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SS ZC416 Mathematical Foundations for Data Science
Calculus Refresher

Objectives

Calculus Refresher

– Convex functions

– Maxima & Minima

Maxima & Minima

Finding Absolute Minima / Maxima

Any local maximum or minimum of $f(x)$ occurs at

- A critical point, OR the boundary OR a point where $f'(x)$ is discontinuous or does not exist.

To find the absolute minimum or maximum of $f(x)$, we just need to find all of these points and compare the values of $f(x)$ at these points.

Example

Minimize / Maximize $y = 2\sqrt{x} + \sqrt{1-x}$ over the interval $[0,1]$

$y' = \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{1-x}}$

Solving for $y' = 0$ we have that $\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{1-x}}$ which implies that $4(1-x) = x$, giving $x = \frac{4}{5}$.

When $x = \frac{4}{5}$, $y = 2\sqrt{\frac{4}{5}} + \sqrt{1 - \frac{4}{5}} = \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \sqrt{5}$. The critical point is $(\frac{4}{5}, \sqrt{5})$.

The boundary points are $(0,1)$ and $(1,2)$

Compute the function at all three points. We will see that

- y is maximized at $(\frac{4}{5}, \sqrt{5})$ and minimized at $(0,1)$.

The Second Derivative Test

How can we tell if a critical point is a local maximum or a local minimum?

If $f''(x) > 0$, it's a local minimum

If $f''(x) < 0$, it's a local maximum

Otherwise, the second derivative test is inconclusive.

Convex and Concave Functions

Convex Function

- A function bends upwards if it has an increasing first derivative
- f‘’(x) > 0
- If a function is convex at x = a then any line segment joining points on the curve near a will be above (a, f(a))

Concave Function

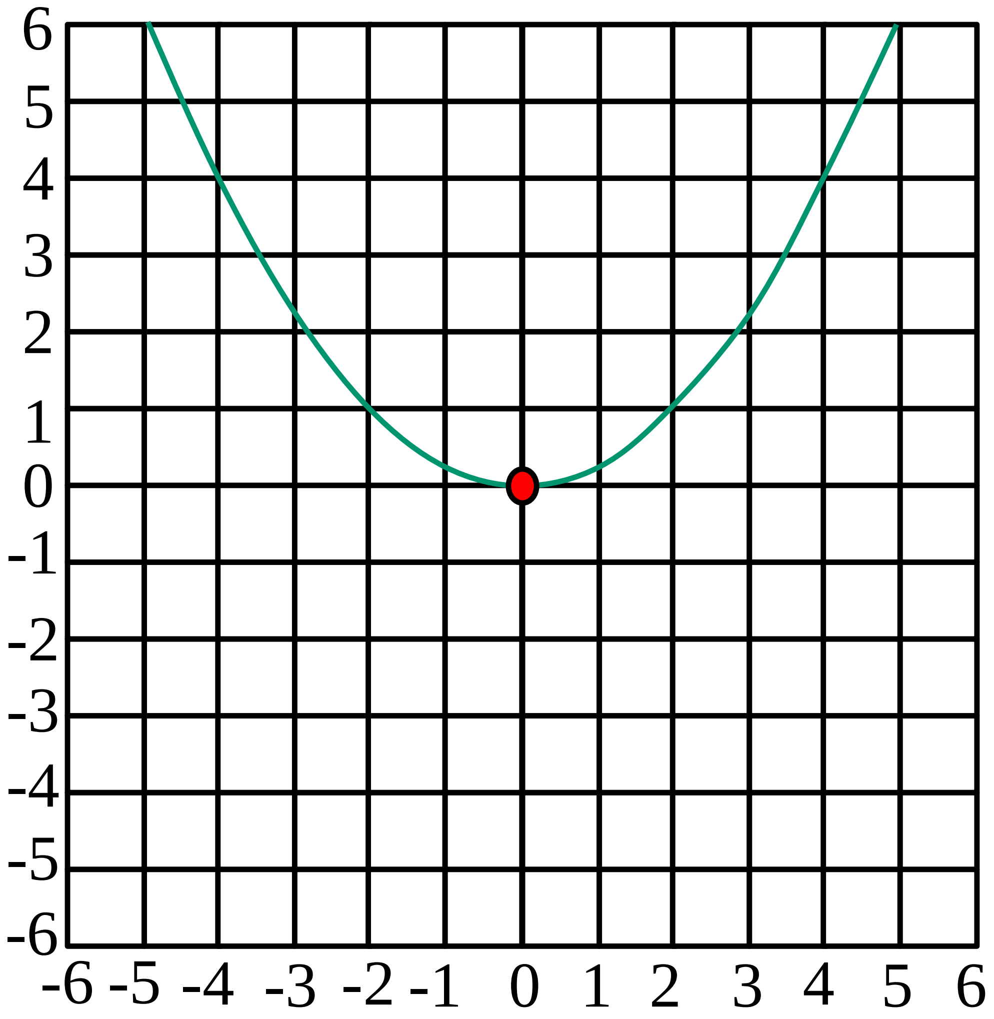
- A function bends downwards if it has an decreasing first derivative
- f‘’(x) < 0
- If a function is convex at x = a then any line segment joining points on the curve near a will be below (a, f(a))

