Eastern Economy Edition

Differential Equations and Their Applications

Second Edition

Zafar Ahsan

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Preface

The first edition of *Differential Equations and Their Applications* has been favourabley received by a large number of users. The second edition reflects the suggestions and experiences of these users to whom I am extremely thankful.

The spirit of this new edition is same as that of the previous one. As far as possible, the efforts have been made to keep this new edition free from typographic and other errors. Few additions have been made, which are in accordance wih the UGC curriculum (2001) in mathematics for all Indian universities. In Chapter 5, the series solution of a differential equation has been included and the method of Frobenius for solving a differential equation has been discussed in detail with the help of a number of examples. The series solutions for some of the differential equations of special functions, namely, Bessel, Legendre and hypergeometric equations have also been derived in this chapter. The major change of this edition is the inclusion of calculus of variations as Chapter 10. The calculus of variations has a number of applications in any field of study, where optimization is needed, e.g. the path of a guided missile, economic growth, pest control, spread of a contagious disease, cancer chemotherapy and immune system, etc. This chapter deals with the methods of finding the extremals of a given functional and thus leads to the solution of differential equations. Some of the applications of the calculus of variations have also been given.

For a successful publication of the book many other persons are involved, I wish to thank all of them personally: my friend Professor J.L. Lopez Bonilla, Instituto Politecnico Nacional, Mexico, for fruitful suggestions and pointing out a few errors in the first edition; my friend and colleague Professor Mursaleen, for critically reading the contents of Chapter 10 and correcting a number of errors (these fellows are, of course, not responsible for any error that remains) and finally my publisher, Prentice-Hall of India, in particular, the editorial and production team for their nice efforts to bring out the new edition so quickly. Lastly, I wish to thank my family members for lots of encouragement and patience they have shown during the preparation of the material for this new edition.

Any suggestions for further improvement shall be welcomed.

Zafar Ahsan

Preface to the First Edition

It is an incontrovertible fact that differential equations form the most important branch of modern mathematics and, in fact, occupy the position at the centre stage of both pure and applied mathematics. This is obvious because the mathematical understanding of any physical situation usually consists of the following steps:

1. Understanding the various parameters of the situation and then making a rough mathematical model;
2. Posing a corresponding precise mathematical problem, and analyzing it, trying to find an exact or approximate solution;
3. Comparing the result with the experimental data to check the validity of the model.

Step 2 is nothing but forming the corresponding differential equations and then evolving techniques of pure mathematics to arrive at the solution.

It is also obvious that differential equations form the basis of applied mathematics. As far as their role in pure mathematics is concerned, attempts to get their exact solutions lead us to the Existence Theorems and the Theory of Functions, Differential Geometry, and deep results in Functional Analysis.

It is impossible to describe all the roles played by differential equations in pure and applied mathematics in a single book. However, the author's objective in writing this book is two-fold:

1. To provide the reader with an easier and systematic way of solving ordinary and partial differential equations;
2. To find the possible applications of differential equation in such diverse areas as biology, physiology medicine and economics, along with the applications in physical and engineering sciences.

In Chapter 1, a brief introduction to the definitions and terminology is presented. This is followed by a discussion about how differential equations arise naturally from geometrical and physical points of view.

Chapter 2 discusses about the different methods for solving differential equations of first order and first degree. Chapter 3 covers differential equations of first order but not of first degree.

Chapter 4 includes a wide variety of problems which are chosen from different disciplines. The applications of first-order differential equations to carbon dating and radioactivity, mixture problem, estimation of time of death, absorption of drugs in cell, the problem of epidemiology, the motion of a rocket, the path of a guided missile, electric circuits, chemical reactions, are just a few of the attractions of this chapter.

In Chapter 5, the higher-order linear differential equations, the concepts of solutions, and the methods of obtaining these solutions are analyzed. Also, a brief account is given regarding the most frequently occurring second-order differential equations that lead to the development of an exciting branch of mathematics, viz. Special Functions.

Chapter 6 covers the applications of higher-order differential equations in rectilinear motion, resonance, the hanging cable, civil and electrical engineering, economics, cardiology and detection of diabetes.

Chapter 7 deals with simultaneous differential equations and their applications. It first focusses on the basic definition and the methods of solution and then on their applications.

The Laplace transforms and their applications constitute Chapter 8. The chapter gives the solution of the tautochrone problem and analyzes the theory of automatic control and servomechanics, along with some other applications of transform methods for solving differential equations.

Chapter 9 provides an elementary treatment of partial differential equations and their applications. The applications covered include vibration of string and membranes, heat conduction equation, transmission lines and nuclear fission.

The emphasis throughout the text is on solving problems. At the end of each chapter, a carefully selected set of problems is given. The answers to each problem set are provided at the end of the book. More than 330 problems have been solved completely while the number of unsolved problems is 480.

This book can be used as a textbook for the undergraduate and postgraduate students of science and is also suitable for engineering students of various universities.

In preparing the book, I have consulted many standard works. I am indebted to the authors of those works. I wish to express my gratitude to (Late) Prof. S. Izhar Husain whose excellent training has stimulated and enlivened my interest in mathematics and inspired me to write this book.

I also wish to thank my colleague and friend, Dr. M.M.R. Khan, whose suggestions and comments proved fruitful and clarified many points. My thanks are also due to Mr. S. Fazal Hasnain Naqvi for the excellent typing of the manuscript. The partial financial assistance provided by Aligarh Muslim University for the preparation of the manuscript is gratefully acknowledged. Finally, I wish to express my sincere thanks to the publisher, Prentice-Hall of India, in particular, the editorial and production team, for their meticulous processing of the manuscript.

Any suggestions or comments for improving the contents will be warmly appreciated.

Zafar Ahsan

1

Basic Concepts

1.1 INTRODUCTION

In this text, we deal with differential equations and their applications. It is well known that differential equations are very useful to students of applied sciences. It may be worthwhile to list the various differential equations which have arisen in the different fields of engineering and the sciences. Such listing is intended to indicate to students that differential equations can be applied to many practical fields, even though it must be emphasized that the subject is of great interest in itself. The differential equations have been compiled from those occurring in advanced textbooks and research journals. Some of these equations are as follows:

$$xy'' + y' + xy = 0 \tag{1}$$

$$\frac{d^2x}{dt^2} = -kx \tag{2}$$

$$\frac{d^2I}{dt^2} + 5\frac{dI}{dt} + 8I = 100\sin 20t \tag{3}$$

$$EIy^{\text{iv}} = w(x) \tag{4}$$

$$y' = \frac{W}{H}\sqrt{1+y'^2} \tag{5}$$

$$v + m\frac{dv}{dm} = v^2 \tag{6}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{7}$$

$$\frac{\partial V}{\partial t} = k\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \tag{8}$$

$$\frac{\partial^2 V}{\partial t^2} = a^2\frac{\partial^2 V}{\partial x^2} \tag{9}$$

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = F(x, y) \tag{10}$$

Equation (1) arises in the field of mechanics, heat, electricity, aerodynamics, stress analysis, and so on.

Equation (2) has a wide range of application in the field of mechanics, in relation to simple harmonic motion, as in small oscillations of a simple pendulum. It could, however, arise in many other allied areas.

Equation (3) determines the current I as a function of time t in an alternating current circuit.

Equation (4) is an important equation in civil engineering in the theory of bending and deflection of beams.

Equation (5) arises in problems concerning suspension cables.

Equation (6) occurs in problems on rocket flight.

Equation (7) is the famous Laplace's equation which occurs in heat, electricity, aerodynamics, potential theory, gravitation and many other fields.

Equation (8) is found in the theory of heat conduction as well as in the diffusion of neutrons in an atomic pile for the production of nuclear energy. It also occurs in the study of Brownian motion.

Equation (9) is used in connection with the vibration of strings, bars, membranes, as well as in the propagation of electric signals and nuclear reactors.

Equation (10) is widely used in the theory of stress analysis (in the theory of slow motion of viscous fluid and the theory of an elastic body).

These are a few of the many equations which could occur and a few of the fields from which they are taken. Studies of differential equations such as these by pure mathematicians, applied mathematicians, theoretical and applied physicists, chemists, engineers and other scientists throughout the years have led to the conclusion that there are certain definite methods by which many of these equations can be solved. The history of these discoveries is, in itself, extremely interesting. However, there are many unsolved equations; some of them are of great importance. The use of modern giant calculating machines, in determining the solution of such equations which are vital for research involving national defence as well as many other endeavours, is still in progress.

It is one of the aims of this book to provide an introduction to some of the important real life problems appearing in many areas of science with which most of the researchers should be acquainted.

1.2 DEFINITION AND TERMINOLOGY

1.2.1 Differential Equations

An equation involving independent and dependent variables and the derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Apart from Eqs. (1)–(10), the following relations are also some of the examples of differential equations:

$$\frac{dy}{dx} = \sin x + \cos x \tag{11}$$

$$y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{\dfrac{dy}{dx}} \tag{12}$$

$$k\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \tag{13}$$

$$y = x\frac{dy}{dx} + k\left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right] \tag{14}$$

$$\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^4 = e^t \tag{15}$$

$$\frac{\partial^3 v}{\partial t^3} = k\left(\frac{\partial^2 v}{\partial x^2}\right)^2 \tag{16}$$

There are two main classes of differential equations:

(i) Ordinary differential equations.
(ii) Partial differential equations.

Ordinary differential equations. A differential equation which involves derivatives with respect to a single independent variable is known as an *ordinary differential equation.*

Equations (1)–(6) and (11)–(15) are examples of ordinary differential equations.

Partial differential equations. A differential equation which contains two or more independent variables and partial derivatives with respect to them is called a *partial differential equation.* Equations (7)–(10) and (16) are examples of partial differential equations.

1.2.2 Order of Differential Equations

The order of the highest order derivative involved in a differential equation is called the *order* of a differential equation.

Equations (6), (11), (12) and (14) are of the first order; Eqs. (1)–(3), (5), (7)–(9) and (13) are of the second order; while Eqs. (15) and (16) and Eqs. (4) and (10) are equations of the third and fourth orders, respectively.

1.2.3 Degree of a Differential Equation

The *degree* of a differential equation is the degree of the highest order derivative present in the equation, after the differential equation has been made free from the radicals and fractions as far as the derivatives are concerned.

Equations (1)–(11), except Eq. (5), are of degree one. Equations (5) and (12)–(14) can respectively be written as

$$(y'')^2 = \frac{W^2}{H^2}(1 + y'^2)$$

$$y\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + k$$

$$k^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

$$\left(y - x\frac{dy}{dx}\right)^2 = k^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

The above equations are of the second degree.

1.3 LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS

A differential equation in which the dependent variables and all its derivatives present occur in the first degree only and no products of dependent variables and/or derivatives occur is known as a *linear differential equation.* A differential equation which is not linear is called a *nonlinear differential equation.* Thus, Eq. (11) is a linear equation of order one and Eqs. (2) and (7) are linear equations of order two. Equations (12)–(16) are nonlinear equations.

1.4 SOLUTION OF A DIFFERENTIAL EQUATION

A *solution* of a differential equation is a relation between the dependent and independent variables, not involving the derivatives such that this relation and the derivatives obtained from it satisfies the given differential equation. For example, $y = ce^{2x}$ is a solution of the differential equation $dy/dx - 2y = 0$, because $dy/dx = 2ce^{2x}$ and $y = ce^{2x}$ satisfy the given differential equation.

1.5 ORIGINS AND FORMATION OF DIFFERENTIAL EQUATIONS

In the discussion that follows we shall see how specific differential equations arise not only out of consideration of families of geometric curves, but also how differential equations result from an attempt to describe, in mathematical terms, physical problems in science and engineering. It would not be too presumptive to state that differential equations form the basis of subjects such as physi·s and electrical engineering, and even provide an important working tool in such diverse fields as biology, physiology, medicine, statistics, sociology, psychology and economics. Both theoretical and applied differential equations are active fields of current research. Several of the examples and problems in this section will serve as previews of topics discussed in Chapters 4 and 7–9.

1.5.1 Differential Equation of a Family of Curves

Suppose we are given an equation containing n arbitrary constants. Then by differentiating it successively n times we get n equations more containing n arbitrary constants and derivatives. Now by eliminating n arbitrary constants from the above $(n + 1)$ equations and obtaining an equation which involves derivatives upto the nth order, we get a differential equation of order n. We now work out in detail some examples to illustrate the method of forming differential equations.

Example 1.1 Find the differential equation of the family of curves $y = c_1e^{2x} + c_2e^{-2x}$, where c_1 and c_2 are arbitrary constants.

Solution Given

$$y = c_1e^{2x} + c_2e^{-2x} \tag{17}$$

Differentiating Eq. (17) twice with respect to x, we get

$$\frac{dy}{dx} = 2c_1e^{2x} - 2c_2e^{-2x} \tag{18}$$

$$\frac{d^2y}{dx^2} = 4c_1e^{2x} + 4c_2e^{-2x} = 4(c_1e^{2x} + c_2e^{-2x}) \tag{19}$$

From Eqs. (17) and (19), we obtain

$$\frac{d^2y}{dx^2} - 4y = 0 \tag{20}$$

Thus, the two arbitrary constants c_1 and c_2 have been eliminated from Eqs. (17)–(19). Hence Eq. (20) is the required equation of the family of curves given by Eq. (17).

Example 1.2 Find the differential equation corresponding to the family of curves $y = c(x - c)^2$, where c is an arbitrary constant.

Solution Given

$$y = c(x - c)^2 \tag{21}$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 2c(x - c) \tag{22}$$

or

$$\left(\frac{dy}{dx}\right)^2 = 4c^2(x-c)^2 \tag{23}$$

Dividing Eq. (23) by (21), we obtain

$$\frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 4c \quad \text{or} \quad c = \frac{1}{4y}\left(\frac{dy}{dx}\right)^2$$

Substituting this value of c in Eq. (22), we get

$$\frac{dy}{dx} = 2\cdot\frac{1}{4y}\left(\frac{dy}{dx}\right)^2\left[x - \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\right]$$

or

$$2y = \frac{dy}{dx}\left[x - \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\right]$$

or

$$8y^2 = 4xy\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3$$

which is the required differential equation for the family of curves (21).

Example 1.3 Find the differential equation that describes the family of circles passing through the origin.

Solution The general form of the circles passing through the origin is

$$(x-h)^2 + (y-k)^2 = \left(\sqrt{h^2+k^2}\right)^2$$

or

$$x^2 - 2xh + y^2 - 2ky = 0 \tag{24}$$

Using implicit differentiation twice, we find

$$x - h + yy' - ky' = 0 \tag{25}$$

or

$$1 + yy'' + (y')^2 - ky'' = 0 \tag{26}$$

Now, Eq. (24) yields

$$h = \frac{x^2 + y^2 - 2ky}{2x}$$

Putting this in Eq. (25), we get

$$x - \frac{x^2 + y^2 - 2ky}{2x} + yy' - ky' = 0 \tag{27}$$

Now, solving Eq. (27) for k, we obtain

$$k = \frac{x^2 - y^2 + 2xyy'}{2(xy' - y)} \tag{28}$$

Substituting this value in Eq. (26) and simplifying, we get the following nonlinear differential equation:

$$1 + yy'' + (y')^2 - \frac{x^2 - y^2 + 2xyy'}{2(xy' - y)} y'' = 0$$

or

$$(x^2 + y^2)y'' + 2[(y')^2 + 1]\,(y - xy') = 0$$

which is the required differential equation that describes the family of circles passing through the origin.

Example 1.4 Find the differential equation of all circles of radius r.

Solution The equation of all circles of radius r is given by

$$(x - h)^2 + (y - k)^2 = r^2 \tag{29}$$

where h and k are the coordinates of the centre and are taken as arbitrary constants.

Differentiating Eq. (29) with respect to x, we get

$$(x - h) + (y - k)\frac{dy}{dx} = 0 \tag{30}$$

Again, differentiating Eq. (30) with respect to x, we obtain

$$1 + (y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \tag{31}$$

Equation (31) yields

$$(y - k) = -\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \tag{32}$$

Substituting this value of $(y - k)$ in Eq. (30), we get

$$(x - h) = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}} \tag{33}$$

Now, substituting the values of $(x - h)$ and $(y - k)$ from Eqs. (33) and (32), respectively, in Eq. (29), we obtain

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2}+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}=r^2$$

or

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=r^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the differential equation of all circles of radius r.

Example 1.5 Show that the differential equation of a general parabola is

$$\frac{d^2}{dx^2}\left[\left(\frac{d^2y}{dx^2}\right)^{-2/3}\right]=0.$$

Solution The equation of a general parabola is

$$a^2x^2 + 2abxy + b^2y^2 + 2gx + 2fy + c = 0$$

or

$$(ax + by)^2 + 2gx + 2fy + c = 0 \tag{34}$$

Differentiating Eq. (34) with respect to x, we get

$$\frac{dy}{dx}=-\frac{a^2x+aby+g}{abx+b^2y+f} \tag{35}$$

Differentiating Eq. (35) with respect to x, we obtain

$$\frac{d^2y}{dx^2}=-\frac{(af-bg)^2}{(abx+b^2y+f)^3}$$

Thus

$$\left(\frac{d^2y}{dx^2}\right)^{-1/3}=\left[-\frac{(abx+b^2y+f)^3}{(af-bg)^2}\right]^{1/3}=-\frac{abx+b^2y+f}{(af-bg)^{2/3}}$$

or

$$\left(\frac{d^2y}{dx^2}\right)^{-2/3}=\frac{(abx+b^2y+f)^2}{(af-bg)^{4/3}}=A(abx+b^2y+f)^2$$

where $A = 1/(af - bg)^{4/3}$. Therefore,

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)^{-2/3} = 2A(abf - b^2g), \qquad \left[\text{Using Eq. (35) for } \frac{dy}{dx}\right]$$

or

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)^{-2/3} = \frac{2b}{(af - bg)^{1/3}} = \text{constant} = k \text{ (say)}$$

Differentiating with respect to x, yields

$$\frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right)^{-2/3} = 0$$

which proves the result.

Example 1.6 The equation to a system of confocal ellipse is

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} = 1$$

where k is an arbitrary constant. Find the corresponding differential equation.

Solution Given

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} = 1 \tag{36}$$

Differentiating Eq. (36) with respect to x, we get

$$\frac{2x}{a^2 + k} + \frac{2y}{b^2 + k}\frac{dy}{dx} = 0 \tag{37}$$

Denote $dy/dx = p$; then from Eq. (37)

$$\frac{2x}{a^2 + k} + \frac{2y}{b^2 + k}p = 0$$

Thus

$$k = \frac{-pya^2 - b^2x}{x + py}$$

Also

$$a^2 + k = \frac{(a^2 - b^2)\,x}{x + py}, \qquad b^2 + k = \frac{-(a^2 - b^2)\,py}{x + py}$$

Substituting these values of $(a^2 + k)$ and $(b^2 + k)$ in Eq. (36), after simplification, we have

$$(x^2 - y^2) + xy\left(p - \frac{1}{p}\right) = a^2 - b^2$$

as the required differential equation, where $p = dy/dx$.

Example 1.7 Find the differential equation corresponding to the family of curves $x^2 + y^2 + 2c_1x + 2c_2y + c_3 = 0$, where c_1, c_2 and c_3 are arbitiary constants.

Solution The given equation of the curve is

$$x^2 + y^2 + 2c_1x + 2c_2y + c_3 = 0 \tag{38}$$

Differentiating Eq. (38) with respect to x, we get

$$2x + 2y\frac{dy}{dx} + 2c_1 + 2c_2\frac{dy}{dx} = 0 \tag{39}$$

Differentiating again, we obtain

$$2 + 2\left(y\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{dy}{dx}\right) + 2c_2\frac{d^2y}{dx^2} = 0$$

or

$$1 + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + c_2\frac{d^2y}{dx^2} = 0 \tag{40}$$

Differentiating Eq. (40) with respect to x yields

$$(y + c_2)\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\frac{d^2y}{dx^2} = 0 \tag{41}$$

From Eq. (40)

$$c_2\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2} - 1$$

and Eq. (41) gives

$$c_2\frac{d^3y}{dx^3} = -y\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\frac{d^2y}{dx^2}$$

Dividing these two equations, we get

$$\frac{d^3y}{dx^3}\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

REMARK. From the above examples we observe that Examples 1.2 and 1.6 have only one arbitrary constant and the differential equations thus formed are of the first order. On the other hand, Examples 1.1, 1.3 and 1.4 contain two arbitrary constants and the resulting differential equations are of the second order, while Example 1.7 contains three arbitrary constants and we get the third order differential equation. Thus we see that the number of arbitrary constants in a solution of a differential equation depends upon the order of the differential equation and is the same as its order. Hence, a differential equation of n order will contain n arbitrary constants.

1.5.2 Physical Origins of Differential Equations

In the above discussions we have seen how differential equations arise through geometrical consideration and, in this section, with the help of some examples, we shall look for the physical origins of differential equations.

Example 1.8 We know that freely falling objects, close to the surface of the earth, accelerate at a constant rate g. Acceleration is the derivative of velocity, which in turn, is the derivative of the distance s. Thus, if we assume that the upward direction is positive, the equation

$$\frac{d^2s}{dt^2} = -g$$

is the differential equation governing the vertical distance that the falling body travels. The negative sign is used since the weight of the body is a force directed opposite to the positive direction.

Now, if we assume that a stone is tossed off the roof of a building of height s_0 (Fig. 1.1) with an initial upward velocity v_0, then we have to solve the differential equation

$$\frac{d^2s}{dt^2} = -g, \qquad 0 < t < t_1$$

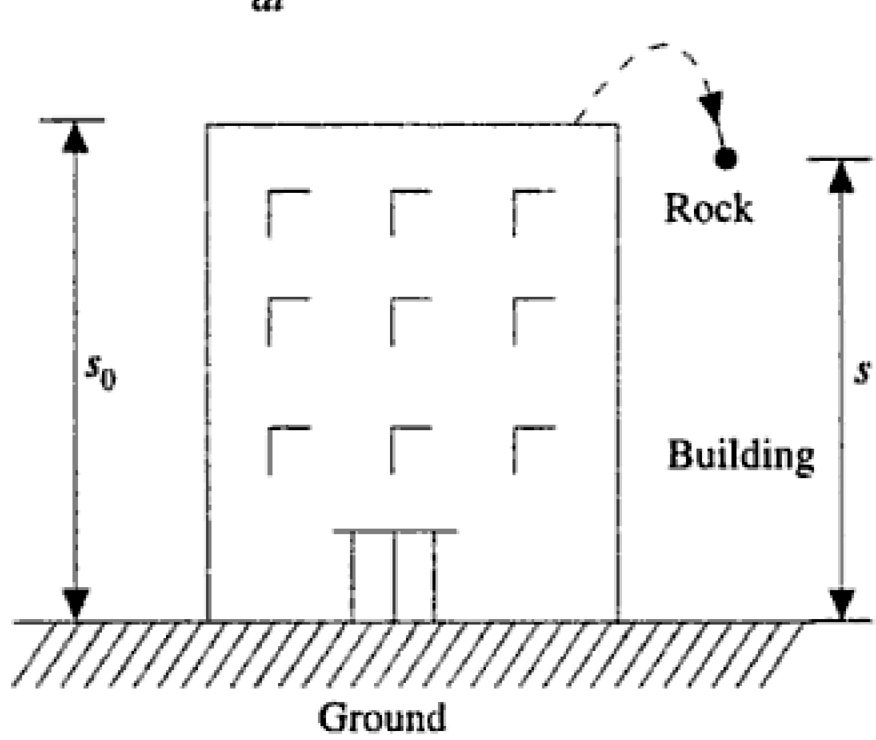

Fig. 1.1 A stone is tossed off from the roof of a building.

with initial conditions $s(0) = s_0$, $s'(0) = v_0$. Here $t = 0$ is taken to be the initial time when the stone leaves the roof of the building and t_1 is the time required to hit the ground. As the stone is thrown upward, it would be assumed that $v_0 > 0$. The formulation of the problem does not include the force of air resistance acting on the body.

Example 1.9 To find the vertical displacement $x(t)$ of a mass attached to a spring (see Fig. 1.2) we use two different laws: Newton's second law of motion and Hooke's law. The former states that the net force acting on the system in motion is $F = ma$, where m is the mass and a is the acceleration, while the latter states that the restoring force of a stretched spring is proportional to the elongation $s + x$, i.e. the restoring force is $k(s + x)$, where $k > 0$ is a constant. In Fig. 1.2b, s is the elongation of the spring after the mass m has been attached and the system hangs at rest in the equilibrium position. When the system is in motion, the variable x represents a directed distance of the mass beyond the equilibrium position (see Fig. 1.2c).

