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SS ZC416 Mathematical Foundations for Data Science
Graphs

Objectives

- Introduction to Graphs
  - Undirected & Directed Graphs
  - Simple Graphs & Multigraphs
  - Special Graphs
  - Adjacency Matrix & Incidence Matrix
  - Connectedness
  - Euler Paths & Circuits
  - Hamiltonian Paths & Circuits

Graphs

- An undirected graph G = (V, E) consists of V, a nonempty of vertices or nodes and E, a set of edges. Each edge has either one or two vertices associated with it, called its end points. An edge is said to connect its endpoints.
- A graph with an infinite vertex set is called infinite graph. Other wise it is called finite graph.
- Usually, graphs are represented by pictures, using a point for each vertex and a line for each edge.

Example – Simple Graph

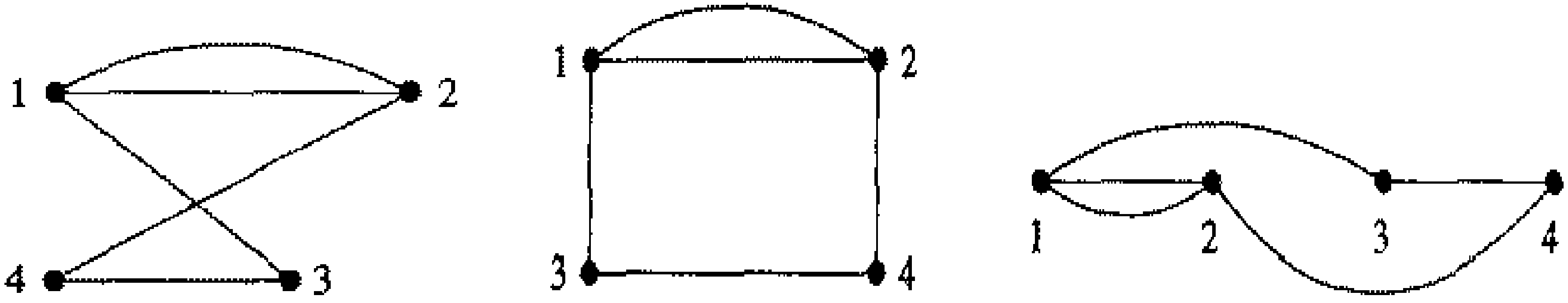
G = (V, E) is an undirected graph where

- Let V = {1, 2, 3, 4} is the set of vertices
- E = {e1 = (1, 2), e2 = (4, 3), e3 = (1, 3), e4 = (2, 4) is the set of undirected edges

Example - Multigraph

G = (V, E) is an undirected graph where

- Let V = {1, 2, 3, 4} is the set of vertices
- E = {e1 = (1, 2), e2 = (4, 3), e3 = (1, 3), e4 = (2, 4), e5 = (1, 2) is the set of undirected edges
- The edge (1, 2) is said to have multiplicity 2



Definitions – Graphs

- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph
- Graphs that may have multiple edges connecting the same vertices called multigraphs
- If there are m different edges associated to the same unordered pair of vertices {u, v} we say it is an edge of multiplicity m
- Edges that connect a vertex to itself are called loops
- Graphs that include loops and possibly multiple edges connecting the same pair of vertices are called pseudographs

Directed Graphs

- A directed graph (V, E) consists of a nonempty set of vertices V and a set of directed edges E each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (*u*, *v*) is said to start at *u* and end at *v*.
- A directed graph with no loops and no multiple edges is called a simple directed graph.
- If there are m directed edges to an ordered pairs (u, v) we say that (u, v) is an edge of multiplicity *m* and such graphs are called multiple directed graphs.
- A graph with both directed and undirected edges is called a mixed graph.

Terminology - Summary

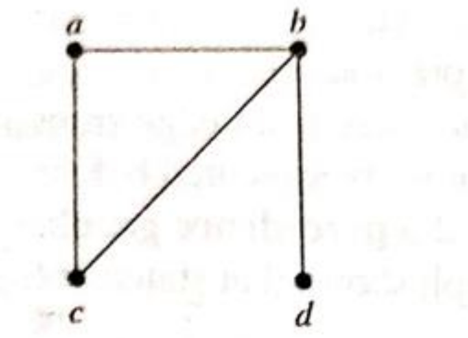
TABLE 1 **Graph Terminology.**

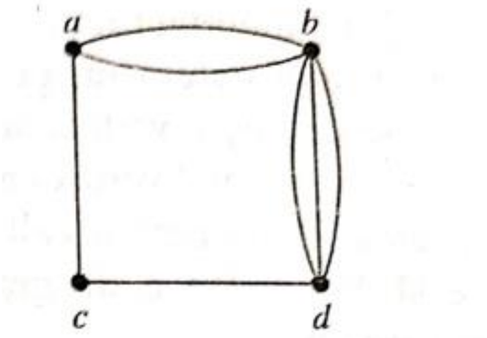
Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

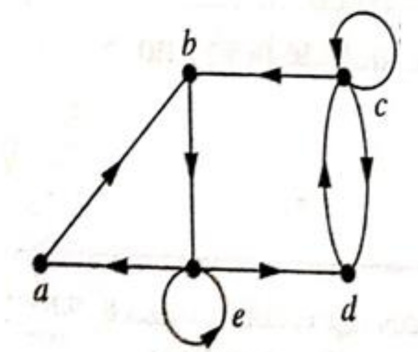
Examples

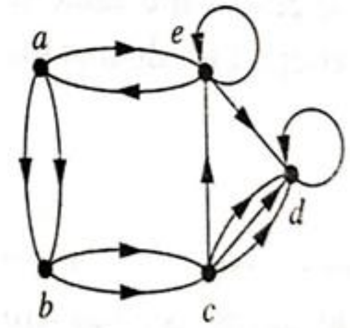
For each graph, determine
- Whether it is a directed or undirected graph
- Whether it has multiple edges or not
- Whether it has loops or not.

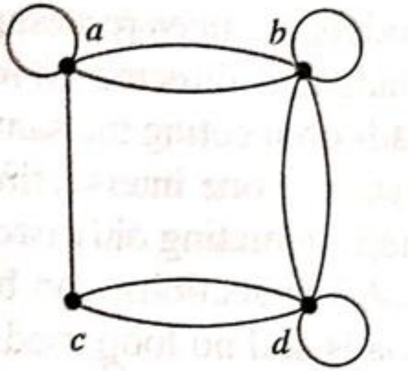
And therefore determine the type of the graph

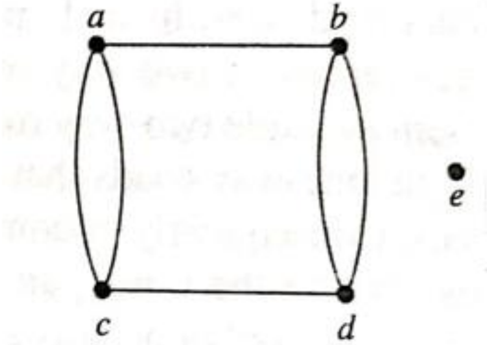


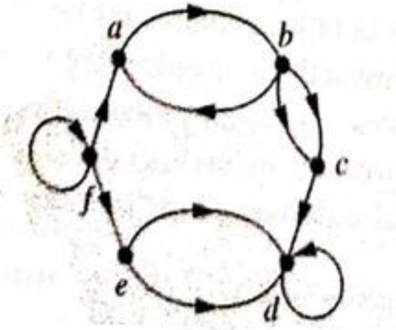












Basic Terminology

- Two vertices u and v in an undirected graph G are called adjacent in G if u and v are endpoints of an edge in G.
- The edge e = {u, v} is stb incident with the vertices u and v.
- The edge e is also said to connect u and v.