Example

$f(x) = \frac{x^2}{4}$, f'(x) = 0.5x, f''(x) = 0.5

The critical point is (0, 0) since f'(0) = 0
Since f''(0) > 0, the origin is an absolute minimum

Convex at x = 0



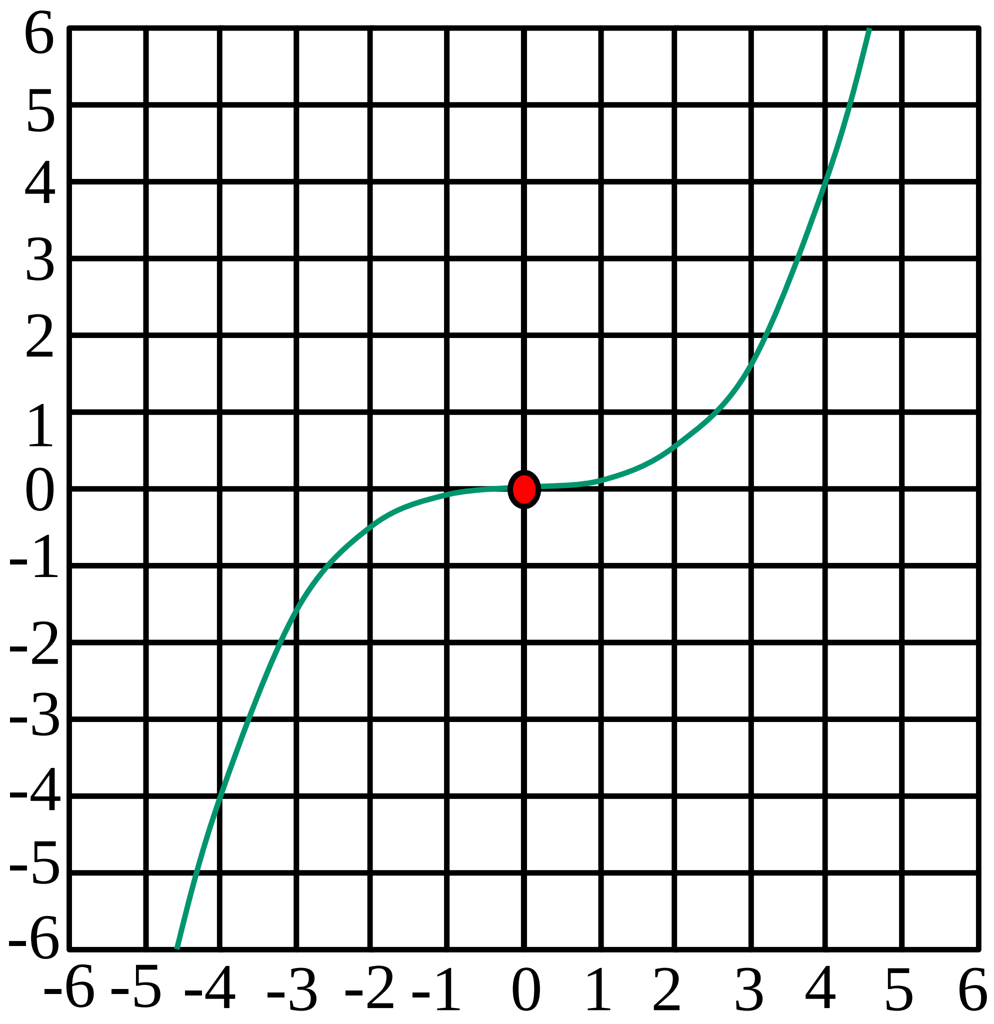
Example

$$f(x) = \frac{x^3}{16}$$

f'(x) = 0 ⇒ x = 0

The critical point is (0, 0)