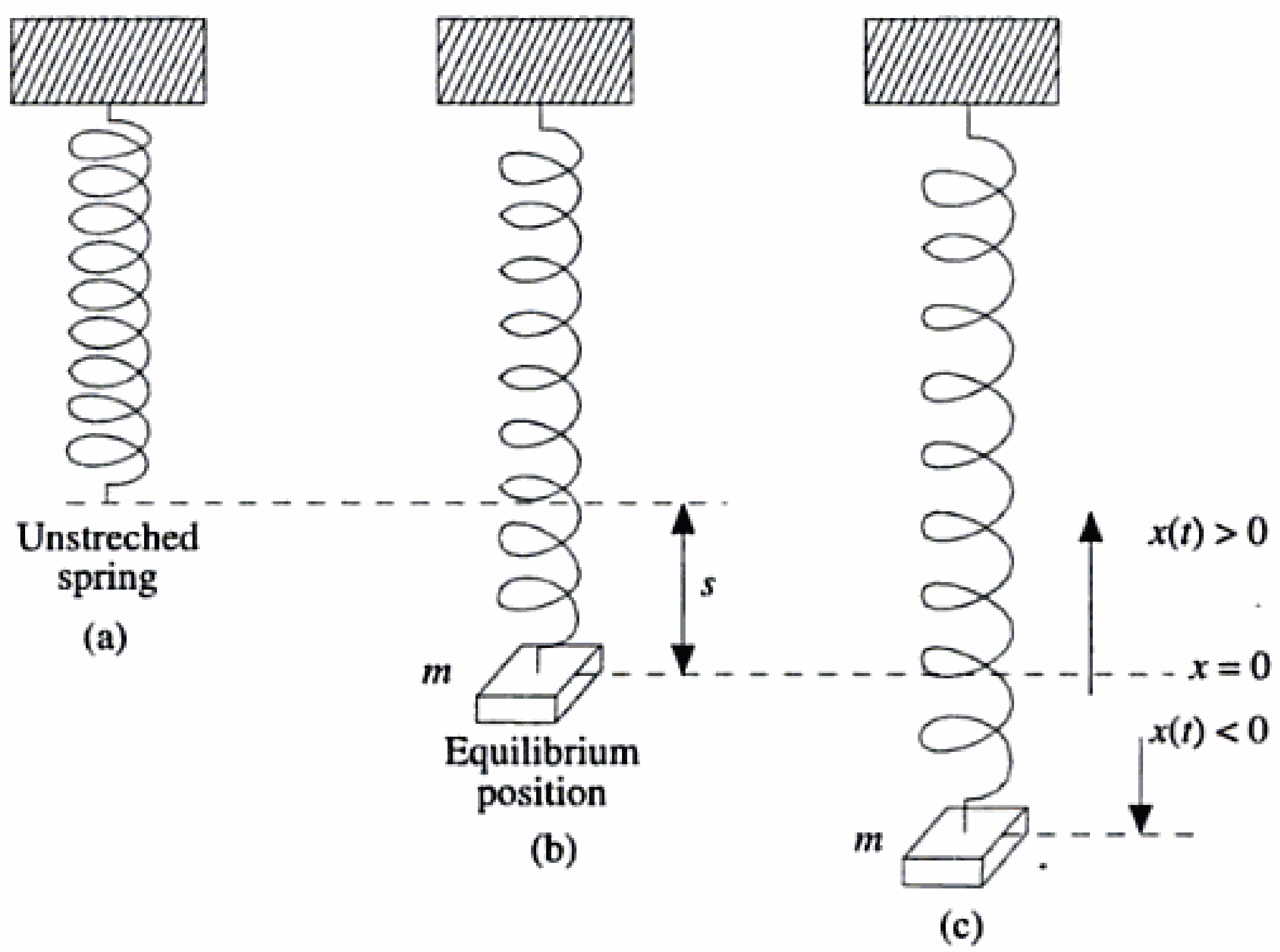

Fig. 1.2 Motion of a stretched spring—Hooke's law.

It will be proved in Chapter 6 that the net force acting on the mass is $F = -kx$, when the system is in motion. Thus, in the absence of damping and other external forces, the differential equation of the vertical motion through the centre of gravity of the mass can be obtained by equating

$$m\frac{d^2x}{dt^2} = -kx$$

where the negative sign denotes the restoring force of the spring acting opposite to the direction of motion, i.e. towards the equilibrium position. Often, we write the differential equation as

$$\frac{d^2x}{dt^2} + w^2x = 0 \tag{42}$$

where $w^2 = k/m$.

Units. Three commonly used system of units are summarized in Table 1.1. In each unit the unit for time is second(s).

Table 1.1 Systems of Units

Quantity	FPS	MKS	CGS
Force	pound (lb)	newton (N)	dyne
Mass	slug	kilogramme (kg)	gram (g)
Distance	foot (ft)	metre (m)	centimetre (cm)
Acceleration due to gravity g	32 ft/s^2	9.8 m/s^2	980 cm/s^2

The gravitational force exerted by the earth on a body of mass m is called its weight W. In the absence of air resistance, the only force acting on a freely falling body is its weight. Hence, from Newton's second law, the mass m and weight W are related by $W = mg$. For example, in engineering system a mass of 1/4 slug corresponds to an 8 lb weight. Since $m = W/g$, a 64 lb weight corresponds to a mass of 64/32 = 2 slugs. In the CGS system, a weight of 2450 dynes has a mass of 2450/980 = 2.5 g. In the MKS system, a weight of 50 N has a mass of 50/9.8 = 5.1 kg. We note 1 N = 10^5 dynes = 0.2247 lb.

In the following example, we shall derive the differential equation which governs the motion of a simple pendulum.

Example 1.10 A mass m having weight W is suspended from the end of a rod of length l. For vertical motion (see Fig. 1.3), we want to determine the displacement angle θ, measured from the vertical, as a function of time (we consider $\theta > 0$ to the right of OP and $\theta < 0$ to the left of OP). We know that an arc s of a circle of radius l is related to the central angle θ through the formula $s = l\theta$. Hence, the angular acceleration is $a = d^2s/dt^2 = l(d^2\theta/dt^2)$. From Newton's second law we have

$$F = ma = ml\frac{d^2\theta}{dt^2}$$

From Fig. 1.3, we see that the tangential component of the force due to the weight W is $mg \sin \theta$. Neglecting the mass of the rod, we have

$$ml\frac{d^2\theta}{dt^2} = -mg \sin \theta$$

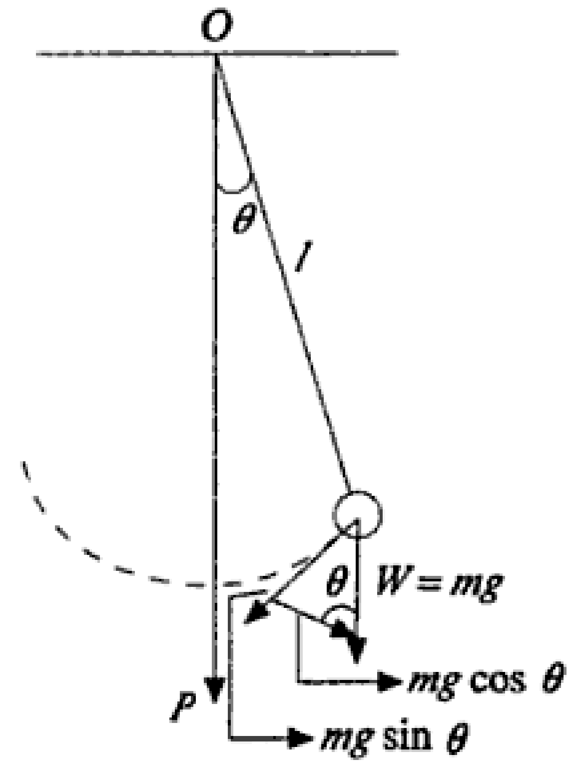

Fig. 1.3 Motion of a simple pendulum.

or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0 \tag{43}$$

The nonlinear differential Eq. (43) cannot be solved in terms of the elementary functions; so we make further simplifying assumptions. If the angular displacements are not too large, then $\sin \theta \approx \theta$ and Eq. (43) can be replaced by the second order linear differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\ \theta = 0 \tag{44}$$

If we put $w^2 = g/l$, we see that Eq. (42) has exactly the same structure as the differential Eq. (44) describing the motion of a spring. The fact that one basic differential equation can describe many diverse physical or even economic pheomena is a common occurrence in the study of applicable mathematics.

Example 1.11 Consider the single loop series circuit containing an inductor, resistor and capacitor (Fig. 1.4). Kirchhoff's second law states that the sum of the

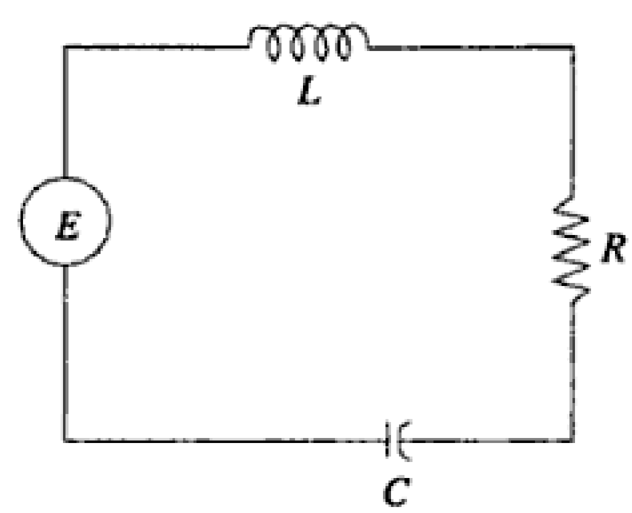

Fig. 1.4 Single loop series circuit.

voltage drops across each part of the circuit is same as the impressed voltage $E(t)$. If $q(t)$ denotes the charge of the capacitor at any time t, then $i(t)$ is given by $i = dq/dt$.

We also know that the voltage drops across:

$$\text{an inductor} = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

$$\text{a capacitor } = \frac{1}{C}q$$

$$\text{a resistor} = iR = R\frac{dq}{dt}$$

where L, C and R are constants known as inductance, capacitance and resistance, respectively. Therefore, to determine $q(t)$ we have to solve the second order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \tag{45}$$

REMARK. In the above example, the initial conditions $q(0)$ and $q'(0)$ represent the charge on the capacitor and the current in the circuit respectively, at $t = 0$. Also, the impressed voltage $E(t)$ is said to be an electromotive force, or emf. In other words, emf causes the current to flow. Table 1.2 and Fig. 1.5 show the basic units of measurements and various circuit elements used in circuit analysis.

Table 1.2 Units of Circuit Analysis

Quantity	Units
Impressed voltage or emf V	volt (V)
Inductance L	henry (H)
Capacitance C	farad (F)
Resistance R	ohm (Ω)
Charge $q(Q)$	coulomb (C)
Current $i(I)$	ampere (A)

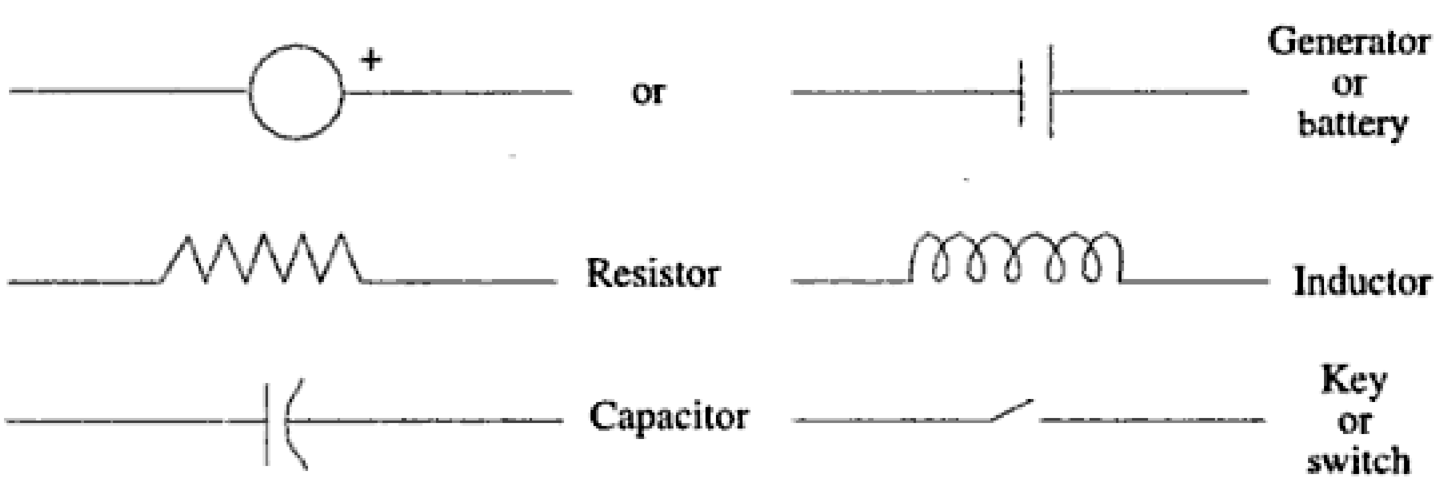

Fig. 1.5 Various circuit elements.

Example 1.12 It seems acceptable to expect that the rate at which a population P expands is proportional to the population which is present at any time t. Roughly speaking, the more people there are, the greater will be the increase in population. Thus one model for population growth is given by the differential equation

$$\frac{dP}{dt} = kP \tag{46}$$

where k is the constant of proportionality. Since we also expect the population to expand, we must have $dP/dt > 0$, and thus $k > 0$.

Example 1.13 In the spread of a contagious disease, for example, the influenza virus, it is reasonable to assume that the rate, dx/dt, at which the disease spreads is proportional not only to the number of people, $x(t)$, who have counteracted the disease, but also to the number of people, $y(t)$, who have not yet been exposed. That is

$$\frac{dx}{dt} = kxy \tag{47}$$

where k is the constant of proportionality. If one infected person is introduced into a fixed population of n people, then x and y are related by

$$x + y = n + 1 \tag{48}$$

From Eq. (48), Eq. (47) yields

$$\frac{dx}{dt} = kx(n + 1 - x) \tag{49}$$

with the initial condition as $x(0) = 1$.

REMARK. The nonlinear first order differential Eq. (49) is a special case of a more general equation

$$\frac{dP}{dt} = P(a - bP) \tag{50}$$

where a and b are constants, known as *logistic equation* (cf. Chapter 4). The solution of this equation is very important in ecological, sociological and managerial sciences.

Example 1.14 *Newton's law of cooling* states that the time rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. If $T(t)$ denotes the temperature of the body at any time t, and T_0 is the constant temperature of the outside medium, then

$$\frac{dT}{dt} = k(T - T_0) \tag{51}$$

where k is the usual constant of proportionality. When $T_0 = 0$, then Eq. (51) reduces to Eq. (46). However, in this case $T(t)$ is decreasing and so we require $k < 0$.

Example 1.15 When interest is compounded continuously, the rate at which an amount of money S grows is proportional to the amount of money present at any time. That is

$$\frac{dS}{dt} = rS \tag{52}$$

where r is the rate of interest annually. Both dS/dt and r are rates. The ratio $(dS/dt)/S$ is called the growth rate, specific growth rate, relative growth rate or average growth rate. This is analogous to the population growth of an earlier Example (1.12). The rate of growth is large when the amount of money present in the account is also very large. Geometrically, this means that the tangent line is steep when S is large (see Fig. 1.6).

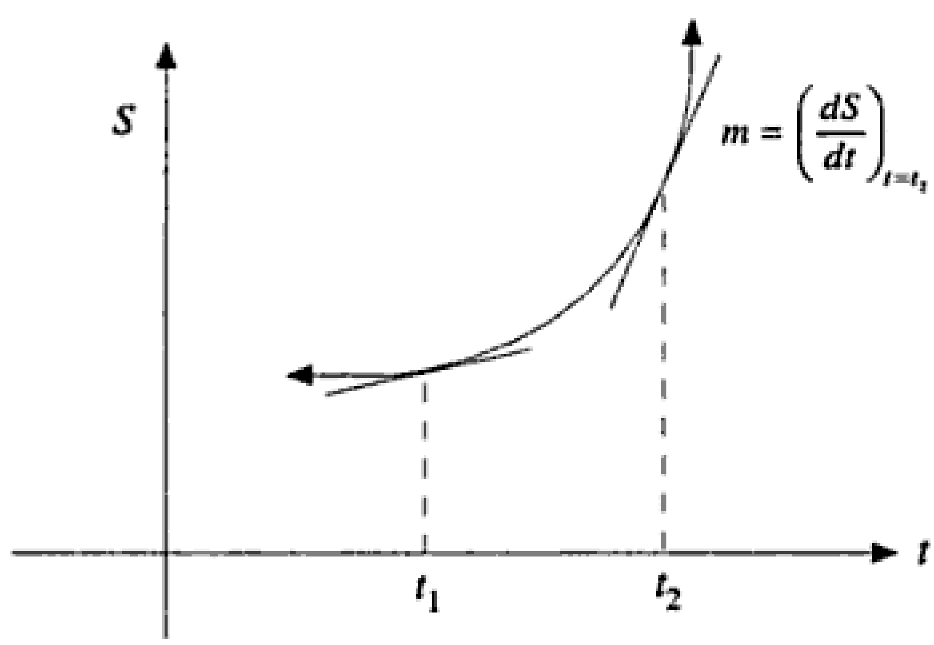

Fig. 1.6 Graph of amount of money that grows versus time—compound interest.

Equation (52) can be obtained by using the definition of the derivative as follows: Let $S(t)$ be the amount accrued in a savings account after t years when the annual rate of interest r is compounded continuously. If Δt represents an increase in t, then the interest obtained in the time span $(t + \Delta t) - t$ is the difference in the amount accrued:

$$S(t + \Delta t) - S(t) \tag{53}$$

Since

$$\text{interest} = \text{rate} \times \text{time} \times \text{principal} \tag{54}$$

we can approximate the interest earned in this same period of time either by

$$r\Delta t S(t) \tag{55}$$

or by

$$r\Delta t S(t + \Delta t) \tag{56}$$

For the actual interest Eqs. (53), (55) and (56) are the lower and upper bounds, respectively, i.e.

$$r\Delta t S(t) \le S(t + \Delta t) - S(t) \le r\Delta t S(t + \Delta t)$$

or by

$$rS(t) \le \frac{S(t+\Delta t) - S(t)}{\Delta t} \le rS(t+\Delta t)$$

Now, taking limit as $\Delta t \to 0$, we have

$$rS(t) \le \lim_{\Delta t \to 0} \frac{S(t+\Delta t) - S(t)}{\Delta t} \le rS(t)$$

which implies that

$$\lim_{\Delta t \to 0} \frac{S(t+\Delta t) - S(t)}{\Delta t} = rS(t)$$

or

$$\frac{dS}{dt} = rS$$

which is same as Eq. (52).

Example 1.16 Let a suspended wire hang under its own weight (Fig. 1.7a). This could be a long telephone wire between two posts. We want to determine the differential equation that describes the shape of the hanging wire.

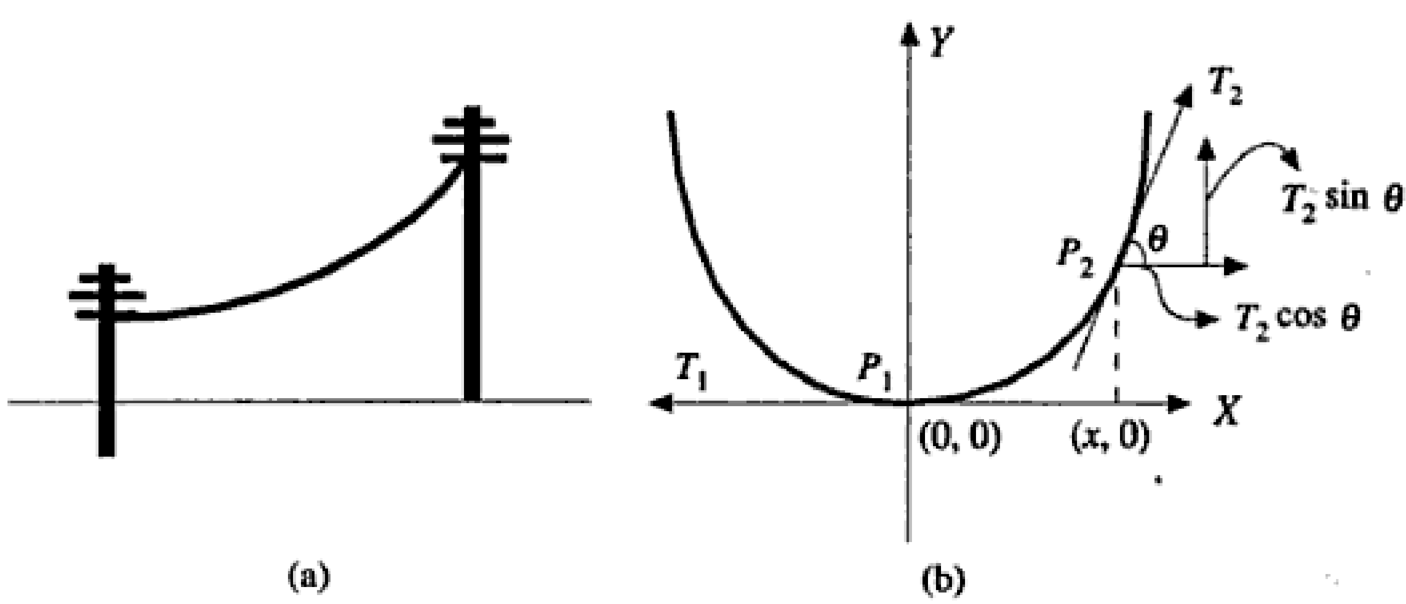

Fig. 1.7 (a) Telephone wire between two posts; (b) Schematic diagram of a portion of wire between the lowest point P_1 and an arbitrary point P_2.

Let us examine only a portion of the wire between the lowest point P_1 and any arbitrary point P_2 (Fig. 1.7b). Three forces are acting on the wire: the weight of the segment P_1P_2 and the tensions T_1 and T_2 in the wire at P_1 and P_2, respectively. If W is the linear density and s is the length of the segment P_1P_2, its weight is Ws.

The horizontal and vertical components of tension T_2 are $T_2 \cos\theta$ and $T_2 \sin\theta$, respectively. Due to the equilibrium, we have

$$|T_1| = T_1 = T_2 \cos\theta$$

$$Ws = T_2 \sin \theta$$

By dividing these two equations, we get

$$\tan \theta = \frac{Ws}{T_1} \quad \text{or} \quad \frac{dy}{dx} = \frac{Ws}{T_1} \tag{57}$$

Now, as the length of the segment P_1P_2 is

$$s = \int_0^s \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

it follows that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{58}$$

Differentiating Eq. (57) with respect to x and using Eq. (58), we get

$$\frac{d^2y}{dx^2} = \frac{W}{T_1}\frac{ds}{dx} = \frac{W}{T_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{59}$$

From Fig. 1.7, we conclude that the shape of the hanging wire is a parabola. However, this is not the case. A wire or heavy rope hanging under its own weight takes on the shape of a hyperbolic cosine and we know that the graph of a hyperbolic cosine is called a *catenary*, which stems from the word 'catena' meaning 'chain'. The Romans used the word catena to mean a dog leash.

Example 1.17 Two masses m_1 and m_2 are connected to two springs A and B having k_1 and k_2 as spring constants, as illustrated in Fig. 1.8. Let $x_1(t)$ and $x_2(t)$ represent the vertical displacement of the masses from their equilibrium positions. When the system is in motion, spring B is subject to both an elongation and a compression, hence its net elongation is $x_2 - x_1$. From Hooke's law, it follows that springs A and B exert forces $-kx_1$ and $k_2(x_2 - x_1)$, respectively, on m_1. The net force acting on m_1 is then $-k_1x_1 + k_2(x_2 - x_1)$. By Newton's second law, we have

$$m_1 \frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1) \tag{60}$$

Similarly, the net force exerted on mass m_2 is due to the elongation of B only, i.e. $-k_2(x_2 - x_1)$. Thus

$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1) \tag{61}$$

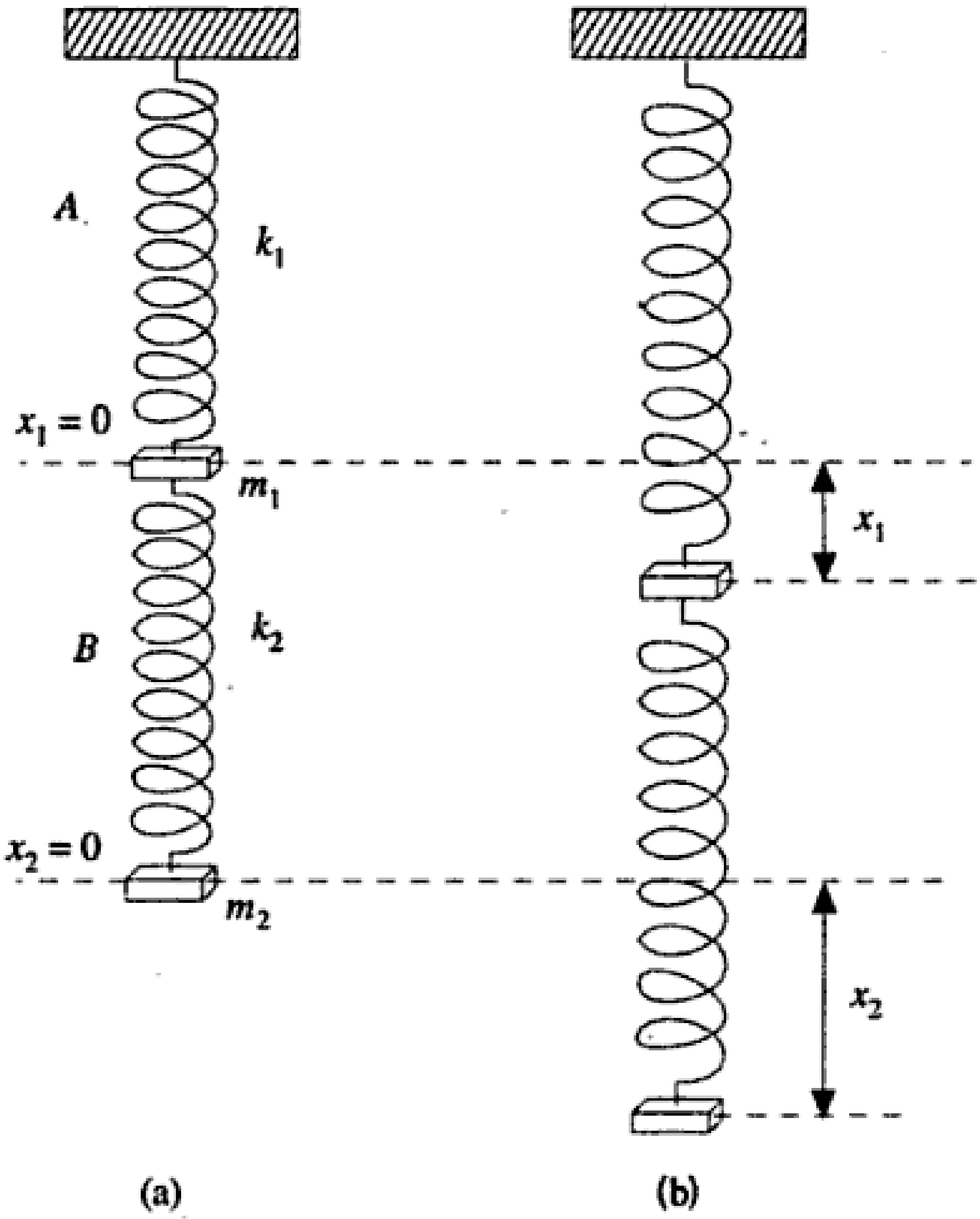

Fig. 1.8 Two masses connected to two springs: (a) Equilibrium position; (b) Position during motion.

