IS-ZC444: ARTIFICIAL INTELLIGENCE

Lecture-10: ALPHA-BETA Pruning, Constraint Satisfaction

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Recap: Adversarial Search (game)

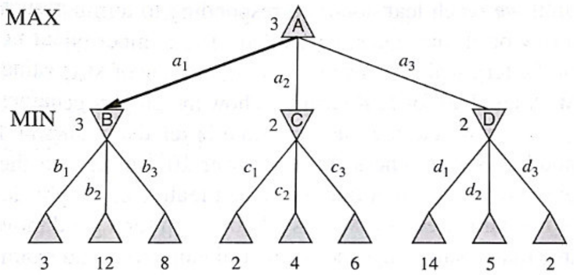
Agents having conflicting goals in competitive multiagent environment

- Deterministic, fully-observable, turn-taking, two-player, zero-sum
- Chess has roughly branching factor 35, moves 50 so tree search space is $35^{100} = 10^{154}$ however, graph has 10^{40} nodes
- Finding optimal move is infeasible but, needs an ability to decide

Game is between MAX and MIN (MAX moves first)

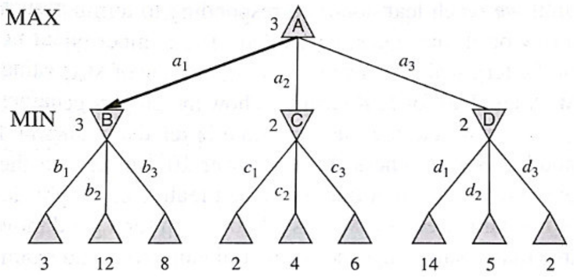
- S_0: the initial state
- PLAYER(s): defines which player has move to start
- ACTIONS(s): returns set of legal moves in a state
- RESULT(s, a):termination model defining result of a move
- TERMINAL_TEST(s): is true when game is over
- UTILITY(s, p): utility function defining reward (for chess +1,0,1/2)

Recap: Two half moves is one **ply**



[1] utility value for MAX of being in corresponding state (assuming then onwards both player play optimally)

Recap: Two half moves is one **ply**



Given the game tree, optimal strategy can be determined from **minimax value** of each node.

$$MINIMAX(s) = \begin{cases} UTILITY(s) & \text{if } TERMINAL\_TEST(s) \\ argmax_{a \in Actions(s)} MINIMAX(RESULT(s, a)) & \text{if } PLAYER(s) = MAX \\ argmin_{a \in Actions(s)} MINIMAX(RESULT(s, a)) & \text{if } PLAYER(s) = MIN \end{cases}$$

Action a_1 is the optimal choice [1]

(essentially optimizing worst-case outcome for MAX)

[1] utility value for MAX of being in corresponding state (assuming then onwards both player play optimally)

MINIMAX Algorithm

Returns the action corresponding to best move

function MINIMAX-DECISION(*state*) **returns** *an action*
  return arg max$_{a \in \text{ACTIONS}(s)}$ MIN-VALUE(RESULT(*state*, *a*))

---

function MAX-VALUE(*state*) **returns** *a utility value*
  if TERMINAL-TEST(*state*) **then return** UTILITY(*state*)
  $v \leftarrow -\infty$
  for each *a* **in** ACTIONS(*state*) **do**
    $v \leftarrow$ MAX(v, MIN-VALUE(RESULT(s, a)))
  return v

---

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Recursion proceeds all the way down to the leaves. Time complexity $O(b^m)$ that is impractical but provides a basis of solution.