f''(0) = 0 ⇒ Inconclusive

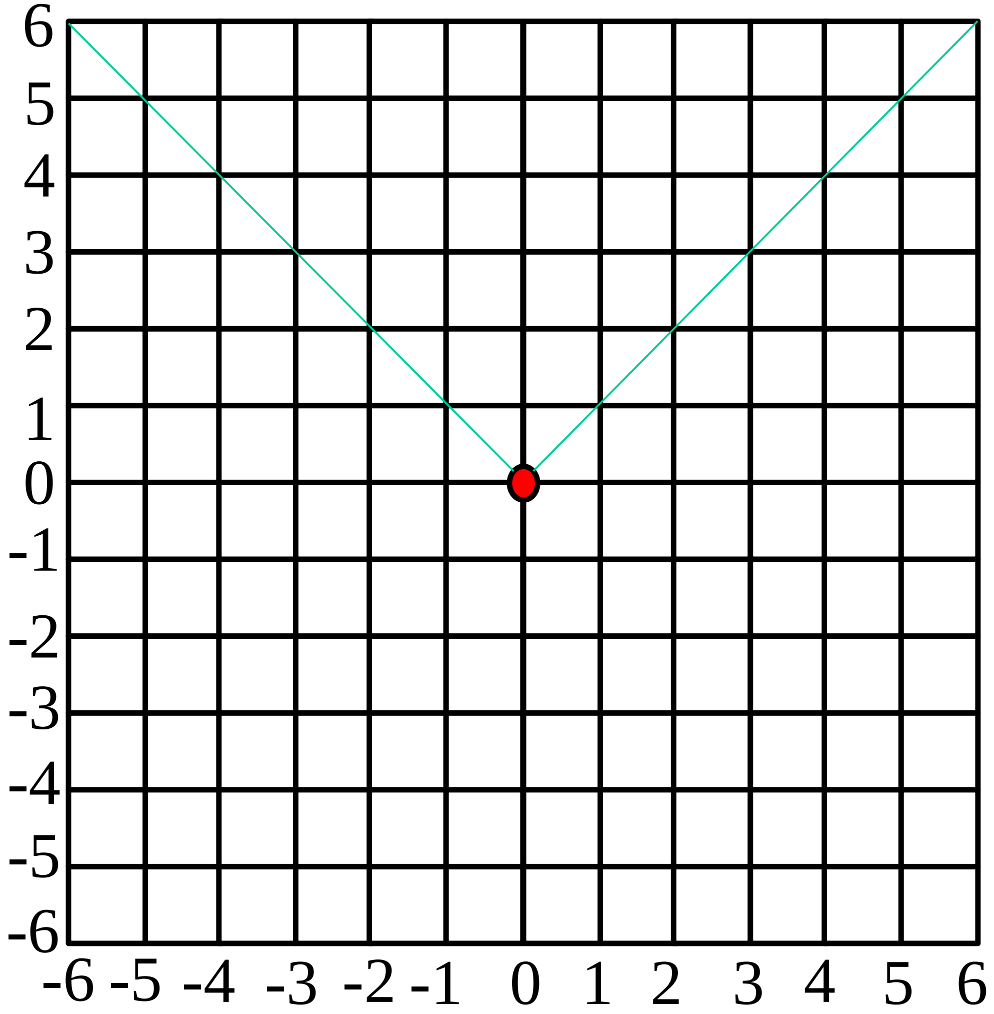


Example

$f(x) = |x|$

f'(0) does not exist

But convex at x = 0



Example

$f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3$

$f'(x) = 0 \Rightarrow x = +1 \text{ or } -1$

