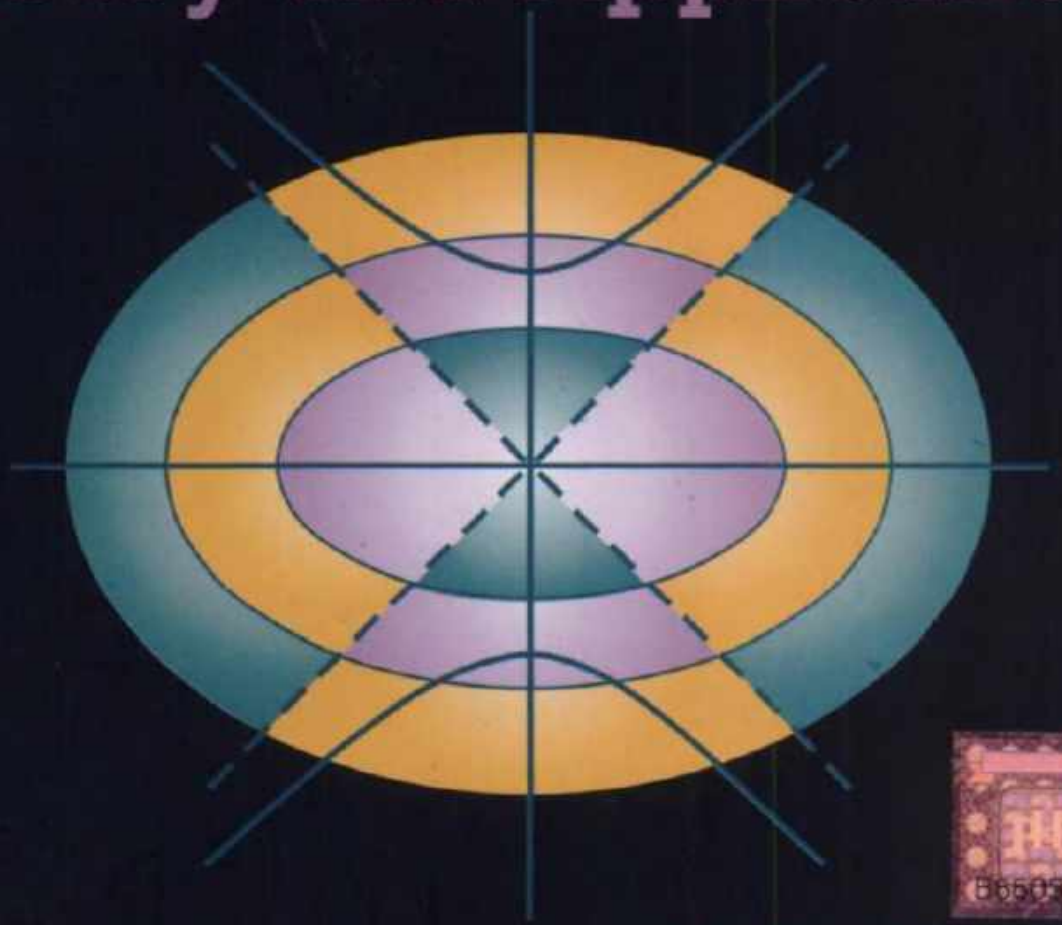
Eastern Economy Edition

Second Edition

Complex Variables

Theory and Applications



H.S. Kasana

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COMPLEX VARIABLES

Theory and Applications

SECOND EDITION

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Preface

This book is an outcome of the lectures delivered by me at Birla Institute of Technology and Science (BITS), Pilani. However, writing of the text started in 1991 when I was on a visiting assignment at Uppsala University, Sweden. Since then, the process of teaching the course has been continued at Thapar Institute of Engineering and Technology, Patiala. While teaching this course I felt the need of a suitable text on complex variables for imparting comprehensive knowledge of various concepts and problem-solving techniques to students.

This book is designed to meet the requirement of graduate and undergraduate students in the science and engineering disciplines. I have tried teaching the contents of this book as a core course in one semester with four hours per week. In order to write a relatively brief text on a vast subject, I had to make a number of choices about inclusion and exclusion of certain topics. In spite of this constraint, a wide selection of topics has been achieved. The basic concepts and fundamental techniques have been emphasized while highly specialized topics and methods have not been given prominence.

The level of the text assumes that the reader is acquainted with elementary real analysis. Enough care has been taken so that the reader may attempt different type of problems. Each chapter consists of a number of unsolved problem sets. Standard problems have been specially designed for this text. Geometric interpretation of the results wherever necessary has been inducted for making the analysis more accessible. The book has been written in a fairly sequential manner so that understanding of most chapters requires familiarity with the preceding ones. However, Chapter 9 may be taken up directly after Chapter 4 without any difficulty.

This book has been made possible through academic interactions with Prof. M.G. Nadkarni (University of Mumbai), Prof. G.S. Srivastava (IIT-Roorkee), Prof. Matts Essén (Uppsala University, Sweden). During the course of writing the text I have received valuable contributions from my colleagues, Profs. B. Singh, P. Singh, M.C. Datta and S.P. Yadava, all of BITS, Pilani.

At this moment I cannot forget my old friends, G. Indira, Rashmi Mishra, Kompella Kalyani and Harendra Singh Nehra, who all requested me a number of times to contribute a book on complex variables. To honour the sentiments of these friends this book has been so fashioned that it caters to those who use complex variables and need its applications at some stage or the other in their professional pursuits.

More solved and unsolved problems have been added to the second edition of the book. I have modified and improved the proofs of various results in view of the latest developments in the field of complex analysis.

H.S. Kasana

List of Symbols

$\mathbb{N}$	set of natural numbers
$\mathbb{I}$	set of integers
$\mathbb{Q}$	set of rational integers
$\mathbb{R}$	real number system
$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$	set of all ordered pair of real numbers
$\mathbb{C}$	complex numbers system
$\text{Re } z$	real part of $z = x + iy$
$\text{Im } z$	imaginary part of $z = x + iy$
$\lvert z \rvert$	modulus of $z = x + iy$
$\arg z$	set of all real values of $\theta : z = re^{i\theta}$
Arg	values of θ: $-\pi < \theta \leq \pi$
$\mathbb{I}_0$	set of nonzero integers
$\mathbb{I}_+$	set of positive integers
$\mathbb{R}_+$	set of positive real numbers
$N_\delta(z_0) = \{z : \lvert z - z_0 \rvert < \delta\}$	δ-neighbourhood of z_0
$N_\delta(z_0)\backslash z_0 = \{z : 0 < \lvert z - z_0 \rvert < \delta\}$	deleted δ-neighbourhood of z_0
∂S	boundary of the set S
$\bar{S}$	clouser of the set S, $\bar{S} = S \cup \partial S$
D/Ω	domain
$\sup S$	least upper bound of the set S
$\inf S$	greatest lower bound of the set S
$u \equiv u(x, y)$	real component of $f = u + iv$
$v \equiv (x, y)$	imaginary component of $f = u + iv$
$\gamma \equiv \gamma(t) = x(t) + iy(t)$	path in the complex plane
$\Rightarrow$	implies (gives)
$\Leftarrow$	implied by
$\Leftrightarrow$	if and only if
$\frac{\partial}{\partial z}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$
$\frac{\partial}{\partial \bar{z}}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$
f_z	$\frac{\partial f}{\partial z}$
$f_{\bar{z}}$	$\frac{\partial f}{\partial \bar{z}}$
$\ln x$	natural logarithm
lim sup	limit superior
lim inf	limit inferior
$\exists$	there exists
$[z_1 z_2]$	line segment joining z_1 and z_2

$\gamma_1 + \gamma_2$	sum of the paths γ_1 and γ_2
e^z	exponential function of z
$\log z = \ln\lvert z\rvert + i \arg z$	logarithmic function of z
$\operatorname{Log} z = \ln z + i \operatorname{Arg} z$	principal logarithmic function of z
$\eta(\gamma, a)$	winding number of a path γ with respect to a
$\operatorname{Res}_{z=a} f(z)$	residue of $f(z)$ at $z = a$
PV	Principal Value (Cauchy)
O	big oh
o	small oh
$F(p)$	Laplace transform of f
$n(r)$	number of zeros of f in $\lvert z\rvert \leq r$
ρ	order of an entire function
σ	type of an entire function

1

Algebra of Complex Numbers

This chapter starts with the theme how the real number system is extended to the complex number system. Various inequalities, polar and exponential forms and powers and roots of the complex numbers are included. A brief exposition of point set topology of the complex plane follows. At the end of the chapter, the extended complex plane is introduced.

1.1 COMPLEX NUMBERS

The equation $x^2 + 5 = 0$ has no solution in the real number system, i.e. there does not exist any real number which satisfies $x^2 = -5$. This prompted mathematicians to enlarge the real number system. However, it seems quite reasonable to impose the requirement that any number system suitable for computational purpose should permit us to solve this equation.

For solving such equations the real number system was enlarged to complex number system and G. Cardano was the first Italian mathematician who found the formula for cubic equations. The term *complex number* was introduced by C.F. Gauss, a German mathematician. Later on, the subject was enriched by the original work of A.L. Cauchy, B. Riemann, K. Weierstrass, and others.

Here, we develop the concept of complex numbers through algebra. The counting numbers $\{1, 2, 3, \ldots\}$, known as natural numbers, denoted by $\mathbb{N}$ constituted the first set conceived by civilization. It was observed that the difference of two natural numbers may not be a natural number. Hence the system $\mathbb{N}$ was extended to the set of integers $\mathbb{I}$ by taking $\mathbb{N} \times \mathbb{N}$ and defining the operations of addition and subtraction. Further, from the viewpoint of including the operation of division, the integers were enlarged by considering $\mathbb{I} \times \mathbb{I}$ to $\mathbb{Q}$, the set of rational numbers. In a similar fashion, the real number system $\mathbb{R}$ is extended to the complex number system by defining appropriate operations on $\mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^2$.

Consider $\mathbb{R}^2$, the set of all ordered pairs of real numbers (x, y) defined by

$$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}.$$

Here 'ordered' means that (x, y) and (y, x) are different unless $x = y$ and, (x, y), a nonzero element in $\mathbb{R}^2$, means that at least one of the members of the ordered pair is nonzero.

We define the operations of addition (+) and multiplication (·) in $\mathbb{R}^2$ as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{1.1}$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2). \tag{1.2}$$

The set $\mathbb{R}^2$ forms a commutative group with respect to addition (+) defined in (1.1). Also, the set of all nonzero elements in $\mathbb{R}^2$ forms a commutative group with respect to multiplication (·) defined in (1.2). The reader can prove these statements easily.

Proposition 1. $\mathbb{R}^2$ forms a field under addition and multiplication as defined in (1.1) and (1.2), respectively.

Proof. (a) $(\mathbb{R}^2, +)$ is a commutative group.

(b) $(\mathbb{R}^2\backslash\{(0, 0)\}, \cdot)$ is a commutative group.

It remains to show that (·) is distributive over (+). We observe

$$\begin{aligned}(x_1, y_1) \cdot [(x_2, y_2) + (x_3, y_3)] &= (x_1, y_1) \cdot (x_2 + x_3, y_2 + y_3)\\ &= (x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3, x_1y_2 + x_1y_3 + y_1x_2 + y_1x_3)\\ &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) + (x_1x_3 - y_1y_3, x_1y_3 + y_1x_3)\\ &= (x_1, y_1) \cdot (x_2, y_2) + (x_1, y_1) \cdot (x_3, y_3).\end{aligned}$$

Similarly, it can be argued that

$$[(x_1, y_1) + (x_2, y_2)] \cdot (x_3, y_3) = (x_1, y_1) \cdot (x_3, y_3) + (x_2, y_2) \cdot (x_3, y_3).$$

Definition 1. A complex number is defined as the ordered pair (x, y) of real numbers, $z = (x, y)$ satisfying the following rules for addition and multiplication:

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2),$$

$$z_1z_2 = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2),$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

Also, (i) two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal $\Leftrightarrow x_1 = x_2$ and $y_1 = y_2$; (ii) a complex number $z = (x, y)$ is nonzero if at least one of the numbers x and y is nonzero.

In view of definition of complex numbers and Proposition 1, it is obvious that the set of all complex numbers $\mathbb{C}$ forms a field with respect to addition and multiplication. Actually, this is a translation of Proposition 1. Although, it is not our purpose to study fields or any other algebraic structure in detail, it is worthwhile to prove that some familiar properties of $\mathbb{C}$ are the consequences of field axioms.

It is trivial to prove that for any real numbers a and b, we have

$$(a, 0) + (b, 0) = (a + b, 0); \qquad (a, 0) \cdot (b, 0) = (ab, 0).$$

This motivates the idea that complex numbers of the form $(a, 0)$ have the same arithmetic properties as the corresponding real numbers a. We can therefore identify the ordered pair $(a, 0)$ by the real number a. This identification gives us the real field as a subfield of the complex field.

Definition 2. The imaginary unit i (*iota*) is defined as

$$i = (0, 1).$$

Proposition 2. $i^2 = -1$.

Proof. $i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0 - 1, 0 + 0) = (-1, 0) \equiv -1$.

Proposition 3. Every complex number $z = (x, y)$ can be written as $z = x + iy$ or $x + yi$, where x and y are real and imaginary parts of z and denoted by Re z and Im z, respectively.

Proof. We have

$$z = (x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1)\,(y, 0) = x + iy.$$

Similarly, $z = x + yi$.

With the introduction of i, Definition 1 for addition and multiplication of complex numbers can be translated as

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$
$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

From now onwards $x + iy$ or $x + yi$ will have the same meaning. For $z = (x, y)$, the additive inverse will be defined by $-z = (-x, -y)$ since $z + (-z) = (-z) + z = 0$. Further, if $z \neq 0$, then the multiplicative inverse is given by solving the equations

$$(x + iy)(a + ib) = 1 = (a + ib)(x + iy)$$

or

$$ax - by = 1 \quad \text{and} \quad bx + ay = 0.$$

These linear simultaneous equations have the unique solution

$$a = \frac{x}{x^2 + y^2} \qquad \text{and} \qquad b = -\frac{y}{x^2 + y^2}.$$

Denoting $a + ib$ by z^{-1}, we get

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right) \qquad \text{or} \qquad z^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}. \tag{1.3}$$

Consequences of $\mathbb{C}$ to be a field are:

(i) Additive identity is unique $z = 0 \equiv (0, 0)$.

(ii) Additive inverse of $z = (x, y)$ as $z = (-x, -y)$ is unique.

(iii) Cancellation laws hold for addition;

$$z + z_1 = z + z_2 \Rightarrow z_1 = z_2,$$
$$z_1 + z = z_2 + z \Rightarrow z_1 = z_2.$$

(iv) The multiplicative identity $z = 1 \equiv (1, 0)$ is unique.

(v) For $z \neq 0$, the multiplicative inverse z^{-1} is unique.

(vi) If $z \neq 0$, then

$$zz_1 = zz_2 \Rightarrow z_1 = z_2,$$

$$z_1 z = z_2 z \Rightarrow z_1 = z_2.$$

(vii) If $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$.

Subtraction

This is the inverse process of addition, z_2 is subtracted from z_1 as

$$z_1 - z_2 = z_1 + (-z_2) = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2).$$

Division

If $z_2 \neq 0$, then division by z_2 is defined as

$$\frac{z_1}{z_2} = z_1 z_2^{-1}.$$

In this way, for $z_1 = 1$, $1/z_2 = z_2^{-1}$, we can write

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = z_1 \frac{1}{z_2}.$$

So, we have the following simple results:

$$(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}, \quad \text{since } (z_1 z_2)(z_1^{-1} z_2^{-1}) = 1 \text{ and } (z_1^{-1} z_2^{-1})(z_1 z_2) = 1,$$

$$\frac{1}{z_1 z_2} = (z_1 z_2)^{-1} = z_1^{-1} z_2^{-1},$$

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2) z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3}.$$

Now, $\mathbb{R}^2$ has been identified as $\mathbb{C}$ provided addition and multiplication of elements of $\mathbb{R}^2$ are defined by (1.1) and (1.2). For definition, other than (1.1) and (1.2), we shall not permit $\mathbb{R}^2$ to be identified by $\mathbb{C}$. Note that

$$\text{Re } z_1 + \text{Re } z_2 = \text{Re}(z_1 + z_2), \tag{1.4}$$

$$\text{Im } z_1 + \text{Im } z_2 = \text{Im}(z_1 + z_2). \tag{1.5}$$

Geometric representation

Every complex number can be represented geometrically as a point in xy-plane. Now, we shall call this plane as complex plane or argand diagram. Introducing a rectangular coordinate system in the plane, we can identify the complex number $z = x + iy$ with the point $P = (x, y)$. Clearly, the set of all real numbers $(x, 0)$ corresponds to the x-axis, called the real axis, and the set of all purely imaginary numbers $(0, y)$ corresponds to the y-axis, and hence called the imaginary axis, while the set of all imaginary numbers corresponds

to all points which do not lie on the real axis. The origin identifies the complex number $0 = 0 + i0$. The complex plane will be termed z-plane, w-plane,..., depending on the letters z, w,..., used to denote the generic complex number.

The nonnegative number $|z|$, called the modulus or absolute value of $z = (x, y)$ is the distance of the complex number from the origin, and hence $|z| = \sqrt{(x^2 + y^2)}$ (see Fig. 1.1).

We shall frequently use the following simple inequalities:

$$x \le |x| \le \sqrt{x^2 + y^2} \Rightarrow \text{Re } z \le |\text{Re } z| \le |z|,$$

$$y \le |y| \le \sqrt{x^2 + y^2} \Rightarrow \text{Im } z \le |\text{Im } z| \le |z|.$$

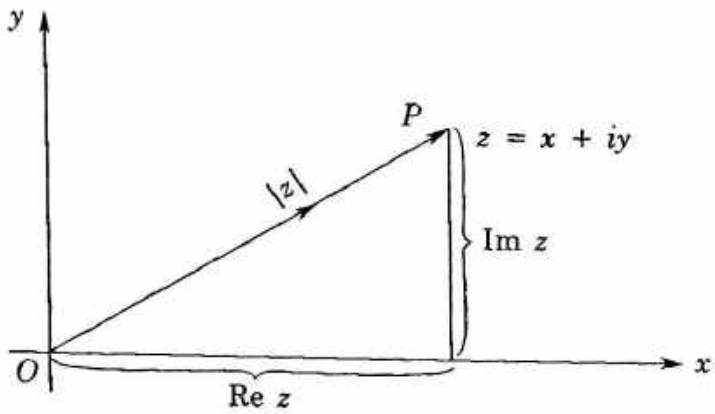

Figure 1.1

Conjugate

If $z = x + iy$ is any complex number, then its complex conjugate denoted by $\bar{z}$ is $\bar{z} = x - iy$ or $\bar{z} = (x, -y)$. Obviously, $\bar{z}$ is the mirror image of the complex point z into real axis. This indicates that $z = \bar{z} \Leftrightarrow z$ is purely a real number. Also, $\bar{\bar{z}} = z$.

Following are the easy consequences of the above definition:

(a) $\overline{z + w} = \bar{z} + \bar{w}$;

(b) $\overline{zw} = \bar{z}\bar{w}$;

(c) $z + \bar{z} = 2 \text{ Re } z$;

(d) $z - \bar{z} = 2i \text{ Im } z$;

(e) $z\bar{z}$ is real and positive unless $z = 0$;

(f) $\overline{z - w} = \bar{z} - \bar{w}$;

(g) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\bar{z}}{\bar{w}}$.

Another entirely equivalent way of representing a complex number $z = x + iy$ is to use the vector $\overrightarrow{OP}$ joining the origin O of the complex plane to the point $P = (x, y)$, instead of using the point P itself. We discuss the sum and difference of two complex numbers as follows:

(i) *Sum*. Since a complex number $z = (x, y)$ can be represented by a vector in the

xy-plane, the sum of two complex numbers may be given by a vector which is diagonal of the parallelogram and whose sides are represented by these vectors, cf., Fig. 1.2(a).

(ii) *Difference*. Similarly, the difference of two complex numbers is the vector $z_1 - z_2$ joining z_2 to z_1, cf., Fig. 1.2(b).

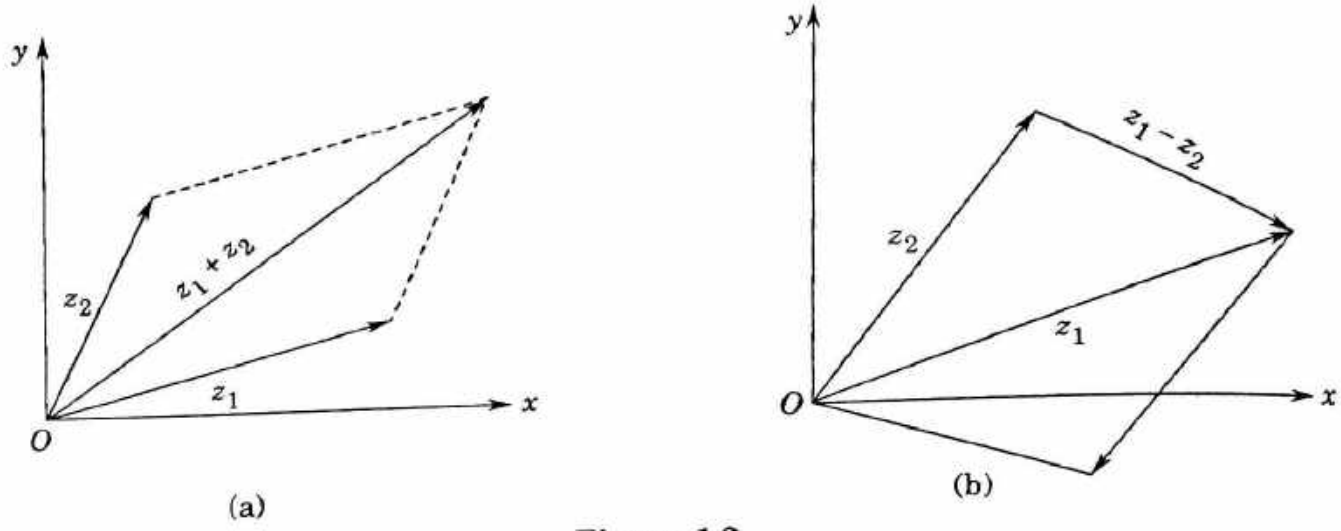

Figure 1.2

Thus, $|z_1 - z_2|$ is the distance between the points z_1 and z_2 or the length of the line segment joining z_2 to z_1. In view of this, $|z| = r$ is the set of all complex numbers which are at a fixed distance r from the origin. It means that all such numbers lie on the circle of radius r centred at the origin. C: $|z| = 1$ will be termed unit circle centred at the origin. Obviously,

$$z\bar{z} = |z|^2 .$$

The roots of complex numbers and different equations will be discussed in Section 1.4 and subsequent chapters. Here, we show how the relationships among the coefficients of quadratic equation influence the nature of roots.

Example 1. Under what conditions on complex constants α and β, will the quadratic equation $z^2 + \alpha z + \beta = 0$ have (a) real roots, (b) one of the roots on the unit circle $|z| = 1$?

Suppose the quadratic equation has real roots, say $z = x$. Then,

$$x^2 + \alpha x + \beta = 0 \qquad \text{and} \qquad x^2 + \bar{\alpha} x + \bar{\beta} = 0 .$$

Eliminating x, we get

$$(\beta - \bar{\beta})^2 - \alpha(\beta - \bar{\beta})(\alpha - \bar{\alpha}) + \beta(\alpha - \bar{\alpha})^2 = 0$$

and hence the required condition is

$$(\text{Im } \beta)^2 + (\text{Im } \alpha)[\text{Im}(\alpha\bar{\beta})] = 0.$$

For the second part, let z_1 be the root such that $|z_1| = 1$. So,

$$z_1^2 + \alpha z_1 + \beta = 0.$$

Since $|z_1| = 1$ implies $z_1\bar{z}_1 = 1$, multiplying by $\bar{z}_1$, we have

$$z_1 + \alpha + \beta\bar{z}_1 = 0$$

and after conjugation,

$$\bar{z}_1 + \bar{\alpha} + \bar{\beta}z_1 = 0.$$

Eliminating $\bar{z}_1$ from these equations, we obtain

$$(1 - |\beta|^2)z_1 = (\alpha - \bar{\alpha}\beta).$$

The condition $|z_1| = 1$ yields the required condition as

$$|\alpha - \bar{\alpha}\beta| = |1 - |\beta|^2|.$$

Note. As yet we have not defined a relation of the form $z_1 < z_2$ if z_1 and z_2 are arbitrary complex numbers. We reason this as follows:

Suppose we were able to define an order relation $<$ in $\mathbb{C}$ satisfying the axioms: (i) If $z < w$, then for every z_0 we have $z + z_0 < w + z_0$; (ii) if $z > 0$, $w > 0$, then $zw > 0$. If $z > 0$, we call z positive: if $z < 0$, z is negative. Since $i \neq 0$ we must have either $i > 0$ or $i < 0$. Let us assume $i > 0$. This implies $i^2 > 0$ or $-1 > 0$. Also, if $i < 0$, then $-i > 0$ and again $-1 = (-i)(-i) > 0$, which is not true in an ordered field $\mathbb{R}$.

PROBLEM SET 1.1

1. Determine all real x and y which satisfy the given relation

(a) $x + iy = |x + iy|$; (b) $x + iy = (x + iy)^2$; (c) $\Sigma_{k=0}^{100} i^k = x + iy$.

2. Show that $\operatorname{Im} iz = \operatorname{Re} z$ and $\operatorname{Re} iz = -\operatorname{Im} z = |z|^2 \operatorname{Im} z^{-1}$.

3. Prove that

(a) $(z_1z_2)(z_3z_4) = (z_1z_3)(z_2z_4)$;

(b) $\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$;

(c) $\overline{z_1z_2 \cdots z_n} = \bar{z}_1\bar{z}_2 \cdots \bar{z}_n$.

4. If $z_1z_2z_3 = 0$, show that least one of $z_k = 0$, $k = 1, 2, 3$.

5. For $z_4 \neq 0$, verify that

$$\frac{z_1 + z_2 + z_3}{z_4} = \frac{z_1}{z_4} + \frac{z_2}{z_4} + \frac{z_3}{z_4}.$$

6. Solve the equation: $|z| - z = 2 + i$.

7. Under what conditions on complex constants α and β, will the quadratic equation $z^2 + \alpha z + \beta = 0$ have purely imaginary roots?

8. Show that the condition for the cubic equation

$$z^4 + az^3 + bz^2 + cz + d = 0, \qquad a, b, c, d \in \mathbb{R}\backslash\{0\}$$

to have a purely imaginary root is $a^2d + c^2 = abc$.

9. Prove that a quadrilateral is a parallelogram iff its diagonals bisect each other.

1.2 TRIANGLE INEQUALITY AND ITS APPLICATIONS

The $|z|$ has already been defined. We also have $|z|^2 = z\bar{z}$ which is used to prove various results of this section.

Proposition 4. Let z and w be any complex numbers. Then

(a) $|z| > 0$ unless $z = 0$;

(b) $|\bar{z}| = |z| = |-z| = |-\bar{z}|$;

(c) $|zw| = |z||w|$;

(d) $|z/w| = |z|/|w|$;

(e) $|z + w| \leq |z| + |w|$ (*triangle inequality*).

Proof. (a), (b) and (c) are the direct consequences of the definition $|z|^2 = z\bar{z}$.

(d) Note that

$$\left|\frac{z}{w}\right|^2 = \left(\frac{z}{w}\right)\overline{\left(\frac{z}{w}\right)} = \left(\frac{z}{w}\frac{\bar{z}}{\bar{w}}\right) = \frac{|z|^2}{|w|^2} \Rightarrow \left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

(e) To prove this, we proceed as

$$\begin{aligned}
|z + w|^2 &= (z + w)\overline{(z + w)} = (z + w)(\bar{z} + \bar{w}) \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \\
&= |z|^2 + 2\mathrm{Re}\,(z\bar{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\bar{w}| + |w|^2 \\
&= |z|^2 + 2|z||w| + |w|^2 \\
&= (|z| + |w|)^2.
\end{aligned}$$

Taking the nonnegative square root of both sides, we get the triangle inequality.

The geometrical meaning of the triangle inequality (e) is merely a statement that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. Note that the inequality (e) is actually an equality when z, w and O are collinear (see Fig. 1.2(a)).

The triangle inequality can be generalized immediately to the case of n complex numbers, using mathematical induction,

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Now, we derive some well-known inequalities encountered frequently in the text.

Proposition 5. We have

(a) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$;

(b) $||z_1| - |z_2|| \leq |z_1 + z_2|$;

(c) $(|x| + |y|)/\sqrt{2} \le |z| \le |x| + |y|$;

(d) $\left(\Sigma_{k=1}^{n} |z_k w_k|\right)^2 \le \left(\Sigma_{k=1}^{n} |z_k|^2\right)\left(\Sigma_{k=1}^{n} |w_k|^2\right)$ (*Schwarz inequality*).

Proof. (a) For the second part of the inequality in (a),

$$|z_1 - z_2| \le |z_1| + |-z_2| = |z_1| + |z_2|.$$

For the first part, we proceed as

$$|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$$

or

$$|z_1| - |z_2| \le |z_1 - z_2|. \tag{1.6}$$

Interchanging the role of z_1 and z_2 in (1.6), we have

$$|z_2| - |z_1| \le |z_2 - z_1| = |-(z_1 - z_2)| = |z_1 - z_2|$$

or

$$-(|z_1| - |z_2|) \le |z_1 - z_2|. \tag{1.7}$$

(1.6) and (1.7) combine to give (a).

(b) Replacing z_2 by $-z_2$ in (a), (b) follows at once. Inequality (b), of course implies that the length of one side of a triangle is greater than or equal to the difference of the other two sides.

(c) For the second part, we proceed as

$$|z| = |x + iy| \le |x| + |iy| = |x| + |i||y| = |x| + |y|, \text{ since } |i| = 1.$$

For the first part of the inequality, we use the arithmetic mean–geometric mean inequality, i.e.

$$|x|^2 + |y|^2 \ge 2|x||y|.$$

Thus,

$$|x|^2 + |y|^2 + 2|x||y| \le 2(|x|^2 + |y|^2) = 2(x^2 + y^2)$$

or

$$(|x| + |y|)^2 \le 2|z|^2.$$

Taking the nonnegative square root of both sides, we have $|x| + |y| \le \sqrt{2}\,|z|$.

(d) For every real number x,

$$\sum_{k=1}^{n} (|z_k|x + |w_k|)^2 \ge 0 \Leftrightarrow \left(\sum_{k=1}^{n} |z_k|^2\right)x^2 + 2\left(\sum_{k=1}^{n} |z_k||w_k|\right)x + \sum_{k=1}^{n} |w_k|^2 \ge 0.$$

If $ax^2 + bx + c \ge 0\ \forall\ x \in \mathbb{R}$, then $b^2 \le 4ac$. Applying this result to the right side of the expression, (d) is immediate.

Example 2. Find the upper bound of the function

$$\left|\frac{z^2+3}{z^2-z-6}\right|, \qquad \text{when } |z| = 1.$$

For this we utilize inequalities (a) and (b) of Proposition 5 to have

$$\left|\frac{z^2+3}{z^2-z-6}\right| = \frac{|z^2+3|}{|(z+2)(z-3)|} \le \frac{|z|^2+3}{||z|-2||\,|z|-3|} = \frac{1+3}{|-1||-2|} = 2.$$

PROBLEM SET 1.2

1. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ and interpret this result geometrically, and hence deduce that

(a) $\left|\alpha + \sqrt{\alpha^2-\beta^2}\right| + \left|\alpha - \sqrt{\alpha^2-\beta^2}\right| = |\alpha+\beta| + |\alpha-\beta|;\ \alpha, \beta \in \mathbb{C};$

(b) $|z_1| + |z_2| = \left|\dfrac{z_1+z_2}{2} + u\right| + \left|\dfrac{z_1+z_2}{2} - u\right|,$

where z_1 and z_2 are any complex numbers and $u = \sqrt{z_1 z_2}$.

Suggestion. For part (b), show that the square of both the sides are equal. Also, see the effect of replacing z_1 and z_2 by $z - z_1$ and $z - z_2$, respectively.

2. Find the upper bound of the functions

(a) $\left|\dfrac{z-2}{z^2+2iz-3}\right|$, when $|z| < \dfrac{1}{2}$; (b) $\left|\dfrac{1}{z^4-4z^2+3}\right|$, when $|z| = 2$.

3. Give an alternative proof of the triangle inequality.

Suggestion. Use the inequality (1.4) for

$$\operatorname{Re}\frac{z_1}{z_1+z_2} \quad \text{and} \quad \operatorname{Re}\frac{z_2}{z_1+z_2}$$

4. For $p > 0$ and z_1, z_2 in $\mathbb{C}$, prove that

$$2\operatorname{Re}(z_1\bar{z}_2) \le p|z_1|^2 + p^{-1}|z|^2$$

and hence establish the inequality

$$|z_1+z_2|^2 \le (1+p)|z_1|^2 + \left(1+\frac{1}{p}\right)|z_2|^2.$$

5. If n is a positive integer, show that $|\operatorname{Im} z^n| \le n|\operatorname{Im} z||z|^{n-1}$.

Suggestion. Use the identity

$$\frac{z^n - (\bar{z})^n}{z-\bar{z}} = z^{n-1} + z^{n-2}\bar{z} + \cdots + (\bar{z})^{n-1}.$$

6. For the distinct complex numbers z and a:

(a) Establish the inequality,

$$\left|\frac{z^n - a^n}{z - a}\right| \le \frac{|z|^n - |a|^n}{|z| - |a|}, \qquad n = 1, 2, \ldots$$

(b) If the assumption $|z| > |a|$ is added in (a), show that

$$n\left|\frac{a}{z}\right|^{n-1} < \frac{|z|}{|z| - |a|}, \qquad n = 2, 3, \ldots .$$

7. Let z_1, z_2, z_3 be the vertices of a triangle on the complex plane. Then show that the area of the triangle is given by $\Sigma(z_2 - z_3)|z_1|^2/(4iz_1)$, and hence prove that z_1, z_2, z_3 are collinear, if and only if

$$\sum_{k=1}^{3} \alpha_k z_k = 0 \quad \text{when} \quad \sum_{k=1}^{3} \alpha_k = 0,\ \alpha_k \in \mathbb{R}$$

8. For any complex numbers $z_1, z_2, \ldots, z_n$, prove the inequality:

$$|z_1 + z_2 + \cdots + z_{n-1} + z_n| \ge |z_n| - |z_1| - |z_2| - \cdots - |z_{n-1}|.$$

Further, if $|z_n| > |z_1| + |z_2| + \cdots + |z_{n-1}|$, deduce the result

$$\frac{1}{|z^4 + 5z^3 + 8z^2 + 11z + 90|} \le \frac{1}{4000} \quad \text{when} \quad |z| = 10.$$

Suggestion. Write $z_n = (z_1 + z_2 + \cdots + z_n) - (z_1 + z_2 + \cdots + z_{n-1})$, and apply the triangle inequality twice.

9. The inequality in Proposition 5(c) can be written as $|z| \le |x| + |y| \le \sqrt{2}\,|z|$. Show that the upper bound $\sqrt{2}|z|$ cannot be further improved.

Suggestion. With $x = y$, we get a contradiction.

10. Let $|z_k| < 1$, $k = 1, 2, \ldots, n$. Show that

(a) $\left|\sum_{k=1}^{n} \alpha_k z_k\right| < 1$, if $\sum_{k=1}^{n} |\alpha_k| = 1$;

(b) $\left|\sum_{k=1}^{n} \alpha_k z_k\right| < \sqrt{n}$, if $\sum_{k=1}^{n} |\alpha_k|^2 = 1$.

11. If $z = \alpha + i\beta$, α and β are integers, show that

(a) $|1 + z + \cdots + z^n| \ge |z|^n$ if $\alpha > 0$;

(b) $|1 + z + \cdots + z^n| \le |z|^n$ if $\alpha < 0$.

Suggestion. For $\alpha = \operatorname{Re} z > 0$, $|z - 1| < |z|$, and hence

$$\left|\frac{z^{n+1} - 1}{z - 1}\right| > \left|\frac{z^{n+1} - 1}{z}\right| \ge |z|^n - \frac{1}{|z|}.$$

This implies that $|1 + z + z^2 + \cdots + z^n| > |z|^n - |z|^{-1}$. Now that $|z|^{-1} < 1$ and the left side of this inequality is an integer, we can neglect the term $|z|^{-1}$, if we replace the symbol $>$ by $\geq$. Analysis for (b) is similar.

12. Generalize the cosine law: $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1 \bar{z}_2)$ as follows:

$$\left|\sum_{k=1}^{n} z_k\right|^2 = \sum_{k=1}^{n} |z_k|^2 - 2\,\mathrm{Re}\left(\sum_{1\leq k<j\leq n} z_k \bar{z}_k\right)$$

Use this law to prove Lagrange's identity

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} |w_k|^2 - \sum_{1\leq k<j\leq n} \left|z_k \bar{w}_j - z_j \bar{w}_k\right|^2$$

and hence deduce the Schwarz inequality. Under what conditions does the equality hold in the Schwarz inequality?

Suggestion. For Lagrange's identity, replace z_k by $z_k w_k$ in the cosine law.

13. For $p > 1$, satisfying

$$\frac{1}{p} + \frac{1}{q} = 1$$

establish the Hölder's inequality

$$\left|\sum_{k=1}^{n} z_k w_k\right| \leq \left(\sum_{k=1}^{n} |z_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |w_k|^q\right)^{1/q}.$$

Notice that the Schwarz inequality is a special case of $p = q = 2$.

14. Let $p \geq 1$. If $z_1, z_2, \ldots, z_n$ and $w_1, w_2, \ldots, w_n$ are complex numbers, show that

$$\left(\sum_{k=1}^{n} |z_k + w_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |z_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |w_k|^p\right)^{1/p}.$$

This result is usually referred to as Minkowski's inequality.

Suggestion. The result is trivial for $p = 1$ using $|z_k + w_k| \leq |z_k| + |w_k|$. Using the same inequality for $p > 1$, we have

$$\begin{aligned}\sum_{k=1}^{n} |z_k + w_k|^p &\leq \sum_{k=1}^{n} |z_k|\,|z_k + w_k|^{p-1} + \sum_{k=1}^{n} |w_k|\,|z_k + w_k|^{p-1} \\ &\leq \left(\sum_{k=1}^{n} |z_k|^{1/p}\right)\left(\sum_{k=1}^{n} |z_k + w_k|^{pq-q}\right)^{1/q} \\ &\quad + \left(\sum_{k=1}^{n} |w_k|^{1/p}\right)\left(\sum_{k=1}^{n} |z_k + w_k|^{pq-q}\right)^{1/q}.\end{aligned}$$

Here we have used the Hölder's inequality twice. Dividing both sides by

$$\left(\sum_{k=1}^{n} |z_k + w_k|^{pq-q}\right)^{1/q}$$

we obtain the desired result, since $qp - q = p$ and $1 - 1/q = 1/p$.

15. Prove *Ptolemy's theorem*. If $z_k (k = 1, 2, 3, 4)$ are the successive vertices of a quadrangle inscribed in a circle, then

$$|(z_2 - z_1)(z_4 - z_3)| + |(z_3 - z_2)(z_4 - z_1)| = |z_3 - z_1|\,|z_4 - z_2|.$$

1.3 POLAR AND EXPONENTIAL FORMS

Let r and θ be the polar coordinates of a point $z = (x, y)$. For $z \neq 0$, let $x = r \cos \theta$, $y = r \sin \theta$. Then z can be written as $z = r(\cos \theta + i \sin \theta)$, where $r = |z|$, a unique positive real number and $\theta = \tan^{-1}(y/x)$ is a multivalued function measured in radians as shown in Fig. 1.3.

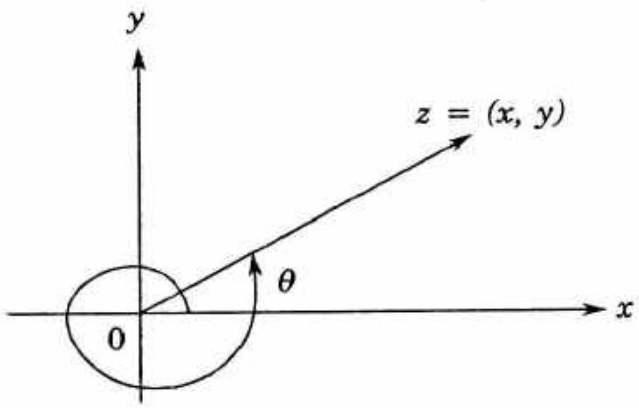

Figure 1.3

Consider the point $z = 1 - i\sqrt{3}$. Then, its polar form is

$$1 - i\sqrt{3} = 2\left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right].$$

Observe that any one of the infinite values

$$\theta = -\frac{\pi}{3} + 2n\pi, \qquad n \in \mathbb{I} \text{ (set of integers)}$$

can also be used to denote $1 - i\sqrt{3}$. For instance $(n = 1)$,

$$1 - i\sqrt{3} = 2\left(\cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}\right).$$

The number θ is called the argument or amplitude of z and is denoted by arg z, the angle measured in radians which the radius vector makes with the positive real axis. It is infinite valued function and all its values differ by an integral multiple of 2π.

The specific value of arg z satisfying $-\pi < \arg z \le \pi$ is called the *principal value* of arg z, and we denote it by Arg z.

Some precaution is needed while inverting $\tan\theta = y/x$, i.e. $\tan(\arg z) = y/x$. The function arctan (inverse of tan function) is multivalued. However, there is one and only one value of $\tan^{-1}(y/x)$, say α, satisfying the inequality

$$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$$

and we call α the principal value of $\tan^{-1}(y/x)$. Now, invert $\tan(\arg z) = (y/x)$ as

$$\operatorname{Arg} z = \begin{cases} \tan^{-1}\dfrac{y}{x}, & x > 0, \\ \tan^{-1}\dfrac{y}{x} + \pi, & x < 0, y \ge 0, \\ \tan^{-1}\dfrac{y}{x} - \pi, & x < 0, y < 0. \end{cases}$$

Moreover, when y/x becomes infinite, we have

$$\operatorname{Arg} z = \begin{cases} \dfrac{\pi}{2}, & x = 0, \quad y > 0, \\ -\dfrac{\pi}{2}, & x = 0, \quad y < 0. \end{cases}$$

Clearly, $\tan\theta = y/x$, where the quadrant containing the point must be specified. As $\tan\theta = 1$ for $1 + i$ and $-1 - i$. Thus, with specification of the quadrant, $\theta = \pi/4$ for $1 + i$ and $\theta = -3\pi/4$ for $-1 - i$. So,

$$z = 1 + i, \operatorname{Arg} z = \frac{\pi}{4}, \qquad z = 1 - i, \operatorname{Arg} z = -\frac{\pi}{4},$$

$$z = -1 + i, \operatorname{Arg} z = \frac{3\pi}{4}, \qquad z = -1 - i, \operatorname{Arg} z = -\frac{3\pi}{4}.$$

For more exposition, we determine the modulus and principal argument of $z = 1 + \cos 2\theta + i \sin 2\theta$, $-\pi < \theta \le \pi$. Since $z = 2\cos\theta(\cos\theta + i\sin\theta)$, we should not be in haste to say that $2\cos\theta$ is the modulus of z as $|z|$ is nonnegative and value of $2\cos\theta$ may be negative depending on θ. By carefully analyzing all situations, we note that

$\lvert z\rvert$	Arg z
$-2\cos\theta$	$-\pi < \theta < -\pi/2$
$2\cos\theta$	$-\pi/2 < \theta < \pi/2$
$-2\cos\theta$	$\pi/2 < \theta \le \pi$

For $\theta = \pm\pi/2$, $z = 0$ and Arg z is indeterminate or not defined.

Remarks. 1. $\bar{z} = r(\cos\theta - i\sin\theta)$.

2. $\arg z = \operatorname{Arg} z + 2n\pi$, $n \in \mathbb{I}$.
3. If z is negative real number, then $\operatorname{Arg} z = \pi$ (by convention).
4. Note that the argument of 0 is undefined.

Proposition 6. For any complex numbers z_1 and z_2, we have

$$\arg z_1 z_2 = \arg z_1 + \arg z_2. \tag{1.8}$$

Proof. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

Note that θ_1 and θ_2 can be any value of $\arg z_1$ and $\arg z_2$, respectively, and it is clear from above that $\theta_1 + \theta_2$ is a value of $\arg z_1 z_2$.

If, on the other hand, $\arg z_1 z_2$ and $\arg z_1$ are specified as

$$\arg z_1 z_2 = \theta_1 + \theta_2 + 2n\pi, \qquad n \in \mathbb{I},$$
$$\arg z_1 = \theta_1 + 2n_1\pi, \qquad n_1 \in \mathbb{I},$$

then the result

$$\theta_1 + \theta_2 + 2n\pi = \theta_1 + 2n_1\pi + \theta_2 + 2(n - n_1)\pi,$$

holds provided we choose $\arg z_2$ as $\theta_2 + 2(n - n_1)\pi$.

Similarly, if $\arg z_1 z_2$ and $\arg z_2$ are specified, then $\arg z_1$ can be chosen to maintain equality in (1.8).

Using the induction process, the above proposition can be extended to any finite set of nonzero complex numbers as

$$\arg(z_1 z_2 \cdots z_n) = \arg z_1 + \arg z_2 + \cdots + \arg z_n.$$

The relation $\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$ may or may not be true for all z_1 and z_2. For example, consider $z_1 = i$ and $z_2 = -1$. Then $\operatorname{Arg} z_1 = \pi/2$ and $\operatorname{Arg} z_2 = \pi$. Now, $z_1 z_2 = -i$, $\operatorname{Arg} z_1 z_2 = -\pi/2$. Hence, $\operatorname{Arg} z_1 z_2 \neq \operatorname{Arg} z_1 + \operatorname{Arg} z_2$ as $-\pi/2 \neq (\pi/2) + \pi$.

We have seen that for $z \neq 0$, the multiplicative inverse of $z = (x, y)$ is given by (1.3). Similarly, in polar form,

$$z^{-1} = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]. \tag{1.9}$$

Clearly, $zz^{-1} = z^{-1}z = 1$ in both cases.

Also, for $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, using (1.9),

$$\begin{aligned}\frac{z_1}{z_2} &= z_1 z_2^{-1}\\ &= r_1(\cos\theta_1 + i\sin\theta_1)\frac{1}{r_2}[\cos(-\theta_2) + i\sin(-\theta_2)]\\ &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].\end{aligned}$$

With this result, if $z_2 \neq 0$, we can establish

$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

Note that $\arg(z - z_0) = \alpha$ represents a line passing through z_0 with slope $\tan\alpha$. Clearly,

$$\phi = \arg\frac{z_3 - z_1}{z_2 - z_1}$$

gives the angle at the vertex z_1 where the lines joining z_2 to z_1 and joining z_3 to z_1 meet (see Fig. 1.4).

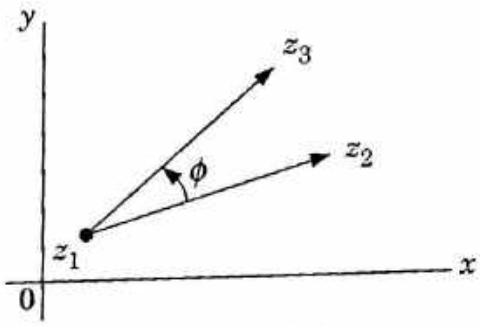

Figure 1.4

Example 3. Given $z_1z_2 \neq 0$, show that

(a) $\text{Re}(z_1\bar{z}_2) = |z_1||z_2| \Leftrightarrow \arg z_1 - \arg z_2 = 2n\pi,\ n \in \mathbb{I}$;

(b) $|z_1 - z_2| = |z_1| + |z_2| \Leftrightarrow \arg z_1 - \arg z_2 = (2n + 1)\pi,\ n \in \mathbb{I}$

and interpret these results geometrically.

Since $z_1z_2 \neq 0$ implies $z_1 \neq 0$ and $z_2 \neq 0$, we can use the polar form of complex numbers. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$z_1\bar{z}_2 = r_1r_2[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

Hence, for part (a), we have

$$\begin{aligned}\text{Re}(z_1\bar{z}_2) = |z_1||z_2| &\Leftrightarrow r_1r_2\cos(\theta_1 - \theta_2) = r_1r_2\\ &\Leftrightarrow \cos(\theta_1 - \theta_2) = 1\\ &\Leftrightarrow \theta_1 - \theta_2 = 2n\pi\\ &\Leftrightarrow \arg z_1 - \arg z_2 = 2n\pi,\ n \in \mathbb{I}.\end{aligned}$$

This envisages the fact that both complex numbers lie along the same ray (passing through the origin) in the complex plane.

For part (b), we note that

$$\begin{aligned}|z_1 - z_2| = |z_1| + |z_2| &\Leftrightarrow |z_1 - z_2|^2 = (|z_1| + |z_2|)^2\\ &\Leftrightarrow z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\\ &\Leftrightarrow -2\text{Re}(z_1\bar{z}_2) = 2|z_1||z_2|\\ &\Leftrightarrow \cos(\theta_1 - \theta_2) = -1\\ &\Leftrightarrow \theta_1 - \theta_2 = (2n + 1)\pi,\ n \in \mathbb{I}.\end{aligned}$$

Geometrically, this tells us that both complex numbers lie along the same ray through the origin but in the opposite direction.

It is interesting to observe that the assertions of the above example may also be interpreted as (a) $\text{Re}(z_1\bar{z}_2) = |z_1||z_2| \Leftrightarrow z_1/z_2$ is a positive real number; (b) $|z_1 - z_2| = |z_1| + |z_2| \Leftrightarrow z_1/z_2$ is a negative real number.

Exponential form. The symbol $e^{i\theta}$ or $\exp(i\theta)$, called *exponential* of $i\theta$, is defined by

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This relation is known as *Euler's formula*. If $z \neq 0$, then $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ and this is the exponential form of a complex number. By straightforward multiplication of $e^{i\theta_1} = \cos\theta_1 + i\sin\theta_1$ and $e^{i\theta_2} = \cos\theta_2 + i\sin\theta_2$, we have

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

Remarks. 1. If $\theta_1 = \theta$ and $\theta_2 = -\theta$ in the above relation, then we have $e^{i\theta}e^{i(-\theta)} = e^{i(\theta-\theta)} = e^{i0} = 1 \Rightarrow e^{i(-\theta)} = 1/e^{i\theta}$. Thus, writing $e^{i(-\theta)}$ as $e^{-i\theta}$, we observe $e^{-i\theta} = 1/e^{i\theta}$.

2. If $\theta_1 = \theta_2 = \theta$, then $(e^{i\theta})^2 = e^{2i\theta}$. By mathematical induction it can be extended to $(e^{i\theta})^n = e^{in\theta}$, where $n = 0, 1, 2, \ldots$.

3. Since $\overline{e^{i\theta}} = e^{-i\theta}$, we have $\bar{z} = re^{-i\theta}$.

4. The multiplicative inverse of $z = re^{i\theta}$ in exponential form is given by $z^{-1} = (1/r)e^{-i\theta}$.

Two complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ are equal if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2n\pi$, $n \in \mathbb{I}$.

The equation $|z| = R$ written as $z = Re^{i\theta}$, $0 \le \theta \le 2\pi$ represents a circle having counterclockwise rotation (positive direction) of radius R centred at the origin. Thus,

$$|z - z_0| = R, \qquad z = z_0 + Re^{i\theta}, \qquad 0 \le \theta \le 2\pi$$

gives a circle of radius R centred at z_0 with positive orientation. This form is parametric representation of a circle and is shown in Fig. 1.5.

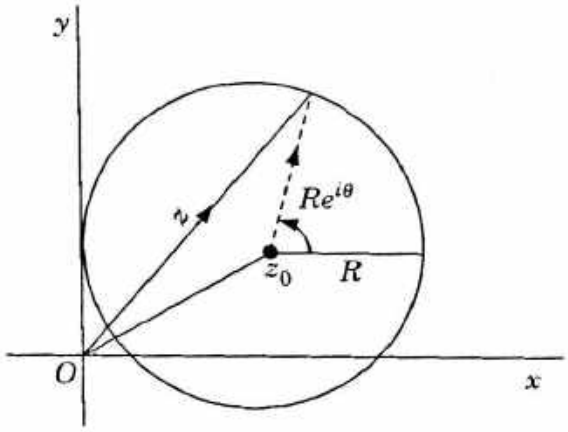

Figure 1.5

The circle $|z - z_0| = R$ can also be written in the form:

$$|z|^2 - z_0\bar{z} - \bar{z}_0 z + |z_0|^2 - R^2 = 0. \tag{1.10}$$

Thus, the negative of the coefficient of $\bar{z}$ gives the centre, and the radius is obtained by subtracting the constant terms in (1.10) from the square of absolute value of the centre and then taking the positive square root.

Let us analyze the general equation

$$az\bar{z} + b\bar{z} + \bar{b}z + c = 0, \quad a \ge 0, \quad c \in \mathbb{R}, \quad b \in \mathbb{C}.$$

If $a \neq 0$, this represents a circle

$$|z|^2 + \frac{b}{a}\bar{z} + \frac{\bar{b}}{a}z + \frac{c}{a} = 0.$$

Comparing with (1.10), the centre of the circle is $-b/a$ and the radius is $\sqrt{|b|^2 - ac}/a$. However, for $a = 0$, this produces

$$b\bar{z} + \bar{b}z + c = 0,$$

which is the equation of a line $\alpha x + \beta y + c = 0$, where $\alpha = b + \bar{b}$ and $\beta = -(b - \bar{b})i$. Clearly, β is real.

Another way of writting the equation of a straight line passing through a point a in complex plane can be given by a parametric equation

$$z = a + bt,$$

where $b \neq 0$ is a complex number and $t \in \mathbb{R}$. This can also be written as $\text{Im}[(z - a)/b] = 0$. Two equations

$$z = a + bt \quad \text{and} \quad z = a' + b't$$

represent the same line if and only if $(a' - a)$ and b' are real multiplies of b. These lines are parallel when b' is real multiple of b, and are directed in the same direction if b' is a positive multiple of b. The direction of a line can be identified by arg b, and the angle between these lines is arg (b'/b); note that it depends on the order in which the lines are named. The lines are orthogonal if b'/b is purely imaginary.

It is to be noted that a directed line $a + bt$ determines a right half-plane consisting of all points z (not on the line) such that

$$\text{Im}\left(\frac{z-a}{b}\right) < 0.$$

The set of all points satisfying the above condition is on the right side of an observer moving along the line in the direction b. Hence, the left half-plane with respect to this line is the set of points satisfying

$$\text{Im}\left(\frac{z-a}{b}\right) > 0.$$

Example 4. Every complex number on the circle $|z| = r$ can be expressed as

$$z = r\frac{1+it}{1-it}, \quad r > 0, \qquad t \in \mathbb{R}.$$

Every complex number on $|z| = r$, can be represented in exponential form $z = re^{i\theta}$, $-\pi < \theta \leq \pi$. For $\theta = \pi$, $z = -r$. In case $-\pi < \theta < \pi$, we write

$$z = re^{i\theta} = r\frac{e^{i\theta/2}}{e^{-i\theta/2}} = r\frac{\cos(\theta/2) + i\sin(\theta/2)}{\cos(\theta/2) - i\sin(\theta/2)} = r\frac{1 + i\tan(\theta/2)}{1 - i\tan(\theta/2)}$$

or

$$z = r\frac{1+it}{1-it}, \qquad t = \tan\frac{\theta}{2}.$$

Example 5. Prove that all the circles passing through a and $1/\bar{a}$ are orthogonal to the circle $|z| = 1$.

Consider the system of all the circles with centre at z_0,

$$|z - z_0|^2 = r^2,$$

where r is a real parameter. Given that it passes through a and $1/\bar{a}$

$$|a - z_0|^2 = r^2 \quad \text{and} \quad |1 - \bar{a}z_0|^2 = |a|^2r^2$$

or

$$|a|^2 + |z_0|^2 - 2\mathrm{Re}(\bar{a}z_0) = r^2 \quad \text{and} \quad 1 + |a|^2|z_0|^2 - 2\mathrm{Re}(\bar{a}z_0) = |a|^2r^2.$$

Subtracting one from the other, we have

$$(1 - |a|^2)(1 + r^2 - |z_0|^2) = 0.$$

Thus, either $|a| = 1$ or $|z_0|^2 = 1 + r^2$. The first is not permissible and the second gives the condition when the circles intersect orthogonally.

Definition 3. Two points p and q are called *inverse* with respect to the circle of radius ρ and centre z_0 if z_0, p and q lie, in that order, on the same line and $|p - z_0||q - z_0| = \rho^2$.

In other words p and q are symmetric with respect to the circle $|z - z_0| = \rho$.

Example 6. Find the inverse of a point p with respect to the circle $|z - z_0| = \rho$.

Let q be the inverse point of p with respect to the circle $|z - z_0| = \rho$. Then, z_0, p and q lie on the same line, it follows that $p = z_0 + r_1e^{i\phi}$, $q = z_0 + r_2e^{i\phi}$ (see Fig. 1.6) and p and q are inverse to each other, then $r_1r_2 = \rho^2$ or

$$(p - z_0)(\bar{q} - \bar{z}_0) = \rho^2. \tag{1.11}$$

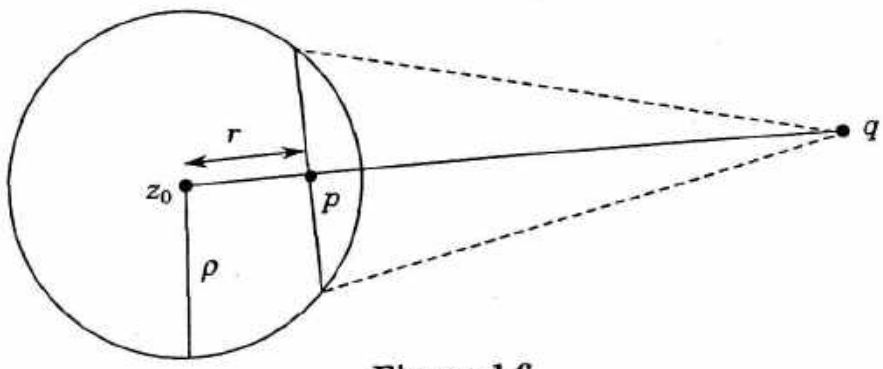

Figure 1.6

Equation (1.11) ensures

$$q = z_0 + \frac{\rho^2}{\bar{p} - \bar{z}_0}.$$

Remarks. 1. The mathematical description of two inverse points is

$$p = z_0 + re^{i\phi}, \qquad q = z_0 + \frac{\rho^2}{r} e^{i\phi}. \tag{1.12}$$

So, if a is any point in $|z| < 1$, then its inverse point with respect to the circle $|z| = 1$ is $1/\bar{a}$.

2. As $r \to 0$, $p = z_0$ and $q = \infty$. This is consistent with the geometric description of inverse points that show that q recedes to ∞, p tends to the centre. Thus, the centre of a circle and ∞ are inverse points.

3. Let z be any point on the circle $z = z_0 + \rho e^{i\theta}$. Then using the relations (1.12), we have

$$\frac{z-p}{z-q} = \frac{(z-z_0)-(p-z_0)}{(z-z_0)-(q-z_0)} = \frac{\rho e^{i\theta} - re^{i\phi}}{\rho e^{i\theta} - (\rho^2/r)e^{i\phi}} = \frac{r}{\rho}\,\frac{\rho e^{i\theta} - re^{i\phi}}{re^{i\theta} - \rho e^{i\phi}}$$

whereupon

$$\left|\frac{z-p}{z-q}\right| = \frac{r}{\rho} < 1,$$

when p and q lie inside and outside the circle $|z - z_0| = \rho$, respectively.

Example 7. Show that the set of all complex numbers satisfying

$$\left|\frac{z-p}{z-q}\right| = k, \quad 0 < k \le 1 \tag{1.13}$$

represents a circle (line), with inverse points p and q, and the point p is inside the circle.

From the given condition and $k \neq 1$,

$$|z-p|^2 = k^2|z-q|^2 \Leftrightarrow z\bar{z} - p\bar{z} - \bar{p}z + |p|^2 = k^2(z\bar{z} - q\bar{z} - \bar{q}z + |q|^2).$$

or

$$|z|^2 - \frac{p - qk^2}{1-k^2}\bar{z} - \frac{\bar{p} - \bar{q}k^2}{1-k^2}z + \frac{|p|^2 - k^2|q^2|}{1-k^2} = 0. \tag{1.14}$$

Equation (1.14) is the equation of a circle with centre

$$z_0 = \frac{p - qk^2}{1-k^2}$$

and radius

$$\rho = \sqrt{\frac{|p-qk^2|^2}{(1-k^2)^2} - \frac{|p|^2 - k^2|q|^2}{1-k^2}}$$

$$= \frac{k}{1-k^2}\sqrt{|p|^2 + |q|^2 - \bar{p}q - \bar{q}p}$$

$$= \frac{k|p-q|}{1-k^2}.$$

Using $z_0 = (p - qk^2)/(1 - k^2)$, compute the values

$$p - z_0 = \frac{k^2(q-p)}{1-k^2}, \qquad \overline{q} - \overline{z}_0 = \frac{\overline{q} - \overline{p}}{1-k^2}$$

which guarantee $(p - z_0)(\overline{q} - \overline{z}_0) = \rho^2$, the definition of inverse points. Remark 3 ensures that p is inside the circle.

Indeed, if $k = 1$, (1.13) reduces to $|z - p| = |z - q|$, which states that z is equidistant from p and q; the locus of such points is a straight line which is the perpendicular bisector of the line segment joining p and q. In the case of straight line, p and q are inverse points and q is the image of p in this line. In particular, the points z_0 and $\overline{z_0}$ are inverse (symmetric) with respect to the real axis.

Note that $k > 1$ is possible in Example 7, but in this situation the positions of p and q will interchange with respect to the circle for which these are inverse points. To adopt the same analysis, we interchange the numerator and denominator of (1.13).

One important observation is made as multiplication of a nonzero complex number by $e^{i\alpha}$ rotates the radius vector by an angle of magnitude α with no change in absolute value.

For $z \neq 0$, let $z = r(\cos\theta + i\sin\theta)$. We have $e^{i\alpha} = \cos\alpha + i\sin\alpha$, and hence

$$ze^{i\alpha} = r\,[\cos(\theta + \alpha) + i\sin(\theta + \alpha)].$$

This shows that θ is rotated by α. To be more specific, multiplication of a nonzero complex number by i rotates the z-vector by an angle of magnitude $\pi/2$.

Example 8. The points P, Q, R represent the numbers z_1, z_2, z_3, respectively, and the angles of the triangle PQR at Q and R are equal with the angle at $P = \alpha$. Prove that

$$(z_3 - z_2)^2 = 4(z_1 - z_3)(z_2 - z_1)\sin^2\left(\frac{\alpha}{2}\right).$$

We observe that when the vector PQ is rotated in the counterclockwise direction with an angle α it coincides with the vector PR, and hence

$$z_3 - z_1 = (z_2 - z_1)e^{i\alpha}. \tag{1.15}$$

On simplification, this yields

$$z_3 - z_2 = (z_2 - z_1)\left(-2\sin^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$$

$$= 2i(z_2 - z_1)\sin\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right),$$

or, on squaring,

$$(z_3 - z_2)^2 = -4(z_2 - z_1)^2 e^{i\alpha}\sin^2\left(\frac{\alpha}{2}\right).$$

Substituting the value of $e^{i\alpha}$ from (1.15), we have the desired result.

PROBLEM SET 1.3

1. Prove the relations:

(a) $\text{Arg } z_1z_2 = \text{Arg } z_1 + \text{Arg } z_2 + 2n\pi, \quad n = 0, \pm 1;$

(b) $\text{Arg } \dfrac{z_1}{z_2} = \text{Arg } z_1 - \text{Arg } z_2 + 2n_1\pi, \quad n_1 = 0, \pm 1.$

Find the set in the complex plane for which $n = 0$.
Suggestion. Define

$$n = \begin{cases} 0, & -\pi < \text{Arg } z_1 + \text{Arg } z_2 \le \pi, \\ 1, & -2\pi < \text{Arg } z_1 + \text{Arg } z_2 \le -\pi, \\ -1, & -\pi < \text{Arg } z_1 + \text{Arg } z_2 \le 2\pi. \end{cases}$$

The part (b) may be similarly treated.

2. If $|z| \le 1$, $|w| \le 1$, show that $|z - w|^2 \le (|z| - |w|)^2 + (\text{Arg } z - \text{Arg } w)^2$.

3. Let z and z_0 be any two complex numbers. Then show that

$$\arg(z - z_0) = -\arg \overline{z - z_0}$$

and hence $\text{Arg}(z - z_0) = -\text{Arg } \overline{z - z_0} \Leftrightarrow z - z_0$ is not a negative real number.

4. Using purely geometric argument, show that $|z - 1| \le ||z| - 1| + |z||\text{Arg } z|$, and hence deduce the inequality

$$|z - w| \le ||z| - |w|| + |z||\text{Arg } z - \text{Arg } w + 2n_1\pi|, \qquad n_1 = 0, \pm 1.$$

5. Let z_1, z_2, z_3 be distinct nonzero complex numbers such that $|z_1| = |z_2| = |z_3| = 1$. Then show that

$$2 \arg \frac{z_3 - z_2}{z_3 - z_1} = \arg \frac{z_2}{z_1}.$$

6. Given that $z_1z_2 \ne 0$, show that for any integer n:

(a) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg z_1 - \arg z_2 = 2n\pi;$

(b) $|z_1 - z_2| = ||z_1| - |z_2|| \Leftrightarrow \arg z_1 - \arg z_2 = 2n\pi;$

(c) $z_1\bar{z}_2 + \bar{z}_1z_2 = 0 \Leftrightarrow \arg z_1 - \arg z_2 = (2n + 1)\pi/2;$

(d) $z_1\bar{z}_2 - \bar{z}_1z_2 = 0 \Leftrightarrow \arg z_1 - \arg z_2 = n\pi.$

Interpret these results geometrically. Try to find the alternative form for assertions (c) and (d) to draw the same geometrical significance.

7. If a and b are complex numbers, show that

(a) $|a - b|^2 \le (1 + |a|^2)(1 + |b|^2);$

(b) If $a \ne 0$, then $|a + b| = |a| + |b| \Leftrightarrow b/a$ is a nonnegative real number;

(c) $|a - b| = |1 - \bar{a}b| \Leftrightarrow |a| = 1$ or $|b| = 1$, for which a and b is the inequality $|a - b| < |1 - \bar{a}b|$ valid?

8. Two nonzero complex numbers z_1 and z_2 have the same moduli $\Leftrightarrow$ there are complex numbers c_1 and c_2 such that $z_1 = c_1c_2$ and $z_2 = c_1\bar{c}_2$.

9. Let z_1, z_2, z_3 be the vertices of a triangle on the complex plane. If the

(a) triangle is equilateral, then $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$;

(b) triangle is isosceles and right-angled at z_3, then $z_1^2 + z_2^2 + 2z_3^2 = 2(z_1 + z_2)z_3$.

Is the converse true in part (a)? Also locate the positions of the vertex z_3 in (a) and (b) when z_1 and z_2 are conjugate to each other with $z_1 = 1 + i$.

10. Given three complex numbers z_1, z_2, z_3 such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, show that these complex numbers are vertices of an equilateral triangle inscribed in the unit circle centred at the origin.

11. Let z_1, z_2, z_3 and z_1', z_2', z_3' be the vertices of the triangles A and B, respectively. If A and B are similar, show that

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1' & z_2' & z_3' \end{vmatrix} = 0.$$

12. Find the complex numbers z which satisfy the equation

$$|z - 2 + 2i| = 1$$

and have the least and largest absolute value.

13. If three numbers z_1, z_2, z_3 lie on one side of a line passing through the origin in the complex plane, show that $1/z_1$, $1/z_2$, $1/z_3$ also lie on one side of the line. Moreover, establish the facts:

$$z_1 + z_2 + z_3 \neq 0, \qquad z_1^{-1} + z_2^{-1} + z_3^{-1} \neq 0.$$

14. Let A, B, C, D be the vertices of a rhombus in the Argand plane such that affixes of the vertices are the complex numbers z_1, z_2, z_3, z_4, respectively and $\angle CBA = \pi/3$. Then show that $2z_3 = z_1(1 - i\sqrt{3}) + z_2(1 + i\sqrt{3})$ and $2z_4 = z_1(3 - i\sqrt{3}) + z_2(-1 + i\sqrt{3})$.

15. Prove that the line through the points z_1 and z_2 is perpendicular to the line through the points z_3 and z_4 if and only if

$$\arg \frac{z_1 - z_2}{z_3 - z_4} = \pm \frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{I}.$$

16. Prove that the diagonals of a rhombus are orthogonal.

Suggestion. Take one of the vertices at the origin and use Problems 8 and 15.

17. Find the

(a) inverse point of $1 + i$ with respect to the circle $|z + 1 - 2i| = 2$.

(b) family of all circles which are orthogonal to $|z| = 1$ and $|z - 1| = 4$.

18. If p and q are the inverse points of the circle $az\bar{z} + b\bar{z} + \bar{b}z + c = 0$, show that

$$ap\bar{q} + b\bar{q} + \bar{b}p + c = 0$$

Suggestion. $\qquad (p + b/a)(\bar{q} + \bar{b}/a) = (b^2 - ac)/a^2.$

19. Describe the family of curves in the z-plane determined by the relation:

$$\operatorname{Arg} \frac{z - a}{z - b} = k \; (-\pi < k \leq \pi).$$

1.4 POWERS AND ROOTS

It is evident from the preceding section that whenever a complex number is written in polar or exponential form it means that we are dealing with nonzero complex numbers.

Definition 4. For any nonzero complex number z, we define

$$z^0 = 1, \qquad z^{n+1} = z^n \cdot z, \qquad \text{if } n \geq 0,$$
$$z^{-n} = (z^{-1})^n \quad \text{if } z \neq 0, \qquad n > 0.$$

Proposition 7. For any complex number $z = re^{i\theta}$ and $n = 0, \pm 1, \pm 2, \ldots$, we have

$$z^n = r^n e^{in\theta}. \tag{1.16}$$

Proof. For $n = 0$, it is trivially true with the definition $z^0 = 1$ and for $n = 1, 2, \ldots$, it can be proved by mathematical induction. For $n = -1, -2, \ldots$, let $n = -m$, $m = 1, 2, \ldots$. Now, using Definition 4, we observe that

$$z^n = z^{-m} = (z^{-1})^m = \left(\frac{1}{r}e^{-i\theta}\right)^m = \left(\frac{1}{r}\right)^m e^{-im\theta} = r^{-m}e^{i(-m\theta)} = r^n e^{in\theta}.$$

For $r = 1$, (1.16) reduces to $(e^{i\theta})^n = e^{in\theta}$, i.e.

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta), \qquad n \in \mathbb{I}, \tag{1.17}$$

which is known as *De-Moivre's formula* for the integral index.

Remark. We may define $z^{-m} = (z^m)^{-1}$, $m = 1, 2, \ldots$ to prove Proposition 7 as

$$z^n = z^{-m} = (z^m)^{-1} = \frac{1}{z^m} = \frac{1}{r^m e^{im\theta}} = r^{-m}e^{i(-m\theta)} = r^n e^{in\theta}.$$

Now, we state some applications of this formula with a brief sketch of the proofs.

1. If n is a positive integer, then we write the binomial expansion of the left-hand side expression of (1.17) to have

$$\cos n\theta + i \operatorname{isn} n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\,(i\sin\theta)^k.$$

Comparing the real and imaginary parts, we get

$$\cos n\theta = \sum_{k=0,\text{even}}^{n} (-1)^{k/2} \binom{n}{k} \cos^{n-k}\theta \sin^k\theta, \tag{1.18}$$

$$\sin n\theta = \sum_{k=1,\text{odd}}^{n} (-1)^{(k-1)/2} \binom{n}{k} \cos^{n-k}\theta \sin^k\theta. \tag{1.19}$$

2. From De-Moivre's formula with $z = e^{i\theta}$,

$$z^n = \cos n\theta + i\sin n\theta \quad \text{and} \quad z^{-n} = \cos n\theta - i\sin n\theta$$

and hence,

$$z^n + z^{-n} = 2\cos n\theta \quad \text{and} \quad z^n - z^{-n} = 2i\sin n\theta.$$

These expressions are used to obtain $\cos^n\theta$, when we expand $(z + z^{-1})^n$ by binomial expansion. For instance,

$$(z + z^{-1})^5 = (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1})$$

which becomes

$$(2 \cos \theta)^5 = 2 \cos 5\theta + 10 \cos 3\theta + 20 \cos\theta$$

or

$$\cos^5\theta = 2^{-4}[\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta].$$

Similarly, the expansion of $(z - z^{-1})^n$ gives $\sin^n\theta$. The functions with higher powers of $\cos \theta$ and $\sin \theta$ can be integrated by using these simplified expressions.

3. This formula is also very useful in finding the sum of sine and cosine series. What we are going to develop is the most general method of finding the sum of series containing the trigonometric functions:

$$a_0 \sin \alpha + a_1\sin(\alpha + \beta) + a_2 \sin(\alpha + 2\beta) + \cdots$$

and its associated series

$$a_0 \cos \alpha + a_1 \cos(\alpha + \beta) + a_2 \cos(\alpha + 2\beta) + \cdots.$$

Denote the sum of sine series by S and write the corresponding cosine series and denote its sum by C. Then, evaluate $C + iS$. Finally, after some calculations compare the real and imaginary parts on both sides to determine C and S.

Example 9. Find the sum of the series

$$1 + \cos \alpha + \cos 2\alpha + \cos 3\alpha + \cdots + \cos n\alpha$$

and its associated sine series.

Writing

$$C = 1 + \cos \alpha + \cos 2\alpha + \cdots + \cos n\alpha,$$
$$S = \sin \alpha + \sin 2\alpha + \cdots + \sin n\alpha,$$

we have

$$C + iS = 1 + e^{i\alpha} + e^{i2\alpha} + \cdots + e^{in\alpha}$$

$$= \frac{1 - e^{i(n+1)\alpha}}{1 - e^{i\alpha}}$$

$$= \frac{e^{i(n+1)\alpha/2}[e^{-i(n+1)\alpha/2} - e^{i(n+1)\alpha/2}]}{e^{i\alpha/2}(e^{-i\alpha/2} - e^{i\alpha/2})}$$

$$= e^{in\alpha/2} \frac{\sin [(n+1)\alpha/2]}{\sin (\alpha/2)}.$$

Comparing real and imaginary components,

$$C = \frac{\cos (n\alpha/2) \sin [(n+1)\alpha/2]}{\sin (\alpha/2)}, \qquad S = \frac{\sin (n\alpha/2) \sin [(n+1)\alpha/2]}{\sin (\alpha/2)}, \qquad 0 < \alpha < 2\pi.$$

On simplifying, these expressions take the forms

$$C = \frac{1}{2} + \frac{\sin(n+1/2)\alpha}{2\sin(\alpha/2)},$$

$$S = \frac{1}{2}\cot\left(\frac{\alpha}{2}\right) - \frac{\cos(n+1/2)\alpha}{2\sin(\alpha/2)}.$$

For $\theta = 0$ or 2π, $C = n + 1$ and $S = 0$.

Theorem 1. For $z_0 \neq 0$, there exists n values of z which satisfy the equation: $z^n = z_0$

Proof. The equation $z^n = z_0$ can be written as $z = (z_0)^{1/n}$. Let $z_0 = r_0 e^{i\theta_0}$, $-\pi < \theta_0 \leq \pi$ and $z = re^{i\theta}$ be any value of z. Then

$$(re^{i\theta})^n = r_0 e^{i\theta_0} \Rightarrow r^n e^{in\theta} = r_0 e^{i\theta_0}.$$

Using the equality of two complex numbers in exponential form, we can write

$$r^n = r_0 \quad \text{and} \quad n\theta = \theta_0 + 2k\pi, \quad k \in \mathbb{I}.$$

This implies

$$r = \sqrt[n]{r_0}, \qquad \theta = \frac{\theta_0}{n} + \frac{2k\pi}{n},$$

where $\sqrt[n]{r_0}$ denotes the positive nth root of r_0.

In this way we observe that all values of z given by

$$z = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right], \qquad k = 0, 1, 2, \ldots, n-1 \tag{1.20}$$

satisfy the given equation $z^n = z_0$. These are termed *nth roots of* z_0.

For $k = 0, 1, 2, \ldots, n-1$ in Eq. (1.20), we get different n roots. The root corresponding to $k = 0$, i.e. $C = \sqrt[n]{r_0}\, e^{i\theta_0/n}$ is termed the *principal root*. Thus,

$$C, C\omega, C\omega^2, \ldots, C\omega^{n-1}, \qquad \omega = e^{i2\pi/n}$$

are n distinct roots of a nonzero complex number.

Here, it is must to mention that for values of k other than $0, 1, \ldots, n-1$, the roots start repeating.

Geometrically, $\sqrt[n]{r_0}$ is the length of each radius vector representing the nth root. The roots lie at the vertices of a regular polygon of n sides that is inscribed in the circle of radius $\sqrt[n]{r_0}$ centred at the origin. These roots are equally spaced with difference of angle $2\pi/n$.

Sum of the roots:

$$\begin{aligned} C + C\omega + C\omega^2 + \cdots + C\omega^{n-1} &= C(1 + \omega + \omega^2 + \cdots + \omega^{n-1}) \\ &= \frac{C(1-\omega^n)}{1-\omega}, \qquad \omega \neq 1 \\ &= 0, \qquad \text{since } \omega^n = 1. \end{aligned}$$

Product of the roots:

$$\begin{aligned} C \cdot C\omega \cdot C\omega^2 \cdots C\omega^{n-1} &= C^n(\omega \cdot \omega^2 \cdots \omega^{n-1}) \\ &= C^n \omega^{1+2+\cdots+(n-1)} \\ &= r_0 e^{i\theta_0} \omega^{n(n-1)/2} \\ &= (-1)^{n-1} z_0, \qquad n \in \mathbb{N}. \end{aligned}$$

Remark. Alternatively, factor the polynomial $z^n - z_0$ (see Section 4.8) as

$$z^n - z_0 = (z - \rho_0)(z - \rho_1) \cdots (z - \rho_{n-1}),$$

where $\rho_0, \rho_1, \ldots, \rho_{n-1}$ are n distinct roots of $z^n - z_0 = 0$. If we compare the coefficients of z^{n-1}, and the constant term on both sides of the factored form, the sum and the product of the roots are immediately available.

The case $z_0 = 1$ is interesting. The roots of the equation $z^n = 1$ are called the nth roots of unity, and all the roots can be expressed as $1, \omega, \omega^2, \ldots, \omega^{n-1}$ and

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$$

Comparing the real and imaginary parts on both sides, we have the important formulas:

$$\sum_{k=1}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = -1 \quad \text{and} \quad \sum_{k=1}^{n-1} \sin\left(\frac{2k\pi}{n}\right) = 0.$$

Example 10. Find all the roots of $(-8 - 8\sqrt{3}i)^{1/4}$ and exhibit them geometrically.

First, write the complex number in exponential form $z_0 = 16\, e^{-i2\pi/3}$. Taking $r_0 = 16$, $\theta_0 = -2\pi/3$ and $n = 4$ in (1.20), we have

$$2\exp\left[i\left(\frac{-\pi}{6} + \frac{k\pi}{2}\right)\right].$$

For $k = 0, 1, 2, 3$, all the four roots are

$$2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]; \qquad 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right);$$

$$2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right); \qquad 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right);$$

or $\sqrt{3} - i$, $1 + i\sqrt{3}$, $-\sqrt{3} + i$, $-1 - i\sqrt{3}$.

These roots lie at the vertices of a square inscribed in a circle of radius 2 centred at the origin, also are equally spaced with difference of angle $\pi/2$ (see Fig. 1.7).

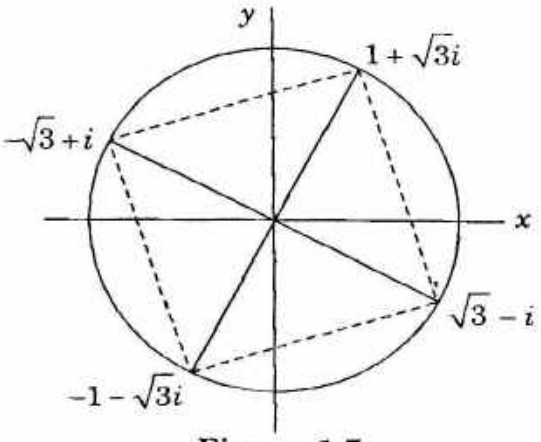

Figure 1.7

Example 11. Establish the relation:

$$\frac{n}{2^{n-1}} = \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right), \qquad n \geq 2.$$

Let $1, \rho_1, \rho_2, \ldots, \rho_{n-1}$ be the nth roots of unity, when $\rho_k = e^{i2\pi k/n}$, $k = 1, 2, \ldots, n-1$. Then,

$$z^n - 1 = (z-1)(z-\rho_1)(z-\rho_2)\cdots(z-\rho_{n-1}).$$

Dividing both sides by $z - 1$ and letting $z \to 1$,

$$n = (1-\rho_1)(1-\rho_2)\cdots(1-\rho_{n-1})$$

and hence

$$n = (1-\bar{\rho}_1)(1-\bar{\rho}_2)\cdots(1-\bar{\rho}_{n-1}).$$

Thus,

$$\begin{aligned} n^2 &= \prod_{k=1}^{n-1} (1-\rho_k)(1-\bar{\rho}_k) \\ &= \prod_{k=1}^{n-1} (1-e^{2\pi ik/n})(1-e^{-2\pi ik/n}) \\ &= \prod_{k=1}^{n-1} \left[2\left(1-\cos\frac{2k\pi}{n}\right)\right] \\ &= \prod_{k=1}^{n-1} \left[4\sin^2\left(\frac{k\pi}{n}\right)\right] \end{aligned}$$

or

$$n^2 = 2^{2(n-1)} \prod_{k=1}^{n-1} \sin^2\left(\frac{k\pi}{n}\right), \qquad n > 1.$$

Taking the nonnegative square of both sides, we have the desired result.

The roots of the quadratic equation $az^2 + bz + c = 0$, $a \neq 0$ are given by

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}.$$

Note that here we have not used $\pm$ in front of $(b^2 - 4ac)^{1/2}$ as in the case of a real quadratic equation, since in the complex case $(b^2 - 4ac)^{1/2}$ automatically gives two distinct roots.

Moreover, for clarity of notation $\sqrt{r}$ has been reserved to exhibit a unique nonnegative real number while $r^{1/2}$ will give two values.

PROBLEM SET 1.4

1. Write in the form $z = x + iy$: (a) $(1 - \sqrt{3}i)^{72}$; (b) $[(1 + i\sqrt{3})/(1 - i)]^{30}$.

2. Using De-moivre's formula, show that

(a) $$\cos^n\theta = \frac{1}{2^{n-1}}\left[\sum_{k=0}^{[n/2]}\binom{n}{k}\cos(n-2k)\theta + R_n\right],$$

where $$R_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \dfrac{n!}{2[(n/2)!]^2}, & \text{if } n \text{ is even;} \end{cases}$$

(b) $$\sin^n\theta = \begin{cases} \dfrac{1}{(2i)^{n-1}}\left[\displaystyle\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}\sin(n-2k)\theta\right], & n = 1, 3, \ldots \\ \dfrac{1}{(2i)^{n-1}i^n}\left[\displaystyle\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}\cos(n-2k)\theta + \dfrac{(-1)^{[n/2]}n}{2[(n/2)!]^2}\right], & n = 0, 2, \ldots \end{cases}$$

and hence deduce the trigonometric identities:

(a) $\sin^5\theta = 2^{-4}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$;

(b) $\sin^6\theta = -2^{-5}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$;

(c) $\sin^5\theta\cos^5\theta = 2^{-9}(\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta)$.

3. Use (1.18) and (1.19) to write

(a) $\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$;

(b) $\sin 5\theta = 5\cos^4\theta\sin\theta - 20\cos^2\theta\sin^3\theta + \sin^5\theta$;

(c) $$\tan 6\theta = \frac{2(3\tan\theta - 10\tan^3\theta + 3\tan^5\theta)}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}.$$

4. Find the sum of the finite series

(a) $\cos\alpha + \cos 3\alpha + \cos 5\alpha + \cdots$ up to n terms;

(b) $\sin^2\alpha + \sin^2(2\alpha) + \sin^2(3\alpha) + \cdots$ up to n terms.

5. Verify the following sums

(a) $$n\sin\theta + \frac{n(n-1)}{2!}\sin 2\theta + \cdots + \sin n\theta = 2^n\cos^n\left(\frac{\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right);$$

(b) $$\frac{1}{2}\sin\theta + \frac{1}{2^2}\sin 2\theta + \frac{1}{2^3}\sin 3\theta + \frac{1}{2^4}\sin 4\theta + \cdots = \frac{2\sin\theta}{5 - 4\cos\theta}.$$

6. If $(1 + z)^n = p_0 + p_1z + p_2z^2 + \cdots + p_nz^n$, $n \in \mathbb{N}$, find the value of $p_0 + p_3 + p_6 + \cdots + p_{3[n/3]}$, when $n = 6t$; $n = 6t + 1$ and $n = 6t + 3$; $t \in \mathbb{N}$. Also, verify that

$$2^{n/2}\cos\frac{n\pi}{4} = \sum_{k=0}^{[n/2]}(-1)^k p_{2k},$$

where $[n/2]$ denotes the integral part of $n/2$.

7. For any real x and natural number n, show that

$$e^{i2n\cot^{-1}x}\left(\frac{ix+1}{ix-1}\right)^n = 1.$$

8. Determine all the nth roots of unity and find the sum and product of these roots. What geometric fact does this sum express? Also, verify the identity

$$\sum_{k=0}^{n-1}\left|z_1+\omega^k z_2\right|^2 = n\left(|z_1|^2+|z_2|^2\right)$$

For $n = 2$, this is a parallelogram identity.

9. If $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the nth roots of unity, calculate:

(a) $1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1}$;
(b) $1 + 4\omega + 9\omega^2 + \cdots + n^2\omega^{n-1}$;
(c) $1 - \omega + \omega^2 - \cdots + (-1)^{n-1}\omega^{n-1}$.

Also, verify the identity

$$(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1}) = 1 + z + z^2 + \cdots + z^{n-1}.$$

10. If $\rho_1, \rho_2, \ldots, \rho_n$ are the nth roots of $z_0 = r_0e^{i\theta_0}$, determine

(a) $\displaystyle\sum_{k=1}^{n}\rho_k^h,\ h \in \mathbb{R}$; (b) $\displaystyle\sum_{i=1}^{n}(-1)^{k-1}\rho_k^h,\ h \in \mathbb{R}$; (c) $\displaystyle\sum_{1\le k<j\le n}(-1)^{k-1}\rho_k^h$,

Suggestion. For part (b), consider $(\rho_1 + \rho_2 + \cdots + \rho_n)^2 = 0$.

11. In each of the following cases, find all of the roots (in Cartesian form) and exhibit them geometrically, and point out which is the principal root:

(a) $(1-\sqrt{3}\,i)^{1/2}$; (b) $(-8i)^{1/3}$; (c) $(-5-6i)^{1/2}$; (d) $(-16)^{1/4}$.

12. Find all the roots of the equations:

(a) $z^3 - z^2 + z - 1 = 0$; (b) $z^3 + z^2 + z + 1 = 0$, (c) $z^6 + z^3 + 1 = 0$;
(d) $z^4 - (1-z)^4 = 0$; (e) $z^7 + z^4 + z^3 + 1 = 0$. (f) $z^n = \bar{z}$

13. Solve the equation $z^7 + 1 = 0$ and deduce that

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}.$$

Also, find the quadratic equation which has $\omega + \omega^2 + \omega^4$ and $\omega^3 + \omega^5 + \omega^6$ as its roots, where $\omega = e^{2\pi i/7}$ is one of the roots of $z^7 - 1 = 0$.

14. Prove that for $n \ge 1$,

(a) $\displaystyle\prod_{k=0}^{n-1}\tan^2\left[\frac{(2k+1)\pi}{4n}\right] = 1$; (b) $\displaystyle\prod_{k=0}^{n-1}\sec^2\left[\frac{(2k+1)\pi}{4n}\right] = 2^{2n-1}$.

Suggestion. Solve the equation $(z+1)^{2n} + (z-1)^{2n} = 0$. The roots are

$$z = \pm i\tan\left[\frac{(2k+1)\pi}{4n}\right],\ k = 0, 1, \ldots, n-1$$

so that

$$(z+1)^{2n}+(z-1)^{2n}=2\prod_{k=0}^{n-1}\left[z^2+\tan^2\left(\frac{(2k+1)\pi}{4n}\right)\right].$$

The assertions now follow on taking $z = 0$ and $z = 1$.

15. For what values of p, q, r and s the roots of the equation

$$z^4 + pz^3 + qz^2 + rz + s = 0$$

lie on the vertices of a rectangle so that one of the vertices is at origin.

Suggestion. Consider $(z + 1)^4 = 1$ and equate the coefficients.

16. If z_1, z_2, z_3 lie on the unit circle, show that

(a) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \dfrac{3}{2}$ if $z_1 + z_2 + z_3 = 0$;

(b) $\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma = 18 \sin(\alpha + \beta + \gamma)$, if $z_1 + 2z_2 + 3z_3 = 0$. where, α, β, γ are the principal arguments of z_1, z_2, z_3, respectively.

17. Show that the roots of the equation $z^2 + az + b = 0$ are both of unit modulus $\Leftrightarrow$ $|a| \le 2$, $|b| = 1$, $\arg b = 2 \arg a$.

18. Show that Eq. (1.18) can also be written as

(a) $\cos n\theta = \displaystyle\sum_{k=0}^{[n/2]}\binom{n}{2k}\cos^{n-2k}\theta\,(\cos^2\theta - 1)^k$;

(b) If x is restricted to lie in the interval $-1 \le x \le 1$ and write $x = \cos\theta$, where $0 \le \theta \le \pi$, then with the help of part (a) point out how the equation

$$T_n(x) = \cos(n \cos^{-1}x)$$

gives rise to polynomials of degree n in the variable x.

These polynomials are called Chebyshev polynomials and play a key role in approximation theory and numerical analysis. Basically, these came into existence as solutions of second order linear equations.

1.5 REGIONS IN THE COMPLEX PLANE

Here we introduce elementary topology of the complex plane. To make the section self explanatory and keeping in view the analysis in Chapter 4 and onwards, we also recall some concepts on the real line.

By a *neighbourhood* or *circular neighbourhood* of a point z_0 in complex plane $\mathbb{C}$, we mean a set of points given by

$$S = \{z \in \mathbb{C} : |z - z_0| < \delta,\ \delta > 0\}.$$

Frequently, we shall denote S by $N_\delta(z_0)$ and $N_\delta(z_0)\backslash z_0 = \{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$ will be referred to as deleted neighbourhood of z_0.

A point $z_0 \in S$ is said to be an *interior point* of S if there exists a neighbourhood of z_0 which is contained in S, and z_0 will be an *exterior point* of S if there exists a neighbourhood of z_0 which contains no point of S. Moreover, z_0 is the boundary point of

the set if each deleted neighbourhood of z_0 intersects S and its complement. Note that a boundary point does not need to be in the set. The set of all boundary points of S is called the *boundary* of S, and is denoted by ∂S.

Open set. A set S in $\mathbb{C}$ is said to be *open* if corresponding to each point of S there exists a neighbourhood which is contained in S.

In view of the above definition it is trivial to say that a set is open if and only if all its points are interior points. A set S is said to be closed if its compliment S^c is open. There are sets which are neither open nor closed and also, there are sets which are both open and closed. The sets ϕ and $\mathbb{C}$ are both open and closed. The set $A = \{z : 0 < |z - z_0| \leq r\}$ is neither open nor closed. Further, we mention two facts: (i) the intersection of a finite number of open sets is open; (ii) the arbitrary union of open sets is open.

Speaking in a simple manner, an open set does not have any of its boundary points, while a closed set contains all its boundary points. Moreover, the set $\overline{S} = S \cup \partial S$ will be termed *closure* of S.

Limit point. A point z_0 is said to be a *limit point* of S if each deleted neighbourhood of z_0 intersects S.

Remarks. 1. Accumulation point and cluster point are synonym terms.

2. If a point is not the limit point of a set but belongs to the set then it is classified as isolated point. For example, $1 + i$ is the isolated point of the set $S = \{z : |z| = 1\} \cup \{1 + i\}$.

Theorem 2. A set S is closed $\Leftrightarrow$ it contains all its limit points.

Proof. Let S be a closed set. Assume a be a limit point of S which is not in S. This means that $a \in S^c$ and since S^c is open there exists a neighbourhood $N_\delta(a) \subset S^c$. Thus, $N_\delta(a) \cap S = \phi$. This contradicts that a is a limit point of S. Thus, our assumption is not valid and every limit point of S belongs to S.

To prove the sufficient part we are given that S contains all its limit points. Let $z \in S^c$. Then $z \notin S$. Hence z cannot be a limit point of S and these conditions imply $N_\delta(z) \cap S = \phi$. This implies $N_\delta(z) \subset S^c$. Since z is arbitrary, it follows that S^c is open.

Before introducing connected sets in the complex plane we recollect the definition of star-like sets and convex sets. It is simple to establish that the equation of a line in the complex plane joining two points a and b can be written as

$$z = (1 - t)a + tb, \qquad t \in \mathbb{R}.$$

For obvious reasons $t \geq 0$ yields a half line emanating from a and pointing towards b. Similarly, $t \leq 0$ is the half line emanating from b and pointing towards a. Conclusively, $0 \leq t \leq 1$ is the line segment joining a and b.

A set S in $\mathbb{C}$ is said to be *star-like* or *star-shaped* at a if for every $z \in S$, the line segment joining a to z lies entirely in S. If the point a is not mentioned, then star-like means with respect to the origin.

A set S in $\mathbb{C}$ is said to be convex if a line segment joining any pair of points of S lies in S. Mathematically, we write

$$(1 - t)z_1 + tz_2 \in S \ \forall \ z_1, z_2 \in S, \ t \in [0, 1].$$

Note that every star-like set may not be convex. An open disc, half-planes, and rectangles are examples of star-like sets and hence convex sets. An example of a star-like set but not convex set is $\mathbb{C}\backslash\{z = x, x > 0\}$, i.e. the plane minus the positive x-axis.

Connected set. A set S in $\mathbb{C}$ is said to be connected if it cannot be written as $A \cup B$ such that $\overline{A} \cap B = \phi$ or $A \cap \overline{B} = \phi$.

A simple consequence of this definition is that an open connected set in the complex plane is connected if it cannot be written as a union of two disjoint open sets. By convention the empty set ϕ is connected.

Path. By a path we mean a continuous complex function defined on the interval $[0, 1]$. Hence a path is $\gamma : [0, 1] \mapsto \mathbb{C}$ or

$$\gamma(t) = x(t) + iy(t), \qquad 0 \le t \le 1.$$

Pathwise connected set. Let $S \subseteq \mathbb{C}$. We say S is pathwise or arcwise connected if for any pair of points a and b in S there exists a path γ defined on $[0, 1]$ such that

$$\gamma(0) = a, \qquad \gamma(1) = b; \qquad \text{and} \qquad \gamma(t) \in S \quad \forall\ 0 < t < 1.$$

Equivalently, we may also say that a set $S \subseteq \mathbb{C}$ is pathwise connected if each pair of points can be joined by a polygonal arc (consisting of line segments joined end to end) that lies entirely in S. Observe that there may be different paths joining the same points as $\gamma_1(t) = t + it$ and $\gamma_2(t) = t + it^2$ are line segment and parabolic curve, respectively which join the points $(0, 0)$ and $(1, 1)$.

Example 12. Show that $S = \{z : \text{Im } z = 0\}$ is pathwise connected.

For any two points a, b in S, define

$$\gamma(t) = (1 - t)a + bt, \qquad \text{Im } a = 0, \qquad \text{Im } b = 0,$$

where γ is a real continuous function. Obviously, $\gamma(0) = a$, $\gamma(1) = b$ and moreover, this is line segment joining a and b such that $\text{Im } \gamma(t) = 0$, and hence $\gamma(t) \in S\ \forall\ t \in [0, 1]$, i.e. S is pathwise connected.

Theorem 3. Let A and B be two pathwise connected sets in complex plane $\mathbb{C}$ such that $A \cap B \neq \phi$. Then $A \cup B$ is pathwise connected.

Proof. Our aim is to find a continuous complex function joining any pair of points in $A \cup B$. Let $a, b \in A \cup B$ such that a and b are not in the same set. Since $A \cap B \neq \phi$, there exists a $z \in A \cap B$ (see Fig. 1.8).

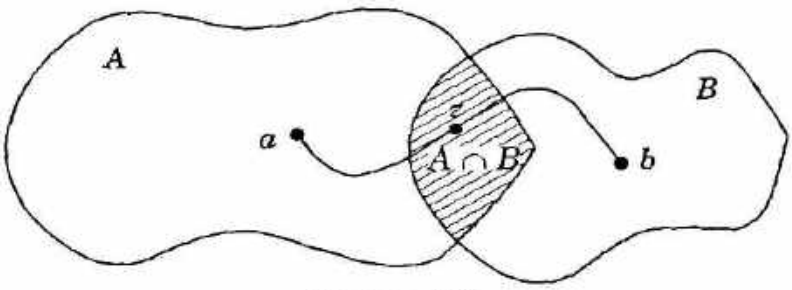

Figure 1.8

Now, $a, z \in A$ and A is pathwise connected so there exists a continuous function γ_1 such that $\gamma_1(0) = a$, $\gamma_1(1) = z$ and $\gamma_1(t) \in A$. Further, $z, b \in B$ and B is pathwise connected so we have a path γ_2 such that $\gamma_2(0) = z$, $\gamma_2(1) = b$ and $\gamma_2(t) \in B$.

Define

$$\gamma(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2, \\ \gamma_2(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Here, γ is a combined path with the properties $\gamma(0) = a$, $\gamma(1) = b$ and $\gamma(t) \in A \cup B$. Obviously, γ is continuous. Hence $A \cup B$ is connected.

Remarks. 1. Every convex set in $\mathbb{C}$ is arcwise connected, since a line segment joining any two points lies in the set but the converse may not be true. For instance, the set $\{z : |z| > 1\}$ is pathwise connected but it is not convex.

2. Pathwise connected sets are always connected but the converse may not be true. Let

$$A = \{(x, y), 0 < x \le 1, y = \sin(1/x)\},$$
$$B = \{(x, y), y = 0, -1 \le x \le 0\}.$$

Then $S = A \cup B$ is connected but not pathwise connected.

However, for open sets both concepts are equivalent. In this regard we quote a result which we shall not prove:

An open set in $\mathbb{C}$ is connected $\Leftrightarrow$ it is pathwise connected.

Suppose E is a bounded set of real numbers and if there exists a real number α such that (i) α is an upper bound of E; (ii) if $\beta < \alpha$ then β is not an upper bound of E. Then we say that α is the least upper bound (l.u.b.) of E (that there is at most one such α which is clear from (ii)) or the supremum of E, and we write

$$\alpha = \sup E.$$

The sup E may or may not belong to E. For $E = (0, 1)$, $\sup E = 1 \notin E$, but for $E = [0, 1]$, sup E belongs to the set.

The greatest lower bound (g.l.b.), or infimum of a set which is bounded below is defined in the same manner. The statement

$$\alpha = \inf E$$

means that α is a lower bound of E and that no $\beta > \alpha$ is a lower bound of E. If E is unbounded above and bounded below then $\sup E = \infty$ and, for unbounded below and bounded above we have $\inf E = -\infty$. Note that max E and min E are always members of E by definition, provided they exist.

Bounded set. A set S in complex plane is *bounded* if each point of S lies in some $|z| = R$. A set which is not bounded is called unbounded set.

Here, we mention an important property of bounded infinite sets. This will be utilized to prove various results in subsequent chapters, refer Section 4.9.

Bolzano–Weierstrass property. Every bounded infinite set in the complex plane has at least one limit point.

Compact set. A set $S \subset \mathbb{C}$ is said to be *compact* if it is bounded and closed.

Remark. The above definition is Heine–Borel theorem which will serve our most of the purposes. A collection Λ of open sets is called the covering of a given set S if

$$S \subseteq \bigcup_{A \in \Lambda} A.$$

If every open covering of a set S admits a finite covering, i.e. there are open sets $A_1, A_2, \ldots, A_n$ such that

$$S \subseteq \bigcup_{i=1}^{n} A_i$$

then S is said to be compact. This is the basic definition of a compact set.

Domain. A nonempty open connected set in $\mathbb{C}$ is called a *domain*.

It is to be noted that every neighbourhood of a point in the complex plane is a domain. The bounded domain means it is bounded as a set. A domain together with some, none or all of its boundary points will be referred to as *region*. Thus, every domain is a region but converse may not be true.

Example 13. For any complex number a, sketch the following set in z-plane.

$$S = \{z : |z - a| + |z + a| = 2c\}.$$

First, we note that the set is empty if $|a| > c$, since

$$|2a| = |(z - a) - (z + a)| \leq |z - a| + |z + a| \leq 2c.$$

Suppose $z \in S$. Then

$$|2z| = |(z - a) + (z + a)| \leq |z - a| + |z + a| \leq 2c.$$

So,

$$\max_{z \in S} |z| = c.$$

The maximum is attained at the point $ce^{i\theta}$, where $\theta = \arg(a/c)$. Now, S is the locus of a point which moves such that the sum of its distance from fixed points a and $-a$ remains constant. Hence, S is an ellipse with foci at $\pm a$ (see Fig. 1.9(a)).

In particular, if a is a real number, then taking $z = x + iy$ in the definition of S and simplifying the expression, we get

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - a^2} = 1.$$

This is the equation of ellipse centred at the origin and its semi-major and semi-minor axes equal to c and $\sqrt{(c^2 - a^2)}$, respectively, as shown in Fig. 1.9(b).

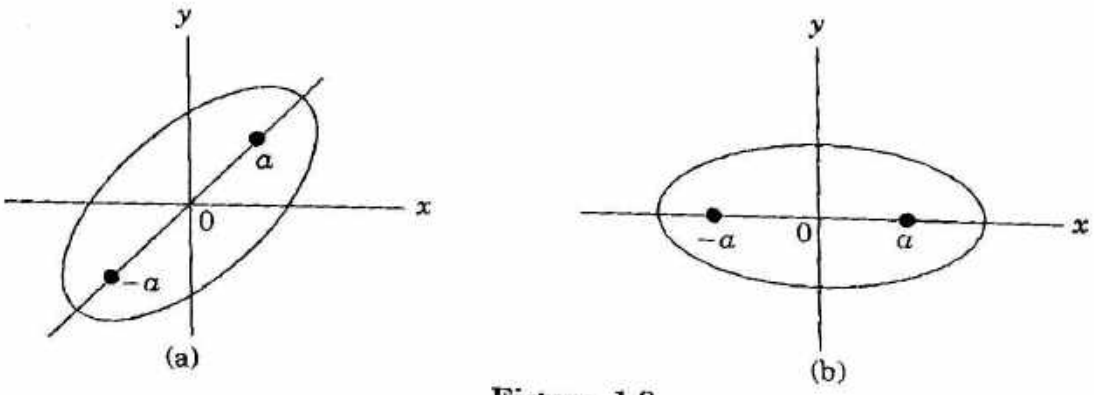

Figure 1.9

Example 14. Using the above definitions, we study the following set:

$$S = \left\{\left(\frac{1}{m} + i\frac{1}{n}\right) : m, n \in \mathbb{I}_0\right\},$$

where $\mathbb{I}_0$ denotes the set of nonzero integers.

(i) Clearly, S is bounded, since $S \subseteq \{z : |z| \leq \sqrt{2}\}$;

(ii) The set of limit points is given by $\{0, 1/n, i/n, n \in \mathbb{I}_0\}$;

(iii) It is not open as any neighbourhood about a point of the set is not contained in S. Moreover, it is not closed as it does not contain its limit point 0. Thus, S is neither open nor closed. Clearly, it is not compact;

(iv) This set is not convex as any line segment connecting two points of the set is not contained in the set. Further, if we connect any two points by a polygonal line, then there are points on the line which are not in S. So, S is not pathwise connected.

1.6 EXTENDED COMPLEX PLANE

By the *extended complex number system* $\mathbb{C}_\infty$, we shall mean the complex plane $\mathbb{C}$ along with a symbol ∞ which satisfies the following properties:

(a) If $z \in \mathbb{C}$, then we have $z + \infty = z - \infty = \infty$, $z/\infty = 0$;

(b) If $z \in \mathbb{C}$, but $z \neq 0$, then $z \cdot \infty = \infty$ and $z/0 = \infty$;

(c) $\infty + \infty = \infty \cdot \infty = \infty$;

(d) $\dfrac{\infty}{z} = \infty \, (z \neq \infty)$.

The set $\mathbb{C} \cup \{\infty\}$ is called the *extended complex plane* and will be denoted by $\mathbb{C}_\infty$.

To visualize the point at infinity, one can think of the complex plane as passing through the equator of the unit sphere centred at the origin $z = 0$. Now, we establish a one-one correspondence between the points on the sphere and the points on the plane. Then to each point A in the plane there corresponds a unique point A' on the sphere. The point A' is the point where the line joining A to the north pole N intersects the sphere. Conversely, corresponding to each point A' on the sphere (except the north pole) there exists a unique point A in the plane. By defining that the north pole N corresponds to the point at infinity, we conclude that there exists a one-one correspondence between the points on the sphere and the points in the extended complex plane. This sphere is known as *Riemann sphere* and the correspondence is *stereographic projection*, (see Fig. 1.10).

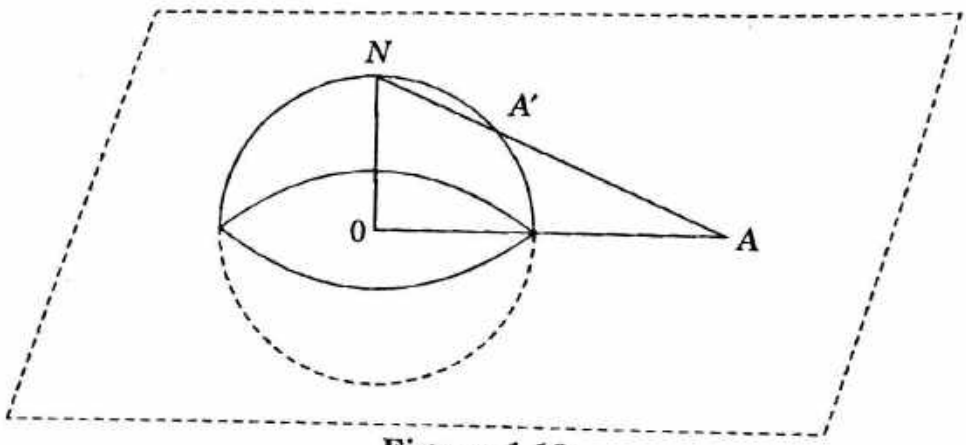

Figure 1.10

Let the sphere be

$$x_1^2 + x_2^2 + x_3^2 = 1 \tag{1.21}$$

and the plane of projection be $x_3 = 0$. We take $N = (0, 0, 1)$.

For any point $A' = (x_1, x_2, x_3)$ on the sphere we have point $A = (x, y, 0)$ in the complex plane where NA meets the plane of projection. Since the points $(0, 0, 1)$, (x_1, x_2, x_3), $(x, y, 0)$ are on a straight line, we have

$$\frac{x_1}{x} = \frac{x_2}{y} = \frac{x_3 - 1}{-1}.$$

This implies

$$x = \frac{x_1}{1 - x_3}, \quad y = \frac{x_2}{1 - x_3} \tag{1.22}$$

and hence

$$z = x + iy = \frac{x_1 + ix_2}{1 - x_3}. \tag{1.23}$$

In view of (1.21) and (1.23),

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}.$$

This implies

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Further, computation yields

$$x_1 = \frac{z + \bar{z}}{z\bar{z} + 1} = \frac{z + \bar{z}}{|z|^2 + 1}, \qquad x_2 = \frac{z - \bar{z}}{i(z\bar{z} + 1)} = \frac{z - \bar{z}}{i(|z|^2 + 1)}.$$

These equations and (1.22) establish one-one correspondence between the complex numbers in the extended complex plane and points on the sphere. It is geometrically obvious that stereographic projection transforms every straight line in z-plane into a circle on S which passes through $(0, 0, 1)$.

Let us prove the assertion that a circle on the sphere corresponds to a circle or straight line in the z-plane. Note that a circle on the sphere lies on the plane

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0,$$

where it is assumed that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $0 \le \alpha_0 < 1$.

Converting this equation in terms of z and $\bar{z}$, we get

$$\alpha_1(z + \bar{z}) - \alpha_2 i(z - \bar{z}) + \alpha_3(|z|^2 - 1) = \alpha_0(|z|^2 + 1)$$

or

$$(\alpha_0 - \alpha_3)(x^2 + y^2) - 2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0.$$

For $\alpha_0 \neq \alpha_3$, this represents a circle, and for $\alpha_0 = \alpha_3$, it reduces to a line.

PROBLEM SET 1.6

1. Sketch the region in the z-plane represented by the set of points. Also, state which of these sets are bounded:

(a) $\mathrm{Re}(\bar{z} - 1) = 2$; (b) $|z - 2 + i| \le 1$;

(c) $|2z + 3| > 4$;
(d) $z^2 + \bar{z}^2 = 2$;
(e) $\{z : |z - 1| + |z + 1| = 4\}$;
(f) $\mathrm{Re}(1/z) \le 1/2$;
(g) $\mathrm{Re}(z^2) > 0$;
(h) $\{z : |z - i| - |z + i| = 2\}$;
(i) $\mathrm{Re}[(z - z_1)/(z - z_2)] = 0$;

2. Show that (a) the arbitrary union of open sets is open; (b) the finite intersection of open sets is open. With the help of De-Morgan's law, translate the assertions (a) and (b) for closed sets.

3. Determine whether the following sets are open, closed, neither open nor closed, both open and closed. Also, decide which are domains:

(a) $\mathrm{Im}\, z > 1$;
(b) $\mathrm{Im}\, z = 1$;
(c) $0 \le \arg z \le \pi/4$;
(d) $|z - 4| \ge |z|$;
(e) $|z + i| + |z - i| \le 4 \cap |\arg z| < \pi/4$;
(f) $\{z = x + iy,\ x, y \in \mathbb{Q}\}$;
(g) $0 \le \arg z \le \pi/4 \cup |z| < 1$;
(h) $0 < |z| < 2 \backslash \{1\}$.

4. If, each neighbourhood of a point a intersects a set S at infinitely many points, prove that a is an accumulation point of S.

5. Find the accumulation points of the sets:

(a) $-\pi < \arg z < \pi,\ z \ne 0$;
(b) $|\mathrm{Re}\, z| < |z|$;
(c) $i^n,\ n \in \mathbb{N}$;
(d) $i(z + \bar{z})(z - \bar{z}) > 4$;
(e) $\{z = x + iy : x \in \mathbb{I}, y \in \mathbb{R}\}$;
(f) $i^n/n,\ n \in \mathbb{N}$;
(g) $0 \le \arg z < \pi/2$;
(h) $(-1)^n (1 + i)(n - 1)/n,\ n \in \mathbb{N}$;
(i) $ni^n/(n + 1),\ n \in \mathbb{N}$.

6. Prove that a set S is closed $\Leftrightarrow S = \bar{S}$.

7. A set S in $\mathbb{C}$ is unbounded $\Leftrightarrow$ every neighbourhood of infinity contains at least one point of S.

8. Prove that every finite set in $\mathbb{C}$ is closed. What can you say about countable subsets of $\mathbb{C}$?

9. Which of the following sets are connected:

(a) $\{z : |z| < 1\} \cup \{z : |z - 1| < 1\}$;
(b) $\{z : -1 < |z| \le 1\} \cup \{2\}$;
(c) $\{z : -\pi/4 < \arg z \le \pi/4\}$;
(d) $\{z : |z^2 - 1| < 2\}$.

10. Show that the set $\{z : |z - z_0| < \delta, \delta > 0\}$ is convex and hence pathwise connected.

11. Discuss Example 13 when c is any complex number such that

$$|z - a| + |z + a| = 2|c|.$$

12. Suppose a cube has its vertices on the Riemann sphere and its edges parallel to the coordinate axes. Find the stereographic projection of the vertices.

13. Describe the position of the following points in relation to z in complex plane, as viewed on the Riemann sphere

$$\bar{z},\ z^{-1},\ iz,\ -iz.$$

2

Analytic Functions

In this chapter, we shall define the functions of a complex variable and discuss the basic notions of limit, continuity, and differentiability. However, our attention will be focused on developing the theory of analytic functions, as these functions have a wide range of applications in complex analysis. The chapter concludes by giving a brief exposition of harmonic functions and their relationship with the analytic functions.

2.1 FUNCTIONS OF A COMPLEX VARIABLE

Let S be a subset of complex plane. By a function f from S to $\mathbb{C}$, we mean a rule which assigns to each $z \in S$ a unique value $f(z) \in \mathbb{C}$. Symbolically, we write $w = f(z)$, $z \in S$ or

$$f : S \to \mathbb{C}. \tag{2.1}$$

We say f is a complex function of a complex variable. It is customary to locate $f(z) \in \mathbb{C}$ in another complex plane called the w-plane. S is called the domain of f (it is a set, not necessarily open and connected) and $S_1 = \{f(z) : z \in S\}$, range of f. Note that the codomain $\mathbb{C}$ of f in (2.1), may be any subset of $\mathbb{C}$. When S is a subset of real line, we say that f is a complex function of a real variable (see definition of a path, Section 1.5).

Every function $f(z)$ can be written as $f(z) = u(x, y) + iv(x, y)$ or $f = u + iv$, e.g. $f(z) = z^2 \Leftrightarrow f(z) = x^2 - y^2 + i2xy$ and $f(z) = z^3 + z + 1 \Leftrightarrow f(z) = x^3 - 3xy^2 + x + 1 + i(3x^2y - y^3 + y)$. Here, $u(x, y)$ and $v(x, y)$ are real functions of two variables and will be referred to as real and imaginary components of $f(z)$, respectively.

When the domain of definition is not specified, we take the largest possible set as the domain for f on which f is well defined.

For the function $f(z) = 1/(z^2 + 1)$, the domain of definition is $S = \mathbb{C} - \{i, -i\}$. In the case of $f(z) = \text{Arg } z$, the domain of definition is $S = \mathbb{C} - \{0\}$.

Multivalued functions

Generalization of a concept of a function is the multivalued function. These functions take more than one value at some or all points of the domain of definition. Though, set theoretically, such associations are not functions. By abuse of language we call these type

of associations as *multivalued functions*. To study such functions we take one of the possible values at each point of domain. For example,

$$f(z) = z^{1/2} = \pm\sqrt{r}e^{i\theta/2} \quad (-\pi < \theta \leq \pi).$$

This is a multivalued function and has two values $\pm\sqrt{r}e^{i\theta/2}$ $(-\pi < \theta \leq \pi)$. If we choose only the positive values of $\sqrt{r}$ and write

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi),$$

then the above function is single valued in the specified domain. Moreover, we may add 0 to the domain of definition (as 0 is the only square root of zero) to write

$$f(z) = \begin{cases} \sqrt{r}e^{i\theta/2}, & |z| > 0, \quad -\pi < \theta < \pi, \\ 0, & z = 0. \end{cases}$$

Also, every function can also be written as $f(z) = u(r, \theta) + iv(r, \theta)$. For example, let

$$f(z) = z + \frac{1}{z}, \qquad z \neq 0.$$

Then

$$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

Whenever we speak of a function we shall mean unless otherwise stated explicitly, a *single-valued function*. Suppose f is a complex function defined on a set A such that

$$f(z_1) = f(z_2) \Rightarrow z_1 = z_2 \ \forall\ z_1, z_2 \in A$$

or, equivalently

$$z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2) \ \forall\ z_1, z_2 \in A$$

then the mapping $w = f(z)$ is called *one-to-one*. If the mapping $w = f(z)$ is one-to-one then the function is called *univalent*. Note that a function f is said to be univalent at point z_0 if it is univalent in some neighbourhood of z_0.

Suppose $f : A \to B$ is univalent and onto $B \subseteq \mathbb{C}$. Then, we can define an inverse function as

$$z = f^{-1}(w) \quad \text{if } w = f(z).$$

It is to be noted that the inverse function is also univalent on B.

Suppose we have a function f with domain D and another function g with domain D_1 and further, let for every $z \in D$, $f(z) \in D_1$. Then, for every $z \in D$, the composition of f with g or superposition of g on f is the function gf defined by

$$(gf)(z) = g(f(z)).$$

We make further observations for $f(z) = u(x, y) + iv(x, y)$ with reference to conjugation.

Define

$$\overline{f(z)} = u(x, y) - iv(x, y)$$

and

$$f(\bar{z}) = u(x, -y) + iv(x, -y).$$

Thus, in general, $\overline{f(z)}$ and $f(\bar{z})$ are different functions. For instance, if $f(z) = (1+i)z$, we write $f(z)$ as

$$f(z) = (x-y) + i(x+y)$$

so that

$$\overline{f(z)} = (x-y) - i(x+y) = (1-i)\bar{z},$$

and

$$f(\bar{z}) = (x+y) + i(x-y) = (1+i)\bar{z}.$$

Example 1. Show that the function

$$f(z) = 2z + z^2, \qquad |z| < 1$$

is univalent in the domain.

Let z_1 and z_2 be any two points in $|z| < 1$. Then

$$f(z_1) = f(z_2) \Rightarrow (z_1 - z_2)(z_1 + z_2 + 2) = 0 \Rightarrow z_1 - z_2 = 0$$

as $z_1 + z_2 + 2 \neq 0$ in $|z| < 1$. This is clear from the fact

$$\text{Re}(z_1 + z_2 + 2) = \text{Re } z_1 + \text{Re } z_2 + 2 > -1 - 1 + 2 = 0.$$

Thus, f is one-one in $|z| < 1$ and hence univalent in the domain.

Note. The above function is not univalent in the whole complex plane since the choice of $z_1 = 1$, $z_2 = -3$ makes $z_1 + z_2 + 2 = 0$, i.e. $f(z_1) = f(z_2)$ even though z_1 and z_2 are different.

Here, we give a brief exposition of sequences as these are very much needed in discussing the important basic concepts of limits and continuity in this chapter and series in Chapter 5.

By a sequence of complex numbers we mean a one-one function from the set of positive integers to the set of complex numbers, i.e.

$$f : \mathbb{N} \to \mathbb{C}.$$

The values $f(1), f(2), \ldots, f(n)$ are called the first, the second and the nth term of the sequence, respectively, and usually denoted by $z_1, z_2, \ldots, z_n$. We shall prefer to write the infinite sequence $f(1), f(2), \ldots, f(n), \ldots$ as $\{z_n\}_{n=1}^{\infty}$ or, if the context is clear, simply by $\{z_n\}$. Sometimes $\mathbb{N}$ is replaced by $\mathbb{I}_0$ (set of nonnegative integers).

A sequence $\{z_n\}$ is said to converge to z_0 (or has a limit z_0) if for each $\varepsilon > 0$ there exists a positive integer N such that

$$|z_n - z_0| < \varepsilon \quad \text{whenever } n \geq N. \tag{2.2}$$

Symbolically, we write

$$\lim_{n \to \infty} z_n = z_0.$$

Geometrically, this means that, for sufficiently large values of n, the points z_n lie in given ε neighbourhood of z_0 (Fig. 2.1). Since we can choose ε as small as we please, it follows that the points z_n become closer and closer to z_0 as their subscripts increase.

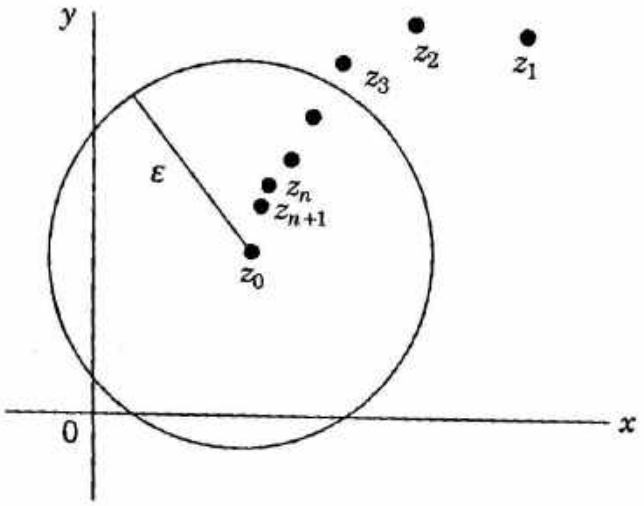

Figure 2.1

A sequence which is not convergent is defined as divergent. Moreover, we say that a sequence $\{z_n\}$ is bounded if there exists a number $M > 0$ such that $|z_n| \le M$ for all n.

Proposition 1. If $\lim_{n\to\infty} z_n = z_0$, then (a) $|z_n| \to |z_0|$ as $n \to \infty$; (b) the sequence $\{z_n\}$ is bounded.

Proof. (a) By the given hypothesis, we have for given $\varepsilon > 0$, there exists a positive integer N such that (2.2) holds. Since $||z_n| - |z_0|| \le |z_n - z_0|$, the inequality (2.2) yields

$$||z_n| - |z_0|| < \varepsilon \quad \text{whenever } n \ge N \tag{2.3}$$

and this ensures (a).

(b) From (2.3), we have

$$|z_0| - \varepsilon < |z_n| < |z_0| + \varepsilon \quad \text{whenever } n \ge N.$$

This implies that the sequence $\{z_n\}$ is bounded.

The sequence $\{2 + i/n\}$ is convergent and its limit is 2. Further, the sequence $\{(-1)^n + i/n\}$ diverges. Here, -1 and 1 are its limit points but these are not limit of the sequence as the limit of a sequence is always unique (see Section 5.1). This is a good example of a sequence which is bounded and the sequence of moduli converges but the sequence is not convergent. So, it supports the case that the converse statements of Proposition 1 may or may be true. However, with some additional condition on $\{z_n\}$ the converse of Proposition 1(a) holds (see Problem Set 2.1).

Let us briefly introduce the idea of subsequences.

Given a sequence $\{a_n\}$, consider a sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\{a_{n_k}\}$ is called a *subsequence* of $\{a_n\}$. If $\{a_{n_k}\}$ converges, its limit is called *subsequential limit.*

For example, if $1 + i(-1)^n/n$ is a sequence, then $1 + i/2n$ is one of its subsequences.

2.2 LIMITS

In this section, we discuss the limit concept for complex-valued functions.

Definition 1. Let $f(z)$ be a function defined in some neighbourhood of z_0. The f is said to have limit w_0 at z_0 if, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta. \tag{2.4}$$

Symbolically, we write

$$\lim_{z \to z_0} f(z) = w_0$$

or, formally this is also used in the form

$$f(z) = w_0 + \varepsilon(z) \ \forall\, z : 0 < |z - z_0| < \delta$$

with the condition that $\varepsilon(z) \to 0$ as $z \to z_0$.

This definition says that for each ε neighbourhood $|w - w_0| < \varepsilon$ of w_0, there exists a deleted δ neighbourhood of z_0 such that every point z in this has an image w lying in ε neighbourhood (Fig. 2.2).

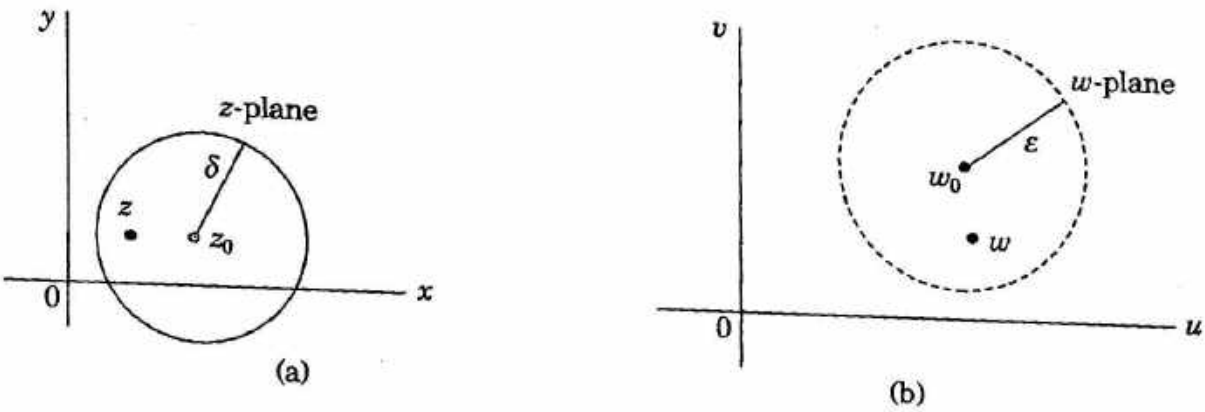

Figure 2.2

Since (2.4) is true for all $z \in \{z : 0 < |z - z_0| < \delta\}$, $z \to z_0$ means that z approaches z_0 through any path. Hence the limit w_0 is independent of path. Note that δ is not unique, and even more to say, the function may not be defined, where we are interested to evaluate limits.

Remarks. 1. Using $||f(z)| - |w_0|| \le |f(z) - w_0|$, in (2.4), we have

$$||f(z)| - |w_0|| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta.$$

This proves the result $\lim_{z \to z_0} |f(z)| = |w_0|$. Moreover, the function is bounded in some deleted neighbourhood of z_0 since

$$|w_0| - \varepsilon < |f(z)| < |w_0| + \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta.$$

2. If $\lim f(z) \to w_0$ as $z \to z_0$ exists, then it is unique (see Problem Set 2.1).

3. If $\lim_{z \to z_0} f(z) = w_0$, then $\lim_{z \to z_0} \overline{f(z)} = \overline{w}_0$ (see Problem Set 2.1).

Now, we prove some results concerning limits.

Example 2. Using the definition of limit, show that

$$\lim_{z \to z_0} az^2 + bz + c = az_0^2 + bz_0 + c,$$

where a, b, c are complex constants.

To verify this limit we have to find a $\delta > 0$, for given $\varepsilon > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |(az^2 + bz + c) - (az_0^2 + bz_0 + c)| < \varepsilon. \tag{2.5}$$

Now,

$$\begin{aligned} |(az^2 + bz + c) - (az_0^2 + bz_0 + c)| &= |a(z^2 - z_0^2) + b(z - z_0)| \\ &= |(z - z_0)(a(z + z_0) + b)| \\ &\le |z - z_0|(|a|\,|z + z_0| + |b|). \end{aligned} \tag{2.6}$$

From (2.6), note that $|(az^2 + bz + c) - (az_0^2 + bz_0 + c)| < \varepsilon$ holds true if we assume

$$|z - z_0|(|a|\,|z + z_0| + |b|) < \varepsilon.$$

Restricting $|z - z_0| < 1$, implies

$$\begin{aligned} |a|\,|z + z_0| + |b| &= |a|\,|z - z_0 + 2z_0| + |b| \\ &\le |a|(|z - z_0| + 2|z_0|) + |b| \\ &< |a|(1 + 2|z_0|) + |b|. \end{aligned}$$

If we choose

$$\delta = \min\left\{1, \frac{\varepsilon}{|a|(1 + 2|z_0|) + |b|}\right\},$$

then (2.5) holds.

Example 3. In case $f(z)$ is given in $u(x, y) + iv(x, y)$ form, let us verify

$$\lim_{z \to 2+3i} x + i(x + y^2) = 2 + 11i.$$

Given $\varepsilon > 0$, we have to find a $\delta > 0$ such that

$$|x + i(x + y^2) - (2 + 11i)| < \varepsilon \quad \text{whenever } 0 < |z - (2 + 3i)| < \delta.$$

Consider the expression

$$\begin{aligned} |x + i(x + y^2) - (2 + 11i)| &= |(x - 2) + i(x - 2) + i(y^2 - 9)| \\ &\le |x - 2| + |i(x - 2)| + |i(y - 3)(y + 3)| \\ &\le 2|x - 2| + |(y - 3)(y + 3)|. \end{aligned}$$

Then $|x + i(x + y^2) - (2 + 11i)| < \varepsilon$ will be satisfied if we assume

$$2|x - 2| + |(y - 3)(y + 3)| < \varepsilon$$

or

$$2|x - 2| < \frac{\varepsilon}{2} \quad \text{and} \quad |(y - 3)(y + 3)| < \frac{\varepsilon}{2}.$$

The first of these is, of course satisfied when $|x-2| < \varepsilon/4$. To determine the condition on y such that the second one holds, we restrict y so that $|y-3| < 1$ and observe that

$$|y+3| = |y-3+6| \le |y-3| + 6 < 7.$$

Hence, if $|y-3| < \min\{1, \varepsilon/14\}$, it follows that

$$|y-3|\,|y+3| < \frac{\varepsilon}{14}7 = \frac{\varepsilon}{2}.$$

An appropriate value of δ is now easily seen by setting

$$|x-2| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |y-3| < \min\left\{1, \frac{\varepsilon}{14}\right\}:$$

and consequently,

$$\delta = \min\left\{1, \frac{\varepsilon}{14}\right\}.$$

Remark. Since δ is not unique, if we proceed as

$$\begin{aligned}|x+i(x+y^2)-(2+11i)| &= |(x-2)+i(x-2)+i(y^2-9)| \\ &= |(1+i)(x-2)+i(y-3)(y+3)| \\ &\le |(1+i)(x-2)| + |(y-3)(y+3)| \\ &= \sqrt{2}\,|x-2| + |(y-3)|\,|(y+3)|.\end{aligned}$$

Let

$$\sqrt{2}\,|x-2| < (1-\alpha)\varepsilon \quad \text{and} \quad |y-3|\,|y+3| < \alpha\varepsilon, \qquad \alpha > 0$$

or

$$|x-2| < \frac{(1-\alpha)\varepsilon}{\sqrt{2}} \quad \text{and} \quad |y-3|\,|y+3| < \alpha\varepsilon. \tag{2.7}$$

Then, for this choice, $|x^2 + i(x+y^2) - (2+11i)| < \varepsilon$ is satisfied. Restricting $|y-3| < 1$, we get

$$|y+3| = |y-3+6| \le |y-3| + 6 < 7.$$

Hence, if $|y-3| < \min\{1, \alpha\varepsilon/7\}$, the second inequality in (2.7) is satisfied,

$$|y-3|\,|y+3| < \frac{\alpha\varepsilon}{7}7 = \alpha\varepsilon.$$

Now,

$$\frac{(1-\alpha)\varepsilon}{\sqrt{2}} = \frac{\alpha\varepsilon}{7} \Rightarrow \alpha = \frac{7}{7+\sqrt{2}}.$$

Finally, we choose

$$\delta = \min\left\{1, \frac{\varepsilon}{7+\sqrt{2}}\right\}.$$

It is to be noted that both values of δ are justified. However, the value obtained in the last analysis for the same $\varepsilon > 0$ covers larger sets to satisfy $|x+i(x+y^2)-(2+11i)| < \varepsilon$.

Theorem 1. Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $w_0 = p + iq$. Then

$$\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow \lim_{(x, y) \to (x_0, y_0)} u(x, y) = p \quad \text{and} \quad \lim_{(x, y) \to (x_0, y_0)} v(x, y) = q. \tag{2.8}$$

Proof. Necessary condition: Let $\lim_{z \to z_0} f(z) = w_0$. Then, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta$$

or

$$|u(x, y) + iv(x, y) - (p + iq)| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta$$

or

$$|u(x, y) - p + i(v(x, y) - q)| < \varepsilon \quad \text{whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Using the facts: $\operatorname{Re} z \le |z|$, $\operatorname{Im} z \le |z|$, the above inequality leads to

$$|u(x, y) - p| < \varepsilon \quad \text{whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta,$$

$$|v(x, y) - q| < \varepsilon \quad \text{whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

These inequalities imply the limits in right side of (2.8).

Sufficient condition: Given that $u(x, y) \to p$ and $v(x, y) \to q$ as $(x, y) \to (x_0, y_0)$. Hence, for given $\varepsilon > 0$, we have

$$|u(x, y) - p| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1,$$

$$|v(x, y) - q| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2.$$

Choosing $\delta = \min\{\delta_1, \delta_2\}$, we note

$$\begin{aligned} |f(z) - w_0| &= |u(x, y) + iv(x, y) - p - iq| \\ &\le |u(x, y) - p| + |v(x, y) - q| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta. \end{aligned}$$

Thus, $f(z) \to w_0$ as $z \to z_0$.

Theorem 2. Suppose that $\lim_{z \to z_0} f(z) = \ell$ and $\lim_{z \to z_0} g(z) = m$. Then,

(a) $\lim_{z \to z_0} f(z) \pm g(z) = \ell \pm m$;

(b) $\lim_{z \to z_0} cf(z) = c\ell$; c a complex constant;

(c) $\lim_{z \to z_0} f(z)g(z) = \ell m$;

(d) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\ell}{m}$ provided $m \ne 0$.

Proof. (a) Since $\lim_{z \to z_0} f(z) = \ell$ and $\lim_{z \to z_0} g(z) = m$, we have, for given $\varepsilon > 0$, there exist positive numbers δ_1 and δ_2 such that

$$|f(z) - \ell| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |z - z_0| < \delta_1,$$

$$|g(z) - m| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |z - z_0| < \delta_2.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|f(z) + g(z) - \ell - m| \le |f(z) - \ell| + |g(z) - m|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta.$$

(b) Its proof is trivial.

(c) Let us write

$$f(z)g(z) - \ell m = m[f(z) - \ell] + \ell[g(z) - m] + [f(z) - \ell][g(z) - m]. \tag{2.9}$$

Since $f(z) \to \ell$ and $g(z) \to m$ as $z \to z_0$, for $\varepsilon > 0$, we have

$$|f(z) - \ell| < \sqrt{\varepsilon} \quad \text{whenever } 0 < |z - z_0| < \delta_1,$$

$$|g(z) - m| < \sqrt{\varepsilon} \quad \text{whenever } 0 < |z - z_0| < \delta_2.$$

We now let $\delta = \min\{\delta_1, \delta_2\}$. Thus, for $\delta > 0$,

$$|[f(z) - \ell][g(z) - m]| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta.$$

Therefore, $\lim_{z \to z_0} [f(z) - \ell][g(z) - m] = 0$.

We now apply (a) and (b) to (2.9) to conclude

$$\lim_{z \to z_0} f(z)g(z) - \ell m = 0.$$

(d) As $\lim_{z \to z_0} g(z) = m$, $m \neq 0$, we have, for given $\varepsilon > 0$, $\exists$ a $\delta_1 > 0$ such that

$$|g(z) - m| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta_1.$$

Since $m \neq 0$, choice of $\varepsilon = |m|/2$ reduces the above inequality to

$$\frac{|m|}{2} < |g(z)| < \frac{3|m|}{2} \quad \text{whenever } 0 < |z - z_0| < \delta_1,$$

and its further application yields

$$\left| \frac{1}{g(z)} - \frac{1}{m} \right| = \left| \frac{g(z) - m}{g(z)m} \right| = \frac{|g(z) - m|}{|g(z)||m|} \le \frac{2|g(z) - m|}{|m|^2}. \tag{2.10}$$

But $\varepsilon > 0$ is arbitrary, so $|m|\varepsilon/2$ is positive and there is a number $\delta_2 > 0$ such that

$$|g(z) - m| < \frac{|m|^2}{2}\varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta_2. \tag{2.11}$$

For $\delta = \min\{\delta_1, \delta_2\}$, (2.10) and (2.11) combine to produce the result

$$\left| \frac{1}{g(z)} - \frac{1}{m} \right| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta.$$

What we have just shown is that

$$\lim_{z \to z_0} \frac{1}{g(z)} = \frac{1}{m}.$$

Having already proved the product law [see part (c)], we get the quotient law by applying the product law to $f(z)$ and $1/g(z)$ as follows:

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} f(z) \lim_{z \to z_0} \frac{1}{g(z)} = \frac{\ell}{m}.$$

With the intention of giving more clarity to the concept of limit we are now prepared to give an alternative definition. This definition is due to M. Heine, while δ-ε definition was developed by A.L. Cauchy. However, it will be shown that both definitions are equivalent.

Definition 2. A complex function $f(z)$ defined in some neighbourhood of z_0 is said to have limit w_0 at z_0 if, for every sequence $\{z_n\}$ converging to z_0, we have

$$\lim_{n \to \infty} f(z_n) = w_0.$$

In the above definition it is tacitly assumed that f is defined for all z_n. The next step is to establish the fact that both definitions are equivalent.

Proposition 2. Definitions 1 and 2 are equivalent.

Proof. Suppose $f(z)$ has a limit w_0 at z_0 in accordance with Definition 1, then we have to establish that w_0 is also the limit by Definition 2. For given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta.$$

Let $\{z_n\}$ be any sequence converging to z_0. Then, for this $\delta > 0$, there exists a positive integer N such that

$$|z_n - z_0| < \delta \quad \text{whenever } n \geq N.$$

Both the statements are combined for given $\varepsilon > 0$ to have

$$|f(z_n) - w_0| < \varepsilon \quad \text{whenever } |z_n - z_0| < \delta \quad \text{for all } n \geq N.$$

Hence $f(z_n) \to w_0$ as $n \to \infty$.

To prove the converse, i.e. Definition 2 $\Rightarrow$ Definition 1. We prove the result by negation. Assume that $f(z)$ has no limit in view of Definition 1. This means there exists at least one $\varepsilon > 0$ such that for all $\delta > 0$,

$$|f(z) - w_0| \geq \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta.$$

Since $z_n \to z_0$ as $n \to \infty$, it follows that for this $\delta > 0$

$$|z_n - z_0| < \delta \quad \text{whenever } n \geq N.$$

Thus, for $\varepsilon > 0$,

$$|f(z_n) - w_0| \geq \varepsilon \quad \forall\, n \geq N.$$

This contradicts the convergence of $\{f(z_n)\}$. This means that Definition 2 has failed.

Limits involving point at infinity. Next we consider the concepts of limits on the extend complex number system, denoted by the symbol ∞ (see Section 1.6).

Definition 3 (Limit at infinity). Let f be defined on an unbounded set E the in the complex plane. If, for a given $\varepsilon > 0$, there exists an $R > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever } z \in E \text{ and } |z| > R,$$

then we say that $f(z) \to w_0$ as $z \to \infty$, and write

$$\lim_{z \to \infty} f(z) = w_0 \quad \text{or} \quad \lim_{|z| \to \infty} f(z) = w_0.$$

The set of the forms $\{z : |z| > R\}$ and $\{z : |z| > R\} \cup \{\infty\}$ are called *deleted neighbourhood* and *neighbourhood of infinity*, respectively.

Example 4. Show that $\lim_{z \to \infty} 1/z^2 = 0$.

The function $1/z^2$ is defined everywhere in $E = \mathbb{C} \setminus \{0\}$. For given $\varepsilon > 0$, note that

$$\left|\frac{1}{z^2}\right| < \varepsilon \Leftrightarrow |z| > \frac{1}{\sqrt{\varepsilon}}$$

so, in this case, choose $R = 1/\sqrt{\varepsilon}$.

Definition 4 (Infinite limit). Let f be defined in some neighbourhood of a point z_0. If, for every $R > 0$, there exists a $\delta > 0$ such that

$$|f(z)| > R \quad \text{whenever} \quad 0 < |z - z_0| < \delta,$$

then we say that $f(z) \to \infty$ as $z \to z_0$, and write

$$\lim_{z \to z_0} f(z) = \infty \quad \text{or} \quad \lim_{z \to z_0} |f(z)| = \infty.$$

Definition 5. Let f be defined on an unbounded set E of complex plane. If, for given $R > 0$, there exists a $K > 0$ such that

$$|f(z)| > R \quad \text{whenever } |z| > K,$$

then we say that $f(z) \to \infty$ as $z \to \infty$, and write

$$\lim_{z \to \infty} f(z) = \infty \quad \text{or} \quad \lim_{|z| \to \infty} |f(z)| = \infty.$$

Whenever we discuss limits in extended plane it is more convenient to study the equivalent limits in finite plane by appropriately defining as

$$\lim_{z \to \infty} f(z) = w \Leftrightarrow \lim_{z \to 0} f\left(\frac{1}{z}\right) = w; \tag{2.12}$$

$$\lim_{z \to z_0} f(z) = \infty \Leftrightarrow \lim_{z \to z_0} \frac{1}{f(z)} = 0; \tag{2.13}$$

$$\lim_{z \to \infty} f(z) = \infty \Leftrightarrow \lim_{z \to 0} \frac{1}{f(1/z)} = 0. \tag{2.14}$$

Remark. The validity of the above equivalent forms can be established with the help of Definitions 3 to 5.

PROBLEM SET 2.1

1. Determine the domain and range of the functions:

(a) $\dfrac{z}{z+\bar{z}}$; (b) $\dfrac{1}{1-|z|^2}$; (c) $z + \operatorname{Arg}\left(\dfrac{1}{z}\right)$; (d) $\dfrac{iz}{z-\bar{z}}$.

2. Which of the following functions is univalent:

(a) $f(z) = z^2 + 3z + 1,\ |z| < 1$; (b) $g(z) = z + z^2,\ |z| < 1$;

(c) $T(z) = \dfrac{az+b}{cz+d},\ z \neq -\dfrac{d}{c}$; (d) $K(z) = \dfrac{z}{1-z^2},\ |z|<1$.

3. A point z is called a fixed point of the function $w = f(z)$ if $f(z) = z$. Use this definition to show that if z is a fixed point of $w = f(z)$, then it is also a fixed point of the inverse function $z = g(w)$.

4. If $\lim_{z\to z_0} f(z) = w_0$, prove that the limit is unique.

5. If $\lim_{z\to z_0} f(z) = w_0$, show that $\lim_{z\to z_0} \overline{f(z)} = \bar{w}_0$.

6. Using the definition, verify the following limits.

(a) $\lim\limits_{z\to i} (z^2 + 2) = 1$; (b) $\lim\limits_{z\to i} z^4 = 1$;

(c) $\lim\limits_{z\to 1} z^3 = 1$; (c) $\lim\limits_{z\to 1+i} 3x + i4y = 3 + 4i$.

7. If $g(z)$ is bounded and $\lim_{z\to z_0} f(z) = 0$, show that $\lim_{z\to z_0} f(z)g(z) = 0$.

8. Let $\lim_{z\to z_0} f(z) = w_0$, $w_0 \neq 0$. Then show that there is a deleted neighbourhood of z_0 in which

$$\frac{|w_0|}{3} < |f(z)| < \frac{5|w_0|}{3}.$$

9. If $\lim_{z\to 0} f(z) = l$ and $\lim_{z\to\infty} g(z) = m$, prove that there exists a circle centred at the origin in which the function $f(z) + g(1/z)$ is bounded.

10. By using the properties (2.12), (2.13) and (2.14), justify the following limits:

(a) $\lim\limits_{z\to\infty} \dfrac{iz^2}{(z-1)^2} = i$; (b) $\lim\limits_{z\to 2} \dfrac{3}{(z-2)^2} = \infty$; (c) $\lim\limits_{z\to\infty} \dfrac{z^3-1}{z^2+1} = \infty$.

11. Give reasons why the limits involving the point at infinity (if existing) are unique.

12. If $T(z) = (az + b)/(cz + d)$, $ad - bc \neq 0$, show that (using the definition of limit)

(a) $\lim\limits_{z\to\infty} T(z) = \infty$, if $c = 0$;

(b) $\lim\limits_{z\to\infty} T(z) = \dfrac{a}{c}$, if $c \neq 0$;

(c) $\lim\limits_{z\to -d/c} T(z) = \infty$, if $c \neq 0$.

13. Prove Theorem 2 using Theorem 1 and Definition 2 independently.

2.3 CONTINUITY

With the help of discussion on limits we introduce the concept of continuity as:

Definition 6. Let $f(z)$ be a complex function defined in some neighbourhood of z_0 (including z_0). Then $f(z)$ function is said to be continuous at z_0 if, for every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever } |z - z_0| < \delta.$$

Symbolically, we write

$$\lim_{z \to z_0} f(z) = f(z_0).$$

This means, in case of continuity at a point, the limiting value and the functional value at the point have the same value.

A function $f(z)$ is continuous on a set S if it is continuous at every point of S. If a function is not continuous at z_0, then we say that the function is discontinuous at z_0 or z_0 is the point of discontinuity.

The following are the easy consequences of Definition 6:

1. If $f(z)$ is continuous at z_0, then using Theorem 1, we have the corresponding result for continuous functions:

$$\lim_{z \to z_0} f(z) = f(z_0) \Leftrightarrow \lim_{(x, y) \to (x_0, y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x, y) \to (x_0, y_0)} v(x, y) = v_0,$$

where $f(z_0) = u_0 + iv_0$. This adds the information that component functions $u(x, y)$ and $v(x, y)$ are also continuous at $z_0 = (x_0, y_0)$.

2. If the function $f(z)$ is continuous, so are $|f(z)|$, $f(\bar{z})$ and $\overline{f(z)}$.

3. When $f(z)$ is continuous at z_0, Definition 2 says

$$\lim_{n \to \infty} f(z_n) = f(z_0)$$

for every sequence $\{z_n\}$ converging to z_0.

4. This also suggests the formal expression:

$$f(z_n) = f(z_0) + \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$ such that $z_n \to z_0$ as $n \to \infty$.

Suppose that f and g are continuous functions at the point z_0. Then the following functions are continuous at z_0:

(a) The sum $f(z) + g(z)$;

(b) The difference $f(z) - g(z)$;

(c) Their product $f(z)g(z)$;

(d) Their quotient $f(z)/g(z)$, provided $g(z_0) \neq 0$.

These assertions are the direct consequences of Theorem 2, just replacing the limiting values by the corresponding functional values.

Polynomials

An expression of the form

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad a_n \neq 0,$$

where $a_0, a_1, \ldots, a_n$ are constants, is termed polynomial of degree n. In particular, every

constant is a polynomial and its degree is 0. However, the degree of the constant polynomial 0 is undefined. If $P(z_0) = 0$ for some z_0, then we say that z_0 is a zero or root of the polynomial equation $P(z) = 0$.

Quotients of polynomials,

$$\frac{a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0}{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0}, \quad a_m \neq 0,\ b_n \neq 0$$

are called *rational functions* and are well defined for all z where the denominator does not vanish.

Remark. For formal reasons, if 0 is regarded as a polynomial, its degree is set equal to $-\infty$.

Example 5. Show that a polynomial is continuous at each point of the complex plane.

Let $P(z)$ be a polynomial of degree n given by

$$w = P(z) = a_0 + a_1 z + \cdots + a_n z^n,\ a_n \neq 0.$$

Assume z_0 to be an arbitrary point in the complex plane. Note that a_0 is a constant function and $\lim_{z\to\infty} a_0 = a_0$. If $a_1 \neq 0$, then we can show that $\lim_{z\to 0} a_1 z = a_1 z_0$ with the choice of $\delta = \varepsilon/|a_1|$. Now, by the property (c), $a_k z^k$, $k = 0, 1, 2, \ldots, n$ are continuous functions. Further, using the property (a), the finite sum is a continuous function and hence a polynomial is a continuous function in the entire complex plane.

Similar argument can be supplied to claim that

$$f(z) = e^x y + i \sin(x^2 - 2xy^3)$$

is continuous in $\mathbb{C}$.

Suppose $\lim_{z\to 0} f(z) = w_0$ and the function is undefined at z_0. This means that the function is discontinuous at z_0. However, by defining $f(z_0) = w_0$, the function can be made continuous at z_0. Such type of discontinuity is called the removable discontinuity.

Example 6. Consider the function

$$f(z) = \frac{z - ib}{z^2 + b^2}.$$

and discuss its continuity at the point $z = ib$.

The function is undefined at $z = ib$, but

$$\lim_{z\to ib} f(z) = \lim_{z\to ib} \frac{z - ib}{z^2 + b^2} = -\frac{i}{2b}.$$

Hence, if we define

$$f(z) = \begin{cases} \dfrac{z - ib}{z^2 + b^2}, & z \neq ib, \\ -\dfrac{i}{2b}, & z = ib, \end{cases}$$

then the function is continuous at $z = ib$.

Example 7. The function defined as

$$f(z) = \begin{cases} \dfrac{1}{z}, & z \neq 0, \\ \infty, & z = 0, \\ 0, & z = \infty, \end{cases}$$

is continuous in the extended complex plane.

For $z \neq 0$ or ∞, we have to determine a $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow \left| \frac{1}{z} - \frac{1}{z_0} \right| < \varepsilon.$$

This is true if we assign

$$\frac{|z - z_0|}{|z_0 z|} < \varepsilon.$$

Now, restricting $|z - z_0| < 1$, we note that

$$\frac{1}{|zz_0|} = \frac{1}{|(z - z_0)z_0 + z_0^2|} \leq \frac{1}{|-|z_0||z - z_0| + |z_0|^2|} < \frac{1}{|z_0||1 - |z_0||}.$$

Choose $\delta = \min\{1, |z_0||1 - |z_0||\varepsilon\}$. The limits at $z = 0$ and $z = \infty$ may be discussed by converting into an equivalent form in finite plane.

Next, we illustrate some consequences of continuity.

Proposition 3. A continuous function of a continuous function is continuous.

Proof. Let $f(z)$ be a continuous function at z_0. Then $f(z)$ is defined in some neighbourhood of z_0. Suppose $g(z)$ is a function which is defined on f-image of this neighbourhood. Given that $g(z)$ is continuous at $w_0 = f(z_0)$. Then, for given $\varepsilon > 0$, there exists a $\gamma > 0$ such that

$$|g(f(z)) - g(f(z_0))| < \varepsilon \quad \text{whenever } |f(z) - f(z_0)| < \gamma. \tag{2.15}$$

Now, $f(z)$ is continuous at z_0. Hence, for this $\gamma > 0$ there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \gamma \quad \text{whenever } |z - z_0| < \delta. \tag{2.16}$$

Equations (2.15) and (2.16) combine to give

$$|(gf)(z) - (gf)(z_0)| < \varepsilon \quad \text{whenever } |z - z_0| < \delta.$$

Hence the composite function gf is continuous at z_0.

Proposition 4. If a function $f(z)$ is continuous at z_0 and $f(z_0) \neq 0$, then there exists a neighbourhood of z_0 in which $f(z)$ is nonzero.

Proof. Given that $f(z)$ is continuous at z_0. Then, for given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever } |z - z_0| < \delta. \tag{2.17}$$

Also, given $f(z_0) \neq 0$, i.e. $|f(z_0)| \neq 0$. Suppose $f(z) = 0$ at some point in $N_\delta(z_0)\backslash\{z_0\}$. Then from (2.17), $|f(z_0)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, on choosing $\varepsilon = |f(z_0)|/2$ this is contradicted. This concludes $f(z) \neq 0 \ \forall \ z \in N_\delta(z_0)$.

Proposition 5. If a function $f(z)$ is continuous on a bounded and closed set $S \subset \mathbb{C}$, then minimum and maximum of $|f(z)|$ exist on S.

Proof. Given that $f(z) = u(x, y) + v(x, y)$ is continuous on S. This implies that the component functions $u(x, y)$ and $v(x, y)$ are continuous on S. Hence,

$$|f(z)| = \sqrt{[u(x, y)]^2 + [v(x, y)]^2}$$

is a real-valued continuous function on the bounded and closed set S. By real calculus, $|f(z)|$ attains its maximum and minimum on S.

Let $f(z)$ be a continuous function in a region Ω. Then, for each point $z \in \Omega$ and given $\varepsilon > 0$, we can find a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Here, normally, δ is a function of ε and the point z_0. Finding $\delta > 0$, depending on ε but not on the particular point z_0, gives rise to the concept of uniformly continuous function.

Uniform continuity. A function $f(z)$ is said to be uniformly continuous on a set S if, for given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z_1) - f(z_2)| < \varepsilon \quad \text{whenever } |z_1 - z_2| < \delta \ \forall \ z_1, z_2 \in S.$$

Here, the choice of δ is independent of z_1 and z_2 in S.

Theorem 3. Let $f(z)$ be a continuous function on a closed and bounded set S in a complex plane. Then it is uniformly continuous on S.

Proof. Suppose f is not uniformly continuous on S. Then, there exists an $\varepsilon > 0$ and two sequences p_n and q_n corresponding to p and q in S such that for every n,

$$|p_n - q_n| < \frac{1}{n} \quad \text{and} \quad |f(p_n) - f(q_n)| \geq \varepsilon. \tag{2.18}$$

Since S is closed and bounded, every sequence $\{p_n\}$ contains a subsequence p_{n_k} converging to $z_0 \in S$. Let q_{n_k} be the corresponding subsequence of $\{q_n\}$. Then, $q_{n_k} \to z_0$ as $k \to \infty$ because

$$|q_{n_k} - z_0| < |q_{n_k} - p_{n_k}| + |p_{n_k} - z_0| \quad \text{(using the triangle inequality).}$$

Hence, for subsequences $\{p_{n_k}\}$ and $\{q_{n_k}\}$, (2.18) implies

$$|p_{n_k} - q_{n_k}| < \frac{1}{n_k} \quad \text{and} \quad |f(p_{n_k}) - f(q_{n_k})| \geq \varepsilon. \tag{2.1}$$

for every k. Since f is continuous at z_0, we have

$$f(q_{n_k}) \to f(z_0) \quad \text{and} \quad f(p_{n_k}) \to f(z_0) \quad \text{as } k \to \infty.$$

This contradicts (2.19). Hence f is uniformly continuous.

There are uniformly continuous functions defined on a set which are not bounded and closed. The function z is uniformly continuous in $\mathbb{C}$ but z^2 is not uniformly continuous in $\mathbb{C}$. However, z^2 is uniformly continuous on $|z| < R$.

The following example will induce more insight on this concept.

Example 8. Show that $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$, but the function $g(z) = 1/z$ is not uniformly continuous in this region.

Suppose $f(z)$ is uniformly continuous in the region. Then, for given $\varepsilon > 0$ we must be able to find $\delta > 0$ such that $|z^2 - z_0^2| < \varepsilon$ whenever $|z - z_0| < \delta$, where δ depends only on ε and not on the particular choice of the point z_0. If z and z_0 are any points in $|z| < 1$, then

$$|z^2 - z_0^2| = |z + z_0||z - z_0| \le (|z| + |z_0|)|z - z_0| < 2|z - z_0|.$$

Thus, if we choose $\delta = \varepsilon/2$, it follows that $|z^2 - z_0^2| < \varepsilon$. This ensures that $f(z) = z^2$ is uniformly continuous in $|z| < 1$.

For the second case, assume that $g(z) = 1/z$ is uniformly continuous in $|z| < 1$. This means that for given $\varepsilon > 0$ we can find $\delta > 0$, say between 0 and 1, such that $|g(z) - g(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$ for all z and z_0 in the given region. Fix $z = \delta$ and $z_0 = \delta/(1 + \varepsilon)$. Clearly, the points z and z_0 are in $|z| < 1$ and we also observe that

$$|z - z_0| = \left|\delta - \frac{\delta}{1+\varepsilon}\right| = \frac{\varepsilon}{1+\varepsilon}\delta < \delta.$$

However,

$$\left|\frac{1}{z} - \frac{1}{z_0}\right| = \left|\frac{1}{\delta} - \frac{1+\varepsilon}{\delta}\right| = \frac{\varepsilon}{\delta} > \varepsilon \quad \text{since } 0 < \delta < 1.$$

Thus, we reach at a contradiction and hence the function $g(z) = 1/z$ cannot be uniformly continuous in the region.

PROBLEM SET 2.2

1. If a function f is continuous in a domain D and takes only the integral values, show that f is constant.
2. If a function f is continuous in a domain D, prove that the functions $|f(z)|$, $f(\bar{z})$ and $\overline{f(z)}$ are also continuous in the domain.
3. Define the function

$$f(z) = \begin{cases} \dfrac{z \operatorname{Re} z}{|z|}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

   Prove that $f(z)$ is continuous in the entire complex plane.
4. Let $S = \{z : |z| = 1\}$. Then show that the function $f(z) = \operatorname{Arg} z$ is continuous from $S \backslash \{-1\}$ to $(-\pi, \pi)$. Further, generalize this result by showing that $\operatorname{Arg} z$ is continuous on $\mathbb{C} - \operatorname{Re} z \le 0 (y = 0)$.

2.4 DIFFERENTIABILITY

Now, we are ready to introduce derivative of a complex valued function.

Definition 7. Let $f(z)$ be a function defined in a neighbourhood of z_0. $f(z)$ is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists and is finite}$$

The limit in (2.20) is denoted by $f'(z_0)$ and we say $f(z)$ is differentiable at z_0.

Further, we say a function is differentiable on a set S if it is differentiable at each point of the set.

Remarks 1. From (2.20), we observe that $f(z)$ has a derivative at z_0 if $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon, \text{ whenever}$$

5. Show that a function $f(z)$ is continuous at a point $z_0 \Leftrightarrow$ for every sequence of complex numbers $\{z_n\}$ converging to z_0, $\lim_{n\to\infty} f(z_n) = f(z_0)$.

6. Using Theorem 2, prove that $f \pm g$, $f \cdot g$ and f/g, $g \neq 0$ are continuous functions, where f and g are continuous functions.

7. Is the function

$$\frac{z^2 + (2-i)z - 2i}{z - i}$$

continuous at $z = i$? If not, can it be made continuous by redefining at $z = i$? Do you agree that the same situtation occurs for the function

$$f(z) = \frac{z^3 + (1+i)z^2 + (2+i)z + 2}{z - i}$$

at the same point?

8. Prove that

$$f(z) = \frac{z^3 + 1}{z^3 + 9}$$

is continuous and bounded in the region $|z| \leq 2$.

9. On what curve Arg z is discontinuous if it is defined as the value of the arg z satisfying the inequality $|z| - 2\pi < \arg z \leq |z|$.

   Suggestion. Note that Arg z is discontinuous on the negative real axis when it is defined as the value of arg z satisfying $\pi < \arg z \leq \pi$.

10. Prove that $f(z) = 3z - 1$ is uniformly continuous in the region $|z| \leq 7$.

11. Consider the function $f(z) = 1/z^2$. Prove that (a) $f(z)$ is not uniformly continuous in the region $|z| \leq 1$ but (b) it is uniformly continuous in the region $1/2 \leq |z| \leq 1$.

12. Show that the continuous image of a connected set is connected.

13. Show that the continuous image of a compact set is compact.

14. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, be a polynomial of degree n.

    (a) If $a_k \in \mathbb{R}$, $k = 0, 1, \ldots, n$ and $s = re^{i\theta}$ is a root of $P(z) = 0$, show that $\bar{s}$ is also a root, and also deduce that $z^2 - 2r\cos\theta + r^2$ is a factor of $P(z)$.

    (b) If the coefficients in $P(z)$satisfy the condition $0 < a_0 < a_1 < \cdots < a_n$, show that $P(z) = 0 \Rightarrow |z| < 1$.

    (c) If $\omega = e^{i2\pi/m}$, $m \geq 2$ integer, show that

$$P(1) + P(\omega) + P(\omega^2) + \cdots + P(\omega^{m-1}) = mP(0).$$

Suggestion. For part (b), consider $(z-1)P(z)$ and show that $|z| \geq 1 \Rightarrow P(z) \neq 0$.

2.4 DIFFERENTIABILITY

Now, we are ready to introduce derivatives of complex-valued functions.

Definition 7. Let $f(z)$ be a function defined in some neighbourhood of z_0. We say that $f(z)$ is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists and is finite.} \tag{2.20}$$

The limit in (2.20) is denoted by $f'(z_0)$ and we say $f'(z_0)$ is the derivative of $f(z)$ at z_0.

Further, we say a function is differentiable on a set S in complex plane if it is differentiable at each point of the set.

Remarks 1. From (2.20), we observe that $f(z)$ has a derivative $f'(z_0)$ at z_0 if, for given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon, \quad \text{whenever} \quad |z - z_0| < \delta.$$

Define

$$\eta(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0), & z \neq z_0, \\ 0, & z = z_0. \end{cases}$$

We note that $\lim_{z \to z_0} \eta(z) = \eta(z_0)$. This implies that $\eta(z)$ is continuous at z_0, and hence $|\eta(z)| < \varepsilon$ whenever $|z - z_0| < \delta$. Thus, we get an explicit expression for $f(z)$ in the form

$$f(z) = f(z_0) + (z - z_0)f'(z) + (z - z_0)\eta(z) \qquad \text{for } |z - z_0| < \delta. \tag{2.21}$$

2. Replacing $z - z_0$ by Δz in (2.20), we have

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0).$$

Moreover, we drop the suffix from z_0 and usually write

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z). \tag{2.22}$$

3. For $w = f(z)$, let $w + \Delta w = f(z + \Delta z)$. Then (2.22) explains

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z).$$

Example 9. $f(z) = z^3 + 2z^2 + i$. Find $f'(z)$ and verify it using the definition of limit.

$$\begin{aligned} \frac{f(z + \Delta z) - f(z)}{\Delta z} &= \frac{[(z + \Delta z)^3 + 2(z + \Delta z)^2 + i] - (z^3 + 2z^2 + i)}{\Delta z} \\ &= 3z^2 + 4z + 3z\Delta z + 2\Delta z + (\Delta z)^2. \end{aligned}$$

Taking limits, we get $f'(z) = 3z^2 + 4z$.

Verification. For given $\varepsilon > 0$, we must find $\delta > 0$ such that

$$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-f'(z)\right| < \varepsilon \quad \text{whenever} \quad |\Delta z| < \delta.$$

Consider

$$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-f'(z)\right| = |3z+2+\Delta z|\,|\Delta z|.$$

Restricting $|\Delta z| < 1$, we observe that

$$|3z+2+\Delta z| \le 3|z|+2+|\Delta z| < 3(1+|z|).$$

If we choose $\delta = \min\{1, \varepsilon/3(1+|z|)\}$, then the verification is over.

Proposition 6. Every differentiable function is continuous.

Proof. Let $f(z)$ be differentiable at some point z_0. Then, writing

$$f(z)-f(z_0) = \frac{f(z)-f(z_0)}{z-z_0}(z-z_0).$$

Since $f'(z_0)$ exists by hypothesis, on taking the limits, it follows that

$$\begin{aligned}\lim_{z\to z_0}[f(z)-f(z_0)] &= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}(z-z_0)\\ &= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\lim_{z\to z_0}(z-z_0)\\ &= f'(z_0)\cdot 0\\ &= 0.\end{aligned}$$

Thus, $\lim_{z\to z_0} f(z)-f(z_0) = 0$ or $\lim_{z\to z_0} f(z) = f(z_0)$. So, $f(z)$ is continuous at z_0.

The converse statement of Proposition 6 is not true as shown in the following example.

Example 10. Consider the function $f(z) = |z|$ and prove that it is nowhere differentiable in $\mathbb{C}$ but continuous everywhere.

For verifying continuity, let z_0 be an arbitrary point in $\mathbb{C}$. Then, we have to find a $\delta > 0$, for given $\varepsilon > 0$ such that $|z-z_0| < \delta \Rightarrow ||z|-|z_0|| < \varepsilon$. Noting the inequality, $||z|-|z_0|| < |z-z_0|$, if we choose $\delta = \varepsilon$, then the verification is over.