Degree of a Vertex

- The degree of a vertex in an undirected graph is the number of edges incident with it.
  - deg(v) = the degree of the vertex
- A vertex with degree 0, is called isolated vertex.
- A vertex is pendant iff it has degree one.
- A loop at a vertex contributes twice to the degree of that vertex.

Handshaking Theorem

Let G = (V, E) be an undirected graph with e edges. Then

$$\sum deg(vi) = 2e$$

Terminology – Digraphs

- If (u, v) is an edge in a directed graph G, u is initial vertex of (u, v) and v is called the terminal or end vertex (u, v).
- In a directed graph, the in-degree of a vertex v, is the number of edges with v as their terminal vertex and is denoted by $deg^{-}(v)$.
- In a directed graph, the out-degree of a vertex v, is the number of edges with v as their initial vertex and is denoted by $deg^{+}(v)$.

Fact

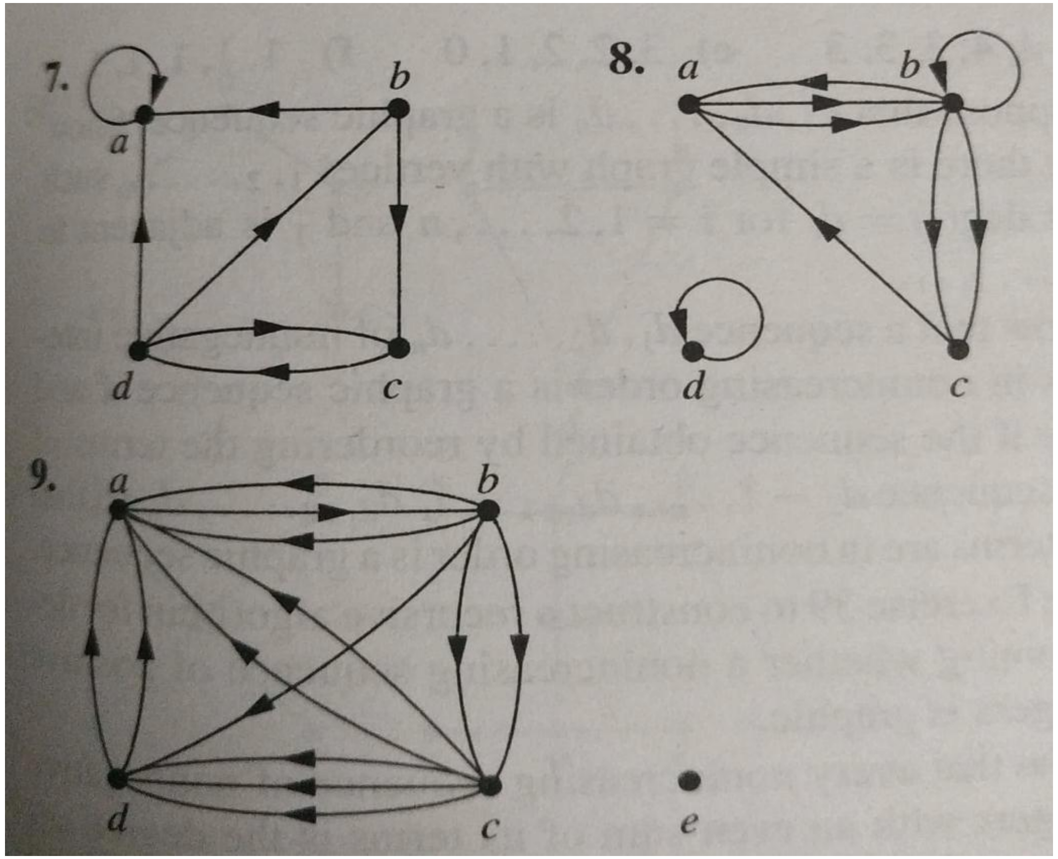
- A directed loop at the vertex v contributes one for its in-degree and one for out-degree.

- Let G = (V, E) be a graph with directed edges. Then

$$\sum \deg^{-}(v) = \sum \deg^{+}(v) = |E|$$

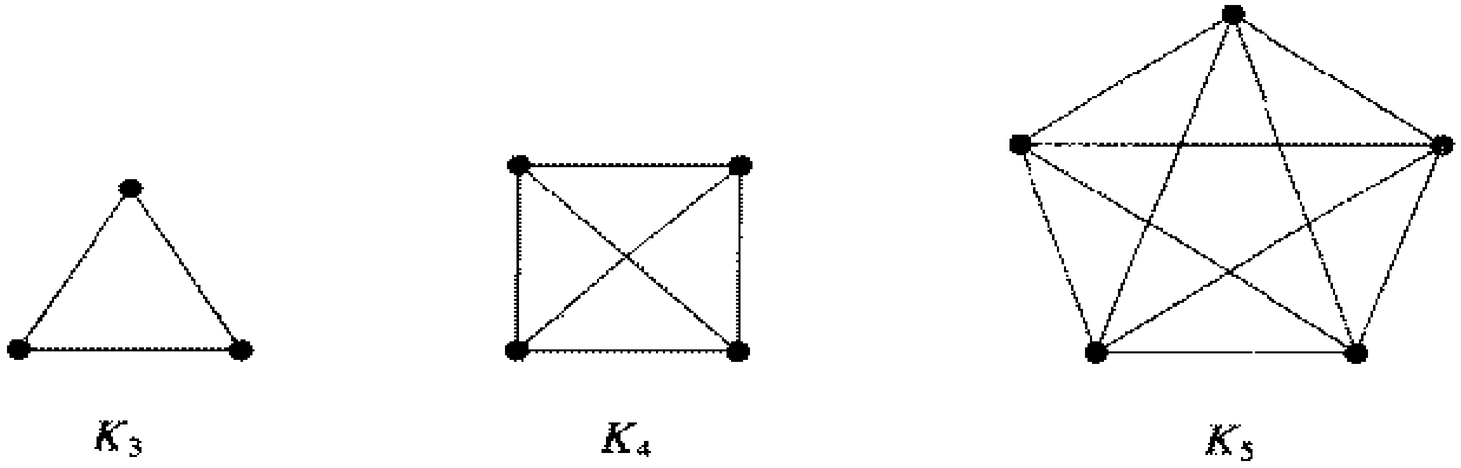
Example

- Find the in-degree and out-degree of each vertex for the given directed multigraph. Determine the sum of in-degree and out-degree of the vertices and show that they are equal to the number of edges.



