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# BITS Pilani presentation

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# SS ZC416 Mathematical Foundations for Data Science
# Linear Algebra

# Recap

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- Determinant of a Square Matrix
- Inverse of a Square Matrix using
  - RREF
  - Determinants

# Linear Independence of Vectors

# Linear Independence of Vectors

Consider v1 = $\begin{bmatrix}1\\2\end{bmatrix}$, v1 = $\begin{bmatrix}2\\2\end{bmatrix}$, v1 = $\begin{bmatrix}-3\\-2\end{bmatrix}$

Verify that v1 – 2 v2 – v3 = 0

Verify that there is no non-zero scalars α1 and α2 such that α1 * v1 + α2 * v2 = 0

Hint: Note that if such α's exist then we can solve $\begin{bmatrix}1 & 2\\2 & 2\end{bmatrix}\begin{bmatrix}\alpha 1\\ \alpha 2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

# Linear Independence of Vectors

Given any set of m vectors $a_{(1)}, \dots, a_{(m)}$ (with the same number of components)

A linear combination of these vectors: $c_1 a_{(1)}, + c_2\, a_{(2)}, + \dots + c_m a_{(m)}$
where $c_1, c_2, \dots, c_m$ are any scalars.

Now consider the equation $c_1 a_{(1)}, + c_2\, a_{(2)}, + \dots + c_m a_{(m)} = 0$

This vector equation holds if all $c_j$'s zero

If this is the only m-tuple of scalars for which the liner combination equals 0, then

- The vectors are said to form a linearly independent set or,
- The vectors are linearly independent.

Otherwise, these vectors are stb **linearly dependent**.

# Linear Dependence

If the set of vectors are linearly dependent, then

- At least one of the vectors as a linear combination of the other vectors.
- For instance, if $c_1 a_{(1)}, + c_2\, a_{(2)}, + \ldots + c_m a_{(m)} = 0$ with $c_1 \neq 0$, we can solve for $\mathbf{a}_{(1)}$:

$$\mathbf{a}_{(1)} = k_2\, \mathbf{a}_{(2)} + \ldots + k_m \mathbf{a}_{(m)} \text{ where } k_j = -c_j / c_1.$$

(Some $k_j$'s may be zero. Or even all of them, namely, if $\mathbf{a}_{(1)} = \mathbf{0}$).

# Linear Independence and Dependence

Why is linear independence important?

Well, if a set of vectors is linearly dependent, then we can get rid of at least one or perhaps more of the vectors until we get a linearly independent set.
This set is then the smallest “truly essential” set with which we can work.
Thus, we cannot express any of the vectors, of this set, linearly in terms of the others.

The information in the larger dataset is now contained in a smaller dataset

# Example

Consider three vectors

$$a_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} -6 \\ 42 \\ 24 \\ 54 \end{bmatrix}, a_3 = \begin{bmatrix} 21 \\ -21 \\ 0 \\ -15 \end{bmatrix}$$

- Show a1 and a2 are linearly independent vectors
- a1, a2 and a3 are not linearly independent since 6 a1 – 0.5 a2 – a3 = 0

# Rank of a Matrix

The **rank (A)** is the maximum number of linearly independent row vectors of **A**.

# Example

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad \text{(given)}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{array}{l} \text{Row 2} + 2\ \text{Row 1} \\ \text{Row 3} - 7\ \text{Row 1} \end{array}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Row 3} + \frac{1}{2}\ \text{Row 2.}$$

Rank A = 2

# Linear Independence and Matrix Rank

Consider p vectors that each have n components.
Let A be the matrix formed with these vectors as row vectors

- These vectors are linearly independent if rank A = p
- However, these vectors are linearly dependent if that matrix has rank less than p.
- Rank A = equals the Max. number of linearly independent **column** vectors of **A**.
  - Hence **A** and its transpose $\mathbf{A}^T$ have the same rank.
- If $n < p$, then these vectors are linearly dependent.