For differentiability, we proceed as

$$\begin{aligned}\frac{|z+\Delta z|-|z|}{\Delta z} &= \frac{|z+\Delta z|-|z|}{\Delta z}\,\frac{|z+\Delta z|+|z|}{|z+\Delta z|+|z|}\\ &= \frac{|z+\Delta z|^2-|z|^2}{\Delta z(|z+\Delta z|+|z|)}\\ &= \frac{\bar{z}\Delta z+z\overline{\Delta z}+\Delta z\overline{\Delta z}}{\Delta z(|z+\Delta z|+|z|)}\end{aligned}$$

$$= \frac{\bar{z}}{|z+\Delta z|+|z|} + \frac{z}{|z+\Delta z|+|z|}\frac{\overline{\Delta z}}{\Delta z} + \frac{\overline{\Delta z}}{|z+\Delta z|+|z|}.$$

Taking the limits, we get

$$\lim_{\Delta z \to 0} \frac{|z+\Delta z|-|z|}{\Delta z} = \frac{\bar{z}}{2|z|} + \lim_{\Delta z \to 0} \frac{z}{|z+\Delta z|+|z|}\frac{\overline{\Delta z}}{\Delta z}$$

$$= \begin{cases} \dfrac{\bar{z}}{2|z|} + \dfrac{z}{2|z|}, & \text{parallel to the } x\text{-axis}, \\[2ex] \dfrac{\bar{z}}{2|z|} - \dfrac{z}{2|z|}, & \text{parallel to the } y\text{-axis}. \end{cases}$$

Hence the function is nowhere differentiable in the complex plane.

By the definition of derivative we can show that for $f(z) = c$, $f'(z) = 0$ and for $f(z) = z^n$, $n \neq 0$, $f'(z) = nz^{n-1}$.

Proposition 7. If $f(z)$ and $g(z)$ have derivatives at z, then

(a) $\dfrac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$;

(b) $\dfrac{d}{dz}[cf(z)] = cf'(z)$, where c is some complex constant;

(c) $\dfrac{d}{dz}[f(z) \cdot g(z)] = f'(z)g(z) + f(z)g'(z)$;

(d) $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{f'(z)g(z) - g'(z)f(z)}{[g(z)]^2}$, provided $g(z) \neq 0$.

Proof. These differentiation formulas can be verified by making use of the definition:

$$\frac{d}{dz}[f(z)] = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z).$$

Theorem 4 (Chain rule). Suppose that f has a derivative at z_0 and g has a derivative at $f(z_0)$. Then the composite function gf has a derivative at z_0. If $F(z) = (gf)(z) = g(f(z))$, then

$$F'(z_0) = g'(f(z_0))f'(z_0), \tag{2.23}$$

where g' is the derivative of g with respect to $f(z)$ at z_0.

Proof. Given that $f(z)$ is differentiable at z_0 and g is differentiable at $w_0 = f(z_0)$. Then, in view of (2.21),

$$f(z) - f(z_0) = [f'(z_0) + \varepsilon(z)](z - z_0), \tag{2.24}$$

$$g(w) - g(w_0) = [g'(w_0) + \eta(w)](w - w_0), \tag{2.25}$$

where $\varepsilon(z) \to 0$ as $z \to z_0$ and $\eta(w) \to 0$ as $w \to w_0$. Put $w = f(z)$ and substitute (2.24) into (2.25). If $z \neq z_0$,

$$\frac{F(z) - F(z_0)}{z - z_0} = [g'(f(z_0)) - \eta(f(z))][f'(z_0) + \varepsilon(z)]. \tag{2.26}$$

$$= \frac{\bar{z}}{|z+\Delta z|+|z|} + \frac{z}{|z+\Delta z|+|z|}\frac{\overline{\Delta z}}{\Delta z} + \frac{\overline{\Delta z}}{|z+\Delta z|+|z|}.$$

Taking the limits, we get

$$\lim_{\Delta z \to 0} \frac{|z+\Delta z|-|z|}{\Delta z} = \frac{\bar{z}}{2|z|} + \lim_{\Delta z \to 0} \frac{z}{|z+\Delta z|+|z|}\frac{\overline{\Delta z}}{\Delta z}$$

$$= \begin{cases} \dfrac{\bar{z}}{2|z|} + \dfrac{z}{2|z|}, & \text{parallel to the } x\text{-axis}, \\[2ex] \dfrac{\bar{z}}{2|z|} - \dfrac{z}{2|z|}, & \text{parallel to the } y\text{-axis}. \end{cases}$$

Hence the function is nowhere differentiable in the complex plane.

By the definition of derivative we can show that for $f(z) = c$, $f'(z) = 0$ and for $f(z) = z^n$, $n \neq 0$, $f'(z) = nz^{n-1}$.

Proposition 7. If $f(z)$ and $g(z)$ have derivatives at z, then

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(b) $\dfrac{d}{dz}[cf(z)] = cf'(z)$, where c is some complex constant;

(c) $\dfrac{d}{dz}[f(z) \cdot g(z)] = f'(z)g(z) + f(z)g'(z)$;

(d) $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{f'(z)g(z) - g'(z)f(z)}{[g(z)]^2}$, provided $g(z) \neq 0$.

Proof. These differentiation formulas can be verified by making use of the definition:

$$\frac{d}{dz}[f(z)] = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z).$$

Theorem 4 (Chain rule). Suppose that f has a derivative at z_0 and g has a derivative at $f(z_0)$. Then the composite function gf has a derivative at z_0. If $F(z) = (gf)(z) = g(f(z))$, then

$$F'(z_0) = g'(f(z_0))f'(z_0), \tag{2.23}$$

where g' is the derivative of g with respect to $f(z)$ at z_0.

Proof. Given that $f(z)$ is differentiable at z_0 and g is differentiable at $w_0 = f(z_0)$. Then, in view of (2.21),

$$f(z) - f(z_0) = [f'(z_0) + \varepsilon(z)](z - z_0), \tag{2.24}$$

$$g(w) - g(w_0) = [g'(w_0) + \eta(w)](w - w_0), \tag{2.25}$$

where $\varepsilon(z) \to 0$ as $z \to z_0$ and $\eta(w) \to 0$ as $w \to w_0$. Put $w = f(z)$ and substitute (2.24) into (2.25). If $z \neq z_0$,

$$\frac{F(z) - F(z_0)}{z - z_0} = [g'(f(z_0)) - \eta(f(z))][f'(z_0) + \varepsilon(z)]. \tag{2.26}$$

As differentiability of f forces f to be continuous at z_0, i.e. $\lim_{z\to z_0} \eta(f(z)) = \eta(f(z_0)) = 0$. Therefore, (2.23) follows from (2.26).

2.5 CAUCHY-RIEMANN EQUATIONS

We have seen that every complex valued function can be expressed as $f(z) = u(x, y) + iv(x, y)$, where $u(x, y) \equiv u$ and $v(x, y) \equiv v$ are real functions of two variables. In the sequel, we shall be frequently dealing with partial derivatives of u and v. For the sake of convenience and space, we shall use the notation:

$$u_x,\ u_y,\ u_{xx},\ u_{yy},\ u_{xy} \text{ for } \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial^2 x}, \frac{\partial^2 u}{\partial^2 y}, \frac{\partial^2 u}{\partial y \partial x}, \text{ respectively.}$$

Proposition 8. Suppose that $f(z) = u(x, y) + iv(x, y)$ and $f'(z)$ exists at $z_0 = x_0 + iy_0$. Then the first order partial derivatives of u and v at (x_0, y_0) satisfy

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \tag{2.27}$$

and, moreover,

$$f'(z_0) = u_x + iv_x \quad \text{or} \quad f'(z_0) = v_y - iu_y.$$

Equations (2.27) are known as Cauchy–Riemann equations (C-R equations).

Proof. Given that $f(z) = u(x, y) + iv(x, y)$ has derivative at z_0. Hence

$$\lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0).$$

The limit $f'(z_0)$ is finite and whatever path we adopt to let $\Delta z \to 0$ it must be unique. Suppose $\Delta z \to 0$ through real values (parallel to the x-axis). Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x\to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right] \\ &= u_x + iv_x. \end{aligned} \tag{2.28}$$

Let $\Delta z \to 0$ through imaginary values (parallel to y-axis)

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y\to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \\ &= -i \lim_{\Delta y\to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta y} \\ &= -i \lim_{\Delta y\to 0} \left[\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} + i\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y}\right] \\ &= -iu_y + v_y. \end{aligned} \tag{2.29}$$

Equations (2.28) and (2.29) give $f'(z_0)$. On equating these, we also have (2.27).

It should also be noted that the C-R equations are necessary conditions for the existence of derivative but these are not sufficient. In other words, if $f(z)$ is differentiable at z_0, then the C-R equations are satisfied at z_0 but the converse may not be true.

Consider the following example.

Example 11. For the function $f(z)$, defined by

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

show that the C-R equations are satisfied at (0, 0) but the function is not differentiable at $0 + i0$.

Note that

$$\begin{aligned} f'(0) &= \lim_{z \to 0} \frac{f(z) - f(0)}{z} \\ &= \lim_{z \to 0} \left(\frac{\bar{z}}{z} \right)^2 \\ &= \lim_{(x, y) \to (0, 0)} \left(\frac{x - iy}{x + iy} \right)^2 \\ &= \begin{cases} 1, & z \to 0 \text{ along the } x\text{-axis}, \\ 1, & z \to 0 \text{ along the } y\text{-axis}, \\ -1, & z \to 0 \text{ along the line } y = x. \end{cases} \end{aligned}$$

As $f'(0)$ turns up with different values for different paths, $f'(0)$ does not exist, i.e. f is not differentiable at $0 + i0$.

To check validity of the C-R equations at (0, 0), write $f(z) = u(x, y) + iv(x, y)$ and $f(0) = 0 + i0$, $u(0, 0) = 0$, $v(0, 0) = 0$. Then

$$f(z) = \frac{(\bar{z})^3}{z\bar{z}} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - i3x^2y - 3xy^2 + iy^3}{x^2 + y^2},$$

and
$$u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2}, \qquad v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2},$$

Then
$$u_x = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = 1,$$

$$u_y = \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = 0,$$

$$v_x = \lim_{\Delta x \to 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = 0,$$

$$v_y = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = 1.$$

Thus, we observe that $u_x = v_y$ and $u_y = -v_x$ at $z = (0, 0)$.

To make the C-R equations as sufficient, an additional condition on partial derivatives is induced and this is the condition of continuity. In reference to this we prove:

Theorem 5. Let the function $f(z) = u(x, y) + iv(x, y)$ be defined in some neighbourhood of $z = a + ib$. Suppose that the first order partial derivatives of u and v are continuous at (a, b) and satisfy the C-R equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

then $f'(z)$ exists.

Proof. To start the proof, we define $\Delta w = f(z + \Delta z) - f(z)$. Thus, $\Delta w = \Delta u + i\Delta v$, where

$$\Delta u = u(a + \Delta x, b + \Delta y) - u(a, b),$$
$$\Delta v = v(a + \Delta x, b + \Delta y) - v(a, b).$$

Since the first order partial derivatives of u and v are continuous at (a, b), applying the increment theorem

$$\Delta u = u_x(a, b)\Delta x + u_y(a, b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y,$$
$$\Delta v = v_x(a, b)\Delta x + v_y(a, b)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0$ as $\Delta x, \Delta y \to 0$. Hence, $\Delta w = \Delta u + i\Delta v$ is evaluated by

$$\Delta w = u_x(a, b)\Delta x + u_y(a, b)\Delta y + iv_x(a, b)\Delta x + iv_y(a, b)\Delta y$$
$$+ (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y.$$

Given that the C-R equations are satisfied at (a, b), replacing $u_y(a, b)$ and $v_y(a, b)$ by $-v_x(a, b)$ and $u_x(a, b)$, respectively, in the above expression, it follows that

$$\Delta w = (\Delta x + i\Delta y)[u_x(a, b) + iv_x(a, b)] + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y. \tag{2.30}$$

Dividing both sides of (2.30) by $\Delta z = \Delta x + i\Delta y$, we obtain

$$\frac{\Delta w}{\Delta z} = u_x(a, b) + iv_x(a, b) + \frac{(\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y}{\Delta x + i\Delta y}. \tag{2.31}$$

Since

$$\left| \frac{(\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y}{\Delta x + i\Delta y} \right| \le |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| + |\varepsilon_4|,$$

on proceeding to limits as $(\Delta x, \Delta y) \to (0, 0)$ in (2.31), we get

$$f'(z) = u_x(a, b) + iv_x(a, b).$$

Remark. Note that somewhere in the above analysis we have used the inequalities

$$|\Delta x| \le \sqrt{(\Delta x)^2 + (\Delta y)^2}, \qquad |\Delta y| \le \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Also, try to search the steps where continuity of the first order partial derivatives has been utilized in the proof.

Polar form of C-R equations. Now, we transform the C-R equations in Cartesian form: $u_x = v_y$, $u_y = -v_x$ into polar coordinates by changing the coordinate system from (x, y) to (r, θ). We use a result from the theory of partial differentiation.

If $f(x, y)$, $x = g(r, \theta)$, $y = h(r, \theta) \in C'$, then

$$\text{(i)} \quad \frac{\partial}{\partial r}[f(g, h)] = f_1(g, h)g_1(r, \theta) + f_2(g, h)h_1(r, \theta),$$

$$\text{(ii)} \quad \frac{\partial}{\partial \theta}[f(g, h)] = f_1(g, h)g_2(r, \theta) + f_2(g, h)h_2(r, \theta),$$

where C' is the class of functions whose first order partial derivatives are continuous and f_i, g_i are the mean partial derivatives of f and g with respect to the ith variable.

It is understood, of course, that the domain for (r, θ) and the one for (x, y) must be such that the functions g, h can be substituted in $f(x, y)$ to form

$$f(g(r, \theta), h(r, \theta)).$$

Using the above result for $u(x, y)$, $x \equiv (r, \theta)$, $y \equiv (r, \theta)$

$$u_r = u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r} \quad \text{and} \quad u_\theta = u_x \frac{\partial x}{\partial \theta} + u_y \frac{\partial y}{\partial \theta}.$$

Utilizing the transformation $x = r \cos \theta$ and $y = r \sin \theta$, these are converted to

$$u_r = \cos \theta\, u_x + \sin \theta\, u_y \quad \text{and} \quad u_\theta = -r \sin \theta\, u_x + r \cos \theta\, u_y. \tag{2.32}$$

Likewise for $v(x, y)$,

$$v_r = \cos \theta\, v_x + \sin \theta\, v_y \quad \text{and} \quad v_\theta = -r \sin \theta\, v_x + r \cos \theta\, v_y. \tag{2.33}$$

Further, using $u_x = v_y$ and $u_y = -v_x$ in (2.33), we have

$$v_r = -\cos \theta\, u_y + \sin \theta\, u_x \quad \text{and} \quad v_\theta = r \sin \theta\, u_y + r \cos \theta\, u_x. \tag{2.34}$$

Comparing (2.32) and (2.34), we get

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad \frac{1}{r} u_\theta = -v_r. \tag{2.35}$$

These are the polar form of the C-R equations.

For deriving the expression for $f'(z_0)$ in polar system, we solve the equations in (2.32) for u_x to have

$$u_x = \cos \theta\, u_r - \frac{\sin \theta}{r} u_\theta. \tag{2.36}$$

Likewise, in view of (2.33) for v_x,

$$v_x = \cos \theta\, v_r - \frac{\sin \theta}{r} v_\theta. \tag{2.37}$$

Applying the C-R equations, (2.36) and (2.37) yield

$$u_x = \cos \theta\, u_r + \sin \theta\, v_r \quad \text{and} \quad v_x = \cos \theta\, v_r - \sin \theta\, u_r.$$

Hence

$$f'(z_0) = u_x + iv_x = \cos \theta\, u_r + \sin \theta\, v_r + i(\cos \theta\, v_r - i \sin \theta\, u_r) = e^{-i\theta}(u_r + iv_r),$$

or, if u_r and v_r are replaced by v_θ/r and $-u_\theta/r$, then

$$f'(z_0) = e^{-i\theta}\left(\frac{v_\theta}{r} - i\frac{u_\theta}{r}\right) = \frac{-i}{re^{i\theta}}(u_\theta + iv_\theta) = \frac{-i}{z_0}(u_\theta + iv_\theta).$$

What we just discussed is the analysis of Proposition 8 for polar system, and a companion theorem to Theorem 5 is as follows.

Theorem 6. Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be defined in some neighbourhood of a nonzero point $z_0 = r_0e^{i\theta_0}$. If the first order partial derivatives of u and v with reference to r and θ are continuous at z_0 and satisfy (2.35), then $f'(z_0)$ exists.

To illustrate the discussion so far on polar system, we solve an example.

Example 12. It is shown that the function $1/z^4$, $z \neq 0$ is differentiable in indicated domains and $f'(z)$ is evaluated.

Since $f(z) = 1/z^4$, $z \neq 0$, it is convenient to deal with the polar system. Writing $f(z) = r^{-4}(\cos 4\theta - i \sin 4\theta)$, we have

$$u = \frac{\cos 4\theta}{r^4}, \qquad v = -\frac{\sin 4\theta}{r^4}.$$

Thus,

$$u_r = -\frac{4\cos 4\theta}{r^5}, \qquad u_\theta = -\frac{4\sin 4\theta}{r^4}, \qquad v_r = \frac{4\sin 4\theta}{r^5}, \qquad v_\theta = -\frac{4\cos 4\theta}{r^4}.$$

The first partial derivatives of u and v are continuous functions as being rational continuous functions (with nonvanishing denominator). Also, the C-R equations

$$u_r = \frac{-4\cos 4\theta}{r^5} = \frac{1}{r}v_\theta \quad \text{and} \quad \frac{1}{r}u_\theta = -\frac{4\sin 4\theta}{r^5} = -v_r$$

are satisfied. This concludes that $f'(z)$ exists and its value is

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{-4\cos 4\theta}{r^5} + i\frac{4\sin 4\theta}{r^5}\right) = -\frac{4}{(re^{i\theta})^5} = -\frac{4}{z^5}.$$

PROBLEM SET 2.3

1. Consider the function $f(z) = |z|^2$. Show that $f(z)$ is continuous in the whole complex plane but is not differentiable in $\mathbb{C}$ except at the origin. Using this result, discuss the differentiability of the function $|z + i|^2$.
2. Determine the points where the functions

   (a) $f(z) = z\text{Re } z$; (b) $g(z) = \begin{cases} \dfrac{\text{Re } z}{z^2}, & z \neq 0, \\ 0, & z = 0. \end{cases}$

   are not differentiable.
3. Show that when $f(z) = x^3 + i(1-y)^3$, it is justified to write $f'(z) = 3x^2$ only when $z = i$.

4. Derive the Cartesian form of the C-R equations assuming that the polar form is known.

5. Show that the functions defined by

$$\text{(a) } f(z) = \begin{cases} \dfrac{(1+i)x^3-(1-i)y^3}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2=0 \end{cases} \qquad \text{(b) } g(z) = \begin{cases} e^{-z^{-4}}, & z=0 \\ 0, & z=0 \end{cases}$$

Prove that for the both functions the C-R equations are satisfied at the origin but the functions are not differentiable there.

6. For the function defined by $f(z) = \sqrt{|xy|}$, show that the C-R equations are satisfied at $z = 0$, but the function is not differentiable at this point.

7. Show that the functions

(a) $f(z) = \sqrt{r}\, e^{i\theta/2}$, $(r > 0, -\pi < \theta < \pi)$,

(b) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$, $(r > 0, 0 < \theta < 2\pi)$

are differentiable in the indicated domains. Also, find $f'(z)$ in each case.

8. Let $f(z)$ be a real function of a complex variable z. Then show that either f has the derivative zero or the derivative does not exist.

9. Verify the truth of the statement: A complex function $f(z)$ is differentiable at $z_0 \Leftrightarrow$ for every complex sequence $\{z_n\}$ on which $f(z)$ is defined such that $z_n \to z_0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{f(z_n)-f(z_0)}{z_n - z_0} = f'(z_0).$$

2.6 ANALYTIC FUNCTIONS

The concept of analytic function is the heart of complex analysis. Unlike differentiable functions these functions have many additional properties which will be discussed at length in Chapters 4, 6 and 11.

Definition 8. A complex function $f(z)$ is said to be analytic at a point z_0 if it is differentiable in some neighbourhood of z_0.

A function is said to be *analytic* in a domain D if it is analytic at each point of D. The terms holomorphic and regular are used with identical meanings in the literature. To prove that a function is analytic in a domain (open and connected set), we shall only show that it is differentiable at each point of the domain. Why?

Remarks. 1. We say that $f(z)$ is analytic on a closed subset S of $\mathbb{C}$, if $f(z)$ is the restriction of an analytic function defined on a domain D containing S.

2. Keeping track of Exercise 8, Problem Set 2.3, we are assured that real functions

of complex variable are nowhere analytic unless these are constant valued. Hence, Re z, Im z, $|z|$, $|z|^2$, etc. are nowhere analytic, since these are real valued but not constant on any domain in complex plane.

The definition envisages that when we are interested to verify the analyticity of a function at a point z_0, it is supposed to prove: (i) first order partial derivatives of u and v are continuous in some neighbourhood of $z_0 = (x_0, y_0)$; (ii) C-R equations are satisfied in this neighbourhood.

Example 13. Show that the function $f(z) = 1/(z-1)$ is analytic at $z = 1 + i$.

In view of Definition 8, we exhibit that (i) u_x, u_y, v_x, v_y are continuous in some neighbourhood of $z = (1, 1)$; (ii) C-R equations are satisfied in this neighbourhood of $z = (1, 1)$.

$$f(z) = \frac{1}{z-1} = \frac{1}{x-1+iy} = \frac{x-1}{(x-1)^2+y^2} - i\frac{y}{(x-1)^2+y^2}.$$

Hence,

$$u(x, y) = \frac{x-1}{(x-1)^2+y^2}, \qquad v(x, y) = -\frac{y}{(x-1)^2+y^2};$$

Then

$$u_x = \frac{y^2-(x-1)^2}{[y^2+(x-1)^2]^2}, \qquad u_y = -\frac{2y(x-1)}{[(x-1)^2+y^2]^2};$$

$$v_x = \frac{2y(x-1)}{[(x-1)^2+y^2]^2}, \qquad v_y = \frac{y^2-(x-1)^2}{[(x-1)^2+y^2]^2}.$$

Since u_x, u_y, v_x, v_y are rational functions of the form $p(x, y)/q(x, y)$, where $p(x, y)$ and $q(x, y)$ being polynomial functions are continuous and $q(x, y)$ is nonzero in $|z-(1+i)| < \delta$, $0 < \delta < 1$, it follows that these partial derivatives are continuous in this neighbourhood. Obviously,

$$u_x = v_y \;\forall\; z = (x, y)\colon |z-(1+i)| < \delta,$$

$$u_y = -v_x \;\forall\; z = (x, y)\colon |z-(1+i)| < \delta.$$

This ensures that the function $f(z) = 1/(z-1)$ is analytic at $z = 1 + i$.

The sum, the product and the quotient of analytic functions are analytic as well (see Problem Set 2.4). In particular, the quotient of two polynomials $P(z)/Q(z)$ is analytic in a domain, provided $Q(z)$ does not vanish in the domain. More interesting is the fact that composition (superposition) of analytic functions is analytic too.

Theorem 7. Let $f(z)$ be analytic in a domain D such that $f(D) \subset D_1$. If a function g is analytic in the domain D_1, and $F(z) = (gf)(z)$, then $F(z)$ is analytic in D.

Proof. Let z_0 be an arbitrary point in D. In Section 2.5, Eq. (2.19) reveals that $F(z)$ is differentiable at z_0 and moreover, $F'(z_0) = g'(f(z_0))f'(z_0)$. Since z_0 is an arbitrary point in D, it follows that $F(z)$ is differentiable and hence analytic in D.

Example 14. Let $f(z) = 1/(z-1)$ and $g(z) = z^2$. Discuss the analyticity of the composite function gf.

The function $f(z) = 1/(z-1)$ is analytic in the domain $D = \mathbb{C}\backslash\{1\}$ and $g(z)$ is analytic in $\mathbb{C}$. Now, $f(D) = \mathbb{C}\backslash\{0\}$ is contained in $\mathbb{C}$, the domain of g.

$$\frac{d}{dz}[(gf)(z)] = g'(f(z))f'(z) = \frac{2}{z-1}\frac{-1}{(z-1)^2} = \frac{-2}{(z-1)^3}, \quad z \neq 1.$$

Throughout the text whenever we may use the term *analytic function* without reference to a specific domain, we shall mean a function which is analytic in a domain Ω. Let us briefly introduce here the entire function. A detailed discussion on this function will be given in Chapters 6 and 11.

Entire function. A complex function f is said to be entire (sometimes integral) if it is analytic in the whole complex plane.

Very obviously, the sum and the product of two or more entire functions are entire functions. Moreover, the superposition of an entire function is again an entire function.

The polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$; and the function $f(z) = e^x(\cos y + i \sin y)$ are entire functions. The verification is left as an exercise for the reader.

Singular point or singularity. If a function $f(z)$ fails to be analytic at a point z_0 but in every neighbourhood of z_0 there exist at least one point where the function is analytic, then z_0 is said to be a singular point or singularity of $f(z)$.

The function $f(z) = 1/z$ is analytic except at $z = 0$ and in each deleted neighbourhood $f(z)$ is analytic, so by definition $z = 0$ is a singular point. For the function

$$f(z) = \frac{z^3+7}{(z^2-2z+2)(z-3)},$$

$z = 3$, $z = 1+i$ and $z = 1-i$ are singularities, since the function is analytic except at $z = 1+\mathrm{i}$, $z = 1-i$ and $z = 3$.

Proposition 9. If a function $f(z)$ is analytic in a domain D, then a function $g(z)$ defined on $D^* = \{z : \bar{z} \in D\}$ by $g(z) = \bar{f}(\bar{z})$ is analytic as a function of z in D^*.

Proof. We have

$$\begin{aligned}\frac{g(z+h)-g(z)}{h} &= \frac{\bar{f}(\overline{z+h})-\bar{f}(\bar{z})}{h}\\ &= \frac{\bar{f}(\bar{z}+\bar{h})-\bar{f}(\bar{z})}{h}\\ &= \overline{\left(\frac{f(\bar{z}+\bar{h})-f(\bar{z})}{\bar{h}}\right)}.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{h\to 0}\frac{g(z+h)-g(z)}{h} &= \lim_{\bar{h}\to 0}\overline{\left(\frac{f(\bar{z}+\bar{h})-f(\bar{z})}{\bar{h}}\right)}\\ &= \overline{\left(\lim_{\bar{h}\to 0}\frac{f(\bar{z}+\bar{h})-f(\bar{z})}{\bar{h}}\right)}\\ &= \overline{\left(\frac{df(\bar{z})}{dz}\right)}.\end{aligned}$$

Thus, $g(z)$ has a derivative at z and the derivative is equal to the complex conjugate of the derivative of f at $\bar{z}$. Since this holds for all $z \in D^*$, g is analytic in D^*.

Proposition 10. If $f'(z) = 0$ in a domain Ω, then $f(z)$ is constant in Ω.

Proof. Let $f(z) = u(x, y) + iv(x, y)$. Now, $f'(z) = 0 \Rightarrow u_x + iv_x = 0 \Rightarrow u_x = 0 = v_x$ for all (x, y) in Ω. Moreover, using the C-R equations, $u_y = v_y = 0$ for all (x, y) in Ω. So, the gradient vector $\nabla u = (u_x, u_y) = (0, 0)$ is zero. This means that the directional derivative of $u(x, y)$ is zero in all directions. Hence $u(x, y)$ is constant along a line segment joining two points. Since Ω is connected, every pair of points can be joined by a polygonal line lying in D. Thus, $u(x, y)$ is constant along all line segments lying in D, and hence for all (x, y) in Ω.

By the parallel arguments, $v(x, y)$ is constant for all (x, y) in Ω. This concludes that the function $f(z) = u(x, y) + iv(x, y)$ is constant in Ω.

Example 15. Prove that $f(z) = \sin x \cosh y - i \cos x \sinh y$ is nowhere analytic.

The real and imaginary parts of $f(z)$ are $u = \sin x \cosh y$, $v = -\cos x \sinh y$. Hence

$$u_x = v_y \Rightarrow \cos x \cosh y = -\cos x \cosh y \Rightarrow 2 \cos x \cosh y = 0,$$
$$u_y = -v_x \Rightarrow \sin x \sinh y = -\sin x \sinh y \Rightarrow 2 \sin x \sinh y = 0.$$

Thus,

$$\cos x \cosh y = 0 \Rightarrow \cos x = 0 \text{ as } \cosh y \neq 0$$
$$\Rightarrow x = \left(n - \frac{1}{2}\right)\pi \quad \text{or} \quad z = \left(n - \frac{1}{2}\right)\pi + iy,\ y \in \mathbb{R};$$

and

$$\sin x \sinh y = 0 \Rightarrow \sin x = 0 \quad \text{or} \quad \sinh y = 0$$
$$\Rightarrow x = n\pi \quad \text{or} \quad y = 0$$
$$\Rightarrow z = n\pi + iy,\ \ y \in \mathbb{R} \quad \text{or} \quad z = x,\ x \in \mathbb{R}.$$

Now, the C-R equations are satisfied on the set

$$[(x + i0) \cup (n\pi + iy)] \cap \left[\left(n - \frac{1}{2}\right)\pi + iy\right] = \left(n - \frac{1}{2}\right)\pi, \quad n \in \mathbb{I}.$$

The set of points, where $f(z)$ is differentiable is given by $z = (n - 1/2)\pi$, i.e. $x = (n - 1/2)\pi$, $n \in \mathbb{I}$. But in any neighbourhood of $(n - 1/2)\pi$, the function is not differentiable. Hence at these points where the C-R equations are satisfied, the function is not analytic. This ensures that $f(z)$ is not analytic anywhere.

PROBLEM SET 2.4

1. In each of the following cases, determine the set of all singular points.

(a) $\dfrac{z-1}{z^2+1}$; (b) $\dfrac{z^2+i}{(z-3)(z^2-2z+3)}$; (c) $\dfrac{z^3-1}{z(z^2-1)}$.

(d) $|z|^2 + \dfrac{1}{z}$; (e) $\dfrac{z+4}{z^2-iz+2}$; (f) $\operatorname{Re} z + \operatorname{Arg} z$.

State why the function is analytic in the domain except at singular points in cases (a), (b) and (c).

2. Let $f(z)$ be an analytic function defined in a domain D. Show that $f(z)$ must be constant if one of the following statements is true for all $z \in D$.

   (a) $f(z)$ is real valued;

   (b) $\overline{f(z)}$ is analytic;

   (c) $|f(z)|$ is constant;

   (d) Re $f(z)$ or Im $f(z)$ are constant functions;

   (e) arg f is constant;

   (f) Re $f(z) = [\text{Im } f(z)]^2$.

3. Show that the function $g(z) = \sqrt{r}\, e^{i\theta/2}$ $(r > 0, -\pi < \theta < \pi)$ is analytic in its domain of definition and $g'(z) = 1/[2g(z)]$.

   Using this result, show that the composite function $g(2z - 2 + i)$ is analytic except for the half-line $x \le 1 (y = -1/2)$ with derivative as

$$\frac{d}{dz}[g(2z-2+i)] = \frac{1}{g(2z-2+i)}.$$

4. Show that the following functions are nowhere analytic:

   (a) $f(z) = (xy^2 + 2) + iy$; (b) $f(z) = e^y \cos x + 2ie^y \sin x$;

   (c) $f(z) = e^x(\cos y - i \sin y)$. (d) $f(z) = z + |z|^2$

   Suggestion. Verify that the C-R equations are satisfied nowhere or at points which have no neighbourhood in which the function is differentiable.

5. Let $f(z)$ and $g(z)$ be analytic functions in a domain D. Then show that the following functions are analytic in D:

   (a) $c_1 f(z) + c_2 g(z)$, c_1 and c_2 are any constants;

   (b) $f(z)g(z)$;

   (c) $f(z)/g(z)$, $g(z) = 0$ in D.

6. If $f(z)$ is an entire function, show that $f(e^z)$ and $f(z^2)$ are entire functions.

7. Restate Theorem 7 to discuss the analyticity of the composite function $f(g(z))$ and then work out Example 14 to find $(fg)'$.

8. Let $f(z) = u + iv$ be analytic in a domain D. Prove that $|f(z)|^2$ can be written as Jacobian of transformations of u and v with respect to x and y.

9. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain Ω. Define the family of level curves

$$u(x, y) = c_1 \quad \text{and} \quad v(x, y) = c_2,$$

   where c_1 and c_2 are real numbers. In particular, verify this result for $f(z) = \sin x \cosh y + i \cos x \sinh y$.

2.7 HARMONIC FUNCTIONS

Harmonic functions play an important role in many engineering problems where the solution of the Laplace equation is involved in the analysis. For problems arising out of engineering mathematics, these functions are popular in the name of potential functions.

Definition 9. A real function h of two variables is said to be harmonic in a domain D if, throughout D it has second order continuous partial derivatives, and satisfies the Laplace equation

$$h_{xx} + h_{yy} = 0. \tag{2.38}$$

Example 16. Verify that the function $h(x, y) = 3x^2y - y^3 + 4$ is harmonic in $\mathbb{C}$.

Since $h_{xx} = 6y$, $h_{yy} = -6y$, it is obvious that the second order partial derivatives of $h(x, y)$ are continuous and satisfy (2.38). Thus, $h(x, y)$ is harmonic.

Theorem 8. If a function $f(z) = u + iv$ is analytic in a domain D, then its component functions u and v are harmonic in D.

Proof. Use the result, "If a function $f(z) = u + iv$ is analytic in a given domain D, then the derivatives of all orders are analytic in D". Hence, u and v have continuous partial derivatives of all orders in D. This result will be proved later in Chapter 4.

$f(z) = u + iv$ is analytic in $D \Leftrightarrow u_x, u_y, v_x, v_y$ are continuous and $u_x = v_y$ and $u_y = -v_x$ in D.

Again, by the quoted result: u_{xx}, u_{yy}, v_{xx} and v_{yy} are continuous in D and $u_{xx} = v_{yx}$, $u_{yy} = -v_{xy}$ (using C-R equations). Hence (on addition) $u_{xx} + u_{yy} = 0$, for all (x, y) in D. This implies that $u(x, y)$ is harmonic in D.

Similarly, it can be proved that $v(x, y)$ is harmonic in D.

Harmonic conjugate. The function q is said to be a harmonic conjugate of p if p and q are harmonic and the first order partial derivatives of p and q satisfy the C-R equations.

Of course, this use of the word "conjugate" is different here from that employed in defining $\bar{z}$, the conjugate of z.

Theorem 9. A function $f(z) = u + iv$ is analytic in $D \Leftrightarrow v$ is a harmonic conjugate of u.

Proof. Let $f(z) = u + iv$ be analytic in D. Then, the first order partial derivatives of u and v satisfy the C-R equations and just by Theorem 8, u and v are harmonic functions. Hence v is a harmonic conjugate of u.

Conversely, if v is the harmonic conjugate of u, then u and v are the harmonic functions and the first order partial derivatives of u and v satisfy the C-R equations, i.e. $u_x = v_y$ and $u_y = -v_x$. Moreover, if u and v are harmonic, it means that their second order partial derivatives and hence the first order partial derivatives are continuous. This implies that $f(z) = u + iv$ is analytic.

Now, we construct an analytic function whose real or imaginary component is given. In view of the above theorem, real or imaginary component which we use to construct the analytic function must be harmonic. The following proposition is a further step in this direction.

Proposition 11. Let $u(x, y)$ be harmonic in some neighbourhood of the point (x_0, y_0). Then, there exists a conjugate harmonic function $v(x, y)$ defined in the neighbourhood, and $f(z) = u(x, y) + iv(x, y)$ is an analytic function.

Proof. The harmonic function u and its conjugate harmonic function v will satisfy the C-R equations $u_x = v_y$ and $u_y = -v_x$. The function $v(x, y)$ is constructed in two steps. First integrate v_y (which is equal to u_x) with respect to y, treating x as constant, i.e.

$$v(x, y) = \int u_x(x, y)dy + \phi(x), \tag{2.39}$$

where $\phi(x)$ is a function of x only, and hence the partial derivative of $\phi(x)$ with respect to y is zero. Secondly, we are able to find $\phi'(x)$ by differentiating (2.39) with respect to x, and replacing v_x by $-u_y$ on the left side, i.e.

$$-u_y(x, y) = \frac{d}{dx}\int u_x(x, y)dy + \phi'(x). \tag{2.40}$$

Since u is harmonic, all terms except those involving x (only) in Eq. (2.40) cancel out, and a formula for $\phi'(x)$ will purely be a function of x. Now, the integration of $\phi'(x)$ can be used to have $\phi(x)$, and this we insert in Eq. (2.39) to obtain the desired analytic function $f(z) = u + iv$.

Example 17. Find an analytic function whose real part is $u(x, y) = 2xy + 2x$.

First we note that $u(x, y)$ is harmonic and hence it is feasible to construct an analytic function $f(z) = u(x, y) + iv(x, y)$. Since $f(z)$ is analytic, the first order partial derivatives of component functions u and v satisfy the C-R equations, i.e. $u_x = v_y$ and $u_y = -v_x$,

$$v_y = 2y + 2 \quad \text{since } u_x = v_y. \tag{2.41}$$

Integrating (2.41) with respect to y, treating x as constant, we have

$$v = y^2 + 2y + \phi(x), \tag{2.42}$$

where $\phi(x)$ is a function of x. Hence

$$v_x = \phi'(x) = -2x, \text{ since } u_y = -v_x.$$

Thus, $\phi'(x) = -2x$ and hence, $\phi(x) = -x^2 + c$, where c is some real constant. Now, (2.42) determines v by

$$v = y^2 + 2y - x^2 + c,$$

giving rise to

$$f(z) = 2xy + 2x + i(-x^2 + y^2 + 2y + c).$$

Example 18. Transform the Laplace equation $u_{xx} + u_{yy} = 0$ into polar coordinates, and show that

$$u(r, \theta) = e^{-\theta}\cos(\ln r), \qquad r > 0, \qquad 0 < \theta < 2\pi$$

is harmonic. Also, find the harmonic conjugate of $u(r, \theta)$ and the corresponding analytic function.

Given that $u(x, y)$ is harmonic function, let $v(x, y)$ be the harmonic conjugate of u. Then, using the transformation $x = r\cos\theta$, $y = r\sin\theta$, the C-R equations in polar form have been obtained in Section 2.5 as

$$ru_r = v_\theta, \qquad u_\theta = -rv_r.$$

Differentiating these equations partially with respect to r and θ, respectively, we have

$$u_r + ru_{rr} = v_{\theta r}, \tag{2.43}$$

$$u_{\theta\theta} = -rv_{r\theta}. \tag{2.44}$$

Note that $v_{r\theta} = v_{\theta r}$. We add (2.43) and (2.44) to have Laplace equation in polar form:

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0. \tag{2.45}$$

The function $u(r, \theta) = e^{-\theta}\cos(\ln r)$ is harmonic, since

$$u_r = \frac{-e^{-\theta}\sin(\ln r)}{r} \Rightarrow ru_r = -e^{-\theta}\sin(\ln r)$$

or

$$u_r + ru_{rr} = -\frac{e^{-\theta}\cos(\ln r)}{r} \Rightarrow r^2u_{rr} + ru_r = -e^{-\theta}\cos(\ln r)$$

and

$$u_\theta = -e^{-\theta}\cos(\ln r) \Rightarrow u_{\theta\theta} = e^{-\theta}\cos(\ln r).$$

Second-order partial derivations of u with reference to r and θ are continuous in the specified domain and Eq. (2.45) is satisfied and hence, $u(r, \theta)$ is harmonic.

Construction. Use $ru_r = v_\theta$ and $u_\theta = -rv_r$. Hence

$$v_\theta = -e^{-\theta}\sin(\ln r)$$

On integration (with respect to θ) this gives

$$v = e^{-\theta}\sin(\ln r) + \phi(r).$$

Now,

$$v_r = e^{-\theta}\frac{\cos(\ln r)}{r} + \phi'(r) \Rightarrow -\frac{u_\theta}{r} = e^{-\theta}\frac{\cos(\ln r)}{r} + \phi'(r).$$

Thus, $\phi'(r) = 0 \Rightarrow \phi(r) = c$, some real constant. So, the required function is

$$f(z) = e^{-\theta}\cos(\ln r) + i[e^{-\theta}\sin(\ln r) + c].$$

Why have we excluded the line $\theta = 0$ from the discussion? Comment.

Partial derivative in relation to z and $\bar{z}$

In Sections 2.5–2.7, we studied the C-R equations, analyticity, Laplace equation and harmonic functions. In our analysis, we freely used partial derivatives of u and v with respect to x and y. Here, these aspects are viewed with reference to independent variables z and $\bar{z}$ (ignoring the fact that z and $\bar{z}$ are conjugate to each other).

Every complex function $f(z)$ of a complex variable $z = x + iy$ can be written as

$$f(x + iy) = u(x, y) + iv(x, y),$$

where $u(x, y)$ and $v(x, y)$ are real functions of independent variables x and y. This means that $f(x + iy)$ may be treated as functions of two variables, i.e. $f(x, y)$ with the understanding that it is given by adding real function $u(x, y)$ and i times of real function $v(x, y)$. Note that

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}. \tag{2.46}$$

First, we derive the complex form of C-R equations. For this purpose, we apply the chain rule in calculus to $f(x, y)$, where x and y are functions of z and $\bar{z}$. In view of (2.46), we have

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Let us define the differential operators

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{2.47}$$

Applying these operators to $f = u + iv$, we have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z}\right) = \frac{1}{2}[u_x + v_y + i(v_x - u_y)],$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}\right) = \frac{1}{2}[u_x - v_y + i(u_y + v_z)].$$

If the first-order partial derivatives of u and v satisfy the C-R equations, then

$$f_{\bar{z}} \equiv \frac{\partial f}{\partial \bar{z}} = 0. \tag{2.48}$$

This is known as the *complex form* of C-R equations. The condition in (2.48) completely characterizes the analytic functions, i.e. we have proved a fact that every analytic function is independent of $\bar{z}$.

In a similar way, if $f(z)$ is an analytic function in some domain and the second differential operator in (2.47) is applied to $f(z) = u - iv$, then

$$\bar{f}_z \equiv \frac{\partial \bar{f}}{\partial z} = 0.$$

This suggests that the complex conjugate $\bar{f}$ of an analytic function may be considered as a function of $\bar{z}$ and, even more to say, $\bar{f}(\bar{z})$ is analytic (see Proposition 9). Therefore, we can denote $\overline{f(z)}$ simply by $\bar{f}(\bar{z})$ and write

$$\overline{f(x+iy)} = \bar{f}(x - iy) \tag{2.49}$$

so that

$$u(x, y) = \frac{1}{2}[f(x+iy) + \bar{f}(x-iy)]. \tag{2.50}$$

It is reasonable to expect that (2.49) is a formal identity, and then it holds when x and y are complex. On substituting $x = z/2$, $y = z/2i$, we obtain

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}\left[f\left(\frac{z}{2} + i\frac{z}{2i}\right) + \bar{f}\left(\frac{z}{2} - i\frac{z}{2i}\right)\right] = \frac{1}{2}[f(z) + \bar{f}(0)].$$

This leads to

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0)$$

and from (2.50),

$$u(0, 0) = \frac{1}{2}[f(0) + \bar{f}(0)] = \operatorname{Re} f(0).$$

So,

$$\overline{f(0)} = \operatorname{Re} f(0) - i \operatorname{Im} f(0) = u(0, 0) - ic,$$

where c is some real number. Consequently, a function $f(z)$ may be expressed as

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ic. \tag{2.51}$$

Equation (2.51) can be utilized to construct analytic functions, without using the integration process, when its real or imaginary component is given and hence for finding the conjugate harmonic functions when one of the harmonic functions is given.

Consider again the function $u(x, y) = 2xy + 2x$ of Example 17. The direct application of (2.49) yields

$$f(z) = 2\left(2\frac{z}{2}\frac{z}{2i} + 2\frac{z}{2}\right) + ic,$$

or,

$$f(z) = -iz^2 + 2z + ic, \qquad (c \text{ is some real constant}).$$

We are moving to find the complex form of Laplace equation. Suppose $u(x, y)$ is a harmonic function. Then, formally, treating z and $\bar{z}$ as independent variables,

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}(u_x - iu_y).$$

This gives

$$\begin{aligned}\frac{\partial^2 u}{\partial \bar{z}\,\partial z} &= \left[\frac{1}{2}\frac{\partial}{\partial x}(u_x - iu_y)\frac{\partial x}{\partial \bar{z}} + \frac{1}{2}\frac{\partial}{\partial y}(u_x - iu_y)\frac{\partial y}{\partial \bar{z}}\right] \\ &= \frac{1}{4}[(u_{xx} + u_{yy}) + i(u_{yx} - u_{xy})] = 0 \\ &= 0.\end{aligned}$$

The function u, being harmonic, satisfies the Laplace equation and the second order partial derivatives are continuous. Thus,

$$\frac{\partial^2 u}{\partial \bar{z}\partial z} = 0$$

is the *complex form* of Laplace equation.

The preceding discussion reveals that the Laplacian operator has the equivalent form

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial \bar{z}\,\partial z}. \tag{2.52}$$

Example 19. Let $f(z)$ be an analytic function. Then prove that:

(a) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2;$

(b) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2 = 2|f'(z)|^2.$

(a) Considering (2.49) and (2.52), we have

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 &= 4\frac{\partial^2}{\partial \bar{z}\,\partial z}[f(z)\overline{f(z)}] \\
&= 4\frac{\partial^2}{\partial \bar{z}\,\partial z}[f(z)\bar{f}(\bar{z})] \\
&= 4\frac{\partial}{\partial z}\left[f(z)\,\bar{f}'(\bar{z})\right] \\
&= 4\left[f'(z)\bar{f}'(\bar{z})\right] \\
&= 4\left[f'(z)\overline{f'(z)}\right] \\
&= 4\,|f'(z)|^2.
\end{aligned}$$

Similarly, the part (b) can be verified by writing

$$\text{Re } f(z) = \frac{f(z) + \overline{f(z)}}{2}$$

and proceeding on the lines of analysis as for part (a).

Note. Example 19 may also be solved with the information $|f|^2 = u^2 + v^2$. Use the fact that u and v are harmonic functions.

Remark. Suppose a complex function $f(z)$, is given in the form $f(z) = u(x, y) + iv(x, y)$. To write this as function of z, we note that

$$f(x + iy) = u(x, y) + iv(x, y).$$

Assigning $y = 0$, we have

$$f(x) = u(x, 0) + iv(x, 0). \tag{2.53}$$

Also,

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Now, $y = 0 \Leftrightarrow z = \bar{z}$. So,

$$f(z) = u(z, 0) + iv(z, 0). \tag{2.54}$$

Clearly, (2.53) and (2.54) suggest that simply by putting $y = 0$ in $f(z) = u(x, y) + iv(x, y)$ results in an expression involving a function of x only. Thus, replacing x by z in the resulting expression will yield the desired function in terms of z.

Milne–Thomson method. Finally, we describe the Milne–Thomson method for constructing analytic functions when the real or imaginary component is given and hence finding harmonic conjugates. For the time being, let us take the following result as guaranteed until we reach Chapter 4.

If a function f is analytic in a domain then f can be integrated in the domain using antiderivative, i.e. by finding $F(z)$ such that $F'(z) = f(z)$.

Suppose a harmonic function $u(x, y)$ is given. Then, we have to construct an analytic function $f(z) = u + iv$. Since

$$f'(z) = u_x(x, y) - iu_y(x, y),$$

we can write $f'(z)$ in terms of z and the integration of $f'(z)$ then gives the desired function to some additive complex constant.

Consider (for the last time) the function $u(x, y) = 2xy + 2x$ of Example 17. So,

$$f'(z) = u_x - iu_y = 2y + 2 - i2x.$$

This suggests that

$$f'(z) = -2iz + 2.$$

Integrating the above equation, we get

$$f(z) = -iz^2 + 2z + c,$$

where c is some complex constant.

After converting $f(z)$ into the form $u(x, y) + iv(x, y)$, it will yield the harmonic conjugate of $u(x, y) = 2xy + 2x$.

PROBLEM SET 2.5

1. For what values of a the function $e^{ax} \cos y$ is harmonic?
2. If $p(x, y)$ and $q(x, y)$ are harmonic functions in a domain Ω, prove that
   (a) $c_1p + c_2q$ is harmonic in Ω, here c_1 and c_1 are constants;
   (b) $p_y - q_x + i(p_x + q_y)$ is analytic in Ω.
3. Let v be the harmonic conjugate of u. Establish the truth of the statements:
   (a) If u is the harmonic conjugate of v, then $f(z) = u + iv$ is constant;
   (b) u may not be the harmonic conjugate of v, give an example;
   (c) u is the harmonic conjugate of $-v$;

(d) $-u$ is the harmonic conjugate of v;

(e) If V is also the harmonic conjugate of u, then $v - V$ is a constant function;

(f) $dv = u_x dy - u_y dx$.

4. Show that the functions

(a) $u(x, y) = -x^3 + 3xy^2 + 2y + 1$; (b) $u(x, y) = \cos x \sinh y$;

(c) $u(x, y) = e^x \cos y + y$; (d) $u(x, y) = (x + y)/(x^2 + y^2)$

are harmonic. Find their harmonic conjugates and construct the corresponding analytic functions.

5. If $f(z)$ is a nonvanishing analytic function in some domain D, show that $\ln|f(z)|$ is harmonic in D.

6. Let $f(z)$ be an analytic. Then, for $n \in \mathbb{I}_0$, generalize Example 19, in the form

(a) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2|f(z)|^{n-2}|f'(z)|^2$;

(b) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^n = n(n-1)|f(z)|^{n-2}|f'(z)|^2$.

7. Let $f(z)$ be an analytic function such that $f'(z) \neq 0$. Then show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\ln|f'(z)| = 0.$$

Moreover, if $|f'(z)|$ is the product of a function of x and a function of y, prove that $f'(z) = \exp(\alpha z^2 + \beta z + \gamma)$, where $\alpha \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{C}$.

8. Let $f(z) = u(x, y) + iv(x, y)$ be analytic. Then show that

(a) $f(z) = 2u\left(\dfrac{z}{2}, -\dfrac{iz}{2}\right) + \text{constant}$; (b) $f(z) = 2iv\left(\dfrac{z}{2}, -\dfrac{iz}{2}\right) + \text{constant}$.

and hence find $f(z) = u + iv$, if

(c) $u = x^4 - 6x^2y^2 + y^4$; (d) $v = 2x(y + 1) - 4$;

(e) $u = 4xy + x + 1$, $f(1) = 1 + i$; (f) $v = \sinh x \cos y$, $f(i\pi/2) = 0$.

9. Let $f(z) = u + iv$ be an analytic function. Find $f(z)$ (as a function of z), when

(a) $u - v = (x - y)(x^2 + 4xy + y^2)$; (b) $2u + 3v = 13(x^2 - y^2) + 2x + 3y$.

Suggestion. (a) $f(z) = u + iv \Rightarrow (1 + i)f(z) = u - v + i(u + v)$. Construct $(1 + i)f(z)$ with $U = u - v$ as its real part; (b) $(2 - 3i)f(z) = 2u + 3v + i(2v - 3u)$.

10. Find all the harmonic functions of the form $u = \phi(\sqrt{(x^2 + y^2)})$ that are not constants, where the second order partial derivatives of ϕ are continuous.

Suggestion. Assume $t = \sqrt{(x^2 + y^2)}$. Thus, $u_x = x\phi'(t)/t$, $u_y = y\phi'(t)/y$. Compute u_{xx} and u_{yy} and put into the Laplace equation to yield a differential equation whose solution is the required function.

11. Find the most general cubic form

$$u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

of harmonic functions and determine the corresponding analytic function for which $u(x, y)$ is a real part.

12. The real part $u(x, y)$ is given for an analytic function $f(z)$ in some domain. Give an alternate proof of Proposition 11 as follows:

(a) Let (a, b) and (x, y) be two points in the domain. Define

$$v(x, y) = \int_a^x -u_y(t, b)\,dt + \int_b^y u_x(x, t)\,dt.$$

Assume that the line segment joining (a, b) to (x, b) or (a, y) and joining (x, b) or (a, y) to (x, y) lie in the domain. Then show that $v(x, y)$ is the harmonic conjugate of $u(x, y)$.

(b) Use part (a) to find $f(z)$ when its real part is

$$u(x, y) = F\left(x + \sqrt{x^2 + y^2}\right)$$

13. Find the numbers a, b and c so that the function $x^3 + ax^2y + bxy^2 + cy^3$ is an entire function.

Suggestion. Check $f(x + i0)$.

14. Let $u(x, y)$ and $v(x, y)$ be a pair of harmonic functions, and let $\phi(u, v)$ be the harmonic function of the variables u and v. Prove that $\phi[u(x, y), v(x, y)]$ is a harmonic function of the variables x and y.

15. Given that $f(z)$ is analytic in a domain Ω. Without recourse to Proposition 9, show that $\overline{f(\bar{z})}$ is analytic.

16. Is it possible to construct an analytic function in some domain having the real part as $x(x + 2y) + 3$?, if not, why?

3

Elementary Functions

We start by introducing the exponential function of a complex variable. With the help of this function, trigonometric, hyperbolic and logarithmic functions are defined and their properties are discussed. Multivalued functions have been added to facilitate the discussion on branches of logarithmic function, inverse of trigonometric and hyperbolic functions and complex power of complex valued functions.

3.1 INTRODUCTION

Most of the specific functions encountered in elementary analysis belong to a class widely known as *elementary* functions. To have an appropriate exposition of this class we start with recalling that an algebraic function is a polynomial, a rational function, or more generally any function $w = f(z)$ that satisfies the equation

$$P_n(z)w^n + P_{n-1}(z)\,w^{n-1} + \cdots + P_1(z)w + P_0(z) = 0,$$

where each $P_k(z)$ is polynomial of degree k. The set of elementary functions consists of the algebraic functions; the elementary transcendental (or nonalgebraic) functions, i.e. exponential, logarithmic, trigonometric, inverse trigonometric, hyperbolic functions and their inverses; and all others that can be constructed by adding, subtracting, multiplying, dividing or forming a function of a function. Thus,

$$w = \tan\left(\frac{ze^z + \tan^{-1} z}{\sin z \cos 2z - \log z}\right)$$

is an elementary function.

Beyond elementary functions there exist the higher transcendental functions, often called *special functions*. The field of special functions was cultivated with great devotion by the rehonoured mathematicians of the eighteenth and nineteenth centuries—Euler, Gauss, Jacobi, Abel, Weierstrass, Riemann, Hermite and Poincaré, among others. However, we shall not concentrate our attention on the study of these functions. Some of the well known of these are the gamma function (Section 11.4, Chapter 11), the Riemann zeta function (Exercise 16, Probelm Set 5.2), the Gauss hyper-geometric function and the elliptic function, etc.

Special functions vary rather widely with respect to their origin, nature and applications. However, a large group of these functions occurred as solutions of second order linear differential equations of mathematical physics. They are also of great importance through the theory of orthogonal expansions.

Periodic functions

A function f defined in a domain Ω is said to be periodic if there is a nonzero complex number p such that whenever $z \in \Omega$ and $z + p \in \Omega$, $f(z+p) = f(z)$. In this case p is said to be the period of f.

A constant function is a trivial example of a periodic function. In Sections 3.2 and 3.3 we shall observe that exponential and sine functions are nontrivial examples of periodic functions. Note that we have defined period for any complex function not necessarily elementary function.

Clearly, if p is a period of a function, so is np for any integer n provided $np \in \Omega$; we tacitly assume this hereafter. Let f be a nonconstant analytic and periodic function with period p. Then, for each n, np is also its period. All the points np, $n \in \mathbb{I}$ lie on the same straight line joining the origin to the point p. Among all such periods there is a nonzero point in the upper half-plane ($\text{Im } z \geq 0$) and on this line which is a period of f, and also is nearest to the origin. Such a period is called a *primitive* period or *fundamental* period of f. Thus, all the periods of f lying on this line are the integral multiples of the primitive period.

Let P be the set of all periods of an analytic function and let p be its primitive period. If p is its only primitive period, then the function is said to be simply periodic. There exist analytic functions which are not simply periodic and we classify them doubly periodic due to the following fact: A periodic function can have at most two primitive periods, e.g. elliptic functions. The proof of this fact is beyond the scope of this text. Henceforth, the period of a function would mean its primitive period, until specifically mentioned otherwise. Thus, throughout this text whenever we say that p is the period of a function, it would mean that p is its fundamental or primitive period.

3.2 EXPONENTIAL FUNCTION

Definition 1. The exponential function e^z or $\exp(z)$ of a complex variable $z = x + iy$ is defined by

$$e^z = e^x e^{iy}. \tag{3.1}$$

Remarks. 1. In view of Definition 1 and Euler's formula, $e^z = e^x(\cos x + i \sin y)$.

2. e^z is periodic with period $2\pi i$, since

$$e^{z+2i\pi} = e^x e^{i(y+2\pi)} = e^x e^{iy} = e^z.$$

However, $2n\pi i$; $n \in \mathbb{I}$ is any period.

3. $e^z \neq 0$, $\forall\, z \in \mathbb{C}$, since $|e^z| = e^x > 0\ \forall\, x \in \mathbb{R}$.

For $z = x + i0$, (3.1) reduces to the well familiar exponential function of a real variable. The first order partial derivatives of real and imaginary components of $\exp(z)$ are continuous and satisfy the C-R equations in the whole complex plane. Using $f'(z) = u_x + iv_x$, we have

$$\frac{d}{dz}(e^z) = e^z \ \forall z \in \mathbb{C}.$$

Hence, e^z is an entire function.

Proposition 1. For any complex numbers z_1 and z_2, we have

$$e^{z_1+z_2} = e^{z_1}e^{z_2}. \tag{3.2}$$

Proof. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then, using Definition 1 and commutativity

$$e^{z_1+z_2} = e^{x_1+x_2+i(y_1+y_2)} = e^{x_1+x_2}e^{i(y_1+y_2)} = e^{x_1}e^{iy_1}e^{x_2}e^{iy_2} = e^{z_1}e^{z_2}.$$

The above proposition can be extended to

$$e^{z_1+z_2+\cdots+z_n} = e^{z_1}e^{z_2}\cdots e^{z_n}. \tag{3.3}$$

The substitution $z_1 = z$, $z_2 = -z$ in (3.2) gives $e^{-z} = 1/e^z$ and this observation leads to the result

$$e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}.$$

Proposition 2. The equation $e^z = w$ has infinitely many solutions.

Proof. Since $w \neq 0$, we can write $w = |w|e^{i\phi}$, where $\phi = \operatorname{Arg} w$. Thus, $e^z = w \Leftrightarrow e^x e^{iy} = |w|e^{i\phi}$. Recalling equality of two complex numbers in exponential form (see Section 1.3), $e^x = |w|$ and $y = 2n\pi + \phi$, $n \in \mathbb{I}$. This reveals that the numbers

$$z = \ln|w| + i(2n\pi + \phi), \qquad n \in \mathbb{I}$$

satisfy the equation.

Thus, the map $w = e^z$ is a many-one mapping and this is due to the periodicity of the exponential function. The detailed exposition of mapping by exponential functions will be presented in Section 9.5.

Example 1. Solve the equation $e^{2z-1} = 1 + i$.

Writing $1 + i$ in the exponential form $\sqrt{2}\,e^{i\pi/4}$, we immediately arrive at

$$2z - 1 = \ln\sqrt{2} + i\left(2n\pi + \frac{\pi}{4}\right).$$

Hence, the solution set is given by

$$z = \frac{1}{2} + \frac{1}{4}\ln 2 + i\left(n + \frac{1}{8}\right)\pi, \qquad n \in \mathbb{I}.$$

PROBLEM SET 3.1

1. Find the limit of e^{iz} when (a) $x \to \infty$; (b) $y \to \infty$.
2. Does $\lim_{z\to\infty} e^{-z^2}$ exist? If not, why? Explain.

3. Solve the equations:

(a) $e^{iz+1} + 2 = 0$; (b) $ie^{3z} - e^{2z} + ie^{z} = 1$; (c) $e^{2z} - (2-i)e^{z} - 2i = 0$.

4. Show that (a) $|\exp(z^2)| \le \exp(|z|^2)$; (b) $|e^z - 1| \le e^{|z|} - 1 \le |z|e^{|z|}$.

5. Verify the equation $\exp(nz) = [\exp(z)]^n$, $n \in \mathbb{I}$.

Suggestion. Take $z_1 = z_2 = \cdots = z_n = z$ in (3.3).

6. Show that (a) $\exp(\bar{z}) = \overline{\exp(z)}$; (b) $\exp(i\bar{z}) = \overline{\exp(iz)} \Leftrightarrow z = n\pi$, $n \in \mathbb{I}$.

7. Find the primitive period of (a) e^{iz} and (b) e^{2z+1}.

8. Let $f(z)$ be analytic in a domain D such that $f'(z) = f(z)\ \forall\ z \in D$. Then show that $f(0) = 1$ if and only if $f(z) = \exp(z)$.

Suggestion. Consider $g(z) = f(z)/\exp(z)$ and observe that $g'(z) = 0$ in D.

9. Show that the function $e^{\bar{z}}$ is nowhere analytic.

10. Use Milne–Thomson method to construct an analytic function $f(z) = u + iv$, when $u + v = e^x(\cos y + \sin y) + 2(x + y) + 1$.

11. Define a complex function g by

$$g(z) = \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Show that $g(z)$ is not continuous at $z = 0$ by taking limits along the real and imaginary axes. Comment on the outcome for $x = 0$ when g is a real function of the real variable x.

12. Consider the function $f(z) = e^{-1/z}$, $z \neq 0$ and prove that:

(a) $f(z)$ is bounded but not continuous on the region $0 < |z| < 1$, $|\arg z| \le \pi/2$;

(b) $f(z)$ is continuous but not uniformly continuous on the domain $0 < |z| < 1$, $|\arg z| < \pi/2$;

(c) $f(z)$ is uniformly continuous on the sector $0 < |z| < 1$, $|\arg z| \le \alpha < \pi/2$.

13. Show that the Rolle's theorem fails to hold for the function $e^{iz} - 1$, even though the conditions of the theorem appear to be satisfied with respect to the two values $z = 0$ and $z = 2\pi$. Explain.

3.3 TRIGONOMETRIC FUNCTIONS

Motivated by the expressions $2i \sin x = e^{ix} - e^{-ix}$ and $2 \cos x = e^{ix} + e^{-ix}$, we introduce trigonometric functions of a complex variable $z = x + iy$ as follows:

Definition 2. The functions sine and cosine of a complex variable z are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{3.4}$$

Remarks. 1. Since e^{iz} and e^{-iz} are entire functions, it follows that $\sin z$ and $\cos z$ are entire functions, being linear combinations of entire functions (see Problem Set 2.4). So, finding the derivatives of e^{iz} and e^{-iz}, it can be shown that

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

Similarly,

$$\frac{d}{dz}(\cos z) = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.$$

2. From (3.4), we define

$$e^{iz} = \cos z + i \sin z \qquad \forall\, z \in \mathbb{C}.$$

3. The functions $\sin z$ and $\cos z$ are 2π periodic, since

$$\sin(z + 2\pi) = \sin z \quad \text{and} \quad \cos(z + 2\pi) = \cos z.$$

However, $2n\pi$ is any period.

For $z = 0 + iy$, (3.4) reveals that $\sin(iy) = i \sinh y$ and $\cos(iy) = \cosh y$. Also,

$$\sin(z + \pi/2) = \cos z, \qquad \cos(z + \pi/2) = -\sin z,$$
$$\sin(z + \pi) = -\sin z, \qquad \cos(z + \pi) = -\cos z,$$
$$\overline{\sin z} = \sin \bar{z}, \qquad \overline{\cos z} = \cos \bar{z}.$$

Using the definition, it is easy to establish the following trigonometric identities:

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2;$$
$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2.$$

Various identities involving the trigonometric functions of real variable carry over to complex analysis with slight modifications: (i) $\sin 2z = 2 \sin z \cos z$; (ii) $\cos 2z = \cos^2 z - \sin^2 z$; (iii) $\sin^2 z + \cos^2 z = 1$, etc.

The function $\sin z$ can be written as

$$\begin{aligned}\sin(x + iy) &= \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + i \cos x \sinh y. \end{aligned} \tag{3.5}$$

Considering expression (3.5), we have

$$|\sin z|^2 = \sin z\, \overline{\sin z} = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

which further produces the identities

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \quad \text{and} \quad |\sin z|^2 = \cosh^2 y - \cos^2 x. \tag{3.6}$$

The identities in (3.6) yield the inequalities

$$\sinh y \le |\sin z| \le \cosh y.$$

Similarly,

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \quad \text{and} \quad |\cos z|^2 = \cosh^2 y - \sin^2 x. \tag{3.7}$$

Again, the identities in (3.7) give

$$\sinh y \le |\cos z| \le \cosh y.$$

These inequalities establish the fact that $\sin z$ and $\cos z$ are unbounded functions.

Example 2. Prove that the equation $\tan z = z$ has only real roots.

Writing the equation in the form $\tan(x + iy) = x + iy$ and equating the real and imaginary parts, we have

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = x, \qquad \frac{\sinh 2y}{\cos 2x + \cosh 2y} = y$$

and hence

$$\frac{\sin 2x}{x} = \frac{\sinh 2y}{y}. \tag{3.8}$$

Note that for all nonzero real θ,

$$|\sin \theta| < |\theta|, \qquad |\sinh \theta| > |\theta|.$$

Therefore, if neither x nor y vanishes, the absolute value of the left-side of (3.8) is less than 2 while the absolute value of the right-side is greater than 2. This shows that at least one of x, y must vanish. Suppose $y = 0$, then $\tan z = z$ assumes the form

$$\tan x = x.$$

Now, we draw the graph of $y = \tan x$ and $y = x$ and see that the line intersects the tangent curve at a point in the interval $(\pi, 3\pi/2)$. Again the function $\tan x$ is π periodic and this gives intersection at other points in the intervals $n\pi < x < (n + 1/2)\pi$, $n \in \mathbb{I}$.

Note that, if $x = 0$ and $y \neq 0$, then

$$\tanh y = y.$$

This does not have any solution except $y = 0$. Thus the assertion follows.

Zeros of sin z and cos z. We observe that $\sin z = 0 \Leftrightarrow z = n\pi$, $n \in \mathbb{I}$. When $z = n\pi$, $\sin n\pi = 0$ is obvious from the definition. In case $\sin z = 0$, we have $\sin^2 x + \sinh^2 y = 0$ and hence $\sin x = 0$ and $\sinh y = 0$. Now, $x = n\pi$ and $y = 0$ is the solution set of these equations, i.e. $z = n\pi$, $n \in \mathbb{I}$. Also,

$$\cos z = 0 \Leftrightarrow \sin\left(z + \frac{\pi}{2}\right) = 0 \Leftrightarrow z = \left(n + \frac{1}{2}\right)\pi, \qquad n \in \mathbb{I}.$$

Example 3. Find all the zeros and periods of the function $\cos(2iz + 13)$.

As $\cos(2iz + 13) = 0 \Leftrightarrow 2iz + 13 = (n + 1/2)\pi$, so all the zeros are given by

$$z = \frac{(2n-1)\pi + 26}{4} i, \qquad n = 0, \pm 1, \ldots$$

The function $\cos z$ is 2π periodic. Let the period of the given function be k. Since $\cos[2i(z + k) + 13] = \cos(2iz + 13 + 2ik)$, it follows that $2ik = 2\pi \Rightarrow k = -i\pi$, and hence the required period is πi.

Also, other trigonometric functions can be expressed in terms of sine and cosine functions, viz.

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}, \qquad \sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}. \tag{3.9}$$

The singularities of $\tan z$ and $\sec z$ are the zeros of $\cos z$, i.e. $z = (n + 1/2)\pi$. Except these points $\tan z$ and $\sec z$ are analytic everywhere and their derivatives are

$$\frac{d}{dz}(\tan z) = \sec^2 z, \qquad \frac{d}{dz}(\sec z) = \sec z \tan z.$$

Parallel reasoning is associated with $\cot z$ and $\csc z$.

$\tan z$ and $\cot z$ are π-periodic, since

$$\tan(z + \pi) = \tan z, \qquad \cot(z + \pi) = \cot z.$$

The mapping properties of these functions, with useful applications, are explained in Chapter 9.

PROBLEM SET 3.2

1. Establish the fact that $\sin z$ and $\cos z$ are unbounded functions.

2. Show that $\cos z_1 = \cos z_2 \Leftrightarrow z_1 + z_2$ or $z_1 - z_2$ is an integral multiple of 2π. Write the consequence of this result if z_1 and z_2 are replaced by $z_1 - \pi/2$ and $z_2 - \pi/2$, respectively.

Suggestion. $\cos z_1 - \cos z_2 = 0 \Leftrightarrow -2\sin\dfrac{z_1 + z_2}{2}\sin\dfrac{z_1 - z_2}{2} = 0.$

3. Determine the primitive period of the functions:

(a) $\sin^2 z$; (b) $|\cos z|$;

(c) $e^{iz} + \cos 2z$; (d) $\sin(1 + z/2) + \cos(1 + z/5)$.

4. Find the period and all the zeros of the functions:

(a) $\sin(iz + 2)$; (b) $\cos(z - 2i)$;

(c) $\tan(iz + 3m)$, $m \in \mathbb{I}$; (d) $\cot z$.

5. Show that (a) $\sin i\bar{z} = \overline{\sin iz} \Leftrightarrow z = n\pi i$; (b) $\cos i\bar{z} = \overline{\cos iz}$.

6. Solve the equations:

(a) $\sin z = \cosh 4$; (b) $\cos z = 2$;

(c) $e^z \sin z + i(2 \sin z - e^z) + 2 = 0$;

(d) $\tan z + \cot z = 2$.

Suggestion. For part (c): $(e^z + 2i)(\sin z - i) = 0$.

7. Prove that the functions $\sin\bar{z}$ and $\cos\bar{z}$ are nowhere analytic.

8. If $|\sin z| \le 1$, show that z assumes only real values.

9. Using any method to construct an analytic function $f(z) = u + iv$ such that

$$u - v = \frac{\sin x + \cos x - e^{-y}}{2(\cos x - \cosh y)} \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = 0.$$

10. Show that $\tan(\pi/6 + i\alpha) = x + iy \Leftrightarrow x^2 + y^2 + 2x/\sqrt{3} = 1$.

11. If $\tan(x + iy) = i$, comment on the nature of x and y.

3.4 HYPERBOLIC FUNCTIONS

Motivated by the equations $2 \sinh x = e^x - e^{-x}$ and $2 \cosh x = e^x + e^{-x}$, we now introduce hyperbolic trigonometric functions of a complex variable $z = x + iy$ as follows:

Definition 3. The hyperbolic of sine and cosine of a complex variable z are defined by

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{3.10}$$

Remarks. 1. Since e^z and e^{-z} are entire functions, it follows that $\sinh z$ and $\cosh z$ are entire functions (being linear combinations of entire functions). Also,

$$\frac{d}{dz}(\sinh z) = \cosh z \quad \text{and} \quad \frac{d}{dz}(\cosh z) = \sinh z.$$

2. The hyperbolic functions $\sinh z$ and $\cosh z$ are $2\pi i$ periodic, since

$$\sinh(z + 2\pi i) = \sinh z, \qquad \cosh(z + 2\pi i) = \cosh z.$$

When $z = 0 + iy$, Eq. (3.10) implies $\sinh(iy) = i \sin y$ and $\cosh(iy) = \cos y$. Also,

$$-i \sinh iz = \sin z, \qquad \cosh iz = \cos z.$$

Further, replacing z by $-iz$, we have the very useful identities:

$$-i \sin iz = \sinh z \quad \text{and} \quad \cos iz = \cosh z.$$

Using the definition, it is easy to establish the trigonometric identities:

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2;$$
$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2.$$

Proceeding on the same lines as for trigonometric functions, one can easily obtain

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y,$$
$$|\cosh z|^2 = \sinh^2 x + \cos^2 y.$$

Now, we solve an equation involving hyperbolic functions. It will be simpler to work out the analysis through trigonometric functions.

Example 4. Solve the equation $\sinh z = i$.

Note that $\sinh z = i \Rightarrow -i \sin iz = i$. This means that $\sin(-y + ix) = -1$, or

$$-\sin y \cos ix + \cos y \sin ix = -1$$

or

$$-\sin y \cosh x + i \cos y \sinh x = -1.$$

Comparing the real and imaginary components, we get the pair of equations:

$$\sin y \cosh x = 1 \quad \text{and} \quad \cos y \sinh x = 0.$$

The second of these equations is true for $y = (n + 1/2)\pi$, $n \in \mathbb{I}$ or $x = 0$. Note that for $y = (n + 1/2)\pi$, the first equation implies

$$\cosh x = -1 \quad \text{when} \quad n = \pm 1, \pm 3, \ldots$$

This is not permissible. Hence, we try $y = (2n + 1/2)\pi$ and find $\cosh x = 1$ which is certainly true for $x = 0$. Hence, the set of all complex numbers $z = x + iy$, which satisfies both equations is given by

$$z = \left(2n + \frac{1}{2}\right)\pi i, \qquad n \in \mathbb{I}.$$

Zeros of hyperbolic functions. The functions

$$\sinh z = 0 \Leftrightarrow -i \sin iz = 0 \Leftrightarrow iz = n\pi \Leftrightarrow z = n\pi i \in \mathbb{I},$$

and

$$\cosh z = 0 \Leftrightarrow z = \left(n + \frac{1}{2}\right)\pi i, \qquad n \in \mathbb{I}.$$

Other hyperbolic functions are given in terms of hyperbolic functions of sine and cosine such as

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{cosech} z = \frac{1}{\sinh z}. \tag{3.11}$$

The singularities of $\tanh z$ and $\operatorname{sech} z$ are the zeros of $\cosh z$, i.e. $z = (n + \frac{1}{2})\pi i$. Except at these points, $\tanh z$ and $\operatorname{sech} z$ are analytic everywhere and their derivatives are

$$\frac{d}{dz}(\tanh z) = \operatorname{sech}^2 z, \quad \frac{d}{dz}(\operatorname{sech} z) = \operatorname{sech} z \tanh z.$$

Parallel reasoning is associated with $\coth z$ and $\operatorname{cosech} z$.

Note that $\tanh z$ and $\coth z$ are πi-periodic, since

$$\tanh(z + \pi i) = \tanh z, \qquad \coth(z + \pi i) = \coth z.$$

The mapping properties of hyperbolic functions are studied by utilizing the relations of these functions with the corresponding trigonometric functions.

PROBLEM SET 3.3

1. Verify the trigonometric identities:

(a) $-i \sinh iz = \sin z$; (b) $\cosh iz = \cos z$;

(c) $\sinh^2 z + \cosh^2 z = \cosh 2z$. (d) $\sinh 2z = 2\sinh z \cosh z$

2. Show that:

(a) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$; (b) $|\sinh x| \le |\cosh z| \le \cosh x$.

3. Find all the roots of the equations:

(a) $\cosh z = \dfrac{1}{2}$; (b) $\sinh z = -i$; (c) $\tanh z - \coth z + 2i = 0$

4. Find the period and all the zeros of the functions:

(a) $\tanh(z + \sqrt{3})$; (b) $\sinh(iz + 2)$; (c) $|\sinh z|$.

5. Show that $\sinh z$ and $\cosh z$ are unbounded functions.

6. If $|\sinh z| \le 1$ for all z, show that z is either purely imaginary or zero.

7. Show that $\cosh z_1 = \cosh z_2 \Leftrightarrow z_1 + z_2$ or $z_1 - z_2$ is an even integer multiple of πi.
8. Show that $\sinh \bar{z}$ is nowhere analytic but it is differentiable at some points. Locate these points in the complex plane.
9. Construct an analytic function $f(z) = u(x, y) + iv(x, y)$ for which the real part is given by $u(x, y) = (e^x + \sinh x)\cos y - y$.
10. How are the complex numbers z_1 and z_2 related so that the equations

$$\sinh z_1 = \sinh z_2 \quad \text{and} \quad \cosh z_1 = \cosh z_2$$

are valid simultaneously.

11. Explain why is $\cos(e^z)$ entire. State why the real part of this function must be harmonic everywhere.

3.5 MULTIVALUED FUNCTION AND ITS BRANCHES

All elementary functions of subsequent sections belong to this class. We have already given a brief introduction to such type of functions in Section 2.1. In this section, we shall introduce some more concepts concerning these functions which are helpful in giving comprehensive treatment of the functions discussed in the remaining part of this chapter.

Let us recall that a function $w = f(z)$ is multivalued when for given z, we may find more than one value of w. Thus, a function f is said to be single valued if f satisfies

$$f(z) = f[z(r, \theta)] = f[z(r, \theta + 2\pi)]$$

otherwise, f is classified as multivalued function.

For analytical properties of a multivalued function, we consider domains in which these functions are single valued.

Branch. By branch of a multivalued function $f(z)$ defined on S we mean a single valued function $F(z)$ which is analytic in some domain $D \subset S$ at each point of which $F(z)$ is one of the values of $f(z)$.

For the multivalued function $f(z) = z^{1/2}$, we define

$$z^{1/2} = \sqrt{r}e^{i\theta/2}, \qquad r > 0,\ \alpha < \theta < \alpha + 2\pi, \quad \alpha \in \mathbb{R} \tag{3.12}$$

with component functions

$$u(r, \theta) = \sqrt{r}\cos\frac{\theta}{2} \qquad \text{and} \qquad v(r, \theta) = \sqrt{r}\sin\frac{\theta}{2}$$

is single valued and continuous on $|z| > 0$, $\alpha < \arg z < \alpha + 2\pi$, as illustrated in Fig. 3.1.

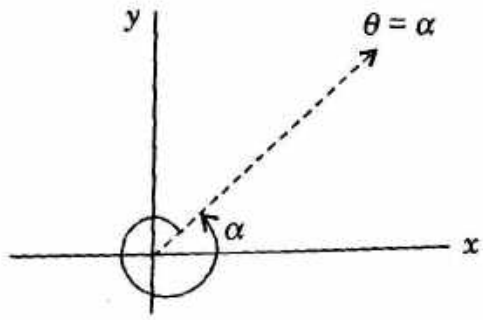

Figure 3.1

Evidently, the function in (3.12) is not continuous on the line $\theta = \alpha$ as there are points arbitrarily close to z at which the values of $v(r, \theta)$ are nearer to $\sqrt{r}\sin(\alpha/2)$ and also points such that the values of $v(r, \theta)$ are nearer to $-\sqrt{r}\sin(\alpha/2)$. Interestingly, we claim that the function in (3.12) is not only continuous but also analytic in the specified domain as the first order partial derivatives of u and v are continuous in this domain and moreover, satisfy C-R equations (in polar form): $ru_r = v_\theta$, $u_\theta = -rv_r$. Furthermore,

$$\frac{d}{dz}(z^{1/2}) = e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} + i\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}\right) = \frac{1}{2z^{1/2}}.$$

In the foregoing discussion we have seen that the function in (3.12) is a branch of the multivalued function $z^{1/2}$. The function

$$z^{1/2} = \sqrt{r}e^{i\theta}, \qquad |z| > 0, -\pi < \theta < \pi$$

is called the *principal branch* of the multivalued function $z^{1/2}$.

Branch cut. A *branch* cut is a portion of a line or curve, which is introduced in order to define the branch of a multivalued function.

As an example, the origin and the line $\theta = \alpha$ form the branch cut for the function in (3.12).

Branch point. A multivalued function $f(z)$ defined on S is said to have a *branch point* at z_0 in $\mathbb{C}$ if, when z describes an arbitrary small circle about z_0, then for every branch F of f, $F(z)$ does not return to its original value.

Thus, $z = 0$ is the branch point of $z^{1/2}$. When moving from any point $z = re^{i\theta}$ about the origin, for one complete circuit to reach again at z, we change arg z by 2π. For original position $z = e^{i\theta}$, we have $w = \sqrt{r}e^{i\theta/2}$ and after one complete circuit $w = \sqrt{r}e^{i(\theta+2\pi)/2} = -\sqrt{r}e^{\theta/2}$. Thus, w has not returned to its original value and the occurrence of this phenomenon is explained as a change in branch has occurred. Since a complete circuit about $z = 0$ altered the branch of the function, $z = 0$ is a branch point.

Geometrically, a point common to all branch cuts is recognized as a branch point of a multivalued function.

Example 5. Let $f(z) = (z^2 + 1)^{1/2}$. Then, we find its branch points, branch line and show that a complete circuit around these points produces no change in the branches of the function.

We have $w = (z^2 + 1)^{1/2} = [(z + i)(z - i)]^{1/2}$. Then

$$\arg w = \frac{1}{2}\arg(z - i) + \frac{1}{2}\arg(z + i),$$

so that

$$\text{Change in arg } w = \frac{1}{2}[\text{Change in arg}(z + i)] + \frac{1}{2}[\text{Change in arg}(z - i)].$$

Let C be a circuit enclosing the point i but not the point $-i$ [see Fig. 3.2(a)].

The point z goes once counterclockwise around C and this causes

$$\text{Change in arg}(z - i) = 2\pi \qquad \text{and} \qquad \text{Change in arg}(z + i)] = 0$$

so that

$$\text{Change in } \arg w = \pi.$$

Hence, w does not return to its original value, i.e. a change in branch has occurred. Since the closed circuit about $z = i$ altered the branch of the function, $z = i$ is a branch point. Similarly, if C is a circuit enclosing $z = -i$, but not $z = i$ then we can show that $z = -i$ is another branch point.

The part of the y-axis $y \le -1$ $(x = 0) \cup y \ge 1 (x = 0)$ or line segment $-1 \le y \le 1$ $(x = 0)$ constitutes the branch line.

If C encloses both the branch points $z = \pm i$ and as z moves counterclockwise around C, as in Fig. 3.2(b), then

$$\text{Change in } \arg(z - i) = 2\pi \quad \text{and} \quad \text{Change in } \arg(z + i) = 2\pi$$

So,

$$\text{Change in } \arg w = 2\pi.$$

Hence, a complete circuit around both the branch points does not change the branch.

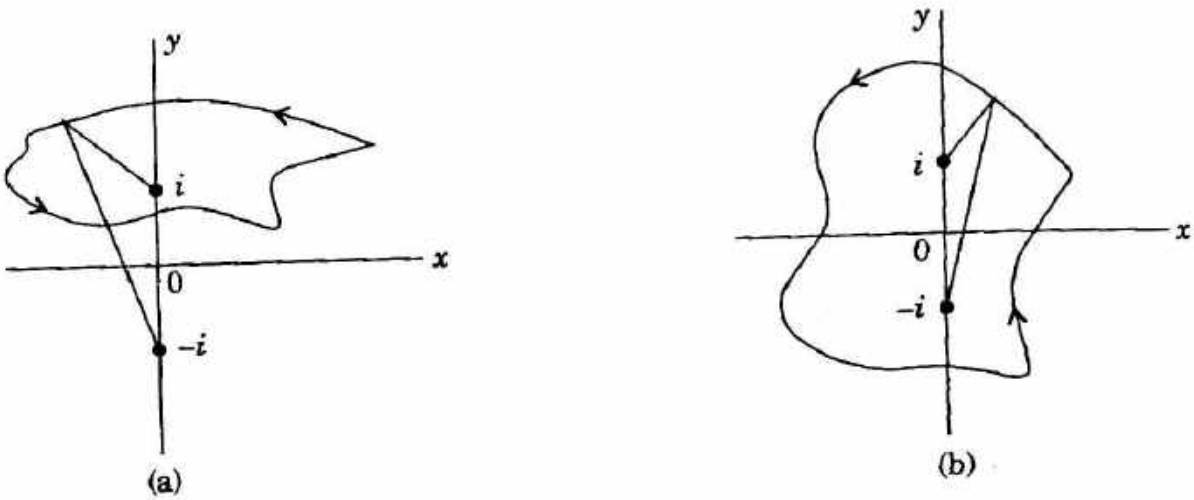

Figure 3.2

3.6 LOGARITHMIC FUNCTION

We, now introduce the logarithmic function of a complex variable. By Proposition 2, $e^z = w, \Rightarrow z = \ln|w| + i(2n\pi + \phi)$. This motivates us to define

$$\log w = \ln |w| + i(2n\pi + \phi), \quad \text{where Arg } w = \phi.$$

Now, we observe that

$$\exp(\log z) = z \qquad \forall\, z \ne 0.$$

However,

$$\log(\exp z) = \log(e^x e^{iy}) = \ln e^x + i(2n\pi + y) = x + iy + i2n\pi = z + 2n\pi i.$$

But, if the number $z = x + iy$ is confined to the strip $-\pi < y \le \pi$, then

$$\log(\exp z) = \ln e^x + i\text{Arg}\,(e^z) = x + iy = z,$$

when the domain of definition is restricted to the horizontal strip $(-\pi < \text{Im } z \le \pi)$. This means that $\log z$ may be used as the inverse function of the exponential function.

Definition 4. For $z \neq 0$, the logarithmic function of a complex variable z is defined as

$$\log z = \ln|z| + i(2n\pi + \phi), \tag{3.13}$$

where $\phi = \text{Arg } z$ and n is any integer.

Remarks. 1. In view of the above definition, $\log z$ is a multivalued function. The principal value of $\log z$ denoted by $\text{Log } z$ is obtained by putting $n = 0$ in (3.13). Thus,

$$\text{Log } z = \ln|z| + i \text{ Arg } z. \tag{3.14}$$

Note that (3.13) is reduced to

$$\log z = \text{Log } z + i2n\pi, \qquad n \in \mathbb{I}.$$

Equation (3.14) tells that $\text{Log } z$ is a single-valued function and it coincides with the natural logarithm when z is a positive real number, i.e. $\text{Log } x = \ln x$ when $z = x$.

2. Since $\arg z = \text{Arg } z + 2n\pi$, (3.13) is reformulated as

$$\log z = \ln|z| + i \arg z$$

or

$$\log z = \ln r + i\theta, \qquad \text{when } r = |z| \text{ and } \theta = \arg z. \tag{3.15}$$

Because of (3.14), note that $\log i = i(2n + 1/2)\pi$, $n \in \mathbb{I}$, $\log(-1) = i(2n + 1)\pi$, $n \in \mathbb{I}$ and in particular, $\text{Log } i = i\pi/2$, $\text{Log } (-1) = i\pi$.

Let α be a real number such that the values of θ in (3.15) satisfy $\alpha < \theta < \alpha + 2\pi$. Then the function

$$\log z = \ln r + i\theta, \qquad r > 0, \quad \alpha < \theta < \alpha + 2\pi \tag{3.16}$$

with component functions

$$u(r, \theta) = \ln r \qquad \text{and} \qquad v(r, \theta) = \theta.$$

is single valued and continuous on the domain $\{z : |z| > 0, \alpha < \arg z < \alpha + 2\pi\}$ (see Fig. 3.1). Obviously, the function in (3.16) is not continuous on the line $\theta = \alpha$ as there are points arbitrary close to z at which the values of $v(r, \theta)$ are nearer to α and also points such that the values of $v(r, \theta)$ are nearer to $\alpha + 2\pi$. Moreover, we claim that the function in (3.16) is not only continuous but also analytic in the specified domain as the first order partial derivatives of u and v are continuous in this domain and satisfy the C-R equations: $ru_r = v_\theta$, $u_\theta = -rv_r$. Furthermore,

$$\frac{d}{dz}(\log z) = e^{-i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}, \qquad r > 0, \quad \alpha < \arg z < \alpha + 2\pi.$$

In particular,

$$\frac{d}{dz}(\log z) = \frac{1}{z}, \qquad r > 0, \quad -\pi < \text{Arg } z < \pi.$$

The properties

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

and

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

are the direct consequences of the definition of logarithmic function and the relations: $\arg z_1z_2 = \arg z_1 + \arg z_2$, $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ (see Section 1.3). Again, the property

$$\text{Log}(z_1z_2) = \text{Log } z_1 + \text{Log } z_2$$

does not hold true always. The choice $z_1 = z_2 = -1$ serves a simple example for the job. As $z_1z_2 = 1$ and $\text{Log } z_1z_2 = \text{Log } 1 = 0$ but $\text{Log } z_1 + \text{Log } z_2 = 2\pi i$.

With the discussion in Section 3.5, we observe that, for $\alpha \in \mathbb{R}$,

$$\log z, \qquad |z| > 0, \qquad a < \arg z < \alpha + 2\pi$$

is any branch of the logarithmic function. The line $\theta = \alpha$ including the origin is the branch line and $z = 0$ is the branch point. Now, for $\theta = -\alpha$

$$\log z, \qquad |z| > 0, \quad -\pi < \arg z < \pi$$

is the principal branch denoted by Log z.

PROBLEM SET 3.4

1. If $\text{Log } \sin(\theta + i\phi) = \alpha + i\beta$, show that $2 \cos 2\theta = e^{2\phi} + e^{-2\phi} - 4e^{2\alpha}$.
2. Find the branch points and construct the branch lines for the functions:
   (a) $f(z) = [z/(1-z)]^{1/2}$; (b) $f(z) = (z^2 - 4)^{1/3}$; (c) $f(z) = \log(z - z^2)$.
3. Prove that $z = \pm i$ are the branch points of the function $f(z) = (z^2 + 1)^{1/3}$.
4. Find the general value of (a) $\log(-1 + \sqrt{3}\,i)$; (b) $\log i$; (c) $\log e$.
5. For nonzero complex numbers z_1, z_2, show that $\log z_1z_2 = \log z_1 + \log z_2$.
6. Let $zz_2 \neq 0$. Then show that $\text{Log } z_1z_2 = \text{Log } z_1 + \text{Log } z_2 + 2n\pi i$; $n = 0, \pm 1$. Mention the set for which $n = -1$.
   Suggestion. Use Eq. (3.14) and Exercise 1, Problem Set 1.3.
7. Show that:
   (a) $\text{Log}(1 + i)^2 = 2\,\text{Log}(1 + i)$, but $\text{Log}(-1 + i)^2 \neq 2\,\text{Log}(-1 + i)$.
   (b) $\text{Log}(1 + i\sqrt{3})^2 = 2\,\text{Log}(1 + i\sqrt{3})$, but $\text{Log}(-1 + i\sqrt{3})^2 \neq 2\,\text{Log}(-1 + i\sqrt{3})$.
8. Prove that:
   (a) If $\log z = \ln r + i\theta$ $(r > 0,\ \pi/4 < \theta < 9\pi/4)$, then $\log i^2 = 2 \log i$;
   (b) If $\log z = \ln r + i\theta$ $(r > 0,\ 3\pi/4 < \theta < 11\pi/4)$, then $\log i^2 \neq 2 \log i$.
9. Show that the function
   (a) $\text{Log}(z - i)$ is analytic except on the half line $y = 1 (x \leq 0)$;
   (b) $[\text{Log}(z + 4)]/(z^2 + i)$ is analytic everywhere except at the points $\pm(1 - i)/\sqrt{2}$ and on the portion $x \leq -4 (y = 0)$ of the real axis.
10. Verify that $z = i$ is the only solution of the equation $\log z = (\pi/2)i$.
11. Find the branch point, the principal branch and the branch cut of the function $(az + b)^{1/2}$, where a and b are nonzero real constants.
12. Let $f(z) = \sqrt{z}$ denote the principal branch and $g(z) = z^2$. Then show that the composition function $(gf)(z)$ is analytic everywhere in the complex plane while $f(z)$ is analytic only in $(|z| > 0, -\pi < \text{Arg } z < \pi)$. Does this violate the chain rule?

13. Construct a branch of $\log(z+4)$ that is analytic at the point $z = -5$ and takes on the values $7\pi i$ there.

3.7 COMPLEX EXPONENTS

Definition 5. When $z \neq 0$ and the exponent c is any complex number, the function z^c is defined by means of the equation

$$z^c = \exp(c \log z), \tag{3.17}$$

where $\log z$ denotes the multivalued logarithmic function.

In view of Definition 5, z^c is a multivalued function, provided c is not an integer. If $\log z$ is replaced by $\operatorname{Log} z$ in (3.17), we write $z^c = e^{c \operatorname{Log} z}$ and this is accepted as the principal value of z^c. The principal branch and any other branch of the multivalued function z^c are obtained from $\operatorname{Log} z$ and $\log z$, respectively.

According to Definition 5, the exponential function with base c, where c is any nonzero complex constant is written as

$$c^z = \exp(z \log c). \tag{3.18}$$

Note that e^z is, in general, a multivalued function according to Definition 5 but the usual interpretation of e^z occurs when the principal value of logarithm is taken. Hence

$$e^z = e^{z \log e} = e^{z \ln e} = e^z.$$

By (3.18), c^z is a multivalued function. However, it has no branch point. Every branch is a single-valued continuous function defined in the whole finite plane (and not on some domain whose boundary consists of certain curves joining the branch points). In fact, after making a circuit along any simple closed curve, we return to the original complex number (perhaps with a different value of argument). Thus, c^z (c is a fixed complex number) is an entire function.

By definition, $z^c = \exp(c \log z)$ is a multivalued function and hence for any real number α, the branch

$$\log z = \ln r + i\theta, \qquad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$$

is a single valued and analytic function in the specified domain. When this branch is used, the function z^c is single valued and analytic. The derivative of such a branch is obtained as

$$\frac{d}{dz}(z^c) = \frac{d}{dz}\exp[(c \log z)] = \exp(c \log z)\frac{c}{z} = c\frac{\exp(c \log z)}{\exp(\log z)} = cz^{c-1}.$$

Example 6. Find all the values of (a) i^{-2i}; (b) $(1-i)^{1+i}$. Also, specify the principal values.

Using (3.17), for part (a),

$$\begin{aligned} i^{-2i} &= \exp(-2i \log i) \\ &= \exp\left[-2i^2\left(2n\pi + \frac{\pi}{2}\right)\right] \\ &= \exp[(4n+1)\pi], \qquad n \in \mathbb{I}. \end{aligned}$$

Principal value: e^{π}.

For part (b),

$$\begin{aligned}(1-i)^{1+i} &= e^{(1+i)\log(1-i)} \\ &= e^{\log(1-i)}e^{i[\ln\sqrt{2}+i(2n-1/4)\pi]} \\ &= (1-i)\,e^{(2n+1/4)\pi}\,e^{i\ln\sqrt{2}}, \quad n \in \mathbb{I}.\end{aligned}$$

Principal value: $(1-i)e^{\pi/4}e^{i\ln\sqrt{2}}$.

Remark. Some of the rules for exponents carry over from the real case. If $z \neq 0$, and c and d are complex numbers then using Definition 5 we may verify the validity of the following identities:

(a) $z^{-c} = \dfrac{1}{z^c}$.

(b) $z^c z^d = z^{c+d}$.

(c) $\dfrac{z^c}{z^d} = z^{c-d}$.

(d) $(z^c)^d = z^{cd}e^{i2\pi nd}, \quad n \in \mathbb{I}$.

PROBLEM SET 3.5

1. Find all the values of

(a) $(-i)^i$; (b) $(-1)^{1/n}$; (c) $(1+i)^i$; (d) $(1+i)^{1-i}$;

(e) $i^{\sin i}$; (f) $\mathrm{Re}(\sqrt{2})^i$; (g) $\mathrm{Im}(\cos i)^i$; (h) i^{i^i}

2. Find the principal value of (a) $(-1-\sqrt{3}\,i)^{3\pi i}$; (b) $(-1)^{\log(1+i)}$.

3. Prove that the principal root and the principal value of $z^{1/n}$ are the same.

4. Assuming that $z \neq 0$, show that $z^{m/n}$, $m, n \in \mathbb{I}$ has n distinct values if and only if m and n are relatively prime to each other.

Using this result, solve the equations:

(a) $z^{2/3} = -1 + \sqrt{3i}$; (b) $z^{4/3} = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$.

5. Let $z_1 z_2 \neq 0$ and s be a complex number. Prove that if all of the powers involved are principal values, then $(z_1 z_2)^s = z_1^s z_2^s e^{2\pi i s n}$, where n is defined as in Exercise 1, Problem Set 1.3.

In view of this result, explain what is wrong with the following?

$$1 = 1^{1/2} = [(-1)(-1)]^{1/2} = (-1)^{1/2}(-1)^{1/2} = i \cdot i = -1.$$

6. If the principal value of z^i is considered, prove that $|z^i| < e^{\pi}$ for all complex numbers $z \neq 0$.

7. Prove that $\mathrm{Log}(z^w) = w(\mathrm{Log}\, z + 2\pi n i)$, where n is an integer.

8. Find the singular points and the branch cut of the function $\tanh z\, \mathrm{Log}\,(z-1+i)$.

9. Let $f(z) = z^z$. Considering the principal branch of $f(z)$, evaluate $f'(2i)$ and write $\operatorname{Re} f(z)$. Repeat this problem when $g(z) = z^{\log z}$.
10. If $i^z = z$, show that $|z|^2 = e^{-(4n+1)y}$, where $z = x + iy$ and n is an integer.
11. If $i^{i^{\cdot^{\cdot^{\infty}}}} = A + iB$ and only the principal values are considered, show that $A^2 + B^2 = e^{-\pi B}$ and $\tan(\pi A/2) = B/A$.

    Suggestion. $i^{A+iB} = e^{(A+iB)\log i} = e^{(A+iB)i\pi/2} = A + iB$. Compare the real and imaginary parts on both sides and proceed.

3.8 INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Definition 6. The inverse of sine and cosine functions of a complex variable $z = x + iy$ are defined by

$$w = \sin^{-1}z, \quad \text{when} \quad z = \sin \omega; \qquad \text{and} \qquad w = \cos^{-1}z, \quad \text{when } z = \cos w. \tag{3.19}$$

Proposition 3. The inverse trigonometric functions are multivalued functions.

Proof. By definition of $\sin^{-1}z$ in (3.19),

$$\begin{aligned} w = \sin^{-1}z &\Rightarrow z = \sin w \\ &\Rightarrow z = \frac{e^{iw} - e^{-iw}}{2i} \\ &\Rightarrow (e^{iw})^2 - 2ize^{iw} - 1 = 0 \\ &\Rightarrow e^{iw} = iz + (1 - z^2)^{1/2} \\ &\Rightarrow iw = \log[iz + (1 - z^2)^{1/2}] \\ &\Rightarrow \sin^{-1}z = -i \log[iz + (1 - z^2)^{1/2}]. \end{aligned} \tag{3.20}$$

Similarly, it can be shown that

$$\cos^{-1}z = -i \log[z + i(1 - z^2)^{1/2}]. \tag{3.21}$$

Also,

$$\begin{aligned} w = \tan^{-1}z &\Rightarrow z = \tan w \\ &\Rightarrow z = \frac{(e^{iw} - e^{-iw})/2i}{(e^{iw} + e^{-iw})/2} \\ &\Rightarrow iz = \frac{e^{2iw} - 1}{e^{2iw} + 1} \\ &\Rightarrow e^{2iw} = \frac{1 + iz}{-iz + 1} \\ &\Rightarrow w = \frac{1}{2i} \log\left(\frac{i - z}{i + z}\right) \\ &\Rightarrow \tan^{-1}z = \frac{i}{2} \log\left(\frac{i + z}{i - z}\right). \end{aligned} \tag{3.22}$$

In view of (3.20–3.22), we observe that $\sin^{-1}z$, $\cos^{-1}z$, $\tan^{-1}z$, have been expressed in terms of the logarithmic function. Since the logarithmic function is multivalued, it ensures that these functions too are multivalued.

When z is replaced by $1/z$, in (3.20), (3.21) and (3.22) the corresponding expressions for $\operatorname{cosec}^{-1}z$, $\sec^{-1}z$ and $\cot^{-1}z$ are available, respectively.

The derivates of $\sin^{-1}z$ and $\cos^{-1}z$ depend on the values chosen for square roots. It happens due to being multivalued. So,

$$\frac{d}{dz}(\sin^{-1}z) = \frac{1}{(1-z^2)^{1/2}}, \qquad \frac{d}{dz}(\cos^{-1}z) = -\frac{1}{(1-z^2)^{1/2}}.$$

The derivative of $\tan^{-1}z$ does not depend upon the way it is made single valued.

$$\frac{d}{dz}(\tan^{-1}z) = \frac{1}{1+z^2}.$$

Example 7. Solve the trigonometric equation: $\tan z = 2$.

In view of (3.22) the solution set is given by $z = \tan^{-1}2$.

$$\begin{aligned}
\tan^{-1}2 &= \frac{i}{2}\log\left(\frac{i+2}{i-2}\right) \\
&= \frac{i}{2}\log\left(-\frac{3+4i}{5}\right) \\
&= \frac{i}{2}\left[i\left(2n\pi - \pi + \tan^{-1}\frac{4}{3}\right)\right] \\
&= \frac{1}{2}\left((2n+1)\pi - \tan^{-1}\frac{4}{3}\right), \qquad n \in \mathbb{I},
\end{aligned}$$

where $\operatorname{Arg}[-3/5-(4/5)i] = \tan^{-1}(4/3) - \pi$ is a unique number.

Inverse hyperbolic functions

Inverse of hyperbolic trigonometric functions are defined in an analogous manner. These functions are also multivalued, since

$$\begin{aligned}
\sinh^{-1}z &= \log\,[z + (z^2+1)^{1/2}]; \\
\cosh^{-1}z &= \log\,[z + (z^2-1)^{1/2}]; \\
\tanh^{-1}z &= \frac{1}{2}\log\frac{1+z}{1-z}.
\end{aligned}$$

When z is replaced by $1/z$, in the above three relations, the corresponding expressions for $\operatorname{cosech}^{-1}z$, $\operatorname{sech}^{-1}z$ and $\coth^{-1}z$ are available, respectively.

Remark. While finding the solution of trigonometric equations involving hyperbolic and tangent functions, it is rather more convenient to deal with the corresponding inverse functions. The reader will realize this in Exercises 5 and 6, Problem Set 3.6.

PROBLEM SET 3.6

1. Find all the values of (a) $\sin^{-1}(-i)$; (b) $\cosh^{-1}(-1)$ and mention their principal values.
2. If $u + iv = \sin^{-1}(x + iy)$, prove that

   (a) $\dfrac{x^2}{\cosh^2 v} + \dfrac{y^2}{\sinh^2 v} = 1$; (b) $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$.

3. Separate into real and imaginary parts the function: $\cos^{-1}(\cos\theta + i \sin\theta)$, where θ is an acute angle and the principal values of inverse functions are involved.
4. Establish the identities

   (a) $\sin^{-1}z + \cos^{-1}z = (2n + 1/2)\pi$; (b) $\tan^{-1}z + \cot^{-1}z = (n - 1/2)\pi$,

   where n is an integer.
5. Solve the trigonometric equations: (a) $\sin z = 10$; (b) $\sinh z = 2$, by two methods.

   Suggestion. First by equating the real and imaginary parts in that equation and secondly by using the expressions for $\sin^{-1} z$ and $\sinh^{-1} z$.
6. Solve the equations:

   (a) $\sinh(5z + i) = -\sqrt{3}\,i$; (b) $\cosh(z - 1) = -1$;

   (c) $\tanh\left(\dfrac{z-3}{2}\right) = -1 + i$; (d) $\cot z = 2i$.

7. Show that if $z \neq 0$ and a is a real number, then $|z^a| = |z|^a$, where the principal value of $|z|^a$ is to be taken.

   Suggestion. $|z^a| = |e^{a[\ln|z| + i(2n\pi+\phi)]}| = e^{a \ln|z|}|e^{i(2n\pi+\phi)a}| = e^{a \ln|z|}$.
8. Find the branch points and the branch line of the function $f(z) = \cos^{-1}z$.
9. Determine the branch points of the function $\tan^{-1}(z^2 + 2z + 1)$.
10. Prove that

$$\tan\left[i \log\left(\frac{a - ib}{a + ib}\right)\right] = \frac{2ab}{a^2 - b^2}.$$

    Suggestion. Replace z by $2ab/(a^2 - b^2)$ in (3.22).
11. Express $\sinh^{-1}z$, $\cosh^{-1}z$ and $\tanh^{-1}z$ in terms of the logarithmic function.

4

Complex Integration

We start with defining the integration of complex-valued functions of a real variable and then, extend the idea to integration of complex-valued functions of a complex variable. The contents of this chapter include the Cauchy–Goursat theorem, the Cauchy integral formula and the maximum modulus principle and their applications in different directions.

4.1 INTRODUCTION

Definition 1. Let $w(t)$ be a complex-valued function of a real variable $t \in [a, b]$. Then, we define

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt, \tag{4.1}$$

where $w(t) = u(t) + iv(t)$, and the individual integrals on the right-hand side exist.

Remarks. 1. From (4.1), we have

$$\text{Re} \int_a^b w(t)dt = \int_a^b \text{Re } w(t)dt \quad \text{and} \quad \text{Im} \int_a^b w(t)dt = \int_a^b \text{Im } w(t)dt.$$

2. The existence of the integrals of u and v in (4.1) is ensured if these functions are piecewise continuous on $[a, b]$. A real function is said to be piecewise continuous on $[a, b]$, if $[a, b]$ can be broken up into a finite number of subintervals in which the function is continuous. The function $w(t)$ is piecewise continuous if the component functions u and v are piecewise continuous.

3. The integrals over the unbounded intervals are defined in a similar manner.

Proposition 1. Let $w(t)$ be a complex-valued function integrable on $[a, b]$. Then

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt. \tag{4.2}$$

Proof. If $a = b$ or $\int_a^b w(t)\,dt = 0$, (4.2) is obvious. Let $a < b$. Then we may assume that the value of the integral is a nonzero complex number $r_0 e^{i\theta_0}$ as

$$\int_a^b w(t)\,dt = r_0 e^{i\theta_0}. \tag{4.3}$$

Then

$$r_0 = e^{-i\theta_0} \int_a^b w(t)\,dt = \int_a^b e^{-i\theta_0} w(t)\,dt.$$

Since, r_0 is positive, the integral on the extreme right is positive. Using the fact $\operatorname{Re} z \le |z|$, we have

$$r_0 = \int_a^b \operatorname{Re}[e^{-i\theta_0} w(t)]\,dt \le \int_a^b |e^{-i\theta_0} w(t)|\,dt = \int_a^b |w(t)|\,dt.$$

But from (4.3),

$$|r_0| \le \left|\int_a^b w(t)\,dt\right|. \tag{4.4}$$

This is what we asserted.

Similarly, we can prove the inequality

$$\left|\int_a^\infty w(t)\,dt\right| \le \int_a^\infty |w(t)|\,dt,$$

provided both the integrals exist.

A continuous complex function γ defined on a closed and bounded interval $[a, b]$ is called a *path* or an arc in the complex plane. Moreover, (i) the path is closed if $\gamma(a) = \gamma(b)$; (ii) the path is simple (Jordan arc) if it does not cross itself, i.e. $\gamma_1(t_1) \ne \gamma_2(t_2)$, $\forall\, t_1, t_2 \in (a, b)$; (iii) the range of γ is the image of $[a, b]$. A closed path is *multiply connected* if it intersects itself more than once.

For example $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$ is a simple closed path. Note that $\gamma_1(t) = e^{it}$, $0 \le t \le 4\pi$ is a closed path with the same graph, but it is not simple as it winds up two times about the origin.

The path $\gamma(t) = x(t) + iy(t)$ is smooth if $x'(t)$ and $y'(t)$ are continuous on $[a, b]$, where at a and b we take right and left derivatives, respectively and the derivative of γ is written as

$$\gamma'(t) = x'(t) + iy'(t).$$

Let the paths γ_1 and γ_2 be defined on $[a, b]$ and $[c, d]$, respectively, such that the terminal point of γ_1 is the initial point of γ_2, i.e. $\gamma_1(b) = \gamma_2(c)$. Then $(\gamma_1 + \gamma_2)$, sum of the two paths or the combined arc constituted by joining γ_1 and γ_2 is written as

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t), & a \le t \le b, \\ \gamma_2(t + c - b), & b \le t \le b + d - c. \end{cases} \tag{4.5}$$

Definition 2. A path γ is said to be *piecewise smooth* if there exists a partition $\mathbb{P}$ of $[a, b]$ such that $a = t_0 < t_1 < \cdots < t_n = b$ and γ is smooth on each subinterval $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$.

For piecewise smooth path, $\gamma'(t)$ exists on $[a, b]$, except at points $t_0, t_1, \cdots, t_n$ where the right and left derivatives exist but may not be equal. Since $\gamma'(t)$ assumes arbitrary values at these points, there is no loss of generality while dealing with problems on integration, if we define $\gamma'(t_i) = 0$.

Definition 3. Let γ and α be two paths defined on $[a, b]$ and $[c, d]$, respectively. We say α is *reparametrization* of γ if there exists a one-one continuous increasing function $\phi : [c, d] \to [a, b]$ such that $\phi(c) = a$, $\phi(d) = b$ and $\alpha = \gamma\phi$, i.e.

$$\alpha(t) = \gamma(\phi(t)), \qquad c \le t \le d. \tag{4.6}$$

Definition 4. A contour is a piecewise smooth path, i.e. a curve consisting of a finite number of smooth arcs joined end to end.

The circle $z(\theta) = z_0 + Re^{i\theta}$, $0 \le \theta \le 2\pi$ about the point z_0 is a simple closed contour. Also, the curve defined by

$$\gamma(x) = \begin{cases} x + i4, & 0 \le x \le 2, \\ x + ix^2, & 2 \le x \le 3 \end{cases}$$

is the contour between the points (0, 4) and (3, 9) made of two smooth arcs $\gamma_1(x) = x + i4$ joining the points (0, 4) and (2, 4) and $\gamma_2(x) = x + ix^2$ joining the points (2, 4) and (3, 9).

A path γ is *rectifiable* if it has finite arc length. Every piecewise smooth path (contour) is rectifiable and its arc length is defined as follows:

Length of arc. Let $\gamma(t)$ be a smooth arc. Then the real-valued function

$$|\gamma'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable over $[a, b]$ and the length of the arc γ is defined as

$$L = \int_a^b |\gamma'(t)|\,dt. \tag{4.7}$$

Remark. It can be shown that L remains unchanged under reparametrization.

Let α be reparametrization of γ. Then (4.6) holds. Hence, by definition,

$$\int_c^d |\alpha'(t)|\,dt = \int_c^d |\gamma'(\phi(t))\phi'(t)|\,dt$$

$$= \int_{\phi(c)}^{\phi(d)} |\gamma'(\mu)| d\mu, \qquad \mu = \phi(t)$$

$$= \int_a^b |\gamma'(t)| dt.$$

The introduction is terminated by mentioning that the fundamental theorem of calculus is applicable to complex functions of a real variable. Let the complex functions $w(t)$ and $W(t)$ be continuous on $[a, b]$ and

$$w(t) = u(t) + iv(t), \qquad W(t) = U(t) + V(t).$$

If $w(t) = W'(t)$ for $a \le t \le b$, then $u(t) = U'(t)$ and $v(t) = V'(t)$. In view of (4.1),

$$\begin{aligned}\int_a^b w(t)dt &= \int_a^b U'(t)dt + i\int_a^b V'(t)dt \\ &= U(t)|_a^b + iV(t)|_a^b \\ &= [U(b) + iV(b)] - [U(a) + iV(a)] \\ &= W(b) - W(a).\end{aligned}$$

4.2 CONTOUR INTEGRAL

Let $f(z)$ be a piecewise continuous function defined on a contour $\gamma = x(t) + iy(t)$, $a \le t \le b$. Then the integral of f along $\gamma \equiv \gamma(t)$ is defined as

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt. \tag{4.8}$$

This integral is sometimes referred to as *complex line integral*. Note that

$$\text{Re}\left[\int_\gamma f(z)dz\right] \neq \int_\gamma \text{Re}\,[f(z)]\, dz.$$

For instance, this statement can be verified by taking $\gamma(t) = it$, $t \in [0, 1]$ and $f(z) = 1$.

Based on the definition in (4.8), we derive some properties of contour integrals.

1. The value of the integral in (4.8) remains unchanged under reparametrization.

Suppose $\alpha \equiv \alpha(t) = \gamma(\phi(t))$, where $\phi : [c, d] \to [a, b]$ is one-one continuous increasing function such that $\phi(c) = a$ and $\phi(d) = b$. Then, by definition of contour integral,

$$\int_\alpha f(z)\, dz = \int_c^d f(\alpha(t))\alpha'(t)\, dt$$

$$= \int_c^d f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt$$

$$= \int_{\phi(c)}^{\phi(d)} f(\gamma(\mu))\gamma'(\mu)d\mu$$

$$= \int_a^b f(\gamma(\mu))\gamma'(\mu)d\mu$$

$$= \int_\gamma f(z)dz.$$

2. If $\gamma(t)$, $a \le t \le b$ describes the contour γ, then $\gamma(-t), -b \le t \le -a$ gives the contour $-\gamma$, and hence, we may define

$$-\gamma(t) = \gamma[ta + (1-t)b], \quad 0 \le t \le 1.$$

Now, it is proved that

$$\int_{-\gamma} f(z)\ dz = -\int_\gamma f(z)dz.$$

Clearly,

$$\int_{-\gamma} f(z)\ dz = \int_0^1 f(\gamma(ta + (1-t)b))\gamma'(at + (1-t)b)(a-b)dt$$

$$= \int_b^a f(\gamma(s))\gamma'(s)ds$$

$$= -\int_\gamma f(z)dz.$$

3. If $\gamma = \gamma_1 + \gamma_2$, then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

Using (4.5) and the definition of contour integral, we observe that

$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_a^{b+d-c} f[(\gamma_1 + \gamma_2)](t)]\ (\gamma_1 + \gamma_2)'\ (t)dt$$

$$= \int_a^b f(\gamma_1(t))\gamma_1'(t)dt + \int_b^{b+d-c} f(\gamma_2(t + c - b))\gamma_2'(t + c - b)dt$$

$$= \int_a^b f(\gamma_1(t))\gamma_1'(t)dt + \int_c^d f(\gamma_2(t))\gamma_2'(t)dt$$

$$= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

4. Let z_0 be any fixed complex number. Then

$$\int_\gamma z_0 f(z)dz = z_0 \int_\gamma f(z)dz.$$

5. Also,

$$\int_\gamma [f(z) \pm g(z)]dz = \int_\gamma f(z)dz \pm \int_\gamma g(z)dz.$$

Remarks. 1. If the arc γ is subdivided into a finite number of sub-arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, then property (3) can be extended to the form

$$\int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz.$$

2. Let $f_1, f_2, \ldots, f_n$ be piecewise continuous functions defined on a contour γ. Then, property (5) has the form,

$$\int_\gamma (f_1 + f_2 + \cdots + f_n) = \int_\gamma f_1 + \int_\gamma f_2 + \cdots + \int_\gamma f_n.$$

3. If γ is a contour and $f(z)$ is continuous on $\gamma(t)$, it is possible to define contour integral as the limit of a sum

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\gamma(t_k))[\gamma(t_k) - \gamma(t_{k-1})].$$

Example 1. Integrate the functions $f(z) = x + y - 3ix^3$ and $g(z) = x + (y+2)i$ between the two points $z = 0$ and $z = 1 + i$ along differed paths.

Let the point $z = 1 + i$ be marked by B in the complex plane. Then to reach at B we choose two paths:

(i) O to A along the x-axis and then from A to B parallel to the y-axis;

(ii) Directly from A to B along the line $y = x$.

We begin with the function $f(z)$. Using the definition of contour integral we calculate the integral values of $f(z)$ along different paths between 0 and $1 + i$ (see Fig. 4.1) as follows:

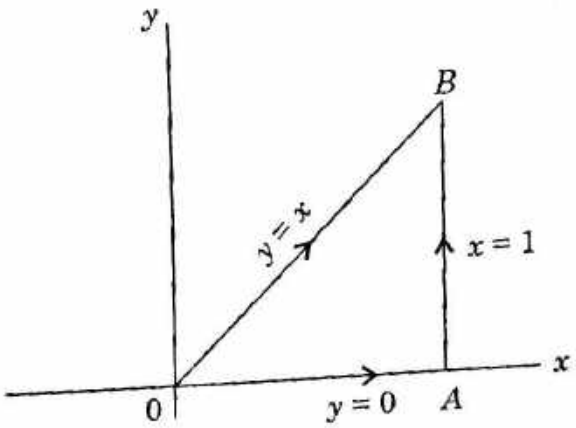

Figure 4.1

Along $OA + AB$, the paths are $z(x) = x$, $0 \le x \le 1$ and $z(y) = 1 + iy$, $0 \le y \le 1$. Then $f[\gamma(t)]$ assumes

$$f[z(x)] = f(x) = x - 3ix^3 \quad \text{and} \quad f[z(y)] = f(1 + iy) = 1 + y - 3i,$$

respectively along OA and OB. Now, the application of (4.8) yields

$$\int_0^1 (x - 3ix^3)dx + i\int_0^1 (y + 1 - 3i)dy = \frac{1}{2} - \frac{3i}{4} + \frac{i}{2} + i + 3 = \frac{14 + 3i}{4}.$$

Along OB, the path is $z(x) = x + ix$, $0 \le x \le 1$, and $f(\gamma(x)) = f(x + ix) = 2x - 3ix^3$. So,

$$(1 + i)\int_0^1 (2x - 3ix^3)dx = \frac{7 + i}{4}.$$

Similarly, for the function $g(z)$, we observe that along $OA + AB$,

$$\int_0^1 (x + 2i)dx + i\int_0^1 1 + (y + 2)i\, dy = \frac{1}{2} + 2i - \frac{5}{2} + \frac{i}{2} = -2 + 3i$$

and along OB,

$$(1 + i)\int_0^1 x + (x + 2)i\, dx = -2 + 3i.$$

Note that the integrals of $f(z)$ along different paths between two fixed points are different while the integrals of $g(z)$ along different paths between these points are same. This is not an accident but a useful phenomenon which will be studied at length in the next section.

Proposition 2 (ML-inequality). Let $f(z)$ be a piecewise continuous function defined on a contour γ. Then

$$\left|\int_\gamma f(z)dz\right| \le ML,$$

where L is the length of the contour defined by (4.7) and $M = \sup_{z\in\gamma}|f(z)|$.

Proof. Using (4.8), Proposition 1 and (4.7), we have

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right|$$

$$\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\leq M\int_a^b |\gamma'(t)|\,dt = ML, \quad M = \sup_{a\leq t\leq b} |f(\gamma(t))|.$$

Example 2. Find the upper bound of the integral (without evaluating the integral)

$$\left|\int_C \frac{(z^2+3)e^{iz}\,\mathrm{Log}\,z}{z^2-2}\,dz\right|, \qquad C = \left\{z : z = 2e^{i\theta}, 0\leq\theta\leq\frac{\pi}{3}\right\}.$$

We observe that

$$|f(z)| = \left|\frac{(z^2+3)\,e^{iz}\,\mathrm{Log}\,z}{z^2-2}\right|$$

$$\leq \frac{(|z|^2+3)\,e^{-y}\,|\ln r + i\,\mathrm{Arg}\,z|}{||z|^2-2|}$$

$$\leq \frac{7(\ln 2+\pi/3)}{2}, \qquad \text{since } e^{-y}\leq 1 \text{ for } 0\leq y\leq\sqrt{3}$$

$$= \frac{7(3\ln 2+\pi)}{6}.$$

Also,

$$L = \int_0^{\pi/3} 2\,d\theta = \frac{2\pi}{3}.$$

Hence

$$ML = \frac{7(3\ln 2+\pi)\pi}{9}.$$

Example 3. Find the upper bound of the integral

$$\left|\int_{C_n} \frac{1}{z^2\cos z}\,dz\right|,$$

where $C_n = \{(x, y), x = \pm n\pi, y = \pm n\pi\}$, $n = 1, 2, \ldots$

The relation $|\cos z|^2 = \cos^2 x + \sinh^2 y$ in Section 3.3 ensures that on the horizontal sides

$$|\cos z| \geq |\cos x| \geq 1,$$

while on the vertical sides of the square

$$|\cos z| \geq |\sinh y| \geq \sinh \pi$$

Combining both the inequalities we reach at

$$2|\cos z| \geq 1 + \sinh \pi$$

for all points z on the boundary of the square (see Fig. 4.2).

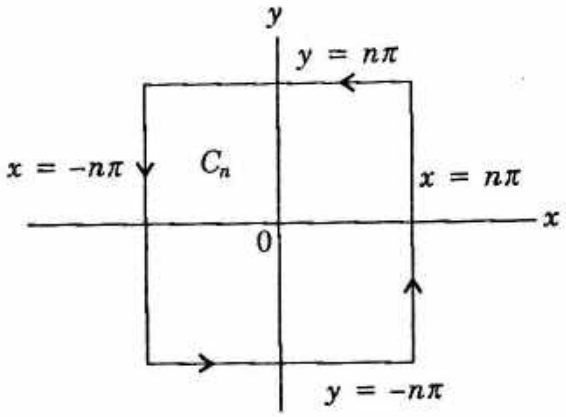

Figure 4.2

The perimeter of the square is $8n\pi$ and $|z| \geq n\pi$ on the boundary of the square. Thus, in view of the ML inequality,

$$\left| \int_{C_n} \frac{1}{z^2 \cos z} dz \right| \leq \frac{8n\pi}{n^2\pi^2} \frac{2}{1+\sinh \pi} = \frac{16}{n\pi(1+\sinh \pi)}. \tag{4.9}$$

Inequality (4.9) leads to an important result

$$\lim_{n \to \infty} \int_{C_n} \frac{1}{z^2 \cos z} dz = 0.$$

If the equation $\gamma(t)$ $(a \leq t \leq b)$ represents a smooth arc and $\gamma'(t) \neq 0$ in $a < t < b$ then the unit tangent vector

$$T = e^{i\theta} = \frac{\gamma'(t)}{|\gamma'(t)|}. \tag{4.10}$$

is well defined for all t in the open interval (a, b), with the angle of inclination $\arg \gamma'(t)$. When T turns, it does so continuously as the parameter t varies over the interval $a < t < b$. Here, for a particular value of t, $\gamma(t)$ is interpreted as a radius vector. Hence, the direction of the tangent is determined by $\arg \gamma'(t)$ at each point of the open interval.

Note that, if $\gamma'(t_0) = 0$ in (4.10), the tangent at z_0 is not defined. In case $\gamma'(t_0) = 0$ and if the direction of the tangent at t_0 exists, then it is given by the limit

$$T = \lim_{t \to t_0} \frac{\gamma'(t_0)}{|\gamma'(t_0)|}.$$

Example 4. Let a curve $\gamma(t)$ be defined by

$$\gamma(t) = t + i(t+1)^2, \qquad -2 \le t \le 2.$$

Then find the direction of the tangent at $t = 0$.

Since $\gamma'(t) = 1 + 2i(t+1)$ and $\gamma(0) \neq 0$, the direction of tangent at $t = 0$ is given by

$$T = e^{i\theta} = \frac{\gamma'(0)}{|\gamma'(0)|} = \frac{1+2i}{\sqrt{5}}.$$

Thus, the angle between the direction of tangent and the positive real axis is $\tan^{-1}2$.

To supplement this section we incorporate two more different definitions on contour integrals.

$$\int_\gamma f(z)\,|dz| = \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt, \tag{4.11}$$

$$\int_\gamma f(z)\,\overline{dz} = \int_\gamma \overline{f(z)\,dz}.$$

Equation (4.11) defines an integral with respect to arc length. All properties of contour integral (4.8) are followed by (4.11), except property (2).

Writing $f(z) = 1$ in (4.11), the length of γ defined in (4.7) can also be written as

$$L = \int_\gamma |dz|.$$

The points on any simple closed contour γ are the boundary points of two different domains, one of which is the interior of γ and is bounded. The other, which is the exterior of γ, is unbounded. This is the *Jordan curve theorem*, and its proof is not given here.

PROBLEM SET 4.1

1. Suppose $w(t) = u(t) + iv(t)$ is a continuous complex function defined on the interval $[-a, a]$, show that

$$\int_{-a}^{a} w(t)\,dt = \begin{cases} 2\displaystyle\int_0^a w(t)\,dt, & \text{when } w(t) \text{ is even,} \\ 0, & \text{when } w(t) \text{ is odd.} \end{cases}$$

2. Let $w(t)$ be a continuous complex function defined on the interval $[a, b]$. By considering the simple function $w = e^{it}$ on the interval $[0, 2\pi]$, show that it is not always true that there is a complex number c in the interval $a < t < b$ such that

$$\int_a^b w(t)\,dt = w(c)(b-a).$$

This shows that the mean value theorem for definite integrals in real calculus does not apply to complex-valued functions.

3. What curves are represented by the following functions?

(a) $(1+2i)t,\ 0 \le t \le 4$; (b) $1-i-2e^{it},\ 0 \le t \le \pi$;

(c) $t+it^2,\ -1 \le t \le 2$; (d) $\cos t + 2i \sin t,\ -\pi < t < \pi$.

4. Represent the following functions in the form $z = z(t)$:

(a) $|z-3+4i| = 4$; (b) $y = 1/x$ from $(1, 1)$ to $(3, 1/3)$;

(c) $y = x^2$ from $(0, 0)$ to $(2, 4)$; (d) $4(x-1)^2 + 9(y+2)^2 = 36$.

5. Write the parametric equation of the contour $C = C_1 + C_2$, where C_1 is circular arc in first quadrant joining 2 and $2i$, and C_2 is the line segment in second quadrant joining $2i$ and -2.

6. If γ is the contour defined by $\gamma(t) = x(t) + iy(t)$, $a \le t \le b$, show that there exists a contour γ_1 defined on $[0, 1]$ such that

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz.$$

7. Evaluate $\int_\gamma f(z)dz$, where γ is the arc from $z = -1-i$ to $z = 1+i$ consisting of a line segment from $(-1, -1)$ to $(0, 0)$ and portion of the curve $y = x^3$ from $(0, 0)$ to $(1, 1)$, and

$$f(z) = \begin{cases} 1, & y < 0, \\ 4y, & y > 0. \end{cases}$$

8. Find the value of the integral

$$\int_0^{1+2i} (3x^2 - y + ix^3)dz$$

along the real axis from $z = 0$ to $z = 1$ and then, along a line parallel to the imaginary axis from $z = 1$ to $z = 1+2i$.

9. Evaluate $\int_\gamma \bar{z}\,dz$, when

(a) γ is the parabola $y = x^2$ from 0 to $1+i$;

(b) γ is the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ between the points $(0, 0)$ and $(a\pi, 2a)$.

10. Let C_ρ denote the circle $|z| = \rho (0 < \rho < 1)$ with positive orientation, and suppose that $f(z)$ is analytic in the disc $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, there is a constant $M > 0$ independent of ρ such that

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \le 2\pi M \sqrt{\rho}\,.$$

11. Find the upper bound of the integral (without evaluating the integral)

$$\left| \int_C \frac{(z-1)e^{2z} \operatorname{Log} z}{z^2 - 7}\, dz \right|, \qquad C = \left\{ z : z = e^{i\theta}, \frac{\pi}{3} \le \theta \le \frac{\pi}{2} \right\}.$$

12. Let C denote the circle $|z| = 2$, described in counterclockwise direction. Considering the branch $\log z = \ln|z| + i\theta$ $(|z| > 0,\ 0 < \theta < 2\pi)$, show that

$$\left| \int_C \frac{z^{-i}}{z^2 - 1}\, dz \right| \le \frac{4\pi e^{2\pi}}{3}\,.$$

13. Let C be the arc of the circle $|z| = 3/2$ from $z = (3/2)i$ to $z = -3/2$ that lies in the second quadrant. Without calculating the integral, find the value of A such that

$$\left| \int_C \frac{3z - 2e^{5i\bar{z}}}{3z^2 + 2}\, dz \right| \le A.$$

14. Find the upper bound of the integral

$$\left| \int_{C_n} \frac{1}{z^2 \sin z}\, dz \right|,$$

where $C_n = \{(x, y),\ x = \pm(n + 1/2)\pi,\ y = \pm(n + 1/2)\pi\}$, $n = 0, 1, \ldots$ Also, verify that the value of the integral tends to zero as $n \to \infty$.

15. Use the ML-inequality to show that

$$\left| \int_C \frac{z}{z^2 + 1}\, dz \right| \le \frac{1}{2},$$

where C is the straight line segment from 2 to $2 + i$.

Suggestion. Note that $|z| \le \sqrt{5}$ for all z on the segment. Also, $|z^2+1| = |z+i||z-i|$, and now take help of the geometric argument.

16. Suppose $P(z)$ and $Q(z)$ are polynomials of degree n and m, respectively, such that $m \ge n+2$. Then show that

$$\lim_{R\to\infty} \int_C \frac{P(z)}{Q(z)}\, dz = 0,$$

where C is the circle $|z| = R$.

Suggestion. Use the ML inequality.

17. Prove the following assertions:

(a) $\displaystyle\int_{-\gamma} f(z)|dz| = \int_{\gamma} f(z)|dz|$; (b) $\displaystyle\left|\int_{\gamma} f(z)\,dz\right| \le \int_{\gamma} |f(z)||dz|$.

and using (a) verify that

$$\int_{|z|=1} |z-1||dz| = 8$$

18. Explain how contour integrals studied in complex analysis and line integrals in calculus are different. How are they similar?

19. Let γ: $[-2, 1] \mapsto \mathbb{C}$ be defined by $\gamma(t) = (1+i)(1+t)^2$. Then, find the direction of the tangent at the origin and the point -1.

20. Give an example of a curve $\gamma(t)$ such that $\gamma'(0) = 0$, even then the tangent to $\gamma(t)$ at the origin has a definite direction.

4.3 PRIMITIVES

It has been observed in Example 1, that the function

$$f(z) = y + x - i3x^3$$

has different integral values along different paths joining $z_1 = 0$ and $z_2 = 1+i$, while the function

$$g(z) = x + (y+2)i$$

has the same integral values along different paths joining these points. Why this happens is the outcome of our next proposition.

Proposition 3. Suppose that a function f is continuous in a domain D. Then the following statements are equivalent:

(a) f has antiderivative (primitive or indefinite integral) F in D;

(b) the integrals of $f(z)$ along any path (lying entirely in D) between any two fixed points in D is independent of path;

(c) the integral of $f(z)$ along every closed contour in D is zero.

Proof. (a) $\Rightarrow$ (b). Let us first assume that

$$\gamma = x(t) + iy(t), \text{ where } a \le t \le b$$

is a smooth path such that $\gamma(a) = z_0$ and $\gamma(b) = z_1$. Given that $F'(z) = f(z)$ for all $z \in D$. By definition

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b F'(\gamma(t))\gamma'(t)dt.$$

Now, using the first fundamental theorem of integral calculus (see Section 4.1)

$$\int_\gamma f(z)dz = [F(\gamma(t))]_a^b = F(\gamma(b)) - F(\gamma(a)) = F(z_1) - F(z_0),$$

i.e. the integral is independent of path. Thus, we have proved (b) for a smooth closed path in D. This result is, of course, true for any piecewise smooth closed path (contour) lying in D. A contour γ may be considered as made of a finite number of smooth arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$ joined end to end such that each γ_k is a smooth arc which connects z_{k-1} to z_k, $k = 1, 2, \ldots, n$. Thus,

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz = \sum_{k=1}^{n} [F(z_k) - F(z_{k-1})] = F(z_n) - F(z_0).$$

Again, the integral is independent of path, and hence (b) is established from (a).

(b) $\Leftrightarrow$ (c). Let γ_1 and γ_2 be two contours lying in D and which connect two fixed points z_1 and z_0 as illustrated in Fig. 4.3. Then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz \Leftrightarrow \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = 0$$

or

$$\int_{\gamma_1} f(z)dz + \int_{-\gamma_2} f(z)dz = 0 \Leftrightarrow \int_{\gamma_1-\gamma_2} f(z)dz = 0,$$

where $\gamma_1 - \gamma_2$ is a closed contour.

(c) $\Rightarrow$ (a). To prove this we show that (c) $\Rightarrow$ (b) $\Rightarrow$ (a). Let γ be any closed contour in D such that $\int_\gamma f(z)\,dz = 0$. Suppose z_1 and z_0 are any two fixed points on γ and let γ_1 and γ_2 be two different paths joining these points, (see Fig. 4.3). Now, $\gamma_1 - \gamma_2$ is a closed path and hence the analysis of assertion (b) implies

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

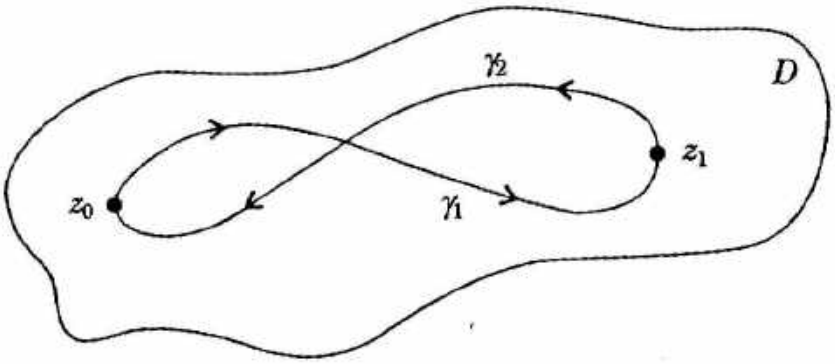

Figure 4.3

Thus, the integral is independent of path and it depends only on the position of the points. So, it is justified to define

$$F(z) = \int_{z_0}^{z} f(s)ds, \tag{4.12}$$

where z is an arbitrary but fixed point in D. Now, take a line segment as shown in Fig. 4.4, joining z and $z + \Delta z$.

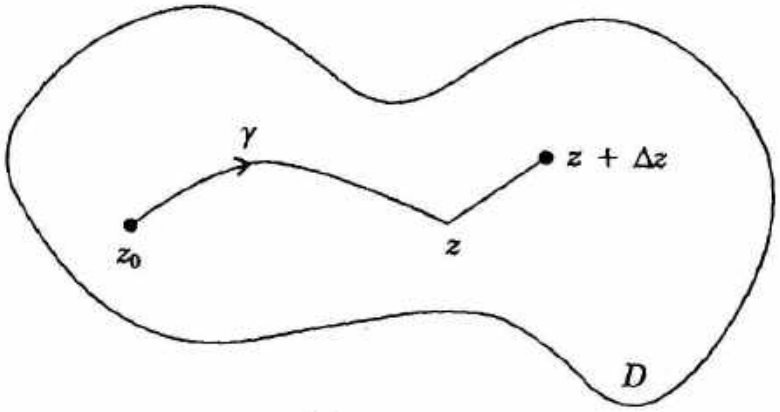

Figure 4.4

Then

$$f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)\, ds. \tag{4.13}$$

From (4.12) and (4.13), we have

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(s) - f(z)]ds.$$

Since $f(z)$ is continuous in D, for given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(s) - f(z)| < \varepsilon \qquad \text{whenever } |s - z| < \delta. \tag{4.14}$$

Consequently, if $z + \Delta z$ is close enough to z so that $|\Delta z| < \varepsilon$, in view of (4.14),

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \varepsilon \quad \text{whenever } |\Delta z| < \delta$$

or

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z).$$

Since z is arbitrary, $F'(z) = f(z)\ \forall\ z \in D$, it follows that f has the antiderivative F in D.

4.4 CAUCHY-GOURSAT THEOREM

This section is mainly devoted to the proof of the well-known Cauchy–Goursat theorem. Later sections explain its extensions in different situations and also explain its various applications.

Basically, Cauchy proved the following result:

Proposition 4. Let $f(z)$ be analytic in a domain D and f' be continuous in D. Then

$$\int_\gamma f(z)dz = 0$$

for every simple closed contour γ in D.

Proof. Let $[a, b]$ be the parametric interval of γ. Then, by definition of the contour integral

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Now, $f(z) = u(x, y) + iv(x, y)$ and $\gamma(t) = x(t) + iy(t)$, the integrand on the right in the above expression is the product of $u[x(t), y(t)] + iv[x(t), y(t)]$ and $x'(t) + iy'(t)$, and hence

$$\int_\gamma f(z)dz = \int_a^b (ux' - vy')dt + i \int_a^b (uy' + vx')dt$$

or

$$\int_\gamma f(z)dz = \int_\gamma u\,dx - v\,dy + i \int_\gamma v\,dx + u\,dy. \tag{4.15}$$

Since f' is continuous, the first order partial derivatives of u and v are continuous. By using the Green's theorem, integral (4.15) can be written as

$$\int_\gamma f(z)dz = \iint_R (-v_x - u_y)\,dx\,dy + i \iint_R (u_x - v_y)dx\,dy, \tag{4.16}$$

where R is the region enclosed by γ.

In view of the Cauchy–Riemann equations: $u_x = v_y$, $u_y = -v_x$ and therefore the value of the integral on the right side of (4.16) is zero.

Later on, Goursat dropped the condition of continuity for f' and proved the result which is widely known as the Cauchy–Goursat theorem.

Simply and multiply connected domains

Definition 5. A domain D is said to be *simply connected* if every simple closed contour in D contains points of D only.

Geometrically, every simple closed contour in D can be squeezed to a point without going outside the domain.

A set interior to a simple closed contour and a set of interior points of a circle or ellipse or rectangle are simply connected domains. Moreover, the infinite strip $1 < \text{Re}\, z < 2$ is also simply connected.

Definition 6. A domain which is not simply connected is called the *multiply connected* domain.

The annular region between two concentric circles is multiply connected.

From now onwards, Ω is an open set in the complex plane which is simply connected unless otherwise mentioned specifically.

Remark. Note that the annular domain $\{z : 1<|z|< 2\}$ is multiply connected, however, the domain $\{z : 1 < |z| < 2, \text{Re}\, z \notin (1, 2)\}$ is simply connected.

The Cauchy–Goursat theorem is stated as: "If a function $f(z)$ is analytic in a domain Ω, then

$$\int_\gamma f(z)dz = 0 \tag{4.17}$$

for every simple closed contour γ in Ω".

We prove this result in stages. First, we take domain as a triangle in the complex plane and the extend the result to any convex polygon. Finally, we reach any simple connected domain.

Proposition 5. Let f be analytic in a domain D containing a plane triangle Δ. Let $\partial\Delta$ be the boundary of the triangle. Then

$$\int_{\partial\Delta} f(z)dz = 0. \tag{4.18}$$

Proof. Let a, b, c be the vertices of Δ and let a', b', c' be the midpoints of the sides $[ab]$, $[bc]$ and $[ca]$, respectively. By joining the midpoints of the sides we subdivide the triangle into four congruent triangles. Consider the triangles ${\Delta_1}^k$, $k = 1, 2, 3, 4$, which are formed by the ordered triplets (a, c', b'), (a', b', c'), (c', b, a'), (a', c, b'), as shown in Fig. 4.5.

Since integrals for the sides of the triangle (a', b', c'), being in opposite direction cancel out in pairs, we have

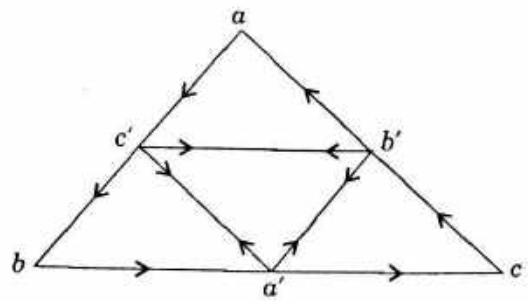

Figure 4.5

$$\int_{\partial\Delta} f(z)dz = \int_{\partial\Delta_1^1} f\,dz + \int_{\partial\Delta_1^2} f\,dz + \int_{\partial\Delta_1^3} f\,dz + \int_{\partial\Delta_1^4} f\,dz,$$

and further, by triangle inequality,

$$\left|\int_{\partial\Delta} f(z)dz\right| \le \left|\int_{\partial\Delta_1^1} f\,dz\right| + \left|\int_{\partial\Delta_1^2} f\,dz\right| + \left|\int_{\partial\Delta_1^3} f\,dz\right| + \left|\int_{\partial\Delta_1^4} f\,dz\right|.$$

Now, we choose an integral on the right that is having the largest absolute value and call it Δ_1 and its boundary (path) by $\partial\Delta_1$. So,

$$\left|\int_{\partial\Delta} f(z)dz\right| \le 4\left|\int_{\partial\Delta_1} f\,dz\right|.$$

Further, we subdivide the triangle Δ_1 as before and select a triangle Δ_2 of subdivision with boundary $\partial\Delta_2$ for which

$$\left|\int_{\partial\Delta_1} f(z)dz\right| \le 4\left|\int_{\partial\Delta_2} f\,dz\right|,$$

i.e.

$$\left|\int_{\partial\Delta} f(z)dz\right| \le 4^2\left|\int_{\partial\Delta_2} f\,dz\right|.$$

Continuing in this fashion, we get a sequence of triangles $\Delta_1, \Delta_2, \ldots$ such that $\Delta \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \ldots$ and

$$\left|\int_{\partial\Delta} f(z)dz\right| \le 4^n\left|\int_{\partial\Delta_n} f\,dz\right|, \qquad n = 1, 2, \ldots \tag{4.19}$$

Now, let L be the length of Δ. Then, the path $\partial\Delta_1$ has the length $L_1 = L/2$, the path $\partial\Delta_2$ has the length $L_2 = L_1/2 = L/4$, etc. Clearly, the length of $\partial\Delta_n$ is $L_n = (1/2^n)L$. Let z_0 be the point that belongs to each of these triangles, a fact guaranteed by the Cantor

intersection theorem which we shall not discuss here. Given that f is analytic at z_0, so for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon \quad \text{whenever } |z-z_0| < \delta$$

or

$$|f(z)-f(z_0)-(z-z_0)f'(z_0)| < \varepsilon|z-z_0| \quad \text{whenever } |z-z_0| < \delta.$$

Hence,

$$\int_{\partial\Delta_n} [f(z)-f(z_0)-(z-z_0)f'(z_0)]\,dz = \int_{\partial\Delta_n} f(z)\,dz - \int_{\partial\Delta_n} f(z_0)\,dz$$

$$- \int_{\partial\Delta_n} zf'(z_0)\,dz + \int_{\partial\Delta_n} z_0 f'(z)\,dz.$$

Now, $\partial\Delta_n$ is a closed path, z_0 is fixed (interior to Δ_n) and the derivative of z is continuous, we utilize Cauchy's theorem to conclude that all integrals on the right side are zero except $\int_{\partial\Delta_n} f(z)\,dz$. So,

$$\left|\int_{\partial\Delta_n} f(z)dz\right| = \left|\int_{\partial\Delta_n} f(z)-f(z_0)-(z-z_0)f'(z_0)dz\right|$$

$$\leq \int_{\partial\Delta_n} |f(z)-f(z_0)-(z-z_0)\,f'(z_0)|\,|dz|$$

$$\leq \varepsilon \int_{\partial\Delta_n} |z-z_0|\,|dz|.$$

We take n so large so that $\Delta_n \subseteq \{z : |z-z_0| < \delta\}$. In view of ML inequality (see Section 4.2), this implies

$$\left|\int_{\partial\Delta_n} f(z)dz\right| \leq \varepsilon L_n \cdot L_n = \varepsilon L_n^2.$$

Using this in (4.19) and substituting the value of L_n in terms of L, we have

$$\left|\int_{\partial\Delta} f(z)dz\right| \leq 4^n \left|\int_{\partial\Delta_n} f(z)dz\right| \leq 4^n \varepsilon L_n^2 = \varepsilon L^2.$$

Since $\varepsilon > 0$ is arbitrary, (4.18) holds.

Corollary 1. Let $f(z)$ be analytic in $\Omega\backslash\{p\}$ and continuous on Ω. Then

$$\int_\gamma f(z)\ dz = 0,$$

where γ is the parameterized boundary of a plane triangle contained in Ω.

Proof. Let a, b, c be the vertices of the triangle. If $p \notin \Delta$, then the assertion follows from Proposition 5. Moreover, if a, b, c are collinear then the assertion is trivially true. The point p may have different positions within and on Δ as shown in Fig. 4.6.

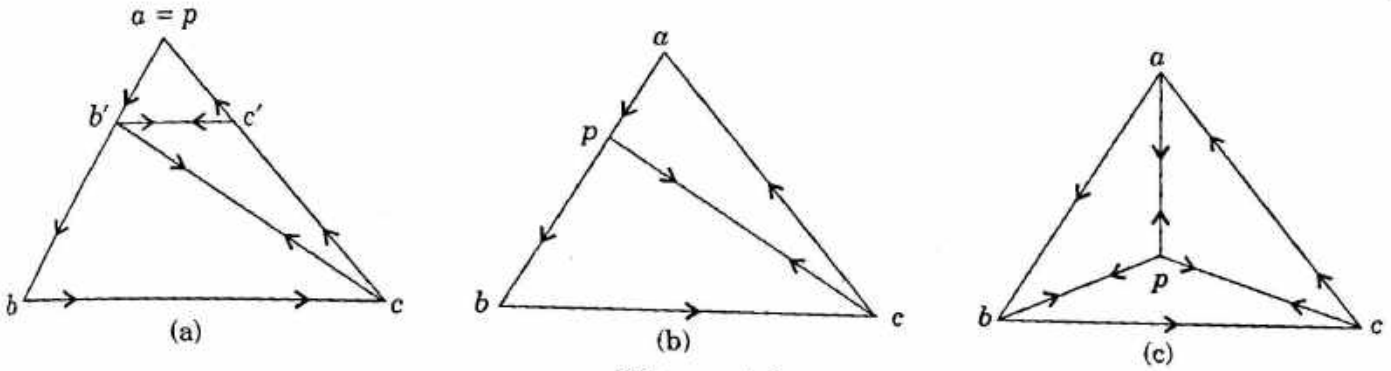

Figure 4.6

For Fig. 4.6(a), let p be on the vertex a. Take two points close to a as b' and c' on the sides $[ab]$ and $[ac]$, respectively. Then, we have

$$\int_{[abca]} f(z)dz = \int_{[ab'c'a]} f(z)dz + \int_{[b'bcb']} f(z)dz + \int_{[cc'b'c]} f(z)dz.$$

Given $f(z)$ is continuous at $a = p$, there exists a neighbourhood of p in which the function is bounded, i.e. $|f(z)| \le M$. Thus, for the first integral,

$$\left| \int_{[ab'c'a]} f(z)dz \right| \le M \times \text{perimeter of the triangle } ab'c'.$$

If b' and c' tend to p, the above integral reduces to zero and in view of Proposition 5,

$$\int_{[b'bcb']} f(z)dz = 0 \quad \text{and} \quad \int_{[cc'a'c]} f(z)dz = 0$$

Hence,

$$\int_{[abca]} f(z)dz = 0.$$

For Figs. 4.6(b) and 4.6(c), if p is on one side of or interior to Δ, then using arguments for Fig. 4.6(a), we have

$$\int_{[abca]} f(z)dz = \int_{[apca]} f(z)dz + \int_{[pbcp]} f(z)dz = 0$$

and

$$\int_{[abca]} f(z)dz = \int_{[abpa]} f(z)dz + \int_{[bcpb]} f(z)dz + \int_{[pcap]} f(z)dz = 0.$$

So, for all situations the integral around the triangle $abca$ is zero.

Remark. If Λ is a quadrilateral in a domain Ω, it can be divided into two triangles Δ and Δ' such that

$$\int_{\Lambda} f(z)dz = \int_{\Delta} f(z)dz + \int_{\Delta'} f(z)dz = 0.$$

In general, we have come to the conclusion that if $f(z)$ is analytic in $\Omega\backslash\{p\}$ and continuous on Ω, then (4.17) holds along each simple closed polygonal path Λ, since the simple closed polygon can be decomposed into triangles.

Thus, the Cauchy–Goursat theorem for a simple closed polygonal path is established. Our subsequent discussion is its extension for any piecewise smooth simple closed path (closed contour). Before doing this we prove an interesting result where the domain is an open convex set or open star-shaped domain (for definition see Section 1.5).

Theorem 1 (For open convex domain). Suppose Ω is a convex open set. Let f be analytic in $\Omega \backslash\{a\}$ and continuous on Ω. Then,

$$\int_{\gamma} f(z)dz = 0, \tag{4.20}$$

where γ is a simple closed contour contained in the domain.

Proof. Fix $a \in \Omega$. Since Ω is convex, every line segment joining a to any point z of the domain lies entirely in Ω. So, it is legitimate to define

$$F(z) = \int_{a}^{z} f(s)\, ds.$$

Let z_0 be arbitrary but a fixed point in the domain. If $z \in \Omega$, the triangle with vertices a, z, z_0 lies in Ω (see Fig. 4.7), the directed line segment joining a and z has been marked by $[az]$.

As a is one of the vertices of the triangle, Corollary 1 is used to have

$$F(z) - F(z_0) = \int_{[az]} f(s)ds - \int_{[az_0]} f(s)ds$$

$$= \int_{[az]} f(s)ds + \int_{[z_0 a]} f(s)ds$$

$$= \int_{[az]+[z_0a]} f(s)ds$$

$$= \int_{[z_0z]} f(s)ds, \quad \text{since} \int_{[az]+[zz_0]+[z_0a]} f(s)ds = 0$$

$$= \int_{[z_0z]} f(s)ds.$$

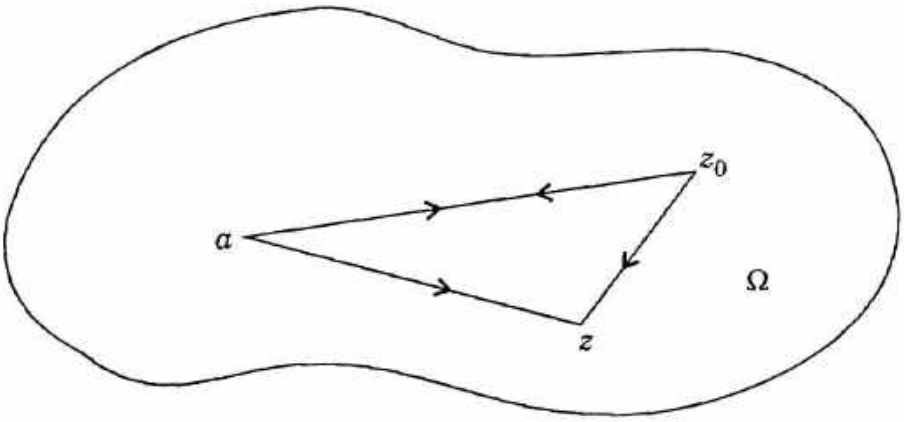

Figure 4.7

Hence,

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0z]} f(s)ds - \frac{f(z_0)}{z-z_0}\int_{[z_0z]} ds$$

$$= \frac{1}{z-z_0}\int_{[z_0z]} [f(s)-f(z_0)]ds.$$

Since f is continuous, given $\varepsilon > 0$, we can find $\delta > 0$ such that $|f(s)-f(z_0)| < \varepsilon$ whenever $|s-z_0| < \delta$. Thus, we have

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \le \frac{1}{|z-z_0|}\int_{[z_0z]} |f(s)-f(z_0)|\,|ds|$$

$$< \varepsilon \quad \text{if } |z-z_0| < \delta.$$

This ensures that $F'(z_0) = f(z_0)$, and moreover,

$$F'(z) = f(z) \ \forall\ z \in \Omega,$$

as z_0, being arbitrary but fixed.

So, $f(z)$ has an antiderivative, and immediate application of Proposition 3, Section 4.3 gives (4.20) for all simple closed contours γ in Ω.

Remark. In particular, theorem 1 is valid for $|z| < R$, since every open disc is a convex domain, and is also valid for star-shaped domains (Exercise 10, Problem Set 4.2).

Now, we are in a position to prove this theorem for a general case in which Ω may be any open simply connected set in the complex plane.

Theorem 2 (For open connected domain). Let $f(z)$ be analytic in a domain Ω. Then

$$\int_{\gamma} f(z)dz = 0,$$

where γ is the simple closed contour contained in the domain.

Proof. Let γ be a simple closed curve in Ω. Construct a closed polygon Λ in Ω with vertices $z_1, z_2, \ldots, z_n = z_0$ and each z_j, $j = 1, 2, \ldots, n$ lying on γ (see Fig. 4.8).

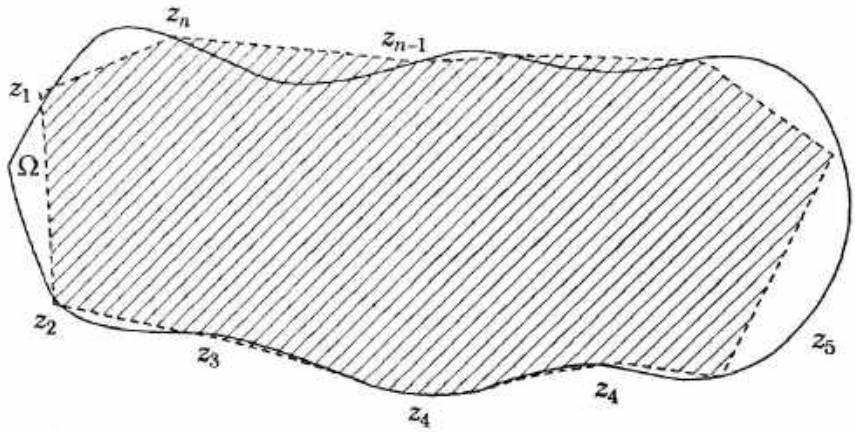

Figure 4.8

By the given condition, f is analytic within and on γ and hence for every $z \in \gamma$ there exists a $\delta > 0$ such that f is analytic on the closed disc $|z - z_0| \leq \delta$. Thus, f is analytic on a closed set $D_1 \subset \Omega$, where D_1 contains all the closed discs. Moreover, f is uniformly continuous on D_1. So, for given $\varepsilon > 0$ and sufficiently large n, we have

$$|f(z) - f(z_k)| < \frac{\varepsilon}{2L}, \qquad k = 1, 2, \ldots, n, \tag{4.21}$$

where z is on the line segment $[z_{k-1} z_k]$ and L is the length of γ. Further,

$$\begin{aligned}
\int_{\Lambda} f(z)dz &= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} f(z)dz \\
&= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [f(z) - f(z_k) + f(z_k)]dz \\
&= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [f(z) - f(z_k)]dz + \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}).
\end{aligned}$$

Since, by the remark following Corollary 1,

$$\int_{\Lambda} f(z)dz = 0,$$

it follows that

$$\sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [f(z) - f(z_k)]dz. \tag{4.22}$$

By (4.21) and (4.22), we have

$$\left|\sum_{k=1}^{n} f(z_k)(z_k - z_{k-1})\right| \leq \frac{\varepsilon}{2L} \sum_{k=1}^{n} |z_k - z_{k-1}| < \frac{\varepsilon}{2}.$$

Proceeding to limits as $n \to \infty$ and considering Remark (3), Section 4.2,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}) = \int_{\gamma} f(z)\, dz.$$

Since ε is arbitrary, (4.17) is established. This proves the theorem.

Corollary 2. Let f be analytic in a domain $\Omega\backslash\{a\}$ and continuous on Ω. Then

$$\int_{\gamma} f(z)\, dz = 0,$$

where γ is a simple closed contour contained in the domain.

Proof. Construct a simple polygonal path which has a on one of its vertices or on any side or interior to the path. Then, by the remark following Proposition 5 the integral along this polygonal path is zero, a fact used to prove Theorem 2. The rest of the proof proceeds on the same lines as in Theorem 2.

4.5 EXTENSIONS OF CAUCHY-GOURSAT THEOREM

So far we have considered the path of integration to be a simple closed contour and domain as simply connected. The following theorems are further extensions of the Cauchy–Goursat theorem to any closed contour and arbitrary domains.

Theorem 3. If a function $f(z)$ is analytic in a simply connected domain D, then

$$\int_{\gamma} f(z)\, dz = 0. \tag{4.23}$$

for every closed contour γ lying in D.

Proof. If γ is a simple closed contour and lies in D, the function f is analytic at each point within and on γ; and the Cauchy–Goursat theorem ensures that (4.23) holds.

Furthermore, if γ is closed and intersects itself a finite number of times, n (say), it consists of a finite number of simple closed contours, as shown in Fig. 4.9, for the case $n = 5$.

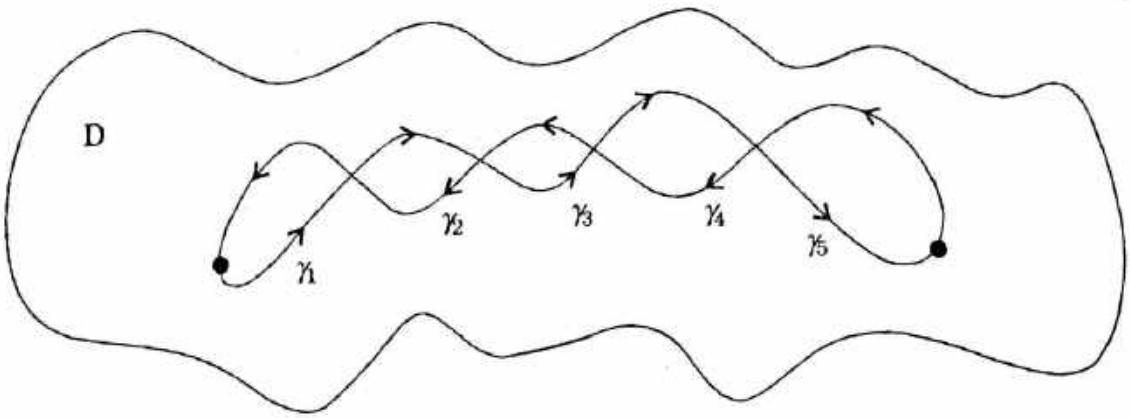

Figure 4.9

Applying the Cauchy–Goursat theorem to each of these simple closed contours, we have

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz + \cdots + (-1)^{k+1}\int_{\gamma_k} f(z)\,dz, \quad k = 1, 2, \ldots, n-1.$$

Each $\gamma_1, \gamma_2, \ldots, \gamma_n$ is a simple closed contour and the Cauchy–Goursat theorem ensures that the integral around each γ_k, $k = 1, 2, \ldots, n$ is zero and hence along γ, i.e.

$$\int_{\gamma} f(z)\,dz = 0.$$

The third case arises when we consider any closed contour intersecting itself at infinite number of points. But this is beyond the scope of this text.[1]

Corollary 3. If $f(z)$ is analytic throughout a domain D, then f has an antiderivative.

Proof. Given that $f(z)$ is analytic in D. Then for every closed contour γ in D.

$$\int_{\gamma} f(z)\,dz = 0.$$

Now, $f(z)$ is analytic in $D \Rightarrow f(z)$ is continuous in D. By Proposition 3, $f(z)$ has an antiderivative $F(z)$ in D such that $F'(z) = f(z)$ for all z in the domain.

Note. This corollary is very much useful for evaluating integrals along any path by using antiderivatives. It is a point of order (while evaluating integrals using antiderivatives) that $F(z)$ must also be analytic in a domain containing the path.

We have not proved Theorem 3 for a general path which intersects itself at infinite number of points. However, to support our argument we demonstrate the logic with an example illustrated as follows:

Example 5. Construct a path which intersects itself infinite number of times and then show that Theorem 3 is applicable.

1. For a detailed proof of this assertion, refer to *Complex Variables* by A.I. Markushevich.

Let us define the path C_1

$$y(x) = \begin{cases} x^3 \sin \dfrac{\pi}{x}, & 0 < x \le 1, \\ 0, & x = 0. \end{cases}$$

This means path $C_1 : z(x) = x + ix^3 \sin(\pi/x)$ and it is a continuous arc defined on $[0, 1]$, since

$$\left| x^3 \sin\left(\frac{\pi}{x}\right) \right| \le x^3 \to 0 \text{ as } x \to 0.$$

Moreover,

$$y'(x) = \begin{cases} 3x^2 \sin\left(\dfrac{\pi}{x}\right) - \pi x \cos\left(\dfrac{\pi}{x}\right), & 0 < x \le 1, \\ 0, & x = 0. \end{cases}$$

Thus, the arc is smooth as $y'(x)$ happens again to be continuous and intersects the positive x-axis at $z = 1, 1/2, 1/3, \ldots$, and origin $z = 0$ as depicted in Fig. 4.10.

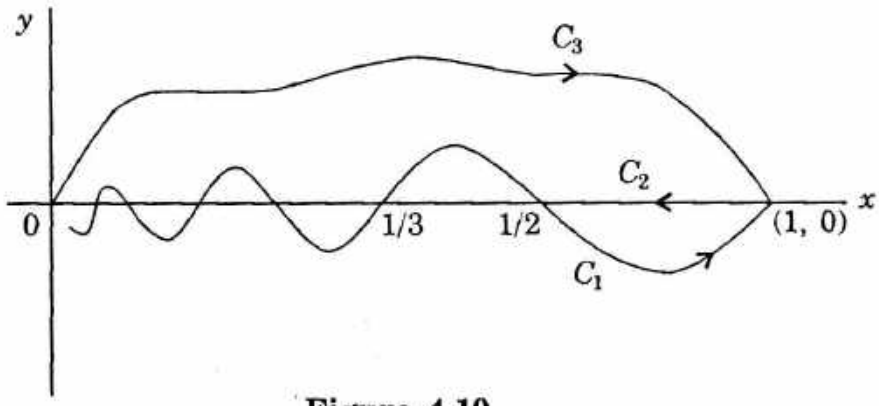

Figure 4.10

Let $C = C_1 + C_2$, where C_2 is part of the real axis directed from (1, 0) to the origin. Clearly, C is a closed contour intersecting itself infinite number of times. Suppose $f(z)$ is an analytic function in a domain containing C and C_3, where C_3 is a path above C connecting the origin and (1, 0). Now,

$$\int_{C_3} f(z)\, dz = -\int_{C_2} f(z)\, dz.$$

Also,

$$\int_{C_3} f(z)\, dz = \int_{C_1} f(z)\, dz.$$

Keeping in view these relations, we have

$$\int_C f(z)\, dz = \int_{C_1 + C_2} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz = 0$$

which verifies Theorem 3.

Example 6. Evaluate the integral $\int_C z^{1/2}\,dz$ $(r > 0,\ 0 < \theta < 2\pi)$, by using:

(a) Definition of the integral and that C is the arc: $\{z : z = 3e^{i\theta},\ 0 \le \theta \le \pi\}$;

(b) Antiderivative and that C is an arc in the upper half-plane that joins the points 3 and –3.

For part (a), although the branch $z^{1/2}$ $(r > 0,\ 0 < \theta < 2\pi)$ is not defined at the initial point $z = 3$ of C, the integral of this branch nevertheless exists. It is due to the reason that

$$f[z(\theta)] = \sqrt{3}\,e^{i\theta/2} = \sqrt{3}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right), \qquad z(\theta) = 3e^{i\theta}, \quad 0 \le \theta \le \pi$$

is piecewise continuous on $[0, \pi]$. If we define $f(3) = \sqrt{3}$, then by observing

$$\lim_{\theta\to 0^+} u(3,\ \theta) = \sqrt{3} \quad \text{and} \quad \lim_{\theta\to 0^+} v(3,\ \theta) = 0,$$

we conclude that $f(z)$ is piecewise continuous on $[0, \pi]$. Using (4.8), we have

$$\int_C z^{1/2}\,dz = \int_0^{\pi} \sqrt{3}\,e^{i\theta/2}\ 3ie^{i\theta} = 2\sqrt{3}\left[e^{i3\theta/2}\right]_0^{\pi} = -2\sqrt{3}\,(1 + i).$$

(b) Here, we use that branch which is analytic in the domain contained in upper half-plane, so that Corollary 3 may be used (see Fig. 4.11).

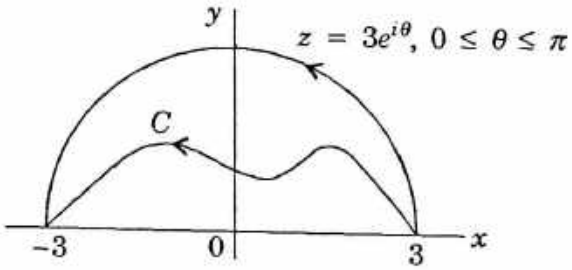

Figure 4.11

Let us take the branch

$$z^{1/2}\left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right).$$

This branch is analytic within and on γ and hence $z^{1/2}$ has an antiderivative $(2/3)z^{3/2}$. So,

$$\int_C z^{1/2}\,dz = \int_3^{-3} z^{1/2}\,dz = \frac{2}{3}\left[z^{3/2}\right]_3^{-3} = \frac{2}{3}\left[r\sqrt{r}\,e^{i3\theta/2}\right]_0^{\pi} = -2\sqrt{3}(1+i).$$

Example 7. Show that

$$\int_{-1}^{1} z^i\,dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where z^i denotes the principal branch: $z^i = \exp(i\ \mathrm{Log}\ z)$ $(|z| > 0,\ -\pi < \theta < \pi)$ and path

of integration is any contour from $z = -1$ to $z = 1$ except for its end points that lie above the real axis.