Complete Graph

The Complete graph on n vertices denoted by k_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

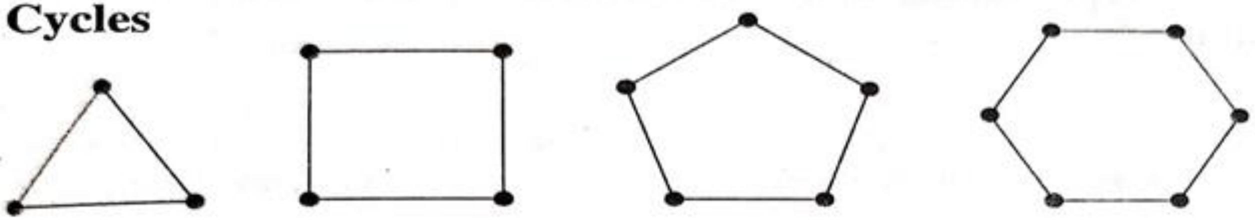


Other Special Graphs

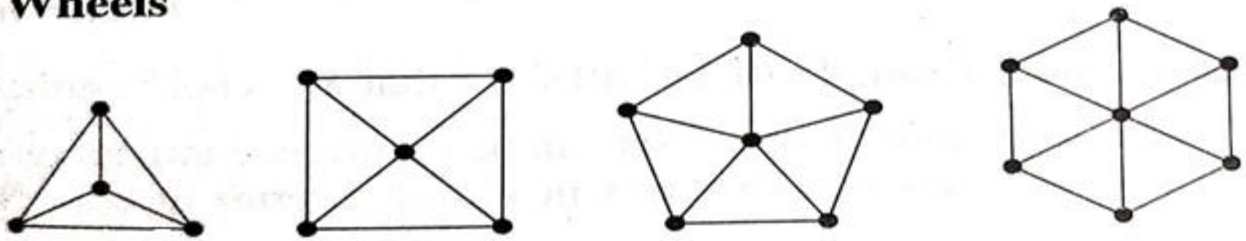
- The Cycle graph $C_n, n \geq 3$ consists of n vertices in which all the edges forms a cycle.
- The Wheel graph $W_n,$ is obtained by adding an additional vertex to the cycle graph $C_n\, n \geq 3,$ and connect this new vertex to each of the n vertices in C_n by new edges.
- The n-dimensional hypercube or n-cube denoted by Q_n is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if the bit strings they represent differ only in one position

Examples

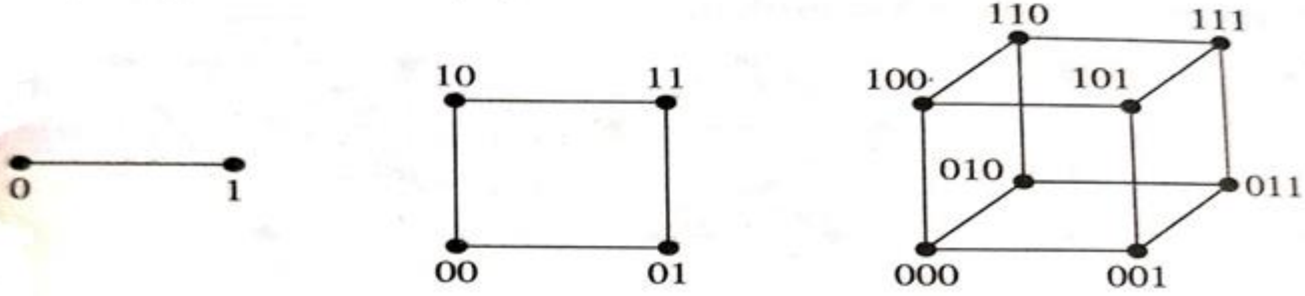
Cycles



Wheels



***n*-cube Q_n for $n = 1, 2,$ and 3.**



Bipartite Graphs

- A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 and no two vertices are adjacent in V_1 or in V_2
- We call the pair (V_1, V_2) a bipartition of the vertex set V of G.
- A simple graph G is called **complete bipartite graph** $K_{m,n}$ if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 of m and n vertices respectively such that there is an edge between two vertices iff one vertex is in the first subset and the other vertex is in the second subset and no two vertices are adjacent in V_1 or in V_2.

Examples

Draw the graphs: K_7, $K_{2,6}$, C_7, W_7

Complete Bipartite Graphs

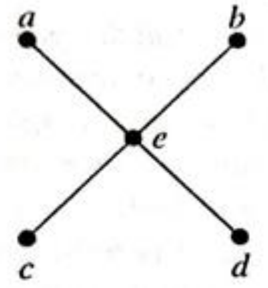
A simple graph is bipartite iff it is possible to assign one of the two colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

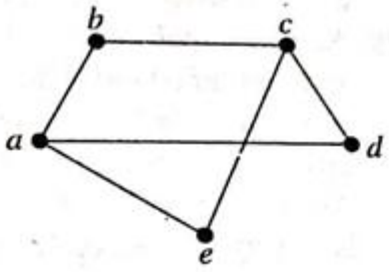
To see whether a graph is bipartite

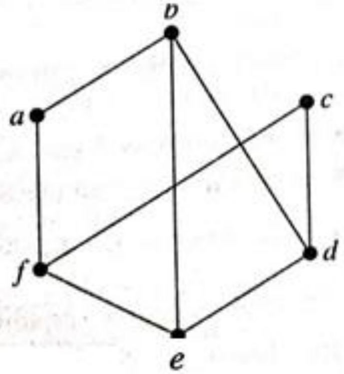
1. Start with one vertex & colour it say Red
2. All vertices adjacent to this vertex must be coloured a different colour, say Blue
3. The vertices adjacent to these Blue vertices must be coloured Red.
4. And so on
5. If there is a clash, then it is not bipartite

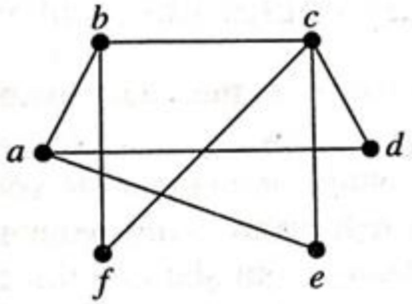
Examples

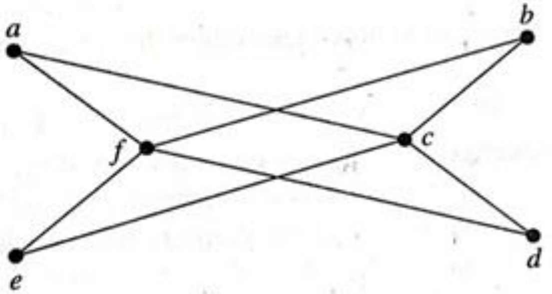
Determine whether the following graphs are bipartite









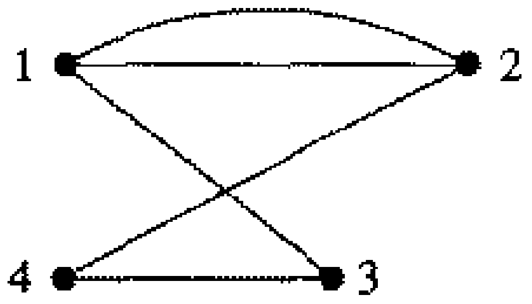


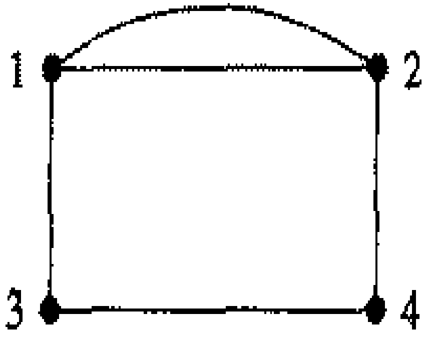
Degree Sequence

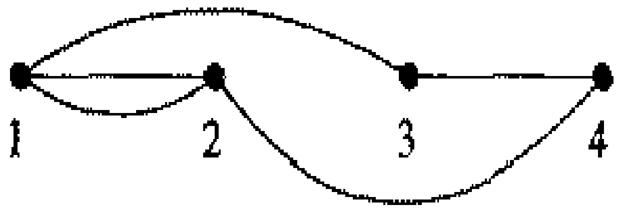
The degree sequence of an undirected graph is the sequence of the degrees of the vertices of the graph in non increasing order.

