method
– Convex functions
– Maxima & Minima

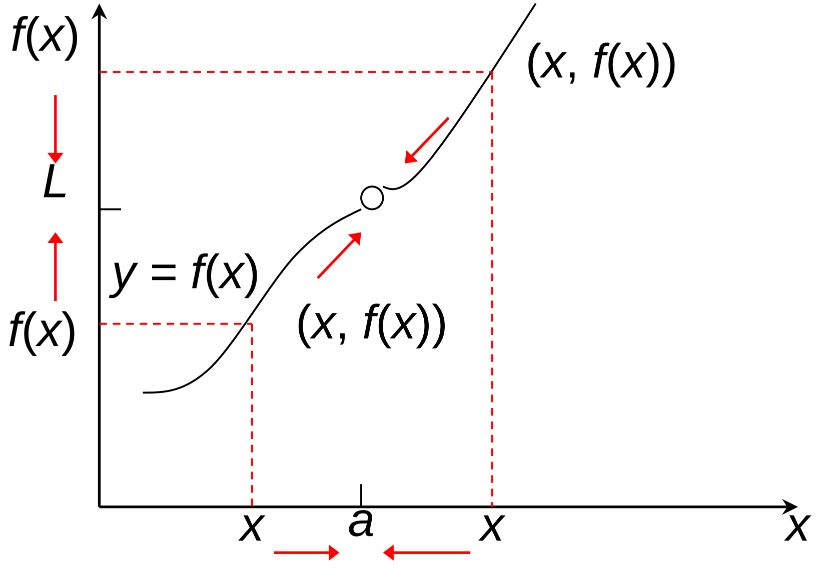
Limits

Definition of Limit of a Function

Suppose that the function *f*(*x*) is defined for all values of *x* near *a*, but not necessarily at *a*. If as *x* approaches *a* (without actually attaining the value *a*), *f*(*x*) approaches the number *L*, then we say that *L* is the limit of *f*(*x*) as *x* approaches *a*, and write

$$\lim_{x \to a} f(x) = L$$

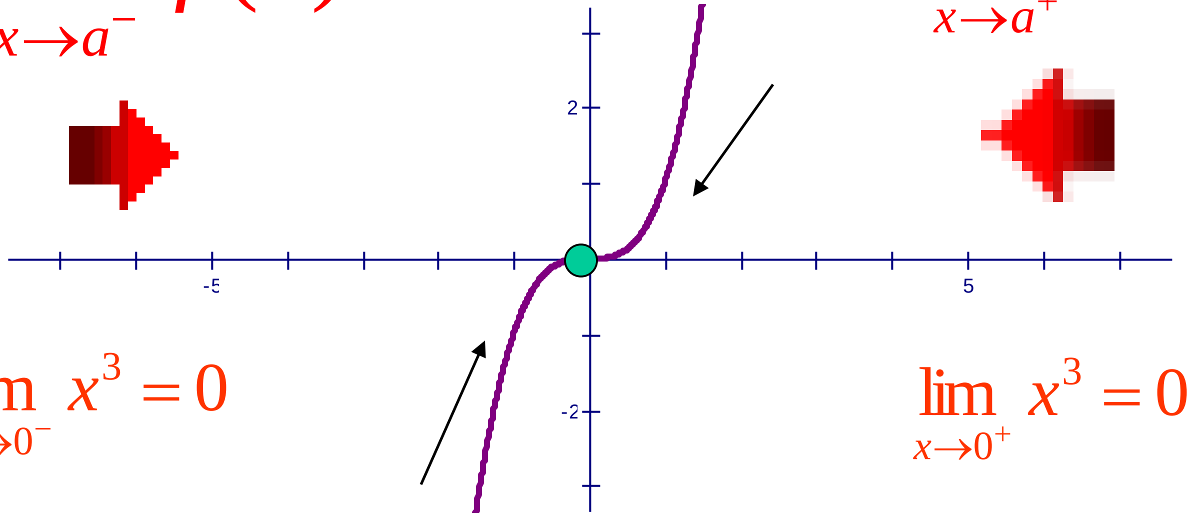
No matter how *x* approaches *a*, *f*(*x*) approaches *L*



Left & Right Hand Limits

$$\lim_{x \to a^-} f(x)$$

$$\lim_{x \to a^+} f(x)$$

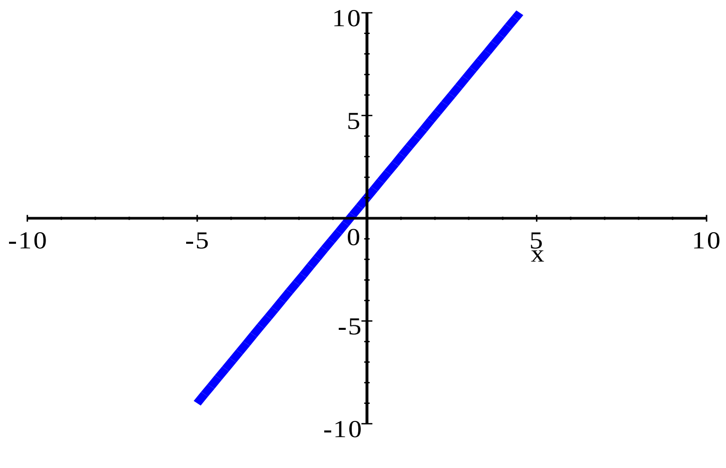


$$\lim_{x \to 0^-} x^3 = 0$$

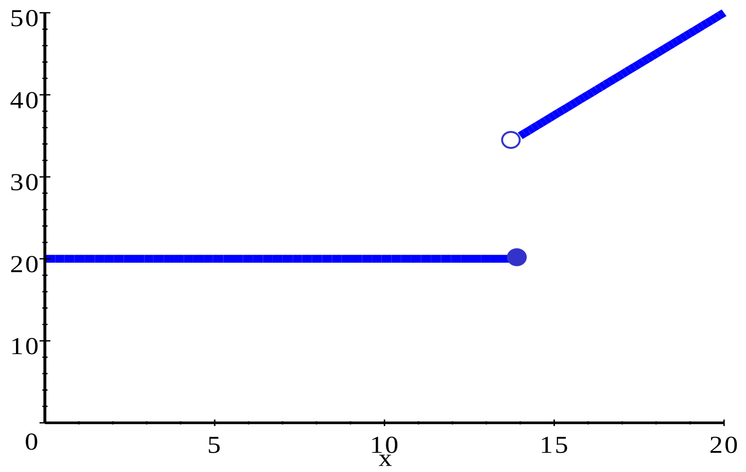
$$\lim_{x \to 0^+} x^3 = 0$$

Theorem

f(x) has a limit as **x** approaches **a** iff $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a} f(x) = L$

