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SS ZC416 Mathematical Foundations for Data Science
Relations

Objectives

- Relation
  - Operations on relations
  - Matrix representation
  - Digraph representation
- Lattice
  - Poset
  - Hasse Diagram

Motivation

Binary Relations

Binary relations represent relationships between the elements of two sets.

Let A and B be sets.

- A binary relation from A to B is a subset of A X B.

If $(a, b) \in R$, then

- we use the notation a R b
- a is said to be related to b by R.

Power Set of a Set

Relation on the set A is a relation from A to A; The relation is a subset of A x A

Consider A = {a, b}

A x A = {(a, a), (a, b), (b , a), (b, b)}

Subsets:

- ∅, {(a, a)}, {(a, b)}, {(b, a)}, {(b, b)},
- {(a, a), (a, b)}, {(a, a), (b, a)}, {(a, a), (b, b)}, {(a, b), (b, a)}, {(a, b), (b, b)}, {(b, a), (b, b)},
- {(a, a), (a, b), (b, a)}, {(a, a), (a, b), (b, b)}, {(a, a), (b, b), (b, b)}, {(b, a), (a, b), (b, b)},
- {(a, a), (a, b), (b, a), (b, b)}

# of subsets: 1 + 4 + 6 + 4 + 1 = 16 = 2^{n^2}, where n = |A|

Domain & Range of a Relation

- If $R \subseteq A \; X \; B$, then R is a relation from A to B

- Domain of R is given by
  dom R = {a | a ∈ A, (a, b) ∈ R for some b ∈ B}
- Domain of $R \subseteq A$

- Range of R is given by
  ran R = {b / b∈ B, (a, b) ∈ R for some a∈ A}
- Range of $R \subseteq B$

Examples

1. List the ordered pairs in the relation R from

- A = {0,1,2,3,4} to B= {0,1,2,3} where (a, b) ∈ R iff

  (i) a divides b

  (ii) a > b

  Also find domain and range of R.

2. Which of the following relations contain the ordered pairs

- (1,1),(1,2),(2,1), (1,-1),(2,2)

$R_1 = \{(a, b) \mid a \leq b\}$

$R_2 = \{(a, b) \mid a + b \leq 3\}$

Inverse of a Relation

Let R be a relation from A to B.

R^{-1}, the inverse of relation R, is a relation from B to A defined by

- R^{-1} = {(b, a) | (a, b) ∈ R}

Let R be a relation from A to B where

- A = {1, 2, 3, 4} and B = {x, y, z} and R = {(1,x),(2,y),(3,z),(4,x)}
- Then R^{-1}= {(x,1), (y,2), (z,3) (x,4)}

Example

$R1 = \{ (a, b) | a <= b \}$
$R2 = \{ (a, b) | a > b \}$
$R3 = \{ (a, b) | a = b \text{ or } a = -b \}$
$R4 = \{ (a, b) | a = b \}$
$R5 = \{ (a, b) | a = b + 1 \}$
$R6 = \{ (a, b) | a + b <= 3 \}$

Consider the following ordered pairs: (1, 1), (1, 2), (2, 1), (1, -1), (2, 2)

- For each ordered pair, identify the relation to which it belongs.

Properties of Relations

Reflexive Property

- A relation R on a set *A* is called **reflexive** if (a, a) ∈ R for ∀ a ∈ *A.*
- A relation R is reflexive if every element is related to itself.

Example:

- Divides, ≤, ≥, ⊆, ⊇
- Note: Sometimes Divides is denoted by ***/*** in these notes

Example

Consider the following relations on {1, 2, 3, 4}

R1 = {(1, 1), {1, 2). (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}

R2 = {(1, 1), {1, 2), (2, 1)}

R3 = {(3, 4)}

R4 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}

R5 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}

Which relations are reflexive?

Symmetric Property

A relation R on a set *A* is called **symmetric**

- if (b, a) ∈ R whenever (a, b) ∈ R
-
- Set of integers and the relation is “Equal to”
- Set of integers and the relation is “Divides”

Example

Consider the following relations on {1, 2, 3, 4}

R1 = {(1, 1), {1, 2). (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}

R2 = {(1, 1), {1, 2), (2, 1)}

R3 = {(3, 4)}

R4 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}

R5 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}

Which relations are symmetric?

Transitive Property

A relation R on a set *A* is called **transitive**

- if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ for every $a, b, c \in A$.

Example

- divides, $<$, $>$, $\subset$, $\supset$.

