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SS ZC416 Mathematical Foundations for Data Science
Linear Algebra

Recap

Recap

- Matrix Notation – A_{mxn}
- Vectors – Column / Row vectors
- Special matrices
- Scalar multiplication
- Matrix operations
  - Multiplication
- Verification using Excel

Linear System of Equations
Gauss Elimination

Linear System of Equations

n unknowns $x_1, \dots, x_n$ & **m equations**

(1)

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m .
\end{aligned}
$$

- Linear: Each variable x_j appears in the first power only
- Coefficients: $a_{11}, \dots, a_{mn}$ are given numbers,
- $b_1, \dots, b_m$:
  - **Homogeneous**: All the bj are zero
  - **Non-homogeneous**: If at least one bj is not zero

Solution

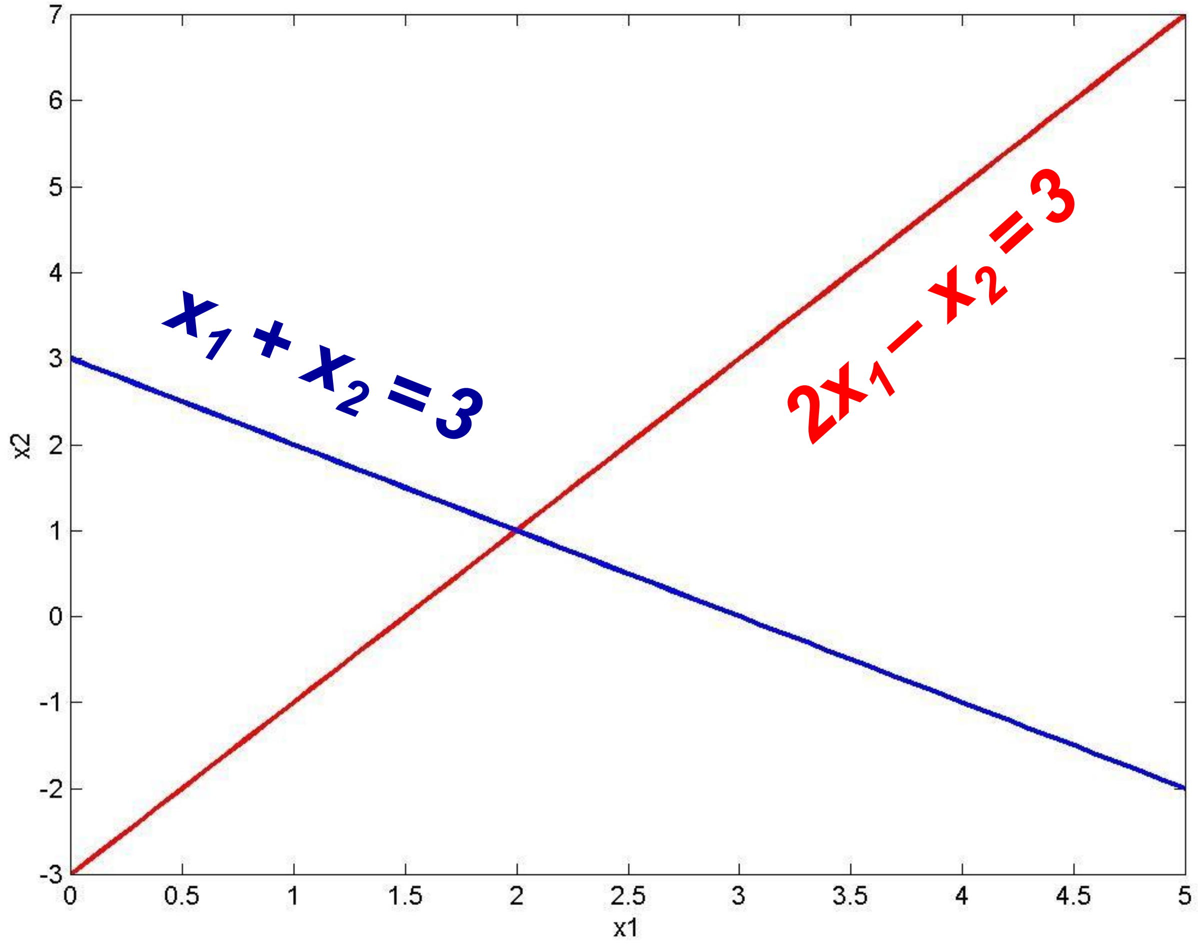
A **solution** is a set of numbers $x_1, \ldots, x_n$ that satisfies all the m equations.