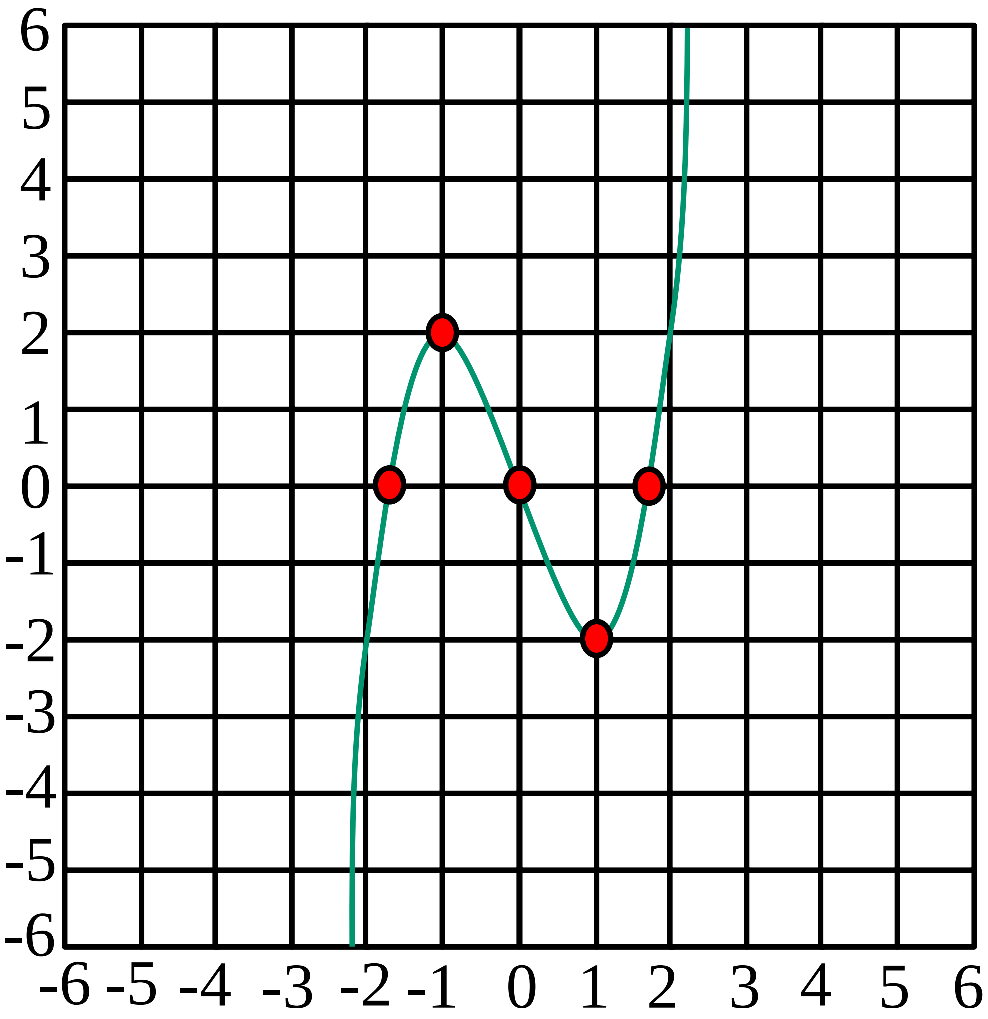
$f''(x) = 6x$

$f''(+1) = 6 > 0 \Rightarrow$ f is a minimum at $x = +1$

$f''(-1) = -6 < 0 \Rightarrow$ f is a maximum at $x = -1$

Convex at x = 1

Concave at x = -1



Example

$f(x) = \sqrt{x^2+1}$:

Recall

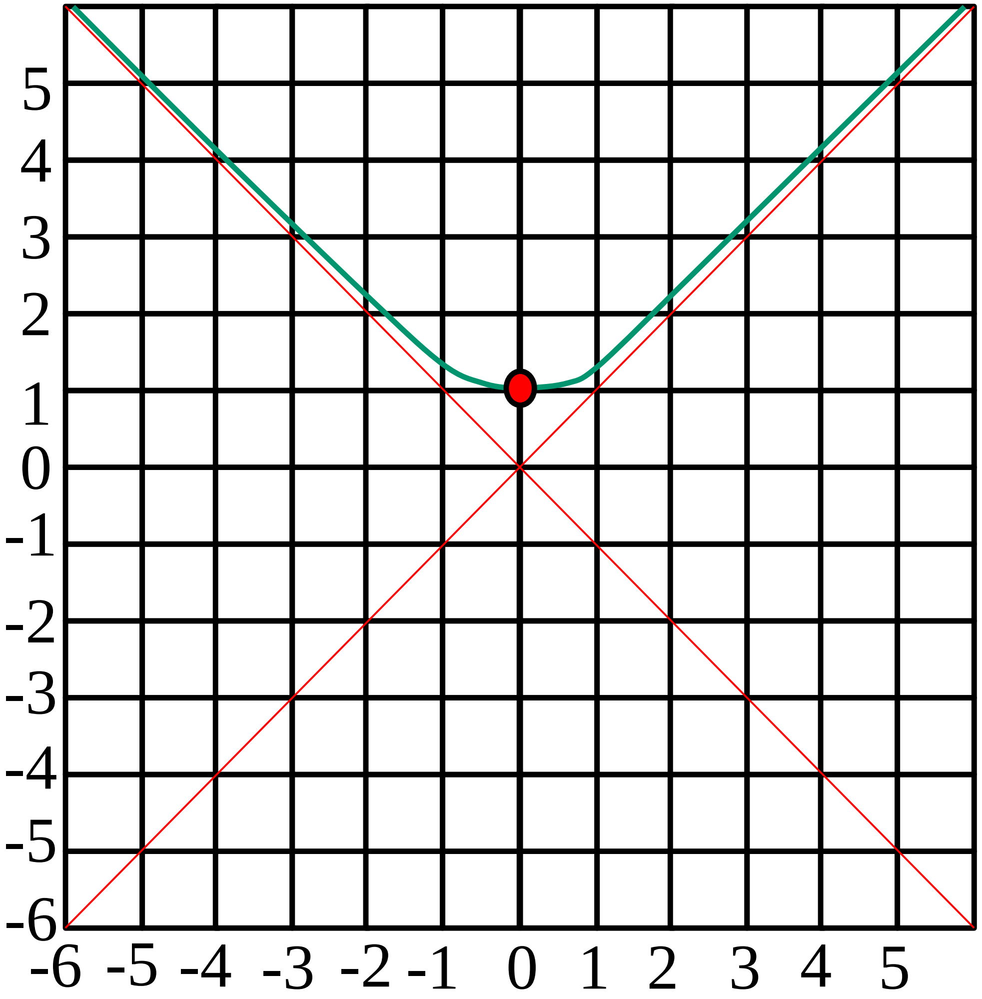
$f'(x) = \frac{x}{\sqrt{x^2+1}}$ is negative if x < 0