In other words, the motion of the coupled system is represented by the simultaneous second order differential equations

$$\left.\begin{aligned} m_1x_1'' &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2x_2'' &= -k_2(x_2 - x_1) \end{aligned}\right\} \tag{62}$$

Example 1.18 Under some circumstances a falling body A of mass m (such as a man hanging from a parachute) experiences air resistance proportional to its instantaneous velocity $v(t)$, see also Example 4.58. Assuming that the downward motion is positive, the sum of the forces acting on the body A is

$$mg - kv \tag{63}$$

where k is the constant of proportionality and the negative sign indicates that the resistance acts in the direction opposite to the motion (see Fig. 1.9).

Newton's second law can be written as

$$F = ma = m\frac{dv}{dt} \tag{64}$$

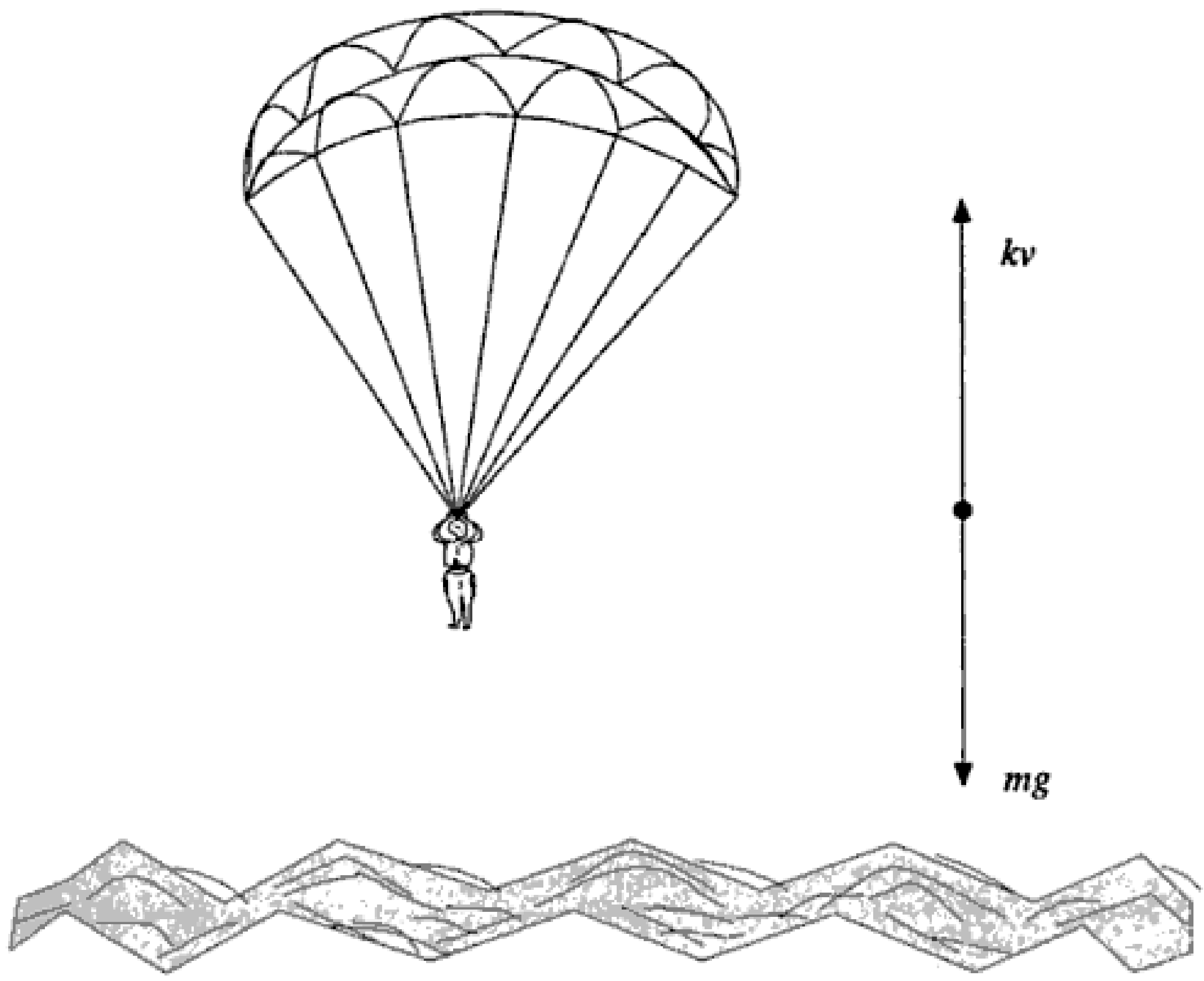

Fig. 1.9 Motion of a paratrooper.

Equating Eqs. (63) and (64), the differential equation for the velocity of the falling body is

$$m\frac{dv}{dt} = mg - kv \qquad \text{or} \qquad \frac{dv}{dt} + \frac{k}{m}v = g$$

Example 1.19 A rocket is shot vertically upward from the surface of the earth. After all its fuel has been expended, the mass of the rocket is a constant, say, m. We can find out the differential equation for distance y from the earth's centre to the rocket at any time after the burnout.

From Newton's second law of motion and his law of gravitation, we have

$$m\frac{d^2y}{dt^2} = -k_1\frac{mM}{y^2}$$

where M is the mass of the earth and k_1 is the constant of proportionality. This equation can also be written as

$$\frac{d^2y}{dt^2} = -\frac{k}{y^2} \tag{65}$$

where $k = k_1M$. On the surface of the earth, $y = R$ so that $k_1mM/y^2 = mg$ will become $k_1M = gR^2$, i.e. $k = gR^2$. If $t = 0$ is the time at which burnout takes place, then

$y(0) = R + y_0$, where y_0 is the distance from the earth's surface to the rocket at the time of burnout, and $y'(0) = V_B$ is the corresponding velocity at that time.

Example 1.20 A rocket is shot vertically upward from the ground with an initial velocity v_0 (Fig. 1.10). Equation (65) can also be used to give more information about the motion of the rocket. We have

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$$

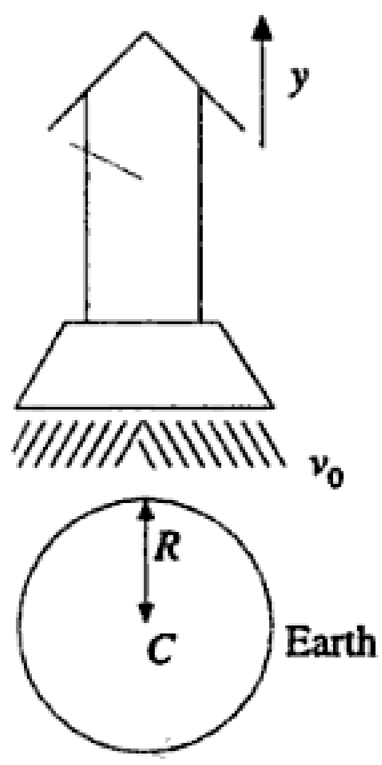

Fig. 1.10 Motion of a rocket.

and Eq. (65) becomes a first order differential equation in v, i.e.

$$v\frac{dv}{dy} = -\frac{k}{y^2}$$

whose solution is

$$\frac{v^2}{2} = \frac{k}{y} + c$$

where c is the constant of integration. Since, $v = v_0$ at $y = R$, we have

$$\frac{v^2}{2} = \frac{k}{y} - \frac{k}{R} + \frac{v_0^2}{2} \tag{66}$$

But $k = gR^2$. Thus, Eq. (66) yields

$$v^2 = \frac{2gR^2}{y} + v_0^2 - 2gR \tag{67}$$

Note that as y increases, v decreases. In particular, if $v_0^2 - 2gR < 0$, then there must be some value of y for $v = 0$; the rocket stops and returns to the earth under the influence of gravity. However, if $v_0^2 - 2gR \geq 0$, then $v > 0$ for all values of y. Hence, we have $v_0 \geq \sqrt{(2gR)}$. This v_0 is known as the escape velocity of the rocket. Taking $R = 4000$ miles, $g = 32$ ft/s^2, 1 ft = 1/5280 miles, 1 s = 1/3600 hr, we have $v_0 \geq 25{,}067$ miles/h (refer also Example 4.56).

1.6 GENERAL, PARTICULAR AND SINGULAR SOLUTIONS

A solution which contains a number of arbitrary constants equal to the order of the differential equation is called the *general solution* or *complete solution* of the differential equation. A solution obtained from a general solution by giving particular values to the constants is called a *particular solution.* For example, $y = c_1e^{2x} + c_2e^{-2x}$ is a general solution and if we take $c_1 = 4$ and $c_2 = 0$, we get $y = 4e^{2x}$ as the particular solution.

A solution which cannot be derived from the general solution but still is a solution of the given differential equation is called a *singular solution.* For example, $x^2 + y^2 = a^2$ is the singular solution of

$$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

and is such that is cannot be derived from the general solution $y = cx + a\sqrt{(1 + c^2)}$ by assigning values to the arbitrary constant c.

REMARKS 1. In counting the arbitrary constants in the general solution, it must be seen that they are independent and are not equivalent to a lesser number of constants. Thus, the solution $y = c_1 \cos x + c_2 \sin (x + c_3)$ appears to have three constants but they are actually equivalent to two, because

$$\begin{aligned} c_1 \cos x + c_2 \sin (x + c_3) &= c_1 \cos x + c_2 \sin x \cos c_3 + c_2 \cos x \sin c_3 \\ &= (c_1 + c_2 \sin c_3) \cos x + c_2 \cos c_3 \sin x \\ &= A \cos x + B \sin x \end{aligned}$$

Hence, three constants c_1, c_2 and c_3 are equivalent to two constants A and B. Thus, constants c_1, c_2 and c_3 are not independent.

2. The general solution of a differential equation can have more than one form, but arbitrary constants in one form will be related to arbitrary constants in another form. Thus, $y = c_1 \cos (x + c_2)$ and $y = c_3 \sin x + c_4 \cos x$ are both solutions of the differential equation $d^2y/dx^2 + y = 0$. Each is a general solution containing two arbitrary constants. Expanding the first equation and comparing it with the second, we get

$$c_1 \cos c_2 = c_4, \qquad -c_1 \sin c_2 = c_3$$

$$c_1 = \sqrt{c_3^2 + c_4^2}, \qquad c_2 = -\tan^{-1}\frac{c_3}{c_4}$$

which shows the relationship between the constants appearing in two solutions.

EXERCISES

Find the differential equation of the family of curves:

1. $y = c_1x + 2.$

2. $y = c_1e^{-x}.$

3. $y^2 = c_1(x + 1).$

4. $c_1y^2 + 4y = 2x^2.$

5. $y = c_1 + c_2e^x.$

6. $y = e^x(c_1 \cos x + c_2 \sin x).$

7. $y = c_1 \sin \omega t + c_2 \cos \omega t$, where ω is a constant to be eliminated.

8. $y = c_1 \sinh kt + c_2 \cosh kt$, where k is a constant to be eliminated.

9. $y = c_1e^{4x} + c_2xe^{4x}.$

10. $y = c_1 + c_2 \log x.$

11. $y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$

12. $r = c_1 (1 + \cos \theta).$

13. $xy = c_1e^x - c_2e^{-x} + x^2.$

14. $y = c_1e^{2x} + c_2e^{-3x} + c_3e^x.$

15. $y^2 = 4c_1(x + c_1).$

16. $y = \frac{c_1}{x} + c_2.$

17. Find the differential equation of a family of straight lines passing through the origin.

18. Find the differential equation of a family of circles passing through the origin wtih its centre on the x-axis.

19. Find the differential equation of a family of circles whose centres are on the y-axis and touch the x-axis.

20. Find the differential equation of a family of parabolas whose vertex is at the origin but the focus is on the x-axis.

21. Find the differential equation of all parabolas whose axes are parallel to the y-axis.

22. Find the differential equation of a family of parabolas whose vertex and focus are on the x-axis.

23. A series circuit contains a resistor and an inductor (Fig. 1.11).

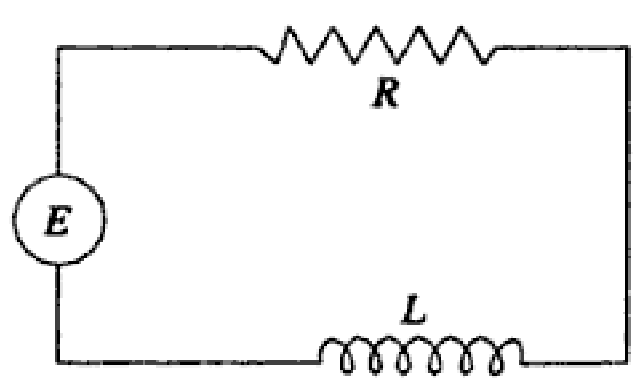

Fig. 1.11

Determine the differential equation for the current $i(t)$, if the resistance is R, the inductance is L, and the impressed voltage is $E(t)$.

24. Obtain the differential equation for the velocity v of a body of mass m falling vertically downward through a medium offering a resistance proportional to the square of the instantaneous velocity.

25. A drug is infused into a patient's bloodstream at a constant rate r g/s. Simultaneously, the drug is removed at a rate proportional to the amount $x(t)$ of the drug present at any time. Find the differential equation describing the amount $x(t)$.

26. Light strikes a plane curve C such that all beams L parallel to the y-axis are reflected to a single point O. Obtain the differential equation for the function $y = f(x)$ describing the shape of the curve.

[*Hint*: It is known that the angle of incidence is equal to the angle of reflection. Also, from Fig. 1.12, we see that inclination of the tangent line from the horizontal at $P(x, y)$ is $(\pi/2 - \theta)$ and set $\phi = 2\theta$].

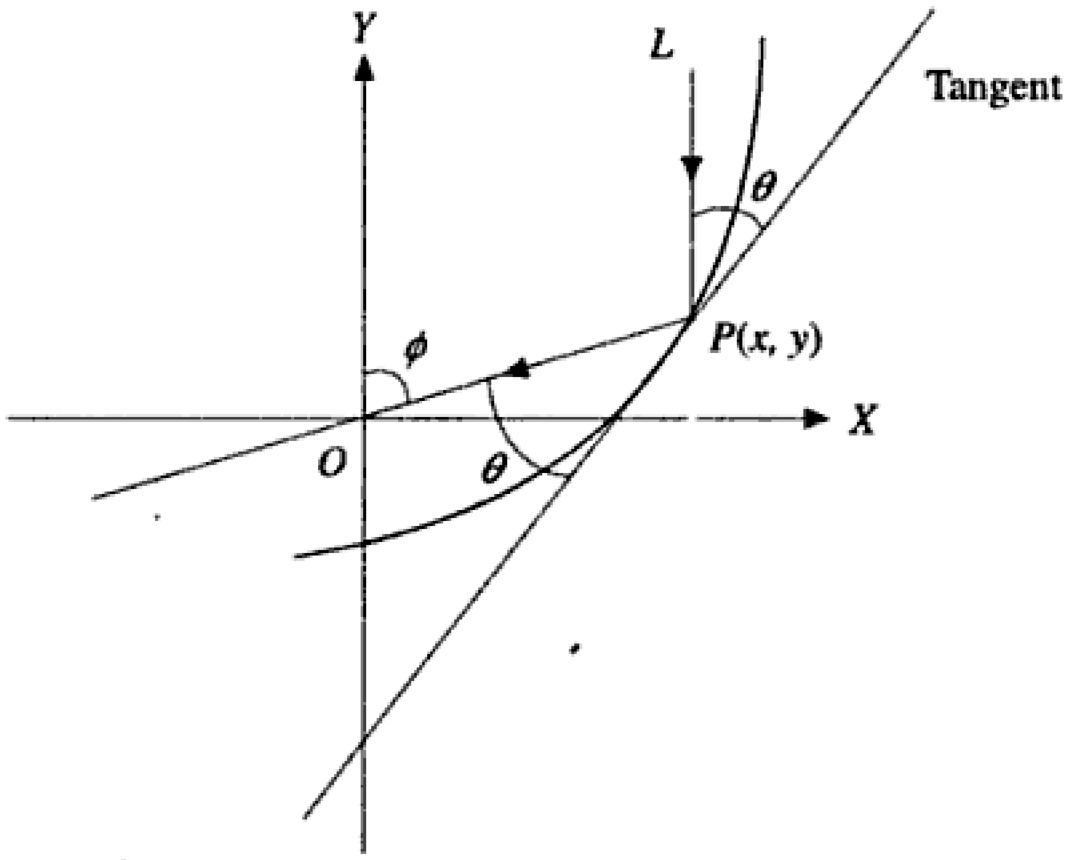

Fig. 1.12

27. A canonical tank (Fig. 1.13) loses water out of an orifice at its bottom. If the cross-sectional area of the orifice is $(1/4)\text{ft}^2$, obtain the differential equation representing the height h of the water at any time.

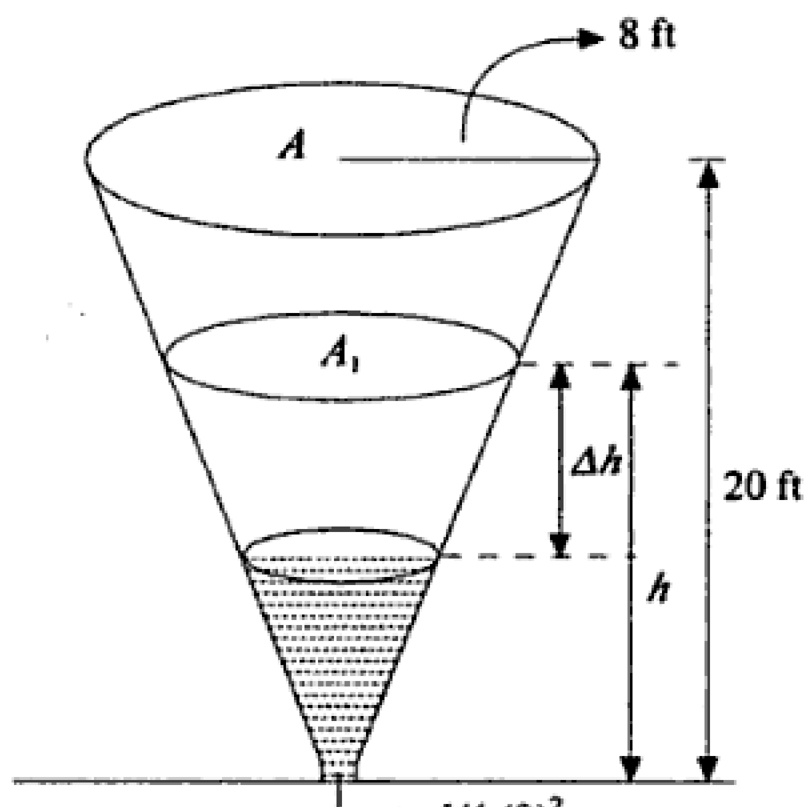

Fig. 1.13

28. A man M, starting from the origin moves in the direction of the positive x-axis pulling a weight along the curve C (known as the tractrix). The weight, initially located on the y-axis at (O, s), is pulled by a rope of constant length s which is kept taut throughout the motion (Fig. 1.14). Find the differential equation of the path of the motion.

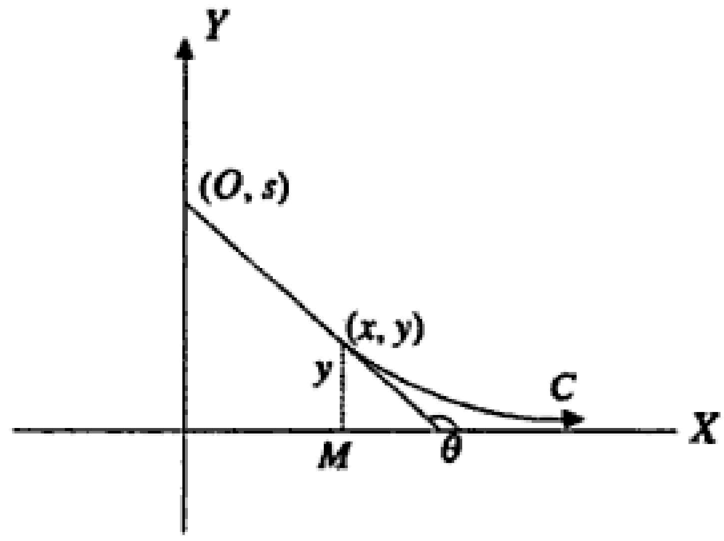

Fig. 1.14

2

Differential Equations of First Order and First Degree

2.1 INTRODUCTION

In Chapter 1, we discussed different ways of obtaining, from a given relation between x, y and constants, a relation between x, y and the derivatives of y with respect to x. In this and the subsequent chapters we shall consider the inverse problem, viz., from a given relation between x, y and the derivatives of y, how we shall find a relation between the variables themselves.

An ordinary differential equation of the first order and first degree is of the form

$$\frac{dy}{dx} + f(x,y) = 0 \tag{1}$$

which is sometimes conveniently written as

$$Mdx + Ndy = 0 \tag{2}$$

where M and N are functions of x and y, or constants. Equation (2) cannot be solved in every case; however, the solution exists if these equations belong to any of the standard forms discussed here.

2.2 EQUATIONS IN WHICH VARIABLES ARE SEPARABLE

If the differential Eq. (2) can be put in the form $f_1(x)dx + f_2(y)dy = 0$, we say that the variables are separable and the solution obtainable by integrating is

$$\int f_1(x)\,dx + \int f_2(y)\,dy = c$$

Example 2.1 Solve $(1 - x)dy + (1 - y)dx = 0$.

Solution The given differential equation can be written as

$$\frac{dx}{1-x} + \frac{dy}{1-y} = 0$$

Integration yields

$$\int \frac{dx}{1-x} + \int \frac{dy}{1-y} = c \quad \text{or} \quad -\log(1-x) - \log(1-y) = -\log c_1$$

where $c = -\log c_1$. Hence, the solution is

$$\log (1 - x)(1 - y) = \log c_1 \quad \text{or} \quad (1 - x)(1 - y) = c_1$$

Example 2.2 Solve $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$.

Solution The above equation can be written as

$$\frac{dy}{y(1-ay)} = \frac{dx}{x+a}$$

Resolving the left-hand side into partial fractions, we get

$$\left(\frac{a}{1-ay} + \frac{1}{y}\right)dy = \frac{dx}{x+a}$$

Integrating, we get

$$[-\log (1 - ay) + \log y] = \log (x + a) + \log c$$

or

$$\frac{\log y}{1-ay} = \log [c(x + a)] \quad \text{or} \quad \frac{y}{1-ay} = c(x + a) \quad \text{or} \quad y = c(x + a)(1 - ay)$$

which is the required solution.

Example 2.3 Solve $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$.

Solution The given equation may be written as

$$(\sin y + y \cos y)dy = x(2 \log x + 1)dx$$

Integrating, we have

$$\int \sin y\, dy + \int y \cos y\, dy = 2\int x \log x\, dx + \int x\, dx$$

or

$$-\cos y + y \sin y + \cos y = 2\left(\frac{1}{2}x^2 \log x - \frac{1}{4}x^2\right) + \frac{x^2}{2} + c$$

or

$$y \sin y = x^2 \log x + c$$

which is the required solution.

Example 2.4 Solve $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$.

Solution The given equation can be written as

$$\frac{3e^x}{1-e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating, we get

$$-3 \log (1 - e^x) + \log (\tan y) = \log c$$

or

$$\log [(1 - e^x)^{-3} \tan y] = \log c$$

or

$$\tan y = c(1 - e^x)^3$$

which is the required solution.

Example 2.5 Solve $\dfrac{dy}{dx} + x^2 = x^2 e^{3y}$.

Solution The given equation can be written as

$$\frac{dy}{e^{3y} - 1} = x^2 dx \quad \text{or} \quad \frac{e^{-3y} dy}{e^{-3y}(e^{3y} - 1)} = x^2 dx$$

or

$$\frac{e^{-3y} dy}{1 - e^{-3y}} = x^2 dx$$

Integrating, we obtain

$$\frac{1}{3} \log (1 - e^{-3y}) = \frac{1}{3} x^3 + c$$

or

$$\log (1 - e^{-3y}) = x^3 + c_1, \qquad c_1 = 3c$$

or

$$1 - e^{-3y} = e^{x^3 + c_1} = e^{x^3} e^{c_1} = c_2 e^{x^3}, \qquad c_2 = e^{c_1}$$

or

$$1 - \frac{1}{e^{3y}} = c_2 e^{x^3}$$

or

$$e^{3y} - 1 = c_2 e^{x^3} e^{3y} = c_2 e^{x^3 + 3y}$$

which is the required solution.

Example 2.6 Solve $(x + y)^2 \dfrac{dy}{dx} = a^2$.