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

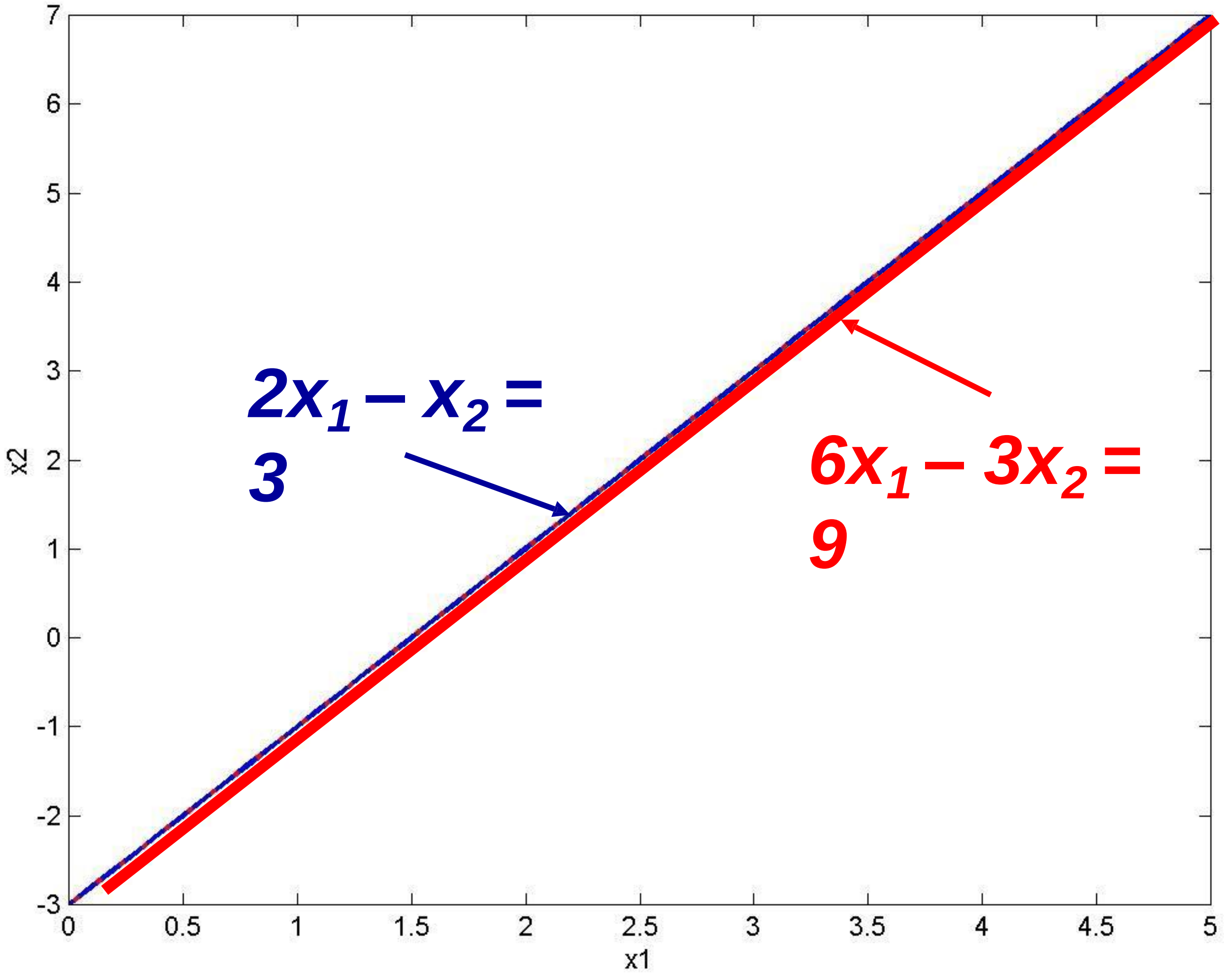
A **solution vector** is a vector **x** whose components form a solution.

A homogeneous system always has at least the **trivial solution** $x_1 = 0, \ldots, x_n = 0$.

