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SS ZC416 Mathematical Foundations for Data Science
Review of Functions

Objectives

Quick Review of Functions

– Prelude to Derivatives

Functions & Relations

Given two sets A and B

- $A \times B = \{(a, b): \forall a \in A, \forall b \in B\}$
- A relation $\subseteq A \times B$
- A function $f \subseteq A \times B$

Example

Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$

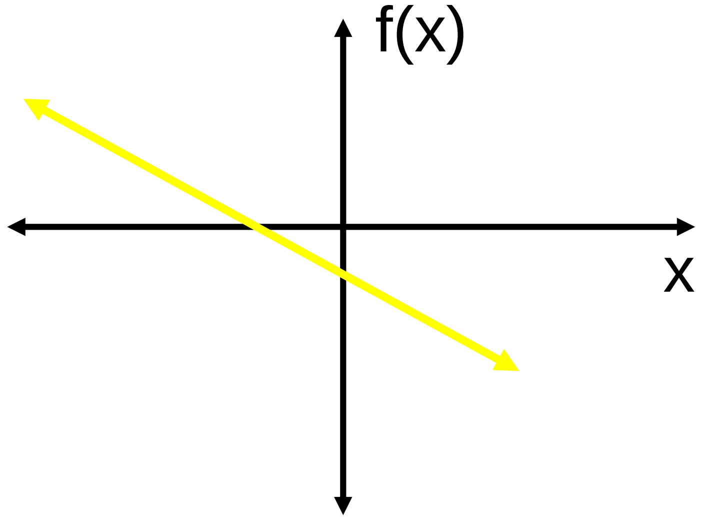
- $A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z)\}$
- Possible Relations:
  $R = \{(1, x), (2, x), (3, x)\}$, $S = \{(1, x), (1, y), (2, z), (3, y)\}$, $T = \{(1, z), (2, y)\}$
- Possible functions:
  R is a function, S and T are not

Functions

How do you describe the yellow function?

$f(x) = -(1/2)x - 1/2$

So what is a function?



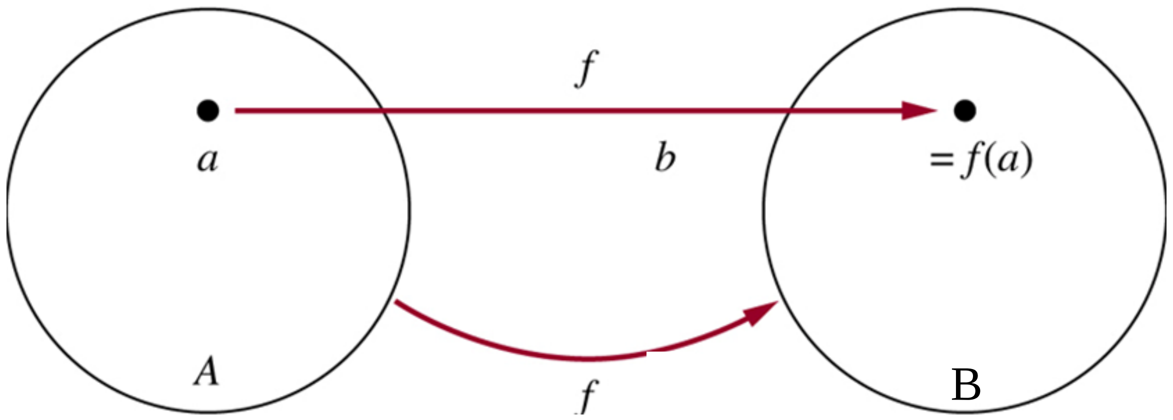
Notations & Terminology

A **function** f from set A to set B

- Is an assignment of **exactly** one element of B to each element of A.

We write

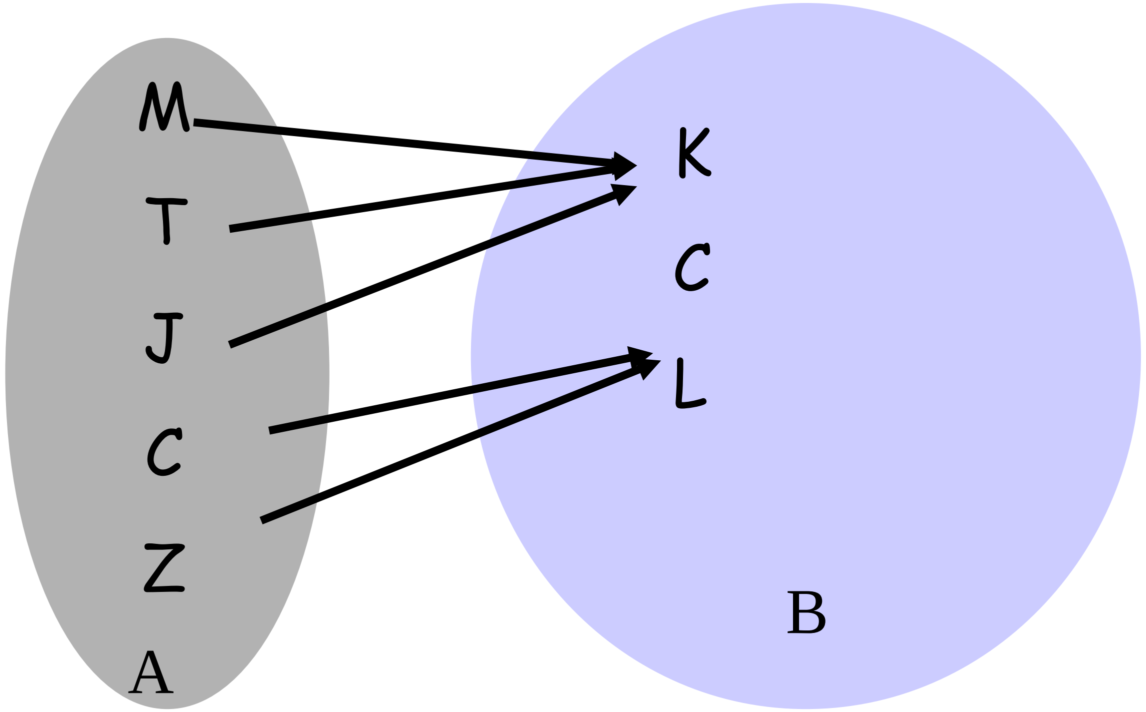
- f(a) = b if b is the element of B assigned by f to the element a of A.
- $f : A \rightarrow B$.
- f is also called a **mapping** or a **transformation**
- f is stb a relation that contains exactly one (a, b) for every a ∈ A



Discrete Functions

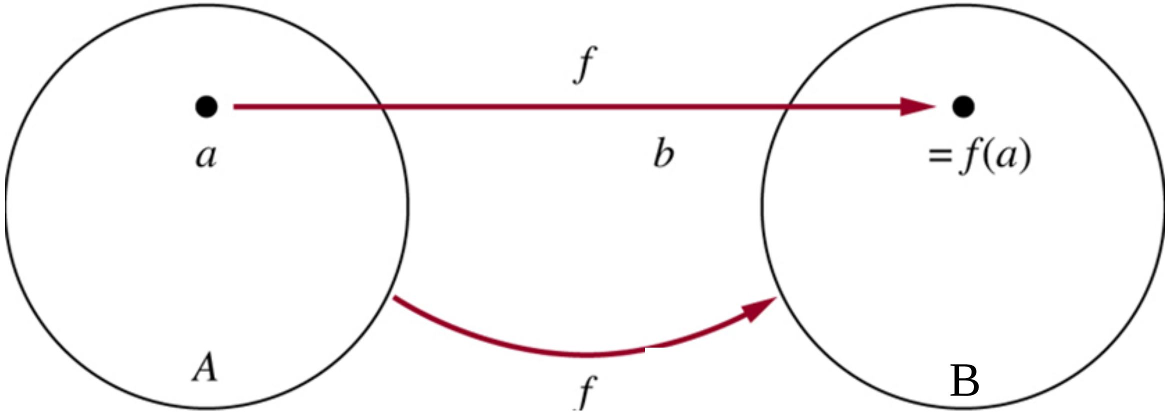
A = {M, T , J , C , Z} & B = { K, C, L}

f: A $\rightarrow$ B where f = {(M. K), (T, K), (J, K), (C, L), (Z, L)}



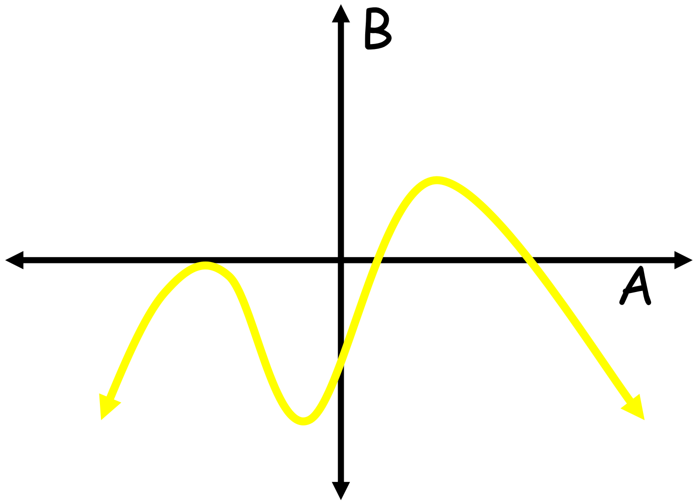
Functions – Domain & Co-Domain

A - Domain of f
B- Co-Domain of f



A More Mathematical Definition

A function $f : A \rightarrow B$ is a subset of $A \times B$ where $\forall\, a \in A, \exists!\, b \in B$ and $<a, b> \in f$.