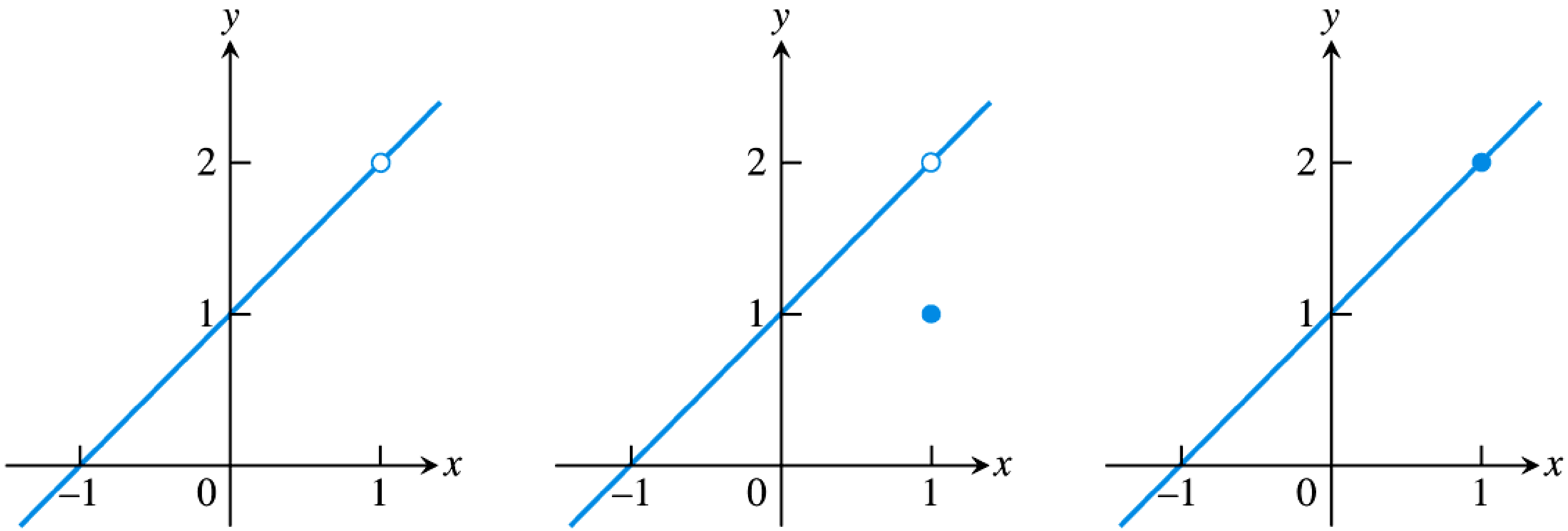


The limit exists as x approaches 2



The limit at x = 15 does not exist

Example



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

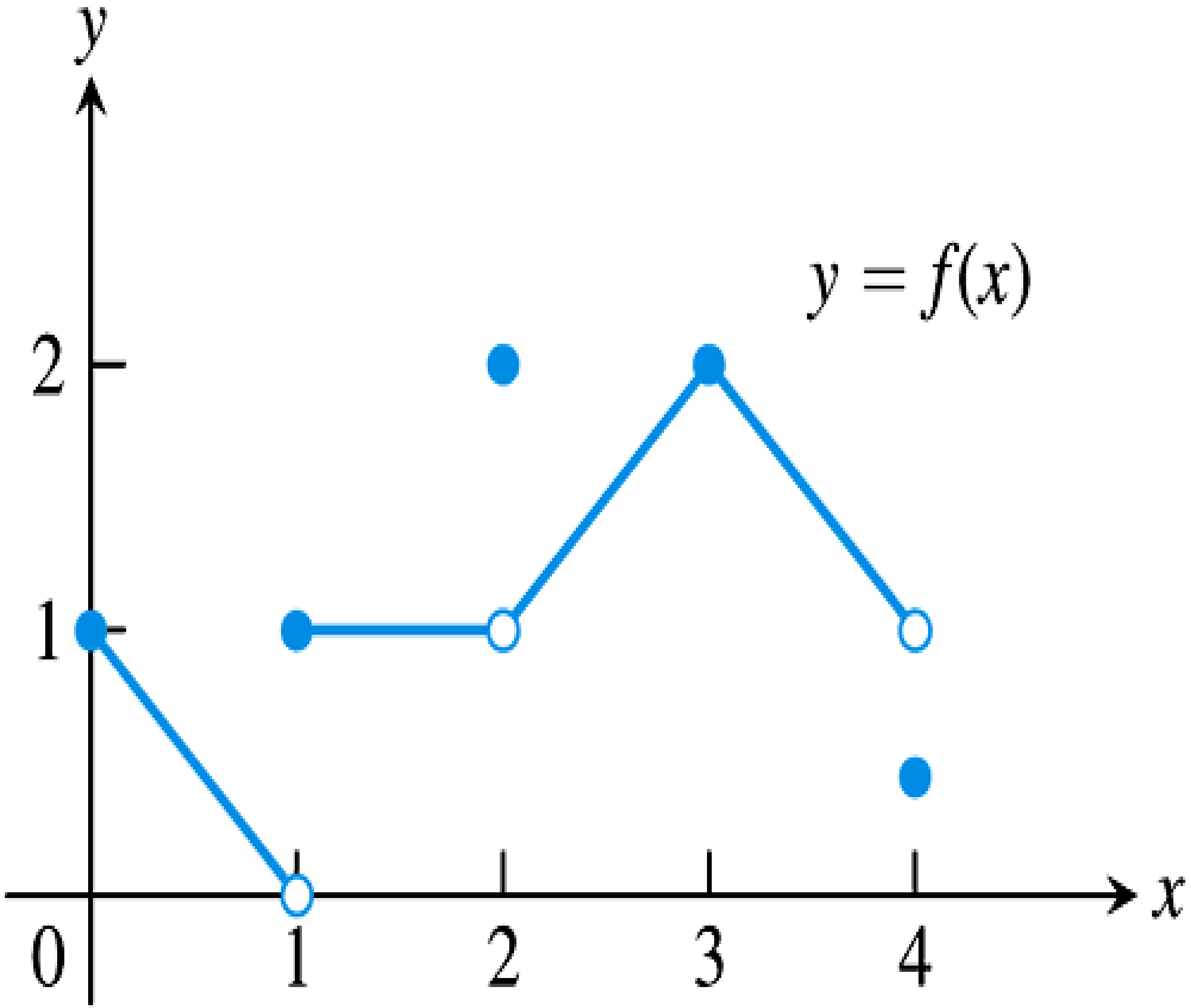
(c) $h(x) = x + 1$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = \lim_{x \to 1} f(x) = 2$$

The limit can exist even when the function is not defined at a point or has a value different from the limit

Example

- $\lim\limits_{x \to 1} f(x)$ DNE
- $\lim\limits_{x \to 2} f(x) = 1$
- $\lim\limits_{x \to 3} f(x) = 2$
- $\lim\limits_{x \to 4^-} f(x)$ DNE

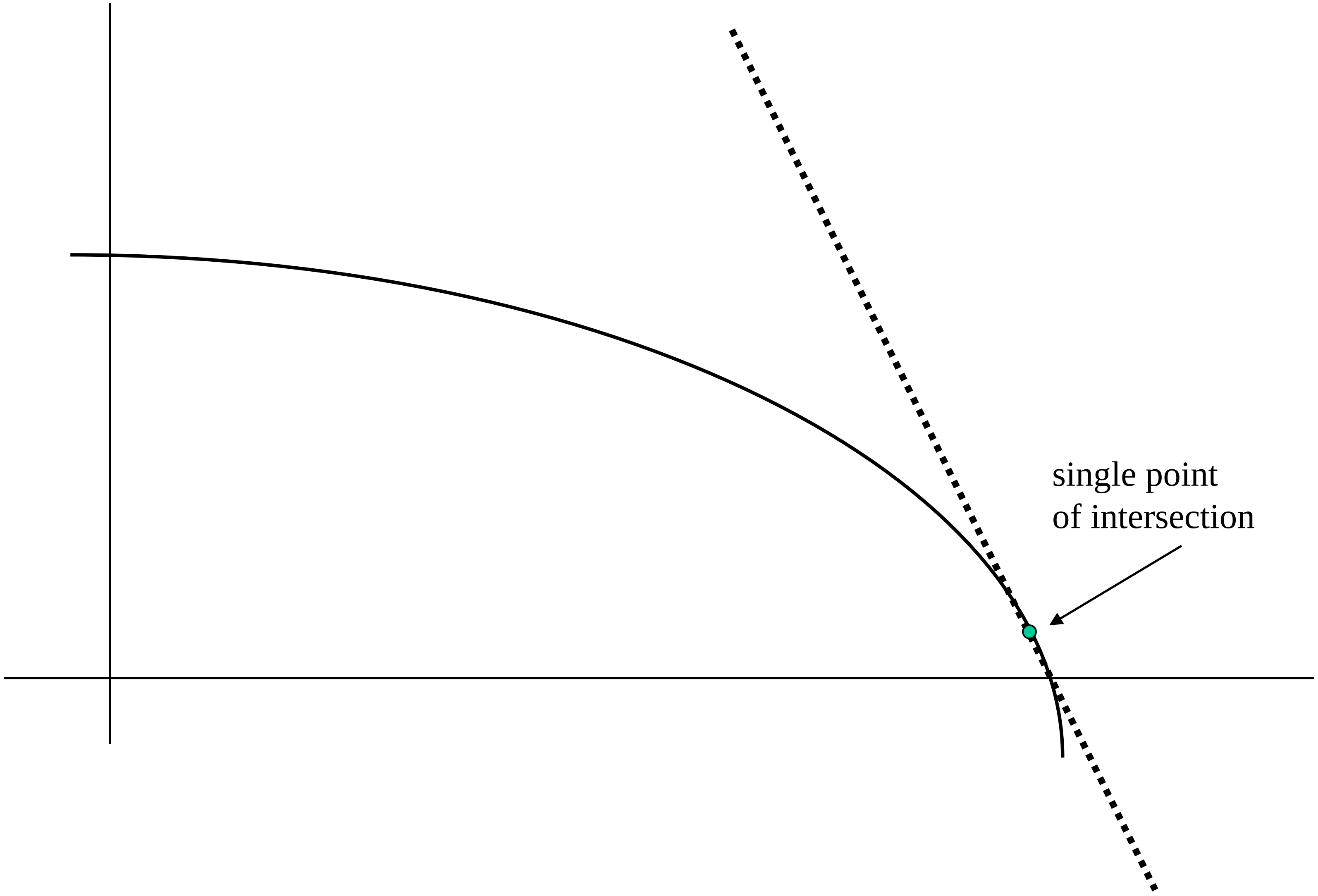


What is a Derivative?