Taking the appropriate branch of z^i, i.e. $\{z : |z| > 0, -\pi/2 < \arg z < 3\pi/2\}$ and using the antiderivative, we have

$$\begin{aligned}
\int_{-1}^{1} z^i \, dz &= \left[\frac{z^{i+1}}{i+1}\right]_{-1}^{1} \\
&= \frac{1}{1+i}\left[\exp\left((1+i)\log z\right)\right]_{-1}^{1} \\
&= \frac{1}{1+i}\left[\exp\left((1+i)0\right) - \exp\left((1+i)i\pi\right)\right] \\
&= \frac{1+e^{-\pi}}{1+i} \\
&= \frac{1+e^{-\pi}}{2}(1-i).
\end{aligned}$$

In our next result, the Cauchy–Goursat theorem is extended in such a way that requires integrals along the boundary of a multiply connected domain.

Theorem 4. Suppose that (a) γ is a simple closed contour, described in the counterclockwise direction; (b) γ_k $(k = 1, 2, \ldots, n)$ are finite number of simple closed contours, all described in the clockwise direction, that are interior to γ and whose interiors are disjoint.

If a function $f(z)$ is analytic throughout the closed region consisting of all points within and on γ except for the points interior to each γ_k, then

$$\int_{\gamma} f(z)\, dz + \sum_{k=1}^{n} \int_{\gamma_k} f(z)\, dz = 0 \tag{4.24}$$

or

$$\int_{B} f(z)\, dz = 0,$$

where B is the oriented boundary of the domain so that points of the domain lie left when B is traversed.

Proof. To establish this we introduce a polygonal path L_1 consisting of a finite number of line segments joined end to end to connect the outer contour γ to the inner contour γ_1. We introduce another polygonal path L_2 which connects γ_1 to γ_2 and continue in this manner with L_{n+1} connecting γ_n to γ. From Fig. 4.12, it is clear that the simple closed contours Γ_1 and Γ_2 are formed, each consisting of polygonal paths L_k or $-L_k$ and pieces of γ and γ_k, and each is described in such a way that the points of the domain lie to the left when we move along the path.

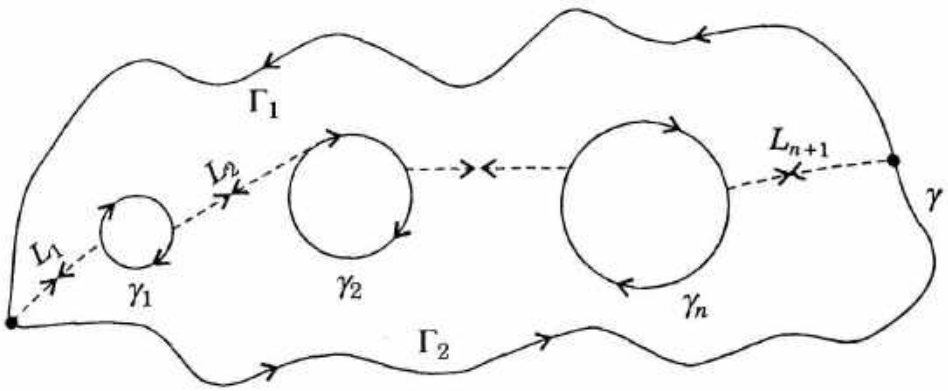

Figure 4.12

Applying the Cauchy–Goursat theorem to f on Γ_1 and Γ_2,

$$\int_{\Gamma_1} + \int_{\Gamma_2} = 0.$$

The integral along each L_k, being in opposite directions cancel each other and only the integrals along γ and γ_k remain, and therefore

$$\int_{\gamma} f(z)\,dz + \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz = 0$$

as asserted in (4.24).

Example 8. We claim that

$$\int_B \frac{e^{iz}}{(z^2+9)\sin z}\,dz = 0,$$

where B is the oriented boundary consisting of the circle $|z| = 2$ described in positive direction and the circle $|z| = 1$ having the negative orientation.

The singularities due to functions in the denominator are $z = n\pi$, $n \in \mathbb{I}$ and $z = \pm 3i$ which are outside B except 0 which is inside the inner circle $|z| = 1$. Hence, by the above theorem the integral around B is zero.

The following corollary is an important consequence of the above theorem. If a singularity falls interior to a simple closed contour then it can be continuously deformed into another simple closed contour without changing the value of the integral. This result is known as the *principle of deformation of path*.

Corollary 4. Let γ_1 and γ_2 denote positively oriented simple closed contours, where γ_2 is interior to γ_1. If $f(z)$ is analytic in the closed region consisting of these contours and all points between them, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof. Consider Fig. 4.13, and apply Theorem 4 for the contour $\gamma = \gamma_1 - \gamma_2$ to have the desired result.

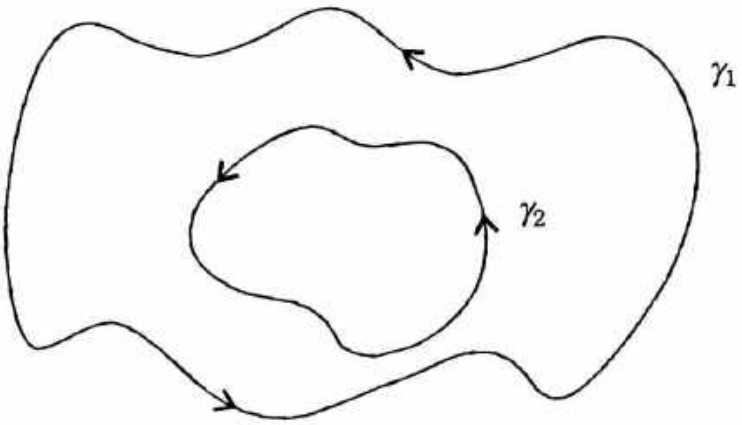

Figure 4.13

Example 9. Use the principle of deformation to show that for $n \in \mathbb{I}$,

$$\int_\gamma \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 2\pi i, & n=1, \\ 0, & n \neq 1. \end{cases}$$

where γ is any positively oriented simple closed contour enclosing z_0.

As z_0 is the singularity inside the path, we take a circle C, $z - z_0 = Re^{i\theta}$, $0 \le \theta \le 2\pi$, interior to γ. Using deformation of path, we have

$$\int_\gamma \frac{1}{(z-z_0)^n}\,dz = \int_C \frac{1}{(z-z_0)^n}\,dz.$$

Now the definition of contour integral is applied to evaluate the integral on the right side of the above equation. Note that $z'(\theta) = Rie^{i\theta}$.

For $n = 1$, the integral on the right side gives

$$\int_0^{2\pi} \frac{Rie^{i\theta}}{Re^{i\theta}}\,d\theta = 2\pi i,$$

whereas, for $n \neq 1$,

$$\int_0^{2\pi} \frac{Rie^{i\theta}}{R^n e^{in\theta}}\,d\theta = \frac{i}{R^{n-1}} \int_0^{2\pi} e^{-i(n-1)\theta}\,d\theta = \frac{1}{(n-1)\,R^{n-1}}\left[-e^{-i(n-1)\theta}\right]_0^{2\pi} = 0.$$

PROBLEM SET 4.2

1. Let $\log z = \ln r + i\theta \quad (r > 0,\ 0 < \theta < 2\pi)$.

(a) By using the appropriate branch, show that

$$\int_{-2i}^{2i} \frac{dz}{z} = -\pi i,$$

when the path of integration from $-2i$ to $2i$ is the left-hand half of the circle $|z| = 2$.

(b) Using the result in part (a), also show that

$$\int_C \frac{dz}{z} = 2\pi i,$$

where C is the positively oriented circle $|z| = 2$.

2. Let C denote the entire positively oriented boundary of the half-disc $0 \le r \le 1$, $0 \le \theta \le \pi$ and let $f(z)$ be a continuous function defined on that half-disc by writing $f(0) = 0$ and using the branch $f(z) = \sqrt{r}\, e^{i\theta/2}$ $(r > 0,\ -\pi/2 < \theta < 3\pi/2)$, of the multivalued function $z^{1/2}$. Show that

$$\int_C f(z)\, dz = 0$$

by evaluating separately the integrals of $f(z)$ over the semi-circle and the two radii which constitute C. Why is the Cauchy–Goursat theorem not applied?

3. Prove that

$$\int_{|z|=1} a^z dz = 0$$

for any single-valued branch of a^z.

4. For $\alpha \in \mathbb{R}$ and the circle C: $|z - z_0| = R$ (with positive orientation), show that

$$\int_C (z - z_0)^{\alpha-1} = \begin{cases} 2i\alpha^{-1}R^{\alpha} \sin \alpha\pi, & \alpha \neq 0, \\ 2\pi i, & \alpha = 0, \end{cases}$$

where the principal branch of integrand and the principal value of R^{α} are taken. Is the value of this integral affected by choice of the branch for the integrand? If so, compute its value for the branch $|z| > 0$, $0 < \arg z < 2\pi$.

5. If a function f is analytic in $\Omega \backslash \{a_1, a_2, \ldots, a_n\}$ and continuous on Ω, show that

$$\int_\gamma f(z)\, dz = 0,$$

where γ is the parameterized boundary of a plane triangle $\Delta \subset \Omega$.

Suggestion. Use the induction process in Corollary 2.

6. Let f be analytic in $\Omega\backslash\{a_1, a_2, \ldots, a_n\}$ and continuous on Ω, where Ω is open and convex. Let p and q be any two points in Ω and let γ be any contour joining these points. Then show that

$$\int_{\gamma} f(z)\,dz = \int_{[pq]} f(z)\,dz.$$

7. Suppose Ω is a convex open set in $\mathbb{C}$. Let f be analytic in $\Omega\backslash\{a_1, a_2, \ldots, a_n\}$ and continuous on Ω. Then

$$\int_{\gamma} f(z)\,dz = 0,$$

where γ is a simple closed contour contained in the domain.

8. Let D be star-shaped at a, and f be analytic in D. Then there exists an analytic function $F(z)$ in D such that $F'(z) = f(z)$ in D. In particular, (4.20) holds.

9. Can an open convex set be replaced by a simply connected domain in Exercise 7? If so, analyze the reasons.

10. Let $f(z) = (z^2 + 5z + 6)/(z - 2)$. Can the Cauchy–Goursat theorem be applied to evaluate $\int_C f(z)\,dz$

(a) when the path of integration C is a circle of radius 3 with origin as centre, and

(b) when C is a unit circle centred at origin?

Explain your answer in each case.

11. Apply the Cauchy–Goursat theorem to infer that $\int_C f(z)\,dz = 0$

(a) $\displaystyle\int_C \operatorname{Log}(z+2)\,dz$; (b) $\displaystyle\int_C \frac{e^{-z}}{z-\pi/2}\,dz$,

where $C = \{z : |z| = 1\}$.

12. Find the value of the integral

$$\int_{\gamma} \frac{1}{(z-2-i)^n}\,dz, \quad n \in \mathbb{I},$$

where γ is the boundary of the rectangular region: $0 \le x \le 3$, $0 \le y \le 2$, described in a positive sense.

13. Let B denote the boundary of the domain between the ellipse $\{z : |z-3| + |z+3| = 10\}$ and the rectangle whose sides lie along the lines $x = \pm 2$, $y = \pm 1$. Assume that B is oriented so that the points of the domain lie to the left of B. State why?

$$\int_B f(z)\,dz = 0,$$

when

(a) $f(z) = \dfrac{1}{2z^2+1}$; (b) $f(z) = \dfrac{z^3+1}{\sinh(z/2)}$.

14. If $f(z + w) = f(z) + f(w)\ \forall\ z, w \in \mathbb{C}$ and $f(z)$ is analytic in $\mathbb{C}$, show that $f(z) = cz$, where c is some constant.

15. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0$ and $a_0, a_1, \cdots, a_n$ are complex constants. Then verify that:

(a) $\displaystyle \frac{1}{2\pi i}\int_\gamma \frac{zP'(z)}{P(z)}\,dz = -\frac{a_{n-1}}{a_n}$; (b) $\displaystyle \frac{1}{2\pi i}\int_\gamma \frac{z^2P'(z)}{P(z)}\,dz = \frac{a_{n-1}^2 - 2a_n a_{n-2}}{a_n^2}$,

where γ is a simple closed contour enclosing all zeros of $P(z)$.

Suggestion. Use formal division and then Example 9.

4.6 WINDING NUMBER

Here, we introduce the concept of winding number (or index) of a closed path with respect to a point.

Definition 7. Let γ be a closed contour and z be a point not on the path γ. Then, the *index* or *winding number* $n(\gamma, z)$ of γ with respect to z is given by the integral

$$\eta(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{s-z}\,ds\,. \tag{4.25}$$

Geometrically speaking, the winding number counts the number of rounds of a path around the point. It is a positive integer for positive oriented contours and a negative integer for contours with negative orientation. If the point z is enclosed by a simple closed contour, then $\eta(\gamma, z) = 1$ (see Example 9) and, if z is outside the path, then $\eta(\gamma, z) = 0$.

The following two properties of the winding number are the direct consequences of the definition. If γ is the sum of two closed contours γ_1 and γ_2 in a complex plane, then for every $z \notin \gamma$,

(i) $\eta(\gamma, z) = \eta(\gamma_1, z) + \eta(\gamma_2, z)$; and

(ii) $\eta(-\gamma, z) = -\eta(\gamma, z)$.

Figure 4.14 shows how the index number changes with respect to the orientation and nature of the curves.

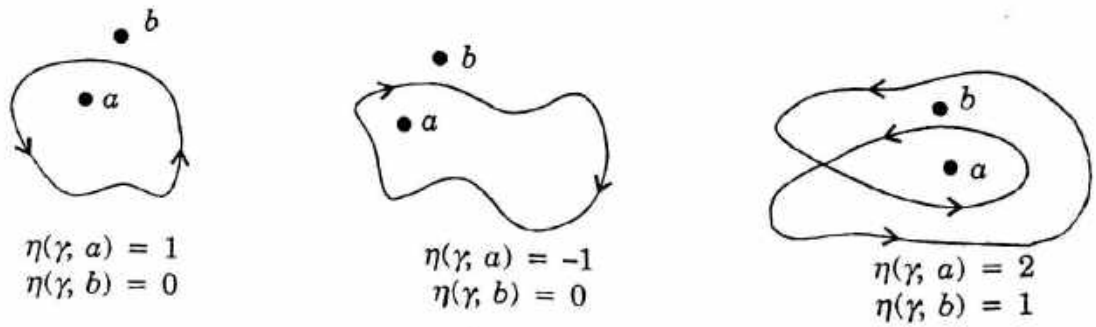

Figure 4.14

Proposition 6. Let γ be a closed contour and z be a point not on γ. Then, $\eta(\gamma, z)$ is an integer (depending on γ and on z).

Proof. Suppose γ has a domain $[a, b]$. Then, using the definition of contour integral

$$\int_\gamma \frac{1}{s-z}\,ds = \int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\,dt.$$

Define a complex-valued function on $[a, b]$ by the equation

$$F(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t)-z}\,dt, \quad a \le x \le b.$$

To prove the proposition, we must show that $F(b) = 2\pi i n$ for some integer n. Now, F is continuous on $[a, b]$ and has a derivative

$$F'(x) = \frac{\gamma'(x)}{\gamma(x)-z} \tag{4.26}$$

at each point of continuity of γ'. Therefore, the function defined by

$$G(t) = e^{-F(t)}[\gamma(t)-z], \qquad t \in [a, b]$$

is also continuous on $[a, b]$. Moreover, at each point of continuity of γ', using (4.26), we have

$$G'(t) = e^{-F(t)}\gamma'(t) - F'(t)e^{-F(t)}[\gamma(t)-z] = 0.$$

Therefore, $G'(t) = 0$ for each $t \in [a, b]$ except (possibly) for a finite number of points. By continuity, G is constant throughout $[a, b]$. Hence $G(b) = G(a)$. In other words, we have

$$e^{-F(b)}[\gamma(b)-z] = \gamma(a)-z.$$

Since $\gamma(b) = \gamma(a) \neq z$, we find

$$e^{-F(b)} = 1,$$

which implies $F(b) = 2\pi i n$, where n is any integer.

4.7 CAUCHY INTEGRAL FORMULA

The Cauchy integral formula exposes a remarkable fact about analytic functions. The values of the analytic function at all interior points to a simple closed contour in a domain are completely determined by its values on the contour. Moreover, the values of each derivative of an analytic function are also determined by the values of the function on the contour.

Theorem 5. Let $f(z)$ be analytic in a domain Ω and let γ be a simple closed contour in Ω, taken in the positive sense. Then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\,ds, \quad z \notin \gamma. \tag{4.27}$$

Proof. For $z \in \Omega\backslash\gamma$, define a function g on Ω as follows:

$$g(s) = \begin{cases} \dfrac{f(s)-f(z)}{s-z}, & s \neq z, \\ f'(z), & s = z. \end{cases}$$

Clearly, (i) g is continuous on Ω and the point z is removable discontinuity; (ii) g is analytic in $\Omega\backslash\{z\}$, being quotient of two analytic functions (see Fig. 4.15).

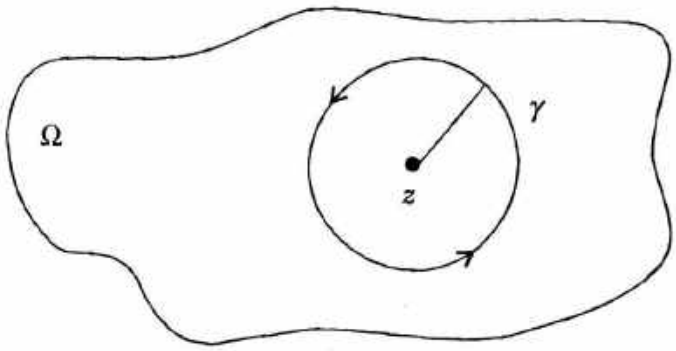

Figure 4.15

Hence, by Corollary 2,

$$\int_\gamma g(s)\, ds = 0.$$

Substituting for $g(s)$, we get

$$\int_\gamma \frac{f(s)-f(z)}{s-z}\, ds = 0,$$

which means

$$\int_\gamma \frac{f(s)}{s-z}\, ds - \int_\gamma \frac{f(z)}{s-z}\, ds = 0 \qquad \text{or} \qquad \int_\gamma \frac{f(s)}{s-z}\, ds = \int_\gamma \frac{f(z)}{s-z}\, ds\,.$$

Since γ is positively oriented and simple, we use Example 5, Chapter 4 to have

$$\int_\gamma \frac{f(s)}{s-z}\, ds = f(z)\int_\gamma \frac{1}{s-z}\, ds = 2\pi i f(z).$$

The proof is complete.

Remarks. 1. Under the hypothesis of Theorem 5, if γ is any closed contour, then (4.27) takes the form

$$\eta(\gamma, z) f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\, ds\,, \quad z \notin \gamma. \tag{4.28}$$

This immediately follows by inducting the definition of winding number (see (4.25)) in the last line of proof for the Cauchy integral formula.

2. If z is outside the closed contour γ, then $\eta(\gamma, z) = 0$ and the value of the integrals in (4.28) and hence in (4.27) is zero.

3. The Cauchy integral formula (4.27) can be extended to be applicable for multiply connected domains. Let us consider the case of a doubly connected domain bounded by two simple closed curves γ_1 and γ_2, each having positive orientation and γ_1 being inside γ_2. Take a point z interior to the domain. Suppose γ is a positively oriented simple closed contour enclosing the point z and is completely contained in the domain (see Fig. 4.16).

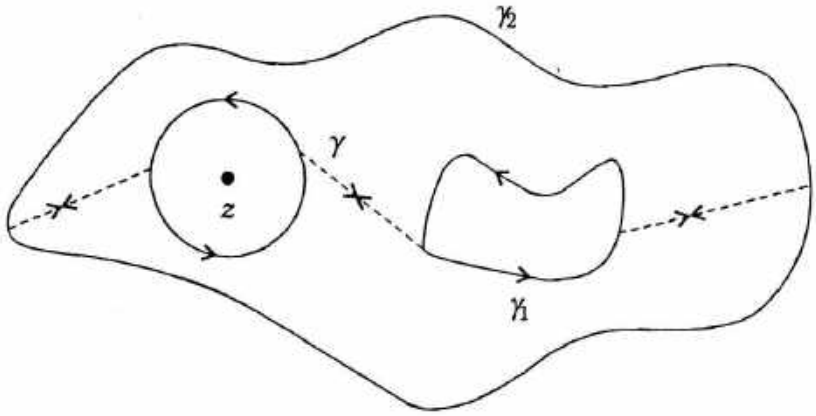

Figure 4.16

Observe that the function $f(s)/(s-z)$, where $s \in \gamma \cup \gamma_1 \cup \gamma_2$, is analytic throughout the closed region consisting of all points within and on γ_2 except for the points interior to γ_1 and at z interior to γ. Then, by the Cauchy–Goursat theorem for multiple connected domains (refer Theorem 4), we have

$$\int_{\gamma_2} \frac{f(s)}{s-z}\,ds - \int_{\gamma_1} \frac{f(s)}{s-z}\,ds - \int_{\gamma} \frac{f(s)}{s-z}\,ds = 0.$$

But according to the Cauchy integral formula, the value of the last integral is $2\pi i f(z)$. Hence,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(s)}{s-z}\,ds. \tag{4.29}$$

The expression (4.29) is the Cauchy integral formula for doubly connected domains.

Example 10. With the help of the Cauchy integral formula, evaluate the integral

$$\int_{|z|=1} \frac{\cos 2\pi z}{(2z-1)(z-3)}\,dz,$$

where $|z| = 1$ is a positively oriented circle.

As $z = 1/2$ is interior to the circle, we choose the integrand

$$f(s) = \frac{\cos 2\pi s}{2(s-3)}.$$

Comparing the integral

$$\int_{|s|=1} \frac{\cos 2\pi s}{2(s-3)} \frac{1}{s-1/2}\, ds$$

with (4.27), the value of the integral is $2\pi i f(1/2)$, i.e.

$$\frac{2\pi i \cos \pi}{2(1/2-3)} = \frac{2\pi i}{5}.$$

Derivatives of an analytic function

Unlike real functions, the derivatives of all orders of an analytic function are analytic. We shall elaborate this further now.

Theorem 6. Let $f(z)$ be analytic in a domain Ω and let γ be a simple closed contour in Ω, taken in a positive sense. Then, for all points z interior to γ,

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^2}\, ds. \tag{4.30}$$

Proof. Take any point z interior to γ and choose h such that $z+h$ is also interior to γ. To verify (4.30) it is sufficient to prove that for given $\varepsilon > 0$, their exists a $\delta > 0$ such that

$$\left| \frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^2}\, ds \right| < \varepsilon \quad \text{whenever } |h| < \delta.$$

Now, in view of the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{s-z}\, ds,$$

where z is any arbitrary but fixed point interior to γ, we have

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i h} \int_\gamma \left(\frac{1}{s-z-h} - \frac{1}{s-z} \right) f(s) ds$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{(s-z-h)(s-z)}\, ds.$$

Hence,

$$\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i} \int_\gamma \frac{f(s)ds}{(s-z)^2} = \frac{1}{2\pi i} \int_\gamma \left[\frac{1}{(s-z-h)(s-z)} - \frac{1}{(s-z)^2} \right] f(s)\, ds$$

$$= \frac{h}{2\pi i} \int_\gamma \frac{f(s)}{(s-z-h)(s-z)^2}\, ds$$

or

$$\left|\frac{f(z+h)-f(z)}{h}-\frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^2}\,ds\right| = \frac{|h|}{2\pi}\left|\int_\gamma \frac{f(s)}{(s-z-h)(s-z)^2}\,ds\right|. \tag{4.31}$$

Let d be the smallest distance of z from the path γ. Again, $f(s)$ is continuous on $\gamma \Rightarrow |f(s)| \le M$, $\forall\ s$ on γ and $|s-z-h| \ge \big||s-z|-|h|\big| = d-|h|$ and $|s-z| \ge d$. Using these observations in (4.31), we have

$$\left|\frac{f(z+h)-f(z)}{h}-\frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^2}\,ds\right| \le \frac{|h|\,ML}{(d-|h|)d^2},$$

where L is the length of γ. Hence,

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^2}\,ds.$$

Since z is arbitrary, the expression (4.30) is valid $\forall\ z$ interior to γ.

Theorem 7. Let $f(z)$ be analytic in a domain Ω and let γ be a simple closed contour in Ω, taken in positive sense. Then, for all points z interior to γ,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^{n+1}}\,ds, \qquad n = 0, 1, \ldots. \tag{4.32}$$

Proof. We prove the result by mathematical induction. For $n = 0$ and $n = 1$, (4.32) is Cauchy integral formula and Theorem 6, respectively. Assume that (4.32) holds for $n = m$. Let z be an arbitrary point interior to γ and $z + h$ be another point in some neighbourhood z such that $z + h$ also lies interior to γ. Note that

$$\begin{aligned}
\frac{f^m(z+h)-f^m(z)}{h} &= \frac{m!}{2\pi i h}\int_\gamma f(s)\left[\frac{1}{(s-z-h)^{m+1}}-\frac{1}{(s-z)^{m+1}}\right]ds \\
&= \frac{m!}{2\pi i h}\int_\gamma \frac{f(s)}{(s-z)^{m+1}}\left[\left(1-\frac{h}{s-z}\right)^{-m-1}-1\right]ds \\
&= \frac{m!}{2\pi i h}\int_\gamma \frac{f(s)}{(s-z)^{m+1}}\left[\frac{h(m+1)}{s-z}+\frac{(m+1)(m+2)h^2}{2!(s-z)^2}+\cdots\right]ds \\
&= \frac{m!}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^{m+1}}\left[\frac{m+1}{s-z}+\frac{(m+1)(m+2)h}{2!(s-z)^2}+\ldots\right]ds.
\end{aligned}$$

Taking the limit as $h \to 0$, we have

$$\lim_{h\to 0} \frac{f^{(m)}(z+h) - f^{(m)}(z)}{h} = \frac{m!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^{m+1}} \frac{m+1}{s-z}\, ds$$

or

$$f^{(m+1)}(z) = \frac{(m+1)!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^{m+2}}\, ds\,.$$

This proves (4.32) for $n = m + 1$, if it is true for $n = m$, and hence the result.

Summarizing the outcomes of Theorems 6 and 7, we conclude that "If a function is analytic at a point, then its derivatives of all orders are also analytic functions at that point". From this statement, we also conclude "If a function is analytic at a point, then the component functions u and v have continuous partial derivatives of all orders at this point".

Let $f(z) = u(x, y) + iv(x, y)$ be analytic at a point $z = (x, y)$. The analyticity of f' ensures the continuity of f' at (x, y) and

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

This explains the continuity of first-order partial derivatives at (x, y). Again, f'' is analytic and continuous at $z = (x, y)$ such that

$$f''(z) = u_{xx} + v_{xx} = v_{yx} - iu_{yx}$$

or

$$f''(z) = v_{xy} - iu_{xy} = -u_{yy} - iv_{yy}$$

and so on.

Example 11. If f is analytic within and on a simple closed contour γ and z_0 is a point interior to γ, prove that

$$\frac{1}{n!} \int_\gamma \frac{f^{(n)}(z)}{z - z_0}\, dz = \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

The function $f(z)$ is analytic within and on γ implies $f^{(n)}(z)$ is analytic within and on γ, and hence applying the Cauchy integral formula to $f^{(n)}$, we have

$$f^{(n)}(z_0) = \frac{1}{2\pi i} \int \frac{f^{(n)}(z)}{z - z_0}\, dz\,.$$

Comparing this result with the expression (4.32), the required result follows.

Theorem 8 (Morera). If a function $f(z)$ is continuous throughout a domain D and

$$\int_\gamma f(z)\, dz = 0$$

for every closed contour γ in D, then $f(z)$ is analytic throughout D.

Proof. Given that $f(z)$ is continuous in D. The condition $\int_\gamma f(z)\,dz = 0 \Rightarrow$ integral between any two points of D is independent of path. It is equivalent (see Section 4.3) to saying that $f(z)$ has antiderivative $F(z)$ in D, i.e.

$$F'(z) = f(z) \quad \forall\, z \in D.$$

This implies that $F(z)$ is analytic in D. Again, $F(z)$ is analytic in D and Theorm 6 ensures that F' is also analytic in D. But $F'(z) = f(z)$, and hence $f(z)$ is analytic in D.

Corollary 5. Suppose f is continuous on a domain Ω and analytic in $\Omega\backslash\{a\}$ for some $a \in \Omega$. Then f is analytic in Ω.

Proof. By Corollary 2,

$$\int_\gamma f(z)\,dz = 0$$

for every closed contour contained in Ω, and the rest of the result is immediate in view of Morera's theorem.

The section is winded up with the inclusion of an important result which has applications in conformal maps to be discussed in Chapter 9.

Proposition 7. Let f be analytic in a domain D and let $f'(z_0) \neq 0$ for some $z_0 \in D$. Then there exists a neighbourhoold of z_0 on which f is one-one (univalent).

Proof. Choose $R > 0$ such that the disc $|z - z_0| < R$ is contained in D. Since $f(z) - zf'(z_0)$ is the primitive of $f'(z) - f'(z_0)$, it follows that for any distinct z, z_0 in $|z_0 - z_0| < R$,

$$f(z) - f(z_0) = \int_{z_0}^{z} [f'(z) - f'(z_0)]dz + (z - z_0)f'(z_0)$$

The continuity of f' implies that for given $\varepsilon = |f'(z_0)|/2$, there is $\delta > 0$ such that

$$|f'(z) - f'(z_0)| < \frac{|f(z_0)|}{2} \qquad \text{whenever } |z - z_0| < \delta < R.$$

Using the previous expression and this inequality, we have

$$\begin{aligned} |f(z) - f(z_0)| &\geq |(z - z_0)f'(z_0) + \int_{z_0}^{z} |f'(z) - f'(0)|dz \\ &> |(z - z_0)|\,|f'(z_0)| - |z_0 - z_0|\,\frac{|f(z_0)|}{2} \\ &> |(z - z_0)|\,\frac{|f'(z_0)|}{2}. \end{aligned}$$

Since $f'(z_0) \neq 0$ and z and z_0 are distinct, the proposition follows.

Remark. We have also proved an additional property in Proposition 7 that $f(z) \neq 0$ in $|z - z_0| < \delta$.

PROBLEM SET 4.3

1. Evaluate the integrals

(a) $\displaystyle\int_C \frac{\cos e^z}{z(z+2)}\,dz$; (b) $\displaystyle\int_C \frac{\sinh z}{z^2(z-2)}\,dz$;

where $C = \{z : |z| = 1\}$, taken in the positive sense.

2. Compute

$$\int_\gamma \frac{e^{i\pi z}(z+i)^2 \cos nz}{z^2-1}\,dz,$$

where $\gamma = \{z : |z| = 2\cos\theta, -\pi/2 \le \theta \le \pi/2\}$.

3. Compute

$$\int_{|z|=1} \frac{|dz|}{|z-2|^2}.$$

Suggestion. Use $z\bar{z} = 1$ and $|dz| = -i\,dz/z$ and apply the Cauchy integral formula.

4. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{az}+\cosh bz}{z^{n+1}}\,dz = \begin{cases} \dfrac{a^n+2b^n}{n!} & n = 0, 2, 4, \ldots, \\[2ex] \dfrac{a^n}{n!}, & n = 1, 3, 5, \ldots, \end{cases}$$

where γ is any positively oriented closed contour around the origin.

5. Let

$$g(p) = \int_\gamma \frac{z^2-z+1}{z-p}\,dz,$$

where $\gamma = \{z : |z-3| + |z+3| = 10\}$ is the positively oriented simple contour. Then, write the expression for $g''(p)$ and hence determine $g''(2)$ and $g'(5i)$.

6. By using the integral representation for $f^{(n)}(z)$,

(a) show that

$$\frac{1}{2\pi i}\int_\gamma \frac{c^n e^{cz}}{n!\,z^{n+1}}\,dz = \left(\frac{c^n}{n!}\right)^2,$$

where γ is any positively oriented closed contour enclosing the origin.

(b) With the help of part (a), deduce

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2c\cos\theta}\,d\theta = \sum_{n=1}^{\infty}\left(\frac{c^n}{n!}\right)^2.$$

7. Let $f(z)$ be a continuous function in a domain Ω. For any positive integer n, define

$$g_n(z) = \int_\gamma \frac{f(s)}{(s-z)^n}\, ds, \qquad z \notin \gamma$$

where γ is any simple closed contour in D. Then show that $g_n(z)$ is analytic in Ω and satisfies

$$g_n'(z) = n g_{n+1}(z).$$

8. Let $u(x, y)$ be a harmonic function in a domain D. Then prove that the partial derivatives of all orders of $u(x, y)$ exist.

9. Legendre polynomial $P_n(z)$ is defined by

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2-1)^2.$$

(a) Show that

$$P_n(z_0) = \frac{1}{2^{n+1}\pi i} \int_\gamma \frac{(z^2-1)^2}{(z-z_0)^{n+1}}\, dz,$$

where γ is any positively oriented simple closed contour enclosing the point z_0. Also, verify that $P_n(z)$ satisfies

$$(1-z^2)\frac{d^2P_n}{dz^2} - 2z\frac{dP_n}{dz} + n(n+1)P_n = 0.$$

This is referred to as Schlaefli's representation for $P_n(z)$, or Schlaefli's formula.

(b) Prove that $P_n(z_0)$ may also be represented by

$$P_n(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \left(z_0 + \sqrt{z_0^2-1}\cos\theta\right)^n d\theta.$$

Suggestion. To prove the part (b), consider a circle at centre z_0 and radius $\sqrt{|z_0^2-1|}$. Then the equation for C is

$$|z-z_0| = \sqrt{|z_0^2-1|} \quad \text{or} \quad z = z_0 + \sqrt{|z_0^2-1|}\, e^{i\theta}, \qquad 0 \le \theta \le 2\pi.$$

Use this in part (a) to have

$$P_n(z_0) = \frac{1}{2^n \pi i} \int_0^{2\pi} \frac{\left[\left(z_0 + \sqrt{z_0^2-1}\, e^{i\theta}\right)^2 - 1\right]^n \sqrt{z_0^2-1}\, i e^{i\theta}}{\left(\sqrt{z_0^2-1}\, e^{i\theta}\right)^{n+1}}\, d\theta.$$

Further simplification gives the desired result.

10. Prove that the map $z \mapsto \eta(\gamma, z)$ is a continuous function for all $z \notin \gamma$, where γ is a closed curve.

11. Let f be continuous on a domain Ω and analytic in $\Omega\backslash\{a\}$ and let γ be a closed contour in Ω. Then, for all $z \notin \Omega\backslash\gamma$,

$$\eta(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\,ds.$$

12. Evaluate the integral

$$\int_C \frac{e^{\pi z}}{z(z+4)}\,dz,$$

where $C = \{z : z = 2e^{-i4\theta},\ 0 \le \theta \le 4\pi\}$.

13. If $f(z + w) = f(z)f(w)\ \forall\ z, w \in \mathbb{C}$ and $f(z)$ is analytic in $\mathbb{C}$ such that $f(z) \neq 0\ \forall\ z$, show that $f(z) = e^{cz}$, where c is some constant.

14. Let $f(z) = u + iv$ be an entire function such that $u_y(x, y) = 0$. Then show that $f(z)$ is of the form $Az + B$, where A is real and B is complex constant.

15. Let Ω be a simply connected domain and let $[a, b]$ be an interval on real line. Let $f(z, t)$ and its partial derivative $f_z(z, t)$ be continuous for all $z \in \Omega$ and all $t \in [a, b]$. Then

$$F(z) = \int_a^b f(z, t)dt$$

is analytic in Ω, and

$$F'(z) = \int_a^b f_z(z, t)dt.$$

This is the Leibnitz rule.[2] Use this rule to give a shorter and simple proof of Theorems 6 and 7.

16. Prove the Cauchy integral formula for multiple connected domains.

17. Let $C : |z| = 3$ be the positively oriented circle. Then, apply the Cauchy integral formula for multiple connected domain to evaluate the integral

$$\int_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)}\,dz.$$

18. Let D be a domain enclosed by a simple closed contour γ with positive sense and let $f(z)$ be analytic in exterior of D such that $\lim_{z\to\infty} f(z) = A$. Then prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\,ds = A, \qquad \text{if } z \in D$$

and

$$\frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\,ds = -f(z) + A, \qquad \text{if } z \notin D.$$

2. R. Nevalinna and V. Paaterio, *Introduction to Complex Analysis*, Addison-Wesley, Reading, Massachusetts.

This result is sometimes called the *Cauchy integral formula for unbounded domain*, and can be proved using the transformation $w = 1/z$, $\eta = 1/s$.

4.8 CONSEQUENCES OF THE CAUCHY INTEGRAL FORMULA

In this section, we derive some useful results which are direct consequences of the Cauchy integral formula.

Cauchy's inequality

Suppose $f(z)$ is analytic in a domain Ω and $C = \{z : |z-a| = R\}$ contained in Ω. Then

$$|f^{(n)}(a)| \leq \frac{n!\,M_R}{R^n}, \qquad n = 0, 1, 2, \ldots, \tag{4.33}$$

where $M_R = \max_{z\in C}|f(z)|$.

Proof. As an application of the Cauchy's integral formula, we have shown that if $f(z)$ is analytic in a domain Ω and C is the circle $|z-a| = R$ contained in Ω, then derivatives of all orders exist in interior of C, and hence (4.32) is applied to have

$$\begin{aligned}
|f^{(n)}(a)| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \\
&= \frac{n!}{2\pi}\left|\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \\
&\leq \frac{n!}{2\pi}\left|\int_C \frac{|f(z)|}{|(z-a)|^{n+1}}|\,dz|\right| \\
&\leq \frac{n!}{2\pi}\frac{2\pi R M_R}{R^{n+1}} \\
&= \frac{n!M_R}{R^n}.
\end{aligned}$$

Example 12. Let $f(z)$ be an analytic function within and on $|z-2| = 3$, taken in positive sense. Suppose $|f(z)|$ has maximum value 2 on $|z-2| = 3$, then find the upper bound of $|f^{(4)}(2)|$.

Using Cauchy's inequality (4.33), we have

$$|f^{(4)}(2)| \leq \frac{4!2}{3^4} = \frac{48}{81}.$$

Theorem 9 (Liouville). A bounded entire function is constant.

Proof. The function f is entire $\Rightarrow f$ is analytic for all points in the complex plane. Let z be any arbitrary point. Hence, by Cauchy's inequality,

$$|f^{(n)}(z)| \leq \frac{n!\,M_R}{R^n}.$$

Further, $|f(z)|$ is bounded means $M_R \leq M$. So,

$$|f'(z)| \leq \frac{n!\,M}{R}.$$

Since this is true for all $R > 0$, letting $R \to \infty$, we have

$$|f'(z)| = 0 \Rightarrow f'(z) = 0 \qquad \text{for all } z.$$

Hence, $f(z)$ is constant.

Corollary 6. A nonconstant entire function is unbounded.

Proof. This is the statement of Liouville's theorem in reverse direction.

Corollary 7. Let $f(z)$ be an entire function. If $u(x, y) = \text{Re } f(z)$ is bounded for all (x, y) in complex plane, then $u(x, y)$ and $v(x, y)$ are constant functions.

Proof. Let $|u(x, y)| \leq u_0$ for all (x, y) in $\mathbb{C}$. Consider the function

$$g(z) = e^{f(z)}.$$

Given that $f(z)$ is entire, implies that $g(z)$ is entire. Moreover, $|g(z)| = e^{u(x, y)} \leq e^{u_0}$, i.e. $|g(z)|$ is bounded $\Rightarrow g(z)$ is constant (by Liouville's theorem). Thus, $e^{f(z)}$ is constant, which means $f(z)$ is constant. Ultimately, $u(x, y)$ and $v(x, y)$ are constant functions.

Example 13. Suppose $f(z)$ and $g(z)$ are entire functions; $g(z)$ is never zero and $|f(z)| \leq |g(z)|$ for all z. Show that there is a constant c such that $f(z) = c\,g(z)$.

Let us define the function

$$h(z) = \frac{f(z)}{g(z)}.$$

Since $g(z) \neq 0$ for all z, it follows that $h(z)$ is well defined for every complex number and is an entire function, being quotient of two entire functions. Also, by the given hypothesis

$$|h(z)| = \left|\frac{f(z)}{g(z)}\right| = \frac{|f(z)|}{|g(z)|} \leq 1.$$

Thus, $h(z)$ is a bounded entire function. Now, by Liouville's theorem,

$$h(z) = c \text{ (some constant)} \Rightarrow \frac{f(z)}{g(z)} = c$$

and hence the result.

Theorem 10 (Fundamental theorem of algebra). Let

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0.$$

Then there exists a point z_0 in $\mathbb{C}$ such that $P(z_0) = 0$.

Proof. Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $n \geq 1$ and $a_n \neq 0$. Assume that $P(z)$ has no zeros in $\mathbb{C}$, i.e. $P(z) \neq 0$ for all points in complex plane. Consider the function

$$Q(z) = \frac{1}{P(z)}.$$

Since $P(z)$ is a polynomial of degree n, it is a nonconstant entire function, and hence, by Corollary 6, $P(z)$ is unbounded. This means that

$$\lim_{z\to\infty} P(z) \to \infty \qquad \text{or} \qquad \lim_{z\to\infty} Q(z) \to 0.$$

Thus, there is $R > 0$ (see Definition 3, Chapter 2) such that

$$|Q(z)| \le 1 \quad \text{whenever } |z| > R.$$

But, $Q(z)$ is bounded in $|z| \le R$, being continuous. This concludes that $Q(z)$ is bounded everywhere. By Liouville's theorem, $Q(z)$ is constant. This implies that $P(z)$ is constant. Thus, we arrive at a contradiction, and hence the result.

Corollary 8. A polynomial of degree n has exactly n zeros (roots) counting multiplicity.

Proof. The fundamental theorem of algebra asserts that a polynomial of degree n has at least one zero. As a consequence of this theorem we show that a polynomial of degree n has exactly n zeros with due regard to counting multiplicity. Consider the nth degree polynomial of the form:

$$P_n(z) = b_0 + b_1 z + \cdots + z^n, \qquad n \ge 1.$$

Since $P_n(z) = 0$ has at least one complex number z_1 such that $P_n(z_1) = 0$. By division algorithm there exists a polynomial of degree $(n-1)$ such that

$$P_n(z) = (z - z_1)P_{n-1}(z).$$

Thus, if $n > 1$ we can apply the fundamental theorem again to conclude that there is another z_2 such that $P_{n-1}(z_2) = 0$ and hence by the above reasoning we can factor out the linear factor $z - z_2$ from $P_{n-1}(z)$, leaving behind a polynomial $P_{n-2}(z)$. Proceeding in this way, we can express $P(z)$ uniquely as a product of linear factors

$$P_n(z) = \prod_{k=1}^{n} (z - z_k). \tag{4.34}$$

In case of multiple roots, $z_1, z_2, \cdots, z_n$ may not be distinct.

Let us prove some results concerning the zeros of a polynomial and its derivative. The concept of convex hull of a set will make the analysis more simple.

Definition 8 (Convex hull). The convex hull of a set D in $\mathbb{C}$ is the intersection of all convex sets containing D.

It is not difficult to show that the convex hull of D is the smallest convex set containing D (see Exercise 11, Problem Set 4.4). By convention, the convex hull of empty set is the set $\{0\}$. The convex hull of D is D itself provided D is convex.

Proposition 8 (Gauss). Let $P(z)$ be a polynomial of degree $n \ge 1$. Then every zero of $P'(z)$ also lies in the convex hull of the set of zeros of $P(z)$.

Proof. Using the factorization (4.34), we have for $z \ne z_k$,

$$\frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k} = \sum_{k=1}^{n} \frac{\overline{z - z_k}}{|z - z_k|^2}$$

or

$$\overline{\left(\frac{P'(z)}{P(z)}\right)} = \sum_{k=1}^{n} \frac{z - z_k}{|z - z_k|^2}.$$

Suppose $c \in \mathbb{C}$ is such that $p(c) \neq 0$ and $p'(c) = 0$, then the above equality implies

$$c = \frac{\sum_{k=1}^{n} z_k |c - z_k|^{-2}}{\sum_{k=1}^{n} |c - z_k|^{-2}}$$

or

$$c = \sum_{k=1}^{n} \lambda_k z_k, \quad \lambda_j = \frac{|c - z_j|^{-2}}{\sum_{k=1}^{n} |c - z_k|^{-2}}, \qquad j = 1, 2, \cdots, n.$$

This shows that if $z_1, z_2, \cdots, z_n$ are the zeros of $P(z)$, then every zero c of $P'(z)$ can be written as the convex linear combination of these zeros as

$$c = \sum_{k=1}^{n} \lambda_k z_k, \qquad \sum_{k=1}^{n} \lambda_k = 1, \quad \lambda_k \geq 0.$$

In case $P(c) = 0 = P'(c)$, the above construction is trivial. For this situation, take $\lambda_k = 1$ for $k = j$ and zero for the remaining indices.

Theorem 11 (Luca). If all zeros of a polynomial $P_n(z)$ lie in a half-plane, then the zeros of the derivative also lie in the same half-plane.

Proof. If $z_1, z_2, \ldots, z_n$ are the zeros of the polynomial $P_n(z)$, then by Proposition 8, all the zeros of $P'(z)$ are in the convex hull of the set $\{z_1, z_2, \cdots, z_n\}$. Given that $z_1, z_2, \cdots, z_n$ are in some half-plane. Every half-plane is convex and hence contains the smallest convex set, i.e. the convex hull of $\{z_1, z_2, \cdots, z_n\}$.

This proves the theorem.

Definition 9 (Mean-value property). A complex function $f(z)$ defined in a domain D is said to have mean value property, if for every $a \in D$ and $r > 0$ such that $N_r(a) = \{z : |z - a| \leq r\} \subseteq D$, and

$$\frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) dt = f(a). \tag{4.35}$$

This means that $f(a)$ is the mean value (average) of $f(z)$ on the circle $|z - a| = r$.

Theorem 12 (Gauss mean value theorem). If $f(z)$ is an analytic function in a domain D, then $f(z)$ has mean value property.

Proof. Let a be an arbitrary point in D such that $\{z : |z - a| = r, r > 0\} \subseteq D$. Applying the Cauchy integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}}\, rie^{it}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt.$$

Corollary 9. If $f = u + iv$ is analytic in a domain Ω, then the component functions u and v have mean value property in the domain, i.e.

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})dt \quad \text{and} \quad v(a) = \frac{1}{2\pi}\int_0^{2\pi} v(a+re^{it})dt.$$

Proof. The proof is an immediate consequence of (4.35) when we compare real and imaginary parts on both sides.

Remark. Very interesting to mention here is the fact that for any nonconstant analytic function $f(z)$, the function $|f(z)|$ does not satisfy the mean value property. This fact generated a vast scope for studying a larger class of functions.

PROBLEM SET 4.4

1. Let D be rectangular region $\{(x, y) : |x| \le 4,\ |y| \le 3\}$. Suppose f is analytic in D and satisfies $|f(z)| \le 1$ on ∂D, then show that

$$|f'(0)| \le \frac{14}{9\pi}.$$

2. Let f be analytic in $|z| < 5$, and suppose that $|f(s)| \le 10$ for all points on the circle $|s-1| = 3$. Find a bound for $|f^{(3)}(0)|$.
3. Let $f(z)$ be an analytic function in a domain D and let $a \in D$. Then show that

$$f^{(n)}(a) = \frac{1}{2\pi r^{n+1}}\int_0^{2\pi} f(a+re^{it})e^{-int}\,dt,$$

where $\{z : |z-a| \le r\} \subseteq D$.

4. Show that Cauchy's inequality reduces to equality if and only if $f(z)$ is a constant multiple of a power of z.
5. Give an alternative proof (without using Cauchy's inequality) of Liouville's theorem based on the Cauchy integral formula.

Suggestion. Consider the integral

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\,dz,$$

where $a \neq b$, $|a| < R$, $|b| < R$, and take the limit as $R \to \infty$.

6. Let $f(z)$ be analytic in an open unit disc $D : |z| < 1$ and let $C : |z| < \rho, 0 < \rho < 1$ be contained in D. Then verify the inequality

$$|f^{(n)}(0)| \leq e(n+1)!,$$

where $\max_{z\in C}|f(z)| = 1/(1-\rho)$.

Suggestion. By Cauchy's inequality,

$$|f^n(0)| \leq \frac{n!}{(1-\rho)\rho^n}.$$

Now, minimize the right side which occurs at $\rho = n/(n+1)$.

7. Let $f(z)$ be an analytic function in a domain Ω, and let a be an arbitrary point in Ω. Then establish the Cauchy–Hadamard inequality

$$\limsup_{n\to\infty} \sqrt[n]{\frac{|f^n(a)|}{n!}} \leq \frac{1}{d},$$

where $d = \inf\{|a-z| : z \in \partial\Omega\}$ is the distance of a from the boundary $\partial\Omega$.

8. Suppose $f(z)$ is an entire function such that $|f(z)| \leq |z|^2$, $f(1) = 2i$ and $f(2) = i$. Find $f(z)$.

9. Let $f(z)$ be an entire function which satisfies one of the conditions for all $z \in \mathbb{C}$:

(a) Re $f(z)$ or Im $f(z)$ has upper bounds;

(b) $|f(z) - a| \geq M$, $M > 0$, any complex α;

(c) Re $f(z)$ or Im $f(z)$ has no zeros;

(d) $f(z)$ has no zeros and $f(z) \to c \neq 0$ as $z \to \infty$.

Prove that $f(z)$ is constant.

10. The function $\sin x$ of the real variable possesses derivatives of all orders for every value of x. It is also bounded, since $|\sin x| \leq 1$ for all x; yet it is not constant. Explain this in the context of Liouville's theorem.

11. Let D be a nonempty set in $\mathbb{C}$. Then show that the convex hull of D is the smallest convex set containing D in $\mathbb{C}$.

12. Suppose $u(x, y)$ is a bounded harmonic function in the whole complex plane. Then show that $u(x, y)$ is constant.

13. Use the Cauchy integral formula to prove the Cayley–Hamilton theorem: A square matrix over $\mathbb{C}$ satisfies its own characteristic equation.[3]

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3. Refer to C.A. McCarthy, *Amer. Math. Monthly*, 82(1975), 390–391.

14. Suppose $a_0 + a_1 z + \cdots + a_n z^n = 0$, $a_n \neq 0$, is a polynomial equation with integer coefficients. Then show that a real rational number p/q (having no common factor) is a root of the equation if p and q are factors of a_0 and a_n, respectively.

15. Based on the result of the above problem, detect all the roots of the polynomial equation $36z^3 - 33z^2 - 23z + 10 = 0$.

16. Prove that

$$\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)d\theta = 2\pi.$$

Suggestion. Apply formula (4.35) to the function $\cos z$.

17. Prove that if $f(z)$ is a nonvanishing analytic function in a domain Ω, and if Ω contains the circle $C : |z - z_0| = \rho$ and its interior, then

$$\ln|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(z_0 + \rho e^{i\theta})|d\theta.$$

18. Prove that all the zeros of a polynomial $P(z)$ having negative real part and positive imaginary part ensure that all the zeros of $P'(z)$ lie in the second quadrant.

19. Find the mean value of $x^2 - y^2 + 2y$ over the circle $|z - 5 + 2i| = 3$.

20. Use Corollary 9 to evaluate

$$\int_0^{\pi} \ln|\sin\theta|d\theta.$$

Suggestion. Consider $f(z) = \log(1 + z)$ and the appropriate branch.

4.9 MAXIMUM MODULI OF FUNCTIONS

Let D be a bounded domain in $\mathbb{C}$ and let ∂D denote the boundary of D so that $\overline{D} = D \cup \partial D$ is a bounded and closed domain in the complex plane. First, we prove the maximum modulus theorem for continuous complex functions satisfying the mean value property. This is more general class of functions and includes analytic functions.

To prove the main result of this section, we need the following lemma.

Lemma. Let $g(x)$ be a real continuous function defined on $[a, b]$ such that $g(x) \geq 0$ $\forall$ $x \in [a, b]$. If

$$\int_a^b g(t)dt = 0,$$

then $g(x) = 0$ for all $x \in [a, b]$.

Proof. Define the function

$$\phi(x) = \int_a^x g(t)\, dt \quad \text{for all } x \in [a, b].$$

Clearly, $\phi(x)$ is differentiable and

$$0 \le \phi(x) \le \int_a^b g(t)\, dt = 0.$$

So, $\phi(x) = 0$ for all x and $\phi'(x) = 0$. Since $\phi'(x) = g(x)$, this concludes that $g(x) = 0$ for all x.

Theorem 13. If $f(z)$ is a nonconstant continuous complex function defined in a domain D and satisfies the mean value property, then $|f|$ does not attain its maximum value in D unless it is a constant.

Proof. Suppose $|f|$ attains its maximum value at some point $a \in D$, i.e.

$$|f(z)| \le |f(a)| \;\; \forall\, z \in D \tag{4.36}$$

Now, we show that f is constant in D. Since D is open, there exists $r > 0$ such that

$$C = \{z : |z-a| = r,\, r > 0\} \subset D.$$

Using (4.36) and the mean value property of $f(z)$, we note that

$$|f(a)| \le \left| \frac{1}{2\pi i} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta \right| \le \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{i\theta})|\, d\theta = |f(a)|$$

so,

$$\frac{1}{2\pi} \int_0^{2\pi} \left[|f(a)| - |f(a + re^{i\theta})| \right] d\theta = 0.$$

Further, in view of the Lemma, we have

$$|f(a + re^{i\theta})| = |f(a)| \qquad \text{for } 0 \le \theta \le 2\pi.$$

This equation holds for all circles $|z-a| = s$, $0 < s \le r$, and hence $|f(z)|$ is constant on $|z-a| \le r$. So far, we have proved that $f(z)$ is constant within and on circles centred at a and contained in D. Now, we prove that f is constant on the whole domain.

Take an arbitrary point b in D. Since D is open and connected, there exists a continuous function γ defined on $[0, 1]$ such that

$$\gamma(0) = a,\; \gamma(1) = b,\; \gamma(t) \in D$$

for all $t \in [0, 1]$, as shown in Fig. 4.17.

Form the composite function $g(t) = f(\gamma(t))$. Clearly,

$$g(0) = f(\gamma(0)) = f(a) \quad \text{and} \quad g(1) = f(\gamma(1)) = f(b).$$

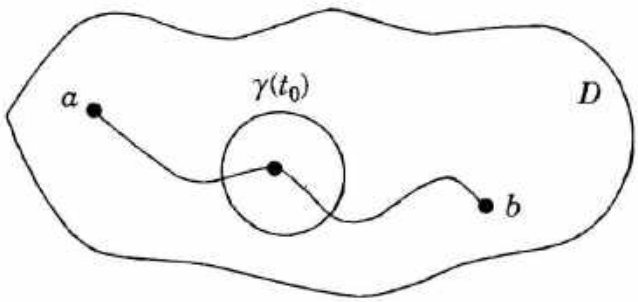

Figure 4.17

So, g is a continuous function connecting $f(a)$ and $f(b)$.

Consider the set

$$E = \{t : f(\gamma(t)) = f(a),\ t \in [0, 1]\}.$$

Let t_0 be the supremum of E. Then $f(\gamma(t_0)) = f(a)$ is obvious by the following reasoning: Take a sequence $\{t_n\}$ in E such that $t_n \to t_0$ as $n \to \infty$ (Bolzano–Weierstrass property). As $g = f\gamma$ is continuous (being composite of continuous functions), we have

$$f(\gamma(t_n)) \to f(\gamma(t_0)) \text{ as } t_n \to t_0.$$

Now, by the definition of E, $f(\gamma(t_n)) = f(a)\ \forall\ n$. Thus,

$$f(a) = f(\gamma(t_0)).$$

Thus, at $\gamma(t_0)$ the function has maximum value $|f(a)|$, again, by using the mean value property and the above analysis, we can find a circle $|z - \gamma(t_0)| < \delta$, $\delta > 0$ in which the function remains constant, i.e.

$$f(a) = f(\gamma(t_0)) = f(z)\ \forall\ |z - \gamma(t_0)| < \delta.$$

This means $t_0 \in E$. Moreover, $t_0 = 1$. If $t_0 \neq 1$, i.e. $t_0 < 1$, there exists t' such that $t_0 < t' < 1$ and

$$\gamma(t') \in N_\delta(\gamma(t_0)) = \{z : |z - \gamma(t_0)|\}.$$

Then, $f(\gamma(t')) = f(a)$. This implies that t' is supremum of E, a contradiction. Therefore, $t_0 = 1$ and

$$f(a) = f(\gamma(1)) = f(b).$$

Since b is arbitrary, f is constant.

Proposition 9. Let $f(z)$ be a continuous complex function on $\overline{D}$ and have the mean value property, then $|f|$ attains its maximum on ∂D.

Proof. If the function is constant on $\overline{D}$, there is nothing to prove. Suppose f is not constant. The function $|f|$ is continuous on $\overline{D}$, a bounded and closed set. It attains its maximum and minimum values somewhere in $\overline{D}$. By Theorem 13, $|f|$ does not attain its maximum value in D. In other words, $|f|$ must attain its maximum value somewhere on ∂D, the boundary of D.

All that we have done so far in this section are the proofs of the maximum modulus principle (Theorem 13) and maximum modulus theorem (Proposition 9) applicable for continuous complex functions having mean value property. Now, we state and prove these results for analytic functions.

Proposition 10 (Maximum modulus principle). If $f(z)$ is analytic in a domain D, then $|f|$ does not attain its maximum value in D unless it is a constant.

Proof. The function f is analytic in D and this implies that f is continuous and satisfies the mean value property in D (see Corollery 9). Now, the rest follows from Theorem 13.

Corollary 10 (Maximum modulus theorem). Suppose $f|z|$ is analytic in D and continuous on ∂D. Then the maximum of $|f(z)|$, which is always reached, occurs somewhere on the boundary of D and never in the interior.

Remarks. The alternative and simple proofs of the maximum modulus principle or theorem which do not need the mean value property (Theorem 13) can be seen as Exercise 11, Problem Set 6.3 or Exercise 18, Problem Set 7.3.

Example 14. Let $u(x, y)$ be a harmonic function in D and continuous on ∂D. Then show that the maximum of $|u(x, y)|$ occurs somewhere on ∂D.

In case $u(x, y)$ is constant, there is nothing to prove. Suppose $u(x, y)$ is a nonconstant harmonic function in D we can find a harmonic conjugate $v(x, y)$ of $u(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is a nonconstant analytic function in D and continuous on ∂D. Now, we show that the component function $u(x, y)$ attains its maximum value on ∂D.

Consider a function $g(z) = e^{f(z)}$, where $f = u + iv$. Clearly, $g(z)$ is a nonconstant analytic function in D and continuous on ∂D. Hence, maximum of $|g| = |e^{f(z)}| = e^{u(x,y)}$ occurs somewhere on boundary of D. Since $e^{u(x,y)}$ is an increasing function of $u(x, y)$, it follows that the maximum of $u(x, y)$ occurs on ∂D.

Proposition 11 (Minimum modulus principle). Let $f(z)$ be analytic in D such that $f(z) \neq 0 \ \forall \ z \in D$. Then $|f(z)|$ does not attain its minimum unless it is a constant.

Proof. Since $f(z) \neq 0$ for all $z \in D$, it follows by the given hypothesis that the function $1/f(z)$ is analytic in D. By the maximum modulus principle $|1/f(z)|$ does not attain its maximum unless it is a constant, i.e. $|f(z)|$ does not attain its minimum value unless it is a constant.

Corollary 11 (Minimum modulus theorem). Let $f(z)$ be an analytic function in D and continuous on ∂D such that $f(z) \neq 0 \ \forall \ z \in D$. Then $|f(z)|$ attains its minimum value somewhere on ∂D.

Proof. If $f(z) = 0$ for some $z \in \partial D$, then $\min|f(z)| = 0$. In case $f(z) \neq 0$ for any $z \in \partial D$, then with the given hypothesis it is noted that $f(z) \neq 0$ for all $z \in D \cup \partial D$. Now, consider the function $g(z) = 1/f(z)$. Clearly, $g(z)$ is continuous on ∂D and analytic in interior of D. Hence, maximum of $|g(z)|$ occurs somewhere on ∂D. This means minimum of $|f(z)|$ occurs somewhere on ∂D.

Example 15. Let $f(z)$ be a nonconstant analytic function defined in a domain $D = \{z : |z| < r\}$ and continuous on ∂D such that $|f(z)| > m$ on ∂D. If $|f(0)| < m$, show that there exists at least one zero of $f(z)$ in $|z| < r$.

Suppose $f(z) \neq 0$ in $|z| < r$. Since $f(z)$ is a nonconstant analytic function in $|z| < r$ and a continuous on $|z| = r$, it follows that the minimum of $|f(z)|$ occurs somewhere on $|z| = r$. But this contradicts the hypothesis $|f(0)| < m$ and $|f(z)| > m$ on $|z| = r$. Hence, $f(z)$ vanishes for at least one point in $|z| < r$.

Example 16. If $f(z) = z^2 + 2$, then determine the minimum value of $|f(z)|$ over the closed region $|z| \leq 1$.

Since the zeros of $f(z)$, namely, $z = \pm\sqrt{2}i$ are outside the disc $|z| \leq 1$, it follows from the minimum modulus theorem that the minimum of $|f(z)|$ occurs on the boundary $|z| = 1$. Hence, our problem is actually not a problem of two independent variables x and y ranging over $|z| = 1$, but rather a problem involving a single variable, the argument θ of a variable ranging over $|z| = 1$. Setting $x = \cos\theta$, $y = \sin\theta$, we note at once that

$$\begin{aligned} |f(z)| &= |z^2 + 2| \\ &= \sqrt{(x^2 - y^2 + 2)^2 + (2xy)^2} \\ &= \sqrt{(\cos 2\theta + 2)^2 + \sin^2(2\theta)} \\ &= \sqrt{4\cos 2\theta + 5} \end{aligned}$$

The extreme values of $|f(z)|$ are thus determined by the condition

$$\frac{d|f(z)|}{d\theta} = -\frac{8\sin 2\theta}{2\sqrt{4\cos 2\theta + 5}} = 0.$$

Hence, $\theta = 0, \pi/2, \pi, 3\pi/2$ are the values where the extremum occurs. The values $\pi/2$ and $3\pi/2$ correspond to minimum value 1.

PROBLEM SET 4.5

1. Suppose D is a domain and ∂D is its boundary. Let $f(z)$ be a nonconstant function defined on $D \cup \partial D$ such that $|f(z_0)| > m$ for some $z_0 \in D$ and $|f(z)| \leq m$ on ∂D. Then,
   (a) if $f(z)$ is analytic in D, show that there exists at least one point on ∂D where the function $f(z)$ is not continuous;
   (b) if $f(z)$ is continuous on ∂D, show that there exists at least one point in D where the function $f(z)$ is not analytic.
2. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D and continuous on ∂D. Then show that the functions
   (a) $(x^2 + y^2)e^{u(x,y)}$; (b) $(u^2 + v^2)e^{u(x,y)}$;
   (c) $(\sin^2 x + \sinh^2 y)e^{u(x,y)}$; (d) $(\cos^2 x + \sinh^2 y)e^{u(x,y)}$.
   will have the maximum on the boundary of the domain.

Suggestion. Define $g(z)$ whose real part corresponds to these functions and then apply the maximum modulus theorem.

3. Find the minimum and maximum values for each of the functions in the specified regions.

(a) $\sin z$ on $0 \le x, y \le 2\pi$; (b) $\cos z$ on $0 \le x,y \le 2\pi$; (c) e^z on $|z| \le 1$.

Discuss parts (a) and (b) when the region is $|z| \le 1$.

4. Let $f(z)$ be analytic in D and continuous on ∂D. Let (a) $g(z) = e^{-f(z)}$; (b) $h(z) = e^{if(z)}$. Then comment on the outcomes when the maximum modulus principle is applied to the functions $g(z)$ and $h(z)$.

5. Suppose $u(x, y)$ is a nonconstant harmonic function in a domain D and continuous on ∂D. Then show that $u(x, y)$ achieves its minimum value on ∂D.

6. Let $f(z)$ be an analytic function in a bounded domain D and continuous on $D \cup \partial D$. If $f(z) = k$ for all z on ∂D, then show that either $f(z)$ is constant or vanishes at least once in D.

7. Let $f(z)$ and $g(z)$ be analytic functions in a domain D and continuous on ∂D. If $f(z)$ and $g(z)$ coincide on the boundary of D, i.e. $f(z) = g(z)\ \forall\ z \in \partial D$, then show that $f(z)$ and $g(z)$ also coincide in D.

Suggestion. Apply Problem 6 to the function $F(z) = f(z) - g(z)$.

8. Suppose $u(x, y)$ is a harmonic function in a domain D such that $u(x, y)$ is constant on ∂D. Then show that $u(x, y)$ is constant on $D \cup \partial D$.

9. Let $u_1(x, y)$ and $u_2(x, y)$ be two harmonic functions defined on a domain D such that $u_1(x, y) = u_2(x, y)$ for all (x, y) on ∂D. Then prove that $u_1(x, y) = u_2(x, y)$ for all (x, y) in D as well.

10. Let $f(z)$ and $g(z)$ be nonvanishing continuous functions on $\bar{D}$ and analytic in D. Suppose that $f(z) = g(z)$ on ∂D. Then show that $f(z) = cg(z)$ for all $z \in \bar{D}$, where c is a fixed complex number such that $|c| = 1$.

11. Find all entire functions $f(z)$ such that $|f(z)| = 1$ on $|z| = 1$.

5

Sequences, Series and Products

To begin with we will discuss the convergence of sequences and infinite series whose terms are complex numbers and then switch over to sequences and series having their terms as complex-valued functions defined in some domain of the complex plane. We will also emphasize the notion of uniform convergence. Finally, this chapter will provide an introduction to infinite products.

5.1 SEQUENCES

Let us recall the definition of convergence of a sequence.

Definition 1. A sequence $\{z_n\}$ is said to converge to z_0 (as n approaches infinity) if, for each $\varepsilon > 0$ there exists a positive integer N such that

$$|z_n - z_0| < \varepsilon \quad \text{whenever } n \geq N. \tag{5.1}$$

Symbolically, we write

$$\lim_{n\to\infty} z_n = z_0.$$

A sequence which is not convergent is defined to be divergent.

If $\lim_{n\to\infty} z_n = z_0$, we have (a) $|z_n| \to |z_0|$ as $n \to \infty$; (b) the sequence $\{z_n\}$ is bounded. These facts have been established as Proposition 1 in Section 2.1.

Theorem 1. Let $z_n = x_n + iy_n$ and $z_0 = x_0 + iy_0$. Then,

$$\lim_{n\to\infty} z_n \to z_0 \Leftrightarrow \lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad \lim_{n\to\infty} y_n = y_0.$$

Proof. ($\Rightarrow$) Let $\lim_{n\to\infty} z_n = z_0$. Then, for given $\varepsilon > 0$ there exists N such that

$$|z_n - z_0| < \varepsilon \quad \forall\, n \geq N$$

or

$$|(x_n - x_0) + i(y_n - y_0)| < \varepsilon \quad \forall\, n \geq N.$$

Thus,

$$|x_n - x_0| < \varepsilon \quad \text{and} \quad |y_n - y_0| < \varepsilon \quad \forall\, n \geq N.$$

($\Leftarrow$) Given that $\{x_n\}$ and $\{y_n\}$ converge to x_0 and y_0, respectively. Hence, for given $\varepsilon > 0$ there exist integers N and N' such that

$$|x_n - x_0| < \frac{\varepsilon}{2} \quad \forall\, n \geq N \qquad \text{and} \qquad |y_n - y_0| < \frac{\varepsilon}{2} \quad \forall\, n \geq N'.$$

Choose $n_0 = \max\{N, N'\}$. Then

$$|z_n - z_0| \leq |x_n - x_0| + |y_n - y_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n \geq n_0.$$

Proposition 1. The limit of a convergent sequence is unique.

Proof. Let z and z' be two distinct limits of a sequence $\{z_n\}$. This means $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} z_n = z'$ and $z \neq z'$. Thus for given $\varepsilon > 0$, we have

$$|z_n - z| < \frac{\varepsilon}{2} \quad \forall\, n \geq N \qquad \text{and} \qquad |z_n - z'| < \frac{\varepsilon}{2} \quad \forall\, n \geq N'.$$

Fix $n_0 = \max\{N, N'\}$. These inequalities assert

$$|z - z'| = |(z - z_n) + (z_n - z')| \leq |z_n - z| + |z_n - z'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n \geq n_0.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $z = z'$.

Theorem 2. Let $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$. Then,

(a) $\lim\limits_{n\to\infty} (z_n \pm w_n) = z \pm w$;

(b) $\lim\limits_{n\to\infty} cz_n = cz$; c (a complex constant);

(c) $\lim\limits_{n\to\infty} z_n w_n = zw$;

(d) $\lim\limits_{n\to\infty} \dfrac{z_n}{w_n} = \dfrac{z}{w}$, provided $w \neq 0$.

Proof. (a) Since $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$, for given $\varepsilon > 0$ there exist positive integers N and N' such that

$$|z_n - z_0| < \frac{\varepsilon}{2} \quad \forall\, n \geq N \qquad \text{and} \qquad |w_n - w_0| < \frac{\varepsilon}{2} \quad \forall\, n \geq N'.$$

Choose $n_0 = \max\{N, N'\}$. Then,

$$|z_n + w_n - z_0 - w_0| \leq |z_n - z_0| + |w_n - w_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n \geq n_0.$$

(b) This is trivial.

(c) Let $\varepsilon > 0$ be arbitrary. Write,

$$z_n w_n - zw = z(w_n - w) + w(z_n - z) + (z_n - z)(w_n - w). \tag{5.2}$$

Since $z_n \to z$ and $w_n \to w$ as $n \to \infty$, we have

$$|z_n - z| < \sqrt{\varepsilon} \quad \forall\, n \geq N_1 \qquad \text{and} \qquad |w_n - w| < \sqrt{\varepsilon} \quad \forall\, n \geq N_2.$$

We now let $n_0 = \max\{N_1, N_2\}$. Then, for all $n \geq n_0$,

$$|(z_n - z)(w_n - w)| < \varepsilon.$$

Hence, $\lim_{n\to\infty}(z_n - z)(w_n - w) = 0$. We now apply (a) and (b) to (5.2) to conclude that

$$\lim_{n\to\infty} z_n w_n - zw = 0.$$

(d) Given $\lim_{n\to\infty} w_n = w$, $w \neq 0$. For given $\varepsilon > 0$, there exists integer $N_1 > 0$ such that

$$|w_n - w| < \varepsilon \text{ whenever } n \geq N_1.$$

Since $w \neq 0$, the choice of $\varepsilon = |w|/2$ reduces the above inequality to

$$\frac{|w|}{2} < |w_n| < \frac{3|w|}{2} \text{ whenever } n \geq N_1,$$

and its further application yields

$$\left|\frac{1}{w_n} - \frac{1}{w}\right| = \left|\frac{w - w_n}{ww_n}\right| = \frac{|w - w_n|}{|w||w_n|} \leq \frac{2|w_n - w|}{|w|^2}. \tag{5.3}$$

But ε is positive, so $|w|/2 > 0$, and there is a positive integer N_2 such that

$$|w_n - w| < \frac{|w|^2}{2}\varepsilon \text{ whenever } n \geq N_2. \tag{5.4}$$

For $n_0 = \max\{N_1, N_2\}$, (5.3) and (5.4) combine to produce the result

$$\left|\frac{1}{w_n} - \frac{1}{w}\right| < \varepsilon \text{ whenever } n \geq n_0.$$

What we have just shown is that

$$\lim_{n\to\infty} \frac{1}{w_n} = \frac{1}{w}.$$

Having already proved the product law [see part (c)], we get the quotient law by applying the product law to $\{z_n\}$ and $\{1/w_n\}$ as follows:

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \lim_{n\to\infty} z_n \lim_{n\to\infty} \frac{1}{w_n} = \frac{z}{w}.$$

Next, we briefly introduce the idea of subsequences.

Definition 2. Given a sequence $\{a_n\}$, consider a sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\{a_{n_k}\}$ is called a *subsequence* of $\{a_n\}$. If $\{a_{n_k}\}$ converges, its limit is called subsequential limit.

Theorem 3. A sequence $\{a_n\}$ of complex numbers converges to p if and only if every subsequence converges to p.

Proof. Let $a_n \to p$ and consider any subsequence $\{a_{n_k}\}$. For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies $|a_n - p| < \varepsilon$. Since $\{n_k\}$ is an increasing subsequence there is an integer M such that $n_k \geq N$ for $k \geq M$. Hence, $k \geq M$ implies

$$|a_{n_k} - p| < \varepsilon.$$

This proves that $a_{n_k} \to p$ as $k \to \infty$.

The converse statement holds trivially, since $\{a_n\}$ is itself a subsequence.

Remarks. 1. Any point z_0 is a limit point of the sequence $\{z_n\}$ if there exists a subsequence that converges to z_0. Equivalently, if a sequence converges, then the limit of the sequence is the limit point.

2. Every bounded sequence $\{z_n\}$ has at least one limit point, a useful principle known as the Bolzano–Weierstrass property for sequences. We shall not prove this principle, but it will be needed later to prove many results.

Example 1. Show in two different ways that the sequence $z_n = -2 + i(-1)^n/n^2$, $n = 1, 2, \ldots$ converges to -2. However, the sequence $\{\text{Arg } z_n\}$ does not converge to Arg (-2).

Using Definition 1, i.e. (5.1),

$$|z_n - (-2)| = \left| i\frac{(-1)^n}{n^2} \right| = \frac{1}{n^2} < \varepsilon \quad \text{whenever } n \geq \frac{1}{\sqrt{\varepsilon}}.$$

So, $N = [1/\sqrt{\varepsilon}] + 1$, where $[x]$ denotes the largest integer contained in x.

Using Theorem 1,

$$x_n = -2, \qquad y_n = \frac{(-1)^n}{n^2} \Rightarrow \lim_{n\to\infty} x_n = -2 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0,$$

so that $\lim_{n\to\infty} z_n = -2 + i0 = -2$.

Now, we observe that

$$\text{Arg } z_{2n} = -\tan^{-1}\left(\frac{1}{8n^2}\right) + \pi, \qquad \text{Arg } z_{2n+1} = \tan^{-1}\frac{1}{2(2n+1)^2} - \pi.$$

which gives

$$\lim_{n\to\infty} \text{Arg } z_{2n} = \pi \quad \text{and} \quad \lim_{n\to\infty} \text{Arg } z_{2n+1} = -\pi.$$

So, the sequence Arg z_n does not converge.

Remark. Note that the sequence $\{\arg z_n\}$ converges to arg (-2). To verify this, let ϕ_n denote the unique value of arg z_n satisfying the inequality

$$\frac{\pi}{2} < \phi_n < \frac{3\pi}{2}.$$

Then the sequence converges to arg (-2), i.e. π, since

$$\phi_{2n} = -\tan^{-1}\left(\frac{1}{8n^2}\right) + \pi, \quad \phi_{2n+1} = \tan^{-1}\frac{1}{2(2n+1)^2} - \pi + 2\pi.$$

For a more general case, see Exercise 8, Problem Set 5.1.

Cauchy condition for sequences

A sequence $\{z_n\}$ is said to satisfy the Cauchy condition if, for given $\varepsilon > 0$, there exists a positive integer N such that

$$|z_m - z_n| < \varepsilon \quad \text{whenever } n \geq N \text{ and } m \geq N$$

or

$$|z_{n+p} - z_n| < \varepsilon \quad \text{whenever } n \geq N; \quad p = 1, 2, \ldots \tag{5.5}$$

Usually, the sequences satisfying the Cauchy condition are called Cauchy sequences. Utilizing the Cauchy condition, we prove a useful result as follows.

Theorem 4. Let $\{z_n\}$ be a sequence in $\mathbb{C}$. Then $\{z_n\}$ converges if and only if it satisfies the Cauchy condition.

Proof. ($\Rightarrow$) Suppose $z_n \to z_0$ as $n \to \infty$. For given $\varepsilon > 0$, there exists N such that

$$|z_n - z_0| < \frac{\varepsilon}{2} \ \forall\, n \geq N.$$

It follows that for all $n \geq N$ and $p = 1, 2, \ldots,$

$$\begin{aligned} |z_{n+p} - z_n| &= |(z_{n+p} - z_0) + (z_0 - z_n)| \\ &\leq |z_{n+p} - z_0| + |z_n - z_0| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

($\Leftarrow$) Suppose (5.5) is satisfied. Let $\varepsilon = 1$. By the given condition,

$$|z_{n+p} - z_n| < 1 \quad \text{whenever } n \geq N; \qquad p = 1, 2, \ldots$$

starting from a certain value of n. Fixing one of these values of n, say n_0, we find that

$$\left| z_{n_0+p} - z_{n_0} \right| < 1 \quad \text{whenever } p = 1, 2, \ldots$$

So, all the points $z_{n_0+1}, z_{n_0+2}, \ldots$ belong to the neighbourhood $N_1(z_{n_0})$, i.e., lie interior to the circle with a centre z_{n_0} and radius 1. Clearly, there is a neighbourhood of the point $z = 0$ with a radius large enough to contain both $N_1(z_{n_0})$ and the finite number of points $z_1, z_2, \ldots, z_{n_0}$. Thus, the sequence $\{z_n\}$ is bounded, and hence by the Bolzano–Weierstrass property (see Remark 2 of Theorem 3) this sequence has a limit point.

It remains to show that $\{z_n\}$ cannot have two different limit points. Suppose the contrary is true, i.e., ξ and ξ' be two distinct limit points of the sequence $\{z_n\}$. Choosing $\varepsilon = |\xi - \xi'|/3$, we find that the inequality

$$|z_{n+p} - z_n| < \frac{1}{3}|\xi - \xi'| \tag{5.6}$$

holds true for all sufficiently large n, say $n > N$. On the other hand, each of the neighbourhoods $N(\xi; \varepsilon)$ and $N(\xi'; \varepsilon)$ must contain infinitely many terms of the sequence $\{z_n\}$. Let z_{n_1} be the term with index larger than N belonging to $N(\xi; \varepsilon)$, and let $z_{n_1+p_1}$ be

the term with index larger than N belonging to $N(\xi'; \varepsilon)$. Then, it is obvious from Fig. 5.1 that

$$|z_{n_1+p_1} - z_{n_1}| > \frac{1}{3}|\xi - \xi'|$$

which contradicts the inequality (5.6). This contradiction implies that $\{z_n\}$ has only one limit point and hence it converges.

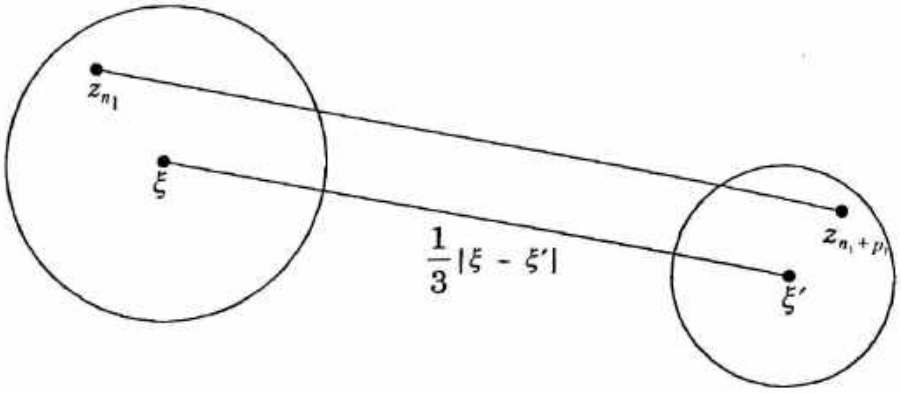

Figure 5.1

5.2 SERIES

The discussion on the convergence of infinite series whose terms are complex numbers starts with the assumption that the reader is well acquainted with the notion of convergence of sequences of complex numbers.

Definition 3. An infinite series $\Sigma_{n=1}^{\infty} a_n$ converges to the sum s if the sequence $s_n = \Sigma_{k=1}^{n} a_k$, $n = 1, 2, \ldots$ of partial sums converges to s; and we write $s_n \to s$ as $n \to \infty$ or

$$\sum_{n=1}^{\infty} a_n = s. \tag{5.7}$$

An infinite series for which $\{s_n\}$ diverges is classified as divergent (or a series which is not convergent).

Remark. For a convergent series, if $s_n \to s$ as $n \to \infty$, we should have written this as

$$s = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k.$$

However, we have denoted s by $\Sigma_{n=1}^{\infty} a_n$ in (5.7). This can be done since the letter k used in $\Sigma_{k=1}^{\infty}$ is a 'dummy variable' and can be replaced by any convenient letter.

The following facts are very important to note:

1. The necessary condition for a series to be convergent is that its nth term should approach zero, since $a_n = s_n - s_{n-1} \to 0$ as $n \to \infty$.

2. The terms of convergent series of complex numbers are bounded.

3. We can always add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence. However, in case of convergent series its sum will change.

4. It is also permissible to change the index of any series without changing its convergence, for example

$$\sum_{n=1}^{\infty} a_n, \qquad \sum_{n=1-k}^{\infty} a_{n+k}, \qquad \sum_{n=1+k}^{\infty} a_{n-k}$$

are the different representations of the same infinite series. Here k is any positive integer. Indexing is done just to simplify the expressions.

5. As the limit of $\{s_n\}$ is unique, it is inferred that the sum of a convergent series is also unique.

Let $\Sigma_{n=1}^{\infty} a_n$ be a series with complex terms and write $a_n = b_n + ic_n$, where b_n and c_n are real. The series $\Sigma_{n=1}^{\infty} b_n$ and $\Sigma_{n=1}^{\infty} c_n$ are called, respectively, the *real* and *imaginary parts of* $\Sigma_{n=1}^{\infty} a_n$. In situations involving complex series, it is often convenient to treat the real and imaginary parts separately. Of course, convergence of both $\Sigma_{n=1}^{\infty} b_n$ and $\Sigma_{n=1}^{\infty} c_n$ implies convergence of $\Sigma_{n=1}^{\infty} a_n$. Conversely, convergence of $\Sigma_{n=1}^{\infty} a_n$ implies convergence of both $\Sigma_{n=1}^{\infty} b_n$ and $\Sigma_{n=1}^{\infty} c_n$. This assertion is proved in the following result.

Theorem 5. Suppose that $a_n = b_n + ic_n$ and $s = b + ic$. Then

$$\sum_{n=1}^{\infty} a_n = s \Leftrightarrow \sum_{n=1}^{\infty} b_n = b \quad \text{and} \quad \sum_{n=1}^{\infty} c_n = c.$$

Proof. Let

$$s_n = \sum_{k=1}^{n} a_k, \qquad s_n = p_n + iq_n, \qquad p_n = \sum_{k=1}^{n} b_k, \qquad q_n = \sum_{k=1}^{n} c_k.$$

Then, using Definition 3 and Theorem 1,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n = s &\Leftrightarrow s_n \to s \text{ as } n \to \infty \\ &\Leftrightarrow p_n \to b \text{ and } q_n \to c \quad \text{as } n \to \infty \\ &\Leftrightarrow \sum_{n=1}^{\infty} b_n = b \quad \text{and} \quad \sum_{n=1}^{\infty} c_n = c.\end{aligned}$$

When the limit function of the series is unknown or difficult to compute, it is convenient to use the Cauchy condition for deciding the nature of the series.

Proposition 2 (Cauchy condition). The series $\Sigma_{n=0}^{\infty} a_n$ converges $\Leftrightarrow$ for every $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$ implies

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \varepsilon \quad \text{for each } p = 1, 2, \ldots$$

Proof. Let $s_n = \Sigma_{k=1}^{n} a_k$. Write

$$s_{n+p} - s_n = a_{n+1} + a_{n+2} + \cdots + a_{n+p}.$$

Now, by the Cauchy condition (5.5) for sequence $\{s_n\}$, the result follows at once.

Definition 4. A series $\Sigma_{n=1}^{\infty} a_n$ is said to converge absolutely if $\Sigma_{n=1}^{\infty} |a_n|$ converges.

Every absolutely convergent series is convergent, since

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}|.$$

Moreover, if $\Sigma_{n=1}^{\infty} a_n$ converges but $\Sigma_{n=1}^{\infty} |a_n|$ diverges, then the series is said to be conditionally convergent, for example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n i}{n+1}$$

is conditionally convergent. The significance of absolutely convergent series lies in the fact that these series can be rearranged in any order even then all such series converge to the same sum. Also, the sum and difference of two absolutely convergent series are absolutely convergent. These facts are not always true for conditionally convergent series.

Definition 5. Let $f: I_+ \to I_+$ be a one-to-one function on I_+, the set of positive integers. Let $\Sigma_{n=1}^{\infty} a_n$ and $\Sigma_{n=1}^{\infty} b_n$ be two series such that

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_{f(n)}. \tag{5.8}$$

Then $\Sigma_{n=1}^{\infty} b_n$ is said to be a rearrangement of $\Sigma_{n=1}^{\infty} a_n$.

Remark. From (5.8), $a_n = b_{f^{-1}(n)}$, and hence $\Sigma_{n=1}^{\infty} a_n$ is also a rearrangement of $\Sigma_{n=1}^{\infty} b_n$.

Theorem 6. Let $\Sigma_{n=1}^{\infty} a_n$ be an absolutely convergent series having sum s. Then, every rearrangement of $\Sigma_{n=1}^{\infty} a_n$ also converges absolutely and has the sum s.

Proof. Let $\{b_n\}$ be defined by (5.8). Then,

$$|b_1| + |b_2| + \cdots + |b_n| = |a_{f(1)}| + |a_{f(2)}| + \cdots + |a_{f(n)}| \leq \sum_{n=0}^{\infty} |a_n|.$$

So, $\Sigma_{n=1}^{\infty} |b_n|$ has bounded partial sums (with monotonicity). Thus, $\Sigma_{n=1}^{\infty} b_n$ converges absolutely.

To show that $\Sigma_{n=1}^{\infty} b_n = s$, let

$$p_n = b_1 + b_2 + \cdots + b_n \quad \text{and} \quad q_n = a_1 + a_2 + \cdots + a_n.$$

For given $\varepsilon > 0$, choose N so that

$$|q_N - s| < \frac{\varepsilon}{2} \quad \text{and} \quad |a_{N+1}| + |a_{N+2}| + \cdots < \frac{\varepsilon}{2}.$$

Then,

$$|p_n - s| \le |p_n - q_N| + |q_N - s| < |p_n - q_N| + \frac{\varepsilon}{2}.$$

Choose M such that

$$\{1, 2, \ldots, N\} \subseteq \{f(1), f(2), \ldots, f(M)\}.$$

Then $n \ge M$ implies $f(n) \ge N$, and for such N we have

$$\begin{aligned} |p_n - q_N| &= |b_1 + b_2 + \cdots + b_n - (a_1 + a_2 + \cdots + a_N)| \\ &= |a_{f(1)} + a_{f(2)} + \cdots + a_{f(n)} - (a_1 + a_2 + \cdots + a_N)| \\ &\le |a_{N+1}| + |a_{N+2}| + \cdots < \frac{\varepsilon}{2}, \end{aligned}$$

since all the terms $a_1, a_2, \ldots, a_N$ cancel out in the subtraction. Hence, $n \ge M$ implies $|p_n - s| < \varepsilon$ and this means $\Sigma_{n=1}^{\infty} b_n = s$.

Now, we explain some tests which are well established for infinite series in real calculus.

Proposition 3 (Comparison test). If

1. $|a_n| \le |b_n|$ except for finitely many n's;
2. $\Sigma_{n=1}^{\infty} b_n$ converges absolutely;

then $\Sigma_{n=1}^{\infty} a_n$ converges absolutely.

Proof. Let $\varepsilon > 0$ be given. Then, there exists $m, n \ge N$ such that

$$|s_n - s_m| \le \sum_{n=m+1}^{n} |a_n| \le \sum_{n=m+1}^{n} |a_n| < \varepsilon.$$

This observation proves the proposition.

Corollary 1. Let $|b_n| \le |a_n|$ and suppose $\Sigma_{n=1}^{\infty} |b_n|$ diverges. Then, $\Sigma_{n=1}^{\infty} |a_n|$ diverges.

Proof. If $\Sigma_{n=1}^{\infty} |a_n|$ converges, then by comparison test $\Sigma_{n=1}^{\infty} |b_n|$ converges, a contradiction to the given hypothesis.

In subsequent tests we shall be using the concept of limit superior (upper limit) and limit inferior (lower limit). So, we digress from this discussion on infinite series to recall these concepts.

Given a sequence of real numbers $\{a_n\}$, we say $\limsup_{n\to\infty} a_n = a_0$ if and only if, for given $\varepsilon > 0$

(i) $a_n \le a_0 + \varepsilon$ for all $n \ge n_0$;
(ii) $a_n > a_0 - \varepsilon$ for infinitely many values of n (a subsequence of n).

Here, a_0 has been assumed finite. If a_0 is not finite, then $\limsup_{n\to\infty} a_n = -\infty$ or ∞.

In an analogous manner, we can define limit inferior, just by reversing the inequalities or defining it as follows:

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n).$$

Moreover, for $a_n > 0$,

$$\frac{1}{p} = \limsup_{n\to\infty} a_n \Leftrightarrow p = \liminf_{n\to\infty}\frac{1}{a_n}.$$

The limit superior and limit inferior are equal if and only if the sequence converges or diverges to $+\infty$ or $-\infty$. When a_n and b_n are real sequences bounded below, the following relations are the direct consequences of the definition

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \le \liminf_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n;$$

$$\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \le \limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Moreover, if $a_n, b_n > 0$ for all n and both the limits are finite or infinite, then

$$\limsup_{n\to\infty}(a_n b_n) \le \limsup_{n\to\infty} a_n \limsup_{n\to\infty} b_n.$$

Proposition 4 (Ratio test). Given a series $\Sigma_{n=1}^{\infty} a_n$ of nonzero complex numbers, let

$$\ell = \liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|, \qquad L = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

Then,

(a) the series converges absolutely if $L < 1$;
(b) the series diverges if $\ell > 1$.
(c) the test fails for $\ell \le 1 \le L$.

Proof. Assume that $L < 1$ and choose r so that $L < r < 1$. The definition of L implies the existence of positive integer N such that $|a_{n+1}/a_n| < r$ if $n \ge N$. Since $r = r^{n+1}/r^n$, it means

$$\frac{|a_{n+1}|}{r^{n+1}} < \frac{|a_n|}{r^n} < \frac{|a_N|}{r^N}$$

and hence

$$|a_n| < cr^n, \qquad \text{if } n \ge N,$$

where $c = |a_N|r^{-N}$. Now, the series converges by the comparison test.

To prove the second assertion, we simply observe that $\ell > 1$ implies $|a_{n+1}| > |a_n|$ for all $n \ge N$ and for some N and hence $\lim_{n\to\infty} a_n \ne 0$.

The fact that the test remains inconclusive is supported by the examples:

$$\sum_{n=1}^{\infty}\frac{1}{n}, \quad \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

In both cases $\ell = L = 1$, but $\Sigma_{n=1}^{\infty} n^{-1}$ diverges and $\Sigma_{n=1}^{\infty} n^{-2}$ converges.

Proposition 5 (Cauchy root test). Given a series $\Sigma_{n=1}^{\infty} a_n$ of complex numbers, let

$$\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then,

(a) for $\rho < 1$, the series coverges absolutely;
(b) for $\rho > 1$, the series diverges;
(c) for $\rho = 1$, the test fails.

Proof. Assume that $\rho < 1$ and choose r so that $\rho < r < 1$. The definition of ρ implies the existence of positive integer N such that $|a_n| < r^n$ if $n \geq N$. Now, the series converges by comparison test.

To prove the second assertion, we simply observe that $\rho > 1$ implies $|a_n| > 1$ for infinitely many n's, and hence we have $\lim_{n\to\infty} a_n \neq 0$.

The test remains inconclusive for $\rho = 1$ is supported by the examples we have just mentioned in the ratio test.

The root test is more powerful than the ratio test. That is, whenever the root test is inconclusive, so is the ratio test. But there are examples where the ratio test fails yet the root test is conclusive. The argument is supported by the following well known inequalities

$$\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \leq \liminf_{n\to\infty} \sqrt[n]{|a_n|} \leq \limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}. \tag{5.9}$$

We give a brief sketch of the proof for the inequalities in (5.9). Let

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{|a_n|}\right| = L.$$

If $L = \infty$, there is nothing to prove. Suppose $0 \leq L < \infty$. Then, by definition of lim sup, for given $\varepsilon > 0$ there exists a positive integer m such that

$$\frac{|a_{n+1}|}{|a_n|} < L + \varepsilon \quad \text{for all } n \geq m.$$

Writing the above inequality for $m, m+1, \ldots, n-1$, we note that

$$\begin{aligned} |a_{m+1}| &< (L+\varepsilon)|a_m| \\ |a_{m+2}| &< (L+\varepsilon)|a_{m+1}| \\ &\vdots \\ |a_n| &< (L+\varepsilon)|a_{n-1}|. \end{aligned}$$

Multiplying these inequalities column-wise, we have

$$|a_n| < |a_m|(L+\varepsilon)^m \quad \text{for all } n \geq m$$

or

$$|a_n|^{1/n} < |a_m|^{1/n}(L+\varepsilon)^{m/n} \quad \text{for all } n \geq m.$$

Since $\varepsilon > 0$ is arbitrary, taking limits

$$\limsup_{n\to\infty} |a_n|^{1/n} \le L.$$

Similarly, by using lim inf,

$$\liminf_{n\to\infty} |a_n|^{1/n} \ge \ell.$$

Thus,

$$\ell \le \liminf_{n\to\infty} |a_n|^{1/n} \le \limsup_{n\to\infty} |a_n|^{1/n} \le L.$$

The following example illustrates how the ratio test fails but the root test is conclusive.

Example 2. Discuss the convergence of the series $\Sigma_{n=1}^{\infty} a_n$, where

$$a_{2n-1} = \left(\frac{1+i}{3\sqrt{2}}\right)^{2n-1} \quad \text{and} \quad a_{2n} = \left(\frac{1-i}{2\sqrt{2}}\right)^{2n}.$$

By the definition of a_n's, we have

$$|a_n|^{1/n} = \begin{cases} 1/3 & \text{if } n = 1, 3, \ldots, \\ 1/2 & \text{if } n = 2, 4, \ldots, \end{cases}$$

So,

$$\rho = \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{2}.$$

In case, the ratio test is applied, then

$$\left|\frac{a_{2n}}{a_{2n-1}}\right| = \frac{1}{3}\left(\frac{3}{2}\right)^{2n} \to \infty \quad \text{as } n \to \infty,$$

whereas

$$\left|\frac{a_{2n+1}}{a_{2n}}\right| = \frac{1}{3}\left(\frac{2}{3}\right)^{2n} \to 0 \quad \text{as } n \to \infty.$$

Thus, ratio test gives no information while root test guarantees the convergence.

All tests discussed so far help us to determine the absolute convergence of a series with complex terms. It is also important to have tests for determining the convergence when the series might not converge absolutely. The next discussion will comprise these tests which make use of partial summation formula due to Abel.

Lemma (Partial summation formula). If $\{a_n\}$ and $\{b_n\}$ are two sequences, define

$$A_n = \sum_{k=1}^{n} a_k.$$

Then we have the identity

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k). \tag{5.10}$$

Proof. Writing $A_0 = 0$, we have

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1}.$$

This explains well that $\Sigma_{n=1}^{\infty} a_n b_n$ converges provided that both $\Sigma_{n=1}^{\infty} A_n(b_{n+1} - b_n)$ and $\{A_n b_{n+1}\}$ converge.

The partial summation formula is analogous to the formula for integration by parts,

$$\int_1^n a(x)b(x)\,dx = A(x)b(x)\big|_1^n - \int_1^n A(x)b'(x)\,dx, \quad A(x) = \int a(x)\,dx.$$

Theorem 7 (Dirichlet test). Suppose that

1. the partial sums of $\Sigma_{n=1}^{\infty} a_n$ are bounded;
2. $b_n \downarrow$ and $b_n \to 0$ as $n \to \infty$.

Then $\Sigma_{n=1}^{\infty} a_n b_n$ converges.

Proof. Let $A_n = a_1 + a_2 + \cdots + a_n$ and assume $|A_n| \le M$ for all n. Then, in view of hypothesis $b_n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} A_n b_{n+1} = 0.$$

In view of (5.10), we need now only to show that $\Sigma_{n=1}^{\infty} A_n(b_{n+1} - b_n)$ is convergent.

Since

$$|A_n(b_{n+1} - b_n)| \le M|b_n - b_{n+1}|, \qquad b_n \downarrow 0$$

and $\Sigma_{n=1}^{\infty} |b_n - b_{n+1}|$ is convergent telescoping series, hence the comparison test implies absolute convergence of $\Sigma_{n=1}^{\infty} A_n(b_{n+1} - b_n)$.

Abel's test. Suppose that

1. $\Sigma_{n=1}^{\infty} a_n$ converges;
2. $\{b_n\}$ is a monotonic convergent sequence.

Then $\Sigma_{n=1}^{\infty} a_n b_n$ converges.

Proof. Convergence of $\Sigma_{n=1}^{\infty} a_n$ and $\{b_n\}$ implies that $\lim_{n\to\infty} A_n b_{n+1}$ exists, since $A_n = a_1 + a_2 + \cdots + a_n$ is bounded.

The rest of the proof proceeds exactly on the lines of Theorem 7.

PROBLEM SET 5.1

1. Using the definition of limit, show that

$$\lim_{n\to\infty} \frac{1}{n^2} + \frac{n}{n+1} i = i.$$

2. Determine the limit points and the limit of the following sequences:

(a) $(-1)^n \dfrac{i}{n^2}$; (b) $\dfrac{n}{n+1} + i\dfrac{n^2}{n^2+2n+1}$;

(c) $\sqrt{n+2i} - \sqrt{n+i}$; (d) $\sqrt{n}\left(\sqrt{n+2i} - \sqrt{n+i}\right)$.

3. If $\lim_{n\to\infty} z_n = z_0$, show that

(a) $\lim\limits_{n\to\infty} \overline{z_n} = \overline{z_0}$ (b) $\lim\limits_{n\to\infty} \dfrac{z_1 + z_2 + \cdots + z_n}{n} = z_0$.

Write the more general version of result (b).

4. Construct a sequence $\{z_n\}$ which does not converge, although the limit in assertion (b) of the preceding problem tends to zero.

5. Using Theorem 1, prove Theorem 2.

6. Suppose that $\lim_{n\to\infty} z_n = 0$ and $\{b_n\}$ is a bounded sequence. Show that $\lim_{n\to\infty} z_n b_n = 0$.

7. Prove that for a sequence $\{a_n\}$ if $\lim_{n\to\infty} a_{2n} \to \ell$ and $\lim_{n\to\infty} a_{2n+1} \to \ell$, then $a_n \to \ell$ as $n \to \infty$, i.e. if the sequences with even subscripts and odd subscripts converge to the same limit, then the sequence is convergent.

8. Let $\lim_{n\to z_0} z_n \to z_0 \neq 0$. Then prove that $\arg z_n$ converges to $\arg z_0$. Is it possible to replace arg by Arg in this problem?

9. Let $\{r_n\}$ and $\{\theta_n\}$ be two convergent sequences of real numbers such that

$$\lim_{n\to\infty} r_n = r_0 \quad \text{and} \quad \lim_{n\to\infty} \theta_n = \theta_0.$$

Then show that $\lim_{n\to\infty} r_n e^{i\theta_n} = r_0 e^{i\theta_0}$.

Suggestion: Apply Theorem 2(c) to the sequences $\{r_n\}$ and $\{e^{i\theta_n}\}$. This result tells us that the simultaneous convergence of the sequence $\{|z_n|\}$ and $\{\arg z_n\}$ is sufficient for convergence of the sequence $\{z_n\}$.

10. If $\Sigma_{n=0}^{\infty} z_n$ converges, show that

$$\left|\sum_{n=0}^{\infty} z_n\right| \leq \sum_{n=0}^{\infty} |z_n|.$$

11. Prove that every bounded sequence in $\mathbb{C}$ has a convergent subsequence.

12. Show that (a) the sum of a convergent series is unique; (b) the terms of a convergent series are bounded, but the converse is not true.

13. Let $\Sigma_{n=0}^{\infty} z_n = s$ and $\Sigma_{n=0}^{\infty} w_n = t$. Then prove that

$$\sum_{n=0}^{\infty} z_n \pm \sum_{n=0}^{\infty} w_n = s \pm t \quad \text{and} \quad \sum_{n=0}^{\infty} cz_n = cs,$$

where c is any complex constant.

14. Show that if $\Sigma_{n=0}^{\infty} z_n = s$, then $\Sigma_{n=0}^{\infty} \overline{z_n} = \overline{s}$

15. If $A = a_1 + b_1 + \cdots$ and $B = b_1 + b_2 + \cdots$ are convergent, show that
$$AB = a_1b_1 + (a_2b_1 + a_1b_2) + (a_3b_1 + a_2b_2 + a_1b_3) + \cdots,$$
provided that at least one of the series is absolutely convergent.

16. Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $a_n = b_{n+1} - b_n$ for $n = 1, 2, \ldots$. Then prove that $\Sigma_{n=0}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} b_n$ exists, in which case we have
$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} b_n - b_1.$$
This is termed the *telescoping series*.

17. Prove that if the series $\Sigma_{n=1}^{\infty} z_n$ converges absolutely, then so do the series:

(a) $\displaystyle\sum_{n=1}^{\infty} z_n^2$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{z_n}{1+z_n}$ $(z_n \neq 1)$; (c) $\displaystyle\sum_{n=1}^{\infty} \frac{z_n^2}{1+z_n^2}$

Is the converse true in case (a)?

18. Prove that if $\Sigma_{n=1}^{\infty} z_n$ converges, then $\Sigma_{n=1}^{\infty} z_n/n$ converges.
Suggestion. A consequence of Abel's test.

19. Prove that
$$\sum_{n=1}^{\infty} |a_n|^2 < \infty \Rightarrow n^{-1} \sum_{k=1}^{n} \sqrt{k}\, |a_k| \text{ is bounded.}$$
Suggestion. Use the Schwarz inequality, Section 1.2.

20. Let $c_n = a_n + ib_n$, where $a_n = (-1)^n/\sqrt{n}$, $b_n = 1/n^2$. Show that $\Sigma_{n=0}^{\infty} c_n$ is conditionally convergent.

21. Given an absolutely convergent series $\Sigma_{n=1}^{\infty} a_n$ such that $|a_n| \geq |a_{n+1}| > 0$, prove that $\lim_{n\to\infty} n|a_n| = 0$.
Suggestion. Use the Cauchy condition, and first set $p = n$, and then $p = n + 1$.

22. Prove that the series $\Sigma_{n=1}^{\infty} \log(1 + a_n)$ and $\Sigma_{n=1}^{\infty} a_n$ converge absolutely in a simultaneous manner.
Suggestion. This is an immediate consequence of the fact
$$\lim_{z\to 0} \frac{\log(1+z)}{z} = 1.$$

23. If $\Sigma_{n=1}^{\infty} na_n$ converges, show that $\Sigma_{n=1}^{\infty} a_n$ converges.

5.3 SEQUENCES OF FUNCTIONS

In Section 5.1, we discussed the sequences of complex numbers. This section deals with sequence $\{f_n\}$ whose terms are complex-valued functions having a common domain

of definition in the complex plane. For each z in the domain, we can find another sequence $\{f_n(z)\}$ whose terms are the corresponding function values. Let S denote the set of z for which this second sequence converges. The function defined by

$$\lim_{n\to\infty} f_n(z) = f(z), \quad \text{if } z \in S$$

is called the *limit function of the sequence* $\{f_n\}$, and we say that $\{f_n(z)\}$ converges pointwise to f on S. For example

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)z = z$$

for all z. But in order to have

$$\left|\left(1+\frac{1}{n}\right)z - z\right| = \frac{|z|}{n} < \varepsilon$$

it is essential that $N > |z|/\varepsilon$. Such N exists for each z. But the requirement cannot be met simultaneously for all z.

Before we introduce the concept of uniform convergence, let us enquire whether the limit function is continuous when each $f_n(z)$ is continuous in the domain. This phenomenon is answered by uniform convergence.

Definition 6 (Uniform convergence). A sequence of functions $\{f_n\}$ is said to *converge uniformly* to f on a set S, if for every $\varepsilon > 0$, there exists an N (depending on ε only) such that $n \geq N$ implies

$$|f_n(z) - f(z)| < \varepsilon \quad \text{for each } z \text{ in } S.$$

Symbolically, we write

$$f_n \to f \quad \text{uniformly on } S.$$

A sequence $\{f_n\}$ is uniformly bounded on S if there exists an $M > 0$ such that $|f_n(z)| \leq M$ for all z in S and for all n. The positive number M is called the *uniform bound* for $\{f_n\}$.

In the theory of analytic functions, we shall observe that uniform convergence happens to be more significant than pointwise convergence.

Theorem 8. Assume that $f_n \to f$ uniformly on S. If each f_n is continuous at a point c of S, then the limit function is also continuous at c.

Proof. If c is an isolated point of S, there is nothing to prove (since f is automatically continuous there). Suppose c is a limit point of S. By hypothesis, for every $\varepsilon > 0$ there is an integer M such that $n \geq M$ implies

$$|f_n(z) - f(z)| < \frac{\varepsilon}{3} \quad \text{for each } z \text{ in } S.$$

Since f_M is continuous at c, there is a neighbourhood $N_\delta(c)$ such that $z \in N_\delta(c) \cap S$ implies

$$|f_M(z) - f_M(c)| < \frac{\varepsilon}{3}.$$

But
$$|f(z)-f(c)| \le |f(z)-f_M(z)| + |f_M(z)-f_M(c)| + |f_M(z)-f(c)|.$$
If $z \in N_\delta(c) \cap S$, each term on the right in the above inequality is less than $\varepsilon/3$, and hence,
$$|f(z)-f(c)| < \varepsilon.$$
This completes the proof.

Remark. Uniform convergence of $\{f_n\}$ is sufficient but not necessary to transmit continuity from the individual terms to the limit function. There are sequences of complex terms which converge to continuous functions even though convergence is not uniform. It should be noted that the above theorem is utilized as a *negative test for uniform convergence.*

Example 3. Show that the sequence $f_n(z) = |z|^n$ converges on $|z| \le 1$, but not uniformly.

The limit function $f(z)$ is not continuous at $z = 1$ since
$$f(z) = \begin{cases} 0, & |z| < 1, \\ 1, & |z| = 1. \end{cases}$$
This means that the sequence $\{f_n\}$ converges pointwise but not uniformly on $|z| \le 1$.

Before proceeding further we recall compact sets, that is the sets which are bounded and closed in the complex plane.

Theorem 9 (Weierstrass theorem for sequences). Let $\{f_n\}$ be a sequence of analytic functions defined on a domain Ω such that $f_n \to f$ uniformly on every compact subset of Ω. Then f is analytic in Ω.

Moreover, as $n \to \infty$, the sequence of derivatives $\{f_n^{(k)}(z)\}$, $k = 1, 2, \ldots$, converges uniformly to $f^{(k)}(z)$ on every compact subset of Ω.

Proof. Let γ be a closed contour such that the region enclosed by γ is contained in Ω. Clearly, all points on γ form a compact set S. Since $f_n \to f$ uniformly on every compact subset of Ω, it follows that for given $\varepsilon > 0$, there exists a positive integer N such that $n \ge N$ implies
$$|f_n(z)-f(z)| < \varepsilon \quad \forall\, z \in \gamma. \tag{5.11}$$
Using (5.11) and the fact that each f_n is analytic, we have
$$\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| &= \left|\int_\gamma [(f(z)-f_n(z)) + f_n(z)]dz\right| \\
&\le \left|\int_\gamma [f(z)-f_n(z)]dz\right| + \left|\int_\gamma f_n(z)dz\right| \\
&\le \int_\gamma |f(z)-f_n(z)|\,|dz| \\
&< \varepsilon \times \text{length of } \gamma.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_\gamma f(z)\, dz = 0$$

for every closed contour γ (as γ was chosen arbitrarily) lying in Ω. Now, by Morera's theorem, the assertion is proved.

Since $f_n(z)$ is analytic, it follows that each $f_n^{(k)}(z)$ is analytic in the given domain. Let $\gamma : |z - z_0| = r$ be a circle in Ω. Then, by the Cauchy–Goursat theorem,

$$\int_\gamma f_n^{(k)}(z)\, dz = 0.$$

Further,

$$\left|\int_\gamma f^{(k)}(z)\, dz\right| = \left|\int_\gamma \left\{[f^{(k)}(z) - f_n^{(k)}(z)] + f_n^{(k)}(z)\right\} dz\right|$$

$$\leq \left|\int_\gamma [f^{(k)}(z) - f_n^{(k)}(z)]\, dz\right| + \left|\int_\gamma f_n^{(k)}(z)\, dz\right|$$

$$\leq \int_\gamma |f^{(k)}(z) - f_n^{(k)}(z)|\, |dz| < \varepsilon \times \text{length of } \gamma.$$

Again, the Morera's theorem ensures the analyticity of the limiting function $f^{(k)}$ in Ω. This completes the proof.

Remark. If $f_n \to f$ converges uniformly in a domain Ω, then it is not necessary that $f_n^{(k)} \to f^{(k)}$ uniformly in Ω. For example, consider the sequence

$$f_n(z) = \frac{z^{n+1}}{n(n+1)}.$$

This converges uniformly in $|z| < 1$, since

$$|f_n(z)| \leq \left|\frac{z^{n+1}}{n(n+1)}\right| \leq \frac{1}{n(n+1)}.$$

But the derived sequence

$$f_n''(z) = z^{n-1}$$

does not converge uniformly. This is why we talk of every compact subset in the statement of the theorem.

Theorem 10 (Cauchy condition). Let f_n be a sequence of functions defined on a set S. Then, $f_n \to f$ uniformly on $S \Leftrightarrow$ for every $\varepsilon > 0$ there exists an N such that for all $m, n \geq N$ and for all $z \in S$,

$$|f_n(z) - f_m(z)| < \varepsilon.$$

Proof. Assume $f_n \to f$ uniformly on S. Then, for given $\varepsilon > 0$ we can find $N(\varepsilon)$ so that $n \geq N(\varepsilon)$ implies

$$|f_n(z) - f(z)| < \frac{\varepsilon}{2} \quad \text{for each } z \in S.$$

Taking $m > N(\varepsilon)$, we also have

$$|f_m(z) - f(z)| < \frac{\varepsilon}{2},$$

and hence $|f_n(z) - f_m(z)| < \varepsilon$.

Conversely, suppose that the Cauchy condition is satisfied. Then, for each $z \in S$, the sequence $\{f_n(z)\}$ converges (Cauchy condition for sequence of complex numbers). Let

$$f(z) = \lim_{n\to\infty} f_n(z).$$

We have to show that $f_n \to f$ uniformly on S.

If $\varepsilon > 0$ be given, then by the given hypothesis there exists N so that $m, n > N$ implies

$$|f_n(z) - f_m(z)| < \varepsilon \quad \text{for each } z \in S.$$

Let us take $m = n + p$. Then,

$$|f_n(z) - f_{n+p}(z)| < \varepsilon \quad \text{for each } p = 1, 2, \ldots$$

and every z in S. Therefore,

$$\lim_{p\to\infty} |f_n(z) - f_{n+p}(z)| \leq \varepsilon.$$

Hence, $n \geq N$ implies $|f_n(z) - f(z)| < \varepsilon$ for every z in S.

5.4 SERIES OF FUNCTIONS

Let $\{f_n(z)\}$ be a sequence of complex-valued functions defined on a set S in $\mathbb{C}$. Then the expression

$$\sum_{n=1}^{\infty} f_n(z) \tag{5.12}$$

is an infinite series whose terms are functions. As in the case of (5.7), we say (5.12) converges if and only if the sequence of partial sums $s_n(z) = \Sigma_{k=1}^{n} f_k(z)$ converges for each z in S. When the series converges, we write

$$\lim_{n\to\infty} s_n(z) = s(z).$$

The function $s(z)$ is called the *sum function* of the series at z.

Remark. The corresponding ratio and root tests are discussed below:

Let

$$\lim_{n\to\infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| = |r(z)|.$$

The series (5.12) converges absolutely for those values of z for which $|r(z)| < 1$ and diverges for those values for which $|r(z)| > 1$. The test is inconclusive for all z such that $|r(z)| = 1$. Let

$$\lim_{n\to\infty} \sqrt[n]{|f_n(z)|} = |p(z)|.$$

The series (5.12) converges absolutely for those values of z for which $|p(z)| < 1$ and diverges for those values for which $|p(z)| > 1$. The test is inconclusive for all z satisfying $|p(z)| = 1$.

Example 4. Test the convergence of the series

$$1 + \frac{1}{2^2}\left(\frac{z-1}{z+1}\right) + \frac{1}{3^2}\left(\frac{z-1}{z+1}\right)^2 + \cdots.$$

We apply the ratio test and obtain

$$\left|\frac{f_{n+1}(z)}{f_n(z)}\right| = \left(\frac{n}{n+1}\right)^2 \left|\frac{z-1}{z+1}\right|.$$

As $n \to \infty$, the expression on the right approaches to $|(z-1)/(z+1)|$. In view of the ratio test, the series converges absolutely for all z satisfying

$$|z-1| < |z+1|,$$

i.e. for all z in the right-half plane. The series diverges for $|z-1| > |z+1|$, i.e. in the left-half plane. The test gives no information for all z satisfying $|z-1| = |z+1|$ and this is the imaginary axis.

For points on the imaginary axis the corresponding series of absolute values

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots,$$

is convergent. Thus, the series converges absolutely for Re $z \geq 0$ and diverges for Re $z < 0$.

To apply the Dirichlet test effectively on infinite series with its terms as functions, we must be acquainted with a few series having bounded partial sums. Of course, all convergent series have this property. Now, we quote an example of a divergent series, whose partial sums are bounded. This is the geometric series $\Sigma_{n=1}^{\infty} z^n$, with $|z| = 1$, i.e. $z = e^{ix}$, where x is real. The formula for partial sums of this series is of fundamental importance in the theory of Fourier series.

Example 5. For every real $x \neq 2n\pi$, $n \in \mathbb{I}$, show that

$$\sum_{k=1}^{n} e^{ikx} = \frac{\sin(nx/2)}{\sin(x/2)} e^{i(n+1)x/2}. \tag{5.13}$$

Following the technique of Example 9 in Chapter 1, the identity (5.13) can be established. Note that the above identity yields the estimate

$$\left|\sum_{k=1}^{n} e^{ikx}\right| \leq \frac{1}{|\sin(x/2)|}.$$

By considering the real and imaginary parts of (5.13), we obtain

$$\sum_{k=1}^{n} \cos kx = -\frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}},$$

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\left(\frac{nx}{2}\right)\sin\left(\frac{n+1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}.$$

Using (5.13), we can also write

$$\sum_{k=1}^{n} e^{i(2k-1)x} = e^{-ix}\sum_{k=1}^{n} e^{i2kx} = \frac{\sin nx}{\sin x}e^{inx}, \tag{5.14}$$

as an identity valid for every $x \neq 2n\pi$. Taking the real and imaginary parts of (5.14), we obtain

$$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin(2nx)}{2\sin x},$$

$$\sum_{k=1}^{n} \sin(2k-1)x = \frac{\sin^2(nx)}{\sin x}.$$

These formulas occur frequently in the theory of Fourier series.

Example 6. Show that the logarithmic series

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{z^n}{n}$$

converges on $|z| = 1$, except possibly at $z = -1$.

Let us write $a_n = (-1)^{n-1}e^{in\theta}$ for $z = e^{i\theta}$, $-\pi < \theta < \pi$. Note that

$$|a_1 + a_2 + \cdots + a_n| = \left|\sum_{k=1}^{n} (-1)^{k-1} e^{ik\theta}\right| = \left|\frac{e^{i\theta} - (-1)^n e^{i(n+1)\theta}}{1+e^{i\theta}}\right| = \leq \frac{2}{|1+e^{i\theta}|}.$$

Hence, the partial sums are bounded as the last expression on the right side is independent of n. Writing $b_n = 1/n$, we note that $b_n \downarrow$ and $\lim_{n\to\infty} b_n = 0$. Now, applying the Dirichlet test, the required assertion follows.

For $z = -1$, the series turns to be $-\Sigma_{n=1}^{\infty} 1/n$ and this is known to be divergent.

Definition 7 (Uniform convergence). The series (5.12) converges uniformly on the set S if the sequence of partial sums $\{s_n(z)\}$ converges uniformly on S.

Proposition 6 (Cauchy condition). The infinite series $\Sigma_{n=1}^{\infty} f_n(z)$ converges uniformly on $S \Leftrightarrow$ for every $\varepsilon > 0$ there is an integer N such that $k \geq N$ implies

$$\left| \sum_{k=N}^{N+p} f_k(z) \right| < \varepsilon \quad \text{for each } p = 0, 1, 2, \ldots, z \text{ in } S.$$

Proof. Define $s_N(z) = \Sigma_{k=1}^{N} f_k(z)$ and apply Theorem 14 with $m = N + p$.

Proposition 7 (Weierstrass M-test). Let $\{M_n\}$ be a sequence of nonzero real numbers such that

$$|f_n(z)| \leq M_n \quad \text{for each } n = 1, 2, \ldots, z \text{ in } S.$$

Then the series $\Sigma_{n=1}^{\infty} f_n(z)$ converges uniformly on S if $\Sigma_{n=1}^{\infty} M_n$ converges.

Proof. We have

$$\left| \sum_{k=n}^{n+p} f_k(z) \right| \leq \sum_{k=n}^{n+p} M_k .$$

Now, apply the Cauchy condition (Proposition 6) for the series and Proposition 2, in conjunction with the above inequality.

Remark. Note that the M-test also establishes absolute convergence.

Example 7. Show that the series

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{1 - z^{2n}}$$

converges uniformly in the region $|z| \leq r, r < 1$.

We have

$$\left| \frac{z^{2n}}{1 - z^{2n}} \right| \leq \frac{|z|^{2n}}{1 - |z|^{2n}} < \frac{r^{2n}}{1 - r^{2n}}, \quad \text{when } r < 1.$$

Now, the series

$$\sum_{n=0}^{\infty} \frac{r^{2n}}{1 - r^{2n}}$$

is convergent for $0 \leq r < 1$. This can be verified by ratio test since

$$\frac{a_{n+1}}{a_n} = r^2 .$$

Hence, by Weierstrass M-test the given series is uniformly convergent for $|z| \leq r, r < 1$.

Now, we have two analogous results for the series of functions parallel to Theorems 8 and 9 for sequences of functions.

Proposition 8. Suppose $\Sigma_{n=1}^{\infty} f_n(z)$ converges uniformly to $f(z)$ on a set S. If each $f_n(z)$ is continuous on S, then the sum function is continuous.

Proof. Consider the sequence

$$s_n(z) = \sum_{k=1}^{k} f_k(z)$$

of nth partial sums. By definition, $s_n(z)$ converges uniformly to $s(z)$, the sum function of the series in S. Each $s_n(z)$ is continuous in S, hence, by Theorem 8, the limit function $s(z)$ is continuous.

Theorem 11 (Weierstrass theorem for series). Suppose that

1. $\Sigma_{n=1}^{\infty} f_k(z)$ converges uniformly to $f(z)$ on every compact subset of a domain Ω;
2. Each $f_n(z)$ is analytic in Ω.

Then the sum function $f(z)$ is analytic in the domain. Moreover, the series can be differentiated term by term any number of times, i.e.

$$\sum_{k=1}^{\infty} f_k^{(m)}(z) = f^{(m)}(z), \qquad z \in \Omega, \quad m = 1, 2, \ldots$$

and each differentiated series is uniformly convergent on every compact subset of Ω.

Proof. The proof proceeds on the lines of Theorem 9, provided we deal with the sequence of partial sums $s_n(z) = \Sigma_{k=1}^{n} f_k(z)$. Hence, the details are omitted.

Remark. One should not forget to mention the compactness of Theorem 11, e.g. the geometric series in Section 6.1 converges in $|z| < 1$, but not uniformly. However, it does converge uniformly on every compact subset of the unit disc $|z| < 1$, in particular on every disc $|z| \leq r$, $r < 1$.

Every time it is not possible to apply the Weierstrass M-test for verifying uniform convergence. When such a situation arises, we suggest using the following two tests due to Hardy. For the sake of space, we are constrained to give only the statements without their proofs.

Test-I. If, in a closed and bounded region D the series

(i) $\Sigma\, u_n(z)$ has uniformly bounded partial sums;
(ii) $\Sigma\, [(v_n(z) - v_{n+1}(z)]$ is uniformly and absolutely convergent;
(iii) $v_n(z) \to 0$ uniformly as $n \to \infty$;

then the series $\Sigma\, [u_n(z) - v_n(z)]$ converges uniformly in D.

Test-II. If, in a closed and bounded region D the series

(i) $\Sigma\, u_n(z)$ converges uniformly;

(ii) $\Sigma\,(v_n(z) - v_{n+1}(z))$ is absolutely convergent and has partial bounded sums; and

(iii) $v_0(z)$ is bounded,

then the series $\Sigma\,[u_n(z) - v_n(z)]$ converges uniformly in D.

PROBLEM SET 5.2

1. Consider the sequence of functions defined by

$$f_n(z) = \frac{1}{1+nz}, \qquad n = 0, 1, 2, \ldots$$

Decide the region of convergence and uniform convergence.

2. Where does each of the following sequences converge uniformly?

(a) $\left\{\dfrac{1}{1+z^n}\right\}$ (b) $\left\{\dfrac{z^n}{1+z^{2n}}\right\}$ (c) $\left\{\dfrac{\sin nz}{n}\right\}$

3. Let f_n be a sequence of continuous functions defined on a domain D such that

$$\int_\gamma f_n(z)dz = 0, \qquad n = 1, 2, \ldots$$

for every closed contour γ in D. If $f_n \to f$ uniformly on D, show that f is analytic in D.

4. If $f_n \to f$ uniformly on S and each $f_n(z)$ is bounded. Then prove that $f(z)$ is uniformly bounded on S.

5. If a sequence $\{f_n(z)\}$ is uniformly convergent on a compact set D, and every term $f_n(z)$ is continuous on D, show that $f_n(z)$ is uniformly bounded on D.

6. Let $\{f_n(z)\}$ be a uniformly convergent sequence on a set E. Then, show that $\{|f_n(z)|\}$ is also uniformly convergent. Moreover, if there exists a constant $M > 0$ such that $|\phi(z)| \le M$ for all $z \in E$, prove that the sequence $\{\phi(z)\, f_n(z)\}$ is uniformly convergent on E.

7. Assume that $f_n \to f$ uniformly on S, $g_n \to g$ uniformly on S. Prove that $f_n \pm g_n$ converges uniformly on S.

8. Suppose that $f_n \to f$ uniformly on D and $\{f_n\}$ is uniformly bounded on D. Let g be continuous on $|z| \le M$ and define $p_n(z) = g[f_n(z)]$. Prove that $p_n \to p$ uniformly on S, where $p(z) = g[f(z)]$.

9. For what values of z the following series converge and diverge?

(a) $z + \dfrac{1}{2}\dfrac{z^3}{3} + \dfrac{1}{2}\dfrac{3}{4}\dfrac{z^5}{x^5} + \dfrac{1}{2}\dfrac{3}{4}\dfrac{5}{6}\dfrac{z^7}{7} + \cdots$ (b) $1 + \dfrac{2z}{2!} + \dfrac{3^2 z^2}{3!} + \dfrac{4^3 z^3}{4!} + \cdots$

(c) $z + \dfrac{2^2 z^2}{2!} + \dfrac{3^3 z^3}{3!} + \dfrac{4^4 z^4}{4!} + \cdots$ (d) $\sqrt{\dfrac{1}{2}} + \sqrt{\dfrac{2}{5}}z + \sqrt{\dfrac{3}{10}}z^2 + \sqrt{\dfrac{4}{17}}z^3 + \cdots$

(e) $\dfrac{z}{z+1}+\dfrac{z^2}{z+2}+\dfrac{z^3}{z+3}+\cdots$

(f) $\dfrac{1}{|z|}+\dfrac{1}{|z|^2}+\dfrac{1}{|z|^3}+\dfrac{1}{|z|^4}+\cdots$

10. Find the sum and region of convergence of the series

$$\frac{1}{z(z+1)}+\frac{1}{(z+1)(z+2)}+\frac{1}{(z+2)(z+3)}+\cdots$$

11. Find the region of convergence of the series

$$\sum_{n=1}^{\infty}\frac{(z-i)^n}{(n+1)(n+2)}.$$

12. Discuss the following series with reference to uniform and absolute convergence.

$$\frac{1}{1+|z|}-\frac{1}{2+|z|}+\frac{1}{3+|z|}-\cdots$$

13. Show that the series

$$\sum_{n=1}^{\infty}\frac{z^n}{3^n+1}, \qquad |z|\le r,$$

where $r < 3$ converges uniformly in the indicated domain.

14. Show that the series

$$2\sin\left(\frac{1}{3z}\right)+4\sin\left(\frac{1}{9z}\right)+\cdots+2^n\sin\left(\frac{1}{3^n z}\right)+\cdots$$

converges absolutely for all nonzero complex numbers, but does not converge uniformly near $z = 0$.

15. If the series $\Sigma_{n=1}^{\infty} f_n(z)$ converges uniformly on every compact subset of a domain D and $\phi(z)$ be any function bounded in D, show that the series $\Sigma_{n=1}^{\infty}\phi(z)f_n(z)$ converges uniformly on every compact subset of the domain.

16. Determine the region in which the series $\Sigma_{n=1}^{\infty} n^{-z}$ converges. Does it converge uniformly in some domain? Find this domain.

Suggestion. This is the Riemann zeta function, denoted by $\zeta(z)$ and converges for $\operatorname{Re} z > 1$, since

$$|n^{-z}| = |e^{-z\log n}| = e^{-\operatorname{Re} z\ln n} = n^{-\operatorname{Re} z}.$$

Note that for $\operatorname{Re} z \ge \delta$, $\delta > 1$, the Riemann zeta function gives rise to an analytic function.

17. Show that $\Sigma_{n=0}^{\infty} e^{inz}$ converges for $\operatorname{Im} z > 0$.

18. Use the formula for geometric series with $z = re^{i\theta}$, where $r < 1$, to prove that

$$\sum_{n=0}^{\infty} z^n = \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2},$$

and hence verify the following sums:

(a) $\displaystyle\sum_{n=0}^{\infty} r^n \cos n\theta = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}$;

(b) $\displaystyle\sum_{n=0}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$.

19. Give the example of a divergent series of functions whose partial sums are bounded.

Suggestion. See Example 5.

20. Considering the logarithmic series of Example 6, for $-\pi < \theta < \pi$, verify the following Fourier series expansions:

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos n\theta}{n} = \ln\left(2\cos\frac{\theta}{2}\right)$; (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin n\theta}{n} = \frac{\theta}{2}$.

Comment on the outcome of (a) and (b) for $\theta = 0$ and $\theta = \pi/2$, respectively.

21. Let $\alpha \in \mathbb{C}$ be fixed such that $|\alpha| > 1$. Discuss the convergence of the series

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{\alpha^{|n|}}.$$

22. Test the following series for convergence:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} \tan\frac{z}{2^n}$; (b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2} \frac{z^n}{1 + z^n}$.

Suggestion. For (a), take $u_n = z/2^n$, $v_n = (1/z)\tan(z/2^n)$, and for (b), $u_n = 1/n^2$, $v_n = z^n/(1 + z^n)$. Now, apply the Hardy's first and second tests, respectively.

5.5 INFINITE PRODUCTS

Infinite products have close analogy with infinite series concerning convergence of different types and have immense applications in developing theory of entire functions and the gamma function in Chapter 11.

Definition 8. By an infinite product we mean an expression of the form

$$\prod_{n=1}^{\infty} u_n, \tag{5.15}$$

where u_n are nonzero complex numbers.

Let

$$P_n = u_1 u_2 \cdots u_n$$

be partial products of n factors from starting. Then the infinite product (5.15) is said to converge if the sequence of partial products $\{P_n\}$ converges to a nonzero limit and the limit of the sequence is called value of the infinite product, i.e. if

$$\lim_{n\to\infty} P_n = u \neq 0,$$

we write

$$\prod_{n=1}^{\infty} u_n = u.$$

An infinite product which does not converge to a nonzero (finite) limit is said to be divergent. Since

$$u_n = \frac{\prod_{k=1}^{n} u_k}{\prod_{k=1}^{n-1} u_k},$$

it follows that the necessary condition for the convergence of the infinite product (5.15) is the $\lim_{n\to\infty} u_n = 1$.

This suggests that we write the general term of the infinite product in the form

$$u_n = 1 + a_n.$$

where the necessary condition for convergence becomes $\lim_{n\to\infty} a_n = 0$.

The phrase *diverges to zero* may not be appealing to the reader at first sight. However, it will be clear at a later stage by observing the phenomenon that the nature of infinite product $\Pi(1 + a_n)$, $a_n \neq -1$ is completely characterized by the series $\Sigma \log(1 + a_n)$. Note that $a_n \to -1$ causes the series to diverge. Moreover, there are other justifications to exclude the zero. Suppose $u = 0$ is permitted, then any infinite product with one factor as zero would converge and the convergence will not depend on the whole sequence of factors. Very often, we will define some functions as infinite products and it may be possible to do so even if the function has zeros. Now, we elaborate the topic with some results.

Remark. If only a finite number of the terms $a_n = -1$, while the rest of the infinite product converges, the infinite product is said to converge. Note that for $a_n = a$ for all n and $|a| < 1$, $\prod_{k=1}^{\infty} z_n = 0$, although $\lim a_n = a \neq 0$.

Example 8. Show that

(a) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)+\cdots = \frac{1}{2}$;

(b) $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)+\cdots$ diverges.

For part (a), consider

$$a_{n-1} = 1 - \frac{1}{n^2} = \frac{n^2-1}{n^2} = \frac{(n-1)(n+1)}{n^2}, \qquad n \geq 2.$$

Hence,

$$P_n = \frac{1\cdot 3}{2^2}\cdot\frac{2\cdot 4}{3^2}\cdot\frac{3\cdot 5}{4^2}\cdots\frac{(n-1)(n+1)}{n^2} = \frac{1}{2}\left(1+\frac{1}{n}\right).$$

This implies $P_n \to 1/2$ as $n \to \infty$, and by Definition 8, the infinite product converges to 1/2.

For part (b), note that

$$P_n = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)$$

$$= \frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n} = n+1 \to \infty \quad \text{as} \quad n \to \infty.$$

Thus, the infinite product in part (b) diverges.

Theorem 12. The necessary and sufficient condition for the convergence of an infinite product is that, for every $\varepsilon > 0$, there exists a positive integer N such that

$$\left|\frac{P_{n+m}}{P_n} - 1\right| < \varepsilon \;\forall\; n \geq N, \qquad m = 1, 2, \ldots, \tag{5.16}$$

where P_n is the partial product of n factors of the infinite product.

Proof. Suppose P_n converges to a finite nonzero number. Then, a positive number α can be found such that

$$|P_n| > \alpha \quad \text{for} \quad n \geq N_1.$$

Also, for given $\varepsilon > 0$, we can find N_2 such that $n \geq N_2$ implies

$$|P_{n+m} - P_n| < \alpha\varepsilon, \qquad m = 0, 1, 2, \ldots$$

Hence, for $n \geq N = \max\{N_1, N_2\}$,

$$\left|\frac{P_{n+m}}{P_n} - 1\right| < \frac{\alpha\varepsilon}{|P_n|} < \varepsilon.$$

To prove the converse, assume that (5.16) holds. It means that for $n \geq N$,

$$|P_{n+m} - P_n| < \varepsilon P_n, \qquad m = 1, 2, \ldots$$

In particular,

$$|P_{N+m} - P_N| < \varepsilon P_N, \qquad m = 1, 2, \ldots$$

This is the Cauchy condition for the convergence of a sequence, refer to Theorem 4. So, $\{P_n\}$ converges, and by definition, an infinite product converges.

Remark. The condition (5.16) is analogous to the Cauchy condition for sequences and infinite series (refer to Sections 5.1 and 5.2).

The practical way of deciding the convergence of an infinite product is contained in our next result.

Theorem 13. The infinite product

$$\prod_{n=1}^{\infty} (1 + a_n) \quad \text{with } a_n \neq -1 \tag{5.17}$$

converges if and only if the infinite series

$$\sum_{n=1}^{\infty} \log (1 + a_n) \tag{5.18}$$

converges, where each term involving the logarithm represents the principal branch.

Proof. Let us define

$$P_n = \prod_{k=1}^{n} (1 + a_k), \qquad S_n = \sum_{k=1}^{n} \log (1 + a_k).$$

In order to prove that the condition is sufficient, suppose that $S_n \to S$. Now,

$$S_n = \log (1 + a_1)(1 + a_2) \cdots (1 + a_n) = \log P_n$$

and we agree on the principal branch in each term, then $P_n = e^{S_n}$, and if $S_n \to S$, it follows that P_n tends to e^S, which is nonzero.

To prove the necessary condition, suppose that $P_n \to P \neq 0$. It is not true that the series (5.18) formed with the principal values converges to the principal value of $\log P$ (see Section 3.6) and hence principal value of P_n may not converge to the principal value of $\log P$. What we wish to show is that it converges to some value of $\log P$. For

$$P_n = (1 + a_1)(1 + a_2) \cdots (1 + a_n),$$

note that

$$\log P_n = \log(1 + a_1) + \log(1 + a_2) + \cdots + \log(1 + a_n) + 2\pi i k_n. \tag{5.19}$$

Now, we show that k_n is not necessarily zero but constant for large values of n. Let β_n and α_n be the imaginary parts of $\log P_n$ and $\log (1 + a_n)$, respectively. Equating imaginary parts of both sides of (5.19), we get

$$\beta_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n + 2\pi k_n.$$

This implies

$$\beta_{n+1} - \beta_n = \alpha_{n+1} + 2\pi (k_{n+1} - k_n). \tag{5.20}$$

Since $1 + a_n = P_n/P_{n-1}$, it follows that

$$\lim_{n\to\infty}(1 + a_n) = \lim_{n\to\infty} \frac{P_n}{P_{n-1}} = 1.$$

Note that we are considering the principal value, and this gives

$$\lim_{n\to\infty} \log(1 + a_n) = \log 1 = 0.$$

The practical way of deciding the convergence of an infinite product is contained in our next result.

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converges if and only if the infinite series

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Proof. Let us define

$$P_n = \prod_{k=1}^{n} (1 + a_k), \qquad S_n = \sum_{k=1}^{n} \log (1 + a_k).$$

In order to prove that the condition is sufficient, suppose that $S_n \to S$. Now,

$$S_n = \log (1 + a_1)(1 + a_2) \cdots (1 + a_n) = \log P_n$$

and we agree on the principal branch in each term, then $P_n = e^{S_n}$, and if $S_n \to S$, it follows that P_n tends to e^S, which is nonzero.

To prove the necessary condition, suppose that $P_n \to P \neq 0$. It is not true that the series (5.18) formed with the principal values converges to the principal value of $\log P$ (see Section 3.6) and hence principal value of P_n may not converge to the principal value of $\log P$. What we wish to show is that it converges to some value of $\log P$. For

$$P_n = (1 + a_1)(1 + a_2) \cdots (1 + a_n);$$

note that

$$\log P_n = \log(1 + a_1) + \log(1 + a_2) + \cdots + \log(1 + a_n) + 2\pi i k_n. \tag{5.19}$$

Now, we show that k_n is not necessarily zero but constant for large values of n. Let β_n and α_n be the imaginary parts of $\log P_n$ and $\log (1 + a_n)$, respectively. Equating imaginary parts of both sides of (5.19), we get

$$\beta_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n + 2\pi k_n.$$

This implies

$$\beta_{n+1} - \beta_n = \alpha_{n+1} + 2\pi (k_{n+1} - k_n). \tag{5.20}$$

Since $1 + a_n = P_n/P_{n-1}$, it follows that

$$\lim_{n\to\infty} (1 + a_n) = \lim_{n\to\infty} \frac{P_n}{P_{n-1}} = 1.$$

Note that we are considering the principal value, and this gives

$$\lim_{n\to\infty} \log(1 + a_n) = \log 1 = 0.$$

This means that the imaginary part of $\lim \log(1 + a_n)$ is zero, and hence $\lim \alpha_n = 0 = \lim \alpha_{n+1}$. Using this in (5.20), and given that $\lim \beta_n = \lim \beta_{n+1}$, we conclude that

$$\lim_{n\to\infty}(k_{n+1} - k_n) = 0 .$$

Hence $\lim k_n$ is some constant which is independent of n. So,

$$\sum_{n=0}^{\infty} \log(1 + a_n) = P - 2\pi k$$

a finite complex number, as asserted.

Absolute convergence. Like infinite series, the rearrangement of factors of absolutely convergent infinite product does not alter its limiting value.

Definition 9. The infinite product $\Pi_{n=1}^{\infty}(1 + a_n)$ is said to be absolutely convergent if the corresponding series (5.18) converges absolutely.

Every absolutely convergent infinite product is convergent. This is the direct consequence of the definition.

Theorem 14. The infinite product $\Pi_{n=1}^{\infty}(1 + a_n)$ is absolutely convergent if and only if the infinite series $\Sigma_{n=0}^{\infty} a_n$ is absolutely convergent.

Proof. Note that

$$\lim_{z\to 0}\frac{\log(1+z)}{z} = 1 .$$

Since $a_n \to 0$, we can replace z by a_n and observe that for a given $\varepsilon > 0$,

$$(1-\varepsilon)|a_n| < |\log(1 + a_n)| < (1 + \varepsilon)|a_n|, \qquad n > N. \tag{5.21}$$

In view of (5.21) the series $\Sigma_{n=1}^{\infty}\log(1 + a_n)$ and $\Sigma_{n=0}^{\infty} a_n$ are simultaneously convergent. Thus, in view of Theorem 13, we have completed the proof of Theorem 14.

Corollary 2. If $-1 < a_n \le 1$, then the infinite product $\Pi_{n=1}^{\infty}(1 + a_n)$ and the series $\Sigma_{n=1}^{\infty} a_n$ converge or diverge together.

Proof. First, let $0 \le a_n \le 1$. Then, in view of Theorem 14, the infinite product $\Pi_{n=1}^{\infty}(1 + a_n)$ is absolutely convergent if and only if the infinite series $\Sigma_{n=0}^{\infty} a_n$ is convergent. Since $a_n \ge 0$, both concepts are equivalent.

To prove that $-1 < a_n \le 0$, let $a_n = -b_n$. Then $b_n \ge 0$, and again, by the above reasoning the result follows.

Corollary 3. If $a_n \ne -1$ and $\Sigma_{n=1}^{\infty} a_n^2$, $a_n \in \mathbb{R}$ converges, then the infinite product $\Pi_{n=1}^{\infty}(1 + a_n)$ and the series $\Sigma_{n=1}^{\infty} a_n$ converge or diverge together.

Proof. Since $a_n \to 0$ as $n \to \infty$, for given $\varepsilon = 1/4$, we can find n such that $n \ge N$,

$$a_n^2 < \frac{1}{4} \quad \text{or} \quad |a_n| < \frac{1}{2}.$$

Using (5.21), we have

$$\frac{|a_n|}{2} < |\log(1 + a_n)| < \frac{3|a_n|}{2}.$$

Now the result is obvious.

Infinite product of functions. The foregoing discussion is restricted to infinite product of complex numbers. However, we have to pay a lot of attention to studies of infinite products whose factors are functions of a complex variable. The presence of zeros may create some difficulty, which may be overcome by considering only sets on which only a finite number of factors may vanish. Actually, these finite factors may be separated out for the proper functioning of the remaining infinite product. All theorems and propositions proved so far may be extended to infinite products with complex functions as their factors with some modifications.

Definition 10 (Uniform convergence). An infinite product

$$\prod_{n=1}^{\infty} [1 + a_n(z)], \tag{5.22}$$

where each $v_n(z)$ is a function defined on a given set E, is said to be uniformly convergent on E, if the sequence of partial products

$$P_n(z) = [1 + a_n(z)][1 + a_2(z)] \cdots [1 + a_n(z)]$$

is uniformly convergent.

Let us quote two results (without proof), which are helpful in determining the uniform convergence of infinite products.

Result 1. Suppose every term of the sequence $\{a_n(z)\}$ is analytic in a domain Ω, and suppose that the infinite series

$$\sum_{n=1}^{\infty} \ln [1 + a_n(z)]$$

is uniformly convergent on every compact subset of Ω. Then the infinite product (5.22) converges uniformly to a nonvanishing analytic function $f(z)$ on every compact subset of the domain.

Result 2. Suppose every term of the sequence $\{a_n(z)\}$ is analytic in a domain Ω, and suppose that none of the terms $a_n(z)$ takes the value -1 at any point of Ω. Moreover, there is a convergent series $\Sigma_{n=1}^{\infty} M_n$ whose terms are nonnegative constants, such that

$$|a_n(z)| \le M_n$$

for all $z \in \Gamma$ and all n exceeding a certain integer $N > 0$. Then the infinite product (5.22) converges uniformly to a nonvanishing analytic function.

Remarks. Result 2 also provides a way to test the absolute convergence of infinite product (5.22) with complex functions as its factors defined on Ω.

Example 9. Discuss the convergence of

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Let $a_n(z) = -z^2/n^2$. Then $|a_n(z)| = |z|^2/n^2$, and the series

$$\sum_{n=0}^{\infty} |a_n(z)| \le |z|^2 \sum_{n=0}^{\infty} \frac{1}{n^2}$$

converges for all finite values of z (see Result 2). Hence, by Theorem 14, the infinite product is absolutely convergent and thus convergent.

Example 10. Prove that

(a) $\displaystyle\prod_{n=1}^{\infty} \left(1 - \frac{z}{c+n}\right) e^{z/n}$

converges absolutely for all z, provided that c is not a negative integer, and that

(b) $\displaystyle\prod_{n=1}^{\infty} \left[1 - \left(\frac{nz}{n+1}\right)^n\right]$

converges absolutely in $|z| < 1$.

For (a), note that

$$\left(1 - \frac{z}{c+n}\right) e^{z/n} = \left(1 - \frac{z}{c+n}\right)\left(1 + \frac{z}{n} + \frac{z^2}{2!n^2} + \frac{z^3}{3!n^3} + \cdots\right)$$

$$= 1 + \frac{c}{n(n+c)} z + \frac{n-c}{2n^2(n+c)} z^2 + \cdots$$

Here,

$$a_n = 1 + \frac{c}{n(n+c)} z + \frac{n-c}{2n^2(n+c)} z^2 + \cdots$$

Comparing with $b_n = 1/n^2$, it follows that

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = |cz|$$

a finite quantity. Therefore, $\Sigma_{n=1}^{\infty}(a_n)$ is absolutely convergent. This implies that the infinite product $\Pi(1 + a_n)$ is absolutely convergent.

For (b),

$$a_n(z) = -\left(\frac{nz}{n+1}\right)^n.$$

The product is convergent if the series

$$\sum_{n=1}^{\infty} -\left(\frac{n}{n+1}\right)^n z^n$$

is absolutely convergent. Since $\lim_{n\to\infty} |a_n(z)|^{1/n} = z$, it follows that the series $\Sigma_{n=1}^{\infty} a_n(z)$ and the infinite product $\Pi(1 + a_n(z))$ are absolutely convergent. For $|z| \ge 1$, the product is divergent.

Example 11. Discuss the convergence of the infinite product

$$\prod_{n=1}^{\infty} \frac{z+z^{2n}}{1+z^2}.$$

Here,

$$1+a_n = \frac{z+z^{2n}}{1+z^{2n}} \Rightarrow a_n = \frac{z-1}{1+z^{2n}}.$$

When $|z| > 1$, the series $\Sigma_{n=1}^{\infty} z^{-2n}$ converges, and since,

$$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} \left| \frac{z-1}{1+z^{2n}} \right| < \sum_{n=0}^{\infty} \left| \frac{1}{z^{2n}} \right|$$

the series $\Sigma_{n=1}^{\infty} a_n$ is absolutely convergent. Thus, the given product is absolutely convergent.

For $|z| < 1$,

$$1+a_n = \frac{z(1-z^{2n-1})}{1+z^{2n}} \to z \quad \text{as } n \to \infty.$$

This means that a_n does not tend to zero. Hence the product is divergent.

If $z = -1$, every factor is zero and $a_n \to -1$ as $n \to \infty$ and hence the series $\Sigma_{n=1}^{\infty} a_n$ is divergent. It follows that the $\Pi(1+a_n)$ is divergent.

Let us prove a result on uniform convergence which finds applications in the construction of entire functions with prescribed zeros (see Section 11.5, Chapter 11).

Proposition 9. Let $\{a_n\}$ be a sequence of complex numbers (finite repetition permitted) such that $a_n \to \infty$ as $n \to \infty$. Then, there exists integers $M_n \geq 0$ such that the infinite product

$$\prod_{n=1}^{\infty} \left(1-\frac{z}{a_n}\right) \exp\left[\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right] \tag{5.23}$$

is uniformly convergent on every compact subset of $\mathbb{C}$.

Proof. Fix $R > 0$. Let $z : |z| \leq R$. Choose n so large, or to say choose $n > N$ such that

$$|a_n| > \varepsilon R$$

Since $|z/a_n| < 1$, we have

$$\text{Log}\left(1-\frac{z}{a_n}\right) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \cdots - \frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n} - \cdots$$

The series on the right side converges uniformly (being power series) to the function on the left side for all $|z| \leq R$. Consider the remainder series

$$r_n(z) = \text{Log}\left(1-\frac{z}{a_n}\right) + \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}$$

$$= -\frac{1}{M_n+1}\left(\frac{z}{a_n}\right)^{M_n+1} - \frac{1}{M_n+2}\left(\frac{z}{a_n}\right)^{M_n+2} - \cdots$$

We can easily obtain the estimate

$$|r_n(z)| \le \frac{1}{M_n+1}\left(\frac{R}{|a_n|}\right)^{M_n+1}\left(1-\frac{R}{|a_n|}\right)^{-1}.$$

Suppose that the series

$$\sum_{n=0}^{\infty}\frac{1}{M_n+1}\left(\frac{R}{|a_n|}\right)^{M_n+1}$$

Now, choose positive integers M_n, so that

$$M_n + 1 \ge \ln(n^2) \ \forall\ n.$$

Then

$$\left|\frac{R}{a_n}\right|^{M_n+1} < \frac{1}{e^{M_n+1}} < \frac{1}{n^2}.$$

In this way the series

$$\sum_{n=1}^{\infty}\frac{1}{M_n+1}\left|\frac{R}{a_n}\right|^{M_n+1}$$

converges and serves as a majorant for $\Sigma_{n=1}^{\infty} r_n(z)$. Thus, $\Sigma_{n=1}^{\infty} r_n(z)$ is absolutely and uniformly convergent in $|z| \le R$ (see Proposition 7), and in view of continuity of the exponential function this implies that the product (5.23) represents an analytic function in $|z| < R$ (see also Result 2). However, uniform convergence is not affected when $|a_n| \ge eR$ is taken into account. Since R is fixed but arbitrary, the product (5.23) gives rise to an entire function.

PROBLEM SET 5.3

1. Investigate the convergence of

(a) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{3/2}}\right)$; (b) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)$;

(c) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{n}{\sqrt{n^2+1}}\right)$; (d) $\displaystyle\prod_{n=1}^{\infty}(1+\cot^{-1}n^2)$.

2. Show that the infinite product

$$\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)\cdots$$

converges absolutely and uniformly on any compact set D which contains none of the points $\pm1, \pm2, \ldots$, whereas the product

$$\left(1-\frac{z}{1}\right)\left(1-\frac{z}{1}\right)\left(1-\frac{z}{2}\right)\left(1+\frac{z}{2}\right)\cdots$$

converges uniformly but not absolutely in the same region.

3. Discuss the convergence of the infinite products:

(a) $\prod_{n=1}^{\infty}\left(1+\frac{z}{n\pi}\right)e^{-z/n\pi}$;

(b) $\prod_{n=1}^{\infty}\left(\frac{n-z}{n+z}e^{2z/n}\right)$;

(c) $\prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n+1}z}{n}\right)$;

(d) $\prod_{n=1}^{\infty}\left(1+\frac{e^{inz}}{n^2}\right)$.

4. Prove that

$$\prod_{n=1}^{\infty}\frac{n^z+n^2+1}{n^z+n^2-1}$$

represents an analytic function in the domain $\operatorname{Re} z > 2$.

Suggestion. Here $a_n(z) = 2/(n^z + n^2 - 1)$. Take $\operatorname{Re} z \geq 2 + \delta$, $\delta > 0$, and apply the M-test to establish uniform convergence, since

$$|a_n(z)| \leq \frac{2}{n^{2+\delta} - n^2 + 1} \quad \forall \operatorname{Re} z \geq 2 + \delta.$$

5. If $w_n(z)$, $n = 1, 2, \ldots$, are the analytic functions in a domain D and $\Sigma_{n=1}^{\infty} w_n(z)$ converges uniformly in D, show that $\Pi_{n=1}^{\infty}[1 + w_n(z)]$ converges to an analytic function in D.

6. Where does each of the following infinite products converge absolutely?

(a) $\prod_{n=1}^{\infty}(1-z^n)$; (b) $\prod_{n=1}^{\infty}\cos\left(\frac{z}{n}\right)$; (c) $\prod_{n=1}^{\infty}\frac{\sin(z/n)}{z/n}$; (d) $\prod_{n=1}^{\infty}\left(1-\frac{1}{n^z}\right)$

7. Find the value of the infinite product

$$\prod_{n=0}^{\infty}(1+z^{2^n}); \quad |z|<1.$$

8. Let $\{a_n(z)\}$ be a sequence defined on a domain Ω such that no $a_n(z)$ is identically zero. If

$$\sum_{n=1}^{\infty} a_n(z) - 1$$

converges absolutely and uniformly on compact subsets of Ω, show that

$$\prod_{n=1} a_n(z)$$

converges to an analytic function in Ω.

Suggestion. This exercise is some what similar to Theorem 14 when a_n is replaced by $a_n - 1$. Here the product factors are complex functions.

6

Series Expansions

In Chapter 5, we have studied series whose terms are complex functions. This chapter is devoted to a specific infinite series known as power series. In Sections 6.1–6.2 we start by defining a power series and discuss its uniform convergence followed by term-by-term integration and differentiation. In Sections 6.3–6.5, we deal with Taylor series, zeros and Laurent series of analytic functions, respectively. The last two sections include solutions of different problems based on multiplication and division of power series and the theory developed in Sections 6.1–6.5.

6.1 POWER SERIES

Power series are very useful in dealing with various problems in engineering and this has made it necessary to study these series separately. These series arise in solutions of second-order differential equations where coefficients are variables. Power series methods are considered efficient tools for solving various physical and engineering problems.

Definition 1. Given a sequence $\{a_n\}$ of complex numbers the series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{6.1}$$

is called the *power series* about the point z_0. In particular, the series

$$\sum_{n=0}^{\infty} a_n z^n \tag{6.2}$$

is a power series about the origin. There is no loss of generality if we study the various properties of series (6.2), since generalization of results to be applicable for (6.1) in most cases merely requires the substitution of $(z-z_0)$ for z.

We define

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}\,. \tag{6.3}$$

Equation (6.3) is known as the *Hadamard formula*. The quantity R is called the *radius of convergence* of the power series. Since

$$\limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|z|}{R},$$

it follows by the Cauchy root test that (6.2) converges absolutely for $|z| < R$ and diverges for $|z| > R$. We define $|z| = R$ as the circle of convergence.

In situations in which $\lim_{n\to\infty}|a_{n+1}|/|a_n|$ exists, it is simpler to compute the radius of convergence using the ratio formula. Recalling Eq. (5.9), Section 5.2,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|. \tag{6.4}$$

Example 1. Find the radius of convergence of the series

(a) $\sum_{n=0}^{\infty} n^n z^n$; (b) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$; (c) $\sum_{n=0}^{\infty} \frac{z^n}{n}$; (d) $\sum_{n=1}^{\infty} \frac{z^{2n}}{2^n}$.

For (a), consider (6.3), then

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} n = \infty \Rightarrow R = 0.$$

For (b), using (6.4), we get

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0 \Rightarrow R = \infty.$$

For (c),

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1 \Rightarrow R = 1.$$

In case of (d), $\lim|a_{n+1}|/|a_n|$ does not exist, hence the formula (6.4) cannot be applied. For this we proceed through the definition. Note that

$$a_n = \begin{cases} 0, & n = 2k-1, \\ 2^{-n}, & n = 2k. \end{cases} \quad k = 1, 2, \ldots$$

Now,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{k\to\infty} |2^{-2k}|^{1/(2k)} = \frac{1}{2}$$

and

$$\liminf_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{k\to\infty} |a_{2k-1}|^{1/(2k-1)} = 0.$$

Hence, $|z|^2 = 2$, i.e. $R = \sqrt{2}$.

Note. The procedure is simplified if z^2 is replaced by s. Then, $|z|^2 = |s| = R$.

Equation (6.3) is known as the *Hadamard formula*. The quantity R is called the *radius of convergence* of the power series. Since

$$\limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|z|}{R},$$

it follows by the Cauchy root test that (6.2) converges absolutely for $|z| < R$ and diverges for $|z| > R$. We define $|z| = R$ as the circle of convergence.

In situations in which $\lim_{n\to\infty}|a_{n+1}|/|a_n|$ exists, it is simpler to compute the radius of convergence using the ratio formula. Recalling Eq. (5.9), Section 5.2,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|. \tag{6.4}$$

Example 1. Find the radius of convergence of the series

(a) $\sum_{n=0}^{\infty} n^n z^n$; (b) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$; (c) $\sum_{n=0}^{\infty} \frac{z^n}{n}$; (d) $\sum_{n=1}^{\infty} \frac{z^{2n}}{2^n}$.

For (a), consider (6.3), then

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} n = \infty \Rightarrow R = 0.$$

For (b), using (6.4), we get

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0 \Rightarrow R = \infty.$$

For (c),

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1 \Rightarrow R = 1.$$

In case of (d), $\lim|a_{n+1}|/|a_n|$ does not exist, hence the formula (6.4) cannot be applied. For this we proceed through the definition. Note that

$$a_n = \begin{cases} 0, & n = 2k-1, \\ 2^{-n}, & n = 2k. \end{cases} \quad k = 1, 2, \ldots$$

Now,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{k\to\infty} |2^{-2k}|^{1/(2k)} = \frac{1}{2}$$

and

$$\liminf_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{k\to\infty} |a_{2k-1}|^{1/(2k-1)} = 0.$$

Hence, $|z|^2 = 2$, i.e. $R = \sqrt{2}$.

Note. The procedure is simplified if z^2 is replaced by s. Then, $|z|^2 = |s| = R$.

6.2 UNIFORM CONVERGENCE OF POWER SERIES

In Chapter 5, we have already defined uniform convergence of general infinite series. Here, we illustrate some applications of this concept with reference to power series.

Proposition 1. If z_1 is a point interior to the circle of convergence $|z| = R$ of a power series $\Sigma_{n=0}^{\infty} a_n z^n$, then the series is uniformly convergent in the closed disc $|z| \le |z_1|$.

Proof. We simply observe that $|z| \le |z_1| < R$. Hence,

$$|a_n z^n| \le |a_n|\,|z_1^n| \quad \text{for each} |z| \le |z_1|,$$

and the Weierstrass M-test is applicable with the choice $M_n = |a_n|\,|z_1|^n$. For then

$$\sum_{n=0}^{\infty} M_k < \infty$$

using the definition of radius of convergence.

Example 2. Prove that the geometric series $\Sigma_{n=0}^{\infty} z^n$ converges uniformly in any closed disc $|z| \le r < 1$. However, it is not uniformly convergent in its whole disc of convergence $|z| < 1$.

The radius of convergence of the geometric series $\Sigma_{k=0}^{\infty} a_n x^n$ is 1. Hence, by Proposition 1, this converges uniformly in every closed interval $|x| \le r < 1$.

To prove that the convergence is not uniform in $|z| < 1$, consider

$$s(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \sum_{k=0}^{n-1} z^k + \sum_{k=n}^{\infty} z^k = s_n(z) + \frac{z^n}{1-z}.$$

Note that

$$|s_n(z) - s(z)| = \left|\frac{z^n}{1-z}\right| = \frac{|z|^n}{|1-z|}.$$

For a given real K (no matter how large) and n we can always find a z in the unit disc $|z| < 1$ such that

$$\left|\frac{z^n}{1-z}\right| = \frac{|z|^n}{|1-z|} > K,$$

simply by taking z sufficiently close enough to 1. Hence, no single $N(\varepsilon)$ will suffice to make $|s_n(z) - s(z)|$ smaller than a given $\varepsilon > 0$ throughout the whole disc. By definition this shows that the convergence of the geometric series in $|z| < 1$ is not uniform.

Proposition 2. Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ and $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$ converge for $|z| < R_1$ and $|z| < R_2$, respectively. If there exists a sequence $\{z_k\}$ in $0 < |z| < \delta < \min\{R_1, R_2\}$ such that $z_k \to 0$ as $k \to \infty$ and $f(z_k) = g(z_k)$ for all k, then $a_n = b_n$ for all n.

Proof. The functions f and g are continuous on $|z| \le \delta$, being analytic on larger sets. This implies

$$\lim_{n\to\infty} f(z_k) = \lim_{n\to\infty} g(z_k) \Rightarrow f(0) = g(0).$$

But, $f(0) = a_0$ and $g(0) = b_0$. Hence, $a_0 = b_0$. Now, to complete the proof, we use induction. Assume that

$$a_n = b_n \quad \text{for } n = 0, 1, \ldots, m-1.$$

Note that

$$\begin{aligned} f(z_k) = g(z_k) &\Rightarrow \sum_{n\ge m} a_n z_k^n = \sum_{n\ge m} b_n z_k^n \\ &\Rightarrow \sum_{n\ge m} a_n z_k^{n-m} = \sum_{n\ge m} b_n z_k^{n-m}, \quad \text{since } z_k \ne 0 \\ &\Rightarrow F(z_k) = G(z_k) \end{aligned}$$

where

$$F(z) = \sum_{n\ge m} a_n z^{n-m}, \qquad G(z) = \sum_{n\ge m} b_n z^{n-m}.$$

Note that the radius of convergence of

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=m}^{\infty} a_n z^{n-m}.$$

are same. Also, the radius of convergence of $\Sigma_{n=0}^{\infty} b_n z^n$ and $\Sigma_{n\ge m}^{\infty} b_n z^{n-m}$ are same. This ensures that $F(z)$ and $G(z)$ are continuous on $|z| \le \delta$. Thus

$$\begin{aligned} F(z_k) = G(z_k) &\Rightarrow \lim_{k\to\infty} F(z_k) = \lim_{k\to\infty} G(z_k) \\ &\Rightarrow F(0) = G(0) \\ &\Rightarrow a_m = b_m. \end{aligned}$$

This concludes that $a_n = b_n$ for every n.

Theorem 1. Let γ denote any contour interior to the circle of convergence of the power series $f(z) = \Sigma_{k=0}^{\infty} a_k z^k$, and let $g(z)$ be any continuous function defined on γ. Then the series formed by multiplying each term of the power series by $g(z)$ can be integrated term-by-term as

$$\int_\gamma g(z) f(z)\, dz = \sum_{k=0}^{\infty} a_k \int_\gamma g(z) z^k\, dz.$$

Proof. Writing the power series as

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = s_n(z) + \rho_n(z),$$

where $s_n(z)$ is the nth partial sum and $\rho_n(z)$ is the remainder of the given series after the nth term. Since $f(z)$ and also the finite sum $s_n(z)$ are continuous on γ, the integral of $\rho_n(z)$ must exist on γ, and utilizing the continuity of $g(z)$ on γ,

$$\int_\gamma g(z)f(z)dz = \sum_{k=0}^{n-1} a_k \int_\gamma g(z)z^k dz + \int_\gamma g(z)\rho_n(z)dz. \tag{6.5}$$

Let L denote the length of the contour γ. In view of the uniform convergence of the series for each $\varepsilon > 0$, there exists a positive integer N such that $n \geq N$ implies

$$|s_n(z) - f(z)| = |\rho_n(z)| < \varepsilon \quad \forall\, z \text{ on } \gamma.$$

Again, being continuous, $|g(z)| \leq M$ on γ. Since N is independent of z, we find that for $n \geq N$,

$$\left| \int_\gamma g(z)\rho_n(z)\, dz \right| < ML\varepsilon.$$

Letting $n \to \infty$,

$$\lim_{n\to\infty} \left| \int_\gamma g(z)\rho_n(z)\, dz \right| = 0.$$

Using (6.5), we have

$$\int_\gamma g(z)f(z)dz = \lim_{n\to\infty} \sum_{k=0}^{n-1} a_k \int_\gamma g(z)z^k dz.$$

as asserted by the theorem.

Corollary 1. The power series $f(z) = \Sigma_{k=0}^{\infty} a_k z^k$ can be integrated term-by-term along any contour γ contained inside its circle of convergence, i.e.

$$\int_\gamma f(z)dz = \sum_{k=0}^{\infty} a_k \int_\gamma z^k dz.$$

Proof. The proof merely requires the choice of $g(z) = 1$ for all z on γ in Theorem 1. It can be easily verified (using the ratio test) that the integrated power series has the same radius of convergence as that of the original series.

Interior to its circle of convergence each power series gives rise to a continuous function. The proof of this simple assertion is left to reader (see Proposition 8 in Section 5.4). However, we prove much more than this in the next theorem.

Theorem 2. Let the radius of convergence of $\Sigma_{n=0}^{\infty} a_n z^n$ be R, $R > 0$. Then the function defined by the equation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in $|z| < R$, and moreover,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Proof. Let z be an arbitrary point, and let γ be any positively oriented simple closed contour enclosing and contained in $|z| < R$. Then, by Corollary 1,

$$\int_\gamma f(z)dz = \sum_{n=0}^{\infty} a_n \int_\gamma z^n dz = 0$$

for every closed contour γ lying in $|z| < R$. Now, by Morera's theorem, $f(z)$ is analytic in the domain $|z| < R$.

We define the function

$$g(s) = \frac{1}{2\pi i(s-z)^2}.$$

Clearly, $g(s)$ is continuous on γ.

Since $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ is analytic within and on γ, we can write

$$\int_\gamma g(s)f(s)ds = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^2}\,ds = f'(z) \tag{6.6}$$

with the application of Cauchy integral formula. Also, by Theorem 1,

$$\begin{aligned}\int_\gamma g(s)f(s)ds &= \sum_{n=0}^{\infty} a_n \int_\gamma g(s)s^n ds \\ &= \sum_{n=0}^{\infty} a_n \frac{1}{2\pi i}\int_\gamma \frac{s^n}{(s-z)^2}\,ds \\ &= \sum_{n=0}^{\infty} a_n \frac{d}{dz} z^n \\ &= \sum_{n=1}^{\infty} a_n n z^{n-1}. \end{aligned} \tag{6.7}$$

Comparing (6.6) with (6.7), the theorem is proved.