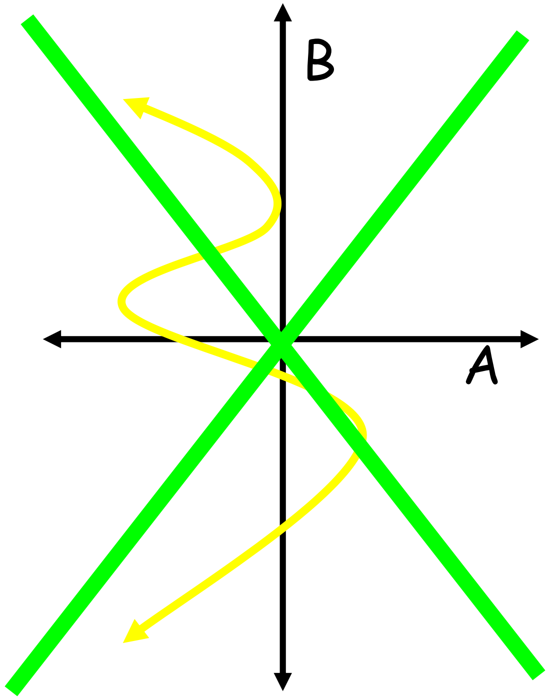
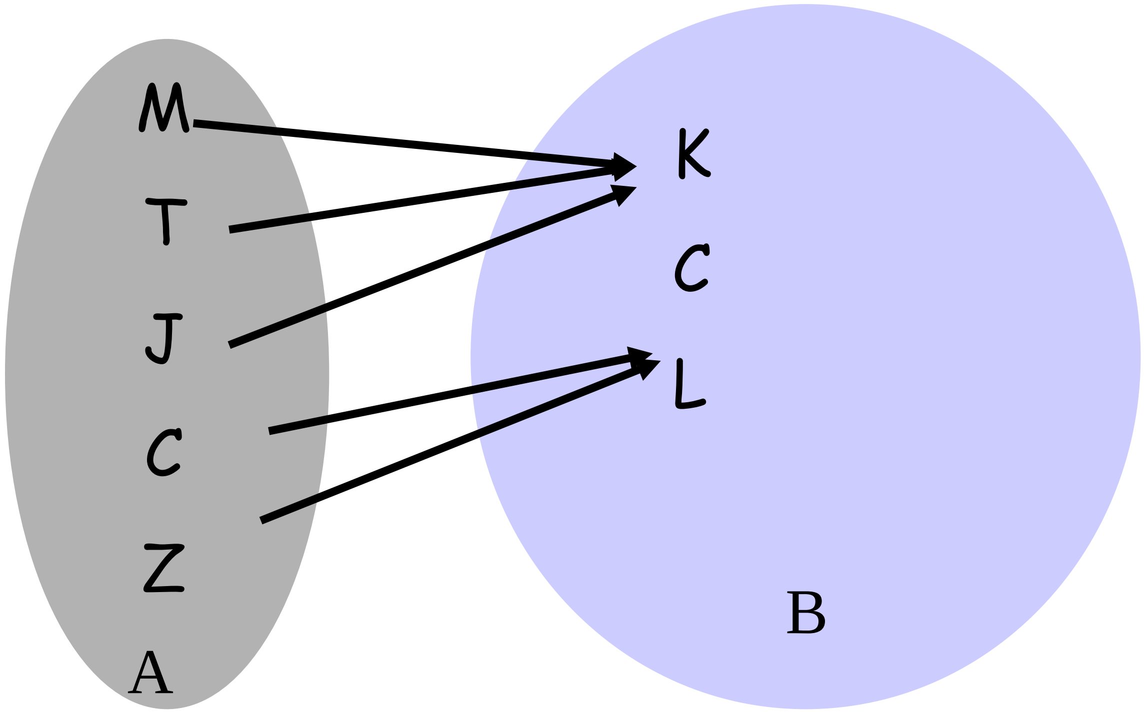


Image & Preimage

For any set $S \subseteq A$, $\text{image}(S) = \{b : \exists a \in S, f(a) = b\}$
Range of f = image(A)

So, image({M, T}) = {K} & image(A) = B - {C}

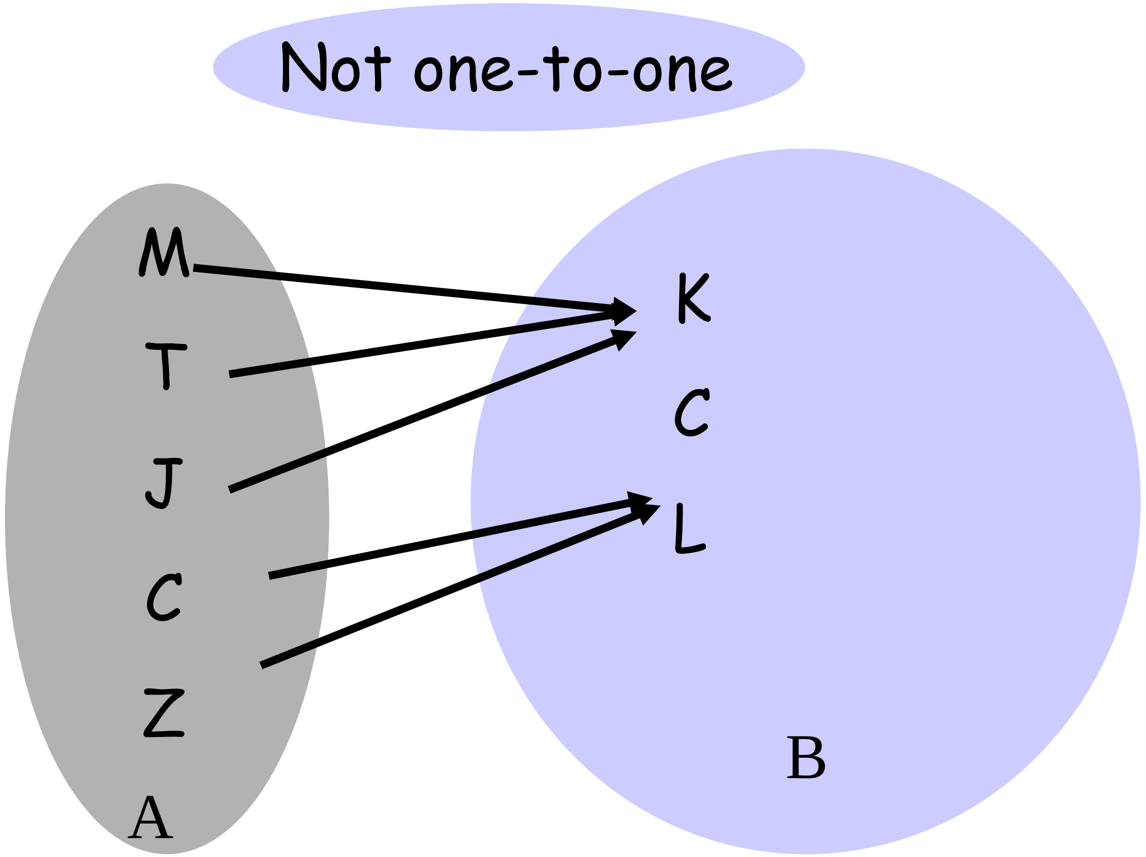


Functions – Injection

A function f: A → B is one-to-one (injective, an injection) if

- $\forall a,b,c, (f(a) = b \wedge f(c) = b) \rightarrow a = c$

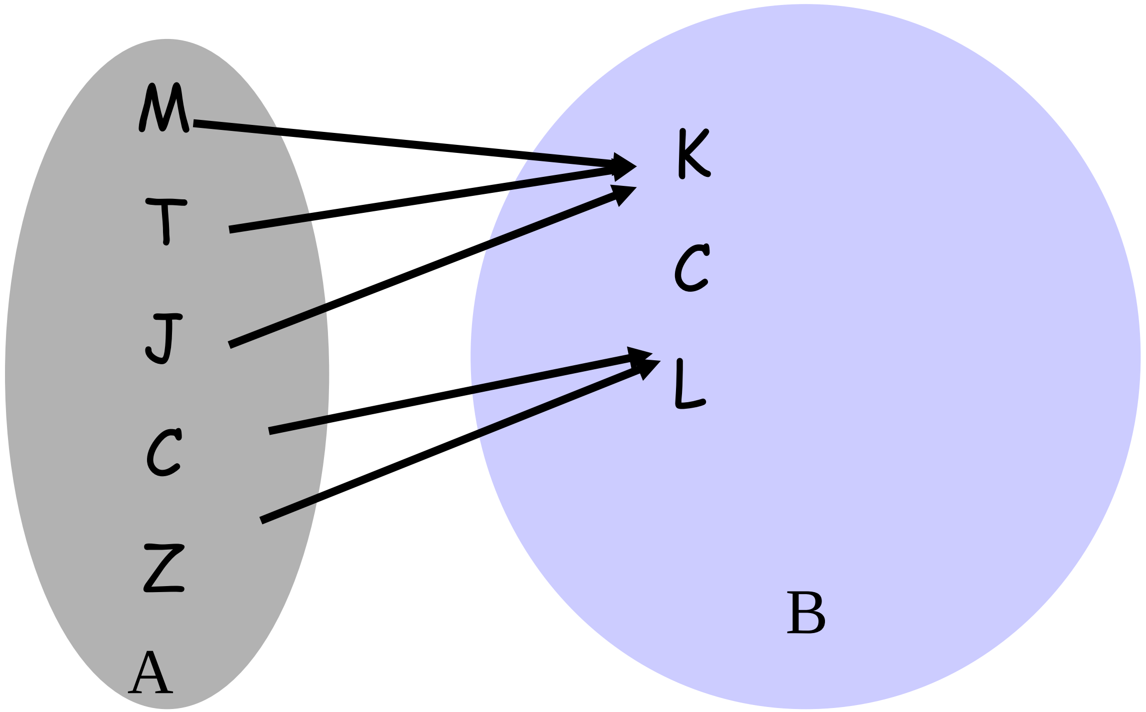
Every $b \in B$ has at most 1 preimage.



Functions – Surjection

A function f: A $\rightarrow$ B is onto (surjective, a surjection) if $\forall b \in B, \exists a \in A \; f(a) = b$

Every $b \in B$ has at least 1 preimage.

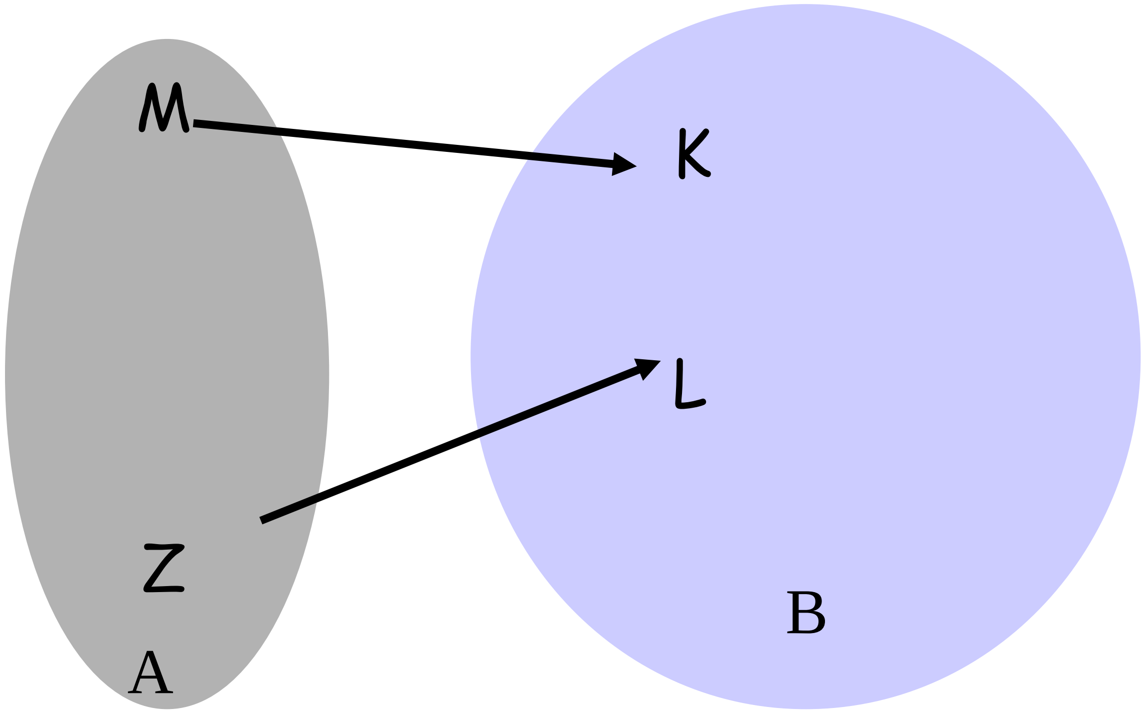


Functions – Bijection

A function f: A → B is bijective if it is one-to-one and onto.
Every b ∈ B has exactly 1 preimage.