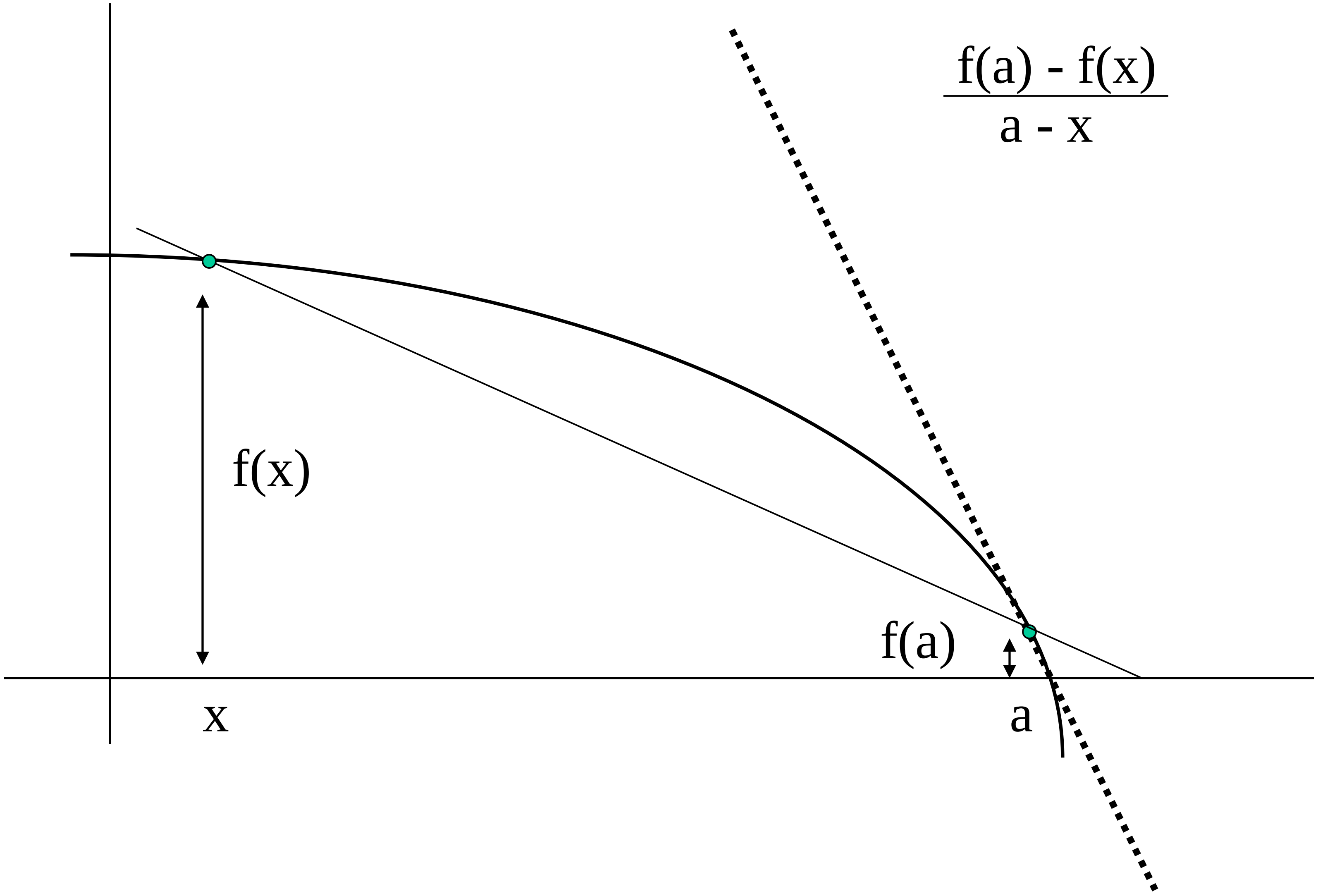
What is a Derivative?

- A function
- The rate of change of a function
- The slope of the line tangent to the curve