Solution Put $x + y = v$, then

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

With this, the given equation takes the form

$$v^2 \left(\frac{dv}{dx} - 1 \right) = a^2$$

or

$$\frac{v^2}{v^2+a^2}\,dv = dx$$

or

$$\left(1-\frac{a^2}{v^2+a^2}\right)dv = dx$$

Integrating, we get

$$v - a^2\frac{1}{a}\tan^{-1}\frac{v}{a} = x + c$$

or

$$(x+y) - a\tan^{-1}\frac{x+y}{a} = x + c$$

or

$$y = a\tan^{-1}\frac{x+y}{a} + c$$

which is the required solution.

Example 2.7 Solve $\dfrac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

Solution Let $x + y = v$, then

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

Substituting these values in the given equation, we get

$$\frac{dv}{dx} = 1 + \cos v + \sin v$$

$$= 2\cos^2\frac{v}{2} + 2\sin\frac{v}{2}\cos\frac{v}{2}$$

$$= 2\cos^2\frac{v}{2}\left(1+\tan\frac{v}{2}\right)$$

or

$$\frac{dv}{2\cos^2\frac{v}{2}\left(1+\tan\frac{v}{2}\right)} = dx$$

or

$$\frac{\frac{1}{2}\sec^2\frac{v}{2}}{1+\tan\frac{v}{2}}\,dv = dx$$

which, on integration, yields

$$\log\left(1+\tan\frac{v}{2}\right) = x + c$$

or

$$\log\left(1+\tan\frac{x+y}{2}\right) = x + c$$

which is the required solution.

Example 2.8 Solve $\dfrac{x+y-a}{x+y-b}\dfrac{dy}{dx} = \dfrac{x+y+a}{x+y+b}$.

Solution Let $x + y = v$, then

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

Substituting these values in the given equation, we get, after simplification, the relation

$$\frac{v^2 - va + vb - ab}{v^2 - ab}\,dv = 2dx$$

or

$$\left[1+\frac{v(b-a)}{v^2-ab}\right]dv = 2dx$$

Integrating, we get

$$v + \frac{b-a}{2}\int\frac{2v}{v^2-ab}\,dv = 2x + c$$

or

$$v + \frac{1}{2}(b-a)\log\,(v^2 - ab) = 2x + c$$

or

$$x + y + \frac{1}{2}(b-a)\log\,[(x+y)^2 - ab] = 2x + c$$

or

$$(b-a)\log\,[(x+y)^2 - ab] = 2(x - y + c)$$

which is the required solution.

2.3 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equation $Mdx + Ndy = 0$ is said to be *homogeneous* if it can be put in the form

$$\frac{dy}{dx} = \frac{\phi(x, y)}{\psi(x, y)} \tag{3}$$

where $\phi(x, y)$ and $\psi(x, y)$ are homogeneous functions of the same degree (say, n). Taking x^n common both from the numerator and the denominator of Eq. (3), we have

$$\frac{dy}{dx} = \frac{x^n \phi(x/y)}{x^n \psi(x/y)} = F\left(\frac{y}{x}\right) \tag{4}$$

Such types of equations can be solved by the substitution $y = vx$. Equation (4), thus becomes

$$v + x\frac{dv}{dx} = F(v) \qquad \text{or} \qquad \frac{dv}{F(v) - v} = \frac{dx}{x}$$

The variables have now been separated and the solution is

$$\int \frac{dv}{F(v) - v} = \log x + c$$

After the integration v should be replaced by y/x to get the required solution.

Example 2.9 Solve $(x^2 - y^2)dx + 2xydy = 0$.

Solution The given equation can be written as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \tag{5}$$

which is a homogeneous differential equation. Putting $y = vx$ in (5), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2xvx} = \frac{v^2 - 1}{2v}$$

Separating the variables, we have

$$\frac{2v}{v^2 + 1}dv = -\frac{dx}{x}$$

Integrating, we obtain

$$\log(v^2 + 1) = -\log x + \log c$$

or

$$\log(v^2 + 1) + \log x = \log c \qquad \text{or} \qquad \log[(v^2 + 1)x] = \log c$$

or

$$(v^2 + 1)x = c \qquad \text{or} \qquad (y^2 + x^2) = cx$$

which is the required solution.

Example 2.10 Solve $(x - y)^2dx + 2xydy = 0$.

Solution The given equation can be written as

$$(x-y)^2 + 2xy\frac{dy}{dx} = 0$$

Putting $y = vx$ in this equation, we get

$$(x - vx)^2 + 2x(vx)\left(v + x\frac{dv}{dx}\right) = 0$$

or

$$\frac{dx}{x} + \frac{2vdv}{3v^2 - 2v + 1} = 0$$

or

$$\frac{dx}{x} + \frac{(1/3)(6v-2) + (2/3)}{3v^2 - 2v + 1}dv = 0$$

or

$$\frac{dx}{x} + \frac{(1/3)6v-2}{3v^2 - 2v + 1}dv + \frac{(2/3)dv}{v^2 - (2/3)v + (1/3)} = 0$$

or

$$\frac{dx}{x} + \frac{1}{3}\frac{6v-2}{3v^2-2v+1}dv + \frac{2}{3}\frac{dv}{\left(v-\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} = 0$$

Integrating, we get

$$\log x + \frac{1}{3}\log(3v^2 - 2v + 1) + \frac{2}{3}\frac{3}{\sqrt{2}}\tan^{-1}\frac{v-(1/3)}{\sqrt{2}/3} = c$$

or

$$3\log x + \log(3v^2 - 2v + 1) + 3\sqrt{2}\tan^{-1}\frac{3v-1}{\sqrt{2}} = 3c$$

or

$$\log[x^3(3v^2 - 2v + 1)] + 3\sqrt{2}\tan^{-1}\frac{3v-1}{\sqrt{2}} = 3c$$

Putting $v = y/x$, after simplification, we get

$$\log(3y^2x - 2yx^2 - x^3) + 2\tan^{-1}\frac{3y-x}{x\sqrt{2}} = 3c$$

which is the required solution.

Example 2.11 Solve $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy.$

Solution The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2y - 2xy^2}{x^3 - 3x^2y} = \frac{xy - 2y^2}{x^2 - 3xy}$$

Putting $y = vx$ and separating the variables, we get

$$\frac{1 - 3v}{v^2}\,dv = \frac{dx}{x} \qquad \text{or} \qquad \frac{1}{v^2}\,dv = \frac{dx}{x} + \frac{3}{v}\,dv$$

Integrating, we obtain

$$-\frac{1}{v} = \log x + 3 \log v + \log c$$

or

$$-\frac{x}{y} = \log x + 3 \log \frac{y}{x} + \log c$$

$$= \log\left[x\left(\frac{y}{x}\right)^3 c\right]$$

$$= \log \frac{cy^3}{x^2}$$

or

$$\frac{x}{y} = \log\left(\frac{x^2}{cy^3}\right) \qquad \text{or} \qquad e^{x/y} = \frac{x^2}{cy^3}$$

or

$$x^2 = cy^3e^{x/y}$$

which is the required solution.

Example 2.12 Solve $xdy - ydx = \sqrt{x^2 + y^2}\,dx.$

Solution The given equation can be written as

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$$

Substituting $y = vx$ in this equation and separating the variables, we get

$$\frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

Integrating, we obtain

$$\log\,[v + \sqrt{v^2 + 1}\,] = \log x + \log c$$

or

$$y+\sqrt{y^2+x^2}=cx^2 \qquad \left(\text{since } v=\frac{y}{x}\right)$$

This is the required solution.

Example 2.13 Solve $x\dfrac{dy}{dx}=y(\log y-\log x+1)$.

Solution Substitution of $y = vx$ in the given equation and separation of variables yield

$$\frac{dv}{v\log v}=\frac{dx}{x}$$

which on integration gives

$$\log(\log v)=\log x+\log c=\log cx$$

or

$$\log v=cx \qquad \text{or} \qquad v=e^{cx} \qquad \text{or} \qquad y=xe^{cx}$$

which is the required solution.

2.4 DIFFERENTIAL EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

If the differential equation is of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'} \tag{6}$$

it can be reduced to a homogeneous differential equation as follows:

Put

$$x=X+h, \qquad y=Y+k \tag{7}$$

where X and Y are new variables and h and k are constants yet to be chosen. From Eq. (7)

$$dx=dX, \qquad dy=dY$$

Equation (6), thus reduces to

$$\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')} \tag{8}$$

In order to have Eq. (8) as a homogeneous differential equation, choose h and k such that the following equations are satisfied:

$$\left.\begin{aligned} ah+bk+c&=0\\ a'h+b'k+c'&=0\end{aligned}\right\} \tag{9}$$

Now, Eq. (8) becomes

$$\frac{dY}{dX}=\frac{aX+bY}{a'X+b'Y} \tag{10}$$

which is a homogeneous differential equation and can be solved by putting $Y = vX$.

REMARK. Solving Eq. (9), we obtain

$$\frac{h}{bc' - b'c} = \frac{k}{a'c - ac'} = \frac{1}{ab' - a'b}$$

If $ab' - a'b = 0$, the above method does not apply. In such cases, we have

$$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{t}$$

Equation (6) now becomes

$$\frac{dy}{dx} = \frac{ax + by + c}{t(ax + by) + c'} \tag{11}$$

which can be solved by putting $ax + by = v$ so that

$$a + b\,\frac{dy}{dx} = \frac{dv}{dx}.$$

Hence, Eq. (11) takes the form

$$\frac{dv}{dx} = a + b\,\frac{v + c}{tv + c'}$$

Now separate the variables and integrate to get the required solution.

Example 2.14 Solve $(2x + y - 3)dy = (x + 2y - 3)dx$.

Solution The given differential equation is

$$\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$$

Putting $x = X + h$, $y = Y + k$, we get

$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)} \tag{12}$$

Choose h and k such that

$$h + 2k - 3 = 0, \qquad 2h + k - 3 = 0$$

Solving these equations, we get $h = k = 1$. Equation (12) can now be written as

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \tag{13}$$

where $X = x - 1$ and $Y = y - 1$.

Putting $Y = vX$ in Eq. (13) and simplifying, we get

$$\frac{2 + v}{1 - v^2}\,dv = \frac{dX}{X}$$

Resolving into partial fraction and integrating, we get

$$\int\left(\frac{1}{2}\frac{1}{1+v}+\frac{3}{2}\frac{1}{1-v}\right)dv=\int\frac{dX}{X}$$

or

$$\frac{1}{2}\log(1+v)-\frac{3}{2}\log(1-v)=\log X+\log c$$

or

$$\log\left[\frac{1+v}{(1-v)^3}\right]=2\log Xc=\log(cX)^2$$

or

$$\frac{1+v}{(1-v)^3}=(cX)^2$$

Now substituting $v = Y/X = (y - 1)/(x - 1)$, we get

$$(x + y - 2) = c^2(x - y)^3$$

which is the required solution.

Example 2.15 Solve $(2x - y + 1)dx + (2y - x - 1)dy = 0$.

Solution The given equation can be written as

$$\frac{dy}{dx}+\frac{2x-y+1}{2y-x-1}=0$$

Putting

$$x = X + h,\ y = Y + k \tag{14}$$

the given equation reduces to

$$\frac{dY}{dX}+\frac{2X-Y+(2h-k+1)}{2Y-X+(2k-h-1)}=0 \tag{15}$$

Choose h and k, so that

$$2h - k + 1 = 0, \qquad 2k - h - 1 = 0$$

which on solving give rise to $h = -1/3$, $k = 1/3$.

From Eq. (14),

$$x = X-\frac{1}{3}, \qquad y = Y+\frac{1}{3} \tag{16}$$

Thus, Eq. (15) reduces to

$$\frac{dY}{dX}=\frac{2X-Y}{X-2Y}$$

Putting $Y = vX$ and simplifying, we get

$$\frac{2v-1}{v^2-v+1}dv + \frac{2}{X}dX = 0$$

Integrating, we have

$$\log(v^2 - v + 1) + 2\log X = \log c$$

or

$$X^2(v^2 - v + 1) = c$$

From Eq. (16) and $v = Y/X$, the solution is given by

$$3x^2 + 3y^2 - 3xy - 3y + 3x + 1 - 3c = 0$$

Example 2.16 Solve $\dfrac{dy}{dx} = \dfrac{x-y+3}{2x-2y+5}$.

Solution Here, the coefficients of x and y, in the numerator and denominator of the right-hand side are proportional. Therefore, we put

$$x - y + 3 = v \tag{17}$$

Differentiating Eq. (17), we get

$$\frac{dy}{dx} = 1 - \frac{dv}{dx}$$

Substituting these values in the given differential equation, we get

$$\frac{2v-1}{v-1}\,dv = dx$$

or

$$\left(2 + \frac{1}{v-1}\right)dv = dx$$

Integration yields

$$2v + \log(v - 1) = x + c$$

or

$$2(x - y + 3) + \log(x - y + 3 - 1) = x + c$$

or

$$(x - 2y) + \log(x - y + 2) = c_1 \qquad (c_1 = c - 6)$$

which is the required solution.

Example 2.17 Solve $(2x + 4y + 3)\dfrac{dy}{dx} = x + 2y + 1$.

Solution The given equation can be written as

$$\frac{dy}{dx} = \frac{(x+2y)+1}{2(x+2y)+3} \tag{18}$$

Put $x + 2y = v$, so that

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{dv}{dx} - 1\right)$$

Equation (18), thus becomes

$$\frac{2v+3}{4v+5}\,dv = dx$$

or

$$\frac{1}{2}\left(1 + \frac{1}{4v+5}\right)dv = dx$$

Integrating, we have

$$\frac{1}{2}v + \frac{1}{2}\frac{1}{4}\log(4v+5) = x + c$$

or

$$\log(4v + 5) = 8x + 8c - 4v$$

or

$$\log(4x + 8y + 5) = 4(x - 2y) + 8c$$

or

$$4x + 8y + 5 = e^{8c}e^{4(x-2y)}$$

or

$$4x + 8y + 5 = c_1 e^{4(x-2y)} \qquad (c_1 = e^{8c})$$

which is the required solution.

2.5 LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear if the dependent variable y and its differential coefficients occur only in the first degree and are not multiplied together. The linear differential equation of the first order is of the form

$$\frac{dy}{dx} + Py = Q \tag{19}$$

where P and Q are constants or functions of x alone.

To solve such an equation, multiply both sides of Eq. (19) by $e^{\int P dx}$. Then, we get

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = e^{\int P dx}Q$$

or

$$\frac{d}{dx}\left(ye^{\int P dx}\right) = e^{\int P dx}Q$$

Integrating both sides, we get

$$ye^{\int P dx} = \int Qe^{\int P dx}dx + c$$

which is the complete solution of Eq. (19).

REMARKS. 1. The factor $e^{\int P dx}$, on multiplying by which the left-hand side of Eq. (19) becomes the exact differential coefficient of some function of x and y, is called the integrating factor of the given differential equation.

2. Sometimes a given differential equation becomes linear if we take x as the dependent variable and y as the independent variable, i.e. it can be written in the form

$$\frac{dx}{dy} + P_1 x = Q_1$$

where P_1 and Q_1 are constants or functions of y alone. In this case, the solution is

$$xe^{\int P_1 dy} = \int Q_1 e^{\int P_1 dy} dy + c$$

Example 2.18 Solve $(1 + x)\dfrac{dy}{dx} - xy = 1 - x$.

Solution The given equation can be written as

$$\frac{dy}{dx} - \frac{x}{1+x} y = \frac{1-x}{1+x}$$

Here

$$P = -\frac{x}{1+x} = \frac{1}{1+x} - 1, \qquad Q = \frac{1-x}{1+x}$$

The integrating factor (I.F.) is thus, given by

$$\text{I.F.} = e^{\int P dx} = \exp\left[\int \left(\frac{1}{1+x} - 1\right) dx\right] = e^{\log(1+x) - x}$$

Therefore, the solution of the given differential is

$$ye^{\int P dx} = \int Q\, e^{\int P dx}\, dx + c$$

or

$$y(1 + x)\, e^{-x} = \int \frac{1-x}{1+x}(1+x)\, e^{-x} dx + c$$

$$= \int (e^{-x} - xe^{-x})\, dx + c$$

$$= xe^{-x} + c$$

or

$$y(1 + x) = x + ce^{x}$$

Example 2.19 Solve $ydx - xdy + \log x dx = 0$.

Solution The given equation can be written as

$$\frac{dy}{dx} - \frac{1}{x} y = \frac{1}{x} \log x$$

Here

$$P = -\frac{1}{x},\ Q = \frac{1}{x}\log x$$

Therefore,

$$\text{I.F.} = e^{\int P\,dx} = e^{-\int dx/x} = e^{-\log x} = \frac{1}{x}$$

Hence, the solution is

$$y e^{\int P dx} = \int Q\, e^{\int P dx} dx + c$$

or

$$y\frac{1}{x} = \int \left(\frac{1}{x}\log x\right)\frac{1}{x}\,dx + c$$

Putting, $\log x = t$ and integrating by parts the integral on right-hand side, we get

$$y = cx - (1 + \log x)$$

Example 2.20 Solve $(1-x^2)\dfrac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$.

Solution The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x(1-x^2)^{1/2}}{1-x^2}$$

where

$$\text{I.F.} = e^{\int P dx} = \exp\left(\int \frac{2x}{1-x^2}\,dx\right) = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

Therefore, the solution of the given differential equation is

$$y e^{\int P dx} = \int Q e^{\int P dx}\,dx + c$$

or

$$y\frac{1}{1-x^2} = \int \frac{x(1-x^2)^{1/2}}{1-x^2}\,\frac{1}{1-x^2}\,dx + c$$

Putting, $t = 1 - x^2$ and integrating the integral on the right-hand side, we get

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c$$

or

$$y = \sqrt{1-x^2} + c(1-x^2)$$

Example 2.21 Solve $\sin 2x\dfrac{dy}{dx} - y = \tan x$.

Solution The given equation can be written as

$$\frac{dy}{dx} - \operatorname{cosec} 2xy = \frac{\tan x}{\sin 2x} = \frac{1}{2}\sec^2 x$$

Here

$$\text{I.F.} = e^{\int P dx} = e^{-\int \operatorname{cosec} 2x\, dx} = \exp\left[\log(\tan x)^{-1/2}\right] = \frac{1}{\sqrt{\tan x}}$$

Therefore, the solution of the given equation is

$$y\frac{1}{\sqrt{\tan x}} = \int \frac{1}{2}\sec^2 x \frac{1}{\sqrt{\tan x}}\, dx + c$$

Substituting $t = \tan x$ in the integral on the right-hand side and simplifying, we get

$$y = c\sqrt{\tan x} + \tan x$$

Example 2.22 Solve $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$.

Solution The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

Here

$$\text{I.F.} = e^{\int P dy} = \exp\left(\int \frac{1}{1+y^2}\, dy\right) = e^{\tan^{-1} y}$$

The solution of the given differential equation, therefore, is

$$x e^{\int P dy} = \int Q\, e^{\int P dy} dy + c$$

$$x e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy + c$$

Put $t = \tan^{-1} y$ in the integral on the right-hand side, and integrate, we get

$$x e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$$

Example 2.23 Solve $(x + 2y^3)\dfrac{dy}{dx} = y$.

Solution Write the given equation as

$$\frac{dx}{dy} - \frac{1}{y}x = 2y^2$$

Then

$$\text{I.F.} = e^{\int P dy} = e^{-\int (dy/y)} = e^{-\log y} = \frac{1}{y}$$

The solution is, therefore

$$x e^{\int P dy} = \int Q e^{\int P dy} dy + c$$

or

$$x\frac{1}{y} = \int 2y^2 \frac{1}{y} dy + c$$

or

$$x = y^3 + cy$$

2.6 DIFFERENTIAL EQUATIONS REDUCIBLE TO LINEAR FORM

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \tag{20}$$

where P and Q are constants or functions of x alone and n is constant except 0 and 1, is called a *Bernoulli's equation.**

The solution of Eq. (20) is obtained as follows: Divide Eq. (20) by y^n and put $y^{-n+1} = v$, so that $(-n + 1)y^{-n}\, dy/dx = dv/dx$ and, Eq. (20) thus, reduces to

$$\frac{1}{-n+1}\frac{dv}{dx} + Pv = Q$$

or

$$\frac{dv}{dx} + (1-n)Pv = (1-n)Q$$

which is a linear differential equation in v and can be solved by the method described in Section 2.5.

Example 2.24 Solve $(1 - x^2)\dfrac{dy}{dx} + xy = xy^2$.

Solution The given equation can be written as

$$y^{-2}\frac{dy}{dx} + \frac{x}{1-x^2}y^{-1} = \frac{x}{1-x^2}$$

*Named after James Bernoulli (1654–1705), who studied it in 1695.

Put $y^{-1} = v$, so that $-y^{-2}\, dy/dx = dv/dx$ and the given equation reduces to

$$\frac{dv}{dx} - \frac{x}{1-x^2}v = -\frac{x}{1-x^2}$$

which is a linear equation in v. Its integrating factor is

$$\text{I.F.} = e^{\int P dx} = \exp\left(-\int \frac{x}{1-x^2}dx\right) = e^{(1/2)\log(1-x^2)} = (1-x^2)^{1/2}$$

The required solution of the given equation is

$$ve^{\int P dx} = \int Q e^{\int P dx}\, dx + c$$

or

$$v(1-x^2)^{1/2} = \int -\frac{x}{1-x^2}(1-x^2)^{1/2}\, dx + c$$

In the integral on the right-hand side, put $t = 1 - x^2$ and integrate. We get

$$v(1-x^2)^{1/2} = (1-x^2)^{1/2} + c$$

or

$$\frac{1}{y}(1-x^2)^{1/2} = (1-x^2)^{1/2} + c$$

or

$$(1-y)\sqrt{1-x^2} = cy$$

Example 2.25 Solve $x\dfrac{dy}{dx} + y = y^2 \log x$.

Solution The given equation can be written as

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y} = \frac{1}{x}\log x$$

Putting, $-y^{-1} = v$ and $1/y^2(dy/dx) = dv/dx$ in this equation, we get

$$\frac{dv}{dx} - \frac{1}{x}v = \frac{1}{x}\log x$$

which is a linear equation in v, and the integrating factor is

$$\text{I.F.} = e^{\int P dx} = e^{-\int(dx/x)} = e^{-\log x} = \frac{1}{x}$$

Therefore, the solution is

$$v\frac{1}{x} = \int \frac{\log x}{x}\frac{1}{x}dx + c$$

Put $t = \log x$ in the integral on the right-hand side and integrate, we obtain

$$\frac{v}{x} = c - (1 + \log x)e^{-\log x}$$

or

$$\frac{-1}{xy} = c - (1 + \log x)\frac{1}{x}$$

or

$$1 = (1 + \log x)y - cxy$$

Example 2.26 Solve $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

Solution The given equation can be written as

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \tag{21}$$

Put $\tan y = v$ so that $\sec^2 y(dy/dx) = dv/dx$, and Eq. (21) reduces to

$$\frac{dv}{dx} + 2xv = x^3 \tag{22}$$

which is a linear differential equation. Now

$$\text{I.F.} = e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$$

Multiplying Eq. (22) by I.F. and then integrating, we get

$$ve^{x^2} = \int x^3 e^{x^2} dx + c$$

$$= \int \frac{1}{2} t\, e^t dt + c \qquad (x^2 = t)$$

$$= \frac{1}{2} e^t (t - 1) + c$$

$$= \frac{1}{2} e^{x^2} (x^2 - 1) + c$$

or

$$(\tan y)e^{x^2} = \frac{1}{2} e^{x^2} (x^2 - 1) + c$$

or

$$\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

which is the required solution.

Example 2.27 Solve $y(2xy + e^x)dx - e^x dy = 0$.

Solution The given equation can be written as

$$y^{-2}\frac{dy}{dx} - y^{-1} = 2xe^{-x}$$

Now, put $y^{-1} = v$ so that $-y^{-2}(dy/dx) = dv/dx$ and the above equation reduces to

$$\frac{dv}{dx} + v = -2xe^{-x}$$

which is a linear equation in v, and the solution is

$$ve^{\int P dx} = \int Qe^{\int P dx} dx + c$$

or

$$ve^{\int dx} = -\int 2xe^{-x}e^{\int dx} dx + c$$

or

$$\frac{1}{y}e^{x} = -x^{2} + c$$

2.7 EXACT DIFFERENTIAL EQUATIONS

A differential equation is said to be *exact* if it can be derived from its primitive (general solution) directly by differentiation, without any subsequent multiplication, elimination, etc.

Theorem 2.1 The necessary and sufficient condition for the differential equation $M\,dx + N\,dy = 0$ to be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof *Necessary condition:* Let

$$f(x, y) = c \tag{23}$$

be the general solution of the given differential equation, where c is a constant. The given equation is an exact differential equation if it can be obtained directly by differentiating Eq. (23).

$$d[f(x, y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Therefore,

$$Mdx + Ndy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{24}$$

Equating the coefficients dx and dy in Eq. (24), we get

$$M = \frac{\partial f}{\partial x} \tag{25}$$

$$N = \frac{\partial f}{\partial y} \tag{26}$$

To eliminate the unknown function f, differentiate Eqs. (25) and (26) partially w.r.t. y and x, respectively. Thus

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is the required necessary condition.

Sufficient condition: Let $P = \int M dx$, then $\dfrac{\partial P}{\partial x} = M$ so that $\dfrac{\partial^2 P}{\partial y \partial x} = \dfrac{\partial M}{\partial y}$. But

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 P}{\partial y \partial x} = \frac{\partial^2 P}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)$$

Integrating both sides with respect to x, we get

$$N = \frac{\partial P}{\partial y} + \phi(y)$$

where $\phi(y)$ is a function of y only. Therefore,

$$\begin{aligned} M\,dx + N\,dy &= \frac{\partial P}{\partial x} dx + \left[\frac{\partial P}{\partial y} + \phi(y)\right] dy \\ &= dP + dF(y), \qquad dF(y) = \phi(y)dy \\ &= d[P + F(y)] \end{aligned}$$

which shows that $M\,dx + N\,dy$ is an exact differential and this proves the sufficient part.