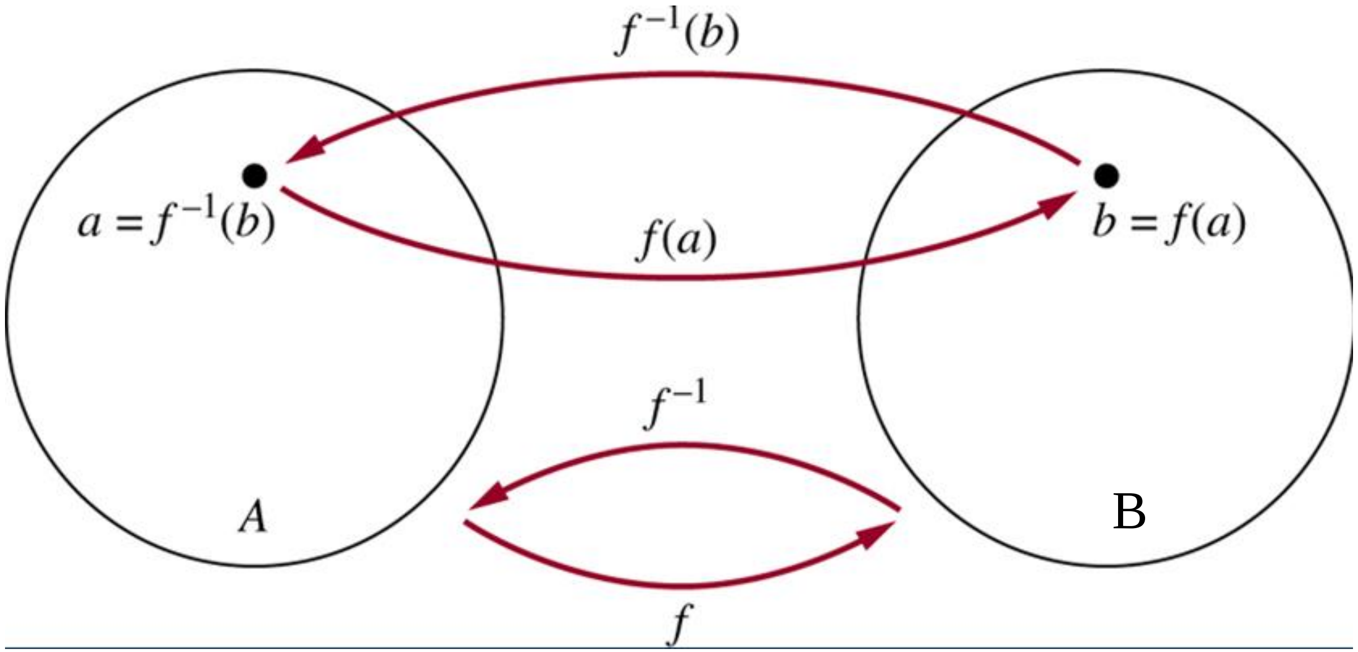
An important implication of this characteristic:

The preimage (f^{-1}) is a function! The function f is invertible & so is f^{-1}



Inverse function

Given f, a one-to-one correspondence from set A to set B, the **inverse function of f** is the function that assigns to an element b belonging to B the unique element a in A such that f(a)=b. The inverse function is denoted f^{-1} . f^{-1} (b)=a, when f(a)=b.



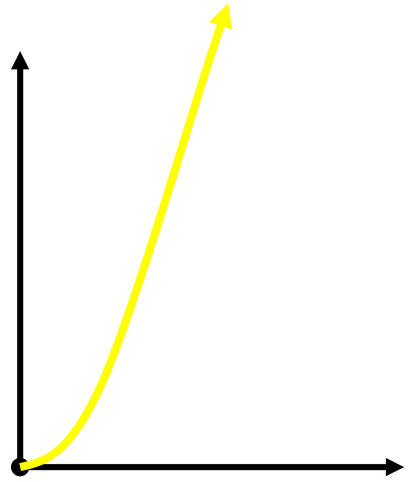
Examples

Suppose f: $R^+ \rightarrow R^+$, $f(x) = x^2$

- Is f one-to-one?
- Is f onto?
- Is f bijective?

Function is invertible

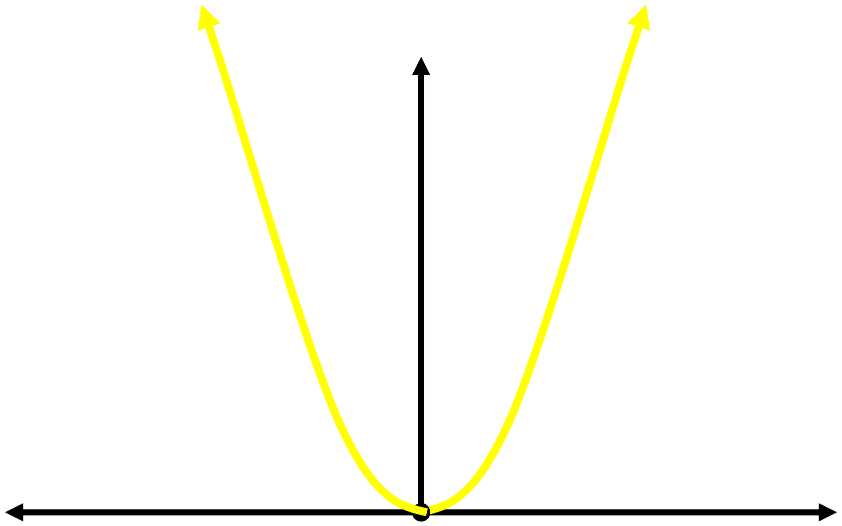
$f^{-1}(y) = \sqrt{y}$



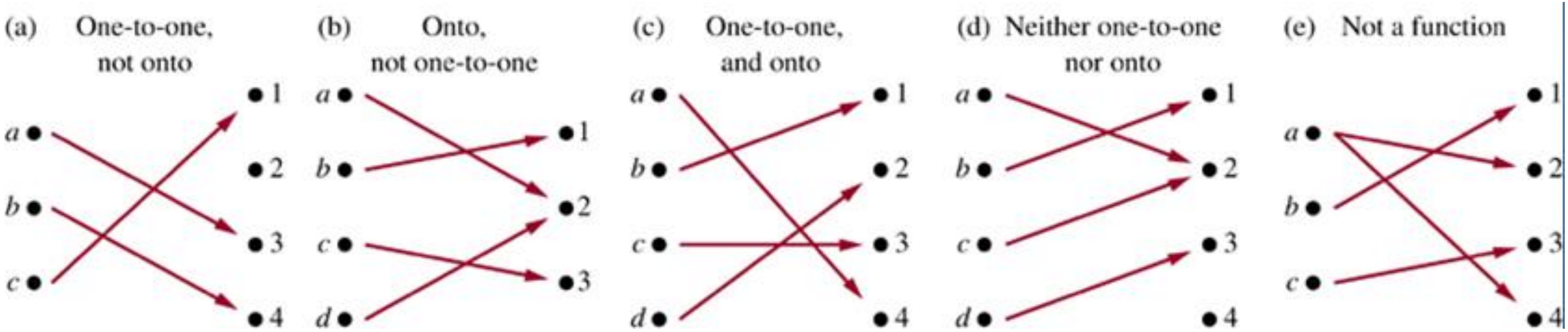
Suppose f: $R \rightarrow R^+$, $f(x) = x^2$.

- Is f one-to-one?
- Is f onto?
- Is f bijective?

Function is not invertible



Examples

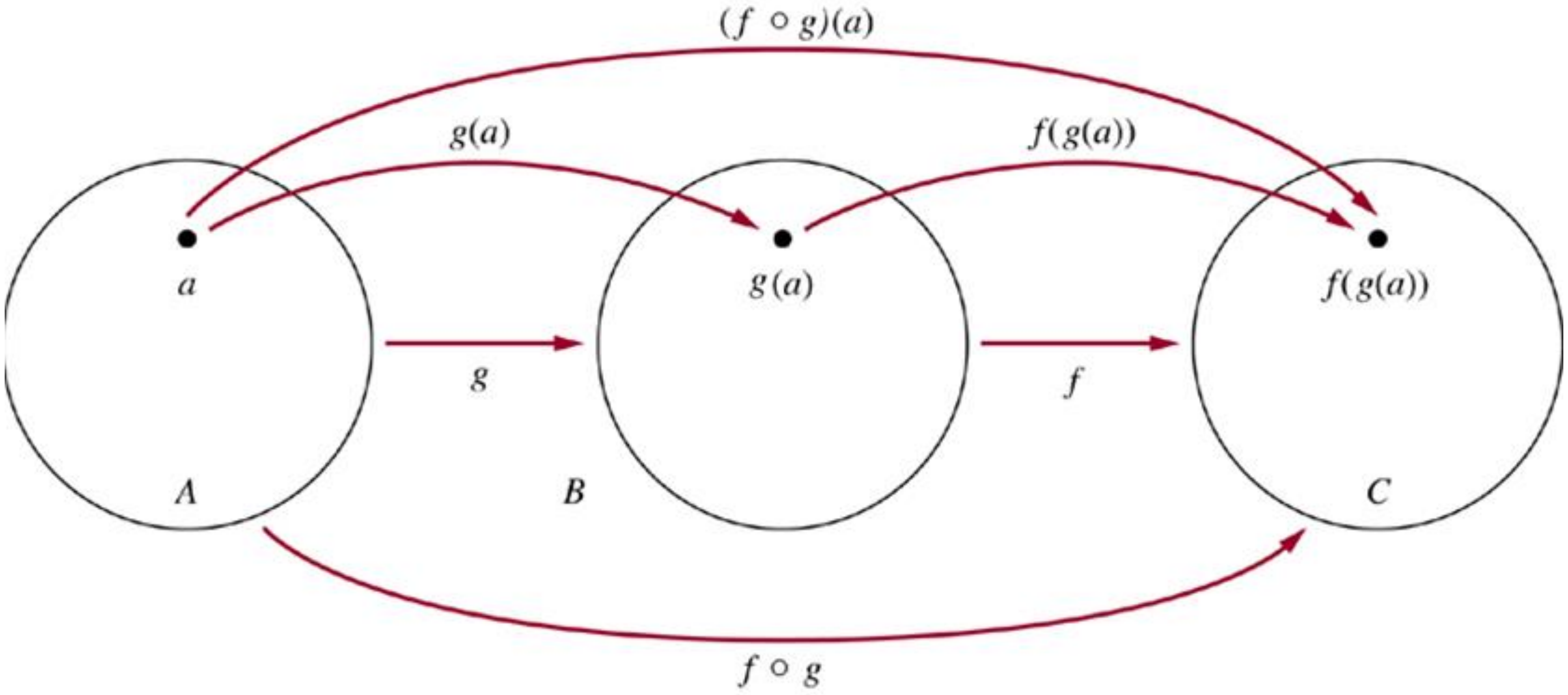

(a) One-to-one, not onto
a
b
c
1
2
3
4
(b) Onto, not one-to-one
a
b
c
d
1
2
3
(c) One-to-one, and onto
a
b
c
d
1
2
3
4
(d) Neither one-to-one nor onto
a
b
c
d
1
2
3
4
(e) Not a function
a
b
c
1
2
3
4

Composition of Functions

Let f: B→C, and g: A→B be functions. Then the composition of f and g is:

(f o g)(x) = f(g(x)): A→C

Note: (f o g) cannot be defined unless the range of g is a subset of the domain of f.