$f'(x) = \frac{x}{\sqrt{x^2+1}}$ positive if x > 0

$f'(x)$ = 0 at x = 0

$f'(x) = \frac{1}{(x^2+1)^{1.5}} > 0$ at x = 0

Convex at x = 1

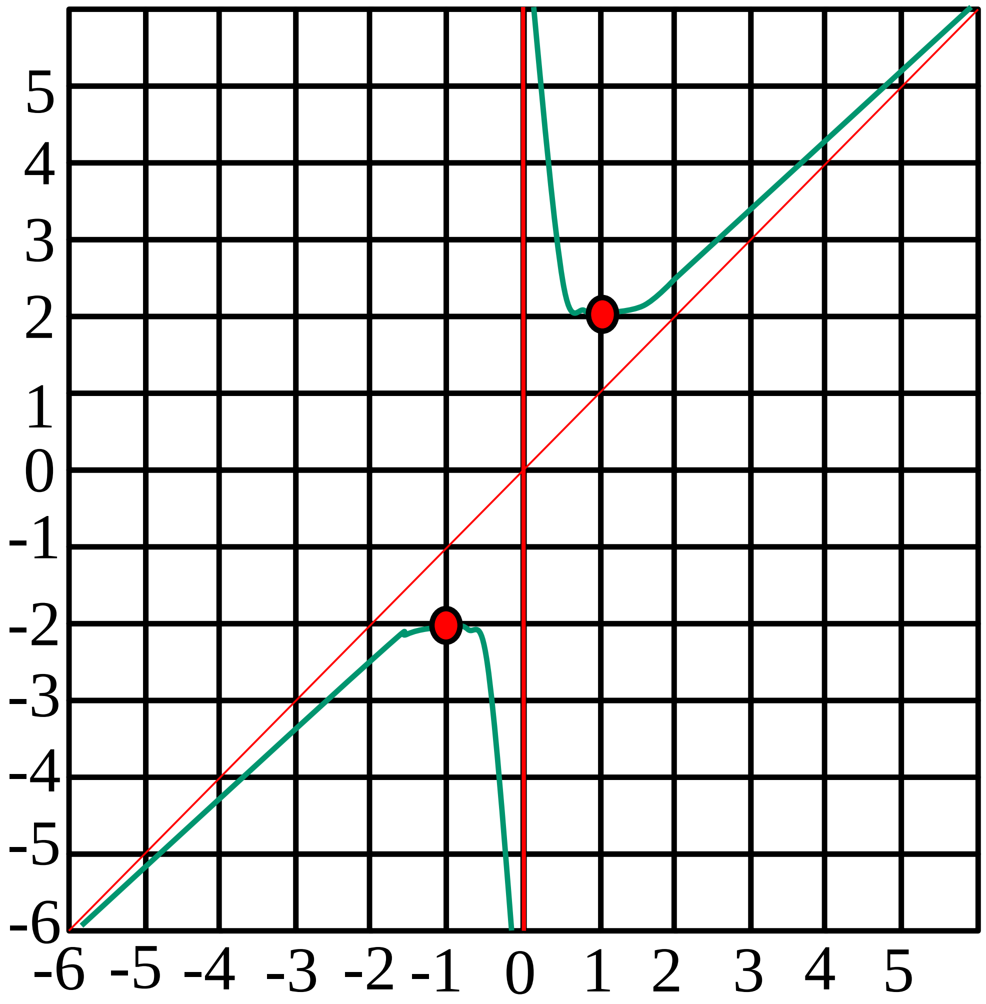


Example

$f(x) = x + \frac{1}{x}$:

- The critical points are $(-1, -2)$ and $(1{,}2)$
- $f'(x) = 1 - \frac{1}{x^2}$ = 0 at x = +1 and -1

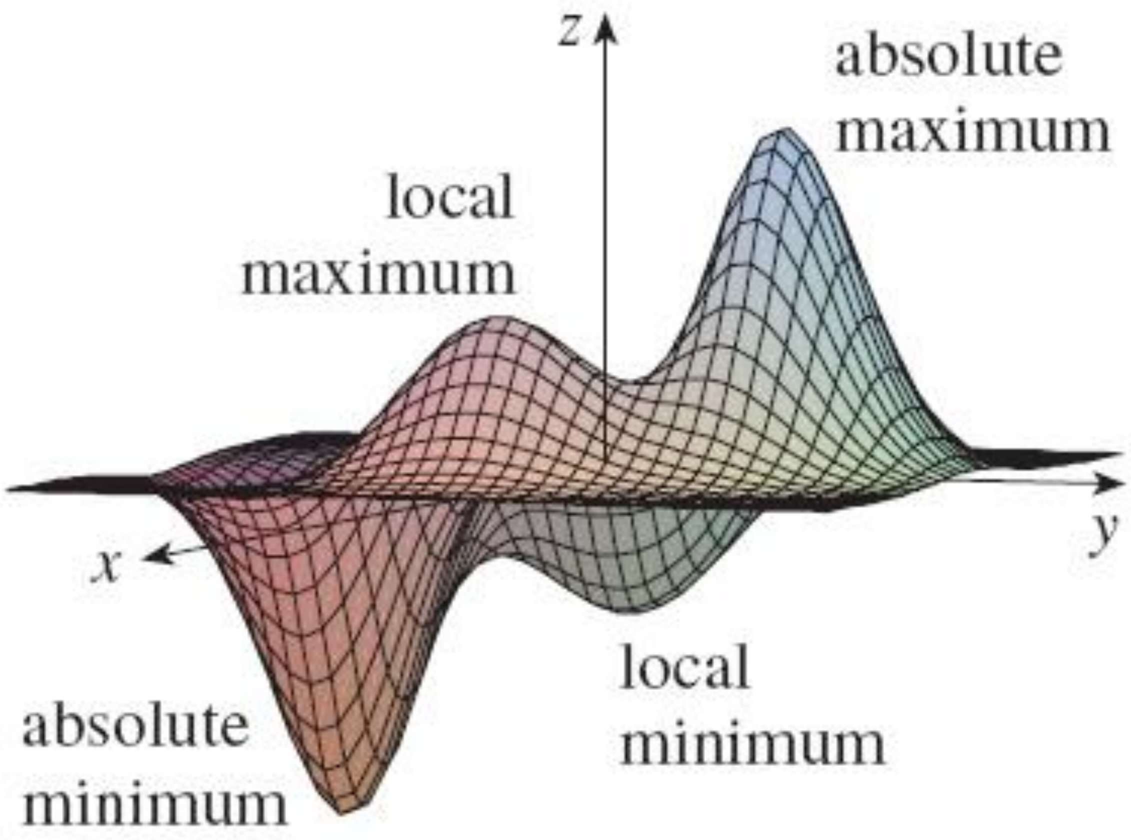
- $f''(x) = 2\frac{1}{x^3}$
- f''(1) > 0 & f''(-1) < 0
- Convex at x = 1
- Concave at x = -1