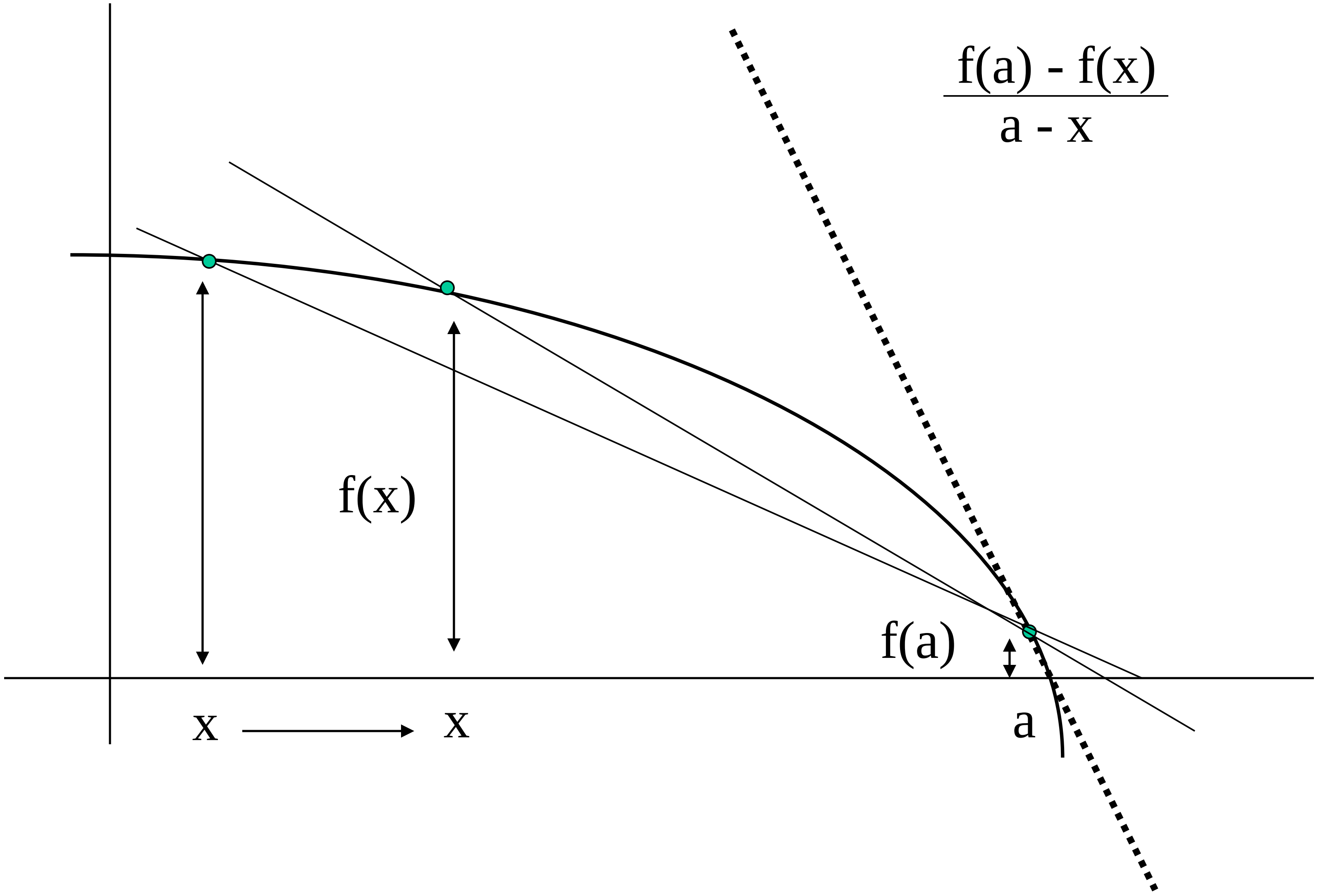
The Tangent Line



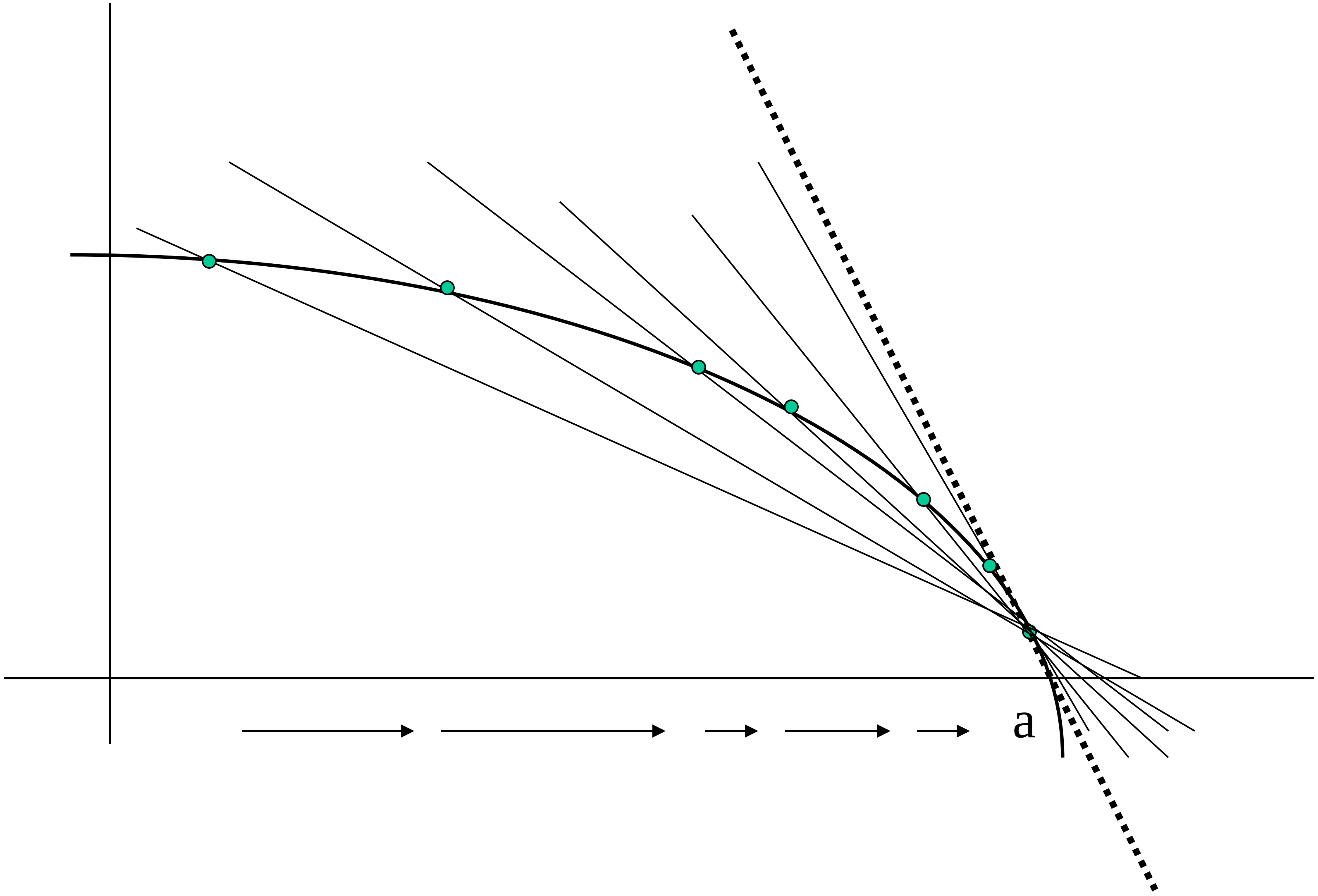
Slope of a Secant Line



Slope of a (Closer) Secant Line

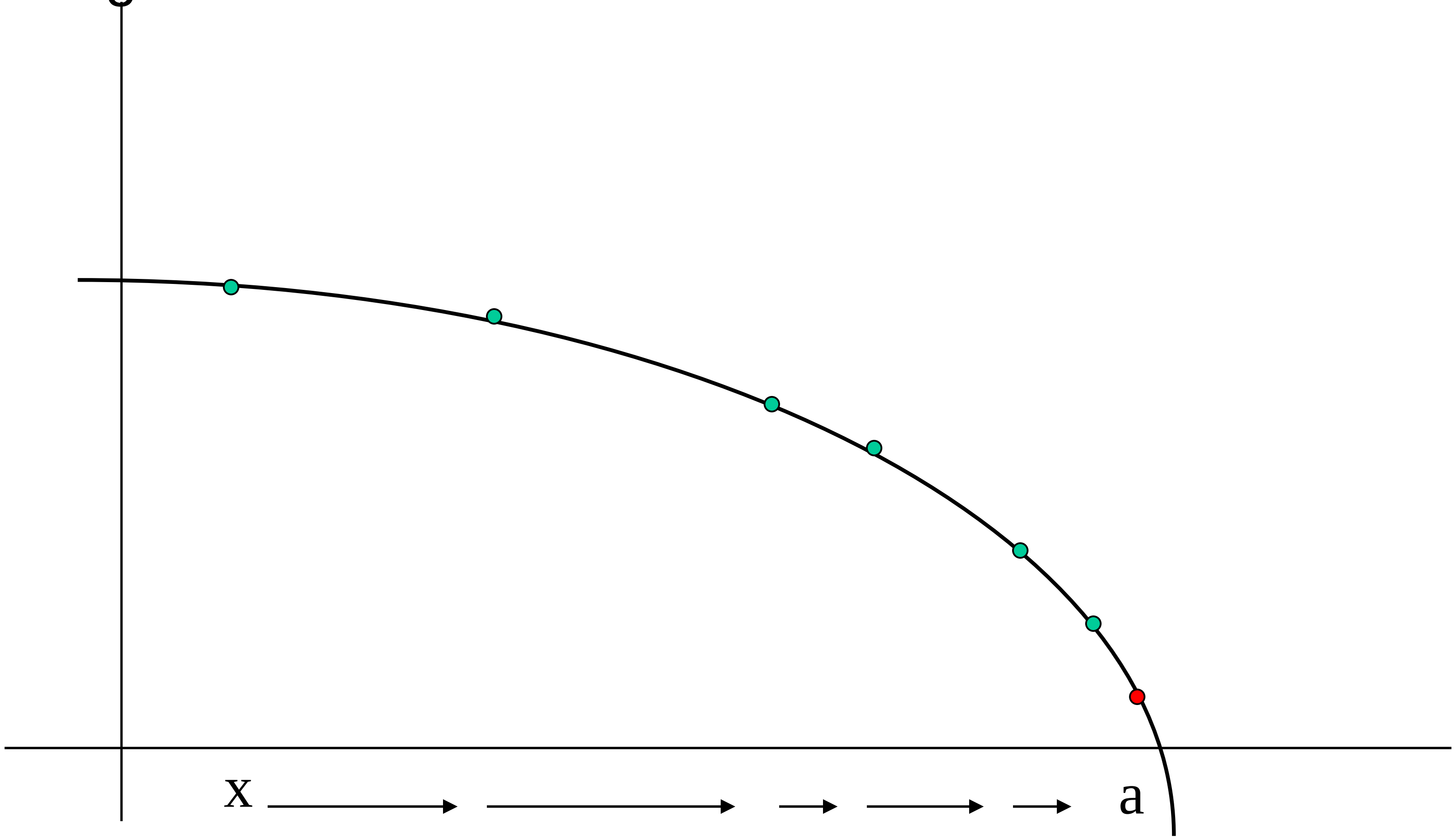


Closer and Closer…



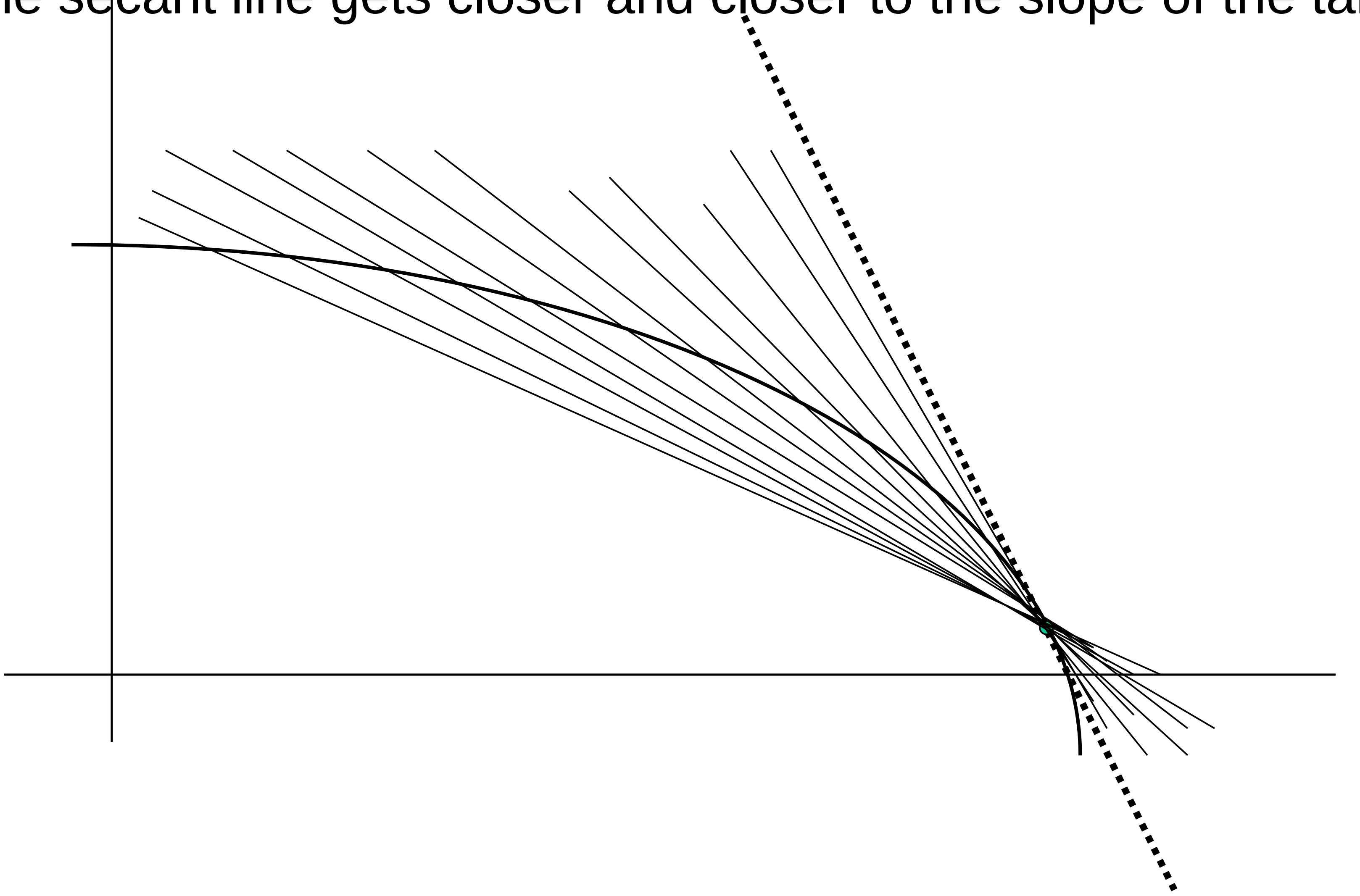
Watch (x, y) ...

As the values of *x* get closer and closer to *a*!