In order to obtain the solution of an exact differential equation, we have to proceed as follows:

(a) Integrate M with respect to x, keeping y as constant.

(b) Integrate with respect to y only those terms of N which do not contain x.

(c) Add the two expressions obtained in (a) and (b) above and equate the result to an arbitrary constant.

In other words, the solution of an exact differential equation is

$$\underset{(y=\text{constant})}{\int M\,dx} + \underset{(\text{terms not having } x)}{\int N\,dy} = c$$

Example 2.28 Solve $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0.$

Solution Here, $M = ax + hy + g$ and $N = hx + by + f$. Thus, $\partial M/\partial y = \partial N/\partial x$, which shows that the given equation is exact. Hence, the required solution is

$$\underset{(y=\text{constant})}{\int M\,dx} + \underset{(\text{terms not having } x)}{\int N\,dy} = c$$

or

$$\int (ax + hy + g)dx + \int (by + f)dy = c_1$$

or

$$\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = c_1$$

or

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \qquad c = -2c_1$$

Example 2.29 Solve $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0.$

Solution The given equation can be written as

$$\left(x - \frac{y}{x^2 + y^2}\right)dx + \left(y + \frac{x}{x^2 + y^2}\right)dy = 0$$

Here

$$M = \frac{x^3 + xy^2 - y}{x^2 + y^2}, \qquad N = \frac{x^2y + y^3 + x}{x^2 + y^2}; \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which show that the given equation is exact. Now

$$\int M\,dx = \int \left(x - \frac{y}{x^2 + y^2}\right)dx = \int x\,dx - y\int \frac{dx}{x^2 + y^2}$$

$$= \frac{x^2}{2} - y\frac{1}{y}\tan^{-1}\frac{x}{y} \tag{27}$$

and

$$\int N\,dy = \int \left(y + \frac{x}{x^2 + y^2}\right)dy = \int y\,dy = \frac{y^2}{2} \tag{28}$$

From Eqs. (27) and (28), the required solution is

$$x^2 - 2\tan^{-1}\frac{x}{y} + y^2 = 2c$$

Example 2.30 Solve $(1 + e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$.

Solution Here

$$M = 1 + e^{x/y}, \qquad N = e^{x/y}\left(1 - \frac{x}{y}\right); \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus, the given equation is exact. Now

$$\underset{(y=\text{constant})}{\int M\,dx} = \int (1 + e^{x/y})dx = x + ye^{x/y}, \qquad \underset{(\text{terms not having } x)}{\int N dy} = 0$$

Therefore, the required solution is

$$\int M\,dx + \int N\,dy = c$$

or

$$x + ye^{x/y} = c$$

Example 2.31 Solve $(\sin x \cos y + e^{2x})\,dx + (\cos x \sin y + \tan y)\,dy = 0$.

Solution Here

$$M = \sin x \cos y + e^{2x}, \qquad N = \cos x \sin y + \tan y, \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which show that the given equation is exact. Now

$$\underset{(y=\text{constant})}{\int M\,dx} = \int (\sin x \cos y + e^{2x})\,dx = -\cos x \cos y + \frac{1}{2}e^{2x}$$

and

$$\underset{(\text{terms not having } x)}{\int N\,dy} = \int \tan y\,dy = \log \sec y$$

Therefore, the solution of the given equation is

$$\frac{1}{2}e^{2x} - \cos x \cos y + \log \sec y = c$$

2.8 INTEGRATING FACTORS

A non-exact differential equation can always be made exact by multiplying it by some functions of x and y. Such a function is called an *integrating factor.* Although a differential equation of the type $Mdx + Ndy = 0$ always has an integrating factor, there is no general method of finding them. Here we shall explain some of the methods for finding the integrating factors.

Method I. In some cases the integrating factor is found by inspection. Using the following few exact differentials, it is easy to find the integrating factors:

(a) $d\left(\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{y^2}$ (b) $d\left(\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2}$

(c) $d(xy) = xdy + ydx$ (d) $d\left(\dfrac{x^2}{y}\right) = \dfrac{2yxdx - x^2dy}{y^2}$

(e) $d\left(\dfrac{y^2}{x}\right) = \dfrac{2xydy - y^2dx}{x^2}$ (f) $d\left(\dfrac{x^2}{y^2}\right) = \dfrac{2xy^2dx - 2x^2ydy}{y^4}$

(g) $d\left(\dfrac{y^2}{x^2}\right) = \dfrac{2x^2ydy - 2xy^2dx}{x^4}$ (h) $d\left(\dfrac{1}{xy}\right) = -\dfrac{xdy + ydx}{x^2y^2}$

(i) $d\left(\log\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{xy}$ (j) $d\left(\log\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{xy}$

(k) $d\left(\tan^{-1}\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{x^2 + y^2}$ (l) $d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2 + y^2}$

(m) $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^xdx - e^xdy}{y^2}$ (n) $d\left(\dfrac{e^y}{x}\right) = \dfrac{xe^ydy - e^ydx}{x^2}$

(o) $d\left(-\dfrac{1}{xy}\right) = \dfrac{xdy + ydx}{x^2y^2}$ (p) $d\left[\dfrac{1}{2}\log(x^2 + y^2)\right] = \dfrac{xdx + ydy}{x^2 + y^2}$

Example 2.32 Solve $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$.

Solution The given equation can be written as

$$(ydx + xdy) + (xy^2dx - x^2ydy) = 0$$

or

$$d(yx) + xy^2dx - x^2ydy = 0 \qquad \text{[see equation in (c) above]}$$

Dividing by x^2y^2, we get

$$-d\left(\frac{1}{xy}\right)+\frac{1}{x}dx-\frac{1}{y}dy=0$$

Integrating, we get

$$-\frac{1}{xy}+\log x-\log y=c$$

or

$$-\frac{1}{xy}+\log\frac{x}{y}=c \qquad \text{or} \qquad \log\frac{x}{y}=c+\frac{1}{xy}$$

which is the required solution.

Example 2.33 Solve $(x^3e^x - my^2)\,dx + m\,xy\,dy = 0.$

Solution The given equation can be written as

$$x^3e^xdx + m(xy\,dy - y^2dx) = 0$$

Dividing by x^3, we get

$$e^x dx+m\frac{xy\,dy-y^2dx}{x^3}=0$$

or

$$e^x dx+\frac{m}{2}\frac{x^2\,2y\,dy-y^2\,2x\,dx}{x^4}=0$$

or

$$e^x dx+\frac{1}{2}m\,d\left(\frac{y^2}{x^2}\right)=0 \qquad \text{[from exact differential (g)]}$$

Integrating, we get

$$e^x+\frac{1}{2}m\frac{y^2}{x^2}=c$$

or

$$2x^2e^x + my^2 = 2cx^2$$

which is the required solution.

Example 2.34 Solve $x\,dy - y\,dx = a(x^2 + y^2)\,dy.$

Solution The given equation can be written as

$$\frac{x\,dy-y\,dx}{x^2+y^2}=a\,dy$$

or

$$d\left(\tan^{-1}\frac{y}{x}\right) = a\,dy \qquad \text{[see exact differential (a)]}$$

Integrating, we get the required solution as

$$\tan^{-1}\frac{y}{x} = ay + c$$

Method II. If the differential equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$, then $1/(Mx + Ny)$ is the integrating factor.

Proof We have

$$Mdx + Ndy = \frac{1}{2}\left[(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right] \tag{29}$$

Multiplying by $1/(Mx + Ny)$, we get

$$\frac{M\,dx + N\,dy}{Mx + Ny} = \frac{1}{2}\left(\frac{y\,dx + x\,dy}{xy} + \frac{Mx - Ny}{Mx + Ny}\,\frac{ydx - xdy}{xy}\right)$$

$$= \frac{1}{2}\left[d(\log xy) + f\left(\frac{y}{x}\right)d\left(\log\frac{x}{y}\right)\right]$$

Since $Mdx + Ndy = 0$ is homogeneous, we obtain

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\left[d(\log xy) + f\left(e^{\log(y/x)}\right)d\left(\log\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[d(\log xy) + F\left(\log\frac{x}{y}\right)d\left(\log\frac{x}{y}\right)\right]$$

which is an exact differential. Therefore, if $1/(Mx + Ny)$ is an integrating factor, then $Mdx + Ndy = 0$ is an exact differential equation.

Example 2.35 Solve $x^2ydx - (x^3 + y^3)dy = 0$.

Solution Here, $Mx + Ny \neq 0 = -y^4$, and the given equation is homogeneous. Thus, the integrating factor is $-1/y^4$. Multiplying the given equation by this factor, we get

$$\frac{-x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

Now

$$M = \frac{-x^2}{y^3}, \qquad N = \frac{x^3}{y^4} + \frac{1}{y}; \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This shows that the resulting differential equation is exact. Thus

$$\underset{(y\,=\,\text{constant})}{\int M dx} = -\int \frac{x^2}{y^3}\,dx = -\frac{x^3}{3y^3}$$

$$\underset{(\text{terms not having } x)}{\int N dy} = \int \frac{dy}{y} = \log y$$

Therefore, the required solution is

$$\int M\,dx + \int N\,dy = c$$

or

$$-\frac{x^3}{3y^3} + \log y = c \qquad \text{or} \qquad x^3 = 3y^3(\log y - c)$$

Example 2.36 Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$.

Solution The given equation is homogeneous and $Mx + Ny = x^2y^2 \neq 0$. The integrating factor is $1/x^2y^2$. Multiplying the given equation by this factor, we get

$$\left(\frac{1}{y}\,dx - \frac{x}{y^2}\,dy\right) - \frac{2}{x}\,dx + \frac{3}{y}\,dy = 0$$

or

$$d\left(\frac{x}{y}\right) - \frac{2}{x}\,dx + \frac{3}{y}\,dy = 0$$

Integrating term by term, we obtain

$$\frac{x}{y} - 2\log x + 3\log y = c$$

which is the required solution.

Method III. If in the differential equation $Mdx + Ndy = 0$, $M = yf_1(xy)$ and $N = xf_2(xy)$. Then $1/(Mx - Ny)$ is an integrating factor.

Proof Multiply Eq. (29) by $1/(Mx - Ny)$ and then proceed in the same way as in method II, the proof follows.

Example 2.37 Solve $(xy + 2x^2y^2)ydx + (xy - x^2y^2)xdy = 0$.

Solution Here, $M = yf_1(xy)$ and $N = xf_2(xy)$. Thus, the integrating factor $= 1/(Mx - Ny) = 1/(3x^3y^3)$. Multiplying the given equation by this factor, we get

$$\frac{1}{3}\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \frac{1}{3}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

which is an exact equation, whose solution is

$$\frac{1}{3}\left(-\frac{1}{xy}+2\log x\right)+\frac{1}{3}(-\log y)=C$$

or

$$2\log x-\log y=\frac{1}{xy}+C_1 \qquad (C_1=3C)$$

Example 2.38 Solve $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$.

Solution Here, $M = yf_1(xy)$, $N = xf_2(xy)$, and thus the integrating factor $1/(2x^2y^2)$. Multiplying the given equation by this factor, we get

$$(ydx+xdy)+\left(\frac{1}{x}dx-\frac{1}{y}dy\right)+\left(\frac{1}{x^2y}dx+\frac{1}{xy^2}dy\right)=0$$

or

$$d(xy)+\left(\frac{dx}{x}-\frac{dy}{y}\right)+\frac{ydx+xdy}{x^2y^2}=0$$

or

$$d(xy)+\frac{dx}{x}-\frac{dy}{y}+\frac{d(xy)}{x^2y^2}=0$$

Integrating term by term, we get

$$xy+\log x-\log y-(1/xy)=c$$

which is the required solution.

Method IV. The equation, $Mdx + Ndy = 0$ has $e^{\int f(x)\,dx}$ as the integrating factc if $1/N(\partial M/\partial y - \partial N/\partial x)$ is a function of x [say $f(x)$].

Proof Multiplying the given equation by $e^{\int f(x)\,dx}$, we get

$$Me^{\int f(x)\,dx}\,dx+Ne^{\int f(x)\,dx}\,dy=0$$

which is of the form $M_1dx + N_1dy = 0$. Now

$$\frac{\partial M_1}{\partial y}=\frac{\partial}{\partial y}[Me^{\int f(x)\,dx}]=\frac{\partial M}{\partial y}e^{\int f(x)dx}$$

and

$$\frac{\partial N_1}{\partial x}=\frac{\partial}{\partial x}[Ne^{\int f(x)dx}]$$

$$=e^{\int f(x)\,dx}\left[Nf(x)+\frac{\partial N}{\partial x}\right]$$

$$= e^{\int f(x)\,dx}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} + \frac{\partial N}{\partial x}\right)$$

$$= \frac{\partial M}{\partial y}\, e^{\int f(x)\,dx}$$

Thus

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

which shows that the given equation is exact.

REMARK: If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = k$ (a const.), then I.F. $= e^{\int k dx}$.

Example 2.39 Solve $(x^2 + y^2)dx - 2xydy = 0$.

Solution Here

$$M = x^2 + y^2, \qquad N = -2xy, \qquad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{2}{x}$$

Thus

$$\text{I.F.} = e^{\int f(x)\,dx} = \exp\left(\int -\frac{2}{x}dx\right) = e^{-2\log x} = \frac{1}{x^2}$$

Multiplying the given equation by $1/x^2$, we get

$$\left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

or

$$dx + d\left(-\frac{y^2}{x}\right) = 0$$

Integrating term by term, we obtain

$$x - \frac{y^2}{x} = C$$

which is the required solution.

Example 2.40 Solve $\left(y + \dfrac{1}{3}y^3 + \dfrac{1}{2}x^2\right)dx + \dfrac{1}{4}(x + xy^2)dy = 0$.

Solution Here

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3}{x}$$

Thus

$$\text{I.F.} = e^{\int f(x)\,dx} = x^3$$

Multiply the given equation by this factor. Then we get

$$2x^5\,dx + (x^4\,dy + 4x^3y\,dx) + \frac{1}{3}(x^4 3y^2\,dy + 4x^3y^3\,dx) = 0$$

or

$$2x^5dx + d(x^4y) + \frac{1}{3}d(x^4y^3) = 0$$

Integrating term by term, we have

$$x^6 + 3x^4y + x^4y^3 = 3C$$

which is the required solution.

Method V. If $1/M(\partial N/\partial x - \partial M/\partial y)$ is a function of y [say $f(y)$], then $e^{\int f(y)\,dy}$ is the integrating factor of $Mdx + Ndy = 0$.

Proof Proceed in a similar way as that of Method IV.

REMARK: If $1/M(\partial N/\partial x - \partial M/\partial y) = k$ (a constant), then $e^{\int k\,dy}$ is the integrating factor.

Example 2.41 Solve $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$.

Solution Here

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y}$$

Thus

$$\text{I.F.} = e^{\int f(y)dy} = y$$

Multiplying the given equation by y, we get

$$(xy^4 + y^2)dx + 2(x^2y^3 + xy + y^5)dy = 0$$

Here

$$M_1 = xy^4 + y^2, \qquad N_1 = 2x^2y^3 + 2xy + 2y^5,$$

Then

$$\underset{(y=\text{constant})}{\int M_1dx} = \frac{1}{2}x^2y^4 + xy^2 \qquad \text{and} \qquad \underset{(\text{terms not having } x)}{\int N_1dy} = \frac{2}{6}y^6$$

Therefore, the required solution is

$$3x^2y^4 + 6xy^2 + 2y^6 = 6C$$

Example 2.42 Solve $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$.

Solution Here

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 6$$

which is a constant, thus I.F. $= e^{\int 6dy} = e^{6y}$. Multiplying the given equation by this factor, we get

$$(xy^2 - x^2)e^{6y}dx + (3x^2y^2 + x^2y - 2x^3 + y^2)e^{6y}dy = 0$$

Here

$$M_1 = xy^2e^{6y} - x^2e^{6y} \qquad \text{and} \qquad N_1 = 3x^2y^2e^{6y} + x^2ye^{6y} - 2x^3e^{6y} + y^2e^{6y}$$

Thus

$$\underset{(y=\text{constant})}{\int M_1 dx} = e^{6y}\left(\frac{1}{2}x^2y^2 - \frac{1}{3}x^3\right)$$

and

$$\underset{(\text{terms not having } x)}{\int N_1 dy} = \int y^2e^{6y}dy = e^{6y}\left(\frac{1}{6}y^2 - \frac{1}{18}y + \frac{1}{108}\right)$$

by integrating by parts. Therefore, the required solution is

$$e^{6y}\left(\frac{1}{2}x^2y^2 - \frac{1}{3}x^3 + \frac{1}{6}y^2 - \frac{1}{18}y + \frac{1}{108}\right) = C$$

Method VI. If the equation $Mdx + Ndy = 0$ is of the form

$$x^ay^b(mydx + nxdy) + x^cy^d(pydx + qxdy) = 0$$

where a, b, c, d, m, n, p and q are constants, then x^hy^k is the integrating factor, where h, k are constants and can be obtained by applying the condition that after multiplication by x^hy^k the given equation is exact.

Example 2.43 Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$.

Solution The given equation can be written as

$$y(y + 2x^2)dx + x(2x^2 - y)dy = 0$$

Let x^hy^k be the integrating factor. Multiplying the given equation by this factor, we have

$$(x^hy^{k+2} + 2x^{h+2}y^{k+1})dx + (2x^{h+3}y^k - x^{h+1}y^{k+1})dy = 0 \tag{30}$$

Here,

$$M = x^hy^{k+2} + 2x^{h+2}y^{k+1}$$
$$N = 2x^{h+3}y^k - x^{h+1}y^{k+1}$$

If Eq. (30) is exact, then $\partial M/\partial y = \partial N/\partial x$, or

$$(k + 2)\, x^h y^{k+1} + 2(k + 1)x^{h+2}y^k = -(h + 1)x^h y^{k+1} + 2(h + 3)\, x^{h+2}y^k$$

Equating the coefficients of $x^h y^{k+1}$ and $x^{h+2}y^k$ on both sides and solving, we get

$$h = -\frac{5}{2}, \qquad k = -\frac{1}{2}$$

Therefore, the integrating factor is

$$x^h y^k = x^{-5/2}y^{-1/2}$$

Multiplying the given equation by this factor, we get

$$(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})dx + (2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2})dy = 0$$

In this equation, we have

$$M_1 = x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}, \qquad N_1 = 2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}$$

and the equation is exact. Also

$$\underset{(y=\text{constant})}{\int M_1 dx} = -\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2}$$

and

$$\underset{(\text{terms not having } x)}{\int N_1 dy} = 0$$

Hence, the required solution of the given equation is

$$\int M_1 dx + \int N_1 dy = C$$

or

$$-\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = C$$

Example 2.44 Solve $(2ydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$.

Solution We can write the given equation as

$$(2y + 6xy^2)dx + (3x + 8x^2y)dy = 0$$

Let $x^h y^k$ be the integrating factor. Multiplying the given equation by this factor, we get

$$(2x^h y^{k+1} + 6x^{h+1}y^{k+2})dx + (3x^{h+1}y^k + 8x^{h+2}y^{k+1})dy = 0$$

If this equation is exact, we must then have $\partial M/\partial y = \partial N/\partial x$. Thus

$$2(k + 1)y^k x^h + 6\,(k + 2)y^{k+1}x^{h+1} = 3(h + 1)x^h y^k + 8(h + 2)x^{h+1}y^{k+1}$$

Equating the coefficients of $x^h y^k$ and $x^{h+1}y^{k+1}$ on both sides and solving, we get $h = 1$, $k = 2$. Thus, the integrating factor is $x^h y^k = xy^2$. Now, the multiplication of the given equation by xy^2 yields

$$(2xy^3 + 6x^2y^4)dx + (3x^2y^2 + 8x^3y^3)dy = 0$$

Here, $M_1 = 2xy^3 + 6x^2y^4$, $N_1 = 3x^2y^2 + 8x^3y^3$, and the equation is exact. Therefore, the solution is

$$\underset{(y\,=\,\text{constant})}{\int M_1 dx} \quad + \quad \underset{(\text{terms not having } x)}{\int N_1 dy} = C$$

or

$$x^2y^3 + 2x^3y^4 = C$$

2.9 CHANGE OF VARIABLES

By suitable substitution we can reduce a given differential equation which does not directly come under any of the forms discussed so far to one of these forms. This procedure of reducing the given differential equation by substitution is called the change of dependent (or independent) variable.

Example 2.45 Solve $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$.

Solution Let $x = r\cos\theta$, and $y = r\sin\theta$. Then $r^2 = x^2 + y^2$ and $y/x = \tan\theta$. Differentiating these relations and substituting the respective value in the given equation, we get after simplification, the relation

$$2rdr = 2a^2 d\theta$$

which on integration yields

$$r^2 = 2a^2\theta + C \quad \text{or} \quad x^2 + y^2 = 2a^2 \tan^{-1}\frac{y}{x} + C$$

which is the required solution.

Example 2.46 Solve $\sec^2 y\left(\dfrac{dy}{dx}\right) + 2x\tan y = x^3$.

Solution Putting $\tan y = v$, the given equation reduces to

$$\frac{dv}{dx} + 2xv = x^3$$

which is a linear differential equation whose solution is

$$ve^{x^2} = \int x^3 e^{x^2} dx + c \quad \text{or} \quad \tan y = \frac{1}{2}(x^2 - 1) + Ce^{-x^2}$$

2.10 TOTAL DIFFERENTIAL EQUATIONS

An ordinary differential equation of the first order and first degree involving three variables is of the form

$$P + Q\frac{dy}{dx} + R\frac{dz}{dx} = 0 \tag{31}$$

where P, Q, R are functions of x and x is the independent variable.

In terms of differentials, Eq. (31) has the form

$$Pdx + Qdy + Rdz = 0 \tag{32}$$

Equation (32) is integrable only when

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \tag{33}$$

To get a solution of Eq. (32), we have the following rule.

Rule: If Eq. (33) is satisfied, take one of the variables, say z, as constant so that $dz = 0$. Then integrate the equation $Pdx + Qdy = 0$. Replace the arbitrary constant appearing in its integral by $\phi(z)$. Now differentiate the integral just obtained with respect to x, y, z. Finally, compare this result with the given differential equation to determine $\phi(z)$.

Example 2.47 Solve $(y^2 + yz)\,dx + (z^2 + zx)\,dy + (y^2 - xy)\,dz = 0$.

Solution Here, $P = y^2 + yz$, $Q = z^2 + zx$, $R = y^2 - xy$. It can be shown that Eq. (33) is satisfied here. Consider z as constant so that the given equation takes the form

$$(y^2 + yz)\,dx + (z^2 + zx)\,dy = 0$$

or

$$\frac{dx}{z(z+x)} + \frac{dy}{y(y+z)} = 0$$

Integrate it to get

$$\log(z + x) + \log y - \log(y + z) = \text{constant}$$

or

$$\frac{y(z+x)}{y+z} = \phi(z) = \text{constant} \tag{34}$$

or

$$y(z + x) - \phi(z)(y + z) = 0$$

Differentiate with respect to x, y, z, to obtain

$$ydx + [z + x - \phi(z)]dy + [y - (y + z)\phi'(z) - \phi(z)]dz = 0 \tag{35}$$

Comparing Eq. (35) with the given equation, we get

$$\frac{y^2+yz}{y}=\frac{z^2+zx}{z+x-\phi(z)}=\frac{y^2-xy}{y-(y+z)\phi'(z)-\phi(z)}$$

The relation

$$\frac{y^2+yz}{y}=\frac{z^2+zx}{z+x-\phi(z)}$$

reduces to Eq. (34) and thus gives no information about $\phi(z)$. Therefore, take

$$\frac{y^2+yz}{y}=\frac{y^2-xy}{y-(y+z)\phi'(z)-\phi(z)}$$

and simplify to get

$$y^2-xy=y^2-xy-(y+z)^2\,\phi'(z) \qquad \text{[use Eq. (34)]}$$

or

$$(y+z)^2\,\phi'(z)=0$$

which gives $\phi'(z)=0$ and $\phi(z)=C$.

Hence, the required solution, from Eq. (34), is

$$y(z+x)=(y+z)C$$

REMARK. Sometimes the integral is obtained simply by regrouping the terms in the given equation as illustrated in the following example.

Example 2.48 Solve $xdx + zdy + (y + 2z)dz = 0$.

Solution After regrouping the terms, the given equation can be written as

$$xdx+(ydz+zdy)+2zdz=0$$

By integrating this equation, we obtain

$$\frac{x^2}{2}+yz+z^2=C$$

2.11 SIMULTANEOUS TOTAL DIFFERENTIAL EQUATIONS

The equation

$$\begin{aligned} Pdx+Qdy+Rdz&=0\\ P'dx+Q'dy+R'dz&=0 \end{aligned} \tag{36}$$

where P, Q, R and P', Q', R' are any functions of x, are called *simultaneous total differential equations*.

(a) If each of the above equation is integrable and has solution $\phi(x, y, z) = C$ and $\psi(x, y, z) = C'$, respectively, then taken these equations together form the solution of (36).

(b) If one or both of the Eq. (36) is not integrable, then we write them as

$$\frac{dx}{QR' - RQ'} = \frac{dy}{RP' - PR'} = \frac{dz}{PQ' - QP'}$$

and solve these by the methods given below.

2.12 EQUATIONS OF THE FORM $dx/P = dy/Q = dz/R$

2.12.1 Method of Grouping

Note that if it is possible to take two fractions $\frac{dx}{P} = \frac{dz}{R}$, from which y can be cancelled or is absent, leaving the equation in x and z only. Then integrate it giving

$$\phi(x, z) = C \tag{37}$$

Again, note that if one variable, say x, is absent or can be cancelled (may be with the help of Eq. 37) from the equation $\frac{dy}{Q} = \frac{dz}{R}$, then integrate it to get

$$\psi(y, z) = C' \tag{38}$$

The two independent solutions, Eqs. (37) and 38, taken together form the complete solution of the given equation.