Example

Consider the set of integers Z

Which of the following are reflexive, symmetric, transitive

R1 = { (a, b) | a<= b}

R2 = { (a, b) | a >b}

R3 = { (a, b) | a = b or a = -b}

R4 = { (a, b) | a = b}

R5 = { (a, b) | a = b+ 1}

R6 = { (a, b) | a + b <= 3 }

Examples

For each of these relations on the set {1,2,3,4} decide whether it is reflexive, symmetric and transitive.

a) {(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)}

b) {(1,1),(1,2),(2,1),(2,2),(3,3)(4,4)}

Determine whether the relation R on the set of integers is reflexive, symmetric, transitive where (**a**, **b**) ∈ R iff **a** is a multiple of **b**

Operations on Relations

Union of Relations

Let R1 and R2 be relations from set A to a set B,

The *union* of R1 and R2 denoted by R1 U R2

- (a, b) ∈ R1 U R2 iff (a, b) ∈ R1 or (a, b) ∈ R2
- R1 U R2 is a relation from A to B

Intersection of Relations

Let R1 and R2 be relations from set A to a set B,

The *intersection* of R1 and R2 denoted by $R1 \cap R2$

- $(a, b) \in R1 \cap R2$ iff $(a, b) \in R1$ and $(a, b) \in R2$

R1 U R2 is a relation from A to B

Example

Consider relations R_1 and R_2 from set A to a set B, where

- A = {1,2,3} and B= {1,2,3,4} and
- R_1 = {(1,2), (2,3), (3,4)}
- R_2 ={(1,1), (1,2), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4)}.

Find

a) R1 ∪ R2

b) R1 ∩ R2

Composition of Relations

Let

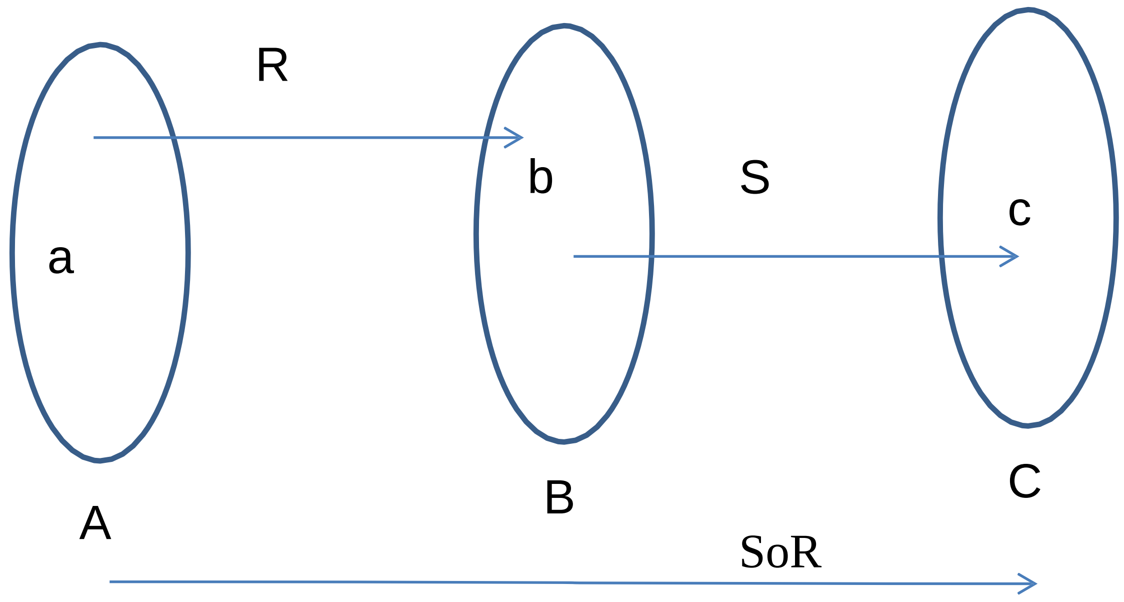
- R be a relation from a set A to a set B and
- S be a relation from B to a set C

The composite of R and S is the relation consisting of

- ordered pairs $(a, c) \in S o R$ iff
- there exists b in B such that $(a, b) \in R$ and $(b, c) \in S$.