# Vector Space

# Vector Space - $R^2$ & $R^n$

Consider $R^2$ consisting of ordered pairs (x, y) in the two-dimensional space

1. Take any two vectors / points. Add them. The result is in $R^2$
2. You can sum in any order
3. You can sum scalar multiples of these two vectors. The result is in $R^2$
4. 0 = (0 0) is the zero vector.
5. 1 times a vector is itself. 1 a = a

Convince yourself that this holds for any $R^n$

# Basis of a Vector Space - $R^2$ & $R^n$

Consider $R^2$

Consider $b_1^T = (1\ 0)$ and $b_2^T = (0\ 1)$.

- Every vector in $R^2$ can be written as a linear combination of these two vectors
- The only linear combination of these two vectors equating to 0 is the trivial solution

We say the (1, 0) and (0, 1) are basis for this vector space.

What about v1 = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, v1 = $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$?

The dimension of $R^2$ equals the size of a basis.

What about $R^n$ ?

# Span & Subspace - $R^n$

Consider $R^3$

Consider $b_1^T = (1\ 0\ 0)$ and $b_2^T = (0\ 1\ 0)$

Span: All linear combinations of these two vectors

- For the example, the span is $\mathrm{R}^2$

Subspace: If the span ≠ Vector Space

# Vector Space

Consider a nonempty set V of vectors where each vector has the same number of components. If

For any two vectors a and b in V and any two real numbers α and β

1. $\alpha a + \beta b \in V$
2. $a + b = b + a$, $a + 0 = a$, $a + (-a) = 0$
3. $\alpha(a + b) = \alpha a + \alpha b$, $(\alpha + \beta)a = \alpha a + \beta a$, $(\alpha\beta)a = \alpha(\beta a)$, $1 a = a$

Example

$R^2$ consisting of ordered pairs (x, y) is a two-dimensional vector space

$R^3$ consisting of ordered triples (x, y, z) is a three-dimensional vector space

$R^n$ consisting of all vectors with n components (n real numbers) has dimension n.

# Span & Subspace

The set of all linear combinations of given vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(p)}$ with the same number of components is called the **span** of these vectors.

Obviously, a span is a vector space.

If in addition, the given vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(p)}$ are linearly independent, then they form a basis for that vector space.

By a **subspace** of a vector space $V$ we mean a nonempty subset of $V$ (including $V$ itself) that forms a vector space with respect to the two algebraic operations (addition and scalar multiplication) defined for the vectors of $V$.

# Examples

Consider (1, 0).  What is the span?

Consider (1, 0) & (0, 1).  What is the span?

Consider ((1, 0) & (1, 1).  What is the span?

# Polynomials of Degree < 3

- Consider B = $\{1, x, x^2\}$
- What is the span?
- Is this a vector space?
- Is B a basis for this vector space?

- What about all polynomials of degree <= n?

# Vector Space of Polynomials

- If F[x] is a set of all polynomials.
- Then F[x] is a vector space.
- F[x] is not a finite dimensional vector space.
- Then the infinite set

  $S = \{1, x, x^2, x^3, \ldots\ldots\ldots\ldots x^n, \ldots\ldots\ldots\ldots\}$ is linearly independent.

- S is a basis of the vector space F[x] over the field of R.

# Null Space of a Matrix – Example

Consider $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & -2 & -1 \\ 1 & 8 & 3 \end{bmatrix}$

We see that $\begin{bmatrix} 1 & 3 & 1 \\ 1 & -2 & -1 \\ 1 & 8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 3y + 1z \\ 1x - 2y - 1z \\ 1x + 8y + 3z \end{bmatrix}$ takes a vector in $R^3$ into $R^3$

The output may be the 0 vector or may not be the 0 vector.