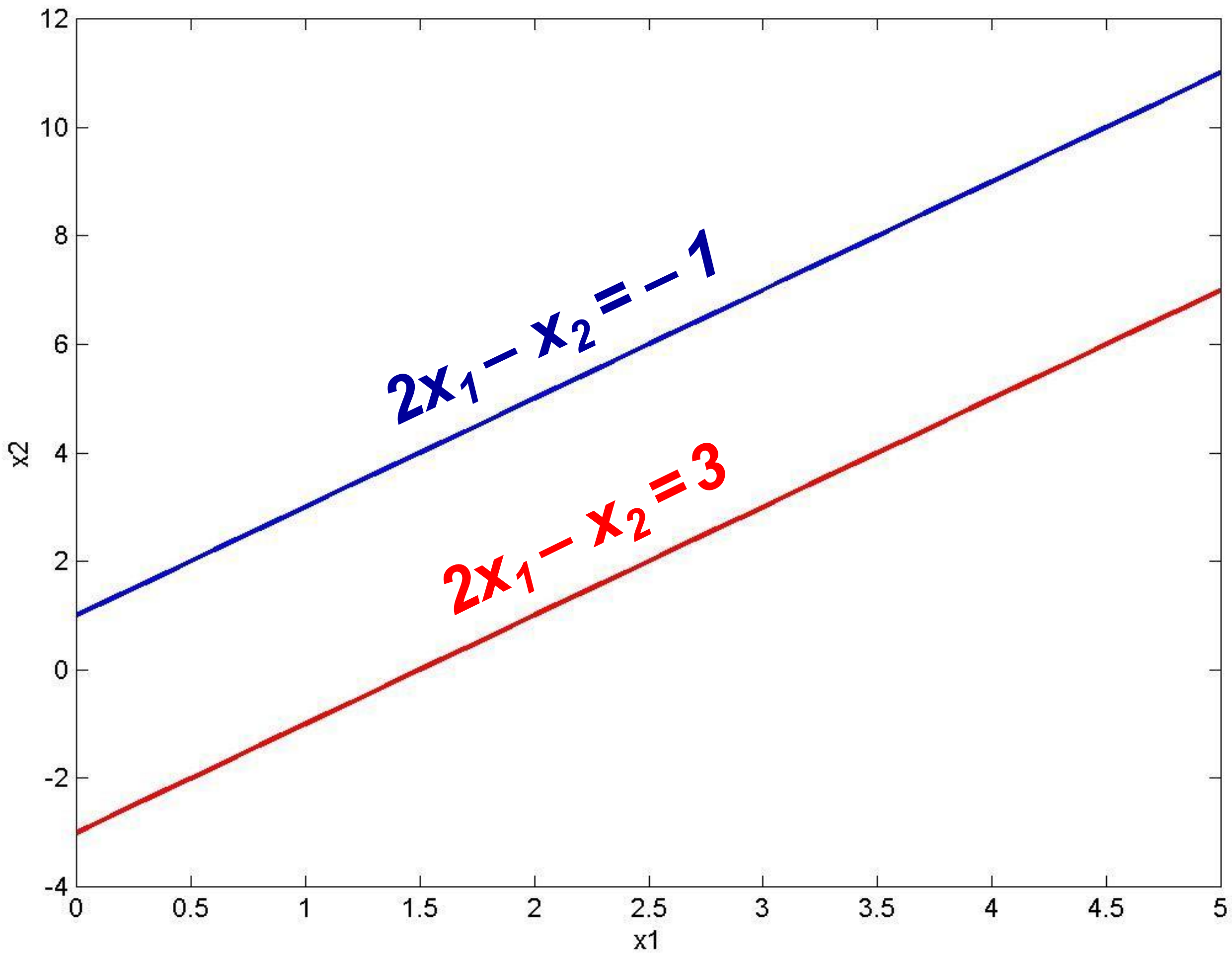
Consistent System: Unique Solution



Consistent System: Infinitely Many Solutions



Inconsistent System: No Solution



Matrix Form of the Linear System

From the definition of matrix multiplication we see that the m equations may be written as a single vector equation

(2) **Ax = b**

where the **coefficient matrix A** = $[a_{jk}]$ is the m × n matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

are column vectors.

The Augmented Matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc:c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Gauss Elimination and Back Substitution

We will use row operations on the augmented matrix and reduce it to a triangular form.

Elementary Row Operations

- Swap the positions of two of the rows
- Multiply one of the rows by a nonzero scalar.
- Add or subtract the scalar multiple of one row to another row.

We then use back substitution to find the solution(s).

Example

Solve the linear system by Gauss Elimination

$$\begin{aligned} x_1 - \quad x_2 + \quad x_3 &= 0 \\ -x_1 + \quad x_2 - \quad x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 \qquad\quad &= 80. \end{aligned}$$

Example – Pivot

Write down the augmented matrix of the system and apply the row operations

Augmented Matrix $\tilde{\mathbf{A}}$

Pivot 1

Eliminate

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

Equations

Pivot 1

Eliminate

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 \quad &= 80. \end{aligned}$$

- Row 2 – Row 1 & Row 4 – 20 Row 1

Example: After the first Pivot

Row 2 – Row 1 & Row 4 – 20 Row 1

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]$$

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 0 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80. \end{aligned}$$

Example

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

Elementary Row Operations

Matrices

- Interchange of two rows
- Addition of a constant multiple of one row to another row
- Multiplication of a row by a **nonzero** constant c

CAUTION!

