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SS ZC416 Mathematical Foundations for Data Science
Eigenvalues & Eigenvectors

Recap

- Linear Algebra
  - Matrix dimensions
  - Matrix operations
  - Linear system of equations
    - Existence & uniqueness
    - Gauss elimination method & Gauss Jordan elimination method
      - Elementary Row Operations & Pivots
  - Inverse of a matrix
  - Rank & Nullity
  - Vector Spaces
  - Linear transformations
  - Inner products

Example 1

Comment on the following matrix multiplications

$$\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 33 \\ 27 \end{bmatrix} \qquad \text{vs.} \qquad \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix} = 10\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

- A eigenvalue problem considers the vector equation

$$Ax = \lambda x$$

- Here A is a given square matrix, λ an unknown scalar, and x an unknown vector. In a matrix eigenvalue problem, the task is to determine λ's and x's

Eigenvector

- Let A be an nxn square matrix

- Consider the Vector Equation

$$AX = \lambda X$$

- X = 0 is a trivial solution

- A non-zero solution is called an eigenvector or characteristic vector of A

Characteristic Equation & Eigenvalue

Let A be an nxn square matrix

- Consider the Vector Equation $AX = \lambda X$
- This is equivalent to $(A - \lambda I)X = 0$
- This implies $\mathrm{Det}(A - \lambda I) = 0$
- The characteristic equation is $\mathrm{Det}(A - \lambda I) = 0$

Remarks

- Eigenvalues are the solution of the characteristic equation
- λ is called eigenvalue or characteristic value or latent root of A
- Geometrically Ax has the same effect as λx
- The transpose A^T of a square matrix A has the same eigenvalues as A.

The Set of Eigenvalue & Eigenvector

- The set of all eigenvalues is called the spectrum of A
- The largest absolute eigenvalue is called the spectral radius of A.
- The set of all eigenvectors corresponding to an eigenvalue λ of A together with **0** forms a vector space called eigenspace of A corresponding to that λ

Multiplicities of Eigenvalues

- Algebraic multiplicity: The multiplicity of λ in the characteristic equation

- The geometric multiplicity of an eigenvalue λ is the nullity$(A - \lambda I)$

- In general Geometric Multiplicity $\leq$ Algebraic Multiplicity

Procedure To Find Eigenvalues

- Build the characteristic matrix (A-λI)
- Develop the characteristic polynomial D(λ)= Det(A-λI)
- Determine the characteristic equation D(λ)= Det(A-λI) = 0.
- The solution of the above equation are the eigenvalues of A
- Solve homogeneous equation (A- λI)X = 0 to find the eigenevectors

Example 2 – Eigenvalues

$$\boldsymbol{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

The characteristic matrix: (A-λI) = $\begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$

The characteristic equation: (-5 – λ)(-2- λ) – 4 = 0

The solution of the characteristic equation:

$\lambda^2 + 7\lambda + 6 = 0 \Rightarrow (\lambda+1)(\lambda+6) = 0 \Rightarrow \lambda = -1$ and $\lambda = -6$

Example 2 – Eigenvectors

Solutions of $D(\lambda) = 0$ are $\lambda_1 = -1$ and $\lambda_2 = -6$ – the eigenvalues of **A**

Eigenvector corresponding to λ = -1:

$$(A-\lambda I)\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-5+1 & 2\\2 & -2+1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0 \text{ or } (A+1)\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-4 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$$

That is, $-4x + 2y = 0$ & $2x - y = 0 \Rightarrow x = 1, y = 2$ or $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector corresponding to the eigenvalue λ = -1

Note: Verify that any scalar múltiple of $\begin{bmatrix}1\\2\end{bmatrix}$ is also an eigenvector

Example 2 – Eigenvectors

Solutions of D(λ) = 0 are $\lambda_1 = -1$ and $\lambda_2 = -6$ – the eigenvalues of **A**

Similarly, eigenvector corresponding to λ = -6:

$$(A-\lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

That is, $x + 2y = 0$ & $2x + 4y = 0 \Rightarrow x = 2, y = -1$ or $\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue λ = -6

Note: Verify that any scalar múltiple of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is also an eigenvector

Example 2 – Conclusion

$\boldsymbol{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

- The characteristic matrix: (A-λI) = $\begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$
- The characteristic equation: (-5 – λ)(-2- λ) – 4 = 0
- Eigenvalues = -1, -6
- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue λ = -1
- $\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue λ = -6
- Algebraic multiplicity of both eigenvalues is 1
- Geometric multiplicity of both eigenvalues is 1