All vectors mapped to the 0 vector is called the Null Space of A

- Consider $v^T = (1, -2, 5)$ or $v^T = (-2, 4, -10)$

The Null Space is a vector space

# Example

| | | | |
|---|---|---|---|
| 1 | 3 | 1 | b1 |
| 1 | -2 | -1 | b2 |
| 1 | 8 | 3 | b3 |

| | | | |
|---|---|---|---|
| 1 | 3 | 1 | b1 |
| 0 | -5 | -2 | b2 - b1 |
| 0 | 5 | 2 | b3-b1 |

| | | | |
|---|---|---|---|
| 1 | 3 | 1 | b1 |
| 0 | 5 | 2 | b3-b1 |
| 0 | -5 | -2 | b2 - b1 |

| | | | |
|---|---|---|---|
| 1 | 3 | 1 | b1 |
| 0 | 1 | 0.4 | (0.2)(b3-b1) |
| 0 | -5 | -2 | b2 - b1 |

| | | | |
|---|---|---|---|
| 1 | 0 | -0.2 | b1-(0.6)(b3-b1) |
| 0 | 1 | 0.4 | (0.2)(b3-b1) |
| 0 | 0 | 0 | b2 - b1 +(b3 - b1) |

Consider $\mathrm{v} = \begin{bmatrix} 0.2\mathrm{t} \\ -0.4\mathrm{t} \\ \mathrm{t} \end{bmatrix}$, where t can take any value. Let $v_1$ be the vector when t = 1

A maps $\mathrm{v}_1$ to the 0 vector.  Also all vectors in the Null Space are multiples of $\mathrm{v}_1$

The dimension of the Null Space of A = 1 = Nullity of A

# Examples

Find the Rank & Nullity of the following matrices

Example

$$\begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

# Rank + Nullity = Number of Columns

Finally, for a given matrix **A** the solution set of the homogeneous system **Ax** = 0 is a vector space, called the **null space** of **A**, and its dimension is called the **nullity** of **A**.

rank **A** + nullity **A** = Number of columns of **A**.

The Rank of A is the dimension of the image of A.

# Inner Products

# Inner Product

Consider the vector space $R^n$

Let a and b be two vectors in this space

The inner product <a, b> is the dot product a.b

That is, if a = (a1, a2, …, an) and b = (b1, b2, …, bn)

Then $<a, b> = a_1*b_1 + a_2*b_2+\ldots+a_n*b_n$

# Remarks

Vectors a and b are orthogonal if <a, b> = 0

The length or **norm** of a vector a defined as $||a|| = \sqrt{< a, a >}$

A vector of norm 1 is called a **unit vector**.

# Exercises

a = (1, 2) and b = (2, -3). Find <a, b>, <a, a> and Norm of a

x = (1, 2, 3) and y = (-1, -2, -3). Find <x, y>, <y, y> and Norm of y

# Linear Transformation

# Linear Transformations

Let X and Y be any vector spaces. To each vector **x** in X we assign a unique vector **y** in Y. Then we say that a **mapping** (or **transformation** or **operator**) of X into Y is given.
Such a mapping is denoted by a capital letter, say F. The vector **y** in Y assigned to a vector **x** in X is called the **image** of **x** under F and is denoted by F(x) [or F**x**, without parentheses].

# Linear Mapping

*F* is called a **linear mapping** or **linear transformation** if, for all vectors **v** and **x** in *X* and scalars *c*,

$$F(\mathbf{v}+\mathbf{x}) = F(\mathbf{v}) + F(\mathbf{x})$$

$$F(c\mathbf{x}) = cF(\mathbf{x}).$$

# Linear Transformation of $R^n$ into $R^m$

Let X = $R^n$ and Y = $R^m$. Let A be m × n matrix **A** = $[a_{jk}]$

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Is a transformation of $R^n$ into $R^m$,

Since **A**(**u** + **x**) = **Au** + **Ax** and **A**(c**x**) = c**Ax,** this transformation is linear.

# Nonsingular Matrix

Suppose **A** is square, $n \times n$

Then $Ax$ maps $R^n$ into $R^n$.

If this **A** is nonsingular, so that $\mathbf{A}^{-1}$ exists
Then $y = Ax$ and $x = A^{-1}y$

Note that both transformations are linear

# Rank & Nullity

If $F: X \rightarrow Y$ is a linear transformation given by

$Y = Ax$

Then $\text{Rank}(F) + \text{Nullity}(F) = \dim X$


# Thank you!!