Maxima & Minima

$z = f(x, y)$

Example


z
absolute
maximum
local
maximum
x
y
absolute
minimum
local
minimum

Local Maxima & Minima

A function of two variables has a local maximum at (a, b)

- if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b).
- This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).
- The number $f(a, b)$ is called a local maximum value.

If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a local minimum at (a, b).

- $f(a, b)$ is a local minimum value.

If the inequalities hold for all points (x, y) in the domain of f, then

- f has an absolute maximum (or absolute minimum) at (a, b).

Critical Point

A point (a, b) is called a critical point (or stationary point) of f if

- $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Example

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$

Then, $f_x(x, y) = 2x - 2$ & $f_y(x, y) = 2y - 6$

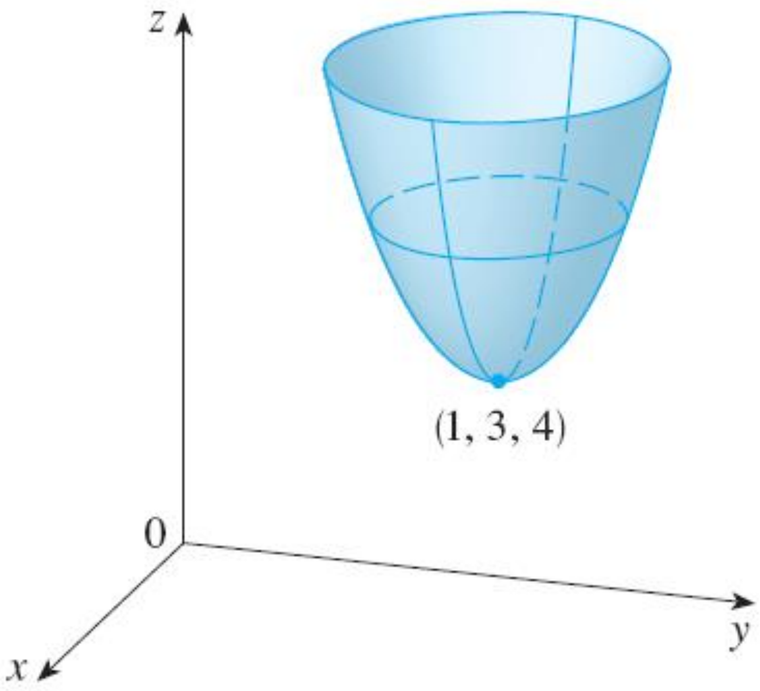
- These partial derivatives are equal to 0 when $x = 1$ and $y = 3$.
- So, the only critical point is (1, 3)

Now $f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$

- Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y.
  - So, $f(1, 3) = 4$ is a local minimum.
  - In fact, it is the absolute minimum of f.

LOCAL MINIMUM

This can be confirmed geometrically from
the graph of *f*, which is the elliptic paraboloid with vertex (1, 3, 4).



Example

Find the extreme values of $f(x, y) = y^2 - x^2$

- Since $f_x = -2x$ and $f_y = 2y$, the only critical point is (0, 0)

Notice

- For points on the x-axis, we have $y = 0 \rightarrow$ So, $f(x, y) = -x^2 < 0$ (if $x \neq 0$).
- For points on the y-axis, we have $x = 0 \rightarrow$ So, $f(x, y) = y^2 > 0$ (if $y \neq 0$).

Thus, every disk with center (0, 0) contains points where f takes positive values as well as points where f takes negative values.

- So, $f(0, 0) = 0$ can't be an extreme value for f.
- Hence, f has no extreme value.

This example illustrates the fact that a function need not have a maximum or minimum value at a critical point.

MAXIMUM & MINIMUM VALUES

$z = y^2 - x^2$.