Examples

Let $f(x) = 2x + 3$; $g(x) = 3x + 2$;

$(f \circ g)(x) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7.$

$(g \circ f)(x) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

As this example shows, $(f \circ g)$ and $(g \circ f)$ are not necessarily equal

- The composition of functions is not commutative.

Exercise

1. $f(x) = 2x + 3$ & $g(x) = 3x + 2$.
- Find $f \circ g$ & $g \circ f$

2. $f(a) = a - 1$ and $g(b) = b^2$
- Find $(f \circ g)(2)$, $(g \circ f)(2)$, $(g \circ f)(x)$, $(f \circ f)(y)$

3. Let $f(x) = ax + b$ & $g(x) = cx + d$, where a, b, c, d are constants
- Determine for which constants a, b, c, d $(f \circ g) = (g \circ f)$

Invertible Functions & Composition

$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a.$

$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f^{-}(a) = b.$

Therefore $(f^{-1} \circ f) = I_A$ and $(f \circ f^{-1}) = I_B$ where I_A and I_B are the identity function on the sets A and B.

$(f^{-1})^{-1} = f$

Some Important Functions

Absolute value: Domain R; Co-Domain = {0} ∪ R^+; $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

Floor function: Domain = R; Co-Domain = Z; $\lfloor x \rfloor$ = largest integer not greater than x

$\lfloor 3.2 \rfloor = 3$; $\lfloor -2.5 \rfloor = -3$

Ceiling function: Domain = R; Co-Domain = Z; $\lceil x \rceil$ = smallest integer greater than x

$\lceil 3.2 \rceil = 4$; $\lceil -2.5 \rceil = -2$

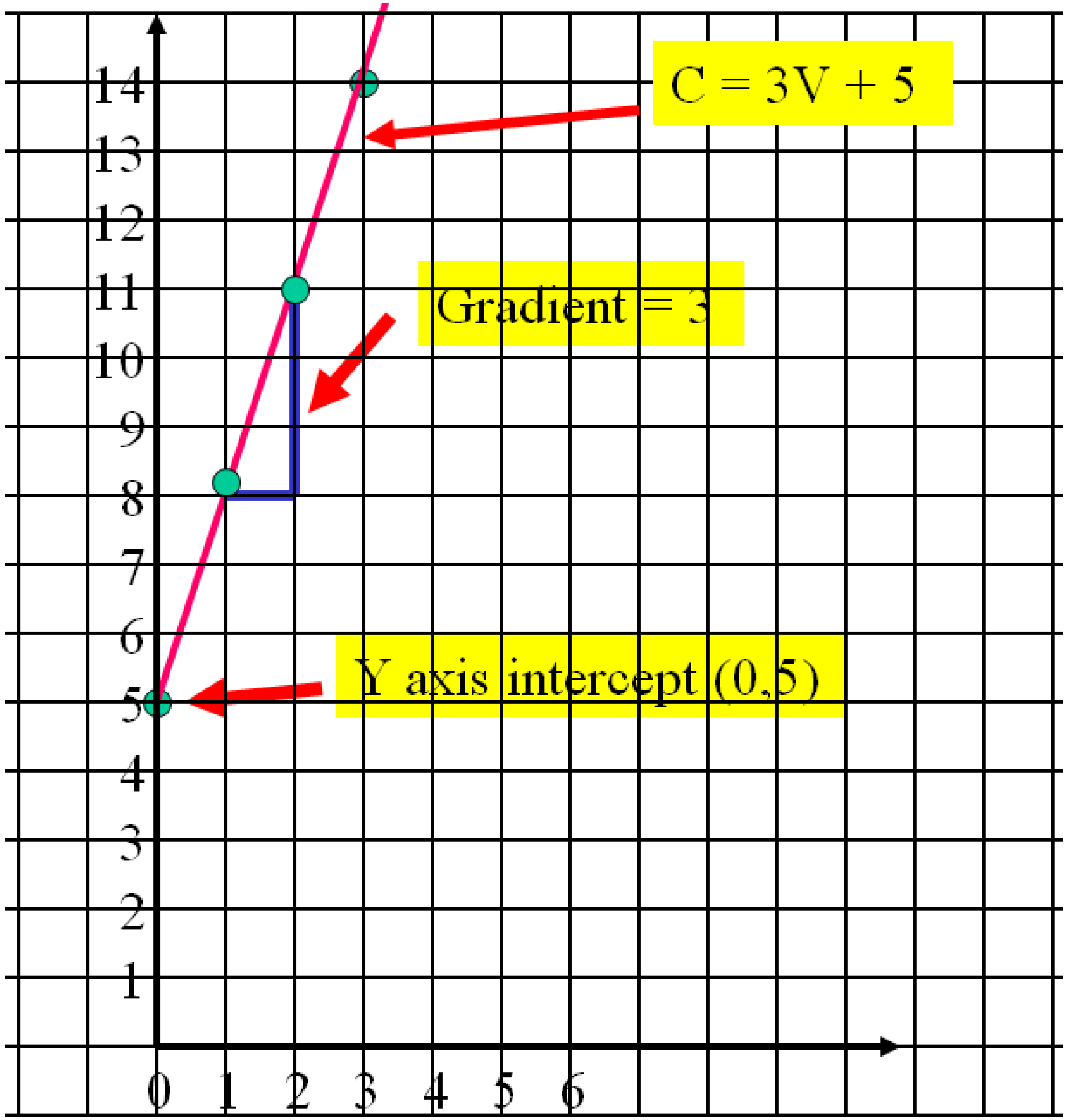
Factorial function: Domain = Range = N; n! = n (n-1)(n-2) …, 3 x 2 x 1

5! = 5 x 4 x 3 x 2 x 1 = 120

0! = 1 by convention.

Straight Lines & Hyperplanes

Example


C = 3V + 5
Gradient = 3
Y axis intercept (0,5)
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0 1 2 3 4 5 6

The Equation Of A Straight Line

To find the equation of any straight line we require to know two things:

(a) The gradient of the line: m

(b) The y axis intercept of the line: c

The equation of a straight line is $y = m x + c$

Examples

Give the gradient and the y axis intercept for each of the following lines.

$y = 6x + 5$, $y = -4x + 2$, $y –x -2$

Straight Line From Two Points

Find the equation of the straight line passing through (4,6) and (6,12)

$m = (y2 – y1) / (x2 – x1) = (12 – 6) / (6 – 4) = 3$

Substitute gradient into $y = m x + c$, That is $y = 3x + c$

Substitute (4, 6) into $y = 3x + c$ gives $c = -6$

Therefore equation of the line is $y = 3x - 6$

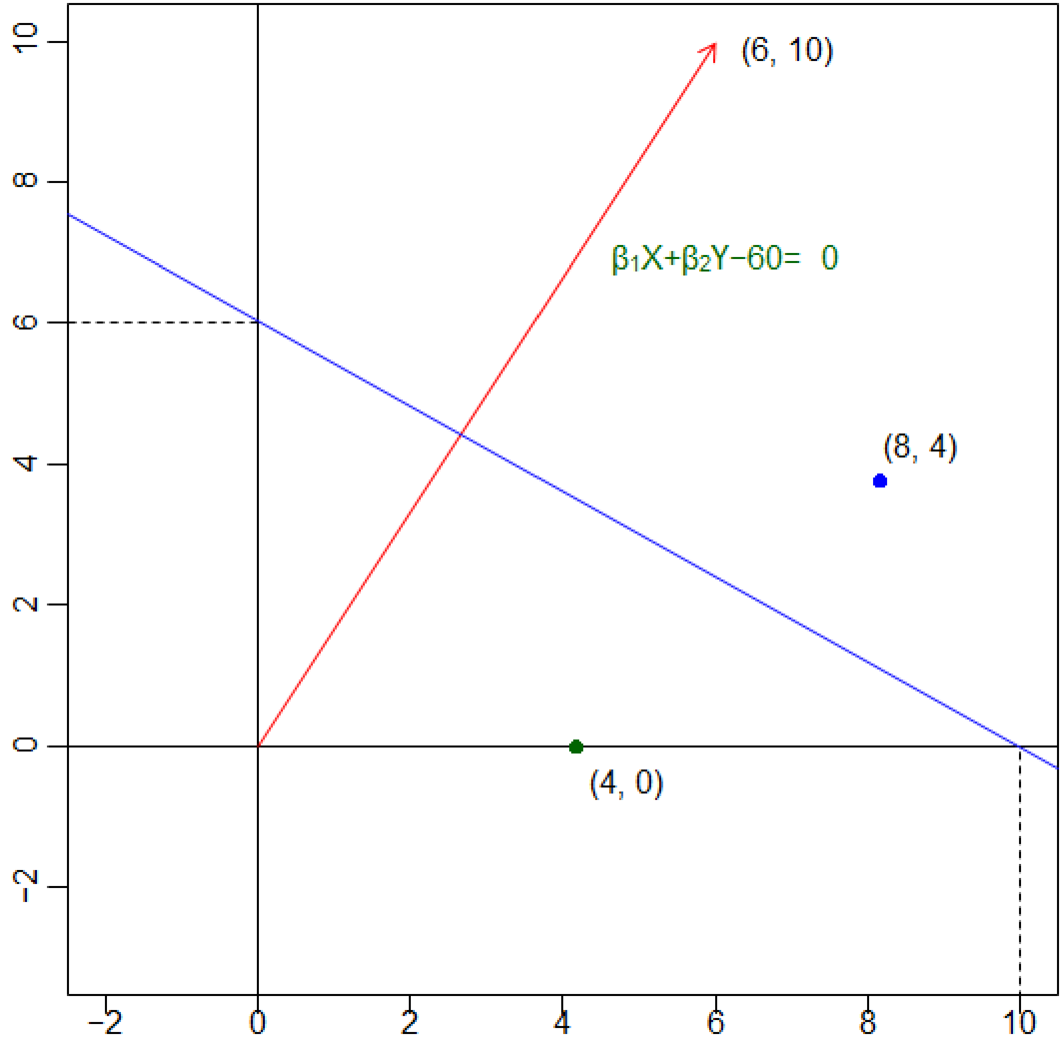
Hyperplane

- A hyperplane in p dimensions is a flat affine subspace of dimension $p - 1$.
- In general the equation for a hyperplane has the form

$$\beta_0 + \beta_1 X_1 + \beta_2 X_2 + . . . + \beta_p X_p = 0$$

- If $p = 2$, a hyperplane is a line.
- If $\beta_0 = 0$, the hyperplane goes through the origin, otherwise "affine"
- The vector $\beta = (\beta_1, \beta_2, \cdot \cdot \cdot , \beta_p)$ is called the normal vector
  — It points in a direction orthogonal to the surface of a hyperplane.

