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SS ZC416 Mathematical Foundations for Data Science
Numeric Linear Algebra

Recap

- Eigenvalues & Eigenvectors
- Orthogonal matrix
- Diagonalization of a matrix

Gauss Elimination Method with Partial Pivoting

$a_{11}x_1$	$+ a_{12}x_2$	$+ a_{13}x_3$	$+ a_{14}x_4$	$= b_1$
$a_{21}x_1$	$+ a_{22}x_2$	$+ a_{23}x_3$	$+ a_{24}x_4$	$= b_2$
a31 x1	$+ a_{32}x_2$	$+ a_{33}x_3$	$+ a_{34}x_4$	$= b_3$
a41 x1	$+ a_{42}x_2$	$+ a_{43}x_3$	$+ a_{44}x_4$	$= b_4$

$$[A \mid b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{array}\right]$$

Elimination of First Column – A Tweak

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{array}\right] \qquad \begin{array}{l} \\ f_{21} = a_{21} / a_{11} \\ f_{31} = a_{31} / a_{11} \\ f_{41} = a_{41} / a_{11} \end{array}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ \boxed{0} & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ \boxed{0} & a'_{32} & a'_{33} & a'_{34} & b'_3 \\ \boxed{0} & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{array}\right] \qquad \begin{array}{l} \\ R2 - f_{21} \times R1 \\ R3 - f_{31} \times R1 \\ R4 - f_{41} \times R1 \end{array}$$

- First equation is pivot equation
- a_{11} is pivot element
- The pivot must be different from zero
- We choose our pivot equation as one that has the absolutely largest in the column.
- This method is called partial pivoting
- The iteration stops when we get an Upper Triangular Matrix
- Back Substitution is then done

Example 1

Solve the system using Gauss elimination method with partial pivoting

$$8x_2 + 2x_3 = -7$$

$$3x_1 + 5x_2 + 2x_3 = 8$$

$$6x_1 + 2x_2 + 8x_3 = 26$$

Example 1 – Solution

$$\begin{aligned} 0x_1 + 8x_2 + 2x_3 &= -7 \\ 3x_1 + 5x_2 + 2x_3 &= 8 \\ 6x_1 + 2x_2 + 8x_3 &= 26 \end{aligned}$$

$$\begin{bmatrix} 0 & 8 & 2 & -7 \\ 3 & 5 & 2 & 8 \\ 6 & 2 & 8 & 26 \end{bmatrix}$$

$$\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 \\ 3x_1 + 5x_2 + 2x_3 &= 8 \\ 0x_1 + 8x_2 + 2x_3 &= -7 \end{aligned}$$

$$\begin{bmatrix} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{bmatrix} \left\{ \text{R2} = \text{R2} - 0.5\ \text{R1} \right.$$

$$\rightarrow \begin{bmatrix} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{bmatrix} \left\{ \text{R2} \leftrightarrow \text{R3} \right.$$

Example – Triangular Form & Back Substitution

$$\begin{bmatrix} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -1.5 \end{bmatrix} \begin{cases} \\ R3 = R3 - 0.5R2 \end{cases}$$

That is

$$\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 \\ 8x_2 + 2x_3 &= -7 \\ 3x_3 &= 1.5 \end{aligned}$$

This implies, $x_3 = 0.5$, $x_2 = -1$, $x_1 = 4$

Exercises

Solve the following system by Gauss elimination with partial pivoting

1. $$\begin{aligned} 2x_1 - 8x_2 &= -4 \\ 6x_1 + 2x_2 &= 14 \end{aligned}$$

2. $$\begin{aligned} 4x_1 + 10x_2 - 2x_3 &= -20 \\ -x_1 - 15x_2 + 3x_3 &= 30 \\ 25x_2 - 5x_3 &= -50 \end{aligned}$$

3. $$\begin{aligned} 5x_1 + 3x_2 + x_3 &= 2 \\ -4x_2 + 8x_3 &= -3 \\ 10x_1 - 6x_2 + 26x_3 &= 0 \end{aligned}$$

Gauss-Jacobi Iterative Method

This is an iterative or indirect method, which start from an approximation to the true solution and the cycle of computations being repeated till the required accuracy is obtained.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Gauss-Jacobi Iterative Method

If the diagonal elements are all non-zero, the first equation can be solve for x_1 , the second equation for x_2 , and the third equation for x_3

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \qquad (1)$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \qquad (2)$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}} \qquad (3)$$

Example 2

Solve the set of equations by Gauss-Jacobi iteration starting from [0,0,0,0].

$$\begin{aligned}
10x_1 - 2x_2 - x_3 - x_4 &= 3 \\
-2x_1 + 10x_2 - x_3 - x_4 &= 15 \\
-x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\
-x_1 - x_2 - 2x_3 + 10x_4 &= -9
\end{aligned}$$