Corollary 2. If $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ converges in $|z| < R$, then the derivatives of all orders exist in $|z| < R$ and

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)\,a_n z^{n-k}, \quad k = 0, 1, 2, \cdots$$

Proof. The proof may be given by the induction process. To verify the fact that the kth derived series has also the same radius of convergence, we note that

$$\left|\frac{n!}{(n-k)!}a_n\right|^{1/n} = \left|\frac{n!}{(n-k)!}\right|^{1/n} |a_n|^{1/n}. \tag{6.8}$$

Since

$$\limsup_{n\to\infty}\left[\frac{n!}{(n-k)!}\right]^{1/n} = \lim_{n\to\infty}\left[n^k\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right]^{1/n}$$

$$= \lim_{n\to\infty} n^{k/n}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right]^{1/n}$$

$$= 1.$$

Thus, (6.8) confirms that the derived series and the original series have the same radius of convergence.

What we have concluded is that every power series represents an analytic function interior to its circle of convergence, and it can be integrated and differentiated term by term. Moreover, the derivatives of all orders of a power series also give rise to analytic functions interior to the same circle of convergence.

A complex function is even if $f(z) = f(-z)$ in D and odd if $f(-z) = -f(z)$ in D. Note that $f(z) = z^2$ is even in $|z| \le 2$ but we cannot say anything about $f(z) = z^2 : z = 2e^{i\theta}$, $0 \le \theta \le \pi$. The domain must be symmetrical such that z and $-z$ must be in D.

Remark. Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$. Then (a) $a_n = 0$ for all $n = 0, 2, 4, \ldots$, if $f(z)$ is an odd function; (b) $a_n = 0$ for all $n = 1, 3, \ldots$, if $f(z)$ is an even function.

For proof we just write the series for $f(z)$ and $f(-z)$. Then addition and subtraction operations imply (a) and (b).

Note that a power series can behave in any of the following three ways on its circle of convergence Γ:

(i) The series may diverge everywhere on Γ, as the geometric series $\Sigma_{n=0}^{\infty} z^n$.
(ii) The series may converge at some of the points of Γ and diverge at other points of Γ, as the logarithmic series $\Sigma_{n=1}^{\infty}(-1)^n z^n/n$, which converges for $z = 1$ and diverges for $z = -1$.
(iii) The series may converge for all points on Γ, as the series $\Sigma_{n=0}^{\infty} z^2/n^2$ which converges for all $|z| = 1$.

Now, we explain in detail what the convergence of a power series at a point $s \in \Gamma$ implies in terms of the behaviour of the sum of the series in a neighbourhood of s.

Theorem 3 (Abel's limit theorem). If $\Sigma_{n=0}^{\infty} a_n$ converges, then the power series $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ with $R = 1$ tends to $f(1)$ as $z \to 1$, provided $|1-z|/(1-|z|)$ remains bounded.

Proof. To prove this we do not lose anything if we assume $\Sigma_{n=0}^{\infty} a_n = 0$. This can be done by adding a constant to a_0. So, $f(1) = 0$, and we write

$$S_n \equiv S_n(1) = a_0 + a_1 + \cdots + a_n$$

and making use of the identity (summation by parts),

$$S_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

$$= S_0 + (S_1 - S_0)z + (S_2 - S_1)z^2 + \cdots + (S_n - S_{n-1})z^n$$

$$= (1-z)(S_0 + S_1 z + \cdots + S_{n-1}z^{n-1}) + S_n z^n.$$

But $S_n z^n \to 0$ as $n \to \infty$ ($f(1) = 0$), $|z| \le 1$) so that we can write

$$f(z) = (1-z)\sum_{n=0}^{\infty} S_n z^n.$$

Since $|1-z|/(1-|z|)$ remains bounded, there exists a constant K such that

$$\frac{|1-z|}{1-|z|} \le K.$$

Again, for given $\varepsilon > 0$, there is a positive integer m such that $n \ge m$ implies $|S_n| \le \varepsilon$.

Finally, we say that the remainder $\Sigma_{n\ge m} S_n z^n$, is then dominated by the geometric series

$$\varepsilon \sum_{n\ge m}^{\infty} |z^n| = \frac{\varepsilon |z|^m}{1-z} < \frac{\varepsilon}{1-|z|}.$$

Hence,

$$|f(z)| \le |1-z| \left| \sum_{n=0}^{m-1} S_n z^n \right| + K\varepsilon.$$

The first term on the right can be made arbitrarily small by choosing z sufficiently close to 1, and we conclude that $f(z) \to 0$ when $z \to 1$, subject to the stated condition. This completes the proof.

PROBLEM SET 6.1

1. Find the radius of convergence of the following power series:

(a) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{z}{2}\right)^n$; (b) $\displaystyle\sum_{n=0}^{\infty} 2^{2n}\left(z-\frac{1}{2}\right)^n$; (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(z-i)^n}{n^2}$; (d) $\displaystyle\sum_{n=1}^{\infty} 4^n z^{3n}$.

2. Determine the radius of convergence of the following power series:

(a) $f(z) = 1 + z + \dfrac{z^2}{2^2} + \dfrac{z^3}{3!} + \dfrac{z^4}{2^4} + \dfrac{z^5}{5!} + \dfrac{z^6}{2^6} + \cdots$;

(b) $g(z) = 1 + z + \dfrac{2^2}{2!}z^2 + \dfrac{3^3}{3!}z^3 + \dfrac{4^4}{4!}z^4 + \cdots$.

Suggestion. Use Eq. (6.3) under the influence of the Sterling formula:

$$n! \sim n^n e^{-n}\sqrt{(2\pi n)} \text{ as } n \to \infty.$$

3. Let $\Sigma_{n=0}^{\infty} a_n z^n$ be a power series such that $a_0 = 0$, $a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$, $n \ge 2$. Then, determine its radius of convergence.

4. Prove the result

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{\frac{|a_n|}{n+1}}$$

and conclude that term-by-term integration does not alter the radius of convergence of the original power series.

5. Let $\Sigma_{n=0}^{\infty} a_n z^n$ be a power series which converges for z_1. Then prove that the power series converges absolutely in the open disc $|z| < |z_1|$.

Suggestion. This means $|z_1| < R$, the radius of convergence. Since the power series converges absolutely for all $|z| < R$, it also converges absolutely on the subset $|z| < |z_1| < R$.

6. If $\Sigma_{n=0}^{\infty} a_n z^n$ and $\Sigma_{n=0}^{\infty} b_n z^n$ are both convergent in $|z| < R$ and have the same sum for each z in the circle of convergence, prove that the two series are identical, i.e. $a_n = b_n$ for all $n = 0, 1, 2, \ldots$

7. Given that the radius of convergence of $\Sigma_{n=0}^{\infty} c_n z^n$ is $R(0 < R < \infty)$, find the radius of convergence of the following series:

(a) $\displaystyle\sum_{n=0}^{\infty} n^k c_n z^n, \quad (k = 0, 1, \ldots);$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{c_n}{n!} z^n;$ (c) $\displaystyle\sum_{n=0}^{\infty} c_n^{\alpha} z^n, \quad \alpha > 0.$

8. The radii of convergence of the power series $\Sigma a_n z^n$ and $\Sigma b_n z^n$ are R_1 and R_2, respectively. What can be said about the radius of convergence of the following power series?

(a) $\displaystyle\sum_{n=0}^{\infty} (a_n + b_n) z^n;$ (b) $\displaystyle\sum_{n=0}^{\infty} a_n b_n z^n;$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{b_n} z^n, \quad b_n \neq 0.$

9. Give an example of two different power series with the same finite radius of convergence such that the radius of convergence of their sum power series is infinite.

10. Show that if the radius of convergence of the series $\Sigma_{n=0}^{\infty} a_n z^n$ is $R > 0$, then the radius of convergence of the series $\Sigma_{n=0}^{\infty} a_n R^n z^n$ is 1. What is the radius of the series $\Sigma_{n=0}^{\infty} b_n z^n$, when $b_n = 1/a_n$, if $a_n \neq 0$, and $b_n = 0$, if $a_n = 0$?

11. If $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ has the radius of convergence 1 and $a_n \geq 0$ such that $\Sigma_{n=0}^{\infty} a_n$ diverges, show that $f(r) \to \infty$ as $r \to 1^-$ along the real axis. Can we drop the condition of divergence for $\Sigma_{n=0}^{\infty} a_n$? If not, support your case by giving an example.

12. Suppose the radius of convergence of $\Sigma_{n=0}^{\infty} c_n z^n$ is 1 and let $c_0 \geq c_1 \geq c_2 \geq \ldots$ such that $\lim_{n\to\infty} c_n = 0$. Then the series converges at every point on the circle of convergence except possibly at 1.

Suggestion. Use the Dirichlet test, Section 5.2, and take help of Example 6, Chapter 5.

13. Using the above problem, determine the region of convergence of the following series:

(a) $\displaystyle\sum_{n=0}^{\infty} z^n;$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n};$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n^2};$ (d) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{\sqrt{n}}.$

14. Find the value(s) for the constant b that will make the radius of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{b^n}{\ln n} z^n$$

equal to 5.

15. Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ have the radius of convergence R, $R > 0$ such that $a_1 \neq 0$. Then f is univalent in $|z| < r$ for some $0 < r \leq R$.

Suggestion. See Proposition 7 in Chapter 4.

16. If $\Sigma_{n=0}^{\infty} a_n z^n$ converges for $z = c \neq 0$, show that it converges absolutely for all $|z| \leq |c|$. If the series diverges for $z = d$, show that it diverges for all $|z| \geq |d|$.

17. For $z = x + iy$, define $e^z = \Sigma_{n=0}^{\infty} z^n/n!$. Then prove that

$$e^z = e^x e^{iy} \quad \forall\, z \in \mathbb{C}.$$

This result has been taken as the definition of exponential function in Chapter 3. Some of the authors define exponential function by power series.

Suggestion. Establish $e^{z_1+z_2} = e^{z_1}e^{z_2}$, and then choose $z_1 = x$ and $z_2 = iy$.

18. Restate the Abel's limit theorem for $|z| = R$ and prove.

19. Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n (a_0 \neq 0)$ be a power series with radius of convergence R, and let z_0 be the zero of $f(z)$ closest to the origin. Then prove that

$$|z_0| \geq \frac{r\,|a_0|}{M(r) + |a_0|}, r < R, \quad M(r) = \max_{|z|=r} |f(z)|.$$

6.3 TAYLOR SERIES

In Section 6.2, we established a result that every power series represents an analytic function interior to its circle of convergence. Now, we prove its converse, widely known as Taylor series, i.e. every analytic function in a domain can be expanded in power series about each point of the domain.

Theorem 4. Suppose that $f(z)$ is analytic throughout a domain Ω and let a be any point in Ω. Then, for all $z : |z-a| < R \subseteq \Omega$, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad |z-a| < R, \tag{6.9}$$

where

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(s)}{(s-a)^{n+1}} ds = \frac{f^{(n)}(a)}{n!}; \qquad n = 0, 1, \ldots \tag{6.10}$$

and γ is any positively oriented circular path with centre at a and radius $r < R$.

Proof. Consider a path $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$, as shown in Fig. 6.1.

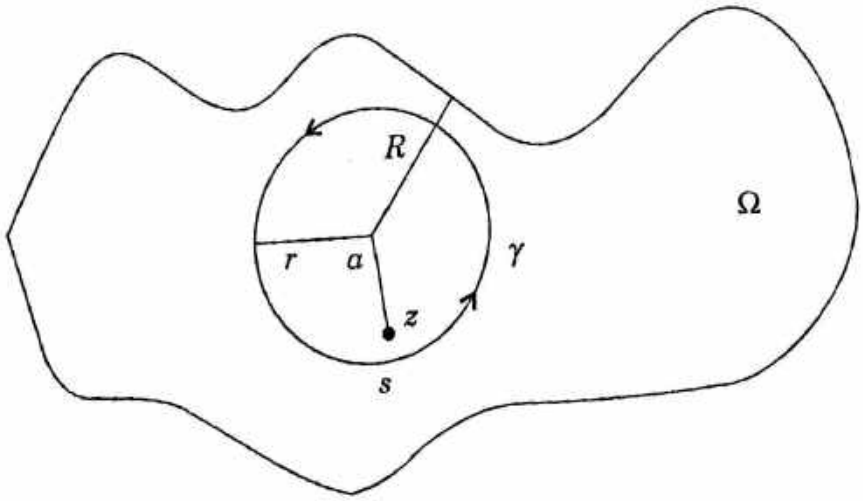

Figure 6.1

Take $z : |z-a| < r < R$. Let $R = \inf\{|z-a| : z \in \text{boundary of } \Omega\}$. Then, by the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}ds,, \quad s \in \gamma$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-a)\left(1-\dfrac{z-a}{s-a}\right)}ds$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-a}\left(1-\frac{z-a}{s-a}\right)^{-1} ds$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-a}\left[\sum_{n=0}^{\infty}\left(\frac{z-a}{s-a}\right)^n\right]ds.$$

Note that $|(z-a)/(s-a)|^n < M^n$, where $0 < M < 1$. Since $\Sigma_{n=0}^{\infty} M^n$ converges, it follows by the Weierstrass M-test that the series $\Sigma_{n=0}^{\infty}[(z-a)/(s-a)]^n$ converges uniformly. This permits us to interchange the summation and integration process (see Corollary 1). Hence

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-a)^n\int_\gamma \frac{f(s)}{(s-a)^{n+1}}ds$$

or

$$f(z) = \sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(z-a)^n = \sum_{n=0}^{\infty} a_n(z-a)^n.$$

The series thus obtained is known as the Taylor series of $f(z)$ about the point a. Note that the point z and $r < R$ are arbitrary.

Corollary 3. Let $f(z)$ be an entire function. Then, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n, \quad |z-a| < \infty.$$

Remark. The Taylor series of an analytic function about the origin, i.e. $a = 0$ is classified as Maclaurin series of that function. Hence (6.9) takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{f^n(0)}{n!}, \quad |z| < R.$$

Example 3. Obtain the Maclaurin series expansion of the following functions:

(a) e^z; (b) $\sin z$; (c) $\cos z$; (d) $\dfrac{1}{1-z}$; (e) $\dfrac{1}{1+z}$.

For (a) $f(z) = e^z$, $f^{(n)}(z) = e^z \Rightarrow f^{(n)}(0) = 1 \ \forall\ n$. Hence

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{6.11}$$

Since the radius of convergence $R = \infty$, the expansion is valid for all z in the complex plane and this is in conformity so that e^z is an entire function.

(b) The function and its derivatives are:

$$\begin{aligned} f(z) &= \sin z & f'(z) &= \cos z \\ f''(z) &= -\sin z, & f'''(z) &= -\cos z \\ &\vdots & &\vdots \\ f^{2n}(z) &= (-1)^n \sin z & f^{2n+1}(z) &= (-1)^n \cos z \end{aligned}$$

So

$$f^{2n}(0) = 0, \qquad f^{2n+1}(0) = (-1)^n.$$

The series has odd powers only. The Taylor theorem yields

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{6.12}$$

Again, $R = \infty$ means that the expansion is valid in $|z| < \infty$.

(c) Similarly,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^2}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad |z| < \infty. \tag{6.13}$$

For other functions we adopt the same procedure to have

$$\text{(d)} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1. \tag{6.14}$$

$$\text{(e)} \quad \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1. \tag{6.15}$$

Note. It often happens to be a formidable task to calculate the coefficients using (6.10). For such type of situations, (6.9) can readily be deduced by starting from

some other series expansions with known coefficients. Hence the reader is advised to commit to memory all series expansions obtained above because of their significant role in developing the series expansions of other functions without differentiating the function, i.e. without evaluating the coefficients a_n's defined by (6.10).

Example 4. Write the series expansions for (a) $\sinh z$, (b) $\cosh z$, and (c) z^3e^{2z}.

Using the expansion of e^z [see (6.11)], we have

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$= \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) + \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right)\right]$$

$$= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.$$

Similarly, using $\cosh z = (e^x + e^{-x})/2$, we have

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad |z| < \infty.$$

Also,

$$z^3e^{2z} = z^3 \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n z^{n+3}}{n!}, \quad |z| < \infty.$$

Remark. Alternatively, for (a) and (b), we may use the expansions (6.12) and (6.13) in conjunction with the identities $\sinh z = -i \sin iz$ and $\cosh z = \cos iz$, respectively.

So far we have worked out examples for Maclaurin series expansion of functions. Now, we find Taylor series expansions for some functions.

Example 5. Find the series expansion of (a) $\cos z$ about $z = \pi/2$ and (b) $\sin z$ about $z = -\pi$.

Replacing z by $z - \pi/2$ in (6.12), we have

$$\sin\left(z - \frac{\pi}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(z - \pi/2)^{2n+1}}{(2n+1)!}$$

or

$$\cos z = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z - \pi/2)^{2n+1}}{(2n+1)!}, \quad |z| < \infty.$$

Also,

$$\sin(z + \pi) = \sum_{n=0}^{\infty} (-1)^n \frac{(z + \pi)^{2n+1}}{(2n+1)!}, \quad |z + \pi| < \infty$$

or

$$\sin z = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z+\pi)^{2n+1}}{(2n+1)!}, \quad |z| < \infty.$$

Example 6. Find a function $f(z)$ which is analytic within and on a unit circle centred at the origin and assumes the values

$$\frac{a - \cos t}{a^2 - 2a\cos t + 1} + i\frac{\sin t}{a^2 - 2a\cos t + 1}, \qquad 0 \le t \le 2\pi, \quad a > 1$$

on its circumference.

Since $f(z)$ is analytic within and on the unit circle, we may expand it in the Taylor series about the origin,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{s^{n+1}}\,ds,$$

where C is the positively oriented unit circle, i.e. $s = e^{it}$, $0 \le t \le 2\pi$ and

$$\begin{aligned} f(s) &= \frac{a - \cos t + i\sin t}{a^2 - 2a\cos t + 1} \\ &= \frac{a - e^{-it}}{a^2 - a(e^{it} + e^{-it}) + 1} \\ &= \frac{a - e^{-it}}{(a - e^{-it})(a - e^{it})} \\ &= \frac{1}{a - e^{it}} \\ &= \frac{1}{a - s}. \end{aligned}$$

So,

$$a_n = \frac{1}{2\pi i}\int_C \frac{1}{(a-s)s^{n+1}}\,ds.$$

The condition $a > 1$ implies that it is outside the unit circle. We apply the Cauchy integral formula for the nth derivative to have

$$a_n = \frac{1}{a^{n+1}}.$$

Hence,

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}} = \frac{1}{a - z}.$$

Here, $z = re^{it}$, $0 \le r \le 1$.

Proposition 3 (Parseval's formula). Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor expansion of an analytic function in a domain $|z| < R$. Show that, for $0 < r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2\,d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}. \tag{6.16}$$

Proof. Given that $f(z)$ is analytic in $|z| < R$, its Taylor series exists about the origin for all $z = re^{i\theta}$, $0 < r < R$,

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}.$$

Also,

$$\overline{f(re^{i\theta})} = \sum_{m=0}^{\infty} \overline{a_m} r^m e^{-im\theta}.$$

This is the series after the conjugation process and change of notation for index n. Due to absolute convergence of both series in $|z| < R$, it is permissible to multiply these. Also, the series for $f(re^{i\theta})$ and $\overline{f(re^{i\theta})}$ are uniformly convergent on $|z| = r$; so is the product series

$$|f(re^{i\theta})|^2 = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{m=0}^{\infty} \overline{a_m} r^m e^{-im\theta}. \tag{6.17}$$

Therefore, term-by-term integration is valid for (6.17), and in view of the result

$$\int_0^{2\pi} e^{i(n-m)\theta}\, d\theta = \begin{cases} 2\pi, & n = m, \\ 0, & n \neq m, \end{cases}$$

we obtain

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta = \sum_{n=0}^{\infty} a_n \overline{a_n} r^{2n} \int_0^{2\pi} d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

In the theory of Fourier series, (6.16) is known as the *Parseval's formula*. Moreover, this result can be extended to the harmonic functions $u(r, \theta)$ and $v(r, \theta)$ as

$$u(r, \theta) = \alpha_0 + \sum_{n=1}^{\infty} r^n(\alpha_n \cos n\theta - \beta_n \sin n\theta),$$

$$v(r, \theta) = \beta_0 + \sum_{n=1}^{\infty} r^n(\beta_n \cos n\theta - \alpha_n \sin n\theta),$$

where $a_n = \alpha_n + i\beta_n$ is the Taylor coefficient for the corresponding analytic function $f(z) = u + iv$.

Example 7. Let $f(z)$ be analytic in a domain Ω. Then, for each $a \in \Omega$ such that the circle $|z - a| = R$, $R > 0$ is contained in the domain, show that

$$f'(a) = \frac{1}{\pi r} \int_0^{2\pi} P_{a,r}(\theta) e^{-i\theta} d\theta,$$

where $P_{a,r}(\theta) = \text{Re}\, f(a + re^{i\theta})$, $0 < r < R$.

The function $f(z)$ is analytic within and on $C: |z-a| = r$. Hence, the Cauchy integral formula for the derivative (see Theorem 6, Section 4.7) ensures that

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz. \tag{6.18}$$

Again, since $f(z)$ is analytic at a, it has the Taylor expansion,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n, \quad |z-a| < R.$$

Set $(z-a) = re^{i\theta}$. Then, from the above expansion

$$f(a+re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}.$$

Taking conjugation,

$$\overline{f(a+re^{i\theta})} = \sum_{n=0}^{\infty} \overline{a_n}\, r^n e^{-in\theta}.$$

Hence,

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{\overline{f(z)}}{(z-a)^2}\,dz &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{r^2 e^{2i\theta}} \sum_{n=0}^{\infty} \overline{a_n}\, r^n e^{-in\theta} r i e^{i\theta}\,d\theta \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty} \overline{a_n} r^{n-1} \int_0^{2\pi} e^{-i(n+1)\theta}\,d\theta \\
&= 0, \quad \text{since} \int_0^{2\pi} e^{-ik\theta}\,d\theta = 0, \quad k = 1, 2, \ldots
\end{aligned} \tag{6.19}$$

Adding (6.18) and (6.19), we obtain

$$\begin{aligned}
f'(a) &= \frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(z)}}{(z-a)^2}\,dz \\
&= \frac{1}{\pi i}\int_C \frac{\operatorname{Re} f(z)}{(z-a)^2}\,dz \\
&= \frac{1}{\pi i}\int_0^{2\pi} \frac{\operatorname{Re} f(a+re^{i\theta})}{r^2 e^{2i\theta}}\, i r e^{i\theta}\,d\theta \\
&= \frac{1}{\pi r}\int_0^{2\pi} e^{-i\theta} \operatorname{Re} f(a+re^{i\theta})\,d\theta.
\end{aligned}$$

Proposition 4. Let $f(z)$ be analytic in a domain containing the disc $|z| \le r$ and let $A(r)$ be the maximum value of $\operatorname{Re} f(z)$ on the circle $|z| = r$. Then,

$$|a_n| r^n \le \max\{4A(r), 0\} - 2 \operatorname{Re} f(0) \quad \forall\, n > 0 \quad \text{and} \quad \forall\, r. \tag{6.20}$$

Proof. For $a_n = \alpha_n + i\beta_n$, $C : z = re^{i\theta}$, $-\pi \le \theta \le \pi$, we have

$$f(z) = \sum_{n=0}^{\infty} (\alpha_n + i\beta_n) r^n e^{in\theta}.$$

Comparing the real parts on both sides,

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n(\alpha_n \cos n\theta - \beta_n \sin n\theta).$$

The above series converges uniformly on the compact set $|z| = r$. Therefore, after multiplying by $\cos n\theta$, we do term-by-term integration to have

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r, \theta) d\theta$$

and

$$\alpha_n r^n = \frac{1}{\pi} \int_{-\pi}^{\pi} u(r, \theta) \cos n\theta \, d\theta \quad (n = 1, 2, \ldots).$$

Similarly, multiplication by $\sin n\theta$ and term-by-term integration yields

$$-\beta_n r^n = \frac{1}{\pi} \int_{-\pi}^{\pi} u(r, \theta) \sin n\theta \, d\theta.$$

Putting the values of α_n and β_n in $a_n r^n = (\alpha_n + i\beta_n) r^n$, we get

$$a_n r^n = \frac{1}{\pi} \int_{-\pi}^{\pi} u(r, \theta) \cos n\theta \, d\theta - i \frac{1}{\pi} \int_{-\pi}^{\pi} u(r, \theta) \sin n\theta \, d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} u(r, \theta) e^{-in\theta} \, d\theta.$$

Hence,

$$|a_n| r^n \le \frac{1}{\pi} \int_{-\pi}^{\pi} |u(r, \theta)| \, d\theta, \qquad n = 1, 2, \ldots.$$

Adding $2\alpha_0$ on both sides, this inequality gives

$$|a_n| r^n + 2\alpha_0 \le \frac{1}{\pi} \int_{-\pi}^{\pi} [|u(r, \theta)| + u(r, \theta)] d\theta.$$

If $u(r, \theta) \le 0$, the integrand is zero, i.e. $A(r) = 0$ and inequality (6.20) is satisfied. For $u(r, \theta) > 0$, the right side does not exceed

$$\frac{1}{\pi} \int_{-\pi}^{\pi} 2A(r) d\theta = 4A(r).$$

Hence, $|a_n| r^n \le 4A(r) - 2\mathrm{Re}\, f(0)$. In any case, in both situations the desired inequality (6.20) is satisfied.

Remark. Note that for $a_0 = \alpha_0 + i\beta_0$, β_0 may not be zero when we deal with the function $v(r, \theta)$.

Taylor series of a composite function. Consider the Taylor series

$$f(z) = a_0 + a_1(z - z_0) + \cdots + a_n(z - z_0)^n + \cdots \tag{6.21}$$

at the point z_0 in the z-plane, and a second Taylor series

$$g(w) = A_0 + A_1(w - w_0) + \cdots + A_n(w - w_0)^n + \cdots \tag{6.22}$$

at the point $w_0 = f(z_0) = a_0$ in the w-plane.

Suppose we replace w by $f(z)$ in (6.22). Then, we obtain a new series expansion

$$g(f(z)) = A_0 + A_1(f(z) - a_0) + \cdots + A_n(f(z) - a_0)^n + \cdots \tag{6.23}$$

corresponding to the composite function $(gf)(z)$. Further, suppose that the algebraic operations called for on the right side of (6.23) are carried out formally. This involves raising $f(z) - a_0$ to various powers after replacing $f(z)$ by the series (6.21), and then grouping and adding all the coefficients of each power $(z - z_0)^n$, $n = 0, 1, \ldots$ in the expansion so obtained. The result is a particular Taylor series of the form

$$(gf)(z) = b_0 + b_1(z - z_0) + \cdots + b_n(z - z_0)^n + \cdots$$

which is said to be obtained by formal substitution of the series (6.21) into the series (6.22).

Example 8. Consider the single-valued branch of the function

$$F(z) = \sqrt{\cos z}$$

and obtain its Taylor series expansion about $z = 0$.

For agreement with the single-valued branch, we take the value 1 at $z = 0$ for the multivalued function $F(z)$. Take

$$f(z) = 1 - \cos z, \qquad gf(z) = [1 - f(z)]^{1/2}.$$

Then, obviously $g[f(z)] = \sqrt{\cos z}$. Note that

$$f(z) = 1 - \cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots, \quad |z| < \infty$$

and

$$gf(z) = [1 - f(z)]^{1/2} = 1 - \frac{f(z)}{2} - \frac{\left[f(z)^2\right]}{8} - \frac{\left[f(z)^3\right]}{16} - \cdots, \quad |f(z)| < 1,$$

Inserting the series for $f(z)$, the outcome is

$$g[f(z)] = 1 - \frac{1}{2}\left(\frac{z^2}{2} - \frac{z^4}{24} + \frac{z^4}{720} - \cdots\right) - \frac{1}{8}\left(\frac{z^2}{2} - \frac{z^4}{24} + \cdots\right)^2 - \frac{1}{16}\left(\frac{z^2}{2} - \cdots\right)^3 - \cdots,$$

where the omitted terms contribute only to the coefficients of $z^8, z^{10}, \ldots$ in the Taylor series of $(gf)(z)$. Thus, writing

$$\left(\frac{z^2}{2} - \frac{z^4}{24} + \cdots\right)^2 = \frac{z^4}{4} - \frac{z^6}{24} + \cdots, \qquad \left(\frac{z^2}{2} - \cdots\right)^3 = \frac{z^6}{8} - \cdots$$

we find that

$$\sqrt{\cos z} = 1 - \frac{1}{2}\left(\frac{z^2}{2} - \frac{z^4}{24} + \frac{z^4}{720} - \cdots\right) - \frac{1}{8}\left(\frac{z^4}{4} - \frac{z^6}{24} + \cdots\right) - \frac{1}{16}\left(\frac{z^6}{8} - \cdots\right)$$

$$= 1 - \frac{z^2}{4} - \frac{z^4}{96} - \frac{19z^6}{5760} - \cdots$$

The above series converges on the disc $|z| < \pi/2$, since $F(z) = \sqrt{\cos z}$ is analytic in this disc, with the derivative

$$\frac{d}{dz}\sqrt{\cos z} = -\frac{\sin z}{2\sqrt{\cos z}}.$$

PROBLEM SET 6.2

1. Find the Maclaurin series expansion of

(a) $\sin z^2$; (b) $z \cosh^2(iz)$; (c) e^{4+z^3}; (d) $\cos^2 z$; (e) $\sin^3 z$.

Suggestion. For (e), use the identity: $\sin 3z = 3 \sin z - 4 \sin^3 z$.

2. Obtain the Taylor's series of

(a) $\cosh 2z$ about the point $z = i\pi$; (b) $\sin z$ about the point $z = 2$.

Suggestion. For (a), set $z = u + i\pi$.

3. Write the Maclaurin series expansion of $\sinh z^2$ and point out why

$$f^{(2n)}(0) = 0, \quad f^{(2n+1)}(0) = 0, \quad n = 0, 1, 2, \ldots$$

4. Making use of successive differentiation, show that

$$e^z \sin z = \sum_{n=0}^{\infty} 2^{n/2} \frac{\sin(n\pi/4)}{n!} z^n.$$

5. Use the Parseval's formula for $|z - z_0| = \rho < R$ to deduce the result

$$\frac{1}{1-\rho^2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1 - 2\rho\cos\theta + \rho^2}\, d\theta, \qquad (0 \le \rho < 1).$$

6. Let $f(z)$ be an analytic function on $|z - z_0| \le R$. Then prove that

$$|a_n| \le \frac{M_R}{R^n}, \quad M_R = \max_{|z-z_0|\le R} |f(z)|.$$

This is known as Cauchy inequality for coefficients of Taylor series, and to derive this use $f^{(n)}(0) = n!a_n$ in (4.33).

7. If $f(z)$ is an entire function such that $|f(z)| = O(|z|^k)$, $k = 0, 1, 2, \ldots$ as $|z| \to \infty$, show that $f(z)$ is a polynomial of degree $\leq k$.

Here, $|f(z)| = O(g(z))$, $|z| \to \infty$ is read as "$f(z)$ is big oh of $g(z)$". It means that there exists a constant $M > 0$ such that $|f(z)| \leq M|g(z)|$ for all sufficiently large z.

Suggestion. Use Cauchy's inequality for coefficients. Exercise 7 is a generalization of Liouville's theorem (Theorem 9, Chapter 4).

8. Let $f(z)$ be an entire function such that $A(r) = O(r^k)$ for $r > r_0$ and $k \in \mathbb{N}$. Then show that $f(z)$ is a polynomial of degree less than or equal to k, and hence deduce that $f(z)$ is constant when $A(r)$ is bounded, and also show that $f(z)$ is constant.

Suggestion. For Problem 8, use (6.20). In Problems 7 and 8, k may be any positive number and in this situation the conclusion reads as "$f(z)$ is a polynomial of degree less than or equal to $[k]$".

9. Suppose $f(z)$ is analytic in a domain Ω containing the disc $|z-2-i| \leq 2$. If $\max|f(z)| = 5$ on $|z-2-i| = 5$, show that in Taylor series expansion

$$f(z) = a_0 + a_1(z-2-i) + a_2(z-2-i)^2 + \ldots$$

the absolute value of the coefficient a_6 is bounded by 5/64.

10. Show that $\cos\sqrt{z}$ is an entire function. However, $\sqrt{z}$ is analytic in a part of the complex plane. Why? Explain. In a similar way, comment on $(\sin\sqrt{z})/\sqrt{z}$.

11. If a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to $f(z)$ in $|z-z_0| < R$, show that it is the Taylor series expansion of $f(z)$ about $z = z_0$, i.e. the Taylor expansion is unique.

12. Considering the appropriate branch of $\sin^{-1}z$, find the expansion

$$(\sin^{-1}z)^2 = \sum_{k=0}^{\infty} \frac{2^2 \cdot 4^2 \ldots (4k-2)^2}{(2k)!} z^{2k}, \qquad |z| < 1.$$

Suggestion. Find the nth derivative at the origin by successive differentiation of the equation $(1-z^2)w'' - zw' = 2$.

13. Prove that for any complex number α,

$$(1+z)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} z^n, \qquad |z| < 1,$$

where $(1+z)^\alpha$ represents the principal branch.

With the application of the above exercise, expand the principal branch of the function $(z+i)^{1/2}$ about the point $z = 0$.

Suggestion. Take $i^{1/2} = (1+i)/\sqrt{2}$.

14. For $|z| < 1$, find the Machaurim series

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots + (-1)^{n-1}\frac{z^n}{n} + \cdots$$

and hence obtain

$$\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z} = \sum_{n=1}^{\infty} \frac{z^{2n+1}}{2n+1}.$$

Also, show that the last series converges in $|z| \le 1$ except for $z = \pm 1$.

Suggestion. See Example 6, Chapter 5.

15. Prove that:

(a) $\dfrac{1}{z^2} = \sum_{n=0}^{\infty} (n+1)(z+1)^n$ for $|z+1| < 1$;

(b) $\dfrac{1}{(1-z^2)^2} = \sum_{n=0}^{\infty} (n+1)z^{2n}$ for $|z| < 1$.

16. Let $f : [a, b] \to \mathbb{C}$ be integrable, and let

$$F(z) = \int_a^b e^{-zt} f(t)dt.$$

Show that $F(z)$ is an entire function and

$$F'(z) = -\int_a^b te^{-zt} f(t)dt.$$

17. Extend Example 7 to nth derivative as

$$f^{(n)}(a) = \frac{n!}{\pi r^n} \int_0^{2\pi} e^{-in\theta} \operatorname{Re} f(a + re^{i\theta})d\theta.$$

18. Let g be continuous on the unit circle $|z| = 1$, and

$$g(e^{it}) = \frac{3 + 2\cos t}{13 + \cos t} - i\frac{2\sin t}{13 + 6\cos t}, \quad 0 \le t \le 2\pi.$$

Then, find a function $f(z)$ which is analytic in unit disc and coincides with $g(z)$ on $|z| = 1$.

19. If $|f(z)| \le M$ in $|z| < R$, show that

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M^2 \quad (0 < r < R)$$

and hence deduce the Liouville's theorem for bounded entire functions.

Suggestion. Use Parseval's formula (6.16).

20. Find the Maclaurin expansion of the following functions:

(a) $\log \cos z$; (b) $\log \dfrac{\sin z}{z}$; (c) $e^{\sin z}$; (d) e^{e^z}.

Also, mention the region of convergence of each expansion.

6.4 ZEROS OF ANALYTIC FUNCTIONS

This section consists of various results on zeros of analytic functions. The main result is the Schwarz lemma.

Definition 2. A function $f(z)$ analytic in $|z-a| < R$ is said to have zero of order $m \in \mathbb{I}_+$ (positive integer) at the point a if $f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$ and $f^{(m)}(a) \neq 0$. For $m = 1$, a is a simple zero of f.

Suppose $f(z)$ is analytic at z_0, then we have for all $|z-z_0| < R$,

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$$

In particular, every polynomial P of degree n is an analytic function in the whole complex plane. The order of any zero of P cannot exceed n.

Proposition 5. Let $f(z)$ be an analytic function at z_0. Then the function $f(z)$ has a zero of order m at $z_0 \Leftrightarrow f(z) = (z-z_0)^m g(z)$, where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Proof. A function $f(z)$ is analytic at z_0 and has a zero of order m if and only if it has the representation

$$f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots, \qquad a_m \neq 0, \quad |z-z_0| < R$$

or

$$\begin{aligned} f(z) &= (z-z_0)^m[a_m + a_{m+1}(z-z_0) + \cdots] \\ &= (z-z_0)^m \sum_{n=m}^{\infty} a_n(z-z_0)^{n-m} \\ &= (z-z_0)^m g(z). \end{aligned} \tag{6.24}$$

Clearly, $g(z_0) = a_m \neq 0$, and $g(z)$ is analytic at z_0 since it is represented by a power series which converges in $|z-z_0| < R$.

Corollary 4. The zeros of an analytic function are isolated.

Proof. Suppose $f(z)$ has a zero of order m at z_0. In view of Proposition 5, we can write $f(z) = (z-z_0)^m g(z)$, where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

The function $g(z)$ is continuous at z_0 and $g(z_0) \neq 0$, it follows that there exists a neighbourhood of z_0, say $|z-z_0| < \delta$ (see Proposition 4, Section 2.3) in which the function is nonzero. Now,

$$f(z) = (z-z_0)^m g(z) \neq 0, \qquad 0 < |z-z_0| < \delta.$$

Thus, a deleted neighbourhood of z_0 is available in which $f(z)$ is nonzero and since z_0 is arbitrary, the assertion is proved.

Corollary 5. If $f(z)$ and $g(z)$ are analytic at z_0 and have zeros of order m and n, respectively, at $z = z_0$, then the product function $h(z) = f(z)g(z)$ has a zero of order $m + n$ at $z = z_0$.

Example 9. Find order of zero of the function $z^3 \sin z$ at the origin.

The function $z^3 \sin z$ can be factored as z^3 and $\sin z$ which have zeros of order $m = 3$ and $n = 1$ at the origin. Hence, by Corollary 5, $z = 0$ is a zero of order 4 of the product function $z^3 \sin z$.

Example 10. Prove that the order of any zero of nth degree polynomial P equals the order of its first nonvanishing derivative.

Let α be zero of order m of P. Then, by (6.24),

$$P(z) = (z - \alpha)^m Q(z),$$

where $Q(\alpha) \neq 0$. On successive differentiation, we obtain

$$P'(z) = m(z - \alpha)^{m-1}Q(z) + (z - \alpha)^m Q'(z);$$

$$P''(z) = m(m - 1)(z - \alpha)^{m-2}Q(z) + 2m(z - \alpha)^{m-1}Q'(z) + (z - \alpha)^m Q''(z).$$

So, $P^{(m)}(z)$ will be of the form

$$P^{(m)}(z) = m!\, Q(z) + \binom{m}{c_1}\frac{m!}{1!}(z - \alpha)Q'(z) + \binom{m}{c_2}\frac{m!}{2!}(z - \alpha)^2 Q''(z) + \cdots + (z - \alpha)^m Q^{(m)}(z).$$

Setting $z = \alpha$, it follows that $P(\alpha) = P'(\alpha) = \cdots = P^{(m-1)}(\alpha) = 0$, whereas $P^{(m)}(\alpha) \neq 0$.

Proposition 6. The number of zeros of an analytic function (not identically zero) interior to a simple closed contour are finite.

Proof. Suppose $f(z)$ has infinite number of zeros inside a region enclosed by a contour. This region is bounded and closed. Hence, by Bolzano Weierstrass property any sequence of zeros $\{a_1, a_2, \ldots\}$ in the region has a limit point in the region. Let a be the limit point of the sequence, i.e. there exists a subsequence $a_{n_k} \to a$ as $k \to \infty$. This implies

$$f(a) = \lim_{k \to \infty} f(a_{n_k}) = 0.$$

So, a is a zero of the function, and in every neighbourhood of a, however small it may be, there are infinitely many zeros and hence a is not isolated which contradicts Corollary 4. Thus, the zeros of an analytic function inside a simple closed contour are finite in number.

Theorem 5. Let f be analytic in a domain Ω, and let S be the set of all zeros of f which has a limit point in Ω. Then $f(z) \equiv 0$ in Ω, i.e. $f(z)$ is identically zero in Ω.

Proof. Let a be limit point of S, the set of all zeros. This means that there exists a sequence a_n of zeros such that $a_n \to a$ as $n \to \infty$. The function f is continuous in Ω and hence for any sequence $\{a_n\}$ converging to a, we have

$$\lim_{n \to \infty} f(a_n) = f(a).$$

But $f(a_n) = 0 \ \forall\ n = 0, 1, \ldots$, so $f(a) = 0$. Since a is a limit point of the sequence $\{a_n\}$, every neighbourhood $|z - a| < \delta$, $\delta > 0$ of a, no matter how small, must have infinitely

many points of $\{a_n\}$. This implies that a is not an isolated zero of $f(z)$. Thus, $f(z)$ is not analytic or $f(z) \equiv 0$ in $|z-a| < \delta$. By hypothesis, $f(z)$ is analytic in Ω and hence on $|z-a| < \delta$. This implies that $f(z) \equiv 0$ in this disc.

To complete the proof we have to show that $f(z) \equiv 0$ on the whole of domain D. For this purpose, we decompose Ω into two sets:

$$E_1 = \{z \in \Omega : z \text{ is a limit point of } S\},$$
$$E_2 = \{z \in \Omega : z \notin E_1\}.$$

Since $z \in E_1$ is a limit point of S, $f(z) \equiv 0$ in a neighbourhood of z. Thus, each point of E_1 is an interior point; so E_1 is a nonempty open set.

Again, let $z' \in E_2$. Since z' is not a limit point of S, by continuity of f at z', there exists a $\delta > 0$ such that $f(z) \neq 0$ throughout $N_\delta(z') \subset \Omega$. This shows that E_2 is open. Since Ω is connected it cannot be written as $E_1 \cup E_2$, union of two disjoint nonempty open sets. Hence, either $E_1 = \phi$ or $E_2 = \phi$. But E_1 cannot be empty as $a \in E_1$. This concludes that $E_2 = \phi$. Thus, $E_1 = \Omega$, and each point of Ω is a limit point of S and hence $f(z) \equiv 0$ in Ω.

Corollary 6 (Uniqueness theorem). Let f and g be analytic functions in a domain Ω and let them coincide on a set which has a limit point in Ω. Then, $f(z) = g(z)$ for every z in Ω.

In other words, an analytic function in a domain is determined by its values on a set which has a limit point in the domain. This corollary is an important result known as *uniqueness theorem*. Note that the uniqueness fails, if we drop the condition that Ω is connected. For example, define $\Omega = \Omega_0 \cup \Omega_1$, Ω_0 and Ω_1 are disjoint open sets; $f(z) = 0, z \in \Omega_0$; $f(z) = 1, z \in \Omega_1$.

Example 11. Prove that, if the set P of all periods of an analytic function f has a limit point, then f is constant.

Proof. Suppose p_0 is a limit point of P. Then there is a sequence p_n in P of distinct points converging to p_0. But,

$$f(p_n) = f(p_0) \quad \text{for all } n,$$

which means that f is constant on the infinite set P together with its limit point. Since f is analytic, this ensures that f is constant everywhere on its domain.

Note. The period set P of a nonconstant analytic function has no limit point.

Proposition 7. Let $f(z)$ be an analytic function in a domain Ω such that for some a in Ω, $f^{(n)}(a) = 0$, $n = 0, 1, 2 \ldots$. Then, $f(z) \equiv 0$ in Ω.

Proof. The function $f(z)$ is analytic at a, and hence it can be expanded in Taylor series about the point a as

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots, \qquad |z-a| < \delta.$$

Given that $f^{(n)}(a) = 0$ for $n = 0, 1, \ldots$. Thus, we have

$$f(z) = 0 \text{ in } |z-a| < \delta.$$

To prove $f(z) = 0$ for all $z \in \Omega$, we split the domain into two sets

$$A = \{a \in \Omega : f^{(n)}(a) = 0,\ n = 0, 1, \ldots\}, \qquad B = \{a \in \Omega : a \notin A\}.$$

The rest of the proof proceeds exactly on the lines of Theorem 5.

Corollary 7. Let f be an analytic function in a domain Ω and $f(z) = 0$ on some arc lying in the domain. Then, $f(z) \equiv 0$ in Ω.

Remark. There are many versions of uniqueness theorem and this corollary is one of these. This is often useful in checking the validity of certain identities in the complex plane which are true in the real number system.

Let us discuss a little about local behaviour of analytic function. The concepts of local maximum and minimum are prominent which are outcomes of the uniqueness theorem.

Definition 3. Let f be defined on a set $S \subset \mathbb{C}$. Then we say that f has local maximum at a point $a \in S$ if there exists an open disc $B(a, r)$ in which $|f(z)| \le |f(a)|$ for all $z \in B(a, r)$.

Similarly, the local minimum of $|f|$ can be defined.

Once the uniqueness theorem is at hand, the proof of Proposition 9 and hence of maximum modulus theorem in Section 4.9, Chapter 4 becomes very simple provided we assume that $f(z)$ is a nonconstant analytic function in a domain D. The proof is reformulated as follows:

> f is analytic in D implies f is continuous and has the mean value property. Suppose that at some point a, $|f(z)|$ assumes the maximum value in some disc $|z - a| < r$ contained in D. Then f is constant on the boundary of a disc centred at a (see Theorem 13, Chapter 4). Now, by the uniqueness theorem, f is constant on the entire domain, a contradiction.

This simple proof of maximum modulus leads to the following corresponding result on local minimum.

Assume that f is analytic and not constant in a domain D. If f has a local minimum at $a \in D$, then $f(a) = 0$.

Example 12. Show that $\cos^2 z - \sin^2 z = \cos 2z$ is true for all $z \in \mathbb{C}$.

Consider the function

$$f(z) = \cos^2 z - \sin^2 z - \cos 2z.$$

The function $f(z)$ is analytic in the whole complex plane, and on the real line we find that

$$\cos^2 x - \sin^2 x = \cos 2x,$$

an identity which is true for all real x. Thus, $f(z) \equiv 0$ for all real z. By uniqueness theorem, $f(z) \equiv 0$ for all z, and hence $\cos^2 z - \sin^2 z - \cos 2z = 0$ for all z, as asserted.

When a function $f(z)$ is analytic in the open disc $|z| < R$ and continuous on the closed disc $|z| \le R$, and in case we know that $|f(z)| \le M$ on $|z| = R$, maximum modulus theorem ensures that $|f(z)| \le M$ on the whole disc. The equality is only possible for constant functions. Therefore, if we know that $f|z|$ assumes some value $< M$ it may be expected that a better estimate can be given, at least pointwise. The results to this effect are very useful. In this direction an important result widely known as the Schwarz lemma is proved:

Theorem 6 (Schwarz lemma). Let $f(z)$ be analytic in $|z| < 1$, with a zero of order n at the origin. Suppose that $|f(z)| \le 1$ for all z in $|z| < 1$. Then,

$$|f(z)| \le |z|^n, \quad |z| < 1, \tag{6.25}$$

$$|f^{(n)}(0)| \le n!. \tag{6.26}$$

Further, equality holds in (6.25) for some $z \neq 0$ or in (6.26) if and only if $f(z)$ is of the form

$$f(z) = cz^n, \qquad |c| = 1 \tag{6.27}$$

Proof. From the Taylor expansion and using the fact $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, it follows that

$$f(z) = \frac{f^{(n)}(0)}{n!}z^n + \frac{f^{(n+1)}(0)}{(n+1)!}z^{n+1} + \cdots, \qquad |z| < 1.$$

For $z = 0$, (6.25) is the hypothesis. Let $z \neq 0$. Then, from the above expansion we have

$$\frac{f(z)}{z^n} = \frac{f^{(n)}(0)}{n!} + \frac{f^{(n+1)}(0)}{(n+1)!}z + \cdots, \qquad 0 < |z| < 1.$$

The series on the right side converges for $z = 0$. Let us define

$$g(z) = \begin{cases} \dfrac{f(z)}{z^n}, & z \neq 0, \\[2ex] \dfrac{f^{(n)}(0)}{n!}, & z = 0. \end{cases}$$

Now $g(z)$ is analytic in disc $|z| < 1$. Let C be the circle $|z| = \rho$, where $0 < \rho < 1$. Then, by the maximum modulus principle,

$$|g(z)| \le \max_{|z|=\rho} |g(z)| = \max_{|z|=\rho} \left|\frac{f(z)}{z^n}\right| \le \frac{1}{\rho^n}, |z| \le \rho.$$

The above inequality is true for all $\rho < 1$, letting $\rho \to 1$, we find that

$$\frac{|f(z)|}{|z|^n} \le 1 \quad \text{or} \quad |f(z)| \le |z|^n.$$

Since $g(0) = f^{(n)}(0)/n!$ and $|g(0)| \le 1$, we immediately have (6.26).

In case $|f(z_0)| = |z_0|^n$ for some z_0 in $0 < |z_0| < R$, then $g(z_0) = 1$. It means that $|g|$ attains its maximum at the interior point z_0. This is only possible when $g(z)$ is constant, say, $g(z) = c$ or $f(z) = cz^n$ for some constant c with $|c| = 1$. This is (6.27).

The case $f^{(n)}(0) = n!$ is treated similarly.

Remark. Originally, the Schwarz lemma was proved for $n = 1$.

Corollaries 8 and 9 are further generalizations of the Schwarz lemma.

Corollary 8. Let $f(z)$ be analytic in $|z| < R$, with a zero of order n at the origin. Suppose that $|f(z)| \le M$ for all z in $|z| < R$. Then,

$$|f(z)| \le \frac{M|z|^n}{R^n}, \quad |z| < R, \tag{6.28}$$

$$|f^{(n)}(0)| \le \frac{Mn!}{R}. \tag{6.29}$$

Further, equality holds in (6.28) for some $z \neq 0$, or in (6.29) if and only if $f(z)$ is of the form

$$f(z) = \frac{Mcz^n}{R^n}, \quad |c| = 1.$$

Proof. Apply the Schwarz lemma to $f(Rz)/M$.

Corollary 9. Let $f(z)$ be analytic in $|z-a| < R$, with a zero of order n at a, and suppose that $|f(z)| \leq M$ for all z in $|z-a| < R$. Then,

$$|f(z)| \leq \frac{M\,|z-a|^n}{R^n}, \quad |z-a| < R, \tag{6.30}$$

$$|f^{(n)}(a)| \leq \frac{Mn!}{R^n}. \tag{6.31}$$

Further, equality holds in (6.30) for some $z_0 \neq a$ or in (6.31) if and only if $f(z)$ is of the form

$$f(z) = \frac{Mc\,|z-a|^n}{R^n}, \quad |c| = 1.$$

Proof. Apply Corollary 8 to the function $g(z) = f(z+a)$ or use the Schwarz's lemma for $h(z) = M^{-1}f(Rz+a)$.

The Schwarz lemma is very helpful to describe one-one onto analytic functions with analytic inverse on the unit disc. Let us define the function

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}, \quad 0 < |a| < 1.$$

This is well defined everywhere except $z = 1/\bar{a}$, which goes to infinity. Note that:

(i) $\phi_a(a) = 0$, $\phi_a(0) = -a$;

(ii) $|\phi_a(z)| < 1 \;\; \forall \; |z| < 1$ (see Problem 7(c), Problem Set 1.3);

(iii) $\phi_{-a}\phi_a(z) = z = \phi_a\phi_{-a}(z)$.

This concludes that ϕ_a maps $|z| < 1$ onto itself in a one-one manner. A one-one function which maps $|z| < 1$ onto itself such that the function and its inverse are analytic in $|z| < 1$ is called an *automorphism* of the unit disc. Thus, ϕ_a and ϕ_{-a} are automorphisms of the unit disc.

Moreover, for any point on the boundary

$$|\phi_a(e^{i\alpha})| = \frac{|e^{i\alpha} - a|}{|1 - \bar{a}e^{i\alpha}|} = \frac{|e^{i\alpha} - a|}{|e^{-i\alpha} - \bar{a}|} = 1.$$

This says that ϕ_a maps $|z| = 1$ onto itself. Any way, the conclusion is that ϕ_a maps $|z| \leq 1$ onto itself in a one-one fashion. Obviously, its inverse has the same properties. An important observation is also made that $\phi_a(z)$ is analytic in $|z| \leq 1$. Thus, we must have an open disc containing $|z| \leq 1$ in which ϕ_a is analytic. Since $z = 1/\bar{a}$ is the only singularity of the function and $|a|^{-1} > 1$, this ensures the existence of a larger open disc

containing $|z| \le 1$. Morover,

$$\phi_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

In particular,

$$\phi'_a(0) = 1-|a|^2 \quad \text{and} \quad \phi'_a(a) = \frac{1}{1-|a|^2}. \tag{6.32}$$

Our main theme in defining ϕ_a is to utilize this function in conjunction with the Schwarz lemma for determining the estimates of $|f(z)|$ and $|f'(z)|$. The following results speak about these estimates.

Proposition 8. Let $f(z)$ be analytic in $|z| < 1$ and $|f(z)| \le 1$ on $|z| < 1$. Then

$$\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}\,f(z)}\right| \le \left|\frac{z-z_0}{1-\bar{z}_0 z}\right|, \tag{6.33}$$

for any z, z_0 inside the unit disc.

Proof. Consider the function

$$g = \phi_{f(z_0)} f \phi_{-z_0}.$$

Note that $g(z)$ is a function from $|z| < 1$ to $|z| < 1$ and is analytic such that

$$g(0) = \phi_{f(z_0)} f \phi_{-z_0}(0) = \phi_{f(z_0)}[f(z_0)] = 0, \qquad \text{[see property (i)]}$$

Thus, the Schwarz's lemma is applicable to $g(z)$, so that

$$|g(z)| \le |z|, \qquad |z| < 1$$
$$|g'(0)| \le 1$$

But by the definition of g, for any z, we have

$$g(z) = \phi_{f(z_0)} f \phi_{-z_0}(z) = \phi_{f(z_0)} f\left(\frac{z+z_0}{1+\bar{z}_0 z}\right) = \frac{f\left(\dfrac{z+z_0}{1+\bar{z}_0 z}\right) - f(z_0)}{1-\overline{f(z_0)}\,f\left(\dfrac{z+z_0}{1+\bar{z}_0 z}\right)}$$

Thus,

$$\left|\frac{f\left(\dfrac{z+z_0}{1+\bar{z}_0 z}\right) - f(z_0)}{1-\overline{f(z_0)}\,f\left(\dfrac{z+z_0}{1+\bar{z}_0 z}\right)}\right| \le |z|. \tag{6.34}$$

Let

$$\frac{z+z_0}{1+\bar{z}_0 z} = \alpha, \quad |\alpha| < 1.$$

Solving this equation,

$$z = \frac{\alpha - z_0}{1-\bar{z}_0 \alpha}.$$

Hence (6.34) yields

$$\left|\frac{f(\alpha)-f(z_0)}{1-\overline{f(z_0)}f(\alpha)}\right| \le \left|\frac{\alpha-z_0}{1-\bar{z}_0\alpha}\right|.$$

Replacing α by z, we get (6.33).

Remarks. 1. The equality will occur in (6.33) when $|g(z)| = |z|$, or $|g'(0)| = 1$. This means that

$$\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)} = c\frac{z-z_0}{1-\bar{z}_0 z}, \qquad |c|=1.$$

On simplification, we get

$$f(z) = \frac{c(z-z_0)+(1-\bar{z}_0 z)f(z_0)}{c(z-z_0)\overline{f(z_0)}+1-\bar{z}_0 z}. \tag{6.35}$$

2. Every univalent function from $|z| < 1$ onto itself which has inverse must be of the form

$$f(z) = e^{i\alpha}\frac{z-z_0}{1-\bar{z}_0 z}, \qquad |z_0|<1.$$

In view of bijective nature of f there is a unique z_0 such that $f(z_0) = 0$. The function g of Proposition 8 is analytic and maps $|z| < 1$ onto itself and $g(0) = 0$. Thus the Schwarz's lemma is applicable to g and $|g(z)| \le |z|$. But g^{-1} is also analytic and bijective on $|z| < 1$ and $g^{-1}(0) = 0$. Apply Schwarz's lemma to g^{-1} to obtain

$$|z| = |g^{-1}(g(z))| \le |g(z)|.$$

Thus $|g(z)| = |z|$ on $|z| < 1$. Thus, by Schwarz's lemma $g(z) = cz$ and this gives the equality (6.35).

3. Assigning $f(z_0) = 0$ in (6.35) we have $f(z) = c\phi_{z0}(z)$ or $|f(z)| = |f_{z0}(z)|$, since $|c| = 1$.

Proposition 9. Let $|f(z)| \le 1$ for $|z| < 1$ and f be analytic in $|z| < 1$. Then

(a) $$|f(z)| \le \frac{|f(0)|+|z|}{1+|f(0)||z|}; \tag{6.36}$$

(b) $$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}. \tag{6.37}$$

Proof. To prove assertion (a), take $z_0 = 0$ in (6.33) to have

$$\left|\frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}\right| \le |z|.$$

It is easy to verify that if $|a| < 1$ and $|b| < 1$, then

$$\frac{|b|-|a|}{1-|a||b|} \le \left|\frac{b-a}{1-\bar{a}b}\right|.$$

Assign $b = f(z)$ and $a = f(0)$, to get

$$\frac{|f(z)| - |f(0)|}{1 - |f(0)|\,|f(z)|} \le \frac{|f(z) - f(0)|}{|1 - \overline{f(0)}f(z)|} \le |z|.$$

Solving the extreme left inequality, we have (6.36).

For assertion (b), we use the outcome $|g'(0)| \le 1$. But for any z

$$g'(z) = \phi'_{f(z_0)}[f(\phi_{-z_0}(z))]\,f'(\phi_{-z_0}(z))\phi'_{-z_0}(z)$$

by the chain rule. Thus, by (6.31)

$$|\phi'_{f(z_0)}(f(z_0))|\,|f'(z_0)|\,|\phi'_{-z_0}(0)| \le 1 \qquad \text{or} \qquad |f'(z_0)| \le \frac{1 - |f(z_0)|^2}{1 - |z_0|^2}.$$

Since z_0 is arbitrary in $|z| < 1$, (6.37) is proved.

Example 13. If $f(z)$ is analytic in $|z| < 1$, $f(0) = 1$ and $\operatorname{Re} f(z) \ge 0$, show that

(a) $\dfrac{1 - |z|}{1 + |z|} \le |f(z)| \le \dfrac{1 + |z|}{1 - |z|}$, $\quad |z| < 1$; (b) $|f'(0)| \le 2$.

First, note that $f(z)$ is a mapping from $|z| < 1$ to $\mathbb{C}$ such that $\operatorname{Re} f(z) \ge 0$. In order to apply Schwarz's lemma, we combine f with the tranformation which maps the right half-plane onto $|z| < 1$. The transformation

$$\phi(z) = \frac{z - 1}{z + 1}$$

maps the right half-plane onto $|z| < 1$, since $\operatorname{Re} z > 0$ implies $|\phi(z)| < 1$; this is evident from the fact that a point in the half-plane $\operatorname{Re} z > 0$ is nearer to $+1$ as compared to -1.

The composite function $g(z) = (\phi f)(z)$ maps $|z| < 1$ onto itself and $g(0) = \phi(1) = 0$. Hence, by Schwarz's lemma,

(i) $|g(z)| \le |z|$,

(ii) $|g'(0)| \le 1$

From (i), it follows that $\operatorname{Re} f(z) > 0$ for all z. Why? Thus

$$|g(z)| = |\phi(f(z))| \le |z|$$

so that

$$\frac{||f(z)| - 1|}{|f(z)| + 1} \le |z|$$

Thus

$$\frac{|f(z)| - 1}{|f(z)| + 1} \le |z| \Rightarrow |f(z)| \le \frac{1 + |z|}{1 - |z|}$$

Similarly,

$$\frac{-(|f(z)| - 1)}{1 + |f(z)|} \le |z| \Rightarrow \frac{1 - |z|}{1 + |z|} \le |f(z)|.$$

For part (b), first note that $g'(z) = \phi'(f(z))f'(z)$. Now, we use (ii) to write

$$|g'(0)| = |\phi'(f(0))f'(0)| \leq 1.$$

This gives

$$|f'(0)| \leq \frac{1}{|\phi'(1)|}.$$

But $\phi'(z) = 2/(z+1)^2$ yields $\phi'(1) = 1/2$ and hence the desired result follows.

Theorem 7 (Borel Carathéodory). Let f be analytic in $|z| \leq R$. Then,

$$M(r) \leq \frac{2r}{R-r}[A(R) - \operatorname{Re} f(0)] + |f(0)|, \tag{6.38}$$

where

$$M(r) = \max_{|z|=r} |f(z)| \quad \text{and} \quad A(r) = \max_{|z|=r} \{\operatorname{Re} f(z)\}.$$

Proof. Let $f(z) = a + ib$ be constant in $|z| \leq R$. Then inserting $M(r) = \sqrt{(a^2+b^2)}$, $A(r) = a$ and $|f(0)| = \sqrt{(a^2+b^2)}$ in (6.38) we get $-\sqrt{(a^2+b^2)} \leq a$, and this is always true.

Suppose $f(z) = u + iv$ is not constant and $f(0) = 0$. Then $A(R) > A(0) = 0$. Consider the function

$$\phi(z) = \frac{f(z)}{2A(R) - f(z)}. \tag{6.39}$$

Then, clearly $2A(R) - f(z) \neq 0$, for all $z : |z| \leq R$, otherwise, $f(z)$ will be a real-valued function of a complex variable contradicting analyticity of $f(z)$. Now, $\phi(0) = 0$ and $\phi(z)$ is analytic in $|z| \leq R$, being quotient of two analytic functions. Note that

$$|\phi(z)|^2 = \frac{u^2 + v^2}{[A(R) - u]^2 + v^2} \leq 1 \quad \text{in } |z| \leq R.$$

Applying Schwarz's lemma to $\phi(z)$, we have

$$|\phi(z)| \leq \frac{1 \cdot r}{R}, \qquad 0 < r < R.$$

Since we assume that $f(0) = 0$, it remains to establish the inequality

$$M(r) \leq \frac{2r}{R-r} A(R). \tag{6.40}$$

From (6.39),

$$f(z) = \frac{2A(R)\phi(z)}{1 + \phi(z)}.$$

Hence,

$$|f(z)| \leq \frac{2rA(R)}{R|1+\phi(z)|} \leq \frac{2rA(R)}{R|1 - |\phi(z)||} \leq \frac{2rA(R)}{R\left(1 - \frac{r}{R}\right)} \leq \frac{2rA(R)}{R-r}.$$

Taking the maximum on the extreme left for $|z| = r$, (6.40) is available.

In case $f(0) \neq 0$, then we utilize the above analysis for the function $f(z) - f(0) = 0$ at $z = 0$ to have

$$\begin{aligned} |f(z)| - |f(0)| &\leq |f(z) - f(0)| \\ &\leq \frac{2r}{R-r} \max_{|z|=R} \operatorname{Re}[f(z) - f(0)] \\ &\leq \frac{2r}{R-r}\left[\max_{|z|=R} \operatorname{Re} f(z) - \operatorname{Re} f(0)\right] \\ &\leq \frac{2r}{R-r}[A(R) - \operatorname{Re} f(0)] \end{aligned}$$

or

$$|f(z)| \leq \frac{2r}{R-r}[A(R) - \operatorname{Re} f(0)] + |f(0)|.$$

Taking maximum on the circle $|z| = r$, we get (6.38).

Remark. Inequality (6.38) is sometimes available in the form of a crude estimate

$$M(r) \leq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|.$$

The reader is advised to note the different forms of (6.38) when it is either applied to max Im $f(z)$ or $f(z)$ is replaced by $f(z) + ik$, where k is a constant.

We noticed in Section 2.1 that $\overline{f(z)}$ and $f(\bar{z})$ may not in general have the same value. Our next result forecasts the conditions under which both these functions coincide.

Theorem 8 (Reflection principle). Let $f(z)$ be analytic in a domain Ω containing a segment of the x-axis and be symmetric to that axis. Then

$$\overline{f(z)} = f(\bar{z}), \qquad z \in \Omega$$

if and only if $f(x)$ is real for each point on the segment of x-axis.

Proof. Suppose $f(x)$ is real on the segment of the x-axis contained in Ω. Let us denote

$$g(z) = \overline{f(\bar{z})}$$

and write

$$f(z) = u(x, y) + iv(x, y), \qquad g(z) = p(x, y) + iq(x, y).$$

Clearly,

$$\overline{f(\bar{z})} = u(x, -y) - iv(x, -y).$$

So,

$$p(x, y) = u(x, -y), \quad q(x, y) = -v(x, -y),$$

and in view of the C-R equations $u_x = v_y$ and $u_y = -v_x$, we have

$$p_x = q_y, \quad p_y = -q_x.$$

This means that p and q satisfy the C-R equations and, moreover, the first-order partial

derivative are continuous. Thus, $g(z)$ is analytic in Ω. The given hypothesis that $f(x)$ is real on a segment of real x-axis implies $v(x, 0) = 0$. Ultimately,

$$g(x) = p(x, 0) = u(x, 0).$$

That is, $g(z) = f(z)$ on the segment of the real axis.

We now refer to the uniqueness theorem for the function $g(z) - f(z)$, which is identically zero on the line segment of the x-axis, and hence

$$g(z) - f(z) \equiv 0 \quad \forall\, z \in \Omega,$$

and this is equivalent to saying $g(z) = f(z)$ or $\overline{f(z)} = f(\bar{z})$, $z \in \Omega$.

To prove the converse part, we assume $f(z) = \overline{f(\bar{z})}$, $z \in \Omega$ holds, and this ensures the relation

$$u(x, y) + iv(x, y) = u(x, -y) - iv(x, -y).$$

In particular, we note that

$$u(x, 0) + iv(x, 0) = u(x, 0) - iv(x, 0).$$

Thus, $v(x, 0) = 0$. Hence $f(x)$ is real on the part of the x-axis contained in the domain.

Example 14. Show that $\overline{z^2 + 2z + \sin z} = \bar{z}^2 + 2\bar{z} + \sin \bar{z}$ for all complex numbers.

Let $f(z) = z^2 + 2z + \sin z$. Then, f is analytic in the entire complex plane and this is symmetric with respect to the x-axis. Why? Further, note that $f(x) = x^2 + 2x + \sin x$ is real when x is real, and by the reflection principle the result is immediate.

Remark. The reflection principle does not work for $iz + 1$, since $ix + 1$ is not real for all real x. In this situation, it can be modified to give another relation (see Problem 12, Problem Set 6.3).

PROBLEM SET 6.3

1. Let $f(z)$ and $g(z)$ be analytic at z_0 and have zeros of order m and n, respectively. What can be said about the zero of $f + g$ at $z = z_0$.
2. Locate the zeros of the following functions and determine their order:

   (a) $(1 + z^2)^4$; (b) $z^8 + z^4$.
3. Find the order of zero at $z = 0$ of the functions:

   (a) $z^2(e^{z^2} - 1)$; (b) $6 \sin z^3 + z^3(z^3 - 6)$.
4. If f has zero of order k at z_0, show that f' has a zero of order $k - 1$ at z_0.
5. If f and g are analytic functions in a domain Ω and $f(z) = g(z)$ on a sequence which has a limit point in Ω, prove that $f(z) = g(z)$ on Ω.
6. If $f(z)$ and $g(z)$ are both analytic in a domain Ω and $f(z)g(z) = 0$ in Ω, prove that either $f(z) \equiv 0$ in Ω or $g(z) \equiv 0$ in Ω.

7. Use the uniqueness theorem to verify the validity of the following trigonometric identities in the complex plane:

(a) $\cosh^2 z - \sinh^2 z = 1$; (b) $1 + \cot^2 z = \operatorname{cosec}^2 z$.

8. The α-points of a function $f(z)$ are defined to be the solution set of the equation $f(z) = \alpha$. Restate and prove Theorem 5 in the form: Let $f(z)$ be analytic in Ω and S be the set of all α-points of $f(z)$ which has a limit point in Ω. Then show that $f(z)$ is a constant function in Ω.

9. Let $f(z)$ be an entire function which satisfies one of the following conditions

(a) $f(z) = f(kz)$, $k \in \mathbb{C}\backslash\{0, 1\}$ is fixed;

(b) $f(z) = f(z + z_1) = f(z + z_2)$, $z_1/z_2 \in \mathbb{R}$.

Prove that $f(z)$ is constant.

10. Utilize Parseval's formula and the uniqueness theorem to give an elegant proof of the maximum modulus principle.

11. Use the reflection principle to show that

(a) $\overline{\sin z} = \sin \bar{z}$; (b) $\overline{\cos z} = \cos \bar{z}$.

12. Let $f(z)$ be analytic in a domain Ω containing a segment of the x-axis and be symmetric to that axis. Then show that $\overline{f(z)} = -f(\bar{z})$, $z \in \Omega$ if and only if $f(x)$ is purely imaginary for each point on the segment of x-axis.

13. Let f be analytic in $|z| < 1$ and $\operatorname{Im} z > 0$ for all z with $f(0) = i$. Show that

$$|f(z)| \le \frac{1+|z|}{1-|z|}.$$

14. Suppose $f(z)$ is analytic in $|z| < R$ and satisfies (i) $|f(z)| \le M$ for all $z : |z| < R$; (ii) $f(z_0) = w_0$ for some z_0 in $|z| < R$ such that $|w_0| < R$. Show that

$$M\left|\frac{f(z)-w_0}{M^2 - \bar{w}_0 f(z)}\right| \le R\left|\frac{z-z_0}{R^2 - \bar{z}_0 z}\right| \quad \text{for } |z| < R.$$

Suggestion. Note that $M^{-1}f(Rz)$ is analytic in $|z| < 1$ and $M^{-1}|f(Rz)| \le 1$ in $|z| < 1$. Now, apply (6.33) to the function $g(z) = M^{-1}f(Rz)$. Finally, translate the result to $|z| < R$, replacing z and z_0 by z/R and z_0/R, respectively.

15. Find an analytic function f on $|z| \le 1$ such that $f(0) = 3$ and $|f(z)| \le 3$ for $|z| < 1$.

Suggestion. Use the result of Problem 14 when equality occurs.

16. Let f be analytic in $|z| < 1$ such that $|f(z)| \le 1$ and $f(1/3) = 0$. Then, estimate $|f(1/7)|$.

17. Let $f: U \to U$ be analytic, where U is the upper-half plane. Then show that

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \le \frac{1}{\operatorname{Im} z}.$$

18. Does there exist an analytic function on $|z| < 1$ with $|f(z)| < 1$ such that $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?

Suggestion. Use (6.37).

19. State and prove a generalization of Theorem 7 in the form

$$\max_{|z|=r} |f^{(n)}(z)| \le \frac{2^{n+1} n! R}{(R-r)^{n+1}} [A(R) + |f(0)|].$$

20. With f as in Proposition 9, show that

$$\frac{1-|f(0)|\,|z|}{1+|f(0)|\,|z|} \le |f(z)| \le \frac{|f(0)|+|z|}{1-|f(0)|\,|z|}.$$

21. Use Schwarz's lemma to prove Liouville's theorem.

Suggestion. M is fixed, and R may be chosen large in Corollary 8.

22. Does there exist an analytic function f from unit disc to itself such that

(i) $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?

(ii) $f(0) = 1/2$ and $f'(0) = 2/3$?

If yes, find the function.

23. If $M = R$ and if f maps the boundary to the boundary, then at any point b with $|b| = M$, where $f'(b)$ exists, prove that one has $|f'(b)| \ge 1$.

Suggestion. The proof follows immediately from the fact that distances to the origin are shrunk under f, and therefore the distances from the boundary are stretched. More precisely, for t real, $0 < t < 1$, we have $|f(tb)| \le tb$, so that $|f(tb) - f(b)| \ge R - tR = |tb - b|$ and from which the required result follows.

6.5 LAURENT SERIES

For the Taylor series expansion of a function $f(z)$ about a point, we require $f(z)$ to be analytic at that point. When $f(z)$ fails to be analytic at a point but it is analytic in some neighbourhood of the point, in such situations $f(z)$ has another series expansion, known as *Laurent series expansion.*

Theorem 9. Suppose that a function f is analytic in the annular domain $R_1 < |z-a| < R_2$. Let C denote any positively oriented circle around a and lying in this domain. Then $f(z)$ can be expanded about any point in the annular domain $R_1 < |z-a| < R_2$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}, \tag{6.41}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-a)^{n+1}}\, ds, \qquad n = 0, 1, \ldots$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-a)^{-n+1}}\,ds, \quad n = 1, 2, \ldots$$

The series in (6.41) is known as Laurent series.

Proof. Consider the domain shown in Fig. 6.2, contained in the annular region $R_1 < r_1 \le |z-a| \le r_2 < R_2$. The innermost dotted circle is $|z| = R_1$ and the outermost dotted circle is $|z| = R_2$.

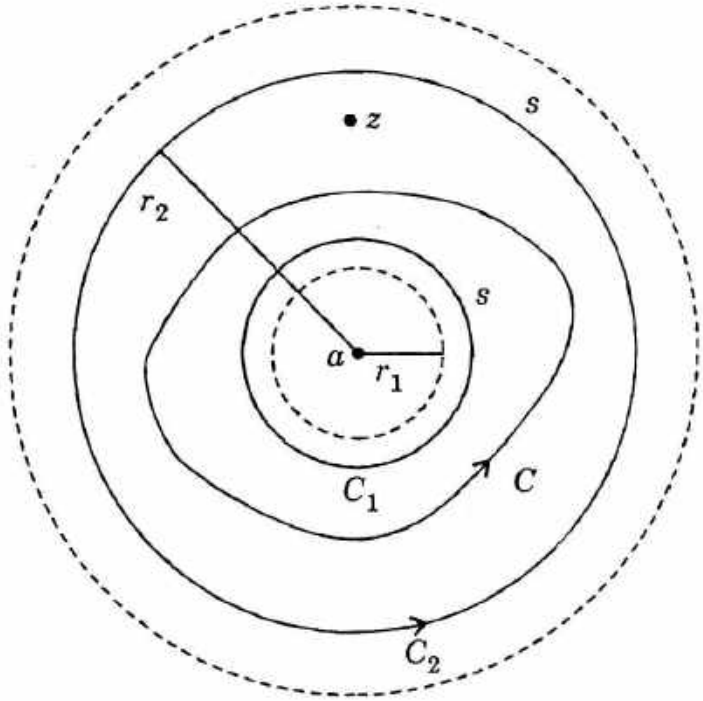

Figure 6.2

Then, by the Cauchy integral formula for a doubly connected domain (see Remark (3), Section 4.7), we have

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}\,ds, \tag{6.42}$$

where $C_1 = \{z : |z-a| = r_1\}$ and $C_2 = \{z : |z-a| = r_2\}$ such that $R_1 < r_1 < r_2 < R_2$.

The factor $1/(s-z)$ in the first integral of (6.42) is the same as was used to derive Taylor's expansion (see Theorem 4), and hence working the same analysis, we have

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds = \sum_{n=0}^{\infty} a_n(z-a)^n, \qquad a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{(s-a)^{n+1}}\,ds. \tag{6.43}$$

The second integral on right side of (6.42) is treated as

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}\,ds = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{-(z-a)\left(1-\dfrac{s-a}{z-a}\right)}\,ds$$

$$= -\frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-a}\left(1-\frac{s-a}{z-a}\right)^{-1} ds$$

$$= -\frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-a}\left[\sum_{n=0}^{\infty}\left(\frac{s-a}{z-a}\right)^n\right] ds$$

$$= -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{C_1} \frac{f(s)(s-a)^n}{(z-a)^{n+1}} ds, \quad \text{since} \left|\frac{s-a}{z-a}\right| < 1$$

$$= -\sum_{n=0}^{\infty} \frac{1}{(z-a)^{n+1}} \frac{1}{2\pi i}\int_{C_1} f(s)(s-a)^n\, ds$$

$$= -\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}, \qquad b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{(s-a)^{-n+1}} ds. \tag{6.44}$$

By the property of deformation of path, C_1 and C_2 may be replaced by C. The values of integrals obtained in (6.43) and (6.44) are substituted in (6.42) to have (6.41).

Remarks. 1. The Laurent series can be written in a unified form. Replacing n by $-n$ in second sum on the right side of (6.38) leads to

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=-1}^{-\infty} b_{-n}(z-a)^n$$

$$= \sum_{n=-\infty}^{\infty} c_n(z-a)^n, \qquad c_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-a)^{n+1}} ds = b_{-n},\ n \in \mathbb{I}. \tag{6.45}$$

2. The integrand in (6.45) can be written as $f(z)(z-a)^{-(n+1)}$. If $f(z)$ is analytic throughout $|z-a| < R_2$, this integrand is also analytic for $n = -1, -2, \ldots$. Hence all the coefficients $a_{-1}, a_{-2}, \ldots$ are zero and expansion (6.45) reduces to Taylor series (6.10) about a.

3. For annular domain $R_1 < 1 < R_2$, the Laurent series is associated with the Fourier series expansion. Let f be an analytic function in a domain $\Omega = \{z : 1-\delta < |z| < 1+\delta\}$, $\delta > 0$. Obviously, Ω contains the unit disc $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Now, by (6.45), for $a = 0$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi i}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\, d\theta. \tag{6.46}$$

In particular, if we take $f(e^{it}) = F(t)$ and $z = e^{it}$ to be on the unit circle, then equations (6.46) yield

$$F(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \qquad a_n = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{-int}\, dt.$$

This is the well known Fourier expansion of $F(t)$ in complex form.

Example 15. Derive the Laurent series of the functions

(a) $f(z) = z^3 \cosh \dfrac{1}{z}$; (b) $f(z) = \dfrac{1}{z^2(1-z)}$.

(a) Using Example 4, we have

$$\cosh \frac{1}{z} = \sum_{n=0}^{\infty} \frac{z^{-2n}}{(2n)!}, \qquad 0 < |z| < \infty$$

or

$$z^3 \cosh \frac{1}{z} = \sum_{n=0}^{\infty} \frac{z^{3-2n}}{(2n)!}$$

$$= z^3 + \frac{z}{2} + \sum_{n=2}^{\infty} \frac{1}{(2n!)z^{2n-3}}$$

$$= \frac{z}{2} + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!z^{2n-1}}, \qquad 0 < |z| < \infty.$$

(b) For the region $0 < |z| < 1$,

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=2}^{\infty} z^{n-2} = \sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2}.$$

Again, consider the region $1 < |z| < \infty$,

$$\frac{1}{z^2(1-z)} = -\frac{1}{z^3\left(1-\dfrac{1}{z}\right)} = -\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}} = -\sum_{n=1}^{\infty}\frac{1}{z^{n+2}}.$$

Example 16. (a) Obtain the Laurent series expansion of

$$f(z) = \frac{z}{z^2 - 4z + 3};$$

(b) Expand the function of (a) in powers of $z-1$ and mention the regions of validity.

After factorizing the denominator and splitting into partial fractions, we have

$$f(z) = \frac{z}{(z-1)(z-3)} = \frac{1}{2}\left(-\frac{1}{z-1} + \frac{3}{z-3}\right).$$

(a) Both the functions are analytic in $D_1: |z| < 1$. So, the Maclaurin series expansion is

$$f(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right] = \frac{1}{2}\sum_{n=0}^{\infty}\left(1 - \frac{1}{3^n}\right)z^n.$$

For domain $D_2 : 1 < |z| < 3$, we have

$$f(z) = \frac{1}{2}\left[-\frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{1-\frac{z}{3}}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right]$$

$$= -\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{z^n}{3^n} + \sum_{n=1}^{\infty}\frac{1}{z^n}\right).$$

In $D_3 : |z| > 3$. Observe that $|3/z| < 1 \Rightarrow |1/z| < 1$. So,

$$f(z) = \frac{1}{2}\left[-\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{3}{z\left(1-\frac{3}{z}\right)}\right]$$

$$= \frac{1}{2z}\left[-\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty}3\left(\frac{3}{z}\right)^n\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(3^{n+1}-1)\frac{1}{z^{n+1}}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(3^n-1)\frac{1}{z^n}.$$

(b) The first component of $(1/2)[-1/(z-1) + 3/(z-3)]$ is already in power of $z-1$. So, we treat only $3/2(z-3)$ as

$$\frac{3}{2(z-3)} = -\frac{3}{4}\frac{1}{\left(1-\frac{z-1}{2}\right)}, \qquad \left|\frac{z-1}{2}\right| < 1$$

$$= -\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n, \qquad |z-1| < 2.$$

So the required expansion is

$$-\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n - \frac{1}{2(z-1)}, \qquad |z-1| < 2.$$

Again, in the domain $|z-1| > 2$, we observe

$$\frac{3}{2(z-3)} = \frac{3}{2(z-1)\left(1-\frac{2}{z-1}\right)}$$

$$= \frac{3}{2(z-1)} \sum_{n=0}^{\infty} \left(\frac{2}{z-1}\right)^n, \qquad \left|\frac{2}{z-1}\right| < 1$$

$$= 3 \sum_{n=1}^{\infty} \frac{2^{n-2}}{(z-1)^n}.$$

Hence, in this case the expansion is

$$-\frac{1}{2(z-1)} + 3\sum_{n=1}^{\infty} \frac{2^{n-2}}{(z-1)^n} = \frac{1}{z-1} + 3\sum_{n=2}^{\infty} \frac{2^{n-2}}{(z-1)^n}.$$

Proposition 10. If the series $\Sigma_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges to $f(z)$ in an annular region $R_1 < |z-z_0| < R_2$, then it is the Laurent expansion of $f(z)$ about $z = z_0$.

Proof. Suppose that we get a different Laurent expansion for $f(z)$ given by

$$f(z) = \sum_{m=-\infty}^{\infty} b_m(z-z_0)^m, \qquad R_1 < |z-z_0| < R_2,$$

where for any positively oriented contour in the annular domain,

$$b_m = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{m+1}}\, dz.$$

Now,

$$b_m = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{m+1}}\, dz$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{1}{(z-z_0)^{m+1}} \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n\, dz$$

$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} a_n \int_\gamma \frac{1}{(z-z_0)^{m-n+1}}\, dz.$$

Note that the term-by-term integration is justified since the series converges uniformly on every bounded and closed subset of annulus. Referring to Example 9, Section 4.5,

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{(z-z_0)^{m-n+1}}\, dz = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

It is clear that

$$\frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} a_n \int_\gamma \frac{1}{(z-z_0)^{m-n+1}}\, dz = a_m.$$

Thus, we have proved that Laurent expansion, whenever it exists, is unique.

Example 17. Let c be an arbitrary complex number. Then show that

$$e^{(c/2)(z-z^{-1})} = \sum_{n=-\infty}^{\infty} a_n z^n, \tag{6.47}$$

where

$$a_n \equiv a_n(c) = \frac{1}{\pi}\int_0^{\pi} \cos(c \sin\theta - n\theta)d\theta.$$

Also, deduce that

$$\frac{1}{\pi}\int_0^{\pi} \cos(c \sin\theta - n\theta)d\theta = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!}\left(\frac{c}{2}\right)^{n+2k}.$$

Clearly, $f(z) = e^{(c/2)(z-z^{-1})}$ is analytic in deleted neighbourhood of $z = 0$. Hence, by the Laurent expansion,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad 0 < |z| < R,$$

where

$$a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(s)}{z^{n+1}}ds,$$

and γ is any positively oriented simple closed contour enclosing the origin.

Due to deformation of path, γ may be taken $z = e^{i\theta}$, $-\pi < \theta \le \pi$. Then,

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{c(e^{i\theta}-e^{-i\theta})/2}\frac{ie^{i\theta}}{e^{i(n+1)\theta}}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ic\sin\theta}e^{-in\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(c\sin\theta - n\theta)}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(c\sin\theta - n\theta)d\theta + i\,\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(c\sin\theta - n\theta)d\theta \\
&= \frac{1}{\pi}\int_0^{\pi} \cos(c\sin\theta - n\theta)d\theta.
\end{aligned}$$

Since integrand is an odd function of θ,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(c\sin\theta - n\theta)d\theta = 0.$$

To deduce the second relation, we formally multiply the two series

$$e^{cz/2} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{c}{2}\right)^j z^j, \qquad e^{-cz^{-1}/2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{c}{2}\right)^k z^{-k}.$$

The result is the so-called double series, whose terms are the possible products of a term from the first series and a term from the second. The fact that each of these series is absolutely convergent ensures that this double series converges to the proper sum regardless of the order of its terms. For each fixed integer $n \geq 0$, we obtain a term of the double series containing z^n, precisely when $j = n + k$; and when all possible values are counted for, the total coefficient of z^n is

$$\sum_{k=0}^{\infty} \frac{1}{(n+k)!}\left(\frac{c}{2}\right)^{n+k} \frac{(-1)^k}{k!}\left(\frac{c}{2}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!}\left(\frac{c}{2}\right)^{n+2k} = a_n(c).$$

This verifies the second assertion.

Remarks. 1. The coefficients $a_n(c)$ are called the Bessel functions of the first kind and have a prominent role in series solutions of second order linear differential equations and many boundary value problems. Note that the function $f(z) = e^{(c/2)(z-(1/z))}$ is called the generating function for the Bessel function.

2. Similar to Example 17, it is easy to show that for $c \in \mathbb{C}$,

$$e^{(c/2)(z+z^{-1})} = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \text{where} \quad a_n \equiv a_n(c) = \frac{1}{\pi}\int_0^{\pi} e^{c\cos\theta}\cos(n\theta)d\theta$$

with

$$\frac{1}{\pi}\int_0^{\pi} e^{c\cos\theta}\cos(n\theta)d\theta = \sum_{k=0}^{\infty} \frac{1}{(n+k)!k!}\left(\frac{c}{2}\right)^{n+2k}.$$

3. A simple method to multiply the two series is given in Example 17, Section 6.7.

PROBLEM SET 6.4

1. Find the different series expansions in certain domains and specify those domains for the function $f(z) = 1/[z(1+z^2)]$.
2. Find the different developments of $4z + 3/[z(z-3)(z+2)]$ in powers of z according to the position of the point z in the plane.
3. Find the Laurent series expansion of the function (in powers of z) in different domains,

$$f(z) = \frac{z}{(z+a)(z+b)}, \qquad |a| < |b|.$$

Also, mention the regions of validity in each case.

4. Expand the following functions about the specified points: (a) $f(z) = 1/(z^2+4)$ in powers of $(z+2i)$; (b) $\tan z$ about $z = \pi/2$ (only the first three nonzero terms)

5. Show that

$$\frac{1}{z^6} = \sum_{k=0}^{\infty} \binom{5+k}{5} \frac{i^5}{(z+i)^{k+6}} \quad \text{when} \quad 1 < |z+i| < \infty.$$

6. Find the different developments of $z^2 + 5/(z-2)(z+3)$ in powers of $(z-1)$ and write explicitly their respective regions of convergence.

7. Show that:

(a) $$\sinh c\left(z+\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n = \frac{1}{\pi}\int_0^{\pi} \sin(2\cos c\theta)\cos n\theta d\theta;$$

(b) $$\cosh\left(z+\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cosh(2\cos\theta)d\theta.$$

8. If $\alpha, \beta \in \mathbb{C}$, show that

$$e^{\alpha z+\beta z^{-1}} = 1 + \sum_{n=-\infty}^{\infty} a_n z^n;$$

where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(\alpha+\beta)\cos\theta}\cos[(\alpha-\beta)\sin\theta - n\theta]d\theta.$$

9. Show that if $c > 0$, then

$$e^{z+c^3/(2z^3)} = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where

$$a_n = \frac{e^{-1/2c}}{2\pi c^n}\int_0^{2\pi} e^{c(\cos\theta+\cos^2\theta)}\cos[c\sin\theta(1-\cos\theta) - n\theta]d\theta.$$

10. Analyzing Problem 6, Problem Set 6.2, state and prove the Cauchy inequality for Laurent series.

11. If $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges to $f(z)$ in $R_1 < |z-z_0| < R_2$, show that the series converges uniformly in the sub-annular region $R_1 < r_1 \le |z-z_0| \le r_2 < R_2$.

Suggestion. Observe that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly in $|z-z_0| \le r_2$, refer to Proposition 1, and by the Weierstrass M-test, the series $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly in $|z-z_0| \ge r_1$.

12. Suppose f is analytic in $r < |z-z_0| < R$, and has the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \quad \text{for all } z: r < |z-z_0| < R.$$

Then, show that

$$\int_\gamma f(z)dz = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z-z_0)^n \, dz,$$

where γ is any contour lying in the annulus.

13. Suppose f is analytic in $r < |z-z_0| < R$, and has the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \quad \text{for all } z : r < |z-z_0| < R.$$

Then, prove that

$$f'(z) = \sum_{n=-\infty}^{\infty} a_n \, n(z-z_0)^{n-1} \quad \text{for all } z : r < |z-z_0| < R.$$

Suggestion. Problems 12 and 13 reveal the fact that term-by-term integration and differentiation are also valid for Laurent series. The proof proceeds on the lines of Theorems 1 and 2.

14. For the function

$$f(z) = \frac{2z^3+1}{z^2+z},$$

find the Taylor series about the point $z = i$, and a Laurent series valid within the interval whose centre is the origin.

15. Let $\{x[n]\}$ be a complex sequence such that the series

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

converges to an analytic function $X(z)$ in some annulus $R_1 < |z| < R_2$. The sum $X(z)$ is called the Z-transform of $\{x[n]\}$. Use formula (6.45) for the coefficients in a Laurent series to show that if the annulus contains the unit disc $|z| = 1$, then the inverse Z-transform is given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\theta})e^{in\theta} \, d\theta.$$

The prominent application of Z-transform is to solve the difference equations.

16. With reference to Problem 15, find $X(z)$, when $n \geq 0$, for the following sequences:

(a) $x[n] = a^n$; (b) $x[n] = n$; (c) $x[n] = na^n$;

(d) $x[n] = \dfrac{1}{n+1}$; (e) $x[n] = \dfrac{1}{n!}$; (f) $x[n] = \begin{cases} 1, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$

17. Let p and q be complex numbers. Use relation (6.47) to prove that

$$a_n(p+q) = \sum_{k=-\infty}^{\infty} a_{n-k}(p)a_k(q).$$

6.6 INTEGRATION AND DIFFERENTIATION OF POWER SERIES

Here, we solve some problems based on the theory developed in Sections 6.1–6.5.

Example 18. In the w-plane, integrate the series $1/w = \Sigma_{n=0}^{\infty}(-1)^n(w-1)^n$, $|w-1| < 1$, along a contour interior to the circle of convergence from $w = 1$ to $w = z$ to obtain

$$\text{Log } z = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n}.$$

We have

$$\int_1^z \frac{1}{w} dw = \int_1^z \left(\sum_{n=0}^{\infty} (-1)^n (w-1)^n \right) dw = \sum_{n=0}^{\infty} (-1)^n \int_1^z (w-1)^n \, dw.$$

The function $1/w$ is analytic in $|w-1| < 1$, and hence using the antiderivative (Corollary 3, Chapter 4), it can be integrated to have

$$[\text{Log } w]_1^z = \sum_{n=0}^{\infty} (-1)^n \left[\frac{(z-1)^{n+1}}{n+1} \right]_1^z$$

or

$$\text{Log } z = \sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^n}{n}, \quad |z-1| < 1.$$

Moreover, replacing z by $1 + z$, we obtain the Maclaurin series for $\text{Log}(1 + z)$ as

$$\text{Log}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}, \qquad |z| < 1.$$

Example 19. Expand the function $1/(z-3)$ in powers of $z-2$ and hence obtain the expression for $1/(z-3)^3$, and also find the series for $(z-2)/(z-3)^3$.

$$\frac{1}{z-3} = -\frac{1}{1-(z-2)} = -\sum_{n=0}^{\infty} (z-2)^n, \qquad |z-2| < 1.$$

Differentiating twice, we get

$$\frac{2}{(z-3)^3} = -\sum_{n=2}^{\infty} n(n-1)(z-2)^{n-2} = -\sum_{n=0}^{\infty} n(n+1)z^n.$$

Hence

$$\frac{1}{(z-3)^3} = -\frac{1}{2}\sum_{n=0}^{\infty} (n+1)(n+2)z^n, \qquad |z-2| < 1. \tag{6.48}$$

For domain $|z-2| > 1$, we observe that

$$\frac{1}{z-3} = \frac{1}{(z-2)\left(1 - \dfrac{1}{z-2}\right)} = \sum_{n=0}^{\infty} \left(\frac{1}{z-2}\right)^{n+1} = \sum_{n=1}^{\infty} \left(\frac{1}{z-2}\right)^n.$$

Differentiating twice, we obtain

$$\frac{2}{(z-3)^3} = \sum_{n=1}^{\infty} n(n+1)(z-2)^{-(n+2)}$$

or

$$\frac{1}{(z-3)^3} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{n(n+1)}{(z-2)^{n+2}}. \tag{6.49}$$

To find the series for $(z-2)/(z-3)^3$, multiply both sides of Eqs. (6.48) and (6.49) by $(z-2)$.

Remark. We have obtained (6.49) by differentiating the Laurent series which is not a power series. But term-by-term differentiation is also permissible for Laurent series, (refer to Problem 13, Problem Set 6.4).

6.7 MULTIPLICATION AND DIVISION OF POWER SERIES

Suppose $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, $|z| < R_1$ and $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$, $|z| < R_2$. Since $f(z)$ and $g(z)$ are analytic in $|z| < R_1$ and $|z| < R_2$, respectively, it follows that $f(z)g(z)$ is analytic in $|z| < R$, $R = \min\{R_1, R_2\}$. Let the power series of the product function $f(z)g(z)$ be written as

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < R = \min\{R_1, R_2\}$$

where the c_n's are determined by

$$c_n = \frac{(fg)^{(n)}(0)}{n!} = \sum_{k=0}^{n} \frac{1}{n!}\frac{n!}{k!(n-k)!} f^{(k)}(0) g^{(n-k)}(0) = \sum_{k=0}^{n} a_k b_{n-k}.$$

Clearly,

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_0 b_1 + a_1 b_0 \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\ &\vdots \\ c_n &= a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0. \end{aligned}$$

This method of multiplication of power series is sometimes referred to as the *Cauchy product*. Thus, based on the above observations, multiplications of these power series can be expressed as

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_n b_{n-k}\right) z^n \qquad \text{or} \qquad \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k} b_n\right) z^n.$$

Remark. Careful attention is needed when one of the power series contains only the even powers of z. Let us explain the product of

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^{2n} \qquad \text{and} \qquad f_2(z) = \sum_{n=0}^{\infty} b_n z^n.$$

What we want to do here is to collect together the nth powers of z from all possible products $a_k z^{2k} b_j z^j$, so $2k + j = n$ and the terms we consider are $a_k z^{2k} b_{n-2k} z^{n-2k}$. The restrictions

are $k \geq 0$ and $n - 2k \geq 0$, i.e. $0 \leq k \leq n/2$ and we see that k varies from 0 to the greatest integer $\leq n/2$. This yields the product formula

$$f_1(z)f_2(z) = \left(\sum_{n=0}^{\infty} a_n z^{2n}\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]} a_k b_{n-2k}\right) z^n,$$

where $[n/2]$ is the standard notation for the greatest integer less than or equal to $n/2$. The radius of convergence is again minimum of both series.

Similarly, if one of the series contains only the odd powers of z, viz.

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^{2n+1} \quad \text{and} \quad f_2(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then the product of these series expansions is written as

$$f_1(z)f_2(z) = \left(\sum_{n=0}^{\infty} a_n z^{2n+1}\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{[(n-1)/2]} a_k b_{n-2k-1}\right) z^n.$$

For one series with odd powers and the other series with even powers, the multiplication is defined in a parallel way, but note that the resulting series will involve only the odd powers of z.

Example 20. Write the power series of the function $e^z \cosh z$.

Since

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad |z| < \infty \quad \text{and} \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty,$$

in view of the above remark the product becomes

$$\left(\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]} \frac{1}{(2k)!(n-2k)!}\right) z^n, \quad |z| < \infty.$$

Example 21. Show that the function defined by

$$f(z) = \begin{cases} \dfrac{\sinh z - z}{z^3}, & z \neq 0, \\[2ex] \dfrac{1}{6}, & z = 0. \end{cases}$$

is an entire function.

Using the power series for $\sinh z$ obtained in Example 4(a), we get for $z \neq 0$, the relation

$$f(z) = \frac{\sinh z - z}{z^3} = \frac{1}{z^3}\sum_{n=1}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad 0 < |z| < \infty.$$

This series converges for all values of z and in particular for $z = 0$. As

$$f(z) = \frac{\sinh z - z}{z^3} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+3)!}, \quad |z| < \infty,$$

we have $f(0) = 1/3!$, the value of the function (given). This tells us that the power series of $f(z)$ converges for all points in the complex plane and hence gives rise to an entire function (refer to Theorem 2).

Example 22. Show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{z}{2} - \frac{5}{6}z^2 + \cdots, \qquad 0 < |z| < 1.$$

Writing

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \qquad |z| < \infty,$$

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots, \qquad |z| < 1.$$

Hence, for $|z| < \infty \cap 0 < |z| < 1$,

$$\begin{aligned}\frac{e^z}{z(z^2+1)} &= \frac{1}{z}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)(1 - z^2 + z^4 - z^6 + \cdots)\\ &= \frac{1}{z}\left(1 + z - \frac{z^2}{2} - \frac{5}{6}z^3 + \cdots\right)\\ &= \frac{1}{z} + 1 - \frac{z}{2} - \frac{5}{6}z^2 - \cdots, \qquad 0 < |z| < 1.\end{aligned}$$

Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

be a power series with radius of convergence $r > 0$, and let

$$g(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots$$

be a second power series with radius of convergence r'. Since we assume that $b_0 \neq 0$, there is a disk $|z| < R$ such that both series converge on this disk. In fact, we need only choose

$$R = \min\{r, r', |z_1|\},$$

where z_1 is the zero of second power series which lies closest to origin.

Thus $f(z)/g(z)$ is an analytic function in $|z| < R$ and hence its Taylor series expansion exists as

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n z^n, \tag{6.50}$$

where d_n's can be obtained by evaluating the derivatives of $f(z)/g(z)$ at $z = 0$, or to get the first few terms we simply divide the numerator function by the denominator function.

Thus, by division of power series we mean the process of finding the power series for the quotient. To determine the series for the quotient, we illustrate an elegant method of finding d_ns. This is known as the *method of undetermined coefficients*. Equation (6.50) is written as

$$(d_0 + d_1 z + \cdots + d_n z^n + \cdots)(b_0 + b_1 z + \cdots + b_n z^n + \cdots) = (a_0 + a_1 z + \cdots + a_n z^n + \cdots).$$

Comparing the coefficients of like powers of z on both sides, we get an infinite set of simultaneous linear equations

$$\begin{aligned} d_0 b_0 &= a_0 \\ d_0 b_1 + d_1 b_0 &= a_1 \\ &\vdots \\ d_0 b_n + d_1 b_{n-1} + \cdots + d_n b_0 &= a_n \\ &\vdots \end{aligned}$$

which determine the unknown coefficients d'_ns.

The problem of solving these equations is facilitated by the fact that the first of $(n+1)$ equations involves only the first $n+1$ unknowns. Solving the first two equations of the set, we obtain

$$d_0 = \frac{a_0}{b_0}, \qquad d_1 = \frac{a_1 b_0 - a_0 b_1}{b_0^2}$$

and so on. Thus, having determined $d_0, d_1, \ldots, d_{n-1}$, we substitute these values into the $(n+1)$st equation, obtaining

$$d_n = \frac{a_n - d_0 b_n - d_1 b_{n-1} - \cdots - d_{n-1} b_1}{b_0}.$$

These successive substitutions can be avoided by using Cramer's rule. The determinant of the coefficient matrix of the system is

$$\Delta = \begin{vmatrix} b_0 & 0 & 0 & \cdots & 0 \\ b_1 & b_0 & 0 & \cdots & 0 \\ b_2 & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_0 \end{vmatrix} = b_0^{n+1} \neq 0$$

and hence

$$d_n = \frac{1}{b_0^{n+1}} \begin{vmatrix} b_0 & 0 & 0 & \cdots & a_0 \\ b_1 & b_0 & 0 & \cdots & a_1 \\ b_2 & b_1 & b_0 & \cdots & a_2 \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & b_{n-1} & b_{n-2} & \cdots & a_n \end{vmatrix}, \quad n = 0, 1, 2, \ldots$$

Example 23. For the function

$$\frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} d_n z^n,$$

we show that $d_0 = d_1 = 1$ and $d_n = d_{n-1} + d_{n-2}$, $n \geq 2$.

Writing

$$(d_0 + d_1 z + d_2 z^2 + \cdots)(1 - z - z^2) = 1.$$

Comparing the coefficients of like powers of z on both sides, we have

$$d_0 = 1, \qquad d_0 \cdot (-1) + d_1 \cdot 1 = 0 \Rightarrow d_0 = d_1 = 1.$$

Now, using

$$d_0 b_n + d_1 b_{n-1} + \cdots + d_{n-2} b_2 + d_{n-1} b_1 + d_n b_0 = 0,$$

and assigning $b_0 = 1$, $b_1 = -1$, $b_2 = -1$, $b_n = 0$, $n \geq 3$, we get

$$d_{n-2}(-1) + d_{n-1}(-1) + d_n = 0.$$

This gives the recursive relations,

$$d_n = d_{n-1} + d_{n-2}, \quad n \geq 2. \tag{6.51}$$

The numbers d_n are called the *Fibonacci numbers*.

The closed form solution of the recursive relation (6.51) under the initial conditions $d_0 = d_1 = 1$ is

$$d_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \tag{6.52}$$

Thus,

$$\frac{1}{1-z-z^2} = \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]z^n, \quad |z| < \alpha.$$

The zeros of $(1-z-z^2)$ are given by

$$z_1 = \frac{-1+\sqrt{5}}{2} \quad \text{and} \quad z_2 = \frac{-1-\sqrt{5}}{2}.$$

The zero of the denominator function closet to the origin is $z_1 = (-1+\sqrt{5})/2$ which determines α, the radius of convergence of $\sum_{n=0}^{\infty} d_n z^n$.

Remark. If $f(z)$ and $g(z)$ have zeros of order m and n at the origin, respectively, then the division process gives power series for $m \geq n$, provided for $m = n$, $z = 0$ is a removable singularity. For $n > m$, be sure to have the Laurent series.

PROBLEM SET 6.5

1. Find the power series of the function $f(z) = (1-z)^{-25}$ and hence obtain the power series expansions of the functions $f_1(z) = (1-z)^{-24}$ and $f_2(z) = (1-z)^{-26}$.
   Suggestion. Use the binomial expansion,

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} z^k,$$

   and then, apply term-by-term integration and differentiation.

2. Find the series expansion of the following functions about the indicated points.

   (a) $\dfrac{e^z}{z^2}$ about $z = i$. (b) $\dfrac{e^{z-1}}{z(1-z)}$ in powers of $z - 1$;

(c) $\dfrac{\sin z - 1}{z^2}$ about $z = 1$; (d) $\dfrac{e^z}{z(z+1)}$ in powers of $z - 2$.

Suggestion. For (b) set $z = u + \pi/2$.

3. Developing $1/\sqrt{1-z^2}$ and integrating term by term, show that

$$\sin^{-1}z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \cdots, \quad |z| < 1.$$

Show that the series represents the principal value of $\sin^{-1}z$.

4. Using the technique of the above problem, find the power series expansion of $\tan^{-1}z$. Also, mention the regions of validity. Using Leibniz rule for alternating series, justify the sum

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{7} - \cdots$$

5. Sum the following power series for $|z| < 1$:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$; (b) $\displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$; (c) $\displaystyle\sum_{n=0}^{\infty} (n+3)z^n$; (d) $\displaystyle\sum_{n=1}^{\infty} (n^2+2n)z^n$.

Suggestion. For (a) and (b), differentiate, while for (c) multiply by z^2 and integrate and treat (d) with the help of (c).

6. Verify the following Fourier series expansions:

(a) $\displaystyle\frac{1}{2}\ln\left(\cot\frac{\theta}{2}\right) = \sum_{n=0}^{\infty} \frac{\cos(2n+1)\theta}{2n+1}, \quad 0 < \theta < \pi;$

(b) $\displaystyle\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1}, \quad 0 < \theta < \pi;$

(c) $\displaystyle e^{\cos\theta}\cos(\sin\theta) = \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!};$

(d) $\displaystyle e^{\cos\theta}\sin(\sin\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n!}.$

Suggestion. For parts (a) and (b), compare the real and imaginary parts of both the sides for $z = e^{i\theta}$ in Problem 14, Problem Set 6.2.

7. Let $f(z)$ and $g(z)$ be analytic functions in a domain D such that for some $a \in D$, $f(a) = g(a) = \cdots = f^{(n-1)}(a) = g^{(n-1)}(a) = 0$, and $g^{(n)}(a) \neq 0$. Then show that

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

This is identified as the L'Hôpital's rule in complex analysis.

8. Determine the constants a, b, c such that

$$\lim_{z\to 0}\frac{ae^z - b\cos z + ce^{-z}}{z\sin z} = 2i.$$

9. Find the expansion of $1/(z^2 \sinh z)$, and then, show that

$$\int_C \frac{dz}{z^2 \sinh z} = \frac{-\pi i}{3},$$

where C is a positively oriented contour enclosing the origin in the region $0 < |z| < \pi$.

10. Show that the function defined by

$$f(z) = \begin{cases} \dfrac{\operatorname{Log} z}{z-1}, & z \neq 1, \\ 1, & z = 1 \end{cases}$$

is analytic throughout the domain $0 < |z| < \infty$, $-\pi < \operatorname{Arg} z < \pi$.

11. Prove that if

$$f(z) = \begin{cases} \dfrac{\sin z}{z^2 - \pi^2}, & z \neq \pm\pi, \\ \dfrac{1}{2\pi}, & z = -\pi, \\ -\dfrac{1}{2\pi}, & z = \pi, \end{cases}$$

then f is an entire function.

12. Use the Cauchy product to find the series expansion of the functions:

(a) $\dfrac{\cosh z}{1-z}$; (b) $e^z \sinh z$; (c) $\dfrac{\sin z}{1-z^2}$; (d) $\cos z \cosh z$.

13. Find the Maclaurin series of the function

$$f(z) = \frac{1}{1 - z + z^2},$$

where the coefficients are given by $a_0 = a_1 = 1$, $a_2 = 0$ and $a_{n+3} + a_n = 0$ for $n \geq 0$.

14. In the expansion

$$\frac{a_0 + a_1 z + a_2 z^2}{b_0 + b_1 z + b_2 z^2 + b_3 z^3} = \sum_{n=0}^{\infty} c_n z^n \qquad (b_0 \neq 0)$$

find c_0, c_1 and c_2, and also find a four-terms recursive relation involving the coefficients c_n, c_{n-1}, c_{n-2} and c_{n-3}, $n \geq 3$.

15. The Maclaurin series

$$\sec z = E_0 - \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 - \frac{E_6}{6!}z^6 + \cdots, \qquad |z| < \frac{\pi}{2}$$

defines Euler's numbers E_{2n}. Show that $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$. Why are all the odd suffix Euler's numbers zero? Explain.

Suggestion. Some authors define Euler's numbers by the relation

$$\operatorname{sech} z = \sum_{n=0}^{\infty} E_{2n} \frac{z^{2n}}{(2n)!}, \qquad |z| < \frac{\pi}{2}.$$

16. The Maclaurin series of

$$\frac{z}{e^z - 1} = B_0 + B_1 z + \frac{B_2}{2!} z^2 + \frac{B_3}{3!} z^3 + \cdots$$

defines Bernoulli numbers. Using the method of undetermined coefficients, show that

(a) $B_1 = -\frac{1}{2}$, $B_n = 0$, $n = 3, 5, \ldots$;

(b) $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, etc.

Suggestion. The power series of $(e^z - 1)/z$ has the value 1 at $z = 0$, the reciprocal function $z/(e^z - 1)$ has a power series expansion in some neighbourhood of the origin if the value of this function is defined 1 at 0, i.e. $B_0 = 1$.

However, the easiest way to compute Bernoulli numbers is to write

$$\frac{z}{e^z - 1} = \frac{z}{2}\left(\frac{e^z + 1}{e^z - 1} - 1\right) = -\frac{z}{2} + \frac{z}{2}\frac{e^z + 1}{e^z - 1}.$$

and noticing that the second term on the right is an even function, so that $B_1 = -1/2$ and $B_n = 0$ if n is odd and > 1. Further, writing the defining equation in the form

$$\left(\frac{B_0}{0!} + \frac{B_1}{1!} z + \frac{B_2}{2!} z^2 + \cdots\right)\left(\frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right) = 1$$

and multiplying the two power series on the left, we conclude by examining the coefficient of z^{n-1} that

$$\binom{n}{0} B_0 + \binom{n}{1} B_1 + \binom{n}{2} B_2 + \cdots + \binom{n}{n-1} B_{n-1} = 0, \qquad n \geq 2,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

By taking $n = 3, 5, 7, 9, \ldots$ in the above equation, we get B_2, B_4, B_6, B_8, etc.

17. Use the result of the preceding problem to write the following Taylor series expansions:

(a) $z \cot z = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}, \qquad |z| < \pi;$

(b) $\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(1 - 2^{2n}) B_{2n}}{(2n)!} z^{2n-1}, \qquad |z| < \pi/2;$

(c) $z \operatorname{csz} z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2^{2n} - 2) B_{2n}}{(2n)!} z^{2n}, \qquad |z| < \pi;$

18. Derive the closed form expression (6.52) for the coefficients d_n's.

7

Singularities and Residues

Chapter 2 introduced the concept of singularity. However, we did not give any description of this concept with reference to residues, poles, zeros, etc. This chapter develops the theory of residue calculus. The relations of zeros and poles have been discussed in Sections 7.3 and 7.4. In Sections 7.5 and 7.6, we introduce meromorphic functions and obtain partial fraction expansions for those functions. Section 7.7 is concerned with the open mapping theorem and its consequences. Finally, we discuss regular points and singularities along with analytic continuation, and the main result obtained is Pringsheim's theorem.

7.1 CLASSIFICATION OF SINGULARITIES

Let us recall that a point z_0 is said to be a singular point or singularity of $f(z)$ if $f(z)$ fails to be analytic at z_0 but is analytic at some point in every neighbourhood of z_0.

Singularities are divided into two classes: (i) Isolated singularity; (ii) nonisolated singularity.

Isolated Singularity

A point z_0 is said to be an *isolated* singularity of a function $f(z)$ if there exists a deleted neighbourhood of z_0, $\{z : 0 < |z - z_0| < \delta\}$ in which the function is analytic.

For the function (i) $f(z) = 1/(z-1)$, $z = 1$ is an isolated singular point, since $f(z)$ is analytic in the open disc $0 < |z-1| < R$, $R > 0$, and for (ii) $g(z) = 1/(z-2)(z-3)$, $z = 2, 3$ are isolated singular points since the function is analytic in the annular domain $2 < |z| < 3$.

Nonisolated Singularity

A singularity which is not isolated is termed *nonisolated* singularity.

For $f(z) = \text{Log } z$, $z = 0$ is a nonisolated singularity, and moreover $(-\infty, 0]$ is the set of all nonisolated singularities of the function. Also, for $g(z) = 1/\sin(1/z)$, $z = 1/(n\pi)$,

$n \in \mathbb{I}$ are the singular points, while 0 is nonisolated singularity as each neighbourhood of the origin contains a singularity.

From now onwards, most of the time we shall be talking about isolated singularities. These are further classified into (i) removable singularity; (ii) pole; and (iii) essential singularity.

If z_0 is an isolated singularity, then in some deleted neighbourhood of z_0 the function is analytic and hence its Laurent expansion exists as

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots, \quad 0 < |z-z_0| < R.$$

The first series on the right side of the Laurent expansion is the *regular part* (Taylor series) while the series with negative powers of $(z-z_0)$ is called the *principal part*. Here R is the distance from z_0 to the nearest singularity of f other than z_0. If z_0 is the only singularity, then $R = \infty$.

Removable singularity. If, in some neighbourhood of an isolated singularity z_0, the Laurent expansion of the function has no principal part, then z_0 is called a *removable* singularity. Moreover, the nomenclature is justified as it can be removed by appropriately defining the function at z_0.

The function

$$f(z) = \frac{\sin z}{z}, \qquad z \neq 0$$

is undefined at $z = 0$. Note that it has a removable singularity at $z = 0$ due to the absence of the principal part in its Laurent series expansion,

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{4!} - \cdots, \qquad 0 < |z| < \infty.$$

However, if we define

$$g(z) = \begin{cases} \dfrac{\sin z}{z} & z \neq 0, \\ 1, & z = 0. \end{cases}$$

then the above series converges to 1 at $z = 0$ and the function $f(z)$ becomes analytic at this point. In a more general setting, if f has a removable singularity at a point z_0, say, then there is a function g analytic at z_0 such that $f(z) = g(z)$ for all z in some deleted neighbourhood of z_0.

Theorem 1 (Riemann). If f is analytic in $\Omega\backslash\{b\}$ and bounded in $0 < |z-b| < r$, then either f is analytic at b or f has a removable singularity at b.

Proof. Consider the function

$$h(z) = \begin{cases} (z-b)^2 f(z), & z \neq b, \\ 0, & z = b. \end{cases}$$

Then,

$$\lim_{k\to 0}\frac{h(b+k)-h(b)}{k} = \lim_{k\to 0}\frac{k^2 f(b+k)}{k} = \lim_{k\to 0} kf(b+k) = 0, \quad \text{since } |f(z)| \le M.$$

By definition, h is differentiable at any other point of domain Ω. Hence, h is analytic in Ω. So, by Theorem 4, Chapter 6, it can be expressed in a power series about each point of Ω. Thus,

$$h(z) = \sum_{n=2}^{\infty} c_n(z-b)^n.$$

Note that the summation index has been started from $n = 2$, since $c_0 = h(b) = 0$ and $c_1 = h'(b) = 0$.

Let $f(b) = c_2$. Then

$$(z-b)^2 f(z) = \sum_{n=2}^{\infty} c_n(z-b)^n$$

or

$$f(z) = \sum_{n=0}^{\infty} c_{n+2}(z-b)^n, \quad |z-b| < r.$$

This proves the assertion.

Pole. If, in the Laurent expansion of the function in some deleted neighbourhood of z_0, the principal part contains a finite number of terms, then the singularity z_0 is called a *pole*.

Suppose $b_m \neq 0$, $b_{m+1} = b_{m+2} = \cdots = 0$, the point z_0 is termed pole of order m. Let us consider the function

$$f(z) = \frac{z^2-2z+3}{z-2} = z + \frac{3}{z-2}, \qquad 0 < |z-2| < \infty$$

The point $z = 2$ is a pole of order 1 or a simple pole.

The function

$$f(z) = \frac{e^z}{(z-1)^3}$$

has a pole of order 3, since

$$\frac{e^z}{(z-1)^3} = \frac{ee^{z-1}}{(z-1)^3} = \frac{e}{(z-1)^3} + \frac{e}{(z-1)^2} + \frac{e}{2!(z-1)} + \cdots, \quad 0 < |z-1| < \infty.$$

Essential singularity. If the principal part of the Laurent expansion of a function $f(z)$ in some deleted neighbourhood of z_0 contains infinitely many terms, then z_0 is called an essential singularity of $f(z)$.

The function $f(z) = e^{1/z}$ has $z = 0$ as an essential singularity, since

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n} + \ldots, \qquad 0 < |z| < \infty.$$

Remark. Let us consider the expansion

$$\sum_{n=0}^{\infty} \frac{z^n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}, \qquad 1 < |z| < 3.$$

This expansion has infinite number of negative powers of z. Even then, $z = 0$ is not an essential singularity. This is so because the domain of convergence is not a deleted neighbourhood of the origin—a sensitive point to keep in mind while deciding essential singularity. In fact, it is the Laurent expansion of the function

$$\frac{2z}{(1-z)(z-3)}$$

in the annular domain $1 < |z| < 3$. Actually, f has simple poles at $z = 1$ and $z = 3$.

7.2 RESIDUES

Suppose $f(z)$ has an isolated singular point z_0. This means $f(z)$ can be expanded in the Laurent series in some deleted neighbourhood of z_0 as

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad 0 < |z-z_0| < R, \tag{7.1}$$

where

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \qquad n = 0, 1, \ldots,$$

$$b_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-z_0)^{-n+1}}\, dz, \qquad n = 1, 2, \ldots,$$

and γ is any positively oriented closed contour interior to the annular domain $0 < |z-z_0| < R$ and enclosing the point z_0. For $n = 1$, we observe that

$$b_1 = \frac{1}{2\pi i} \int_{\gamma} f(z)dz.$$

The number b_1 is defined as residue at $z = z_0$.

This means that the coefficient of $1/(z-z_0)$ in Laurent expansion will be identified by the residue of $f(z)$ at z_0, provided the series expansion is written in the form of Eq. (7.1) which is the most general method of finding residue for any isolated singularity.

Example 1. Classify the nature of the singularity of the function

$$f(z) = \frac{e^{-z}}{(z-2)^4}$$

and compute the residue.

Since $z = 2$ is the singularity, write the Laurent expansion in $0 < |z-2| < R$,

$$f(z) = \frac{e^{-2}e^{-(z-2)}}{(z-2)^4} = e^{-2}\left[\frac{1}{(z-2)^4} - \frac{1}{(z-2)^3} + \frac{1}{2!(z-2)^2} - \frac{1}{3!(z-2)} + \cdots\right].$$

Thus, $z = 2$ is a pole of order 4 and the residue at this point is given by the coefficient of $1/(z-2)$ in the expansion. Therefore, the residue of $f(z)$ at $z = 2$ is $-1/6e^2$.

Example 2. For the function

$$f(z) = \frac{1}{z}\sum_{k=0}^{n} (z+a)^{-k},$$

evaluate the residue for different values of a.

Clearly, for $a = 0$, $\mathrm{Res}_{z=0} f(z) = 1$. For $a \neq 0$, writing the Laurent expansion of $f(z)$ about the point $-a$,

$$f(z) = -\frac{1}{a}\left(1 - \frac{z+a}{a}\right)^{-1} \sum_{k=0}^{n} (z+a)^{-k}, \qquad 0 < |z+a| < |a|$$

$$= -\frac{1}{a}\left[1 + \left(\frac{z+a}{a}\right) + \cdots + \left(\frac{z+a}{a}\right)^{n-1} + \cdots\right] \sum_{k=0}^{n} (z+a)^{-k}.$$

The coefficient of $1/(z+a)$ is

$$-\frac{1}{a}\left(1 + \frac{1}{a} + \cdots + \frac{1}{a^{n-1}}\right)$$

and thus,

$$\mathrm{Res}_{z=-a} f(z) = \begin{cases} -\dfrac{1-a^n}{a^n(1-a)}, & a \neq 1, \\ -n, & a = 1. \end{cases}$$

Remarks. 1. If $f(z)$ has a simple pole at z_0 and $g(z)$ is analytic at z_0 such that $g(z_0) \neq 0$, then

$$\mathrm{Res}_{z=z_0} f(z)\, g(z) = g(z_0)\, \mathrm{Res}_{z=z_0} f(z).$$

Proof. The result immediately follows due to existence of the limit

$$\lim_{z\to z_0} (z-z_0)f(z)g(z) = g(z_0) \lim_{z\to z_0} (z-z_0)f(z) = g(z_0)\, \mathrm{Res}_{z=z_0} f(z).$$

2. If a function $f(z)$ has an isolated singularity at z_0 and $f(z)$ is even in $z-z_0$, i.e. $f(z-z_0) = f(-(z-z_0))$, then $\mathrm{Res}_{z=z_0} f(z) = 0$.

Proof. Suppose that f is even. Then the Laurent expansion of f about z_0 cannot have odd powers of $(z-z_0)$. The assertion follows.

Example 3. Find the residue at $z = 0$ of the function

$$f(z) = \frac{\sin z^2}{z^5 \sin z}.$$

Since $f(z) = f(-z)$, the function is symmetric about singularity $z = 0$, in view of Remark (2) it follows that the residue at the origin is 0.

Theorem 2 (Residue theorem). If γ is a positively oriented simple closed contour within and on which the function is analytic, except at a finite number of singularities $z_k (k = 1, 2, \ldots, n)$ interior to γ, then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z).$$

Proof. Since the singular points $\{z_1, z_2, \ldots, z_n\}$ are finite, they can be separated by circular neighbourhoods (Fig. 7.1), each having a positive orientation. The circles C_k and the boundary of γ form the boundary of a closed region in which the function $f(z)$ is analytic and whose interior is a multiply connected domain.

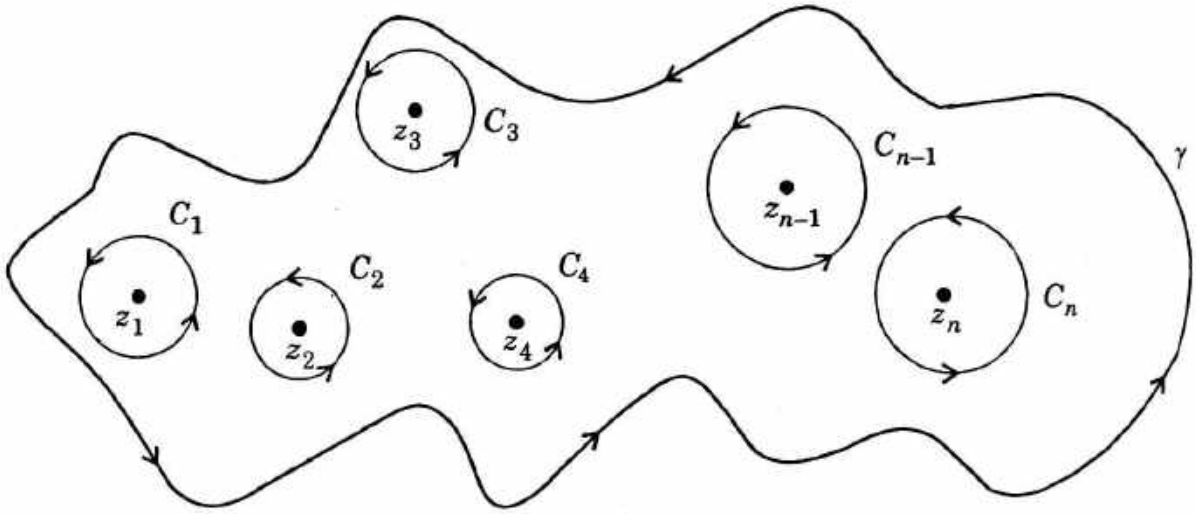

Figure 7.1

Hence, by the Cauchy–Goursat theorem for multiply connected regions (Theorem 4, Chapter 4),

$$\int_\gamma f(z)dz - \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

or

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz$$

or

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z).$$

Note. Residue is not defined for a nonisolated singularity of a function. Also, the number of isolated singular points inside a simple closed contour is finite, a fact used in residue theorem but will be proved later.

7.3 POLES AND ZEROS

The basic method of finding residue at an isolated singularity z_0 is to write the Laurent expansion about z_0 and determine the coefficient of $1/(z-z_0)$. But for poles, in addition to this, we have other simpler methods, where the series expansion is not required.

Proposition 1. The function $f(z)$ has a pole of order m at $z_0 \Leftrightarrow f(z)$ can be written in the form $f(z) = \phi(z)/(z-z_0)^m$, where $\phi(z_0) \neq 0$, $\phi(z)$ is analytic at z_0.

Proof. Since $f(z)$ has a pole of order m, then by definition, for $0 < |z-z_0| < R$,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}, \quad b_m \neq 0$$

or

$$f(z) = \frac{1}{(z-z_0)^m}\left[\sum_{n=0}^{\infty} a_n(z-z_0)^{m+n} + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m+2} + \cdots + b_m\right]$$

$$= \frac{\phi(z)}{(z-z_0)^m}.$$

Clearly, $\phi(z_0) = b_m \neq 0$ and is analytic at z_0, as it has Taylor series expansion about z_0.

Conversely, suppose $f(z)$ can be written in the form: $f(z) = \phi(z)/(z-z_0)^m$, then

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z-z_0) + \frac{\phi''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \cdots$$

or, in $0 < |z-z_0| < R$,

$$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}\frac{1}{(z-z_0)} + \frac{\phi^{(m)}(z_0)}{m!} + \frac{\phi^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots$$

Since $\phi(z_0) \neq 0$, $f(z)$ has a pole of order m with residue,

$$b_1 = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \tag{7.2}$$

In case of simple pole, i.e. $m = 1$, $\operatorname{Res}_{z=z_0} f(z_0) = \phi(z_0)$.

Interestingly, the sufficient part has developed a method of determining residue at z_0. Equation (7.2) is usually applied in the form

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right]. \tag{7.3}$$

Example 4. The function

$$f(z) = \frac{z^2 + 16}{(z-i)^2 (z+3)}$$

has singularities at $z = i$ and $z = -3$. Find the residue at these singularities.

For $z = -3$,

$$\phi(z) = \frac{z^2 + 16}{(z-i)^2}.$$

Here, $\phi(-3) \neq 0$ and $\phi(z)$ is analytic at $z = -3$. Hence the residue at $z = -3$ is given by

$$\phi(-3) = \frac{9 + 16}{(-3-i)^2} = 2 - \frac{3}{2} i.$$

For residue at $z = i$,

$$\phi(z) = \frac{z^2 + 16}{z + 3}.$$

Here, $z = i$ is a pole of order 2. Hence the residue at $z = 2$, in view of (7.2) or (7.3),

$$\phi'(z) = \frac{2z(z+3) - (z^2 + 16)}{(z+3)^2} = \frac{z^2 + 6z - 16}{(z+3)^2}.$$

Putting $z = i$, $\text{Res}_{z=i} f(z) = \phi'(i) = -1 + (3/2)i$.

Let us work out one more example concerning the application of formula (7.2).

Example 5. If $f(z) = (z-a)^{-n} (z-b)^{-m}$, where m, n are positive integers, show that

$$\text{Res}_{z=a} f(z) = -\text{Res}_{z=b} f(z).$$

The given function has a pole of order n at $z = a$. This suggests

$$\phi(z) = (z-b)^{-m}$$

Hence

$$\text{Res}_{z=a} f(z) = \frac{\phi^{(n-1)}(a)}{(n-1)!} = \frac{(-m)(-m-1)\cdots(-m-n+2)}{(n-1)!} (a-b)^{-m-n+1}$$

or

$$\text{Res}_{z=a} f(z) = (-1)^{n-1} \binom{m+n-2}{n-1} (a-b)^{-m-n+1}.$$

Now, the result follows at once by interchanging the role of a and b; and n and m.

Note. For the function $f(z) = (\sin z)/z^4$, we cannot apply the above method, since $\phi(z) = \sin z = 0$ at $z = 0$.

Proposition 2. If a function $f(z)$ is of the form $f(z) = p(z)/q(z)$, $p(z)$ and $q(z)$ are analytic at z_0, $p(z_0) \neq 0$, then at the point $z = z_0$,

$$f(z) = \frac{p(z)}{q(z)} \text{ has a pole of order } m \Leftrightarrow q(z) \text{ has a zero of order } m.$$

This is a combination of Proposition 5 of Section 6.4 and Proposition 1 of this section.

In fact, Proposition 2 gives a very good method of finding the order of any pole. In addition, for $p(z_0) \neq 0$, $q'(z_0) \neq 0$, then $q(z_0) = 0$ indicates a simple zero, and hence in this case $m = 1$, and we can write

$$q(z) = (z - z_0)g(z)$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0. The residue is given by

$$b_1 = \frac{p(z_0)}{g(z_0)}.$$

Since $g(z_0) = q'(z_0)$, we get

$$b_1 = \frac{p(z_0)}{q'(z_0)}, \qquad q'(z_0) \neq 0. \tag{7.4}$$

The formula (7.4) is frequently used for computing residue at simple poles when the function is given in the form $p(z)/q(z)$ and $p(z)$ does not vanish at these poles.

Example 6. For the functions $\cot z$, $\pi \operatorname{cosec}(\pi z)$ and $\tanh z$, determine singularities and calculate the residues.

Writing

$$\cot z = \frac{\cos z}{\sin z},$$

the denominator function has simple zeros at $z = n\pi$, $n \in \mathbb{I}$. Since $\cos n\pi \neq 0$ at $z = n\pi$, in view of Proposition 2, these are the simple poles of $\cot z$. The residue at each pole $z = n\pi$ is the consequence of (7.4),

$$\operatorname{Res}_{z=n\pi} \cot z = \frac{p(z)}{q'(z)} = 1 \Rightarrow \operatorname{Res}_{z=n\pi} \cot z = 1,\ p(z) = \cos z,\ \ q(z) = \sin z.$$

Similarly, we write the residue of following functions:

$$\operatorname{Res}_{z=n} (\pi \operatorname{cosec} \pi z) = \left.\frac{\pi}{(\sin \pi z)'}\right|_{z=n} = \frac{1}{\cos n\pi} = (-1)^n, \quad n \in \mathbb{I},$$

$$\operatorname{Res}_{z=(n+1/2)\pi i} \tanh z = 1, \quad n \in \mathbb{I}.$$

Note. Observe that formula (7.4) cannot be applied to $(\tanh z)/z^2$, even though $z = 0$ is a simple pole of the function. The obvious reason is that at $z = 0$, the numerator vanishes. So, (7.3) is used to get

$$\text{Res}_{z=0}\frac{\tanh z}{z^2} = \lim_{z\to 0}\frac{(z-0)\sinh z}{z^2\cosh z} = 1.$$

However, for other simple poles $z = (n + 1/2)\pi i$, $n \in \mathbb{I}$, (7.4) is applicable as numerator does not vanish at these points. For this situation

$$\text{Res}_{z=(n+1/2)\pi i}\frac{\tanh z}{z^2} = \lim_{z\to(n+1/2)\pi i}\frac{\sinh z}{2z\cosh z + z^2\sinh z} = -\frac{4}{(2n+1)^2\pi^2}.$$

Example 7. Decide the nature of all the singularities of the following functions: (a) $\pi(\cot \pi z)/z^2$; (b) $\pi(\text{cosec}\, \pi z)/z^2$. Also, calculate the residue at these singularities for both the functions

For part (a), the singularities of the function

$$\frac{\pi\cot\pi z}{x^2} = \frac{\pi\cos\pi z}{z^2\sin\pi z}.$$

are $z = n$, $n \in \mathbb{I}$. The origin is a zero of order three of the denominator function as $\cos n\pi \neq 0$, $n \in \mathbb{I}$, the point $z = 0$ is a pole of order three of the function, while $z = n$, $n = \pm 1, \pm 2, \ldots$ are simple poles. Using (7.4), we have

$$\text{Res}_{z=n}\frac{\pi(\cos\pi z)/z^2}{\sin\pi z} = \frac{1}{n^2}, \qquad n = \pm 1, \pm 2, \ldots$$

By finding the coefficient of $1/z$ in the series expansion (use division of power series) or direct application of formula (7.3), we have

$$\text{Res}_{z=0}\frac{\pi\cot\pi z}{z^2} = -\frac{\pi^2}{3}.$$

For part (b), the singularities are same as in (a). The corresponding residues are computed in a similar way

$$\text{Res}_{z=0}\frac{\pi\,\text{cosec}\,\pi z}{z^2} = \frac{\pi^2}{6}, \qquad \text{Res}_{z=n}\frac{\pi\,\text{cosec}\,\pi z}{z^2} = -\frac{1}{n^2}.$$

Example 8. Let γ be a simple closed contour with positive orientation and enclosing the origin. Then show that

$$\int_\gamma e^{z+z^{-1}}\,dz = 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}. \tag{7.5}$$

Let us take $g(z) = e^{1/z}$ and $f(z) = e^z$ in Theorem 1, Section 6.2. Write the power series for $f(z)$, integrate term by term and apply the residue theorem to obtain

$$\int_\gamma e^{z+z^{-1}}\,dz = \int_\gamma \sum_{n=0}^{\infty}\frac{z^n}{n!}e^{1/z}dz$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\int_\gamma z^n e^{1/z}dz$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} 2\pi i \operatorname{Res}_{z=0} z^n e^{1/z}. \tag{7.6}$$

The function inside the integral on the right side of (7.6) has essential singularity $z = 0$, and its residue is given by the Laurent expansion

$$z^n e^{1/z} = z^n + z^{n-1} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}\frac{1}{z} + \cdots, \qquad 0 < |z| < \infty.$$

Thus, the coefficient of $1/z$ gives the residue $1/(n+1)!$,

$$\operatorname{Res}_{z=0} z^n e^{1/z} = \frac{1}{(n+1)!}.$$

Now, inserting this value in (7.6), we get what agrees with (7.5).

Proposition 3. Let $f(z)$ be analytic at z_0 such that $f(z_0) \neq 0$ and $g(z)$ has a zero of order two at z_0. Then

$$\operatorname{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{6f'(z_0)g''(z_0) - 2f(z_0)g'''(z_0)}{3[g''(z_0)]^2}. \tag{7.7}$$

The function $g(z)$ has a zero of order two at z_0. Hence,

$$g(z) = (z - z_0)^2 p(z),$$

where $p(z_0) \neq 0$ and $p(z)$ is analytic at z_0. (see Proposition 5, Chapter 6). Thus,

$$\frac{f(z)}{g(z)} = \frac{f(z)/p(z)}{(z - z_0)^2}.$$

has a pole of order 2, and its residue at z_0 is given by

$$\operatorname{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)p(z_0) - f(z_0)p'(z_0)}{[p(z_0)]^2}. \tag{7.8}$$

From the relation $g(z) = (z - z_0)^2 p(z)$, we have

$$\begin{aligned} g'(z) &= 2(z - z_0)\, p(z) + (z - z_0)^2 p'(z); \\ g''(z) &= 2p(z) + 4(z - z_0)p'(z) + (z - z_0)^2 p''(z); \\ g'''(z) &= 6p'(z) + 6(z - z_0)p''(z) + (z - z_0)^2 p'''(z). \end{aligned}$$

These relations imply $p(z_0) = g''(z_0)/2$ and $p'(z_0) = g'''(z_0)/6$. On incorporating these values in (7.8), we have (7.7).

Example 9. Use the above proposition to find the residue of the function

$$\frac{z^2 + \sin z}{\cos z - 1}.$$

at its all singular points.

Note that $g(z) = \cos z - 1 = 0$ implies that $z = 2n\pi$, $n \in \mathbb{I}$, and $g'(2n\pi) = 0$, but $g''(2n\pi) = -1$. Thus, $g(z)$ has $2n\pi$ zeros each having order two. Moreover, $f(z) = z^2 + \sin z$ is nonzero at these zeros. Also, $f'(2n\pi) = 4n\pi + 1$ and $g'''(2n\pi) = 0$. Inserting these values in (7.7) gives

$$\mathrm{Res}_{z=2n\pi} \frac{z^2 + \sin z}{\cos z - 1} = 2(4n\pi - 1), \quad n \in \mathbb{I}.$$

Let us examine how a branch of a multivalued function affects the residue.

Example 10. Determine the residue of the multivalued function

$$f(z) = \frac{z^{1/2}}{z^2 + 1}$$

at each of its poles.

Note that for a given real number α, the branch

$$\frac{z^{1/2}}{z^2+1}, \qquad (|z| > 0,\ \alpha < \arg z < \alpha + 2\pi)$$

is single-valued and analytic in this domain except the points $z = \pm i$. These points are isolated singularities of the function as simple poles. Hence,

$$\mathrm{Res}_{z=i}\, f(z) = \left.\frac{z^{1/2}}{z+i}\right|_{z=i} = \frac{e^{(1/2)\log i}}{2i} = \frac{e^{i(n+1/4)\pi}}{2i} = \frac{1-i}{2\sqrt{2}} e^{in},$$

where $n \in \mathbb{I}$ is chosen such that $\alpha < (n + 1/4)\pi < \alpha + 2\pi$ is satisfied for the specified value of α, i.e. for the given branch.

Similarly, the residue at $z = -i$ is

$$\mathrm{Res}_{z=-i}\, f(z) = \frac{1+i}{2\sqrt{2}} e^{in}.$$

Remark. In particular, for the principal branch

$$\frac{z^{1/2}}{z^2+1}, \qquad (|z| > 0,\ -\pi < \arg z < \pi),$$

take $\alpha = -\pi$ in Example 10. Hence the residues at i and $-i$ are given by assigning $n = 0$, respectively, as

$$\mathrm{Res}_{z=i}\, f(z) = \frac{1-i}{2\sqrt{2}}, \qquad \mathrm{Res}_{z=-i}\, f(z) = \frac{1+i}{2\sqrt{2}}.$$

Theorem 3 (Casorati–Weierstrass). Suppose $f(z)$ is analytic in a domain $\Omega\backslash\{a\}$ and a is an essential singularity of the function. Then, for any given complex number c and for any $\varepsilon > 0$ there exists a neighbourhood of a such that

$$z \in 0 < |z - a| < \delta \text{ implies } |f(z) - c| < \varepsilon. \tag{7.9}$$

Proof. We prove the theorem by contradiction. Suppose (7.9) does not hold. This

means that there exists a complex number c such that for given $\varepsilon > 0$, we have

$$\forall\, z \in 0 < |z-a| < \delta \Rightarrow |f(z)-c| \geq \varepsilon. \tag{7.10}$$

The function $f(z) - c$ is analytic in $\Omega - \{a\}$ and does not vanish due to (7.10). So, $1/[f(z) - c]$ is analytic in $\Omega - \{a\}$, and by our assumption, $1/|[f(z)-c]| \leq \varepsilon$. This implies $1/[f(z) - c]$ is bounded in $0 < |z - a| < \delta$. In view of the Riemann theorem, the function

$$g(z) = \frac{1}{f(z)-c}$$

is either analytic at a or a is a removable singularity. Let $g(a)$ be defined so that $g(z)$ is analytic at a. Since f cannot be a constant function, neither can g; in view of the Taylor series for g at a either $g(a) \neq 0$, or g has a zero of some finite order at a. Consequently, its reciprocal

$$\frac{1}{g(z)} = f(z) - c$$

is either analytic at a or has a pole there. But this contradicts the hypothesis that $z = a$ is an essential singularity. Thus, condition (7.9) must be satisfied at some point in the given deleted neighbourhood.

Corollary 1. A function $f(z)$ has a pole of order m at $z = a \Leftrightarrow \lim_{z\to a} f(z) = \infty$.

Proof. Since $f(z)$ has a pole of order m at a, in view of Proposition 1, we can write

$$f(z) = \frac{\phi(z)}{(z-a)^m},$$

where $\phi(a) \neq 0$, $\phi(z)$ is analytic at a, and hence taking limit as $z \to a$ we have $f(z) \to \infty$.

To prove the converse, let us assume that z_0 is a pole but $\lim_{z\to a} f(z) \neq \infty$. This means $\lim_{z\to a} |f(z)|$ is finite or $\lim_{z\to a} f(z)$ does not exist. In the first case, $z = a$ is a removable singularity (refer to Theorem 1), and in the second case, it is an essential singularity (see Theorem 3), a contradiction to our hypothesis that $z = a$ is a pole.

Corollary 2. The poles of an analytic function interior to a simple closed contour are finite in number.

Proof. Suppose $f(z)$ has infinite number of poles inside a region enclosed by the contour γ. This region is bounded and closed. Hence, by Bolzano Weierstrass property any sequence of poles $\{a_1, a_2, \ldots\}$ in the region has a limit point in the bounded and closed region. Let a be the limit point of the sequence, i.e. $a_n \to a$ as $n \to \infty$. This implies that

$$f(a) = \lim_{n\to\infty} f(a_n) = \infty.$$

So, a is pole of the function by Corollary 1, and moreover, a is not isolated (every neighbourhood of a contains some a_n) which contradicts the definition of pole. Thus, the poles of an analytic function inside a simple closed contour are finite in number.

Note. The poles of an analytic function are isolated by definition.

If $f(z)$ is analytic and has a pole at $z = a$, then by Corollary 1, $|f(z)| \to \infty$ as $z \to a$ in any manner. The behaviour of a function in the neighbourhood of an essential singularity differs radically from its behaviour in the neighbourhood of a pole. For $z = a$ as the essential singularity, $f(z)$ assumes every value infinite number of times with at most one possible exception in every neighbourhood of $z = a$. This is the Picard theorem which we shall not prove here. However, a result having close analogy with Picard theorem has already been proved as Theorem 2. To convince the reader about this theorem the following example is proposed.

Example 11. Show that $e^{1/z}$ takes every value (in its range) infinitely many times with the exception of zero (Picard exceptional value).

Since $z = 0$ is an essential singularity (illustrated in Section 7.1) and the exponential function vanishes nowhere, this implies that 0 is the exceptional value. Let $c \neq 0$ be any arbitrary complex number. Consider the equation

$$e^{1/z} = c.$$

The solution of the above equation is the set of points given by

$$\frac{1}{z} = \ln|c| + i(2k\pi + \operatorname{Arg} c), \qquad k = 0, \pm 1, \pm 2, \ldots$$

That is,

$$z_k = \frac{1}{\ln|c| + i(2k\pi + \operatorname{Arg} c)}.$$

In particular, an infinite sequence $\{z_n\}$ is obtained such that $z_n \to 0$ as $n \to \infty$ and

$$e^{1/z_n} = c, \qquad n = 1, 2, \ldots$$

For n sufficiently large, all z_n can be made to lie in any neighbourhood about $z = 0$.

PROBLEM SET 7.1

1. Evaluate the following integrals using the residue theorem:

(a) $\displaystyle\int_\gamma \frac{e^{-iz}}{(z+3)(z-i)^2}\, dz, \qquad \gamma = \{z : z = 1 + 2e^{i\theta},\ 0 \le \theta \le 2\pi\};$

(b) $\displaystyle\int_\gamma \frac{z}{(z^2+1)(z-3)^2}\, dz, \qquad \gamma = \{z : |z| = 2\}$, positively oriented;

(c) $\displaystyle\int_\gamma \frac{\sin(3z)}{(z-\pi/4)^4}\, dz, \qquad \gamma = \{(x, y),\ |x| \le 2,\ |y| \le 2\}$, positively oriented;

(d) $\displaystyle\int_\gamma \frac{1}{z(2z-5)(z-4)}\, dz, \qquad \gamma = \{z : |z+2| + |z-2| = 6\}$, positively oriented;

(e) $\int_C \tan(nz)dz, \quad n \in \mathbb{I}_+, \quad C : |z| = 1/n$, positively oriented.

2. Evaluate the integral

$$\int_C \frac{2z^2 - z + 1}{(2z-1)(z+1)^2}\, dz, \qquad C : r = 2\cos\theta,\ 0 \le \theta \le \pi$$

by applying the residue theorem and as well as by using the Cauchy integral formula.

3. If γ is a positively oriented closed contour within and on which the function $f(z)$ is analytic except at a finite number of singularities z_k $(k = 1, 2, \ldots, n)$ interior to γ, show that

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \eta(\gamma, z_k)\, \mathrm{Res}_{z=z_k} f(z),$$

where $\eta(\gamma, z_k)$ is the winding number of γ with respect to z_k. Use this result to compute

$$\int_\gamma \cot z\, dz, \quad z = 4e^{i4\theta}, \quad -\pi \le \theta \le \pi.$$

4. Give an alternative proof of the Riemann theorem, Section 7.1 by using the Laurent expansion in a deleted neighbourhood of $z = b$.

5. Calculate $\mathrm{Res}_{z=a}\, g(z)f'(z)/f(z)$, given that $g(z)$ is analytic at $z = a$, while (a) $f(z)$ has a zero of order k at $z = a$; (b) $f(z)$ has a pole of order k at $z = a$.

6. Locate and classify the singularities of the following functions:

(a) $\dfrac{z - \sin(z-1)}{z-1}$; (b) $\dfrac{\cosh 2z - 1}{z^5}$; (c) $\dfrac{\cot z}{z^2}$; (d) $\dfrac{1}{1+z+z^2+z^3+z^4}$;

(e) $\dfrac{e^{2iz} + \cos z}{(\sin z - 1)\sin z}$ (f) $\dfrac{1+z}{\cos z - 1}$; (g) $\sec\left(\dfrac{1}{z}\right)$; (h) $\dfrac{e^{az}}{1 - e^{-z}}$

7. Suppose f and g are both analytic at z_0 and $f(z_0) \neq 0$, and z_0 is a first order zero for g. Show that

$$\mathrm{Res}_{z=z_0} \frac{f(z)}{[g(z)]^2} = \frac{f'(z_0)g'(z_0) - f(z_0)g''(z_0)}{[g'(z_0)]^3}.$$

8. Classify the singularities of the following functions and determine the residue at $z = 0$,

(a) $\csc^2 z$; (b) $\dfrac{1}{(z^2+z)^3}$; (c) $\dfrac{\sin 2z - 2\sin z}{(\sin z - z)\sin z}$; (d) $\cot z - \dfrac{1}{z}$.

Suggestion. Use the result of the preceding problem.

9. Let $f(z)$ and $g(z)$ be two analytic functions and have zeros of order m and n, respectively, at z_0. Then show that:

(a) $\dfrac{f(z)}{g(z)}$ is analytic at z_0, if $m > n$;

(b) $\dfrac{f(z)}{g(z)}$ has removable singularity at z_0, if $m = n$;

(c) $\dfrac{f(z)}{g(z)}$ has a pole of order $n - m$, if $m < n$.

10. Suppose ϕ is analytic at z_0 and $\phi(z_0) \neq 0$. If a function g has a zero of order three at z_0, show that

$$\operatorname{Res}_{z=z_0} \frac{\phi(z)}{g(z)} = 3\left\{\frac{-\phi''(z_0)}{g'''(z_0)} - \frac{1}{2}\frac{\phi'(z_0)g^{(iv)}(z_0)}{[g'''(z_0)]^2}\right\}.$$

11 Classify the singularities (isolated) of the function $f(z)$ and find the residue when

$$f(z) = \frac{\log z}{(z^2-1)^2}, \qquad |z| > 0, \qquad \frac{\pi}{4} < \arg z < \frac{9\pi}{4}.$$

12. If n is an even integer, locate all the singularities of $\tan^{n-1}(\pi z)$ and find the residue at $z = 1/2$.

13. Use residues to find the partial fraction expansion of the following functions:

(a) $f(z) = \dfrac{z^2 - 7z + 4}{z^2(z+4)}$; (b) $g(z) = \dfrac{z^3 + 3z^2 - z + 1}{z(z+1)^2(z^2+1)}$.

Suggestion. For (a), write

$$\frac{z^2 - 7z + 4}{z^2(z+4)} = \frac{A}{z+4} + \frac{B}{z} + \frac{C}{z^2},$$

where $A = \operatorname{Res}_{z=-4} f(z)$, $B = \operatorname{Res}_{z=0} f(z)$, $C = \operatorname{Res}_{z=0} zf(z)$. In case of (b) the process is similar.

14. If a function f has a pole of order k at z_0, show that z_0 is also a pole of f' but its order increases to $k + 1$.

15. Suppose $f(z)$ has a pole of order m at a and $P(z)$ is a polynomial of degree n. Show that the composite function $(Pf)(z)$ has a pole of order $m + n$ at a. Also, determine the residue at $z = a$ for the product function $Q(z)f(z)$, where $Q(z)$ is a polynomial in $z - a$ of degree n.

16. Suppose f has only finite singularities and $\lim_{z\to\infty} zf(z) = 0$. Show that there is $R > 0$ such that $\{z^2 f(z) : |z| > R\}$ is bounded.

17. Suppose $f(z)$ has an essential singularity. Then prove that there exists at least one sequence z_n such that $\lim_{n\to\infty} f(z_n) = \infty$.

18. Show that the function $\sin(1/z)$ assumes every value with one exception in every neighbourhood of the origin. What is the Picard exceptional value?
19. Prove the residue theorem for multiply connected regions.
20. Comment on the statement "Cauchy integral formula is a special case of the residue theorem".

7.4 BEHAVIOUR OF FUNCTIONS AT INFINITY

We mentioned in Chapter 1 that the complex plane can be extended by adjoining in the set of complex numbers a single point, called infinity in complex plane. Under the inversion transformation $w = 1/z$, the point $z = \infty$ in the extended complex plane corresponds to $w = 0$. Hence it is required to investigate the behaviour of the function for large values of $|z|$ replacing z by $1/z$ and analyzing the behaviour of the function in the neighbourhood of origin.

Definition 1. A function $f(z)$ is said to be analytic at $z = \infty$ if the function $f(1/z)$ is analytic at the origin. Similarly, we say that $f(z)$ has a zero of order m, pole of order n and isolated singularity at ∞ if $f(1/z)$ has a zero of order m, pole of order n and isolated singularity at the origin, respectively.

Example 12. If $f(z)$ is an entire function which has no singularity at infinity, show that $f(z)$ is constant.

Since $f(z)$ is analytic for all finite values z, it can be expanded in power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \qquad |z| < \infty.$$

From the above expansion,

$$f\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \qquad 0 < |z| \le \infty.$$

From the second expansion we conclude that $z = 0$ is an essential singularity, a contradiction to our assumption that $f(1/z)$ is analytic at $z = 0$. This is only possible if $a_n = 0$, $n \ge 1$. Thus,

$$f\left(\frac{1}{z}\right) = a_0 \qquad \text{or} \qquad f(z) = a_0.$$

Example 13. An analytic function $f(z)$ is a polynomial of order k if and only if the only singularity of $f(z)$ in extended complex plane is a pole of order k at ∞.

The only singularity of the function is a pole of order k at infinity. This means that $f(1/z)$ has a pole of order k at 0. Consider the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| < \infty,$$

and replacing z by $1/z$, we observe that

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \qquad 0 < |z| \leq \infty.$$

As $f(1/z)$ has pole of order k at 0, the terms $z^{-(k+1)}, z^{-(k+2)}, \ldots$ should vanish, in other words

$$a_n = 0 \ \forall \ n > k.$$

Thus the second expansion is reduced to

$$f\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \cdots + \frac{a_k}{z^k}, \qquad 0 < |z| \leq \infty.$$

Reverting back to z in the above expansion, $f(z)$ comes out to be a polynomial of degree k.

Example 14. The only singularities of an analytic function $f(z)$ are poles of order 1 and 2 at $z = -1$ and $z = 2$ with residues 1 and 2, respectively at these poles. Determine $f(z)$ if it also satisfies the conditions $f(0) = 7/4$ and $f(1) = 5/2$.

Given that (i) $z = -1$ is pole of order 1 with residue 1; (ii) $z = 2$ is a pole of order 2 with residue 2. This information leads to the formulation of the principal part as

$$\frac{1}{z+1} + \frac{2}{z-2} + \frac{b}{(z-2)^2}.$$

Hence, $f(z)$ can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \frac{1}{z+1} + \frac{2}{z-2} + \frac{b}{(z-2)^2}. \tag{7.11}$$

The function $f(z)$ has no singularity at $z = \infty$. This implies $f(1/z)$ is analytic at $z = 0$. Thus, the regular part of $f(z)$ has no term except some constant. This means that

$$a_n = 0, \qquad n = 1, 2, \ldots$$

With the data available so far, (7.11) turns out to be

$$f(z) = a_0 + \frac{1}{z+1} + \frac{2}{z-2} + \frac{b}{(z-2)^2}. \tag{7.12}$$

Now, using the given conditions $f(0) = 7/4$ and $f(1) = 5/2$ in (7.12), we have two linear equations

$$\frac{7}{4} = a_0 + \frac{b}{4}, \qquad 4 = a_0 + b.$$

with solution $a_0 = 1$ and $b = 3$. Inserting these values in (7.12), the desired function is

$$f(z) = 1 + \frac{1}{z+1} + \frac{2}{z-2} + \frac{3}{(z-2)^2}.$$

Proposition 4. An analytic function $f(z)$ is a rational function if and only if f has no singularity in extended complex plane other than poles.

Proof. First, we show that $f(z)$ can only have finite number of poles, since otherwise the poles of $f(z)$ would have a limit point ξ (possibly at infinity), and then, ξ is certainly a pole and it is nonisolated. This contradicts the definition of a pole.

Assume $a, b, \ldots, k$ are the poles of $f(z)$ with multiplicity $\alpha, \beta, \ldots, \kappa$. Then the function defined by

$$g(z) = f(z)(z-a)^{\alpha}(z-b)^{\beta} \cdots (z-k)^{\kappa}. \tag{7.13}$$

is analytic in $\mathbb{C}$. Therefore,

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| < \infty.$$

If $g(z)$ is analytic at $z = \infty$, then $g(z)$ reduces to a constant (see Example 12), and if $z = \infty$ is a pole, then $g(z)$ is a polynomial, (see Example 13). In both situations, by (7.13), $f(z)$ is a rational function.

Conversely, let $f(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials. Then the zeros of $Q(z)$ are the poles of $f(z)$ (see Proposition 2).

Remark. The reader is advised to note that $\cot z$ has simple poles as its singularities, even though it is not a rational function. Does it violate Proposition 4? if not, comment.

Let $f(z)$ be analytic in the deleted neighbourhood of the point at infinity, so that $f(z)$ has the Laurent expansion of the form

$$f(z) = \cdots + \frac{a_{-m}}{z^m} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots + a_n z^n + \cdots, \quad |z| > R. \tag{7.14}$$

Suppose we integrate $f(z)$ along the circle $C: |z| = \rho$, $\rho > R$, choosing the following convention for the positive direction of traversing C : An observer moving along C in the positive direction finds the exterior of C, and hence the point at infinity, on the left. In other words the positive direction of traversing a curve enclosing infinity is the clockwise direction.

In view of (7.14) and Example 9 in Chapter 4,

$$\int_C f(z)\, dz = \int_C \frac{a_{-1}}{z}\, dz = i a_{-1} \int_0^{-2\pi} d\theta = -2\pi i a_{-1}.$$

This suggests defining the quantity $-a_{-1}$ as the residue at infinity, since, then

$$\int_C f(z)\, dz = 2\pi i \operatorname{Res}_{z=\infty} f(z)$$

in keeping with the residue theorem.

Thus, the residue of $f(z)$ at ∞ is determined by the coefficient of one of the terms of

the regular part of the Laurent expansion of $f(z)$ at ∞, whereas the residue of $f(z)$ at a finite point z_0 is determined by the coefficient of one of the terms of the principal part. While deciding about the regular and principal parts it is to be noted that the expansion must be in the form of domain $0 < |z| < R$, where $R > 0$ is finite. It is significant to mention that the residue of $f(z)$ at ∞ may not be zero even when $f(z)$ is analytic at ∞. For example, $f(z) = 1/z$ is analytic at ∞ and in fact ∞ is a simple zero but

$$\text{Res}_{z=\infty} f(z) = -1 \neq 0.$$

To find residue at $z = \infty$, we use the formula

$$\text{Res}_{z=\infty} f(z) = -a_{-1} = -\text{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right), \tag{7.15}$$

which is the immediate outcome of writing Eq. (7.14) into the form

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \cdots + a_{-m}\, z^{m-2} + \cdots + a_{-2} + \frac{a_{-1}}{z} + \frac{a_0}{z^2} + \frac{a_1}{z^3} + \cdots, \quad 0 < |z| < \frac{1}{R}.$$

Theorem 4. If a function $f(z)$ is analytic except for isolated singular points in extended complex plane, then the sum of all the residues of $f(z)$ equals zero.

Proof. The function can only have a finite number of isolated singular points, since otherwise the singular points of $f(z)$ would have a limit point ξ (possibly at infinity), and then, ξ would be a nonisolated singular point of $f(z)$, contrary to hypothesis. Thus, there is a circle $C : |z| = \rho$, whose radius is so large that all the singularities are contained inside C. Then, by the residue theorem

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z), \tag{7.16}$$

where C is traversed in the usual positive direction for finite points, i.e. the counterclockwise direction. But relative to the point at infinity this is the negative direction of traversing of C and hence the same integral equals

$$\int_C f(z)\, dz = -2\pi i\ \text{Res}_{z=\infty} f(z). \tag{7.17}$$

Subtracting (7.17) from (7.16), we find

$$2\pi i[\text{Res}_{z=z_1} f(z) + \cdots + \text{Res}_{z=z_n} f(z) + \text{Res}_{z=\infty} f(z)] = 0$$

or

$$\text{Res}_{z=z_1} f(z) + \cdots + \text{Res}_{z=z_n} f(z) + \text{Res}_{z=\infty} f(z) = 0 \tag{7.18}$$

as desired.

Example 15. Find the value of the integral

$$\int_{|z|=2} \frac{1}{(z-3)(z^5-1)}\, dz,$$

where $|z| = 2$ is the positively oriented circle.

The singularities of the integrand inside $|z| = 2$ are five roots z_1, z_2, z_3, z_4, z_5 of the equation $z^5 - 1 = 0$. Thus, by the residue theorem

$$\int_{|z|=2} \frac{1}{(z-3)(z^5-1)}\,dz = 2\pi i \sum_{k=1}^{5} \operatorname{Res}_{z=z_k} f(z).$$

To calculate residue at five roots is a lengthy procedure, hence we use Eq. (7.18) to have

$$\sum_{k=1}^{5} \operatorname{Res}_{z=z_k} f(z) + \operatorname{Res}_{z=3} f(z) + \operatorname{Res}_{z=\infty} f(z) = 0.$$

This implies

$$\begin{aligned}\sum_{k=1}^{5} \operatorname{Res}_{z=z_k} f(z) &= -\operatorname{Res}_{z=3} f(z) - \operatorname{Res}_{z=\infty} f(z)\\ &= \left[\frac{1}{z^5-1}\right]_{z=3} + \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)\\ &= -\frac{1}{242} + 0.\end{aligned}$$

Thus, we get

$$\int_{|z|=2} \frac{1}{(z-3)(z^5-1)}\,dz = -\frac{2\pi i}{242} = -\frac{\pi i}{121}.$$

PROBLEM SET 7.2

1. Discuss the nature of the signularities at infinity of the following functions:

   (a) ze^{-z}; (b) $\dfrac{1-e^z}{1+e^z}$; (c) $ze^{1/z}$; (d) $z \operatorname{cosec}\left(\dfrac{1}{z}\right)$.

2. Calculate the residue at $z = \infty$ of the following functions:

   (a) $\dfrac{2-z}{z^2-z}$; (b) $\dfrac{z^4-3z+2}{z^3}$; (c) $\log \dfrac{z-a}{z-b}$; (d) $\sqrt{(z-\alpha)(z-\beta)}$.

3. Calculate $\operatorname{Res}_{z=\infty} f^2(z)$, given that $f(z)$ has a Laurent expansion of the form

$$a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$$

at infinity.

4. If γ is a positively oriented simple closed contour within and on which the function is analytic except for a finite number of singularities $z_k (k = 1, 2, \ldots, n)$ interior to γ and if, in addition, f is analytic exterior to γ, prove that

$$\int_\gamma f(z)\, dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right).$$

Suggestion. This is known as the alternative form of the residue theorem and is useful to evaluate integrals when the number of singularities is large, for this requires residue at a single point. For proof, see (7.15) and (7.18).

5. Using the alternative form of the residue theorem, find the value of the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when $f(z)$ is

(a) $\dfrac{(3z+2)^2}{z(z-1)(2z+5)}$; (b) $\dfrac{z^3 e^{1/z}}{1+z^2}$.

6. Use the alternative form of the residue theorem to establish the result of Problem 16 in Problem Set 4.1.

7. Avoiding excessive calculations evaluate the following integrals

(a) $\displaystyle\int_C \frac{1}{(4z^4-1)(z+2)^2}\, dz$; (b) $\displaystyle\int_C \frac{1}{(z-5)(z^{17}-1)}\, dz$,

where $C: |z| = 3/2$ is a positively oriented circle.

Suggestion. Use Eq. (7.18).

8. Prove or disprove, $\operatorname{Res}_{z=\infty} f(z) = -\lim_{z\to\infty} zf(z)$, provided $f(z)$ is analytic at $z = \infty$.

9. Give an alternative proof of Liouville's theorem based on the residue calculus.

10. Suppose f has only finite singularities and $\lim_{z\to\infty} zf(z) = 0$, show that there exists $R > 0$ such that $z^2 f(z)$ is bounded for $|z| > R$.

11. When $n \in \mathbb{N}$, verify Theorem 4 for the following functions:

(a) $f(z) = \dfrac{z^n e^{1/z}}{1+z}$; (b) $g(z) = \dfrac{1}{(z-3)(z^n-1)}$.

12. Let $f(z)$ be an entire function such that $\lim_{z\to\infty} |f(z)/z| = 0$. Then show that $f(z)$ is a constant. Instead, if we have $\lim_{z\to\infty} |f(z)/z^n| = 0$, what can you say about f?

Using this result verify that every entire function satisfying

$$|f(z)| \le A + B\,|z|^\alpha, \qquad 0 < \alpha < 1 \quad (A, B \text{ are constants})$$

is a constant.

Note. Exercise 12 is a stronger version of the Liouville's theorem.

13. If $f(z)$ is analytic at ∞, show that $f'(\infty) = 0$.

14. Find the Laurent expansion of the function

$$\log\frac{z-a}{z-b}=\sum_{n=1}^{\infty}\frac{b^n-a^n}{nz^n}.$$

at $z=\infty$. In which region is this expansion valid?

Suggestion. Write

$$\log\frac{z-a}{z-b}=\log\left(1-\frac{a}{z}\right)-\log\left(1-\frac{b}{z}\right).$$

7.5 MEROMORPHIC FUNCTIONS

Here, we derive some results based on the Cauchy residue theorem. These have significant applications and pertain to those functions all of whose singularities are poles.

Definition 2. An analytic function, whose only singularities in the finite complex plane are poles, is called a *meromorphic* function.

For example, the rational functions $f(z)=P(z)/Q(z)$ (see Section 2.3), where $P(z)$ and $Q(z)$ are polynomials of certain degree and zeros of $Q(z)$ are the poles of $f(z)$, constitute an important class of meromorphic functions. Note that by definition every analytic function is meromorphic, but the converse is not true.

Other examples of meromorphic functions (beyond rational functions) include $\tan z$, $\cot z$, $\sec z$ and $\csc z$. Clearly, the singularities of $\cot z$ are simple poles $z=n\pi$, $n\in\mathbb{I}$ and ∞ is the limit point of these poles. A meromorphic function does not have essential singularity in finite complex plane.

Having acquaintance with the meromorphic functions, Proposition 4 is restated as,

"A complex function is rational $\Leftrightarrow$ it is meromorphic in $\mathbb{C}_\infty$."

The residue theorem can be used to determine the number of zeros of an analytic function inside a given closed contour γ within and on which $f(z)$ is analytic except for a finite number of poles inside γ.

Proposition 5. If f is analytic within and on a positively oriented simple closed contour γ and does not vanish on γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz=N-P, \tag{7.19}$$

where N is the number of zeros and P is the number of poles of f which lie inside γ, both zeros and poles being counted according to their multiplicity.

Proof. First, we agree that the number of zeros and poles of $f(z)$ within γ is finite, (see Proposition 6, Chapter 6, and Corollary 2).

Suppose $f(z)$ has a zero of order m at $z=a$, then $f(z)$ can be written (see Proposition 5, Chapter 6), as

$$f(z)=(z-a)^m g(z),$$

where $g(a) \neq 0$ and $g(z)$ is analytic at a. Hence,

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}.$$

But, $g'(z)/g(z)$ is analytic at a. This implies $f'(z)/f(z)$ has a simple pole at $z = a$ with residue (logarithmic) m. In this way the residue of $f'(z)/f(z)$ at each zero of $f(z)$ is computed, and let the sum of these residues be denoted by N.

In case $z = b$ is a pole of order n of $f(z)$, then in view of Proposition 1,

$$f(z) = \frac{\phi(z)}{(z-b)^n},$$

where $\phi(z) \neq 0$ and $\phi(z)$ is analytic at b. Differentiation yields

$$\frac{f'(z)}{f(z)} = -\frac{n}{z-b} + \frac{\phi'(z)}{\phi(z)}.$$

This means $f'(z)/f(z)$ has a simple pole at b with residue $-n$, and hence the sum of the residues at each pole of $f(z)$ may be summed up to $-P$, say.