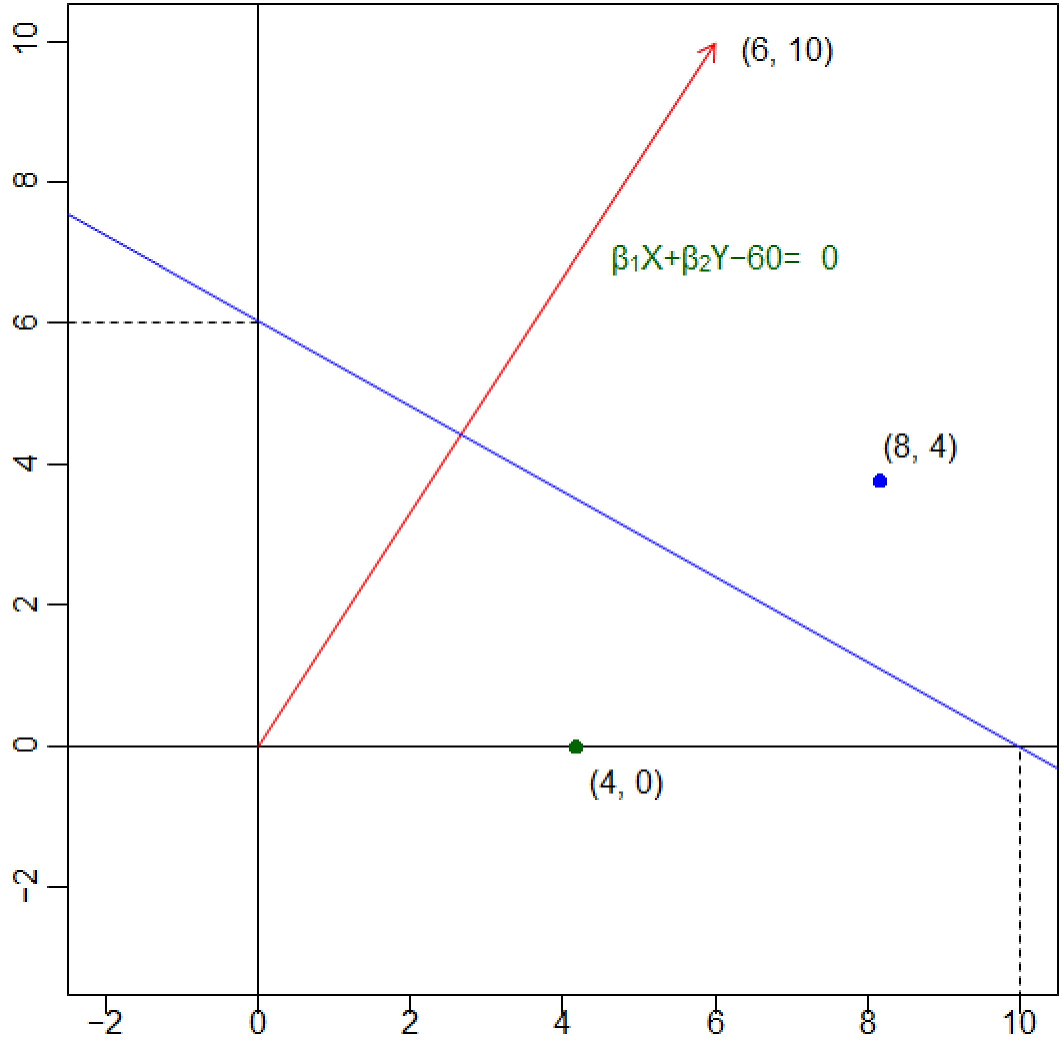
Hyperplane in 2 Dimension Space



Equation of the **Blue Line**:

- Slope $= m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{6-0}{0-10} = -0.6$
- Equation: $\frac{y-y_1}{x-x_1} = m \Rightarrow \frac{y-0}{x-10} = -0.6 \text{ Or } 10y + 6x - 60 = 0$
- **10y + 6x – 60 = 0**
- **This is a hyperplane in the XY plane**

The Normal to the Hyperplane

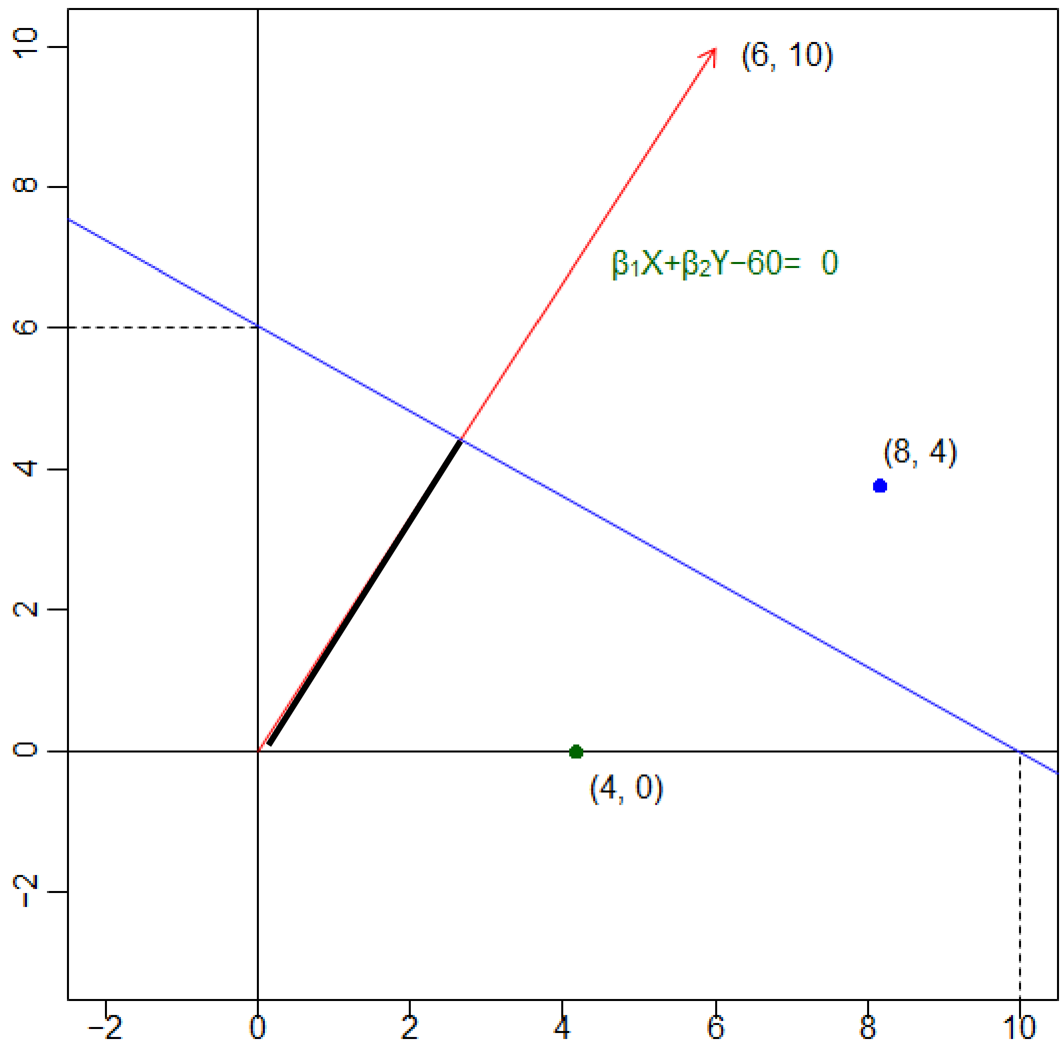


- **The hyperplane: 10y + 6x – 60 = 0**
- The slope of this line: m = –0.6 = – 6/10
- The slope of a perpendicular line: –1/m = 10/6
- The equation of the perpendicular through the origin is

$$\frac{y-0}{x-0} = \frac{10}{6} \Longrightarrow 6y = 10x \text{ or } \mathbf{6y - 10x = 0}$$

- The point (6, 10) is on the perpendicular
- The vector β = (β_1 = 6, β_2 = 10) is the normal vector
- It points in a direction orthogonal to the surface of the **hyperplane**

The Normal Form of the Hyperplane



- The hyperplane: **10y + 6x – 60 = 0**
- The perpendicular through the origin: **6y – 10x = 0**
- The point (6, 10) is on the perpendicular
- The vector $\beta = (\beta_1 = 6, \beta_2 = 10)$ is the normal vector
  - Perpendicular to the surface of the hyperplane
- The Normal Form of the hyperplane:

$$\mathbf{f(X)} = \frac{\mathbf{10}}{\sqrt{\mathbf{136}}}\mathbf{y} + \frac{\mathbf{6}}{\sqrt{\mathbf{136}}}\mathbf{x} - \frac{\mathbf{60}}{\sqrt{\mathbf{136}}} = \mathbf{0}$$

- Note
  - Length of **Perpendicular** to hyperplane = 60/√136
  - f(8, 4) > 0 while f(4, 0) < 0
  - Distance of (8, 4) from line = $\left|\frac{10}{\sqrt{136}}4 + \frac{6}{\sqrt{136}}8 - \frac{60}{\sqrt{136}}\right|$
  - Distance of (4, 0) from line = $\left|\frac{10}{\sqrt{136}}0 + \frac{6}{\sqrt{136}}4 - \frac{60}{\sqrt{136}}\right|$

Polynomial Functions and Their Graphs

Polynomial Functions

A polynomial function has all of its variables with exponents which are positive integers. It is not a polynomial function if a variable has a negative exponent or if the exponent is a fraction.

Examples

Which of these are polynomials?