Exercise

Determine the degree sequence of the following graphs







Regular & Complementary Graphs

- A simple graph is called regular if every vertex of this graph has the same degree.
- A regular graph is called n-regular if every vertex in this graph has degree n.
- Let G = (V, E) be a graph. $G^C = (V, E^C)$, is the complementary graph iff Two vertices are adjacent in G if they are not adjacent in G^C.
- If $K_{|V|}$ is the complete graph on V, then the union of E and E^C = the edges in $K_{|V|}$

Converse of a Digraph

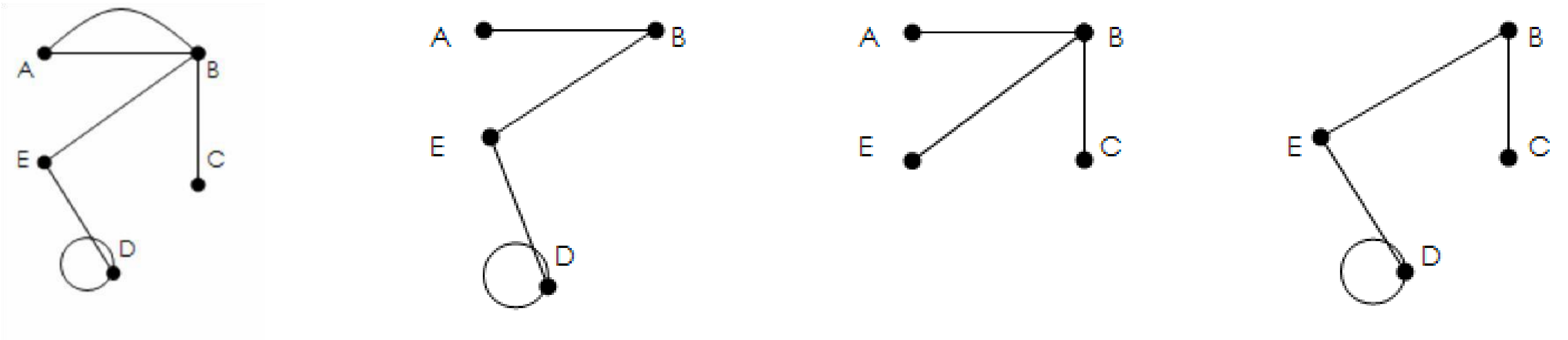
- The converse of a directed graph G = (V, E) denoted G^{conv} is the directed graph (V, F) where the set F of edges of G^{conv} is obtained by reversing the directions of each edge in E.

- $(G^{conv})^{conv} = G$ where G is a directed graph.

Subgraph

- A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.
- A subgraph H of G is a proper subgraph of G if $H \neq G$

Example: (d), (e) and (f) are the sub graphs of (a).



(a) (d) (e) (f)

Union of Two Graphs

- The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

- The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Adjacency Matrices

Suppose G (V, E) is a simple graph where |V| = n

- The Adjacency matrix A of G is an n x n 0-1 matrix A = [a_{ij}]

  where $a_{ij} = \begin{cases} 1, \text{if} \left(v_i, v_j\right) \text{ is an edge in G} \\ 0, \text{if } (v_i, v) \text{ is not an edge in G} \end{cases}$

Observations

- All undirected graphs, have symmetric adjacency matrices.
- Adjacency matrices can also be used to represent undirected graphs with loops.
- Adjacency matrices can also be used to represent undirected graphs with multiple edges, the entry in {i, j} is equals to number of edges that are associated to the two vertices.

Adjacency of a Digraph

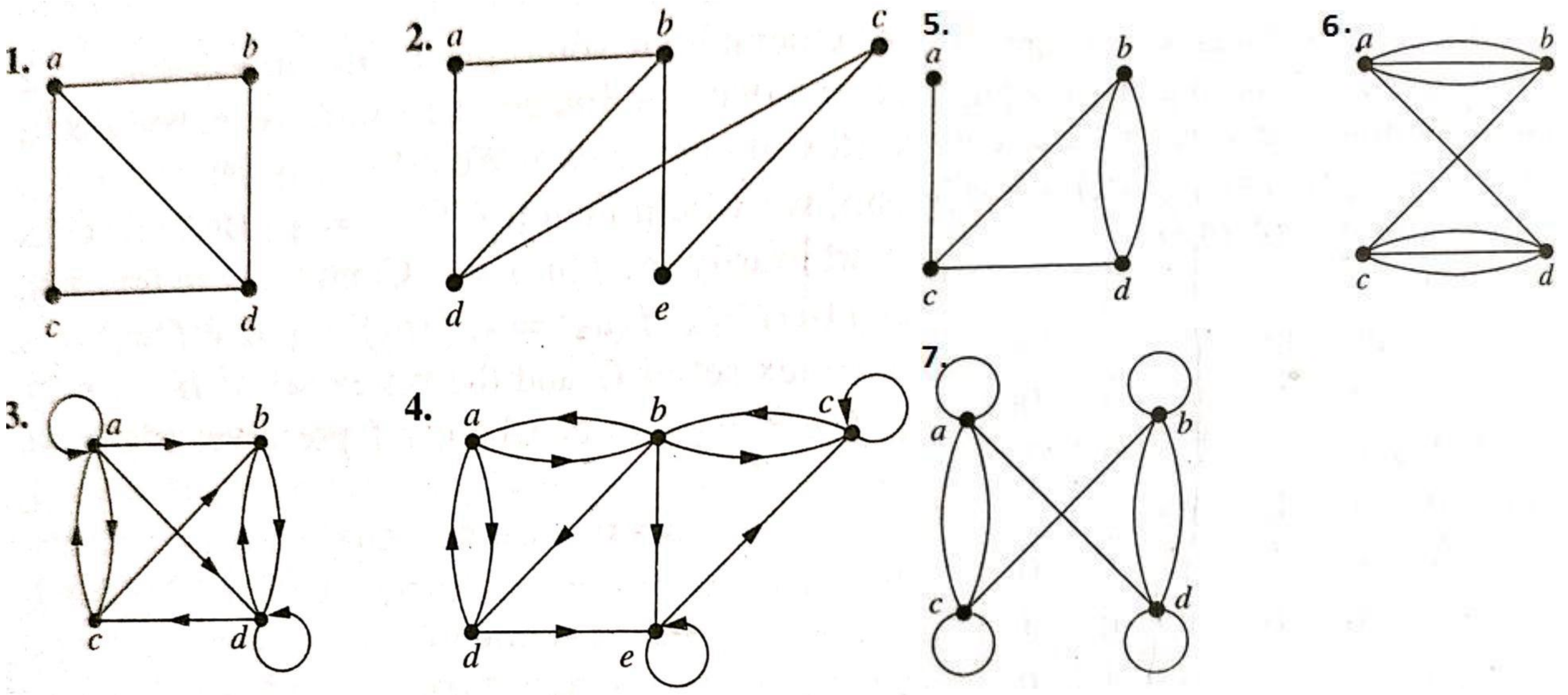
The adjacency matrix for a directed graph is given by A = [a_{ij}], where

$$a_{ij} = \begin{cases} 1, \text{if } (v_i, v_j) \text{ is an edge in G} \\ 0, \text{if } (v_i, v) \text{ is not an edge in G} \end{cases}$$

Adjacency matrices can also be used to represent directed multigraphs where (i, j)th entry is equal to the number of edges that are associated to $(a_i,\ a_j)$.