Example 3

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

- Consider $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix}$
- Characteristic equation: $-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$
-
- Roots: $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$
- Algebraic multiplicity of $\lambda = 5$ is 1
- Algebraic multiplicity of $\lambda = -3$ is 2

Example 3 – Eigenvectors

The eigenvalues of $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ are $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$

Form $(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{0}$ and then use Gauss elimination:

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 3 – Eigenvectors Continued

For $\lambda_1 = 5$: $\mathbf{A} - \lambda_1\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -2/7 & 3/7 \\ 0 & -24/7 & -48/7 \\ 0 & -16/7 & -32/7 \end{bmatrix} \begin{cases} R1 \to \left(-\frac{1}{7}\right) R1 \\ R2 \to R2 - 2R1 \\ R3 \to R3 + R1 \end{cases}$

- $= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{cases} R1 \to R1 + \left(\frac{2}{7}\right) R2 \\ R2 \to (-\frac{7}{24})R2 \\ R3 \to R3 + (\frac{16}{7})R2 \end{cases}$

- Choosing $x_3 = -1$ we have $x_2 = 2$ from 2nd equation and $x_1 = 1$ 1st equation
- So eigenvector is $\mathbf{x}_1 = [1 \quad 2 \quad -1]^T$

Notice:

- Hence, the rank of $\mathbf{A} - \lambda_1\mathbf{I}$ is 2

∴ Algebraic multiplicity of $\lambda_1 = 5$ is 1 & Geometric multiplicity of $\lambda_1 = 5 = 1$

Example 3 – Eigenvectors Continued

For $\lambda = -3$ the characteristic matrix & the Row Echelon Form are:

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{cases} R1 \rightarrow R1 \\ R2 \rightarrow R2 - 2R1 \\ R3 \rightarrow R3 + R1 \end{cases}$$

From $x_1 + 2x_2 - 3x_3 = 0$ we have $x_1 = -2x_2 + 3x_3$

Choosing $x_2 = 1$, $x_3 = 0$ **and** $x_2 = 0$, $x_3 = 1$, we obtain two linearly independent eigenvectors of **A** corresponding to $\lambda = -3$:

$x_2 = [-2\ 1\ 0]^T$ and $x_3 = [3\ 0\ 1]^T$

Nullity?

Notice:

- Hence, the rank of $\mathbf{A} - \lambda_2\mathbf{I}$ is 1

∴ Algebraic multiplicity of λ = -3 is 2 & Geometric multiplicity of λ = -3 is 2

Properties of Eigenvalues

- Product of eigenvalues of a matrix A is equal to determinant of A
- Sum of eigenvalues is Trace A.
- If λ is an eigenvalue of a matrix then $1/\lambda$ is the eigenvalue of A^{-1}
- If A is a square matrix then A and A^T have the same eigenvalues
- If λ is an eigenvalue of a matrix A, then λ^n is the eigenvalue of A^n.

Example 4 – Diagonalization

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

- Characteristic equation: $-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$
- Roots: $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$
- $\lambda = 5$: Eigenvector: $[1 \quad 2 \quad -1]^T$
- $\lambda = -3$: Eigenvectors: $x_2 = [-2\ 1\ 0]^T$ and $x_3 = [3\ 0\ 1]^T$

Since A is a 3x3 matrix and it has 3 eigenvectors, we can DIAGONALIZE A

Consider $\mathbf{X} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \Rightarrow X^{-1} = \begin{bmatrix} 0.125 & 0.25 & -0.375 \\ -0.25 & 0.5 & 0.75 \\ 0.125 & 0.25 & 0.625 \end{bmatrix}$

$$X^{-1}\mathrm{AX} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Diagonalization of a Matrix

A nxn **matrix** to be **diagonalizable** it is both **necessary and sufficient**

- That it admits n linearly independent eigenvectors

Orthogonal Matrix

A, an nxn matrix, is orthogonal iff $A A^T = A^T A = I$

- All columns are pairwise orthogonal
- The determinant of an orthogonal matrix is +1 or –1
- An orthogonal transformation preserves the value of the inner product of vectors **a** and **b**

Example 5

Consider the matrix

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

- Show A is an orthogonal matrix
- Show the columns are pairwise orthogonal

Consider the two vectors $v1 = [1\ 2\ -1]^T$ & $v2 = [0\ 1\ -2]^T$

- Find the inner product of these two vectors
- Show that A preserves this inner product

Example 6

Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

- Fund the eigenvalue of A
- Find the eigenvectors of A
- Are the eigenvectors linearly independent?