Example 2.49 Solve $\dfrac{dx}{z^2y} = \dfrac{dy}{z^2x} = \dfrac{dz}{y^2x}$.

Solution Taking $dx/(z^2y) = dy/(z^2x)$, we get $xdx - ydy = 0$, which on integration, yields

$$x^2 - y^2 = C \tag{39}$$

Now take $dy/(z^2x) = dz/(y^2x)$, to get

$$y^2dy - z^2dz = 0$$

which on integration gives

$$y^3 - z^3 = C' \tag{40}$$

Equations (39) and (40) taken together constitute the required solution of the given equation.

2.12.2 Method of Multipliers

By a proper choice of multipliers l, m, n which are not necessarily constants, we write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

such that $lP + mQ + nR = 0$. Then $ldx + mdy + ndz = 0$ can be solved giving the solution as

$$\phi(x, y, z) = C \tag{41}$$

Again look for another set of multipliers λ, μ, γ such that $\lambda P + \mu Q + \gamma R = 0$, giving

$\lambda dx + \mu dy + \gamma dz = 0$, which on integration gives the solution as

$$\psi(x, y, z) = C' \tag{42}$$

Equations (41) and (42) taken together form the required solution.

Example 2.50 Solve $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{-y(z^2 + x^2)} = \dfrac{dz}{z(x^2 + y^2)}$.

Solution Using the multipliers x, y, z,

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

Thus, $xdx + ydy + zdz = 0$, which on integration gives

$$x^2 + y^2 + z^2 = C \tag{43}$$

Again, using the multipliers $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$,

$$\text{Each fraction} = \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)}$$

$$= \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0}$$

Thus, $\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0$ which becomes, on integration,

$$\log x - \log y - \log z = \text{constant}$$

or

$$yz = C'x \tag{44}$$

Hence, the solution of the given equation is

$$x^2 + y^2 + z^2 = C, \qquad yz = C'x$$

EXERCISES

Solve the following differential equations:

1. $(xy^2 + x)dx + (yx^2 + y)dy = 0$.

2. $(e^y + 1)\cos x dx + e^y \sin x dy = 0$.

3. $\dfrac{dy}{dx} = xy + x + y + 1$.

4. $\dfrac{dy}{dx} = (4x + y + 1)^2$.

5. $\sqrt{(1+x^2+y^2+x^2y^2)}+xy\dfrac{dy}{dx}=0.$

6. $(2ax+x^2)\dfrac{dy}{dx}=a^2+2ax.$

7. $\dfrac{dy}{dx}+\dfrac{x^2+3y^2}{3x^2+y^2}=0.$

8. $x^2y\,dx-(x^3+y^3)dy=0.$

9. $x(x-y)\dfrac{dy}{dx}=y(x+y).$

10. $(1+e^{x/y})dx+e^{x/y}\left(1-\dfrac{x}{y}\right)dy=0.$

11. $x\sin\left(\dfrac{y}{x}\right)\dfrac{dy}{dx}=y\sin\left(\dfrac{y}{x}\right)-x.$

12. $\dfrac{dy}{dx}=\dfrac{y}{x}+\tan\dfrac{y}{x}.$

13. $\dfrac{dy}{dx}=\dfrac{x+y+1}{x-y}.$

14. $(x-y-2)dx=(2x-2y-3)dy.$

15. $\dfrac{dy}{dx}=\dfrac{1-3x-3y}{2(x+y)}.$

16. $(6x+2y-10)\dfrac{dy}{dx}=2x+9y-20.$

17. $\dfrac{dy}{dx}=\dfrac{x+y+1}{2x+2y+3}.$

18. $(7y-3x+3)dy+(3y-7x+7)dx=0.$

19. $x(x-1)\dfrac{dy}{dx}-(x-2)y=x^3(2x-1).$

20. $\cos^3 x\dfrac{dy}{dx}+y\cos x=\sin x.$

21. $x\dfrac{dy}{dx}+2y=x^2\log x.$

22. $\dfrac{dy}{dx}=\dfrac{\sin^2 x}{1+x^3}-\dfrac{3x^2}{1+x^3}y$

23. $\dfrac{dy}{dx}+\dfrac{y}{(1-x)\sqrt{x}}=1-\sqrt{x}.$

24. $(1+y^2)+(x-e^{\tan^{-1}y})\dfrac{dy}{dx}=0.$

25. $(x+\tan y)dy=\sin 2y\,dx.$

26. $xy-\dfrac{dy}{dx}=y^3e^{-x^2}.$

27. $(x+y+1)\dfrac{dy}{dx}=1.$

28. $(2x-10y^3)\dfrac{dy}{dx}+y=0.$

29. $x\dfrac{dy}{dx}+y=xy^3.$

30. $\dfrac{dy}{dx}+\dfrac{y}{x}=y^2\sin x.$

31. $3\dfrac{dy}{dx}+\dfrac{2y}{x+1}=\dfrac{x^3}{y^2}.$

32. $\dfrac{dy}{dx}=\dfrac{x^2+y^2+1}{2xy}.$

33. $\dfrac{dy}{dx}=2y\tan x+y^2\tan^2 x.$

34. $\cos x\,dy=y(\sin x-y)dx.$

35. $x\dfrac{dy}{dx}+y\log y=xye^x$

36. $\dfrac{dy}{dx}=x^3y^3-xy.$

37. $3e^x\tan y+(1-e^x)\sec^2 y\dfrac{dy}{dx}=0.$

38. $(x^2y^3+xy)dy-dx=0.$

39. $(x^2-2xy-y^2)dx-(x+y)^2dy=0.$

40. $y \sin 2x\, dx - (1 + y^2 + \cos^2 x)dy = 0.$

41. $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x - x\sin y)\,dy=0.$

42. $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0.$

43. $ydx - xdy + (1 + x^2)dx + x^2 \sin y\, dy = 0.$

44. $2xy^2dx = e^x(dy - ydx).$

45. $ydx - xdy + \log x dx = 0.$

46. $y(2x^2y + e^x)dx - (e^x + y^3)dy = 0.$

47. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}.$

48. $(x^3y^3 + x^2y^2 + xy + 1)ydx + (x^3y^3 - x^2y^2 - xy + 1)xdy = 0.$

49. $y(1 - xy)dx - x(1 + xy)dy = 0.$

50. $(x^4y^4 + x^2y^2 + xy)ydx + (x^4y^4 - x^2y^2 + xy)xdy = 0.$

51. $(xy \sin xy + \cos xy)ydx + (xy \sin xy - \cos xy)xdy = 0.$

52. $(x^3 - 2y^2)dx + 2xydy = 0.$

53. $(x^2 + y^2 + 2x)dx + 2ydy = 0.$

54. $(2x^2y - 3y^2)dx + (2x^3 - 12xy + \log y)dy = 0.$

55. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0.$

56. $(20x^2 + 8xy + 4y^2 + 3y^3)ydx + 4(x^2 + xy + y^2 + y^3)xdy = 0.$

57. $(3x + 2y^2)ydx + 2x(2x + 3y^2)dy = 0.$

58. $x(4ydx + 2xdy) + y^3(3ydx + 5xdy) = 0.$

59. $x(3ydx + 2xdy) + 8y^4(ydx + 3xdy) = 0.$

60. $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0.$

61. $\dfrac{xdx + ydy}{xdy - ydx} = \sqrt{\dfrac{a^2 - x^2 - y^2}{x^2 + y^2}}.$

62. $(y + z)dx + (z + x)dy + (x + y)dz = 0.$

63. $yzdx - 2xzdy + (xy - zy^3)dz = 0.$

64. $(x + z)^2dy + y^2(dx + dz) = 0.$

65. $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{nxy}.$

66. $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dz}{ly - mx}.$

67. $\dfrac{dx}{y - zx} = \dfrac{dy}{yz + x} = \dfrac{dz}{x^2 + y^2}.$

68. $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)}.$

69. $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}.$

70. Using the substitution $y^2 = v - x$, reduce the equation

$$y^3\frac{dy}{dx} + x + y^2 = 0$$

to the homogeneous form and hence solve the equation.

3

Equations of the First Order but not of the First Degree

The most general form of a differential equation of the first order but not of the first degree (say nth degree) is

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots + P_{n-1}\left(\frac{dy}{dx}\right) + P_n = 0 \qquad (1)$$

or

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0$$

where $p = dy/dx$ and $P_1, P_2, \ldots, P_n$ are functions of x and y. This equation can also be written as

$$F(x, y, p) = 0 \qquad (2)$$

The above equation however cannot be solved in this general form. We will discuss here the situations where a solution of this equation exists. Let us consider two cases:

CASE I. In this case, the first member of Eq. (1) can be resolved into rational factors of the first degree.

CASE II. Here the member cannot be thus factored.

3.1 CASE I

3.1.1 Equations Solvable for p

Suppose a differential equation can be solved for p and is of the form

$$[p - f_1(x, y)]\,[p - f_2(x, y)] \ldots [p - f_n(x, y)] = 0$$

Equating each factor to zero we get equations of the first order and the first degree. Let their solutions be

$$\phi_1(x, y, c_1) = 0, \qquad \phi_2(x, y, c_2) = 0, \ldots, \qquad \phi_n(x, y, c_n) = 0$$

Without any loss of generality, we can write

$$c_1 = c_2 = \cdots = c_n = c$$

as they are arbitrary constants. Therefore, the solution of Eq. (1) can be put in the form

$$\phi_1(x, y, c)\, \phi_2(x, y, c) \cdots \phi_n(x, y, c) = 0$$

Example 3.1 Solve $(p - xy)(p - x^2)(p - y^2) = 0$.

Solution Here, $p = xy, x^2, y^2$. If $p = xy$, then $dy/y = xdx$, which on integration gives $\log y = (1/2)x^2 + c_1$. If $p = x^2$, then integration yields $y = (1/3)x^3 + c_2$. If $p = y^2$, then the solution is $-1/y = x + c_3$. Therefore, the required solution of the given equation is

$$\left(\log y - \frac{1}{2}x^2 - c\right)\left(y - \frac{1}{3}x^3 - c\right)\left(x + \frac{1}{y} + c\right) = 0$$

Example 3.2 Solve $(p + y + x)(xp + y + x)(p + 2x) = 0$.

Solution Here, we have

$$p + y + x = 0, \qquad xp + y + x = 0, \qquad p + 2x = 0$$

If $p + y + x = 0$, then $dy/dx + y + x = 0$. Put $x + y = v$; then this equation becomes

$$\frac{dv}{1-v} = dx \qquad \text{or} \qquad -\log(1 - v) = x + c_1$$

or

$$(1 - v) = e^{-x-c_1} = ce^{-x}$$

or

$$1 - x - y - ce^{-x} = 0 \tag{3}$$

If $xp + y + x = 0$, then $dy/dx + (1/x)y = 1$, whose solution is

$$yx = \frac{1}{2}c_2 - \frac{1}{2}x^2 \qquad \text{or} \qquad 2xy + x^2 - c_2 = 0 \tag{4}$$

Finally, if $p + 2x = 0$, then the solution is

$$y + x^2 - c_3 = 0 \tag{5}$$

From Eqs. (3)–(5), the solution of the given equation is

$$(1 - x - y - ce^{-x})(2xy + x^2 - c)(y + x^2 - c) = 0$$

Example 3.3 Solve $p^2 + 2py \cot x = y^2$.

Solution Solving the given equation for p, we get

$$p = \frac{1}{2}\left[-2y\cot x \pm \sqrt{(4y^2\cot^2 x + 4y^2)}\right] = -y \cot\frac{x}{2},\ y\tan\frac{x}{2}$$

If $p = -y \cot(x/2)$, then by integrating, we have

$$\log y = -2\log\sin\frac{x}{2} + \log A$$

Solving, we get

$$y = \frac{A}{\sin^2(x/2)} \qquad \text{or} \qquad y(1 - \cos x) = 2A = c_1$$

If $p = y \tan (x/2)$, then by integrating, we get

$$\log y = 2 \log \sec \frac{x}{2} + \log B$$

or

$$y(1 + \cos x) = 2B = c_2$$

Therefore, the required solution is

$$[y(1 - \cos x) - c]\,[y(1 + \cos x) - c] = 0$$

Example 3.4 Solve $x^2p^2 + xyp - 6y^2 = 0$.

Solution Solving the given equation for p, we have

$$p = \frac{2y}{x}, \quad -\frac{3y}{x}$$

If $p = 2y/x$, then $\log y = 2 \log x + \log c_1$, or $y = c_1x^2$; and if $p = -3y/x$, then $yx^3 = c_2$. Therefore, the required solution is

$$(y - cx^2)\,(yx^3 - c) = 0$$

Example 3.5 Solve $xy^2(p^2 + 2) = 2py^3 + x^3$.

Solution The given equation can be written as

$$(yp - x)\,[xyp + (x^2 - 2y^2)] = 0$$

If $yp - x = 0$, then integration yields

$$y^2 - x^2 = c_1 \tag{6}$$

If $xyp + x^2 - 2y^2 = 0$, then

$$2y\frac{dy}{dx} - \frac{4y^2}{x} = -2x$$

or

$$\frac{dv}{dx} - \frac{4}{x}v = -2x$$

where $v = y^2$. This is a linear differential equation in v and its solution is

$$\frac{v}{x^4} = c_2 + \frac{1}{x^2}$$

or

$$y^2 = c_2x^4 + x^2 \tag{7}$$

From Eqs. (6) and (7), the required solution is

$$(y^2 - x^2 - c)(y^2 - cx^4 - x^2) = 0$$

Example 3.6 Solve $x^2p^3 + y(1 + x^2y)p^2 + y^3p = 0$.

Solution The given equation can be written as

$$p(x^2p + y)(p + y^2) = 0$$

If $p = 0$, then

$$dy = 0 \qquad \text{or} \qquad y = c_1 \tag{8}$$

If $x^2p + y = 0$, then

$$\frac{dy}{dx} + \frac{y}{x^2} = 0$$

which is a linear equation and the solution is

$$ye^{-1/x} = c_2 \tag{9}$$

If $p + y^2 = 0$, then integration yields

$$xy + c_3y - 1 = 0 \tag{10}$$

Therefore, the required solution is

$$(y - c)(ye^{-1/x} - c)(xy + cy - 1) = 0$$

3.2 CASE II

Equation (2) may have one or more of the following properties:

(a) It may be solvable for y.
(b) It may be solvable for x.
(c) It may not contain either x or y.
(d) It may be homogeneous in x and y.
(e) It may be of the first degree in x and y.

3.2.1 Equations Solvable for y

If the differential equation $f(x, y, p) = 0$ is solvable for y, then

$$y = f(x, p) \tag{11}$$

Differentiating with respect to x, gives

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right) \tag{12}$$

which is an equation in two variables x and p, and it will give rise to a solution of the form

$$F(x, p, c) = 0 \tag{13}$$

The elimination of p between Eqs. (11) and (13) gives a relation between x, y and c, which is the required solution. When the elimination of p between these equations is not easily done, the values of x and y in terms of p can be found, and these together will constitute the required solution.

Example 3.7 Solve $y + px = x^4p^2$.

Solution The given equation is

$$y = x^4p^2 - xp \tag{14}$$

Differentiating with respect to x yields, after simplification,

$$\frac{dp}{p} + 2\frac{dx}{x} = 0$$

which on integration gives

$$p = \frac{c}{x^2}$$

Substitution of this value of p in Eq. (14) gives the required solution as

$$xy + c = c^2x$$

Example 3.8 Solve $y = \sin p - p \cos p$.

Solution Differentiating the given equation with respect to x, we get

$$\sin p \, dp = dx$$

Integrating, we have

$$\cos p = c - x \tag{15}$$

From the given equation, we also have

$$p \cos p = \sin p - y \qquad \text{or} \qquad p = \frac{\sqrt{1-\cos^2 p} - y}{\cos p}$$

or

$$c - x = \cos\left(\frac{\sqrt{1-c^2-x^2+2cx} - y}{c-x}\right)$$

which is the required solution.

Example 3.9 Solve $y = yp^2 + 2px$.

Solution The given equation can be written as

$$y = \frac{2px}{1-p^2} \tag{16}$$

Differentiating Eq. (16) with respect to x, we get

$$\frac{2dp}{p(p^2-1)} = \frac{dx}{x}$$

or

$$\left(\frac{1}{p-1} + \frac{1}{p+1} - \frac{2}{p}\right)dp = \frac{dx}{x}$$

which on integration gives

$$\log (p - 1) + \log (p + 1) - 2 \log p = \log x + \log c$$

or

$$\frac{p^2 - 1}{p^2} = cx \qquad \text{or} \qquad p = \frac{1}{\sqrt{1 - cx}}$$

Substituting this value of p in Eq. (16), we get

$$2x\sqrt{1 - cx} + cxy = 0$$

which is the required solution.

3.2.2 Equations Solvable for x

When the differential Eq. (2) is solvable for x, then we have

$$x = f(y, p)$$

Differentiating with respect to y, gives

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right)$$

from which a relation between p and y may be obtained

$$F(y, p, c) = 0 \text{ (say)}$$

Between this and the given equation, p may be eliminated, or x and y expressed in terms of p as in Section 3.2.1.

Example 3.10 Solve $y^2 \log y = xyp + p^2$.

Solution The given equation can be written as

$$x = \frac{1}{p} y \log y - \frac{p}{y}$$

Differentiating with respect to y, we get, after simplification,

$$\frac{dp}{p} = \frac{dy}{y}$$

which on integration gives

$$\log p = \log y + \log c = \log cy \qquad \text{or} \qquad p = cy$$

Eliminating p from this and the given equation, we obtain

$$\log y = cx + c^2$$

which is the required solution.

Example 3.11 Solve $xp^3 = a + bp$.

Solution Solving for x, we get

$$x = \frac{a}{p^3} + \frac{b}{p^2} \tag{17}$$

Differentiating Eq. (17) with respect to y, we get

$$dy + \left(\frac{3a}{p^3} + \frac{2b}{p^2}\right) dp = 0$$

Integrating it, we have

$$y = \frac{3a}{2p^2} + \frac{2b}{p} + c \tag{18}$$

Equations (17) and (18) constitute the required solution.

Example 3.12 Solve $x = y + a \log p$.

Solution Differentiating the given equation with respect to y, we obtain

$$dy = \frac{a}{1-p}\,dp$$

which on integration gives

$$y = c - a \log (1 - p) \tag{19}$$

From the given equation, we have

$$x = c - a \log (1 - p) + a \log p \tag{20}$$

Equations (19) and (20) give the required solution.

3.2.3 Equations That do not Contain x (or y)

If the equation has the form $f(y, p) = 0$ and is solvable for p, it will then give $dy/dx = \phi(y)$, which is integrable.

If it is solvable for y, then

$$y = F(p),$$

which is the case of Section 3.2.1. When the equation is of the form $f(x, p) = 0$, it will give $dy/dx = \phi(x)$, which is also integrable. But if it is solvable for x, then $x = F(p)$, which is the case of Section 3.2.2.

It may be mentioned that in equations having either of the properties (c) and not solvable for p, on solving for x or y, the differentiation is made w.r.t. absent variable.

By differentiating the equation given in Sections 3.2.1–3.2.3, we have a chance of obtaining a differential equation, by means of which another relation may be found between p and x or y in addition to the original relation. These two relations will then be used either for the elimination of p or for the expression of x and y in terms of p.

3.2.4 Equations Homogeneous in x and y

When the equation is homogeneous in x and y, it can be written as

$$F\left(\frac{dy}{dx}, \frac{y}{x}\right) = 0$$

It is then possible to solve it for dy/dx and proceed as in Section 2.3, or solve it for y/x, and obtain $y = xf(p)$ which is given in Section 3.2.1.

Proceeding as in Section 3.2.1, and differentiating with respect to x, we get

$$p = f(p) + xf'(p)\frac{dp}{dx}$$

from which

$$\frac{dx}{x} = \frac{f'(p)dp}{p - f(p)}$$

where the variables are separated.

3.2.5 Equations of the First Degree in x and y — Clairaut's Equation

When the given Eq. (2) is of the first degree in x and y, then

$$y = xf_1(p) + f_2(p) \tag{21}$$

Equation (21) is known as *Lagrange's equation*. To solve it, we differentiate with respect to x to obtain

$$p = \frac{dy}{dx} = f_1(p) + xf_1'(p) + f_2'(p)\frac{dp}{dx}$$

or

$$\frac{dp}{dx} - \frac{f_1'(p)}{p - f_1(p)}x = \frac{f_2'(p)}{p - f_1(p)} \tag{22}$$

which is a linear equation in x and p, and hence can be solved in the form

$$x = \phi(p, c) \tag{23}$$

Eliminating p from Eqs. (21) and (23), we get the required solution. If it is not possible to eliminate p, then the values of x and y in terms of p can be found from Eqs. (21) and (23), and these will constitute the required solution.

If $f_1(p) = p$ and $f_2(p) = f(p)$, then Eq. (21) reduces to

$$y = px + f(p) \tag{24}$$

Equation (24) is known as *Clairaut's equation*. To solve it, we differentiate with respect to x to obtain

$$p = \frac{dy}{dx} = [x + f'(p)]p' + p$$

or

$$[x + f'(p)]\frac{dp}{dx} = 0 \tag{25}$$

If $dp/dx = 0$, then $p = c$ = constant. Eliminating p between this and Eq. (24), we get

$$y = cx + f(c) \tag{26}$$

which is the required solution of Clairaut's equation.

REMARK. If we eliminate p between

$$x + f'(p) = 0$$

and Eq. (24), we get a solution which does not contain any arbitrary constant, and hence, is not a particular case of solution (26). Such a solution is called *singular solution.*

Sometimes by a suitable substitution, an equation can be reduced to Clairaut's form.

Example 3.13 Solve $(y - px)(p - 1) = p$.

Solution The given equation can be written as

$$y = xp + \frac{p}{p-1}$$

Differentiation yields after simplification

$$\frac{dp}{dx}\left[x - \frac{1}{(p-1)^2}\right] = 0$$

Therefore,

$$\frac{dp}{dx} = 0 \quad \text{or} \quad p = c$$

From this and the given equation, the elimination of p gives

$$(y - cx)(c - 1) = c$$

which is the required solution.

Example 3.14 Solve $p = \log(px - y)$.

Solution The given equation is

$$y = px - e^p.$$

Differentiating with respect to x, we get

$$(x - e^p)\frac{dp}{dx} = 0$$

or

$$\frac{dp}{dx} = 0 \quad \text{or} \quad p = c$$

Eliminating p from this and the given equation, we get the required solution as

$$c = \log\,(cx - y)$$

Example 3.15 Find the general and singular solution of the differential equation $y = px + \sqrt{(a^2p^2 + b^2)}$.

Solution Differentiating the given equation with respect to x, we have

$$[x + a^2p(a^2p^2 + b^2)^{-1/2}]\,\frac{dp}{dx} = 0 \tag{27}$$

If $dp/dx = 0$, then $p = c$. With this and Eq. (27), the general solution is

$$y = cx + \sqrt{a^2c^2 + b^2}$$

Also, from Eq. (27)

$$x + a^2p(a^2p^2 + b^2)^{-1/2} = 0$$

or

$$p = \frac{bx}{a\sqrt{a^2 - x^2}}$$

Using this value of p and Eq. (27), the singular solution is obtained as

$$y^2a^2(a^2 - x^2) = b^2(x^2 + a^2)$$

Example 3.16 Reduce $xyp^2 - (x^2 + y^2 + 1)\,p + xy = 0$ to Clairaut's form and find its singular solution.

Solution Let $x^2 = u$ and $y^2 = v$, then the given equation becomes

$$u\left(\frac{dv}{du}\right)^2 - (u + v - 1)\frac{dv}{du} + v = 0$$

or

$$uP^2 - (u + v + 1)\,P + v = 0, \qquad \text{where } P = \frac{dv}{du}$$

or

$$v = uP + \frac{P}{P-1} \tag{28}$$

which is of Clairaut's form. Differentiating it with respect to u, we get

$$\left[u - \frac{1}{(P-1)^2}\right]\frac{dP}{du} = 0 \tag{29}$$

To get the singular solution, we consider

$$u - \frac{1}{(P-1)^2} = 0$$

which gives $P = 1 + (1/\sqrt{u})$. Putting this in Eq. (28), we get the required solution as

$$y^2 = (x + 1)^2$$

Example 3.17 Solve $(px - y)(py + x) = h^2p$.

Solution Putting $x^2 = u$ and $y^2 = v$, the given equation takes the form

$$u\left(\frac{dv}{du}\right)^2 + (u - v - h^2)\frac{dv}{du} - v = 0$$

or

$$uP^2 + (u - v - h^2)P - v = 0, \qquad \text{where } P = \frac{dv}{du}$$

or

$$v = uP - \frac{h^2P}{P+1}$$

which is of Clairaut's form and has the solution as

$$v = uc - \frac{h^2c}{c+1}$$

where $u = x^2$ and $v = y^2$.

Example 3.18 Solve $y = 2px + y^2p^3$.