Therefore, by the Cauchy residue theorem, (7.19) follows.

Remark. If the function $f(z)$ in Proposition 5 has no poles inside γ, then (7.19) is simplified to

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N. \tag{7.20}$$

Formula (7.20) is very useful to determine certain integrals where the number of zeros is known.

Example 16. Evaluate the integral

$$\int_C \frac{f'(z)}{f(z)}\,dz, \quad \text{when } f(z) = \frac{(z^2+1)^2}{(z^2+3z+2)^3},$$

and C is the circle $|z| = 3$, taken in a positive sense.

The function $f(z)$ has double zero at $z = i$, $z = -i$, poles of order three at $z = -1$, $z = -2$. Hence $N = 2 + 2 = 4$ and $P = 3 + 3 = 6$. The zeros and poles are inside the circle $|z| = 3$. Now, using (7.19)

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = 4 - 6 = -2.$$

Thus, the value of the desired integral is $-4\pi i$.

A generalization of Proposition 5 is stated by:

Let $f(z)$ be analytic within and on a simple closed contour γ, and a_j, $j = 1, 2, \cdots, n$ and $b_k = 1, 2, \cdots, m$ be the zeros and poles of $f(z)$, respectively, not lying on γ. Then, for

a function $g(z)$ which is analytic within and on γ, we prove that

$$\frac{1}{2\pi i}\int_{\gamma} g(z)\,\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} p_j g(a_j) - \sum_{k=1}^{m} q_k g(b_k), \tag{7.21}$$

where p_j is the order of a_j and q_k is the order of b_k.

If $g(z)$ is analytic within and on a simple closed contour γ, then $g(z)f'(z)/f(z)$ has the residue $hg(a)$ at zero a of $f(z)$ of order h and the residue $-hg(a)$, when a is pole of $f(z)$ of order h. These facts lead to formula (7.21).

In particular when $g(z) \equiv 1$, (7.21) reduces to (7.19).

Example 17. For fixed positive integers n and m, compute

$$\int_{|z|=n} z^m \tan z\,dz,$$

where $|z| = n$ is the positively oriented circle.

Here $f(z) = \cos z$ and $f'(z) = -\sin z$. The singularities of $f(z)$ are $(k + 1/2)\pi$, $k \in \mathbb{I}$. Using (7.21), we get

$$\int_{|z|=n} z^m \tan z\,dz = -\int_{|z|=n} z^m\,\frac{-\sin z}{\cos z}\,dz = -\sum_{k=-n}^{n}\left(k+\frac{1}{2}\right)^m.$$

Theorem 5 (Argument principle). If $f(z)$ is analytic within and on a positively oriented simple closed contour γ and does not vanish on γ, then

$$\Delta_\gamma \arg f(z) = 2\pi(N-P), \tag{7.22}$$

where N is the number of zeros and P is the number of poles of $f(z)$ interior to γ, counting multiplicity and $\Delta_\gamma \arg f(z)$ denotes the variation of the argument of $f(z)$ around γ.

Proof. Since

$$\frac{d}{dz}\log f(z) = \frac{f'(z)}{f(z)},$$

for appropriate branch of the logarithmic function, it follows that

$$\int_{\gamma} \frac{f'(z)}{f(z)}\,dz = \Delta_\gamma \log f(z), \tag{7.23}$$

where Δ_γ denotes the variation of $\log f(z)$ around γ. Also,

$$\log f(z) = \ln|f(z)| + i \arg f(z) \Rightarrow \Delta_\gamma \log f(z) = i\Delta_\gamma \arg f(z).$$

This, together with (7.23) implies

$$\int_{\gamma} \frac{f'(z)}{f(z)}\,dz = i\Delta_\gamma \arg f(z). \tag{7.24}$$

Comparing Eqs. (7.19) and (7.24), the desired result follows.

Remark. When $f(z)$ has no poles interior to γ, the argument principle determines N as

$$\Delta_\gamma \arg f(z) = 2\pi N.$$

Example 18. Applying the argument principle, show that the polynomial equation

$$z^4 + 2z^3 + 3z^2 + 4z + 5 = 0$$

has no roots on the real and imaginary axes, and has one complex root in each quadrant.

We analyze the polynomial function

$$f(z) = u + iv = z^4 + 2z^3 + 3z^2 + 4z + 5.$$

For real values, put $z = x$. Then

$$f(x) = x^4 + 2x^3 + 3x^2 + 4x + 5 = (x^2 + 2)(x - 1)^2 + 3 > 0.$$

Since $f(x) > 0$ for all real x, the equation has no real root. Similarly, on the y-axis, take $z = iy$. Then

$$f(iy) = y^4 - 2iy^3 - 3y^2 + 4iy + 5 = 0 \tag{7.25}$$

or

$$(y^4 - 3y^2 + 5) - 2i(y^3 - 2y) = 0$$

Thus, $y^4 - 3y^2 + 5 = 0$ and $y^3 - 2y = 0$. These equations have no common solution in y. So, there are no roots on the y-axis.

Let us search for roots in quadrants. Take the case of the first quadrant. The set of points

$$z = re^{i\theta}, \qquad 0 < \theta < \pi/2, \quad r \to \infty$$

defines the first quadrant $OABO$ as shown in Fig. 7.2, and C denotes the boundary.

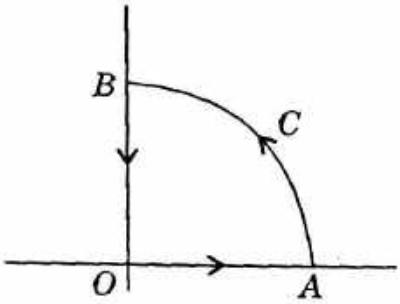

Figure 7.2

Along the line OA: x varies from 0 to ∞, $f(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$ and hence

$$\arg f = \tan^{-1}\left(\frac{v}{u}\right) = \tan^{-1}0 = 0 \Rightarrow \Delta \arg f = 0.$$

Along the arc AB: $z = re^{i\theta}$, $0 < \theta < \pi/2$,

$$f = u + iv = r^4 e^{i4\theta}\left(1 + \frac{2}{re^{i\theta}} + 3\frac{3}{r^2 e^{i2\theta}} + \frac{4}{r^3 e^{i3\theta}} + \frac{5}{r^4 e^{i4\theta}}\right) = r^4 e^{i4\theta}$$

for large values of r. In this situation, $\arg f = 4\theta$ and hence,

$$\Delta \arg f = 4[\theta]_0^{\pi/2} = 2\pi.$$

Along the line BO: $z = iy$, y varies from ∞ to 0 and from (7.25), we have

$$\arg f = \tan^{-1}\frac{2y(2-y^2)}{y^4-3y^2+5}, \qquad \Delta \arg f = \left[\tan^{-1}\frac{2y(2-y^2)}{y^4-3y^2+5}\right]_\infty^0 = 0.$$

This means that as y moves from ∞ to 0 along BO, a point (u, v) starting from any point on the u-axis comes back to some point on the u-axis. The ways in which w moves are given as

y	∞	$\sqrt{2}$	0
(u, v)	$(\infty, -\infty)$	$(3, 0)$	$(5, 0)$
$\tan^{-1}(v/u)$	0	0	0

Also, $\sqrt{2} < y < \infty \Rightarrow u > 0, v < 0$ and $0 < y < \sqrt{2} \Rightarrow u > 0, v > 0$. From this it is clear that y moves from ∞ to 0 along y-axis, while the point (u, v) starting from fourth quadrant reaches the first quadrant. But the curve is parallel to the u-axis through this movement and so, $\Delta \arg f = 0$. Finally,

$$\Delta_C \arg f = \Delta_{OA} \arg f + \Delta_{AB} \arg f + \Delta_{BO} \arg f = 0 + 2\pi + 0 = 2\pi.$$

Now, by the argument principle, the number of zeros in the first quadrant are

$$N = \frac{1}{2\pi}\Delta_C \arg f = 1.$$

This shows that the equation has one root in the first quadrant. Since the complex roots of a polynomial equation with real coefficients occur in conjugate pairs, this also tells us that the second root lies in the fourth quadrant.

The equation, being of fourth degree, has four roots and the remaining two roots lie in the second and third quadrants.

As an application of argument principle, we prove an important result, which compares the number of zeros of two analytic function. This is known as Rouché's theorem.

Theorem 6 (Rouché's theorem). Suppose $f(z)$ and $g(z)$ are analytic within and on a simple closed contour γ and $|f(z)| > |g(z)|$ on γ, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities within γ.

Proof. The hypothesis $|f(z)| > |g(z)| \geq 0$ implies $f(z) \neq 0$ on γ. Also, $f(z) + g(z) \neq 0$ on γ, otherwise $f(z) + g(z) = 0$ for some z on γ ensure that $f(z) = -g(z)$, i.e. $|g(z)| = |f(z)|$ at this point, thus contradicting the hypothesis $|f(z)| > |g(z)|$.

Let N and N' be the number of zeros of $f(z)$ and $f(z) + g(z)$ inside γ, respectively.

Then, by the argument principle, we have

$$2\pi N = \Delta_\gamma \arg f$$

and

$$2\pi N' = \Delta_\gamma \arg(f+g) = \Delta_\gamma \arg f + \Delta_\gamma \arg\left(1+\frac{g}{f}\right).$$

So,

$$2\pi(N'-N) = \Delta_\gamma \arg\left(1+\frac{g}{f}\right).$$

The theorem is thus proved if we show that the right side in the last equation is zero.

Now, $|g(z)| < |f(z)|$ on γ implies $|w-1| < 1$, where $w = 1 + g/f$. As z traverses γ, the point w traverses the closed curve Γ in $|w-1| < 1$, consequently, the image curve Γ does not, then, enclose the origin $w = 0$. Hence, arg $(1 + g/f)$ returns to its original value as z describes γ which means $\Delta_\gamma \arg(1 + g/f) = 0$ and $N = N'$.

Example 19. Using Rouché theorem, find the number of roots of $z^7 - 4z^3 + z - 1$ which lie interior to the unit circle $|z| = 1$.

We define $f(z)$ by involving the terms which give the largest modulus value on $|z| = 1$. Hence, take $f(z) = -4z^3$ and $g(z) = z^7 + z + 1$. Now,

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|z^7+z+1|}{|-4z^3|} \le \frac{|z|^7+|z|+1}{4|z|^3} = \frac{3}{4} < 1 \quad \text{on } |z| = 1.$$

Thus, conditions of Rouché's theorem are satisfied. So $f(z) = -4z^3$ and $f(z) + g(z) = z^7 - 4z^3 + z - 1$ have the same number of zeros inside $|z| = 1$. But $f(z)$ has a zero of order 3 at the origin. Hence, we conclude that $z^7 - 4z^3 + z - 1$ has 3 roots inside $|z| = 1$.

Proposition 6. If $f(z)$ is analytic and univalent in a domain Ω, then $f'(z) \neq 0$ in Ω.

Proof. Assume the contrary that $f'(z_0) = 0$ for some $z_0 \in \Omega$. Consider the Taylor series about z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

By assumption, $a_1 = 0$ $[a_1 = f'(z_0)]$ and hence,

$$f(z) - f(z_0) = \sum_{n=2}^{\infty} a_n (z-z_0)^n$$

has a zero of order $k (k \ge 2)$ at z_0. By hypothesis f is univalent and so f is a nonconstant function. $f(z) - f(z_0)$ is analytic and since the zeros of an analytic function are isolated, we can find a neighbourhood $|z - z_0| \le \rho$, $\rho > 0$, in which $f(z) - f(z_0)$ is not zero (except z_0). Thus, $f(z) - f(z_0) \neq 0$ on the circle $|z - z_0| = \rho$, and hence

$$m = \inf\{|f(z) - f(z_0)| : z \in |z - z_0| = \rho\} \text{ is positive.}$$

Then, for any complex number a such that $0 < |a| < m$, we have

$$|a| < |f(z) - f(z_0)|$$

or

$$|[f(z_0) - f(z_0) - a] - [f(z) - f(z_0)]| < |f(z) - f(z_0)|.$$

Therefore, by the Rouché theorem $f(z) - f(z_0) - a$ and $f(z) - f(z_0)$ have the same number of zeros inside $|z - z_0| = \rho$. But the later function has a zero of order $k (k \geq 2)$ inside this circle. Thus, the equation

$$f(z) = f(z_0) + a$$

is satisfied at two or more points. This contradicts the fact that f is univalent in Ω. This concludes that $f'(z) \neq 0$ for all points in Ω.

Corollary 3. Suppose $f(z)$ is a nonconstant function that is analytic at z_0. If $w_0 = f(z_0)$ and $f(z) - w_0$ have a zero of order n at z_0, show that the given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all a for whch $|a - w_0| < \delta$, the equation $f(z) = a$ has exactly n roots in the disc $|z - z_0| < \varepsilon$.

The proof proceeds exactly on the lines of Proposition 6, and hence omitted.

Remarks. 1. However, the converse of Proposition 6 is not true. For example $f(z) = e^z$ is analytic everywhere and $f'(z) = e^z \neq 0$ for all complex number. But this is a periodic function with period $2\pi i$ and hence not one-one in $\mathbb{C}$.

2. Rouché's theorem is also stated in the equivalent form:

Suppose $f(z)$ and $g(z)$ are analytic functions within and on a closed contour γ and $|g(z) - f(z)| < |f(z)|$ on γ, then $f(z)$ and $g(z)$ have the same number of zeros, counting multiplicities within γ.

Example 20. Suppose $f(z)$ is analytic in a domain Ω containing the closed unit disc, and $|f(z)| < 1$ if $|z| = 1$. How many solutions does the equation $f(z) = z^n$ have there?

Since

$$|(z^n - f(z)) - z^n| = |f(z)| < 1 = |z^n|,$$

the Rouché theorem ensures that $z^n - f(z)$ and z^n have the same number of zeros inside the unit disc. But z^n has n zeros and hence $z^n - f(z)$ has also n zeros.

Further implication of the Rouché's theorem is the following:

Theorem 7 (Hurwitz). Let $f_n(z)$ be a sequence of analytic functions defined on a domain Ω such that $f_n(z) \neq 0 \ \forall \ z \in \Omega$, $n = 1, 2, \ldots$ Assume that $f_n(z)$ converges uniformly to $f(z)$ on every compact subset of Ω, then either $f(z) \equiv 0$ or $f(z) \neq 0$ for all z in Ω.

Proof. Suppose $f(z)$ is not identically zero in Ω. Then, we have to show that $f(z)$ is never zero in Ω. Assume the contrary to hold, i.e. $f(z_0) = 0$ for some z_0 in Ω. Since the zeros of an analytic function are isolated (Corollary 4, Chapter 6), there exists a deleted neighbourhood $N_\delta(z_0)$ of z_0 in which the function is nonzero. So,

$$f(z) \neq 0, \qquad z \in 0 < |z - z_0| < \delta, \ \delta > 0.$$

In particular, $f(z)$ is nonzero on the punctured disc $0 < |z - z_0| \leq \delta_1$, $\delta_1 < \delta$. Let C: $|z - z_0| = \delta_1$

$$\varepsilon = \min\{|f(z)| : z \in C\}.$$

Since C is bounded and closed (compact), it follows by the given hypothesis that $f_n(z)$ converges uniformly to $f(z)$ on C. Hence, for the above $\varepsilon > 0$, there exists n_0 such that

$$|f_n(z) - f(z)| < \varepsilon \quad \forall\, n > n_0. \tag{7.26}$$

Owing to the definition of ε, $\varepsilon \le |f(z)|$ for $z \in C$. Hence, using (7.26), we have

$$|f_n(z) - f(z)| < |f(z)| \quad \forall\, n > n_0$$

for all points on C. Now, the Rouché's theorem asserts that the functions

$$f(z) \quad \text{and} \quad [f_n(z) - f(z)] + f(z) = f_n(z)$$

have the same number of zeros inside C. But $f(z)$ has a zero at z_0. Hence, $f_n(z)$ must also have a zero inside C. This contradicts the hypothesis of the theorem. Thus, we conclude that $f(z)$ can never be zero in Ω unless it is identically zero.

Remark. Usually, the Hurwitz's theorem is applied in the following form:

"Let $f_n(z)$ be a sequence of analytic functions within and on a simple closed contour γ. Suppose that $f_n(z)$ converges uniformly to a function $f(z)$, not identically zero within γ. If γ passes through no zero of $f(z)$, then there is an integer $N(\gamma)$ such that every function $f_n(z)$ with $n > N(\gamma)$ has the same number of zeros inside γ as the function $f(z)$".

7.6 THE OPEN MAPPING THEOREM

Nonconstant analytic functions have a very nice property of dispatching open sets onto open sets. Mappings with this property are called open maps.

Theorem 8 (Open mapping theorem). If f is a nonconstant analytic function in a domain D, then f is open.

Proof. We have to show that $f(D)$ is open. Take any w_0 in $f(D)$ and write $f(z_0) = w_0$, $z_0 \in D$. Assume that $f'(z_0) \neq 0$. Then, $g(z) = f(z) - f(z_0)$ vanishes at z_0 and $g'(z_0) \neq 0$ ensures that z_0 is a simple zero. Thus, for some $\rho > 0$, $g(z) \neq 0$ in $|z - z_0| \le \rho$ (zeros are isolated). On the circle $|z - z_0| = \rho$, $g(z)$ is continuous and hence has a positive minimum, say m,

$$m = \inf\{|f(z) - f(z_0)| : z \in |z - z_0| = \rho\}.$$

Consider a complex constant a such that $|a| < m$. Then, $|a| < |g(z)|$, and the Rouché theorem implies that $g(z)$ and $g(z) - a$ have the same number of zeros in $|z - z_0| < \rho$. But $g(z)$ has only one zero z_0 in this circle.

Hence, for every number a, $|a| < m$, there is only one value of $g(z) = f(z) - w_0 = a$ corresponding to every z in $|z - z_0| < \rho$. Therefore, for every sufficiently small ρ, there exists a $\delta > 0$ such that the f image of $|z - z_0| < \rho$ is contained in $|w - w_0| < \delta$, $(\delta = m)$, and therefore, $f(D)$ is open.

For $f(z_0) = 0$, a slight modification of the above analysis is required. If the first nonvanishing derivative of $f(z)$ at z_0 is of nth order, then $g(z)$ has a zero of order n at z_0. The rest of the argument goes through as above, but in this case one obtains from the

Rouché theorem the additional information that in $|z-z_0| < \rho$ the values of w, for which $|w-w_0| < m$, will now be taken n times.

Corollary 4. If f is analytic and not constant in a domain D, then $f(D)$ is a domain.

Proof. By the open mapping theorem, $f(D)$ is open. Further the continuous image of a connected set is connected (see Problem 12 in Problem Set 2.2). Thus, $f(D)$ is open and connected, and hence a domain.

Proposition 7. If $f(z)$ is analytic and univalent in a domain Ω, then the inverse map $z = f^{-1}(w)$ is analytic in $f(\Omega)$.

Proof. Given that f is analytic and one-one in Ω. From Proposition 6, $f'(z) \neq 0$ in Ω. Fix w_0 in $f(\Omega)$. Then $f(z_0) = w_0$ for a unique z_0 in w. Let f^{-1} be the inverse of f. Then, for any w in $f(\Omega)$,

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}$$

is well defined for all w in a deleted neighbourhood of w_0 (note that $f(\Omega)$ is open). Taking the limit as $w \to w_0$, it follows that

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)}.$$

Therefore, f^{-1} is also analytic, since w_0 was arbitrary.

A useful application of Proposition 7 and result (7.21) is that we can now give a formula for f^{-1}.

Proposition 8. Suppose f is analytic and one-one (univalent) on a domain Ω. For $a \in \Omega$, let $|z-a| \leq R$ be contained in Ω. Then, for any $w \in f(|z-z_0| < R)$,

$$f^{-1}(w) = \frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)-w}dz,$$

where $C = \{z : |z-z_0| = R\}$ is the positively oriented circle.

Proof. Since f is analytic and one-one, for each $w \in f(|z-z_0)| < R)$, there is a unique z_0 such that $f(z_0) = w$, or the function $\phi(z) = f(z) - w$ has a unique zero (z_0 is isolated) in $|z-z_0| < R$. Let $g(z) = z$. Then $\phi'(z) = f'(z)$. Applying formula (7.21) to $g(z)$ and $\phi(z)$, we have

$$\frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)-w} = g(z_0) = z_0 = f^{-1}(w).$$

Example 21. Let f be analytic at ∞. Then prove that $f(z)$ is univalent at $z = \infty$ if and only if $\text{Res}_{z=\infty} f(z) \neq 0$.

Suppose f is analytic and univalent at ∞. So, $f(1/z)$ is analytic and univalent at 0 and admits a power series expansion

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < R$$

But, by Proposition 6, this happens if and only if $a_1 = f'(0) \neq 0$. Now, the result is immediate in view of the fact that $\text{Res}_{z=\infty} f(z) = -a_1$.

Example 22. Let D be a domain. Then $f : D \to \mathbb{C}_\infty$ is meromorphic and univalent in D if and only if f has only a simple pole in D.

This is a direct consequence of Example 21.

PROBLEM SET 7.3

1. Evaluate

$$\int_C \frac{f'(z)}{f(z)} dz, \qquad f(z) = \frac{z^2 - 1}{(z^2 + z)^2},$$

where C is the circle $|z| = 2$ taken in positive sense.

2. Show that the following functions are meromorphic.

(a) $\dfrac{z}{z^4 + 2}$; (b) $\dfrac{z}{\sin^2 z}$; (c) $\dfrac{e^z - 1 - z}{z^4}$; (d) $\dfrac{1}{2\pi i} \displaystyle\int_C \frac{w}{(w^2 - 2)(w - z)} dw$,

where C is the unit circle centred at the origin. First find the function for $|z| < 1$ and then analytically continue the function to $|z| \geq 1$ (see Section 7.9).

3. Prove that one of the roots of the equation $z^4 + z^3 + 1 = 0$ lies in the first quadrant.

4. Prove that the equation $z^3 + iz + 1 = 0$ has one root in each of the first, second and fourth quadrants.

5. Using Rouché's theorem, develop an alternative proof of the fundamental theorem of algebra.

Suggestion. Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$. Take $f(z) = -a_n z^n$ and $g(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$. Then for the points on the circle $|z| = R$, we have

$$\left|\frac{g(z)}{f(z)}\right| < \frac{|a_0| + |a_1| + \ldots + |a_{n-1}|}{|a_n| R^n},$$

and see what happens when R is large enough.

6. If c is any complex number such that $|c| > \alpha > 0$, show that the equation $cz^n = \alpha^z$ has n roots inside $|z| = 1$, where the principal value of α^z is to be considered.

7. Show that the roots of the equation $z^4 + 6z + 1 = 0$ lie within the circle $|z| < 2$ but one root lies inside the circle $|z| < 3/2$.

Suggestion. (Number of roots inside $|z| < 2$ − number of roots inside $|z| < 3/2) = 1$.

8. Let $\{a_n\}$ be a decreasing sequence of positive real numbers. Then prove that the analytic function $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, $|z| < 1$ has no zeros in $|z| < 1$.

Suggestion. Let $g(z) = \Sigma_{n=0}^{\infty} kz^n$, $k = \sup\{a_1, a_2, \ldots\}$. Apply Rouché's theorem.

9. Let $P(z) = 4z^2 - 3iz^2 + iz - 9$. Use Rouché's theorem to decide the number of zeros in (a) $|z| < 1$; (b) $|z| < 2$.

10. Suppose $f(z)$ and $g(z)$ are analytic in a domain Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on a simple closed contour γ. Then define the function

$$\phi(z) = \frac{g(z)}{f(z)} \quad \text{on } \gamma$$

and proceed to give another proof of Rouché's theorem as:

(a) Since $f(z)$ and $g(z)$ are zero free on γ and hence $f(z) + g(z)$ does not vanish on γ. Let N' be zeros of $f(z) + g(z)$ inside γ. Then

$$N' = \frac{1}{2\pi i} \int_\gamma \frac{f'(z) + g'(z)}{f(z) + g(z)}\, dz.$$

(b) Then, show that

$$\int_\gamma \frac{f'(z) + g'(z)}{f(z) + g(z)}\, dz = \int_\gamma \frac{f'(z)}{f(z)}\, dz + \int_\gamma \frac{\phi'(z)}{1 + \phi(z)}\, dz.$$

(c) Finally, using the analyticity of $\phi(z)$ and expanding $[1 + \phi(z)]^{-1}$ in power series, prove that the second integral on the right side of assertion (b) reduces to 0.

11. Use Rouché's theorem to show that all roots of the equation $e^{iz} + z^2 + 2 = 0$ lie in the upper half-plane.

12. Suppose $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ is a nonconstant function such that $\Sigma_{n=2}^{\infty} n|a_n| \le 1$. Show that f is univalent in the disc $|z| < 1$.

Suggestion. By the given hypothesis, $|a_n| < 1$, and hence

$$\sum_{n=2}^{\infty} |a_n z^n| < \sum_{n=2}^{\infty} |z^n| < \infty, \qquad |z| < 1.$$

Thus, $f(z)$ is analytic in $|z| < 1$. Let $|z_0| < 1$. Again, by $\Sigma_{n=2}^{\infty}\, n|a_n| \le 1$,

$$|\,[f(z) - f(z_0)] - (z - z_0)\,| < |z - z_0|.$$

Now, Rouché's theorem ensures the univalence.

13. Prove that for given $\rho < 1$, the polynomial $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ has no zeros inside the circle $|z| = \rho$, if n is sufficiently large.

14. Prove that for given $\rho > 0$, the function

$$f_n(z) = 1 + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}$$

has all its zeros inside the circle $|z| = \rho$, if n is sufficiently large.

Suggestion. For Problem 11, $P_n(z) \to 1/(1-z)^2$ as $n \to \infty$, $|z| < 1$, and for this problem $f_n(z) \to e^{1/z}$ as $n \to \infty$, $|z| < \rho$. Now, use Hurwitz's theorem restated in the remark.

15. Let f be meromorphic in $\mathbb{C}$ and let there exist a positive integer n, a positive number M and $R > 0$ such that

$$|f(z)| \le M|z|^n \quad \text{for} \quad |z| > R.$$

Then show that $f(z)$ is a rational function.

16. Suppose $f(z)$ is a nonconstant function which is analytic at z_0. If $f(z_0) = w_0$ and $f(z) - w_0$ has a zero of order n at z_0, show that for given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all a, satisfying $|a - w_0| < \delta$, the equation $f(z) = a$ has exactly n roots in the disc $|z - z_0| < \varepsilon$.

17. Let $f(z)$ be analytic within and on a simple closed contour γ. If $f(z)$ is univalent on γ, then $f(z)$ is also univalent in interior of γ.

Suggestion. Consider $\Gamma = f(\gamma)$, image of γ. Let $0 \in I(\Gamma)$ be an arbitrary point, where $I(\Gamma)$ denotes interior of γ and 0 is taken for convenience. Then, by the remark after Theorem 5, the number of zeros of $f(z)$ equals 1, the number of circuits around the point 0 by the point $w = f(z)$ as z traverses γ once in the positive direction, since Γ is a simple closed contour, i.e. $f(z)$ takes every value once, we have incidentally proved that $w = f(z)$ traverses Γ in the same direction as z traverses γ. Thus, we have proved that $f(I(\gamma)) \supset I(\Gamma)$. We have also $\Gamma \cap f(I(\gamma)) = \phi$, since $w \in \Gamma$, $w \in f(I(\gamma))$ implies that some neighbourhood of w is contained in $f(I(\gamma))$, a contradiction what we have just proved. Therefore, $f(I(\gamma)) = I(\Gamma)$. Thus, every point of $I(\Gamma)$ has just one inverse image. It follows that mapping from $I(\gamma)$ to $I(\Gamma)$ is one-one, i.e. $f(z)$ is univalent on $I(\gamma)$.

18. Show that the open mapping theorem implies the maximum modulus principle.

Suggestion. By open mapping theorem $f(D)$ is open, and for any z_0 in D such that $f(z_0) = w_0$, we have $|w - w_0| < \delta$ is contained in $f(D)$ for some $\delta > 0$. Now, in the disc there are points of modulus $> |w_0|$, hence $f(z)$ is not the maximum.

7.7 PARTIAL FRACTION EXPANSIONS

A natural way of thinking occurs when we observe that every rational function can be decomposed into partial fractions. This idea is extended to meromorphic functions. The next application of the residue theorem is in this direction.

Theorem 9. Suppose that the only singularities of f are simple poles $a_1, a_2, \ldots$, arranged in order of increasing absolute value with residues as $b_1, b_2, \ldots$, respectively. Let C_N be circles of radii R_N which do not pass through any pole and on which $|f(z)| < M$, M is independent of N and $R_N \to \infty$ as $N \to \infty$. Then

$$f(z) = f(0) + \sum_{m=1}^{\infty} b_m \left(\frac{1}{z - a_m} + \frac{1}{a_m} \right). \tag{7.27}$$

Proof. Let z be a point different from $a_1, a_2, \ldots$ Then the function

$$\frac{f(s)}{s-z}$$

has simple poles $a_1, a_2, \ldots, z$. The residues at these poles are

$$\operatorname{Res}_{s=a_m} \frac{f(s)}{s-z} = \lim_{s\to a_m}(s-a_m)\frac{f(s)}{s-z} = \lim_{s\to a_m}(s-a_m)f(s)\lim_{s\to a_m}\frac{1}{s-z} = \frac{b_m}{a_m-z};$$

$$\operatorname{Res}_{s=z} \frac{f(s)}{s-z} = \lim_{s\to z}(s-z)\frac{f(s)}{s-z} = \lim_{s\to z} f(s) = f(z).$$

Now, by the residue theorem,

$$\frac{1}{2\pi i}\int_{C_N} \frac{f(s)}{s-z}\,ds = f(z) + \sum_m \frac{b_m}{a_m - z}, \tag{7.28}$$

where summation is taken over all the poles (finite in number) interior to C_N of radius R_N; refer to Fig. 7.3.

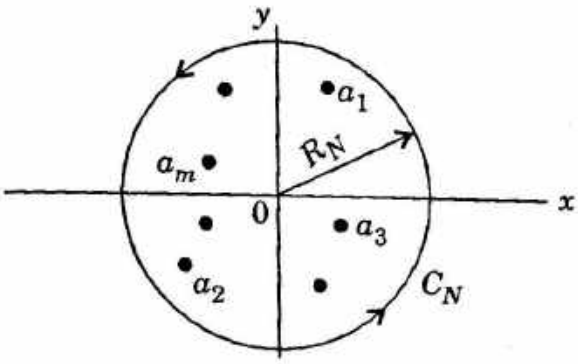

Figure 7.3

Set $z = 0$ in (7.28) and observe that

$$\frac{1}{2\pi i}\int_{C_N} \frac{f(s)}{s}\,ds = f(0) + \sum_m \frac{b_m}{a_m}. \tag{7.29}$$

Subtracting (7.29) from (7.28), we have

$$f(z) - f(0) + \sum_m b_m\left(\frac{1}{a_m - z} - \frac{1}{a_m}\right) = \frac{1}{2\pi i}\int_{C_N} f(z)\left(\frac{1}{s-z} - \frac{1}{s}\right)ds. \tag{7.30}$$

We show that the right-hand side in (7.30) approaches zero as N becomes infinite. For all s on C_N, we have

$$|s-z| \geq |s| - |z| = R_N - |z|.$$

As $|f(s)| < M$ is independent of N, we have

$$\left|\frac{1}{2\pi i}\int_{C_N} f(z)\left(\frac{1}{s-z}-\frac{1}{s}\right)ds\right| = \left|\frac{z}{2\pi i}\int_{C_N}\frac{f(s)}{s(s-z)}\,ds\right|$$

$$\leq \frac{|z|}{2\pi}\frac{M}{R_N(R_N-|z|)}\int_{C_N}|ds|$$

$$= \frac{|z|\,M}{R_N-|z|} \to 0 \quad \text{as } N \to \infty.$$

Proceeding to limits in (7.30), we have

$$f(z)-f(0)+\sum_{m=1}^{\infty} b_m\left(\frac{1}{a_m-z}-\frac{1}{a_m}\right) = 0.$$

This is nothing but (7.27).

Remark. In the above theorem, the circles can be replaced by closed contours C^*_N such that

(i) C^*_N encloses a finite number of poles;
(ii) the interior of C^*_N is contained in the interior of C^*_{N+1};
(iii) the minimum distance R_N of C^*_N from the origin approaches infinity as N approaches infinity; and
(iv) L_N, the length of C^*_N satisfies $L_N = O(R_N)$.

Let us work out some partial fraction expansions of well known functions to illustrate the applications of Theorem 9.

Example 23. Prove that

$$\cot z = \frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}. \tag{7.31}$$

Consider the function

$$f(z) = \cot z - \frac{1}{z} = \frac{z\cos z - \sin z}{z\sin z}.$$

The function $f(z)$ has simple poles $z = n\pi$, $n \in \mathbb{I}$. Moreover, $z = 0$ is a removable singularity, since

$$\lim_{z\to 0}\left(\cot z - \frac{1}{z}\right) = \lim_{z\to 0}\frac{z\cos z - \sin z}{z\sin z} = 0$$

which can be removed by defining the function $f(0) = 0$. Since the numerator is not zero for $z = n\pi$, the residue is given by $p(n\pi)/q'(n\pi) = n\pi\,(\cos n\pi)/n\pi\cos(n\pi) = 1$; see Example 3. Using Theorem 9, we can write

$$\cot z - \frac{1}{z} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{z-n\pi}+\frac{1}{n\pi}\right), \quad n \neq 0.$$

Combining the terms for the indices n and $-n$, we get

$$\cot z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}.$$

The only thing which remains to be established is the boundedness of $f(z)$ on the closed contours C_n containing poles $z = n\pi$. Construct C_n as the square $ABCD$ (depicted

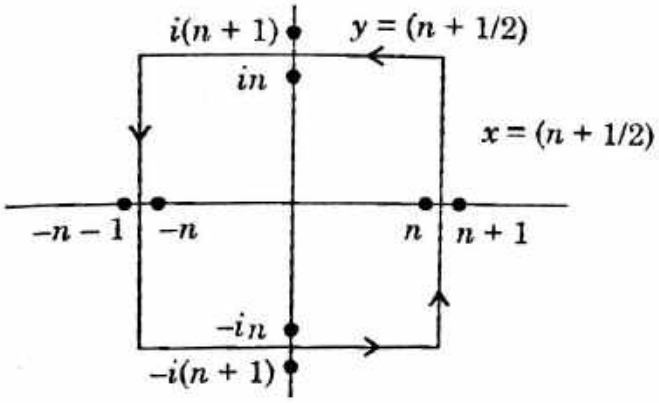

Figure 7.4

in Fig. 7.4) with centre at the origin and corners at the points $(n + 1/2)(\pm 1 \pm i)\pi$.

On the sides parallel to the real axis, $z = x \pm (n + 1/2)\pi i$,

$$\begin{aligned}
|\cot z| &= \left|\frac{\cos[x \pm (n+1/2)\pi i]}{\sin[x \pm (n+1/2)\pi i]}\right| \\
&\le \frac{\cosh(n+1/2)\pi}{\sinh(n+1/2)\pi} \\
&= \frac{1 + e^{-(2n+1)\pi}}{1 - e^{-(2n+1)\pi}} \\
&\le \frac{e^{\pi}+1}{e^{\pi}-1} \\
&< 2,
\end{aligned}$$

while on the sides parallel to the imaginary axis, $z = \pm(n + 1/2)\pi + iy$, and hence

$$|\cot z| = \left|\frac{\cos[\pm(n+1/2)\pi + iy]}{\sin[\pm(n+1/2)\pi + iy]}\right| = \frac{|\sin(iy)|}{|\cos(iy)|} = \frac{e^y - e^{-y}}{e^y + e^{-y}} < 1 \qquad \text{(see Section 3.3.)}$$

Therefore,

$$|\cot z| \le \frac{e^{\pi}+1}{e^{\pi}-1}$$

on every square C_n, i.e. $|\cot z|$ is bounded on C_n. Also, $(1/z) \to 0$ on C_n as $n \to \infty$. Thus, $f(z) = \cot z - 1/z$ is bounded on the square C_n.

Remarks. 1. The relation $\coth z = i \cot iz$ may be utilized to obtain the expansion

$$\coth z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2\pi^2}.$$

2. For our further work in Chapter 8, we show that cosec z is also bounded on each square C_n of Fig. 7.4. Note that $|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y$, (refer to Section 3.3). When z lies on the horizontal side of C_n,

$$|\text{cosec } z|^2 = \frac{1}{\sin^2 x + \sinh^2 (n+1/2)\pi} \le \frac{1}{\sinh^2 (\pi/2)} < 1,$$

whereas on the vertical side of C_n,

$$|\text{cosec } z|^2 = \frac{1}{\sinh^2 (n+1/2)\pi + \sinh^2 y} \le \frac{1}{1 + \sinh^2 \pi} < 1.$$

This proves the fact that $|\text{cosec } z| < 1$ on each C_n.

Example 24. Obtain the partial fraction expansion of sec z.

Note that the function

$$\sec z = \frac{1}{\cos z}$$

has simple poles at $z = (n - 1/2)\pi$, $n \in \mathbb{I}$ and $\sec 0 = 1$. Since the numerator is not zero for $z = (n - 1/2)\pi$, the residue at each of the poles is given by

$$\text{Res}_{z=(n-1/2)\pi} \sec z = -\frac{1}{\sin (n - 1/2)\pi} = (-1)^n.$$

Now, Theorem 9 is applied to write

$$\sec z = 1 + \sum_{n=-\infty}^{\infty} (-1)^n \left[\frac{1}{z - (n - 1/2)\pi} + \frac{1}{(n - 1/2)\pi} \right].$$

Combining the terms for $-n$ and $n + 1$, this assumes the form

$$\sec z = 1 + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2n+1)\pi}{z^2 - (n - 1/2)^2 \pi^2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

Since the last sum is known to be

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \cdots = \frac{\pi}{4},$$

it follows that

$$\sec z = \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)\pi}{z^2 - (n - 1/2)^2 \pi^2}. \tag{7.32}$$

For validity of (7.32), we establish that $\sec z$ is bounded on the closed contours C_n enclosing the poles $z = (n - 1/2)\pi$. Let us take C_n to be a square centred at the origin with vertices $n\pi(\pm 1 \pm i)$ (see Fig. 4.2). We observe that on vertical lines $z = \pm n\pi + iy$,

$$|\sec z| = \frac{1}{|\cos(\pm n\pi + iy)|} = \frac{1}{|\cos iy|} = \frac{1}{\cosh y} \leq 1,$$

while on horizontal lines $z = x \pm n\pi i$, and hence

$$|\sec z| = \frac{1}{|\cos(x \pm n\pi i)|} \leq \left|\frac{1}{\sinh n\pi}\right| \leq \frac{1}{\sinh \pi} < 1 .$$

This implies that for all z on C_n, $|\sec z| \leq 1$.

Remark. In response to the relation $\sec iz = \operatorname{sech} z$, (7.32) gives

$$\operatorname{sech} z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2n-1)\pi}{z^2 + (n - 1/2)^2\pi^2} .$$

Theorem 9 is used to find the partial fraction of a given meromorphic function with simple poles. We go for a more general result whether for the given sequence of poles with arbitrary order there exists a corresponding meromorphic function. This is answered by an interesting development known as *Mittag–Leffler* theorem.

Theorem 10 (Mittag–Leffler). Let $f(z)$ be a meromorphic function in the complex plane with poles $\{a_n\}$ and the corresponding principal parts $\{p_n(z)\}$. Then, there exist polynomials $\{g_n(z)\}$ such that

$$f(z) = p_0(z) + \sum_{n=1}^{\infty} [p_n(z) - g_n(z)] + h(z)$$

holds, and the series

$$\sum_{n=1}^{\infty} [p_n(z) - g_n(z)] .$$

converges uniformly on every bounded set not containing the poles. Here $h(z)$ is an entire function.

Proof. We only give the essential idea behind the proof. Each of the principal parts $\{p_n(z)\}$ can be expanded in a convergent Taylor series about $z = 0$ for $|z| < |a_n|$. It can be shown that enough terms can be taken in this Taylor series so that the polynomials $g_n(z)$ obtained by truncation of the Taylor series of $p_n(z)$ at order z^m

$$g_n(z) = \sum_{k=0}^{m} B_{k,n} z^k$$

ensure that the difference $|p_n(z) - g_n(z)|$ is suitably small. Again, it can be proved that the polynomials $g_n(z)$ of order m gurantee that the series

$$\sum_{n=1}^{\infty} |p_n(z) - g_n(z)|$$

converges uniformly.

Our final example gives an application of partial fraction expansions to obtain the Taylor series expansions. However, these series have already been derived by the division process (see Problem 16, Problem Set 6.5).

Example 25. Comparing the Taylor series expansions through different ways, show that

$$2\left(\frac{2}{\pi}\right)^{2m+1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2m+1}} = (-1)^m \frac{E_{2m}}{(2m)!}, \qquad m = 0, 1, \ldots,$$

where E_{2m} are Euler's numbers. Also, verify the following sums:

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4};$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32};$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \cdots = \frac{5\pi^5}{1536}$$

an so on.

From (7.32), we have

$$\sec z = \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)\pi}{z^2 - (n-1/2)^2\pi^2} = \sum_{n=1}^{\infty} f_n(z),$$

where

$$f_n(z) = (-1)^n \left[\frac{1}{z - (n-1/2)\pi} - \frac{1}{z + (n-1/2)\pi}\right].$$

This series is uniformly convergent on every compact set containing no poles of $\sec z$, in particular on every compact subset of $|z| < \pi/2$ and each of the terms is analytic in $|z| < \pi/2$. Thus, the Taylor series of $\sec z$ may be obtained by adding all the coefficients of the same power of z in the separate Taylor series for each $f_n(z)$. It is easy to have

$$f_n(z) = -(-1)^n \left[\sum_{k=0}^{\infty} \frac{z^k}{[(n-1/2)\pi]^{k+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{[(n-1/2)\pi]^{k+1}}\right]$$

$$= 2(-1)^{n-1} \sum_{m=0}^{\infty} \frac{z^{2m}}{[(n-1/2)\pi]^{2m+1}}, \quad |z| < \left(n - \frac{1}{2}\right)\pi. \tag{7.33}$$

This ensures that expansions (7.33) are valid on $|z| < \pi/2$ for all f_n, $n = 1, 2, \ldots$. Moreover, the expansion (7.33) has only the even powers of z, of course an anticipated result as $\sec z$ is an even function. The coefficient of z^{2m} is

$$2\left(\frac{2}{\pi}\right)^{2m+1} \frac{(-1)^{n-1}}{(2n-1)^{2m+1}}, \quad m = 0, 1, 2, \ldots.$$

It follows that

$$\sec z = \sum_{n=1}^{\infty} f_n(z) = \sum_{m=0}^{\infty} \left[2\left(\frac{2}{\pi}\right)^{2m+1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2m+1}} \right] z^{2m}, \quad |z| < \frac{\pi}{2}. \tag{7.34}$$

Recalling a different Taylor series expansion from Problem 17, Problem Set 6.5

$$\sec z = \sum_{n=1}^{\infty} (-1)^m \frac{E_{2m}}{(2m)!} z^{2m}, \quad |z| < \frac{\pi}{2}. \tag{7.35}$$

which involves Euler numbers $E_{2m}(E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61, \ldots)$. By comparison of expansions (7.34) and (7.35) for $\sec z$, we find that

$$2\left(\frac{2}{\pi}\right)^{2m+1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2m+1}} = (-1)^m \frac{E_{2m}}{(2m)!}, \quad m = 0, 1, \ldots.$$

For $m = 0, 1, 2 \ldots$, in the above expression we get the desired sums.

PROBLEM SET 7.4

1. Verify the expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 + 4n^2\pi^2}.$$

Suggestion. Use alternative expression for $z/(e^z - 1)$, see Problem 18, Problem Set 6.5.

2. Obtain the expansion

$$\text{csz } z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n 2z}{z^2 - n^2\pi^2}.$$

Suggestion. Take $f(z) = \csc z - (1/z)$.

3. Derive the partial fraction expansion

$$\tan z = \sum_{n=1}^{\infty} \frac{2z}{(n-1/2)^2\pi^2 - z^2},$$

and hence deduce

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Suggestion. For sum, use $\lim_{z\to 0} (\tan z)/z = 1$ in the expansion.

4. Using the expansion of $\cot z$, derive the partial fraction expansion of $\pi \cot \pi z$.

5. Use expansions of Problems 2 and 3 to derive the expansions of (a) $\operatorname{cosech} z$ and (b) $\tanh z$.

6. Prove the following version of Mittag–Leffler expansion theorem for simple poles: Let $\{a_n\}$ and $\{b_n\}$ be sequences with a_n distinct, $|a_n| \to \infty$, and m an integer such that

$$\sum_{n=1}^{\infty} \frac{|b_n|}{|a_n|^{m+1}} < \infty.$$

Then the expression

$$p_0(z) + \sum_{n=1}^{\infty} \left(\frac{b_n}{a_n}\right) L\left(\frac{z}{a_n}, m\right) + h(z)$$

represents a meromorphic function whose only singularities are simple poles a_n with residue b_n. Here $p_0(z)$ is the principal part at the origin, $L(w, m) = -w^m/(1-w)$ and $h(z)$ is an entire function.

7. Apply the Mittag–Leffler expansion theorem to expand $\pi^2/\sin^2(\pi z)$ as follows:

(a) Establish that near every integer $z = n$ the function $f(z)$ has the singular part $p_n(z) = 1/(z-n)$;

(b) Show that the series

$$S(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

converges for all $z \neq n$;

(c) Because the series in part (b) converges, explain why the representation

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + h(z)$$

is valid for some entire function $h(z)$;

(d) Show that $h(z)$ is periodic with period 1 by showing that $(\pi/\sin \pi z)^2$ and $S(z)$ are periodic with period 1. Explain why $(\pi/\sin \pi z)^2 - S(z)$ is a bounded function, and show that each term vanishes as $|y| \to \infty$. Thus, conclude that $h(z) = 0$;

(e) What happens when the expression in part (c) is integrated term by term.
Suggestion. This problem pertains to poles of order two.

8. Use the partial expansion to derive the Taylor series of $\cot z - 1/z$ and employ the Taylor series available from Problem 19(a), Problem Set 6.5 to verify the following sums

(a) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$;

(b) $1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$;

(c) $1 + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945}$.

7.8 REGULAR POINTS AND SINGULARITIES

The singular points or singularities of an analytic function have been introduced in Section 2.6, and in greater detail via the Laurent series expansion in Section 7.1. However, we have not discussed anything about the behaviour of boundary points on the circle of convergence. This is done here.

Definition 3. Let $D : |z| < R$ and $f(z)$ be analytic in D. Suppose β is a boundary point of D, i.e. $\beta \in |z| = R$. We say β is a *regular* point of $f(z)$ if there exists a neighbourhood $N(\beta)$ and an analytic function $g(z)$ defined in $N(\beta)$ such that

$$f(z) = g(z) \quad z \in N(\beta) \cap D.$$

Any boundary point of D which is not a regular point is called the *singular* point or *singularity* of the function $f(z)$.

It must be noticed that any point β_0 in open disc D has the characteristic property of a regular point, i.e. there exists a neighbourhood $N(\beta_0)$ (in fact we can find a neighbourhood of β_0 which is contained in D) and an analytic function on $N(\beta_0)$ (in fact $f(z)$ itself) which coincides with $f(z)$ on $N(\beta_0) \cap D$. Thus, the regular points have a broader spectrum than the points of analyticity of a function. This suggests the extension of our definition of regular points; we shall henceforth regard the points of D itself as regular points.

For example, consider $D : |z| < 1$, and let $f(z) = 1 + z + z^2 + z^3 + \cdots$. For every point $\beta \neq 1$ on the boundary we can find a neighbourhood $N(\beta) : |z - \beta| < |1 - \beta|$, and an analytic function $g(z) = 1/(1-z)$ defined on $N(\beta)$ such that $f(z) = g(z)$ on $N(\beta) \cap D$. Hence every point $\beta \neq 1$ on the boundary $|z| = 1$ is a regular point and $\beta = 1$ is a singular point.

Remark. The definition makes it obvious that the set of all regular points (considered on the boundary) is an open subset (may be empty) of the boundary of D. Hence, the singular points on the boundary form a closed set.

Proposition 9. Suppose $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$. Then $f(z)$ has at least one singular point on the boundary $|z| = R$.

Proof. Suppose, on the contrary, $f(z)$ has no singularity on the circle of convergence $|z| = R$, i.e. every boundary point is a regular point. As $|z| = R$ is compact, there are a finite number of open discs $D_1, D_2, \ldots, D_n$ with centres $\beta_1, \beta_2, \ldots, \beta_n$ (Fig. 7.5) and analytic functions $g_j(z)$ such that

$$g_j(z) = f(z) \quad \text{on} \quad D_j \cap D, \quad j = 1, 2, \ldots, n.$$

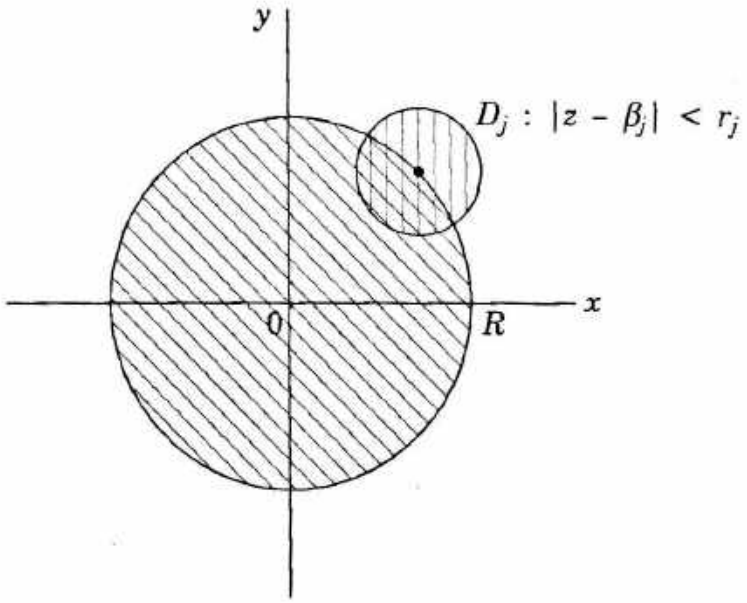

Figure 7.5

If $D_i \cap D_j \neq \phi$, then $V_{ij} = D_i \cap D_j \cap D \neq \phi$, and we have $g_i = f = g_j$ on V_{ij}. Thus g_i and g_j are analytic in $D_i \cap D_j$ and coincide on the subdomain $D_i \cap D_j \cap D$, in view of the uniqueness theorem $g_i = g_j$ on $D_i \cap D_j$. Hence we may define a function h in $\Omega = D_1 \cup D_2 \cup \ldots \cup D_n \cup D$ by

$$h(z) = \begin{cases} f(z), & z \in |z| < R, \\ g_i(z), & z \in D_i. \end{cases}$$

Now, $|z| \leq R$ is contained in Ω and Ω is open, there exists an $\varepsilon > 0$ such that $|z| < R + \varepsilon \subset \Omega$. But $h(z)$ is analytic in Ω, and hence has a power series representation in Ω with radius of convergence $R + \varepsilon$. Again, by uniqueness theorem, $f(z)$ has the power series representation with radius of convergence $R + \varepsilon$, contrary to our assumption.

The next result gives a method for deciding whether or not a given point on the circle of convergence of a power series is a regular point or a singular point.

Theorem 11. Given a power series $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$, let β be a point on the circle $\Gamma : |z| = R$ and let z_1 be a point on the radius $\overline{0\beta}$. Then β is a singular point of $f(z)$ if

$$\Delta = R - |z_1| = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f^{(n)}(z_1)|/n!}}, \tag{7.36}$$

and a regular point of $f(z)$ if

$$\Delta = R - |z_1| < \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f^{(n)}(z_1)|/n!}}, \tag{7.37}$$

Proof. The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R can be differentiated any number of times to have

$$f^{(k)}(z) = \sum_{n=k}^{\infty} a_n n(n-1)(n-2) \cdots (n-k+1)\, z^{n-k}. \tag{7.38}$$

Since $f(z)$ is analytic in $|z| < R$, we can expand $f(z)$ in Taylor series about the point z_1 as

$$\tilde{f}(z) = b_0 + b_1(z-z_1) + \cdots + b_n(z-z_1)^n + \cdots, \quad |z-z_1| < R - |z_1|. \tag{7.39}$$

By the uniqueness theorem and using (7.38),

$$b_k = \frac{f^{(k)}(z_1)}{k!} = \sum_{n=k}^{\infty} \binom{n}{k} a_n z^{n-k}.$$

According to formula (6.3), the radius of convergence of the series (7.39) is

$$r = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|b_n|}}.$$

On the other hand, (7.39) is the Taylor series expansion of $\tilde{f}(z)$ in powers of $z - z_1$ and hence certainly converges on the disc $|z-z_1| < \Delta = R - |z_1|$. It follows that $r \geq \Delta = R - |z_1|$.

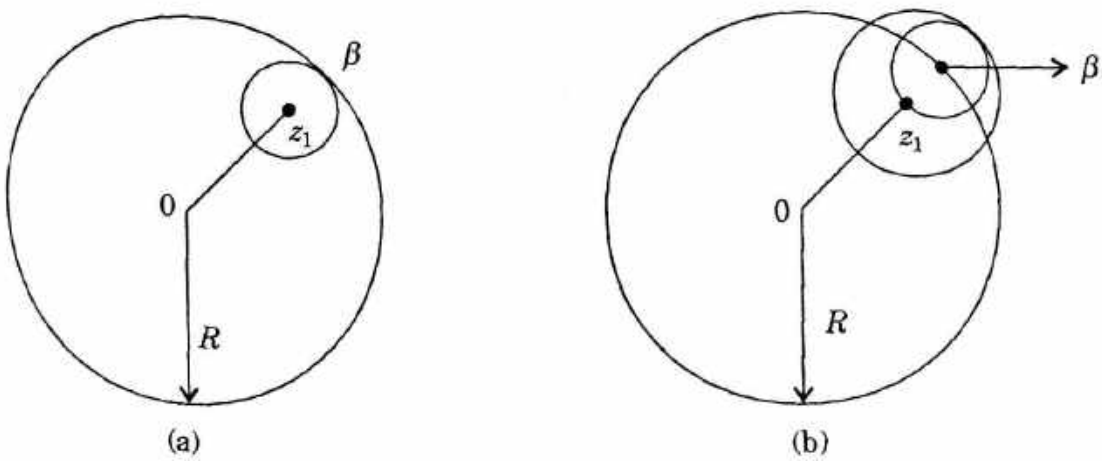

Figure 7.6

First suppose that $r = \Delta$, as shown in Fig. 7.6(a). Then according to Proposition 1, the circle $C : |z-z_1| = r$ contains at least one singular point of $\tilde{f}(z)$, say β. But no point $\beta' \in C \cap \{z : |z| < R\}$ can be a singular point of $\tilde{f}(z)$, since every such β' has a neighbourhood $N(\beta')$ contained in $|z| < R$ such that $\tilde{f}(z)$ and $f(z)$ coincide on $N(\beta') \cap \{z : |z-z_1| < r\}$. Thus, β is a unique singular point of $\tilde{f}(z)$. Moreover, β is also a singular point of $f(z)$, since otherwise there would be function $g(z)$ analytic in some neighbourhood of $N(\beta)$ and coinciding with $f(z)$ on $N(\beta) \cap \{z : |z| < R$. But then $g(z)$ would coincide with $\tilde{f}(z)$ on $N(\beta) \cap \{z : |z-z_1| < \Delta\}$, since $\tilde{f}(z) = f(z)$ on this neighbourhood, contradicting the fact that β is a singular point of $\tilde{f}(z)$. This proves that β is a singular point of $f(z)$ if (7.36) is satisfied.

For the second case $r > \Delta$, the circle $C : |z-z_1| = r$ intersects the circle $|z| = R$, as shown in Fig. 7.6(b). Here there exists a neighbourhood $N(\beta) \subset \{z : |z| < r\}$ in which $\tilde{f}(z)$ reprsents an analytic function coinciding with $f(z)$ on $N(\beta) \cap \{ z : |z| < R\}$. In other words, β is a regular point of $f(z)$ if (7.37) holds, and thus the theorem is proved.

A very good application of Theorem 1 is the following result.

Theorem 12 (Pringsheim). Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, $a_n \geq 0$ be a power series with radius of convergence 1. Then $z = 1$ is a singular point of $f(z)$.

Proof. Suppose $z = 1$ is a regular point of $f(z)$. Assigning $R = 1$ and $z_1 = x(0 < x < 1)$ in (7.46), we have

$$\Delta = 1-x < \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f^{(n)}(x)|/n!}} \tag{7.40}$$

If β is any point on the unit circle $|z| = 1$, let z_1 be the point in which the radius $\overline{0\beta}$ intersects the circle $|z| = x$, so that $|z_1| = x$. Then the distance from z_1 to β is again $\Delta = 1-x$. On the other hand (with the fact $a_n \geq 0$),

$$\begin{aligned}\frac{|f^{(n)}(z_1)|}{n!} &= \left| a_n + \frac{n+1}{1!} a_{n+1} z_1 + \frac{(n+1)(n+2)}{2!} a_{n+2} z_1^2 + \cdots \right| \\ &\leq a_n + \frac{n+1}{1!} a_{n+1} x + \frac{(n+1)(n+2)}{2!} a_{n+2} x^2 + \cdots \\ &= \frac{|f^{(n)}(x)|}{n!},\end{aligned}$$

and hence,

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f^{(n)}(z_1)|/n!}} \geq \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f^{(n)}(x)|/n!}}. \tag{7.41}$$

Combining (7.41) and (7.40), it follows that

$$\Delta = 1 - |z_1| < \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f^{(n)}(z_1)|/n!}},$$

i.e. β is a regular point of $f(z)$. Since β is arbitrary, every point of $|z| = 1$ is a regular point of $f(z)$, contrary to the assumption that $|z| = R$ is the circle of convergence.

Therefore, $z = 1$ must be singular point of $f(z)$, and the theorem is proved.

Remark. Instead of the point 1, consider any point ζ of the unit circle. Then exactly the same arguments lead to the conclusion that ζ is a singular point of the series $f(z) = \sum_{n=0}^{\infty} a_n \zeta^n z^n$ if the numbers $a_n \zeta^n$ are nonnegative real numbers. In fact, it is enough to require only that these numbers be nonnegative real numbers for all n after a certain integer n_0, since writing $f(z)$ in the form

$$f(z) = \sum_{n=0}^{n_0} a_n z^n + \sum_{n=n_0+1}^{\infty} a_n z^n,$$

we note at once that ζ is a singular point of $f(z)$ if and only if ζ is a singular point of the second series on the right.

7.9 ANALYTIC CONTINUATION

Analytic continuation is a very useful concept because it is utilized in making the domain of definition of an analytic function as large as possible. Usually, the domain of an analytic function depends on how it has been defined. For example, every power series gives rise to an analytic function interior to its circle of convergence, a fact established in Section 6.2. The question arises whether an analytic function exists which coincides with power series representation inside the circle of convergence and is defined on a larger domain. The answer to this query is in the affirmative.

Let us examine the analytic continuation of the function

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

This is a power series (geometric series) which converges in $|z| < 1$. For $z = 1$ and $|z| > 1$, the series diverges (see Section 6.2), because outside the circle, the nth term does not tend to 0 as $n \to \infty$ (a necessary condition for the convergence of a series). Further, the above series sums up to the function $f(z) = 1/(1-z)$ valid for $|z| < 1$.

However, the function

$$f_1(z) = \frac{1}{1-z}, \qquad z \neq 1$$

is well defined for all the values except $z = 1$ and is analytic in $\mathbb{C}\backslash\{1\}$. Now, we observe that

$$f(z) = f_1(z) \quad \text{for all} \quad z \in \{z : |z| < 1\} \cap \mathbb{C}\backslash\{1\}.$$

Thus, we say that f_1 is an analytic continuation of f from the domain $|z| < 1$ into the domain $\mathbb{C}\backslash\{1\}$, or the function f has been continued analytically through some function f_1 beyond $|z| < 1$ such that both the functions coincide on $|z| < 1$.

Next, consider the function

$$f_2(z) = \int_0^{\infty} e^{(z-1)t} dt. \tag{7.42}$$

If Re $z < 1$, then

$$f_2(z) = \left.\frac{e^{(z-1)t}}{z-1}\right|_0^\infty = \frac{1}{1-z}.$$

This tells us that the integral (7.42) is convergent in the half-plane Re $z < 1$ and represents the function $1/(1-z)$. In this situation

$$f(z) = f_2(z) \quad \text{for all} \quad z \in |z| < 1 \cap \text{Re } z < 1$$

and call f_2 another analytic continuation of f from the domain $|z| < 1$ into the domain Re $z < 1$. This discussion summarizes how to define the concept.

Definition 4. Let f and f_1 be two functions satisfying that

(i) f is defined and analytic in a domain D;

(ii) f_1 is defined and analytic in a domain D_1 such that $D \cap D_1 \neq \phi$;

(iii) $f(z) = f_1(z)$ for all $z \in D \cap D_1$.

Then f_1 is said to be the *analytic continuation* of f from domain D into domain D_1.

If f_1 is the analytic continuation of f from a domain D into a domain D_1, then the function defined by

$$F(z) = \begin{cases} f(z), & z \in D, \\ f_1(z), & z \in D_1 \end{cases}$$

is analytic in $D \cup D_1$, which is the domain consisting of points of D or D_1. The function F is the analytic continuation into $D \cup D_1$ of either f or f_1; and f and f_1 are called the *function elements* of F, denoted by (f, D) and (f_1, D_1).

Remarks. 1. Suppose f_1 is the analytic continuation of function f from a domain D into D_1. Further, if there is an analytic continuation f_2 of f_1 from a domain D_1 into D_2 (Fig. 7.7), it is not necessarily true that $f_2(z) = f(z)$ for each $z \in D \cap D_2$. This means that such a chain of analytic continuations of a given function $f(z)$ from a domain D may lead to a different function on D_1.

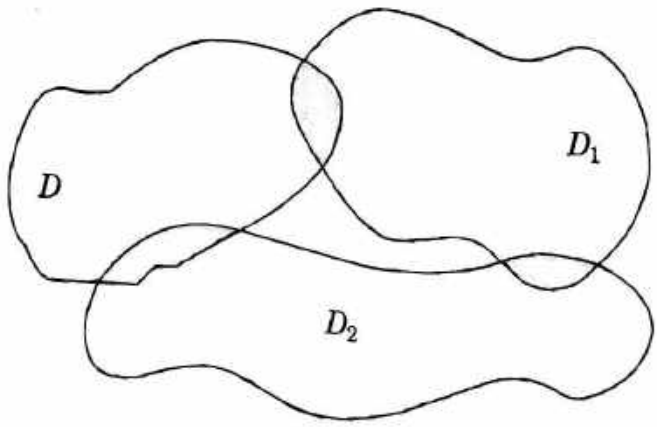

Figure 7.7

2. If D_1 contains D in (ii), then we say that f_1 is the direct analytic continuation of f from D into D_1.

Power series method of analytic continuation. Suppose that a function $f_1(z)$ represented by the Taylor series about a point p_1 in the form

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z - p_1)^n . \tag{7.43}$$

converges in $D_1 : |z-p_1| < R_1$. Take a contour L emerging from the point p_1 and perform analytic continuation as follows:

Let p_2 be any interior point of D_1 such that the arc p_1p_2 lies in the interior to D_1. We then use the series in (7.43) to evaluate the derivatives $f_1^{(n)}(p_2)$ and write the expansion about p_2

$$f_2(z) = \sum_{n=0}^{\infty} b_n (z - p_2)^n = \sum_{n=0}^{\infty} \frac{f_1^{(n)}(p_2)}{n!} (z - p_2)^n . \tag{7.44}$$

Suppose the series in (7.44) converges in $D_2 : |z-p_2| < R_2$. If the circle $|z-p_2| = R_2$ extends beyond D_1, then (7.44) gives an analytic continuation of $f_1(z)$. At the point p_2, the functions $f_1(z)$ and $f_2(z)$ and their derivatives have the same values and these functions coincide on the region $D_1 \cap D_2$ (see Fig. 7.8)

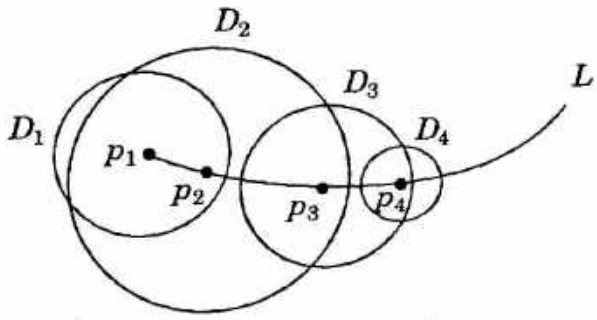

Figure 7.8

The another way of obtaining (7.44) is to write (7.43) in the form

$$f_1(z) = \sum_{n=0}^{\infty} a_n [(z - p_2) + (p_2 - p_1)]^n$$

Now, the expression $[(z-p_2) + (p_2-p_1)]^n$ can be expanded by bionomial theorem and this procedure also yields (7.44).

After the first analytic continuation (7.44), we may proceed to the next analytic continuation by selecting a point p_3 in D_2 and proceeding as above to get a new function element

$$f_3(z) = \sum_{n=0}^{\infty} c_n (z - p_3)^n$$

which is defined in D_3: $|z-p_3| < R_3$ and so on. Here,

$$C_n = \frac{f^{(n)}(p_2)}{n!}.$$

Remarks. 1. While selecting p_2, care should be exercised so that it does not lie on the line segment joining p_1 to any singularity of f_1 on the boundary.

2. We have used the power series method under the assumption that the circle of convergence is not a natural boundary (see Definition 5).

Example 26. Show that the series

$$\text{(a)} \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad \text{and} \quad \text{(b)} \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$$

are analytic continuation of each other

By ratio test, the series (a) converges for $D_1 = \{z : |z| < 2\}$ (Fig. 7.9). This is a geometric series with the sum function

$$f_1(z) = \frac{1/2}{1-z/2} = \frac{1}{2-z}.$$

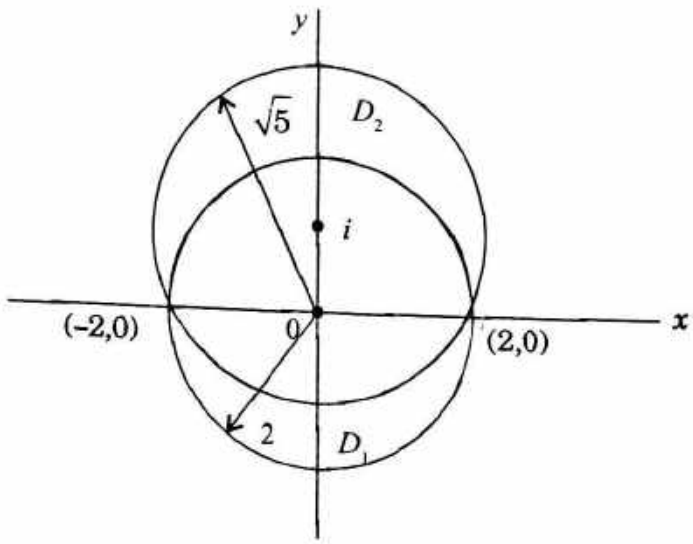

Figure 7.9

Again, by ratio test, the series (b) converges for

$$\left|\frac{z-i}{2-i}\right| < 1,$$

i.e. $D_2 = \{z : |z-i| < \sqrt{5}\}$. Being geometric series it may also be summed up as

$$f_2(z) = \frac{1/(2-i)}{1-(z-2)/(2-i)} = \frac{1}{2-z}.$$

Since both the power series represent the same function in the region common to the interiors of both the circles $|z| < 2$ and $|z-i| < \sqrt{5}$, i.e.

$$f_1(z) = f_2(z) \quad \text{for all } z \in D_1 \cap D_2.$$

This implies that f_1 and f_2 are analytic continuation of each other.

Example 27. Consider the branch of $z^{1/2}$

$$f(z) = \sqrt{r}\, e^{i\theta/2} \quad (r > 0,\ 0 < \theta < \pi)$$

and find the different analytic continuations of this branch.

An analytic continuation of f across the negative real axis into the lower half-plane is

$$f_1(z) = \sqrt{r}\, e^{i\theta/2} \quad \left(r > 0,\ \frac{\pi}{2} < \theta < 2\pi\right).$$

Further, an analytic continuation of f_1 across the positive real axis into the first quadrant is

$$f_2(z) = \sqrt{r}\, e^{i\theta/2} \quad \left(r > 0,\ \pi < \theta < \frac{5\pi}{2}\right).$$

Note that $f_2(z) \neq f(z)$ in the first quadrant; in fact $f_2(z) = -f(z)$.

Definition 5. Let $f(z)$ be analytic in a domain $|z| < R$. If every point of $|z| = R$ is a singular point of f, then $|z| = R$ is said to be the *natural boundary*. In this case $f(z)$ has no analytic extension to any domain which properly contains $|z| < R$.

Example 28. Show that the function represented by the power series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \tag{7.45}$$

cannot be continued analytically.

By root test, the series (7.45) converges in the domain $|z| < 1$ and its radius of convergence is 1. Consequently, $f(z)$ defined by (7.45) is analytic in $|z| < 1$. If $|z| \geq 1$, the series diverges, since

$$\lim_{n\to\infty} |z^{2^n}| \neq 0.$$

Let $z = e^{2\pi ik/2^n}$, $k = 0, 1, \ldots, 2^n - 1$ $(n \in \mathbb{N})$ be the 2^nth roots of unity. It will now be shown that points of the form $z = e^{2\pi ik/2^n}$, where k is any nonegative integer, are singularities of the function. For this we consider the behaviour of $f(z)$ as z moves up to boundary of the circle $|z| = 1$ along the radius through the point $re^{2\pi ik/2^n}$. Write the given series in the form

$$f(z) = 1 + z^2 + \cdots + z^{2^k} + \sum_{n=k+1}^{\infty} z^{2^n} = f_1(z) + f_2(z).$$

Note that $f_1(re^{2\pi ik/2^n})$ is a polynomial in r and tends to a unique limit as $r \to 1_-$. However, for $n > k$,

$$f_2(z) = f_2(re^{2\pi ik/2^n}) = \sum_{n=k+1}^{\infty} r^{2^n} e^{2\pi ik} = \sum_{n=k+1}^{\infty} r^{2^n}$$

The series $\Sigma_{n=k+1}^{\infty} r^{2^n}$ diverges to ∞ as $r \to 1_-$.

Thus, the points $e^{2\pi ik/2^n}$ are singularities of the function, and for large values of n, any neighbourhood of each point on the circle $|z| = 1$ has a singularity. In other words, every arc of $|z| = 1$, however small, will include points of the form $z = e^{2\pi ik/2^n}$. This implies that singularities of $f(z)$ are dense in $|z| = 1$. Evidently, every circle that crosses the circle $|z| = 1$ will also contain points of this form. So $f(z)$ cannot be continued analytically outside $|z| < 1$.

Thus, we have concluded that $|z| = 1$ constitutes the natural boundary of the given power series.

Pringsheim theorem is capable of giving a shorter way of finding the natural boundary as illustrated in our next example.

Example 29. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{2^{n^2}}. \tag{7.46}$$

(a) Show that this function cannot be continued analytically beyond its boundary;
(b) Discuss the continuity and differentiability of the function on the boundary.

For part (a), note that according to the formula (6.3), this series has radius of convergence 1, since

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[2^n]{\frac{1}{2^{n^2}}} = 1.$$

As $a_n \geq 0$, it follows from Theorem 12 that $z = 1$ is a singular point of (7.46). Moreover, in view of the remark following Theorem 12, every point $\zeta = e^{2\pi ik/2^n}$, $k = 0, 1, \ldots, 2^n - 1$ on the unit circle is a singular point of (7.46), since

$$\frac{\zeta^{2^n}}{2^{n^2}} = \frac{e^{2\pi ik}}{2^{n^2}} = \frac{1}{2^{n^2}} > 0.$$

Thus, the set of all singular points is dense in the unit circle $|z| = 1$, and hence constitutes a natural boundary.

(b) Thus in part (a), we have proved that every point of the boundary $|z| = 1$ is a singular point. The interesting result is the fact that these singular points do not prevent the series (7.46) from converging absolutely and uniformly on the closed disc $|z| \leq 1$, and moreover, its sum function is infinitely differentiable on $|z| \leq 1$. Note that

$$\left|\frac{z^{2^n}}{2^{n^2}}\right| \leq \frac{1}{2^{n^2}},$$

if $|z| \le 1$. Therfore, by Weierstrass M-test, series (7.46) converges absolutely and uniformly on $|z| \le 1$. Thus, $f(z)$ is continuous on $|z| \le 1$. Moreover, differentiating series (7.46) term by term k times, we get the series

$$\sum_{n=0}^{\infty} \frac{2^n(2^n-1)\cdots(2^n-k+1)}{2^{n^2}} z^{2n-k},$$

whose terms satisfy

$$\left|\frac{2^n(2^n-1)\cdots(2^n-k+1)}{2^{n^2}}\right| \le \frac{2^{nk}}{2^{n^2}} = \frac{1}{2^{n(n-k)}} \le \frac{1}{2^n},$$

for $n \ge k$ and $|z| \le 1$. This again implies that the derived series is uniformly convergent on $|z| \le 1$, and hence represents the derivative $f^{(k)}(z)$ on $|z| \le 1$. Thus, $f(z)$ is continuous and infinitely differentiable on $|z| \le 1$, which is analytic in the open unit disc $|z| < 1$. Thus, we derive a conclusion that the presence of singularities has no effect on the boundary at least with reference to continuity and differentiability.

Remarks. In fact, to show that every point $\zeta \in |z| = 1$ is singular, we need only to examine the derivatives of $f(z)$ at interior points $z_1 : |z_1| < 1$, by comparing the quantity

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f^{(n)}(z_1)|/n!}}$$

with the distance from z_1 to ζ.

PROBLEM SET 7.5

1. Let $f(z)$ be analytic in a domain Ω and suppose that $f(z) = 0$ at all points on a path joining any two points of Ω. Then show that $f(z) \equiv 0$ in Ω.
2. Let $f_1(z)$ and $f_2(z)$ be analytic functions in a domain Ω, and further suppose that $f_1(z) = f_2(z)$ at all point on a path joining any two points of Ω. Then show that $f_1(z) = f_2(z)$ in Ω.

   Suggestion. Take $F(z) = f_1(z) - f_2(z)$ in Problem 1.
3. Let $f_1(z) = \Sigma_{n=0}^{\infty} z^{n+1}/2^n$. Find an analytic continuation of $f_1(z)$ which converges for $z = 2 + i$ and determine the value of the analytic continuation at this point.
4. Given that an identity $\sin^2 z + \cos^2 z = 1$ holds for all real values of z, show that it holds for all complex values z.
5. Show that the function $g(z) = \Sigma_{k\ge 0} z^{k!}$ cannot be continued analytically across the boundary $|z| = 1$.

   Suggestion. Take the points $e^{2\pi i n/k!}$, $n = 0, 1, \ldots, k! - 1$ on the boundary $|z| = 1$, and follow Example 28.

6. Show that the circle of convergence of the power series

$$f(z) = 1 + z + z^2 + z^4 + z^8 + \cdots$$

is a natural boundary.

7. Show that the function

$$f_1(z) = \frac{1}{z^2 - 1}, \quad (z \neq \pm i)$$

is the analytical continuation of the function

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

into the domain consisting of all the points in the complex plane except $z = \pm i$.

8. Find the analytic continuation of the function

$$f(z) = \int_0^{\infty} te^{-zt} dt \ (\text{Re } z > 0)$$

into the domain consisting of all the points in the complex plane except the origin.

9. Prove that the functions defined by the series

$$f(z) = 1 + az + a^2z^2 + \cdots,$$

$$g(z) = \frac{1}{1-z} - \frac{(1-a)z}{(1-z)^2} + \frac{(1-a)^2 z^2}{(1-z)^3} - \cdots$$

are analytic continuation of each other. Discuss all the possible cases for different values of a.

10. For any $a \in \mathbb{R}$, show that the function $a/(z^2 + a^2)$ is the analytic continuation of the function

$$f(z) = \int_0^{\infty} e^{-zt} \sin at \, dt \quad (\text{Re } z > a)$$

into the domain consisting of all the points in the z-plane except $z = \pm ai$.

11. If an analytic function is continued along two different paths from p to q and we obtain two different values of $f(q)$, then $f(z)$ must have a singularity somewhere between the two paths.

Suggestion. Prove the contrapositive statement: "If we continue an analytic function along two different paths from p to q and obtain the same value $f(q)$, then $f(z)$ has no singularity in the region enclosed by the two paths".

8

Sums and Definite Integrals

The main theme of this chapter is to acquaint the reader with some applications of residue calculus for finding the sum of infinite series and to evaluate certain definite real integrals which are difficult to handle with other methods.

8.1 ESTIMATION OF SUMS

In Chapter 5, we discussed convergence of infinite series. Convergence tests enable us to tell whether a series converges to a finite sum. However, we do not get any idea about the value of the sum. This section explains how the concept of residue calculus is used to estimate the sum of infinite series. The basic idea lies in the fact that the sums

$$\sum_{n=-\infty}^{\infty} f(n), \qquad \sum_{n=-\infty}^{\infty} (-1)^n f(n)$$

may be computed with the help of residues of the function $f(z)g(z)$. Here, $g(z)$ is the auxiliary function which is to be chosen carefully.

Thus, for the first series, we must construct a function whose residues are given by

$$\{f(n) : n \in \mathbb{I}\}.$$

Let $f(z)$ be a function which is analytic except for a finite number of poles $a_1, a_2, \ldots, a_m$ (each is not an integer). Suppose g is any auxiliary function having simple poles at $z = n$, $n \in \mathbb{I}$, such that

$$\text{Res}_{z=n}\, g(z) = 1 \;\forall\; n \in \mathbb{I}.$$

Most commonly used such functions are $\pi \cot \pi z$ and $2\pi i(e^{2\pi i z} - 1)^{-1}$, etc. Hence, for each $n = 0, \pm 1, \pm 2, \ldots$ such that $a_k \neq n$, $k = 1, 2, \ldots, m$, we have

$$\text{Res}_{z=n}\, [f(z)g(z)] = f(n).$$

If C_N is the closed square with positive orientation enclosing the poles of $g(z)$, i.e.

$z = 0, \pm 1, \pm 2, \ldots, \pm N$ (see Fig. 7.4), then by the residue theorem

$$\frac{1}{2\pi i}\int_{C_N} f(z)g(z)dz = \sum_{n=-N}^{n=N} \text{Res}_{z=n}\, f(z)g(z) + \sum_{k=1}^{m} \text{Res}_{z=a_k}\, f(z)g(z) \tag{8.1}$$

With the choice of $g(z)$ as $\pi \cot \pi z$ or $\pi \operatorname{cosec} \pi z$ we shall see that the value of the integral on the left side of (8.1) approaches zero as $N \to \infty$, while the first sum on the right side becomes $\Sigma_{-\infty}^{\infty} f(n)$ or $\Sigma_{-\infty}^{\infty} (-1)^n f(n)$, respectively.

Theorem 1. Let $f(z)$ be a meromorphic function in the complex plane with finite number of poles $a_1, a_2, \ldots, a_m$. Suppose there exist constants M and R such that for $|z| > R$

$$|z^r f(z)| \le M \qquad \text{for a fixed } r > 1. \tag{8.2}$$

Then, for $n \ne a_k$, $k = 1, 2, \ldots, m$, we have

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{m} \text{Res}_{z=a_k}\, f(z)\pi \cot \pi z, \tag{8.3}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{k=1}^{m} \text{Res}_{z=a_k}\, f(z)\pi \operatorname{cosec} \pi z. \tag{8.4}$$

Proof. Equation (8.2) means

$$|f(n)| \le \frac{M}{n^r} \qquad \text{for } |n| > R$$

and hence the series $\Sigma_{-\infty}^{\infty} f(n)$ and $\Sigma_{-\infty}^{\infty} (-1)^n f(n)$ are convergent since the nth term is dominated by Mn^{-r} for large n and $r > 1$. By the given hypothesis $n \ne a_k$, $k = 1, 2, \ldots, m$,

$$\text{Res}_{z=n}\, f(z)\pi \cot \pi z = f(n),$$

$$\text{Res}_{z=n}\, f(z)\pi \operatorname{cosec} \pi z = (-1)^n f(n).$$

Let C_N be a square with vertices at $(N + 1/2)(\pm 1 \pm i)$, enclosing all the poles of f (see Fig. 7.4), where N is a positive integer (we can choose N so large such that $N > |a_k|$ for all $k = 1, 2, \ldots, m$). Then, by residue theorem, for $n \ne a_k$, we have

$$\frac{1}{2\pi i}\int_{C_N} f(z)\pi \cot \pi z\, dz = \sum_{n=-N}^{N} f(n) + \sum_{k=1}^{m} \text{Res}_{z=a_k} f(z)\pi \cot \pi z,$$

$$\frac{1}{2\pi i}\int_{C_N} f(z)\pi \operatorname{cosec} \pi z\, dz = \sum_{n=-N}^{N} (-1)^n f(n) + \sum_{k=1}^{m} \text{Res}_{z=a_k}\, f(z)\pi \operatorname{cosec} \pi z.$$

Now, using boundedness of cot πz (see Example 23, Chapter 7), it follows that

$$\left| \int_{C_N} f(z)\pi \cot \pi z \, dz \right| \leq \sup_{z \in C_N} |f(z)\pi \cot \pi z| \times \text{perimeter of } C_N$$

$$\leq 2\pi \frac{M}{N^r} \times 4(2N+1).$$

Thus, the value of the integral tends to 0 as $N \to \infty$.

Similarly, Remark 2 with Example 23, Chatper 7 ensures that

$$\lim_{N\to\infty} \int_{C_N} f(z)\pi \operatorname{cosec} \pi z \, dz = 0.$$

This proves the theorem.

Remark. Extra precaution is needed when some poles of $f(z)$ and of the auxiliary function coincide. In such a situation, the theory developed in Theorem 1 suggests that the corresponding index n in $\Sigma f(n)$ must be dropped.

Example 1. Find the sum of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

Looking at the nature of the series, we choose $f(z) = 1/z^2$. Using (8.3),

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} f(n) = -\sum \operatorname{Res}_{z=a_k} \frac{\pi \cot \pi z}{z^2},$$

where a_k are the poles of the function $1/z^2$. Note that $z = 0$ is the common pole of auxiliary function $\pi \cot(\pi z)$ and $f(z) = 1/z^2$. Hence, in the sum on the left side, $n \neq 0$. Now, $z = 0$ is a pole of order 3 of the function $\pi(\cot \pi z)/z^2$, and the residue at $z = 0$, computed by division of power series or by using (7.3), is given by [see Example 7(a), Chapter 7]

$$\operatorname{Res}_{z=0} \frac{\pi \cot \pi z}{z^2} = -\frac{\pi^2}{3}.$$

Hence,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Example 2. Find the sum of the series

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2}, \qquad a > 0 \text{ (not an integer)}.$$

Consider the function $f(z) = 1/(z^2 + a^2)$ and the auxiliary function $g(z) = \pi \operatorname{cosec} \pi z$. Then, using (8.4),

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum \operatorname{Res}_{z=a_k} \frac{\pi \operatorname{cosec} \pi z}{z^2 + a^2}.$$

For $z = \pm ai$ are the simple poles of the function $1/(z^2 + a^2)$. Now,

$$\operatorname{Res}_{z=ai} \frac{\pi \operatorname{cosec} \pi z}{z^2 + a^2} + \operatorname{Res}_{z=-ai} \frac{\pi \operatorname{cosec} \pi z}{z^2 + a^2} = -\frac{\pi \operatorname{cosech} \pi a}{a}$$

which gives

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{\pi}{a} \operatorname{cosech} \pi a.$$

PROBLEM SET 8.1

1. Show that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

2. Prove that, for $a > 0$ and not an integer

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a.$$

and hence deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2}, \qquad a > 0.$$

3. Prove that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^2}{32}.$$

Suggestion. Theorem 1 cannot be applied directly, because from the infinite sum the alternate terms are missing. Instead, consider the auxiliary function $\pi(\sec \pi z)/z^3$ (see Problem 7) and integrate it around $\pm N \pm iN$. It has simple poles at $n + 1/2$, $n \in \mathbb{I}$, and also a pole at $z = 0$.

4. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + 1)^2}.$$

5. Evaluate

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a+bn^2}, \ \frac{a}{b} > 0;$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^4+n^2+1};$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4+a^4}, a > 0;$ (d) $\displaystyle\sum_{n=-\infty, n\neq\infty}^{\infty} \frac{1}{n(2n+1)}.$

6. Verify the formula

$$\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^3} = \frac{7\pi^3}{180}.$$

Suggestion. Note

$$\operatorname{Res}_{z=0} \frac{\pi \cot \pi z \coth \pi z}{z^3} = -\frac{7\pi^3}{45}.$$

7. Under the assumptions of Theorem 1, show that

$$\sum_{n=-\infty}^{\infty} f\left(n+\frac{1}{2}\right) = \sum \operatorname{Res}_{z=z_k} f(z)\pi \tan \pi z,$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f\left(n+\frac{1}{2}\right) = \sum \operatorname{Res}_{z=z_k} f(z)\pi \sec \pi z.$$

8. Show that

(a) $1+\dfrac{1}{3^4}+\dfrac{1}{5^4}+\cdots=\dfrac{\pi^4}{96};$ (b) $1-\dfrac{1}{3^5}+\dfrac{1}{5^5}-\cdots=\dfrac{5\pi^5}{1536}.$

8.2 DEFINITE INTEGRALS INVOLVING SINES AND COSINES

These integrals are of the type

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta \quad \text{or} \quad \int_{-\pi}^{\pi} f(\sin\theta, \cos\theta)\,d\theta.$$

Here, θ is the argument of a point z on the unit circle centred at the origin.

Consider the integral in the form (if not, then it can be made by proper substitution)

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta,$$

where $f(x, y)$ is a rational function of x and y. We make the substitution $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, i.e. $d\theta = dz/iz$ and use

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

or

$$\cos\theta = \frac{z + 1/z}{2} \qquad \text{and} \qquad \sin\theta = \frac{z - 1/z}{2i}. \tag{8.5}$$

Thus,

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta = \int_C f\left(\frac{z - 1/z}{2i}, \frac{z + 1/z}{2}\right)\frac{dz}{iz},$$

where C is the unit circle (positively oriented). Using the residue theorem

$$\int_C f\left(\frac{z - 1/z}{2i}, \frac{z + 1/z}{2}\right)\frac{dz}{iz} = 2\pi i \sum \operatorname{Res}_{z=z_k}\left[\frac{f\left(\frac{z-1/z}{2i}, \frac{z+1/z}{2}\right)}{iz}\right].$$

The fact that $f(x, y)$ is a rational function of x, y amounts to finding zeros of a polynomial.

Example 3. Evaluate the integral

$$\int_0^{2\pi} \frac{1}{A + B\sin\theta}\, d\theta, \qquad A^2 > B^2,\ A > 0.$$

Employing the substitution (8.5) with $C : |z| = 1$, and assuming, first that $B \neq 0$,

$$\int_C \frac{1}{A + B\left(\frac{z - 1/z}{2i}\right)}\frac{dz}{iz} = \int_C \frac{2dz}{2iAz + B(z^2 - 1)} = \frac{2}{B}\int_C \frac{dz}{z^2 + (2iA/B)z - 1}.$$

The singularities are the roots of the denominator that satisfy $(z - z_1)(z - z_2) = z^2 + (2iA/B)z - 1$, are given by

$$z_1 = \left(-\frac{A}{B} + \frac{\sqrt{A^2 - B^2}}{B}\right)i, \qquad z_2 = \left(-\frac{A}{B} - \frac{\sqrt{A^2 - B^2}}{B}\right)i.$$

Because $z_1 z_2 = -1$, we find $|z_1||z_2| = 1$. With the information $A^2 - B^2 > 0$ and $A > 0$, it follows that $|z_1| < |z_2|$; hence z_1 lies inside C. Thus, computing the residue of the integrand, we apply the residue theorem to have

$$\frac{2}{B}\int_C \frac{dz}{z^2 + (2iA/B)z - 1} = 2\pi i\frac{2}{B}\frac{1}{z_1 - z_2} = \frac{4\pi i}{B}\frac{B}{2i\sqrt{A^2 - B^2}} = \frac{2\pi}{\sqrt{A^2 - B^2}}.$$

Remark. Example 3 includes two possibilities $A > B > 0$ and $B < 0 < A$ satisfying the condition $A^2 > B^2$. The situation $A < 0$ with $A^2 > B^2$ implies $A < B < 0$ and $A < 0 < B$

and the value of the integral is given by

$$\int_0^{2\pi} \frac{1}{A + B\sin\theta}\, d\theta = -\frac{2\pi}{\sqrt{A^2 - B^2}}, \qquad A^2 > B^2, \quad A < 0.$$

Example 4. Evaluate

$$\int_0^{\pi} \frac{1}{(a + b\cos\theta)^2}\, d\theta, \qquad |a| > |b|.$$

Since the integrand is symmetric in θ, we can write

$$\int_0^{\pi} \frac{1}{(a + b\cos\theta)^2}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{(a + b\cos\theta)^2}\, d\theta. \tag{8.6}$$

As usual the substitution

$$\cos\theta = \frac{z - 1/z}{2} \quad \text{and} \quad d\theta = \frac{dz}{iz}$$

for the unit circle $C : |z| = 1$ is used and the right side of (8.6) is computed as

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{(a + b\cos\theta)^2}\, d\theta &= \int_C \frac{1}{\left[a + b(z^2+1)/(2z)\right]^2}\, \frac{dz}{iz} \\
&= \frac{1}{i}\int_C \frac{4z}{[2az + b(z^2+1)]^2}\, dz \\
&= \frac{4}{ib^2}\int_C \frac{z}{[z^2 + 2(a/b)z + 1]^2}\, dz.
\end{aligned}$$

Singularities of the integrand are poles of order two at

$$z_1 = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}, \qquad z_2 = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}.$$

As $|z_1 z_2| = 1$ and $|z_2| > 1$ (since $|a/b| > 1$), we have $|z_1| < 1$, i.e. z_1 lies inside the unit circle C. Hence,

$$\int_0^{2\pi} \frac{1}{(a + b\cos\theta)^2}\, d\theta = \frac{4}{ib^2}\, 2\pi i\ \mathrm{Res}_{z=z_1} \frac{z}{[z^2 + 2(a/b)z + 1]^2}.$$

Now,

$$\mathrm{Res}_{z=z_1} \frac{z}{[z^2 + 2(a/b)z + 1]^2} = \phi'(z_1), \qquad \phi(z) = \frac{z}{(z - z_2)^2}.$$

Since

$$\phi'(z) = \frac{(z-z_2)^2 - 2(z-z_2)z}{(z-z_2)^4} = \frac{(z-z_2)-2z}{(z-z_2)^3} = -\frac{z+z_2}{(z-z_2)^3},$$

we see that

$$\phi'(z_1) = -\frac{-2a/b}{8(a^2/b^2-1)^{3/2}} = \frac{ab^2}{4(a^2-b^2)^{3/2}},$$

and applying the residue theorem,

$$\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}\,d\theta = \frac{4}{ib^2}2\pi i\frac{ab^2}{4(a^2-b^2)^{3/2}}.$$

Thus,

$$\int_0^{\pi} \frac{1}{(a+b\cos\theta)^2}\,d\theta = \frac{\pi a}{(a^2-b^2)^{3/2}}.$$

Remark. While dealing with such integrals, if the numerator involves a sine or cosine function it is more convenient to replace such a function by the corresponding exponential function and then evaluate the imaginary and real parts of the integral as the case may be.

PROBLEM SET 8.2

1. Evaluate the integrals:

(a) $\displaystyle\int_0^{2\pi} \frac{1}{(3+2\cos\theta)^2}\,d\theta$; (b) $\displaystyle\int_{-\pi}^{\pi} \frac{1}{(1+\sin^2\theta)^2}\,d\theta$;

(c) $\displaystyle\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}\,d\theta$; (d) $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}\,d\theta$.

Suggestion. For part (c) it is convenient to evaluate the imaginary part of the integral

$$\int_0^{2\pi} \frac{i+e^{i\theta}}{3+\cos\theta}\,d\theta.$$

2. Converting into appropriate limits of integration, evaluate the integrals:

(a) $\displaystyle\int_0^{\pi} \frac{1}{1+k\cos\theta}\,d\theta,\quad k^2<1$; (b) $\displaystyle\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta,\quad |a|<1$.

3. Use the relation

$$\int_0^{2\pi} e^{int}\,dt = \begin{cases} 2\pi, & n = 0, \\ 0, & n \neq 0 \end{cases}$$

and the binomial expansion

$$(e^{it} + e^{-it})^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} e^{ikt} e^{-i(2n-k)t}$$

and the other binomial expansion for $(e^{it} - e^{-it})^{2n}$ to derive the integration formulas

$$\int_0^{\pi} \cos^{2n} t\,dt = \int_0^{\pi} \sin^{2n} t\,dt = \frac{\pi}{2^{2n}} \frac{(2n)!}{(n!)^2}, \qquad n = 0, 1, 2, \ldots$$

4. For $a \in \mathbb{R}\backslash[-1, 0]$, evaluate

$$\int_0^{\pi/2} \frac{1}{a + \sin^2 t}\,dt.$$

5. Let a, b, c be real with $a^2 > b^2 + c^2$. Show that

(a) $$\int_0^{2\pi} \frac{dt}{a + b\cos t + c\sin t} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}};$$

(b) $$\int_0^{2\pi} \frac{dt}{(a + b\cos t + c\sin t)^2} = \frac{2\pi a}{\sqrt[3]{a^2 - b^2 - c^2}}.$$

6. If $a^2 > b^2$, $a > 0$ and n is a nonnegative integer, prove that

$$\int_0^{2\pi} \frac{\cos n\theta}{a + b\cos\theta}\,d\theta = \frac{2\pi\left(\sqrt{a^2 - b^2} - a\right)^n}{b^n\sqrt{a^2 - b^2}}$$

and point out how Problem 2(a) is a particular case of this problem. Discuss the situation when $b = 0$.

Suggestion. Evaluate the real part of

$$\int_0^{2\pi} \frac{e^{in\theta}}{a + b\cos\theta}\,d\theta.$$

7. When $a^2 < 1$, apply the calculus of residues to show that

$$\int_0^{2\pi} \frac{\cos n\theta}{1 - 2a\cos\theta + a^2}\,d\theta = \frac{2\pi a^n}{1 - a^2}, \qquad n = 0, 1, 2, \ldots$$

What happens if a is replaced by $-a$? For $n = 2$, this is Problem 2(b).

8. Verify the integral

$$\int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta}\,d\theta = \frac{2\pi(3-\sqrt{5})^n}{\sqrt{5}}, \qquad n = 0, 1, 2, \ldots$$

Suggestion. Compute the real part of

$$\int_0^{2\pi} \frac{(1+e^{i\theta}+e^{-i\theta})^n e^{in\theta}}{3+e^{i\theta}+e^{-i\theta}}\,d\theta, \qquad z = e^{i\theta}.$$

9. For $n = 0, 1, \ldots$, establish the following results:

(a) $\displaystyle\int_0^{2\pi} \cos\theta e^{2\cos\theta}\,d\theta = 2\pi\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}$;

(b) $\displaystyle\int_0^{2\pi} e^{-\cos\theta}\cos(\sin\theta + n\theta)\,d\theta = 2\pi\frac{(-1)^n}{n!}$;

(c) $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!}$.

10. Use Problem 6 to directly evaluate the following integrals:

(a) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$; (b) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$;

(c) $\displaystyle\int_0^{2\pi} \frac{1+2\cos\theta}{5+3\cos\theta}\,d\theta$; (d) $\displaystyle\int_0^{\pi} \frac{\cos^2(2\theta)}{a+b\cos 2\theta}\,d\theta$

11. Deduce the following integrals, using Problem 7.

(a) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{1-2a\cos\theta+a^2}\,d\theta = \pi$;

(b) $\displaystyle\int_0^{\pi} \frac{\cos^2(3\theta)}{1-2a\cos 2\theta+a^2}\,d\theta = \pi\frac{1-a+a^2}{1-a}$.

8.3 IMPROPER INTEGRALS

In real calculus, the improper integral of a continuous function on $[0, \infty)$ is defined by

$$\int_0^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_0^{R} f(x)\,dx, \tag{8.7}$$

provided the limit on the right of (8.7) exists. In this situation, we say that the improper integral converges. If $f(x)$ is continuous on $(-\infty, \infty)$, we define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty} \int_{0}^{R_2} f(x)dx, \tag{8.8}$$

where both integrals exist on the right of (8.8).

We have another value, called the Cauchy principal value defined by

$$\text{PV} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx, \tag{8.9}$$

provided this limit exists.

It can be proved that the convergence of integral in (8.8) implies that its converging value coincides with the Cauchy principal value. However, the converse may not be true. The simple function $f(x) = x$ serves as an example. Its Cauchy principal value is 0 but the integral in (8.8) does not converge. It is very interesting to note that for functions satisfying $f(x) = f(-x)$, i.e. for even functions the converse is also true. Also, when f is even, integrals in (8.8) and (8.9) converge or diverge together and in this situation

$$\int_{0}^{\infty} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx. \tag{8.10}$$

Usually, the integral on the right side of (8.10) will be evaluated as the Cauchy principal value.

The following proposition is fundamental in dealing with these integrals.

Proposition 1. Let Γ be the arc of a circle $|z| = R$ having $\theta_1 \le \theta \le \theta_2$. Suppose $zf(z) \to K$ uniformly as $z \to \infty$. Then,

$$\lim_{R\to\infty} \int_{\Gamma} f(z)dz = i(\theta_2 - \theta_1)K. \tag{8.11}$$

Proof. By the given hypothesis, for given $\varepsilon > 0$, we can find positive real numbers R such that

$$|zf(z) - K| < \varepsilon \quad \text{whenever } |z| > R,$$

or, formally, $zf(z) = K + \varepsilon$, where $\varepsilon \to 0$ as $z \to \infty$. Thus, for sufficiently large R,

$$\begin{aligned}\int_{\Gamma} f(z)dz &= \int_{\Gamma} \frac{K+\varepsilon}{z}\,dz \\ &= \int_{\theta_1}^{\theta_2} (K+\varepsilon)\, i d\theta, \quad z = Re^{i\theta} \\ &= i(\theta_2 - \theta_1)K + \int_{\theta_1}^{\theta_2} i\varepsilon\, d\theta.\end{aligned}$$

Hence,

$$\left| \int_{\Gamma} f(z)dz - i(\theta_2 - \theta_1)K \right| \leq \int_{\theta_1}^{\theta_2} |i\varepsilon| d\theta < \varepsilon(\theta_2 - \theta_1).$$

On taking limits, (8.11) follows.

Improper integrals involving rational functions

These integrals are of the type

$$\int_a^{\infty} f(x)dx; \qquad \int_{-\infty}^{a} f(x)dx; \qquad \int_{-\infty}^{\infty} f(x)dx.$$

In such type of integrals

$$f(x) = \frac{p(x)}{q(x)}, \quad p(x) \neq 0 \text{ on the real axis,}$$

where $p(x)$ and $q(x)$ are polynomials such that

$$\text{degree of } q(x) \geq \text{degree of } p(x) + 2.$$

For evaluating such integrals the procedure is well explained in the following theorem.

Theorem 2. Let $f(x) = p(x)/q(x)$. If $q(x) \neq 0$ for all real x and $n \geq m + 2$, m and n represents the degree of $p(x)$ and $q(x)$, respectively. Then,

$$\text{PV} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{k=1}^{j} \text{Res}_{z=z_k} f(z), \tag{8.12}$$

where $z_1, z_2, \ldots, z_j$ are the poles of $f(z) = p(z)/q(z)$ that lie in the upper half-plane.

Proof. Since there are a finite number of poles of $f(z) = p(z)/q(z)$ that lie in upper half-plane, a real number R can be found such that all these poles lie inside the contour C, which consists of the line segment $-R \leq x \leq R$ of the x-axis together with the upper semi-circle C_R of radius R (positively oriented). Thus, $C = [-R, R] \cup C_R$ (see Fig. 8.1).

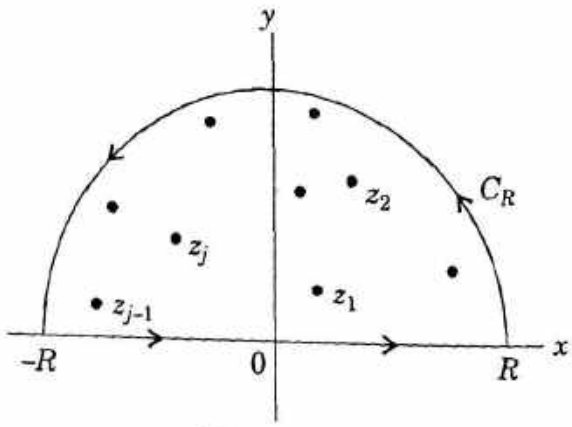

Figure 8.1

Hence,

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \int_{C} f(z)dz,$$

and with the application of the residue theorem,

$$\int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{j} \operatorname{Res}_{z=z_k} f(z)dz - \int_{C_R} f(z)dz. \tag{8.13}$$

The theorem is established if we can show that the integral of $f(z) = p(z)/q(z)$ along C_R on the right side of (8.13) approaches zero as $R \to \infty$. Suppose

$$p(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0, \qquad a_m \neq 0,$$

$$q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0, \qquad b_n \neq 0.$$

Thus,

$$p(z) = z^m(a_m + a_{m-1}z^{-1} + \cdots + a_1 z^{-m+1} + a_0 z^{-m}),$$

$$q(z) = z^n(b_n + b_{n-1}z^{-1} + \cdots + b_1 z^{-n+1} + b_0 z^{-n}).$$

Therefore,

$$\frac{zp(z)}{q(z)} = \frac{z^{m+1}(a_m + a_{m-1}z^{-1} + \cdots + a_1 z^{-m+1} + a_0 z^{-m})}{z^n(b_n + b_{n-1}z^{-1} + \cdots + b_1 z^{-n+1} + b_0 z^{-n})}.$$

The expressions in parentheses of the numerator and denominator approach a_m and b_n as $z \to \infty$. Again, using the condition $n \geq m + 2$, note that

$$\frac{zp(z)}{q(z)} = zf(z) \to 0 \quad \text{as} \quad |z| \to \infty.$$

Thus, using Proposition 1,

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0. \tag{8.14}$$

Let $R \to \infty$ in (8.13). Then, in anticipation of (8.14), (8.12) follows.

Example 5. Evaluate the integral

$$\int_0^\infty \frac{x^2}{x^4+1}\, dx.$$

Since $f(x) = f(-x)$, in view of (8.10), we have

$$\int_0^\infty \frac{x^2}{x^4+1}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx. \tag{8.15}$$

So, we take the corresponding complex function as

$$f(z) = \frac{z^2}{z^4+1} = \frac{p(z)}{q(z)}.$$

Thus,

$$\int_C \frac{z^2}{z^4+1}\,dz = \int_{-R}^{R} \frac{x^2}{x^4+1}\,dx - \int_{C_R} \frac{z^2}{z^4+1}\,dz, \tag{8.16}$$

where C consists of the positively oriented semi-circle C_R and the line segment from $-R$ to R, $R > 1$. The singularities of the function $f(z) = z^2/(z^4+1)$ are the simple poles at

$$e^{i\pi/4}, \quad e^{i3\pi/4}, \quad e^{i5\pi/4}, \quad e^{i7\pi/4}.$$

The singularities $e^{i5\pi/4}$ and $e^{i7\pi/4}$ are outside C (see Fig. 8.1). Residue at $z_1 = e^{i\pi/4}$, and $z_2 = e^{i3\pi/4}$ are given by using (7.6) as

$$\operatorname{Res}_{z=e^{i\pi/4}} f(z) = \left.\frac{p(z)}{q'(z)}\right|_{e^{i\pi/4}} = \left.\frac{1}{4z}\right|_{e^{i\pi/4}} = \frac{e^{-i(\pi/4)}}{4} = \frac{1-i}{4\sqrt{2}},$$

$$\operatorname{Res}_{z=e^{i3\pi/4}} f(z) = \left.\frac{p(z)}{q'(z)}\right|_{e^{i3\pi/4}} = \left.\frac{1}{4z}\right|_{e^{i3\pi/4}} = \frac{e^{-i(3\pi/4)}}{4} = -\frac{1+i}{4\sqrt{2}}.$$

Since $zf(z) \to 0$ as $z \to \infty$, in view of Proposition 1,

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2}{z^4+1}\,dz = 0.$$

Taking $R \to \infty$ in (8.16), we get

$$\text{PV}\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dz = 2\pi i\left(\frac{1-i}{4\sqrt{2}} - \frac{1+i}{4\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$$

So, by Eq. (8.15), we have

$$\int_0^{\infty} \frac{x^2}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}.$$

Example 6. Establish the result

$$\int_0^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx = \frac{(2n)!}{2^{2n+1}(n!)^2}\pi, \qquad n = 0, 1, \ldots$$

Let $f(z) = 1/(1+z^2)^{n+1}$. The singularities of $f(z)$ are the poles of order $n+1$ in the upper half-plane at $z = i$ and obviously, $zf(z) \to 0$ as $z \to \infty$. We apply Theorem 2 to have

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx = 2\pi i \mathrm{Res}_{z=i} f(z).$$

Now,

$$\begin{aligned}\mathrm{Res}_{z=i} \frac{1}{(1+z^2)^{n+1}} &= \frac{1}{n!}\frac{d^n}{dz^n}\left[\frac{1}{(z+i)^{n+1}}\right]_{z=i} \\ &= \frac{(-1)^n(n+1)(n+2)\cdots(n+n)}{n!(2i)^{2n+1}} \\ &= -\frac{(2n)!}{(n!)^2 2^{2n+1}} i.\end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx = -2\pi i \frac{(2n)!}{(n!)^2 2^{2n+1}} i = \frac{(2n)!\pi}{2^{2n}(n!)^2}$$

or

$$\int_{0}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx = \frac{(2n)!}{2^{2n+1}(n!)^2}\pi.$$

Remark. For $x = \tan\theta$, the above integral is deduced to

$$\int_{0}^{\pi/2} \cos^{2n}\theta\, d\theta = \frac{\pi}{2^{2n+1}}\frac{(2n)!}{(n!)^2},$$

a result similar to Problem 3 in Problem Set 8.2.

Improper integrals involving sines and cosines

Improper integrals may involve sine and cosine functions, and this merits a separate discussion on them. These integrals are of the form

$$\int_{-\infty}^{\infty} f(x)\sin ax\, dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x)\cos ax\, dx, \qquad a > 0.$$

Let p and q be polynomials of degree m and n respectively, such that $n \geq m+1$. If $q(x) \neq 0$ for all real x, then

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\cos ax\, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\sin ax\, dx$$

are convergent improper integrals. Basically, these integrals are Fourier type integrals

which are encountered while studying problems concerning Fourier expansion of functions and Fourier transforms. To deal with such type of integrals, we will be using the identities

$$\cos ax = \text{Re } e^{iax} \quad \text{and} \quad \sin ax = \text{Im } e^{iax},$$

and evaluate

$$\text{PV} \int_{-\infty}^{\infty} e^{iax} f(x)dx, \qquad f(x) = \frac{p(x)}{q(x)}.$$

The validity of this integral depends upon the following theorem.

Theorem 3. Let $p(x)$ and $q(x)$ be polynomials with real coefficients of degree m and n, respectively, satisfying $n \geq m + 1$. If $q(x) \neq 0$ for all real x and $a > 0$ and the integrand is of the form $e^{iax} f(x)$, then

$$\text{PV} \int_{-\infty}^{\infty} f(x) \cos ax \, dx = -2\pi \text{ Im } \sum_{k=1}^{j} \text{Res}_{z=z_k} e^{iaz} f(z),$$

and

$$\text{PV} \int_{-\infty}^{\infty} f(x) \sin ax \, dx = 2\pi \text{ Re } \sum_{k=1}^{j} \text{Res}_{z=z_k} e^{iaz} f(z),$$

where $z_1, z_2, \ldots, z_j$ are poles of $f(z) = p(z)/q(z)$ that lie in the upper half-plane and Re and Im stand for real and imaginary parts.

The proof of this theorem is similar to Theorem 2 and hence is omitted. However, we will supply helping results needed for the proof.

Jordan's inequality. We have

$$\frac{2\theta}{\pi} \leq \sin \theta \leq \theta, \quad \text{if} \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Geometrically, the inequality is very obvious if we draw the graph of $y = \sin \theta$ and the line $y = 2\theta/\pi$ between $\theta = 0$ and $\theta = \pi/2$ in the (θ, y)-plane (Fig. 8.2).

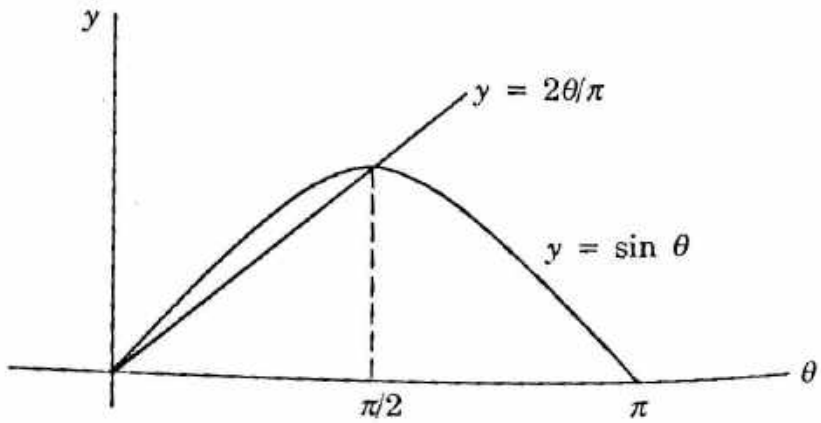

Figure 8.2

The second part of Jordan's inequality is well known. To prove the first part analytically, define the function,

$$f(\theta) = \frac{\sin\theta}{\theta}.$$

Note that

$$f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2}.$$

Since $1 > \sin\theta/\theta > \cos\theta$ for $0 < \theta < \pi/2$, it follows that

$$f'(\theta) < 0 \qquad \text{for } 0 < \theta < \frac{\pi}{2}.$$

Thus, $f(\theta)$ is a decreasing function in the interval $[0, \pi/2]$. Therefore, $f(\pi/2) < f(\theta)$ for $0 < \theta < \pi/2$,

$$\frac{2}{\pi} < \frac{\sin\theta}{\theta} \qquad \text{or} \qquad \frac{2\theta}{\pi} < \sin\theta, \qquad 0 < \theta < \frac{\pi}{2}.$$

Moreover, for $\theta = 0$ or $\theta = \pi/2$, $\sin\theta = 2\theta/\pi$.

Remark. We say that $f(z) \to 0$ uniformly as $R \to \infty$ in C_R (circular arc of radius R) if $|f(z)| \le K_R$, where K_R depends on R (not on arg z) and $K_R \to 0$ as $R \to \infty$.

Jordan's lemma. If $f(z)$ is analytic except at a finite number of singularities and $f(z) \to 0$ uniformly as $z \to \infty$, then

$$\lim_{R\to\infty} \int_\Gamma e^{imz} f(z)dz = 0, \qquad m > 0.$$

where Γ is the semi-circle $|z| = R$, $\operatorname{Im} z \ge 0$.

Proof. We assume that $f(z)$ has no singularities on $\Gamma : |z| = R$ and this can be ensured by taking R large enough. Since $f(z) \to 0$ uniformly as $z \to \infty$, it follows that for given $\varepsilon > 0$, there exists $R > 0$ such that

$$|f(z)| < \varepsilon \qquad \text{for all } z \text{ on } \Gamma.$$

Also, $z = Re^{i\theta} \Rightarrow |dz| = R\,d\theta$, and $|e^{imz}| = e^{-mR\sin\theta}$. Hence, using Jordan's inequality,

$$\begin{aligned}
\left|\int_\Gamma e^{imz} f(z)\,dz\right| &\le \int_\gamma |e^{imz} f(z)|\,|dz| \\
&< \int_0^\pi e^{-mR\sin\theta}\,\varepsilon R\,d\theta \\
&\le 2\varepsilon R \int_0^{\pi/2} e^{-2mR\theta/\pi}\,d\theta \\
&= \frac{\varepsilon\pi}{m}(1 - e^{-mR}) \\
&< \frac{\pi}{m}\varepsilon.
\end{aligned}$$

However $\varepsilon > 0$ is arbitrary, the result follows.

Example 7. Apply the calculus of residues to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx \quad (a > b > 0).$$

We find the real part of the integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx.$$

For this we take the corresponding complex function as

$$e^{iz} f(z), \qquad f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}.$$

Referring to Fig. 8.1, we write

$$\int_{-R}^{R} e^{ix} f(x)\,dx = \int_{C} e^{iz} f(z)\,dz - \int_{C_R} e^{iz} f(z)dz, \tag{8.17}$$

where C is the positively oriented closed contour $C = C_R \cup [-R, R]$, $R > 1$.

Now, the singularities of the integrand are simple poles $z = \pm ai$, $z = \pm bi$, but only two poles ai, bi lie inside C and

$$\begin{aligned}
\text{Res}_{z=ai}\, e^{iz} f(z) &= \lim_{z\to ai} (z-ai)\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} \\
&= \lim_{z\to ai} \frac{e^{iz}}{(z+ai)(z^2+b^2)} \\
&= \frac{e^{-a}}{2ai(-a^2+b^2)},
\end{aligned}$$

and

$$\begin{aligned}
\text{Res}_{z=bi}\, e^{iz} f(z) &= \lim_{z\to bi} (z-bi)\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} \\
&= \lim_{z\to bi} \frac{e^{iz}}{(z^2+a^2)(z+bi)} \\
&= \frac{e^{-b}}{2bi(-b^2+a^2)}.
\end{aligned}$$

Therefore, the sum of the residues is

$$\text{Res}_{z=ai} + \text{Res}_{z=bi} = \frac{1}{2i(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Hence, applying the residue theorem in (8.17),

$$\int_{-R}^{R} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{2\pi i}{2i(a^2-b^2)}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) - \int_{C_R} e^{iz} f(z)\,dz. \tag{8.18}$$

Since $f(z) \to 0$ as $|z| = R \to \infty$, by Jordan's lemma,

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz = 0.$$

Letting $R \to \infty$ in (8.18),

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right).$$

Equating real parts from both sides, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right).$$

PROBLEM SET 8.3

1. Evaluate the following integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+2)(x^2+6)}\,dx;$ (b) $\displaystyle\int_{0}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx.$

(c) $\displaystyle\int_{0}^{\infty} \frac{x^4}{x^6+1}\,dx;$ (d) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\,dx.$

2. Find the Cauchy principal value of the following integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx;$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x+2}{(x^2+a^2)(x^2+b^2)}\,dx;\ a, b \in \mathbb{R}_+.$

3. Show that

(a) $\displaystyle\int_{0}^{\infty} \frac{1}{x^4+a^4}\,dx = \frac{\sqrt{2}\pi}{4a^3},\quad a > 0;$ (b) $\displaystyle\int_{3}^{\infty} \frac{1}{x^2-6x+13}\,dx = \frac{\pi}{4}.$

Suggestion. For part (b), shift x by 3 units to the left.

4. Apply the calculus of residue to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2}; \qquad a, b \in \mathbb{R}_+.$$

5. Show that, for $n = 0, 1, \ldots,$

$$\int_0^{\infty} \frac{1}{(a^2+x^2)^{n+1}}\,dx = \frac{\pi}{(2a)^{2n+1}}\frac{(2n)!}{(n!)^2}.$$

Suggestion. Replace x by x/a in Example 2.

6. Find the Cauchy principal value of the following integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+2x+2}\,dx;$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx.$

7. Prove that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi}{a}e^{-|m|a}, \qquad m \in \mathbb{R},\ a > 0,$$

and hence evaluate the following Fourier type integral

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{a^2+x^2}\,dx.$$

Suggestion. Differentiate the first integral with respect to m under the integral sign by Leibniz rule.

8. Determine the PV of the following integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\,dx;$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+9}\,dx.$

9. By the method of contour integration, show that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\,dx = \frac{\pi}{2a^3}(1+ma)e^{-ma}, \qquad m \geq 0,\ a > 0,$$

and hence evaluate

$$\int_0^{\infty} \frac{x \sin mx}{(x^2+a^2)^2}\,dx.$$

Suggestion. For the second integral, follow the suggestion of Problem 7.

10. Verify the following results:

(a) $\displaystyle\int_0^\infty \frac{\cos 2x}{x^4+x^2+1}\,dx = \frac{\pi e^{\sqrt{6}}}{\sqrt{3}\,\sin(1+\pi/6)}$; (b) $\displaystyle\int_0^\infty \frac{\sin x^2+\cos x^2-1}{x^2}\,dx = 0.$

Suggestion. For (b) consider the integrand $(e^{iz^2}-1)/z^2$.

11. Use the Leibniz rule for successive differentiation and the technique of Example 6 to verify the following formula:

$$\int_0^\infty \frac{\cos x}{(1+x^2)^{n+1}}\,dx = \frac{\pi e^{-1}}{n!\,2^{n+1}}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!\,2^k}, \qquad n=0,1,\ldots .$$

8.4 INTEGRATION ALONG INDENTED CONTOURS

We have already considered integrands which have no poles on the real axis (refer to Theorems 2 and 3). Let us drop this condition. Now the integrand may have poles on the real axis as well as inside the semi-circle C_R, considered earlier. The poles on the real axis are excluded by detouring them with semi-circles of small radii. This procedure is known as *indenting* at a point.

At this stage we digress from indented contours to give a brief idea of other type of improper integrals. In such type of integrals the limits of integration are finite as

$$\int_a^b f(x)\,dx$$

but the function is unbounded or may not be even defined at some point c in the interval of integration. The integration of $f(x)$ is defined by

$$\int_a^b f(x)dx = \lim_{\varepsilon_1\to 0}\int_a^{c-\varepsilon_1} f(x)dx + \lim_{\varepsilon_2\to 0}\int_{c+\varepsilon_2}^{b} f(x)dx, \tag{8.19}$$

where both ε_1 and ε_2 approach zero independently through positive values. In this case, the Cauchy principal value is defined as

$$\text{PV}\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx\right]. \tag{8.20}$$

Again, convergence of (8.19) implies convergence of (8.20), but the converse may not be true. However, for even functions both are equivalent.

To evaluate such integrals we prove some auxiliary results which are needed in this and the next section.

Proposition 2. Let $\lim_{z\to a}(z-a)f(z) = k$ and γ be the arc of the circle $|z-a| = r$ given by $\theta_1 \le \theta \le \theta_2$. Then

$$\lim_{r\to 0}\int_\gamma f(z)\,dz = i(\theta_2 - \theta_1)k. \tag{8.21}$$

Note that k is the residue at a if $z = a$ is a simple pole of $f(z)$.

Proof. Since $\lim_{z\to a}(z-a)f(z) = k$, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|(z-a)f(z) - k| < \varepsilon \quad \text{whenever} \quad 0 < |z-a| < \delta$$

or, formally, $(z-a)f(z) = k + \varepsilon$, where $\varepsilon \to 0$ as $z \to a$.

Hence

$$\int_\gamma f(z)dz = \int_\gamma \frac{k+\varepsilon}{z-a}\,dz = \int_{\theta_1}^{\theta_2} ki\,d\theta + \int_{\theta_1}^{\theta_2} \varepsilon i d\theta = ki(\theta_1 - \theta_2) + \int_{\theta_1}^{\theta_2} \varepsilon i d\theta.$$

Therefore,

$$\left|\int_\gamma f(z)dz - i(\theta_2 - \theta_1)k\right| \le \int_{\theta_1}^{\theta_2} |i\varepsilon| d\theta < \varepsilon(\theta_2 - \theta_1).$$

Let $r \to 0$, (8.21) follows.

Theorem 4. Let $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials with real coefficients of degree m and n, respectively such that $n \ge m + 2$. If $q(z)$ has simple zeros at the points $t_1, t_2, \ldots, t_\ell$ of the real axis, then

$$\text{PV}\int_{-\infty}^{\infty} f(x)dx = 2\pi i\sum_{k=1}^{j} \text{Res}_{z=z_k}\, f(z) + \pi i\sum_{k=1}^{\ell} \text{Res}_{z=t_k}\, f(z), \tag{8.22}$$

where $z_1, z_2, \ldots, z_j$ are the poles of $f(z)$ that lie in the upper half-plane.

The proof of this theorem will appear while elucidating the next theorem. Note that in absence of singularities on the real axis, (8.22) reduces to (8.12).

Theorem 5. Let $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials of degree m and n, respectively such that $n \ge m + 1$, and let $q(z)$ have simple zeros at the points $t_1, t_2, \ldots, t_\ell$ of the real axis. Further, if α is a positive real number and integrand $g(z)$ is of the form

$$g(z) = e^{i\alpha z}f(z), \quad \alpha > 0 \tag{8.23}$$

then

$$\text{PV}\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx = -2\pi\,\text{Im}\sum_{k=1}^{j} \text{Res}_{z=z_k}\, g(z) - \pi\,\text{Im}\sum_{k=1}^{\ell} \text{Res}_{z=t_k}\, g(z), \tag{8.24}$$

$$\text{PV}\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx = 2\pi\,\text{Re}\sum_{k=1}^{j} \text{Res}_{z=z_k}\, g(z) + \pi\,\text{Re}\sum_{k=1}^{\ell} \text{Res}_{z=t_k}\, g(z), \tag{8.25}$$

where $z_1, z_2, \ldots, z_j$ are the poles of $f(z)$ that lie in the upper half-plane.

Proof. Since f has only finite number of poles, we can choose r small enough so that

$$C_k : t_k + re^{i\theta} \quad \text{for } 0 \le \theta \le \pi \quad \text{and} \quad k = 1, 2, \ldots, \ell$$

are disjoint semi-circles and poles $z_1, z_2, \ldots, z_j$ lie in the upper half-plane (see Fig. 8.3).

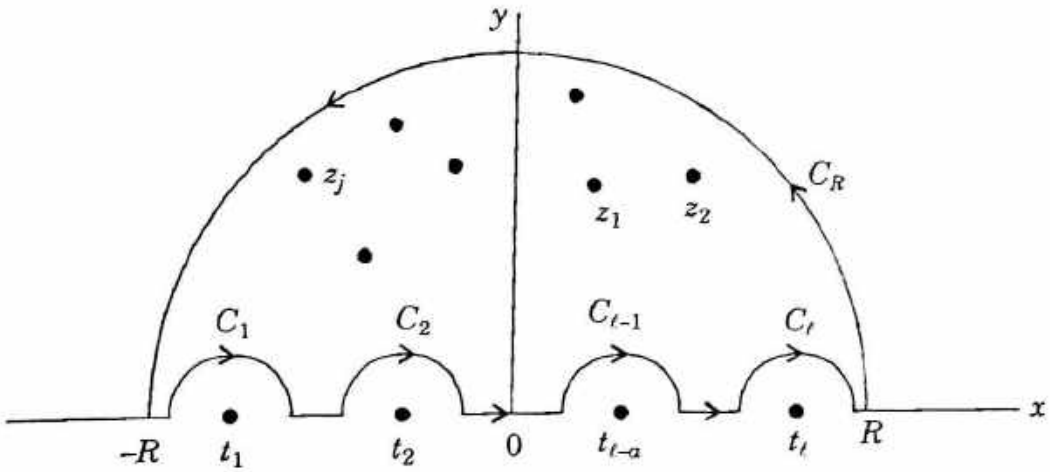

Figure 8.3

Let R be chosen large enough so that all the poles of f lie in upper half-plane under the circle $C_R : z = Re^{i\theta}$ for $0 \le \theta \le \pi$, and the poles on the real axis lie in the interval $[-R, R]$. Let C be the simple positively oriented closed contour that consists of C_R and $-C_1, -C_2, \ldots, -C_\ell$. Now,

$$\int_{I_R} g(x)dx - \sum_{k=1}^{\ell} \int_{C_k} g(z)dz + \int_{C_R} g(z)dz = \int_C g(z)dz, \tag{8.26}$$

where I_R is the portion of the interval $-R \le x \le R$ on the real axis that lies outside the intervals $(t_k - r, t_k + r)$ for $k = 1, 2, \ldots, \ell$. Applying the residue theorem, (8.26) becomes

$$\int_{I_R} g(x)dx = 2\pi i \sum_{k=1}^{j} \text{Res}_{z=z_k}\, g(z) + \sum_{k=1}^{\ell} \int_{C_k} g(z)dz - \int_{C_R} g(z)dz. \tag{8.27}$$

In view of the nature of the function in (8.23), Propositions 1 and 2 imply

$$\lim_{R\to\infty} \int_{C_R} g(z)dz = 0 \quad \text{and} \quad \lim_{r\to 0} \int_{C_k} g(z)dz = i\pi\, \text{Res}_{z=t_k} f(z).$$

Letting $R \to \infty$ and $r \to 0$ in (8.27), and using the above limits, (8.27) gives

$$\text{PV} \int_{-\infty}^{\infty} e^{iax} f(x)dx = 2\pi i \sum_{k=1}^{j} \text{Res}_{z=z_k}\, g(z) + \pi i \sum_{k=1}^{\ell} \text{Res}_{z=t_k}\, g(z),$$

Now (8.24) and (8.25) are available on comparing the real and imaginary parts of this equation.

Remarks. 1. For $a = 0$ and degree of $p(x)$ – degree of $q(x) \geq 2$, (8.27) reduces to (8.22).

2. In absence of singularities on the real axis, (8.24) and (8.25) correspond to the integrals of Theorem 3 in that order.

Example 8. Find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x}{x^3 - 8}\, dx.$$

Define the complex integrand by

$$\frac{z}{z^3 - 8} = dz.$$

The singularities of the integrand are simple poles given as cube roots of 8, and these are

$$t_1 = 2, \qquad z_1 = -1 + i\sqrt{3}, \qquad z_2 = -1 - i\sqrt{3}.$$

Note that $t_1 = 2$ lies on the x-axis and $z_1 = -1 + i\sqrt{3}$ lies in the upper half-plane. Now, Eq. (8.22) gives

$$\begin{aligned} \text{PV} \int_{-\infty}^{\infty} \frac{x}{x^3 - 8}\, dx &= 2\pi i \ \text{Res}_{z=z_1} f(z) + \pi i \ \text{Res}_{z=t_1} f(z) \\ &= \pi i \frac{-1 - i\sqrt{3}}{6} + \pi i \frac{1}{6} \\ &= \frac{\pi\sqrt{3}}{6}. \end{aligned}$$

Example 9. Evaluate

$$\text{PV} \int_{-\infty}^{\infty} \frac{\sin x}{(x-1)(x^2+4)}\, dx.$$

Since it involves the trigonometric function, let us define

$$f(z) = \frac{e^{iz}}{(z-1)(z^2+4)}\, dz$$

so that Theorem 5 can be applied. The singularities of the integrand are simple poles $t_1 = 1$ on the real axis and $z_1 = 2i$ in the upper half-plane. Now, by Eq. (8.25),

$$\begin{aligned} \text{PV} \int_{-\infty}^{\infty} \frac{\sin x}{(x-1)(x^2+4)}\, dx &= 2\pi \ \text{Re}[\text{Res}_{z=z_1} f(z)] + \pi \ \text{Re}[\text{Res}_{z=t_1} f(z)] \\ &= 2\pi \ \text{Re}\left(\frac{-2+i}{20e^2}\right) + \pi \ \text{Re}\left(\frac{\cos 1 + i \sin 1}{5}\right) \\ &= \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right). \end{aligned}$$

Example 10. Show that

$$\int_0^\infty \frac{\sin mx}{x}\,dx = \begin{cases} \dfrac{\pi}{2}, & m > 0, \\ -\dfrac{\pi}{2}, & m < 0. \end{cases}$$

Consider the integrand (so that Theorem 5 can be applied),

$$f(z) = \frac{e^{imz}}{z}.$$

The only singularity of the function is $t_1 = 0$ on the real axis and hence by (8.25), we have

$$\text{PV}\int_{-\infty}^{\infty} \frac{\sin mx}{x}\,dx = \pi\,\text{Re}[\text{Res}_{z=0} f(z)] = \pi.$$

This is the Cauchy principal value and the property $f(x) = f(-x)$ ensures

$$\int_0^\infty \frac{\sin mx}{x}\,dx = \frac{\pi}{2}.$$

Replacing m by $-m$ evaluates the second part.

Remark. Theorems 4 and 5 cannot be applied if $q(z)$ has no simple zeros on the real axis. How we proceed for functions which have poles of order two or more is illustrated in the following example.

Example 11. Apply the calculus of residue to evaluate

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx, \quad a, b \geq 0.$$

We define

$$f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}.$$

The integrand has pole of order 2 at the origin. Consider a contour C consisting of positively oriented semi-circle $C_R : |z| = R$ in the upper half-plane and the line segment $[-R, -r]$, negatively oriented semi-circle $\gamma : |z| = r$, also in the upper half-plane, and the line segment $[r, R]$. Note that

$$\begin{aligned} \lim_{z\to 0} zf(z) &= \lim_{z\to 0} \frac{e^{iaz} - e^{ibz}}{z} \\ &= \lim_{z\to 0} (iae^{iaz} - ibe^{ibz}) \qquad \text{(L'Hôspital's rule)} \\ &= i(a-b). \end{aligned}$$

Using Proposition 2, we have

$$\lim_{z\to 0}\int_{\gamma} z f(z) = -i(\pi - 0)\, i(a-b) = \pi(a-b). \tag{8.28}$$

Again, note that $\lim_{z\to\infty} 1/z^2 = 0$, and hence Jordan's lemma reveals that

$$\lim_{R\to\infty}\int_{C_R} \frac{e^{iaz} - e^{ibz}}{z^2} = 0. \tag{8.29}$$

Evidently, $f(z)$ has no singularity within and on C. Hence, by the residue theorem,

$$\left(\int_r^R + \int_{C_R} + \int_{-R}^{-r} + \int_{\gamma}\right) f(z)dz = \int_C f(z)dz = 0.$$

Let $R \to \infty$ and $r \to 0$ and making use of (8.28) and (8.29), we get

$$\int_0^{\infty} f(x)\,dx + \int_{-\infty}^{0} f(x)\,dx + \pi(a-b) = 0$$

or

$$\int_{-\infty}^{\infty} \frac{e^{iax} - e^{ibx}}{x^2}\,dx = \pi(b-a).$$

Equating the real and imaginary parts, we have

$$\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a).$$

Remark. If a and b are any real numbers, the integral in the above example is written as

$$\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(|b| - |a|).$$

PROBLEM SET 8.4

1. Determine the principal value of the following integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x(x^2 - 4x + 5)}$; (b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^3 - 1}\,dx$.

2. Prove that

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \int_{-\infty}^{\infty} \left(\frac{\cos x}{x}\right)^2 dx = \pi.$$

Suggestion. Consider the integrand $(1-e^{2iz})/z^2$.

3. Find the value of the following integrals:

(a) $\displaystyle\int_0^{\infty} \frac{\sin \pi x}{x(1-x^2)}\, dx;$ (b) $\displaystyle\int_0^{\infty} \frac{\cos x}{a^2-x^2}\, dx, \quad a > 0.$

4. Show that for $m > 0$, $a \in \mathbb{R}$,

$$\int_0^{\infty} \frac{\sin mx}{x(a^2+x^2)}\, dx = \frac{\pi}{2a^2}(1-e^{-m|a|}).$$

5. By contour integration, prove that

(a) $\displaystyle\int_0^{\infty} \frac{\sin x^2}{x} dx = \frac{\pi}{4};$ (b) $\displaystyle\int_0^{\infty} \frac{\sin^3 x}{x^3} dx = \frac{3\pi}{8}$

and deduce the result of Example 3 by substituting $x^2 = t$ in part (a).

Suggestion. Consider the complex function e^{iz^2}/z for (a).

6. Show that

$$\int_0^{\infty} \frac{e^{\cos x} \sin(\sin x)}{x}\, dx = \frac{\pi(e-1)}{2}.$$

Suggestion. Take $f(z) = (\exp e^{iz})/z$ and the contour of Example 10.

7. Apply the calculus of residue to evaluate

$$\int_0^{\infty} \frac{\sin mx}{x(x^2+1)^2}\, dx, \quad m \in \mathbb{R}.$$

8.5 OTHER TYPES OF CONTOURS

In all cases of improper integrals discussed so far we have considered the contour to be a half-circle in the upper half-plane or indented at some point on the real axis. However, some of the integrals may not be evaluated with the help of such type of contours. Solely, the choice of contour depends on the nature of the function to be integrated. Now, we illustrate by examples how real integrals may be evaluated by an appropriate choice of the

contours. A more or less rectangular path is an appropriate choice for functions of the type

$$\int_{-\infty}^{\infty} f(e^x)\{g(x) - g(x + 2\pi i)\}dx.$$

Example 12. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx, \qquad 0 < a < 1.$$

Here the contour γ consists of a rectangle with corners $\pm R$, $\pm R + 2\pi i$. The only singularity of the corresponding complex integrand $e^z/(1 + e^z)$ is the simple pole at πi (Fig. 8.4).

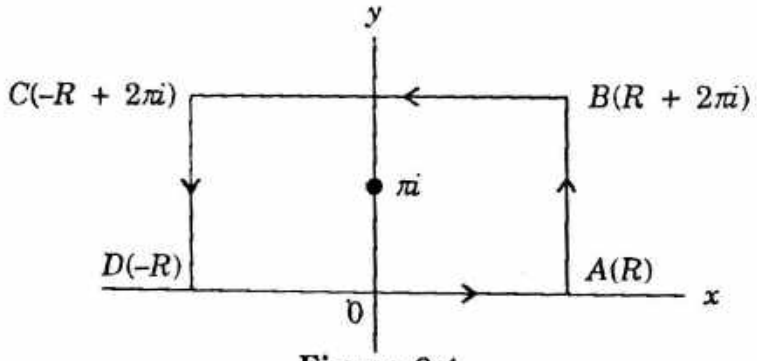

Figure 8.4

Now,

$$\text{Res}_{z=i\pi} \frac{e^{az}}{1+e^z} = \left.\frac{e^{az}}{e^z}\right|_{z=\pi i} = -e^{ia\pi}.$$

Note that on the vertical lines AB and CD,

$$\left|\frac{e^{az}}{1+e^z}\right| \le \frac{e^{aR}}{e^R - 1} \quad \text{and} \quad \left|\frac{e^{az}}{1+e^z}\right| \le \frac{e^{-aR}}{1-e^{-R}}.$$

As $R \to \infty$, these expressions approach zero. Now, applying the residue theorem, we get

$$\int_C \frac{e^{az}}{1+e^z}dz = \int_{-R}^{R} \frac{e^{ax}}{1+e^x}dx + \int_{AB} \frac{e^{az}}{1+e^z}dz + \int_{BC} \frac{e^{az}}{1+e^z}dz + \int_{CD} \frac{e^{az}}{1+e^z}dz = -2\pi i e^{i\pi a}.$$

Let $R \to \infty$. Then

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx - \int_{-\infty}^{\infty} \frac{e^{a(x+2\pi i)}}{1+e^x}dx = -2\pi i e^{i\pi a}.$$

since the integrals along AB and CD vanish as $R \to \infty$ and $z = x + 2\pi i$ along BC. Thus,

$$(1 - e^{i2\pi a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = -2\pi i e^{i\pi a}.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{2\pi i}{e^{i\pi a} - e^{-i\pi a}} = \frac{\pi}{\sin a\pi}.$$

When using a contour integration, it is sometimes essential to employ special properties of the integrand, as illustrated below.

Example 13. Verify the Fresnel integrals

$$\int_0^{\infty} \cos x^2 \, dx = \int_0^{\infty} \sin x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

First we mention the value of Gauss error integral

$$\int_0^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$$

which will be needed in the procedure. Consider the integral

$$\int_{\gamma} e^{-z^2} \, dz,$$

where γ is the arc constituted by segment OA of positive real axis taken from origin, arc $C = Re^{i\theta}$, $0 \le \theta \le \pi/4$ and line BO, $z = re^{i\pi/4}$, $0 \le r \le R$ as shown in Fig. 8.5.

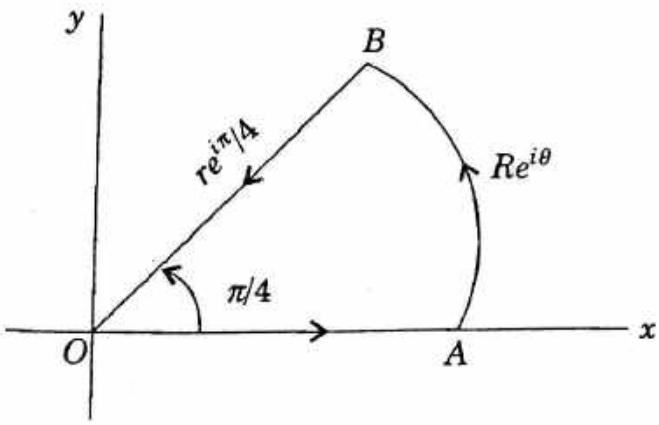

Figure 8.5

Hence, we have

$$\int_0^R f(x)\,dx + \int_C f(z)\,dz + \int_R^0 f(re^{i\pi/4})\,d(re^{i\pi/4}) = 0. \tag{8.30}$$

With the application of Jordan's inequality, note that

$$\left|\int_C f(z)\,dz\right| = \left|\int_0^{\pi/4} f(Re^{i\theta})iRe^{i\theta}\,d\theta\right|$$

$$\leq R\int_0^{\pi/4} e^{-R^2\cos 2\theta}\,d\theta$$

$$\leq R\int_0^{\pi/4} e^{-R^2(1-4\theta/\pi)}\,d\theta$$

$$= \frac{\pi(1-e^{-R^2})}{4R},$$

since $\cos 2\theta \geq 1 - 4\theta/\pi$ for $0 \leq \theta \leq \pi/4$. Thus,

$$\lim_{R\to\infty}\int_C f(z)\,dz = 0. \tag{8.31}$$

Also,

$$\int_R^0 f(re^{i\pi/4})\,d(re^{i\pi/4}) = -e^{-i\pi/4}\int_0^R e^{-ir^2}dr. \tag{8.32}$$

Using (8.31) and (8.32) and proceeding to limits in (8.30), we get

$$\int_0^\infty e^{-x^2}dx - e^{-i\pi/4}\int_0^\infty e^{ix^2}dx = 0.$$

So, on inserting the value of Gauss error integral,

$$\int_0^\infty e^{ix^2}dx = \frac{\sqrt{\pi}}{2}e^{i\pi/4}.$$

Comparing the real and imaginary components, the Fresnel integrals are immediate.

Example 14. Evaluate

$$\int_0^\infty \frac{1}{x^3+a^3}dx, \qquad a > 0.$$

Since the integral is on $(0, \infty)$, we cannot use a contour like that of Fig. 8.3. The first difficulty to adopt the path is that the integrand is not even, and secondly $x < 0$ is a singularity. Nevertheless there is a symmetry that can be employed, namely $(xe^{2\pi i/3})^3 = x^3$. This suggests using the contour as depicted in Fig. 8.6, where C_R is the sector $Re^{i\theta}$, $0 \leq \theta \leq 2\pi/3$.

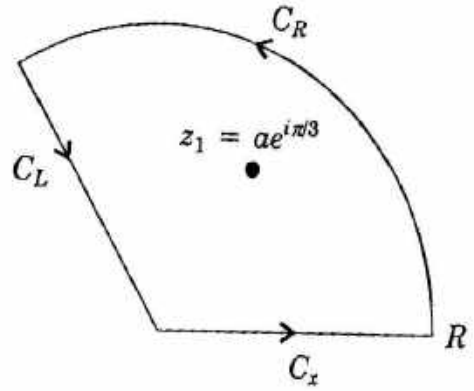

Figure 8.6

We, therefore, have

$$\left(\int_{C_L} + \int_{C_x} + \int_{C_R}\right)\frac{dz}{z^3+a^3} = \int_C \frac{dz}{z^3+a^3} = 2\pi i \sum_j \operatorname{Res}_{z=z_j} \frac{1}{z^3+a^3}.$$

The only pole inside C is one root of $z^3 = -a^3$ and this is $z_1 = ae^{i\pi/3}$, and

$$\operatorname{Res}_{z=z_1} \frac{1}{z^3+a^3} = \frac{1}{3z^2}\bigg|_{z=z_1} = \frac{e^{-2\pi i/3}}{3a^2}.$$

The integral on C_R tends to zero because of Proposition 1. The integral on C_L evaluated by making the substitution $z = re^{2\pi i/3}$ (orientation is to be observed)

$$\int_{C_L} \frac{dz}{z^3+a^3} = \int_R^0 \frac{e^{2\pi i/3}}{r^3+a^3} = -e^{2\pi i/3}\int_0^R \frac{dr}{r^3+a^3}.$$

Taking into account the contribution from $C_x (0 \le x \le R)$ and from C_L, we have

$$(1-e^{2\pi i/3})\int_0^R \frac{dr}{r^3+a^3} = 2\pi i \frac{e^{-2\pi i/3}}{3a^2}.$$

Letting $R \to \infty$,

$$\int_0^\infty \frac{dx}{x^3+a^3} = \frac{2\pi}{3a^2}\frac{e^{-2\pi i/3}}{1-e^{2\pi i/3}} = -\frac{\pi}{3a^2}\frac{2i}{e^{i\pi/3}-e^{-i\pi/3}}e^{-i\pi} = \frac{2\pi}{3\sqrt{3}a^2}.$$

PROBLEM SET 8.5

1. Work out the general form of the Fresnel integrals of Example 13 as

$$\int_0^\infty \cos(ax^2)\,dx = \int_0^\infty \sin(ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \qquad a \in \mathbb{R}.$$

2. Show that

$$\int_0^\infty e^{-x^2}\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}.$$

Suggestion. Consider the contour of Example 13.

3. Prove that

$$\int_0^\infty \frac{\cosh ax}{\cos x}\,dx = \frac{\pi}{2\cos(a\pi/2)}, \quad |a| < 1.$$

Suggestion. Use a rectangular contour with vertices $\pm R$, $\pm R + \pi i$.

4. Find the value of the integral

$$\int_0^\infty \frac{\sin ax}{1+e^x}\,dx.$$

5. Use a rectangular contour to evaluate

$$\int_{-\infty}^\infty \frac{e^x}{e^{2x}+e^{2a}}\,dx.$$

6. By integrating the function $z/(a - e^{iz})$, $a > 0$ about a rectangle with vertices $\pm\pi$, $\pm\pi + iR$, prove that

$$\int_0^\pi \frac{\sin x}{1+a^2-2ax}\,dx = \begin{cases} \dfrac{\pi}{a}\ln(1+a), & 0<a<1, \\[2ex] \dfrac{\pi}{a}\ln\dfrac{1+a}{a}, & a>1. \end{cases}$$

7. Apply residue calculus with appropriate path to show that

$$\int_{-\infty}^\infty \frac{\cos mx}{e^x+e^{-x}}\,dx = \frac{\pi}{2}\sec\left(\frac{m\pi}{2}\right).$$

8. Use a rectangular contour with corners $\pm R$ and $\pm R + i\pi/m$ and appropriate indentation to evaluate the integral

$$\int_0^\infty \frac{x}{\sin mx}\,dx, \quad m \neq 0.$$

9. If $|q| < \pi$, prove that

$$\int_{-\infty}^\infty e^{ipx}\frac{\sinh qx}{\sinh \pi x} = \frac{\sin q}{\cos p + \cosh q}.$$

10. Use a sector contour with radius R, as depicted in Fig. 8.4, centred at the origin with the angle $0 \le \theta \le 2\pi/5$ to find, for $a > 0$

$$\int_0^\infty \frac{1}{x^5 + a^5}\,dx = \frac{\pi}{5a^4 \sin(\pi/5)}.$$

11. Generalize the result of Problem 10 to evaluate

$$\int_0^\infty \frac{1}{x^n + 1}\,dx, \qquad n \in \mathbb{N}.$$

Suggestion. Take a path from 0 to R to $Re^{i2\pi/n}$ to 0. Also refer to the third deduced result of Example 15.

8.6 INTEGRATION THROUGH A BRANCH CUT

The residue theorem is also utilized to evaluate real integrals when the associated complex integrand is multivalued and hence has a branch cut. For instance, consider

$$\frac{x^\alpha p(x)}{q(x)}, \quad \alpha \in \mathbb{R}.$$

The associated complex function involves z^a and hence is multivalued. First, specify the branch that is to be used. Let a be a real number with $0 < a < 1$. Then, usually, we use the branch

$$z^\alpha = e^{\alpha \log z} = e^{\alpha(\ln r + i\theta)} \ (r > 0,\ 0 < \theta < 2\pi).$$

The function z^a is analytic in the domain $r > 0$, $0 < \theta < 2\pi$ (see Section 3.7).

Theorem 6. Let p and q be polynomials of degree m and n, respectively such that $n \ge m + 2$. If $q(x) \ne 0$ for $x > 0$ and q has a zero of order at most 1, then

$$\int_0^\infty x^a f(x)\,dx = \frac{2\pi i}{1 - e^{ia2\pi}} \sum_{k=1}^{j} \operatorname{Res}_{z=z_k} z^a f(z), \qquad 0 < a < 1, \tag{8.33}$$

where $z_1, z_2, \ldots, z_j$ are the nonzero poles of $f(z) = p(z)/q(z)$.

Proof. The function $z^a p(z)/q(z)$ is not defined at 0. We can however avoid 0 by considering the following contour:

Let C denote the positively oriented simple closed contour that consists of the portions of the circle $\gamma : |z| = \varepsilon$ and $\Gamma : |z| = R$ and the horizontal segments joining them, as shown in Fig. 8.7. This is called the *keyhole* contour.

A small value of ε and a large value of R can be selected so that all nonzero poles $z_1, z_2, \ldots, z_j$ of $f(z) = p(z)/q(z)$ lie inside C. Note that

$$\int_\varepsilon^R x^a f(x)dx + \int_\Gamma z^a f(z)dz - \int_\varepsilon^R x^a e^{ia2\pi} f(x)dx - \int_\gamma z^a f(z)dz = \int_C z^a f(z)dz, \tag{8.34}$$

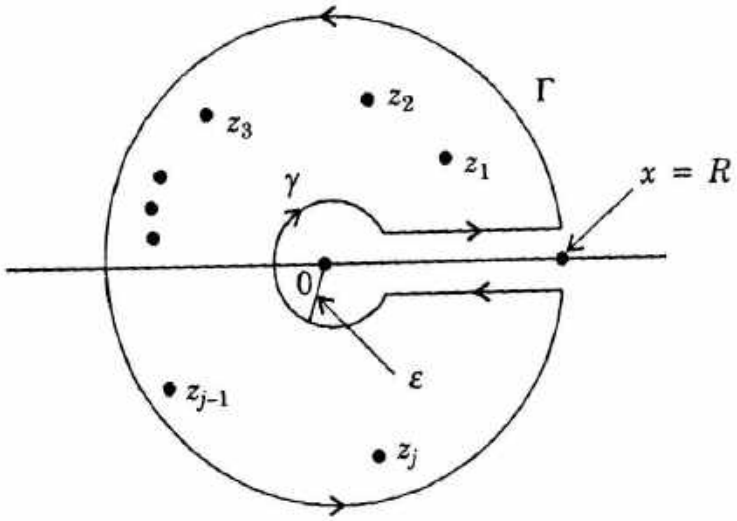

Figure 8.7

where we use $\theta = 0$ and $\theta = 2\pi$ along the upper and lower horizontal segments, respectively, of the cut annulus, since owing to the location of the branch cut, the sum of the integrals on each side of the cut do not cancel.

Obviously, due to $0 < a < 1$ and $n \geq m + 2$,

$$\lim_{z\to 0} z^{a+1} f(z) = \lim_{z\to 0} z^{a+1} f(z) = 0.$$

In view of the above conditions, Propositions 2 and 1 show that

$$\lim_{R\to\infty} \int_{\Gamma} z^a f(z)\,dz = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_{\gamma} z^a f(z)\,dz = 0.$$

Apply the residue theorem and let $\varepsilon \to 0$ in Eq. (8.34)

$$(1 - e^{ia2\pi}) \int_0^R x^a f(x)\,dx = 2\pi i \sum_{k=1}^{j} \operatorname{Res}_{z=z_k} z^a f(z) - \int_{\Gamma} z^a f(z)\,dz$$

or

$$\int_0^R x^a f(x)\,dx = \frac{2\pi i}{1 - e^{ia2\pi}} \sum_{k=1}^{j} \operatorname{Res}_{z=z_k} z^a f(z) + \frac{1}{1 - e^{ia2\pi}} \int_{\Gamma} z^a f(z)\,dz. \tag{8.35}$$

Letting $R \to \infty$, Eq. (8.33) is established.

Example 15. Verify the integral

$$\int_0^{\infty} \frac{x^{-a}}{x+1}\,dx = \frac{\pi}{\sin a\pi}, \qquad 0 < a < 1. \tag{8.36}$$

The integral is improper not only because the upper limit of integration is infinite but also because the integrand is discontinuous at $x = 0$. However, from the real calculus it can be established that the integral exists.

Let us consider the complex function

$$\frac{z^{1-a}}{z(z+1)} = z^{1-a} f(z), \qquad 0 < 1 - a < 1, \qquad f(z) = \frac{1}{z(z+1)}.$$

This function is of the form considered in Theorem 6, and satisfies:

$$\lim_{z\to 0} z^{2-a} f(z) = 0 \qquad \lim_{z\to\infty} z^{2-a} f(z) = 0,$$

the conditions used in the proof. Moreover, $f(z)$ has a nonzero pole at $z_1 = -1$. Using (8.33), for $1-a$, we find that

$$\int_0^\infty \frac{x^{1-a}}{x(x+1)}\,dx = \frac{2\pi i}{1-e^{i2(1-a)\pi}} \operatorname{Res}_{z_1=-1} z^{1-a} f(z).$$

or

$$\begin{aligned}\int_0^\infty \frac{x^{-a}}{x+1}\,dx &= \frac{2\pi i}{1-e^{i2(1-a)\pi}} \operatorname{Res}_{z_1=-1} \frac{z^{-a}}{z+1} \\ &= \frac{2\pi i}{1-e^{-i2a\pi}} e^{-ia\pi} \\ &= \frac{2\pi i}{e^{ia\pi} - e^{-ia\pi}} \\ &= \frac{\pi}{\sin a\pi}.\end{aligned}$$

Deductions. We mention some important deductions of the integral in (8.36).

1. Replacing x by $1/x$, we have

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1. \tag{8.37}$$

2. Further, the substitution $x = e^t$ converts (8.37) to Example 12.
3. For $x = t^n$ and $a = m/n$ (not an integer), $n, m \in \mathbb{R}_+$: $n > m$, (8.37) becomes

$$\int_0^\infty \frac{x^{m-1}}{x^n+1}\,dx = \frac{\pi}{n \sin(m\pi/n)}.$$

In particular, for $n = 4$ and $m = 1-a$,

$$\int_0^\infty \frac{x^{-a}}{1+x^4}\,dx = \frac{\pi}{4} \operatorname{cosec}\left[\frac{\pi}{4}(1-a)\right], \quad -3 < a < 0.$$

Also, for $n = 2$, $m = 1/2$,

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x^2)}\,dx = \frac{\pi}{\sqrt{2}}\cdot$$

Example 16. By contour integration evaluate the integral

$$\int_0^\infty \frac{x^{1/6}\ln x}{(1+x)^2}\,dx.$$

Consider the integrand

$$f(z) = \frac{z^{1/6}\log z}{(1+z)^2}\,dz.$$

The singularities of the integrand are due to the multivalued function

$$z^{1/6}\log z = e^{(1/6)\log z}\log z$$

and pole of order two at $z = -1$ contributed by $1/(1+z)^2$.

To avoid singularities due to multivalued function we take the branch $|z| > 0$, $0 < \arg z < 2\pi$ and integrate around the keyhole contour C of Fig. 8.7. The residue at $z = -1$ is evaluated by

$$\lim_{z\to -1}\phi'(z), \qquad \phi(z) = z^{1/6}\log z.$$

For the above branch,

$$\phi'(z) = \frac{z^{-5/6}}{6}(6+\log z).$$

Hence,

$$\mathrm{Res}_{z=-1} f(z) = \phi'(-1) = \frac{e^{(-5/6)i\pi}}{6}(6+i\pi)$$

or, after simplification, we have

$$\mathrm{Res}_{z=-1} f(z) = \frac{1}{12}[(\pi - 6\sqrt{3}) - i(\sqrt{3}\,\pi + 6)].$$

Applying the residue theorem

$$\left(\int_\varepsilon^R + \int_\Gamma + \int_R^\varepsilon + \int_\gamma\right) f = \frac{\pi}{6}[(6+\sqrt{3}\,\pi) + i(\pi - 6\sqrt{3})]. \tag{8.38}$$

Here Γ and γ are positively and negatively oriented semi-circles, respectively.

Since

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} z\log z\cdot\lim_{z\to 0}\frac{z^{1/6}}{(1+z)^2} = 0, \quad \lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{\log z}{z}\cdot\lim_{z\to\infty}\frac{z^{7/6}}{(1+z)^2} = 0,$$

Propositions 2 and 1 show that

$$\lim_{z\to\infty}\int_{\Gamma} f(z)dz = 0, \qquad \lim_{z\to 0}\int_{\gamma} f(z)dz = 0.$$

Let $\varepsilon \to 0$ and $R \to \infty$ in (8.38). Then,

$$\int_0^\infty \frac{x^{1/6}\log x}{(1+x)^2}\,dx - \int_0^\infty e^{(\ln x + i2\pi)/6}(\ln x + i2\pi)\,dx = \frac{\pi}{6}[(6+\sqrt{3}\,\pi) + i(\pi - 6\sqrt{3})].$$

Equating real and imaginary parts on both sides of the above equation, we get

$$\frac{1}{2}A + \sqrt{3}\,\pi B = \frac{\pi}{6}(\sqrt{3}\,\pi + 6) \quad \text{and} \quad \frac{\sqrt{3}}{2}A + \pi B = \frac{\pi}{6}(\pi - 6\sqrt{3}),$$

where

$$A = \int_0^\infty \frac{x^{1/6}\ln x}{(1+x)^2}\,dx \quad \text{and} \quad B = \int_0^\infty \frac{x^{1/6}}{(1+x)^2}\,dx.$$

Solving these equations,

$$A = 2\pi - \frac{\pi^2}{\sqrt{3}}.$$

Example 17. Evaluate

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx.$$

Consider

$$\int_C \frac{(\log z)^2}{x^2+1}\,dz,$$

where $C = [\varepsilon, R] \cup C_R \cup [-R, -\varepsilon] \cup C_\varepsilon$ is the contour depicted in Fig. 8.8, and take the branch $|z| > 0$, $-\pi/2 < \arg z < 3\pi/2$.

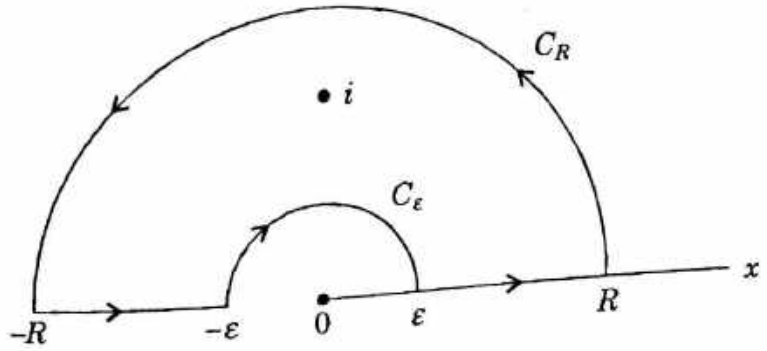

Figure 8.8

We have

$$\int_{\varepsilon}^{R} \frac{(\ln x)^2}{x^2+1}dx + \int_{C_R} \frac{(\log z)^2}{z^2+1}dz + \int_{-R}^{-\varepsilon} \frac{(\log x)^2}{x^2+1}dx + \int_{C_\varepsilon} \frac{(\log z)^2}{z^2+1}dz = \int_{C} \frac{(\log z)^2}{z^2+1}dz.$$

Note that

$$\int_{-R}^{-\varepsilon} \frac{(\log x)^2}{x^2+1}dx = -\int_{R}^{\varepsilon} \frac{[\log(-x)]^2}{x^2+1}dx = \int_{\varepsilon}^{R} \frac{[\log(-x)]^2}{x^2+1}dx.$$

Thus,

$$\int_{\varepsilon}^{R} \frac{(\log x)^2}{x^2+1}dx + \int_{\varepsilon}^{R} \frac{[\ln(-x)]^2}{x^2+1}dx + \left(\int_{C_R} + \int_{C_\varepsilon}\right)\frac{(\log z)^2}{z^2+1}dz = 2\pi i \operatorname{Res}_{z=i} \frac{(\log z)^2}{z^2+1}.$$

Propositions 2 and 1 show that $\int_{C_\varepsilon} \to 0$ as $\varepsilon \to 0$ and $\int_{C_R} \to 0$ as $R \to \infty$. Thus the above equation simplifies, and

$$2\int_{0}^{\infty} \frac{(\ln x)^2}{x^2+1}dx + 2i\pi \int_{0}^{\infty} \frac{\ln x}{x^2+1}dx - \pi^2 \int_{0}^{\infty} \frac{dx}{x^2+1} = -\frac{\pi^3}{4}.$$

But

$$\pi^2 \int_{0}^{\infty} \frac{1}{x^2+1}dx = \pi^2[\tan^{-1}x]_0^\infty = \frac{\pi^3}{2}.$$

Hence, we have

$$2\int_{0}^{\infty} \frac{(\ln x)^2}{x^2+1}dx + 2i\pi \int_{0}^{\infty} \frac{\ln x}{x^2+1}dx = \frac{\pi^3}{4},$$

whereupon, by taking the real and imaginary parts

$$\int_{0}^{\infty} \frac{(\ln x)^2}{x^2+1}dx = \frac{\pi^3}{8}, \qquad \int_{0}^{\infty} \frac{\ln x}{x^2+1}dx = 0.$$

PROBLEM SET 8.6

1. Prove that

(a) $\displaystyle\int_{0}^{\infty} \frac{\ln x}{a^2+x^2}dx = \frac{\pi \ln a}{2a}, \quad a > 0;$ (b) $\displaystyle\int_{0}^{\infty} \frac{\ln x}{(1+x^2)^2}dx = -\frac{\pi}{4}.$

2. Evaluate the integral

$$\int_{0}^{\infty} \frac{\ln x}{1+x^2}dx$$

Suggestion. Use the branch $(\log z)/(1+z^2)$, $(|z| > 0, -\pi/2 < \arg z < 3\pi/2)$. However, this is a by-product of Example 17.

3. Find the Cauchy principal value of

$$\int_0^\infty \frac{x^{a-1}}{1-x}\,dx, \qquad 0 < a < 1.$$

Suggestion. Consider $z^{a-1}/(1-z)$, and the principal branch.

4. Show that

$$\int_0^\infty \frac{x^{2a-1}}{1+x^2}\,dx = \frac{\pi}{2\sin \pi a}, \qquad 0 < a < 1.$$

5. Use Theorem 6 to evaluate the integral

$$\int_0^\infty \frac{x^q}{(1+x^2)^2}\,dx, \qquad -1 < q < 3.$$

6. Using the method of integration through branch cut, prove that

$$\int_0^\infty \frac{x^{p-1}}{x^2+x+1}\,dx = \frac{2\pi}{\sqrt{3}}\cos\left[\frac{\pi(2p+1)}{6}\right]\cos \pi p, \qquad 0 < p < 2.$$

7. The following integrals require different branches of the logarithmic functions for their evaluation:

(a) $\displaystyle\int_0^\infty \frac{\sqrt[3]{x}\,\ln x}{1+x^2}\,dx$; (b) $\displaystyle\int_0^\infty \frac{\sqrt[3]{x}}{(x+3)(x+1)}\,dx$.

Write the appropriate branches and then evaluate these integrals.

8. Applying the calculus of residue and considering the appropriate branch of the multivalued function occurring in the analysis, evaluate

$$\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\,dx.$$

Suggestion. Define the corresponding complex function: $f(z) = \log(z+i)/(1+z^2)$, and find the PV and then compare the real and imaginary parts.

9. Verify the following integral:

$$\int_0^\infty \frac{(\ln x)^2}{x^2+x+1}\,dx = \frac{16\pi^3}{81\sqrt{3}}.$$

10. If $|p| < 1$ and $|\alpha| < \pi$, show that

$$\int_0^\infty \frac{x^{-p}}{x^2 + 2x\cos\alpha + 1}\,dx = \frac{\pi}{\sin p\pi}\,\frac{\sin p\alpha}{\sin\alpha}.$$

Justify the conditions imposed on p and α.

11. The beta function of two real variables is the following integral

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \qquad (p > 0,\ q > 0).$$

Use the transformation $x = 1/(1+t)$ and Example 15 to show that

$$\beta(p, 1-p) = \frac{\pi}{\sin \pi p} \quad (0 < p < 1).$$

12. Show that

$$\int_0^\infty \frac{\cot^{-1} x}{1+x^2}\,dx = \frac{\pi^2}{2}.$$

Suggestion. This is a by-product of the analysis involved in Problem 8 of this Problem Set.

13. Use Example 15 to verify the integral

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\operatorname{cosec}\left[\frac{(2m+1)\pi}{2n}\right].$$

14. Let $f(z)$ be a rational function with no poles on the positive real axis and, further, suppose that

$$\lim_{z\to 0} z^a f(z) = \lim_{z\to\infty} z^a f(z) = 0.$$

Then

$$\int_0^\infty x^{a-1} f(x)\,dx = \frac{\pi}{\sin a\pi}\sum \operatorname{Res}\left[(-z)^{a-1} f(z)\right],$$

where the sum is taken over all the poles of $f(z)$.

Suggestion. This is an alternative technique to deal with integrals of the type (8.33).

9

Conformal Mappings

In this chapter we study mappings which transform different curves and regions from one complex plane to the other complex plane with reference to size and orientation. In particular, we develop the theory of bilinear transformations. We also discuss how different curves and figures in z-plane are carried over to the w-plane by elementary functions.

9.1 INTRODUCTION

Let $w = f(z)$ be a complex function. Corresponding to each point z_0, we have a point $w_0 = f(z_0)$. Both z_0 and w_0 are complex numbers. This suggests that for finding the image of points or curves in the complex plane we need another complex plane to sketch the image region and for this we shall identify these planes as z-plane and w-plane. In this way the function f assigns to each value z in its domain a value $w = f(z)$, called the image point, in w-plane.

Now, we discuss some elementary mappings which will be used frequently to explain the various concepts of conformal mappings.

Linear transformation

The mapping

$$w = z + b \tag{9.1}$$

is a translation. If $b = \alpha + i\beta$, then the image of the point $z = x + iy$ under map (9.1) is translated to $w = x + \alpha + i(y + \beta)$. Hence the image of any region or curve in the z-plane is just the translated region or curve in the w-plane, respectively. The mapping

$$w = az, \qquad a \neq 0 \tag{9.2}$$

is a rotation. If $a = |a|e^{it}$ and $z = re^{i\theta}$, then we have $w = |a|re^{i(\theta+t)}$. This means that the transformation expands (contracts) the radius vector representing z by the factor $|a|$ and rotates it through an angle $t = \arg a$. The image of a given region is, therefore, geometrically similar to the region in the w-plane. The general linear transformation

$$w = az + b \tag{9.3}$$

is a combination of (9.1) and (9.2). Hence it is an expansion or contraction and rotation, followed by translation. Obviously, $|a| > 1$ and $|a| < 1$ give expansion and contraction, respectively.