- These operations are for rows, **not for columns**!
- They correspond to the following **(see next slide):**

Equations

Interchange of two equations
Addition of a constant multiple of one equation to another equation
Multiplication of an equation by a **nonzero** constant c

Row Equivalent Systems

We now call a linear system S_1 **row-equivalent** to a linear system S_2 if S_1 can be obtained from S_2 by (finitely many!) row operations.

Theorem: *Row-equivalent linear systems have the same set of solutions.*

This justifies Gauss elimination method

Consistent Linear Systems

A linear system is called

- **Overdetermined** if it has more equations than unknowns
- **Determined** if $m = n$

Consistent if it has at least one solution

A unique solution or an infinitely many solutions

- **Inconsistent** if it has no solutions at all

Row Echelon Form

At the end of the Gauss elimination the form of the coefficient matrix, the augmented matrix, and the system itself are called the **row echelon form**. In it, rows of zeros, if present, are the last rows, and, in each nonzero row, the leftmost nonzero entry is farther to the right than in the previous row.
The two systems **Ax** = **b** and **Rx** = **f** are equivalent: if either one has a solution, so does the other, and the solutions are identical.

In our example, the coefficient matrix and its augmented in row echelon form are shown below

$$\left[\begin{array}{rrr:r} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - \quad x_2 + \quad x_3 &= \quad 0 \\ 10x_2 + 25x_3 &= \quad 90 \\ -95x_3 &= -190 \\ 0 &= \quad 0. \end{aligned}$$

The Row Echelon Form (REF)

$$\left[\begin{array}{cccccc:c}
r_{11} & r_{12} & \cdots & \cdots & \cdots & r_{1n} & f_1 \\
 & r_{22} & \cdots & \cdots & \cdots & r_{2n} & f_2 \\
 & & \ddots & \cdots & \cdots & \vdots & \vdots \\
 & & & r_{rr} & \cdots & r_{rn} & f_r \\
 & & & & & & f_{r+1} \\
 & & & & & & \vdots \\
 & & & & & & f_m
\end{array}\right]$$

At the end of the Gauss elimination (before the back substitution), the row echelon form of the augmented matrix is as above.

- All entries in the blue triangle and blue rectangle are zero.

The Rank of a Matrix

$$
\left[\begin{array}{cccccc:c}
r_{11} & r_{12} & \cdots & \cdots & \cdots & r_{1n} & f_1 \\
 & r_{22} & \cdots & \cdots & \cdots & r_{2n} & f_2 \\
 & & \ddots & \cdots & \cdots & \vdots & \vdots \\
 & & & r_{rr} & \cdots & r_{rn} & f_r \\
 & & & & & & f_{r+1} \\
 & & & & & & \vdots \\
 & & & & & & f_m
\end{array}\right]
$$

Ax = b

Rank of R = # of nonzero rows, *r*, in the row-reduced coefficient matrix **R**

Rank of A = Rank of R

Note: Rank of A = Rank of R has nothing much to do with the system of equations

Rank – Example 1

$$\left[\begin{array}{cccccc:c} r_{11} & r_{12} & \cdots & \cdots & \cdots & r_{1n} & f_1 \\ & r_{22} & \cdots & \cdots & \cdots & r_{2n} & f_2 \\ & & \ddots & \cdots & \cdots & \vdots & \vdots \\ & & & r_{rr} & \cdots & r_{rn} & f_r \\ & & & & & & f_{r+1} \\ & & & & & & \vdots \\ & & & & & & f_m \end{array}\right]$$

$x_1 + x_2 = 3$ & $2x_1 - x_2 = 3$

$$\left[\begin{array}{cc:c} 1 & 1 & 3 \\ 2 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc:c} 1 & 1 & 3 \\ 0 & -3 & -3 \end{array}\right] \Rightarrow x_2 = 1 \ \& \ x_1 = 2$$