ALPHA-BETA Pruning

- Number of nodes to examine in minimax search is exponential in the depth of tree $O(b^m)$.
- Sometime we can make it $O(b^{m/2})$ using **alpha-beta pruning**

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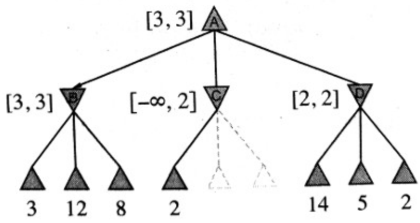

Consider two unevaluated successor of node C have value x and y

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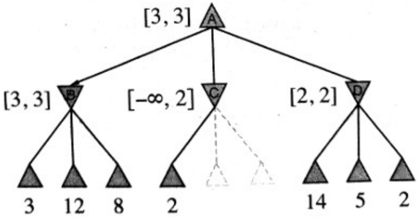

Consider two unevaluated successor of node C have value x and y

MINIMAX(root)
= max(min(3,12,8), min(2,x,y), min(14,5,2))

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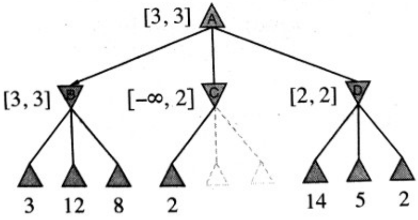

Consider two unevaluated successor of node C have value x and y

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= max(3, min(2,x,y), 2)

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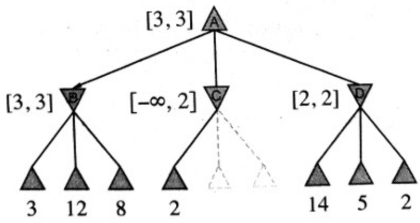

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= max(3, min(2,x,y), 2)
= max(3, z, 2) where z=min(2,x,y) $\leq$ 2

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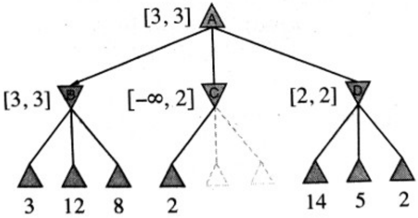

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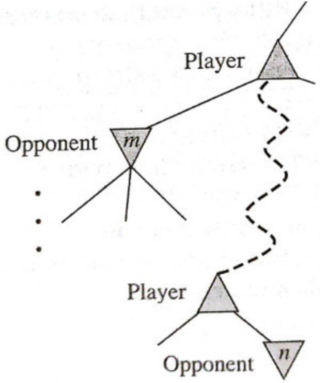
= max(3, z, 2) where z=min(2,x,y)$\leq$ 2
= 3

ALPHA-BETA Pruning

- Alpha-beta pruning can be applied to trees of any depth, and it is often possible to prune entire subtree rather than just leaves.



If *m* is better than *n* for player then we would never go to *n* in play

α	$=$	value of best choice (highest) found so far for MAX
β	$=$	value of best choice (lowest) found so far for MIN

ALPHA-BETA Pruning

```
function ALPHA-BETA-SEARCH(state) returns an action
   v ← MAX-VALUE(state, −∞, +∞)
   return the action in ACTIONS(state) with value v
```

```
function MAX-VALUE(state, α, β) returns a utility value
   if TERMINAL-TEST(state) then return UTILITY(state)
   v ← −∞
   for each a in ACTIONS(state) do
      v ← MAX(v, MIN-VALUE(RESULT(s,a), α, β))
      if v ≥ β then return v
      α ← MAX(α, v)
   return v
```

```
function MIN-VALUE(state, α, β) returns a utility value
   if TERMINAL-TEST(state) then return UTILITY(state)
   v ← +∞
   for each a in ACTIONS(state) do
      v ← MIN(v, MAX-VALUE(RESULT(s,a) , α, β))
      if v ≤ α then return v
      β ← MIN(β, v)
   return v
```

ALPHA-BETA Pruning

function ALPHA-BETA-SEARCH(*state*) **returns** an action
  $v \leftarrow$ MAX-VALUE(*state*, $-\infty, +\infty$)
  return the *action* in ACTIONS(*state*) with value v