We conclude that the image of any region in the z-plane under the transformation (9.3) has the same geometry except that it is expanded or contracted, rotated and translated.

Example 1. Find the image of the region $y > 1$ under the map $w = (1-i)z$.

We write the transformation $w = (1-\mathrm{i})z$ in the form $z = (1/2)(1+\mathrm{i})w$. Denoting $w = u + iv$ and $z = x + iy$, we get

$$x + iy = \frac{1}{2}(1+i)(u+iv) \Leftrightarrow x = \frac{u-v}{2}, \quad y = \frac{u+v}{2}.$$

Hence the region $y > 1$ is mapped on the region: $u + v > 2$ in the w-plane. When we move on the line $y = 1$ from i to $1 + i$, the region remains left to the observer; and this causes movement on the image line $u + v = 2$ from $w = 1 + i$ to $w = 2$, and hence the image region is taken left to the observer, as illustrated in Fig. 9.1.

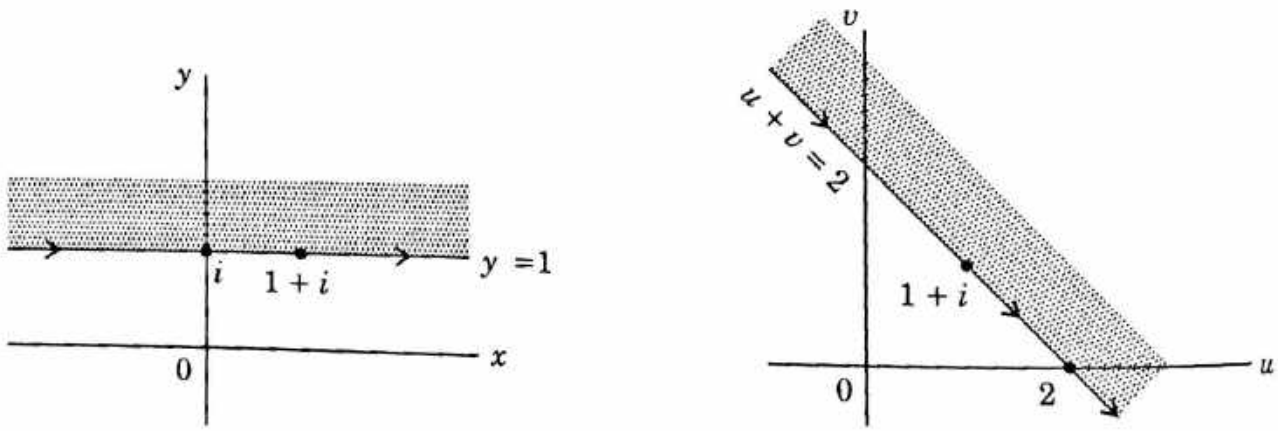

Figure 9.1

Transformation $w = 1/z$

The map $w = 1/z$ establishes a one-one correspondence between the nonzero points of the z and w plane. Obviously, we have $|w| = |1/z| \Rightarrow |w|\,|z| = 1$. This explains that the interior of the circle $|z| = 1$ is mapped onto the exterior of the circle $|w| = 1$, $|z| = 1$ is mapped on $|w| = 1$, and the exterior of the circle $|z| = 1$ is mapped onto the interior of the circle $|w| = 1$. By defining the function $T(z) = 1/z$ on $\mathbb{C} \cup \{\infty\}$,

$$T(z) = \begin{cases} 0, & z = \infty, \\ \infty, & z = 0, \\ 1/z, & \text{otherwise.} \end{cases}$$

With this definition, $T(z)$ is a continuous function in the extended complex plane.

Writing $w = u + iv$ and $z = x + iy$, the following relations are obtained:

$$u = \frac{x}{x^2 + y^2} \qquad v = -\frac{y}{x^2 + y^2}, \tag{9.4}$$

$$x = \frac{u}{u^2 + v^2} \qquad y = -\frac{v}{u^2 + v^2}. \tag{9.5}$$

Relations (9.4) are used for finding the image of curves and figures from w-plane to z-plane, while (9.5) are used for the reverse process.

Proposition 1. The map $w = 1/z$ transforms circles and lines into circles and lines.

Proof. Let a, b, c, d be real numbers such that $b^2 + c^2 > 4ad$. Then

$$a(x^2 + y^2) + bx + cy + d = 0 \tag{9.6}$$

represents the equation of a circle in the z-plane. Using relations (9.5) in (9.6), we have

$$d(u^2 + v^2) + bu - cv + a = 0. \tag{9.7}$$

Note that (9.7) is a circle or line in the w-plane depending on the nature of real constants.

From (9.6) and (9.7), we conclude the following.

(a) A circle not passing through the origin ($a \neq 0$, $d \neq 0$) in the z-plane is mapped on a circle in w-plane not passing through the origin.

(b) A circle passing through the origin ($a \neq 0$, $d = 0$) in the z-plane is mapped on a line in the w-plane not passing through the origin.

(c) A line not passing through the origin ($a = 0$, $d \neq 0$) in the z-plane is mapped on a circle in the w-plane passing through the origin.

(d) A line passing through the origin ($a = 0$, $d = 0$) in the z-plane is mapped on a line in the w-plane passing through the origin.

Remarks. 1. Why have we taken $b^2 + c^2 > 4ad$ in Eq. (9.6) is evident by completing the square and expressing as

$$\left(x + \frac{b}{2a}\right)^2 + \left(y + \frac{c}{2a}\right)^2 = \frac{b^2 + c^2 - 4ad}{4a^2}.$$

For $a = 0$, the condition $b^2 + c^2 > 0$ means that both b and c are not simultaneously zero.

2. This proposition can also be proved by taking the general form of the circle (line) as

$$Az\bar{z} + B\bar{z} + \bar{B}z + C = 0,\ A \geq 0,\ C \in \mathbb{R},\ B \in \mathbb{C},$$

and its image is obtained by substituting $z = 1/w$.

Example 2. Show that when a circle is transformed into a circle under the map $w = 1/z$, the centre of the original circle is never mapped onto the centre of the image circle.

The image of the circle $a(x^2 + y^2) + bx + cy + d = 0$ with centre $(-b/2a, -c/2a)$ such that $b^2 + c^2 > 4ad$ under the $w = 1/z$ is given by (9.7) as $d(u^2 + v^2) + bu - cv + a = 0$ (see Proposition 1). The centre of image circle is $(-b/2d, c/2d)$.

Now, the image of $(-b/2a, -c/2a)$, the centre of the original circle under $w = 1/z$ is given by

$$u = \frac{-b/(2a)}{(b^2+c^2)/(4a^2)}, \qquad v = \frac{c/(2a)}{(b^2+c^2)/(4a^2)}$$

or

$$u = \frac{-2ab}{b^2+c^2}, \qquad v = \frac{2ac}{b^2+c^2}.$$

If we assume that the centre of the circle is mapped on the centre of the image circle, then

$$\frac{-2ab}{b^2+c^2} = \frac{-b}{2d} \Rightarrow b^2 + c^2 = 4ad, \text{ not permissible.}$$

Similarly,

$$\frac{2ac}{b^2+c^2} = \frac{c}{2d} \Rightarrow b^2 + c^2 = 4ad, \text{ not permissible.}$$

Hence, the centre of the original circle is never mapped on the centre of the image circle.

Example 3. Find the image of the infinite strip $0 < y < 1/(2c)$ under the map $w = 1/z$.

Using relations (9.5), we have

$$0 < y \Rightarrow 0 < -\frac{v}{u^2+v^2} \Rightarrow v < 0,$$

and

$$y < \frac{1}{2c} \Rightarrow -\frac{v}{u^2+v^2} < \frac{1}{2c} \Rightarrow u^2 + v^2 + 2cv > 0.$$

Hence, the image of infinite strip $0 < y < 1/(2c)$ is given by

$$v < 0 \quad \text{and} \quad u^2 + (v+c)^2 > c^2.$$

The image region lies below the v-axis in the w-plane and this is the exterior of the circle with centre $(0, -c)$ and radius c, as shown in Fig. 9.2.

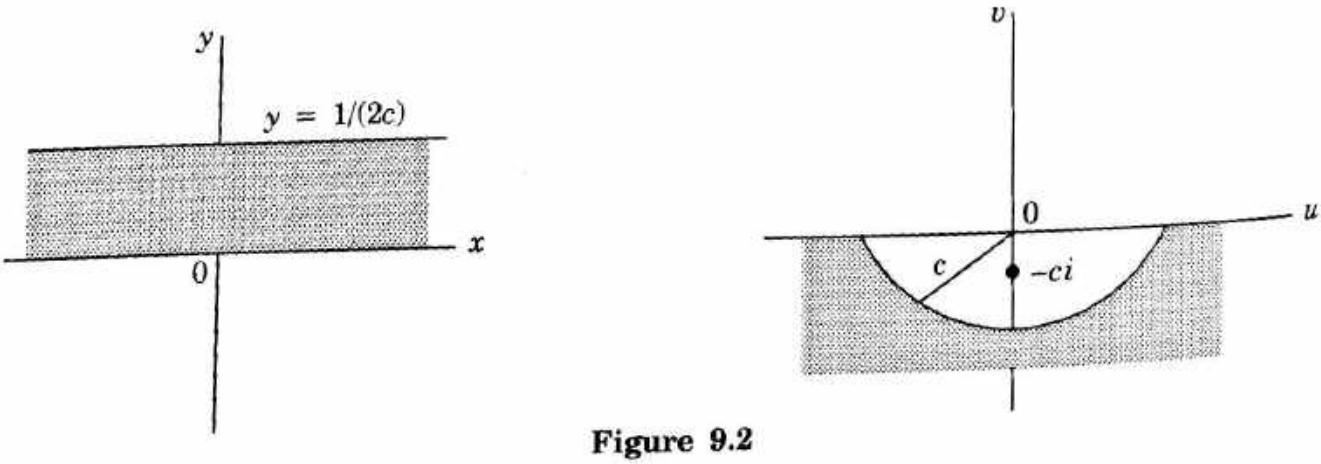

Figure 9.2

The map $w = z^2$

To study this map we write $w = z^2$ as

$$u + iv = (x+iy)^2 = x^2 - y^2 + i2xy,$$

which implies

$$u = x^2 - y^2, \qquad v = 2xy. \tag{9.8}$$

We use relations (9.8) to find the image of certain curves and regions under the map $w = z^2$.

Example 4. Find the image of vertical strip $0 \le x \le c$, $y \ge 0$ under the map $w = z^2$.

Let $x = \alpha$, $0 < \alpha < c$ be any line in the region and let (α, y) be a movable point on this line. Thus, the image of this point is given by the relations

$$u = a^2 - y^2, \qquad v = 2\alpha y, \qquad y \ge 0. \tag{9.9}$$

Eliminating y in Eq. (9.9), we have

$$v^2 = -4\alpha^2(u - \alpha^2).$$

This is the equation of a parabola with the vertex at $(\alpha^2, 0)$ and the focus at the origin. Since v increases as y increases, the point (α, y) moves upwards on the line and the image point moves left on the curve A'. The image of $x = c$ is C (Fig. 9.3).

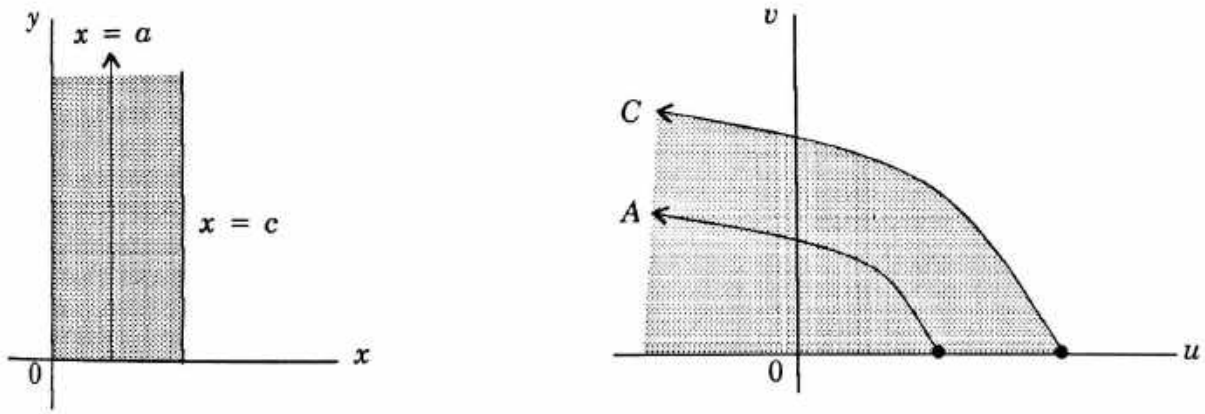

Figure 9.3

Further, the image of the boundary line $x = 0$ $(y \ge 0)$ of the region is found to be $u = -y^2$, $v = 0$. This means that the negative real of the w-plane axis is the image of the positive y-axis in the z-plane. This shows that as $x = \alpha$ moves to the left, parabolic curves try to shrink and in the limiting case the image of $x = 0$ $(y \ge 0)$ reduces to the negative real u-axis. In this way it is observed that the image of all half lines $x = \alpha (y \ge 0)$ fills up the semi-parabolic region as shown in Fig. 9.3.

Remarks. 1. The observations in Example 4 can be extended to verify that the image of the line $y = b$ under the map $w = z^2$ is the parabola

$$v^2 = 4b^2(u + b^2),$$

whose vertex is at $(-b^2, 0)$ and the focus is at the origin. It will be seen that the half-plane $\operatorname{Im} z > 0$ is mapped on the whole w-plane cut along the positive real axis.

2. The two families of parabolas

$$v^2 = -4a^2(u - a^2) \quad \text{and} \quad v^2 = 4b^2(u + b^2)$$

corresponding to two orthogonal families of straight lines

$$x = \pm a \qquad \text{and} \qquad y = \pm b$$

for different values of a and b, form an orthogonal set. The reader can verify that at the intersection points $(a^2 - b^2, 4a^2b^2)$ of parabolas $m_1m_2 = -1$, where m_1 and m_2 are the slopes of tangents for the first and second parabola, respectively, at these points.

Example 5. Show that the image of the closed triangular region formed by the lines $y = \pm x$ and $x = 1$ is the closed parabolic region bounded by the segment $-2 \le v \le 2$ of the v-axis on the right portion of a parabola $v^2 = -4(u - 1)$ under the map $w = z^2$.

From (9.8), we note the following observations:

(i) The line $y = x$, $0 \le y \le 1$ is mapped on $u = 0$, $v = 2y^2$. The point $(1, 1)$ is mapped onto $(0, 2)$;

(ii) The line $y = -x$, $-1 \le y \le 0$ is mapped on $u = 0$, $v = -2y^2$ and the point $(1, -1)$ is mapped onto $(0, -2)$;

(iii) The line $x = 1$ is mapped on $v^2 = -4(u - 1)$, a parabola with vertex at $(1, 0)$ and focus at the origin. The parabola meets the v-axis at $(0, 2)$ and $(0, -2)$. Any point on the line $x = \alpha$, $0 < \alpha < 1$, is mapped on some point interior to the parabola. The mapped region is shown in Fig. 9.4.

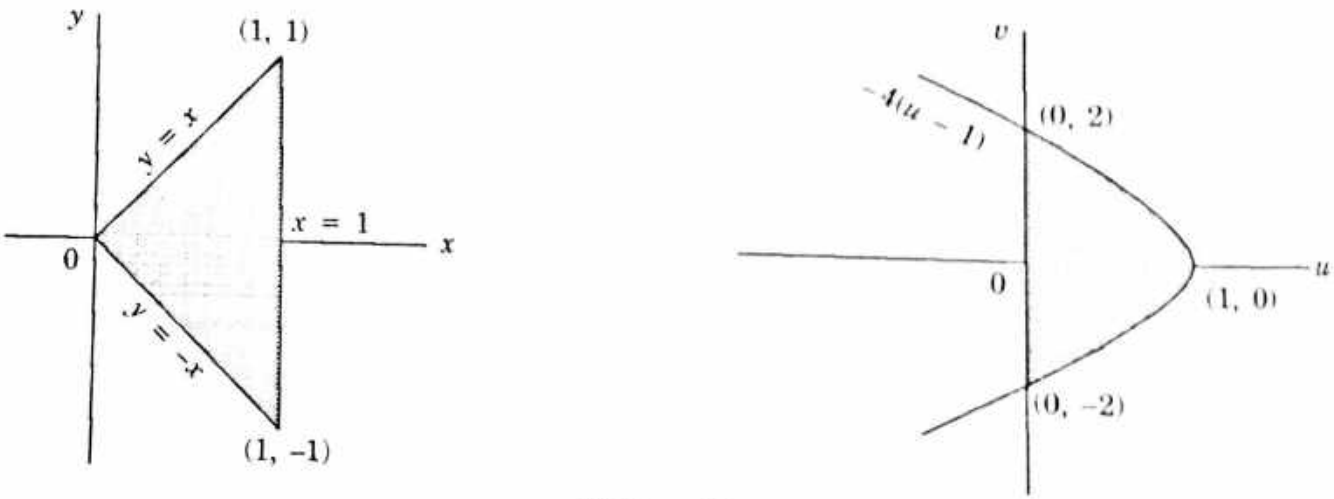

Figure 9.4

Up till now we have described the map $w = z^2$ by using the rectangular coordinates. This system is especially useful when we discuss images of lines, parabolas and hyperbolas in the z-plane. However, when the images of sectors, circles, and quadrants are needed it will be convenient to deal with the polar system. Let $z = re^{i\theta}$ and $w = \rho e^{i\phi}$. Then, $\rho e^{i\phi} = r^2 e^{i2\theta}$ tells us that the image of any nonzero point is found by squaring the modulus of z and doubling the value of arg z, i.e. $|w| = r^2$ and arg $w = 2$ arg z.

The points $z = r_0 e^{i\theta}$ on the circle $|z| = r_0$ are mapped onto the points $w = r_0^2 e^{i2\theta}$ of the circle $|w| = r_0^2$. Also, the first quadrant $r \ge 0$, $0 \le \theta \le \pi/2$, in the z-plane is mapped on the upper half w-plane. The point $w = 0$ is the image of $z = 0$.

Moreover, $w = z^2$ maps the upper half-plane $r \ge 0$, $0 \le \theta \le \pi$ onto the entire w-plane. But in this case mapping is not onto as both positive and negative real axes in the z-plane

are mapped onto the positive real axis in the w-plane. The sector $0 \le \arg z \le \pi/4$ is mapped onto the first quadrant in w-plane.

Example 6. Show that the image of the unit circle $|z| = 1$ under the transformation $w = 2z + z^2$ is a cardioid.

The transformation can be written as $w + 1 = (z + 1)^2$. Let $w + 1 = \rho e^{i\phi}$. Then, $|z| = 1$, i.e. $z = e^{i\theta}$ is transformed into the curve

$$\rho e^{i\phi} = (e^{i\theta} + 1)^2 = [(1 + \cos\theta) + i\sin\theta]^2 = 2(1 + \cos\theta)e^{i\theta}.$$

Hence, $\rho = 2(1 + \cos\theta)$, $\phi = \theta$. Eliminating θ, we note that

$$\rho = 2(1 + \cos\phi)$$

is the equation of the image curve, which is a cardioid. The desired cardioid is obtained by shifting the curve along the real axis in the w-plane by one unit to the left. The image curve is

$$\rho' = 2(1 + \cos\phi) - 1,$$

as shown in Fig. 9.5.

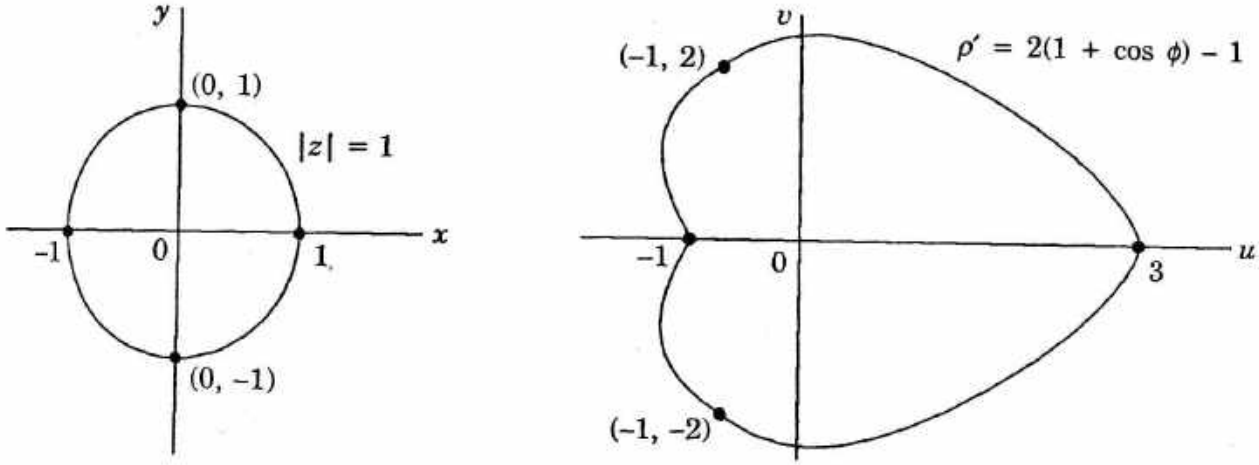

Figure 9.5

Finally, we comment a little about the transformation $w = z^n$, $n > 2$. This map is similar to $w = z^2$. Here,

$$w = z^n, \qquad \rho e^{i\phi} = r^n e^{in\theta}.$$

This transformation maps the region $r \ge 0$, $0 \le \arg z \le \pi/n$ onto the upper-half w-plane.

PROBLEM SET 9.1

1. Find and sketch the image of the region $y > -1$ under the map $w = (1 - i)z + 2i$.
2. Find the image of the semi-infinite strip $x > 0$, $1 < y < 2$, under the map $w = iz + 1$.

3. Construct a linear transformation which (a) carries i onto $-i$ and maps $1+2i$ onto itself; (b) maps $|z-1| = 1$ onto $|w-3i| = 2$.
4. Find the image of the strip $x > 0$, $0 < y < 1$ under the map $w = i/z$.
5. By writing $w = \rho e^{i\phi}$, show that the map $w = i/z$ maps the hyperbola $x^2 - y^2 = 1$ onto the lemniscate $\rho^2 = 2\cos 2\phi$.
6. Given a triangle Δ in the z-plane with vertices i, $1-i$, $1+i$, determine the triangle Δ' into which Δ is mapped under the transformations (a) $w = 3z + 4 - 2i$; (b) $w = iz + 2 - i$. What is the relationship between Δ and Δ' in each case?
7. Let Δ be the triangular region bounded by $x = 1$, $y = 1$ and $x + y = 1$. What is the image of this region under the map $w = z^2$?
8. Prove that the function $w = z^2 - 1$ maps one loop of the lemniscate $r^2 = 2\cos\theta$.
9. Show that the function $w = z^2$ transforms the circle
   (a) $|z-2| = 2$ onto the cardioid $\rho = 8(1+\cos\theta)$ in the w-plane;
   (b) $|z-a| = c$, $a, c \in \mathbb{R}$, onto the limacon in the w-plane. Comment when $a = c$.
   Suggestion. For part (b), use $\rho e^{i\phi} = (a + ce^{i\theta})^2 \Leftrightarrow \rho e^{i\phi} - a^2 + c^2 = ce^{i\theta}(ce^{i\theta} + ce^{-i\theta} + 2a)$. Now, compare the modulus and the argument on both sides.
10. Show that the transformation $w = (z+i)^2$ maps the interior of the circle $|z| = 1$ onto the exterior of the parabola
$$\rho^{-1} = 2(1-\cos\phi), \qquad w = \rho e^{i\phi} \text{ in the } w\text{-plane.}$$
11. Consider the function
$$w = \frac{4}{(z+1)^2}.$$
   Then find the image of the unit circle $|z| = 1$ under this map.
12. Discuss the application of the transformation $w = z^2$ to the area in the first quadrant of the z-plane bounded by the axes and the circles $|z| = a$, $|z| = b$, $a > b > 0$.
13. Show that for the transformation discussed in Example 6, the vertices of an equilateral triangle in the unit circle of the z-plane correspond to the point of contact of parallel tangents of the cardioid.
14. Show that the function $x^2 - y^2 + 2y$ under the transformation $z = w^2$ is harmonic in the w-plane.
15. Determine a transformation which rotates the ellipse $x^2 + xy + y^2 = 2$, clockwise so that it reduces to its principal axes. Also, calculate the length of semi-major and semi-minor axes.
   Suggestion. Assume $w = e^{i\alpha}z$. Find α so that the coefficient of uv in the image curve is zero.

9.2 CONFORMALITY THEOREM

The conformal mappings play an important role in the study of various physical phenomena defined on domains and curves of arbitrary shape. Smaller portions of these domains and curves are conformally mapped by analytic functions to well known domains and curves. Before introducing conformality, it is essential to be acquainted with the angle of rotation of curves under a transformation.

Angle of rotation

Let $\gamma \equiv z(t)$ be a smooth curve given by

$$z(t) = x(t) + iy(t), \qquad a \le t \le b.$$

The curve $z(t)$ is smooth if $x'(t)$ and $y'(t)$ are continuous and $z'(t) = x'(t) + iy'(t) \neq 0 \ \forall$ $t \in (a, b)$. It is to be noted that increasing values of the parameter t will give the positive direction of the curve γ.

Let $f(z)$ be a function defined on γ. Then,

$$w = f(z) = f(z(t)), \qquad a \le t \le b$$

is the parametric representation of the image of γ under the map $w = f(z)$. Suppose γ passes through z_0 (Fig. 9.6). Assume that f is analytic at z_0 and $f'(z_0) \neq 0$. Then, by the chain rule,

$$w'(t_0) = f'(z(t_0))z'(t_0), \qquad a \le t_0 \le b, \quad \text{and } z_0 = z_0(t_0)$$

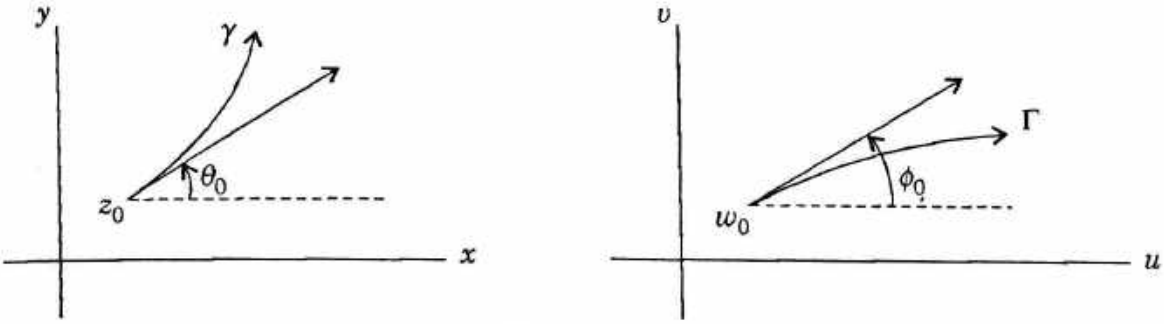

Figure 9.6

Since $f'(z(t_0)) \neq 0$, it follows that

$$\arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(t_0)$$

or

$$\theta_0 = \varphi_0 + \theta_0, \tag{9.10}$$

where θ_0 is the angle of inclination of the directed tangent to γ at z_0 and is a value of $\arg z'(t_0)$. Also, ϕ_0 is the angle of inclination of the directed tangent to Γ (image curve), whereas the rotation of γ is a value of $\arg f'(z_0)$, which we have denoted by φ_0.

Example 7. Verify, by taking some particular curve, the angle of rotation under the transformation $w = z^2$ at the point $2 + i$.

By definition, the angle of rotation for the function $f(z) = z^2$ at $2 + i$ is given by

$$\arg f'(2+i) = \arg(4+2i) = \tan^{-1}\left(\frac{1}{2}\right) + 2n\pi.$$

To verify the angle of rotation for a particular curve we assume a simple curve $y = x - 1$, i.e. $z(x) = x + i(x-1), -\infty < x < \infty$ (C-curve). Clearly, $\theta = \pi/4$ is the angle which curve C makes with the x-axis. The image of $y = x - 1$ under the transformation $w = z^2$ is given by using the relations $u = x^2 - y^2$ and $v = 2xy$ as

$$v = 2\,\frac{u+1}{2}\,\frac{u-1}{2} = \frac{u^2-1}{2} \ (\Gamma\text{-curve}).$$

The image of the point $(2 + i)$ is $(3 + 4i)$ in the w-plane. Now, for the Γ-curve,

$$\phi = \frac{dv}{du} = \tan^{-1}3 \quad \text{at } 3 + 4i.$$

So, the angle of rotation $(\phi - \theta)$ is

$$\tan^{-1}3 - \frac{\pi}{4} = \tan^{-1}3 - \tan^{-1}1 = \tan^{-1}\left(\frac{3-1}{1+3}\right) = \tan^{-1}\frac{1}{2}.$$

This is one of the values of $\tan^{-1}(1/2) + 2n\pi$ and hence the angle of rotation is verified.

Definition 1. A mapping $f(z)$ is said to be *conformal* at a point z_0 if it preserves the angle between the oriented curves passing through z_0 in magnitude as well as in sense.

Geometrically speaking, the images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at z_0, both in magnitude and direction.

A mapping $w = f(z)$ is said to be conformal in a domain D if it is conformal at each point of the domain.

Theorem 1. A mapping $w = f(z)$ is conformal at a point z_0 if it is analytic at z_0 and $f'(z_0) \neq 0$.

Proof. Let C_1 and C_2 be two smooth curves passing through z_0 in the z-plane and let θ_1, θ_2 be the angle of inclinations of the directed tangents to C_1 and C_2, respectively at z_0 (Fig. 9.7). Since $f'(z_0) \neq 0$, (9.10) takes the form

$$\phi_1 = \varphi_0 + \theta_1 \quad \text{for } C_1$$

and

$$\phi_2 = \varphi_0 + \theta_2 \quad \text{for } C_2,$$

where ϕ_1 and ϕ_2 are the angle of inclinations of the directed tangents to image curves Γ_1 and Γ_2, respectively at w_0 (Fig. 9.7). Hence,

$$\phi_2 - \phi_1 = \theta_2 - \theta_1.$$

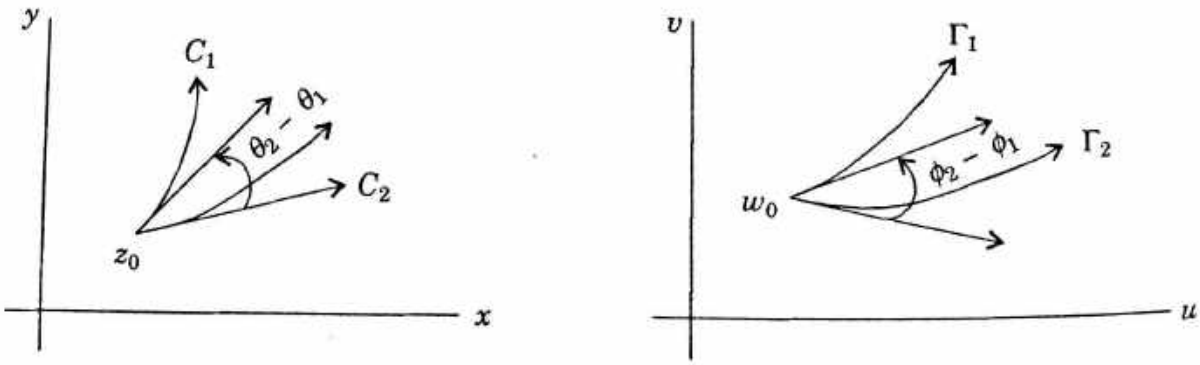

Figure 9.7

This implies that the angle from Γ_1 to Γ_2 is the same as the angle from C_1 to C_2 in magnitude as well as in direction.

In view of Theorem 1, $f(z) = e^z$ is conformal everywhere, since e^z is an entire function and $f'(z) = e^z \neq 0$ for all z.

Also, $w = z^2 - z + 1$ is conformal except $z = 1/2$, since it is analytic everywhere and $f'(1/2) = 0$.

Remarks. 1. If $w = f(z)$ is conformal at z_0 and a function $\xi = g(f(z))$ defined in some neighbourhood of $f(z_0)$ is conformal at this point, then the above theorem can be applied to prove that the composite map gf is conformal at z_0. We therefore conclude that any region Ω in the z-plane is mapped conformally on a region Ω' in the ξ-plane. Detailed analysis is left to the reader.

2. If $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$, then the function is actually conformal in a neighbourhood of z_0. Analyticity at a point implies analyticity in some neighbourhood of z_0 and moreover, by the continuity of $f'(z)$ a neighbourhood can be found in which $f'(z)$ is nonzero.

The *scale factor* of the conformal map $w = f(z)$ at a point $z = z_0$ is given by

$$|f'(z_0)| = \left|\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}\right| = \lim_{z\to z_0}\left|\frac{f(z) - f(z_0)}{z - z_0}\right|.$$

Clearly, $|f'(z)|$ represents an expansion if $|f'(z)| > 1$, and contraction if $|f'(z)| < 1$. Although expressions $\arg f'(z)$ and scale factor $|f'(z)|$ vary point to point, it follows from conformality that these values are approximately $\arg f'(z_0)$ and $|f'(z_0)|$ at all points nearer to z_0. This means that the image of a small region in a neighbourhood of z_0 conforms to the original region in the sense that it has approximately the same shape. A large region may, however, be transformed into a region that bears no resemblance with the original one.

The scale factor of $w = z^2$ at $2 + i$ is

$$|f'(2 + i)| = |4 + 2i| = 2\sqrt{5}.$$

Example 8. Verify conformality at a point $z = 1$ without using the above theorem for the map $w = 1/z$.

We take two simple curves $y = 0$ and $y = x - 1$ in the z-plane and establish the fact that the mapping preserves the angle between these curves and that between the image curves, i.e. both the angles have the same orientation and magnitude.

Clearly, $\theta_2 - \theta_1 = \pi/4$. The image of $y = 0$ and $y = x - 1$ is given by

$$\frac{-v}{u^2 + v^2} = 0 \quad \text{or} \quad v = 0 \qquad (\Gamma_1\text{-curve}),$$

$$\frac{-v}{u^2 + v^2} = \frac{u}{u^2 + v^2} - 1 \quad \text{or} \quad u^2 + v^2 - u - v = 0 \qquad (\Gamma_2\text{-curve}).$$

For the circle $u^2 + v^2 - u - v = 0$, we have

$$\frac{dv}{du} = \frac{1 - 2u}{2v - 1} = \tan^{-1}1 \quad \text{at } w = (1, 0).$$

Thus, the angle between the image curves is $\phi_2 - \phi_1 = \pi/4$. This ensures the conformality at $z = 1$.

Definition 2. A mapping that preserves the magnitude of the angle between two smooth arcs but not the sense is called an *isogonal mapping*.

The map $w = \bar{z}$ is an isogonal map, since it gives the mirror image of curves in the x-axis. Here, the angle between the two curves is preserved in magnitude but the direction of angle between the image curves is reversed.

Definition 3. Let $f(z)$ be a nonconstant analytic function in a domain D. If $f'(z_0) = 0$ for some z_0 in D, then z_0 is called the *critical point* of the transformation.

Clearly, $z = 0$ is the critical point of the map $w = z^2 + 1$.

Example 9. Find the critical points of the mapping $w = e^{2z} - 2iz + 3$.

Since $f(z) = e^{2z} - 2iz + 3$ is nonconstant and analytic $\forall\ z \in \mathbb{C}$, it follows that its critical points are the values of z such that $f'(z) = 0$. Hence

$$2e^{2z} - 2i = 0 \Leftrightarrow e^{2z} = i \Leftrightarrow 2z = i\left(2n\pi + \frac{\pi}{2}\right).$$

The set of critical points: $z = n\pi + \pi/4, \quad n \in \mathbb{I}$.

Remark. Constant map has no critical points and its scale factor is undefined.

Conformal maps exhibit an important property of having local inverse in some neighbourhood of a point, where the mapping is conformal. This is our next result.

Theorem 2. Let $f(z)$ be a conformal at z_0. Then, $f(z)$ has a local inverse at z_0.

Proof. The transformation $w = f(z)$ has a local inverse at z_0 means that there exists a unique transformation $z = g(w)$ defined and analytic in some neighbourhood N of $w_0 = f(z_0)$ such that $g(w_0) = z_0$ and $f(g(w)) = w\ \forall\ w \in N$.

Clearly, $f(g(w)) = w \Rightarrow f'(g(w))g'(w) = 1 \Rightarrow g'(w) = 1/f'(z)$. This shows that g is also conformal at w_0 (since g is analytic and $g'(w_0) \neq 0$ as $f'(z_0) \neq 0$).

Let us verify the existence of such an inverse. The function $f(z)$ is conformal at z_0 if $f(z)$ is analytic at z_0 (or in some neighbourhood of z_0) and $f'(z_0) \neq 0$. Hence the function $f(z) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ have continuous partial derivatives of all orders and the first order derivatives satisfy the C-R equations in some neighbourhood of $z_0 = (x_0, y_0)$.

Now, the pair of equations

$$u = u(x, y), \qquad v = v(x, y)$$

defines a map from this neighbourhood of z_0 to uv-plane. Moreover, the Jacobian of transformation

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x = (u_x)^2 + (v_x)^2 \neq 0$$

at the point (x_0, y_0), since $f'(z_0) \neq 0$.

From a result in real calculus: If $u(x, y)$ and $v(x, y)$ are continuous functions in some neighbourhood of a point (x_0, y_0) and the Jacobian of transformation is nonvanishing, then the local inverse of u and v exists.

Applying this result, we have a unique continuous transformation

$$x = x(u, v), \quad y = y(u, v), \quad \text{if } u_0 = u(x_0, y_0),\ v_0 = v(x_0, y_0)$$

defined in a neighbourhood of (u_0, v_0). Also,

$$x_u = \frac{1}{J} v_y, \quad x_v = -\frac{1}{J} u_y, \quad y_u = -\frac{1}{J} v_x, \quad y_v = \frac{1}{J} u_x.$$

So, for $w = u + iv$, $w_0 = u_0 + iv_0$ and $g(w) = x(u, v) + iy(u, v)$, the transformation $z = g(w)$ is the required local inverse. Since $u + iv = u(x, y) + iv(x, y)$ and $z = x + iy = x(u, v) + iy(u, v) = g(w)$.

Remarks. 1. Proposition 7, Chapter 4 can also be utilized to prove Theroem 2. Actually, for the proof of the theorem we have used the fact that $f'(z_0) \neq 0$, a converse of Theorem 1 (see Problem 10, Problem Set 9.2).

2. The short and simple proof of Theorem 2 is also the direct consequence of Proposition 7, Chapter 7.

What happens to the above theorem when the mapping f is not conformal at z_0, i.e. the condition $f'(z_0) \neq 0$ is dropped?

Proposition 2. Let $f(z)$ be analytic at z_0 such that $f'(z_0) = 0$. If $f'(z)$ has a zero of order $k-1$, $k = 1, 2, \ldots$ at z_0. Then the mapping $w = f(z)$ magnifies the angle at the vertex z_0 by the factor k.

Proof. By the given hypothesis f' has zero of order $k-1$. This means

$$f'(z_0) = f''(z_0) = \cdots = f^{(k-1)}(z_0) = 0.$$

Since $f(z)$ is analytic at z_0, it has the representation

$$f(z) = f(z_0) + a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots$$

So, we may further write

$$\begin{aligned} f(z) - f(z_0) &= (z - z_0)^k\ (a_k + a_{k+1}(z - z_0) + \cdots), \quad a_k \neq 0 \\ &= (z - z_0)^k g(z), \end{aligned} \tag{9.11}$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Consequently, if $w = f(z)$ with $w_0 = f(z_0)$, then using (9.11), we obtain

$$\arg\,(w - w_0) = \arg\,[f(z) - f(z_0)] = k\ \arg\,(z - z_0) + \arg\,g(z). \tag{9.12}$$

Let C be a smooth arc that passes through z_0. If $z \to z_0$ along C, then $w \to w_0$ along the image curve Γ and the angles of inclination of tangents to C and Γ are given by the following limits:

$$\theta_0 = \lim_{z \to z_0} \arg(z - z_0) \quad \text{and} \quad \phi_0 = \lim_{w \to w_0} \arg(w - w_0).$$

From Eqs. (9.11) and (9.12), it follows that

$$\phi_0 = k\theta_0 + \delta, \qquad \delta = \arg\,g(z_0). \tag{9.13}$$

Let C_1 and C_2 be two smooth curves that pass through z_0, and let Γ_1 and Γ_2 be their image curves, respectively. Then, from Eq. (9.13),

$$\phi_2 - \phi_1 = k(\theta_2 - \theta_1).$$

This implies that the angle from Γ_1 to Γ_2 is k-times the angle from C_1 to C_2 in magnitude. However, the orientation is preserved.

PROBLEM SET 9.2

1. Determine the angle of rotation at the point $1-i$ for the transformation $w = z^2$. Illustrate it for some particular curve. Also, find the scale factor of this transformation at this point.
2. Find the images of the lines $x = 0$ and $y = x - 1$ under the transformation $w = i/z$. Sketch all curves, determine the corresponding directions along them and verify the conformality at the point $z = -i$ (without using the conformality theorem).
3. Discuss the scale factor and the angle of rotation for the map $w = z^n$, $n \in \mathbb{I}$, for $z_0 \neq 0$.
4. Determine the critical points of $w = \sin z$.
5. Prove that the transformation $T(z) = az + b$ is conformal at infinity, if $a \neq 0$.
6. Let $w = f(z)$ be a conformal in a domain Ω and $f'(z) \neq 0$. Then show that $f(z)$ is analytic in Ω.
7. Find the local inverse of the transformation $w = z^2$ at the points: (a) $z_0 = 2$, (b) $z_0 = -2$, and (c) $z_0 = -i$.
8. Show that if $z = g(w)$ is the local inverse of a conformal transformation $w = f(z)$, then $g'(w) = 1/f'(z)$ in some neighbourhood N_δ, $\delta > 0$ where g is analytic.
9. Let γ be a smooth arc in a domain D throughout which $w = f(z)$ is conformal and let Γ denote the image of γ under the transformation. Then show that Γ is smooth.
10. Let $f(z)$ be a conformal map at z_0. Then prove that $f'(z_0) \neq 0$.
11. Suppose $f(z)$ is an entire function. If $f(z)$ is univalent in $\mathbb{C}$, show that it is conformal in the entire plane and will be of the form $a_0 + a_1 z$, $a_1 \neq 0$.

9.3 BILINEAR TRANSFORMATION

The map

$$w = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0, \tag{9.14}$$

is called *bilinear* transformation or *linear fractional* transformation or sometimes *Möbius* transformation. This can also be written as

$$Azw + Bz + Cw + D = 0, \qquad AD - BC \neq 0 \tag{9.15}$$

and conversely. Since (9.15) is linear in z as well as in w, (9.14) deserves to be termed bilinear transformation.

If $ad - bc = 0$, then (9.14) reduces to a constant as

$$w = \frac{a(z + b/a)}{c(z + d/c)} = \frac{a}{c} \quad \text{as} \quad ad - bc = 0 \Leftrightarrow \frac{b}{a} = \frac{d}{c}.$$

A constant function is not linear and hence $ad - bc \neq 0$ is the necessary condition for (9.14) to be a bilinear transformation. Also, note that the behaviour of $T(z)$ is not affected when a, b, c and d are multiplied by a constant and hence the expression for $T(z)$ may be normalized by taking $ad - bc = 1$. Suppose $ad - bc = \alpha$, then multiplication by $\sqrt{\alpha}$ of a, b, c, d will do the job. This is obvious from the fact that every bilinear transformation has two normal representations, obtained from each other by changing the signs of the coefficients.

Example 10. Show that the bilinear transformation

$$T(z) = \frac{az + b}{cz + d}, \qquad ad - bc = 1$$

is conformal except at one point and find its scale factor at any point z.

If $c = 0$, then the bilinear transformation reduces to the entire linear transformation

$$T(z) = \alpha z + \beta, \qquad \alpha = \frac{a}{d}, \quad \beta = \frac{b}{d}.$$

For $\alpha \neq 0$, $T'(z)$ is a nonzero constant; so $T(z)$ is conformal in the whole complex plane. The scale factor is $|\alpha|$. If $\alpha = 1$, then $\arg \alpha = 2n\pi$, $n \in \mathbb{I}$. It implies that both rotation and expansion produce no effect and in this case the transformation reduces to $z + \beta$. This obviously displaces the whole plane by the vector β. On the other hand, if $\alpha \neq 1$ the transformation can be written as

$$T(z) - z_0 = \alpha(z - z_0),$$

where z_0 is determined by the equation: $z_0 = \alpha z_0 + \beta$.

This is equivalent to a rotation of the whole plane by $\arg \alpha$ about $z_0 = \beta/(1 - \alpha)$ together with a uniform magnification by the factor α relative to the point z_0.

Next, suppose that $c \neq 0$. In this situation,

$$T'(z) = \frac{ad - bc}{(cz + d)^2} = \frac{1}{c^2} \frac{1}{(z - \delta)^2}$$

exists if $z \neq \delta$, where $\delta = -d/c$. Hence the mapping is conformal at all finite points except possibly at $z = \delta$. The tangents to curves through any point $z \neq \delta$ are rotated through an angle equal to

$$\arg T'(z) = \arg \frac{1}{|c|^2} - 2 \arg (z - \delta),$$

with the magnification at z that equals

$$|T'(z)| = \frac{1}{|c|^2} \frac{1}{|z - \delta|^2}.$$

The angle through which tangents are rotated has the same value for all points with equal $\arg (z - \delta)$, i.e. along any ray drawn from δ, otherwise, it varies from point to point. In

general, the magnification varies with z, but it has the same value for all points with equal values of $|z-z_0|$, a circle with centre δ. In particular, the magnification is 1 at every point of the circle with the equation

$$|z-z_0| = \frac{1}{|c|}.$$

Proposition 3. Every bilinear transformation maps circles and lines into circles and lines (a line is a circle with infinite radius).

Proof. Let $T(z)$ be any bilinear transformation given by

$$T(z) = \frac{az+b}{cz+d}, \qquad ad-bc \neq 0$$

If $c = 0$, then

$$T(z) = \frac{a}{d}z + \frac{b}{d} = Az + B, \qquad A = \frac{a}{d}, \quad B = \frac{b}{d}.$$

Clearly, $Az + B$, being linear, maps circles and lines into circles and lines.

If $c \neq 0$, then

$$\begin{aligned} T(z) &= \frac{a(z+d/c)}{c(z+d/c)} + \frac{b}{cz+d} - \frac{ad}{c(cz+d)} \\ &= \frac{a}{c} + \frac{bc-ad}{c(cz+d)} \\ &= \frac{a}{c} + \frac{bc-ad}{c^2}\,\frac{1}{z+d/c}. \end{aligned}$$

Assigning

$$z_1 = z + \frac{d}{c}, \qquad z_2 = \frac{1}{z_1}, \qquad z_3 = \frac{bc-ad}{c^2}\,z_2$$

we get

$$T(z) = \frac{a}{c} + z_3.$$

The above transformations are of the form

$$w_1 = z + \alpha, \qquad w_2 = \frac{1}{z}, \qquad w_3 = \beta z.$$

This establishes the fact that every bilinear transformation is the resultant of bilinear transformations with simple geometric imports.

Thus a bilinear transformation maps circles and lines into circles.

Example 11. Show that the line $3y = x$ is mapped onto the circle under the bilinear transformation

$$w = \frac{iz+2}{4z+i}.$$

Find the centre and radius of the image circle.

To find the image of $3y = x$, we need values of x and y in terms of u and v, and hence solve $w = (iz + 2)/(4z + i)$ for z,

$$4zw + iw = iz + 2 \Rightarrow (4w - i)z = 2 - iw \Rightarrow z = \frac{2 - iw}{4w - i}.$$

Writing $z = x + iy$ and $w = u + iv$, we have

$$\begin{aligned} x + iy &= \frac{v + 2 - iu}{4u + i(4v - 1)} \\ &= \frac{(v + 2 - iu)[4u - i(4v - 1)]}{16u^2 + (4v - 1)^2} \\ &= \frac{9u - i(4u^2 + 4v^2 + 7v - 2)}{16u^2 + (4v - 1)^2}. \end{aligned}$$

On comparing, we get

$$x = \frac{9u}{16u^2 + (4v - 1)^2}, \qquad y = -\frac{4u^2 + 4v^2 + 7v - 2}{16u^2 + (4v - 1)^2}.$$

Inserting these values in the equation $3y = x$, we find

$$\frac{9u}{16u^2 + (4v - 1)^2} = -\frac{3(4u^2 + 4v^2 + 7v - 2)}{16u^2 + (4v - 1)^2} \Rightarrow u^2 + v^2 + \frac{3}{4}u + \frac{7}{4}v - \frac{1}{2} = 0.$$

This is the equation of the circle with centre $(-3/8, -7/8)$ and radius $(3/4)\sqrt{5/2}$.

We add more information by claiming that $T(z) = (az + b)/(cz + d)$ is a one-one and onto mapping from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. For this, we define

$$T(z) = \begin{cases} \infty, & z = \infty, c = 0, \\ a/c, & z = \infty, c \neq 0, \\ \infty, & z = -d/c, c \neq 0, \\ T(z), & \text{otherwise.} \end{cases}$$

This definition makes the function $T(z)$ continuous on the extended plane.

Further, we discuss two important properties of the transformation $T(z)$.

(a) The inverse of a bilinear transformation is also bilinear.

(b) The composition of two bilinear transformations is also bilinear.

Since $T(z)$ is one-one and onto mapping from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ it follows that T^{-1} exists and is given by

$$w = T(z) \Leftrightarrow z = T^{-1}(w).$$

Solving $w = (az + b)/(cz + d)$ for z, we get

$$z = \frac{-dw + b}{cw - a}, \qquad ad - bc \neq 0$$

or

$$T^{-1}(w) = \frac{-dw+b}{cw-a}.$$

Again, for $T^{-1}(w)$, $ad-bc \neq 0$, hence T^{-1} is bilinear.

To verify (b) we assume two bilinear transformations $T(z)$ and $S(z)$ defined by

$$T(z) = \frac{az+b}{cz+d}, \qquad ad-bc \neq 0,$$

$$S(z) = \frac{pz+q}{rz+s}, \qquad ps-qr \neq 0.$$

Consider the composition

$$\begin{aligned}(ST)(z) = S(T(z)) &= S\left(\frac{az+b)}{cz+d}\right) \\ &= \frac{p[(az+b)/(cz+d)]+q}{r[(az+b)/(cz+d)]+s} \\ &= \frac{paz+bp+qcz+qd}{raz+rb+scz+sd} \\ &= \frac{(pa+qc)z+bp+dq}{(ar+cs)z+br+ds}.\end{aligned}$$

Now,

$$\begin{aligned}(pa+qc)(br+ds)-(bp+dq)(ar+cs) &= (ad-bc)ps-(ad-bc)rq \\ &= (ad-bc)(ps-qr) \neq 0.\end{aligned}$$

Hence ST is a bilinear transformation. Similarly, it can be shown that TS is also a bilinear transformation. Thus, the composition of two linear fractional transformations in either way is a bilinear transformation.

Example 12. Show that a bilinear transformation maps inverse points with respect to a circle or line onto inverse points with respect to the image circle and the image line.

The equation of a circle with p and q as inverse points can be written as (see Example 6 in Chapter 1),

$$\left|\frac{z-p}{z-q}\right| = k, \qquad k \neq 0 \tag{9.16}$$

The image of this circle under the bilinear transformation

$$w = \frac{az+b}{cz+d}$$

is obtained by inserting

$$z = \frac{dw-b}{-cw+a}$$

into Eq. (9.16) to have

$$\left|\frac{\dfrac{dw-b}{-cw+a}-p}{\dfrac{dw-b}{-cw+a}-q}\right| = k$$

or

$$\left|\frac{w(d+cp)-(ap+b)}{w(d+cq)-(aq+b)}\right| = k.$$

This is of the form

$$\left|\frac{w-p'}{w-q'}\right| = \left|\frac{d+cq}{d+cp}\right| k,$$

where

$$p' = \frac{ap+b}{cp+d} = \left.\frac{az+b}{cz+d}\right|_{z=p}, \qquad q' = \frac{aq+b}{cq+d} = \left.\frac{az+b}{cz+d}\right|_{z=q}.$$

Thus,

$$\left|\frac{w-p'}{w-q'}\right| = k', \quad k' = \left|\frac{d+cq}{d+cp}\right| k. \tag{9.17}$$

Note that p' and q' are the inverse points of the image circle, and in addition, these points are the images of p and q, respectively.

For $k = 1$, (9.17) represents a line which is the perpendicular bisector to the line joining p and q. Obviously, in this situation $k' = 1$.

In dealing with Example 12, we have also proved that a bilinear transformation maps circles and lines into circles and lines, an alternative proof of Proposition 3 of this section.

Let us consider the bilinear transformations

$$T_1(z) = \frac{z}{z+1}; \qquad T_2(z) = \frac{z+2}{z+3}.$$

We observe

$$T_1T_2(z) = T_1(T_2(z)) = T_1\left(\frac{z+2}{z+3}\right) = \frac{z+2}{2z+5},$$

$$T_2T_1(z) = T_2(T_1(z)) = T_2\left(\frac{z}{z+1}\right) = \frac{3z+2}{4z+3}.$$

Thus,

$$T_1T_2(z) \neq T_2T_1(z).$$

This indicates that bilinear maps do not follow the commutative property. Assuming identity transformation $I(z) = z$ as a particular case of bilinear transformation, it may be claimed that the set of all bilinear maps form a noncommutative group with respect to composition of maps (refer to Problem 18, Problem Set 9.3).

Definition 4. A bilinear transformation $T(z)$ has a fixed point (invariant point) z_0 if $T(z_0) = z_0$.

Proposition 4. Every bilinear transformation (except the identity map) has at most two fixed points.

Proof. If $T(z)$ has a fixed point z, then $T(z) = z$ or

$$\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + dz = az + b \Leftrightarrow cz^2 - (a-d)z - b = 0.$$

The last equation is quadratic in z and hence can have at most two roots.

For the identity map, $I(z) = z$ for all points in the domain of definition. Hence every point of the domain is a fixed point.

This completes the proof.

Proposition 5. If a bilinear transformation $w = T(z)$ has exactly two fixed points z_1 and z_2, then for some nonzero constant k they satisfy the equation

$$\frac{w-z_1}{w-z_2} = k\frac{z-z_1}{z-z_2}. \tag{9.18}$$

Moreover, if $T(z)$ has only one fixed point z_1, then it can be written as

$$\frac{1}{w-z_1} = k' + \frac{1}{z-z_1}, \qquad k' \neq 0. \tag{9.19}$$

Proof. Let z_1 and z_2 be the given fixed points of the bilinear transformation $w = (az+b)/(cz+d)$, and hence these are the roots of the equation $cz^2 - (a-d)z - b = 0$. This means

$$cz_1^2 - (a-d)z_1 - b = 0 \Leftrightarrow cz_1^2 - az_1 = b - dz_1; \tag{9.20}$$

$$cz_2^2 - (a-d)z_2 - b = 0 \Leftrightarrow cz_2^2 - az_2 = b - dz_2. \tag{9.21}$$

Utilizing (9.20), note that

$$\begin{aligned} w - z_1 &= \frac{az+b}{cz+d} - z_1 \\ &= \frac{az+b-z_1(cz+d)}{cz+d} \\ &= \frac{(a-z_1c)z+b-dz_1}{cz+d} \\ &= \frac{(a-z_1c)z+cz_1^2-az_1}{cz+d} \\ &= \frac{(a-cz_1)(z-z_1)}{cz+d}. \end{aligned}$$

Similarly, looking at (9.21),

$$w - z_2 = \frac{(a-cz_2)(z-z_2)}{cz+d}.$$

Hence,

$$\frac{w-z_1}{w-z_2} = \frac{a-cz_1}{a-cz_2}\cdot\frac{z-z_1}{z-z_2} = k\frac{z-z_1}{z-z_2},$$

where $k = (a-cz_1)/(a-cz_2)$.

For the second part, z_1 is the only fixed point. Then, the equation $cz^2-(a-d)z-b=0$ has only one root z_1, say. So, $cz_1^2-(a-d)z_1-b=0 \Leftrightarrow cz_1^2-az_1=b-dz_1$, and z_1 (being the repeated root) is given by

$$z_1 = \frac{a-d}{2c} \Leftrightarrow a-cz_1 = d+cz_1. \tag{9.22}$$

From the previous analysis,

$$\frac{1}{w-z_1} = \frac{cz+d}{(a-cz_1)(z-z_1)}.$$

which on simplification (adding and subtracting cz_1 in numerator and using (9.22)) gives

$$\frac{1}{w-z_1} = k' + \frac{1}{z-z_1}, \quad k' = \frac{c}{a-cz_1} = \frac{2c}{a+d}.$$

Remarks. 1. Equations (9.18) and (9.19) are known as the *normal* form or *canonical* form of a bilinear transformation.

2. The bilinear transformation with two fixed points such that $|k| = 1$, $k \neq 1$ is said to be elliptic, while for $k > 0$ it is termed hyperbolic. The bilinear transformations with one fixed point are classified as parabolic.

3. Once z_1 and z_2 are known for (9.18) or (9.19), the fact that $z = -d/c$ maps $\omega = \infty$ facilitates the computation of k or k'.

A bilinear transformation that is neither hyperbolic, elliptic, nor parabolic is called *loxodromic*, i.e. it has two fixed points and satisfies the condition $k = ae^{i\alpha}$, $\alpha \neq 0$, $a \neq 1$.

Example 13. Find all the bilinear transformations which have fixed points as -1 and 1.

From (9.13) and Definition 4, we have

$$T(1) = 1 \Rightarrow a+b = c+d \Rightarrow a+b-c-d = 0,$$
$$T(-1) = -1 \Rightarrow -a+b = -(-c+d) \Rightarrow -a+b-c+d = 0.$$

On solving these equations, we get $a = d$, $b = c$. Hence,

$$T(z) = \frac{az+b}{bz+a}, \qquad a^2-b^2 \neq 0.$$

Note. All bilinear transformations which have fixed points as -1 and 1 may also be directly computed using Proposition 5.

Cross ratio. The next step is to develop the specific bilinear transformation which maps three distinct points in the extended z-plane onto three distinct points in the extended w-plane. For this purpose we introduce the concept of cross ratio.

Definition 5. For three distinct complex numbers z_1, z_2, z_3 in $\mathbb{C}_\infty$, the cross ratio of four points z, z_1, z_2, z_3 is defined to be

$$(z, z_1, z_2, z_3) = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}. \tag{9.23}$$

If one of the numbers in (9.23) is replaced by infinity, say z_3, then

$$(z, z_1, z_2, \infty) = \lim_{z_3\to\infty} \frac{(z-z_1)(z_2/z_3-1)}{(z_1-z_2)(1-z/z_3)} = -\frac{z-z_1}{z_1-z_2}.$$

This means that the factors involving z_3 are replaced by -1. To recollect cross ratio (9.23), we write differences of z, z_1, z_2, z_3 in cyclic order $z-z_1, z_1-z_2, z_2-z_3, z_3-z$ and then put them in the numerator and the denominator alternately.

Remarks. 1. The four points z_1, z_2, z_3, z_4 can be ordered (permuted) in $4! = 24$ ways. Thus, 24 cross products are possible. But the reader is advised to verify that only six of these are different.

2. The cross ratio (z, ∞, z_2, z_3) is real $\Leftrightarrow z, z_1 = \infty, z_2, z_3$ are on line with ∞. For this assertion, we observe that

$$\begin{aligned}(z, \infty, z_2, z_3) \text{ is real} &\Leftrightarrow -\frac{z_2-z_3}{z_3-z} = \frac{1}{t}, \quad \text{for some real } t\\ &\Leftrightarrow t(z_2-z_3) = z-z_3\\ &\Leftrightarrow (1-t)z_3 + tz_2 = z.\end{aligned}$$

Thus, z lies on the line passing through z_2 and z_3.

Theorem 3. The cross ratio is invariant under bilinear transformation.

Proof. Let the bilinear transformation be defined by

$$w = T(z) = \frac{az+b}{cz+d}, \qquad ad-bc = 1$$

such that $w_k = T(z_k)$, $k = 1, 2, 3$, then we have to show that

$$(T(z), T(z_1), T(z_2), T(z_3)) = (z, z_1, z_2, z_3). \tag{9.24}$$

Since z_k corresponds to w_k, assuming $ad-bc = 1$, we have

$$w - w_k = \frac{z-z_k}{(cz_k+d)(cz+d)}$$

so that for any pair of (z, w),

$$w-w_1 = \frac{z-z_1}{(cz_1+d)(cz+d)}, \qquad w-w_3 = \frac{z-z_3}{(cz_3+d)(cz+d)}. \tag{9.25}$$

Replacing w by w_2, and z by z_2,

$$w_2-w_1 = \frac{z_2-z_1}{(cz_1+d)(cz_2+d)}, \qquad w_2-w_3 = \frac{z_2-z_3}{(cz_3+d)(cz_2+d)}. \tag{9.26}$$

Equations (9.25) and (9.26) yield,

$$\frac{w-w_1}{w_2-w_3}\,\frac{w_1-w_2}{w_3-w} = \frac{z-z_1}{z_2-z_3}\,\frac{z_1-z_2}{z_3-z}.$$

and this is nothing but

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3).$$

Thus, (9.24) is established.

As an application of this theorem we prove a very interesting result.

Theorem 4. There exists a unique bilinear transformation which maps three given distinct points z_1, z_2 and z_3 in the extended z-plane onto three specified distinct points w_1, w_2 and w_3 in the extended w-plane, respectively.

Proof. Consider the expression

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}. \tag{9.27}$$

This transformation has been obtained in Theorem 3 under the condition that z_k is mapped onto w_k, $k = 1, 2, 3$. However, we give an alternative approach to exhibit the same phenomenon.

Writing (9.27) in the form

$$(w-w_1)(w_2-w_3)(z_1-z_2)(z_3-z) = (w_1-w_2)(w_3-w)(z-z_1)(z_2-z_3), \tag{9.28}$$

we have, for $z = z_1 \Rightarrow w = w_1$ and $z = z_3 \Rightarrow w = w_3$. In the case of $z = z_2$, (9.28) yields

$$(w-w_1)(w_2-w_3) = (w_3-w)(w_1-w_2).$$

The above equation has a unique solution $w = w_2$. Thus, z_1, z_2 and z_3 are mapped onto w_1, w_2 and w_3, respectively. Moreover, (9.28) is a bilinear transformation, since

$$F(z) = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(z_2-z_3)z - z_1(z_2-z_3)}{(z_1-z_2)z + z_3(z_1-z_2)},$$

and since z_1, z_2, z_3 are distinct,

$$z_3(z_1-z_2)(z_2-z_3) - z_1(z_1-z_2)(z_2-z_3) = (z_1-z_2)(z_2-z_3)(z_3-z_1) \neq 0.$$

Hence $F(z)$ is a bilinear transformation.

By similar reasoning,

$$G(w) = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

is a bilinear transformation.

Now, we can write (9.27) as

$$G(w) = F(z) \quad \text{or} \quad w = G^{-1}F(z).$$

The map $G(w)$ is a bilinear transformation and this implies that G^{-1} (inverse map) is bilinear transformation. Again, $G^{-1}F$ is a bilinear transformation (being a composition of two bilinear transformations). Thus, (9.27) which is of the form $w = G^{-1}F(z)$ is a bilinear transformation.

Uniqueness. Let T and S be two bilinear transformations which map z_1, z_2 and z_3 onto w_1, w_2 and w_3, respectively.

$$T(z_1) = w_1, \quad T(z_2) = w_2, \quad T(z_3) = w_3,$$
$$S(z_1) = w_1, \quad S(z_2) = w_2, \quad S(z_3) = w_3.$$

Consider the map $S^{-1}T$. We know that $S^{-1}T$ is a bilinear transformation (being composition of bilinear maps) and

$$(S^{-1}T)(z_1) = S^{-1}(T(z_1)) = S^{-1}(w_1) = z_1;$$

$$(S^{-1}T)(z_2) = S^{-1}(T(z_2)) = S^{-1}(w_2) = z_2;$$
$$(S^{-1}T)(z_3) = S^{-1}(T(z_3)) = S^{-1}(w_3) = z_3$$

which implies that the bilinear map $S^{-1}T$ has three distinct fixed points. But except I, no bilinear transformation can have more than two fixed points. So $S^{-1}T = I$. Similarly, it can be shown that $T^{-1}S = I$ and hence $S^{-1}T = I = T^{-1}S \Rightarrow S = T$, i.e. T is unique.

Example 14. Find the bilinear transformation which (a) maps the points $-1, 0, 1$ onto points $0, i, 3i$; (b) maps $i, 1, -1$ onto $1, 0, \infty$, respectively.

(a) Substituting these points in (9.27), we have

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

which on simplification yields

$$w = -\frac{3i(z+1)}{z-3}.$$

(b) Since $w_3 = \infty$, we take the quotient involving w_3 in (9.27) to be -1. Thus,

$$-\frac{w-1}{1-0} = \frac{(z-i)(1+1)}{(i-1)(-1-z)},$$

and after simplifying,

$$w = i\,\frac{1-z}{1+z}.$$

is the required bilinear transformation.

Example 15. Find a bilinear transformation that maps the crescent-shaped region that lies inside the circle $|z-2| < 2$ and outside the circle $|z-1| > 1$ onto a horizontal strip in the upper half-plane.

For convenience, choose the points $z_1 = 4$, $z_2 = 2 + 2i$, $z_3 = 0$ on the circle $|z-2| = 2$ in the xy-plane, and the image values $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$ in the uv-plane, respectively. The ordered triplet z_1, z_2, z_3 gives the positive orientation as the region of the disc lies to the left. The image points w_1, w_2, w_3 all lie on the extended u-axis, and the region to be mapped should remain left to the observer and it happens to be the upper half-plane $\operatorname{Im} w > 0$. Hence we can use the implicit formula (9.27) to write

$$-\frac{w-0}{0-1} = \frac{(z-4)(2+2i-0)}{(4-2-2i)(0-z)}.$$

After simplification, this yields

$$w = \frac{-iz+4i}{z}.$$

A straightway substitution shows that $z_4 = 1-i$, $z_5 = 2$ and $z_6 = 1+i$ are mapped by the above bilinear transformation on the points $w_4 = -2+i$, $w_5 = i$ and $w_6 = 2+i$. The points w_4, w_5 and w_6 lie on the horizontal strip $v = 1$ in the upper half-plane. As the

region to be mapped lies to the right of z_4, z_5 and z_6, we conclude that the image region is to the right of line Im $w = 1$. Both the regions are shown in Fig. 9.8.

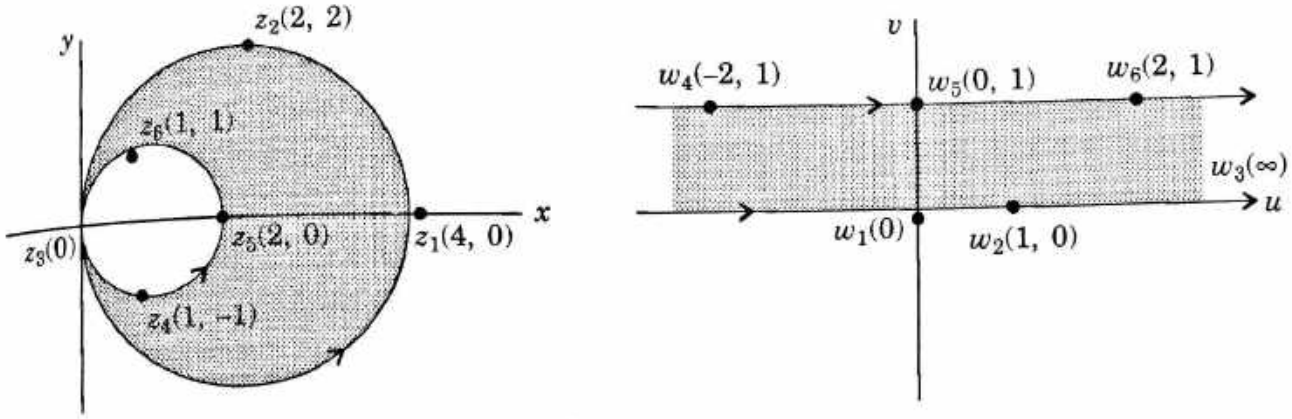

Figure 9.8

PROBLEM SET 9.3

1. Find the image of the line $y = x + 1$ under the transformation $w = (z + i)/(z - 1)$.
2. Find the image of the region Re $z \leq 0$ under the transformation $w = (2z + i)/(iz - 1)$.
3. Show that $w = (z - 1)/(z + 1)$ maps $x > 0$ and the boundary $x = 0$ on $|w| \leq 1$.
4. Let $w = i(1 - z)/(1 + z)$. Then show that Im $w > 0$ if Im $z < 1$.
5. Consider a point z inside a circle C centred at origin and of radius r and another circle C_1 centred at z_0 and of radius R containing the inverse point to z with respect to C. Prove that two successive inversions of the point z about C and C_1, respectively, can be expressed as a bilinear transformation.
6. Determine all bilinear transformations which have fixed points as $-i$ and i.
7. Find the bilinear transformation with fixed points -1 and 1 carrying the points i and $-i$ onto $-i$ and i, respectively.
8. Express the bilinear transformation

$$w = \frac{13iz + 75}{3z - 5i}$$

in the form

$$\frac{w - \alpha}{w - \beta} = k\frac{z - \alpha}{z - \beta},$$

where α, β and k are constants.

Suggestion. Find α and β as fixed points of the bilinear transformation and then evaluate k (see Proposition 5).

9. If the points z_1, z_2, z_3 lie on a line, show that the cross ratio (z, z_1, z_2, z_3) is real $\Leftrightarrow$ either $z = \infty$ or z lies on the line passing through z_1, z_2, z_3.

10. The four points z, z_1, z_2, z_3 in $\mathbb{C}_\infty$ lie on a line or a circle, i.e. are cocylic if and only if their cross ratio (z_2, z, z_3, z_1) is a real number.

Suggestion. Graph a circle with four points $A(z)$, $B(z_1)$, $C(z_2)$, $D(z_3)$ in cyclic order and use the geometrical result: These points are cocylic $\Leftrightarrow \angle ADB = \angle ACB$.

11. Let z_1, z_2, z_3, z_4 be four points in the complex plane such that $z_1z_2 + z_3z_4 = 0$ and $z_1 + z_2 = 0$. Then prove that these points are cocylic.

12. Construct a bilinear transformation which maps (a) $\{\infty, i, 0\}$ onto $\{0, 1, \infty\}$, (b) $\{-1, 0, 1\}$ onto $\{-i, 1, i\}$, (c) $\{0, 1, \infty\}$ onto $\{w_1, w_2, w_3\}$. Show that these maps are unique and further, find the image of $y = c$, $c > 0$ under the transformation (a).

13. Find the bilinear transformation which maps $1, i, -1$ on $i, 0, -i$, respectively. For this transformation, find the images of (a) $|z| \le 1$; (b) $|z| = r$, $r > 1$.

14. Find the condition that $w = (az + b)/(cz + d)$ maps the unit circle in the w-plane into a straight line in the z-plane.

15. With the help of Eq. (9.27), prove that if a bilinear transformation maps each point on the x-axis onto the u-axis, then the coefficients in the transformation are all real, except possibly for a common complex factor.

Suggestion. Map real numbers x_1, x_2, x_3 onto $-1, 0, \infty$.

16. Show that the set of all bilinear transformations forms a non-abelian group with respect to operation as composition of maps.

17. Let $T(z) = (az + b)/(cz + d)$, $ad - bc \neq 0$ be a bilinear transformation other than identity transformations. Show that $T^{-1} = T$ if and only if $d = -bc$.

Suggestion. Write $T^{-1}(z) = T(z)$ and note that $(a + d)[cz^2 + (d - a)z - b] = 0$ justifies the assertion.

18. Show that (9.14) can be written in the form (9.27).

Suggestion. Compute $w = G^{-1}(F(z))$, where F and G are the functions defined in Theorem 4, Section 9.3.

19. For $c > 0$, show that the transformation

$$w = \left(\frac{z - ic}{z + ic}\right)^2$$

maps the right-half of the circle $|z| = c$ into the upper-half of the w-plane. Note that the exterior of the circle $|z| = c$ in the half-plane $\operatorname{Re} x \le 0$ is also mapped into the upper-half of the w-plane.

20. What is the form of bilinear transformation which has α and ∞ as its fixed points? Conclude this transformation if ∞ is the only fixed point.

21. Show that $\int (1/y)\,ds$ is an inverse with respect to bilinear map (9.14), where $ds = \sqrt{(x^2 + y^2)}$.

22. Suppose that a bilinear transformation $T(z)$ has two fixed points z_1 and z_2. Prove that

$$T'(z_1)T'(z_2) = 0.$$

Suggestion. Use (9.18).

9.4 SPECIAL BILINEAR TRANSFORMATIONS

In this section we develop general bilinear transformations which map the specific region in the z-plane onto a given region in the w-plane.

Proposition 6. The set of all bilinear transformations that map the upper half-plane Im $z > 0$ onto $|w| < 1$ and the boundary Im $z = 0$ onto the boundary $|w| = 1$ is given by

$$w = e^{i\alpha}\frac{z - z_0}{z - \bar{z}_0}, \quad \text{Im } z_0 > 0, \quad \alpha \in \mathbb{R}, \tag{9.29}$$

where the point z_0 in the upper half-plane is mapped onto the centre of the unit disc.

Proof. Let the bilinear transformation be

$$w = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

Note that the points z and $\bar{z}$ are symmetric (inverse) with respect to the real axis and must correspond to w and $1/\bar{w}$ inverse with respect to the unit circle (see Example 12).

In particular, we observe that $w = 0$ and $w = \infty$ are the images of the points $z = -b/a$ and $z = -d/c$, respectively, and these, being inverse to each other, can be expressed as

$$-\frac{b}{a} = z_0 \qquad \text{and} \qquad -\frac{d}{c} = \bar{z}_0.$$

for some complex constant z_0.

The constant $c \neq 0$, otherwise the points at ∞ will not be mapped on the boundary $|w| = 1$. So, (9.14) can be expressed as

$$w = \frac{a}{c}\frac{z + b/a}{z + d/c}. \tag{9.30}$$

With this information, the bilinear map (9.30) takes the form

$$w = \frac{a}{c}\frac{z - z_0}{z - \bar{z}_0}.$$

Now, $z = 0$ on the boundary corresponds to the point on the unit circle $|w| = 1$. So,

$$1 = |w| = \left|\frac{a}{c}\right|\left|\frac{-z_0}{-\bar{z}_0}\right| = \left|\frac{a}{c}\right| \Rightarrow \frac{a}{c} = e^{i\alpha}, \quad \alpha \in \mathbb{R}.$$

Further, the transformation assumes the form

$$w = e^{i\alpha}\frac{z - z_0}{z - \bar{z}_0}, \quad \alpha \in \mathbb{R}. \tag{9.31}$$

From (9.30), the point $z = z_0$ corresponds to $w = 0$, and hence z_0 must be in the upper half-plane, i.e. Im $z_0 > 0$. This information is added to have (9.29).

Thus, (9.29) gives the set of all bilinear transformations which map Im $z \geq 0$ onto $|w| \leq 1$ (Fig. 9.9).

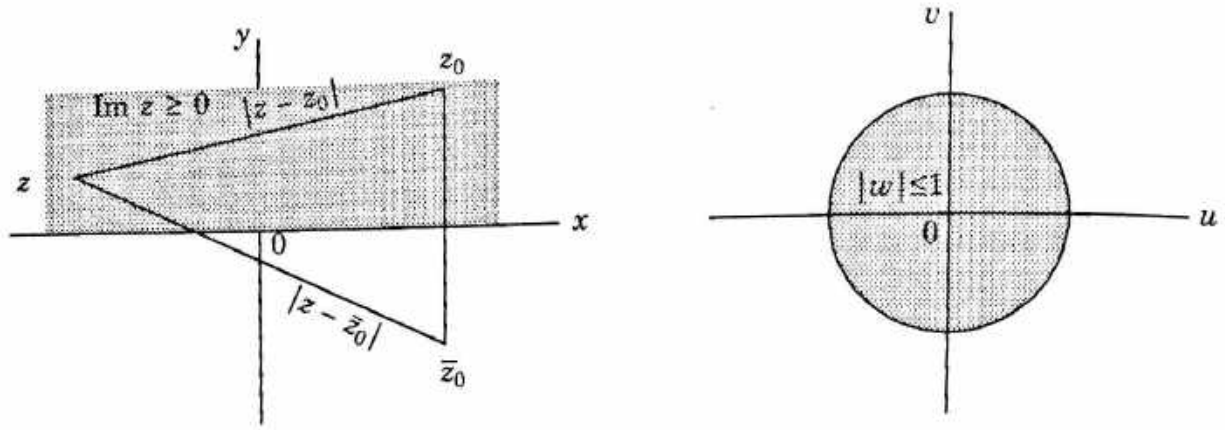

Figure 9.9

Verification. To verify that every transformation of the form represented by (9.29) has the desired property. Taking absolute value of each side in (9.29), observe that

$$|w| = \frac{|z - z_0|}{|z - \bar{z}_0|}.$$

If any point z lies above the real axis, then z and z_0 lie on the same side of the real axis (which is a perpendicular bisector of the line joining z_0 and $\bar{z}_0$). This implies that $|z - z_0| < |z - \bar{z}_0|$ and hence $|w| < 1$. For any z on the real axis $|w| = 1$, as $|z - z_0| = |z - \bar{z}_0|$. Since any bilinear transformation is a one-to-one map on $\mathbb{C}_\infty$, this establishes the fact that transformation (9.29) maps the half-plane Im $z > 0$ onto the disc $|w| < 1$ and the boundary of the half-plane onto the boundary of this unit disc.

Remark. What happens when Im $z_0 > 0$ in (9.29) is replaced by Im $z_0 < 0$?

$$\begin{aligned}
|w| < 1 &\Leftrightarrow |z - z_0| < |z - \bar{z}_0| \\
&\Leftrightarrow |z|^2 + |z_0|^2 - z\bar{z}_0 - \bar{z}z_0 < |z|^2 + |z_0|^2 - z\bar{z}_0 - zz_0 \\
&\Leftrightarrow \text{Re } z\bar{z}_0 > \text{Re } z_0 z \\
&\Leftrightarrow 2y_0 y > 0 \\
&\Leftrightarrow y < 0, \quad \text{since} \quad y_0 < 0.
\end{aligned}$$

Hence the lower half-plane is mapped on $|w| < 1$ and the boundary $y = \text{Im}\, z = 0$ is mapped on $|w| = 1$ such that $z = z_0$ is mapped onto the centre of the disc $|w| \leq 1$.

Example 16. Find the image of Im $z \geq 0$ under the transformation

$$w = \frac{i - z}{i + z}.$$

Let us write the transformation in the form

$$w = e^{i\pi}\,\frac{z-i}{z+i}.$$

Comparing with (9.29), we note that $\alpha = \pi$ and $\operatorname{Im} z_0 = 1 > 0$. Hence this bilinear transformation maps $y \geq 0$ onto $|w| \leq 1$.

Example 17. Find all the bilinear transformations which map the right half-plane $\operatorname{Re} z > 0$ onto the open disc $|w| < 1$ and the boundary $\operatorname{Re} z = 0$ onto the boundary $|w| = 1$.

The right half-plane, $\operatorname{Re} z > 0$ and $\operatorname{Re} z = 0$ are rotated by $\pi/2$ to get the upper half-plane with boundary as $\operatorname{Im} z = 0$. This transformation is

$$f(z) = iz, \quad \operatorname{Re} z \geq 0.$$

The bilinear transformations which map $\operatorname{Im} z \geq 0$ onto $|w| \leq 1$ are given by

$$g(z) = e^{i\alpha}\,\frac{z-z_0}{z-\bar{z}_0}, \qquad \operatorname{Im} z_0 > 0, \quad \alpha \in \mathbb{R}.$$

Hence, the required transformations $g(f(z))$ are of the form

$$g(f(z)) = g(iz) = e^{i\alpha}\,\frac{iz-z_0}{iz-\bar{z}_0}, \quad \operatorname{Im} z_0 > 0, \qquad \alpha \in \mathbb{R}.$$

Proposition 7. The set of all bilinear transformations which map unit disc onto itself is given by

$$w = e^{i\alpha}\frac{z-z_0}{1-\bar{z}_0 z}, \quad |z_0| < 1, \quad a \in \mathbb{R}. \tag{9.32}$$

Such that any point z_0 interior to unit disc is mapped onto the centre of another disc.

Proof. Clearly, $c \neq 0$, otherwise the points at infinity in the two planes would correspond. So, (9.14) can be written in the form of (9.30). We have seen in Example 12 that a bilinear transformation maps a circle and a line of the z-plane into a circle and a line of the w-plane and inverse points are transformed into inverse points.

Now, $w = 0$ and $w = \infty$ are the inverse points of $|w| = 1$ and these are the images of $z = -b/a$ and $-d/c$, respectively (refer to Example 12). Hence, we may write (see Example 6, Chapter 1.1),

$$\frac{-b}{a} = z_0; \quad \frac{-d}{c} = \frac{1}{\bar{z}_0}, \qquad |z_0| < 1.$$

So, (9.30) can be further written as

$$w = \frac{a}{c}\,\frac{z-z_0}{z-1/\bar{z}_0} \quad \text{or} \quad w = -\frac{a\bar{z}_0}{c}\,\frac{z-z_0}{1-\bar{z}_0 z}. \tag{9.33}$$

The point $z = 1$ on the boundary of $|z| = 1$ must correspond to a point on the boundary of $|w| = 1$. Thus, in view of (9.33) and the fact that $|1-z_0| = |1-\bar{z}_0|$, we have

$$1 = |w| = \left|\frac{-a\bar{z}_0}{c}\right|\left|\frac{1-z_0}{\bar{z}_0-1}\right| = \left|\frac{a\bar{z}_0}{c}\right| \quad \text{or} \quad -\frac{a\bar{z}_0}{c} = e^{i\alpha},\ \alpha \in \mathbb{R}.$$

Hence (9.33) becomes (9.32), and this is what we required.

Verification: We show that $|z| \le 1$ is mapped onto $|w| \le 1$. For any z on the unit circle $z = e^{i\theta}$, we observe that

$$w = |e^{i\alpha}|\left|\frac{e^{i\theta}-z_0}{1-\bar{z}_0 e^{i\theta}}\right| = \left|\frac{e^{i\theta}-z_0}{e^{-i\theta}-\bar{z}_0}\right| = 1.$$

Hence $|z| < 1$ is mapped onto one of the complimentary domains of $|w| = 1$. Because $|z_0| < 1$ is mapped onto $w = 0$, this domain is $|w| < 1$.

PROBLEM SET 9.4

1. Let $w = (z+i)/(iz+1)$. Then show that $\operatorname{Im} z \le 0 \Rightarrow |w| \le 1$.

2. Find all transformations which map the sector $0 < \arg z < \pi/3$ on $|w| < 1$.

Suggestion. Sector $0 < \arg z < \pi/3$ is mapped by $g(z) = z^3$ on $y > 0$. Now, $f(g(z))$, where $f(z)$ is given by (9.29), is the required one. Note that $g(f(z))$ is not bilinear.

3. Find all transformations which map the fourth quadrant onto $|w| \le 1$.

Suggestion. The function $h(z) = iz$ maps $x \ge 0$, $y \le 0$ onto $x \ge 0$, $y \ge 0$. Also, $g(z) = z^2$ maps $x \ge 0$, $y \ge 0$ on $y \ge 0$. The region $y \ge 0$ is mapped on $|w| \le 1$ by $f(z)$ in (9.29). Hence the required map: $(fgh)(z)$.

4. Construct the general bilinear transformation which maps the upper half-plane (a) onto itself; (b) onto lower half-plane.

5. Find a bilinear transformation which maps the upper half-plane of the z-plane onto the unit disc in the w-plane in such a way that $z = i$ is mapped on $w = 0$ while the point at infinity is mapped onto $w = -1$.

6. Show that the general bilinear transformation that maps the disc $|z| \le r$ onto the disc $|w| \le r'$ is given by

$$w = rr'e^{i\alpha}\frac{z-\beta}{r^2-\bar{\beta}z}, \qquad |\beta| < r,\ \alpha \in \mathbb{R}.$$

Suggestion. The set of all bilinear transformations which maps $|z| \le 1$ onto $|w| \le 1$ is given by (9.32). To map $|z| \le r$ onto $|w| \le r'$, replace z by z/r and w by w/r' in (9.32) and set $\beta = rz_0$.

7. Show that the region $|z - z_0| \le R$ is mapped conformally onto $|w| \le 1$ by

$$w = e^{i\alpha}\frac{R(z-\beta)}{R^2-(z-z_0)(\bar{\beta}-\bar{z}_0)}, \quad |\beta - z_0| < R,$$

where α is real and β is the point which is transformed onto origin.

Suggestion. Consider (9.32) in the form $w = e^{i\alpha}(z-p)/(1-\bar{p}z)$, $|p| < 1$, $\alpha \in \mathbb{R}$. Now, replace z by $(z-z_0)/R$ and set $z_0 + pR = \beta$.

8. Determine the bilinear transformation which has 1 and -1 as its fixed points and maps $|z| \le 1$ onto $|w| \le 1$.

9. Find an analytic function $w = f(z)$ which maps the region $2 \le y \le x + 1$ in the z-plane onto the disc $|w| \le 1$.

   Suggestion. Take $f_1(z) = z - (1 + 2i)$, $f_2(z) = z^4$, $f_3(z) = e^{i\alpha}(z - z_0)/(z - \bar{z}_0)$, $\operatorname{Im} z_0 > 0$, $\alpha \in \mathbb{R}$. Consider the composition $f_3 f_2 f_1$.

10. Map the upper half-plane $\operatorname{Im} z > 0$ onto the unit disc $|w| < 1$ in such a way that $w(2i = 0)$, $\operatorname{Arg} w'(2i) = 0$.

11. Map the disc $|z - 4i| < 2$ onto the half-plane $v > u$ in such a way that the centre of the disc goes into the point -4, while the point $2i$ on the circumference of the disc goes into the origin.

12. Find the transformation which maps the domain $|z| < 1$, $0 < \arg z < \pi/3$ conformally on $|w| < 1$.

13. Suppose an analytic function $f(z)$ maps a unit disc onto itself. How many fixed points does $f(z)$ have? Locate these points if a nonzero point α interior to one unit disc is mapped onto the origin of the other disc.

14. Show that the transformation

$$w = \left(\frac{z+i}{z-i} + 1\right)^2$$

maps the real axis in the z-plane onto a cardioid in the w-plane. Indicate the region of z-plane which corresponds to the interior of the cardioid.

Suggestion. This transformation can be thought of as the composition of the following maps:

$$\xi = \frac{z - (-i)}{z - (-\bar{i})}, \qquad \zeta = \xi + 1, \qquad w = \zeta^2.$$

15. Find a bilinear transformation which maps $x + y = 1$ onto $|w - 1| = 2$. Is it unique? If not, find the other one. Also carry out the analysis so that the lower half-plane in the z-plane is mapped onto the disc $|w - 1| < 2$.

In Sections 9.1–9.4 we dealt with elementary functions and bilinear maps. To make the chapter self contained, the discussion will be concluded with reasonable exposition of the mappings by some well-known complex functions.

9.5 EXPONENTIAL TRANSFORMATION

The map $w = e^z$ can be written as $\rho e^{i\phi} = e^x e^{iy}$. Thus,

$$\rho = e^x, \qquad \phi = y + 2k\pi, \qquad k \in \mathbb{I};$$

and hence the above transformation from z to w can be expressed as $\rho = e^x$, $\phi = y$. Note that this transformation is conformal for every point of the z-plane, since the derivative of e^z does not vanish anywhere.

The image of a typical point (c_1, y) on any vertical line $x = c_1$ has the polar coordinates $\rho = e^{c_1}$ and $\phi = y$. The point moves counterclockwise around the circle as z moves upward. Thus, the image of the line is the entire circle and every point on the circle is the image of infinite number of points on the line which are 2π spaced. Similarly, a horizontal line $y = c_2$ is mapped in a one-to-one manner on the ray $\phi = c_2$. As the point moves on the line from left to right, the image point (e^x, c_2) moves on the ray $\phi = c_2$ away from the origin.

Thus, the vertical and horizontal line segments are mapped onto portion of circles and rays, respectively. Since e^z is $2\pi i$ periodic, it maps every horizontal strip

$$n2\pi \le y \le (n+1)2\pi$$

onto a region of the w-plane with slit in the same manner. This means that the points z, which differ from one another by $2\pi i$, are therefore mapped onto the same point (Fig. 9.10).

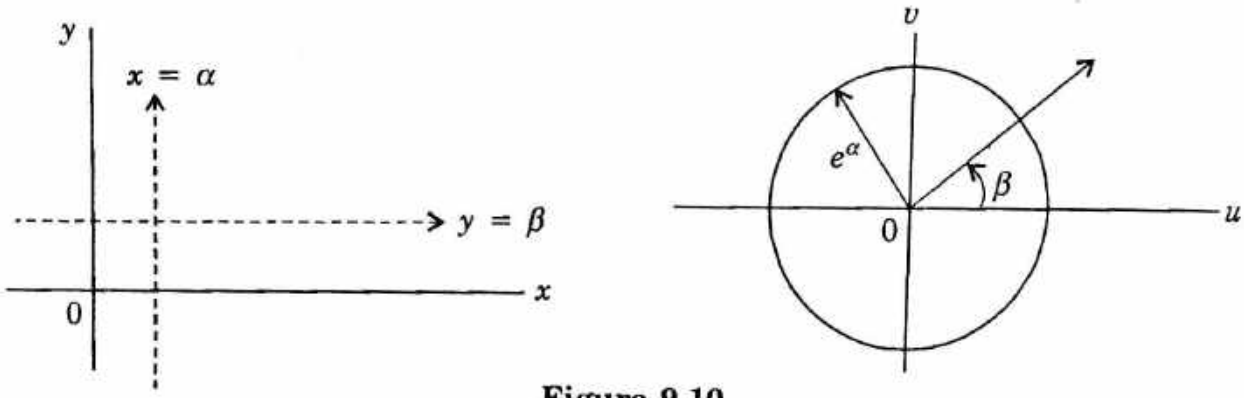

Figure 9.10

Example 18. Find the image of the line $z = (1 + ia)t + ib$, $-\infty < t < \infty$ under the transformation $w = e^z$, given that the line $z = (1 + ia)t + ib$ is not parallel to one of the coordinate axes.

Here, a and b are fixed real numbers and $a \neq 0$ is the slope of the line and b is the y-intercept. This can be verified by eliminating t from the parametric equation $x = t$ and $y = at + b$.

The image of this line under $w = e^z$ is the curve

$$w = e^{t+i(at+b)} = e^t[\cos(at+b) + i\sin(at+b)].$$

Therefore,

$$|w| = \rho = e^t, \qquad \phi = \arg w = at + b + 2k\pi,$$

and eliminating the parameter t, it follows that

$$\rho = e^{(\phi-b-2k\pi)/a}.$$

If we set $\theta = \phi - 2k\pi$, then $\rho = ce^{\theta/a}$, where $c = e^{-b/a}$. This is the equation of logarithmic spiral in the polar form. Since, the mapping is conformal and the line intersects all the lines parallel to the x-axis at the angle $\tan^{-1}a$, it follows that the spiral intersects the image of all these lines at the same angle, i.e. all the rays emanating from the origin are intersected at an angle $\tan^{-1} a$ by logarithmic spiral (Fig. 9.11).

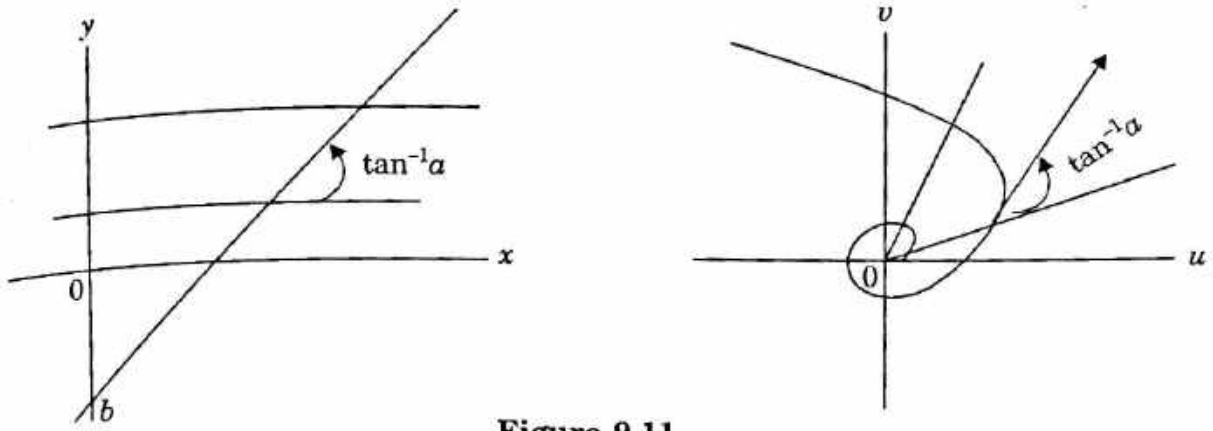

Figure 9.11

Example 19. Find the image of the rectangular region $a \le x \le b$, $c \le y \le d$ under the map $w = e^z$.

The vertical line segment $AD: x = a$, $c \le y \le d$ is mapped on the arc $\rho = e^a$, $c \le \phi \le d$ which is labelled as $A'D'$. Similarly, the vertical line segment BC: $x = b$, $c \le y \le d$, is mapped on the larger arc $\rho = e^b$, $c \le \phi \le d$ which is labelled as $B'C'$. The horizontal line segments AB and DC are mapped on the rays $A'B'$ and $D'C'$, respectively. The mapping is one-to-one if $d - c < 2\pi$. Figure 9.12 shows the desired region.

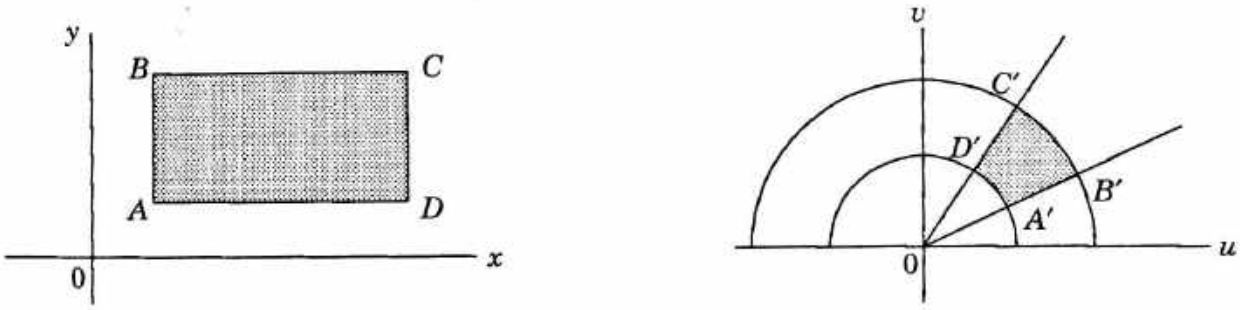

Figure 9.12

Example 20. Show that the transformation

$$w = f(z) = \frac{e^z - i}{e^z + i}$$

is one-to-one conformal mapping of the horizontal strip $0 < y < \pi$ onto the disc $|w| < 1$. Furthermore, x-axis is mapped onto the lower semi-circle bounding the disc, and the line $y = \pi$ is mapped onto the upper semi-circle.

The function $w = f(z)$ is the composition of the exponential function

$$Z = e^z$$

followed by the bilinear transformation

$$w = \frac{Z - i}{Z + i}.$$

The image of the horizontal strip $0 < y < \pi$ under the mapping $Z = e^z$ is the upper half-plane $\text{Im } Z > 0$; the x-axis is mapped onto the positive X-axis; and the line $y = \pi$ is mapped onto the negative X-axis. The bilinear transformation $w = (Z-i)/(Z+i)$ maps the upper half-plane $\text{Im } Z > 0$ on the disc $|w| < 1$, (refer to Proposition 6); the positive X-axis is mapped onto the lower semi-circle and the negative X-axis is mapped onto the upper semi-circle. Figure 9.13 illustrates the composite mapping.

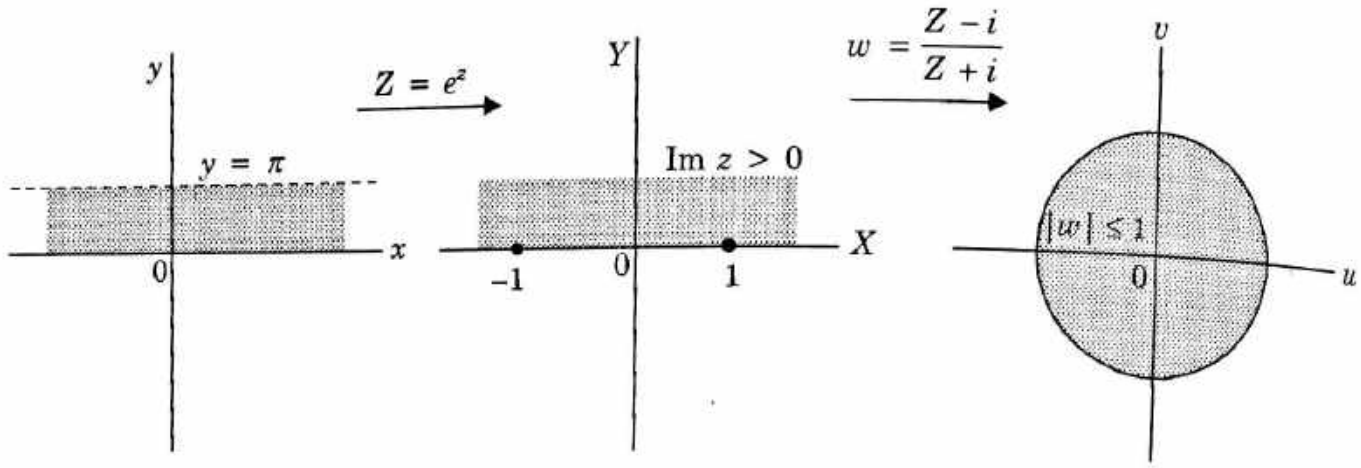

Figure 9.13

9.6 TRIGONOMETRIC TRANSFORMATION

Let us consider the map $w = \sin z$. Writing $w = u + iv$ and $z = x + iy$, we get

$$u + iv = \sin x \cosh y + i \cos x \sinh y.$$

Hence

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y. \tag{9.34}$$

Since the map $w = \sin z$ is a periodic function, this mapping is not one-to-one in the entire complex plane.

Eliminating x from (9.34), we have

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1. \tag{9.35}$$

Thus, the line $y = c_2$ is mapped on confocal ellipses with foci $w = \pm 1$.

However, the elimination of y from the same set of equations yields

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1. \tag{9.36}$$

In this case, the line $x = c_1$ is mapped on confocal hyperbolas.

The real axis $y = 0$ corresponds to $v = 0\ (-1 \le u \le 1)$ and the imaginary axis $x = 0$ corresponds to $u = 0$.

Remarks. 1. The function

$$w = \cos z = \sin\left(z + \frac{\pi}{2}\right)$$

defines a translation of $\sin z$ map to the right through $\pi/2$ units.

2. Further, in view of the relation $\sinh z = -i \sin iz$ (see Section 3.4), note that

$$w = \sinh z = -i \sin iz$$

is the combination of the following mappings:

$$Z = iz, \qquad W = \sin Z, \qquad w = -iW.$$

Hence, any figure in the z-plane is rotated by $\pi/2$ and then we find its image under the map $\sin z$. Finally, rotation of this image by $-\pi/2$ gives the desired image under $\sinh z$.

The transformation

$$w = \cosh z$$

is, likewise, essentially a cosine transformation, since $\cosh z = \cos iz$.

In view of the above remarks our concentration will remain focussed on developing the mapping properties of the function $\sin z$.

Example 21. Find the image of the domain $-\pi/2 \le x \le \pi/2$ and $-c \le y \le c$ under the transformation $w = \sin z$.

The segment of the real axis $y = 0$ $(-\pi/2 \le x \le \pi/2)$ is mapped on the segment of the real axis $v = 0$ $(-1 \le u \le 1)$ in the w-plane. Since $f'(z) = \cos z$ is zero at $z = \pm\pi/2$, the mapping is not conformal at these two points.

From (9.34), we see that the line $x = -\pi/2$ maps onto $v = 0$ $(u \le -1)$ and the line $x = \pi/2$ maps onto $v = 0$ $(u \ge 1)$.

The line segment $y = c > 0$, $-\pi/2 \le x \le \pi/2$, maps onto the portion of the ellipse (9.35) in the upper half-plane and the line segment $y = -c$, $-\pi/2 \le x \le \pi/2$, maps onto the portion of the same ellipse but in the lower half-plane. The foci of these ellipses $w = \pm 1$ are independent of c. Consequently, if we let c vary, we obtain a family of confocal ellipses.

The rectangular region defined by $-\pi/2 < x < \pi/2$ and $-c < y < c$ is mapped onto the interior of the ellipse (9.35). Note that the image of the boundary consists of the ellipse and the two segments of the u-axis as shown in Fig. 9.14 for $c = 1$. The image points of the points on the vertical parts of the boundary coincide in pairs.

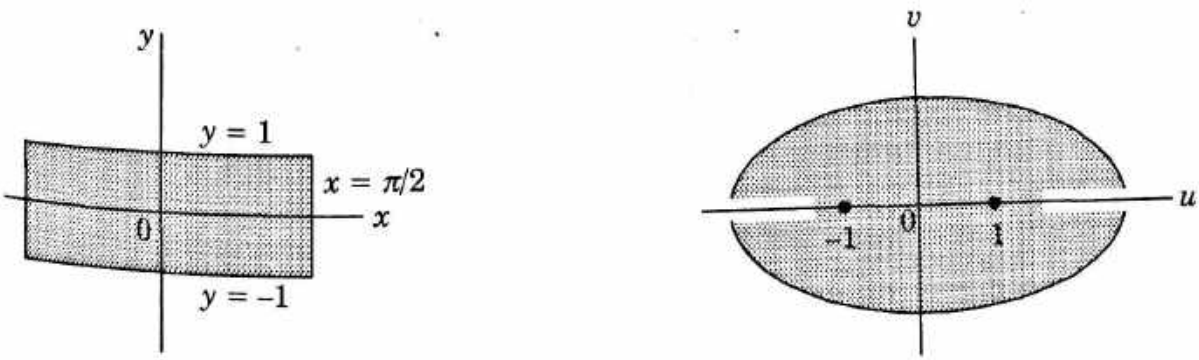

Figure 9.14

Example 22. Find the image of the $-\pi < x < \pi$ and $c < y < d$, $c > 0$ under the transformation $w = \sin z$.

Following the reasoning of the above example the rectangle $-\pi < x < \pi$, $c < y < d$, maps onto the elliptic ring cut along the negative v-axis (Fig. 9.15). The straight lines x = constant map onto confocal hyperbolas (9.36) which intersect these ellipses at right angles. The line segment of y-axis $x = 0$ $(c < y < d)$ maps onto the line segment of v-axis $u = 0$ $(\sinh c < v < \sinh d)$.

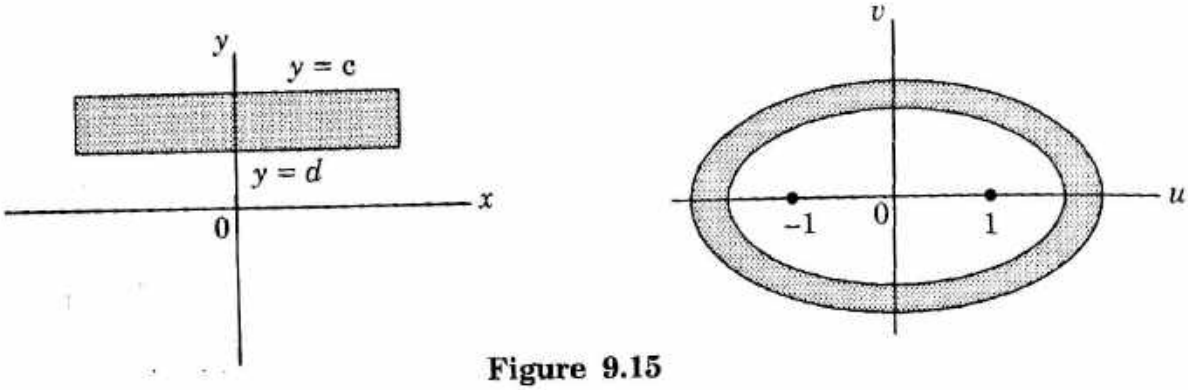

Figure 9.15

Example 23. Show that the transformation $w = \tan z$ is a one-to-one conformal mapping of the vertical open strip $|x| < \pi/4$ onto the unit disc $|w| < 1$.

We write

$$w = \tan z = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{-ie^{2iz} + i}{e^{2iz} + 1}.$$

From the above relation the function $w = \tan z$ can be considered as the composition of the map

$$Z = e^{i2z}$$

followed by the bilinear transformation

$$w = \frac{-iZ + i}{Z + 1}.$$

Now, the map $Z = \exp(i2z)$ maps the open vertical strip $|x| < \pi/4$ one-to-one and onto the right half-plane $\text{Re}\, Z > 0$.

Then the bilinear transformation $w = (-iZ + i)/(Z + 1)$ maps the right half-plane $\text{Re}\, Z > 0$ one-to-one and onto the disc $|w| < 1$ as shown in Fig. 9.16.

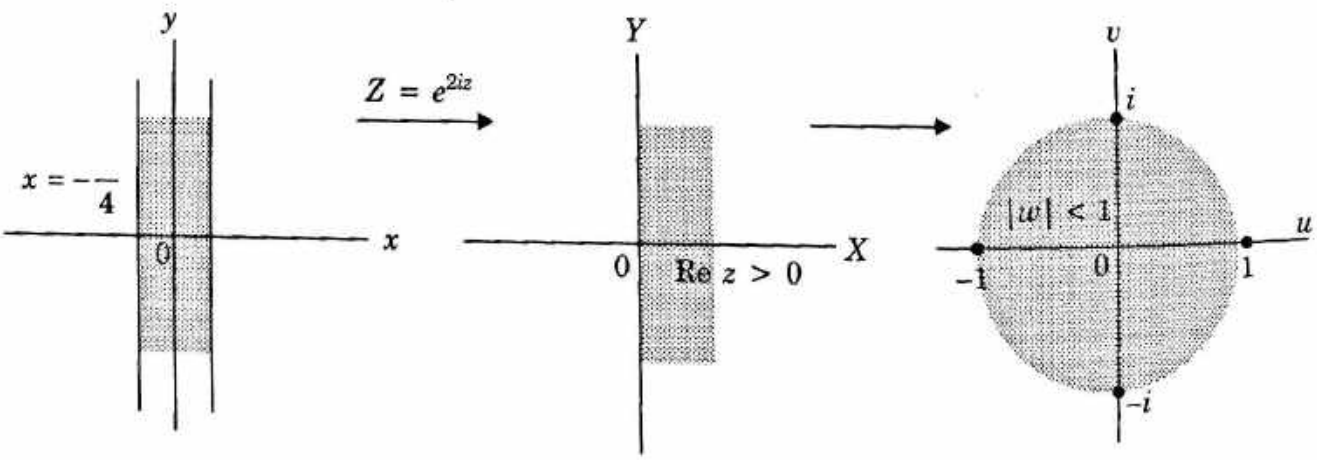

Figure 9.16

9.7 THE MAP $(z + 1/z)$

The transformation

$$w = z + \frac{1}{z}, \qquad (z \neq 0),$$

is known as Joukowski's transformation. This is used to map the exterior of the profile of an airplane wing on the exterior of a nearly circular region. Since

$$f'(z) = 1 - \frac{1}{z^2} = \frac{(z+1)(z-1)}{z^2},$$

the mapping is conformal except at the points $z = 0, \pm 1$. The points $z = 1$ and $z = -1$ correspond to $w = 2$ and $w = -2$, respectively. Solving the transformation, we have

$$z = \frac{1}{2}\{w + [(w+2)(w-2)]^{1/2}\}.$$

Hence z is a double-valued function of w. To every value of w other than ± 2, there corresponds two values of z. The mapping can be made one-one if we construct a two sheeted Riemann surface connecting the branch points $w = -2$ and $w = 2$ crosswise. On setting $w = u + iv$ and $z = re^{i\theta}$, we get

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}} = \left(r + \frac{1}{r}\right)\cos\theta + \left(r - \frac{1}{r}\right)\sin\theta.$$

Hence

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \qquad v = \left(r - \frac{1}{r}\right)\sin\theta. \tag{9.37}$$

Eliminating θ, we obtain

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1, \qquad \text{where } a = r + \frac{1}{r}, \qquad b = \left|r - \frac{1}{r}\right|. \tag{9.38}$$

Equation (9.37) reveals that the unit circle $|z| = 1$ is mapped onto the cut from $w = -2$ to $w = 2$ and this line segment is described twice as θ varies from 0 to 2π.

The circles $|z| = r$, $r \neq 1$ in the z-plane are mapped on ellipses (9.38) in the w-plane, as shown in Fig. 9.17. The principal axes of ellipses lie along the u and v axes in the w-plane and have lengths $2a$ and $2b$, respectively. These ellipses are confocal with foci $(\pm 2, 0)$, since $a^2 - b^2 = 4$ is independent of r. It is to be noted that two circles $|z| = r$ and the $|z| = 1/r$ are mapped on the same ellipse in the w-plane corresponding to the two sheets of the Riemann surface.

Hence we conclude that the region $r < |z| < r'$ is mapped conformally into the region by the two confocal ellipses.

Eliminating r from (9.37), we get

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4. \tag{9.39}$$

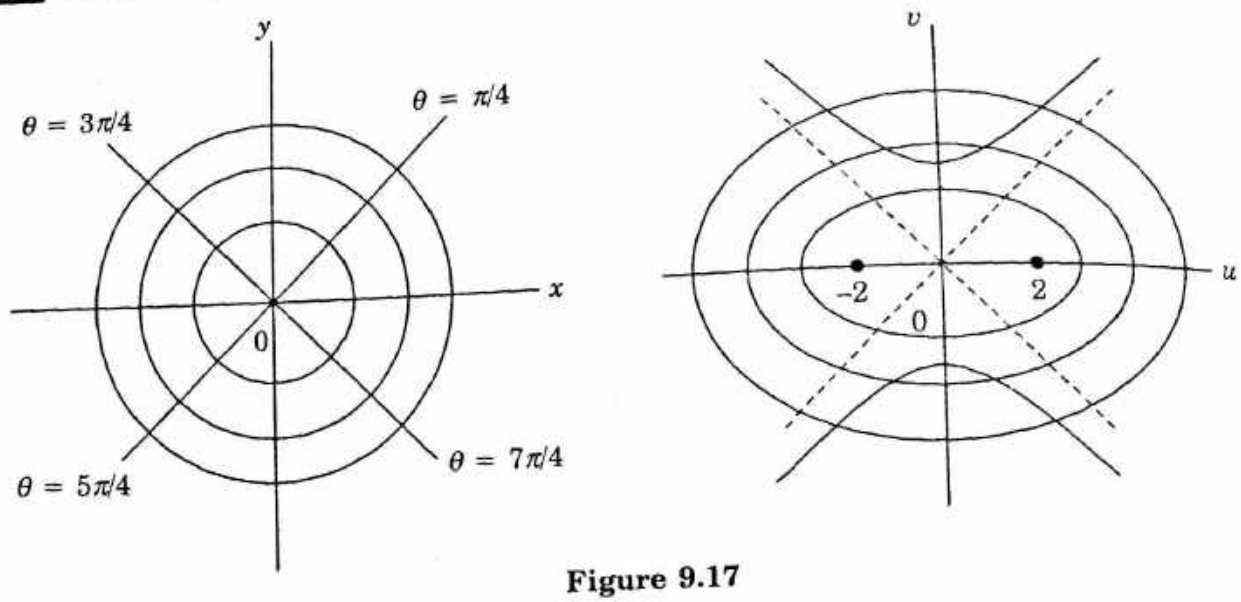

Figure 9.17

This reveals that the ray θ = constant in the z-plane is mapped onto the hyperbola (9.39) with foci $(\pm 2, 0)$ in the w-plane which intersects the ellipses orthogonally. Hence the image of the line $\theta = \alpha$, $0 < \alpha < \pi/2$ is the right-hand branch of the hyperbola, since $u > 0$ for this range of α. Any other pair of the rays $\theta = \alpha$ and $\theta = \alpha + \pi$ is mapped onto the two branch of the same hyperbola.

The image of the ray $\theta = 0$ is given by the equation $u = (r + 1/r)$ and $v = 0$. Since $(r + 1/r) \geq 2$ for all values of r, it follows that the image is the part of the u-axis $u \geq 2$. As r varies from 0 to ∞ the image segment is described twice. The image of the ray $\theta = \pi/2$ is given by the equations $u = 0$ and $v = (r - 1/r)$. As r varies from 0 to ∞ this represents the complete imaginary axis in the w-plane. Similar is the case for $\theta = -\pi/2$. Thus, the y-axis corresponds to the v-axis.

The image of the line $\theta = \pi$ is given by $u = -(r + 1/r)$, $v = 0$. Thus, the image is the part of the real u axis $-\infty < u < -2$ in the w-plane.

The image of the wedge $\alpha < \theta < \beta$, where $0 < \alpha, \beta < \pi/2$ is the region bounded by the two branches lying to the right of the imaginary axis of the hyperbolas.

Example 24. Let C be a circle in the z-plane having its centre on the real axis which passes through $z = 1$ and encloses $z = -1$ as an interior point. Determine the image of C under the transformation $w = z + 1/z$.

First, we observe that

$$\left.\frac{dw}{dz}\right|_{z=1} = 1 - \left.\frac{1}{z^2}\right|_{z=1} = 0, \qquad \left.\frac{d^2w}{dz^2}\right|_{z=1} \neq 0.$$

This informs that $z = 1$ is a zero of order two and mapping is not conformal at this point. So, angles with vertices at $z = 1$ are doubled by the transformation at the image point $w = 2$ (see Proposition 2). In particular, the exterior angle at $z = 1$ of C, i.e. π is magnified twice at the image point $w = 2$. So, the exterior angle to the image curve C' is 2π. Hence C' has sharp tail at $w = 2$. The other points of C' can be found directly (Fig. 9.18).

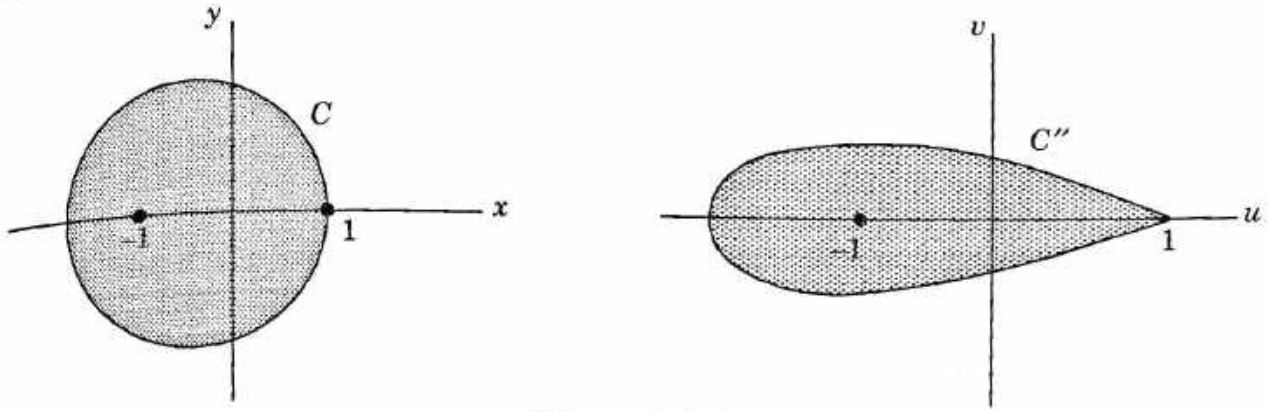

Figure 9.18

In this case C encloses the circle $|z| = 1$ which under the transformation maps on the slit $-2 \le u \le 2$. Thus as C approaches $|z| = 1$, the image curve C' approaches the line joining $w = -2$ to $w = 2$.

Example 25. Let the circle C of Example 2 be moved so that its centre is in the upper half-plane but that is still passes through $z = 1$ and encloses $z = -1$ as its interior point. Determine the image of C under the transformation $w = z + 1/z$.

As in Example 24, since the point $z = 1$ is the critical point, we again obtain the sharp tail at $w = 2$ (Fig. 9.19). If C does not entirely enclose the circle $|z| = 1$ the image C' will not entirely enclose the image of $|z = 1|$ (which is the slit joining -2 to 2). Instead, C' will enclose that portion of the slit which corresponds to the part of $|z| = 1$ inside C. The appearance of C' is depicted in Fig. 9.19.

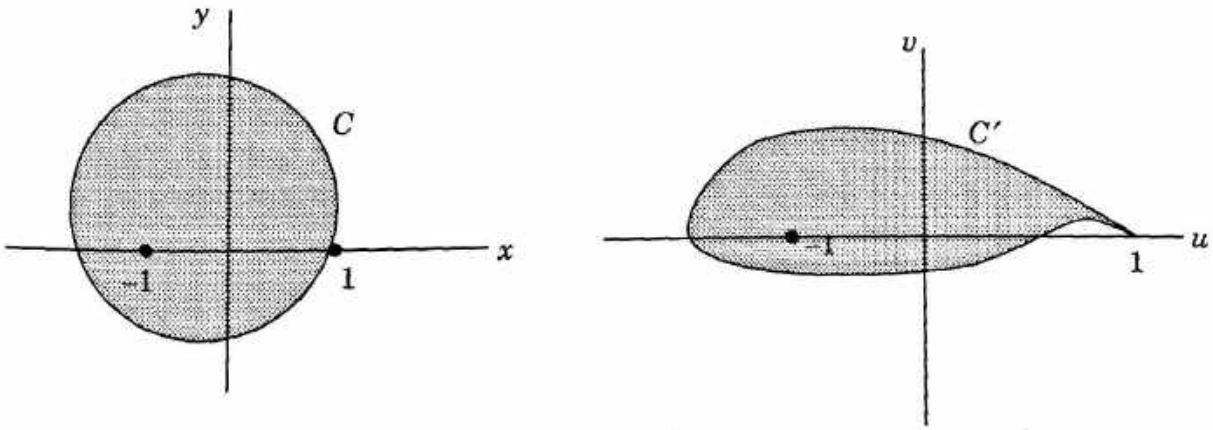

Figure 9.19

The fact that C' resembles the cross-section of wing of an airplane, called an airfoil. For this reason, these shapes are called Joukowski *airfoils* or *profiles*.

In the end we would like to mention a result (without proof) which tells us why conformal maps have large applications. The engineers and scientists have arbitrary domains in their practical problems. It is possible to map in a one-to-one manner these domains onto unit disc with the construction of appropriate functions.

Riemann mapping theorem. Let Ω be a simple connected domain in the complex plane (except the plane itself). Then there exists a conformal map which maps Ω in a one-to-one manner onto the unit disc $|w| < 1$.

Remarks. 1. There exists no conformal map that maps the entire complete plane onto unit disc. This is a consequence of Liouville's theorem. How?

2. In the Riemann mapping theorem, we may equally say that $w = f(z)$ maps Ω onto the upper half-plane. Why?

3. It must be emphasized that Riemann mapping theorem is applicable to only simple connected open sets and says nothing about the behaviour of the mapping function on the boundary.

PROBLEM SET 9.5

1. Find the image of the region $x \le 0$ and $0 \le y \le \pi$ under the mapping $w = e^z$. Sketch the regions.
2. Find the image of the region $-\pi/2 < x < \pi/2$ and $0 < y < 1$ under the map $w = e^{2iz}$.
3. What is the image under the map $w = z + e^z$ of the closed strip $-\pi \le y \le \pi$?
4. Show that under the mapping $w = c \sin z$, $c \in \mathbb{R}$, the region $0 \le x \le \pi/2$ is mapped into the first quadrant. Is the mapping conformal at $z = \pi/2$? Justify.
5. Sketch the image of the region $0 \le x \le \pi/2$, $y \ge 0$ mapped by the transformation $w = \sin^2 z$.
6. Find an analytic function which maps the sector $0 < \arg z < \pi/3$ onto $u > 1$.
   Suggestion. Composite map hgf, where $f(z) = z^3$, $g(z) = -iz$, $h(z) = z + 1$.
7. Describe the mapping $w = \cosh z$ and find the image of $x \ge 0$, $0 \le y \le \pi$ under this transformation.
8. Find the image of the infinite strip $-\pi/2 \le y \le \pi/2$ under the map $\sinh z$.
9. Find the image of the straight lines $x = 1$ and $y = 1$ under the map $w = \tan z$.
10. Consider the map

$$w = \tan^2\left(\frac{z}{2}\right).$$

and show that the vertical strip $0 \le x \le \pi/2$ in the z-plane is mapped onto the boundary and inside of the unit circle $|w| = 1$, cut along the slit on the real axis from -1 to 0 in the w-plane.

Suggestion. Write the transformation in the form

$$w = \frac{1 - \cos z}{1 + \cos z}$$

and observe that any point on the line $x = \pi/2$ is mapped onto

$$|w| = \left|\frac{1 - i\sinh y}{1 + i\sinh y}\right| = 1.$$

Also, y-axis: $x = 0$ is mapped onto

$$w = \frac{1 - \cosh y}{1 + \cosh y}.$$

Clearly, w is real and assumes (twice) values in the interval $-1 \le u \le 0$. The only thing that remains to be verified is that any line $x = \alpha$, $0 < \alpha < \pi/2$, inside the strip is mapped interior to $|w| = 1$.

11. Describe the transformation $w = \cosh z$ in terms of the transformation $f(z) = \sin z$ and comment on rotation and translation effects.

12. Show that the transformation $w = z + 1/z$ can be written as

$$\frac{w-2}{w+2} = \left(\frac{z-1}{z+1}\right)^2.$$

13. Show that the transformation

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

maps half-plane $\operatorname{Re} z > 0$ with semi-circle of radius 1 centred at origin removed onto a half-plane.

14. Let C be circle in the z-plane having its centre above the real axis which passes through $z = -1$ and encloses $z = 1$ as the interior point. Determine the image of C under the transformation $w = z + 1/z$.

10

Laplace Transform

We begin this chapter by introducing Laplace transform and its properties. Next, we prove the convolution theorem and derive complex inversion formula which enables to determine the inverse Laplace transform of complicated functions. In the last two sections Laplace transform is applied to solve differential and integral equations.

10.1 INTRODUCTION

First we define the Laplace transform and discuss the conditions under which Laplace transform of different functions exist.

Definition 1. The *Laplace transform* of a function $f(t)$ of a real variable $t \in (0, \infty)$ is a function $F(p)$ of a complex variable p defined by

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{10.1}$$

whenever the improper integral exists. In a symbolic way (10.1) is written as

$$F(p) = L[f(t)]. \tag{10.2}$$

The function f may be real valued or complex valued but certainly t is a real variable. The argument p is complex. Throughout our discussion on the Laplace transform, lowercase letters are reserved for functions and capital letters for their corresponding Laplace transforms. If the context is clear, $L[f]$ will also be used to denote the Laplace transform of a function f.

The convergence of improper integral on the right side of (10.1) depends on the nature of the function f as well as on the parameter p. For example, if $f(t) = e^{t^2}$, the Laplace transform of this function does not exist, i.e. the integral in (10.1) is not convergent for any value of p. This can easily be verified by writing the integral in the form

$$L[e^{t^2}] = \int_0^\infty e^{t^2 - pt}\, dt = \int_0^\infty e^{(t-p/2)^2 - p^2/4}\, dt.$$

Obviously, the last integral is divergent by comparison with $\int_0^\infty e^t\,dt$. However, we shall demonstrate in the ensuing discussion that $L[t^n]$ exists if $\operatorname{Re} p > 0$, the real part of p. Let us clarify the query: for what type of functions does the Laplace transform exist?

Before answering the above query, we evaluate the Laplace transform of some simple functions based on the definition.

(a) For $a > -1$,

$$L[t^a] = \int_0^\infty e^{-pt} t^a\,dt.$$

On rescaling $pt = x \Rightarrow p\,dt = dx$, and recalling the gamma function, we obtain

$$L[t^a] = \int_0^\infty \frac{e^{-x}}{p}\left(\frac{x}{p}\right)^a dx = \frac{1}{p^{a+1}}\int_0^\infty e^{-x}x^a\,dx = \frac{\Gamma(a+1)}{p^{a+1}}. \tag{10.3}$$

In particular, if $a = n$ is a nonnegative integer in (10.3), we have

$$L[t^n] = \frac{n!}{p^{n+1}}, \qquad L[1] = \frac{1}{p}, \qquad L[t] = \frac{1}{p^2}. \tag{10.4}$$

(b) For $\operatorname{Re} p > a$,

$$L[e^{at}] = \int_0^\infty e^{-pt}e^{at}\,dt = \int_0^\infty e^{-(p-a)t}\,dt = \frac{1}{p-a}[-e^{-(p-a)t}]_0^\infty.$$

Using the fact $e^{-(p-a)t} \to 0$ as $t \to \infty$, when $\operatorname{Re} p > a$,

$$L[e^{at}] = \frac{1}{p-a}. \tag{10.5}$$

(c) For $\operatorname{Re} p > 0$, integrating by parts, we have

$$\begin{aligned} L[\sin at] &= \int_0^\infty e^{-pt}\sin at\,dt \\ &= \left[-\frac{e^{-pt}}{p}\sin at\right]_0^\infty + \frac{a}{p}\int_0^\infty e^{-pt}\cos at\,dt \\ &= \frac{a}{p}\left[-\frac{e^{-pt}}{p}\cos at\right]_0^\infty - \frac{a^2}{p^2}\int_0^\infty e^{-pt}\sin at\,dt \\ &= \frac{a}{p^2} - \frac{a^2}{p^2}L[\sin at] \end{aligned}$$

or

$$\left(1 + \frac{a^2}{p^2}\right)L[\sin at] = \frac{a}{p^2}$$

and hence

$$L[\sin at] = \frac{a}{p^2 + a^2}. \tag{10.6}$$

Similarly,

$$L[\cos at] = \frac{p}{p^2 + a^2}, \qquad \text{Re } p > 0; \tag{10.7}$$

$$L[\sinh at] = \frac{a}{p^2 - a^2}, \qquad \text{Re } p > |a|; \tag{10.8}$$

$$L[\cosh at] = \frac{p}{p^2 - a^2}, \qquad \text{Re } p > |a|. \tag{10.9}$$

Now, we discuss the conditions under which the integral in (10.1) is convergent, i.e. its value is finite. This improper integral is defined by (see Section 8.3)

$$\int_0^\infty e^{-pt} f(t)\, dt = \lim_{b\to\infty} \int_0^b e^{-pt} f(t)\, dt. \tag{10.10}$$

The integral in (10.10) converges if

$$\int_0^\infty |e^{-pt} f(t)| dt$$

converges, and we say that integral (10.1) converges absolutely and hence converges. Finally, if we are able to search a nonnegative function $g(t)$ such that

$$|e^{-pt} f(t)| \le g(t) \qquad \text{and} \qquad \int_0^\infty g(t) < \infty,$$

then we conclude that (10.1) converges due to comparison test.

For this analysis the function $f(t)$, defined for all $t \ge 0$, must be piecewise continuous on $[0, \infty)$. By piecewise continuity we mean that a function is continuous on every finite subinterval $0 \le t \le b$, except possibly at a finite number of points where there are jump discontinuities, i.e. $f(x+)$ and $f(x-)$ exist but are unequal. Note that at the origin, we take only $f(0+)$.

If $f(t)$ is piecewise continuous for $t \ge 0$, then for the existence of integral (10.1) we observe the behaviour of the integrand $e^{-pt} f(t)$ for large t. In order that the integrand $e^{-pt} f(t)$ approaches rapidly to zero, we shall assume that $f(t)$ is of exponential order.

Definition 2. A function $f(t)$ defined on $[0, \infty)$ is said to be of *exponential order* if there exist constants M and α such that

$$|f(t)| \le Me^{\alpha t}, \qquad t \ge 0. \tag{10.11}$$

This inequality needs only to be satisfied, of course, at those points of the non-negative t-axis, where f is defined.

The constant functions are of exponential order, as can be seen by setting $\alpha = 0$ in (10.11). The functions

$$t^n, \quad e^{at}, \quad \sin at, \quad t^n \sin at, \quad t^n e^{at} \sin at$$

for n to be a positive integer, are of exponential order. We will prove that t^n is of exponential order, and for other functions the same procedure may be adopted.

Example 1. Show that the function t^n is of exponential order.

Since, for every $\alpha > 0$,

$$\lim_{t\to\infty} \frac{t^n}{e^{\alpha t}} = 0,$$

for given $\varepsilon = 1 > 0$ there exists some $t > t_0$ such that

$$\left|\frac{t^n}{e^{\alpha t}}\right| < 1 \quad \forall\, t > t_0.$$

But t^n is bounded on finite interval $[0, t_0]$. Thus,

$$|t^n| < Me^{\alpha t} \quad \text{for} \quad t \geq 0.$$

Theorem 1. Let f be a piecewise continuous function of exponential order defined on $[0, \infty)$. Then, there exists a real number α such that the integral

$$\int_0^\infty e^{-pt} f(t)dt$$

converges for $\operatorname{Re} p > \alpha$.

Proof. Since f is of exponential order, it follows that

$$|f(t)| \leq Me^{\alpha t} \quad \forall\, t \in [0, \infty)$$

for some constant M and a real number α; this implies

$$|e^{-pt} f(t)| \leq M e^{-(p_1-\alpha)t}, \qquad p = p_1 + ip_2.$$

Hence

$$|F(p)| = \left|\int_0^\infty e^{-pt} f(t)\, dt\right| \leq \int_0^\infty |e^{-pt} f(t)|\, dt \leq M \int_0^\infty e^{-(p_1-\alpha)t}\, dt.$$

But for $p_1 < \alpha$,

$$\int_0^\infty e^{-(p_1-\alpha)t} dt = \lim_{b\to\infty} \int_0^b e^{-(p_1-\alpha)t}\, dt = \frac{1}{p_1 - \alpha}, \tag{10.12}$$

and therefore,

$$\int_0^\infty e^{-pt} f(t)dt \quad \text{converges absolutely for } \operatorname{Re} p > \alpha$$

and hence converges.

Remarks. 1. If the integral in Theorem 1 converges for some $\operatorname{Re} p = p_0$, then it converges for all $\operatorname{Re} p > p_0$.

2. In view of Eq. (10.12), $F(p) \to 0$ as $p \to \infty$. This gives a useful bit of information that any function $\phi(p)$ for which the $\lim_{p\to\infty} \phi(p)$ either does not exist or is nonzero, it may be immediately forecast that the concerned function is not the Laplace transform of any function of exponential order. So, p^2, $\sin p$, e^p cannot be Laplace transformed. This condition is not sufficient as $p^{-1} \sin p$ is not the Laplace transform of any function even though it goes to zero on all values with some arithmetic progression.

3. In Theorem 1, convergence of the integral has been established with the help of absolute convergence in the domain $\text{Re}\, p > \alpha$. However, its convergence may happen to hold in a larger domain than that of absolute convergence. For example, if we take $f(t) = e^t \cos e^t$, then

$$\int_0^\infty e^{-pt} e^t \cos e^t \, dt = \int_0^\infty u^{-p} \cos u \, du$$

converges for $\text{Re}\, p > 0$, but it converges absolutely for $\text{Re}\, p > 1$.

In general, for any function f, let α_0 be the greatest lower bound of the set of all real numbers α such that $L[f(t)]$ exists for $\text{Re}\, p > \alpha$. Then $L[f(t)]$ does not exist for any p with $\text{Re}\, p < \alpha_0$. Thus, the domain of definition of $L[f(t)]$ is given by $\text{Re}\, p > \alpha_0$. The line $\text{Re}\, p = \alpha_0$ is called the abscissa of convergence.

Also, Theorem 1 gives a sufficient condition for the existence of Laplace transform of any function. However, this condition is not necessary as there exist functions which are neither piecewise continuous nor of exponential order even then the Laplace transform may exist.

For example, the function $t^{-1/2}$ is having discontinuity of infinite order at $t = 0$, since $\lim_{t\to 0^+} t^{-1/2} = \infty$. Note that this discontinuity is different from that which we require for piecewise continuity. Nevertheless, we have for $\text{Re}\, p > 0$,

$$L[t^{-1/2}] = \int_0^\infty e^{-pt} t^{-1/2} \, dt = \frac{1}{\sqrt{p}} \int_0^\infty e^{-x} x^{-1/2} \, dx = \frac{\Gamma(1/2)}{\sqrt{p}} = \sqrt{\frac{\pi}{p}}. \tag{10.13}$$

Equation (10.3) also yields (10.13) for $a = -1/2$. Note that whenever nonintegral index of p is involved we consider the principal value.

Linearity. The Laplace transform L is a linear operator on the vector space V of all functions whose Laplace transforms exist. This means that, if $L[f(t)] = F(p)$ for $\text{Re}\, p > \alpha_1$ and $L[g(t)] = G(p)$ for $\text{Re}\, p > \alpha_2$, then, for all constants a and b,

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)], \quad \text{Re}\, p > \alpha = \max\{\alpha_1, \alpha_2\}. \tag{10.14}$$

Let $L[f(t)]$ and $L[g(t)]$ be defined for $\text{Re}\, p > \alpha_1$ and $\text{Re}\, p > \alpha_2$, respectively. Then choose $\alpha = \max\{\alpha_1, \alpha_2\}$ so that both $L[f(t)]$ and $L[g(t)]$ are defined for $\text{Re}\, p > \alpha$. Now,

$$L[af(t) + bg(t)] = \int_0^\infty [af(t) + bg(t)]e^{-pt} \, dt$$

$$= a \int_0^\infty f(t)e^{-pt} \, dt + b \int_0^\infty g(t)e^{-pt} \, dt = F(p) + G(p),$$

which agrees with (10.14).

As an immediate consequence of linearity, formulas (10.8) and (10.9) may conveniently be derived using (10.5) and the relations:

$$\sinh at = \frac{e^{at} - e^{-at}}{2}, \qquad \cosh at = \frac{e^{at} + e^{-at}}{2}.$$

L is one-one. Consider the operator equation $L[f(t)] = F(p)$, where $F(p)$ is analytic for $\operatorname{Re} p > \alpha$. At the moment we do know that this operator equation has a solution for given $F(p)$ which is analytic for $\operatorname{Re} p > \alpha$. However, if the operator equation has a solution for given $F(p)$, the obvious question is whether it is unique. The following theorem answers this query in affirmative.

Theorem 2. (Lerch's theorem) Let $f(t)$ and $g(t)$ be piecewise continuous functions of exponential order on $[0, \infty)$ such that $L[f(t)] = L[g(t)]$, $\operatorname{Re} p > \alpha$. Then, $f(t) = g(t)$ for all t, except at the points of discontinuity.

Definition 3. Two functions f and g are said to be equivalent on $[0, \infty)$, if $f(t) = g(t)$ for all t, except at the points where the functions are not continuous.

Considering the equivalent functions in V, the vector space of all real functions which are piecewise continuous and of exponential order, as the same functions that the Lerch's theorem ensures that the operator equation (10.2) has a unique solution. Thus, L is a one-to-one linear transformation from vector space V to the space of all analytic functions (which are the Laplace transforms of some functions).

Proposition 1. If $f(t)$ is piecewise continuous and of exponential order with period a, then

$$F(p) = \frac{1}{1 - e^{-ap}} \int_0^a e^{-pt} f(t)\, dt. \tag{10.15}$$

Proof. Given that $f(t + a) = f(t)$ for all $t \in [0, \infty)$. By definition

$$L[f] = \int_0^\infty e^{-pt} f(t) dt$$

$$= \int_0^a e^{-pt} f(t) dt + \int_a^{2a} e^{-pt} f(t)\, dt + \cdots + \int_{na}^{(n+1)a} e^{-pt} f(t) dt + \cdots$$

Set $x + na = t$ in the $(n + 1)$st integral of the above series to have

$$\int_{na}^{(n+1)a} e^{-pt} f(t) dt = \int_0^a e^{-p(x+na)} f(x + na) dx = e^{-nap} \int_0^a e^{-px} f(x) dx,$$

the last expression follows from the periodicity of f. Hence

$$L[f] = \int_0^a e^{-px} f(x) dx + e^{-ap} \int_0^a e^{-px} f(x) dx + \cdots + e^{-nap} \int_0^a e^{-px} f(x) dx + \cdots$$

$$= (1 + e^{-ap} + e^{-2ap} + \cdots + e^{-nap} + \cdots) \int_0^a e^{-px} f(x)dx$$

$$= \frac{1}{1 - e^{-ap}} \int_0^a e^{-pt} f(t)dt .$$

Remark. Let $f(t)$ be piecewise continuous and of exponential order on $[0, \infty)$, and assume that $f(t + \pi) = -f(t)$. Then the Laplace transform of $f(t)$ is given by (10.15) just replacing a by 2π. Why?

Example 2. Find the Laplace transform of the function $|\sin t|$.

Note that the function $|\sin t|$ is π periodic, so applying formula (10.15), we have

$$L[|\sin t|] = \frac{1}{1 - e^{-\pi p}} \int_0^\pi e^{-pt} \, |\sin t| dt.$$

Since $|\sin t| = \sin t$ on $[0, \pi]$, on integrating by parts it follows that, for Re $p > 0$,

$$L[|\sin t|] = \frac{1}{1 - e^{-\pi p}} \int_0^\pi e^{-pt} \sin t \, dt = \frac{1}{1 - e^{\pi p}} \frac{1 + e^{-\pi p}}{1 + p^2} = \frac{1}{1 + p^2} \frac{1 + e^{-\pi p}}{1 - e^{-\pi p}} .$$

10.2 PROPERTIES OF LAPLACE TRANSFORM

We discuss below some properties of the Laplace transform which are useful in developing further theory and giving us a shorter way of computing the Laplace transform, thus avoiding the direct use of formula (10.1).

1. Shifting the variable *p*. If $F(p)$, Re $p > \alpha$ is the Laplace transform of $f(t)$, then

$$L[e^{at} f(t)] = F(p - a), \quad \text{Re } p > a + \alpha. \tag{10.16}$$

This means that the substitution of $p - a$ for p in the transform corresponds to the multiplication of the original function by e^{at}. This result known as *first shifting* formula is verified below.

$$L[e^{at} f(t)] = \int_0^\infty e^{-pt} e^{at} f(t)dt = \int_0^\infty e^{-(p-a)t} f(t)dt = F(p - a), \qquad \text{Re } p > a + \alpha.$$

Example 3. Find the Laplace transform of the functions (i) $t^n e^{3t}$; (ii) $e^{at} \sin at$.

For part (i), let $f(t) = t^n$ and from (10.4), $F(p) = n!/p^{n+1}$. Hence, using (10.16), we have

$$L[t^n e^{3t}] = \frac{n!}{(p - 3)^{n+1}}, \qquad \text{Re } p > 3.$$

For part (ii), take $f(t) = \sin at$ and from (10.6), $F(p) = a/(p^2 + a^2)$. Hence, using (10.16), we have

$$L[e^{at} \sin at] = \frac{a}{(p-a)^2 + a^2}, \qquad \text{Re } p > a.$$

2. Multiplication by t^n. If $F(p)$ is the Laplace transform of f, then

$$L[(-1)^n t^n f(t)] = F^{(n)}(p). \tag{10.17}$$

So, the nth differentiation of the transform with respect to p corresponds to the multiplication of the original function by $(-1)^n t^n$.

To show this, we proceed as

$$F(p) = \int_0^\infty e^{-pt} f(t)dt.$$

Differentiate the above equation with respect to p. Note that differentiation involves under the sign of integration, and this is justified by Leibniz rule. So,

$$F'(p) = -\int_0^\infty e^{-pt}\, t f(t)dt = L[-t f(t)], \qquad \text{Re } p > \alpha.$$

This proves the result for $n = 1$. On further differentiation

$$F''(p) = \int_0^\infty e^{-pt}\, t^2 f(t)dt = L[t^2 f(t)], \qquad \text{Re } p > \alpha$$

which makes the result valid for $n = 2$. The order of growth of $f(t)$ does not change with the multiplication by t^n. Hence, by induction (10.17) can be proved.

Here we have also proved the fact that $F(p)$ is analytic in the domain Re $p > \alpha$.

Example 4. Find the Laplace transform of the functions: (i) $t^3 e^{-2t}$; (ii) $t^2 \cosh 3t$.

For part (i), let $f(t) = e^{-2t}$ and from (10.5) $F(p) = 1/(p + 2)$. Hence, using (10.17), we have

$$L[t^3 e^{-2t}] = (-1)^3 \frac{d^3}{dp^3}\left(\frac{1}{p+2}\right) = \frac{6}{(p+2)^4}, \qquad \text{Re } p > -2.$$

For part (ii), take $f(t) = \cosh 3t$ and from (10.8), $F(p) = p/(p^2 - 9)$. Hence, using (10.17), we have

$$L[t^2 \cosh 3t] = (-1)^2 \frac{d^2}{dp^2}\left(\frac{p}{p^2-9}\right) = \frac{2p(p^2+22)}{(p^2-9)^3}, \qquad \text{Re } p > 3.$$

3. Division by t. If $L[f(t)] = F(p)$, Re $p > \alpha$, and the limit of $f(t)/t$, as t approaches 0 from the right exists, then

$$L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(x)dx. \tag{10.18}$$

provided the integral on the right side exists.

To prove this we start with the right side

$$\int_p^\infty F(x)\,dx = \int_p^\infty \left[\int_0^\infty e^{-xt} f(t)\,dt\right] dx.$$

Since t and x are independent variables, changing the order of integration, we have

$$\begin{aligned}\int_p^\infty \left[\int_0^\infty e^{-xt} f(t)\,dt\right] dx &= \int_0^\infty f(t) \int_p^\infty e^{-xt}\,dx\,dt \\ &= \int_0^\infty f(t)\left(-\frac{e^{-xt}}{t}\right)_p^\infty dt \\ &= \int_0^\infty e^{-pt}\frac{f(t)}{t}\,dt \\ &= L\left[\frac{f(t)}{t}\right].\end{aligned}$$

Further, if we write (10.18) as

$$\int_0^\infty e^{-pt}\frac{f(t)}{t}\,dt = \int_p^\infty F(x)dx,$$

and let $p \to 0$, we have

$$\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(p)dp. \tag{10.19}$$

Formula (10.19) is useful to evaluate integrals.

Example 5. Show that

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

By the direct use of (10.19), we have

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{1}{1+p^2}\,dp = [\tan^{-1} p]_0^\infty = \frac{\pi}{2}.$$

Note that in Section 8.3, we evaluated this integral by residue calculus.

To evaluate definite integrals the idea of Eq. (10.19) is further extended to cover a larger class of functions. That is to say, if a definite integral includes as a factor of the

integrand, a power t^{-n} or a quotient $a/(a^2+t^2)$ or $t/(a^2+t^2)$, the value of the integral can sometimes be obtained by expressing that factor itself as the integral (10.1) and then interchanging the order of integration. For example, if the integral

$$\int_0^\infty \frac{\cos mt}{a^2+t^2}\,dt, \qquad m \in \mathbb{R}, a \in \mathbb{R}_+.$$

is denoted by $f(m)$, then the change of order of integration yields

$$\int_0^\infty \cos mt\left(\frac{1}{a}\int_0^\infty e^{-tx}\sin ax\,dx\right)dt = \frac{1}{a}\int_0^\infty \sin ax\left(\int_0^\infty e^{-xt}\cos mt\,dt\right)dx.$$

Thus,

$$f(m) = \frac{1}{a}\int_0^\infty \frac{x\sin ax}{m^2+x^2}\,dx,$$

which on rescaling as $ax = mt$ gives $f'(m) = -af(m)$. This is a first order linear differential equation with the solution: $f(m) = ce^{-am}$. Since $f(0) = \pi/2a$, c can be computed. This has also been evaluated as Problem 7, Problem Set 8.3 by using residue calculus.

4. Laplace transform of derivatives. This property reveals that the Laplace transform of higher derivatives can be written in terms of the Laplace transform of the original function with some additive constants. This nature of the Laplace transform has varied applications for solving differential equations.

Suppose $y(t)$ is continuous on $(0, \infty)$ and $y'(t)$ is a piecewise continuous function of exponential order. Then,

$$L[y'] = \int_0^\infty e^{-pt}y'\,dt = [e^{-pt}y]_0^\infty + p\int_0^\infty e^{-pt}y\,dt = pL[y] - y(0), \quad \text{Re } p > \alpha. \tag{10.20}$$

Further, if y'' is also of exponential order, Eq. (10.20) can be extended to

$$L[y''] = L[(y')'] = pL[y'] - y'(0) = p^2L[y] - py(0) - y'(0).$$

Similarly, with the assumption that $y, y', \ldots, y^{(n)}$ are piecewise continuous and of exponential order, the above procedure may be continued up to n-times, or by using induction, to obtain

$$L[y^{(n)}] = p^nL[y] - p^{n-1}y(0) - p^{n-2}y'(0) - \cdots - y^{n-1}(0), \quad \text{Re } p > \alpha. \tag{10.21}$$

Remark. While deriving (10.20), the fact $\lim_{t\to\infty} ye^{-pt} = 0$ has been used. However, it is true since (by assumption) y is of exponential order and deciding p such that $\text{Re } p > \alpha$. Also, note that $y^{(n)}(0)$, $n = 0, 1, \ldots$, means $y^{(n)}(0+)$.

5. Laplace transform of indefinite integrals. If $f(t)$ is a piecewise continuous function of exponential order, then

$$L\left[\int_a^t f(u)\,du\right] = \frac{F(p)}{p} - \frac{1}{p}\int_0^a f(u)\,du, \qquad a \ge 0, \text{ Re } p > \max\{\alpha, 0\}. \tag{10.22}$$

Since $f(t)$ is a piecewise continuous function and of exponential order, it follows that

$$g(t) = \int_a^t f(u)\,du$$

is continuous and of exponential order and, moreover, $g'(t) = f(t)$, except possibly at points where $f(t)$ has jump discontinuities. This implies that $g'(t)$ is a piecewise continuous function of exponential order and hence (10.21) assists us to write

$$L[g'(t)] = pL[g(t)] - g(0). \tag{10.23}$$

By definition

$$g(0) = -\int_0^a f(u)\,du$$

and hence, Eq. (10.23) takes the form

$$L[g(t)] = \frac{F(p)}{p} - \frac{1}{p}\int_0^a f(u)du, \qquad a \geq 0,\ \text{Re}\ p > \max\{\alpha, 0\}.$$

In particular,

$$L\left[\int_0^t f(u)\,du\right] = \frac{F(p)}{p}, \qquad \text{Re}\ p > \max\{\alpha, 0\}. \tag{10.24}$$

In general (10.24) is written as

$$L\left[\int_0^t \cdots \int_0^t \int_0^t f(x)\,dx\,dx \cdots dx\right] = \frac{F(p)}{p^n},$$

where the integral is iterated n times.

Example 6. Use the above property to evaluate

$$L\left[t^2 \int_0^t x \sin x\,dx\right].$$

Since

$$L[x \sin x] = -\frac{d}{dp}\left(\frac{1}{1+p^2}\right) = \frac{2p}{(1+p^2)^2},$$

In view of (10.24),

$$L\left[\int_0^t x \sin x\,dx\right] = \frac{2}{(1+p^2)^2}.$$

Thus,

$$L\left[t^2 \int_0^t x \sin x\,dx\right] = \frac{d^2}{dp^2}\left[\frac{2}{(1+p^2)^2}\right] = -\frac{8(5p^2-1)}{(1+p^2)^4}.$$

Unit step function. For $a \geq 0$, let us introduce the function,

$$u_a(t) = \begin{cases} 0, & 0 \leq t \leq a, \\ 1, & t > a. \end{cases}$$

The function $u_a(t)$ is called the *unit step* function or sometimes the Heaviside function. Let $f(t)$ be defined on $[0, \infty)$. Then the graph of $f(t)$ can be shifted by a units to the right, if we write

$$u_a(t)\, f(t-a) = \begin{cases} 0, & t < a, \\ f(t-a), & t > a. \end{cases}$$

Figure 10.1 shows that the graph of $f(t) = \sin t$, for $t > 0$ is shifted by 2 units to the right for the function $u_2(t) \sin (t-2)$. Now, we extract a property due to the influence of the unit step function. This is known as the *second shifting* formula or t-shifting.

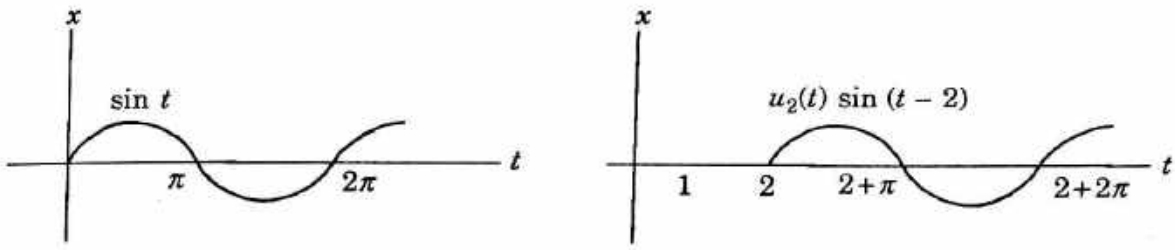

Figure 10.1

Let $f(t)$ be a piecewise continuous function of exponential order. Then

$$L[u_a(t)f(t-a)] = e^{-ap}F(p). \tag{10.25}$$

The factor e^{-ap} on the right side of Eq. (10.25) is called the time delaying factor. To prove this we proceed as

$$\int_0^\infty e^{-pt}\, u_a(t)f(t-a)dt = \int_a^\infty e^{-pt}f(t-a)dt = e^{-ap} \int_0^\infty e^{-pt}f(t)dt = e^{-ap}F(p).$$

In particular for $f(t) = 1$, Eq. (10.25) reduces to

$$L[u_a(t)] = \frac{e^{-ap}}{p}, \qquad L[u_0(t)] = \frac{1}{p}.$$

Example 7. Find the Laplace transform of the function

$$f(t) = \begin{cases} 2, & 0 < t < \pi, \\ 0, & \pi < t < 2\pi, \\ \sin t, & t \geq 2\pi. \end{cases}$$

To apply directly (10.25), we express $f(t)$ in terms of the unit step function.

For $0 < t < \pi$, we take $2u_0(t)$. For $t > \pi$, we want 0, so we must subtract the step function $2u_\pi(t)$. Then, we have $2u_0(t) - 2u_\pi(t)$, for $t > \pi$. This is alright until we reach 2π where we want $\sin t$ to enter; so we add $u_{2\pi}(t) \sin t$. This procedure determines

$$f(t) = 2u_0(t) - 2u_\pi(t) + u_{2\pi}(t) \sin t, \quad t > 2\pi.$$

Now, due to the periodicity of $\sin t$, we have

$$f(t) = 2u_0(t) - 2u_\pi(t) + u_{2\pi}(t) \sin(t - 2\pi), \quad t > 0.$$

Therefore,

$$L[f(t)] = \frac{2}{p} - \frac{2e^{-\pi p}}{p} + \frac{e^{-2\pi p}}{p^2 + 1}.$$

Unit impulse function. In mechanical systems, the *impulse* of a force $f(t)$ over a time interval, say, $a \le t \le a + k$ is defined to be the integral of $f(t)$ from a to $a + k$. Similarly, in an electric circuit the impulse is the integral of electromotive force applied to the circuit, integrated from a to $a + k$. We are interested in studying the systems in which k is very small ($k \to 0$), i.e. the impulse of a force acting for a very short time.

Let a be a fixed positive number. Then consider the function $\delta_k(t - a)$ defined by

$$\delta_k(t - a) = \begin{cases} 1/a, & a \le t \le a + k, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that its impulse is 1, since

$$\int_0^\infty \delta_k(t - a)dt = 1.$$

The unit *impulse* function is obtained by letting the interval $[a, a + k]$ reduce to zero, i.e.

$$\delta(t - a) = \lim_{k \to 0} \delta_k(t)(t - a).$$

It is not a function in the strict sense of calculus, but so called a "generalized function". In the limiting case, when $k \to 0$, we have

$$\delta(t - a) = \begin{cases} \infty, & t = a, \\ 0, & \text{otherwise}. \end{cases} \quad \text{and} \quad \int_0^\infty \delta(t - a)dt = 1.$$

Any ordinary function which is everywhere 0, except at a single point, must have integral value 0, but its integral turns out to be 1. Although $\delta(t - a)$ is called the unit impulse function or the Dirac–delta function, we are at liberty to call it a distribution, and the theory of distribution justifies $\delta(t - a)$ as if it were a function. Anyway in impulse problems it is convenient to operate on $\delta(t - a)$ as if it were a function.

Example 8. Show that $L[\delta(t - a)] = e^{-ap}$.

By definition

$$L[\delta_k(t - a)] = \int_0^\infty \delta_k(t - a)e^{-pt}\, dt = \int_a^{a+k} \frac{1}{k} e^{-pt}\, dt = \frac{e^{-ap} - e^{-(a+k)p}}{pk} = e^{-ap} \frac{1 - e^{-kp}}{kp}.$$

Letting $k \to 0$ and using L'Hôspital's rule, we obtain

$$L[\delta(t-a)] = \lim_{k\to 0} L[\delta_k(t-a)] = \lim_{k\to 0} e^{-ap}\,\frac{1-e^{-pk}}{pk} = e^{-ap}.$$

Remark. Note that $a = 0$ means that the impulse is applied in the beginning and then $L[\delta(t)] = 1$, but it does not mean that the inverse Laplace transform of 1 is the unit impulse function. Why?

10.3 CONVOLUTION OF TWO FUNCTIONS

If $L[f(t)] = F(p)$ and $L[g(t)] = G(p)$, what is the inverse transform of $F(p)G(p)$. In addition to settlement of this question, we give some applications of convolution in subsequent sections.

Definition 4. The convolution of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(t-x)g(x)\,dx. \tag{10.26}$$

Remark. With substituting $t - x = u$ in (10.26), it is evident that $f * g = g * f$. Thus, convolution satisfies the commutative property.

Theorem 3 (Convolution). Let $F(p)$ and $G(p)$ be the Laplace transforms of f and g, respectively. Then

$$L[f * g] = L[f(t)]\; L[g(t)]. \tag{10.27}$$

Proof. To prove this result, we have

$$L[f * g] = \int_0^\infty e^{-pt}\left[\int_0^t f(t-x)g(x)dx\right]dt = \int_0^\infty\int_0^t e^{-pt}f(t-x)\,g(x)dx\,dt\,.$$

On changing the order of integration, we have

$$L[f * g] = \int_0^\infty g(x)\left[\int_x^\infty e^{-pt}f(t-x)\,dt\right]dx.$$

The region of integration is the first half of the first quadrant. Let us introduce a new variable by defining $v = t - x$, so that $v \geq 0$, $t = v + x$ and (x being fixed during the integration of the expression within the brackets) $dt = dv$ (refer to Fig. 10.2).

This enables us to write

$$L[f * g] = \int_0^\infty g(x)\left[\int_0^\infty e^{-p(v+x)}f(v)\,dv\right]dx = \int_0^\infty e^{-pv}f(v)\,dv\int_0^\infty e^{-px}g(x)\,dx$$
$$= L[f(t)]L[g(t)].$$

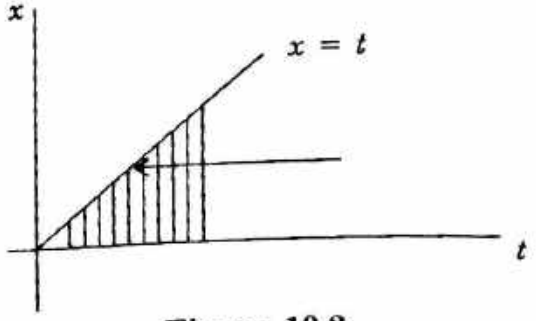

Figure 10.2

Error function. The error function is defined by

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx.$$

Naturally, the complimentary error function is

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2}\, dx.$$

***Example* 9.** Prove that

$$L[\operatorname{erf}(\sqrt{t})] = \frac{1}{p} \frac{1}{\sqrt{p+1}};$$

Use (10.13) and the first shifting property to have

$$L\left[\frac{e^{-t}}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{p+1}}.$$

With the help of the above result and formula (10.27) in reverse order, we have

$$\frac{1}{p} \frac{1}{\sqrt{p+1}} = L\left[1 * \left(\frac{e^{-t}}{\sqrt{\pi t}}\right)\right] = L\left[\frac{1}{\sqrt{\pi}} \int_0^t \frac{e^{-u}}{\sqrt{u}}\, du\right] = L[\operatorname{erf}(\sqrt{t})], \qquad u = x^2.$$

PROBLEM SET 10.1

1. Let f be piecewise continuous on $[0, \infty)$. Then prove that
   (a) f is of exponential order if, for some constant α, such that $\lim_{t\to\infty} [f(t)/e^{\alpha t}] = 0$;
   (b) f is not of exponential order if $\lim_{t\to\infty} [f(t)/e^{\alpha t}] = \infty$, for all real numbers α.
2. As $\lim_{t\to\infty} [t^{-1/2}/e^{\alpha t}] = 0$ for $\alpha > 0$, even then $t^{-1/2}$ is not of exponential order. Does it violate assertion (a) of the preceding problem?
3. Show that the function t^t is not of exponential order.
   Suggestion. Write $t^t = e^{t \ln t}$ and use the preceding problem.

4. If $L[f(t)] = F(p)$, $\operatorname{Re} p > \alpha$, show that

$$L[f(at)] = \frac{1}{a}F\left(\frac{p}{a}\right), \qquad \operatorname{Re} p > a\alpha.$$

This is change of scale property.

5. If f and g are piecewise continuous and of exponential order on $[0, \infty)$, show that (a) $f+g$, (b) $f-g$, (c) kf (k is some constant), and (d) fg are all piecewise continuous and of exponential order.

6. Determine which of the following functions are piecewise continuous on the interval $[0, \infty)$. (a) $f(t) = 0$, if t is rational and $f(t) = 1$, if t is irrational; (b) $f(t) = \exp t^2$; (c) $f(t) = (t+2)/(t-3)$; (d) $f(t) = 0$, if t is a natural number, otherwise $f(t) = 1$.

7. Prove that the integral for the function $f(t) = e^t \sin e^t$ converges absolutely for $\operatorname{Re} p > 1$. Does it converge in a larger domain? If yes, mention this domain.

8. Without using the definition evaluate the Laplace transform of the following functions:

(a) $\sin^2 t$; (b) $\cos^3 t$; (c) $|\cos t|$; (d) $\sin at \cosh at$.

9. Find the Laplace transform of the following functions and mention their domain of existence (abscissa of convergence):

(a) $(t+1)^4$; (b) $6t^3 e^{-t} + e^{2t} \sin 4t$;

(c) $\cosh at \sinh at$; (d) $te^{2t} \sinh 2t$.

10. Find the Laplace transform of each of the following functions:

(a) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 + 1, & t \geq 1; \end{cases}$ (b) $f(t) = \begin{cases} \sin t, & t \leq 2\pi, \\ 0, & t > 2\pi; \end{cases}$

(c) $f(t) = \begin{cases} 0, & t < 2, \\ 8 - 3t, & 2 \leq t \leq 3, \\ t - 4, & t \geq 3; \end{cases}$ (d) $f(t) = \begin{cases} 0, & 0 \leq t \leq \pi/2, \\ \cos t, & \pi/2 \leq t \leq 3\pi/2, \\ 1, & t > 3\pi/2. \end{cases}$

11. Evaluate:

(a) $L[t^{3/2}]$; (b) $L\left[t\int_{\pi}^{t} x \sin x \, dx\right]$;

(c) $L[\sin \sqrt{t}]$; (d) $L\left[\dfrac{\cos \sqrt{t}}{\sqrt{t}}\right]$.

Suggestion. For (c) use the power series expansion and for (d) use the outcome of (c) in conjunction with

$$\frac{d}{dt}(\sin \sqrt{t}) = \frac{\cos \sqrt{t}}{2\sqrt{t}}.$$

12. Verify the following transform results:

(a) $L[[t]] = \dfrac{1}{p(e^p - 1)}$; (b) $L[t - [t]] = \dfrac{e^2 - 1 - p}{p^2(e^p - 1)}$;

(c) $L\left[\dfrac{\sin 2t}{t}\right] = \cot^{-1}\dfrac{p}{2}$; (d) $L\left[\dfrac{1 - \cos 2t}{t^2}\right] = 2\cot^{-1}\left(\dfrac{p}{2}\right) + \dfrac{p}{2}\log\left(\dfrac{p^2}{p^2 + 4}\right)$,

where $[t]$ denotes the maximum integer contained in t.

13. For $n = 0, 1, 2, \ldots$, let

$$f(t) = \begin{cases} 1, & 2n \le t \le 2n + 1, \\ 0, & 2n + 1 < t < 2n + 2. \end{cases}$$

Find the Laplace transform of this function.

Suggestion. The function has its period 2. Apply Proposition 1.

14. For the function $g(t)$ defined for developing the proof of property 5, establish the fact that it is piecewise continuous and of exponential order.

15. Use Laplace transform methods to evaluate the following integrals:

(a) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$; (b) $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$;

(c) $\displaystyle\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$; (d) $\displaystyle\int_0^\infty e^{-\sqrt{x}}\,dx$.

16. Show that

(a) $\displaystyle\int_0^\infty \frac{\cos t}{\sqrt{t}}\,dt = \int_0^\infty \frac{\sin t}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{2}}$; (b) $\displaystyle\int_0^\infty \cos t^2\,dt = \int_0^\infty \sin t^2\,dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Suggestion. For part (a), the comments following Example 5, lead to the integral, $\int_0^\infty (1 + t^2)^{-1} t^{-1/2}\,dt = \pi/\sqrt{2}$ (see also, the deduced results of Example 15, Chapter 8). For (b), set $t^2 = x$ and use (a).

17. Sketch the functions:

(a) $f(t) = (k/a)t$, $0 < t < a$, $f(t + a) = f(t)$ [Sawtooth function], and

(b) $g(t) = kn$, $an < t < an + a$, $n = 0, 1, 2, \ldots$ [Staircase function]

and obtain their Laplace transforms.

18. Show that the convolution operation is associative and satisfies the distributive law over the operation of addition of functions.

19. Like error function there are other familiar functions. We define a few of them.

(a) $Ei(x) = \displaystyle\int_x^\infty \frac{e^{-t}}{t}\,dt$ (Exponential integral);

(b) $Si(x) = \int_x^\infty \frac{\sin t}{t}\, dt$ (Sine integral);

(c) $Ci(x) = \int_x^\infty \frac{\cos t}{t}\, dt$ (Cosine integral);

(d) $P_n(x) = \frac{e^x}{n!}(x^n e^{-x})^{(n)}, \quad n = 0, 1, \ldots, x \in \mathbb{R}$ (Laguerre's polynomial).

Determine the Laplace transform of these functions.