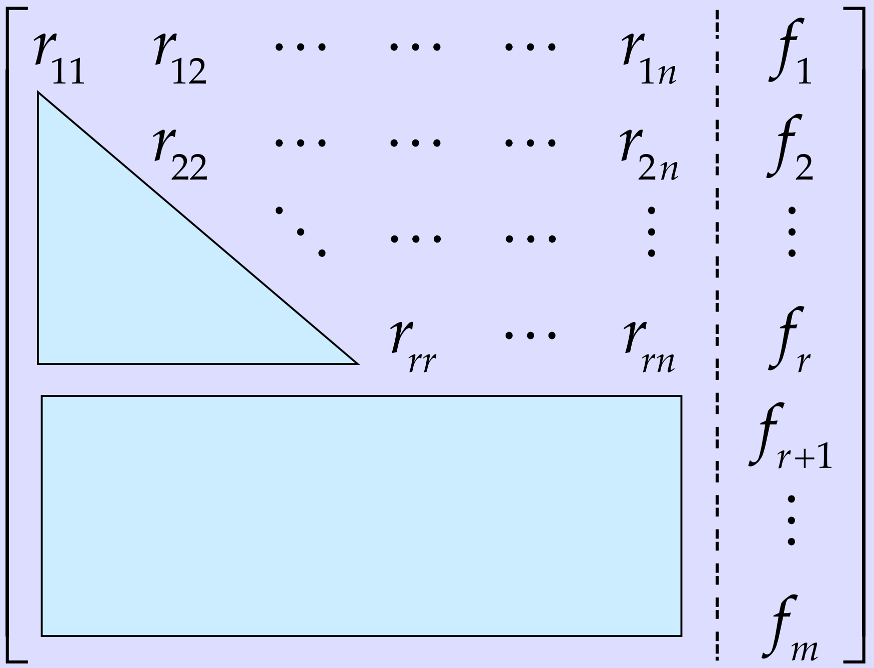
Rank (R) = 2

$\Rightarrow$ Rank (A) = 2

Since Rank (A) = Number of equations

$\Rightarrow$ Unique Solution

Rank – Example 2



$2x_1 - x_2 = 3$ & $6x_1 - 3x_2 = 9$

$$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 6 & -3 & 9 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

Rank (R) = 1

$\Rightarrow$ Rank (A) = 1

Since Rank(A) < # of equations & $f_2 = 0$, Infinite no. of solutions

Rank – Example 3

$$
\left[\begin{array}{cccccc|c}
r_{11} & r_{12} & \cdots & \cdots & \cdots & r_{1n} & f_1 \\
 & r_{22} & \cdots & \cdots & \cdots & r_{2n} & f_2 \\
 & & \ddots & \cdots & \cdots & \vdots & \vdots \\
 & & & r_{rr} & \cdots & r_{rn} & f_r \\
 & & & & & & f_{r+1} \\
 & & & & & & \vdots \\
 & & & & & & f_m
\end{array}\right]
$$

$2x_1 - x_2 = 1$ & $2x_1 – x_2 = 3$

$$
\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 2 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]
$$

Rank (R) = 1

$\Rightarrow$ Rank (A) = 1

Since Rank (A) < Number of equations & $f_2 \neq 0$

Inconsistent system

Consistent & Inconsistent Systems

Consider Ax = B & Rx = f

Note:
- Dim (A) = m x n
- m equations and n unknowns
- Let rank (R) = r

- **No solution.** If r < m & at least one of $f_{r+1}, \ldots, f_m$ is not 0, then **Rx** = **f** is inconsistent & consequently **Ax** = **b** is inconsistent.
- **Unique solution.** r = m
- **Infinitely many solutions.** r < m and all the numbers $f_{r+1}, f_{r+2}, \ldots, f_m$ are zero

Consistent: Either r = m, or r < m and all the numbers $f_{r+1}, f_{r+2}, \ldots, f_m$ are zero

Reduced Row Echelon Form -RREF

$$\begin{bmatrix} 1 & 0 & 0 & f_1 \\ 0 & 1 & 0 & f_2 \\ 0 & 0 & 0 & f_3 \\ 0 & 0 & 0 & f_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & f_1 \\ 0 & 1 & 0 & f_2 \\ 0 & 0 & 1 & f_3 \\ 0 & 0 & 0 & f_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & f_1 \\ 0 & 1 & 0 & f_2 \\ 0 & 0 & 1 & f_3 \end{bmatrix}$$

- The matrix in the above form is called RREF
- No. of non-zero rows in RREF is the rank of the matrix.

Gauss Jordan Elimination Method

- We use Row Echelon Form in Gauss Elimination method.
  - We then use Back Substitution to solve the system.
- We use Reduced Row Echelon Form in Gauss Jordan elimination method.
  - We don’t need back substitute to solve the system.

Example

Solve the linear system by Gauss Elimination / Gauss Jordan elimination method.

$$x + y + z = 6$$

$$3x + 3y + 4z = 20$$

$$2x + y + 3z = 13$$

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Thank you!!