---

function MAX-VALUE(*state*, α, β) **returns** *a utility value*
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  $v \leftarrow -\infty$
  for each a **in** ACTIONS(*state*) **do**
    $v \leftarrow$ MAX(v, MIN-VALUE(RESULT(s,a), α, β))
    if $v \geq \beta$ **then return** v
    $\alpha \leftarrow$ MAX(α, v)
  return v

---

function MIN-VALUE(*state*, α, β) **returns** *a utility value*
  if TERMINAL-TEST(*state*) **then return** UTILITY(*state*)
  $v \leftarrow +\infty$
  for each a **in** ACTIONS(*state*) **do**
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    if $v \leq \alpha$ **then return** v
    $\beta \leftarrow$ MIN(β, v)
  return v

Order matters. So, examine likely to be best successor first.

ALPHA-BETA Pruning

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Is it possible?

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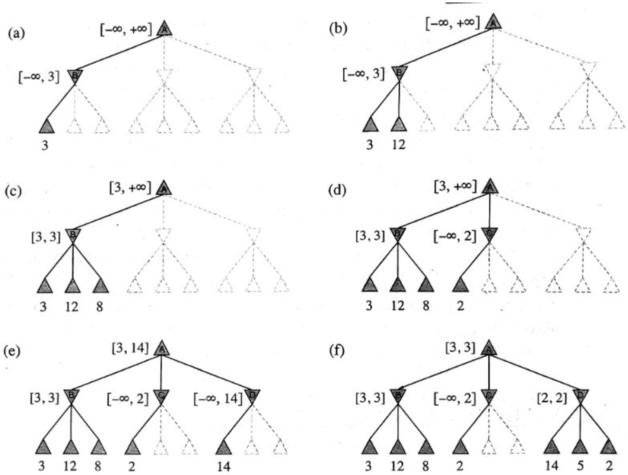
Order matters. So, examine likely to be best successor first.

Is it possible?
No

In-action: ALPHA-BETA Pruning



Move Ordering

- With perfect ordering we need to examine only $O(b^{m/2})$ nodes
- When successors are examined in random order it needs to examine $O(b^{3m/4})$ nodes
- In chess, a strategy that capture $\rightarrow$ threat $\rightarrow$ forward $\rightarrow$ backward, gets you to within about a factor of 2 of the best case $O(b^{m/2})$
- Try first the move that were found useful in past (killer move)
- Iterative deepening could help (adds constant fraction time)
- Hash table of previously seen positions (**transposition table**) can restrict re-computation of states

Imperfect Real-time Decision

- Generating entire search space is overkill
- Covering till large depth is not possible
- Idea is to cutoff the search earlier

$$H\text{-}MINIMAX(s,d) = \begin{cases} Eval(s) & \text{if } CUTOFF\_TEST(s,d) \\ argmax_{a \in Actions(s)} H\text{-}MINIMAX(RESULT(s,a), d+1) & \text{if } PLAYER(s) = MAX \\ argmin_{a \in Actions(s)} H\text{-}MINIMAX(RESULT(s,a), d+1) & \text{if } PLAYER(s) = MIN \end{cases}$$

Evaluation Function

- Returns the estimate of expected utility
- Performance would deeply depend on it
- Eval(win) $\geq$ Eval(draw) $\geq$ Eval(lose)
- Computation should be quick
- Value should relate to actual chance of winning
- Can use features (Number of pawns, Number of queens, ...)
- Can have various categories (all pawn vs one pawn, etc) Suppose experience suggests 72% of states encountered in the two-pawn vs. one pawn category lead to win, 20% lose, and 8% in draw. then eval could be

$$0.72 \times 1 + 0.20 \times 0 + 0.08 \times 0.5 = 0.76$$

- **Weighted linear function** are also possible

$$Eval(s) = w_1 \times f_1(s) + w_2 \times f_2(s) + w_3 \times f_3(s) + \ldots$$

Important Considerations

- **Cutting off search**: having fixed depth may not be a good idea. **Quiescent** states are unlikely to have wild swing so expand till it. **Horizon effect** is difficult to eliminate so **singular extension**[2] could be used
- **Forward Pruning**: cutting off the some search without further consideration. (we do not evaluate all possibilities **beam search**, we only use some in our mind) This may be fatal.
- **Search Vs Lookup** We have some policy on how to start and finish well. Use them as lookup