20. Find the Laplace transform of the triangular wave of period $2a$ given by

$$f(t) = \begin{cases} t, & 0 < t < a, \\ 2a - t, & a < t < 2a. \end{cases}$$

Sketch this function when $f(t) \equiv 0$ on $(a, 2a)$, and determine its Laplace transform.

21. Write the unit impulse function as the sum of two unit step functions and determine the Laplace transform of the Dirac-delta function to match the result (10.26).

22. If $F(p)$ is the Laplace transform of $f(t)$, show that

(a) $\lim_{t\to 0} f(t) = \lim_{p\to\infty} pF(p)$; (b) $\lim_{t\to\infty} f(t) = \lim_{p\to 0} pF(p)$;

provided that all limits exist.

Assertions (a) and (b) are known as *initial* and *final value theorems*, respectively.

23. Prove that

(a) $L[\delta(t-k)] = e^{-kp}, \quad k \geq 0$ (b) $L[\mathrm{erf}(t)] = \frac{e^{p^2/4}}{p}\mathrm{erfc}\left(\frac{p}{2}\right), \quad p \in \mathbb{R}_+.$

Suggestion. Use (10.24) for $p > 0$, to have

$$L\left[\int_0^t e^{-x^2}dx\right] = \frac{L[e^{-x^2}]}{p} = \frac{e^{p^2/4}}{p}\int_0^\infty e^{-(x+p/2)^2}dx = \frac{e^{p^2/4}}{p}\int_{p/2}^\infty e^{-u^2}du.$$

10.4 INVERSE LAPLACE TRANSFORM

The solution of the operator equation

$$L[f(t)] = F(p),$$

whenever it exists is called the *inverse Laplace transform* of $F(p)$, and in a symbolic way we define

$$f(t) = L^{-1}[F].$$

What we have just defined is the following: If $F(p)$ is the Laplace transform of the function $f(t)$, then the function $f(t)$ is the inverse Laplace transform of $F(p)$.

Since L is a linear operator, it can be shown that L^{-1} is also linear.

Remark. The properties 1–3, 5–6 and convolution theorem are also used to determine the inverse Laplace transform when written in the equivalent forms:

1. $L^{-1}[F(p-a)] = e^{at}f(t)$;
2. $L^{-1}[F^{(n)}(p)] = (-1)^n t^n f(t)$;
3. $L^{-1}\left[\int_p^\infty F(x)\,dx\right] = \dfrac{f(t)}{t}$;
4. $L^{-1}\left[\dfrac{F(p)}{p}\right] = \int_0^t f(u)du$;
5. $L^{-1}[e^{-ap}F(p)] = u_a(t)f(t-a)$;
6. $L^{-1}[F(p)G(p)] = f * g$.

Now, we use some of the above companion formulas to find the inverse Laplace transform. The most general method of finding the inverse transform of any function is to decompose the given function into partial fractions. Using linearity of L^{-1}, we write the inverse transform of each fraction involved.

Example 10. Find the inverse Laplace transform of the following functions:

(a) $\dfrac{p}{(p+4)(p^2+4)}$; (b) $\dfrac{2p}{(p^2+4)^2}$.

For (a), by partial fractions,

$$\frac{p}{(p+4)(p^2+4)} = \frac{1}{5}\left(-\frac{1}{p+4} + \frac{1}{p^2+4} + \frac{p}{p^2+4}\right).$$

Now, using linearity of L^{-1} and the above remark, we have

$$L^{-1}\left[\frac{p}{(p+4)(p^2+4)}\right] = \frac{1}{5}\left\{-L^{-1}\left[\frac{1}{p+4}\right] + L^{-1}\left[\frac{1}{p^2+4}\right] + L^{-1}\left[\frac{p}{p^2+4}\right]\right\}.$$

Hence

$$L^{-1}\left[\frac{p}{(p+4)(p^2+4)}\right] = \frac{1}{5}\left(-e^{-4t} + \frac{\sin 2t}{2} + \cos 2t\right).$$

For (b), using property 2, we obtain

$$L^{-1}\left[\frac{2p}{(p^2+4)^2}\right] = -L^{-1}\left[\frac{d}{dp}\left(\frac{1}{p^2+4}\right)\right] = \frac{t\sin 2t}{2}.$$

Example 11. Find the inverse Laplace transform of the function

$$p \log\left(\frac{p+1}{p-1}\right) - 2.$$

Let the given function be the Laplace transform of $f(t)$, i.e.

$$L[f(t)] = p \log\left(\frac{p+1}{p-1}\right) - 2.$$

Then

$$\begin{aligned} L[t^2 f(t)] &= [p \log(p+1) - p \log(p-1) - 2]'' \\ &= \left[\log(p+1) + \frac{p}{p+1} - \log(p-1) - \frac{p}{p-1}\right]' \\ &= \frac{1}{p+1} + \frac{1}{(p+1)^2} - \frac{1}{p-1} + \frac{1}{(p-1)^2} \end{aligned}$$

Taking the inverse transform, we have

$$t^2 f(t) = e^{-t} + e^{-t}t - e^{t} + e^{t}t \quad \text{or} \quad \frac{1}{t^2}[e^{t}(t-1) + e^{-t}(t+1)].$$

Sometimes it is not an easy task to decompose an expression into partial fractions. In such situations we use the convolution theorem in the equivalent form:

$$L^{-1}[F(p)G(p)] = L^{-1}[F(p)] * L^{-1}[G(p)]. \tag{10.28}$$

Example 12. Use convolution theorem to find the inverse transform of the function

$$\frac{1}{p^2(p^2+1)}.$$

Since $L[t] = 1/p^2$ and $L[\sin t] = 1/(p^2+1)$, in view of (10.28), we have

$$L^{-1}\left[\frac{1}{p^2(p^2+1)}\right] = L^{-1}\left[\frac{1}{p^2}\right] * L^{-1}\left[\frac{1}{p^2+1}\right] = t * \sin t = t - \sin t.$$

For transforms involving e^{-ap}, the use of the unit impulse function facilitates the computations.

Example 13. Find the inverse transform of the function

$$F(p) = \frac{2}{p^2} - \frac{2e^{-2p}}{p^2} - \frac{4e^{-2p}}{p} + \frac{pe^{-\pi p}}{p^2+1}.$$

Operating L^{-1} on both the sides and using the second shifting formula in reverse direction, we get

$$\begin{aligned} L^{-1}[F(p)] &= 2t - 2u_2(t)(t-2) - 4u_2(t) + u_\pi(t) \cos(t-\pi) \\ &= 2t - 2u_2(t)t + u_\pi(t) \cos(t-\pi). \end{aligned}$$

Thus,

$$f(t) = \begin{cases} 2t, & 0 < t < 2, \\ 0, & 2 < t < \pi, \\ -\cos t, & t > \pi. \end{cases}$$

10.5 COMPLEX INVERSION FORMULA

In Section 10.4, we evaluated $L^{-1}[F(p)]$ by expressing $F(p)$ as a sum of functions which are the Laplace transforms of known functions or we applied the shifting formula, differentiation of transform, unit impulse function, convolution theorem and other properties of Laplace transform in the reverse direction. However, in some cases, it may not be possible to employ the above methods. To face such type of situations, here we develop the complex inversion formula which helps us to determine the inverse Laplace transform as complex line integral with the help of residue calculus. First we prove:

Proposition 2. Let f be a piecewise continuous function of exponential order defined on $[0, \infty)$. Suppose $L[f(t)] = F(p)$, $\text{Re } p > \alpha$, for some α. Then $F(p)$ is analytic in the domain $\Omega : \text{Re } p > \alpha$.

Proof. Given that $L[f) = F(p)$, $\text{Re } p > \alpha$, for some α, i.e.

$$F(p) = \int_0^\infty e^{-pt} f(t)dt, \qquad \text{Re } p > \alpha.$$

Since differentiation is valid under the sign of integral (Leibniz rule), it follows that

$$F'(p) = -\int_0^\infty e^{-pt} tf(t)dt.$$

The function $tf(t)$ is piecewise continuous and of exponential order (see Problem 5, Problem Set 10.1). This ensures the existence of $F'(p)$ for $\text{Re } p > \alpha$. Thus, F is differentiable for every p in the domain $\text{Re } p > \alpha$, and hence analytic.

Let $F(p)$ be analytic in the half-plane $\text{Re } p > \alpha$ and let $F(p) \to 0$ as $p \to \infty$. Consider $C = C_1 + C_2$, a closed contour consisting of a vertical segment C_1 from $\alpha + iR$ to $\alpha - iR$ and C_2 a semi-circle having positive orientation with radius R and centred at α, as shown in Fig. 10.3. Take any point p interior to C. Then, by the Cauchy integral formula

$$F(p) = \frac{1}{2\pi i}\int_C \frac{F(z)}{z-p}\,dz = \frac{1}{2\pi i}\int_{\alpha+iR}^{\alpha-iR} \frac{F(z)}{z-p}\,dz + \frac{1}{2\pi i}\int_{C_2} \frac{F(z)}{z-p}\,dz. \tag{10.29}$$

Assuming R to be sufficiently large, note that for z on C_2,

$$|z-p| = |(z-\alpha)-(p-\alpha)| \geq |z-\alpha| - |p-\alpha| = R - |p-\alpha|. \tag{10.30}$$

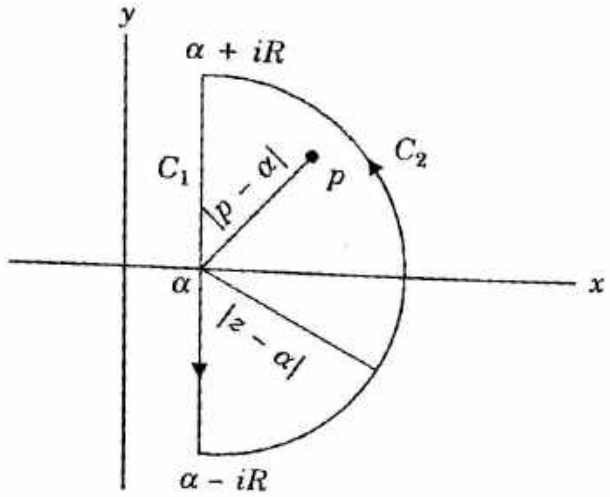

Figure 10.3

By assumption $F(z) \to 0$ as $z \to \infty$ through the domain Re $p > \alpha$. So, for given $\varepsilon > 0$, there exists R_ε such that

$$|F(z)| < \varepsilon \quad \text{whenever} \quad |z| \geq R_\varepsilon \text{ in } \Omega.$$

Again, $|z| \geq |z - \alpha| - \alpha$ (property of sides of a triangel) implies

$$|F(z)| < \varepsilon \quad \text{whenever} \quad |z - \alpha| \geq \alpha + R_\varepsilon \text{ in } \Omega.$$

Thus, by using (10.30), we have

$$\left| \int_{C_2} \frac{F(z)}{z - p}\, dz \right| < \frac{\pi R \varepsilon}{R - |p - \alpha|}, \quad R \geq \alpha + R_\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{R \to \infty} \int_{C_2} \frac{F(z)}{z - p}\, dz = 0.$$

Taking limit as $R \to \infty$ in (10.29), we get

$$\begin{aligned} F(p) &= \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{F(z)}{p - z}\, dz \\ &= \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} F(z) \left[\int_0^\infty e^{-(p-z)t}\, dt \right] dz \\ &= \int_0^\infty e^{-pt} \left[\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{zt} F(z)\, dz \right] dt. \end{aligned}$$

The change in the order of integration, being justified, is taken as granted.

Summarizing the above discussion we have proved the following result widely known as the complex inversion formula or Bromwich's integral formula.

Theorem 4. Let $F(p)$, the Laplace transform of $f(t)$, be analytic in the right half-plane Ω : Re $p > \alpha$ and $F(p) \to 0$ as $p \to \infty$ in Ω. Then

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp, \qquad c > \alpha, \tag{10.31}$$

where α is large enough so that all singularities (finite) of $F(p)$ lie in the half-plane Re $p < \alpha$.

The function $f(t)$ retrieved in (10.31) is referred to as the inverse Laplace transform and the usual function space for $f(t)$ are those functions satisfying (10.11).

Finding the inverse Laplace transform now reduces to computing the integral in (10.31). The contour from $c - i\infty$ to $c + i\infty$ is referred to as *Bromwich contour* and c is taken to the right of all singularities (poles, essential singularity, branch points) in order to ensure (10.32). Note that closing the contour to the right will yield $f(t) = 0$ for $t < 0$. Also, note $L^{-1}[F(p)]$ obtained by (10.31) is independent of c which can be assigned any value greater than α such that $F(p)$ is analytic in the right half-plane.

The complex line integral (10.31) is evaluated by using the residue theorem. To see how it is done, we assume that $F(p)$ has a finite number of singularities in the left half-plane Re $p \le c$. We then let $C = C_1 + C_2$ be the closed contour consisting of the vertical line segment C_1 from $c - iR$ to $c + iR$ and the semi-circle C_2 of radius R centred at Re $p = c$ and lying to the left of C_1, as shown in Fig. 10.4.

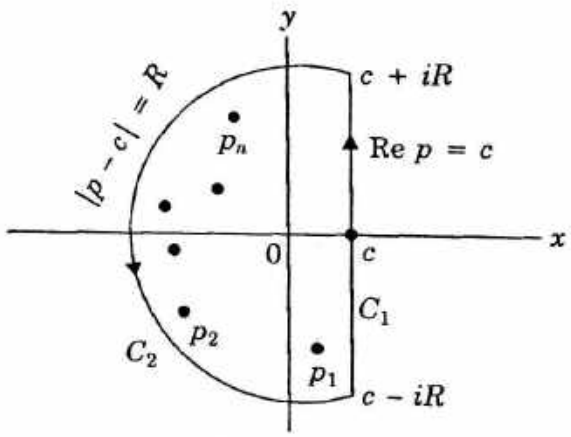

Figure 10.4

The radius R is to be taken large enough so that C encloses all the singularities of the function $e^{pt}F(p)$. Hence, by the residue theorem

$$\frac{1}{2\pi i}\int_{C_1} e^{pt}F(p)dp = \frac{1}{2\pi i}\int_{C} e^{pt}F(p)dp - \frac{1}{2\pi i}\int_{C_2} e^{pt}F(p)dp$$

$$= \sum_{k=1}^{n} \text{Res}_{p=p_k} e^{pt}F(p) - \frac{1}{2\pi i}\int_{C_2} e^{pt}\, F(p)dp \tag{10.32}$$

where $p_1, p_2, \ldots, p_n$ are the singularities of the function $F(p)$ inside C.

Since $F(p) \to 0$ as $p \to \infty$, Jordan's lemma (see Section 8.3) is used to have

$$\lim_{R\to\infty} \int_{C_2} e^{pt} F(p)\,dp = 0.$$

Note that $p = \alpha + Re^{i\theta}$, for some $\alpha > 0$ and $\theta(\pi/2 \le \theta \le 3\pi/2)$. Hence, taking the limit as $R \to \infty$, we obtain from (10.31) and (10.32),

$$L^{-1}\,[F(p)] = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{pt} F(p)dp = \sum_{k=1}^{n} \mathrm{Res}_{p=p_k} e^{pt} F(p). \tag{10.33}$$

Remarks. 1. The analysis to derive the Bromwich integral formula reveals that the sufficient condition $F(p) \to 0$ as $p \to \infty$ may be taken as

$$|\,F(p)\,| < \frac{M}{|\,p\,|^k}, \qquad \mathrm{Re}\ p > \alpha,$$

where M, $k \in \mathbb{R}_+$, the set of positive real numbers.

2. For the existence of inverse Laplace transform these conditions are sufficient but not necessary (see Problem 16, Problem Set 10.2).

Example 14. Evaluate the inverse Laplace transform of the function

$$\frac{2p+3}{(p-2)(p+1)^2}.$$

The partial fraction decomposition of the function $F(p) = (2p+3)/(p-2)(p+1)^2$ is time consuming and hence it is convenient to apply the complex inversion formula which is our last resort. The function $F(p)$ is analytic in $\mathrm{Re}\,p > 2$ and tends to zero uniformly as p approaches to infinity.

The singularities of the function $e^{pt}F(p)$ are a simple pole at $p_1 = 2$ and a pole of order two at $p_2 = -1$, and hence,

$$\mathrm{Res}_{p_1=2}\ e^{pt}F(p) = \lim_{p\to 2}\left[\frac{e^{pt}(2p+3)}{(p+1)^2}\right] = \frac{7}{9}e^{2t},$$

$$\mathrm{Res}_{p_2=-1}\ e^{pt}F(p) = \lim_{p\to -1}\left\{\frac{d}{dp}\left[\frac{e^{pt}(2p+3)}{p-2}\right]\right\} = -\frac{(7+3t)e^{-t}}{9}.$$

Inserting these values into (10.33), we get

$$f(t) = \frac{7}{9}e^{2t} - \frac{1}{9}(7+3t)e^{-t}.$$

In what follows we discuss the case of finding the inverse Laplace transform of multivalued functions.

Example 15. Find the inverse Laplace transform of the function

$$F(p) = p^{-a} \qquad (0 < a < 1).$$

As p^{-a} is multivalued we take the branch cut along the negative real axis (Fig. 10.5).

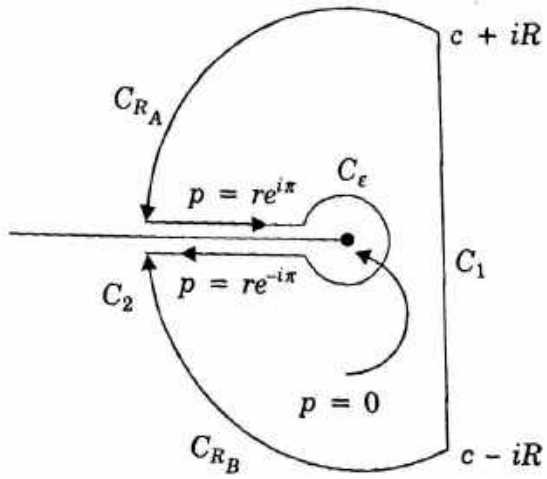

Figure 10.5

Assign $f(t) = 0$ for $t < 0$ when we close to the right. Closing to te left yields

$$f(t) + \left(\int_{C_{R_A}} + \int_{C_{R_B}} + \int_{p=re^{i\pi}} + \int_{p=re^{-i\pi}} + \int_{C_\varepsilon} \right) \frac{F(p)}{2\pi i} e^{pt} dp = 0.$$

The right-side vanishes because there is no singularities enclosed by the contour.

The integrals $\int_{R_A}$, $\int_{R_B}$ and $\int_{C_\varepsilon}$ vanish as $R \to \infty$ (on $C_{R_A} : p = c + Re^{i\theta}$, $\pi/2 < \theta < \pi$; on $C_{R_B} : p = c + Re^{i\theta}$, $-\pi < \theta < -\pi/2$, and $\varepsilon \to 0$ (on $C_e : p = \varepsilon e^{i\theta}$, $-\pi < \theta < \pi$). On C_ε, we have

$$\left| \int_{C_\varepsilon} \frac{F(p)}{2\pi i} e^{pt} dp \right| \le \int_{-\pi}^{\pi} \varepsilon^{1-a} \frac{e^{\varepsilon t}}{2\pi} d\theta \to 0 \qquad \text{as } \varepsilon \to 0.$$

Hence,

$$f(t) - \frac{1}{2\pi i} \int_{r=\infty}^{0} r^{-a} e^{-ia\pi} e^{-rt} dr - \frac{1}{2\pi i} \int_{r=0}^{\infty} r^{-a} e^{ia\pi} e^{-rt} dr = 0$$

or

$$f(t) = \frac{e^{ia\pi} - e^{-ia\pi}}{2\pi i} \int_0^\infty r^{-a} e^{-rt} dr.$$

Rescaling by $rt = u$ and recalling gamma function, we have

$$f(t) = \frac{\sin a\pi}{\pi} t^{a-1} \Gamma(1-a).$$

Remark. This analysis required that $F(p)$ has to have a finite number of singularities. Even if the number of singularities is infinite, formula (10.33) works well. But straightforward continuous limit $R \to \infty$ may pass through one of these singularities. To avoid such situations we take different contours as illustrated in the following example.

Example 16. Evaluate

$$L^{-1}\left[\frac{\cosh x\sqrt{p}}{p\cosh\sqrt{p}}\right], \quad 0 < x < 1.$$

First, let us understand that $p = 0$ is not a branch point (use the Taylor expansion), but it is a simple pole. The other singularities of the function are the zeros of $\cosh\sqrt{p}$, i.e.

$$\sqrt{p} = \left(n + \frac{1}{2}\right)\pi i \qquad \text{or} \qquad p = -\left(n + \frac{1}{2}\right)^2 \pi^2,$$

and these are simple poles.

Let us choose the Bromwich contour C_m as shown in Fig. 1.4 which is closed from the left. The semi-circle C_2 is centred at the origin with radius $R_m = m^2\pi^2$, $m \in \mathbb{I}_+$ so that the contour does not pass through any of the poles.

To find the residue of

$$\frac{e^{pt}\cosh x\sqrt{p}}{p\cosh\sqrt{p}}, \quad 0 < x < 1$$

at simple poles we use the formula $P(z_0)/Q'(z_0)$, $Q'(z_0) \neq 0$ (refer to Section 7.3). Thus, we have

$$\text{Res}_{p=0} = \left.\frac{e^{pt}\cosh x\sqrt{p}}{\cosh\sqrt{p} + (\sqrt{p}/2)\sinh\sqrt{p}}\right|_{p=0} = 1,$$

and

$$\text{Res}_{p=-(n+1/2)^2\pi^2} = \left.\frac{e^{pt}\cosh x\sqrt{p}}{\cosh\sqrt{p} + (\sqrt{p}/2)\sinh\sqrt{p}}\right|_{p=-(n+1/2)^2\pi^2}$$

$$= (-1)^{n+1}\frac{4e^{-(n+1/2)^2\pi^2 t}\cos(n+1/2)\pi x}{\pi(2n+1)}.$$

Hence

$$\frac{1}{2\pi i}\int_{C_m}\frac{e^{pt}\cosh x\sqrt{p}}{p\cosh\sqrt{p}}dp = 1 + \frac{4}{\pi}\sum_{n=0}^{m}\frac{(-1)^{n+1}}{2n+1}e^{-(n+1/2)^2\pi^2 t}\cos\left(n+\frac{1}{2}\right)\pi x.$$

Taking the limit as $m \to \infty$ and noting the fact that the integral around C_2 approaches zero, we find

$$L^{-1}\left[\frac{\cosh x\sqrt{p}}{\cosh\sqrt{p}}\right] = 1 + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}}e^{-(n+1/2)^2\pi^2 t}\cos\frac{(2n-1)\pi x}{2}.$$

are mapped onto the positive real axis in the w-plane. The sector $0 \le \arg z \le \pi/4$ is mapped onto the first quadrant in w-plane.

Example 6. Show that the image of the unit circle $|z| = 1$ under the transformation $w = 2z + z^2$ is a cardioid.

The transformation can be written as $w + 1 = (z + 1)^2$. Let $w + 1 = \rho e^{i\phi}$. Then, $|z| = 1$, i.e. $z = e^{i\theta}$ is transformed into the curve

$$\rho e^{i\phi} = (e^{i\theta} + 1)^2 = [(1 + \cos\theta) + i\sin\theta]^2 = 2(1 + \cos\theta)e^{i\theta}.$$

Hence, $\rho = 2(1 + \cos\theta)$, $\phi = \theta$. Eliminating θ, we note that

$$\rho = 2(1 + \cos\phi)$$

is the equation of the image curve, which is a cardioid. The desired cardioid is obtained by shifting the curve along the real axis in the w-plane by one unit to the left. The image curve is

$$\rho' = 2(1 + \cos\phi) - 1,$$

as shown in Fig. 9.5.

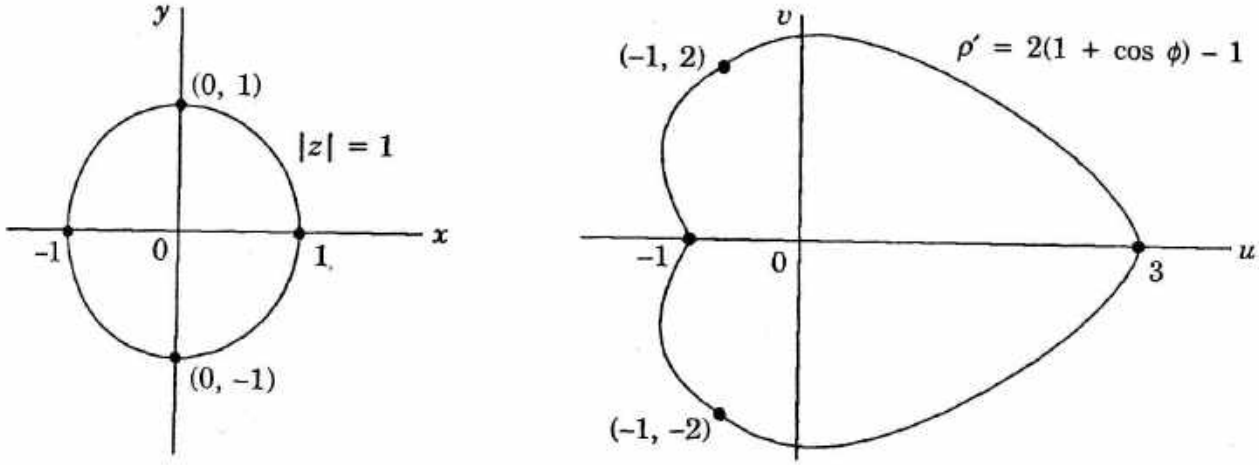

Figure 9.5

Finally, we comment a little about the transformation $w = z^n$, $n > 2$. This map is similar to $w = z^2$. Here,

$$w = z^n, \qquad \rho e^{i\phi} = r^n e^{in\theta}.$$

This transformation maps the region $r \ge 0$, $0 \le \arg z \le \pi/n$ onto the upper-half w-plane.

PROBLEM SET 9.1

1. Find and sketch the image of the region $y > -1$ under the map $w = (1 - i)z + 2i$.
2. Find the image of the semi-infinite strip $x > 0$, $1 < y < 2$, under the map $w = iz + 1$.

3. Construct a linear transformation which (a) carries i onto $-i$ and maps $1+2i$ onto itself; (b) maps $|z-1| = 1$ onto $|w-3i| = 2$.
4. Find the image of the strip $x > 0$, $0 < y < 1$ under the map $w = i/z$.
5. By writing $w = \rho e^{i\phi}$, show that the map $w = i/z$ maps the hyperbola $x^2 - y^2 = 1$ onto the lemniscate $\rho^2 = 2\cos 2\phi$.
6. Given a triangle Δ in the z-plane with vertices i, $1-i$, $1+i$, determine the triangle Δ' into which Δ is mapped under the transformations (a) $w = 3z + 4 - 2i$; (b) $w = iz + 2 - i$. What is the relationship between Δ and Δ' in each case?
7. Let Δ be the triangular region bounded by $x = 1$, $y = 1$ and $x + y = 1$. What is the image of this region under the map $w = z^2$?
8. Prove that the function $w = z^2 - 1$ maps one loop of the lemniscate $r^2 = 2\cos\theta$.
9. Show that the function $w = z^2$ transforms the circle
   (a) $|z-2| = 2$ onto the cardioid $\rho = 8(1+\cos\theta)$ in the w-plane;
   (b) $|z-a| = c$, $a, c \in \mathbb{R}$, onto the limacon in the w-plane. Comment when $a = c$.
   Suggestion. For part (b), use $\rho e^{i\phi} = (a + ce^{i\theta})^2 \Leftrightarrow \rho e^{i\phi} - a^2 + c^2 = ce^{i\theta}(ce^{i\theta} + ce^{-i\theta} + 2a)$. Now, compare the modulus and the argument on both sides.
10. Show that the transformation $w = (z+i)^2$ maps the interior of the circle $|z| = 1$ onto the exterior of the parabola
$$\rho^{-1} = 2(1-\cos\phi), \qquad w = \rho e^{i\phi} \text{ in the } w\text{-plane.}$$
11. Consider the function
$$w = \frac{4}{(z+1)^2}.$$
   Then find the image of the unit circle $|z| = 1$ under this map.
12. Discuss the application of the transformation $w = z^2$ to the area in the first quadrant of the z-plane bounded by the axes and the circles $|z| = a$, $|z| = b$, $a > b > 0$.
13. Show that for the transformation discussed in Example 6, the vertices of an equilateral triangle in the unit circle of the z-plane correspond to the point of contact of parallel tangents of the cardioid.
14. Show that the function $x^2 - y^2 + 2y$ under the transformation $z = w^2$ is harmonic in the w-plane.
15. Determine a transformation which rotates the ellipse $x^2 + xy + y^2 = 2$, clockwise so that it reduces to its principal axes. Also, calculate the length of semi-major and semi-minor axes.
   Suggestion. Assume $w = e^{i\alpha}z$. Find α so that the coefficient of uv in the image curve is zero.

9.2 CONFORMALITY THEOREM

The conformal mappings play an important role in the study of various physical phenomena defined on domains and curves of arbitrary shape. Smaller portions of these domains and curves are conformally mapped by analytic functions to well known domains and curves. Before introducing conformality, it is essential to be acquainted with the angle of rotation of curves under a transformation.

Angle of rotation

Let $\gamma \equiv z(t)$ be a smooth curve given by

$$z(t) = x(t) + iy(t), \qquad a \le t \le b.$$

The curve $z(t)$ is smooth if $x'(t)$ and $y'(t)$ are continuous and $z'(t) = x'(t) + iy'(t) \neq 0 \ \forall$ $t \in (a, b)$. It is to be noted that increasing values of the parameter t will give the positive direction of the curve γ.

Let $f(z)$ be a function defined on γ. Then,

$$w = f(z) = f(z(t)), \qquad a \le t \le b$$

is the parametric representation of the image of γ under the map $w = f(z)$. Suppose γ passes through z_0 (Fig. 9.6). Assume that f is analytic at z_0 and $f'(z_0) \neq 0$. Then, by the chain rule,

$$w'(t_0) = f'(z(t_0))z'(t_0), \qquad a \le t_0 \le b, \quad \text{and } z_0 = z_0(t_0)$$

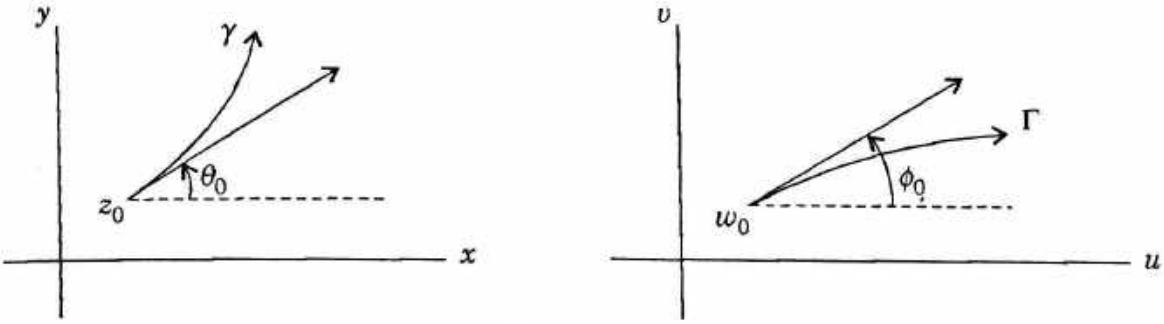

Figure 9.6

Since $f'(z(t_0)) \neq 0$, it follows that

$$\arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(t_0)$$

or

$$\theta_0 = \varphi_0 + \theta_0, \tag{9.10}$$

where θ_0 is the angle of inclination of the directed tangent to γ at z_0 and is a value of $\arg z'(t_0)$. Also, ϕ_0 is the angle of inclination of the directed tangent to Γ (image curve), whereas the rotation of γ is a value of $\arg f'(z_0)$, which we have denoted by φ_0.

Example 7. Verify, by taking some particular curve, the angle of rotation under the transformation $w = z^2$ at the point $2 + i$.

By definition, the angle of rotation for the function $f(z) = z^2$ at $2 + i$ is given by

$$\arg f'(2+i) = \arg(4+2i) = \tan^{-1}\left(\frac{1}{2}\right) + 2n\pi.$$

To verify the angle of rotation for a particular curve we assume a simple curve $y = x - 1$, i.e. $z(x) = x + i(x-1), -\infty < x < \infty$ (C-curve). Clearly, $\theta = \pi/4$ is the angle which curve C makes with the x-axis. The image of $y = x - 1$ under the transformation $w = z^2$ is given by using the relations $u = x^2 - y^2$ and $v = 2xy$ as

$$v = 2\,\frac{u+1}{2}\,\frac{u-1}{2} = \frac{u^2-1}{2} \ (\Gamma\text{-curve}).$$

The image of the point $(2 + i)$ is $(3 + 4i)$ in the w-plane. Now, for the Γ-curve,

$$\phi = \frac{dv}{du} = \tan^{-1}3 \quad \text{at } 3 + 4i.$$

So, the angle of rotation $(\phi - \theta)$ is

$$\tan^{-1}3 - \frac{\pi}{4} = \tan^{-1}3 - \tan^{-1}1 = \tan^{-1}\left(\frac{3-1}{1+3}\right) = \tan^{-1}\frac{1}{2}.$$

This is one of the values of $\tan^{-1}(1/2) + 2n\pi$ and hence the angle of rotation is verified.

Definition 1. A mapping $f(z)$ is said to be *conformal* at a point z_0 if it preserves the angle between the oriented curves passing through z_0 in magnitude as well as in sense.

Geometrically speaking, the images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at z_0, both in magnitude and direction.

A mapping $w = f(z)$ is said to be conformal in a domain D if it is conformal at each point of the domain.

Theorem 1. A mapping $w = f(z)$ is conformal at a point z_0 if it is analytic at z_0 and $f'(z_0) \neq 0$.

Proof. Let C_1 and C_2 be two smooth curves passing through z_0 in the z-plane and let θ_1, θ_2 be the angle of inclinations of the directed tangents to C_1 and C_2, respectively at z_0 (Fig. 9.7). Since $f'(z_0) \neq 0$, (9.10) takes the form

$$\phi_1 = \varphi_0 + \theta_1 \quad \text{for } C_1$$

and

$$\phi_2 = \varphi_0 + \theta_2 \quad \text{for } C_2,$$

where ϕ_1 and ϕ_2 are the angle of inclinations of the directed tangents to image curves Γ_1 and Γ_2, respectively at w_0 (Fig. 9.7). Hence,

$$\phi_2 - \phi_1 = \theta_2 - \theta_1.$$

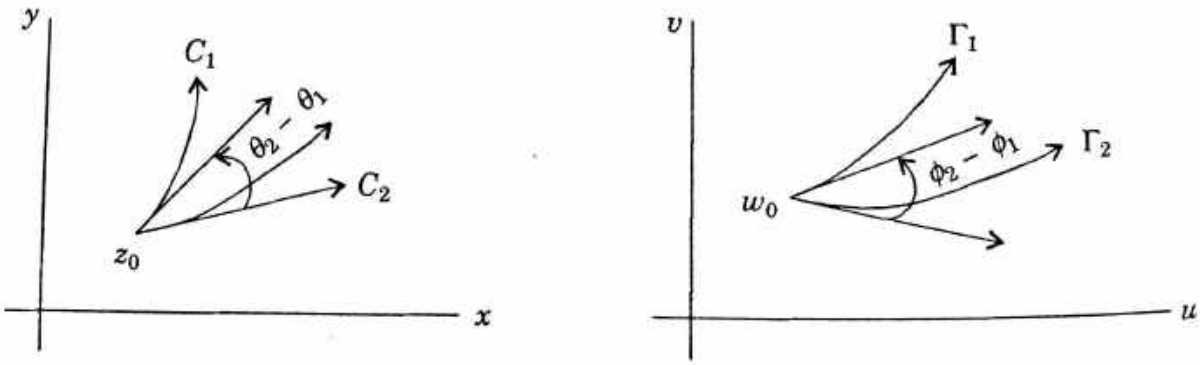

Figure 9.7

This implies that the angle from Γ_1 to Γ_2 is the same as the angle from C_1 to C_2 in magnitude as well as in direction.

In view of Theorem 1, $f(z) = e^z$ is conformal everywhere, since e^z is an entire function and $f'(z) = e^z \neq 0$ for all z.

Also, $w = z^2 - z + 1$ is conformal except $z = 1/2$, since it is analytic everywhere and $f'(1/2) = 0$.

Remarks. 1. If $w = f(z)$ is conformal at z_0 and a function $\xi = g(f(z))$ defined in some neighbourhood of $f(z_0)$ is conformal at this point, then the above theorem can be applied to prove that the composite map gf is conformal at z_0. We therefore conclude that any region Ω in the z-plane is mapped conformally on a region Ω' in the ξ-plane. Detailed analysis is left to the reader.

2. If $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$, then the function is actually conformal in a neighbourhood of z_0. Analyticity at a point implies analyticity in some neighbourhood of z_0 and moreover, by the continuity of $f'(z)$ a neighbourhood can be found in which $f'(z)$ is nonzero.

The *scale factor* of the conformal map $w = f(z)$ at a point $z = z_0$ is given by

$$|f'(z_0)| = \left|\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}\right| = \lim_{z\to z_0}\left|\frac{f(z) - f(z_0)}{z - z_0}\right|.$$

Clearly, $|f'(z)|$ represents an expansion if $|f'(z)| > 1$, and contraction if $|f'(z)| < 1$. Although expressions $\arg f'(z)$ and scale factor $|f'(z)|$ vary point to point, it follows from conformality that these values are approximately $\arg f'(z_0)$ and $|f'(z_0)|$ at all points nearer to z_0. This means that the image of a small region in a neighbourhood of z_0 conforms to the original region in the sense that it has approximately the same shape. A large region may, however, be transformed into a region that bears no resemblance with the original one.

The scale factor of $w = z^2$ at $2 + i$ is

$$|f'(2 + i)| = |4 + 2i| = 2\sqrt{5}.$$

Example 8. Verify conformality at a point $z = 1$ without using the above theorem for the map $w = 1/z$.

We take two simple curves $y = 0$ and $y = x - 1$ in the z-plane and establish the fact that the mapping preserves the angle between these curves and that between the image curves, i.e. both the angles have the same orientation and magnitude.

Clearly, $\theta_2 - \theta_1 = \pi/4$. The image of $y = 0$ and $y = x - 1$ is given by

$$\frac{-v}{u^2 + v^2} = 0 \quad \text{or} \quad v = 0 \qquad (\Gamma_1\text{-curve}),$$

$$\frac{-v}{u^2 + v^2} = \frac{u}{u^2 + v^2} - 1 \quad \text{or} \quad u^2 + v^2 - u - v = 0 \qquad (\Gamma_2\text{-curve}).$$

For the circle $u^2 + v^2 - u - v = 0$, we have

$$\frac{dv}{du} = \frac{1 - 2u}{2v - 1} = \tan^{-1}1 \quad \text{at } w = (1, 0).$$

Thus, the angle between the image curves is $\phi_2 - \phi_1 = \pi/4$. This ensures the conformality at $z = 1$.

Definition 2. A mapping that preserves the magnitude of the angle between two smooth arcs but not the sense is called an *isogonal mapping*.

The map $w = \bar{z}$ is an isogonal map, since it gives the mirror image of curves in the x-axis. Here, the angle between the two curves is preserved in magnitude but the direction of angle between the image curves is reversed.

Definition 3. Let $f(z)$ be a nonconstant analytic function in a domain D. If $f'(z_0) = 0$ for some z_0 in D, then z_0 is called the *critical point* of the transformation.

Clearly, $z = 0$ is the critical point of the map $w = z^2 + 1$.

Example 9. Find the critical points of the mapping $w = e^{2z} - 2iz + 3$.

Since $f(z) = e^{2z} - 2iz + 3$ is nonconstant and analytic $\forall\ z \in \mathbb{C}$, it follows that its critical points are the values of z such that $f'(z) = 0$. Hence

$$2e^{2z} - 2i = 0 \Leftrightarrow e^{2z} = i \Leftrightarrow 2z = i\left(2n\pi + \frac{\pi}{2}\right).$$

The set of critical points: $z = n\pi + \pi/4, \quad n \in \mathbb{I}$.

Remark. Constant map has no critical points and its scale factor is undefined.

Conformal maps exhibit an important property of having local inverse in some neighbourhood of a point, where the mapping is conformal. This is our next result.

Theorem 2. Let $f(z)$ be a conformal at z_0. Then, $f(z)$ has a local inverse at z_0.

Proof. The transformation $w = f(z)$ has a local inverse at z_0 means that there exists a unique transformation $z = g(w)$ defined and analytic in some neighbourhood N of $w_0 = f(z_0)$ such that $g(w_0) = z_0$ and $f(g(w)) = w\ \forall\ w \in N$.

Clearly, $f(g(w)) = w \Rightarrow f'(g(w))g'(w) = 1 \Rightarrow g'(w) = 1/f'(z)$. This shows that g is also conformal at w_0 (since g is analytic and $g'(w_0) \neq 0$ as $f'(z_0) \neq 0$).

Let us verify the existence of such an inverse. The function $f(z)$ is conformal at z_0 if $f(z)$ is analytic at z_0 (or in some neighbourhood of z_0) and $f'(z_0) \neq 0$. Hence the function $f(z) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ have continuous partial derivatives of all orders and the first order derivatives satisfy the C-R equations in some neighbourhood of $z_0 = (x_0, y_0)$.

Now, the pair of equations

$$u = u(x, y), \qquad v = v(x, y)$$

defines a map from this neighbourhood of z_0 to uv-plane. Moreover, the Jacobian of transformation

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x = (u_x)^2 + (v_x)^2 \neq 0$$

at the point (x_0, y_0), since $f'(z_0) \neq 0$.

From a result in real calculus: If $u(x, y)$ and $v(x, y)$ are continuous functions in some neighbourhood of a point (x_0, y_0) and the Jacobian of transformation is nonvanishing, then the local inverse of u and v exists.

Applying this result, we have a unique continuous transformation

$$x = x(u, v), \quad y = y(u, v), \quad \text{if } u_0 = u(x_0, y_0),\ v_0 = v(x_0, y_0)$$

defined in a neighbourhood of (u_0, v_0). Also,

$$x_u = \frac{1}{J} v_y, \quad x_v = -\frac{1}{J} u_y, \quad y_u = -\frac{1}{J} v_x, \quad y_v = \frac{1}{J} u_x.$$

So, for $w = u + iv$, $w_0 = u_0 + iv_0$ and $g(w) = x(u, v) + iy(u, v)$, the transformation $z = g(w)$ is the required local inverse. Since $u + iv = u(x, y) + iv(x, y)$ and $z = x + iy = x(u, v) + iy(u, v) = g(w)$.

Remarks. 1. Proposition 7, Chapter 4 can also be utilized to prove Theroem 2. Actually, for the proof of the theorem we have used the fact that $f'(z_0) \neq 0$, a converse of Theorem 1 (see Problem 10, Problem Set 9.2).

2. The short and simple proof of Theorem 2 is also the direct consequence of Proposition 7, Chapter 7.

What happens to the above theorem when the mapping f is not conformal at z_0, i.e. the condition $f'(z_0) \neq 0$ is dropped?

Proposition 2. Let $f(z)$ be analytic at z_0 such that $f'(z_0) = 0$. If $f'(z)$ has a zero of order $k-1$, $k = 1, 2, \ldots$ at z_0. Then the mapping $w = f(z)$ magnifies the angle at the vertex z_0 by the factor k.

Proof. By the given hypothesis f' has zero of order $k-1$. This means

$$f'(z_0) = f''(z_0) = \cdots = f^{(k-1)}(z_0) = 0.$$

Since $f(z)$ is analytic at z_0, it has the representation

$$f(z) = f(z_0) + a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots$$

So, we may further write

$$\begin{aligned} f(z) - f(z_0) &= (z - z_0)^k\ (a_k + a_{k+1}(z - z_0) + \cdots), \quad a_k \neq 0 \\ &= (z - z_0)^k g(z), \end{aligned} \tag{9.11}$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Consequently, if $w = f(z)$ with $w_0 = f(z_0)$, then using (9.11), we obtain

$$\arg\,(w - w_0) = \arg\,[f(z) - f(z_0)] = k\ \arg\,(z - z_0) + \arg\,g(z). \tag{9.12}$$

Let C be a smooth arc that passes through z_0. If $z \to z_0$ along C, then $w \to w_0$ along the image curve Γ and the angles of inclination of tangents to C and Γ are given by the following limits:

$$\theta_0 = \lim_{z \to z_0} \arg(z - z_0) \quad \text{and} \quad \phi_0 = \lim_{w \to w_0} \arg(w - w_0).$$

From Eqs. (9.11) and (9.12), it follows that

$$\phi_0 = k\theta_0 + \delta, \qquad \delta = \arg\,g(z_0). \tag{9.13}$$

Let C_1 and C_2 be two smooth curves that pass through z_0, and let Γ_1 and Γ_2 be their image curves, respectively. Then, from Eq. (9.13),

$$\phi_2 - \phi_1 = k(\theta_2 - \theta_1).$$

This implies that the angle from Γ_1 to Γ_2 is k-times the angle from C_1 to C_2 in magnitude. However, the orientation is preserved.

PROBLEM SET 9.2

1. Determine the angle of rotation at the point $1-i$ for the transformation $w = z^2$. Illustrate it for some particular curve. Also, find the scale factor of this transformation at this point.
2. Find the images of the lines $x = 0$ and $y = x - 1$ under the transformation $w = i/z$. Sketch all curves, determine the corresponding directions along them and verify the conformality at the point $z = -i$ (without using the conformality theorem).
3. Discuss the scale factor and the angle of rotation for the map $w = z^n$, $n \in \mathbb{I}$, for $z_0 \neq 0$.
4. Determine the critical points of $w = \sin z$.
5. Prove that the transformation $T(z) = az + b$ is conformal at infinity, if $a \neq 0$.
6. Let $w = f(z)$ be a conformal in a domain Ω and $f'(z) \neq 0$. Then show that $f(z)$ is analytic in Ω.
7. Find the local inverse of the transformation $w = z^2$ at the points: (a) $z_0 = 2$, (b) $z_0 = -2$, and (c) $z_0 = -i$.
8. Show that if $z = g(w)$ is the local inverse of a conformal transformation $w = f(z)$, then $g'(w) = 1/f'(z)$ in some neighbourhood N_δ, $\delta > 0$ where g is analytic.
9. Let γ be a smooth arc in a domain D throughout which $w = f(z)$ is conformal and let Γ denote the image of γ under the transformation. Then show that Γ is smooth.
10. Let $f(z)$ be a conformal map at z_0. Then prove that $f'(z_0) \neq 0$.
11. Suppose $f(z)$ is an entire function. If $f(z)$ is univalent in $\mathbb{C}$, show that it is conformal in the entire plane and will be of the form $a_0 + a_1 z$, $a_1 \neq 0$.

9.3 BILINEAR TRANSFORMATION

The map

$$w = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0, \tag{9.14}$$

is called *bilinear* transformation or *linear fractional* transformation or sometimes *Möbius* transformation. This can also be written as

$$Azw + Bz + Cw + D = 0, \qquad AD - BC \neq 0 \tag{9.15}$$

and conversely. Since (9.15) is linear in z as well as in w, (9.14) deserves to be termed bilinear transformation.

If $ad - bc = 0$, then (9.14) reduces to a constant as

$$w = \frac{a(z + b/a)}{c(z + d/c)} = \frac{a}{c} \quad \text{as} \quad ad - bc = 0 \Leftrightarrow \frac{b}{a} = \frac{d}{c}.$$

A constant function is not linear and hence $ad - bc \neq 0$ is the necessary condition for (9.14) to be a bilinear transformation. Also, note that the behaviour of $T(z)$ is not affected when a, b, c and d are multiplied by a constant and hence the expression for $T(z)$ may be normalized by taking $ad - bc = 1$. Suppose $ad - bc = \alpha$, then multiplication by $\sqrt{\alpha}$ of a, b, c, d will do the job. This is obvious from the fact that every bilinear transformation has two normal representations, obtained from each other by changing the signs of the coefficients.

Example 10. Show that the bilinear transformation

$$T(z) = \frac{az + b}{cz + d}, \qquad ad - bc = 1$$

is conformal except at one point and find its scale factor at any point z.

If $c = 0$, then the bilinear transformation reduces to the entire linear transformation

$$T(z) = \alpha z + \beta, \qquad \alpha = \frac{a}{d}, \quad \beta = \frac{b}{d}.$$

For $\alpha \neq 0$, $T'(z)$ is a nonzero constant; so $T(z)$ is conformal in the whole complex plane. The scale factor is $|\alpha|$. If $\alpha = 1$, then $\arg \alpha = 2n\pi$, $n \in \mathbb{I}$. It implies that both rotation and expansion produce no effect and in this case the transformation reduces to $z + \beta$. This obviously displaces the whole plane by the vector β. On the other hand, if $\alpha \neq 1$ the transformation can be written as

$$T(z) - z_0 = \alpha(z - z_0),$$

where z_0 is determined by the equation: $z_0 = \alpha z_0 + \beta$.

This is equivalent to a rotation of the whole plane by $\arg \alpha$ about $z_0 = \beta/(1 - \alpha)$ together with a uniform magnification by the factor α relative to the point z_0.

Next, suppose that $c \neq 0$. In this situation,

$$T'(z) = \frac{ad - bc}{(cz + d)^2} = \frac{1}{c^2} \frac{1}{(z - \delta)^2}$$

exists if $z \neq \delta$, where $\delta = -d/c$. Hence the mapping is conformal at all finite points except possibly at $z = \delta$. The tangents to curves through any point $z \neq \delta$ are rotated through an angle equal to

$$\arg T'(z) = \arg \frac{1}{|c|^2} - 2 \arg (z - \delta),$$

with the magnification at z that equals

$$|T'(z)| = \frac{1}{|c|^2} \frac{1}{|z - \delta|^2}.$$

The angle through which tangents are rotated has the same value for all points with equal $\arg (z - \delta)$, i.e. along any ray drawn from δ, otherwise, it varies from point to point. In

general, the magnification varies with z, but it has the same value for all points with equal values of $|z-z_0|$, a circle with centre δ. In particular, the magnification is 1 at every point of the circle with the equation

$$|z-z_0| = \frac{1}{|c|}.$$

Proposition 3. Every bilinear transformation maps circles and lines into circles and lines (a line is a circle with infinite radius).

Proof. Let $T(z)$ be any bilinear transformation given by

$$T(z) = \frac{az+b}{cz+d}, \qquad ad-bc \neq 0$$

If $c = 0$, then

$$T(z) = \frac{a}{d}z + \frac{b}{d} = Az + B, \qquad A = \frac{a}{d}, \quad B = \frac{b}{d}.$$

Clearly, $Az + B$, being linear, maps circles and lines into circles and lines.

If $c \neq 0$, then

$$\begin{aligned} T(z) &= \frac{a(z+d/c)}{c(z+d/c)} + \frac{b}{cz+d} - \frac{ad}{c(cz+d)} \\ &= \frac{a}{c} + \frac{bc-ad}{c(cz+d)} \\ &= \frac{a}{c} + \frac{bc-ad}{c^2}\,\frac{1}{z+d/c}. \end{aligned}$$

Assigning

$$z_1 = z + \frac{d}{c}, \qquad z_2 = \frac{1}{z_1}, \qquad z_3 = \frac{bc-ad}{c^2}\,z_2$$

we get

$$T(z) = \frac{a}{c} + z_3.$$

The above transformations are of the form

$$w_1 = z + \alpha, \qquad w_2 = \frac{1}{z}, \qquad w_3 = \beta z.$$

This establishes the fact that every bilinear transformation is the resultant of bilinear transformations with simple geometric imports.

Thus a bilinear transformation maps circles and lines into circles.

Example 11. Show that the line $3y = x$ is mapped onto the circle under the bilinear transformation

$$w = \frac{iz+2}{4z+i}.$$

Find the centre and radius of the image circle.

To find the image of $3y = x$, we need values of x and y in terms of u and v, and hence solve $w = (iz + 2)/(4z + i)$ for z,

$$4zw + iw = iz + 2 \Rightarrow (4w - i)z = 2 - iw \Rightarrow z = \frac{2 - iw}{4w - i}.$$

Writing $z = x + iy$ and $w = u + iv$, we have

$$\begin{aligned} x + iy &= \frac{v + 2 - iu}{4u + i(4v - 1)} \\ &= \frac{(v + 2 - iu)[4u - i(4v - 1)]}{16u^2 + (4v - 1)^2} \\ &= \frac{9u - i(4u^2 + 4v^2 + 7v - 2)}{16u^2 + (4v - 1)^2}. \end{aligned}$$

On comparing, we get

$$x = \frac{9u}{16u^2 + (4v - 1)^2}, \qquad y = -\frac{4u^2 + 4v^2 + 7v - 2}{16u^2 + (4v - 1)^2}.$$

Inserting these values in the equation $3y = x$, we find

$$\frac{9u}{16u^2 + (4v - 1)^2} = -\frac{3(4u^2 + 4v^2 + 7v - 2)}{16u^2 + (4v - 1)^2} \Rightarrow u^2 + v^2 + \frac{3}{4}u + \frac{7}{4}v - \frac{1}{2} = 0.$$

This is the equation of the circle with centre $(-3/8, -7/8)$ and radius $(3/4)\sqrt{5/2}$.

We add more information by claiming that $T(z) = (az + b)/(cz + d)$ is a one-one and onto mapping from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. For this, we define

$$T(z) = \begin{cases} \infty, & z = \infty, c = 0, \\ a/c, & z = \infty, c \neq 0, \\ \infty, & z = -d/c, c \neq 0, \\ T(z), & \text{otherwise.} \end{cases}$$

This definition makes the function $T(z)$ continuous on the extended plane.

Further, we discuss two important properties of the transformation $T(z)$.

(a) The inverse of a bilinear transformation is also bilinear.

(b) The composition of two bilinear transformations is also bilinear.

Since $T(z)$ is one-one and onto mapping from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ it follows that T^{-1} exists and is given by

$$w = T(z) \Leftrightarrow z = T^{-1}(w).$$

Solving $w = (az + b)/(cz + d)$ for z, we get

$$z = \frac{-dw + b}{cw - a}, \qquad ad - bc \neq 0$$

or

$$T^{-1}(w) = \frac{-dw+b}{cw-a}.$$

Again, for $T^{-1}(w)$, $ad-bc \neq 0$, hence T^{-1} is bilinear.

To verify (b) we assume two bilinear transformations $T(z)$ and $S(z)$ defined by

$$T(z) = \frac{az+b}{cz+d}, \qquad ad-bc \neq 0,$$

$$S(z) = \frac{pz+q}{rz+s}, \qquad ps-qr \neq 0.$$

Consider the composition

$$\begin{aligned}(ST)(z) = S(T(z)) &= S\left(\frac{az+b)}{cz+d}\right) \\ &= \frac{p[(az+b)/(cz+d)]+q}{r[(az+b)/(cz+d)]+s} \\ &= \frac{paz+bp+qcz+qd}{raz+rb+scz+sd} \\ &= \frac{(pa+qc)z+bp+dq}{(ar+cs)z+br+ds}.\end{aligned}$$

Now,

$$\begin{aligned}(pa+qc)(br+ds)-(bp+dq)(ar+cs) &= (ad-bc)ps-(ad-bc)rq \\ &= (ad-bc)(ps-qr) \neq 0.\end{aligned}$$

Hence ST is a bilinear transformation. Similarly, it can be shown that TS is also a bilinear transformation. Thus, the composition of two linear fractional transformations in either way is a bilinear transformation.

Example 12. Show that a bilinear transformation maps inverse points with respect to a circle or line onto inverse points with respect to the image circle and the image line.

The equation of a circle with p and q as inverse points can be written as (see Example 6 in Chapter 1),

$$\left|\frac{z-p}{z-q}\right| = k, \qquad k \neq 0 \tag{9.16}$$

The image of this circle under the bilinear transformation

$$w = \frac{az+b}{cz+d}$$

is obtained by inserting

$$z = \frac{dw-b}{-cw+a}$$

into Eq. (9.16) to have

$$\left|\frac{\dfrac{dw-b}{-cw+a}-p}{\dfrac{dw-b}{-cw+a}-q}\right| = k$$

or

$$\left|\frac{w(d+cp)-(ap+b)}{w(d+cq)-(aq+b)}\right| = k.$$

This is of the form

$$\left|\frac{w-p'}{w-q'}\right| = \left|\frac{d+cq}{d+cp}\right| k,$$

where

$$p' = \frac{ap+b}{cp+d} = \left.\frac{az+b}{cz+d}\right|_{z=p}, \qquad q' = \frac{aq+b}{cq+d} = \left.\frac{az+b}{cz+d}\right|_{z=q}.$$

Thus,

$$\left|\frac{w-p'}{w-q'}\right| = k', \quad k' = \left|\frac{d+cq}{d+cp}\right| k. \tag{9.17}$$

Note that p' and q' are the inverse points of the image circle, and in addition, these points are the images of p and q, respectively.

For $k = 1$, (9.17) represents a line which is the perpendicular bisector to the line joining p and q. Obviously, in this situation $k' = 1$.

In dealing with Example 12, we have also proved that a bilinear transformation maps circles and lines into circles and lines, an alternative proof of Proposition 3 of this section.

Let us consider the bilinear transformations

$$T_1(z) = \frac{z}{z+1}; \qquad T_2(z) = \frac{z+2}{z+3}.$$

We observe

$$T_1T_2(z) = T_1(T_2(z)) = T_1\left(\frac{z+2}{z+3}\right) = \frac{z+2}{2z+5},$$

$$T_2T_1(z) = T_2(T_1(z)) = T_2\left(\frac{z}{z+1}\right) = \frac{3z+2}{4z+3}.$$

Thus,

$$T_1T_2(z) \neq T_2T_1(z).$$

This indicates that bilinear maps do not follow the commutative property. Assuming identity transformation $I(z) = z$ as a particular case of bilinear transformation, it may be claimed that the set of all bilinear maps form a noncommutative group with respect to composition of maps (refer to Problem 18, Problem Set 9.3).

Definition 4. A bilinear transformation $T(z)$ has a fixed point (invariant point) z_0 if $T(z_0) = z_0$.

Proposition 4. Every bilinear transformation (except the identity map) has at most two fixed points.

Proof. If $T(z)$ has a fixed point z, then $T(z) = z$ or

$$\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + dz = az + b \Leftrightarrow cz^2 - (a-d)z - b = 0.$$

The last equation is quadratic in z and hence can have at most two roots.

For the identity map, $I(z) = z$ for all points in the domain of definition. Hence every point of the domain is a fixed point.

This completes the proof.

Proposition 5. If a bilinear transformation $w = T(z)$ has exactly two fixed points z_1 and z_2, then for some nonzero constant k they satisfy the equation

$$\frac{w-z_1}{w-z_2} = k\frac{z-z_1}{z-z_2}. \tag{9.18}$$

Moreover, if $T(z)$ has only one fixed point z_1, then it can be written as

$$\frac{1}{w-z_1} = k' + \frac{1}{z-z_1}, \qquad k' \neq 0. \tag{9.19}$$

Proof. Let z_1 and z_2 be the given fixed points of the bilinear transformation $w = (az+b)/(cz+d)$, and hence these are the roots of the equation $cz^2 - (a-d)z - b = 0$. This means

$$cz_1^2 - (a-d)z_1 - b = 0 \Leftrightarrow cz_1^2 - az_1 = b - dz_1; \tag{9.20}$$

$$cz_2^2 - (a-d)z_2 - b = 0 \Leftrightarrow cz_2^2 - az_2 = b - dz_2. \tag{9.21}$$

Utilizing (9.20), note that

$$\begin{aligned} w - z_1 &= \frac{az+b}{cz+d} - z_1 \\ &= \frac{az+b-z_1(cz+d)}{cz+d} \\ &= \frac{(a-z_1c)z+b-dz_1}{cz+d} \\ &= \frac{(a-z_1c)z+cz_1^2-az_1}{cz+d} \\ &= \frac{(a-cz_1)(z-z_1)}{cz+d}. \end{aligned}$$

Similarly, looking at (9.21),

$$w - z_2 = \frac{(a-cz_2)(z-z_2)}{cz+d}.$$

Hence,

$$\frac{w-z_1}{w-z_2} = \frac{a-cz_1}{a-cz_2}\cdot\frac{z-z_1}{z-z_2} = k\frac{z-z_1}{z-z_2},$$

where $k = (a-cz_1)/(a-cz_2)$.

For the second part, z_1 is the only fixed point. Then, the equation $cz^2-(a-d)z-b=0$ has only one root z_1, say. So, $cz_1^2-(a-d)z_1-b=0 \Leftrightarrow cz_1^2-az_1=b-dz_1$, and z_1 (being the repeated root) is given by

$$z_1 = \frac{a-d}{2c} \Leftrightarrow a-cz_1 = d+cz_1. \tag{9.22}$$

From the previous analysis,

$$\frac{1}{w-z_1} = \frac{cz+d}{(a-cz_1)(z-z_1)}.$$

which on simplification (adding and subtracting cz_1 in numerator and using (9.22)) gives

$$\frac{1}{w-z_1} = k' + \frac{1}{z-z_1}, \quad k' = \frac{c}{a-cz_1} = \frac{2c}{a+d}.$$

Remarks. 1. Equations (9.18) and (9.19) are known as the *normal* form or *canonical* form of a bilinear transformation.

2. The bilinear transformation with two fixed points such that $|k| = 1$, $k \neq 1$ is said to be elliptic, while for $k > 0$ it is termed hyperbolic. The bilinear transformations with one fixed point are classified as parabolic.

3. Once z_1 and z_2 are known for (9.18) or (9.19), the fact that $z = -d/c$ maps $\omega = \infty$ facilitates the computation of k or k'.

A bilinear transformation that is neither hyperbolic, elliptic, nor parabolic is called *loxodromic*, i.e. it has two fixed points and satisfies the condition $k = ae^{i\alpha}$, $\alpha \neq 0$, $a \neq 1$.

Example 13. Find all the bilinear transformations which have fixed points as -1 and 1.

From (9.13) and Definition 4, we have

$$T(1) = 1 \Rightarrow a+b = c+d \Rightarrow a+b-c-d = 0,$$
$$T(-1) = -1 \Rightarrow -a+b = -(-c+d) \Rightarrow -a+b-c+d = 0.$$

On solving these equations, we get $a = d$, $b = c$. Hence,

$$T(z) = \frac{az+b}{bz+a}, \qquad a^2-b^2 \neq 0.$$

Note. All bilinear transformations which have fixed points as -1 and 1 may also be directly computed using Proposition 5.

Cross ratio. The next step is to develop the specific bilinear transformation which maps three distinct points in the extended z-plane onto three distinct points in the extended w-plane. For this purpose we introduce the concept of cross ratio.

Definition 5. For three distinct complex numbers z_1, z_2, z_3 in $\mathbb{C}_\infty$, the cross ratio of four points z, z_1, z_2, z_3 is defined to be

$$(z, z_1, z_2, z_3) = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}. \tag{9.23}$$

If one of the numbers in (9.23) is replaced by infinity, say z_3, then

$$(z, z_1, z_2, \infty) = \lim_{z_3\to\infty} \frac{(z-z_1)(z_2/z_3-1)}{(z_1-z_2)(1-z/z_3)} = -\frac{z-z_1}{z_1-z_2}.$$

This means that the factors involving z_3 are replaced by -1. To recollect cross ratio (9.23), we write differences of z, z_1, z_2, z_3 in cyclic order $z-z_1, z_1-z_2, z_2-z_3, z_3-z$ and then put them in the numerator and the denominator alternately.

Remarks. 1. The four points z_1, z_2, z_3, z_4 can be ordered (permuted) in $4! = 24$ ways. Thus, 24 cross products are possible. But the reader is advised to verify that only six of these are different.

2. The cross ratio (z, ∞, z_2, z_3) is real $\Leftrightarrow z, z_1 = \infty, z_2, z_3$ are on line with ∞. For this assertion, we observe that

$$\begin{aligned}(z, \infty, z_2, z_3) \text{ is real} &\Leftrightarrow -\frac{z_2-z_3}{z_3-z} = \frac{1}{t}, \quad \text{for some real } t\\ &\Leftrightarrow t(z_2-z_3) = z-z_3\\ &\Leftrightarrow (1-t)z_3 + tz_2 = z.\end{aligned}$$

Thus, z lies on the line passing through z_2 and z_3.

Theorem 3. The cross ratio is invariant under bilinear transformation.

Proof. Let the bilinear transformation be defined by

$$w = T(z) = \frac{az+b}{cz+d}, \qquad ad-bc = 1$$

such that $w_k = T(z_k)$, $k = 1, 2, 3$, then we have to show that

$$(T(z), T(z_1), T(z_2), T(z_3)) = (z, z_1, z_2, z_3). \tag{9.24}$$

Since z_k corresponds to w_k, assuming $ad-bc = 1$, we have

$$w - w_k = \frac{z-z_k}{(cz_k+d)(cz+d)}$$

so that for any pair of (z, w),

$$w-w_1 = \frac{z-z_1}{(cz_1+d)(cz+d)}, \qquad w-w_3 = \frac{z-z_3}{(cz_3+d)(cz+d)}. \tag{9.25}$$

Replacing w by w_2, and z by z_2,

$$w_2-w_1 = \frac{z_2-z_1}{(cz_1+d)(cz_2+d)}, \qquad w_2-w_3 = \frac{z_2-z_3}{(cz_3+d)(cz_2+d)}. \tag{9.26}$$

Equations (9.25) and (9.26) yield,

$$\frac{w-w_1}{w_2-w_3}\,\frac{w_1-w_2}{w_3-w} = \frac{z-z_1}{z_2-z_3}\,\frac{z_1-z_2}{z_3-z}.$$

and this is nothing but

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3).$$

Thus, (9.24) is established.

As an application of this theorem we prove a very interesting result.

Theorem 4. There exists a unique bilinear transformation which maps three given distinct points z_1, z_2 and z_3 in the extended z-plane onto three specified distinct points w_1, w_2 and w_3 in the extended w-plane, respectively.

Proof. Consider the expression

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}. \tag{9.27}$$

This transformation has been obtained in Theorem 3 under the condition that z_k is mapped onto w_k, $k = 1, 2, 3$. However, we give an alternative approach to exhibit the same phenomenon.

Writing (9.27) in the form

$$(w-w_1)(w_2-w_3)(z_1-z_2)(z_3-z) = (w_1-w_2)(w_3-w)(z-z_1)(z_2-z_3), \tag{9.28}$$

we have, for $z = z_1 \Rightarrow w = w_1$ and $z = z_3 \Rightarrow w = w_3$. In the case of $z = z_2$, (9.28) yields

$$(w-w_1)(w_2-w_3) = (w_3-w)(w_1-w_2).$$

The above equation has a unique solution $w = w_2$. Thus, z_1, z_2 and z_3 are mapped onto w_1, w_2 and w_3, respectively. Moreover, (9.28) is a bilinear transformation, since

$$F(z) = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(z_2-z_3)z - z_1(z_2-z_3)}{(z_1-z_2)z + z_3(z_1-z_2)},$$

and since z_1, z_2, z_3 are distinct,

$$z_3(z_1-z_2)(z_2-z_3) - z_1(z_1-z_2)(z_2-z_3) = (z_1-z_2)(z_2-z_3)(z_3-z_1) \neq 0.$$

Hence $F(z)$ is a bilinear transformation.

By similar reasoning,

$$G(w) = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

is a bilinear transformation.

Now, we can write (9.27) as

$$G(w) = F(z) \quad \text{or} \quad w = G^{-1}F(z).$$

The map $G(w)$ is a bilinear transformation and this implies that G^{-1} (inverse map) is bilinear transformation. Again, $G^{-1}F$ is a bilinear transformation (being a composition of two bilinear transformations). Thus, (9.27) which is of the form $w = G^{-1}F(z)$ is a bilinear transformation.

Uniqueness. Let T and S be two bilinear transformations which map z_1, z_2 and z_3 onto w_1, w_2 and w_3, respectively.

$$T(z_1) = w_1, \quad T(z_2) = w_2, \quad T(z_3) = w_3,$$
$$S(z_1) = w_1, \quad S(z_2) = w_2, \quad S(z_3) = w_3.$$

Consider the map $S^{-1}T$. We know that $S^{-1}T$ is a bilinear transformation (being composition of bilinear maps) and

$$(S^{-1}T)(z_1) = S^{-1}(T(z_1)) = S^{-1}(w_1) = z_1;$$

$$(S^{-1}T)(z_2) = S^{-1}(T(z_2)) = S^{-1}(w_2) = z_2;$$
$$(S^{-1}T)(z_3) = S^{-1}(T(z_3)) = S^{-1}(w_3) = z_3$$

which implies that the bilinear map $S^{-1}T$ has three distinct fixed points. But except I, no bilinear transformation can have more than two fixed points. So $S^{-1}T = I$. Similarly, it can be shown that $T^{-1}S = I$ and hence $S^{-1}T = I = T^{-1}S \Rightarrow S = T$, i.e. T is unique.

Example 14. Find the bilinear transformation which (a) maps the points $-1, 0, 1$ onto points $0, i, 3i$; (b) maps $i, 1, -1$ onto $1, 0, \infty$, respectively.

(a) Substituting these points in (9.27), we have

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

which on simplification yields

$$w = -\frac{3i(z+1)}{z-3}.$$

(b) Since $w_3 = \infty$, we take the quotient involving w_3 in (9.27) to be -1. Thus,

$$-\frac{w-1}{1-0} = \frac{(z-i)(1+1)}{(i-1)(-1-z)},$$

and after simplifying,

$$w = i\,\frac{1-z}{1+z}.$$

is the required bilinear transformation.

Example 15. Find a bilinear transformation that maps the crescent-shaped region that lies inside the circle $|z-2| < 2$ and outside the circle $|z-1| > 1$ onto a horizontal strip in the upper half-plane.

For convenience, choose the points $z_1 = 4$, $z_2 = 2 + 2i$, $z_3 = 0$ on the circle $|z-2| = 2$ in the xy-plane, and the image values $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$ in the uv-plane, respectively. The ordered triplet z_1, z_2, z_3 gives the positive orientation as the region of the disc lies to the left. The image points w_1, w_2, w_3 all lie on the extended u-axis, and the region to be mapped should remain left to the observer and it happens to be the upper half-plane $\operatorname{Im} w > 0$. Hence we can use the implicit formula (9.27) to write

$$-\frac{w-0}{0-1} = \frac{(z-4)(2+2i-0)}{(4-2-2i)(0-z)}.$$

After simplification, this yields

$$w = \frac{-iz+4i}{z}.$$

A straightway substitution shows that $z_4 = 1-i$, $z_5 = 2$ and $z_6 = 1+i$ are mapped by the above bilinear transformation on the points $w_4 = -2+i$, $w_5 = i$ and $w_6 = 2+i$. The points w_4, w_5 and w_6 lie on the horizontal strip $v = 1$ in the upper half-plane. As the

region to be mapped lies to the right of z_4, z_5 and z_6, we conclude that the image region is to the right of line Im $w = 1$. Both the regions are shown in Fig. 9.8.

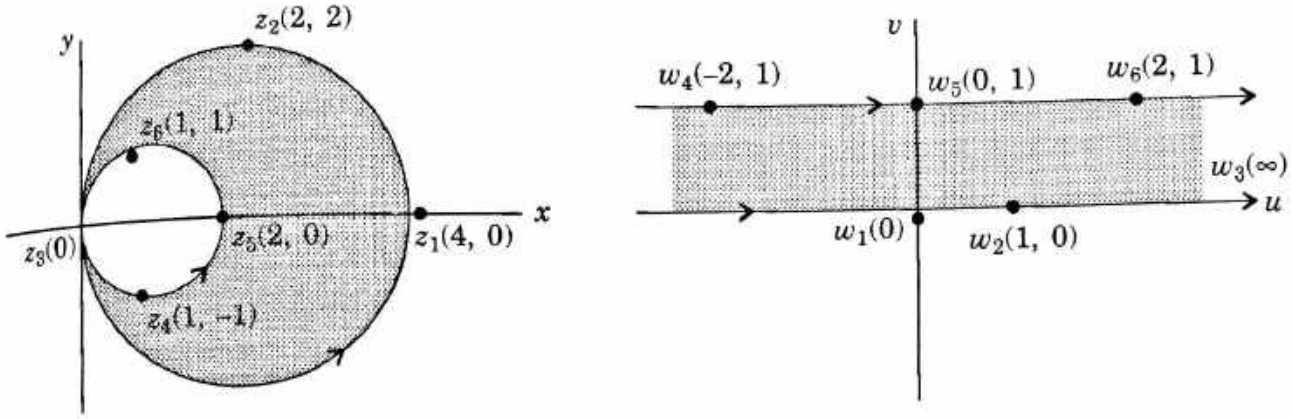

Figure 9.8

PROBLEM SET 9.3

1. Find the image of the line $y = x + 1$ under the transformation $w = (z + i)/(z - 1)$.
2. Find the image of the region Re $z \leq 0$ under the transformation $w = (2z + i)/(iz - 1)$.
3. Show that $w = (z - 1)/(z + 1)$ maps $x > 0$ and the boundary $x = 0$ on $|w| \leq 1$.
4. Let $w = i(1 - z)/(1 + z)$. Then show that Im $w > 0$ if Im $z < 1$.
5. Consider a point z inside a circle C centred at origin and of radius r and another circle C_1 centred at z_0 and of radius R containing the inverse point to z with respect to C. Prove that two successive inversions of the point z about C and C_1, respectively, can be expressed as a bilinear transformation.
6. Determine all bilinear transformations which have fixed points as $-i$ and i.
7. Find the bilinear transformation with fixed points -1 and 1 carrying the points i and $-i$ onto $-i$ and i, respectively.
8. Express the bilinear transformation

$$w = \frac{13iz + 75}{3z - 5i}$$

in the form

$$\frac{w - \alpha}{w - \beta} = k\frac{z - \alpha}{z - \beta},$$

where α, β and k are constants.

Suggestion. Find α and β as fixed points of the bilinear transformation and then evaluate k (see Proposition 5).

9. If the points z_1, z_2, z_3 lie on a line, show that the cross ratio (z, z_1, z_2, z_3) is real $\Leftrightarrow$ either $z = \infty$ or z lies on the line passing through z_1, z_2, z_3.

10. The four points z, z_1, z_2, z_3 in $\mathbb{C}_\infty$ lie on a line or a circle, i.e. are cocylic if and only if their cross ratio (z_2, z, z_3, z_1) is a real number.

Suggestion. Graph a circle with four points $A(z)$, $B(z_1)$, $C(z_2)$, $D(z_3)$ in cyclic order and use the geometrical result: These points are cocylic $\Leftrightarrow \angle ADB = \angle ACB$.

11. Let z_1, z_2, z_3, z_4 be four points in the complex plane such that $z_1z_2 + z_3z_4 = 0$ and $z_1 + z_2 = 0$. Then prove that these points are cocylic.

12. Construct a bilinear transformation which maps (a) $\{\infty, i, 0\}$ onto $\{0, 1, \infty\}$, (b) $\{-1, 0, 1\}$ onto $\{-i, 1, i\}$, (c) $\{0, 1, \infty\}$ onto $\{w_1, w_2, w_3\}$. Show that these maps are unique and further, find the image of $y = c$, $c > 0$ under the transformation (a).

13. Find the bilinear transformation which maps $1, i, -1$ on $i, 0, -i$, respectively. For this transformation, find the images of (a) $|z| \le 1$; (b) $|z| = r$, $r > 1$.

14. Find the condition that $w = (az + b)/(cz + d)$ maps the unit circle in the w-plane into a straight line in the z-plane.

15. With the help of Eq. (9.27), prove that if a bilinear transformation maps each point on the x-axis onto the u-axis, then the coefficients in the transformation are all real, except possibly for a common complex factor.

Suggestion. Map real numbers x_1, x_2, x_3 onto $-1, 0, \infty$.

16. Show that the set of all bilinear transformations forms a non-abelian group with respect to operation as composition of maps.

17. Let $T(z) = (az + b)/(cz + d)$, $ad - bc \neq 0$ be a bilinear transformation other than identity transformations. Show that $T^{-1} = T$ if and only if $d = -bc$.

Suggestion. Write $T^{-1}(z) = T(z)$ and note that $(a + d)[cz^2 + (d - a)z - b] = 0$ justifies the assertion.

18. Show that (9.14) can be written in the form (9.27).

Suggestion. Compute $w = G^{-1}(F(z))$, where F and G are the functions defined in Theorem 4, Section 9.3.

19. For $c > 0$, show that the transformation

$$w = \left(\frac{z - ic}{z + ic}\right)^2$$

maps the right-half of the circle $|z| = c$ into the upper-half of the w-plane. Note that the exterior of the circle $|z| = c$ in the half-plane $\operatorname{Re} x \le 0$ is also mapped into the upper-half of the w-plane.

20. What is the form of bilinear transformation which has α and ∞ as its fixed points? Conclude this transformation if ∞ is the only fixed point.

21. Show that $\int (1/y)\,ds$ is an inverse with respect to bilinear map (9.14), where $ds = \sqrt{(x^2 + y^2)}$.

22. Suppose that a bilinear transformation $T(z)$ has two fixed points z_1 and z_2. Prove that

$$T'(z_1)T'(z_2) = 0.$$

Suggestion. Use (9.18).

9.4 SPECIAL BILINEAR TRANSFORMATIONS

In this section we develop general bilinear transformations which map the specific region in the z-plane onto a given region in the w-plane.

Proposition 6. The set of all bilinear transformations that map the upper half-plane Im $z > 0$ onto $|w| < 1$ and the boundary Im $z = 0$ onto the boundary $|w| = 1$ is given by

$$w = e^{i\alpha}\frac{z - z_0}{z - \bar{z}_0}, \quad \text{Im } z_0 > 0, \quad \alpha \in \mathbb{R}, \tag{9.29}$$

where the point z_0 in the upper half-plane is mapped onto the centre of the unit disc.

Proof. Let the bilinear transformation be

$$w = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

Note that the points z and $\bar{z}$ are symmetric (inverse) with respect to the real axis and must correspond to w and $1/\bar{w}$ inverse with respect to the unit circle (see Example 12).

In particular, we observe that $w = 0$ and $w = \infty$ are the images of the points $z = -b/a$ and $z = -d/c$, respectively, and these, being inverse to each other, can be expressed as

$$-\frac{b}{a} = z_0 \qquad \text{and} \qquad -\frac{d}{c} = \bar{z}_0.$$

for some complex constant z_0.

The constant $c \neq 0$, otherwise the points at ∞ will not be mapped on the boundary $|w| = 1$. So, (9.14) can be expressed as

$$w = \frac{a}{c}\frac{z + b/a}{z + d/c}. \tag{9.30}$$

With this information, the bilinear map (9.30) takes the form

$$w = \frac{a}{c}\frac{z - z_0}{z - \bar{z}_0}.$$

Now, $z = 0$ on the boundary corresponds to the point on the unit circle $|w| = 1$. So,

$$1 = |w| = \left|\frac{a}{c}\right|\left|\frac{-z_0}{-\bar{z}_0}\right| = \left|\frac{a}{c}\right| \Rightarrow \frac{a}{c} = e^{i\alpha}, \quad \alpha \in \mathbb{R}.$$

Further, the transformation assumes the form

$$w = e^{i\alpha}\frac{z - z_0}{z - \bar{z}_0}, \quad \alpha \in \mathbb{R}. \tag{9.31}$$

From (9.30), the point $z = z_0$ corresponds to $w = 0$, and hence z_0 must be in the upper half-plane, i.e. Im $z_0 > 0$. This information is added to have (9.29).

Thus, (9.29) gives the set of all bilinear transformations which map Im $z \geq 0$ onto $|w| \leq 1$ (Fig. 9.9).

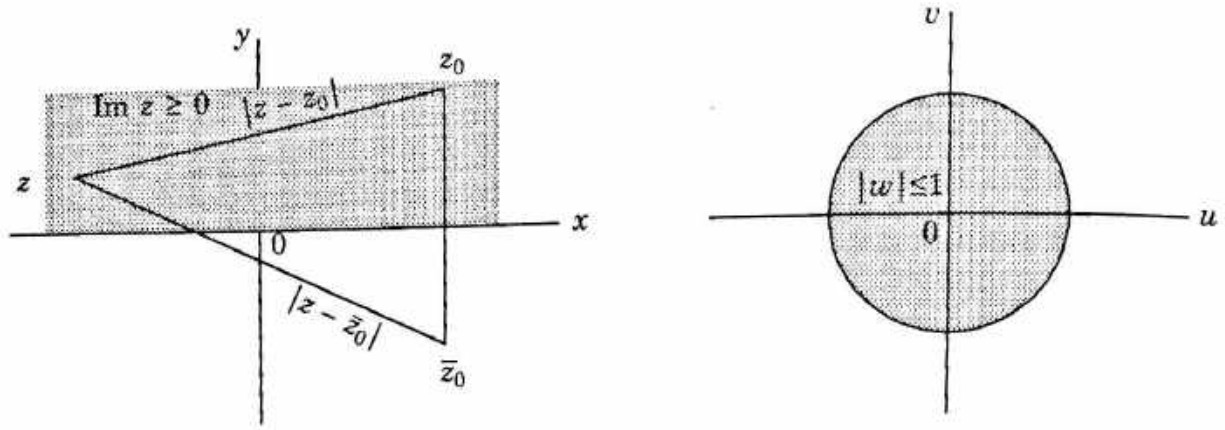

Figure 9.9

Verification. To verify that every transformation of the form represented by (9.29) has the desired property. Taking absolute value of each side in (9.29), observe that

$$|w| = \frac{|z - z_0|}{|z - \bar{z}_0|}.$$

If any point z lies above the real axis, then z and z_0 lie on the same side of the real axis (which is a perpendicular bisector of the line joining z_0 and $\bar{z}_0$). This implies that $|z - z_0| < |z - \bar{z}_0|$ and hence $|w| < 1$. For any z on the real axis $|w| = 1$, as $|z - z_0| = |z - \bar{z}_0|$. Since any bilinear transformation is a one-to-one map on $\mathbb{C}_\infty$, this establishes the fact that transformation (9.29) maps the half-plane Im $z > 0$ onto the disc $|w| < 1$ and the boundary of the half-plane onto the boundary of this unit disc.

Remark. What happens when Im $z_0 > 0$ in (9.29) is replaced by Im $z_0 < 0$?

$$\begin{aligned}
|w| < 1 &\Leftrightarrow |z - z_0| < |z - \bar{z}_0| \\
&\Leftrightarrow |z|^2 + |z_0|^2 - z\bar{z}_0 - \bar{z}z_0 < |z|^2 + |z_0|^2 - z\bar{z}_0 - zz_0 \\
&\Leftrightarrow \text{Re } z\bar{z}_0 > \text{Re } z_0 z \\
&\Leftrightarrow 2y_0 y > 0 \\
&\Leftrightarrow y < 0, \quad \text{since} \quad y_0 < 0.
\end{aligned}$$

Hence the lower half-plane is mapped on $|w| < 1$ and the boundary $y = \text{Im}\, z = 0$ is mapped on $|w| = 1$ such that $z = z_0$ is mapped onto the centre of the disc $|w| \leq 1$.

Example 16. Find the image of Im $z \geq 0$ under the transformation

$$w = \frac{i - z}{i + z}.$$

Let us write the transformation in the form

$$w = e^{i\pi}\,\frac{z-i}{z+i}.$$

Comparing with (9.29), we note that $\alpha = \pi$ and $\operatorname{Im} z_0 = 1 > 0$. Hence this bilinear transformation maps $y \geq 0$ onto $|w| \leq 1$.

Example 17. Find all the bilinear transformations which map the right half-plane $\operatorname{Re} z > 0$ onto the open disc $|w| < 1$ and the boundary $\operatorname{Re} z = 0$ onto the boundary $|w| = 1$.

The right half-plane, $\operatorname{Re} z > 0$ and $\operatorname{Re} z = 0$ are rotated by $\pi/2$ to get the upper half-plane with boundary as $\operatorname{Im} z = 0$. This transformation is

$$f(z) = iz, \quad \operatorname{Re} z \geq 0.$$

The bilinear transformations which map $\operatorname{Im} z \geq 0$ onto $|w| \leq 1$ are given by

$$g(z) = e^{i\alpha}\,\frac{z-z_0}{z-\bar{z}_0}, \qquad \operatorname{Im} z_0 > 0, \quad \alpha \in \mathbb{R}.$$

Hence, the required transformations $g(f(z))$ are of the form

$$g(f(z)) = g(iz) = e^{i\alpha}\,\frac{iz-z_0}{iz-\bar{z}_0}, \quad \operatorname{Im} z_0 > 0, \qquad \alpha \in \mathbb{R}.$$

Proposition 7. The set of all bilinear transformations which map unit disc onto itself is given by

$$w = e^{i\alpha}\frac{z-z_0}{1-\bar{z}_0 z}, \quad |z_0| < 1, \quad a \in \mathbb{R}. \tag{9.32}$$

Such that any point z_0 interior to unit disc is mapped onto the centre of another disc.

Proof. Clearly, $c \neq 0$, otherwise the points at infinity in the two planes would correspond. So, (9.14) can be written in the form of (9.30). We have seen in Example 12 that a bilinear transformation maps a circle and a line of the z-plane into a circle and a line of the w-plane and inverse points are transformed into inverse points.

Now, $w = 0$ and $w = \infty$ are the inverse points of $|w| = 1$ and these are the images of $z = -b/a$ and $-d/c$, respectively (refer to Example 12). Hence, we may write (see Example 6, Chapter 1.1),

$$\frac{-b}{a} = z_0; \quad \frac{-d}{c} = \frac{1}{\bar{z}_0}, \qquad |z_0| < 1.$$

So, (9.30) can be further written as

$$w = \frac{a}{c}\,\frac{z-z_0}{z-1/\bar{z}_0} \quad \text{or} \quad w = -\frac{a\bar{z}_0}{c}\,\frac{z-z_0}{1-\bar{z}_0 z}. \tag{9.33}$$

The point $z = 1$ on the boundary of $|z| = 1$ must correspond to a point on the boundary of $|w| = 1$. Thus, in view of (9.33) and the fact that $|1-z_0| = |1-\bar{z}_0|$, we have

$$1 = |w| = \left|\frac{-a\bar{z}_0}{c}\right|\left|\frac{1-z_0}{\bar{z}_0-1}\right| = \left|\frac{a\bar{z}_0}{c}\right| \quad \text{or} \quad -\frac{a\bar{z}_0}{c} = e^{i\alpha},\ \alpha \in \mathbb{R}.$$

Hence (9.33) becomes (9.32), and this is what we required.

Verification: We show that $|z| \le 1$ is mapped onto $|w| \le 1$. For any z on the unit circle $z = e^{i\theta}$, we observe that

$$w = |e^{i\alpha}|\left|\frac{e^{i\theta}-z_0}{1-\bar{z}_0 e^{i\theta}}\right| = \left|\frac{e^{i\theta}-z_0}{e^{-i\theta}-\bar{z}_0}\right| = 1.$$

Hence $|z| < 1$ is mapped onto one of the complimentary domains of $|w| = 1$. Because $|z_0| < 1$ is mapped onto $w = 0$, this domain is $|w| < 1$.

PROBLEM SET 9.4

1. Let $w = (z+i)/(iz+1)$. Then show that $\operatorname{Im} z \le 0 \Rightarrow |w| \le 1$.

2. Find all transformations which map the sector $0 < \arg z < \pi/3$ on $|w| < 1$.

Suggestion. Sector $0 < \arg z < \pi/3$ is mapped by $g(z) = z^3$ on $y > 0$. Now, $f(g(z))$, where $f(z)$ is given by (9.29), is the required one. Note that $g(f(z))$ is not bilinear.

3. Find all transformations which map the fourth quadrant onto $|w| \le 1$.

Suggestion. The function $h(z) = iz$ maps $x \ge 0$, $y \le 0$ onto $x \ge 0$, $y \ge 0$. Also, $g(z) = z^2$ maps $x \ge 0$, $y \ge 0$ on $y \ge 0$. The region $y \ge 0$ is mapped on $|w| \le 1$ by $f(z)$ in (9.29). Hence the required map: $(fgh)(z)$.

4. Construct the general bilinear transformation which maps the upper half-plane (a) onto itself; (b) onto lower half-plane.

5. Find a bilinear transformation which maps the upper half-plane of the z-plane onto the unit disc in the w-plane in such a way that $z = i$ is mapped on $w = 0$ while the point at infinity is mapped onto $w = -1$.

6. Show that the general bilinear transformation that maps the disc $|z| \le r$ onto the disc $|w| \le r'$ is given by

$$w = rr'e^{i\alpha}\frac{z-\beta}{r^2-\bar{\beta}z}, \qquad |\beta| < r,\ \alpha \in \mathbb{R}.$$

Suggestion. The set of all bilinear transformations which maps $|z| \le 1$ onto $|w| \le 1$ is given by (9.32). To map $|z| \le r$ onto $|w| \le r'$, replace z by z/r and w by w/r' in (9.32) and set $\beta = rz_0$.

7. Show that the region $|z - z_0| \le R$ is mapped conformally onto $|w| \le 1$ by

$$w = e^{i\alpha}\frac{R(z-\beta)}{R^2-(z-z_0)(\bar{\beta}-\bar{z}_0)}, \quad |\beta - z_0| < R,$$

where α is real and β is the point which is transformed onto origin.

Suggestion. Consider (9.32) in the form $w = e^{i\alpha}(z-p)/(1-\bar{p}z)$, $|p| < 1$, $\alpha \in \mathbb{R}$. Now, replace z by $(z-z_0)/R$ and set $z_0 + pR = \beta$.

8. Determine the bilinear transformation which has 1 and -1 as its fixed points and maps $|z| \le 1$ onto $|w| \le 1$.

9. Find an analytic function $w = f(z)$ which maps the region $2 \le y \le x + 1$ in the z-plane onto the disc $|w| \le 1$.

   Suggestion. Take $f_1(z) = z - (1 + 2i)$, $f_2(z) = z^4$, $f_3(z) = e^{i\alpha}(z - z_0)/(z - \bar{z}_0)$, $\operatorname{Im} z_0 > 0$, $\alpha \in \mathbb{R}$. Consider the composition $f_3 f_2 f_1$.

10. Map the upper half-plane $\operatorname{Im} z > 0$ onto the unit disc $|w| < 1$ in such a way that $w(2i = 0)$, $\operatorname{Arg} w'(2i) = 0$.

11. Map the disc $|z - 4i| < 2$ onto the half-plane $v > u$ in such a way that the centre of the disc goes into the point -4, while the point $2i$ on the circumference of the disc goes into the origin.

12. Find the transformation which maps the domain $|z| < 1$, $0 < \arg z < \pi/3$ conformally on $|w| < 1$.

13. Suppose an analytic function $f(z)$ maps a unit disc onto itself. How many fixed points does $f(z)$ have? Locate these points if a nonzero point α interior to one unit disc is mapped onto the origin of the other disc.

14. Show that the transformation

$$w = \left(\frac{z+i}{z-i} + 1\right)^2$$

maps the real axis in the z-plane onto a cardioid in the w-plane. Indicate the region of z-plane which corresponds to the interior of the cardioid.

Suggestion. This transformation can be thought of as the composition of the following maps:

$$\xi = \frac{z - (-i)}{z - (-\bar{i})}, \qquad \zeta = \xi + 1, \qquad w = \zeta^2.$$

15. Find a bilinear transformation which maps $x + y = 1$ onto $|w - 1| = 2$. Is it unique? If not, find the other one. Also carry out the analysis so that the lower half-plane in the z-plane is mapped onto the disc $|w - 1| < 2$.

In Sections 9.1–9.4 we dealt with elementary functions and bilinear maps. To make the chapter self contained, the discussion will be concluded with reasonable exposition of the mappings by some well-known complex functions.

9.5 EXPONENTIAL TRANSFORMATION

The map $w = e^z$ can be written as $\rho e^{i\phi} = e^x e^{iy}$. Thus,

$$\rho = e^x, \qquad \phi = y + 2k\pi, \qquad k \in \mathbb{I};$$

and hence the above transformation from z to w can be expressed as $\rho = e^x$, $\phi = y$. Note that this transformation is conformal for every point of the z-plane, since the derivative of e^z does not vanish anywhere.

The image of a typical point (c_1, y) on any vertical line $x = c_1$ has the polar coordinates $\rho = e^{c_1}$ and $\phi = y$. The point moves counterclockwise around the circle as z moves upward. Thus, the image of the line is the entire circle and every point on the circle is the image of infinite number of points on the line which are 2π spaced. Similarly, a horizontal line $y = c_2$ is mapped in a one-to-one manner on the ray $\phi = c_2$. As the point moves on the line from left to right, the image point (e^x, c_2) moves on the ray $\phi = c_2$ away from the origin.

Thus, the vertical and horizontal line segments are mapped onto portion of circles and rays, respectively. Since e^z is $2\pi i$ periodic, it maps every horizontal strip

$$n2\pi \le y \le (n+1)2\pi$$

onto a region of the w-plane with slit in the same manner. This means that the points z, which differ from one another by $2\pi i$, are therefore mapped onto the same point (Fig. 9.10).

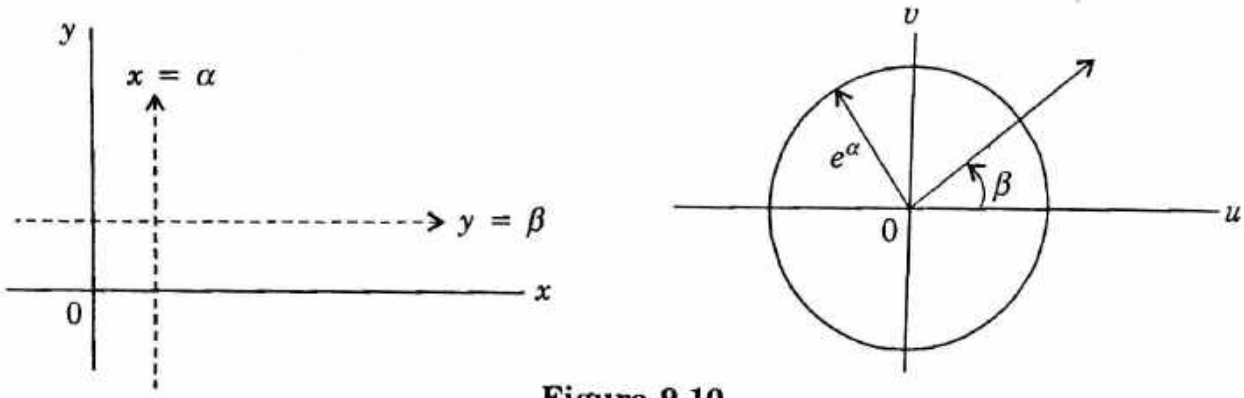

Figure 9.10

Example 18. Find the image of the line $z = (1 + ia)t + ib$, $-\infty < t < \infty$ under the transformation $w = e^z$, given that the line $z = (1 + ia)t + ib$ is not parallel to one of the coordinate axes.

Here, a and b are fixed real numbers and $a \neq 0$ is the slope of the line and b is the y-intercept. This can be verified by eliminating t from the parametric equation $x = t$ and $y = at + b$.

The image of this line under $w = e^z$ is the curve

$$w = e^{t+i(at+b)} = e^t[\cos(at+b) + i\sin(at+b)].$$

Therefore,

$$|w| = \rho = e^t, \qquad \phi = \arg w = at + b + 2k\pi,$$

and eliminating the parameter t, it follows that

$$\rho = e^{(\phi-b-2k\pi)/a}.$$

If we set $\theta = \phi - 2k\pi$, then $\rho = ce^{\theta/a}$, where $c = e^{-b/a}$. This is the equation of logarithmic spiral in the polar form. Since, the mapping is conformal and the line intersects all the lines parallel to the x-axis at the angle $\tan^{-1}a$, it follows that the spiral intersects the image of all these lines at the same angle, i.e. all the rays emanating from the origin are intersected at an angle $\tan^{-1} a$ by logarithmic spiral (Fig. 9.11).

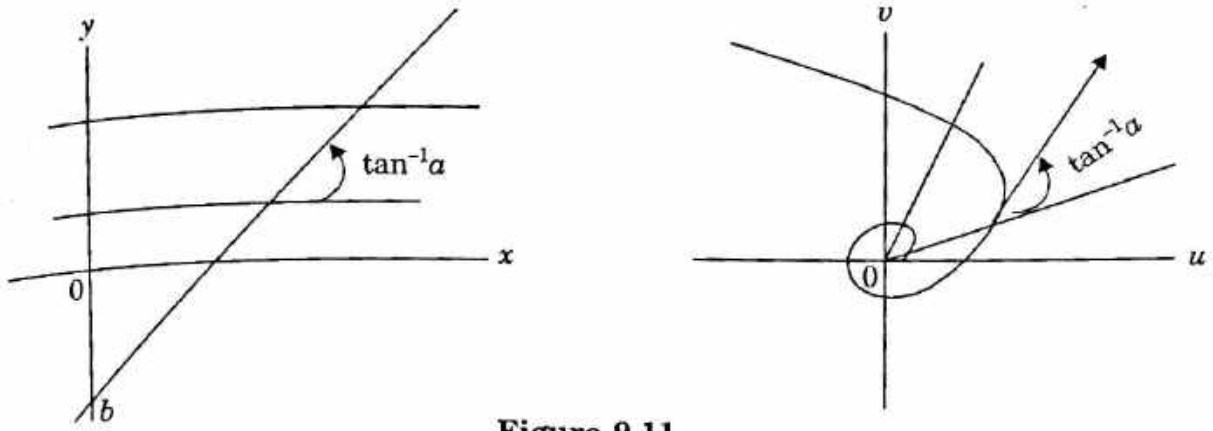

Figure 9.11

Example 19. Find the image of the rectangular region $a \le x \le b$, $c \le y \le d$ under the map $w = e^z$.

The vertical line segment $AD: x = a$, $c \le y \le d$ is mapped on the arc $\rho = e^a$, $c \le \phi \le d$ which is labelled as $A'D'$. Similarly, the vertical line segment BC: $x = b$, $c \le y \le d$, is mapped on the larger arc $\rho = e^b$, $c \le \phi \le d$ which is labelled as $B'C'$. The horizontal line segments AB and DC are mapped on the rays $A'B'$ and $D'C'$, respectively. The mapping is one-to-one if $d - c < 2\pi$. Figure 9.12 shows the desired region.

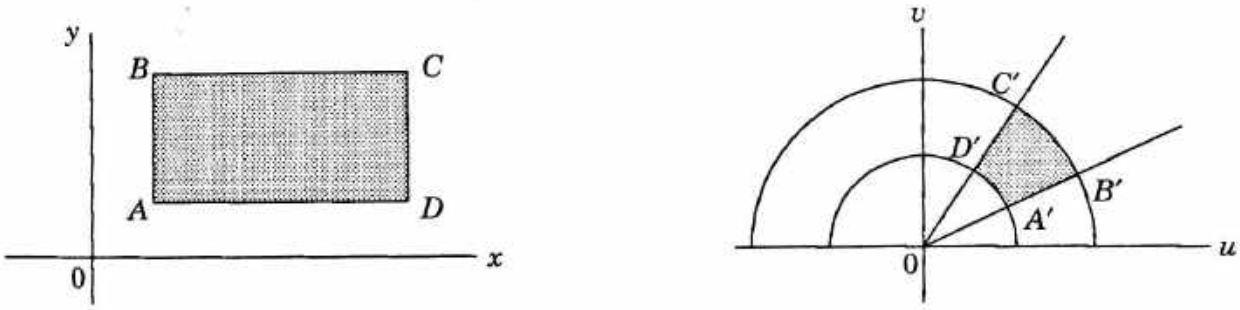

Figure 9.12

Example 20. Show that the transformation

$$w = f(z) = \frac{e^z - i}{e^z + i}$$

is one-to-one conformal mapping of the horizontal strip $0 < y < \pi$ onto the disc $|w| < 1$. Furthermore, x-axis is mapped onto the lower semi-circle bounding the disc, and the line $y = \pi$ is mapped onto the upper semi-circle.

The function $w = f(z)$ is the composition of the exponential function

$$Z = e^z$$

followed by the bilinear transformation

$$w = \frac{Z - i}{Z + i}.$$

The image of the horizontal strip $0 < y < \pi$ under the mapping $Z = e^z$ is the upper half-plane $\text{Im } Z > 0$; the x-axis is mapped onto the positive X-axis; and the line $y = \pi$ is mapped onto the negative X-axis. The bilinear transformation $w = (Z-i)/(Z+i)$ maps the upper half-plane $\text{Im } Z > 0$ on the disc $|w| < 1$, (refer to Proposition 6); the positive X-axis is mapped onto the lower semi-circle and the negative X-axis is mapped onto the upper semi-circle. Figure 9.13 illustrates the composite mapping.

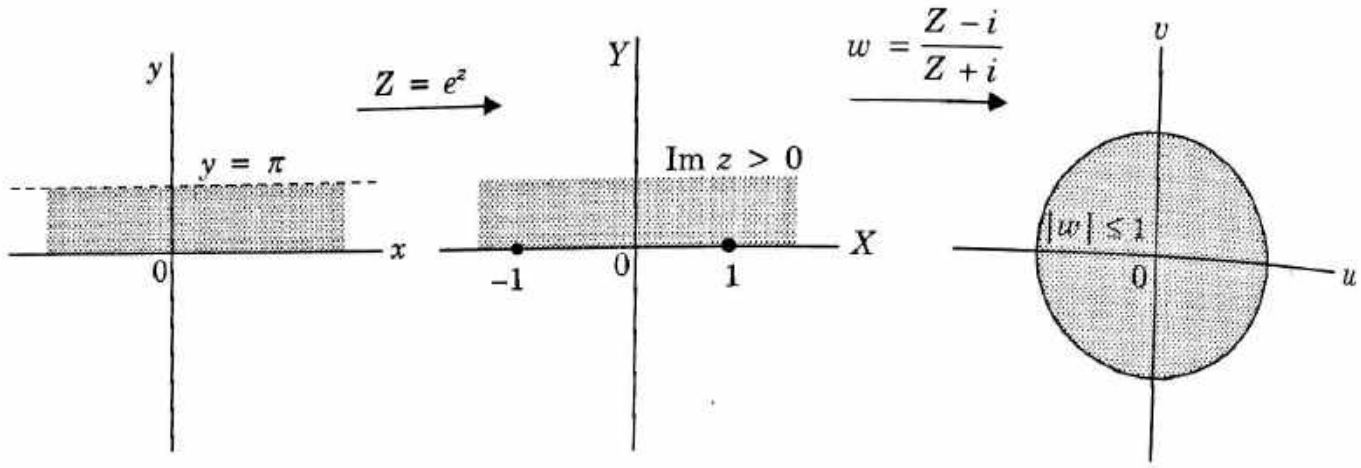

Figure 9.13

9.6 TRIGONOMETRIC TRANSFORMATION

Let us consider the map $w = \sin z$. Writing $w = u + iv$ and $z = x + iy$, we get

$$u + iv = \sin x \cosh y + i \cos x \sinh y.$$

Hence

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y. \tag{9.34}$$

Since the map $w = \sin z$ is a periodic function, this mapping is not one-to-one in the entire complex plane.

Eliminating x from (9.34), we have

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1. \tag{9.35}$$

Thus, the line $y = c_2$ is mapped on confocal ellipses with foci $w = \pm 1$.

However, the elimination of y from the same set of equations yields

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1. \tag{9.36}$$

In this case, the line $x = c_1$ is mapped on confocal hyperbolas.

The real axis $y = 0$ corresponds to $v = 0\ (-1 \le u \le 1)$ and the imaginary axis $x = 0$ corresponds to $u = 0$.

Remarks. 1. The function

$$w = \cos z = \sin\left(z + \frac{\pi}{2}\right)$$

defines a translation of $\sin z$ map to the right through $\pi/2$ units.

2. Further, in view of the relation $\sinh z = -i \sin iz$ (see Section 3.4), note that

$$w = \sinh z = -i \sin iz$$

is the combination of the following mappings:

$$Z = iz, \qquad W = \sin Z, \qquad w = -iW.$$

Hence, any figure in the z-plane is rotated by $\pi/2$ and then we find its image under the map $\sin z$. Finally, rotation of this image by $-\pi/2$ gives the desired image under $\sinh z$.

The transformation

$$w = \cosh z$$

is, likewise, essentially a cosine transformation, since $\cosh z = \cos iz$.

In view of the above remarks our concentration will remain focussed on developing the mapping properties of the function $\sin z$.

Example 21. Find the image of the domain $-\pi/2 \le x \le \pi/2$ and $-c \le y \le c$ under the transformation $w = \sin z$.

The segment of the real axis $y = 0$ $(-\pi/2 \le x \le \pi/2)$ is mapped on the segment of the real axis $v = 0$ $(-1 \le u \le 1)$ in the w-plane. Since $f'(z) = \cos z$ is zero at $z = \pm\pi/2$, the mapping is not conformal at these two points.

From (9.34), we see that the line $x = -\pi/2$ maps onto $v = 0$ $(u \le -1)$ and the line $x = \pi/2$ maps onto $v = 0$ $(u \ge 1)$.

The line segment $y = c > 0$, $-\pi/2 \le x \le \pi/2$, maps onto the portion of the ellipse (9.35) in the upper half-plane and the line segment $y = -c$, $-\pi/2 \le x \le \pi/2$, maps onto the portion of the same ellipse but in the lower half-plane. The foci of these ellipses $w = \pm 1$ are independent of c. Consequently, if we let c vary, we obtain a family of confocal ellipses.

The rectangular region defined by $-\pi/2 < x < \pi/2$ and $-c < y < c$ is mapped onto the interior of the ellipse (9.35). Note that the image of the boundary consists of the ellipse and the two segments of the u-axis as shown in Fig. 9.14 for $c = 1$. The image points of the points on the vertical parts of the boundary coincide in pairs.

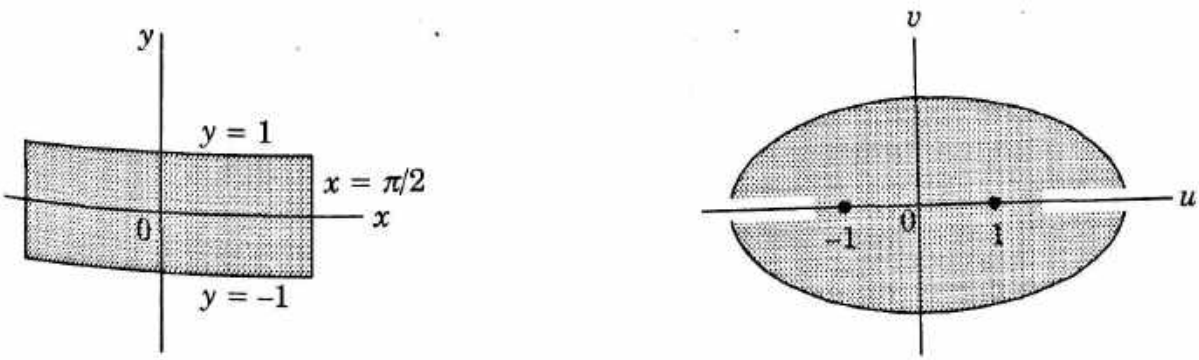

Figure 9.14

Example 22. Find the image of the $-\pi < x < \pi$ and $c < y < d$, $c > 0$ under the transformation $w = \sin z$.

Following the reasoning of the above example the rectangle $-\pi < x < \pi$, $c < y < d$, maps onto the elliptic ring cut along the negative v-axis (Fig. 9.15). The straight lines x = constant map onto confocal hyperbolas (9.36) which intersect these ellipses at right angles. The line segment of y-axis $x = 0$ $(c < y < d)$ maps onto the line segment of v-axis $u = 0$ $(\sinh c < v < \sinh d)$.

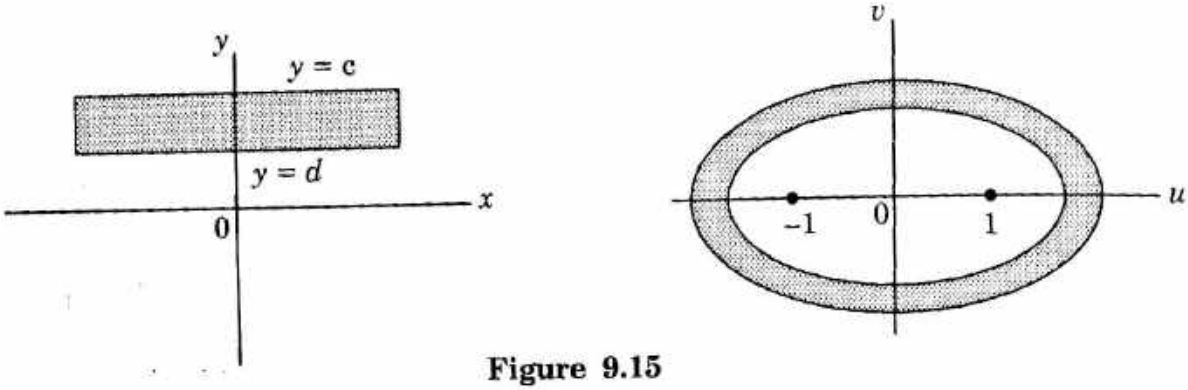

Figure 9.15

Example 23. Show that the transformation $w = \tan z$ is a one-to-one conformal mapping of the vertical open strip $|x| < \pi/4$ onto the unit disc $|w| < 1$.

We write

$$w = \tan z = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{-ie^{2iz} + i}{e^{2iz} + 1}.$$

From the above relation the function $w = \tan z$ can be considered as the composition of the map

$$Z = e^{i2z}$$

followed by the bilinear transformation

$$w = \frac{-iZ + i}{Z + 1}.$$

Now, the map $Z = \exp(i2z)$ maps the open vertical strip $|x| < \pi/4$ one-to-one and onto the right half-plane $\text{Re}\, Z > 0$.

Then the bilinear transformation $w = (-iZ + i)/(Z + 1)$ maps the right half-plane $\text{Re}\, Z > 0$ one-to-one and onto the disc $|w| < 1$ as shown in Fig. 9.16.

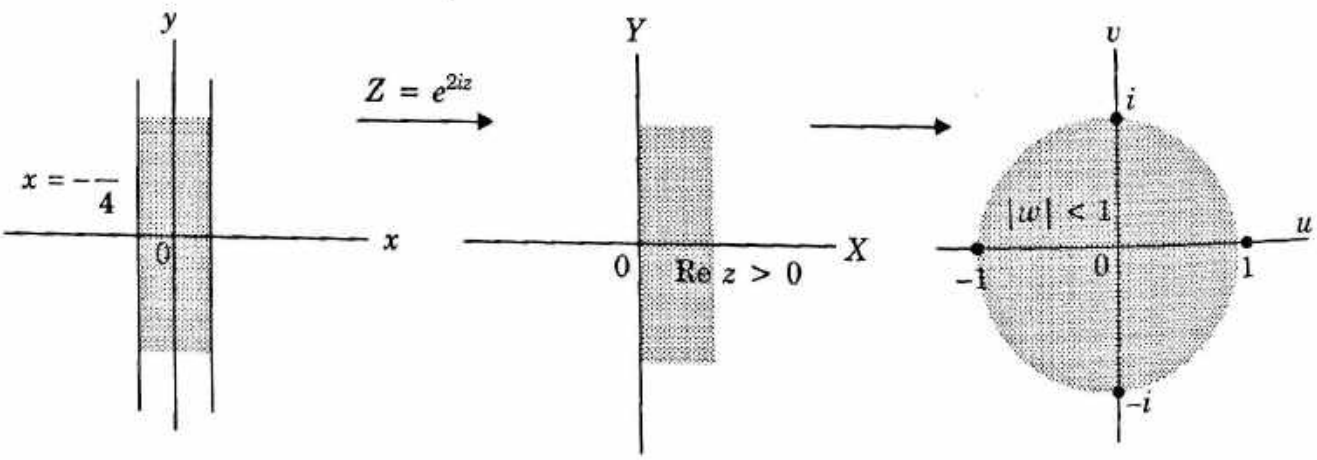

Figure 9.16

9.7 THE MAP $(z + 1/z)$

The transformation

$$w = z + \frac{1}{z}, \qquad (z \neq 0),$$

is known as Joukowski's transformation. This is used to map the exterior of the profile of an airplane wing on the exterior of a nearly circular region. Since

$$f'(z) = 1 - \frac{1}{z^2} = \frac{(z+1)(z-1)}{z^2},$$

the mapping is conformal except at the points $z = 0, \pm 1$. The points $z = 1$ and $z = -1$ correspond to $w = 2$ and $w = -2$, respectively. Solving the transformation, we have

$$z = \frac{1}{2}\{w + [(w+2)(w-2)]^{1/2}\}.$$

Hence z is a double-valued function of w. To every value of w other than ± 2, there corresponds two values of z. The mapping can be made one-one if we construct a two sheeted Riemann surface connecting the branch points $w = -2$ and $w = 2$ crosswise. On setting $w = u + iv$ and $z = re^{i\theta}$, we get

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}} = \left(r + \frac{1}{r}\right)\cos\theta + \left(r - \frac{1}{r}\right)\sin\theta.$$

Hence

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \qquad v = \left(r - \frac{1}{r}\right)\sin\theta. \tag{9.37}$$

Eliminating θ, we obtain

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1, \qquad \text{where } a = r + \frac{1}{r}, \qquad b = \left|r - \frac{1}{r}\right|. \tag{9.38}$$

Equation (9.37) reveals that the unit circle $|z| = 1$ is mapped onto the cut from $w = -2$ to $w = 2$ and this line segment is described twice as θ varies from 0 to 2π.

The circles $|z| = r$, $r \neq 1$ in the z-plane are mapped on ellipses (9.38) in the w-plane, as shown in Fig. 9.17. The principal axes of ellipses lie along the u and v axes in the w-plane and have lengths $2a$ and $2b$, respectively. These ellipses are confocal with foci $(\pm 2, 0)$, since $a^2 - b^2 = 4$ is independent of r. It is to be noted that two circles $|z| = r$ and the $|z| = 1/r$ are mapped on the same ellipse in the w-plane corresponding to the two sheets of the Riemann surface.

Hence we conclude that the region $r < |z| < r'$ is mapped conformally into the region by the two confocal ellipses.

Eliminating r from (9.37), we get

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4. \tag{9.39}$$

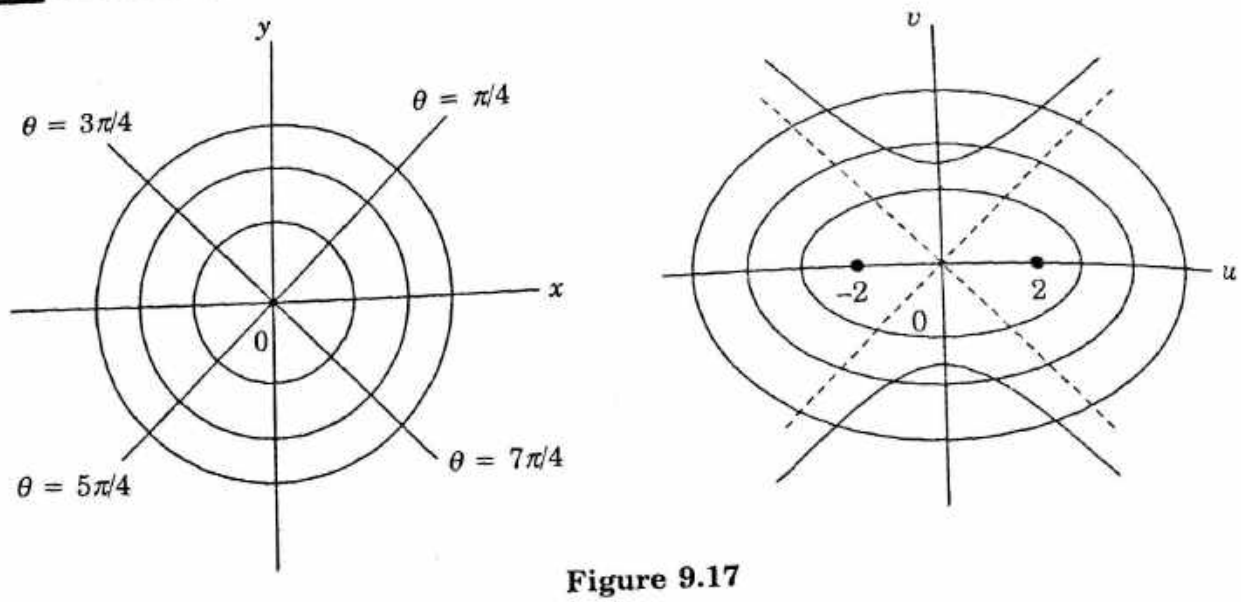

Figure 9.17

This reveals that the ray θ = constant in the z-plane is mapped onto the hyperbola (9.39) with foci $(\pm 2, 0)$ in the w-plane which intersects the ellipses orthogonally. Hence the image of the line $\theta = \alpha$, $0 < \alpha < \pi/2$ is the right-hand branch of the hyperbola, since $u > 0$ for this range of α. Any other pair of the rays $\theta = \alpha$ and $\theta = \alpha + \pi$ is mapped onto the two branch of the same hyperbola.

The image of the ray $\theta = 0$ is given by the equation $u = (r + 1/r)$ and $v = 0$. Since $(r + 1/r) \geq 2$ for all values of r, it follows that the image is the part of the u-axis $u \geq 2$. As r varies from 0 to ∞ the image segment is described twice. The image of the ray $\theta = \pi/2$ is given by the equations $u = 0$ and $v = (r - 1/r)$. As r varies from 0 to ∞ this represents the complete imaginary axis in the w-plane. Similar is the case for $\theta = -\pi/2$. Thus, the y-axis corresponds to the v-axis.

The image of the line $\theta = \pi$ is given by $u = -(r + 1/r)$, $v = 0$. Thus, the image is the part of the real u axis $-\infty < u < -2$ in the w-plane.

The image of the wedge $\alpha < \theta < \beta$, where $0 < \alpha, \beta < \pi/2$ is the region bounded by the two branches lying to the right of the imaginary axis of the hyperbolas.

Example 24. Let C be a circle in the z-plane having its centre on the real axis which passes through $z = 1$ and encloses $z = -1$ as an interior point. Determine the image of C under the transformation $w = z + 1/z$.

First, we observe that

$$\left.\frac{dw}{dz}\right|_{z=1} = 1 - \left.\frac{1}{z^2}\right|_{z=1} = 0, \qquad \left.\frac{d^2w}{dz^2}\right|_{z=1} \neq 0.$$

This informs that $z = 1$ is a zero of order two and mapping is not conformal at this point. So, angles with vertices at $z = 1$ are doubled by the transformation at the image point $w = 2$ (see Proposition 2). In particular, the exterior angle at $z = 1$ of C, i.e. π is magnified twice at the image point $w = 2$. So, the exterior angle to the image curve C' is 2π. Hence C' has sharp tail at $w = 2$. The other points of C' can be found directly (Fig. 9.18).

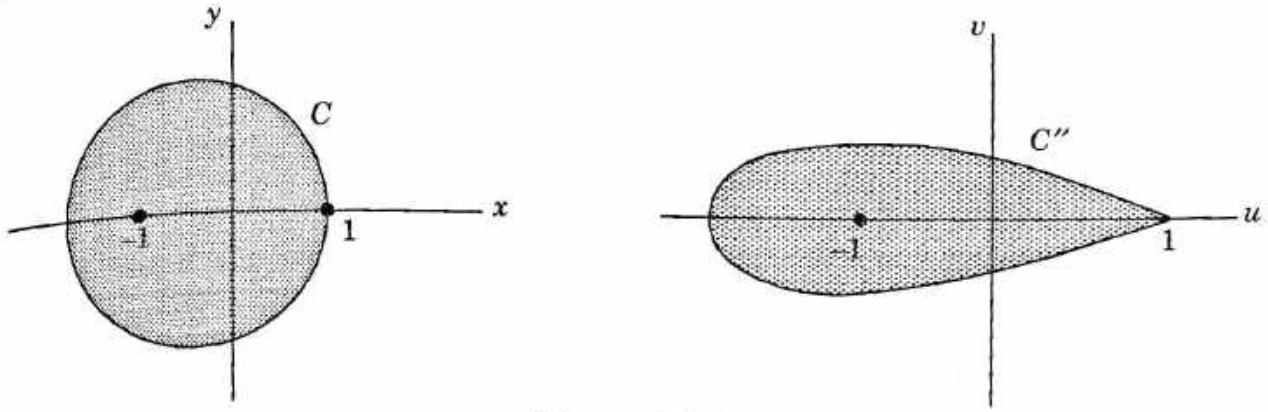

Figure 9.18

In this case C encloses the circle $|z| = 1$ which under the transformation maps on the slit $-2 \le u \le 2$. Thus as C approaches $|z| = 1$, the image curve C' approaches the line joining $w = -2$ to $w = 2$.

Example 25. Let the circle C of Example 2 be moved so that its centre is in the upper half-plane but that is still passes through $z = 1$ and encloses $z = -1$ as its interior point. Determine the image of C under the transformation $w = z + 1/z$.

As in Example 24, since the point $z = 1$ is the critical point, we again obtain the sharp tail at $w = 2$ (Fig. 9.19). If C does not entirely enclose the circle $|z| = 1$ the image C' will not entirely enclose the image of $|z = 1|$ (which is the slit joining -2 to 2). Instead, C' will enclose that portion of the slit which corresponds to the part of $|z| = 1$ inside C. The appearance of C' is depicted in Fig. 9.19.

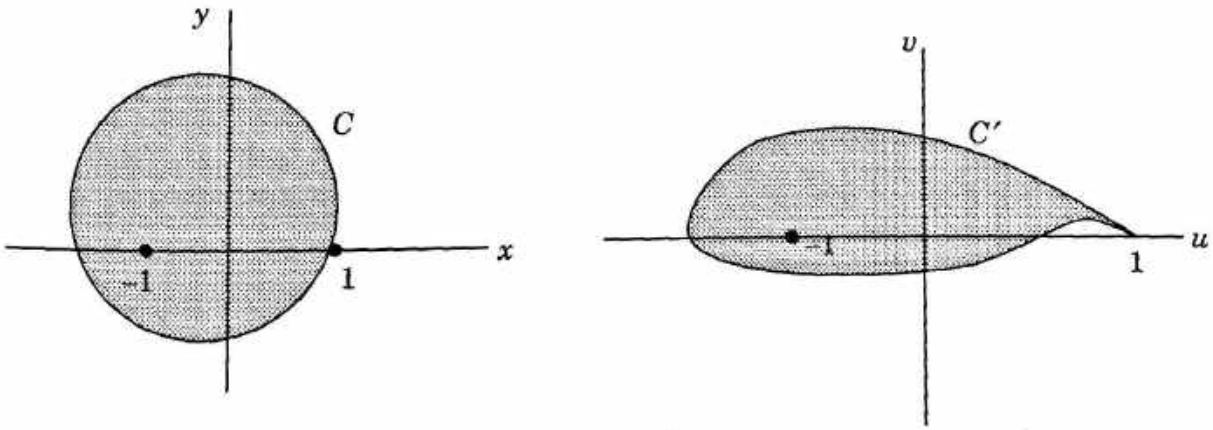

Figure 9.19

The fact that C' resembles the cross-section of wing of an airplane, called an airfoil. For this reason, these shapes are called Joukowski *airfoils* or *profiles*.

In the end we would like to mention a result (without proof) which tells us why conformal maps have large applications. The engineers and scientists have arbitrary domains in their practical problems. It is possible to map in a one-to-one manner these domains onto unit disc with the construction of appropriate functions.

Riemann mapping theorem. Let Ω be a simple connected domain in the complex plane (except the plane itself). Then there exists a conformal map which maps Ω in a one-to-one manner onto the unit disc $|w| < 1$.

Remarks. 1. There exists no conformal map that maps the entire complete plane onto unit disc. This is a consequence of Liouville's theorem. How?

2. In the Riemann mapping theorem, we may equally say that $w = f(z)$ maps Ω onto the upper half-plane. Why?

3. It must be emphasized that Riemann mapping theorem is applicable to only simple connected open sets and says nothing about the behaviour of the mapping function on the boundary.

PROBLEM SET 9.5

1. Find the image of the region $x \le 0$ and $0 \le y \le \pi$ under the mapping $w = e^z$. Sketch the regions.
2. Find the image of the region $-\pi/2 < x < \pi/2$ and $0 < y < 1$ under the map $w = e^{2iz}$.
3. What is the image under the map $w = z + e^z$ of the closed strip $-\pi \le y \le \pi$?
4. Show that under the mapping $w = c \sin z$, $c \in \mathbb{R}$, the region $0 \le x \le \pi/2$ is mapped into the first quadrant. Is the mapping conformal at $z = \pi/2$? Justify.
5. Sketch the image of the region $0 \le x \le \pi/2$, $y \ge 0$ mapped by the transformation $w = \sin^2 z$.
6. Find an analytic function which maps the sector $0 < \arg z < \pi/3$ onto $u > 1$.
   Suggestion. Composite map hgf, where $f(z) = z^3$, $g(z) = -iz$, $h(z) = z + 1$.
7. Describe the mapping $w = \cosh z$ and find the image of $x \ge 0$, $0 \le y \le \pi$ under this transformation.
8. Find the image of the infinite strip $-\pi/2 \le y \le \pi/2$ under the map $\sinh z$.
9. Find the image of the straight lines $x = 1$ and $y = 1$ under the map $w = \tan z$.
10. Consider the map

$$w = \tan^2\left(\frac{z}{2}\right).$$

and show that the vertical strip $0 \le x \le \pi/2$ in the z-plane is mapped onto the boundary and inside of the unit circle $|w| = 1$, cut along the slit on the real axis from -1 to 0 in the w-plane.

Suggestion. Write the transformation in the form

$$w = \frac{1 - \cos z}{1 + \cos z}$$

and observe that any point on the line $x = \pi/2$ is mapped onto

$$|w| = \left|\frac{1 - i\sinh y}{1 + i\sinh y}\right| = 1.$$

Also, y-axis: $x = 0$ is mapped onto

$$w = \frac{1 - \cosh y}{1 + \cosh y}.$$

Clearly, w is real and assumes (twice) values in the interval $-1 \le u \le 0$. The only thing that remains to be verified is that any line $x = \alpha$, $0 < \alpha < \pi/2$, inside the strip is mapped interior to $|w| = 1$.

11. Describe the transformation $w = \cosh z$ in terms of the transformation $f(z) = \sin z$ and comment on rotation and translation effects.

12. Show that the transformation $w = z + 1/z$ can be written as

$$\frac{w-2}{w+2} = \left(\frac{z-1}{z+1}\right)^2.$$

13. Show that the transformation

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

maps half-plane $\operatorname{Re} z > 0$ with semi-circle of radius 1 centred at origin removed onto a half-plane.

14. Let C be circle in the z-plane having its centre above the real axis which passes through $z = -1$ and encloses $z = 1$ as the interior point. Determine the image of C under the transformation $w = z + 1/z$.

10

Laplace Transform

We begin this chapter by introducing Laplace transform and its properties. Next, we prove the convolution theorem and derive complex inversion formula which enables to determine the inverse Laplace transform of complicated functions. In the last two sections Laplace transform is applied to solve differential and integral equations.

10.1 INTRODUCTION

First we define the Laplace transform and discuss the conditions under which Laplace transform of different functions exist.

Definition 1. The *Laplace transform* of a function $f(t)$ of a real variable $t \in (0, \infty)$ is a function $F(p)$ of a complex variable p defined by

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{10.1}$$

whenever the improper integral exists. In a symbolic way (10.1) is written as

$$F(p) = L[f(t)]. \tag{10.2}$$

The function f may be real valued or complex valued but certainly t is a real variable. The argument p is complex. Throughout our discussion on the Laplace transform, lowercase letters are reserved for functions and capital letters for their corresponding Laplace transforms. If the context is clear, $L[f]$ will also be used to denote the Laplace transform of a function f.

The convergence of improper integral on the right side of (10.1) depends on the nature of the function f as well as on the parameter p. For example, if $f(t) = e^{t^2}$, the Laplace transform of this function does not exist, i.e. the integral in (10.1) is not convergent for any value of p. This can easily be verified by writing the integral in the form

$$L[e^{t^2}] = \int_0^\infty e^{t^2 - pt}\, dt = \int_0^\infty e^{(t-p/2)^2 - p^2/4}\, dt.$$

Obviously, the last integral is divergent by comparison with $\int_0^\infty e^t\,dt$. However, we shall demonstrate in the ensuing discussion that $L[t^n]$ exists if $\operatorname{Re} p > 0$, the real part of p. Let us clarify the query: for what type of functions does the Laplace transform exist?

Before answering the above query, we evaluate the Laplace transform of some simple functions based on the definition.

(a) For $a > -1$,

$$L[t^a] = \int_0^\infty e^{-pt} t^a\,dt.$$

On rescaling $pt = x \Rightarrow p\,dt = dx$, and recalling the gamma function, we obtain

$$L[t^a] = \int_0^\infty \frac{e^{-x}}{p}\left(\frac{x}{p}\right)^a dx = \frac{1}{p^{a+1}}\int_0^\infty e^{-x}x^a\,dx = \frac{\Gamma(a+1)}{p^{a+1}}. \tag{10.3}$$

In particular, if $a = n$ is a nonnegative integer in (10.3), we have

$$L[t^n] = \frac{n!}{p^{n+1}}, \qquad L[1] = \frac{1}{p}, \qquad L[t] = \frac{1}{p^2}. \tag{10.4}$$

(b) For $\operatorname{Re} p > a$,

$$L[e^{at}] = \int_0^\infty e^{-pt}e^{at}\,dt = \int_0^\infty e^{-(p-a)t}\,dt = \frac{1}{p-a}[-e^{-(p-a)t}]_0^\infty.$$

Using the fact $e^{-(p-a)t} \to 0$ as $t \to \infty$, when $\operatorname{Re} p > a$,

$$L[e^{at}] = \frac{1}{p-a}. \tag{10.5}$$

(c) For $\operatorname{Re} p > 0$, integrating by parts, we have

$$\begin{aligned} L[\sin at] &= \int_0^\infty e^{-pt}\sin at\,dt \\ &= \left[-\frac{e^{-pt}}{p}\sin at\right]_0^\infty + \frac{a}{p}\int_0^\infty e^{-pt}\cos at\,dt \\ &= \frac{a}{p}\left[-\frac{e^{-pt}}{p}\cos at\right]_0^\infty - \frac{a^2}{p^2}\int_0^\infty e^{-pt}\sin at\,dt \\ &= \frac{a}{p^2} - \frac{a^2}{p^2}L[\sin at] \end{aligned}$$

or

$$\left(1 + \frac{a^2}{p^2}\right)L[\sin at] = \frac{a}{p^2}$$

and hence

$$L[\sin at] = \frac{a}{p^2 + a^2}. \tag{10.6}$$

Similarly,

$$L[\cos at] = \frac{p}{p^2 + a^2}, \qquad \text{Re } p > 0; \tag{10.7}$$

$$L[\sinh at] = \frac{a}{p^2 - a^2}, \qquad \text{Re } p > |a|; \tag{10.8}$$

$$L[\cosh at] = \frac{p}{p^2 - a^2}, \qquad \text{Re } p > |a|. \tag{10.9}$$

Now, we discuss the conditions under which the integral in (10.1) is convergent, i.e. its value is finite. This improper integral is defined by (see Section 8.3)

$$\int_0^\infty e^{-pt} f(t)\, dt = \lim_{b\to\infty} \int_0^b e^{-pt} f(t)\, dt. \tag{10.10}$$

The integral in (10.10) converges if

$$\int_0^\infty |e^{-pt} f(t)| dt$$

converges, and we say that integral (10.1) converges absolutely and hence converges. Finally, if we are able to search a nonnegative function $g(t)$ such that

$$|e^{-pt} f(t)| \le g(t) \qquad \text{and} \qquad \int_0^\infty g(t) < \infty,$$

then we conclude that (10.1) converges due to comparison test.

For this analysis the function $f(t)$, defined for all $t \ge 0$, must be piecewise continuous on $[0, \infty)$. By piecewise continuity we mean that a function is continuous on every finite subinterval $0 \le t \le b$, except possibly at a finite number of points where there are jump discontinuities, i.e. $f(x+)$ and $f(x-)$ exist but are unequal. Note that at the origin, we take only $f(0+)$.

If $f(t)$ is piecewise continuous for $t \ge 0$, then for the existence of integral (10.1) we observe the behaviour of the integrand $e^{-pt} f(t)$ for large t. In order that the integrand $e^{-pt} f(t)$ approaches rapidly to zero, we shall assume that $f(t)$ is of exponential order.

Definition 2. A function $f(t)$ defined on $[0, \infty)$ is said to be of *exponential order* if there exist constants M and α such that

$$|f(t)| \le Me^{\alpha t}, \qquad t \ge 0. \tag{10.11}$$

This inequality needs only to be satisfied, of course, at those points of the non-negative t-axis, where f is defined.

The constant functions are of exponential order, as can be seen by setting $\alpha = 0$ in (10.11). The functions

$$t^n, \quad e^{at}, \quad \sin at, \quad t^n \sin at, \quad t^n e^{at} \sin at$$

for n to be a positive integer, are of exponential order. We will prove that t^n is of exponential order, and for other functions the same procedure may be adopted.

Example 1. Show that the function t^n is of exponential order.

Since, for every $\alpha > 0$,

$$\lim_{t\to\infty} \frac{t^n}{e^{\alpha t}} = 0,$$

for given $\varepsilon = 1 > 0$ there exists some $t > t_0$ such that

$$\left|\frac{t^n}{e^{\alpha t}}\right| < 1 \quad \forall\, t > t_0.$$

But t^n is bounded on finite interval $[0, t_0]$. Thus,

$$|t^n| < Me^{\alpha t} \quad \text{for} \quad t \geq 0.$$

Theorem 1. Let f be a piecewise continuous function of exponential order defined on $[0, \infty)$. Then, there exists a real number α such that the integral

$$\int_0^\infty e^{-pt} f(t)dt$$

converges for $\operatorname{Re} p > \alpha$.

Proof. Since f is of exponential order, it follows that

$$|f(t)| \leq Me^{\alpha t} \quad \forall\, t \in [0, \infty)$$

for some constant M and a real number α; this implies

$$|e^{-pt} f(t)| \leq M e^{-(p_1-\alpha)t}, \qquad p = p_1 + ip_2.$$

Hence

$$|F(p)| = \left|\int_0^\infty e^{-pt} f(t)\, dt\right| \leq \int_0^\infty |e^{-pt} f(t)|\, dt \leq M \int_0^\infty e^{-(p_1-\alpha)t}\, dt.$$

But for $p_1 < \alpha$,

$$\int_0^\infty e^{-(p_1-\alpha)t} dt = \lim_{b\to\infty} \int_0^b e^{-(p_1-\alpha)t}\, dt = \frac{1}{p_1 - \alpha}, \tag{10.12}$$

and therefore,

$$\int_0^\infty e^{-pt} f(t)dt \quad \text{converges absolutely for } \operatorname{Re} p > \alpha$$

and hence converges.

Remarks. 1. If the integral in Theorem 1 converges for some $\operatorname{Re} p = p_0$, then it converges for all $\operatorname{Re} p > p_0$.

2. In view of Eq. (10.12), $F(p) \to 0$ as $p \to \infty$. This gives a useful bit of information that any function $\phi(p)$ for which the $\lim_{p\to\infty} \phi(p)$ either does not exist or is nonzero, it may be immediately forecast that the concerned function is not the Laplace transform of any function of exponential order. So, p^2, $\sin p$, e^p cannot be Laplace transformed. This condition is not sufficient as $p^{-1} \sin p$ is not the Laplace transform of any function even though it goes to zero on all values with some arithmetic progression.

3. In Theorem 1, convergence of the integral has been established with the help of absolute convergence in the domain $\text{Re}\, p > \alpha$. However, its convergence may happen to hold in a larger domain than that of absolute convergence. For example, if we take $f(t) = e^t \cos e^t$, then

$$\int_0^\infty e^{-pt} e^t \cos e^t \, dt = \int_0^\infty u^{-p} \cos u \, du$$

converges for $\text{Re}\, p > 0$, but it converges absolutely for $\text{Re}\, p > 1$.

In general, for any function f, let α_0 be the greatest lower bound of the set of all real numbers α such that $L[f(t)]$ exists for $\text{Re}\, p > \alpha$. Then $L[f(t)]$ does not exist for any p with $\text{Re}\, p < \alpha_0$. Thus, the domain of definition of $L[f(t)]$ is given by $\text{Re}\, p > \alpha_0$. The line $\text{Re}\, p = \alpha_0$ is called the abscissa of convergence.

Also, Theorem 1 gives a sufficient condition for the existence of Laplace transform of any function. However, this condition is not necessary as there exist functions which are neither piecewise continuous nor of exponential order even then the Laplace transform may exist.

For example, the function $t^{-1/2}$ is having discontinuity of infinite order at $t = 0$, since $\lim_{t\to 0^+} t^{-1/2} = \infty$. Note that this discontinuity is different from that which we require for piecewise continuity. Nevertheless, we have for $\text{Re}\, p > 0$,

$$L[t^{-1/2}] = \int_0^\infty e^{-pt} t^{-1/2} \, dt = \frac{1}{\sqrt{p}} \int_0^\infty e^{-x} x^{-1/2} \, dx = \frac{\Gamma(1/2)}{\sqrt{p}} = \sqrt{\frac{\pi}{p}}. \tag{10.13}$$

Equation (10.3) also yields (10.13) for $a = -1/2$. Note that whenever nonintegral index of p is involved we consider the principal value.

Linearity. The Laplace transform L is a linear operator on the vector space V of all functions whose Laplace transforms exist. This means that, if $L[f(t)] = F(p)$ for $\text{Re}\, p > \alpha_1$ and $L[g(t)] = G(p)$ for $\text{Re}\, p > \alpha_2$, then, for all constants a and b,

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)], \quad \text{Re}\, p > \alpha = \max\{\alpha_1, \alpha_2\}. \tag{10.14}$$

Let $L[f(t)]$ and $L[g(t)]$ be defined for $\text{Re}\, p > \alpha_1$ and $\text{Re}\, p > \alpha_2$, respectively. Then choose $\alpha = \max\{\alpha_1, \alpha_2\}$ so that both $L[f(t)]$ and $L[g(t)]$ are defined for $\text{Re}\, p > \alpha$. Now,

$$L[af(t) + bg(t)] = \int_0^\infty [af(t) + bg(t)]e^{-pt} \, dt$$

$$= a \int_0^\infty f(t)e^{-pt} \, dt + b \int_0^\infty g(t)e^{-pt} \, dt = F(p) + G(p),$$

which agrees with (10.14).

As an immediate consequence of linearity, formulas (10.8) and (10.9) may conveniently be derived using (10.5) and the relations:

$$\sinh at = \frac{e^{at} - e^{-at}}{2}, \qquad \cosh at = \frac{e^{at} + e^{-at}}{2}.$$

L is one-one. Consider the operator equation $L[f(t)] = F(p)$, where $F(p)$ is analytic for $\operatorname{Re} p > \alpha$. At the moment we do know that this operator equation has a solution for given $F(p)$ which is analytic for $\operatorname{Re} p > \alpha$. However, if the operator equation has a solution for given $F(p)$, the obvious question is whether it is unique. The following theorem answers this query in affirmative.

Theorem 2. (Lerch's theorem) Let $f(t)$ and $g(t)$ be piecewise continuous functions of exponential order on $[0, \infty)$ such that $L[f(t)] = L[g(t)]$, $\operatorname{Re} p > \alpha$. Then, $f(t) = g(t)$ for all t, except at the points of discontinuity.

Definition 3. Two functions f and g are said to be equivalent on $[0, \infty)$, if $f(t) = g(t)$ for all t, except at the points where the functions are not continuous.

Considering the equivalent functions in V, the vector space of all real functions which are piecewise continuous and of exponential order, as the same functions that the Lerch's theorem ensures that the operator equation (10.2) has a unique solution. Thus, L is a one-to-one linear transformation from vector space V to the space of all analytic functions (which are the Laplace transforms of some functions).

Proposition 1. If $f(t)$ is piecewise continuous and of exponential order with period a, then

$$F(p) = \frac{1}{1 - e^{-ap}} \int_0^a e^{-pt} f(t)\, dt. \tag{10.15}$$

Proof. Given that $f(t + a) = f(t)$ for all $t \in [0, \infty)$. By definition

$$L[f] = \int_0^\infty e^{-pt} f(t) dt$$

$$= \int_0^a e^{-pt} f(t) dt + \int_a^{2a} e^{-pt} f(t)\, dt + \cdots + \int_{na}^{(n+1)a} e^{-pt} f(t) dt + \cdots$$

Set $x + na = t$ in the $(n + 1)$st integral of the above series to have

$$\int_{na}^{(n+1)a} e^{-pt} f(t) dt = \int_0^a e^{-p(x+na)} f(x + na) dx = e^{-nap} \int_0^a e^{-px} f(x) dx,$$

the last expression follows from the periodicity of f. Hence

$$L[f] = \int_0^a e^{-px} f(x) dx + e^{-ap} \int_0^a e^{-px} f(x) dx + \cdots + e^{-nap} \int_0^a e^{-px} f(x) dx + \cdots$$

$$= (1 + e^{-ap} + e^{-2ap} + \cdots + e^{-nap} + \cdots) \int_0^a e^{-px} f(x)dx$$

$$= \frac{1}{1 - e^{-ap}} \int_0^a e^{-pt} f(t)dt .$$

Remark. Let $f(t)$ be piecewise continuous and of exponential order on $[0, \infty)$, and assume that $f(t + \pi) = -f(t)$. Then the Laplace transform of $f(t)$ is given by (10.15) just replacing a by 2π. Why?

Example 2. Find the Laplace transform of the function $|\sin t|$.

Note that the function $|\sin t|$ is π periodic, so applying formula (10.15), we have

$$L[|\sin t|] = \frac{1}{1 - e^{-\pi p}} \int_0^\pi e^{-pt} \, |\sin t| dt.$$

Since $|\sin t| = \sin t$ on $[0, \pi]$, on integrating by parts it follows that, for Re $p > 0$,

$$L[|\sin t|] = \frac{1}{1 - e^{-\pi p}} \int_0^\pi e^{-pt} \sin t \, dt = \frac{1}{1 - e^{\pi p}} \frac{1 + e^{-\pi p}}{1 + p^2} = \frac{1}{1 + p^2} \frac{1 + e^{-\pi p}}{1 - e^{-\pi p}} .$$

10.2 PROPERTIES OF LAPLACE TRANSFORM

We discuss below some properties of the Laplace transform which are useful in developing further theory and giving us a shorter way of computing the Laplace transform, thus avoiding the direct use of formula (10.1).

1. Shifting the variable *p*. If $F(p)$, Re $p > \alpha$ is the Laplace transform of $f(t)$, then

$$L[e^{at} f(t)] = F(p - a), \quad \text{Re } p > a + \alpha. \tag{10.16}$$

This means that the substitution of $p - a$ for p in the transform corresponds to the multiplication of the original function by e^{at}. This result known as *first shifting* formula is verified below.

$$L[e^{at} f(t)] = \int_0^\infty e^{-pt} e^{at} f(t)dt = \int_0^\infty e^{-(p-a)t} f(t)dt = F(p - a), \qquad \text{Re } p > a + \alpha.$$

Example 3. Find the Laplace transform of the functions (i) $t^n e^{3t}$; (ii) $e^{at} \sin at$.

For part (i), let $f(t) = t^n$ and from (10.4), $F(p) = n!/p^{n+1}$. Hence, using (10.16), we have

$$L[t^n e^{3t}] = \frac{n!}{(p - 3)^{n+1}}, \qquad \text{Re } p > 3.$$

For part (ii), take $f(t) = \sin at$ and from (10.6), $F(p) = a/(p^2 + a^2)$. Hence, using (10.16), we have

$$L[e^{at} \sin at] = \frac{a}{(p-a)^2 + a^2}, \qquad \text{Re } p > a.$$

2. Multiplication by t^n. If $F(p)$ is the Laplace transform of f, then

$$L[(-1)^n t^n f(t)] = F^{(n)}(p). \tag{10.17}$$

So, the nth differentiation of the transform with respect to p corresponds to the multiplication of the original function by $(-1)^n t^n$.

To show this, we proceed as

$$F(p) = \int_0^\infty e^{-pt} f(t)dt.$$

Differentiate the above equation with respect to p. Note that differentiation involves under the sign of integration, and this is justified by Leibniz rule. So,

$$F'(p) = -\int_0^\infty e^{-pt}\, t f(t)dt = L[-t f(t)], \qquad \text{Re } p > \alpha.$$

This proves the result for $n = 1$. On further differentiation

$$F''(p) = \int_0^\infty e^{-pt}\, t^2 f(t)dt = L[t^2 f(t)], \qquad \text{Re } p > \alpha$$

which makes the result valid for $n = 2$. The order of growth of $f(t)$ does not change with the multiplication by t^n. Hence, by induction (10.17) can be proved.

Here we have also proved the fact that $F(p)$ is analytic in the domain Re $p > \alpha$.

Example 4. Find the Laplace transform of the functions: (i) $t^3 e^{-2t}$; (ii) $t^2 \cosh 3t$.

For part (i), let $f(t) = e^{-2t}$ and from (10.5) $F(p) = 1/(p + 2)$. Hence, using (10.17), we have

$$L[t^3 e^{-2t}] = (-1)^3 \frac{d^3}{dp^3}\left(\frac{1}{p+2}\right) = \frac{6}{(p+2)^4}, \qquad \text{Re } p > -2.$$

For part (ii), take $f(t) = \cosh 3t$ and from (10.8), $F(p) = p/(p^2 - 9)$. Hence, using (10.17), we have

$$L[t^2 \cosh 3t] = (-1)^2 \frac{d^2}{dp^2}\left(\frac{p}{p^2-9}\right) = \frac{2p(p^2+22)}{(p^2-9)^3}, \qquad \text{Re } p > 3.$$

3. Division by t. If $L[f(t)] = F(p)$, Re $p > \alpha$, and the limit of $f(t)/t$, as t approaches 0 from the right exists, then

$$L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(x)dx. \tag{10.18}$$

provided the integral on the right side exists.

To prove this we start with the right side

$$\int_p^\infty F(x)\,dx = \int_p^\infty \left[\int_0^\infty e^{-xt} f(t)\,dt\right] dx.$$

Since t and x are independent variables, changing the order of integration, we have

$$\begin{aligned}\int_p^\infty \left[\int_0^\infty e^{-xt} f(t)\,dt\right] dx &= \int_0^\infty f(t) \int_p^\infty e^{-xt}\,dx\,dt \\ &= \int_0^\infty f(t)\left(-\frac{e^{-xt}}{t}\right)_p^\infty dt \\ &= \int_0^\infty e^{-pt}\frac{f(t)}{t}\,dt \\ &= L\left[\frac{f(t)}{t}\right].\end{aligned}$$

Further, if we write (10.18) as

$$\int_0^\infty e^{-pt}\frac{f(t)}{t}\,dt = \int_p^\infty F(x)dx,$$

and let $p \to 0$, we have

$$\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(p)dp. \tag{10.19}$$

Formula (10.19) is useful to evaluate integrals.

Example 5. Show that

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

By the direct use of (10.19), we have

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{1}{1+p^2}\,dp = [\tan^{-1} p]_0^\infty = \frac{\pi}{2}.$$

Note that in Section 8.3, we evaluated this integral by residue calculus.

To evaluate definite integrals the idea of Eq. (10.19) is further extended to cover a larger class of functions. That is to say, if a definite integral includes as a factor of the

integrand, a power t^{-n} or a quotient $a/(a^2+t^2)$ or $t/(a^2+t^2)$, the value of the integral can sometimes be obtained by expressing that factor itself as the integral (10.1) and then interchanging the order of integration. For example, if the integral

$$\int_0^\infty \frac{\cos mt}{a^2+t^2}\,dt, \qquad m \in \mathbb{R}, a \in \mathbb{R}_+.$$

is denoted by $f(m)$, then the change of order of integration yields

$$\int_0^\infty \cos mt\left(\frac{1}{a}\int_0^\infty e^{-tx}\sin ax\,dx\right)dt = \frac{1}{a}\int_0^\infty \sin ax\left(\int_0^\infty e^{-xt}\cos mt\,dt\right)dx.$$

Thus,

$$f(m) = \frac{1}{a}\int_0^\infty \frac{x\sin ax}{m^2+x^2}\,dx,$$

which on rescaling as $ax = mt$ gives $f'(m) = -af(m)$. This is a first order linear differential equation with the solution: $f(m) = ce^{-am}$. Since $f(0) = \pi/2a$, c can be computed. This has also been evaluated as Problem 7, Problem Set 8.3 by using residue calculus.

4. Laplace transform of derivatives. This property reveals that the Laplace transform of higher derivatives can be written in terms of the Laplace transform of the original function with some additive constants. This nature of the Laplace transform has varied applications for solving differential equations.

Suppose $y(t)$ is continuous on $(0, \infty)$ and $y'(t)$ is a piecewise continuous function of exponential order. Then,

$$L[y'] = \int_0^\infty e^{-pt}y'\,dt = [e^{-pt}y]_0^\infty + p\int_0^\infty e^{-pt}y\,dt = pL[y] - y(0), \quad \text{Re } p > \alpha. \tag{10.20}$$

Further, if y'' is also of exponential order, Eq. (10.20) can be extended to

$$L[y''] = L[(y')'] = pL[y'] - y'(0) = p^2L[y] - py(0) - y'(0).$$

Similarly, with the assumption that $y, y', \ldots, y^{(n)}$ are piecewise continuous and of exponential order, the above procedure may be continued up to n-times, or by using induction, to obtain

$$L[y^{(n)}] = p^nL[y] - p^{n-1}y(0) - p^{n-2}y'(0) - \cdots - y^{n-1}(0), \quad \text{Re } p > \alpha. \tag{10.21}$$

Remark. While deriving (10.20), the fact $\lim_{t\to\infty} ye^{-pt} = 0$ has been used. However, it is true since (by assumption) y is of exponential order and deciding p such that $\text{Re } p > \alpha$. Also, note that $y^{(n)}(0)$, $n = 0, 1, \ldots$, means $y^{(n)}(0+)$.

5. Laplace transform of indefinite integrals. If $f(t)$ is a piecewise continuous function of exponential order, then

$$L\left[\int_a^t f(u)\,du\right] = \frac{F(p)}{p} - \frac{1}{p}\int_0^a f(u)\,du, \qquad a \ge 0, \text{ Re } p > \max\{\alpha, 0\}. \tag{10.22}$$

Since $f(t)$ is a piecewise continuous function and of exponential order, it follows that

$$g(t) = \int_a^t f(u)\,du$$

is continuous and of exponential order and, moreover, $g'(t) = f(t)$, except possibly at points where $f(t)$ has jump discontinuities. This implies that $g'(t)$ is a piecewise continuous function of exponential order and hence (10.21) assists us to write

$$L[g'(t)] = pL[g(t)] - g(0). \tag{10.23}$$

By definition

$$g(0) = -\int_0^a f(u)\,du$$

and hence, Eq. (10.23) takes the form

$$L[g(t)] = \frac{F(p)}{p} - \frac{1}{p}\int_0^a f(u)du, \qquad a \geq 0,\ \text{Re}\ p > \max\{\alpha, 0\}.$$

In particular,

$$L\left[\int_0^t f(u)\,du\right] = \frac{F(p)}{p}, \qquad \text{Re}\ p > \max\{\alpha, 0\}. \tag{10.24}$$

In general (10.24) is written as

$$L\left[\int_0^t \cdots \int_0^t \int_0^t f(x)\,dx\,dx \cdots dx\right] = \frac{F(p)}{p^n},$$

where the integral is iterated n times.

Example 6. Use the above property to evaluate

$$L\left[t^2 \int_0^t x \sin x\,dx\right].$$

Since

$$L[x \sin x] = -\frac{d}{dp}\left(\frac{1}{1+p^2}\right) = \frac{2p}{(1+p^2)^2},$$

In view of (10.24),

$$L\left[\int_0^t x \sin x\,dx\right] = \frac{2}{(1+p^2)^2}.$$

Thus,

$$L\left[t^2 \int_0^t x \sin x\,dx\right] = \frac{d^2}{dp^2}\left[\frac{2}{(1+p^2)^2}\right] = -\frac{8(5p^2-1)}{(1+p^2)^4}.$$

Unit step function. For $a \geq 0$, let us introduce the function,

$$u_a(t) = \begin{cases} 0, & 0 \leq t \leq a, \\ 1, & t > a. \end{cases}$$

The function $u_a(t)$ is called the *unit step* function or sometimes the Heaviside function. Let $f(t)$ be defined on $[0, \infty)$. Then the graph of $f(t)$ can be shifted by a units to the right, if we write

$$u_a(t)\, f(t-a) = \begin{cases} 0, & t < a, \\ f(t-a), & t > a. \end{cases}$$

Figure 10.1 shows that the graph of $f(t) = \sin t$, for $t > 0$ is shifted by 2 units to the right for the function $u_2(t) \sin (t-2)$. Now, we extract a property due to the influence of the unit step function. This is known as the *second shifting* formula or t-shifting.

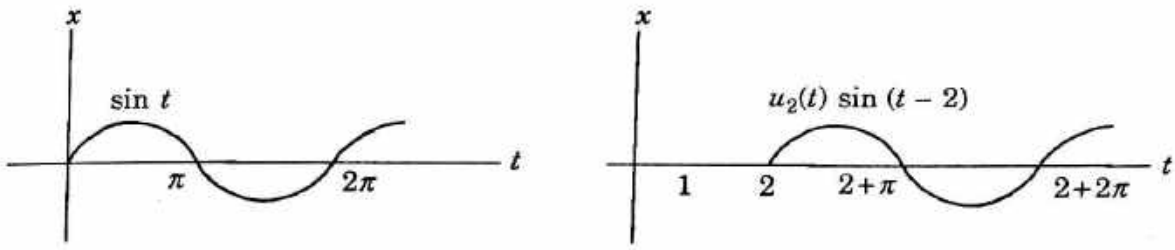

Figure 10.1

Let $f(t)$ be a piecewise continuous function of exponential order. Then

$$L[u_a(t)f(t-a)] = e^{-ap}F(p). \tag{10.25}$$

The factor e^{-ap} on the right side of Eq. (10.25) is called the time delaying factor. To prove this we proceed as

$$\int_0^\infty e^{-pt}\, u_a(t)f(t-a)dt = \int_a^\infty e^{-pt}f(t-a)dt = e^{-ap} \int_0^\infty e^{-pt}f(t)dt = e^{-ap}F(p).$$

In particular for $f(t) = 1$, Eq. (10.25) reduces to

$$L[u_a(t)] = \frac{e^{-ap}}{p}, \qquad L[u_0(t)] = \frac{1}{p}.$$

Example 7. Find the Laplace transform of the function

$$f(t) = \begin{cases} 2, & 0 < t < \pi, \\ 0, & \pi < t < 2\pi, \\ \sin t, & t \geq 2\pi. \end{cases}$$

To apply directly (10.25), we express $f(t)$ in terms of the unit step function.

For $0 < t < \pi$, we take $2u_0(t)$. For $t > \pi$, we want 0, so we must subtract the step function $2u_\pi(t)$. Then, we have $2u_0(t) - 2u_\pi(t)$, for $t > \pi$. This is alright until we reach 2π where we want $\sin t$ to enter; so we add $u_{2\pi}(t) \sin t$. This procedure determines

$$f(t) = 2u_0(t) - 2u_\pi(t) + u_{2\pi}(t) \sin t, \quad t > 2\pi.$$

Now, due to the periodicity of $\sin t$, we have

$$f(t) = 2u_0(t) - 2u_\pi(t) + u_{2\pi}(t) \sin(t - 2\pi), \quad t > 0.$$

Therefore,

$$L[f(t)] = \frac{2}{p} - \frac{2e^{-\pi p}}{p} + \frac{e^{-2\pi p}}{p^2 + 1}.$$

Unit impulse function. In mechanical systems, the *impulse* of a force $f(t)$ over a time interval, say, $a \le t \le a + k$ is defined to be the integral of $f(t)$ from a to $a + k$. Similarly, in an electric circuit the impulse is the integral of electromotive force applied to the circuit, integrated from a to $a + k$. We are interested in studying the systems in which k is very small ($k \to 0$), i.e. the impulse of a force acting for a very short time.

Let a be a fixed positive number. Then consider the function $\delta_k(t - a)$ defined by

$$\delta_k(t - a) = \begin{cases} 1/a, & a \le t \le a + k, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that its impulse is 1, since

$$\int_0^\infty \delta_k(t - a)dt = 1.$$

The unit *impulse* function is obtained by letting the interval $[a, a + k]$ reduce to zero, i.e.

$$\delta(t - a) = \lim_{k \to 0} \delta_k(t)(t - a).$$

It is not a function in the strict sense of calculus, but so called a "generalized function". In the limiting case, when $k \to 0$, we have

$$\delta(t - a) = \begin{cases} \infty, & t = a, \\ 0, & \text{otherwise}. \end{cases} \quad \text{and} \quad \int_0^\infty \delta(t - a)dt = 1.$$

Any ordinary function which is everywhere 0, except at a single point, must have integral value 0, but its integral turns out to be 1. Although $\delta(t - a)$ is called the unit impulse function or the Dirac–delta function, we are at liberty to call it a distribution, and the theory of distribution justifies $\delta(t - a)$ as if it were a function. Anyway in impulse problems it is convenient to operate on $\delta(t - a)$ as if it were a function.

Example 8. Show that $L[\delta(t - a)] = e^{-ap}$.

By definition

$$L[\delta_k(t - a)] = \int_0^\infty \delta_k(t - a)e^{-pt}\, dt = \int_a^{a+k} \frac{1}{k} e^{-pt}\, dt = \frac{e^{-ap} - e^{-(a+k)p}}{pk} = e^{-ap} \frac{1 - e^{-kp}}{kp}.$$

Letting $k \to 0$ and using L'Hôspital's rule, we obtain

$$L[\delta(t-a)] = \lim_{k\to 0} L[\delta_k(t-a)] = \lim_{k\to 0} e^{-ap}\,\frac{1-e^{-pk}}{pk} = e^{-ap}.$$

Remark. Note that $a = 0$ means that the impulse is applied in the beginning and then $L[\delta(t)] = 1$, but it does not mean that the inverse Laplace transform of 1 is the unit impulse function. Why?

10.3 CONVOLUTION OF TWO FUNCTIONS

If $L[f(t)] = F(p)$ and $L[g(t)] = G(p)$, what is the inverse transform of $F(p)G(p)$. In addition to settlement of this question, we give some applications of convolution in subsequent sections.

Definition 4. The convolution of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(t-x)g(x)\,dx. \tag{10.26}$$

Remark. With substituting $t - x = u$ in (10.26), it is evident that $f * g = g * f$. Thus, convolution satisfies the commutative property.

Theorem 3 (Convolution). Let $F(p)$ and $G(p)$ be the Laplace transforms of f and g, respectively. Then

$$L[f * g] = L[f(t)]\; L[g(t)]. \tag{10.27}$$

Proof. To prove this result, we have

$$L[f * g] = \int_0^\infty e^{-pt}\left[\int_0^t f(t-x)g(x)dx\right]dt = \int_0^\infty\int_0^t e^{-pt}f(t-x)\,g(x)dx\,dt\,.$$

On changing the order of integration, we have

$$L[f * g] = \int_0^\infty g(x)\left[\int_x^\infty e^{-pt}f(t-x)\,dt\right]dx.$$

The region of integration is the first half of the first quadrant. Let us introduce a new variable by defining $v = t - x$, so that $v \geq 0$, $t = v + x$ and (x being fixed during the integration of the expression within the brackets) $dt = dv$ (refer to Fig. 10.2).

This enables us to write

$$L[f * g] = \int_0^\infty g(x)\left[\int_0^\infty e^{-p(v+x)}f(v)\,dv\right]dx = \int_0^\infty e^{-pv}f(v)\,dv\int_0^\infty e^{-px}g(x)\,dx$$
$$= L[f(t)]L[g(t)].$$

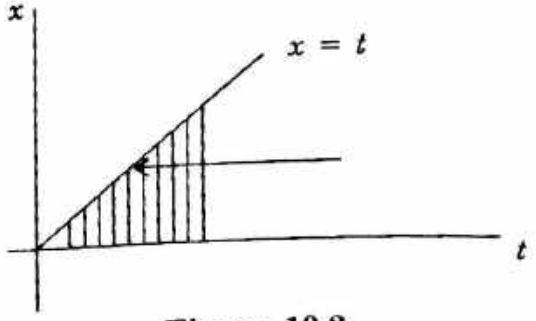

Figure 10.2

Error function. The error function is defined by

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx.$$

Naturally, the complimentary error function is

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2}\, dx.$$

***Example* 9.** Prove that

$$L[\operatorname{erf}(\sqrt{t})] = \frac{1}{p} \frac{1}{\sqrt{p+1}};$$

Use (10.13) and the first shifting property to have

$$L\left[\frac{e^{-t}}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{p+1}}.$$

With the help of the above result and formula (10.27) in reverse order, we have

$$\frac{1}{p} \frac{1}{\sqrt{p+1}} = L\left[1 * \left(\frac{e^{-t}}{\sqrt{\pi t}}\right)\right] = L\left[\frac{1}{\sqrt{\pi}} \int_0^t \frac{e^{-u}}{\sqrt{u}}\, du\right] = L[\operatorname{erf}(\sqrt{t})], \qquad u = x^2.$$

PROBLEM SET 10.1

1. Let f be piecewise continuous on $[0, \infty)$. Then prove that
   (a) f is of exponential order if, for some constant α, such that $\lim_{t\to\infty} [f(t)/e^{\alpha t}] = 0$;
   (b) f is not of exponential order if $\lim_{t\to\infty} [f(t)/e^{\alpha t}] = \infty$, for all real numbers α.
2. As $\lim_{t\to\infty} [t^{-1/2}/e^{\alpha t}] = 0$ for $\alpha > 0$, even then $t^{-1/2}$ is not of exponential order. Does it violate assertion (a) of the preceding problem?
3. Show that the function t^t is not of exponential order.
   Suggestion. Write $t^t = e^{t \ln t}$ and use the preceding problem.

4. If $L[f(t)] = F(p)$, $\operatorname{Re} p > \alpha$, show that

$$L[f(at)] = \frac{1}{a}F\left(\frac{p}{a}\right), \qquad \operatorname{Re} p > a\alpha.$$

This is change of scale property.

5. If f and g are piecewise continuous and of exponential order on $[0, \infty)$, show that (a) $f+g$, (b) $f-g$, (c) kf (k is some constant), and (d) fg are all piecewise continuous and of exponential order.

6. Determine which of the following functions are piecewise continuous on the interval $[0, \infty)$. (a) $f(t) = 0$, if t is rational and $f(t) = 1$, if t is irrational; (b) $f(t) = \exp t^2$; (c) $f(t) = (t+2)/(t-3)$; (d) $f(t) = 0$, if t is a natural number, otherwise $f(t) = 1$.

7. Prove that the integral for the function $f(t) = e^t \sin e^t$ converges absolutely for $\operatorname{Re} p > 1$. Does it converge in a larger domain? If yes, mention this domain.

8. Without using the definition evaluate the Laplace transform of the following functions:

(a) $\sin^2 t$; (b) $\cos^3 t$; (c) $|\cos t|$; (d) $\sin at \cosh at$.

9. Find the Laplace transform of the following functions and mention their domain of existence (abscissa of convergence):

(a) $(t+1)^4$; (b) $6t^3 e^{-t} + e^{2t} \sin 4t$;

(c) $\cosh at \sinh at$; (d) $te^{2t} \sinh 2t$.