Examples

Represent the following graphs with an adjacency matrix.



Examples

Draw an undirected graph with the given adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix

Let G (V, E) be an undirected graph. Suppose that

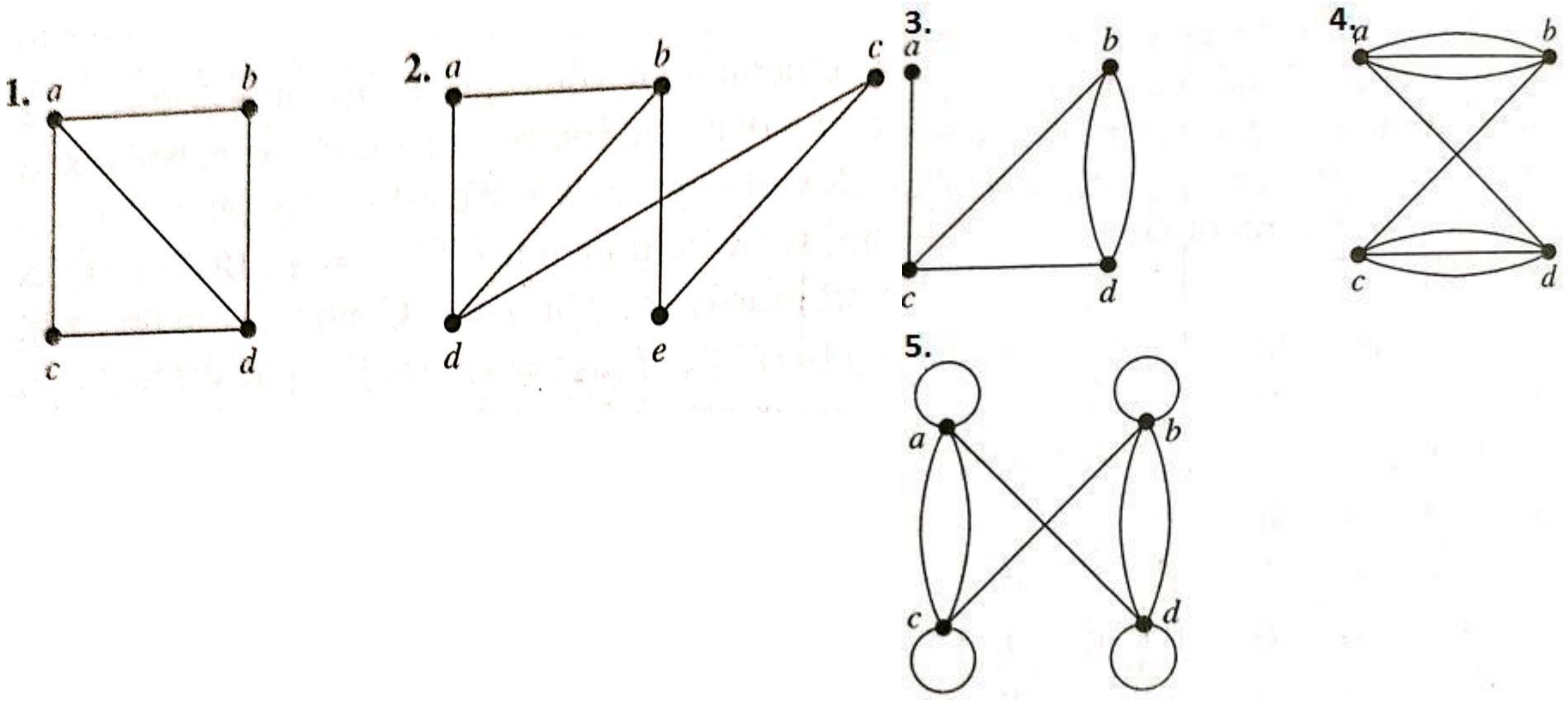
- $v_1, v_2, v_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, v_n$ are the vertices and
- $e_1, e_2, e_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, e_m$ are the edges of G.

The incidence matrix is an n x m matrix $I = [m_{ij}]$ where

$$\mathrm{m_{ij}} = \begin{cases} 1, \text{if } e_j is\ incident\ on\ v_i \\ 0, \text{Otherwise} \end{cases}$$

Examples

Use incidence matrix to represent the given graphs.



Isomorphism of Graphs

The simple graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic if there is a one to one function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(\text{a})$ and $\text{f}(\text{b})$ are adjacent in G_2 for all a and b in V_1.

- Such a function f is called an isomorphism.

Invariant Properties under Isomorphism

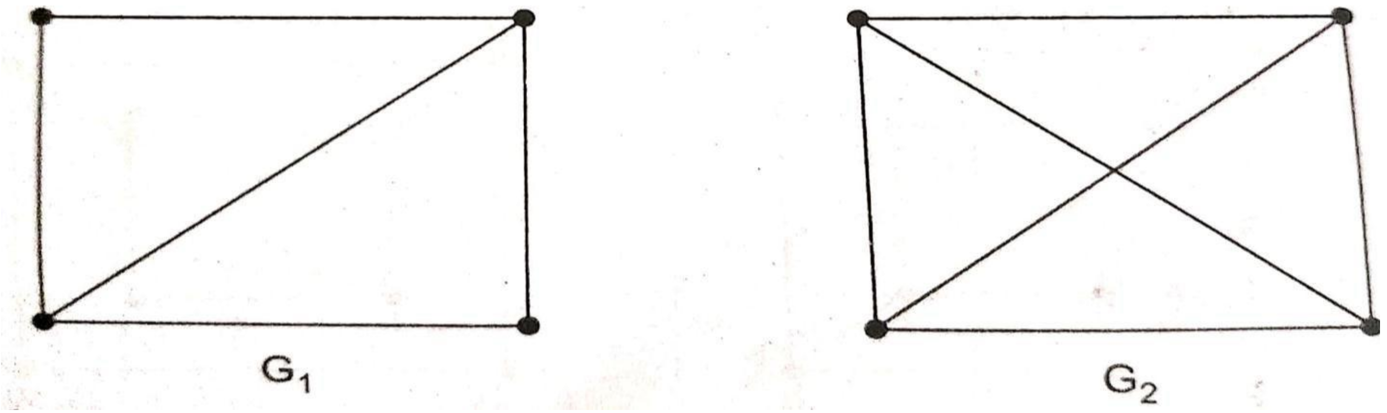
(i) Same number of vertices.
(ii) Same number of edges.
(iii) Same degree sequences.
(iv) Same number of cycles of a particular length.

If any one of the above criterion fails the graphs are not isomorphic. The above rules are only necessary but not sufficient for providing two graphs are isomorphic.

Examples

Determine whether the following graphs are isomorphic.

1.