We denote composite of R and S by $S o R$

SoR



R^2 = R o R

Let R is a relation on A

- The R o R which is again relation on A.
- This composite relation is denoted by R^2.
- Similarly R^3 = (R o R) o R is a relation on A
- The powers R^n, n=1,2,3…….. are defined recursively by
  - $R^1 = R$ and $R^{n+1} = R^n$ o R

Examples

Let R and S be two relations on A={1,2,3} and

- R={(1,1), (1,2), (2,3), (3,1), (3,3)} and
- S={(1,2), (1,3), (2,1), (3,3)}

Compute

(i) $S \circ R$

(ii) $R \circ S$

(iii) R^2

Let A = {1,2,3,4,5} define R on A by aRb iff a+1=b.

- Compute R, R^2, R^3

Reflexive Closure

The reflexive closure of a relation R on a set A = {a, b, c, …} is the relation

- R U {(a, a)} U {(b, b)} U ….

That is, the reflexive closure is the union of R with the Identity relation

Symmetric Closure

The symmetricclosure of a relation R on a set A = {a, b, c, …} is the relation

- $R \cup R^{-1}$

Transitive Closure

The transitive closure of a relation R is the smallest transitive relation containing R.

- We denote it by R^*.

Let X be any finite set containing n elements and R be a relation in X. Then $R^* = R \cup R^2 \cup R^3 \cup \ldots\ldots\ldots\ldots \cup R^n$

Example

Let X = {1, 2, 3, 4} and R ={(1, 2), (2, 3), (3, 4)} be a relation on X.

Find R^*.

Representing Relations using Matrices

Representing Relations using Matrices

A relation between finite sets can be represented using a zero-one matrix $M_R = m_{ij}$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, a_j) \in R \\ 0 \text{if } (a_i, a_j) \notin R \end{cases}$$

Example

Suppose $A = \{1, 2, 3\}$ and $B = \{1, 2\}$ & $R = \{ (a, b) \mid a > b \}$

What is he matrix representation?

Observations

A matrix of a relation on a set which is a square matrix can be used to determine whether the relation has certain properties.

1. R is reflexive if all the elements on the main diagonal of M_R are equal to 1.
2. R is symmetric iff $M_R = {M_R}^T$

Example

1. Represent each of these relations on {1,2,3,4} with a matrix.

   a) {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

   b) {(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)}

2. Determine whether the relations are reflexive and / pr symmetric by your observation from the matrix representation given below

$$1.\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad 2.\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

A Digression

Boolean Variables & Boolean Operations

x is a Boolean variable if it takes the values 0 or 1, that is, False or True

Suppose x and y are Boolean variables. Let $x = 1$, $y = 0$

$x \vee y$ is read as x or y. Therefore $x \vee y = 1$

$x \wedge y$ is read as x and y. Therefore $x \wedge y = 0$

Examples

M is a Boolean matrix if the entries are either 0 or 1

$$\text{Let } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \ \& \ B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \ \& \ C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A \text{ Join } B = A \vee B = \begin{bmatrix} 1\vee 0 & 0\vee 1 \\ 0\vee 0 & 1\vee 1 \\ 1\vee 1 & 0\vee 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \ \& \ A \text{ Meet } B = A\wedge B = \begin{bmatrix} 1\wedge 0 & 0\wedge 1 \\ 0\wedge 0 & 1\wedge 1 \\ 1\wedge 1 & 0\wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{A Boolean Product} = A \odot C = \begin{bmatrix} (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 0) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 0) \\ (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Relations & Boolean Matrices

Operations: Relations & Boolean Matrices

R_1 and R_2 are relations on a set A represented by the matrices M_{R_1} and M_{R_2} respectively,
Then

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$
$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$
$$M_{S \circ R} = M_R \odot M_S$$

Examples

Let R_1 and R_2 are relations on a set A represented by the matrices

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrices representing: $R_1 \cup R_2, R_1 \cap R_2, R_2 \circ R_1, R_1^{-1}$

Warshall's Algorithm

Let M_R be the matrix of the relation R on a set A with n elements. Then the matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_{R^2} \vee \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \vee M_{R^n}$$

Example - Warshall's Algorithm

Let X = {1, 2, 3, 4} and R ={(1, 2), (2, 3), (3, 4)} be a relation on X. Find R^*.

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Representing Relations using Digraphs

Representing Relations using Digraphs

- The relation R on a set A is represented by the directed graph that has the elements of A as its vertices and the ordered pairs (a, b) $\in R$ as edges.

- An ordered pair of the form (*a, a*) is represented using an arc from the vertex *a* to itself. Such an edge is called a loop.

