1

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 4 & 1 & 2 & 2 & 3 \end{array}\right] \xrightarrow[R_3 = R_3 - 4R_1]{R_2 = R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 0 & 5 & -6 & -2 & -5 \\ 0 & 5 & -6 & -2 & -5 \end{array}\right]$$

$$\xrightarrow[\frac{1}{5}R_2]{R_2 =} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 0 & 1 & -1.2 & -0.4 & -1 \\ 0 & 5 & -6 & -2 & -5 \end{array}\right] \xrightarrow[R_3 = R_3 - 5R_2]{R_1 = R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 3.2 & 0.6 & 1 \\ 0 & 1 & 1.2 & -0.4 & -1 \\ 0 & 0 & 4 & 0 & 4 \end{array}\right]$$

$$\xrightarrow[(1/4)R2]{R_3 =} \left[\begin{array}{cccc|c} 1 & 0 & 3.2 & 0.6 & 1 \\ 0 & 1 & 1.2 & -0.4 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow[R_2 = R_2 - 1.2R_3]{R_1 = R_1 - 3.2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0.6 & -2.2 \\ 0 & 1 & 0 & -0.4 & -2.2 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

The 3rd row gives $z = -1$

The 2nd row gives $y = -2.2 + 0.4t$

The 1st row gives $x = -2.2 - 0.6t$

The solution $\begin{bmatrix} -2.2 - 0.6t \\ -2.2 + 0.4t \\ -1 \\ t \end{bmatrix}$ for all values of t

2

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow[R_3 + R_2]{R_3 =}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = \frac{1}{3}R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 + R_2]{R_1 =} \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Null Space $x = (-2/3)z$, $y = (4/3)z$

Or $z \begin{bmatrix} -2/3 \\ 4/3 \\ 1 \end{bmatrix}$

Dim (Null Space) = 1

Range (T) = Col Span (A)

In the RREF, the 3rd column is a linear combination of the first two.

Therefore the basis of Range (T) = Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

Therefore rank (T) = dim (Range (T)) = 2

rank (T) + nullity (T) = dim (V) when $T: V \to W$

2 + 1 = 3

3 $A-\lambda I = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$

$$\det(A-\lambda I) = (-2-\lambda)\left((1-\lambda)(-\lambda)-12\right) - 2(-2\lambda-6)$$
$$-3(-4+1-\lambda)$$

$= (-2-\lambda)(-\lambda+\lambda^2-12)+4\lambda+12+9+3\lambda$

$= 2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 + 9 + 3\lambda$

$= -\lambda^3 - \lambda^2 + 21\lambda + 45$

$\det(A-\lambda I) = 0 \Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\Rightarrow (\lambda+3)^2(\lambda-5) = 0$

Eigenvalues $= -3, -3, 5$

Eigenvector of $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Row 1 gives $x = -2y + 3z$

The eigenvector $\begin{bmatrix} -2y+3z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 3z \\ 0 \\ z \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Eigenvectors for $\lambda = -3$ are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Eigenvector for $\lambda = 5$

$$A - 5I = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \xrightarrow[-\frac{1}{7}R_1]{R_1 =} \begin{bmatrix} 1 & -2/7 & 3/7 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & -2/7 & 3/7 \\ 0 & -24/7 & -48/7 \\ 0 & -16/7 & -32/7 \end{bmatrix} \xrightarrow[\frac{24}{7}R_2]{R_2 =} \begin{bmatrix} 1 & -2/7 & 3/7 \\ 0 & 1 & 2 \\ 0 & -16/7 & -32/7 \end{bmatrix}$$

$$\xrightarrow[R_3 = R_3 + \frac{11}{7}R_2]{R_1 = R_1 + \frac{2}{7}R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Row2 gives $y = -2z$ & Row1 gives $x = -z$

Eigenvector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\therefore$ Eigenvalues are $-3, -3$ & 5

Corresponding eigenvectors are

$\lambda: -3$ $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 5$ $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

4

$$\left[\begin{array}{ccc|c} 0.143 & 0.357 & 2.01 & -5.17 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 11.2 & -4.30 & -0.605 & 4.42 \end{array}\right] \xrightarrow[R_3]{R_1 \leftrightarrow}$$

$$\left[\begin{array}{cccc} 11.2 & -4.30 & -0.605 & 4.42 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 0.143 & 0.357 & 2.01 & -5.17 \end{array}\right] \xrightarrow[\frac{1}{11.2}R_1]{R_1 =}$$

$$\left[\begin{array}{cccc} 1 & -0.3839 & 0.0540 & 0.3946 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 0.143 & 0.357 & 2.01 & -5.17 \end{array}\right] \xrightarrow[R_3 = R_3 - 0.143R_1]{R_2 = R_2 + 1.31R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -0.3839 & -0.0540 & 0.3946 \\ 0 & 0.4081 & 1.9193 & -4.9431 \\ 0 & 0.4119 & 2.0177 & -5.2264 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -0.3839 & -0.0540 & 0.3946 \\ 0 & 0.4119 & 2.0177 & -4.9431 \\ 0 & 0.4081 & 1.9193 & -5.2264 \end{array}\right] \xrightarrow[\frac{1}{.4119}R_2]{R_2 =}$$

$$\left[\begin{array}{ccc|c} 1 & -0.3839 & -0.0540 & 0.3946 \\ 0 & 1 & 4.8985 & -12.6885 \\ 0 & 0.4081 & 1.9193 & -4.9431 \end{array}\right] \xrightarrow[R_3 = R_3 - 0.4081R_2]{R_1 = R_1 + 0.3839R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1.8265 & -4.4765 \\ 0 & 1 & 4.8985 & -12.6885 \\ 0 & 0 & -0.0798 & 0.2351 \end{array}\right] \xrightarrow[\frac{-1}{.0798}R_3]{R_3 =} \left[\begin{array}{ccc|c} 1 & 0 & 1.8265 & -4.4765 \\ 0 & 1 & 4.8985 & -12.6885 \\ 0 & 0 & 1 & -2.9461 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2 - 4.8985 R_3]{R_1 = R_1 - 1.8265 R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9.8576 \\ 0 & 1 & 0 & -1.7430 \\ 0 & 0 & 1 & 2.9461 \end{array}\right]$$

Solution $\begin{bmatrix} -9.8576 \\ -1.7430 \\ 2.9461 \end{bmatrix}$

5 $\text{Det} \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & \\ a_{21} & a_{22} & 0 & 0 & \cdots & \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & \\ a_{n1} & a_{n2} & \cdots & & & a_{nn} \end{bmatrix}$

$a_{11} \, \text{Det} \begin{bmatrix} a_{22} & 0 & \cdots \\ a_{32} & a_{33} & \cdots \\ a_{n2} & a_{n3} & \cdots \end{bmatrix}$

$\therefore \text{Det} = a_{11} \cdots a_{nn}$

If A is a lower triangular matrix

$A - \lambda I$ is also lower "

$\Rightarrow \text{Det}(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$

$\Rightarrow \lambda = a_{11}, a_{22} \cdots a_{nn}$

If A is upper triangular, A^T is lower

And the diagonal entries of $(A - \lambda I) \equiv (A^T - \lambda I)$