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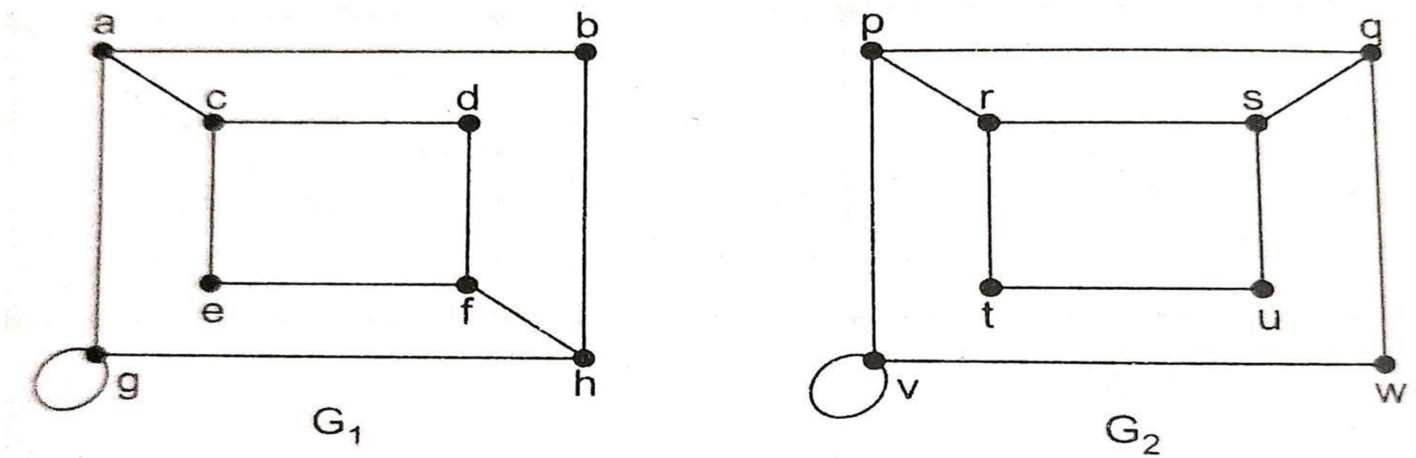
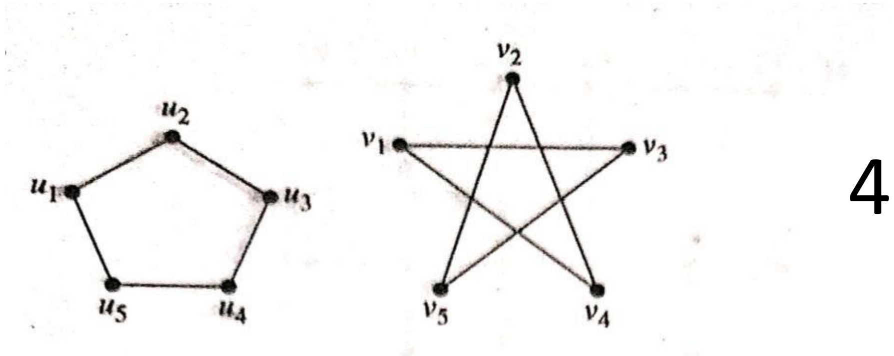


Isomorphic Graphs?

3.



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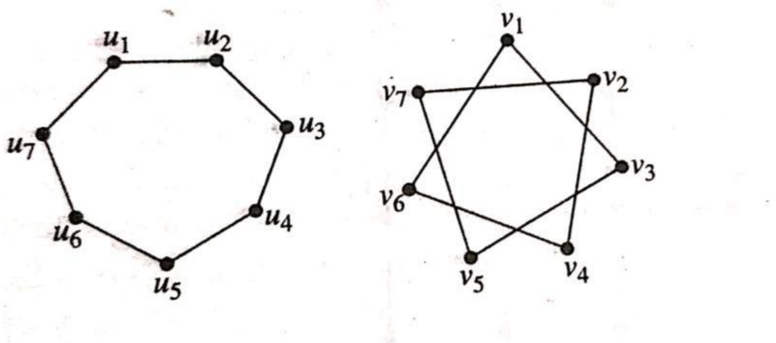

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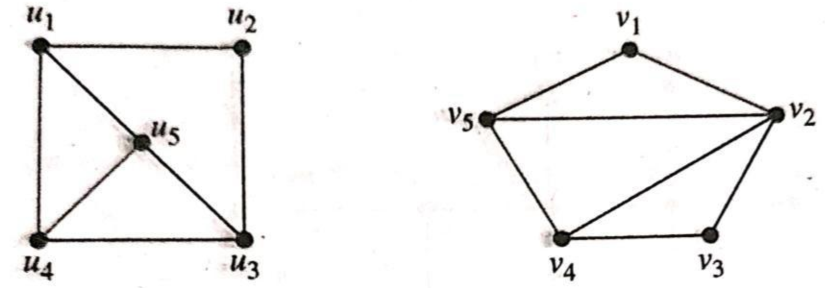

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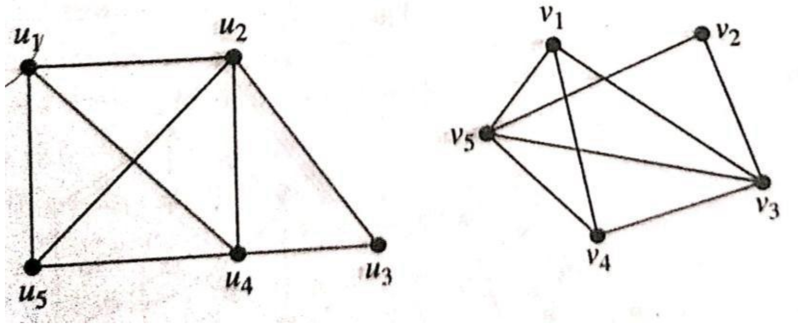
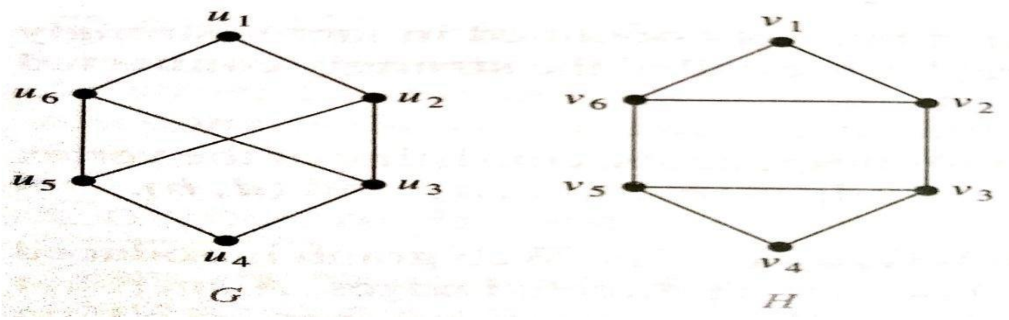


7.



Connectedness

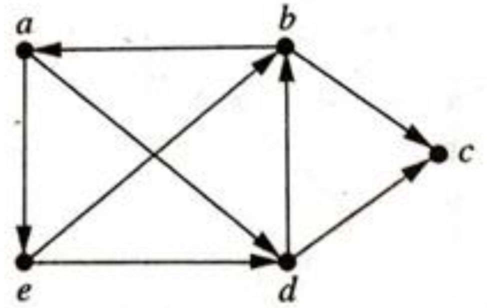
- An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.
- A connected component of a graph G is a connected subgraph of G and is not a proper subgraph of another connected subgraph of G.