Observations

- A relation is reflexive iff there is a loop at every vertex of the directed graph.
- A relation is symmetric iff for every edge between distinct vertices in its digraph there is an edge in the opposite direction.
- A relation is anti symmetric iff there are never two edges in opposite direction between distinct vertices.
- A relation is transitive iff whenever there is an edge from vertex *a* to vertex *b* and an edge between vertex *b* to vertex *c*, there is an edge from *a* to *c*.

Exercise

1. Draw a directed graph that represents the relation $\{(a, a),(a, b),(b, c),(c, b),(c, d),(d, a), (d, b)\}$

2. Draw a directed graph that represents the relation

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \text{ on } \{1,2,3,4\}.$$

Path in a Directed graph

- There is a path form a to b in relation R if there is a sequence of elements
  $a, x_1, x_2, \ldots \ldots \ldots \ldots x_{n-1}, b$ with $(a, x_1), (x_1, x_2), \ldots \ldots (x_{n-1}, b) \in R$.

- There is a path of length n, from a to b where n is a positive integer if and only if (a, b) $\in R^n$.

- A path that begins and ends at the same vertex is called cycle or circuit in a digraph.

Lattices

Poset

- A relation R on a set S is called **partial ordering** or partial order if it is reflexive, anti-symmetric and transitive.
- A set S together with a partial ordering R is called a **partially ordered set** or **poset** and is denoted by (S, R).
- Usually, the poset is denoted by $(S, \preccurlyeq)$

Example:

Let Z be set of integers then (Z, ≤) is a partial order set or poset

A relation R is anti-symmetric if $(\mathrm{a}, \mathrm{b}) \in \mathrm{R} \Rightarrow (\mathrm{b}, \mathrm{a}) \notin \mathrm{R}$ whenevr $\mathrm{a} \neq \mathrm{b}$

Examples

- Show that the “greater than or equal” relation is a partial ordering on the set of integers
- Show that the inclusion relation is poset on the power set of a set S

Hasse Diagrams

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- A graphical representation of a poset where loops and all edges resulting from the transitive property are not shown.
- All the edges are pointed upward and hence we do not have to show the directions of edges.
- An edge is drawn between *x* and *y* only if *y covers x.*

(*y covers x* means there is no z such that $x < z < y$)

Examples

Draw Hasse diagram for divisibility on the set

a) {1, 2, 3, 4, 5, 6}, b) {2, 3, 4, 9, 12 18}, c) {1, 2, 3,6}

Draw Hasse diagram for inclusion on the set P(S) where

(i) S ={a, b} (ii)S ={a, b, c}

Upper & Lower Bounds

Let $(S, \preccurlyeq)$ be a poset with A ⊆S

- u is called an upper bound of set A if $a \preccurlyeq u \ \forall a \in A$
- l is called a lower bound of set A if $l \preccurlyeq a \ \forall a \in A$
- Upper bounds and Lower bounds are not unique

Example

Consider the poset (Z, ≤) and A= {1, 2, 3, 4}

Upper bounds of A are 4,5,6,7…….

Lower bounds of A are 1,0,-1,-2,……..

Greatest Upper & Lower Bounds

Suppose A is a subset of a poset

- The element x is called *least upper bound* (LUB) of A if x is an upper bound that is less than every other upper bond of A.
- The element y is called *greatest lower bound* (GLB) if y is a lower bound that is greater than every other lower bond of A.
- LUB and GLB if exists are unique.

Example

Consider the poset (Z, ≤) and A= {1, 2, 3, 4}

Upper bounds of A are 4,5,6,7…….; LUB of A is 4

Lower bounds of A are 1,0,-1,-2,……..; GLB of A is 1

Lattice

A partially ordered set in which every pair of elements has both a *least upper bound* and a *greater lower bound* is called a *lattice*.

Example

The poset (N, /) is a lattice.