$f(x) = x^4 - x^{-2} + 2x - 9$ **no**

$f(x) = 5x^5 + x^4 - 2\sqrt{x} + 5$ **no**

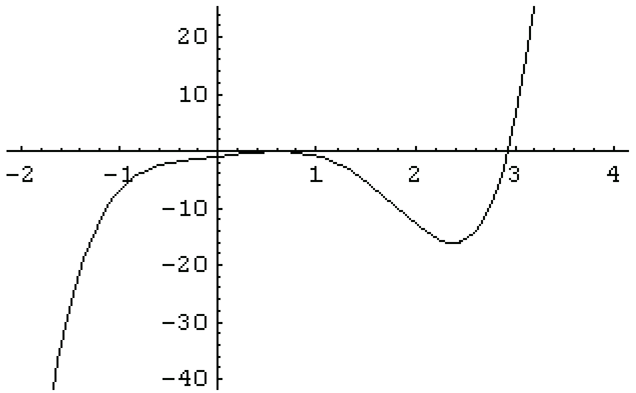
$f(x) = 3x^3 + 2x^2 + 1$ **yes**

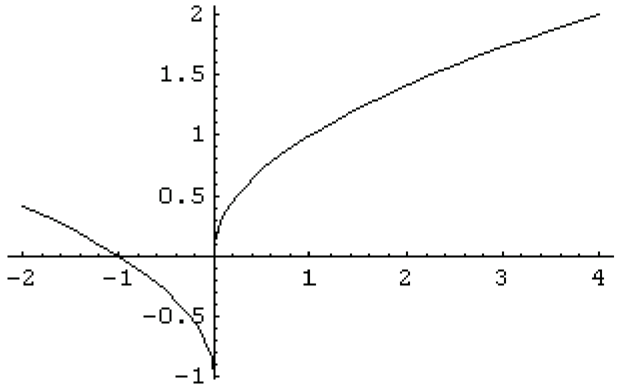
$f(x) = 2x^4(x-2)(x+3)$ **yes**

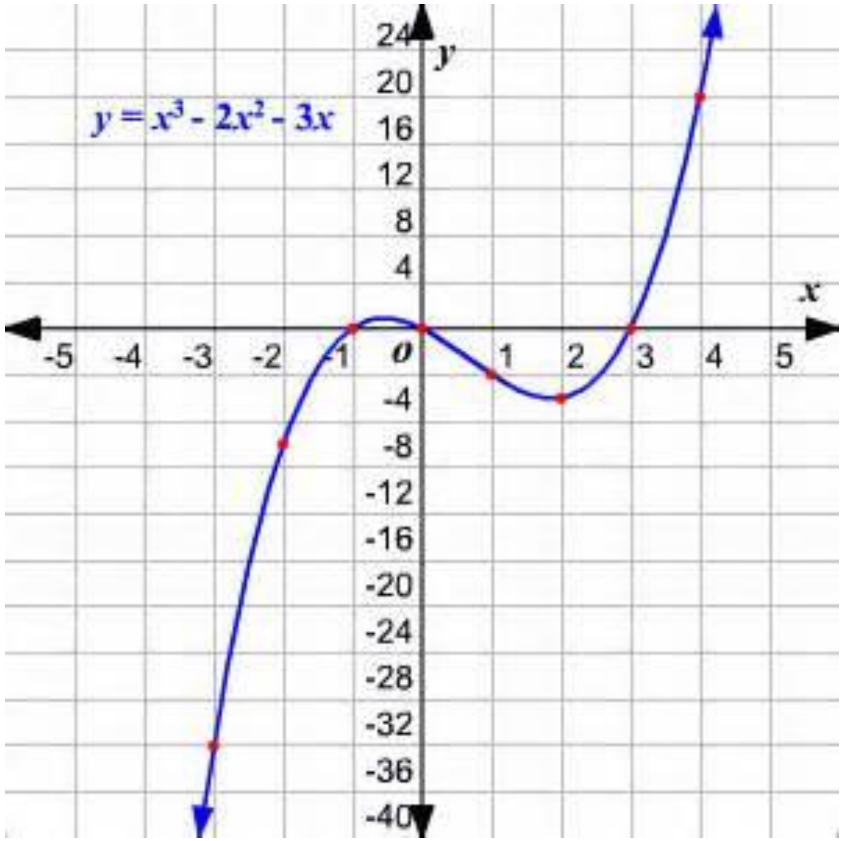
Basic Features of Graphs of Polynomials

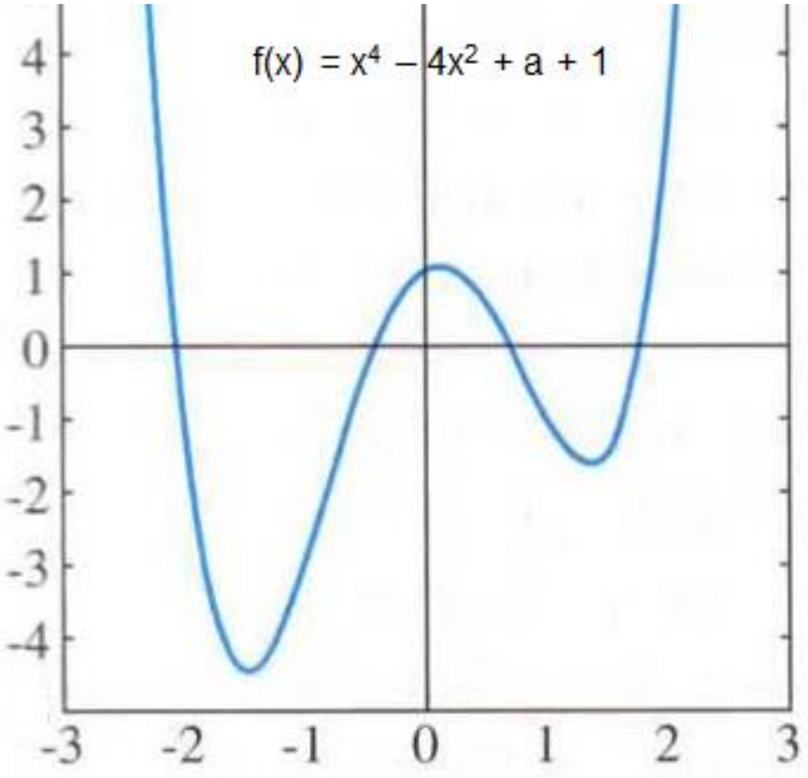
A graph of a polynomial function is continuous

- The graph has no breaks, holes or gaps
- How many zeros are there in each of the polynomials?









The Degree & Leading Coefficient

Polynomial Function	Leading Coefficient	Degree
$f(x) = -2x^5 + 3x^3 - 5x + 1$	-2	5
$f(x) = x^3 + 6x^2 - x + 7$	1	3

End Behaviour

The behavior of the graph of a function to the far left and far right is called its **end behavior**.

Although the graph of a polynomial function may have intervals where it increases or decreases, the graph will eventually rise or fall without bound as it moves far to the left or far to the right.

How can we determine the end behavior of a polynomial function?

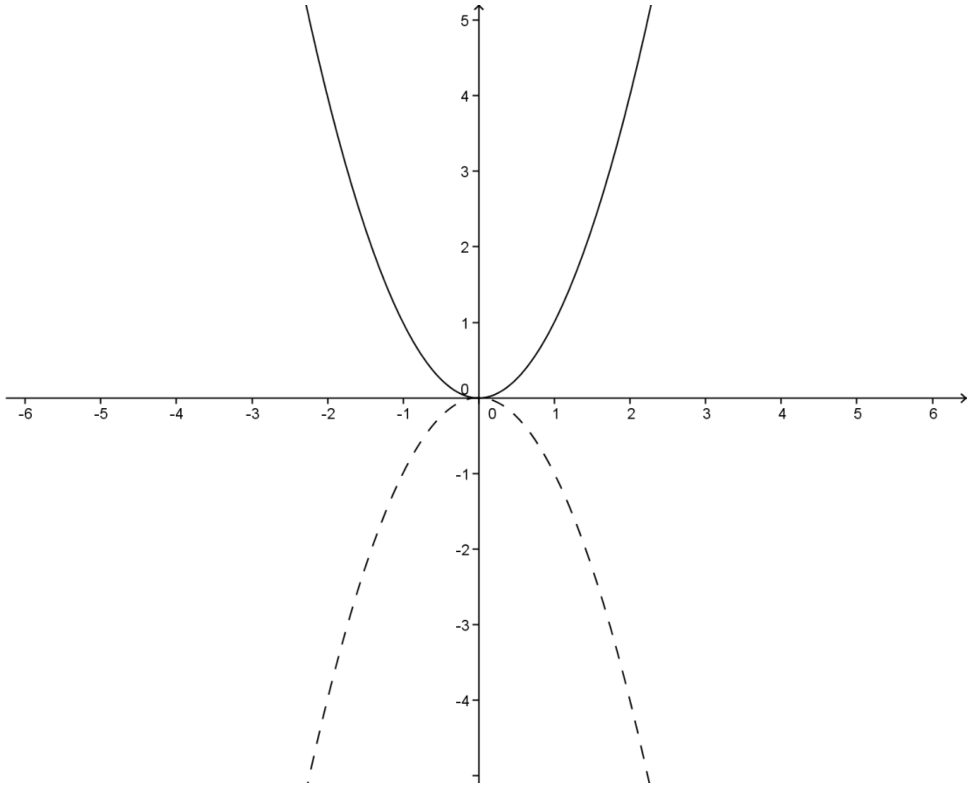
- We look only at the term with the highest degree.

End Behaviour of Even Degree Polynomials

Positive Leading Coefficient: The graph rises to the left and rises to the right (↖, ↗).

Negative Leading Coefficient: The graph falls to the left and falls to the right (↙, ↘).