Watch What the Slope Does...

The slope of the secant line gets closer and closer to the slope of the tangent line...



The Slope of the Tangent

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

as *x* goes to *a*

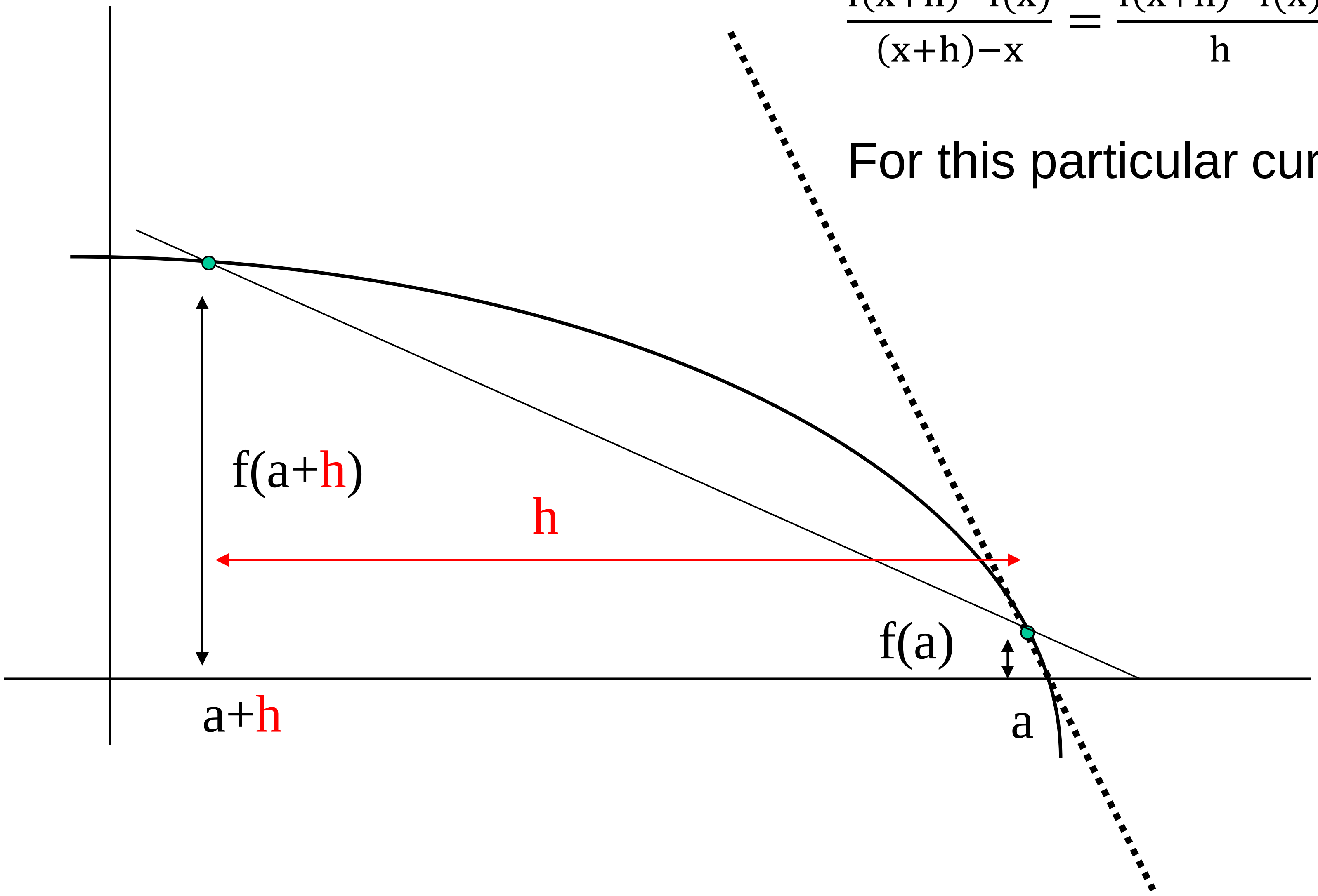
Equation for the slope

Which gives us the exact slope of the line tangent to the curve at *a*!