The figure shows how this is possible.

- The graph of f is the hyperbolic paraboloid

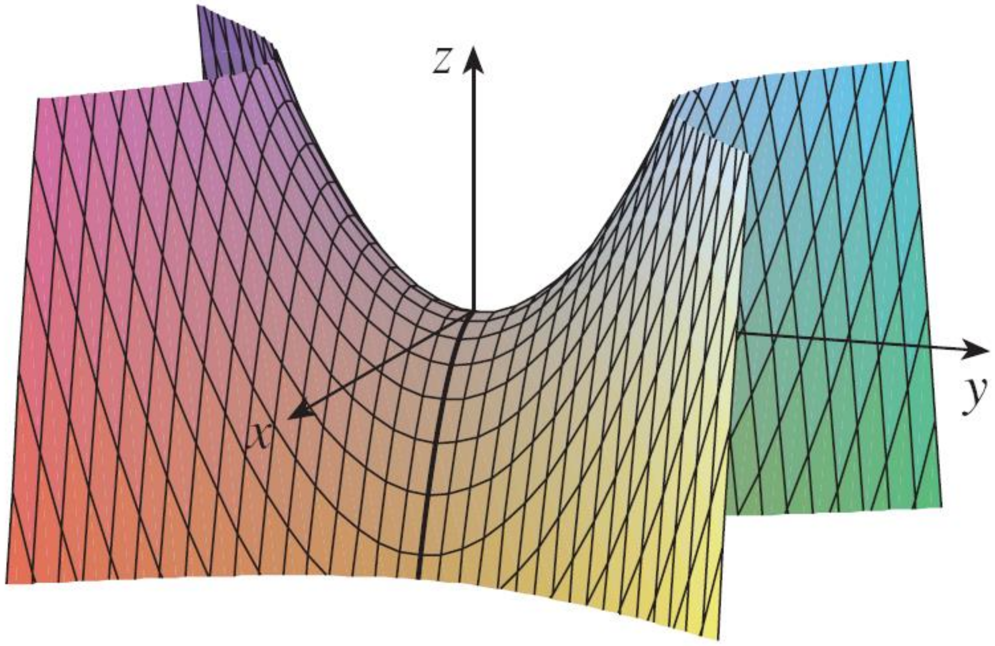
It has a horizontal tangent plane ($z = 0$) at the origin

You can see that $f(0, 0) = 0$ is:

- A maximum in the direction of the x-axis.
- A minimum in the direction of the y-axis

Near the origin, the graph is saddle-shape

So, (0, 0) is called a saddle point of f.



Second Derivative Test

Suppose that:

- $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (that is, (a, b) is a critical point of f)

The second partial derivatives of f are continuous on a disk with center (a, b).

Let $D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$

a) If $D > 0$ and $f_{xx}(a, b) > 0$, f(a, b) is a local minimum.
b) If $D > 0$ and $f_{xx}(a, b) < 0$, f(a, b) is a local maximum.
c) If $D < 0$, f(a, b) is not a local maximum or minimum – a saddle point

If $D = 0$, the test gives no information:

- (a, b) could be a local maximum or local minimum or a saddle point of f.

The Hessian & The Second Derivative Test

To remember the formula for *D*, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - \left(f_{xy}\right)^2$$

a) If $D > 0$ and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum.
b) If $D > 0$ and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum.
c) If $D < 0$, $f(a, b)$ is not a local maximum or minimum – a saddle point

D is the determinant of the Hessian

Example

Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

We first locate the critical points: $f_x = 4x^3 - 4y = 0$ & $f_y = 4y^3 - 4x = 0$

This implies $0 = x^9 - x = x\,(x^8 - 1) = x\,(x^4 - 1)\,(x^4 + 1) = x\,(x^2 - 1)\,(x^2 + 1)\,(x^4 + 1)$

So, there are three real roots: $x = 0, 1, -1$

- The three critical points are: $(0, 0)$, $(1, 1)$, $(-1, -1)$

Now $f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$ & $D(x, y) = f_{xx}\,f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$

Example (Contd)

As $D(0, 0) = -16 < 0$, it follows from case c of the Second Derivatives Test that the origin is a saddle point.

- That is, f has no local maximum or minimum at (0, 0).

As $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, $f(1, 1) = -1$ is a local minimum.

Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, $f(-1, -1) = -1$ is also a local minimum.