Consider X, the matrix formed by the eigenvectors of A

- X is a 3/3 matrix (Why?)
- Compute X^{-1}
- Compute $X^{-1}AX$. Comment!

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$$

Example 6 – Eigenvalues

Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

- Fund the eigenvalues of A
- Find the eigenvectors of A
- Are the eigenvectors linearly independent?

Consider X, the matrix formed by the eigenvectors of A

- X is a 3/3 matrix (Why?)
- Compute X^{-1}
- Compute $X^{-1}AX$. Comment!

Characteristic equation of A:

$\lambda^3 + 0\,\lambda^2 - 12\,\lambda - 16 = 0 \Rightarrow \lambda = -2, -2, 4$

Eigenvectors are solutions of

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$$

Example 6 – Eigenvectors for λ = -2

Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

- Fund the eigenvalue of A
- Find the eigenvectors of A
- Are the eigenvectors linearly independent?

Consider X, the matrix formed by the eigenvectors of A

- X is a 3/3 matrix (Why?)
- Compute X^{-1}
- Compute $X^{-1}AX$. Comment!

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$$

Characteristic equation of A:

$\lambda^3 + 0\,\lambda^2 - 12\,\lambda - 16 = 0 \Rightarrow \lambda = -2, -2, 4$

Eigenvectors are solutions of

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \;\&\; \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

λ = -2:

Gauss Elimination: $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Algebraic multiplicity is 2 & Geometric multiplicity is 2

Eigenvectors

Equation 1 gives solution as $[y-z, y, z]^T$ for any y & z

The eigenvectors can be

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \;\&\; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Example 6 – Eigenvectors for λ = 4

Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

- Fund the eigenvalue of A
- Find the eigenvectors of A
- Are the eigenvectors linearly independent?

Consider X, the matrix formed by the eigenvectors of A

- X is a 3/3 matrix (Why?)
- Compute X^{-1}
- Compute $X^{-1}AX$. Comment!

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$$

Characteristic equation of A:

$\lambda^3 + 0\,\lambda^2 - 12\,\lambda - 16 = 0 \Rightarrow \lambda = -2, -2, 4$

Eigenvectors are solutions of

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \;\&\; \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

λ = 4:

Gauss Elimination: $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Algebraic multiplicity is 1 & Geometric multiplicity is 1

Eigenvectors

Equation 2 gives y = 0.5z & Equation 1 gives x = 0.5z

The eigenvector can be

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Example 6 – Vector Space of Eigenvectors

Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

- Fund the eigenvalue of A
- Find the eigenvectors of A
- Are the eigenvectors linearly independent?

Consider X, the matrix formed by the eigenvectors of A

- X is a 3/3 matrix (Why?)
- Compute X^{-1}
- Compute $X^{-1}AX$. Comment!

Matrix of all eigenvectors

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

Det(X) = 1*(1) -1*(2) + 1*(-1) = -2

Example 6 – Vector Space of Eigenvectors

Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

- Fund the eigenvalue of A
- Find the eigenvectors of A
- Are the eigenvectors linearly independent?

Consider X, the matrix formed by the eigenvectors of A

- X is a 3/3 matrix (Why?)
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Matrix of all eigenvectors

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

Det(X) = 1*(1) -1*(2) + 1*(-1) = -2

$$X^{-1} = \begin{bmatrix} -0.5 & 1.5 & -0.5 \\ 1 & -1 & 0 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}$$

To get points, must show All steps!

Example 6 – Vector Space of Eigenvectors

Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

- Fund the eigenvalue of A
- Find the eigenvectors of A
- Are the eigenvectors linearly independent?

Consider X, the matrix formed by the eigenvectors of A

- X is a 3/3 matrix (Why?)
- Compute X^{-1}
- Compute $X^{-1}AX$. Comment!

Matrix of all eigenvectors

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

Det(X) = 1*(1) -1*(2) + 1*(-1) = -2

$$X^{-1} = \begin{bmatrix} -0.5 & 1.5 & -0.5 \\ 1 & -1 & 0 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

A diagonal matrix with the eigenvalues in the diagonal

Thank you!!