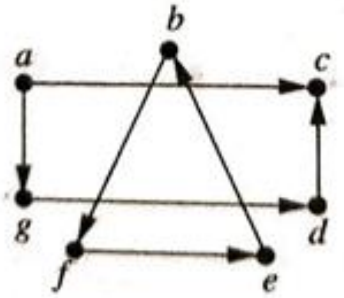
Connectedness in Digraphs

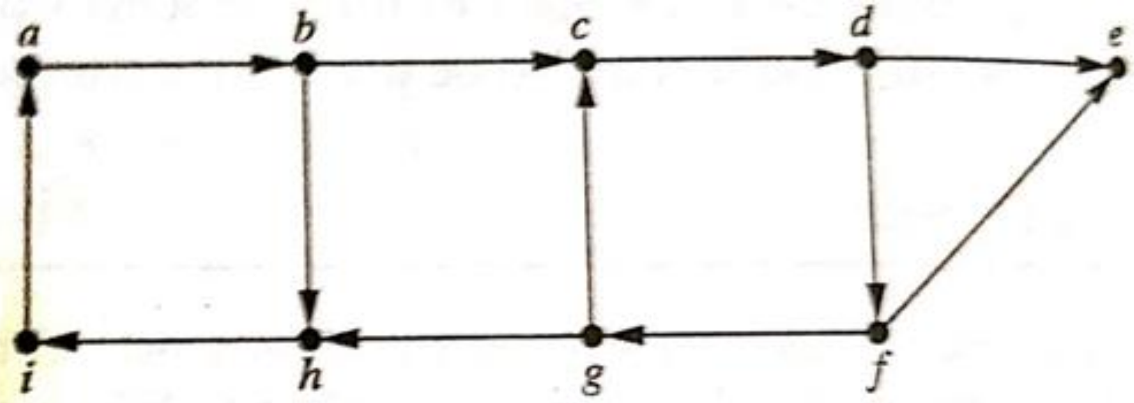
- A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
- A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

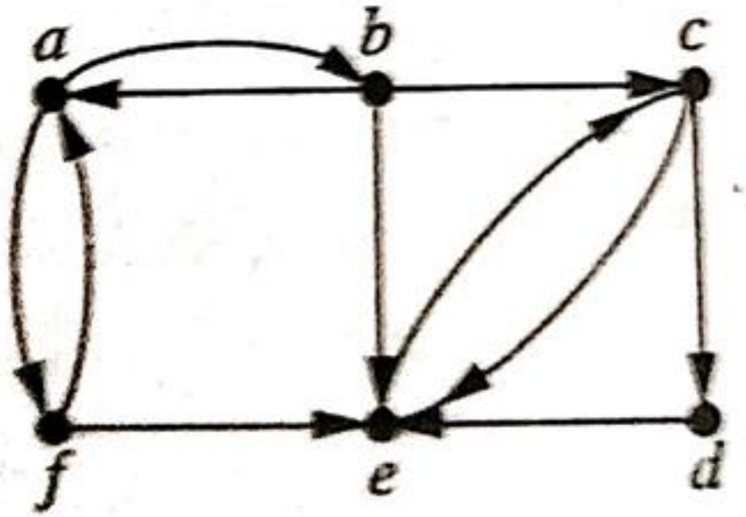
Examples

Which graph is strongly connected, weakly connected, or disconnected.









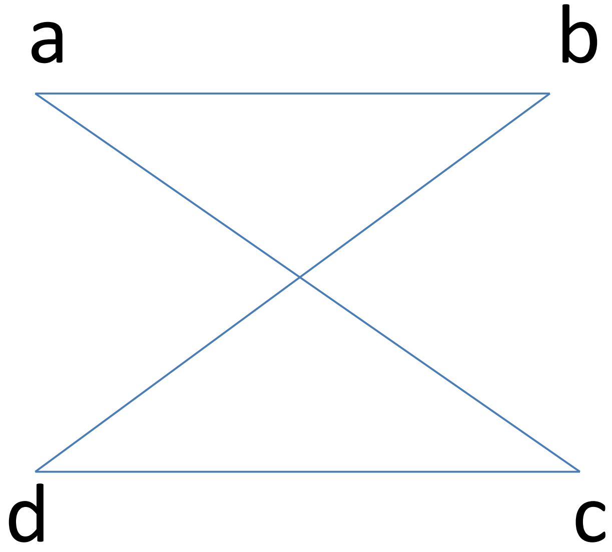
Number of Paths from v_i to v_j

Let G be a graph with adjacency matrix A with directed or undirected edges, with multiple edges and loops allowed.

- The number of different paths of length r from v_i to v_j, where r is a positive integer equals the (i, j)th entry of A^r.

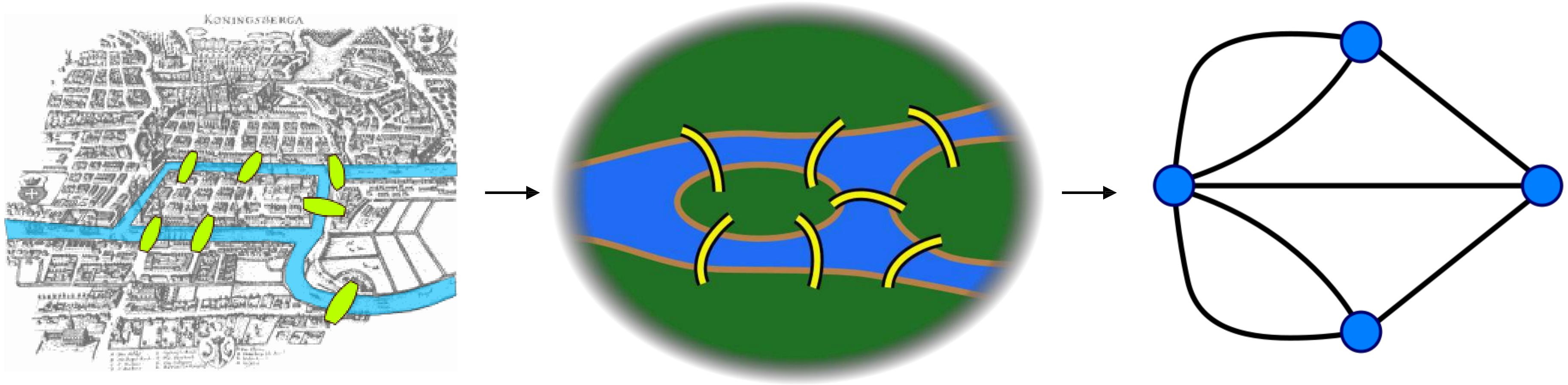
Example

How many paths of length four are there from a to d in graph G?



Seven Bridges of Königsberg

Is it possible to walk with a route that crosses each bridge exactly once?



Euler Paths and Circuits

- An Euler circuit in a graph G is a simple circuit containing every edge of G.
- An Euler path in G is a simple path containing every edge of G.
- A connected multigraph with at least two vertices has an Euler circuit iff each of its vertices has even degree.
- A connected multigraph has an Euler path but not an Euler circuit iff it has exactly two vertices of odd degree.