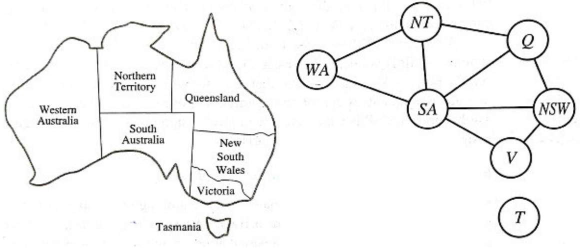
[2] can I take a previous good move

Constraint Satisfaction

- A Constraint Satisfaction Problem (CSP) has three components
  1. X as a set of variables $\{X_1, X_2, X_3, ..., X_n\}$
  2. D as a set of domains $\{D_1, D_2, D_3, ..., D_n\}$
  3. C set of constraints that specify allowable combinations of values
- Each D_i contains allowable set of values $\{v_1, v_2, v_3, ..., v_k\}$ for X_i
- Each C_i contains a pair $<$scope, relation$>$
  such as $<(X_1, X_3), X_1 \neq X_3>$
- To solve CSP it needs state space and a notion of solution
- An assignment of variables such as $(X_1 = v_1, X_2 = v_2, ..., X_n = v_n)$ that does not violates any constraints is called **consistent** or legal solution.
- Our target to have complete assignment that is consistent.

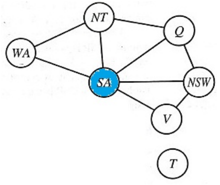
Example: Map Coloring

Given three colors {red,green,blue}, can you color following map such that no two neighboring region have same color.



- X = {WA, NT, Q, NSW, V, SA, T}
- C = {SA≠WA, SA≠NT, SA≠Q, SA≠NSW, SA≠V, WA≠NT, NT≠Q, Q≠NSW, NSW≠V}
- There are many solution to the problem
- It is helpful to visualize as a **constraint graph** (node: variable, edge: participate in constraint)

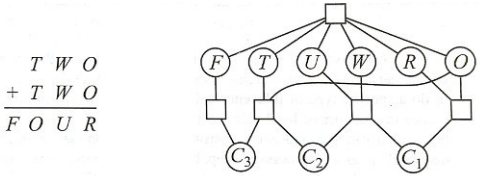
Constraint Propagation



- Once you choose (SA=blue)
- you can conclude that none of its five members could take the color blue
- Without taking advantage of constraint propagation, search needs to consider $3^5 = 243$ assignments for the five neighbors.
- With constraint propagation, it needs $2^5 = 32$ only
- 87% reduction

Example: Cryptarithmethc Puzzle

Consider following addition (we have to find digits)



- Each letter represents a different digit
- We need Alldiff{F,T,U,W,R,O} and
- Additional constraints are

$$\begin{aligned}
O + O &= R + 10 \times C_1 \\
C_1 + W + W &= U + 10 \times C_2 \\
C_2 + T + T &= O + 10 \times C_3 \\
C_3 &= F
\end{aligned}$$

Is it so 234+234=0468?

Preferential Constraints

- Many real world CSPs include **preference constraints**
- Indicating some solution are preferred over other
- Consider university class scheduling problem
  - Apart from absolute constraints such as no professor could simultaneously teach two classes
  - There are some preferential constraints such as Prof. A prefer teaching in morning whereas Prof. B prefer teaching on evening.
  - A solution that schedules Prof. A in evening and Prof. B in morning is still ok
  - But, we do not prefer it
- Such problems are sometimes called **constraint optimization problem** (COP)

Inference in CSP

- In CSP, an algorithm can either
  1. Search or
  2. Do inference: constraint propagation (that reduces number of legal values for another variable)

Enforcing **local consistency** in each part of the graph can cause inconsistency elimination throughout the graph.