First Step

We rewrite the above equations as,

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$$

1^{st} Iteration

Starting Point is X_0 : [0,0,0,0]

Enter these values in the following equations

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$$

We get X_1 : $x_1 = 0.3,\ x_2 = 1.5,\ x_3 = 2.7,\ x_4 = -0.9$

In the next iteration, we will use these values in the same equation

2^{nd} Iteration

$x_1 = 0.3, x_2 = 1.5, x_3 = 2.7, x_4 = -0.9$

Enter these values in the same equations

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$$

And we get

$X_2 : x_1 = 0.78, x_2 = 1.74\ x_3 = 2.7, x_4 = -0.18$

This goes on until the absolute difference between $\|X_i - X_{i+1}\| < \varepsilon$, for some ε

Example 2 – Solution

n	x1	x2	x3	x4
1	0.3	1.5	2.7	-0.9
2	0.78	1.74	2.7	-0.18
3	0.9	1.908	2.916	-0.108
4	0.9624	1.9608	2.9592	-0.036
5	0.9845	1.9848	2.9851	-0.0158
6	0.9939	1.9938	2.9938	-0.006
7	0.9975	1.9975	2.9976	-0.0025
8	0.9990	1.9990	2.9990	-0.0010
9	0.9996	1.9996	2.9996	-0.0004
10	0.9998	1.9998	2.9998	-0.0002
11	0.9999	1.9999	2.9999	-0.0001
12	**1.0**	**2.0**	**3.0**	**0.0**

Exercises

Solve the set of equations by Gauss-Jacobi iteration starting from [1,1,1].

1.
$$4x_1 - x_2 = 21$$
$$-x_1 + 4x_2 - x_3 = -45$$
$$-x_2 + 4x_3 = 33$$

2.
$$4x_1 + 5x_3 = 12.5$$
$$x_1 + 6x_2 + 2x_3 = 18.5$$
$$8x_1 + 2x_2 + x_3 = -11.5$$

Gauss-Seidel Method

This is an iterative or indirect method, which start from an approximation to the true solution and the cycle of computations being repeated till the required accuracy is obtained.

Suppose we have the following system of equations

Ax = b where

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

If the diagonal elements are all non-zero, the first equation can be solve for x_1, second equation for x_2, and third equation for x_3

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \qquad x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \qquad x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Example 3

Solve the set of equations by Gauss-Seidel iteration starting from 0,0,0,0.

$$
\begin{aligned}
10x_1 - 2x_2 - x_3 - x_4 &= 3 \\
-2x_1 + 10x_2 - x_3 - x_4 &= 15 \\
-x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\
-x_1 - x_2 - 2x_3 + 10x_4 &= -9
\end{aligned}
$$

Example 3: Starting Solution

We rewrite the above equations as,

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$$

X_0: [0 0 0 0]

Example 3 – 1st Iteration

X_0: [0 0 0 0]

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \Rightarrow X_1: [0.3\ 0\ 0\ 0]$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \Rightarrow X_1: [0.3\ 1.56\ 0\ 0]$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \Rightarrow X_1: [0.3\ 1.56\ 2.886\ 0]$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \Rightarrow X_1: [0.3\ 1.56\ 2.886\ \text{-}0.1368]$$

X_1: [0.3 1.56 2.886 -0.1368]

Example 3 – 2^{nd} Iteration

X_1: [0.3 1.56 2.886 -0.1368]

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \Rightarrow X_2: [0.8869\ 1.56\ 2.886\ -0.1368]$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \Rightarrow X_2: [0.8869\ 1.9523\ 2.886\ -0.1368]$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \Rightarrow X_2: [0.8869\ 1.9523\ 2.9566\ -0.1368]$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \Rightarrow X_2: [0.8869\ 1.9523\ 2.9566\ -0.0248]$$

X_2: [0.8869 1.9523 2.9566 -0.0248]

Example 3 – Solution

n	x1	x2	x3	x4
1	0.3	1.56	2.886	-0.1368
2	0.8869	1.9523	2.9566	-0.0248
3	0.9836	1.9899	2.9924	-0.0042
4	0.9968	1.9982	2.9987	-0.0008
5	0.9994	1.9997	2.9998	-0.0001
6	0.9999	1.9999	3.0	0.0
7	1.0	2.0	3.0	0.0

Exercises

Use Gauss-Seidel iteration method to obtain the solution starting from 1,1,1.

1. $5x_1+x_2+2x_3=19 \;\; x_1+4x_2-2x_3=-2 \;\; 2x_1+3x_2+8x_3=39$

2. $10x_1+x_2+x_3=6 \;\; x_1+10x_2+x_3=6 \;\; x_1+x_2+10x_3=6$

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Thank you!!