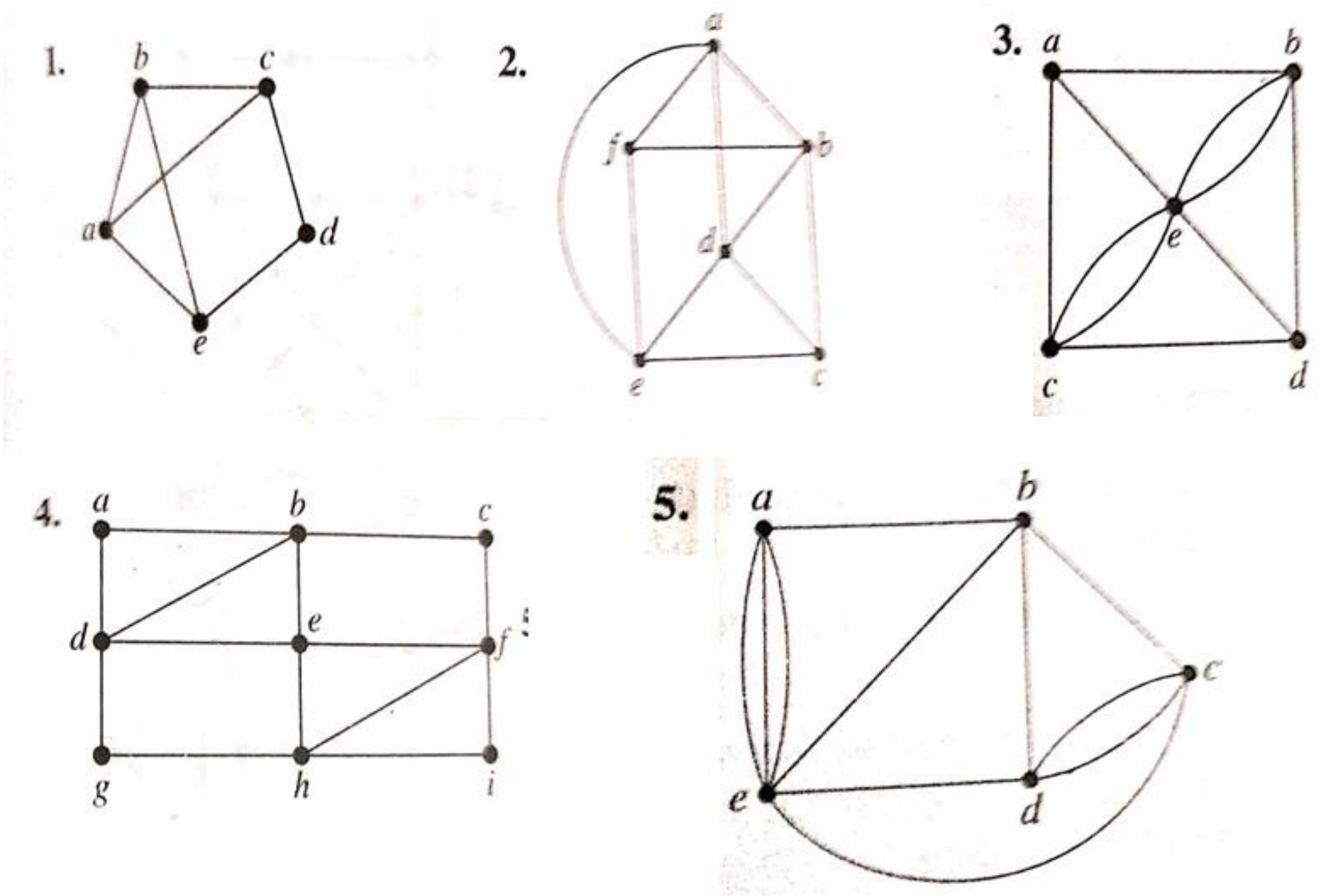
Note

A circuit is same as a cycle

A path or a circuit is simple if it does not contain any edge more than once

Examples

Determine whether it has Euler Path or Circuit. If yes, construct it.



Hamilton Paths and Circuits

- A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path.
- A simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.

Dirac's Theorem & Ore's Theorem

Dirac's theorem

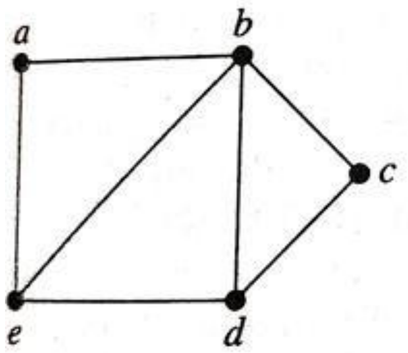
- If G is a simple graph with n vertices with $n \geq 3$, such that the $\deg(v) \geq n/2$ for all v, then G has a Hamilton circuit.

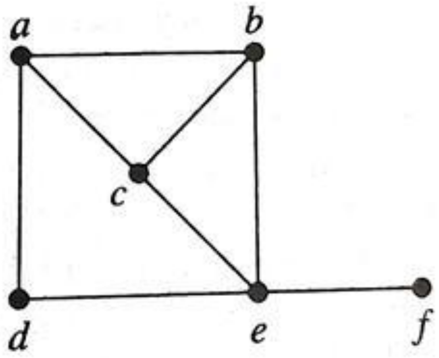
Ore's theorem

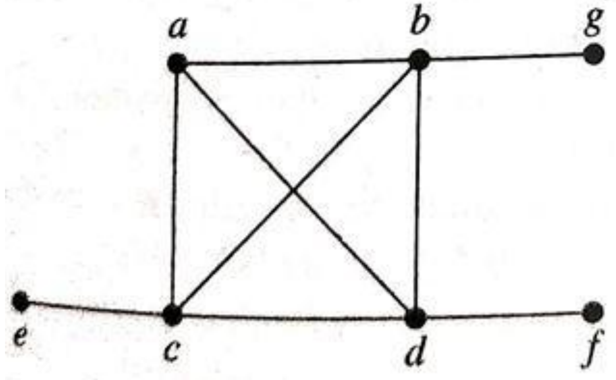
- If G is a simple graph with n vertices with $n \geq 3$, such that $\deg(u) + \deg(v) \geq n$ for every pair of non adjacent vertices u and v in G, then G has a Hamilton circuit

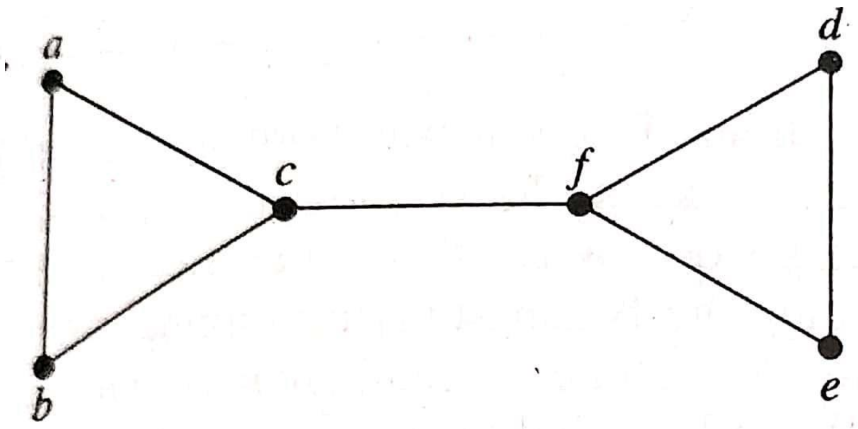
Examples

Does the graph have a Hamilton path or Hamilton circuit? If so, find such a path









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Thank you!!