- **Node consistency:** if all the values in variable's domain satisfy the variable's unary constraints [3]. It is always possible to eliminate all unary constraints by applying *Node consistency*
- **Arc consistency:** X_i is arc consistent with X_j if for every value in current domain D_i there is some value in D_j satisfying binary constraint on (X_i, X_j). Note [4]

[3] If SA do not like green color then use {red,blue} instead of {red,green,blue}

[4] Generalization is possible: using more than two variables in a constraint

AC-3 Algorithm

```
function AC-3(csp) returns false if an inconsistency is found and true otherwise
  inputs: csp, a binary CSP with components (X, D, C)
  local variables: queue, a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if REVISE(csp, X_i, X_j) then
      if size of D_i = 0 then return false
      for each X_k in X_i.NEIGHBORS - {X_j} do
        add (X_k, X_i) to queue
  return true
```

```
function REVISE(csp, X_i, X_j) returns true iff we revise the domain of X_i
  revised ← false
  for each x in D_i do
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j then
      delete x from D_i
      revised ← true
  return revised
```

Takes $O(cd^3)$ time in worst case

Inference in CSP: Path consistency

Arc consistency can help if some domain becomes empty, or size of every domain reduces to 1

- **Path consistency:** two variable set $\{X_i, X_j\}$ is path consistent wrt X_m if for every $\{X_i = a, X_j = b\}$ consistent with constraints on $\{X_i, X_j\}$ there is an assignment to X_m that satisfies constraints on $\{X_i, X_m\}$ and $\{X_m, X_j\}$
- **K-consistency:** for any set of $k - 1$ variables and for any constraint assignment to those variables, a consistent value can always be assigned to any k^{th} variable.
  A CSP is strongly k-consistent is it is k-consistent, k-1-consistent, k-2-consistent, ..., 1-consistent [5]
- **Global Constraints:** like *alldiff*. If all m variable involved in alldiff have only n possible values where $m > n$ then there is no solution.

[5] Finding such considition is hard

Example: Sudoku

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

(a)

Example: Sudoku

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G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

(a)

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

(b)

Example: Sudoku

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

(a)

	1	2	3	4	5	6	7	8	9
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B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

(b)

$$alldiff\{A1, A2, A3, A4, A5, A6, A7, A8, A9\}$$
$$alldiff\{B1, B2, B3, B4, B5, B6, B7, B8, B9\}$$
$$\vdots$$
$$alldiff\{A1, A2, A3, B1, B2, B3, C1, C2, C3\}$$
$$\vdots$$

Backtracking Search for CSP

- When inference only do not work, use search
- CSP with n variable and d domain size can have branching factor nd at top level. Then $(n-1)d$ in next level and so on.
- Tree with $!n.d^n$ leaves get generated (however valid assignments are only d^n)
- It is why we have ignored **commutativity** [6]
- So consider single variable at a node.
- Now we need to **backtrack**, if no legal value is left for assignment