Examples: $f(x) = x^2$ & $f(x) = -x^2$

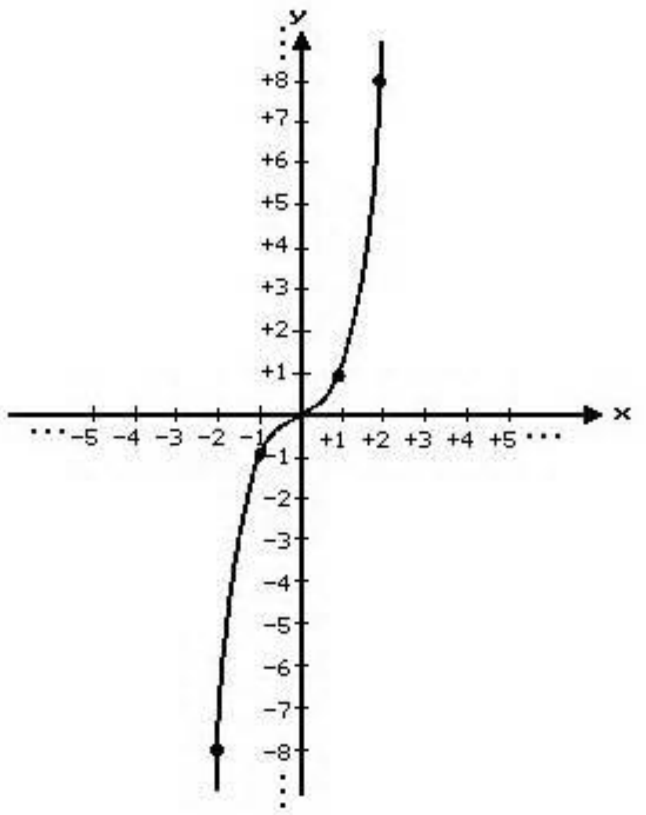


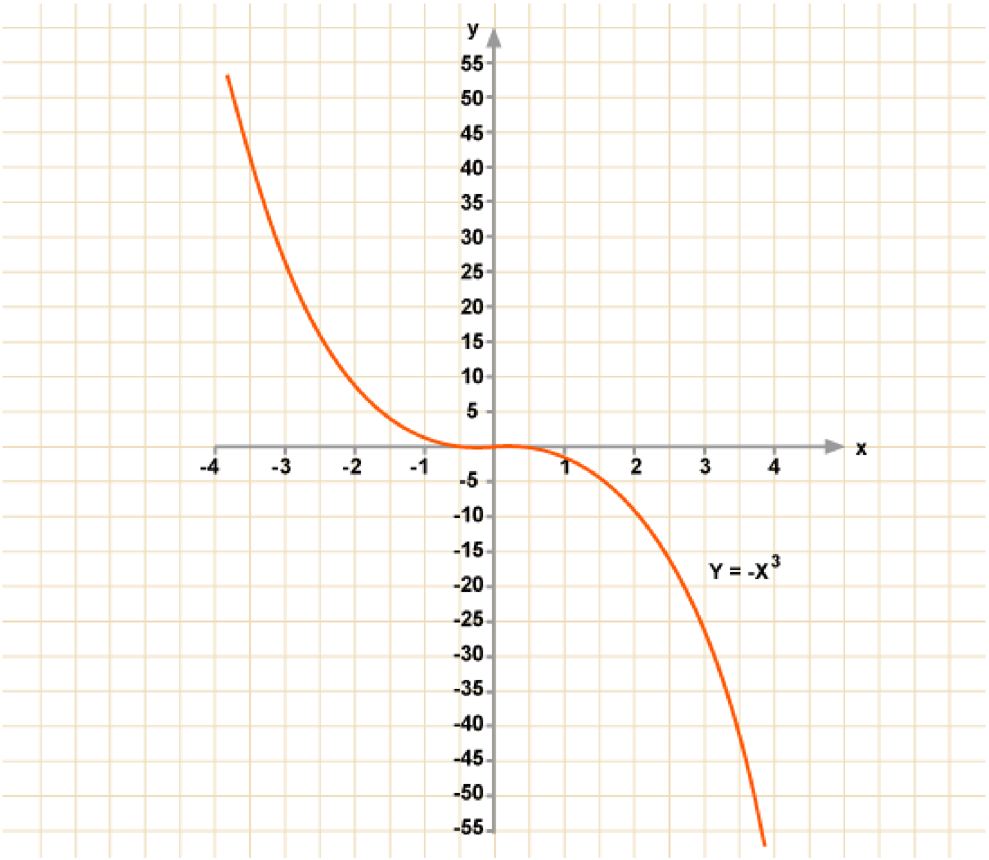
End Behaviour of Odd Degree Polynomials

Positive Leading Coefficient: The graph falls to the left and rises to the right (↙, ↗)

Negative Leading Coefficient: the graph rises to the left and falls to the right (↖, ↘)

Examples: $f(x) = x^3$ & $f(x) = -x^3$





End Behaviour: Examples

$f(x) = x^4 – 4x^2$

- Leading coefficient is positive & Even degree to its variable: (↖, ↗)

$f(x) = x^3 + 3x − x − 3$

- Leading coefficient is positive & Odd degree to its variable: (↙, ↗)

$f(x) = − 2x^3 + 3x^2 − x − 3$

- Leading coefficient is negative & Odd degree to its variable: (↖, ↘).

$f(x) = − 2x^2 + 3x − 10^6$

- Leading coefficient is negative & Even degree to its variable: (↙, ↘)

$f(x) = 3x^3(x − 1)(x + 5)$

- Leading coefficient is positive & Odd degree to its variable: (↙, ↗).

Zeros of Polynomial Functions

It can be shown that for a polynomial function of degree n:

1. The function has, at most, n real zeros.

2. The graph has, at most, $n - 1$ turning points.

- Turning points are points at which the graph changes from increasing to decreasing or vice versa
- Relative maximum / minimum

Multiplicity, Turning Points & *x*-intercepts

Recall (algebraic) multiplicity of a zero:

Example: The zeros of $\mathbf{f(\mathit{x}) = x^2 (x - 2)^2}$ = $x^2 (x - 2)^2 = 0$

- $x^2 = 0$ therefore $x = 0$ with multiplicity of 2
- $(x - 2)^2 = 0$ therefore $x = 2$ with multiplicity of 2

Suppose r is a zero of even multiplicity

- The graph touches the x-axis at r and turns around at r.

Suppose r is a zero of odd multiplicity

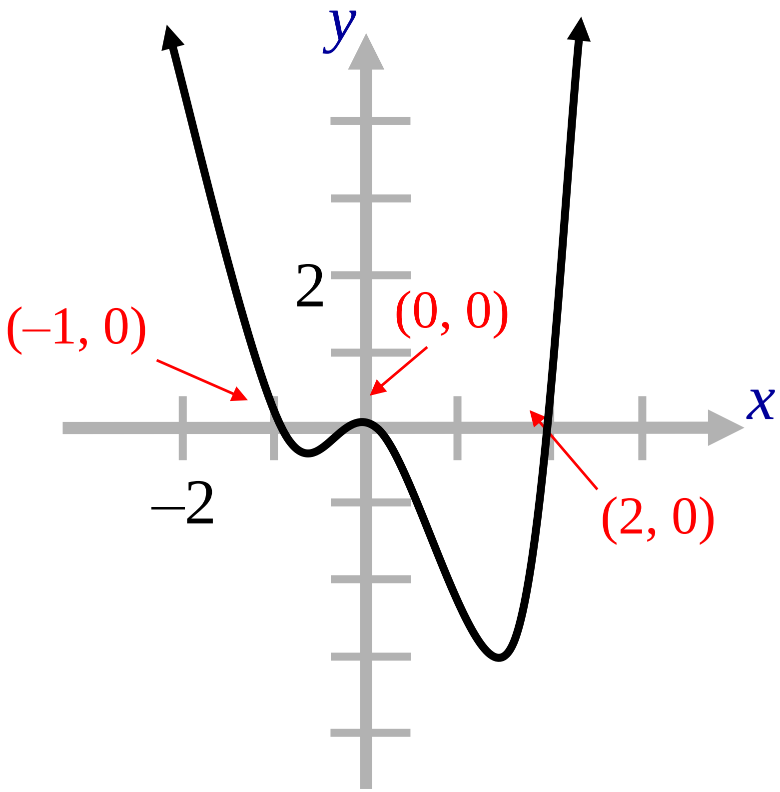
- The graph crosses the x-axis at r.

Zeros & Turning Points: Example

Find all the zeros & Count the turning points of $f(x) = x^4 - x^3 - 2x^2$.

- $f(x) = x^4 - x^3 - 2x^2 = x^2(x + 1)(x - 2)$.

- The real zeros are $x = -1$, $x = 0$, and $x = 2$.
  These correspond to the x-intercepts.



Check out the x-intercepts and the multiplicities

- What happens?

Zeros & Turning Points: Example

$f(x) = -4(x + 2)^2$

Give the multiplicity of each zero. State whether the graph crosses the *x*-axis or touches the *x*-axis and turns around at each zero.

Since x = -2 has an even multiplicity, the graph will touch -2 on the X-Axis and turn

The Intermediate Value Theorem

Let f be a polynomial function

Suppose f(a) and f(b) have opposite signs

- Then there is a value c between a and b for which f(c) = 0
- That is, f(x) = 0 has at least one real root between a and b

Consider $f(x) = x^3 – 2x – 5$

- f(2) = -1 & f(3) = 16
- Therefore there is a root between 2 and 3

Exponential Functions

Exponential Function

The function defined by $f(x) = b^x \ (b > 0)$

- The base is b with exponent *x*
- The domain is the set of all real numbers

Example 1

The exponential function with base 2 is the function:

- $f(x) = 2^x$ with domain $(-\infty, \infty)$.
- $f(3) = 8$, $f(3/2) = 2\sqrt{2}$, $f(0) = 1$

Example 2

The exponential function with base 2 is the function

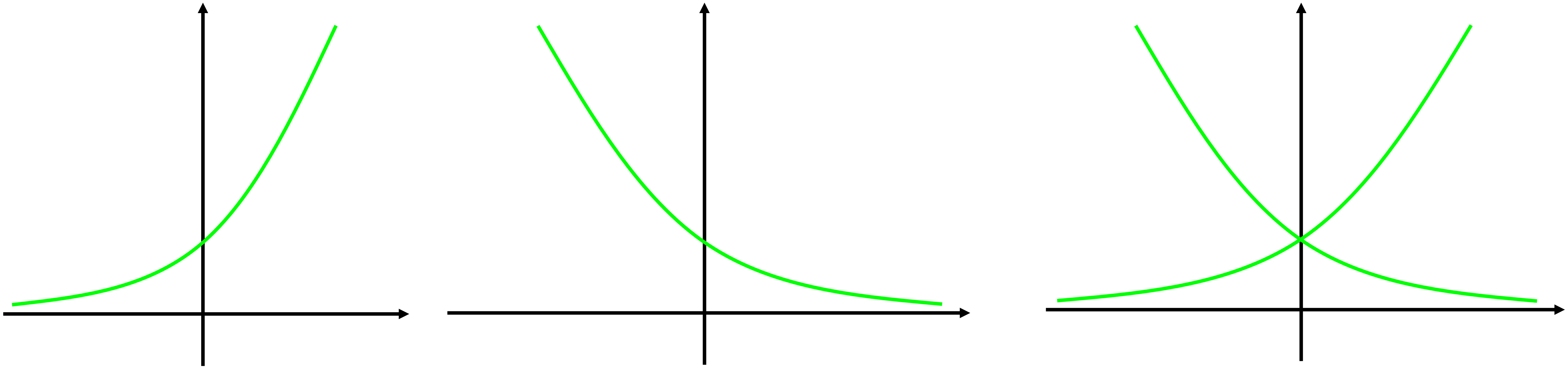
- $f(x) = 2^{-x}$ with domain $(-\infty, \infty)$
- $f(-1) = 0.5, f\left(-\frac{2}{3}\right) = 2^{-2/3} = \frac{1}{\sqrt[3]{4}}$

The Graphs of Exponential Functions

$f(x) = 2^x$. & $f(x) = 2^{-x}$

- Domain: The set of real numbers.
- At $x = 0$ gives $y = 2^0 = 1$, which is the y-intercept.
- There is no x-intercept, since there is no value of x for which $y = 0$

Notice the symmetry!