Least upper bound of two elements is ***l.c.m*** and

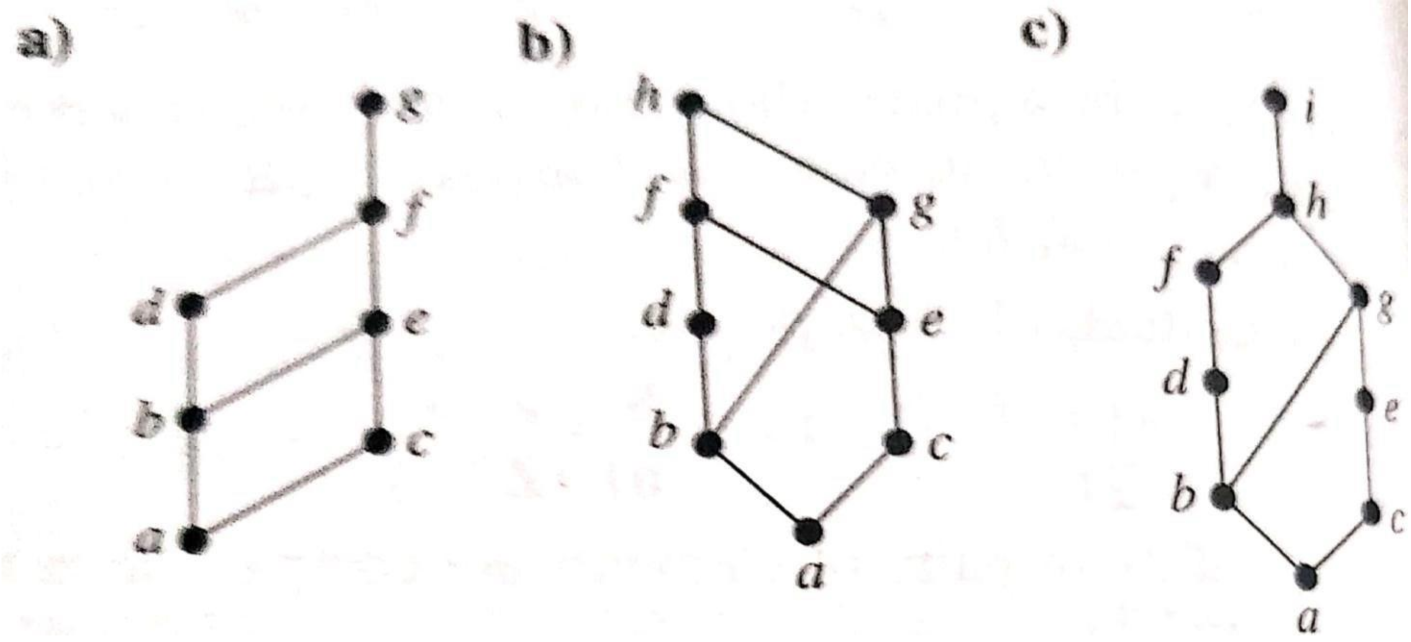
Greatest lower bound of two elements is ***g.c.d***.

Examples

Determine whether these posets are lattices.

a) { {1,3,6,9,12}, /}   b) { {1,5,25,125}, /}

Determine whether the posets with these Hasse diagrams are lattices.



Lexicographic Order

Let $A_1, \preccurlyeq_1$ and $A_2, \preccurlyeq_2$ be two posets.

The lexicographic ordering $\preccurlyeq$ on $A_1 \times A_2$ is defined as if

If $(a_1, a_2)$and $(b_1, b_2) \in A_1 \times A_2$ then $(a_1, a_2) \preccurlyeq (b_1, b_2)$ either if

If $a_1 \preccurlyeq_1 b_1$ **or** if $a_1 = b_1$ and $a_2 \preccurlyeq_2 b_2$

Other Items

Complementary of a Relation

Let R be a relation from A to B.

The complementary relation R from A to B, denoted by R^c, is defined by

$$R^c = \{(a, b) \mid (a, b) \notin R\}$$

Let R be a relation from A to B

- A = {1, 2, 3} and B = {x, y, z} and R = {(1,x),(2,y),(3,z)}

Compute R^c?

Other Operations on Relations

- Given the relations R_1 and R_2 from set A to a set B, then

- $R_1 - R_2 = *(a, b)$ / (a, b) $\in R_1$, (a, b) $\notin R_2$+
- $R_2 - R_1 = *(a, b)$ / (a, b) $\in R_2$, (a, b) $\notin R_1$+
- $R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)$

Examples – Operations on Relations

Consider relations R_1 and R_2 from set A to a set B, where

- A = {1,2,3} and B= {1,2,3,4} and
- R_1 = {(1,2), (2,3), (3,4)}
- R_2 ={(1,1), (1,2), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4)}.