Solution Multiplying the given equation by y and then putting $y^2 = v$, we obtain

$$v = x\frac{dv}{dx} + \frac{1}{8}\left(\frac{dv}{dx}\right)^3 = xP + \frac{1}{8}P^3$$

where $P = dv/dx$. This equation is of Clairaut's form and the solution is

$$v = cx + \frac{1}{8}c^3 \qquad \text{or} \qquad y^2 = cx + \frac{1}{8}c^3$$

EXERCISES

Solve the following differential equations:

1. $p^2 - 7p + 12 = 0.$
2. $yp^2 + (x - y)p - x = 0.$
3. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$
4. $4y^2p^2 + 2xyp(3x + 1) + 3x^3 = 0.$
5. $p^3(x + 2y) + 3p^2(x + y) + (y + 2x)p = 0.$
6. $x^2p^2 - 2xyp + (2y^2 - x^2) = 0.$
7. $xp^2 - 2yp + ax = 0.$
8. $y = 2px + \tan^{-1}(xp^2).$
9. $y = 3x + \log p.$
10. $x^2 + p^2x = yp.$
11. $p = \tan\left(x - \dfrac{p}{1+p^2}\right).$
12. $x^2 = a^2(1 + p^2).$
13. $x = y - p^2.$
14. $p^3 - p(y + 3) + x = 0.$
15. $y = 2px + y^{n-1}p^n.$
16. $y = 2p + 3p^2.$
17. $x(1 + p^2) = 1.$
18. $y^2 + xyp - x^2p^2 = 0.$
19. $y = yp^2 + 2px.$
20. $(x - a)p^2 + (x - y)p - y = 0.$
21. $\sin px \cos y = \cos px \sin y + p.$
22. $xy(y - px) = x + py.$
23. $(x^2 + y^2)(1 + p)^2 - 2(x + y)(1 + p)(x + py) + (x + yp)^2 = 0.$
    [*Hint*: Put $x^2 + y^2 = u$ and $x + y = v$.]
24. Solve $x^2p^2 + yp(2x + y) + y^2 = 0$ by reducing it to Clairaut's form by using the substitution $y = u$ and $xy = v$.
25. Use the transformation $x^2 = u$ and $y^2 = v$ to solve the equation
$$axyp^2 + (x^2 - ay^2 - b)p - xy = 0.$$

4

Applications of First Order Differential Equations

In this chapter, we shall give some applications of the differential equations which appeared in Chapters 2 and 3. The examples worked out here have been chosen from the fields of engineering, physics, chemistry, geology, biological and social sciences, national defence, and so on.

4.1 GROWTH AND DECAY

The initial value problem

$$\frac{dx}{dt} = kx, \qquad x(t_0) = x_0 \tag{1}$$

where k is a constant, occurs in many physical theories involving either growth or decay. For example, in biology it is often observed that the rate at which certain bacteria grow is proportional to the number of bacteria present at any time. Over short intervals of time, the population of small animals, such as rodents, can be predicted quite accurately by the solution of Eq. (1). The constant k can be obtained from the solution of the differential equation by using a subsequent measurement of the population at a time $t_1 > t_0$.

In physics, an initial value problem such as the one shown in Eq. (1) provides a model for approximating the remaining amount of a substance which is disintegrating through radioactivity. Equation (1) can also be used to determine the temperature of a cooling body. In chemistry, the amount of a substance remaining during a reaction is also described by Eq. (1).

We shall now illustrate, with the help of the following solved examples, how Eq. (1) works.

Example 4.1 A culture initially has N_0 number of bacteria. At $t = 1$ hr. the number of bacteria is measured to be $(3/2)N_0$. If the rate of growth is proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple.

Solution The present problem is governed by the differential equation

$$\frac{dN}{dt} = kN \tag{2}$$

subject to $N(0) = N_0$.

Separating the variables in (2) and solving, we have

$$N = N(t) = ce^{kt}$$

At $t = 0$, we have $N_0 = ce^0 = c$ and so $N(t) = N_0e^{kt}$. At $t = 1$, we have $(3/2)\ N_0 = N_0e^k$ or $e^k = 3/2$, which gives $k = \log\ (3/2) = 0.4055$. Thus

$$N(t) = N_0e^{0.4055t}$$

To find the time at which the bacteria have tripled, we solve

$$3N_0 = N_0e^{0.4055t}$$

for t, to get

$$0.4055t = \log 3$$

or

$$t = \frac{\log 3}{0.4055} = 2.71 \text{ hr (approx.)}$$

Note: The function $N(t)$, using the laws of exponents, can also be written as

$$N(t) = N_0(e^k)^t = N_0\left(\frac{3}{2}\right)^t$$

since, $e^k = 3/2$. This latter solution provides a convenient method for computing $N(t)$ for small positive integral values of t; it also shows the influence of the subsequent experimental observation at $t = 1$ on the solution for all time. Also, it may be noticed that the actual number of bacteria present at time $t = 0$ is quite irrelevant in finding the time required to triple the number in the culture. The necessary time to triple, say, 100 or 10,000 bacteria is still approximately 2.71 hours.

As shown in the Fig. 4.1, exponential function e^{kt} increases as t increases for $k > 0$, and decreases as t decreases. Thus, problems describing growth, such as population, bacteria, or even capital, are characterized by a positive value of k, whereas problems involving decay, as in radioactive disintegration, will yield a negative value.

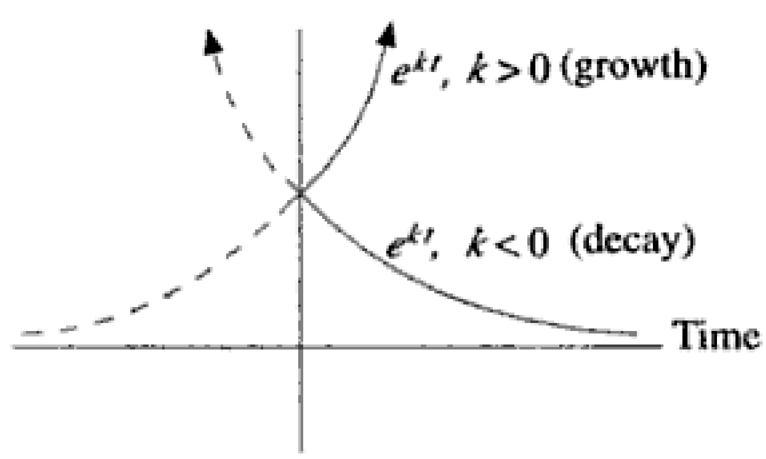

Fig. 4.1 Exponential growth and decay.

Example 4.2 Bacteria in a certain culture increase at a rate proportional to the number present. If the number N increases from 1000 to 2000 in 1 hour, how many are present at the end of 1.5 hours?

Solution The differential equation $dN/dt = kN$ has the solution $N = N_0e^{kt}$. Since, $N = 1000$ when $t = 0$, $N_0 = 1000$. Setting $N = 2000$, and $t = 1$ in $N = 1000\,e^{kt}$, we obtain, $e^k = 2$. Thus, $N(t) = 1000(e^k)^t = 1000(2)^t$, and $N(1.5) = 1000(2)^{1.5} = 2828.4271$.

Example 4.3 In a culture of yeast, the amount A of active yeast grows at a rate proportional to the amount present. If the original amount A_0 doubles in 2 hours, how long does it take for the original amount to triple?

Solution The amount A_0 grows exponentially according to $A = A_0e^{kt}$. Since, $A = 2A_0$ when $t = 2$, $2A_0 = A_0e^{2k}$ and $e^{2k} = 2$. Thus, at time t

$$A = A_0(e^{2k})^{t/2} = A_0(2)^{t/2}$$

Setting $A = 3A_0$ and solving for t, we get

$$3A_0 = A_0(2)^{t/2}$$

or

$$\log 3 = \frac{t}{2}\log 2, \qquad t = \frac{2\log 3}{\log 2} = 3.1699 \text{ hr}$$

Example 4.4 Bacteria in a certain culture increase at a rate proportional to the number present. If the number doubles in one hour, how long does it take for the number to triple?

Solution Let y denote the numbers present at time t. Then the function denoted by $y = f(t)$, satisfies the differential equation

$$\frac{dy}{dt} = ky$$

and the conditions $t = 0$, $y = y_0$; $t = 1$, $y = 2y_0$. Writing the differential equation in linear form, i.e.

$$\frac{dy}{dt} - ky = 0$$

we see that $e^{\int -k\,dt} = e^{-kt}$ is an integrating factor and hence, the equation has the solution

$$ye^{-kt} = \int 0\,dt = c \qquad \text{or} \qquad y = ce^{kt}$$

From the condition $t = 0$, $y = y_0$, we find that $y_0 = c$. The additional condition $t = 1$, $y = 2y_0$ is now used to find the constant of proportionality k. From $2y_0 = y_0e^{k(1)}$, we obtain $k = \log 2$. Hence, $y = y_0e^{t\log 2}$. Substituting $y = 3y_0$ and solving for t yields

$$3y_0 = y_0e^{t\log 2} \quad \text{or} \quad \log 3 = t\log 2$$

Hence, the number will be triple in

$$t = \frac{\log 3}{\log 2} = 1.5849 \text{ hr}$$

4.2 DYNAMICS OF TUMOUR GROWTH

It has been observed experimentally that free-living dividing cells, such as bacteria cells, grow at a rate proportional to the volume of dividing cells at that moment. Let $V(t)$ denote the volume of dividing cells at time t. Then

$$\frac{dV}{dt} = kV \tag{3}$$

for some positive constant k. The solution of Eq. (3) is

$$V(t) = V_0 e^{k(t-t_0)} \tag{4}$$

where V_0 is the volume of dividing cells at time t_0 (initial time). Thus, free-living dividing cells grow exponentially with time, whereas solid tumours do not grow exponentially with time. As the tumour becomes larger, the doubling time of the total tumour volume continuously increases. A number of researchers have shown that the data for many solid tumours is fitted remarkably well, over almost a thousand-fold increase in tumour volume, by the equation

$$V(t) = V_0 \exp\left[\frac{k}{a}(1 - e^{-at})\right] \tag{5}$$

where k and a are positive constants.

Equation (5) is usually known as a *Gompertzian relation*. It states that tumour grows more and more slowly with the passage of time, and that it ultimately approaches the limiting volume $V_0 e^{k/a}$. Medical scientists have long been concerned with explaining this deviation from simple exponential growth. An insight into this problem can be gained by finding a differential equation satisfied by $V(t)$.

Differentiation of Eq. (5) yields

$$\frac{dV}{dt} = V_0 k e^{-at} \exp\left[\frac{k}{a}(1 - e^{-at})\right] = ke^{-at}V \tag{6}$$

Equation (6) can also be arranged as

$$\frac{dV}{dt} = (ke^{-at})V \tag{6a}$$

$$\frac{dV}{dt} = k(e^{-at}V) \tag{6b}$$

With these arrangements of Eq. (6), two theories have been evolved for the dynamics of tumour growth. According to the first theory, the retarding effect of tumour growth is due to an increase in the mean generation time of the cells, without a change in the proportion of reproducing cells. As time goes on, the reproducing cells mature or age, and thus divide more slowly. This theory corresponds to Eq. (6a). On the other hand, the second theory corresponding to Eq. (6b), suggests that the mean generation time of the dividing cells remains constant, and the retardation of growth is due to a loss in reproductive cells in the tumour. One possible explanation for this is that a necrotic region develops in the centre of the tumour. This necrosis appears at a critical size for a particular type of tumour, and thereafter the necrotic "core" increases rapidly as the total tumour mass increases. According to this theory a necrotic core develops because in many tumours the supply of blood, and thus of oxygen and nutrients, is almost completely confined to the surface of the tumour and a short distance beneath it. As the tumour grows, the supply of oxygen to the central core by diffusion becomes more and more difficult resulting in the formation of a necrotic core.

4.3 RADIOACTIVITY AND CARBON DATING

The science of radiogeology applies our knowledge of radioactivity to geology. It is known that uranium 238 undergoes radioactive decay with half-life $T = 4.55$ billion years. During decay, it becomes radium 226 and eventually ends as non-radioactive lead 206. Radioactive dating uses this knowledge to estimate the date of events that took place long ago. One technique uses the ratio of lead to uranium in a rock formation to estimate the time that has elapsed since the lava solidified and formed the mass of the rock. The age of the solar system and hence, of the earth has been estimated by radioactive dating as 4.5 billion years. Other elements such as potassium (half-life 13.9 billion years) and rubidium (half-life 50 billion years) are also used in radioactive dating.

An interesting description of use of Pb 210 or white lead (half-life 22 years) is in determining whether a given oil painting is authentic or a forgery. For instance, it can be proved that the beautiful painting *Disciples at Emmaus* which was bought by the Rembrandt Society of Belgium for $170,000 was a modern forgery (for details see [11]).

An important breakthrough in radiography occurred in 1947 when an American chemist Willard Frank Libby discovered radiocarbon (a radioactive isotope of carbon), designated as carbon-14 (^{14}C). For this discovery and its application to radiogeology and radiochronology, Libby received the Nobel Prize in Chemistry in 1960. The basis of this method is as follows: The atmosphere of the earth is continuously bombarded by cosmic rays. These cosmic rays produce neutrons in the earth's atmosphere, and these neutrons combine with nitrogen to produce ^{14}C. This radiocarbon (^{14}C) is incorporated in carbon dioxide and thus moves through the atmosphere to be absorbed by plants. In turn, radiocarbon is built in animal tissues by eating the plants. In living tissues, the rate of ingestion of ^{14}C exactly balances the rate of disintegration of ^{14}C. When an organism dies, though, it ceases to ingest ^{14}C, its ^{14}C concentration begins to decrease through disintegration of the ^{14}C present. Now, it is a physically accepted fact that the rate of bombardment of the earth's atmosphere by cosmic rays has always been constant. This implies that the original rate of disintegration of ^{14}C in a sample such as charcoal is the same as the rate measured today.* This assumption enables us to find the age of a sample of charcoal.

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*Since the mid-1950's, the testing of nuclear weapons has significantly increased the amount of radiocarbon in our atmosphere. Ironically, this unfortunate state of affairs provides yet another extremely powerful method of detecting art forgeries. Most of the materials of artists, such as linseed oil and canvas paper, come from plants and animals, and so will contain the same concentration of carbon-14 as the atmosphere at the time the plant or animal dies. Thus linseed oil (which is derived from the flax plant) that was produced during the last few years will contain a much greater concentration of ^{14}C than linseed oil produced before 1950.

Let $N(t)$ denote the amount of carbon-14 present in a sample at time t, and N_0 denote the amount present at time $t = 0$ when the sample was formed. If k denotes the decay constant of ^{14}C (half-life 5568 years), then $dN(t)/dt = -kN(t)$, $N(0) = N_0$ and, consequently, $N(t) = N_0e^{-kt}$. Now, the present rate $R(t)$ of disintegration of ^{14}C in the sample is given by $R(t) = kN(t) = kN_0e^{-kt}$, and the original rate of disintegration is $R(0) = kN_0$. Thus, $R(t)/R(0) = e^{-kt}$ so that $t = (1/k) \log [R(0)/R(t)]$. Hence, if we measure $R(t)$, the present rate of disintegration of ^{14}C in the charcoal, and observe that $R(0)$ must equal the rate of disintegration of ^{14}C in a comparable amount of living wood, then we can compute the age t of the charcoal.

We shall work out now some examples about radioactive decay and carbon dating.

Example 4.5 It is found that 22 per cent of the original radiocarbon in a wooden archaeological specimen has decomposed. Use the half-life $T = 5568$ yr. of ^{14}C to compute the number of years since the specimen was a part of a living tree. (This should yield a good estimate of the time elapsed since the specimen, a wooden bowl, was used in an ancient civilization.)

Solution Let C be the amount of ^{14}C present at time t and C_0 the amount present at $t = 0$. Then $C = C_0e^{kt}$ gives $0.78C_0 = C_0e^{kt}$, and finally, $t = \log 0.78/k$. From $T = -\log 2/k = 5568$, we obtain $k = -\log 2/5568$ and, therefore,

$$t = \frac{5568 \log 0.78}{-\log 2} = 1996 \text{ yr. (approx.)}$$

At arbitrary time t,

$$C = C_0 \exp\left(\frac{-\log 2}{5568}\right)t = C_0(2)^{-t/5568}$$

Example 4.6 It is found that 0.5 per cent of radium disappears in 12 years. (a) What percentage will disappear in 1000 years? (b) What is the half-life of radium?

Solution Let A be the quantity of radium in grammes, present after t years. Then dA/dt represents the rate of disintegration of radium. According to the law of radioactive decay, we have

$$\frac{dA}{dt} \propto A \qquad \text{or} \qquad \frac{dA}{dt} = aA$$

Since A is positive and is decreasing, then $dA/dt < 0$ and we see that the constant of proportionality a must be negative. Writing $a = -k$, we get

$$\frac{dA}{dt} = -kA$$

Let A_0 be the amount, in grammes of radium present initially. Then $0.005A_0$ g disappears in 12 years, so that $0.995A_0$ g remains. We thus, have $A = A_0$ at $t = 0$, and $A = 0.995A_0$ g at $t = 12$ (years). The solution of the above equation is $A = Ce^{-kt}$. Since, $A = A_0$ at $t = 0$ and $C = A_0$, hence

$$A = A_0e^{-kt}$$

Also, at $t = 12$ and $A = 0.995A_0$, then

$$0.995A_0 = A_0e^{-12k} \quad \text{or} \quad e^{-12k} = 0.995 \quad \text{or} \quad e^{-k} = (0.995)^{1/12} \tag{7}$$

Hence

$$A = A_0e^{-kt} = A_0(e^{-k})^t = A_0(0.995)^{t/12} \tag{8}$$

or, if we solve for k in Eq. (7), we find $k = 0.000418$, so that

$$A = A_0e^{-0.000418t} \tag{9}$$

Thus, we have

(a) When $t = 1000$, from Eqs. (8) and (9), $A = 0.658A_0$, so that 34.2 per cent will disappear in 1000 years.

(b) The half-life of a radioactive substance is defined as the time it takes for 50 per cent of the substance to disappear. In our case, we have $A = 1/2A_0$ and using Eq. (9), we find $e^{-0.000418t} = 1/2$ or $t = 1672.1770$ years.

Example 4.7 A breeder reactor converts the relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is found that 0.043 per cent of the initial amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope, if the rate of disintegration is proportional to the remaining amount.

Solution Let $A(t)$ denote the amount of the plutonium remaining at any time. Then, the solution of the initial value problem

$$\frac{dA}{dt} = kA, \qquad A(0) = A_0$$

is $A(t) = A_0e^{kt}$.

If 0.043 per cent of the atoms of A_0 have disintegrated then 99.957 per cent of the substance remains. To find k, we solve $0.99957A_0 = A_0e^{15k}$ to get $e^{15k} = 0.99957$ or, $k = \log(0.99957)/15 = -0.00002867$. Hence, $A(t) = A_0e^{-0.00002867t}$.

Now, the half-life is the corresponding value of time for which $A(t) = A_0/2$. Solving for t, we get

$$\frac{A_0}{2} = A_0e^{-0.00002867t}$$

or

$$t = \frac{\log 2}{0.00002867} = 24176.74156 \text{ yr.}$$

Example 4.8 A fossilized bone is found to contain 1/1000 the original amount of ^{14}C. Determine the age of the fossil.

Solution We have

$$A(t) = A_0e^{kt}$$

When $t = 5568$ yr., $A(t) = A_0/2$, from which we can find the value of k as

$$\frac{A_0}{2} = A_0e^{5568k}$$

or

$$k = -\frac{\log 2}{5568} = -0.000124488$$

Therefore, $A(t) = A_0 e^{-0.000124488t}$, where $A(t) = 1/1000$, gives

$$t = \frac{\log 2}{0.000124488} = 55489.32651 \text{ yr}$$

4.4 COMPOUND INTEREST

Interest is defined as a charge for the borrowed money. It is difficult to give a precise definition of interest since there exist many different methods for computing interest. If a principal of P rupees, invested at an interest rate r per annum grows to $P(1 + r)$ rupees in 1 year, $P(1 + r)^2$ rupees in 2 years and $P(1 + r)^t$ rupees in t years, r is called the rate of interest per annum compounded annually. If the interest rate per annum is r and interest is compounded twice a year, P rupees grows to $P(1 + \frac{r}{2})$ rupees in 6 months, $P(1 + \frac{r}{2})^2$ in a year, $P(1 + \frac{r}{2})^3$ rupees in 1.5 years, and $P(1 + \frac{r}{2})^t$ in t years. If P rupees invested at interest rate r per annum with interest compounded k times per year, then the amount a of the original investment at the end of t year is

$$a = P\left(1 + \frac{r}{k}\right)^{kt} = f(t) \tag{10}$$

The quantity r/k is the interest rate applied at each compounding, and e^{kt} is the total number of compoundings in t years. For example, 1 rupee invested at 10 per cent per annum compounded annually ($r = 10$) grows to $1(1 + 0.1)^t = 1.10$ rupee in 1 year. One rupee invested at 10 per cent per annum compounded twice a year ($k = 2$) grows to $1(1 + 0.05)^2 = 1.1025$ rupees in 1 year. In the second instance, 10 per cent is called the nominal interest rate per annum and 10.25 per cent the effective interest rate per annum. Similarly, if the interest is compounded quarterly ($k = 4$), the same rupee becomes $1(1 + 0.025)^4 = 1.1038$ rupee in 1 year and the effective interest rate per annum is 10.38 per cent.

It can be proved easily that for fixed values of P, r and t, the value of a in Eq. (10) increases as k increases. As $k \to \infty$, the value of a does not increase without limit, but rather as

$$\lim_{k\to+\infty} a = \lim_{k\to+\infty} P\left(1 + \frac{r}{k}\right)^{kt} = P \lim_{k\to+\infty}\left[\left(1 + \frac{r}{k}\right)^{k/r}\right]^{rt}$$

$$= P\left[\lim_{k\to+\infty}\left(1 + \frac{r}{k}\right)^{k/r}\right]^{rt} = Pe^{rt}$$

where $e \simeq 2.71828$.

It is reasonable to argue that money should earn interest continuously. But even if the interest is compounded every second, then the function $f(t)$ in Eq. (10) is a step function and is not continuous. In order to have a continuous model for continuous compounding, define A as

$$A = Pe^{rt} = F(t) \tag{11}$$

This definition looks more reasonable as da/dt in Eq. (10) is zero or undefined, while dA/dt is more useful and meaningful. Also, the graph of f in Eq. (10) and F in Eq. (11) are virtually indistinguishable when k is very large. Equation (11), yields on differentiation

$$\frac{dA}{dt} = Pre^{rt} = rA \tag{12}$$

Thus, we see that money invested at compound interest compounded continuously grows according to the differential equation of the organic growth. Conversely, a quantity, such as the number of bacteria in a culture grows according to the compound interest law.

Example 4.9 If Rs. 10,000 is invested at 6 per cent per annum, find what amount has accumulated after 6 years if interest is compounded: (a) annually, (b) quarterly, and (c) continuously.

Solution

(a) $a = 10{,}000(1.06)^6 = \text{Rs. } 14{,}185.19$

(b) $a = 10{,}000\left(1 + \dfrac{0.06}{4}\right)^{4(6)} = 10{,}000(1.015)^{24} = \text{Rs. } 14{,}295.03$

(c) $A = 10{,}000e^{0.36} = \text{Rs. } 14{,}333.29$

Example 4.10 How long does it take for a given amount of money to double at 6 per cent per annum compounded: (a) annually, and (b) continuously?

Solution (a) We have $2P = P(1.06)^t$. Therefore,

$$\log 2 = t \log 1.06 \quad \text{or} \quad t = \frac{\log 2}{\log 1.06} = 11.89566 \text{ yr.}$$

(b) From question $2P = Pe^{0.06t}$. Then

$$\log 2 = 0.06t \quad \text{or} \quad t = \frac{\log 2}{0.06} = 11.55245 \text{ yr.}$$

Example 4.11 A savings and loan company advertises an interest rate per annum of 7.5 per cent compounded continuously. Find the effective interest rate per annum.

Solution The interest for 1 year will be

$$Pe^1 - P = \frac{P}{e^1 - 1}$$

Thus

$$e^{0.075} - 1 = 0.0779$$

and the effective rate per annum is 7.79 per cent. Many companies advertise an effective rate of 7.90 per cent. This is obtained by using a 360-day year and multiply 7.79 by 365/360. A company can thus make a more attractive offer without exceeding the nominal rate 7.5 per cent set by law. This illustrates our statement that there are many ways of calculating the interest.

REMARK. The nominal rate j, which by continuous compounding is equivalent to an effective rate i, is called the *force of interest*. It is useful in comparing various business propositions.