Similarly...

$$\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

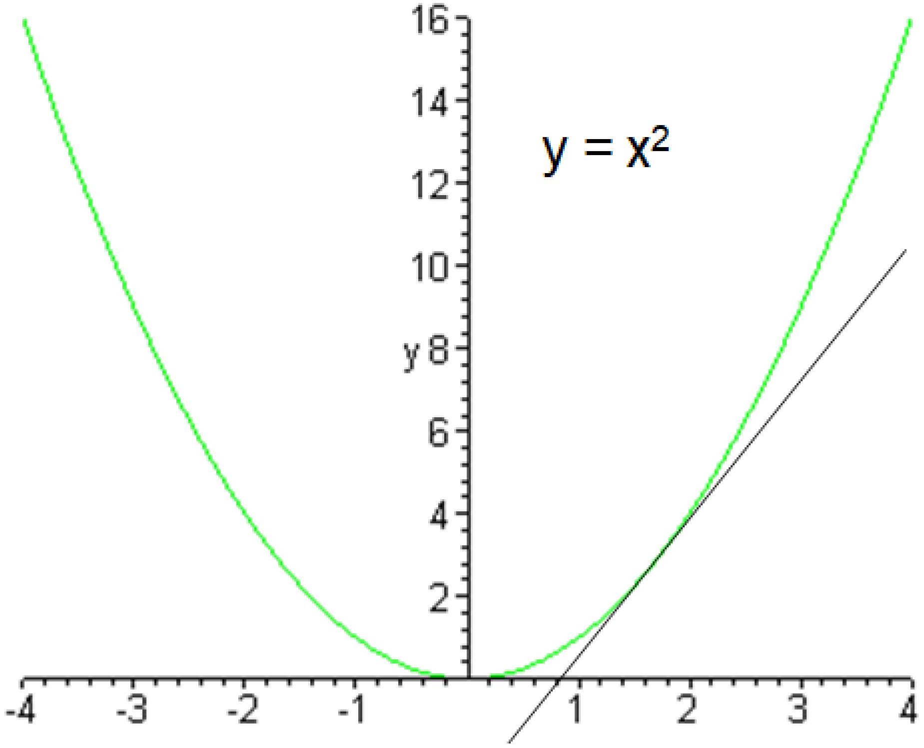
For this particular curve, h is negative



The Derivative

$$\left(\frac{df}{dx}\right)_{x=a} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

A simple example...



We want the slope at x = 2

$$\left(\frac{df}{dx}\right)_{x=2} = f'(2) = \lim_{x\to 2}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 2}\frac{x^2-2^2}{x-2}$$

$$= \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 4$$

Computing Derivatives

Rules of Differentiation

innovate achieve lead

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

Difference Rule: $\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x)+g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Common Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\text{Sin } x) = \text{Cos } x \qquad \frac{d}{dx}(\text{Cos } x) = -\text{Sin } x \qquad \frac{d}{dx}(\text{Tanh } x) = \text{Sech}^2 x$$

Some Hyperbolic Functions

$\text{Sinh } x = \frac{e^x - e^{-x}}{2}$ Cosh $x = \frac{e^x + e^{-x}}{2}$ $\text{Tanh } x = \frac{\text{Sinh } x}{\text{Cosh } x}$ $\text{Sech } x = \frac{1}{\text{Cosh } x}$

Exercises

Differentiate

$x^3 – 2x + 1$ $\quad$ $\text{Sin}(2x^3)$ $\quad$ $\text{Sin}(2x) – \text{Cos}(x^2)$ $\quad$ $f(x) = \text{Sin}(3x) / (1 + x^2)$

e^{3x} $\quad$ e^{-3x} $\quad$ $e^{(2x – 1)}$

$f(x) = e^x / (1 + e^x)$ $\quad$ $f(x) = e^{(1 + 2x)}/(1 – e^{(1 + 2x)})$

$\ln(2x)$ $\quad$ $\ln(x^2 + 2x – 1)$

Critical Points

Critical Points

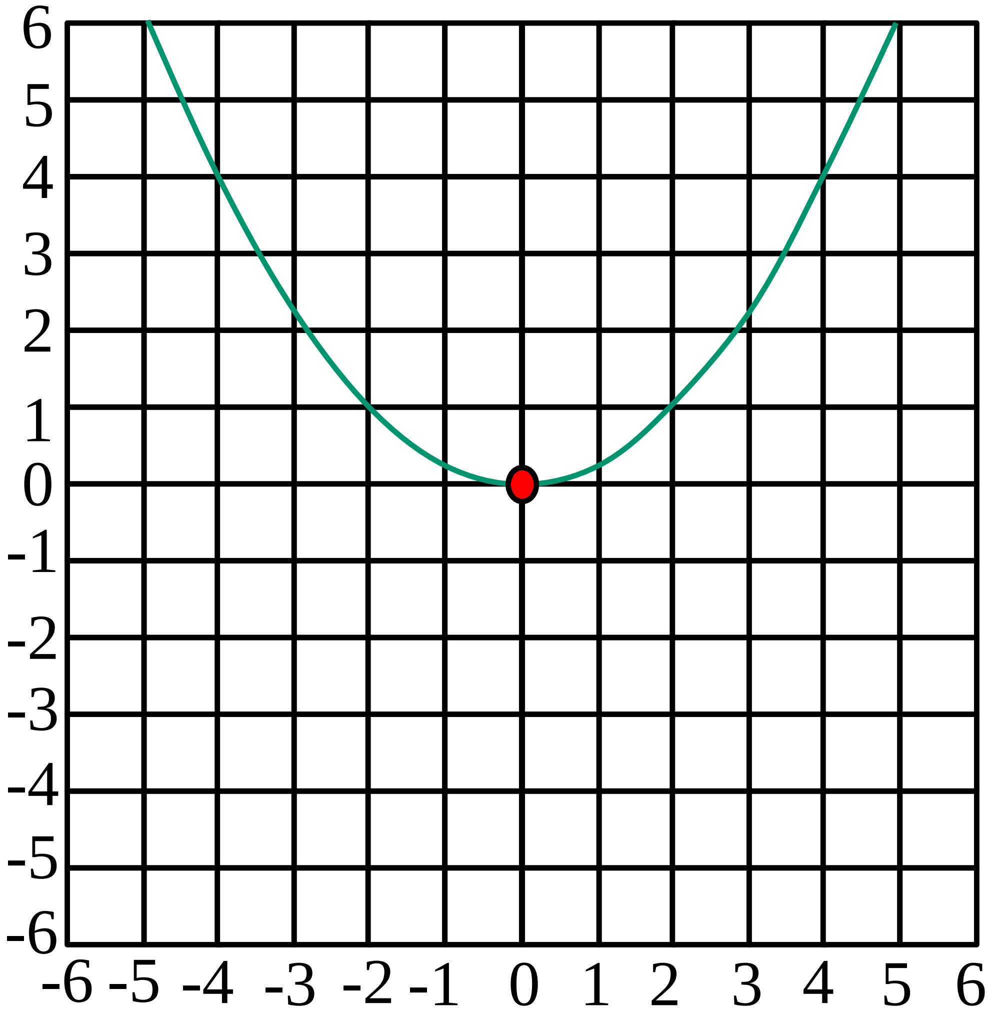
Def: The critical points of a function $f(x)$ are the points where $f'(x) = 0$

Fact: Any local maximum or minimum of $f(x)$ occurs at either a critical point, the boundary, or a point where $f'(x)$ is discontinuous or does not exist.