Find

a) $R_1 \cup R2$

b) $R_1 - R2$

c) $R_1 \oplus R$

d) $R_1 \cap R2$

e) $R_2 - R1$

N-ary Relations

Let A1, A2, …., An be sets. An n-ary relation on these sets is a subset of A1 X A2 X …. X An. The sets A1, A2 …., An are called the domains of the relation, and n is called its degree

Example:
Let R be the relation on N X N X N consisting of triples (a, b, c) where a, b and c are integers with a < b < c

Example:
Let R be the relation on Z X Z X Z consisting of all triples of integers (a, b, c) where a + b = c

Relational data model

A database consists of records, which are n-tuples, made up of fields

Set Partition

- A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

or

- The collection of subsets A_i of the set S forms a partition of S iff

$$A_i \neq \emptyset \text{ for i} \in I$$

$$A_i \cap A_j = \emptyset, i \neq j$$

$$A_i = S$$

Examples – Set Partitions

Which of these collections of subsets are partitions of {1,2,3,4,5,6}?

a) {2,4,6}, {1,3,5}

b) {1,2}, {2,3,4}, {4,5,6}

Which collections of subsets are partitions of {a, b, c, d, e, f, g}?

(i) {a, b} {c, d} {e, f, g} (ii) {a, c, e, g} {b, d} {f}

Equivalence Relations

A relation R on a set A is called an equivalence relation if it is

- reflexive, symmetric and transitive.

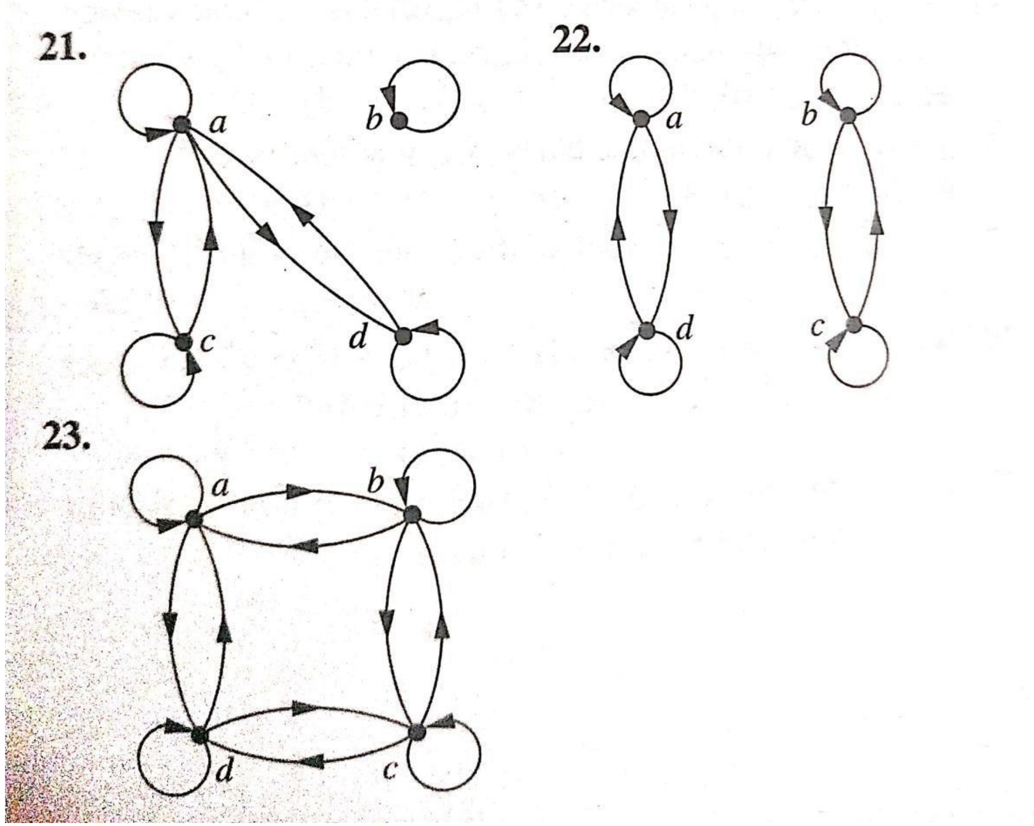
Examples

1. let m be a positive integer with m>1. Then $R=\{(a, b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation.

2. Let R be the relation on the set of real numbers such that aRb iff a-b is an integer. R is an equivalence relation.

Digraphs & Equivalence Relations

Determine whether the relation with the directed graphs shown is an equivalence relation.



Example

Suppose that R is the relation on set of strings of English letters such that A R b if and only if l(a) = l(b), where l(x) is the length of the string x.
Is R is an equivalence relation?

Show that the ‘divides’ relation on the set of positive integers is not an equivalence relation innovate achieve lead

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Thank you!!