[6] No effect of assignment order

Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
  return BACKTRACK({ }, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
     if value is consistent with assignment then
        add {var = value} to assignment
        inferences ← INFERENCE(csp, var, value)
        if inferences ≠ failure then
           add inferences to assignment
           result ← BACKTRACK(assignment, csp)
           if result ≠ failure then
              return result
     remove {var = value} and inferences from assignment
  return failure
```

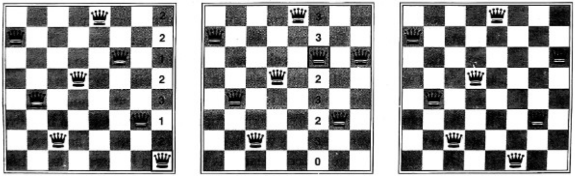
Backtracking Search

- Which variable to choose next? **minimum remaining value** [7] or "fail first"
- **Degree heuristic**, choose one which is involved in many constraints
- Which value to choose? **least constrained value**
- **Forward checking**: interleaving search and inference would help. Whenever a variable X is assigned, forward checking establishes arc consistency for it.
- **Intelligent Backtracking:** Some time it is needed to backtrack upward more than a single step to resolve the inconsistency. Conflicting set is used to find most suitable node.

[7] chose whose domain have fewer entries

Local Search for CSP

Complete state space formulation can also be used for search



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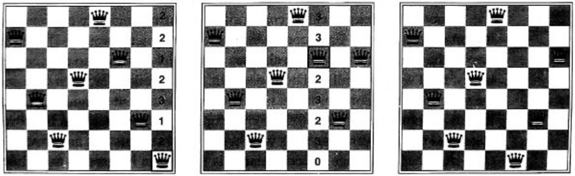


```
function MIN-CONFLICTS(csp, max_steps) returns a solution r failure
   inputs: csp, a constraint satisfaction problem
           max_steps, the number of steps allowed before giving up

   current ← an initial complete assignment for csp
   for i = 1 to max_steps do
      if current is a solution for csp then return current
      var ← a randomly chosen conflicted variable from csp.VARIABLES
      value ← the value v for var that minimizes CONFLICTS(var, v, current, csp)
      set var = value in current
   return failure
```

Local Search for CSP

- **Min-conflict:** can solve Million Queen problem in 50 steps on an average
- **Tabu-search:** keeps a small list of recently visited states and forbidding algorithm to return to these states.
- **Constraint weighting:** starting from weight 1 for each variable; in each step, algorithm choses variable/value such that sum of weights is minimized. Weights of violated variables are incremented.
- Big advantage that the local search can be implemented in online environment.[8]

[8] Airline wants to repair its schedule with minimum changes.

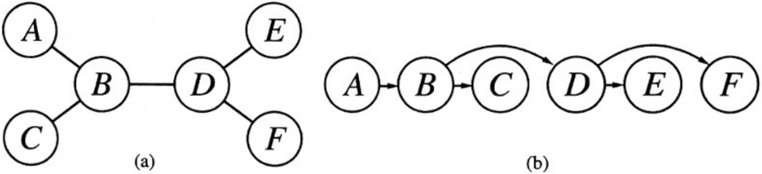
Exploit Problem Structure

Sometime structure of the problem could help to find solution

- **Independent Subproblems:** see that Tasmania is not connected to mainland in Australia map.

  $$\text{Compare } O(d^c n/c) \text{ with } O(d^n) \text{ it is linear}$$

  where each sub problem has c variables
- **Tree structured CSP** is solvable in linear time $O(nd^2)$ with topological sort

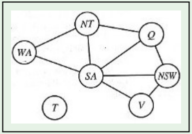


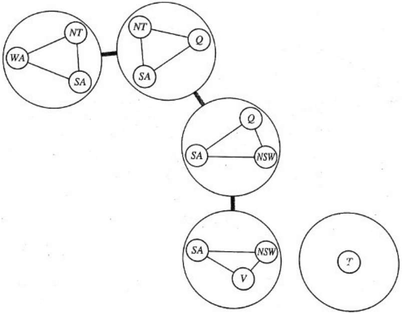
  - **Cut set conditioning:** Assign few, to get tree from remaining vars $O(d^c(n-c)d^2)$
  - Another approach is **tree decomposition**

Tree Decomposition for CSP

Divide the problem in sub-problems

- Every variable in original problem appears at least one of the subproblems
- If two variables are connected by a constraint in original problem, then they must appear together in at least one of the subproblem
- If a variable appears in two subproblems in the tree, it must appear in every subproblem along the path connecting those subproblems





Thank You!

Thank you very much for your attention!

Queries ?

(Reference[9])

[9] 1) Book - *AIMA*, ch-06, Russell and Norvig.