Warning: Critical points are not always local maximums or minimums

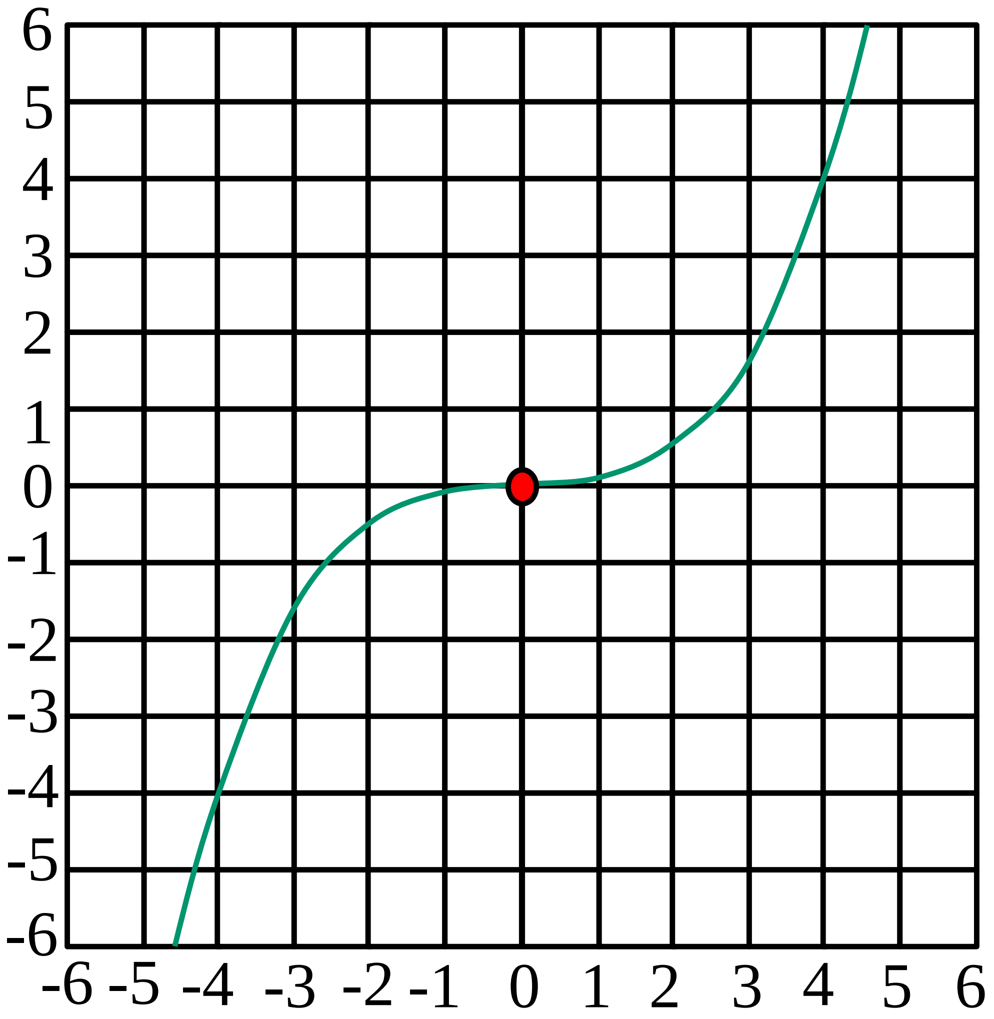
Example

$f(x) = \frac{x^2}{4}$ The critical point (0,0) is a minimum



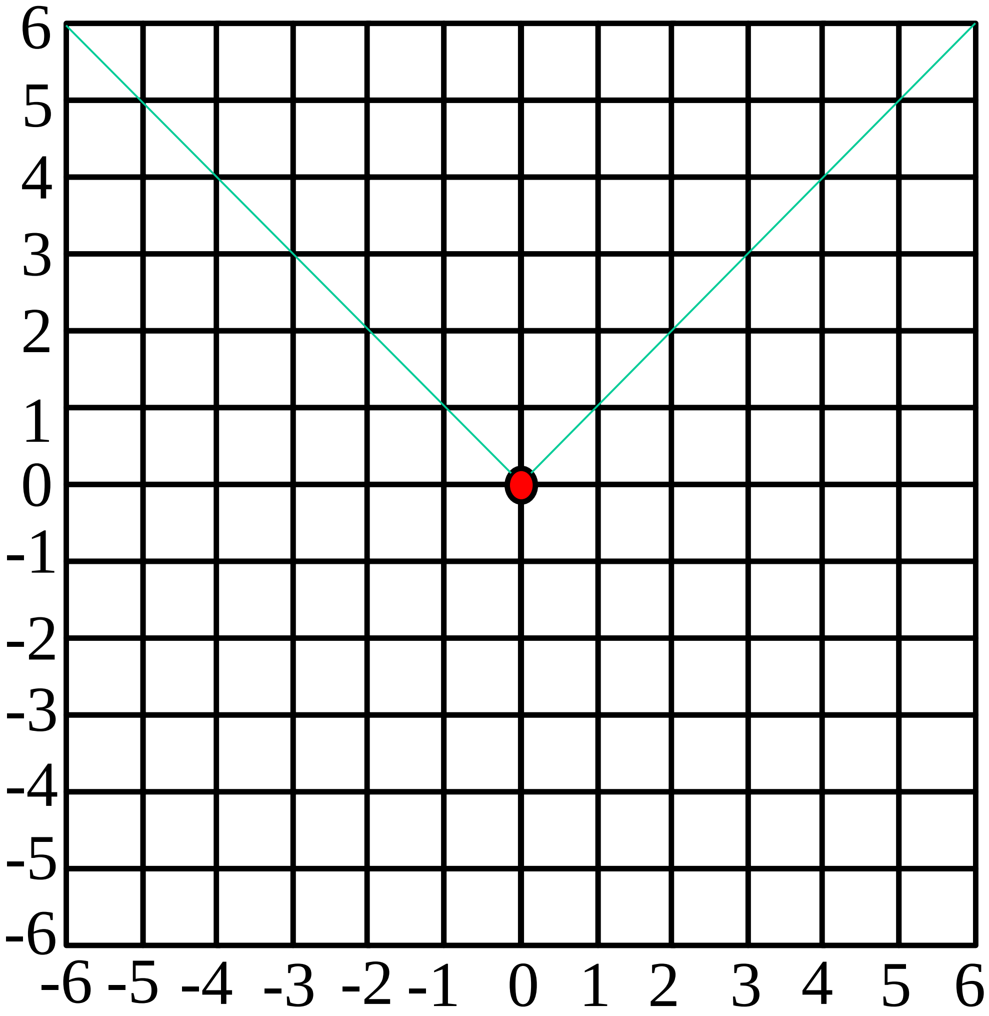
Example

$f(x) = \frac{x^3}{16}$: The critical point (0,0) is not a minimum or maximum



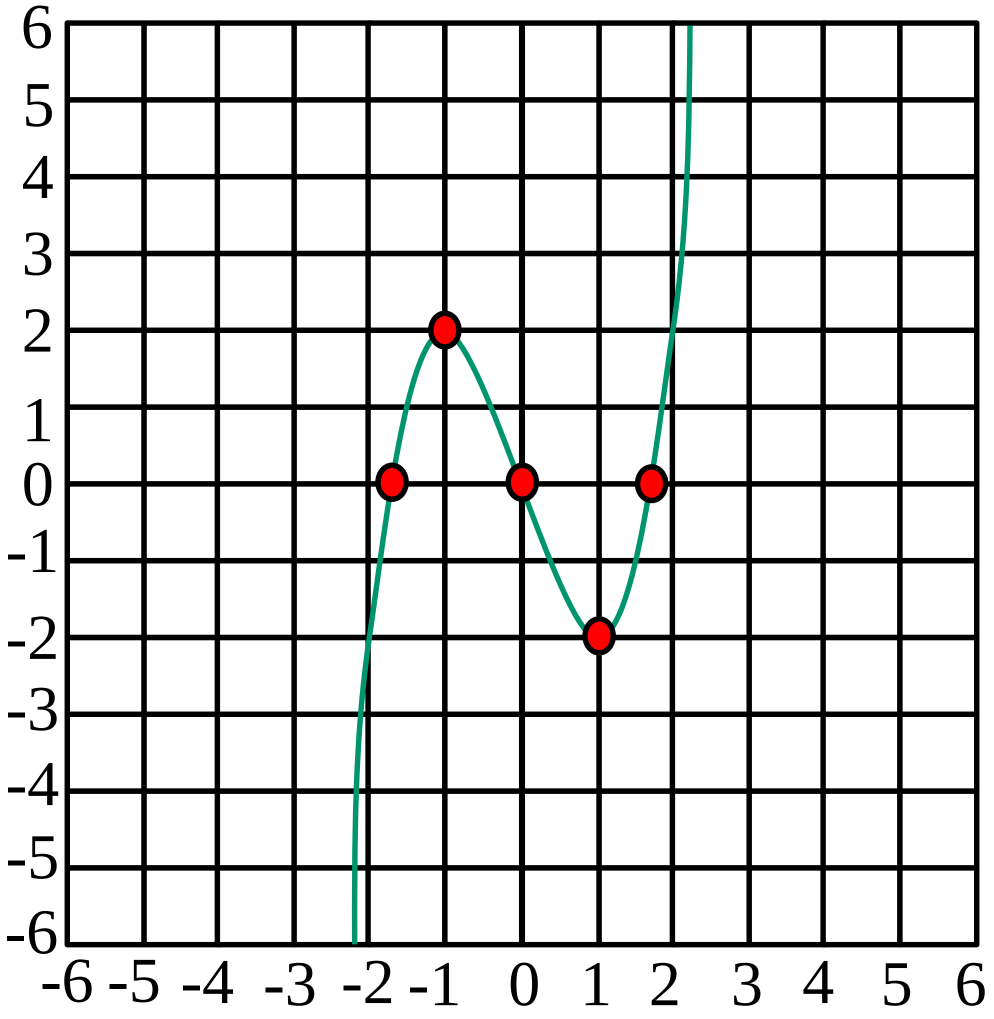
Example

$f(x) = |x|$: (0,0) is a minimum but not a critical point (why?).