4.5 BELT OR CABLE FRICTION

Consider a belt (Fig. 4.2) wrapped around a drum of radius r. Suppose that tensions T_1 and T_0 at B and A, respectively, are such that the belt is on the point of slipping and that $T_1 > T_0$. The tension in the belt varies from $T = T_0$, corresponding to $\theta = 0$ to $T = T_1$, corresponding to $\theta = \angle AOB$, known as the *angle of wrap*. Let T denote the tension at the point P corresponding to a fixed but arbitrary value of θ and $T + \Delta T$ be the tension at a nearby point corresponding to $\theta + \Delta\theta$. The arc of the belt of length $\Delta s = r\Delta\theta$ is in equilibrium under the action of the forces having the magnitudes T, $T + \Delta T$, ΔN, ΔF, where ΔN is the magnitude of the normal force the drum exerts on the section of the belt of length Δs, and ΔF is the magnitude of the force of friction opposing slipping. The force having magnitude $T + \Delta T$ acts at an angle $\Delta\theta$ with the tangential direction t along which the force of magnitude T acts. As the section of the belt of length Δs is in equilibrium, the sum of the components of the four external forces acting on it in any direction must be zero. Choosing the tangential and normal directions t and n, we get

$$T + \Delta F - (T + \Delta T)\cos\Delta\theta = 0$$

$$\Delta N - (T + \Delta T)\sin\Delta\theta = 0$$

Since, the belt is on the point of slipping, we can replace ΔF by $\mu(\Delta N)$, where μ is the coefficient of friction. Now eliminating ΔN, we obtain

$$T + \mu(T + \Delta T)\sin\theta - (T + \Delta T)\cos\theta = 0$$

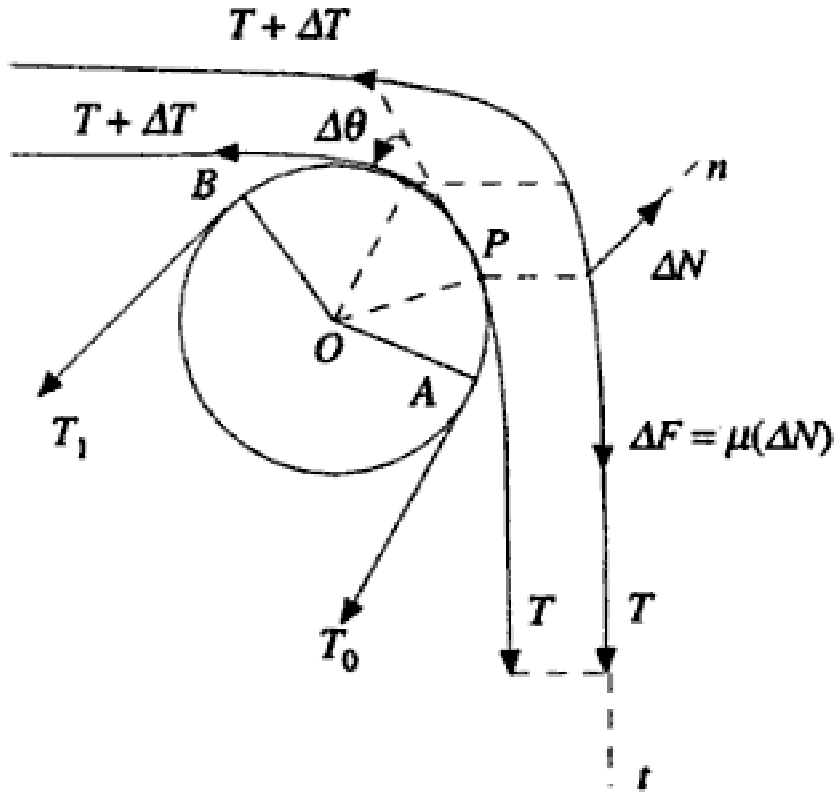


Fig. 4.2 A belt wrapped around a drum of radius r.

which, after division by $\Delta\theta$ and rearrangement, becomes

$$T\frac{1-\cos\Delta\theta}{\Delta\theta}+\mu(T+\Delta T)\frac{\sin\Delta\theta}{\Delta\theta}-\frac{\Delta T}{\Delta\theta}\cos\Delta\theta=0$$

Now taking limit as $\Delta\theta\to 0$, we get

$$\frac{dT}{d\theta}=\mu T \tag{13}$$

which has the solution

$$T=T_0e^{\mu\theta} \tag{14}$$

(The angle θ must be measured in radians because

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

only when x is measured in radians.)

This application has been introduced here with some objectives. First, we want to show how we can derive a differential equation by making assumptions about a complicated physical situation. (For more details about the friction of the belt, see [21]). Second, we want to show that the same mathematical model often governs many apparently unrelated physical situations. We have seen that the differential equation of the organic growth governs the way in which the belt tension varies, as well as the way in which bacteria increase in number, and the way in which money grows at compound interest. As we go on to further problems in this and the coming chapters, we shall come across other examples in which from a mathematical point of view, diverse physical problems and processes are indistinguishable, except for the names given to their significant elements. This point of view helps us (and also many others) to understand and appreciate the tendency of modern mathematics towards abstraction and unification.

Example 4.12 The coefficient of friction between a cable and a cylinder is 0.25. The minimum tension T_0 in the cable when slipping is impending is 60 lb. Find the maximum tension in the cable if the angle of wrap (in radians) is: (a) $\pi/6$; (b) $\pi/2$; (c) π; (d) 2π.

Solution

(a) $T = 60\ e^{\pi/24} \cong 68.4$ lb

(b) $T = 60\ e^{\pi/8} \cong 88.9$ lb

(c) $T = 60\ e^{\pi/4} \cong 131.6$ lb

(d) $T = 60\ e^{\pi/2} \cong 288.6$ lb.

Example 4.13 The coefficient of friction between the fixed horizontal cylinder and the cable is 0.3 (Fig. 4.3). Find what magnitude of the force F is just sufficient: (a) to start the 300 lb body moving upward; (b) to prevent the 300 lb body from moving downward.

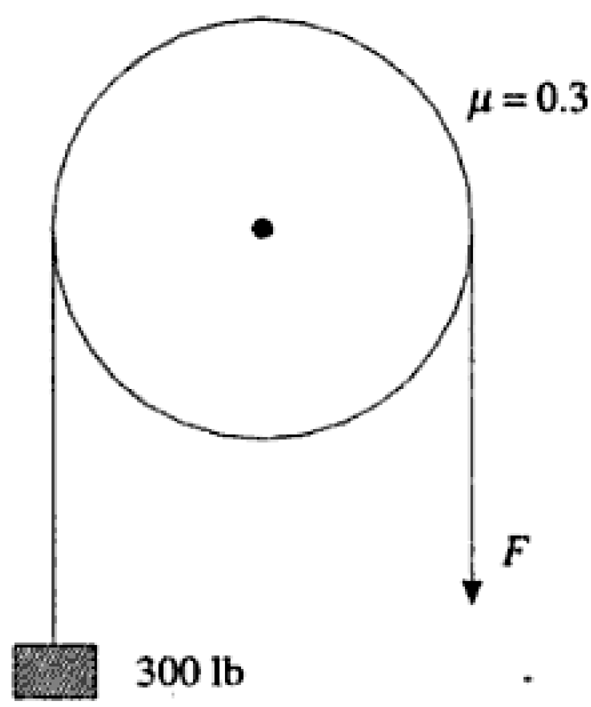


Fig. 4.3 Movement of a cable on a fixed horizontal cylinder.

Solution

(a) $F = 300\ e^{0.3\pi} \cong 769.9$ lb

(b) $300 = F\ e^{0.3\pi}$ or $F = 300\ e^{-0.3\pi} \cong 116.9$ lb.

4.6 TEMPERATURE RATE OF CHANGE (NEWTON'S LAW OF COOLING)

Under certain conditions, the temperature rate of change of a body is proportional to the difference between the temperature T of the body and the temperature T_0 of the surrounding medium. This is known as *Newton's law of cooling*. Here, we shall consider the case in which T_0 remains constant and also suppose that heat flows rapidly enough that the temperature T of the body is the same at all points of the body at a given time t.

If $T = f(t)$ denotes the temperature of the body at time t, then f satisfies the differential equation

$$\frac{dT}{dt} = k(T - T_0) \tag{15}$$

where $k < 0$ (Fig. 4.4). This equation can be solved by separating the variables.

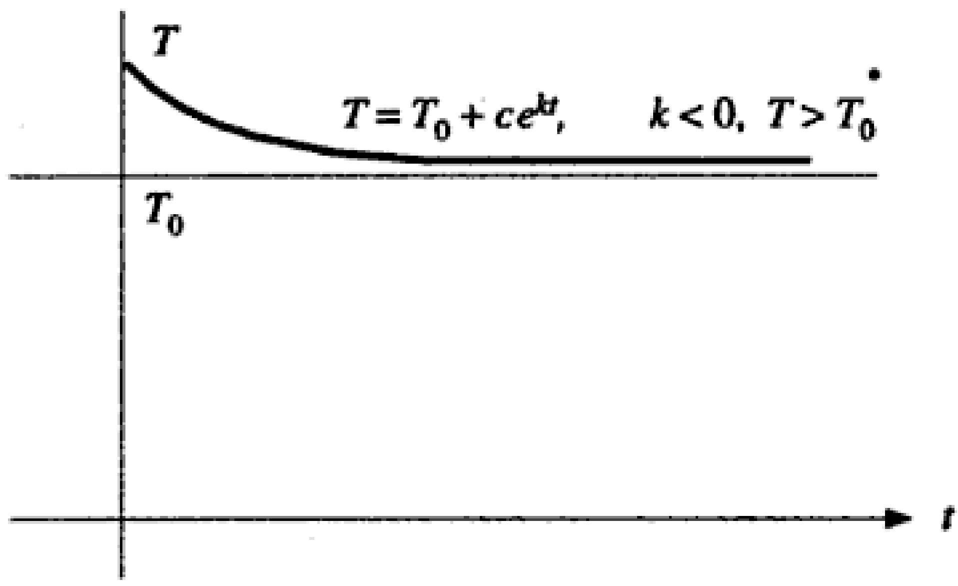


Fig. 4.4 Newton's law of cooling.

Example 4.14 A body whose temperature T is initially 200°C is immersed in a liquid when temperature T_0 is constantly 100°C. If the temperature of the body is 150°C at $t = 1$ minute, what is its temperature at $t = 2$ minutes?

Solution Separating the variables in Eq. (15), we get

$$\frac{dT}{T - 100} = k\,dt$$

and the solution is

$$\log (T - 100) = kt + C \tag{16}$$

When $t = 0$, $T = 200$, we find that $C = \log 100$. Also, at $t = 1$, $T = 150$ and Eq. (16) gives

$$\log 50 = k(1) + \log 100$$

or

$$k = -\log 2$$

Now, substituting $C = \log 100$ and $k = -\log 2$ in Eq. (16), we obtain

$$\log (T - 100) = -t \log 2 + \log 100$$

or

$$T = 100(1 + 2^{-t})$$

Thus, at $t = 2$ min, $T = 125$°C.

REMARK: Consider Eq. (15), which has a solution of the form

$$\log (T - T_0) = kt + \log c \qquad \text{or} \qquad T = T_0 + ce^{kt}$$

Now, as $t \to \infty$, then $T \to T_0$ and, consequently, $dT/dt = k(T - T_0) \to 0$. That is, as t becomes large, the difference between the temperature of the body and the temperature of the surrounding medium approaches zero, and the rate at which the body cools also approaches zero (see Fig. 4.4).

Example 4.15 Water is heated to the boiling point temperature 100°C. It is then removed from heat and kept in a room which is at a constant temperature of 60°C. After 3 minutes, the temperature of the water is 90°C. (a) Find the temperature of water after 6 minutes. (b) When will the temperature of water be 75°C and 61°C?

Solution The differential equation for the present problem is

$$\frac{dT}{dt} = k(T - T_0)$$

whose solution is

$$T = T_0 + ce^{kt} \tag{17}$$

Here, $t = 0$, $T = 100$. Equation (17) then yields ($T_0 = 60$) $c = 40$. Put this value in Eq. (17) with $t = 3$, $T = 90$ to get the value of k as $k = -0.095894024$.

(a) When $t = 6$, Eq. (17) with c and k as above yields

$$T = 60 + 40\, e^{-0.095894024t} = 82.5°\text{C}$$

(b) If $T = 75$°C and $T = 61$°C, then from Eq. (17), we get

$$75° = 60° + 40\, e^{-0.095894024t} \qquad \text{or} \qquad t = 10.2 \text{ min.}$$

and

$$61° = 60° + 40\, e^{-0.095894024t} \quad \text{or} \quad t = 38.5 \text{ min.}$$

Example 4.16 When a cake is removed from an oven, its temperature is measured at 300°F. Three minutes later its temperature is 200°F. How long will it take to cool off to a room temperature of 70°F?

Solution Here, we have to solve the initial value problem

$$\frac{dT}{dt} = k(T - T_0), \qquad T(0) = 300 \tag{18}$$

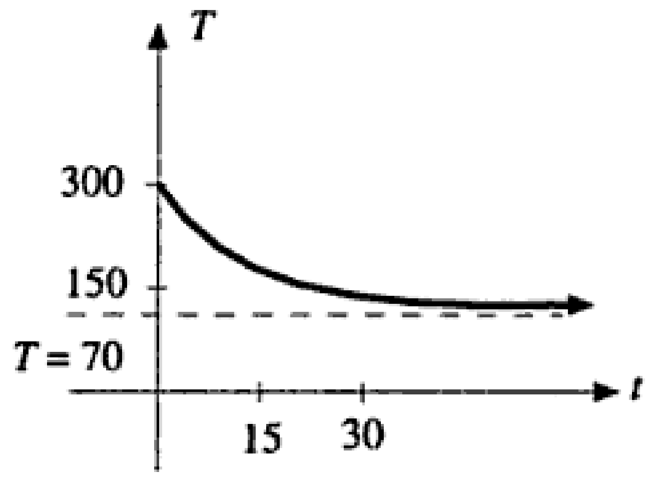
and find the value of k so that $T(3) = 200$. Equation (18), after separating the variables, has a solution of the form

$$T = T_0 + ce^{kt}$$

to give the value of $c = 230$, and thus $k = -0.19018$, so that

$$T(t) = 70 + 230e^{-0.19018t} \tag{19}$$

However, Eq. (19) gives no finite solution to $T(t) = 70$ since $\lim_{t\to\infty} T(t) = 70$. Yet intuitively, we expect the cake will be at room temperature after a long period of time. This period is given in the tabular form in Fig. 4.5(b).



(a)

$T(t)$ (in degree)	t (min)
75	20.1
74	21.3
73	22.8
72	24.9
71	28.6
70.5	32.3

(b)

Fig. 4.5 Cooling time for a cake at room temperature.

Figures 4.5(a) and (b) show that the cake will approximately be at room temperature in about half an hour.

Example 4.17 *Estimation of time of murder.* The body of a murder victim was discovered at 11.00 p.m. The doctor took the temperature of the body at 11.30 p.m., which was 94.6°F. He again took the temperature after one hour when it showed 93.4°F, and noticed that the temperature of the room was 70°F. Estimate the time of death. (Normal temperature of human body = 98.6°F.)

Solution The differential equation governing the present situation is

$$\frac{dT}{dt} = k(T - T_0)$$

whose solution is $T = T_0 + ce^{kt}$.

At $t = 0$, $T = 94.6$ and this gives $c = 24.6$. When $t = 60$ min., $T = 93.4$, and this gives

$$k = \frac{1}{60}\log\left(\frac{23.4}{24.6}\right) = -0.000361988$$

Now, using the values of c, k, $T = 98.6$ and $T_0 = 70$ in the solution of the given differential equation, after simplification we get

$$t = -3.012573443$$

Therefore, the estimated time of death is

$$11.30 - 3.0 = 8.30 \text{ p.m. (approx.)}$$

4.7 DIFFUSION

Let y denote the concentration in mg/cm^3 of a drug or chemical in a small body, and let y_0 be the concentration at time $t = 0$. Suppose, the body is placed in a container or vat in which the concentration of the drug or chemical is a, where $a > y_0$. The concentration in the small body will increase but a remains constant. This is a reasonable assumption if the vat is large and the body is small. *Fick's law of diffusion* states that the time rate of movement of a solute across a thin membrane is proportional to the area of the membrane and to the difference in concentration of the solute on the two sides of the membrane.

From Fick's law, the differential equation satisfied by y is

$$\frac{dy}{dt} = k(a - y) \tag{20}$$

and the solution of Eq. (20) is

$$y = a - ce^{-kt} \tag{21}$$

Thus, the mathematical models for the cooling problem [cf. Eq. (15)] and diffusion problems are essentially the same, except that, here $a > y_0$ and $k > 0$. Consequently, dy/dt is positive for $t \geq 0$ and $y \to a$ as $t \to \infty$ (see Fig. 4.6).

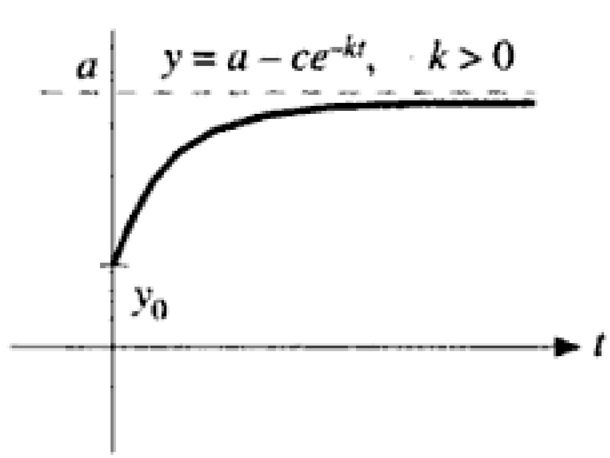


Fig. 4.6 Fick's law of diffusion.

This model is adequate for describing many important phenomena, although it must be modified in a number of physical situations. The body might be a human organ, and often the concentration y_0 is zero. The diffusion model is essentially the model for cooling with the decreasing temperature replaced by an increasing temperature.

Example 4.18 The concentration of the potassium in a kidney is 0.0025 mg/cm^3. The kidney is placed in a vat in which the potassium concentration is 0.0040 mg/cm^3. In 2 hours, the potassium concentration in the kidney is found to be 0.0030 mg/cm^3. What would be the concentration of potassium in the kidney 4 hours after it was placed in the vat? How long does it take for the concentration to reach 0.0035 mg/cm^3? Assume that the vat is sufficiently large and that the vat concentration a = 0.0040 mg/cm^3 remains constant.

Solution This problem can be solved using Eq. (21). Here, $t = 0$, $y = 0.0025$ and $a = 0.0040$. Put these in Eq. (21) to get $c = 0.0015$. Use this value of c with $t = 2$ and $y = 0.0030$ in Eq. (21) to obtain $k = 0.088$. Now the concentration after 4 hours is

$$y = a - ce^{-kt} = 0.004 - (0.0015)e^{(-0.088)(4)}$$

or

$$y = 0.0033 \text{ mg/cm}^3$$

Also, the time required to reach the concentration level $y = 0.0035$ is

$$0.0035 = 0.0040 - (0.0015)e^{-(0.088)t} \quad \text{or} \quad t = 5.42 \text{ hr. (approx.)}$$

Example 4.19 *Intravenous feeding of glucose.* Infusion of glucose into the bloodstream is an important medical technique. To study this process, we define $G(t)$ as the amount of the glucose in the bloodstream of a patient at time t. Assume that the glucose is infused into the bloodstream at a constant rate of k g/min. At the same time, the glucose is converted and removed from the bloodstream at a rate proportional to the amount of the glucose present. Then, the function $G(t)$ satisfies the differential equation

$$\frac{dG}{dt} = k - aG$$

which is a linear equation in G with the integrating factor as e^{at} and, therefore, the solution is

$$G(t) = \frac{k}{a} + ce^{-at}$$

At $t = 0$, $c = G(0) - k/a$, the solution can be written as

$$G(t) = \frac{k}{a} + \left[G(0) - \frac{k}{a}\right]e^{-at}$$

As $t \to \infty$, the concentration of glucose approaches an equilibrium value k/a.

More details about the infusion of the glucose into the veins are given in [23].

Example 4.20 *Nerve excitation.* The cells of a nerve fibre may be conceived as an electric system. The protoplasm contains a large number of different ions, both cations (positive electric charge) and anions (negative electric charge). When an electric current is applied to a nerve fibre, the cations move to the cathode, the anions to the anode, and the electric equilibrium is disturbed. This phenomenon leads to the excitation of the nerve.

Based on the observation that the excitation originates at the cathode, N. Rashevsky, developed a theory which postulates that two different kinds of cations are responsible for the process. One is exciting and the other kind is inhibiting. These two kinds are said to be *antagonistic factors.*

Let $E = E(t)$ be the concentration of the exciting cations and $F = F(t)$ be the concentration of the inhibiting cations near the cathode at any time t. The theory then states that excitation occurs whenever the ratio E/F exceeds a certain value. Denoting this value by C, we have excitation when $E/F \geq C$ and there will be no excitation if $E/F < C$. Let E_0 and F_0 be the concentrations at rest of exciting and inhibiting cations, respectively. When E increases and F remains limited, there is excitation. When E does not increase as fast as F, then there is no excitation.

Let I be the intensity of the stimulant current. For convenience sake, assume that I is constant during a certain time interval. Rashevsky showed that the excitation of nerves can be described by the differential equations

$$\frac{dE}{dt} = JI - K(E - E_0) \qquad \text{and} \qquad \frac{dF}{dt} = LI - M(F - F_0)$$

where J, K, L, M are positive constants.

The above equations can be easily solved for E and F, and finally the ratio E/F determines whether excitation occurs and when. For more details see [9].

4.8 BIOLOGICAL GROWTH

A fundamental problem in biology is that of growth, whether it is the growth of a cell, an organ, a human, a plant or population. We have already dealt with the problem of growth (see Section 4.1), where we saw that the fundamental differential equation was Fq. (2). Now

$$\frac{dN}{dt} = kN \tag{22}$$

has the solution $N = ce^{kt}$, which at $t = 0$ gives $c = N_0$, and

$$N = N_0 e^{kt} \tag{23}$$

From this we see that growth occurs if $k > 0$, while decay (or shrinkage) occurs if $k < 0$. The model given by Eqs. (22) and (23) is called the *Malthusian law of growth*, named after T.R. Malthus (1766–1834). One obvious drawback of Eq. (22) and the corresponding solution [Eq. (23)] is that if $k > 0$ then we have $N \to \infty$ as $t \to \infty$, so that as time passes, growth is unlimited. This conflicts with reality, for after a certain period of time, we know that a cell or individual stops growing having attained a maximum size. We shall now modify Eq. (22) to include these biological facts as follows:

Suppose N denotes the height of a human being (we can also consider the size of a cell) and assume that the rate of change of height depends on the height in a more complicated manner than simple proportionality as shown in Eq. (22). Thus, we have

$$\frac{dN}{dt} = F(N), \qquad N = N_0 \qquad \text{for} \qquad t = 0 \tag{24}$$

where N_0 represents the height at some specified time, $t = 0$, and F is some suitable function but as yet unknown. Since the linear function $F(N) = kN$ is not suitable, we consider a next order of approximation given by a quadratic function $F(N) = aN - bN^2$, where we choose constant $b > 0$ in order to inhibit the growth of N as demanded by reality. Thus, Eq. (24) becomes

$$\frac{dN}{dt} = aN - bN^2, \qquad N = N_0 \qquad \text{for } t = 0 \tag{25}$$

Equation (25) is termed as a *logistic equation* and the growth governed by it is called *logistic growth*. The model represented by this equation is referred to as the *Verhulst-Pearl model*.

92. $y_1 = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4n^2 - 1}$, $y_2 = x^{-1/2} + x^{1/2}$, $c_0 = 1$.

93. $y_1 = x + \frac{1}{15} \sum_{n=1}^{\infty} (2n + 3)(2n + 5) x^{n+1}$,

$y_2 = x^{1/2} + \frac{1}{2} \sum_{n=1}^{\infty} (n + 1)(n + 2) x^{(n+1)/2}$, $c_0 = 1$.

94. $y_1 = 1 + 3 \sum_{n=1}^{\infty} \frac{5^n x^n}{n!(2n + 1)(2n + 3)}$, $y_2 = x^{-3/2} + 10x^{-1/2}$, $c_0 = 1$

95. $y_1 = x^{3/2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{(n+3)/2}}{3^{n-1}(2n - 1)(2n + 1)(2n + 3)}$,

$y_2 = 1 + \frac{2}{3}x + \frac{1}{9}x^2$, $c_0 = 1$

Chapter 6

1. $x = (1/4) \cos 16t$, $v = -4 \sin 16t$, amplitude = (1/4) ft, period = ($\pi/8$) s, frequency = ($8/\pi$) cycles/s, $x = (\sqrt{2}/8)$ ft, $v = -2\sqrt{2}$ ft/s, $a = -32\sqrt{2}$ ft/s^2.

2. (a) $x = \frac{1}{4} \sin (8t)$ ft, $v = 2 \cos (8t)$ ft.

(b) Amplitude $= \frac{1}{4}$ ft, $T = \frac{\pi}{4}$ s, $f = \frac{4}{\pi}$ cycles/s.

(c) 1.89 ft/s, 5.3 ft/s^2.

3. $\frac{\sqrt{2}\pi}{8}$.

4. $x(t) = -\frac{1}{4} \cos 4\sqrt{6}\, t$.

5. $x(t) = \frac{1}{2} \cos 2t + \frac{3}{4} \sin 2t = \frac{\sqrt{13}}{4} \sin (2t + 0.5880)$.

6. (b) $x = \frac{e^{-8t}}{2} (\sin 8t + \cos 8t)$, taking downward as positive.

(c) $x = \frac{\sqrt{2}}{2} e^{-8t} \sin \left(8t + \frac{\pi}{4}\right)$, $A(t) = \frac{\sqrt{2}}{2} e^{-8t}$, $\omega = 8$, $\phi = \frac{\pi}{4}$, quasi period $= \frac{2\pi}{8} = \frac{\pi}{4}$ s.

7. $x = 0.125\, e^{-3t} (3 \sin 4t + 4 \cos 4t) = 0.625\, e^{-3t} (\sin (4t + \phi))$ where $\cos \phi = 3/5$, $\sin \phi = 4/5$, or $\phi = 0.927$ radians $= 53°$.

Second Edition

Differential Equations and Their Applications

Zafar Ahsan

Primarily intended for the undergraduate students in Mathematics, Physics and Engineering, this text gives in-depth coverage of differential equations and the methods of solving them.

The book begins with the basic definitions, the physical and geometric origins of differential equations, and the methods for solving first-order differential equations. Then it goes on to give the applications of these equations to such areas as biology, medical sciences, electrical engineering and economics. The text also discusses, systematically and logically, higher-order differential equations and their applications to telecommunications, civil engineering, cardiology and detection of diabetes, as also the methods of solving simultaneous differential equations and their applications. Besides, the book provides a detailed discussion on Laplace transform and their applications, partial differential equations and their applications to vibration of a stretched string, heat flow, transmission lines, etc., and calculus of variations and its applications.

This book, which is a happy fusion of theory and application, would also be useful to postgraduate students.

New to the Second Edition

- A section (Section 5.11): Series Solution of a Differential Equation—Frobenius Method
- A chapter (Chapter 10): Calculus of Variations and Its Applications.

The Author

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