Properties of Exponential Functions

The exponential function $y = b^x$ $(b > 0, b \neq 1)$ has the following properties:

1. Its domain is $(-\infty, \infty)$.
2. Its range is $(0, \infty)$.
3. Its graph passes through the point $(0, 1)$
4. It is continuous on $(-\infty, \infty)$.
5. It is increasing on $(-\infty, \infty)$ if $b > 1$ and decreasing on $(-\infty, \infty)$ if $b < 1$.

The Base *e*

e: An irrational number whose value is 2.7182818…, plays an important role

It can be shown that

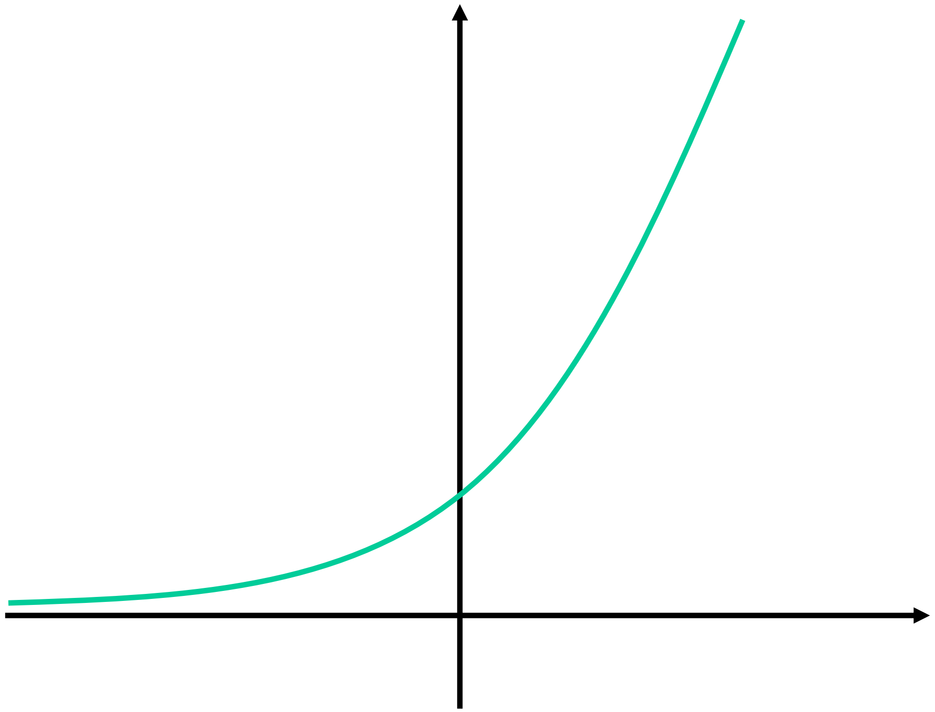
$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m$$

The Graph of $f(x) = e^x$

Since $e^x > 0$ it follows that the graph of $y = e^x$ is similar to the graph of $y = 2^x$.

Consider a few values for x:

x	– 3	– 2	– 1	0	1	2	3
y	0.05	0.14	0.37	1	2.72	7.39	20.09



Graph of $f(x) = e^{-x}$

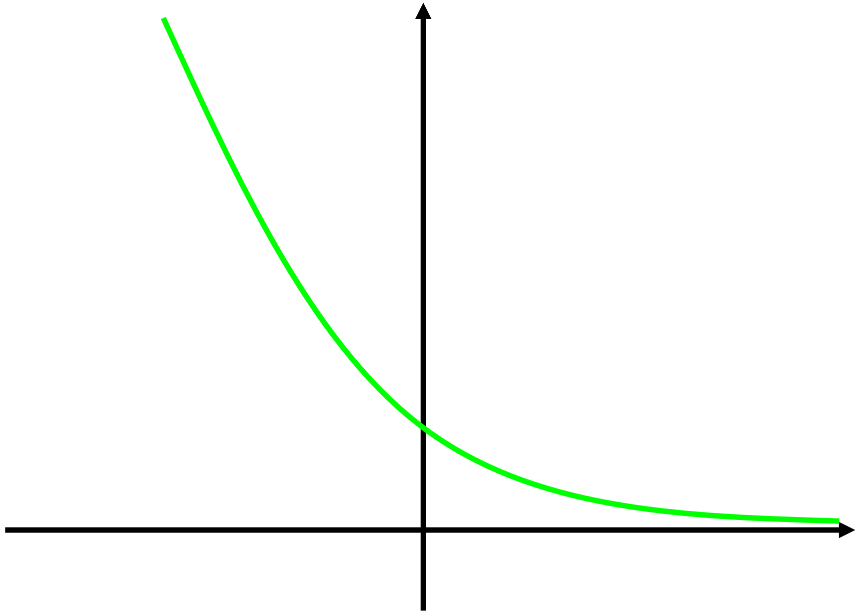
Since $e^{-x} > 0$ it follows that $0 < 1/e < 1$ and so

$f(x) = e^{-x} = 1/e^x = (1/e)^x$ is an exponential function with base less than 1.

Therefore, it has a graph similar to that of $y = (1/2)^x = 2^{-x}$

Consider a few values for x:

x	– 3	– 2	– 1	0	1	2	3
y	20.09	7.39	2.72	1	0.37	0.14	0.05



Logarithm Functions

Logarithms

You may recall that y is called the logarithm of x to the base b, and is denoted $\log_b x$.

- Logarithm of x to the base b

$$y = \log_b x \quad \text{if and only if} \quad x = b^y \qquad (x > 0)$$

Example:

Solve $\log_3 x = 4$ for x:

- By definition, $\log_3 x = 4$ implies $x = 3^4 = 81$.

Logarithmic Notation

Common logarithm: $\log x = \log_{10} x$

Natural logarithm: $\ln x = \log_e x$

Laws of Logarithms

If m and n are positive numbers, then

1. $\log_b mn = \log_b m + \log_b n$
2. $\log_b \frac{m}{n} = \log_b m - \log_b n$
3. $\log_b m^n = n \log_b m$
4. $\log_b 1 = 0$
5. $\log_b b = 1$

Properties of Logarithmic Functions

The logarithmic function

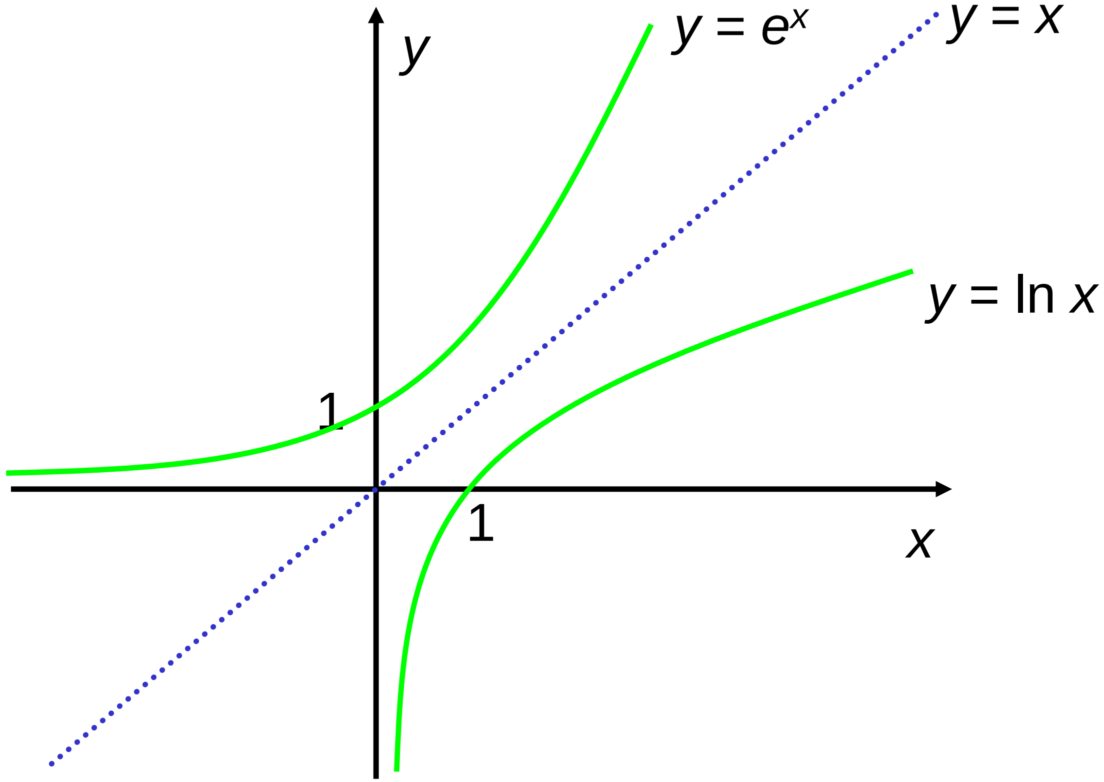
$$y = \log_b x \qquad (b > 0, b \neq 1)$$

has the following properties:

1. Its domain is $(0, \infty)$.
2. Its range is $(-\infty, \infty)$.
3. Its graph passes through the point $(1, 0)$.
4. It is continuous on $(0, \infty)$.
5. It is increasing on $(0, \infty)$ if $b > 1$ and decreasing on $(0, \infty)$ if $b < 1$.

The Graphs of $y = \ln x$ & $y = e^x$

The graph is the mirror image of the graph of $y = e^x$ with respect to the line $y = x$:



Exponential and Logarithmic Functions

- $e^{\ln x} = x$ $(x > 0)$
- $\ln e^{x} = x$ (for any real number x)

Examples

Solve the equation $2e^{x+2} = 5$.

$2e^{x+2} = 5 \Rightarrow e^{x+2} = 5/2 = 2.5 \Rightarrow \ln(e^{x+2}) = \ln(2.5) \Rightarrow (x + 2)\ln(e) = \ln(2.5)$

$\Rightarrow (x + 2) = \ln(2.5) \Rightarrow x = -2 + \ln(2.5) = -1.0837$

Solve the equation $5 \ln x + 3 = 0$.

$5\ln x + 3 = 0 \Rightarrow \ln(x) = -0.6 \Rightarrow x = e^{-0.6} = 0.5488$

Facts

$$\lim_{x \to \infty} \frac{a_0 + \cdots + a_k x^k}{b^x} = 0 \; if \;\; b > 1$$

$$\lim_{x \to \infty} \log(x) = \infty$$

$$\lim_{x \to \infty} \frac{\log(x)}{x} = 0$$

Summation

Summation

The symbol ∑ (Greek letter sigma) is used to denote summation.

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_k$$

- i is the **index of the summation**, and the choice of letter i is arbitrary;
- The index of the summation runs through all integers, with its **lower limit** 1 and ending **upper limit** k.

The limit:

$$\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

Summation

The laws for arithmetic apply to summations

$$\sum_{i=1}^{k}(ca_i + b_i) = c\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i$$

Useful Summations

Sum	Closed Form
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Can you express thefollowing functions as infinite series?

- e^x,
- Sin(x),
- Cos(x)

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Thank you!!