Example

$f(x) = x^3 - 3x$: The critical points of $f(x)$ are $(-1, 2)$ and $(1, -2)$



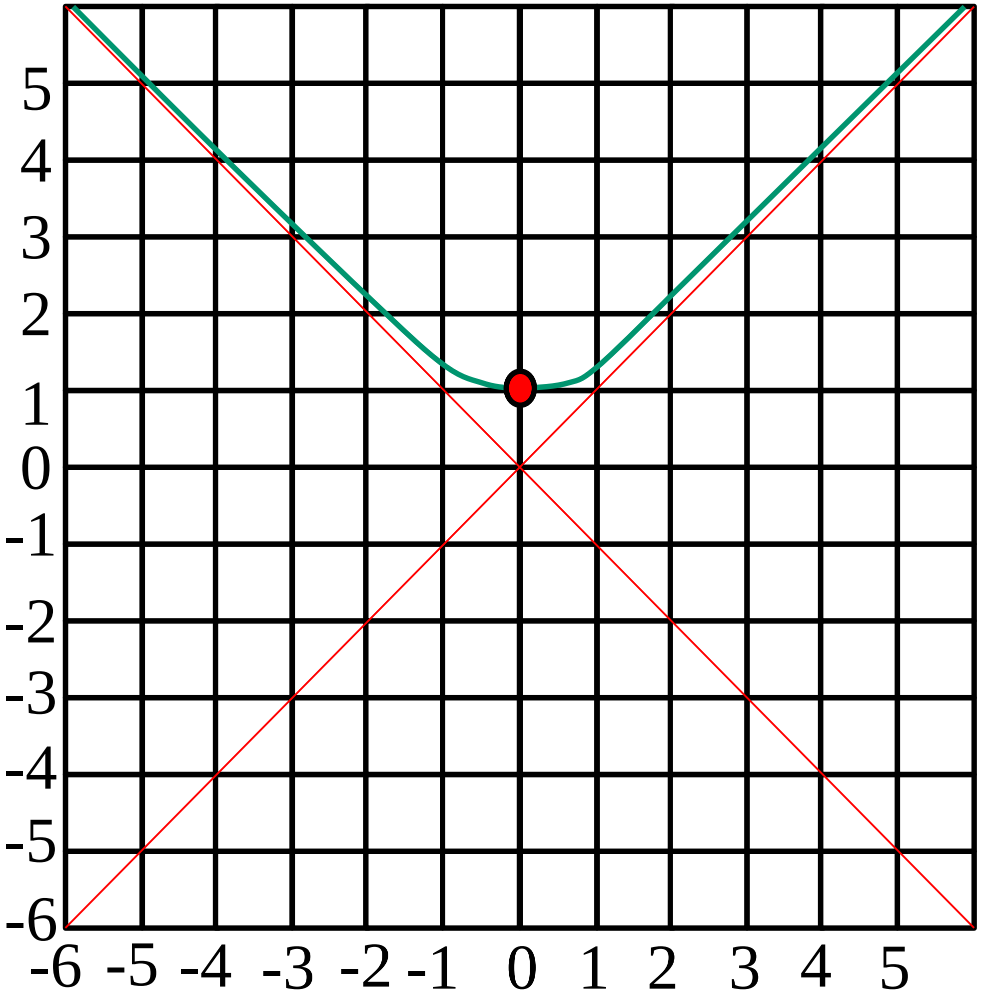
Example

$f(x) = \sqrt{x^2+1}$:

- The only critical point of $f(x)$ is $(0,1)$

1. $f'(x) = \frac{x}{\sqrt{x^2+1}}$ is negative if x < 0
2. $f'(x) = \frac{x}{\sqrt{x^2+1}}$ positive if x > 0

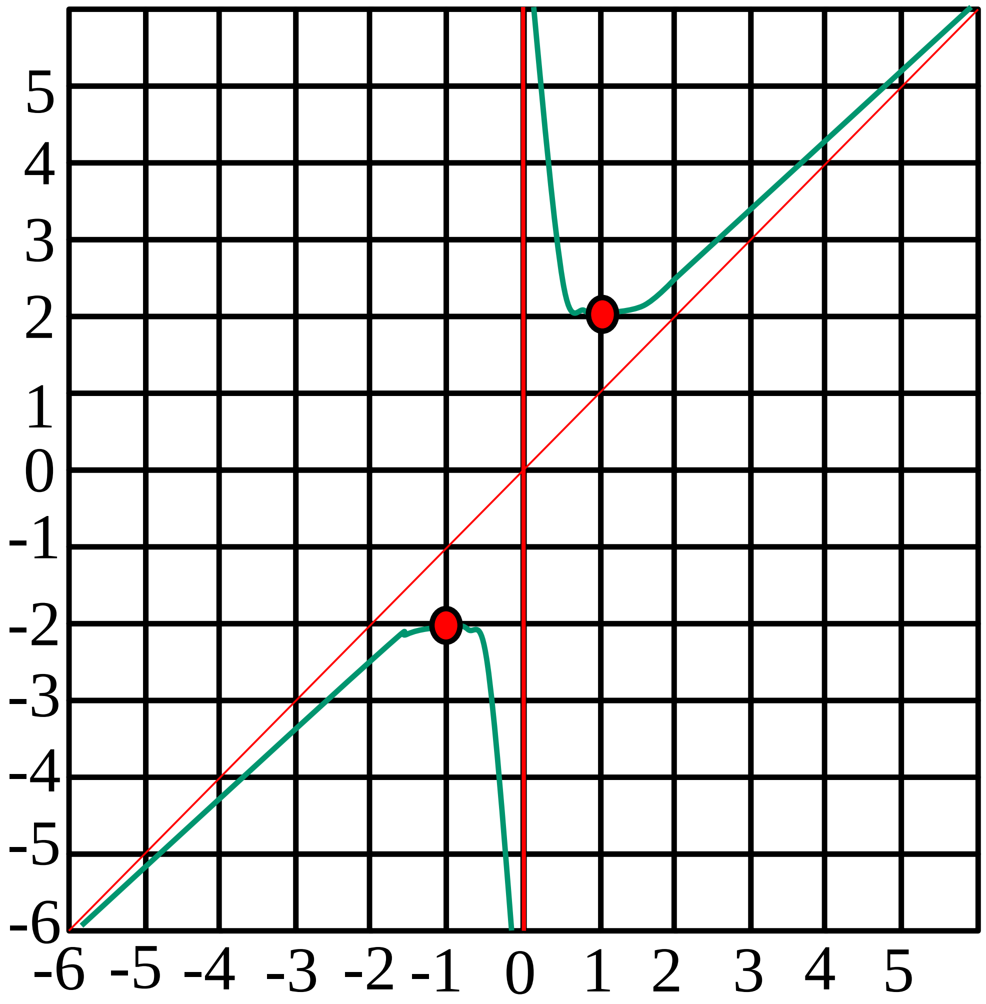
Note the asymptotes



Example

$f(x) = x + \frac{1}{x}$:

- The critical points are $(-1, -2)$ and $(1,2)$
- $f'(x) = 1 - \frac{1}{x^2}$ is positive for $x < -1$
- $f'(x) = 1 - \frac{1}{x^2}$ is negative for $-1 < x < 1$
- $f'(x) = 1 - \frac{1}{x^2}$ is positive for $x > 1$



Asymptotes and Inflection Points

Asymptotes are lines which the functions approaches in some limit.

- In the previous two examples, the red lines were asymptotes for $f(x)$.
- Asymptotes can occur as vertical lines at discontinuities or as limits as $x \to \pm\infty$.

Inflection points are points where $f(x)$ changes from concave up to concave down.

- This usually (but not always) occurs when $f''(x) = 0$