10. Find the Laplace transform of each of the following functions:

(a) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 + 1, & t \geq 1; \end{cases}$ (b) $f(t) = \begin{cases} \sin t, & t \leq 2\pi, \\ 0, & t > 2\pi; \end{cases}$

(c) $f(t) = \begin{cases} 0, & t < 2, \\ 8 - 3t, & 2 \leq t \leq 3, \\ t - 4, & t \geq 3; \end{cases}$ (d) $f(t) = \begin{cases} 0, & 0 \leq t \leq \pi/2, \\ \cos t, & \pi/2 \leq t \leq 3\pi/2, \\ 1, & t > 3\pi/2. \end{cases}$

11. Evaluate:

(a) $L[t^{3/2}]$; (b) $L\left[t\int_{\pi}^{t} x \sin x \, dx\right]$;

(c) $L[\sin \sqrt{t}]$; (d) $L\left[\dfrac{\cos \sqrt{t}}{\sqrt{t}}\right]$.

Suggestion. For (c) use the power series expansion and for (d) use the outcome of (c) in conjunction with

$$\frac{d}{dt}(\sin \sqrt{t}) = \frac{\cos \sqrt{t}}{2\sqrt{t}}.$$

12. Verify the following transform results:

(a) $L[[t]] = \dfrac{1}{p(e^p - 1)}$; (b) $L[t - [t]] = \dfrac{e^2 - 1 - p}{p^2(e^p - 1)}$;

(c) $L\left[\dfrac{\sin 2t}{t}\right] = \cot^{-1}\dfrac{p}{2}$; (d) $L\left[\dfrac{1 - \cos 2t}{t^2}\right] = 2\cot^{-1}\left(\dfrac{p}{2}\right) + \dfrac{p}{2}\log\left(\dfrac{p^2}{p^2 + 4}\right)$,

where $[t]$ denotes the maximum integer contained in t.

13. For $n = 0, 1, 2, \ldots$, let

$$f(t) = \begin{cases} 1, & 2n \le t \le 2n + 1, \\ 0, & 2n + 1 < t < 2n + 2. \end{cases}$$

Find the Laplace transform of this function.

Suggestion. The function has its period 2. Apply Proposition 1.

14. For the function $g(t)$ defined for developing the proof of property 5, establish the fact that it is piecewise continuous and of exponential order.

15. Use Laplace transform methods to evaluate the following integrals:

(a) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$; (b) $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$;

(c) $\displaystyle\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$; (d) $\displaystyle\int_0^\infty e^{-\sqrt{x}}\,dx$.

16. Show that

(a) $\displaystyle\int_0^\infty \frac{\cos t}{\sqrt{t}}\,dt = \int_0^\infty \frac{\sin t}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{2}}$; (b) $\displaystyle\int_0^\infty \cos t^2\,dt = \int_0^\infty \sin t^2\,dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Suggestion. For part (a), the comments following Example 5, lead to the integral, $\int_0^\infty (1 + t^2)^{-1} t^{-1/2}\,dt = \pi/\sqrt{2}$ (see also, the deduced results of Example 15, Chapter 8). For (b), set $t^2 = x$ and use (a).

17. Sketch the functions:

(a) $f(t) = (k/a)t$, $0 < t < a$, $f(t + a) = f(t)$ [Sawtooth function], and

(b) $g(t) = kn$, $an < t < an + a$, $n = 0, 1, 2, \ldots$ [Staircase function]

and obtain their Laplace transforms.

18. Show that the convolution operation is associative and satisfies the distributive law over the operation of addition of functions.

19. Like error function there are other familiar functions. We define a few of them.

(a) $Ei(x) = \displaystyle\int_x^\infty \frac{e^{-t}}{t}\,dt$ (Exponential integral);

(b) $Si(x) = \int_x^\infty \frac{\sin t}{t}\, dt$ (Sine integral);

(c) $Ci(x) = \int_x^\infty \frac{\cos t}{t}\, dt$ (Cosine integral);

(d) $P_n(x) = \frac{e^x}{n!}(x^n e^{-x})^{(n)}, \quad n = 0, 1, \ldots, x \in \mathbb{R}$ (Laguerre's polynomial).

Determine the Laplace transform of these functions.

20. Find the Laplace transform of the triangular wave of period $2a$ given by

$$f(t) = \begin{cases} t, & 0 < t < a, \\ 2a - t, & a < t < 2a. \end{cases}$$

Sketch this function when $f(t) \equiv 0$ on $(a, 2a)$, and determine its Laplace transform.

21. Write the unit impulse function as the sum of two unit step functions and determine the Laplace transform of the Dirac-delta function to match the result (10.26).

22. If $F(p)$ is the Laplace transform of $f(t)$, show that

(a) $\lim_{t\to 0} f(t) = \lim_{p\to\infty} pF(p)$; (b) $\lim_{t\to\infty} f(t) = \lim_{p\to 0} pF(p)$;

provided that all limits exist.

Assertions (a) and (b) are known as *initial* and *final value theorems*, respectively.

23. Prove that

(a) $L[\delta(t-k)] = e^{-kp}, \quad k \geq 0$ (b) $L[\mathrm{erf}(t)] = \frac{e^{p^2/4}}{p}\mathrm{erfc}\left(\frac{p}{2}\right), \quad p \in \mathbb{R}_+.$

Suggestion. Use (10.24) for $p > 0$, to have

$$L\left[\int_0^t e^{-x^2}dx\right] = \frac{L[e^{-x^2}]}{p} = \frac{e^{p^2/4}}{p}\int_0^\infty e^{-(x+p/2)^2}dx = \frac{e^{p^2/4}}{p}\int_{p/2}^\infty e^{-u^2}du.$$

10.4 INVERSE LAPLACE TRANSFORM

The solution of the operator equation

$$L[f(t)] = F(p),$$

whenever it exists is called the *inverse Laplace transform* of $F(p)$, and in a symbolic way we define

$$f(t) = L^{-1}[F].$$

What we have just defined is the following: If $F(p)$ is the Laplace transform of the function $f(t)$, then the function $f(t)$ is the inverse Laplace transform of $F(p)$.

Since L is a linear operator, it can be shown that L^{-1} is also linear.

Remark. The properties 1–3, 5–6 and convolution theorem are also used to determine the inverse Laplace transform when written in the equivalent forms:

1. $L^{-1}[F(p-a)] = e^{at}f(t)$;
2. $L^{-1}[F^{(n)}(p)] = (-1)^n t^n f(t)$;
3. $L^{-1}\left[\int_p^\infty F(x)\,dx\right] = \dfrac{f(t)}{t}$;
4. $L^{-1}\left[\dfrac{F(p)}{p}\right] = \int_0^t f(u)du$;
5. $L^{-1}[e^{-ap}F(p)] = u_a(t)f(t-a)$;
6. $L^{-1}[F(p)G(p)] = f * g$.

Now, we use some of the above companion formulas to find the inverse Laplace transform. The most general method of finding the inverse transform of any function is to decompose the given function into partial fractions. Using linearity of L^{-1}, we write the inverse transform of each fraction involved.

Example 10. Find the inverse Laplace transform of the following functions:

(a) $\dfrac{p}{(p+4)(p^2+4)}$; (b) $\dfrac{2p}{(p^2+4)^2}$.

For (a), by partial fractions,

$$\frac{p}{(p+4)(p^2+4)} = \frac{1}{5}\left(-\frac{1}{p+4} + \frac{1}{p^2+4} + \frac{p}{p^2+4}\right).$$

Now, using linearity of L^{-1} and the above remark, we have

$$L^{-1}\left[\frac{p}{(p+4)(p^2+4)}\right] = \frac{1}{5}\left\{-L^{-1}\left[\frac{1}{p+4}\right] + L^{-1}\left[\frac{1}{p^2+4}\right] + L^{-1}\left[\frac{p}{p^2+4}\right]\right\}.$$

Hence

$$L^{-1}\left[\frac{p}{(p+4)(p^2+4)}\right] = \frac{1}{5}\left(-e^{-4t} + \frac{\sin 2t}{2} + \cos 2t\right).$$

For (b), using property 2, we obtain

$$L^{-1}\left[\frac{2p}{(p^2+4)^2}\right] = -L^{-1}\left[\frac{d}{dp}\left(\frac{1}{p^2+4}\right)\right] = \frac{t\sin 2t}{2}.$$

Example 11. Find the inverse Laplace transform of the function

$$p \log\left(\frac{p+1}{p-1}\right) - 2.$$

Let the given function be the Laplace transform of $f(t)$, i.e.

$$L[f(t)] = p \log\left(\frac{p+1}{p-1}\right) - 2.$$

Then

$$\begin{aligned} L[t^2 f(t)] &= [p \log(p+1) - p \log(p-1) - 2]'' \\ &= \left[\log(p+1) + \frac{p}{p+1} - \log(p-1) - \frac{p}{p-1}\right]' \\ &= \frac{1}{p+1} + \frac{1}{(p+1)^2} - \frac{1}{p-1} + \frac{1}{(p-1)^2} \end{aligned}$$

Taking the inverse transform, we have

$$t^2 f(t) = e^{-t} + e^{-t}t - e^{t} + e^{t}t \quad \text{or} \quad \frac{1}{t^2}[e^{t}(t-1) + e^{-t}(t+1)].$$

Sometimes it is not an easy task to decompose an expression into partial fractions. In such situations we use the convolution theorem in the equivalent form:

$$L^{-1}[F(p)G(p)] = L^{-1}[F(p)] * L^{-1}[G(p)]. \tag{10.28}$$

Example 12. Use convolution theorem to find the inverse transform of the function

$$\frac{1}{p^2(p^2+1)}.$$

Since $L[t] = 1/p^2$ and $L[\sin t] = 1/(p^2+1)$, in view of (10.28), we have

$$L^{-1}\left[\frac{1}{p^2(p^2+1)}\right] = L^{-1}\left[\frac{1}{p^2}\right] * L^{-1}\left[\frac{1}{p^2+1}\right] = t * \sin t = t - \sin t.$$

For transforms involving e^{-ap}, the use of the unit impulse function facilitates the computations.

Example 13. Find the inverse transform of the function

$$F(p) = \frac{2}{p^2} - \frac{2e^{-2p}}{p^2} - \frac{4e^{-2p}}{p} + \frac{pe^{-\pi p}}{p^2+1}.$$

Operating L^{-1} on both the sides and using the second shifting formula in reverse direction, we get

$$\begin{aligned} L^{-1}[F(p)] &= 2t - 2u_2(t)(t-2) - 4u_2(t) + u_\pi(t) \cos(t-\pi) \\ &= 2t - 2u_2(t)t + u_\pi(t) \cos(t-\pi). \end{aligned}$$

Thus,

$$f(t) = \begin{cases} 2t, & 0 < t < 2, \\ 0, & 2 < t < \pi, \\ -\cos t, & t > \pi. \end{cases}$$

10.5 COMPLEX INVERSION FORMULA

In Section 10.4, we evaluated $L^{-1}[F(p)]$ by expressing $F(p)$ as a sum of functions which are the Laplace transforms of known functions or we applied the shifting formula, differentiation of transform, unit impulse function, convolution theorem and other properties of Laplace transform in the reverse direction. However, in some cases, it may not be possible to employ the above methods. To face such type of situations, here we develop the complex inversion formula which helps us to determine the inverse Laplace transform as complex line integral with the help of residue calculus. First we prove:

Proposition 2. Let f be a piecewise continuous function of exponential order defined on $[0, \infty)$. Suppose $L[f(t)] = F(p)$, $\text{Re } p > \alpha$, for some α. Then $F(p)$ is analytic in the domain $\Omega : \text{Re } p > \alpha$.

Proof. Given that $L[f) = F(p)$, $\text{Re } p > \alpha$, for some α, i.e.

$$F(p) = \int_0^\infty e^{-pt} f(t)dt, \qquad \text{Re } p > \alpha.$$

Since differentiation is valid under the sign of integral (Leibniz rule), it follows that

$$F'(p) = -\int_0^\infty e^{-pt} tf(t)dt.$$

The function $tf(t)$ is piecewise continuous and of exponential order (see Problem 5, Problem Set 10.1). This ensures the existence of $F'(p)$ for $\text{Re } p > \alpha$. Thus, F is differentiable for every p in the domain $\text{Re } p > \alpha$, and hence analytic.

Let $F(p)$ be analytic in the half-plane $\text{Re } p > \alpha$ and let $F(p) \to 0$ as $p \to \infty$. Consider $C = C_1 + C_2$, a closed contour consisting of a vertical segment C_1 from $\alpha + iR$ to $\alpha - iR$ and C_2 a semi-circle having positive orientation with radius R and centred at α, as shown in Fig. 10.3. Take any point p interior to C. Then, by the Cauchy integral formula

$$F(p) = \frac{1}{2\pi i}\int_C \frac{F(z)}{z-p}\,dz = \frac{1}{2\pi i}\int_{\alpha+iR}^{\alpha-iR} \frac{F(z)}{z-p}\,dz + \frac{1}{2\pi i}\int_{C_2} \frac{F(z)}{z-p}\,dz. \tag{10.29}$$

Assuming R to be sufficiently large, note that for z on C_2,

$$|z-p| = |(z-\alpha)-(p-\alpha)| \geq |z-\alpha| - |p-\alpha| = R - |p-\alpha|. \tag{10.30}$$

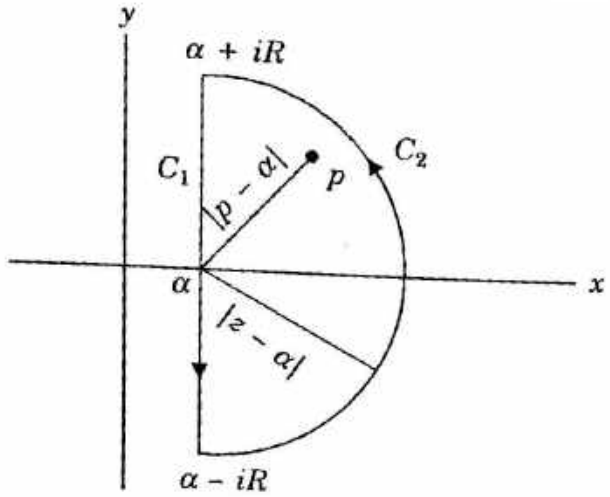

Figure 10.3

By assumption $F(z) \to 0$ as $z \to \infty$ through the domain Re $p > \alpha$. So, for given $\varepsilon > 0$, there exists R_ε such that

$$|F(z)| < \varepsilon \quad \text{whenever} \quad |z| \geq R_\varepsilon \text{ in } \Omega.$$

Again, $|z| \geq |z - \alpha| - \alpha$ (property of sides of a triangel) implies

$$|F(z)| < \varepsilon \quad \text{whenever} \quad |z - \alpha| \geq \alpha + R_\varepsilon \text{ in } \Omega.$$

Thus, by using (10.30), we have

$$\left| \int_{C_2} \frac{F(z)}{z - p}\, dz \right| < \frac{\pi R \varepsilon}{R - |p - \alpha|}, \quad R \geq \alpha + R_\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{R \to \infty} \int_{C_2} \frac{F(z)}{z - p}\, dz = 0.$$

Taking limit as $R \to \infty$ in (10.29), we get

$$\begin{aligned} F(p) &= \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{F(z)}{p - z}\, dz \\ &= \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} F(z) \left[\int_0^\infty e^{-(p-z)t}\, dt \right] dz \\ &= \int_0^\infty e^{-pt} \left[\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{zt} F(z)\, dz \right] dt. \end{aligned}$$

The change in the order of integration, being justified, is taken as granted.

Summarizing the above discussion we have proved the following result widely known as the complex inversion formula or Bromwich's integral formula.

Theorem 4. Let $F(p)$, the Laplace transform of $f(t)$, be analytic in the right half-plane Ω : Re $p > \alpha$ and $F(p) \to 0$ as $p \to \infty$ in Ω. Then

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp, \qquad c > \alpha, \tag{10.31}$$

where α is large enough so that all singularities (finite) of $F(p)$ lie in the half-plane Re $p < \alpha$.

The function $f(t)$ retrieved in (10.31) is referred to as the inverse Laplace transform and the usual function space for $f(t)$ are those functions satisfying (10.11).

Finding the inverse Laplace transform now reduces to computing the integral in (10.31). The contour from $c - i\infty$ to $c + i\infty$ is referred to as *Bromwich contour* and c is taken to the right of all singularities (poles, essential singularity, branch points) in order to ensure (10.32). Note that closing the contour to the right will yield $f(t) = 0$ for $t < 0$. Also, note $L^{-1}[F(p)]$ obtained by (10.31) is independent of c which can be assigned any value greater than α such that $F(p)$ is analytic in the right half-plane.

The complex line integral (10.31) is evaluated by using the residue theorem. To see how it is done, we assume that $F(p)$ has a finite number of singularities in the left half-plane Re $p \le c$. We then let $C = C_1 + C_2$ be the closed contour consisting of the vertical line segment C_1 from $c - iR$ to $c + iR$ and the semi-circle C_2 of radius R centred at Re $p = c$ and lying to the left of C_1, as shown in Fig. 10.4.

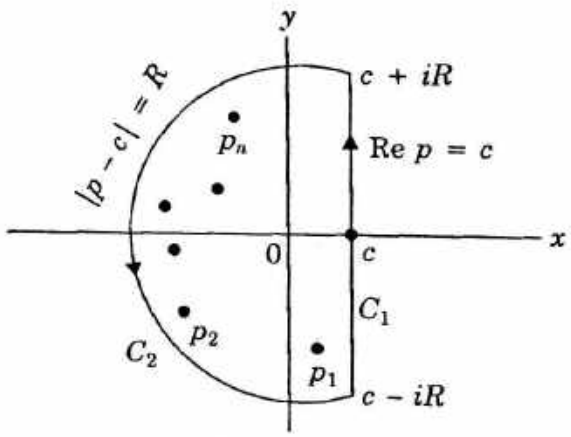

Figure 10.4

The radius R is to be taken large enough so that C encloses all the singularities of the function $e^{pt}F(p)$. Hence, by the residue theorem

$$\frac{1}{2\pi i}\int_{C_1} e^{pt}F(p)dp = \frac{1}{2\pi i}\int_{C} e^{pt}F(p)dp - \frac{1}{2\pi i}\int_{C_2} e^{pt}F(p)dp$$

$$= \sum_{k=1}^{n} \text{Res}_{p=p_k} e^{pt}F(p) - \frac{1}{2\pi i}\int_{C_2} e^{pt}\, F(p)dp \tag{10.32}$$

where $p_1, p_2, \ldots, p_n$ are the singularities of the function $F(p)$ inside C.

Since $F(p) \to 0$ as $p \to \infty$, Jordan's lemma (see Section 8.3) is used to have

$$\lim_{R\to\infty} \int_{C_2} e^{pt} F(p)\,dp = 0.$$

Note that $p = \alpha + Re^{i\theta}$, for some $\alpha > 0$ and $\theta(\pi/2 \le \theta \le 3\pi/2)$. Hence, taking the limit as $R \to \infty$, we obtain from (10.31) and (10.32),

$$L^{-1}\,[F(p)] = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{pt} F(p)dp = \sum_{k=1}^{n} \mathrm{Res}_{p=p_k} e^{pt} F(p). \tag{10.33}$$

Remarks. 1. The analysis to derive the Bromwich integral formula reveals that the sufficient condition $F(p) \to 0$ as $p \to \infty$ may be taken as

$$|\,F(p)\,| < \frac{M}{|\,p\,|^k}, \qquad \mathrm{Re}\ p > \alpha,$$

where M, $k \in \mathbb{R}_+$, the set of positive real numbers.

2. For the existence of inverse Laplace transform these conditions are sufficient but not necessary (see Problem 16, Problem Set 10.2).

Example 14. Evaluate the inverse Laplace transform of the function

$$\frac{2p+3}{(p-2)(p+1)^2}.$$

The partial fraction decomposition of the function $F(p) = (2p+3)/(p-2)(p+1)^2$ is time consuming and hence it is convenient to apply the complex inversion formula which is our last resort. The function $F(p)$ is analytic in $\mathrm{Re}\,p > 2$ and tends to zero uniformly as p approaches to infinity.

The singularities of the function $e^{pt}F(p)$ are a simple pole at $p_1 = 2$ and a pole of order two at $p_2 = -1$, and hence,

$$\mathrm{Res}_{p_1=2}\ e^{pt}F(p) = \lim_{p\to 2}\left[\frac{e^{pt}(2p+3)}{(p+1)^2}\right] = \frac{7}{9}e^{2t},$$

$$\mathrm{Res}_{p_2=-1}\ e^{pt}F(p) = \lim_{p\to -1}\left\{\frac{d}{dp}\left[\frac{e^{pt}(2p+3)}{p-2}\right]\right\} = -\frac{(7+3t)e^{-t}}{9}.$$

Inserting these values into (10.33), we get

$$f(t) = \frac{7}{9}e^{2t} - \frac{1}{9}(7+3t)e^{-t}.$$

In what follows we discuss the case of finding the inverse Laplace transform of multivalued functions.

Example 15. Find the inverse Laplace transform of the function

$$F(p) = p^{-a} \qquad (0 < a < 1).$$

As p^{-a} is multivalued we take the branch cut along the negative real axis (Fig. 10.5).

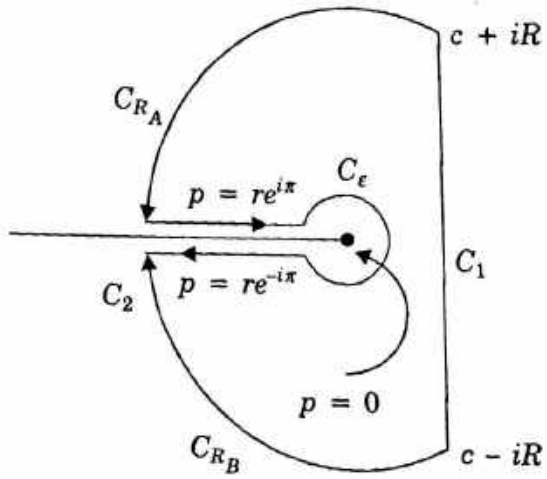

Figure 10.5

Assign $f(t) = 0$ for $t < 0$ when we close to the right. Closing to te left yields

$$f(t) + \left(\int_{C_{R_A}} + \int_{C_{R_B}} + \int_{p=re^{i\pi}} + \int_{p=re^{-i\pi}} + \int_{C_\varepsilon} \right) \frac{F(p)}{2\pi i} e^{pt} dp = 0.$$

The right-side vanishes because there is no singularities enclosed by the contour.

The integrals $\int_{R_A}$, $\int_{R_B}$ and $\int_{C_\varepsilon}$ vanish as $R \to \infty$ (on $C_{R_A} : p = c + Re^{i\theta}$, $\pi/2 < \theta < \pi$; on $C_{R_B} : p = c + Re^{i\theta}$, $-\pi < \theta < -\pi/2$, and $\varepsilon \to 0$ (on $C_e : p = \varepsilon e^{i\theta}$, $-\pi < \theta < \pi$). On C_ε, we have

$$\left| \int_{C_\varepsilon} \frac{F(p)}{2\pi i} e^{pt} dp \right| \le \int_{-\pi}^{\pi} \varepsilon^{1-a} \frac{e^{\varepsilon t}}{2\pi} d\theta \to 0 \qquad \text{as } \varepsilon \to 0.$$

Hence,

$$f(t) - \frac{1}{2\pi i} \int_{r=\infty}^{0} r^{-a} e^{-ia\pi} e^{-rt} dr - \frac{1}{2\pi i} \int_{r=0}^{\infty} r^{-a} e^{ia\pi} e^{-rt} dr = 0$$

or

$$f(t) = \frac{e^{ia\pi} - e^{-ia\pi}}{2\pi i} \int_0^\infty r^{-a} e^{-rt} dr.$$

Rescaling by $rt = u$ and recalling gamma function, we have

$$f(t) = \frac{\sin a\pi}{\pi} t^{a-1} \Gamma(1-a).$$

Remark. This analysis required that $F(p)$ has to have a finite number of singularities. Even if the number of singularities is infinite, formula (10.33) works well. But straightforward continuous limit $R \to \infty$ may pass through one of these singularities. To avoid such situations we take different contours as illustrated in the following example.

Example 16. Evaluate

$$L^{-1}\left[\frac{\cosh x\sqrt{p}}{p\cosh\sqrt{p}}\right], \quad 0 < x < 1.$$

First, let us understand that $p = 0$ is not a branch point (use the Taylor expansion), but it is a simple pole. The other singularities of the function are the zeros of $\cosh\sqrt{p}$, i.e.

$$\sqrt{p} = \left(n + \frac{1}{2}\right)\pi i \qquad \text{or} \qquad p = -\left(n + \frac{1}{2}\right)^2 \pi^2,$$

and these are simple poles.

Let us choose the Bromwich contour C_m as shown in Fig. 1.4 which is closed from the left. The semi-circle C_2 is centred at the origin with radius $R_m = m^2\pi^2$, $m \in \mathbb{I}_+$ so that the contour does not pass through any of the poles.

To find the residue of

$$\frac{e^{pt}\cosh x\sqrt{p}}{p\cosh\sqrt{p}}, \quad 0 < x < 1$$

at simple poles we use the formula $P(z_0)/Q'(z_0)$, $Q'(z_0) \neq 0$ (refer to Section 7.3). Thus, we have

$$\text{Res}_{p=0} = \left.\frac{e^{pt}\cosh x\sqrt{p}}{\cosh\sqrt{p} + (\sqrt{p}/2)\sinh\sqrt{p}}\right|_{p=0} = 1,$$

and

$$\text{Res}_{p=-(n+1/2)^2\pi^2} = \left.\frac{e^{pt}\cosh x\sqrt{p}}{\cosh\sqrt{p} + (\sqrt{p}/2)\sinh\sqrt{p}}\right|_{p=-(n+1/2)^2\pi^2}$$

$$= (-1)^{n+1}\frac{4e^{-(n+1/2)^2\pi^2 t}\cos(n+1/2)\pi x}{\pi(2n+1)}.$$

Hence

$$\frac{1}{2\pi i}\int_{C_m}\frac{e^{pt}\cosh x\sqrt{p}}{p\cosh\sqrt{p}}dp = 1 + \frac{4}{\pi}\sum_{n=0}^{m}\frac{(-1)^{n+1}}{2n+1}e^{-(n+1/2)^2\pi^2 t}\cos\left(n+\frac{1}{2}\right)\pi x.$$

Taking the limit as $m \to \infty$ and noting the fact that the integral around C_2 approaches zero, we find

$$L^{-1}\left[\frac{\cosh x\sqrt{p}}{\cosh\sqrt{p}}\right] = 1 + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}}e^{-(n+1/2)^2\pi^2 t}\cos\frac{(2n-1)\pi x}{2}.$$

Remark. While using (10.33) it is convenient to evaluate the sum of residues of $e^{pt}F(p)$ by the following technique:

If $F(z)$ has a pole of order m at a point z_0 of the real axis, then its Laurent series expansion in the disc $0 < |z - z_0| < R$ has the principal part

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0)$$

and residue of $z = z_0$ is

$$\text{Res}_{z=z_0} e^{zt} F(z) = e^{z_0 t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_m}{(m-1)!}t^{m-1}\right]. \tag{10.34}$$

In case z_0 is of the form $z_0 = \alpha + i\beta$ $(\beta \neq 0)$ and $\overline{F(z)} = F(\bar{z})$ at the points of analyticity of $F(z)$, the conjugate point $\bar{z}_0 = \alpha - i\beta$ is also a pole of order of m. Then,

$$\text{Res}_{z=z_0} e^{zt} F(z) + \text{Res}_{z=\bar{z}_0} e^{zt} F(z) = 2e^{\alpha t}\text{Re}\left\{e^{i\beta t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_m t^{m-1}}{(m-1)!}\right]\right\}. \tag{10.35}$$

when t is real. In particular, if p_0 is a simpel pole $(m = 1)$, expressions (10.34) and (10.35) become

$$\text{Res}_{p=p_0} e^{pt} F(pt) = e^{p_0 t}\text{Res}_{p=p_0} F(p),$$

and

$$\text{Res}_{p=p_0} e^{pt} F(p) + \text{Res}_{p=\bar{p}_0} e^{pt} F(p) = 2e^{\alpha t}\,\text{Re}[e^{i\beta t}\text{Res}_{p=p_0} F(p)],$$

respectively.

PROBLEM SET 10.2

1. Find the inverse Laplace transform of the following functions:

(a) $\dfrac{5}{p^2(p+5)^2}$; (b) $\dfrac{2p+3}{p^2 - 4p + 20}$; (c) $\dfrac{2}{(p+1)(p^2+1)}$;

(d) $\dfrac{3p}{(p^2+1)(p^2+4)}$.

2. Find and sketch the inverse transform of the following functions:

(a) $\dfrac{e^{-3p}}{p^2 + 6p + 10}$; (b) $\dfrac{1 - e^{-\pi p}}{p^2 + 4}$; (c) $\dfrac{3e^{-2p}}{3p^2 + 1}$;

(d) $\dfrac{(2p+1)e^{-\pi p}}{2(p^2 + p - 1)}$.

3. Find $f(t)$ when $F(p)$ is given as

(a) $\log \dfrac{p^2+1}{(p-1)^2}$; (b) $\log\left(\dfrac{p+a}{p+b}\right)$; (c) $\log\left(\dfrac{p^2+1}{p^2}\right)$; (d) $p\log\left(\dfrac{p-1}{p+1}\right)$.

4. Find $f(t)$, if $L[f(t)]$ equals

(a) $\cot^{-1}\dfrac{p}{w}$; (b) $\tan^{-1}\left(\dfrac{1}{p+1}\right)$; (c) $\tan^{-1}\dfrac{a}{p}+\cot^{-1}\dfrac{p}{b}$; (d) $\tan^{-1}\dfrac{2}{p^2}$.

Suggestion. For Problems 3 and 4, use $f(t) = -L^{-1}[F'(p)]/t$ or $f(t) = L^{-1}[F''(p)]/t^2$.

5. Prove that

$$L[t\cosh at] = \frac{p^2+a^2}{(p^2-a^2)^2}$$

and hence evaluate

$$L^{-1}\left[\frac{1}{(p^2-a^2)^2}\right].$$

6. Evaluate following transforms:

(a) $L^{-1}\left[\dfrac{8}{p^4-2p^2}\right]$; (b) $L^{-1}\left[\dfrac{1}{p^2}\dfrac{p+1}{p^2+1}\right]$; (c) $L^{-1}\left[\dfrac{2p^3}{p^4-4}\right]$.

7. Using the convolution and induction process, prove that

$$L^{-1}\left[\frac{1}{(p-a)^n}\right] = \frac{t^{n-1}e^{at}}{(n-1)!}.$$

8. Derive Eqs. (10.34) and (10.35).

9. Show that for any integer $n \geq 1$, $a \neq 0$,

$$L^{-1}\left[\frac{1}{(p^2+a^2)^{n+1}}\right] = \frac{1}{2n}\int_0^t tL^{-1}\left[\frac{1}{(p^2+a^2)^n}\right]dt.$$

10. Use the preceding problem to show that

$$L^{-1}\left[\frac{1}{(p^2+a^2)^{n+1}}\right] = \frac{1}{2^n an!}\int_0^t x\int_0^t x\cdots\int_0^t x\sin ax\,dx\,dx\ldots dx,$$

where the integral is iterated n times.

11. Verify the formulas

(a) $L^{-1}\left[\dfrac{e^{-1/p}}{p}\right] = J_0(2\sqrt{t})$; (b) $L^{-1}\left[\dfrac{1}{p-1}\right] = \dfrac{e^t}{\sqrt{\pi t}}$,

where $J_0(t)$ is the Bessel function of order zero.

Suggestion. For part (a) use the series expansion. For part (b) use the first shifting theorem in $L[t^{-1/2}]$.

12. Prove that

$$L^{-1}\left[\frac{1}{(p^2+2p+5)^{3/2}}\right] = \frac{te^{-t}}{2}J_1(2t),$$

where $J_1(t)$ is the Bessel function of order one.

13. Establish the following results:

(a) $L^{-1}\left[\dfrac{e^{-b\sqrt{p}}}{\sqrt{p}}\right] = \dfrac{1}{\sqrt{\pi}}\dfrac{e^{-b^2/4t}}{t}$;

(b) $L^{-1}\left[\dfrac{e^{-b\sqrt{p}}}{p}\right] = 1 - \operatorname{erf}\left(\dfrac{b}{2\sqrt{t}}\right)$;

(c) $L^{-1}\left[\dfrac{1}{\sqrt{p+a}+\sqrt{p+b}}\right] = \dfrac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}$.

14. Use the complex inversion formula to find the inverse transform of the following functions:

(a) $\dfrac{1}{(p+1)^3(p-2)^2}$; (b) $\dfrac{p^2}{(p^2+4)^2}$; (c) $\dfrac{1}{p(e^p+1)}$; (d) $\dfrac{p}{p^3+1}$.

15. Prove that

(a) $L^{-1}\left[\dfrac{1}{p^2} - \dfrac{1}{\sinh p}\right] = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin n\pi t$;

(b) $L^{-1}\left[\dfrac{\tanh p}{p^2}\right] = 1 - \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos\dfrac{(2n-1)\pi t}{2}, \quad t > 0$;

(c) $L^{-1}\left[\dfrac{1}{p^2\sinh p}\right] = \dfrac{t^2}{2} + \dfrac{2}{\pi^2} + \displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}(1-\cos n\pi t)$;

(d) $L^{-1}\left[\dfrac{1}{p}\sin\dfrac{1}{p}\right] = \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{t^{2n+1}}{[(2n+1)!]^2}$.

16. Deal with the function which does not satisfy the sufficient conditions for the existence of the Bromwich integral (of Section 10.5) as follows:

(a) Show that the inverse Laplace transform of $F_1(p) = \log p$ is $f_1(t) = -1/t$;

(b) Do the same for $F_2(p) = \log(p+1)$ to obtain $f_2(t) = -e^{-t}/t$;

(c) Subtract the results of parts (a) and (b) to verify that the inverse transform of $F(p) = \log[(p+1)/p]$ is the function $f(t) = (1-e^{-t})/t$;

(d) Compute $L^{-1}[\log(1+1/p)]$ directly, using the techniques of Problem 3.

17. First use the complex inversion formula to establish the relation

$$L^{-1}\left[\frac{1}{p\cosh p}\right] = 1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\left(\frac{2n-1}{2}\right)\pi t,$$

and then extend this result as

$$L^{-1}\left[\frac{1}{p^2\cosh p}\right] = t + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(2n-1)^2}\sin\left(\frac{2n-1}{2}\right)\pi t.$$

Suggestion. For extension, apply $L^{-1}[F(p)/p] = \int_0^t f(u)du$.

18. Show that the inverse Laplace transform of the function $F(p) = \log p/(p^2 + \omega^2)$ is

$$f(t) = \frac{\pi}{2\omega}\cos\omega t - \int_0^{\infty}\frac{e^{-rt}}{r^2+\omega^2}dr, \qquad t > 0.$$

11

Special Topics

This chapter is devoted to special topics not treated in earlier chapters. The techniques show how to prove the Cauchy theorem using the concept of homotopy and construct the harmonic functions which take prescribed values on boundary of the domain of definition. Sections 11.3–11.6 include the canonical products, gamma function, theory of entire functions and distribution of zeros. In the end we give a brief exposition of the mapping by multivalued functions and introduce the Schwarz–Christoffel transformation.

11.1 HOMOTOPY AND THE CAUCHY THEOREM

In this section we discuss more general conditions under which we may vary the path γ continuously and even then $\int_\gamma f(z)dz$ remains unchanged when f is analytic in a simply connected domain D and γ is contained in D (see Theorem 2, Chapter 4). This is done by introducing the concept of homotopy of curves in the complex plane and by giving another proof of the famous Cauchy's theorem. Here the Cauchy's theorem means the Cauchy-Goursat theorem of Chapter 4.

Definition 1. Let γ_0 and γ_1 be two paths defined on $[a, b]$ in a domain $D \subset \mathbb{C}$ such that $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$, i.e. the paths have common initial and terminal points. Then γ_0 and γ_1 are said to be homotopic in D if there exists a continuous map F with values in D

$$F\colon [0, 1] \times [a, b] \to \mathbb{C}$$

such that

$$\begin{array}{lll} \text{(i)}\ (0, t) = \gamma_0 & F(1, t) = \gamma_1, & 0 \le t \le 1, \\ \text{(ii)}\ F(s, a) = \gamma_0(a), & F(s, b) = \gamma_0(b), & 0 \le s \le 1. \end{array}$$

Symbolically, we write $\gamma_0 \simeq \gamma_1$ or γ_0 is F homotopic to γ_1. The function F is called a homotopy.

The concept of homotopy has a simple geometric interpretation. For each fixed $s \in [0, 1]$, let

$$\gamma_s(t) = F(s, t).$$

Then $\gamma_s(t)$ can be regarded as an intermediate moving path which starts from γ_0 when $s = 0$ and terminates at γ_1 when $s = 1$. As s varies continuously between 0 and 1, we

have infinitely many paths which start at $\gamma_0(a)$ and terminate at $\gamma_0(b)$, i.e. γ_0 can be continously transformed to γ_1 (Fig. 11.1).

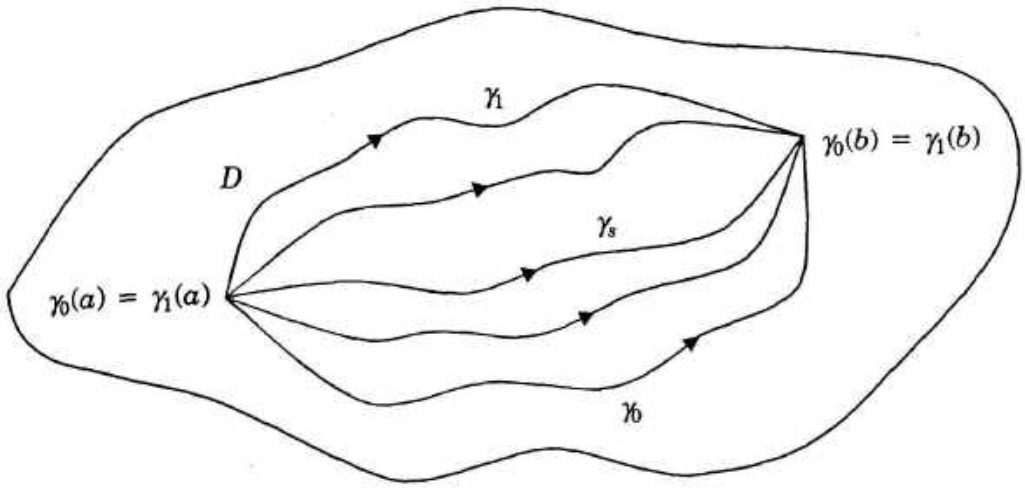

Figure 11.1

Remark. In Definition 1, γ_0 and γ_1 may be closed paths, i.e.

$$\gamma_0(a) = \gamma_0(b), \quad \gamma_1(a) = \gamma_1(b).$$

In this case the condition (ii) of Definition 1 is taken as

$$F(s, a) = F(s, b).$$

Now each γ_s is a closed path and corresponding to s, they form a family of closed paths which start at γ_0 and go to γ_1 (Fig. 11.2).

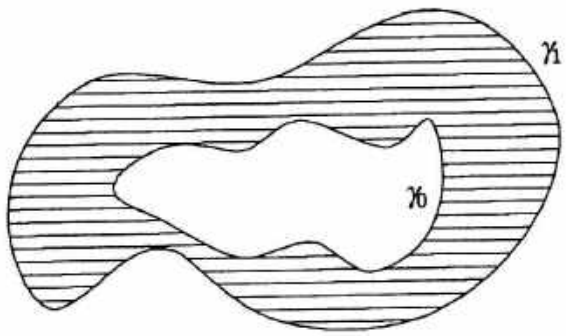

Figure 11.2

Note that if γ_1 is a constant function, so that its graph is a single point, and if $\gamma_0(t)$ is homotopic to γ_1 in D, i.e. $\gamma_1 \simeq \gamma_0$, then we say γ_0 is homotopic to a point in D.

Definition 2. A domain D is said to be simply connected if every simple closed path in D is homotopic to a point (constant function) in D.

If, for each $t \in [a, b]$, the line segment joining $\gamma_0(t)$ and $\gamma_1(t)$ lies in D, then γ_0 and γ_1 are homotopic in D because the function

$$F(s, t) = s\gamma_1(t) + (1-s)\gamma_0(t).$$

serves as a homotopy. In this case we say that γ_0 and γ_1 are linearly homotopic.

Homotopy is an equivalence relation:

(i) $\gamma_0 \simeq \gamma_0$, this is obvious;

(ii) $\gamma_0 \simeq \gamma_1 \Rightarrow \gamma_1 \simeq \gamma_0$. To prove this, if we let F: $\gamma_0 \simeq \gamma_1$ carry γ_0 into γ_1, then

$$H(s, t) = F(1-s, t)$$

is a homotopy carrying γ_1 into γ_0.

(iii) $\gamma_0 \simeq \gamma_1$ and $\gamma_1 \simeq \gamma_2 \Rightarrow \gamma_0 \simeq \gamma_2$. To show this, let F: $\gamma_0 \simeq \gamma_1$ and G: $\gamma_1 \simeq \gamma_2$. Then

$$H(s, t) = \begin{cases} F(2s, t), & 0 \le s \le 1/2 \\ G(2s-1, t) & 1/2 \le s \le 1 \end{cases}$$

is a homotopy carrying γ_0 into γ_2.

Proposition 1 (Polygonal interpolation). Let γ_0 and γ_1 be homotopic paths in a simply connected domain D. Then there exists a finite number of paths $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that

(i) $\alpha_0 = \gamma_0$ and $\alpha_n = \gamma_1$,

(ii) α_j is a polygonal path for $1 \le j \le n-1$,

(iii) α_j is linearly homotopic in D to α_{j+1} for $0 \le j \le n-1$.

Proof. Since γ_0 and γ_1 are homotopic in D, there is a homotopy F satisfying the conditions of Definition 1. Consider the equally spaced partitions

$$\{s_0, s_1, \ldots, s_n\} \text{ of } [0, 1], \qquad \{t_0, t_1, \ldots, t_n\} \text{ of } [a, b]$$

Choose n so large that the image of each rectangle $[s_j, s_{j+1}] \times [t_k, t_{k+1}]$ under F is contained in an open disc D_{jk}. This is possible due to uniform continuity of F.

On the intermediate path γ_{sj} given by

$$\gamma_{sj}(t) = F(s_j, t), \quad 0 < j < n,$$

we inscribe a polygonal path α_j with vertices at the points $F(s_j, t_k)$. That is

$$\alpha_j(t_k) = F(s_j, t_k), \qquad \text{for } k = 0, 1, \ldots, n$$

and α_j is linear on each subinterval $[t_k, t_{k+1}]$ for $0 \le k \le n-1$. We also define $\alpha_0 = \gamma_0$ and $\alpha_n = \gamma_1$ (see Fig. 11.3).

The four vertices $\alpha_j(t_k)$, $\alpha_j(t_{k+1})$, $\alpha_{j+1}(t_k)$, $\alpha_{j+1}(t_{k+1})$ all lie in D_{jk}. Since D_{jk} is pathwise connected, i.e. the line segments joining them also lie in D_{jk}, hence the points

$$s\alpha_{j+1}(t) + (1-s)\alpha_j(t)$$

lie in D_{jk} for each (s, t) in $[0, 1] \times [a, b]$. Therefore all the points of these line segments lie in D for all (s, t) in $[0, 1] \times [a, b]$, so α_{j+1} is linear homotopic to α_j in D.

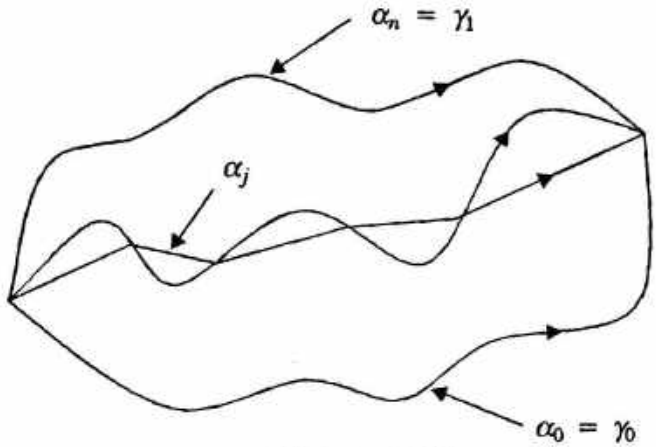

Figure 11.3

Theorem 1. Assume that f is analytic in a domain D, except possibly for a finite number of points, where it is continuous. If γ_0 and γ_1 are homotopic paths in D, we have

$$\int_{\gamma_0} f(z) = \int_{\gamma_1} f(z).$$

Proof. First, consider the case in which γ_0 and γ_1 are linearly homotopic. For each s in [0, 1], let

$$\gamma_s(t) = s\gamma_1(t) + (1-s)\gamma_0(t), \quad t \in [a, b].$$

Then γ_s is a piecewise smooth closed path and its graph lies in D. Note that γ_s can also be written as

$$\gamma_s(t) = \gamma_0(t) + s\alpha(t), \quad \alpha(t) = \gamma_1(t) - \gamma_0(t). \tag{11.1}$$

From (11.1), it obvious that

$$\frac{\partial}{\partial s}[\gamma_s(t)] = \alpha(t), \qquad \frac{\partial}{\partial t}[\gamma_s(t)] = \gamma_0'(t) + s\alpha'(t).$$

Define

$$\phi(s) = \int_{\gamma_s} f = \int_a^b f[\gamma_s(t)]\gamma_0'(t) + s\int_a^b f[\gamma_s(t)]\alpha'(t), \qquad 0 \le s \le 1.$$

We wish to prove that $\phi(0) = \phi(1)$. We prove more and show that ϕ is constant on [0, 1].

Using relations in (11.1) and applying the Leibniz rule for differentiation under the sign of integration, we have

$$\begin{aligned}\phi'(s) &= \int_a^b f'[\gamma_s(t)]\alpha(t)\gamma_0'(t) + s\int_a^b f'[\gamma_s(t)]\alpha(t)\alpha'(t) + \int_a^b f[\gamma_s(t)]\alpha'(t)\\ &= \int_a^b \alpha(t)f[\gamma_s(t)]\gamma_s'(t) + \int_a^b f[\gamma_s(t)]\alpha'(t)\\ &= \alpha(b)f[\gamma s(b)] - \alpha(a)f[\gamma_s(a)]\end{aligned}$$

on applying the integration by parts in the last but one step.

The last expression vanishes because γ_0 and γ_1 are homotopic, $\phi'(s) = 0$ for all $s \in [0, 1]$. Therefore ϕ is constant on $[0, 1]$. This proves the theorem when γ_0 and γ_1 are linearly homotopic.

If they are homotopic in D under a general homotopy F, we interpolate polygonal paths α_j as described in Proposition 1. Since each polygonal path is piecewise smooth, we can repeatedly use the result just proved to obtain

$$\int_{\gamma_0} f = \int_{\alpha_0} f = \int_{\alpha_1} f = \cdots = \int_{\alpha_n} f = \int_{\gamma_1} f.$$

Corollary 1 (Cauchy theorem: Homotopy version). Assume that f is analytic on a simply connected domain D, except possible for a finite number of points, whereas it is continuous. Then for every closed path γ in D,

$$\int_{\gamma} f(z)dz = 0.$$

Proof. Let γ_0 be a constant curve. Then $\gamma'_0(t) = 0$ so that

$$\int_{\gamma_0} f(z)dz = 0.$$

The result follows from Theorem 1 upon taking $\gamma_1 = \gamma$.

Remark. Here, we have proved Theorem 2 of Chapter 4 for any closed path (not just the simple closed path).

11.2 CONSTRUCTION OF HARMONIC FUNCTIONS

How to construct a harmonic function in a given domain when its values are prescribed on the boundary of the domain is the key problem to be discussed in this section. These are known as *Dirichlet problems*.

Before turning to construction of harmonic functions we establish the Poisson integral formula.

Poisson integral formula. Let $u(x, y)$ be a function that is harmonic in a simply connected domain Ω and let $C : |s| = R$ be any circle contained in Ω. Then, u has the integral representation

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(t-\theta) + r^2}\, u(Re^{it})\, dt, \qquad 0 < r < R. \tag{11.2}$$

Proof. Given that $u(x, y)$ is harmonic in Ω, we can find a harmonic function $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic in Ω. Let $C : |s| = R$ be contained in this domain. Then, by the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s - z}\, ds. \tag{11.3}$$

The inverse of the point z with respect to the circle C lies outside the circle and hence again, by the Cauchy integral formula,

$$0 = \frac{1}{2\pi i}\int_C \frac{f(s)}{s - R^2/\bar{z}}\,ds. \tag{11.4}$$

Thus, on subtracting (11.4) from (11.3), we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z} - \frac{1}{s - R^2/\bar{z}}\right) f(s)\,ds \\ &= \frac{1}{2\pi i}\int_C \frac{z - R^2/\bar{z}}{(s-z)(s - R^2/\bar{z})} f(s)\,ds \\ &= \frac{1}{2\pi i}\int_C \frac{|z|^2 - R^2}{(s-z)(s\bar{z} - R^2)} f(s)\,ds \\ &= \frac{1}{2\pi i}\int_C \frac{R^2 - |z|^2}{s(s-z)(\bar{s} - \bar{z})} f(s)\,ds, \quad \text{since } s\bar{s} = R^2. \end{aligned} \tag{11.5}$$

Hence, putting $z = re^{i\theta}$, $s = Re^{it}$ and $\bar{z} = re^{-i\theta}$ in (11.5), we get

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(t-\theta) + r^2} f(Re^{it})\,dt. \tag{11.6}$$

Since $u(x, y)$ is the real part of $f(z)$ and $u(R, t) = U(Re^{it})$ is the real part of $f(Re^{it})$, we compare the real parts on both sides of (11.6) to have (11.2) which is known as the *Poisson integral* formula.

Remarks. 1. Equation (11.2) is also used in the form

$$u(r, \theta) \equiv u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(R, r, t-\theta)u(Re^{it})dt,$$

where

$$P(R, r, t-\theta) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(t-\theta) + r^2}.$$

is defined as the *Poisson kernel.*

2. In case the imaginary components of (11.6) are compared, an expression similar to (11.2) exists for the conjugate harmonic function $v(r, \theta)$.

Here, we describe only the Dirichlet problem for the disc, i.e. that to find a real-valued function $u(x, y)$ that is harmonic in the disc and takes on the boundary values. We assume that the reader is well acquainted with the Fourier series.

For simplicity, we start with the unit disc $D : |z| < 1$. Now, our problem is to find a real-valued function that is harmonic in D and that takes on the boundary values

$$u(\cos t, \sin t) = U(t) \quad \text{for} - \pi < t \leq \pi$$

at points $z = (\cos t, \sin t) = e^{it}$ on the unit circle and for the sake of brevity, we denote $u(e^{it})$ by $U(t)$.

Theorem 2. If $U(\theta)$ has period 2π, and has the Fourier series expansion

$$U(\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

then the solution u to the Dirichlet problem in unit disc is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta),$$

where $z = re^{i\theta}$ denotes a complex number in the disc $|z| \leq 1$.

Proof. Note that for $R = 1$ (also refer to (11.2))

$$\begin{aligned}
\frac{1 - r^2}{1 - 2r\cos(t - \theta) + r^2} &= \frac{1 - re^{-i(t-\theta)} + re^{-i(t-\theta)} - r^2}{1 - r[e^{i(t-\theta)} + e^{-i(t-\theta)}] + r^2} \\
&= \frac{1}{1 - re^{i(t-\theta)}} + \frac{re^{-i(t-\theta)}}{1 - re^{-i(t-\theta)}} \\
&= \sum_{n=0}^{\infty} r^n e^{in(t-\theta)} + \sum_{n=1}^{\infty} r^n e^{-in(t-\theta)} \\
&= 1 + 2\sum_{n=1}^{\infty} r^n \cos n(t - \theta) \\
&= 1 + 2\sum_{n=1}^{\infty} r^n (\cos nt \cos n\theta + \sin nt \sin n\theta). \qquad (11.7)
\end{aligned}$$

On using (11.7) in expression for $u(r, \theta)$ when $R = 1$, we have

$$\begin{aligned}
u(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(t - \theta) + r^2} U(t)\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} U(t)\, dt + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} r^n\, U(t) \cos nt \cos n\theta\, dt \\
&\quad + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} r^n\, U(t) \sin nt \sin n\theta\, dt
\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(t)\,dt + \sum_{n=1}^{\infty} r^n \cos n\theta \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\cos nt\,dt$$

$$+ \sum_{n=1}^{\infty} r^n \sin n\theta \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\sin nt\,dt$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta),$$

where a_n and b_n are the Fourier coefficients of $U(t)$ defined by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\cos nt\,dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\sin nt\,dt, \qquad n = 0, 1, 2, \ldots$$

Remark. What we have concluded is that the problem of determining a harmonic function on a unit disc when its continuous values are prescribed on the boundary just reduces to getting its Fourier series with the understanding that we have to multiply it by r^n, $0 < r \le 1$.

Example 1. Find $u(r, \theta)$ that is harmonic in the unit disc and takes on the values

$$f(\theta) = \frac{\theta}{2} \quad \text{for } -\pi < \theta < \pi.$$

In view of the above remark, first we find the Fourier series of f, which is given by

$$f(\theta) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin n\theta}{n}.$$

Hence, the desired harmonic function is given by

$$u(r, \theta) = \sum_{n=1}^{\infty} (-1)^{n+1} r^n \frac{\sin n\theta}{n}.$$

Observe that this series representation for u takes on the prescribed values at points where f is continuous. The boundary function is discontinuous at $z = -1$ which corresponds to $\pm\pi$; and f was not prescribed at these points. The function $u(r, \theta)$ has been written as the sum of harmonic functions $r^n \sin n\theta$ (convergent series of harmonic functions) and we, therefore, claim that $u(r, \theta)$ is a harmonic function within and on the unit disc.

PROBLEM SET 11.1

1. State and prove the homotopy version of the Cauchy integral formula.

2. For an analytic function f in a simply connected domain D, there is a function F such that $F'(z) = f(z)$ is unique up to an additive constant.

Suggestion. This is another version of the Cauchy theorem.

3. Consider two circular paths defined by

$$\gamma_0(t) = re^{2\pi it}, \qquad \gamma_1(t) = (r+1)e^{2\pi it} \qquad (t \in [0, 1]),$$

where r is fixed. Show that γ_0 and γ_1 are homotopic in a region containing $|z| \leq r + 1$.

4. Let γ_0 and γ_1 be closed paths in a domain D of $\mathbb{C}$ such that $\gamma_0 \simeq \gamma_1$. Then show that

$$\eta(\gamma_0; a) = \eta(\gamma_1; a).$$

5. Consider two circular paths defined by

$$\gamma_0(t) = e^{2\pi it}, \qquad \gamma_1(t) = -e^{2\pi it} \qquad (t \in [0, 1]),$$

Show that γ_0 and γ_1 are not homotopic in $D = \mathbb{C}\backslash\{0\}$.

Suggestion. Check the integral of $f(z) = 1/z$ about γ_0 and γ_1.

6. Find the solution to the given Dirichlet problem in the unit disc D by using the Fourier series expansions for the boundary functions defined as follows:

(a) $f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$; (b) $f(t) = \begin{cases} t + \pi, & -\pi < t < 0 \\ t - \pi, & 0 \leq t \leq \pi \end{cases}$.

11.3 CANONICAL PRODUCTS

This section consists of results obtained on application of infinite products already introduced in Section 5.5.

Proposition 2. Let $f(z)$ be an entire function. Then $f(z)$ never vanishes $\Leftrightarrow f(z) = e^{g(z)}$, where $g(z)$ is an entire function.

Proof. The function $f(z) = e^{g(z)}$ is nonvanishing everywhere is very much obvious.

To prove conversely, consider a function $f'(z)/f(z)$. Since $f(z)$ is an entire function which never vanishes and the derivative $f'(z)$ is also an entire function, it follows that $f'(z)/f(z)$ is well defined for all z and is an entire function. This function has an antiderivative (see Corrollary 3, Chapter 4), $g(z)$ such that

$$g'(z) = \frac{f'(z)}{f(z)}.$$

So, $g'(z)f(z) = f'(z)$, and this ensures that

$$\frac{d}{dz}[f(z)e^{-g(z)}] = f'(z)e^{-g(z)} - g'(z)f(z)e^{-g(z)} = 0.$$

Thus,

$$f(z)e^{-g(z)} = c \qquad \text{or} \qquad f(z) = ce^{g(z)},$$

where c is a constant which may be absorbed in $g(z)$ to have the required assertion.

Remark. If $f(z)$ is an entire function having $a_1, a_2, \ldots, a_n$ as its zeros, then

$$\frac{f(z)}{(z-a_1)(z-a_2)\cdots(z-a_n)}$$

is an entire function which has no zeros. Hence, in view of Proposition 2, it can be written as $e^{g(z)}$, i.e.

$$f(z) = \prod_{k=1}^{n} (z-a_k)\, e^{g(z)}.$$

In case the zeros $a_1, a_2, \ldots a_n$ have multiplicities $p_1, p_2, \ldots p_n$, respectively, then

$$f(z) = \prod_{k=1}^{n} (z-a_k)^{p_k} e^{g(z)}$$

However, we prove a more general result, known as *Weierstrass factorization theorem* which is about constructing an entire function with prescribed zeros.

Theorem 3 (Weierstrass factorization theorem). Let f be an entire function with zeros $a_1, a_2, \ldots$, each zero is counted as often as its multiplicity ($a_n \neq 0$ for any n). Then, f can be expressed in the form

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right), \tag{11.8}$$

where M_n are the positive integers and $g(z)$ is an entire function.

Further, if $f(0) = 0$, then f can be expressed in the form

$$f(z) = z^k e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left[\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right], \tag{11.9}$$

where k is the multiplicity of 0.

Proof. Suppose $f(0) \neq 0$. Then the function

$$P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left[\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right]$$

is an entire function and has zeros at a_n's. Thus, $f(z)/P(z)$ is an entire function without zeros. In view of Proposition 2,

$$\frac{f(z)}{P(z)} = e^{g(z)}$$

for suitable entire function $g(z)$, i.e. $f(z) = e^{g(z)}P(z)$, and hence (11.8) is established.

In case $f(0) = 0$ and 0 has multiplicity k, then $f(z)/z^k P(z)$ is an entire function without zeros and hence by the same reasoning (11.9) is proved.

Corollary 2. Every function which is meromorphic in the whole complex plane is the quotient of two entire functions.

Proof. If $F(z)$ is meromorphic in the whole complex plane, Theorem 3 ensures the construction of an entire $g(z)$ with prescribed zeros which are poles of $F(z)$. Obviously, the product $F(z)g(z)$ is then an entire function $f(z)$, say. Hence,

$$F(z) = \frac{f(z)}{g(z)}.$$

The representation (11.9) is found to be very interesting if it is possible to choose all the M_n equal to each other. The proof of Proposition 9 in Chapter 5 has shown that the product

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left[\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h\right] \tag{11.10}$$

converges and represents an entire function, provided the series

$$\sum_{n=1}^{\infty}\frac{1}{h+1}\left(\frac{R}{|a_n|}\right)^{h+1}$$

converges for all R or $\Sigma(1/|a_n|)^{h+1} < \infty$. Assume that h is the smallest integer for which this series converges; the expression (11.10) is called the *canonical product* associated with the sequence $\{a_n\}$, and h is called the *genus* of the canonical product. Normally, we use the canonical product in the representation of (11.9), which is uniquely determined. If, in a canonical representation $g(z)$ reduces to a polynomial, the function is said to have finite genus, and the genus of $f(z)$ is by definition equal to the degree of this polynomial or to the genus of the canonical product, whichever is larger. For example, an entire function f of genus zero is of the form

$$f(z) = Cz^k\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) \tag{11.11}$$

with $\Sigma\, 1/|a_n| < \infty$. While an entire function f of genus 1 is of the form

$$f(z) = Cz^k e^{az}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{z/a_n} \tag{11.12}$$

with $\Sigma\, 1/|a_n| = \infty$ and $\Sigma\, 1/|a_n|^2 < \infty$.

Note that we have mentioned Eqs. (11.11) and (11.12) for entire functions having zero of order k at the origin. For entire functions without zeros at the origin the term z^k will remain absent from both the equations.

Let us discuss the important phenomenon of entire functions.

Example 2. If $f(z)$ is a nonconstant entire function of genus 0 or 1, which is real on the real axis and has only real zeros, show that the zeros of $f'(z)$ are real, and beteween any two zeros of $f(z)$ lies a zero of $f'(z)$ (and only one).

The entire function f has the form (see Eq. (11.12))

$$f(z) = z^m e^{az+b}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{z/n},$$

where a_n's are real zeros of $f(z)$ and the constant C has been merged with e^{az+b}.

Since $f(z)$ is real for all real, we must have e^{az+b} as real, i.e. constants a and b are real. Taking the logarithmic derivative, we find

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + a + \sum_{n=1}^{\infty}\left(\frac{1}{z-a_n} - \frac{1}{a_n}\right) = Q(z) \quad \text{(say)}$$

so that $f'(z)$ is real, like $f(z)$ is real on the real axis. When $z = x + iy$,

$$\operatorname{Im}\{Q(z)\} = -y\left[\frac{m}{x^2+y^2} + \sum_{n=1}^{\infty}\frac{1}{(x-a_n)+y^2}\right]$$

which vanishes only when $y = 0$, showing that the zeros of $f'(z)$, if any, lie on the real axis. But

$$\frac{d}{dz}\left[\frac{f'(z)}{f(z)}\right] = -\frac{m}{z^2} - \sum_{n=1}^{\infty}\frac{1}{(z-a_n)^2}$$

which is real and negative for real values $z \neq a_n$, $n = 1, 2 \ldots$. Thus, z passes along the real axis from one zero a_n to the next, f'/f steadily decreases from $+\infty$ to $-\infty$ and therefore, f' vanishes exactly once in each iterval (a_n, a_{n+1}), $n = 1, 2, 3, \ldots$.

Similarly, we can prove the result for entire functions of genus zero (see Eq. (11.12)).

As our next application, we derive the canonical representation of the entire functions $\sin z$ and $\cos z$.

Example 3. Write the entire functions $\sin z$ and $\cos z$ as canonical products.

The zeros of $\sin z$ are $z = n\pi$, $n \in \mathbb{I}$. Thus the origin is a simple zero. Since $\Sigma_{n=1}^{\infty} 1/(n\pi)$ diverges and $\Sigma_{n=1}^{\infty} 1/(n^2\pi^2)$ converges, we must take $h = 1$ in (11.10) and apply the Weierstrass factorization theorem to write

$$\sin z = ze^{g(z)}\prod_{n=-\infty}^{\infty}\left(1 - \frac{z}{n\pi}\right)e^{z/n\pi}, \qquad n \neq 0.$$

Combining the factors corresponding to $-n$ and n, we have

$$\sin z = ze^{g(z)}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right). \tag{11.13}$$

To find $g(z)$ we take the logarithmic derivative on both sides and get the expansion

$$\cot z = \frac{1}{z} + g'(z) + \sum_{n=1}^{\infty}\frac{2z}{z^2 - n^2\pi^2}.$$

But this is the partial fraction expansion of $\cot z$ (see Example 23, Chapter 7) with $g'(z) = 0$. Thus, $g(z)$ is constant, and since

$$\lim_{z\to 0}\frac{\sin z}{z} = 1,$$

the relation (11.13) ensures that $e^{g(z)} = 1$. With this value, (11.13) gives the desired result

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right). \tag{11.14}$$

For $\cos z$, the zeros of the function are $z = (n - 1/2)\pi$, $n \in \mathbb{I}$ and hence genus is 1, and we use (11.10) to write

$$\cos z = e^{g(z)} \prod_{n=-\infty}^{\infty} \left[1 - \frac{z}{(n - 1/2)\pi}\right] e^{z/(n-1/2)\pi}.$$

Combining the factors corresponding to $-n$ and $n + 1$, we have

$$\cos z = e^{g(z)} \prod_{n=0}^{\infty} \left[1 - \frac{z^2}{(n + 1/2)^2\, \pi^2}\right].$$

The rest of the analysis follows the above reasoning with the partial fraction expansion of $\tan z$ (see Problem 3, Problem Set 7.4) and $\lim_{z\to 0} \cos z = 1$. Thus, we have

$$\cos z = \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n - 1)^2\pi^2}\right].$$

Remark. In practice, the easiest way of finding the canonical product representation of an entire function $f(z)$ is first to find the partial fraction expansion (see Section 7.7) of $f'(z)/f(z)$ and then to integrate. Since multivalued functions may be involved in the analysis, we may use the principal branch.

11.4 GAMMA FUNCTION

In this section we shall construct a function, called the gamma function or Euler's gamma function which is meromorphic with poles at nonpositive integers, i.e. $z = 0, -1, -2 \ldots$. For this assignment, we introduce functions which have only negative zeros. The simplest function of this kind is

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}. \tag{11.15}$$

Obviously, the function $G(-z)$ has only positive zeros, and moreover, looking at (11.14), we note that

$$zG(z)G(-z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi}. \tag{11.16}$$

Now, the zeros of

$$G(z - 1) = \prod_{n=1}^{\infty} \left(1 + \frac{z - 1}{n}\right) e^{-(z-1)/n} \tag{11.17}$$

are given by $z = 1 - n$, $n = 1, 2, \ldots$.

Proposition 3. Let $G(z)$ be the function defined by (11.15). Then there is a constant γ such that for any z,

$$G(z-1) = ze^{\gamma}G(z). \tag{11.18}$$

Proof. Note that $G(z-1)$ has the same zeros as $G(z)$, and in addition, a simple zero at $z = 0$. By the Weierstrass factorization theorem, we can write

$$G(z-1) = ze^{C(z)}G(z), \tag{11.19}$$

where $C(z)$ is an entire function.

Inserting the values of $G(z)$ and $G(z-1)$ from (11.15) and (11.17), respectively, in (11.18), we get

$$\prod_{n=1}^{\infty}\left[\frac{z-1+n}{n}e^{-(z-1)/n}\right] = ze^{C(z)}\prod_{n=1}^{\infty}\left(\frac{z+n}{n}\right)e^{-z/n}$$

For determining $C(z)$, we take the logarithmic derivative of both sides. This gives the equation

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right) = \frac{1}{z}+C'(z)+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right). \tag{11.20}$$

Replacing n by $(n+1)$, the series on the left side of (11.20) can be written as

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right) &= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right)\\ &= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right).\end{aligned} \tag{11.21}$$

Comparing (11.20) and (11.21), we conclude that $C'(z) = 0$. This means that $C(z)$ is constant, which we denote by γ. Hence, (11.19) is reduced to (11.18)

To determine γ, put $z = 1$ in (11.18) to have $G(0) = e^{\gamma}G(1)$. But by (11.15),

$$G(0) = 1, \qquad G(1) = \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}.$$

So that $G(0) = e^{\gamma}G(1)$ yields

$$\begin{aligned}e^{-\gamma} &= \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}. \qquad (11.22)\\ &= \lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^{e^{-1/k}}\\ &= \lim_{n\to\infty}\exp\left(-1-\frac{1}{2}-\cdots-\frac{1}{n}\right)\prod_{k=1}^{n}\left(1+\frac{1}{k}\right).\end{aligned}$$

Hence

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln(n+1)\right). \tag{11.23}$$

The constant γ is called *Euler's constant*, and its approximate value is 0.57722.

Definition 3. The gamma function is defined by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\frac{1}{G(z)} \tag{11.24}$$

or, if the product representation (11.15) is used for $G(z)$, then

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n}. \tag{11.25}$$

From definition, we observe that $\Gamma(z)$ is well defined in the whole complex plane except at $z = 0, -1, -2, \ldots$, which are simple poles of the function. Hence, $\Gamma(z)$ is meromorphic with these poles but $\Gamma(z)$ has no zeros.

A simple consequence of Definition 3 and Eq. (11.22) is that $\Gamma(1) = 1$.

Properties of $\Gamma(z)$:

1. For $z \neq 0, -1, \ldots$,

$$\Gamma(z+1) = z\Gamma(z). \tag{11.26}$$

In view of (11.18), (11.24) yields

$$\Gamma(z-1) = \frac{e^{-\gamma(z-1)}}{(z-1)G(z-1)} = \frac{e^{-\gamma(z-1)}}{(z-1)ze^{\gamma}G(z)} = \frac{\Gamma(z)}{z-1}$$

or $\Gamma(z) = (z-1)\Gamma(z-1)$, This is equivalent to (11.26). If $n = 1, 2, \ldots$, the repeated use of (11.26) yields an important relation

$$\Gamma(n+1) = n!$$

Thus, the Γ function can be thought of as the generalization of the factorial function.

2. For any z not an integer,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{11.27}$$

Using (11.26) and Definition 3, we have

$$\begin{aligned}\Gamma(1-z)\,\Gamma(z) &= -z\Gamma(-z)\Gamma(z)\\ &= \frac{-ze^{\gamma z}}{-zG(-z)}\frac{e^{-\gamma z}}{zG(z)}\\ &= \frac{1}{zG(z)G(-z)}\end{aligned}$$

Now, in view of (11.16), Eq. (11.27) follows.

3. $\Gamma(1/2) = \sqrt{\pi}$. Set $z = 1/2$ in (11.27).

4. *Gauss formula.* For $z \neq 0, -1, \ldots$,

$$\Gamma(z) = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{n!n^z}{(z+1)(z+2)\cdots(z+n)}. \tag{11.28}$$

Starting from (11.25), we proceed as

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1} e^{z/k}$$

$$= \lim_{n\to\infty} \frac{1}{z} \exp z\left(\sum_{k=1}^{n} \frac{1}{k} - \gamma\right) \prod_{k=1}^{n} \frac{k}{z+k}$$

$$= \lim_{n\to\infty} \frac{n! \exp\left[z\left(\sum_{k=1}^{n} \frac{1}{k} - \log n - \gamma + \log n\right)\right]}{z(z+1)\cdots(z+n)}$$

$$= \lim_{n\to\infty} \frac{n!\, n^z \exp z(1+\frac{1}{2}+\cdots+\frac{1}{n} - \log n - \gamma)}{z(z+1)\cdots(z+n)} \tag{11.29}$$

But according to (11.23),

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n} - \ln n - \gamma\right) = 0.$$

Therefore, (11.29) reduces to (11.28).

5. *Legendre's duplication formula.*

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{11.30}$$

Taking the logarithm of both sides of (11.25) and differentiating, we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(-\frac{1}{z+n} + \frac{1}{n}\right).$$

If differentiation is done again,

$$\frac{d}{dz}\left[\frac{\Gamma'(z)}{\Gamma(z)}\right] = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Replacing z by $2z$,

$$\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right) = 2\sum_{n=0}^{\infty} \frac{1}{(2z+n)^2}.$$

In view of this result, it follows that

$$\frac{d}{dz}\left[\frac{\Gamma'(z)}{\Gamma(z)}\right]+\frac{d}{dz}\left[\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right] = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}+\sum_{n=0}^{\infty}\frac{1}{(z+n+1/2)^2}$$

$$= 4\left[\sum_{n=0}^{\infty}\frac{1}{(2z+2n)^2}+\sum_{n=0}^{\infty}\frac{1}{(2z+2n+1)^2}\right]$$

$$= 4\sum_{n=0}^{\infty}\frac{1}{(2z+n)^2}$$

$$= 2\frac{d}{dz}\left[\frac{\Gamma'(2z)}{\Gamma(2z)}\right].$$

Thus,

$$\frac{d}{dz}\left[\frac{\Gamma'(z)}{\Gamma(z)}+\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right] = 2\frac{d}{dz}\left[\frac{\Gamma'(2z)}{\Gamma(2z)}\right].$$

On integration, we indeed find that

$$\frac{\Gamma'(z)}{\Gamma(z)}+\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)} = 2\frac{\Gamma'(2z)}{\Gamma(2z)}+a,$$

where a is some arbitrary constant.
If integration is repeated,

$$\log\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = \log\Gamma(2z)+az+b,$$

where b is another arbitrary constant. Hence,

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = e^{az+b}\,\Gamma(2z). \tag{11.31}$$

To determine a and b, we substitute $z = 1/2$ and $z = 1$ so that we make use of the results $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, to have

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1) = e^{b+a/2}\,\Gamma(1) \quad \text{and} \quad \Gamma(1)\,\Gamma\left(\frac{3}{2}\right) = e^{b+a}\,\Gamma(2)$$

or

$$e^{b+a/2} = \sqrt{\pi} \quad \text{and} \quad e^{b+a} = \sqrt{\pi}/2$$

or

$$a+2b = \ln\pi \quad \text{and} \quad 2a+2b = \ln\pi - 2\ln 2.$$

The values of a and b available from the solution of these equations are inserted in (11.31) to have *Legendre's duplication formula*.

Example 4. Show that the residue of the gamma function at $-n$ is $(-1)^n/n!$, $n = 0, 1, \cdots$.

By definition of the function $\Gamma(z)$, $z = 0, -1, -2, \ldots, -n, \ldots$ are the simple poles. Moreover, in view of (11.26), we have

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}.$$

By repeated application of this formula, we may write

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \cdots = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}.$$

Hence, residue at $z = -n$ is given by

$$\begin{aligned}
\operatorname{Res}_{z=-n} \Gamma(z) &= \lim_{z\to -n} (z+n) \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n)} \\
&= \lim_{z\to -n} \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n-1)} \\
&= \frac{\Gamma(1)}{-n(-n+1)(-n+2)\cdots(-2)(-1)} \\
&= \frac{1}{(-1)^n n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = \frac{(-1)^n}{n!}.
\end{aligned}$$

Integral representation of $\Gamma(z)$. For $\operatorname{Re} z > 0$, define

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt. \tag{11.32}$$

Proposition 4. The integral in (11.32) is well-defined for $\operatorname{Re} z > 0$ and represents the gamma function $\Gamma(z)$.

Proof. Let $0 < c < \infty$, we can write

$$\int_0^\infty e^{-t} t^{z-1} dt = \int_0^c e^{-t} t^{z-1} dt + \int_c^\infty e^{-t} t^{z-1} dt. \tag{11.33}$$

If the principal value of t^{z-1} is taken, then

$$\left| \frac{e^{-t} t^{z-1}}{e^{-t/2}} \right| = \left| \frac{t^{z-1}}{e^{t/2}} \right| \to 0 \qquad \text{as } t \to \infty.$$

Hence, for large values of c, the second integral on the right side of (11.33) converges, since

$$\left| \int_c^\infty e^{-t} t^{z-1}\, dt \right| \le \int_c^\infty e^{-t/2} dt = 2e^{-c/2}.$$

For the first integral, we find that

$$\left|\int_0^c e^{-t}t^{z-1}\,dt\right| \le \int_0^c |t^{z-1}|\,dt \le \int_0^c t^{\mathrm{Re}(z-1)}\,dt < \infty, \qquad \mathrm{Re}\, z > 0.$$

Thus we have concluded that the integral in (11.32) is well defined for Re $z > 0$.

Now, we show that this integral represents the gamma function. For $x > 0$, we use integration by parts to have

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt = \left[\left(1-\frac{t}{n}\right)^n \frac{t^x}{x}\right]_0^n + \int_0^n \left(1-\frac{t}{n}\right)^{n-1} \frac{t^x}{x}\,dt$$

$$= \frac{1}{x}\int_0^n \left(1-\frac{t}{n}\right)^{n-1} t^x\,dt.$$

If the integration by parts is repeated, then

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt = \frac{1}{x(x+1)}\int_0^n \frac{n-1}{n}\left(1-\frac{t}{n}\right)^{n-2} t^{x+1}dt.$$

After n repetitions, we get

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt = \frac{1}{x(x+1)\cdots(x+n-1)}\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{1}{n}\int_0^n t^{x+n-1}\,dt$$

$$= \frac{1}{x(x+1)\cdots(x+n-1)}\frac{n!}{n^n}\frac{n^{x+n}}{x+n}$$

$$= \frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

On taking limits, it follows that

$$\int_0^\infty e^{-t}t^{x-1}dt = \Gamma(x) \quad \text{(by Gauss formula).}$$

Thus we see that the gamma function coincides with the analytic function

$$\psi(z) = \int_0^\infty e^{-t}t^{z-1}\,dt$$

on the set $L = \{z : z \text{ real} > 0\}$. Since the gamma function is also analytic on $U = \{z : \mathrm{Re}\, z > 0\}$ and the subset L of U contains a limit point of U, it follows that the gamma function coincides with ψ on U.

Finally, we mention the second integral form of gamma function which is the outcome of the substitution $t = x^2$ in (11.32). Thus,

$$\Gamma(z) = 2\int_0^\infty e^{-x^2} x^{2z-1}\, dx, \qquad \text{Re } z > 0.$$

Remark. The integral (11.32) diverges for $\text{Re } z \le 0$ which causes limitation of the gamma function as an analytic function in the domain $\text{Re } z > 0$. However, we may extend this domain of definition to the entire complex plane except the points $z = 0, -1, -2, \ldots$

When integration by parts is done on the integral in (11.32), the outcome is

$$\Gamma(z+1) = z\Gamma(z)$$

or

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad \text{Re}(z+1) > 0.$$

The repeated application of this formula gives

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n)}, \qquad n = 1, 2, \ldots .$$

Obviously, n is the smallest positive integer chosen such that $\text{Re}(z+n+1) > 0$. With this condition the expression on the right in above relation represents an analytic function for all z except $0, -1, -2, \ldots$.

PROBLEM SET 11.2

1. An entire function is said to have α-points if $f(z) = \alpha$ has a solution in $\mathbb{C}$. Use this definition to prove the generalized version of Proposition 2 as follows:
"An entire function $f(z)$ has no α-points $\Leftrightarrow f(z) = \alpha + e^{g(z)}$, where $g(z)$ is an entire function."
Also, generalize the result given in the remark following Proposition 2 as
"Suppose that $z_1, z_2, \ldots, z_n$ are the distinct α-points of an entire function $f(z)$, where z_j is of order $k_j (j = 1, 2, \ldots, n)$. Then $f(z)$ is of the form
$$f(z) = \alpha + (z-z_1)^{k_1} \cdots (z-z_n)^{k_n} e^{g(z)},$$
where $g(z)$ is an entire function.

2. Prove that
$$\text{(a) } \sinh z = \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right); \qquad \text{(b) } \cosh z = \prod_{n=1}^{\infty}\left(1 + \frac{4z^2}{(2n-1)^2\pi^2}\right).$$

3. Write the canonical representation of $\sin(a-z)$, when a is not an integral multiple π.

4. Suppose $a_n \to \infty$ and that the p_n are arbitrary complex numbers. Prove that there exists an entire function $f(z)$ which satisfies $f(a_n) = p_n$.

   Suggestion. Recall the Weierstrass factorization theorem and the Mittag–Leffler theorem.

5. Show that the genus of each of the entire functions

(a) $\cos \sqrt{z}$

(b) $\dfrac{\sin(\pi\sqrt{z})}{\pi\sqrt{z}}$

is zero.

6. Prove Walli's formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n+1)(2n-1)} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots.$$

Also, justify the following product expansions:

(a) $\sqrt{2} = \dfrac{3}{2}\cdot\dfrac{5}{6}\cdot\dfrac{7}{6}\cdot\dfrac{9}{10}\cdot\dfrac{11}{10}\cdot\dfrac{13}{14}+\cdots;$

(b) $\sqrt{3} = \dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{8}{9}\cdot\dfrac{10}{9}\cdot\dfrac{14}{15}\cdot\dfrac{16}{15}\cdots.$

Suggestion. For Walli's formula, use factorization of $\sin z$ with $z = \pi/2$.

7. Use canonical representation for $\sin z$ and $\cos z$ to prove that

$$\cos\left(\frac{nz}{4}\right) - \sin\left(\frac{nz}{4}\right) = \prod_{n=1}^{\infty}\left[\frac{1+(-1)^n z}{2n-1}\right].$$

8. Prove that

$$(2\pi)^{n-1/2}\,\Gamma(z) = n^{z-1/2}\,\Gamma\left(\frac{z}{n}\right)\Gamma\left(\frac{z+1}{n}\right)\cdots\Gamma\left(\frac{z+n-1}{n}\right).$$

9. Verify the following results:

(a) $\Gamma\left(\dfrac{1}{6}\right) = 2^{-1/3}\sqrt{\dfrac{3}{\pi}}\left[\Gamma\left(\dfrac{1}{3}\right)\right]^2;$

(b) $\Gamma\left(\dfrac{1}{7}\right)\Gamma\left(\dfrac{2}{7}\right)\cdots\Gamma\left(\dfrac{6}{7}\right) = \dfrac{8\pi^3}{\sqrt{7}}.$

Suggestion. For (a), use (11.27) and (11.30) for $z = 1/3$, and again (11.27) for $z = 1/6$.

10. Use product expansion of $\sin z$ to prove that

$$\frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}} \to \sqrt{\pi} \text{ as } n \to \infty.$$

Suggestion. First come to Walli's formula and then proceed further. This result is used to establish the Sterling formula.

11. Prove that $\Gamma'(z)/\Gamma(z)$ is a meromorphic function. What are the singularities of this function? Compute residue at these singularities.

12. Is $1/\Gamma(z)$ an entire function? Justify your reasoning.

13. Let $\psi(z)$ be the logarithmic derivative of the gamma function. Prove that

(a) $\text{Res}_{z=-n}\ \psi(z) = -1,\ n \geq 0 \in \mathbb{I}$

(b) $\psi(1) = -\gamma$;

(c) $\psi(z+1) - \psi(z) = \dfrac{1}{z}$;

(d) $\psi(1-z) - \psi(z) = \pi \cot \pi z$.

14. Show that

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-z} t^{z-1}\, dt$$

for $z \neq 0, -1, -2, \cdots$.

15. Prove that

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt \qquad (\text{Re } z > 1),$$

where $\zeta(z)$ is the Riemann zeta function, see Problem 16, Problem Set 5.2.

Suggestion. The Riemann zeta function $\zeta(z)$ is valid for $\text{Re } z > 1$, while the integral representation of $\Gamma(z)$ is valid for $\text{Re } z < 0$. Thus, $\Gamma(z)\zeta(z)$ is valid for $\text{Re } z > 1$. On replacing t by nu, the gamma function becomes

$$\Gamma(z) = n^z \int_0^{\infty} e^{-nu} u^{z-1} du \quad \text{or} \quad \Gamma(z) \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nu} u^{z-1} du.$$

On interchanging the summation and integration, the desired result is immediate.

11.5 ENTIRE FUNCTIONS

In Chapters 2 and 6 and in Section 11.3, we presented a brief exposition of entire functions. These functions play a key role in developing the number theory. To study the growth of these functions, we introduce the concept of order and type and obtain these parameters in terms of coefficients of power series expansions of entire functions.

Growth of entire functions. An entire function $f(z)$ can behave at infinity in three ways:

1. $f(z)$ can have a regular point at infinity, and then, $f(z)$ is identically constant (see Example 12, Chapter 7);
2. $f(z)$ can have a pole of order $k \geq 1$ at infinity, and then $f(z)$ reduces to a polynomial (see Example 13, Chapter 7);
3. $f(z)$ can have an essential singular point at infinity, and then $f(z)$ is said to be an entire transcendental function.

From now on, we shall deal mostly with entire transcendental functions. Note that for these functions $\lim f(z)$ as $z \to \infty$ does not exist. Let us define

$$M(r) = \max_{|z|=r} |f(z)|.$$

$M(r)$ is termed the *maximum modulus* of $f(z)$ on the disc $|z| = r$. In view of the maximum modulus theorem and Liouville's theorem, $M(r)$ is an increasing function of r such that

$$\lim_{n\to\infty} M(r) = \infty$$

Proposition 5 (Generalized maximum modulus principle). Let $f(z)$ be analytic on a bounded domain G with boundary Γ, and suppose that $|f(s)| \le M$, $M > 0$, for all points $s \in \Gamma$ except a finite number of points $\{s_1, s_2, \ldots, s_n\}$, where

$$\limsup_{z\to s_k} |f(z)| < \infty.$$

Then

$$|f(z)| \le M \qquad \forall\, z \in G.$$

Moreover, if $f(z_0) = M$ for some $z_0 \in G$, then $f(z) \equiv M$.

Proof. Consider the functions

$$v_k(z) = \ln|z - s_k|, \qquad k = 1, 2, \ldots$$

Clearly, each $v_k(z)$ is harmonic on G and takes value $-\infty$ at $z = s_k$. For $\bar{G} = G \cup \Gamma$, if

$$R = \sup_{z_1, z_2 \in \bar{G}} |z_1 - z_2|,$$

then, for each k,

$$v_k(z) \le \ln R \quad \forall\, z \in \overline{G}.$$

Therefore, the functions

$$u_k(z) = v_k(z) - \ln R \le 0 \quad \forall\, z \in \overline{G}.$$

Now let $\varepsilon > 0$ be given, and define the real function of a complex variable

$$p_\varepsilon(z) = |f(z)| - M + \varepsilon \sum_{j=1}^{n} u_j(z).$$

Note that

$$\lim_{z\to s} p_\varepsilon(z) = \lim_{z\to s}\left[|f(z)| - M + \varepsilon\sum_{j=1}^{n} u_j(z)\right] \le 0, \qquad s \ne s_k$$

and the relation

$$\lim_{z\to s_k} p_\varepsilon(z) = -\infty \le 0, \qquad k = 1, 2, \ldots, n, \qquad s_k \in \Gamma,$$

since the functions $|f(z)| - M$ and $u_j(z), j \ne k$ are bounded from above in a neighbourhood of s_k, while $u_k(z) \to -\infty$ as $z \to s_k$. Hence

$$p_\varepsilon(z) \le 0 \quad \forall\, z \in G$$

or

$$|f(z)| \le M - \varepsilon \sum_{j=1}^{n} u_j(z).$$

Letting ε approach zero, we obtain

$$|f(z)| \le M, \; z \in G.$$

Finally, we observe that

$$\lim_{z \to s} |f(z)| - M \le 0$$

for every $s \in \Gamma$, including the exceptional values $s_1, s_2, \ldots, s_n$.

Moreover, if $f(z_0) = M$ for some $z_0 \in G$. So, $f(z) - M = 0$ for all $z \in G$, by usual form of maximum modulus principle (see Proposition 10, Chapter 4).

Remark. Roughly speaking, the generalized maximum modulus principle states that mazorant of $|f(z)|$ at all but a finite number of points of the boundary of the domain G also mazorizes $|f(z)|$ at all points of G. This principle is also referred to as Phragmén Lindelöf theorem.

Theorem 4 (Hadamard three-circle theorem). Let $f(z)$ be analytic in the annular region $r_1 \le |z| \le r_3$, and $r_1 < r_2 < r_3$. If M_k is the maximum modulus on the circle $|z| = r_k$, $k = 1, 2, 3$, then

$$M_2^{\ln(r_3/r_1)} \le M_1^{\ln(r_3/r_2)} M_3^{\ln(r_2/r_1)}.$$

Proof. Consider $\phi(z) = z^\lambda f(z)$, where λ is a real constant. If λ is not an integer, then $\phi(z)$ is a multivalued function, and for convenience, we choose the principal branch. For this, we cut the annulus along the negative real axis. By Phragmén Lindelöf theorem maximum of $|\phi(z)|$ (see Problem 16, Problem Set 11.3) is less than or equal to its value on the boundary of the closed annulus. Thus,

$$|z^\lambda f(z)| \le \max\{r_1^\lambda M_1, r_3^\lambda M_3\}.$$

Hence,

$$r_2^\lambda M_2 \le \max\{r_1^\lambda M_1, r_3^\lambda M_3\}. \tag{11.34}$$

Since λ is at our disposal, let us choose it such that

$$r_1^\lambda M_1 = r_3^\lambda M_3.$$

This gives

$$\lambda = -\frac{\ln(M_3/M_1)}{\ln(r_3/r_1)}.$$

With this value of λ, (11.34) yields

$$M_2 \le \left(\frac{r_2}{r_1}\right)^{-\lambda} M_1.$$

Further,

$$\begin{aligned} M_2^{\ln(r_3/r_1)} &\le (r_2/r_1)^{-\lambda \ln(r_3/r_1)} M_1^{\ln(r_3/r_1)} \\ &= (r_2/r_1)^{\ln(M_3/M_1)} M_1^{\ln(r_3/r_1)} \\ &= (M_3/M_1)^{\ln(r_2/r_1)} M_1^{\ln(r_3/r_1)}, \qquad \text{since } a^{\ln b} = b^{\ln a} \end{aligned}$$

$$= M_3^{\ln(r_2/r_1)} M_1^{\ln(r_3/r_1) - \ln(r_2/r_1)}$$

$$= M_1^{\ln(r_3/r_2)} M_3^{\ln(r_2/r_1)}.$$

For more elaboration on the maximum modulus function, let us recall the convex functions.

A real function $f(x)$ is said to be *convex downward* or simply convex, if the curve $y = f(x)$ between x_1 and x_2 lies below the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Mathematically, it is written as

$$f(x) \le \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2).$$

If $\le$ is replaced by $<$, then it is termed strictly convex.

Remarks. From real calculus, we have the following assertions:

1. A convex function is continuous but converse may not be true.
2. The sufficient condition for a function $f(x)$ to be convex is $f''(x) \ge 0$.
3. The convex function of a convex function is convex.

Proposition 6. $\ln M(r)$ is an increasing convex function of $\ln r$.

Proof. From Theorem 4, for $r_1 < r < r_3$, taking logarithm of both sides, we have

$$\ln\left(\frac{r_3}{r_1}\right) \ln M \le \ln\left(\frac{r_3}{r}\right) \ln M_1 + \ln\left(\frac{r}{r_1}\right) \ln M_3.$$

or

$$\ln M(r) \le \frac{\ln r_3 - \ln r}{\ln r_3 - \ln r_1} \ln M(r_1) + \frac{\ln r - \ln r_1}{\ln r_3 - \ln r_1} \ln M(r_3),$$

as asserted.

Mean values. We define the mean value of an entire function as

$$I_2(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta.$$

Proposition 7. $I_2(r)$ is an increasing function, and moreover, $\ln I_2(r)$ is an increasing convex function of $\ln r$.

Proof. By Parseval's formula (refer to Proposition 3 in Chapter 6) we have

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = I_2(r).$$

This clearly implies that $I_2(r_1) \le I_2(r_2)$ for $r_1 \le r_2$.

For the second assertion, note that, for $u = \ln r$ and $I_2 \equiv I_2(r)$,

$$\frac{d}{du}(\ln I_2) = \frac{I_2'}{I_2},$$

where I_2' is the derivative of I_2 with respect to u. Since $u = \ln r \Leftrightarrow r = e^u$, again by Parseval's formula,

$$I_2 = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} |a_n|^2 e^{2nu};$$

or

$$I'_2 = \sum_{n=0}^{\infty} |a_n|^2 2ne^{2nu}, \qquad I''_2 = \sum_{n=0}^{\infty} |a_n|^2 4n^2 e^{2nu}.$$

Hence, using the Schwarz inequality

$$(I'_2)^2 = \left(\sum_{n=0}^{\infty} |a_n|^2 2ne^{2nu}\right)^2 \le \sum_{n=0}^{\infty} |a_n|^2 e^{2nu} \sum_{n=0}^{\infty} |a_n|^2 4n^2 e^{2nu} = I_2 I''_2.$$

Thus, $I_2 I''_2 - (I'_2)^2 \ge 0$, and this ensures that

$$\frac{d^2}{du^2}(\ln I_2) = \frac{d}{du}\left(\frac{I'_2}{I_2}\right) = \frac{(I''_2 I_2 - (I'_2)^2)}{(I_2)^2} \ge 0,$$

a sufficient condition for a function to be convex (see Remark 2 with Theorem 4).

Proposition 8. If $f(z)$ is an entire transcendental function with maximum modulus $M(r)$, then

$$\lim_{r\to\infty} \frac{\ln M(r)}{\ln r} = \infty.$$

Proof. Assume to the contrary that

$$\liminf_{r\to\infty} \frac{\ln M(r)}{\ln r} = \mu < \infty.$$

Then, for given $\varepsilon > 0$, there is an increasing sequence $\{r_k\}$ converging to infinity such that

$$\ln M(r_k) < (\mu + \varepsilon) \ln r_k \qquad \text{or} \qquad M(r_k) < r_k^{\mu+\varepsilon}$$

for every r_k. Therefore, using Cauchy's inequality, we obtain

$$|a_n| \le \frac{M(r_k)}{r_k^n} < r_k^{\mu+\varepsilon-n}.$$

Taking r_k as sufficiently large, it follows that $a_n = 0$ for all $n > \mu + \varepsilon$, and hence $f(z)$ is a polynomial of degree not higher than $[\mu]$.

This proposition reveals that an entire transcendental function grows faster than any fixed positive power of r. It suggests that the exponential function may be utilized to measure the growth of an entire function.

Definition 4. Suppose there exists a positive number k such that

$$M(r) < e^{r^k} \text{ for all } r > r_0.$$

Then the entire function is said to be of finite order ρ defined by

$$\rho = \inf\{k : M(r) < e^{r^k} \ \forall\ r > r_0\}. \tag{11.35}$$

If, for sufficiently large values of r,

$$M(r) > e^{r^k},$$

then $f(z)$ is said to be of infinite order.

For example, the function e^z is of finite order (in fact of order 1) and the function e^{e^z} is of order infinity.

Definition 5. The order ρ of an entire function is defined by

$$\rho = \limsup_{r\to\infty} \frac{\ln \ln M(r)}{\ln r}, \qquad 0 \le \rho \le \infty. \tag{11.36}$$

We show that Definitions 4 and 5 are equivalent. According to (11.35), for given $\varepsilon > 0$, there is a number $R(\varepsilon) > 0$ such that

$$M(r) < e^{r^{\rho+\varepsilon}} \quad \text{for all } r > R(\varepsilon),$$

while on the other hand, there is an increasing sequence $\{r_n\}$ such that

$$M(r_n) > e^{r_n^{\rho-\varepsilon}}.$$

In other words,

$$\frac{\ln \ln M(r)}{\ln r} < \rho + \varepsilon,$$

if $r > R(\varepsilon)$, whereas

$$\frac{\ln \ln M(r_n)}{\ln r_n} > \rho - \varepsilon,$$

for arbitrary large r_n. But this is precisely what we mean by (11.36).

Example 5. Discuss the order of entire functions (a) $f(z) = e^{z^n}$, $n \in \mathbb{N}$; (b) $g(z) = e^{e^z}$.

For part (a): $f(z) = e^{z^n}$, $n \in \mathbb{N}$, note that

$$|f(z)| = e^{\operatorname{Re} z^n} = e^{r^n \cos n\theta} \Rightarrow M(r) = e^{r^n}$$

and hence,

$$\frac{\ln \ln M(r)}{\ln r} = n,$$

thus, $f(z)$ is of order n.

For part (b): $g(z) = e^{e^z}$, we observe that

$$|g(z)| = e^{\operatorname{Re} e^z} = e^{e^r \cos\theta} \Rightarrow M(r) = e^{e^r}.$$

Thus,

$$\frac{\ln \ln M(r)}{\ln r} = \frac{r}{\ln r},$$

and hence, $g(z)$ is of infinite order.

Here, if ρ_1 and ρ_2 are finite orders of entire functions f_1 and f_2, respectively, such that $\rho_1 > \rho_2$, then we say f_1 grows faster than f_2. When both the entire functions are of the same order, it is not possible to compare their growth. For this purpose, we introduce another growth parameter in the following definition.

Definition 6. Let $f(z)$ be an entire function of $\rho(0 < \rho < \infty)$. Suppose that there exists a constant $m > 0$ such that

$$M(r) < e^{mr^\rho}$$

for sufficiently large r. Then $f(z)$ is of type σ defined by

$$\sigma = \inf\{m : M(r) < e^{mr^\rho} \quad \text{for all} \quad r > r_0\}, \qquad 0 \le \sigma \le \infty. \tag{11.37}$$

Parallel to Definition 6, we also have the following definition.

Definition 7. Let $f(z)$ be an entire function of order $\rho(0 < \rho < \infty)$. Then type σ of $f(z)$ is defined as

$$\sigma = \limsup_{r\to\infty} \frac{\ln M(r)}{r^\rho}, \qquad 0 \le \sigma \le \infty. \tag{11.38}$$

Again, it can be verified that Definitions 6 and 7 are equivalent.

Example 6. Find the order and type of the entire function $\sin z$.

Consider the function expressed in the form

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Since, $|\sinh y| \le |\sin z| \le \cosh y$ (refer to Section 3.3), we have

$$\frac{e^r - 1}{2} < \frac{e^r - e^{-r}}{2} \le |\sin z| \le \frac{e^r + e^{-r}}{2} < \frac{e^r + 1}{2}.$$

Hence

$$\frac{e^r - 1}{2} < M(r) < \frac{e^r + 1}{2}.$$

Obviously,

$$\rho = \lim_{r\to\infty} \frac{\ln\ln M(r)}{\ln r} = 1,$$

and

$$\sigma = \lim_{r\to\infty} \frac{\ln M(r)}{r} = 1.$$

Similarly, it can be proved that $\cos z$ is of order 1 and type 1.

Note. Multiplication of an entire function by a polynomial does not change its order.

Note that the series $\Sigma_{n=0}^{\infty} |a_n| r^n$ converges for all finite values of r. Hence the nth term $|a_n| r^n$ tends to 0. This suggests that one of the positive terms $|a_n| r^n$ will be maximum. We define this as *maximum term* and denote it by $\mu(r)$. So,

$$\mu(r) = \max_{n\ge 0} \{|a_n| r^n\}.$$

An inequality is satisfied *asymptotically* with reference to r if this is true for large values of r. Let us develop a formula for the order and type in terms of coefficients when the entire function is represented by a power series. For this work we need the following proposition.

Proposition 9. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ρ. Then, for given $\varepsilon > 0$,

$$\mu(r) \le M(r) < r^{\rho+\varepsilon}\mu(r),$$

where $\mu(r)$ and $M(r)$ have their usual meaning.

Proof. By Cauchy's inequality for coefficients of a power series, $|a_n|r^n \le M(r)$,

$$\mu(r) \le M(r). \tag{11.39}$$

Since $f(z)$ is of the finite order, then for given $\varepsilon > 0$, asymptotically,

$$M(r) < e^{r^{\rho+\varepsilon}},$$

and considering the Cauchy inequality this gives

$$|a_n| < e^{r^{\rho+\varepsilon}} r^{-n}.$$

The minimum on the right side is attained for

$$r = \left(\frac{n}{\rho+\varepsilon}\right)^{1/(\rho+\varepsilon)}.$$

and, therefore, asymptotically

$$|a_n| \le \left[\frac{e(\rho+\varepsilon)}{n}\right]^{n/(\rho+\varepsilon)}. \tag{11.40}$$

This holds for all $n > n_r$. Choose $n_r = [2^{\rho+\varepsilon}e(\rho+\varepsilon)r^{\rho+\varepsilon}]$ and sufficiently large r, then (11.40) ensures

$$|a_n|r^n < 2^{-n}$$

and, hence

$$|f(z)| < \sum_{n=0}^{n_r} |a_n| r^n + \frac{1}{2^{n_r}}.$$

Keeping in memory the definition of $\mu(r)$, it follows that

$$M(r) < [1 + 2^{\rho+\varepsilon}e(\rho+\varepsilon)r^{\rho+\varepsilon}]\mu(r) + 2^{-n_r} \tag{11.41}$$

Taking inequalitites (11.39) and (11.41) together, we may prove what we desire.

Corollary 3. For entire functions of finite order,

$$\ln M(r) \simeq \ln \mu(r) \quad \text{as} \quad r \to \infty$$

and, hence for $0 \le \rho < \infty$,

$$\limsup_{r\to\infty} \frac{\ln\ln\mu(r)}{\ln r} = \rho,$$

while, for $0 < \rho < \infty$,

$$\limsup_{r\to\infty} \frac{\ln \mu(r)}{r^{\rho}} = \sigma, \quad (0 \le \sigma < \infty).$$

Theorem 5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ρ. Then

$$\limsup_{n\to\infty} \frac{\ln n}{\ln |a_n|^{-1/n}} = \rho. \tag{11.42}$$

Proof. Let $\rho < \infty$. Then, for given $\varepsilon > 0$, using (11.35),

$$\ln M(r) < r^{\rho+\varepsilon} \quad \text{for} \quad r > r_0.$$

Therefore, according to Cauchy's inequality $|a_n| r^n \le M(r)$,

$$\ln|a_n| < r^{\rho+\varepsilon} - n \ln r \quad \text{for} \quad r > r_0. \tag{11.43}$$

It is easy to see that the quantity on the right side achieves its minimum value at

$$r = \left(\frac{n}{\rho+\varepsilon}\right)^{1/(\rho+\varepsilon)}.$$

Choosing an integer N so that

$$\left(\frac{n}{\rho+\varepsilon}\right)^{1/(\rho+\varepsilon)} > r_0, \quad \text{for all } n \ge N,$$

and, hence by (11.43),

$$\ln|a_n|^{-1/n} > \frac{1}{\rho+\varepsilon} \ln\left(\frac{n}{\rho+\varepsilon}\right) - \frac{1}{\rho+\varepsilon}$$

or

$$\ln|a_n|^{-1/n} > \frac{\ln n - \ln e(\rho+\varepsilon)}{\rho+\varepsilon}$$

or

$$\frac{\ln n}{\ln|a_n|^{-1/n}} < \frac{\ln n}{\ln n - \ln e(\rho+\varepsilon)}(\rho+\varepsilon).$$

Taking limits and using the fact that $\limsup_{n\to\infty}|a_n|^{-1/n} = \infty$, and noticing that $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_{n\to\infty} \frac{\ln n}{\ln|a_n|^{-1/n}} \le \rho. \tag{11.44}$$

For $\rho = \infty$, this inequality is obvious.

To complete the proof, we show that strict inequality in (11.44) cannot hold. Suppose

$$\limsup_{n\to\infty} \frac{\ln n}{\ln|a_n|^{-1/n}} = \rho', \qquad \rho' < \rho. \tag{11.45}$$

Take $\varepsilon > 0$ such that $\rho' + \varepsilon < \rho$, then from (11.45)

$$|a_n| < \left(\frac{1}{n}\right)^{n/(\rho'+\varepsilon)} \quad \forall\, n > N'$$

Thus,

$$\mu(r) < \max_{n\ge 0}\left[\left(\frac{1}{n}\right)^{n/(\rho'+\varepsilon)} r^n\right].$$

The maximum is attained for $n = r^{\rho+\varepsilon}/e$, so

$$\mu(r) < \exp\left[\frac{r^{\rho'+\varepsilon}}{(\rho'+\varepsilon)e}\right]$$

or

$$\ln \ln \mu(r) < (\rho' + \varepsilon) \ln r - \ln e(\rho' + \varepsilon).$$

In view of Corollary 3, this gives $\rho \le \rho'$, a contradiction to our assumption and hence the theorem.

Theorem 6. Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ be an entire function of order $\rho (0 < \rho < \infty)$ and type σ. Then

$$\frac{1}{\rho e} \limsup_{n\to\infty} n|a_n|^{\rho/n} = \sigma. \tag{11.46}$$

Proof. The proof of this theorem proceeds on the lines of Theorem 5, hence it is enough to give the outlines of the analysis involved.

Suppose σ is finite. Then, in view of (11.37), for given $\varepsilon > 0$,

$$M(r) \le e^{kr^\rho}, \quad k = \sigma + \varepsilon.$$

Applying Cauchy's inequality, this gives

$$|a_n| \le e^{kr^\rho} r^{-n}.$$

The minimum on the right side is obtained for $r = (n/k\rho)^{1/\rho}$, and hence,

$$|a_n| \le \left(\frac{ek\rho}{n}\right)^{n/\rho}.$$

On taking limits, we have

$$\limsup_{n\to\infty} n|a_n|^{\rho/n} \le e\rho\sigma. \tag{11.47}$$

This is true for $\sigma = \infty$.

The proof is complete if we show that strict inequality in (11.47) is not valid.

Let

$$\limsup_{n\to\infty} n|a_n|^{\rho/n} = e\rho\sigma', \quad \sigma' < \sigma.$$

Thus, for $\varepsilon > 0$, $n|a_n|^{\rho/n} \le e\rho(\sigma' + \varepsilon)$ for $n > n_0$. This simplifies to

$$|a_n| \le \left[\frac{e\rho(\sigma'+\varepsilon)}{n}\right]^{n/\rho}$$

which yields

$$\mu(r) \le \max_{n\ge 0}\left[\frac{e\rho(\sigma'+\varepsilon)}{n}\right]^{n/\rho} r^n.$$

One can verify that maximum occurs for $n = \rho(\sigma' + \varepsilon)r^\rho$, and hence

$$\mu(r) \le e^{(\sigma'+\varepsilon)r^\rho} \quad \text{or} \quad \limsup_{r\to\infty} \frac{\ln \mu(r)}{r^\rho} \le \sigma' + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\sigma \le \sigma'$, a contradiction that proves the result.

Remark. The limit superiors in formulas (11.42) and (11.46) are unchanged if a_n is replaced by $(n+1)a_n$. Consequently, the order and type remain unchanged by differentiation.

Example 7. Use Theorems 5 and 6, to find the order and type of the entire function

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{e\rho\sigma}{n}\right)^{n\rho} z^n \quad (\rho > 0,\ \sigma > 0).$$

Given $|a_n| = (e\rho\sigma/n)^{n\rho}$. Thus,

$$\ln|a_n|^{-1/n} = -\rho(e\rho\sigma - \ln n)$$

and using (11.42),

$$\limsup_{n\to\infty} \frac{\ln n}{\ln|a_n|^{-1/n}} = \lim_{n\to\infty} \frac{\ln n}{-\rho(e\rho\sigma - \ln n)} = \frac{1}{\rho}.$$

For determining the type, the application of (11.46) for $1/\rho$ yields

$$\frac{\rho}{e} \limsup_{n\to\infty} n\,|a_n|^{1/n\rho} = \lim_{n\to\infty} \frac{\rho}{e} e\rho\sigma = \rho^2\sigma.$$

So, the order and type of $f(z)$ are $1/\rho$ and $\rho^2\sigma$, respectively.

Example 8. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and $\{b_n\}$ be a complex sequence satisfying $\lim_{n\to\infty} \ln|b_n|/n = 0$. Then, prove that the *Hadamard composition*

$$f(z) = \sum_{n=0}^{\infty} a_n b_n z^n$$

is an entire function with the same order and type as that of $f(z)$.

Since $|b_n|^{-1/n} \to 1$ as $n \to \infty$, it follows that

$$\liminf_{n\to\infty} |a_n b_n|^{-1/n} \geq \liminf_{n\to\infty} |a_n|^{-1/n} \liminf_{n\to\infty} |b_n|^{-1/n} = \infty.$$

Thus, $g(z)$ is an entire function.

For given $\varepsilon > 0$, the given condition asserts

$$-\varepsilon n < \ln|b_n| < \varepsilon n \quad \text{for} \quad n > n_0.$$

Also,

$$\frac{1}{n}\ln|a_n b_n|^{-1} = \frac{\ln|a_n|^{-1}}{n}\left(1 + \frac{\ln|b_n|^{-1}}{\ln|a_n|^{-1}}\right).$$

These two observations imply the asymptotic inequality

$$\frac{\ln|a_n|^{-1/n}}{\ln n} - \varepsilon < \frac{\ln|a_n b_n|^{-1/n}}{\ln n} < \frac{\ln|a_n|^{-1/n}}{\ln n} + \varepsilon.$$

Taking limits and considering (11.42), we find that the order of $f(z)$ and $g(z)$ are same.

Let σ_f and σ_g be the types of $f(z)$ and $g(z)$ respectively. Again, we observe that

$$(1-\varepsilon)|a_n|^{-1/n} < |a_n b_n|^{-1/n} < (1+\varepsilon)|a_n|^{-1/n}.$$

Dividing by $n^{1/\rho}$ and considering formula (11.46), we see that

$$\frac{1}{\rho e \sigma_f} = \frac{1}{\rho e \sigma_g} \Rightarrow \sigma_f = \sigma_g,$$

as asserted.

Our next example decides the nature of the entire function $g(z)$ in Proposition 2, if we add the information that $f(z)$ is of finite order.

Example 9. If $f(z)$ is an entire function of finite order ρ and never vanishes in the finite plane, show that ρ is an integer and $f(z)$ is of the form $f(z) = e^{P(z)}$, where $P(z)$ is a polynomial of degree ρ.

According to Proposition 2, a nonvanishing entire function can be written in the form $f(z) = e^{g(z)}$, where $g(z)$ is an entire function. Since $f(z)$ is of order ρ, then, for given $\varepsilon > 0$, there is a number r_0 such that

$$|f(z)| < e^{r^{\rho+\varepsilon}},$$

and hence,

$$|e^{g(z)}| < e^{r^{\rho+\varepsilon}} \quad \text{or} \quad |e^{\operatorname{Re} g(z)}| < e^{r^{\rho+\varepsilon}}, \quad r > r_0$$

This implies $\operatorname{Re} g(z) < r^{\rho+\varepsilon}$ for large r. Applying the result of Problem 21, Problem Set 6.2, we see that $g(z)$ is a polynomial $P(z)$ of degree not higher than $[\rho+\varepsilon]$, i.e. of degree not higher than $[\rho]$. But Example 1 asserts that the order or $e^{P(z)}$ is n where n is the degree of the polynomial. Therefore, ρ equals n, and $P(z)$ is a polynomial of degree ρ.

This example simplifies the Weierstrass factorization theorem as follows (Theorem 7).

Theorem 7 (Hadamard factorization theorem). Let f be an entire function of finite order ρ with zeros $a_1, a_2, \ldots$, each zero is counted as often as its multiplicity ($a_n \neq 0$ for any n). Then, f can be expressed in the form

$$f(z) = e^{g(z)}P(z),$$

where $g(z)$ is a polynomial of degree not exceeding $[\rho]$ and $P(z)$ is the canonical product formed with the zeros of f. Further, if $f(0) = 0$, then f can be expressed in the form

$$f(z) = z^k e^{g(z)}P(z),$$

where k is the multiplicity of 0.

Proof. Consider the function

$$e^{g(z)} = \frac{f(z)}{P(z)}.$$

The function $f(z)/P(z)$ is an entire function of order ρ and and never vanishes. The above example asserts that $g(z)$ is a polynomial of degree not exceeding $[\rho]$, and hence the result.

Similarly, the second assertion follows.

Remark. We may utilize the Hadamard factorization theorem to derive the canonical representation of $\sin z$ in a shorter way. Since $\sin z$ is an entire function of order 1, write (11.13) as

$$\sin z = ze^{a_0+a_1 z}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right).$$

Now, using the fact that $\lim_{z\to 0}(\sin z)/z = 1$, the constant $a_0 = 0$, and differentiating the above expression, we have that a_1 is also 0. Thus, we get (11.14).

11.6 DISTRIBUTION OF ZEROS

How the zeros of analytic and entire functions influence the growth will be discussed in this section. First, we prove a basic result known as *Jensen's formula*. Its significance lies in the fact that it relates the modulus $|f(z)|$ on a circle to the moduli of the zeros.

Theorem 8 (Jensen's formula). Let $f(z)$ be analytic in $|z| \le R$. Assume that $f(0) \ne 0$ and no zero of $f(z)$ lies on $|z| = r$, $r < R$. If $a_1, a_2, \ldots, a_n$ are the zeros of $f(z)$ in $|z| < R$, counting multiplicity, then

$$\ln \frac{r^n}{|a_1|\,|a_2|\ldots|a_n|} = \frac{1}{2\pi}\int_0^{2\pi} \ln\,|f(re^{i\theta})|\,d\theta - \ln|f(0)|. \tag{11.48}$$

Proof. Consider the function

$$F(z) = f(z)\prod_{k=1}^{n}\frac{r^2-\overline{a_k}z}{r(z-a_k)}. \tag{11.49}$$

Clearly, $F(z)$ is analytic in the domain in which $f(z)$ is analytic and also $F(z) \ne 0$ on $|z| \le r$. Further, $|F(z)| = |f(z)|$ on $|z| = r$, since

$$\left|\prod_{k=1}^{n}\frac{r^2-\overline{a_k}z}{r(z-a_k)}\right| = \prod_{k=1}^{n}\left|\frac{r(r-\overline{a_k}e^{i\theta})}{re^{i\theta}(r-a_k e^{-i\theta})}\right| = 1.$$

As the function $F(z)$ is analytic and nonzero in $|z| < r$, the function $\ln|F(z)|$ is harmonic in $|z| < r$ (refer to Problem 8, Problem Set 2.7). Hence, by the Gauss mean value theorem

$$\ln|F(0)| = \frac{1}{2\pi}\int_0^{2\pi}\ln|f(re^{i\theta})|\,d\theta.$$

Substituting the value of $F(0)$ from the defining expression (11.49) and using the fact that $F(z) = f(z)$ on $|z| = r$, we have

$$\ln|f(0)| + \sum_{k=1}^{n}\ln\frac{r}{|a_k|} = \frac{1}{2\pi}\int_0^{2\pi}\ln\,|f(re^{i\theta})|\,d\theta.$$

This is what we required.

Corollary 4. Under the hypothesis of Theorem 8, we have

$$\int_0^r \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta - \ln|f(0)|,$$

where $n(t)$ is the number of zeros of $f(z)$ in $|z| \le t$.

Proof. By hypothesis, $f(z)$ is an entire function such that $f(0) \neq 0$. Further, suppose that $a_1, a_2, \ldots, a_n$ are the zeros of $f(z)$ in order of their increasing moduli in $|z| < r$ i.e. $|a_1| \le |a_2| \le \cdots \le |a_n|$. Thus, $n(t) = k$ for $|a_k| \le t < |a_{k+1}|$. This implies

$$\begin{aligned}
\ln\left(\frac{r^n}{|a_1|\,|a_2|\cdots|a_n|}\right) &= \ln\left(\frac{|a_2|}{|a_1|}\frac{|a_3|^2}{|a_2|^2}\cdots\frac{|a_n|^{n-1}}{|a_{n-1}|^{n-1}}\frac{r^n}{|a_n|^n}\right)\\
&= \ln\left|\frac{a_2}{a_1}\right| + 2\ln\left|\frac{a_3}{a_2}\right| + \cdots + (n-1)\ln\left|\frac{a_n}{a_{n-1}}\right| + n\ln\left|\frac{r}{a_n}\right|\\
&= \int_{|a_1|}^{|a_2|}\frac{dt}{t} + \int_{|a_2|}^{|a_3|}\frac{2}{t}\,dt + \cdots + \int_{|a_{n-1}|}^{|a_n|}\frac{n-1}{t}\,dt + \int_{|a_n|}^{r}\frac{n}{t}\,dt.
\end{aligned}$$

But the numerical factor in the numerator of the integral from $|a_k|$ to $|a_{k+1}|$ is just the value of the function $n(t)$ in the interval $|a_k| \le t < |a_{k+1}|$. Therefore,

$$\ln\frac{r^n}{|a_1|\,|a_2|\cdots|a_n|} = \int_0^r \frac{n(t)}{t}\,dt.$$

Since $n(t) = 0$, for $0 \le t < |a_1|$, the lower limit of integration may be set at 0.

Remarks 1. Sometimes Corollary 4 is referred to as Jensen's theorem.

2. Normally, Theorem 8 is applied in the form:

$$r^n|f(0)| \le M(r)\ a_1|\,|a_2|\ \ldots\ |a_n|$$

where $M(r)$ has its usual meaning. This inequality is associated with Jensen's name.

Corollary 5. If, in addition to $f(z)$ there is another function $g(z)$ satisfying all the conditions of Theorem 8 and has $b_1, b_2, \ldots, b_m$ zeros in $|z| < r$. Then, for the quotient function $h(z) = f(z)/g(z)$,

$$\int_0^r \frac{n_f(t) - n_g(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \ln|h(re^{i\theta})|\,dt - \ln|h(0)|,$$

where $n_f(t)$ and $n_g(t)$ denote the number of zeros of $f(z)$ and $g(z)$, respectively in $|z| \le t$, and moreover, $f(z)$ and $g(z)$ have no common zeros.

Proof. Following Theorem 8, write the Jensen formula for $f(z)$ and $g(z)$ individually, and subtracting the resulting expressions it immediately follows that

$$\ln|h(0)| = \ln\frac{r^m}{|b_1|\,|b_2|\cdots|b_m|} - \ln\frac{r^n}{|a_1|\,|a_2|\cdots|a_n|} + \frac{1}{2\pi}\int_0^{2\pi}\ln|h(re^{i\theta})|d\theta.$$

Now, the result is obvious in view of Corollary 4.

Theorem 9 (Poisson–Jensen formula). Let $f(z)$ be analytic in $|z| \le R$. Assume that $f(0) \ne 0$ and no zero of $f(z)$ lies on $|z| = R$. If $a_1, a_2, \ldots, a_n$ are the zeros of $f(z)$ in $|z| < R$, then

$$\ln |f(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} \ln |f(Re^{it})|dt$$

$$- \sum_{k=1}^{n} \ln\left|\frac{R^2 - \overline{a_k}z}{R(z - a_k)}\right|, \quad 0 < r < R. \tag{11.50}$$

Proof. Defining the function $F(z)$ as in Theorem 8 and following all initial conclusions we apply Poisson integral formula (refer to Eq. (11.2)) to the harmonic function $\ln|F(z)|$ and obtain

$$\ln|F(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} \ln|F(Re^{it})|dt, \qquad 0 < r < R. \tag{11.51}$$

Since $\ln|F(z)| = \ln|f(z)|$ on $|z| = R$ and by (11.49), it follows that

$$\ln|F(z)| = \ln|f(z)| + \sum_{k=1}^{n} \ln\left|\frac{R^2 - \overline{a_k}z}{R(z - a_k)}\right|.$$

Inserting this in (11.51), we obtain (11.50).

Exponent of convergence. Let $\{z_n\}$ be an arbitrary sequence of nonzero complex numbers such that $|z_1| \le |z_2| \le |z_3| \le \cdots \le |z_n| \to \infty$ as $n \to \infty$, and consider the series

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\alpha}}, \tag{11.52}$$

where α is a nonnegative real number. If the series (11.52) converges for some α_0, then it converges for all $\alpha > \alpha_0$ (real p-series). The greatest lower bound of the values of α for which (11.52) converges is a nonnegative number τ called the *exponent of convergence* (or the *convergence exponent*) of the sequence $\{z_n\}$. If the series (11.52) diverges for every $\alpha > 0$, we say $\tau = \infty$.

Obviously, the convergence exponent of the sequences $\{e^n\}$, $\{n^{1/\tau}\}$ and $\{\ln(n+1)\}$ are 0, τ and ∞, respectively.

Proposition 10. The convergence of exponent τ of a sequence $\{z_n\}$ is given by

$$\tau = \limsup_{n\to\infty} \frac{\ln n}{\ln|z_n|}. \tag{11.53}$$

Proof. Suppose τ is finite. Then the series

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\alpha}}$$

converges for any $\alpha > \tau$. Since the series $\Sigma_{n=1}^{\infty} |z_n|^{-\alpha}$ is convergent, it follows that

$$\limsup_{n\to\infty} \frac{n}{|z_n|^{\alpha}} = 0$$

(see Problem 21, Problem Set 5.1). Therefore, for $\varepsilon = 1$, we note that

$$\alpha > \frac{\ln n}{\ln|z_n|} \qquad \text{for } n > n_0,$$

which implies

$$\alpha \geq \limsup_{n\to\infty} \frac{\ln n}{\ln|z_n|}$$

or

$$\tau \geq \limsup_{n\to\infty} \frac{\ln n}{\ln|z_n|}, \tag{11.54}$$

since α is an arbitrary number exceeding τ.

Next, let α' be an arbitrary number exceeding the right side of (11.53). Then, there exists $n > N = N(\alpha')$, such that

$$\frac{\ln n}{\ln|z_n|} < \alpha' \qquad \text{for all } n > N.$$

Thus, $|z_n|^{-1} < n^{-1/\alpha'}$ for all $n > N$. This means that the series

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\beta}}$$

converges for every $\beta > \alpha'$. It follows from the definition of exponent convergence that $\tau \leq \alpha'$, and hence

$$\tau \leq \limsup_{n\to\infty} \frac{\ln n}{\ln|z_n|} \tag{11.55}$$

because of the definition of α'. Comparing (11.55) and (11.55), we get (11.53).

Moreover, if the right side of (11.55) is finite, so is τ. In other words, if τ is infinite, so is right side of (11.54). This proves the proposition.

Theorem 10. Let $f(z)$ be an entire function of finite order ρ, and $n(r)$ be the number of zeros of $f(z)$ in the closed disc $|z| \leq r$, not including possible zeros at the origin. Then, $n(r) = O(r^{\rho+\varepsilon})$ for large values of r.

Proof. From Corollary 4, we derive the asymptotic inequality

$$\int_0^r \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|\,d\theta - \ln|f(0)|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \ln M(r)\,d\theta - \ln|f(0)| = \ln M(r). \tag{11.56}$$

Since $n(t)$ is an increasing function of t, we also have

$$\int_0^{2r} \frac{n(t)}{t}\,dt \geq \int_r^{2r} \frac{n(t)}{t}\,dt \geq n(r)\ln 2. \tag{11.57}$$

By virtue of (11.56), for sufficiently large r,

$$\int_0^{2r} \frac{n(t)}{t}\,dt \leq \ln M(2r).$$

Combining this with (11.57),

$$n(r)\ln 2 \leq \ln M(2r).$$

But by the definition of order, for given $\varepsilon > 0$, there exists r_0 such that for all $r > r_0$,

$$\ln M(2r) < (2r)^{\rho+\varepsilon} = 2^{\rho+\varepsilon} r^{\rho+\varepsilon}.$$

Thus,

$$n(r) < Kr^{\rho+\varepsilon}, \qquad K = \frac{2^{\rho+\varepsilon}}{\ln 2} > 1$$

for large values of r, and this is precisely what we mean by $n(r) = O(r^{\rho+\varepsilon})$ as $r \to \infty$.

PROBLEM SET 11.3

1. Find the order ρ and type σ of the following entire functions:

   (a) $z^2e^{2z} - e^{3z}$; (b) $e^{(2-i)z^2}$; (c) $e^z \cos z$; (d) $\cos\sqrt{z}$;

   (e) z^4e^{3z}; (f) $e^{5z} - 3e^{2z^3}$; (g) $\cosh z$; (h) $e^z \cos z$.

2. Without recourse to coefficient characterization, find the order and type of the following entire function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(2^m n)!}, \qquad m \in \mathbb{N}$$

3. Find the order and type of the entire function

$$f(z) = \int_0^1 e^{zt^2}\,dt.$$

4. Suppose $f_1(z)$ and $f_2(z)$ are entire functions of orders ρ_1 and ρ_2, respectively, where $\rho_1 \neq \rho_2$. What can be said about the order of

   (a) $f_1(z) + f_2(z)$; (b) $f_1(z)f_2(z)$?

5. Suppose two entire functions $f_1(z)$ and $f_2(z)$ of the same order ρ have types σ_1 and σ_2, respectively, where $\sigma_1 \neq \sigma_2$. What can be said about the order and type of

   (a) $f_1(z) + f_2(z)$; (b) $f_1(z)f_2(z)$?

6. Prove that an entire function $f(z)$ and its derivative $f'(z)$ have the same order and type.

Suggestion. First establish the inequality.

$$\frac{M(r)-|f(0)|}{r} \le M_1(r) \le \frac{M(r)}{R-r}, \qquad R > r,$$

where $M(r)$ and $M_1(r)$ are the maximum modulus functions of $f(z)$ and $f'(z)$, respectively. For an alternative method see the remark following Theorem 6.

7. Using the coefficient characterization formulas for order and type of an entire function, find the order and type of the entire function defined by the following power series:

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{z}{n}\right)^n$; (b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^{a/n} z^n \quad (a > 0)$;

(c) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{n\ln n}\right)^{n/a} z^n \quad (a > 0)$; (d) $\displaystyle\sum_{n=0}^{\infty} e^{-n^2} z^n$.

8. Let $f(z)$ be analytic in $|z| < R$. Assume that f has a zero of order k at the origin. If $a_1, a_2, \ldots, a_n$ are the zeros of $f(z)$ in $|z| < r$, $r < R$ with due regard for multiplicity, show that

$$\ln\frac{r^{n+k}}{|a_1|\,|a_2|\cdots|a_n|} = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|\,d\theta - \ln\frac{|f^{(k)}(0)|}{k!}.$$

Now, modify the equation in Corollary 5 to get

$$\int_0^r \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|\,d\theta - \ln\frac{|f^{(k)}(0)|\,r^k}{k!}.$$

Of course, here $n(t)$ is to be defined as the number of zeros of $f(z)$, different from origin in the disc $|z| \le t$.

Suggestion. Apply Jensen formula (11.48) to the function $f(z)z^{-k}$.

9. Let $f(z)$ be analytic in $|z-a| \le R$. Suppose $f(0) \ne 0$ and no zero of $f(z)$ lies on $|z-a| = R$. Then prove that the number of zeros of $f(z)$ in the circle $|z-a| < \delta R$, $0 < \delta < 1$ does not exceed $-(\ln\delta)^{-1}\ln(M/f(0))$, where $M > 0$ is some constant.

Suggestion. Apply Jensen's formula.

10. Modify the proof of Theorem 8 in case some zeros of $f(z)$ lie on the circle $|z| = r$.

11. Let $f(z)$ be an entire function with maximum modulus $M(r)$, and $f(0) = 1$. If $n(r)$ denotes the number of zeros of $f(z)$ in the closed disc $|z| \le r$, prove that

$$n(r) \le \ln M(er).$$

Suggestion. Follow the technique of deriving inequality (11.56). This result is also valid if $f(z)$ is assumed to be analytic in the disc $|z| \le er$.

12. Prove that $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ is an entire function of order 1 and of minimal type if and only if $\lim_{n\to\infty} \sqrt[n]{n!|a_n|} = 0$.

13. The condition

$$\lim_{n\to\infty} \frac{\ln n}{\ln |a_n|^{-1/n}} = 0$$

characterizes an entire function of order zero. Using this select a_n show that $f(z) = \Sigma_{n=1}^{\infty} a_n z^n$ is an entire function of order zero.

14. The condition

$$\limsup_{n\to\infty} \frac{\ln n}{\ln |a_n|^{-1/n}} = \infty$$

characterizes an entire function of order infinity. Using this select a_n show that $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ is an entire function of order infinity.

15. Use the Hadamard factorization theorem to establish the following canonical representations:

$$\text{(a) } e^z - 1 = ze^{z/2} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4n^2\pi^2}\right); \quad \text{(b) } e^{az} - e^{bz} = (a-b)ze^{(a+b)z/2} \prod_{n=1}^{\infty} \left[1 + \frac{(a-b)^2 z^2}{4n^2\pi^2}\right].$$

16. Prove the following general theorem of Phragmén Lindelöf type: Given a domain G with boundary Γ, let $f(z)$ be analytic in G, and suppose that $\limsup_{z\to s} |f(z)| \le M < \infty$ for every point $s \in \Gamma$ which does not belong to some set $E \subset \Gamma$. Suppose further that there exists a function $w(z)$ satisfying the conditions: (i) $w(z)$ is analytic on G; (ii) $0 < |w(z)| < 1$ for all $z \in G$; (iii) for every $\varepsilon > 0$,

$$\limsup_{z\to p, p\in E} |w(z)|^{\varepsilon} |f(z)| \le M.$$

Then $|f(z)| \le M \ \forall\ z \in G$. Moreover, if $f(z_0) = M$ for some $z_0 \in G$, then $f(z) \equiv M$.

Comment. We can allow $f(z)$ to be multivalued provided $|f(z)|$ is single valued.

17. Prove a refinement of Theorem 10 which is stated as follows: Let $n(r)$ be the number of zeros of $f(z)$ in the disc $|z| \le r$, not including the possible zero at the origin. Then, for any $\varepsilon > 0$,

$$\limsup_{r\to\infty} \frac{n(r)}{r^{\rho+\varepsilon}} \le e(\rho + \varepsilon)$$

if $f(z)$ is of finite order $\rho \ge 0$, while

$$\limsup_{r\to\infty} \frac{n(r)}{r^{\rho}} \le e\rho\sigma$$

if $f(z)$ is of finite order $\rho > 0$ and finite type $\sigma \ge 0$.

11.7 MAPPING BY MULTIVALUED FUNCTIONS

Let us first discuss the nature of the map log z. This is a multivalued function and any branch

$$w = \log z = \ln r + i\theta \qquad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$$

of logarithmic function maps its domain of definition in a one-to-one manner onto the strip $\alpha < v < \alpha + 2\pi$ in the w-plane. This can be seen by writing $z = re^{i\theta_0}$, where $\alpha < \theta_0 < \alpha + 2\pi$ moves outward from the origin along the ray $\theta = \theta_0$. Its image is evidently the point whose rectangular coordinates are given by $(\ln r, \theta_0)$. Hence the image moves to the right along the entire length of the horizontal line $v = \theta_0$, since these lines fill up the strip $\alpha < \theta < \alpha + 2\pi$ as θ_0 varies between α and 2π. Figure 11.4 illustrates the above arguments.

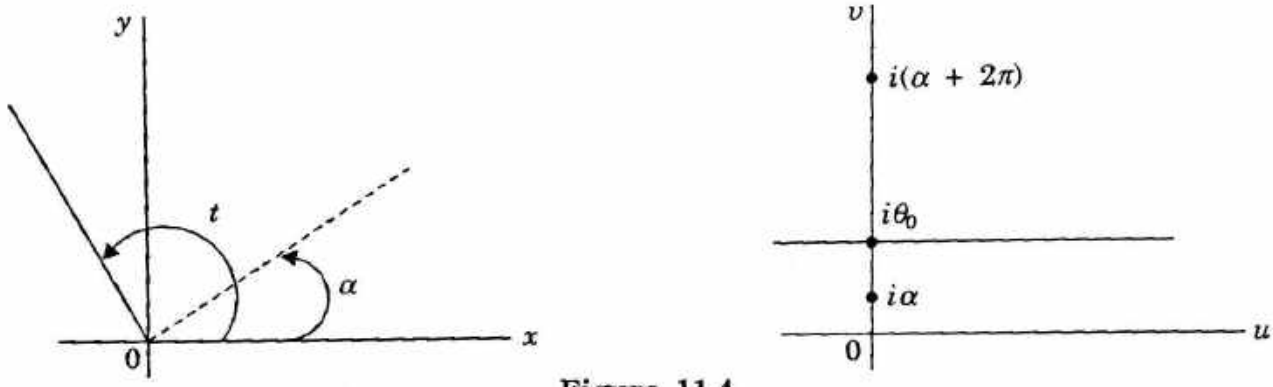

Figure 11.4

Example 10. Show that the transformation

$$w = \text{Log}\,\frac{1+z}{1-z}$$

maps the unit circle $|z| < 1$ on the horizontal strip $-\pi/2 < v < \pi/2$. Further, the upper semi-circle is mapped onto the line $v = \pi/2$ and the lower semi-circle on the line $v = -\pi/2$.

The mapping $w = f(z)$ can be considered composition of the bilinear transformation

$$Z = \frac{1+z}{1-z}$$

followed by the logarithmic mapping

$$w = \text{Log}\ Z.$$

The image of the disc $|z| < 1$ under the map $Z = (1+z)/(1-z)$ is the right half-plane $X > 0$; the upper semi-circle is mapped onto the positive Y-axis; and the lower semi-circle is mapped onto the negative Y-axis. The principal logarithmic function $w = \text{Log}\ Z$, maps the right half-plane on the horizontal strip; the image of the positive Y-axis is the line $v = \pi/2$; and the image of the negative Y-axis is the line $v = -\pi/2$. Figure 11.5 shows the required mapping.

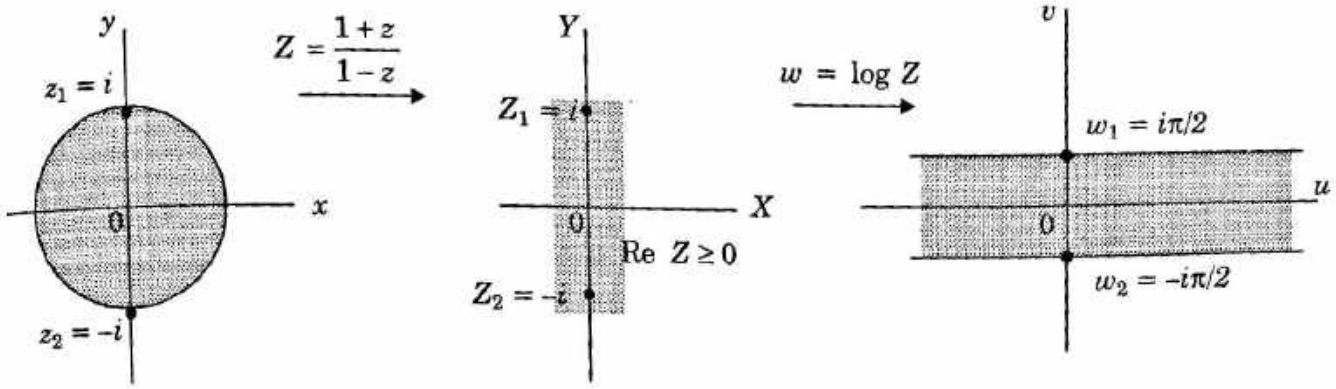

Figure 11.5

The discussion in Section 3.7 reveals that the values of $z^{1/2}$ are the two square roots of z when $z \neq 0$. So, if polar coordinates are used and $z = re^{i\theta}$ $(r > 0, -\pi < \theta \leq \pi)$, then

$$z^{1/2} = \sqrt{r} \exp \frac{i(\theta + 2k\pi)}{2}, \qquad k = 0, 1 \tag{11.58}$$

and the principal root is obtained when $k = 0$. In Section 3.7, we also observed that $z^{1/2}$ can also be written as

$$z^{1/2} = \exp\left(\frac{1}{2}\log z\right), \qquad z \neq 0. \tag{11.59}$$

The principal branch $F_0(z)$ of the double-valued function $z^{1/2}$ is then obtained by taking the principal branch of $\log z$ and writing (see Section 3.6)

$$F_0 = \exp\left(\frac{1}{2}\log z\right) \quad (|z| > 0, -\pi < \theta < \pi)$$

or

$$F_0 = \sqrt{r}\exp\left(\frac{i\theta}{2}\right) \qquad (r > 0, -\pi < \theta < \pi).$$

The right side of this equation is, of course, the same as the right side of Eq. (11.58) for $k = 0$, $-\pi < \operatorname{Arg} z < \pi$. The origin and the ray $\theta = \pi$ form the branch cut for F_0. Images of curves and regions under the transformation $w = z^{1/2}$ may be obtained by writing $w = \rho e^{i\phi}$, where $\rho = \sqrt{r}$ and $\phi = \theta/2$. Arguments are halved by this map, and it is understood that $w = 0$ when $z = 0$. These concepts will become clearer in the following examples.

Example 11. Find the image of the quarter disc $0 \leq r \leq 2$, $0 \leq \theta \leq \pi/2$, under the map $w = z^{1/2}$.

If we take $w = \rho e^{i\phi}$, then we verify that the quarter disc $0 \leq r \leq 2$, $0 \leq \theta \leq \pi/2$ in the z-plane is mapped in a one-to-one manner onto the sector $0 \leq \rho \leq \sqrt{2}$, $0 \leq \phi \leq \pi/4$ in the w-plane. To do this we note that as a point $z = r\exp(i\theta_1)$, $0 \leq \theta_1 \leq \pi/2$, moves

outward from the origin along a radius of length R_1 with the angle of inclination $\theta_1/2$, its image $w = \sqrt{r}\ \exp i\theta_1/2$ moves outward from the origin in the w-plane along a radius R_1' whose length is $\sqrt{2}$ and the angle of inclination is $\theta_1/2$; refer to Fig. 11.6, where another radius R_2 and its image R_2' are also shown. It is now clear from the figure that if the region in the z-plane is thought of as being swept out by a radius, starting with DA and ending with DC, then the region in the w-plane is swept out by the corresponding radius, starting with $D'A'$ and ending with $D'C'$. This establishes a one–one correspondence between the two regions.

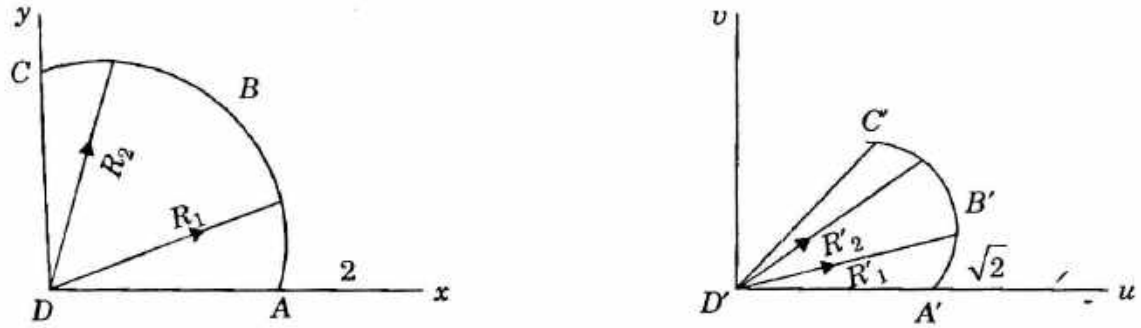

Figure 11.6

11.8 SCHWARZ–CHRISTOFFEL TRANSFORMATION

Many applications of conformal mappings require the construction of a one-to-one conformal mapping from the upper half-plane Im $z > 0$ onto a domain Ω in the w-plane where the boundary consists of straight line segments. Let us consider the case where Ω is the interior of polygon P with vertices $w_1, w_2, \ldots, w_n$ specified with positive orientation. We want to find a function $w = f(z)$ with the property

$$w_k = f(x_k), \qquad k = 1, 2, \ldots, n-1, \qquad w_n = f(x_n = \infty), \tag{11.60}$$

where $x_1 < x_2 < \cdots < x_{n-1} < \infty$.

Two German mathematicians H.A. Schwarz and E.B. Christoffel independently discovered a method for finding f.

Theorem 11 (Schwarz–Christoffel). Let P be a polygon in the w-plane with vertices $w_1, w_2, \ldots, w_n$ and exterior angles α_k, $-\pi < \alpha_k < \pi$, as shown in Fig. 11.7. There exists a one-to-one conformal mapping $w = f(z)$ from the upper half-plane Im $z > 0$ onto Ω (interior of P) that satisfies the boundary condition (11.60). The derivative $f'(z)$ is

$$f'(z) = A(z-x_1)^{-\alpha_1/\pi}\,(z-x_2)^{-\alpha_2/\pi}\cdots(z-x_{n-1})^{-\alpha_{n-1}/\pi} \tag{11.61}$$

and the function can be expressed as an indefinite integral

$$f(z) = A\int (z-x_1)^{-\alpha_1/\pi}\,(z-x_2)^{-\alpha_2/\pi}\cdots(z-x_{n-1})^{-\alpha_{n-1}/\pi}\,dz + B, \tag{11.62}$$

where A and B are complex constants. Two of the points $\{x_k\}$ may be chosen arbitrarily, and the constants A and B determine the size and position of P.

Proof. The proof is based on the fact how much the tangent $\tau = 1 + 0 \cdot i$, which always points to the right at the point $(x, 0)$ must be rotated by the mapping $w = f(z)$ so that the line segment $x_{j-1} < x < x_j$ is mapped onto the edge of polygon P that lies between two points $w_{j-1} = f(x_{j-1})$ and $w_j = f(x_j)$. Since the amount of rotation of the image segment is given by $\arg f'(z)$ (refer to Section 9.2) it specifies $f'(z)$ in terms of values x_j and the amount of rotation α_j that is required at the vertex $w_j = f(x_j)$.

If we let $x_0 = -\infty$ and $x_n = \infty$, then for all real x that lie in the interval $x_{j-1} < x < x_j$ the amount of rotation is given by

$$\arg f'(x) = \arg A - \frac{\alpha_1}{\pi} \arg (x - x_1) - \frac{\alpha_2}{\pi} \arg (x - x_2) - \cdots - \frac{\alpha_{n-1}}{\pi} \arg (x - x_{n-1}).$$

The points $w_1, w_2, \ldots$ corresponding to the vertices of the polygon are mapped on the points $x_1, x_2, \ldots$ of the real axis.

When $z = x_1$ and $x < x_1$,

$$\arg(z - x_1) = \arg(z - x_2) = \cdots = \arg(z - x_{n-1}) = \pi. \tag{11.63}$$

If z moves along the real axis from the left ($x_0 = -\infty$) towards x_1, the point w moves along a side of the polygon towards w_1. As z crosses x_1 from left to right position, $\arg(x - x_1)$ changes its value from π to 0, since $\arg(x - x_1) = \pi$ for $x < x_1$ and $\arg(x - x_1) = 0$ for $x > x_1$, while all other terms in Eq. (11.63) remain unchanged.

The angle of inclination of the tangent vector to the polygon P at the point $w = f(x)$, $x_0 < x < x_1$ is

$$\Gamma_1 = \arg A - \alpha_1 - \alpha_2 - \cdots - \alpha_{n-1}.$$

The angle of inclination of the tangent vector to the polygon P at the point $w = f(x)$, $x_1 < x < x_2$, is

$$\Gamma_2 = \arg A - \alpha_2 - \alpha_3 - \cdots - \alpha_{n-1}.$$

Hence, the argument of $f'(x)$ increases abruptly by $\Gamma_2 - \Gamma_1 = \alpha_1$. It means that the direction (tangent vector) through the point w_1 turns through the angle α_1 with positive sense. Therefore, the exterior angle to the polygon P at the vertex w_1 is given by the angle α_1 and satisfies the inequality $-\pi < \alpha_1 < \pi$.

Similarly, as z moves from left of x_2 to its right, the direction through the point w_2 turns through the angle α_2 in the w-plane. By continuing the process, we observe that as z moves along the real axis through the points $x_1, x_2, \ldots, x_{n-1}$, the image point w moves along the sides of a polygon with vertices $w_1, w_2, \ldots, w_{n-1}$, and vice versa. Of course, the sides of the polygon never cross one another as we have assumed that orientation is positive. Since the sum of the exterior angles of a polygon equals 2π, we have

$$\alpha_n = 2\pi - \alpha_1 - \alpha_2 - \cdots - \alpha_{n-1}$$

so that only $n - 1$ angles need to be specified.

The case $n = 5$ is indicated in Fig. 11.7.

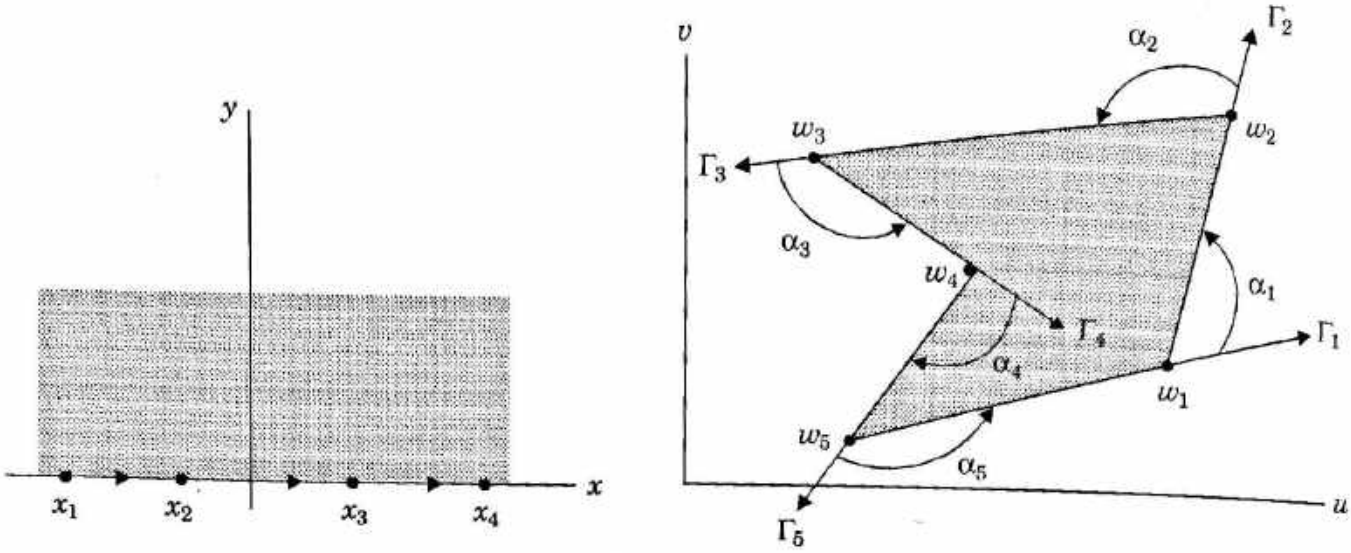

Figure 11.7

The only thing that remains to be verified is that the interior point of a closed polygon in the w-plane is mapped onto a point in the upper half-plane $\operatorname{Im} z > 0$. For this purpose, it is sufficient to show that an interior point of the closed polygon is mapped onto an interior point of unit disc in the z-plane since we are well acquainted with the mapping which maps the interior of unit disc in upper half-plane (refer to Proposition 6 in Chpater 9).

Suppose that the map $w = f(z)$ maps the polygon P in the w-plane onto unit disc in the z-plane and $f(z)$ is analytic within and on unit disc $C : |z| = 1$. For any point a interior to polygon P, by Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_P \frac{1}{w-a}\,dw = 1.$$

Since $w - a = f(z) - a$, it follows that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-a}\,dz = 1. \tag{11.64}$$

The function $f(z) - a$ is analytic interior to C, that there is one zero of $f(z) - a$ interior to C (see Proposition 5, Chapter 7), i.e. there is a point z_0 interior to C such that $f(z_0) = a$. Since a is arbitrary, we conclude that any interior point of polygon P is mapped onto interior of unit disc.

Remarks. 1. The selection of any two point out of $x_1, x_2, \ldots, x_{n-1}$ is at our disposal.

2. We have chosen x_n at infinity in (11.60). That is why the last factor involving x_n in (11.61) and (11.62) is taken as unity. This is justified by defining $A = K/(-x_n)^{-\alpha_n/\pi}$, where K is some constant. In case we involve x_n, then the right side of (11.60) has the form

$$f'(z) = K(z-x_1)^{-\alpha_1/\pi}\,(z-x_2)^{-\alpha_2/\pi}\ldots(z-x_{n-1})^{-\alpha_{n-1}/\pi}\left(\frac{x_n - z}{x_n}\right)^{-\alpha_n/\pi}$$

As $n \to \infty$, the last factor approaches 1. This is equivalent to removal of the last factor.

3. When $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \leq \pi$ occurs. In this case, $\alpha_n > \pi$ and the vertices $w_1, w_2, \ldots, w_n$ cannot form a closed polygon. For such type of situation, formulas (11.61) and (11.62) will determine a mapping from the upper half-plane Im $z > 0$ onto infinite open polygons, i.e. $w_n = \infty$.

Again, the above situation for $n = 5$ is illustrated in Fig. 11.8.

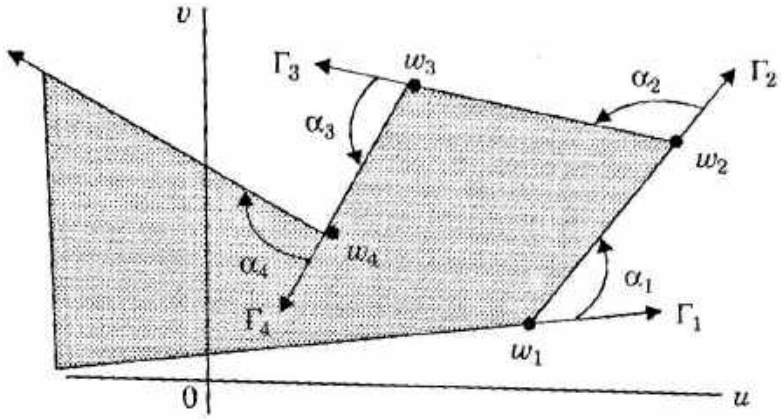

Figure 11.8

4. Formula (11.60) gives $f(z)$ in terms of an indefinite integral. It is also to be noted that these integrals do not represent elementary functions unless the image is an infinite region. Actually, the integral will involve a multivalued function, and a specific branch must be selected to fit the specified boundary values of the problem.

Example 12. Find the mapping of the vertical semi-infinite strip $-\pi/2 < u < \pi/2$, $v > 0$ shown in Fig. 11.9 onto the upper half-plane Im $z > 0$.

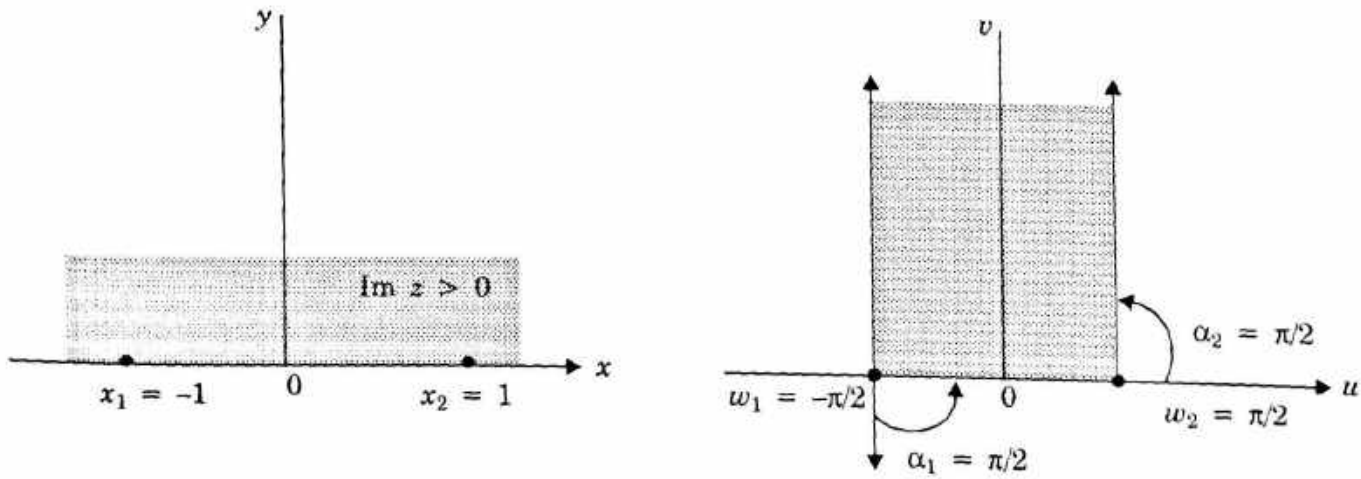

Figure 11.9

If we choose $x_1 = -1$ and $x_2 = 1$, $w_1 = -\pi/2$, $w_2 = \pi/2$, then $\alpha_1 = \pi/2$ and $\alpha_2 = \pi/2$, and Eq. (11.61) becomes

$$f'(z) = A(z+1)^{-\pi/(2\pi)}\,(z-1)^{-\pi/(2\pi)} = \frac{A}{(z^2-1)^{1/2}}.$$

On integration, we have

$$f(z) = Ai\,\sin^{-1}z + B.$$

Using the image values $f(-1) = -\pi/2$ and $f(1) = \pi/2$, we get the linear equations:

$$-\frac{\pi}{2} = -A\frac{i\pi}{2} + B \quad \text{and} \quad \frac{\pi}{2} = A\frac{i\pi}{2} + B.$$

This system gives $A = -i$ and $B = 0$. Hence the required transformation is

$$f(z) = \sin^{-1}z.$$

Example 13. Find a transformation that maps the interior of a triangle with one vertex at the origin in the w-plane onto the upper half of the z-plane.

Let the points $x_1 = 0$ and $x_2 = 1$ correspond to vertices $w_1 = 0$ and $w_2 = 1$, while the third vertex w_3 maps on infinity (see Fig. 11.10). These selections are just a matter of convenience.

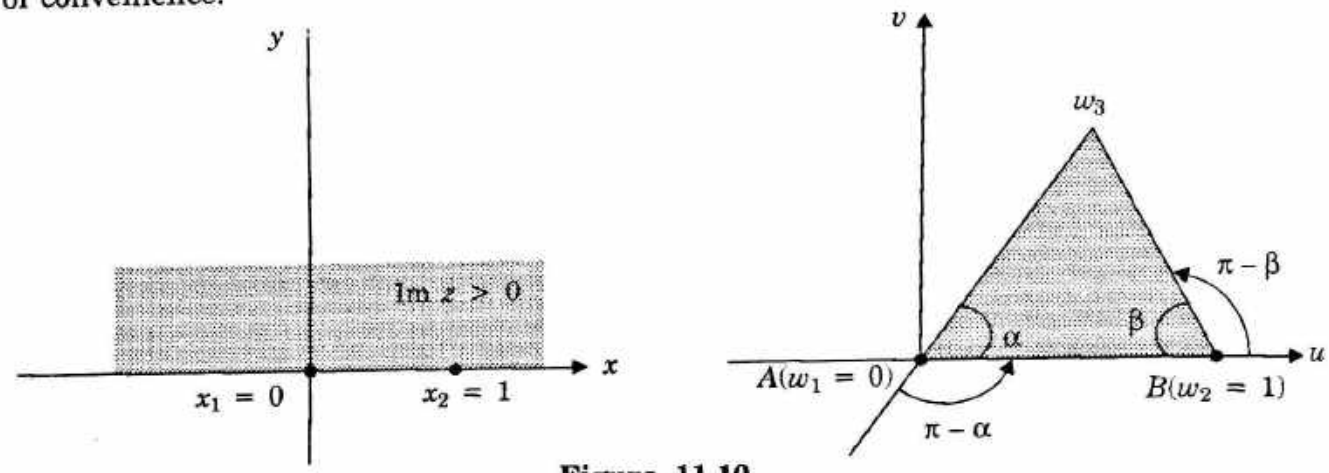

Figure 11.10

Suppose α and β are the angles of the triangle at the vertices A and B, respectively. Here, the exterior angles are $\alpha_1 = \pi - \alpha$ and $\alpha_2 = \pi - \beta$. Then, by using the Schwarz–Christoffel transformation

$$f'(z) = Mz^{(\alpha/\pi-1)}(z-1)^{(\beta/\pi-1)} \quad (M \text{ is constant})$$

Integrating both sides from 0 to any point, we obtain

$$w = C\int_0^z s^{(\alpha/\pi)-1}(1-s)^{(\beta/\pi)-1}ds + D,\ C = (-1)^{(\beta/\pi(-1}M, \tag{11.65}$$

where the principal value of C taken.

As z is from 0 to 1 in the integral, we can write $z = x$, when $s = x > 0$. From (11.65) if $z = 0$, then $w = 0$ and consequently $D = 0$. Also, $w = 1$ maps on $z = 1$. Thus,

$$1 = C\int_0^1 x^{(\alpha/\pi)-1}(1-x)^{(\beta/\pi)-1}dx = C\,\frac{\Gamma(\alpha/\pi)\,\Gamma(\beta/\pi)}{\Gamma[(\alpha+\beta)/\pi]}.$$

Thus, C can be written in terms of gamma function as

$$C = \frac{\Gamma((\alpha+\beta)/\pi)}{\Gamma(\alpha/\pi)\,\Gamma(\beta/\pi)}.$$

Hence the required transformation is given by

$$w = \frac{\Gamma((\alpha+\beta)/\pi)}{\Gamma(\alpha/\pi)\,\Gamma(\beta/\pi)}\int_0^z s^{(\alpha/\pi)-1}(1-s)^{(\beta/\pi)-1}\,ds.$$

PROBLEM SET 11.4

1. Find the image of the semi-infinite strip $0 \le x \le \pi/2$, $y \ge 0$, under the map $\sin z^{1/2}$.
2. When the principal branch of the square root is considered, show that the composite function

$$w = \left(\frac{z-1}{z+1}\right)^{1/2}$$

maps the z-plane, except the line segment $-1 \le x \le 1$, $y = 0$, onto the half-plane $u > 0$.

3. Considering the principal branch of the fourth root of the function $(\sin z)^{1/4}$, show that the infinite strip $-\pi/2 \le x \le \pi/2$, $y > 0$, is mapped onto the part of the first quadrant lying between the line $v = u$ and the u-axis.
4. Let

$$w = \tan^2\left(\frac{\pi}{4}\sqrt{z}\right).$$

Then show that the region within and on $|w| = 1$ is mapped on the boundary and interior of the parabola $y^2 = 4(1-x)$.

Suggestion. Write the transformation in the form

$$w = \frac{1-\cos 2s}{1+\cos 2s} \quad \text{and} \quad s = \frac{\pi}{4}\sqrt{z}.$$

and with $s = a + ib = (\pi/2)\sqrt{r}\,e^{i\theta/2}$. For convenience, we may use the principal branch and note that

$$a = \frac{\pi}{2}\sqrt{r}\cos\frac{\theta}{2}, \qquad b = \frac{\pi}{2}\sqrt{r}\sin\frac{\theta}{2}.$$

Insert in the new form to have $|w| \le 1 \Rightarrow \cos a \cosh b \ge 0$. This gives the curve $r(1 + \cos\theta) \le 2$.

5. Let β_m denote the interior angles of a closed polygon. Then show that

$$\sum_{m=1}^{n} \frac{\beta_m}{\pi} = n - 2.$$

6. Find the transformation which maps the horizontal infinite strip in the first quadrant of the w-plane onto the upper half-plane in the z-plane.
7. Find the transformation which maps the interior of an isosceles triangle in the w-plane onto the upper half z-plane.
8. Find a function which maps the upper half-plane to the interior of a triangle with vertices at $w = 0, 1, i$ corresponding to $z = 0, 1, \infty$ respectively.

Suggestion. Set $\alpha = \pi/2$ and $\beta = \pi/4$ in Example 13.

9. Show that the transformation

$$f(z) = \int_0^z \frac{1}{s(1-s^2)^{3/2}}\,ds$$

maps a square in the w-plane onto the upper half of the z-plane.

10. Determine the function which maps the region of vertical semi-infinite strip of width a in the z-plane to the real axis of the w-plane.

11. Find a transformation which maps a polygon in the w-plane onto the unit circle in the z-plane.

12. Verify that $w = (z^2 - 1)^{1/2}$ maps the upper half-plane $\text{Im } z > 0$ onto the upper half-plane $\text{Im } w > 0$ slit cut along the segment from 0 to i.

13. Show that the function

$$f(z) = \frac{1}{\pi}\sin^{-1}z + \frac{i}{\pi}\sin^{-1}\left(\frac{1}{z}\right) + \frac{1+i}{2}$$

maps the upper half-plane $\text{Im } z > 0$ onto the right angle channel in the first quadrant which is bounded by the coordinate axes and the rays $x \geq 1$, $y = 1$ and $y \geq 1$, $x = 1$.

Suggestion. Choose $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $w_1 = 0$, $w_2 = d$, $w_3 = 1 + i$. Transform α_1, α_2, α_3 on w_1, w_2, w_3, respectively (Fig. 11.11) and use the fact: $d \to \infty$ implies $\alpha_2 \to \pi$ and $\alpha_3 = -\pi/2$. Observe that $\alpha_1 = \pi/2$ yields the required channel.

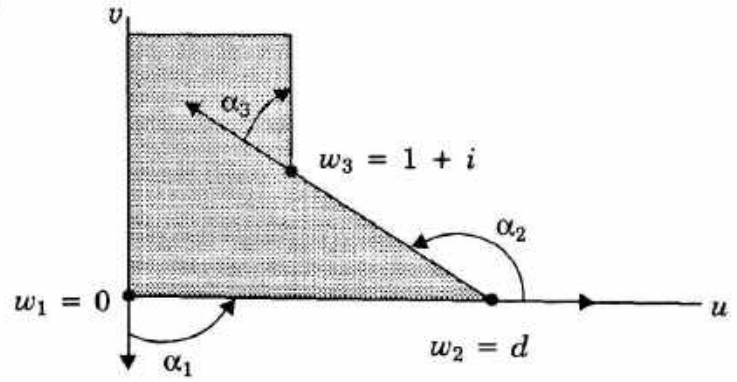

Figure 11.11

14. Let $\gamma = p(t) + iq(t)$, $t \in [0, 1]$ be a curve in the z-plane. Show that the transformation $z = p(w) + iq(w)$ maps γ onto the real axis of w-plane.

Suggestion. This is an interesting result which transforms boundaries in the z-plane onto the real axis of w-plane. Let $w = u + iv$. Then, $p(t) + iq(t) = p(u + iv) + iq(u + iv)$. For $v = 0$, note that $p = p(u)$ and $q = q(u)$ and this is the parametric equation of γ.

15. Based on the above problem, find the transformations which map

(a) ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; (b) circle $x^2 + y^2 = 4$

onto real axis of the w-plane.

Objective-type Questions

1. The value of $e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$ at the point $x = \pi/4$ is

   (a) 1/2 (b) $1/\sqrt{2}$ (c) $\sqrt{2}$ (d) 2

2. Le $n \in \mathbb{I}$. Given that $z_1 z_2 \neq 0$, $\arg z_1 = \theta_1$, $\arg z_2 = \theta_2$. Then $\mathrm{Re}(z_1 \overline{z_2}) = |z_1||z_2|$ $\Leftrightarrow$

   (a) $\theta_1 - \theta_2 = 0$ (b) $\theta_1 - \theta_2 = 2n\pi$
   (c) $\theta_1 - \theta_2 = n\pi$ (d) none of these

3. Let $n \in \mathbb{I}$ and $z_1 z_2 \neq 0$, $\arg z_1 = \theta_1$, $\arg z_2 = \theta_2$. Then $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow$

   (a) $\theta_1 - \theta_2 = 0$ (b) $\theta_1 - \theta_2 = 2n\pi$
   (c) $\theta_1 - \theta_2 = n\pi$ (d) $\theta_1 - \theta_2 = (2n + 1)\pi$

4. The value of $(1 + i)^8$ is _____.

5. Write the value of Arg $(1 - i)$ _____.

6. Let the principal root out of cube roots of a nonzero complex number be $z_0 = re^{i\theta_0}$. Then the other two roots are (i) _____ and (ii) _____.

7. The sum of nth roots of a nonzero complex number is _____.

8. The product of 7 cube roots of unity is

   (a) -1 (b) 0
   (c) 1 (d) none of these

9. The product of 8 cube roots of unity is

   (a) -1 (b) 0
   (c) 1 (d) none of these

10. The set of points $z \in \mathbb{C}$ for which $|z - 2| + |z + 2i| = 4$ is the conic

    (a) hyperbola (b) rectangle
    (c) square (d) ellipse

11. Assume that the curve $|z - z_1| + |z - z_2| = 4$, $z_1 \neq z_2$ exists. This curve is _____.

12. Let $n \in \mathbb{I}$. The set of all singularities of the function $f(z) = z + |z|$ is

(a) only $n\pi$ (b) only $2n\pi$ (c) $\mathbb{R}$ (d) ϕ

13. The set of all limit points of

$$\left\{z : z = \frac{1}{m} + i\frac{1}{n}, n \in \mathbb{I}\right\}$$

are given by _____.

14. The closure of the set $\{z : 0 < \text{Arg } z < \pi/2\}$ is _____.

15. If $f(z) = |z|^2$, then $f'(0)$ is _____.

16. If $f(z) = z \text{ Re } z$, then $f(z)$ is differentiable

(a) at all points z (b) only for $z = 0$
(c) at $z = 1$ (d) at none of these points

17. If $f(z) = z|z|$, then $f(z)$ is differentiable

(a) at all points z (b) only for $z = 0$
(c) at $z = 1$ (d) nowhere

18. If $f(z) = z|z|^2$, then $f(z)$ is differentiable

(a) at all points z (b) at $z = 1$
(c) only for $z = 0$ (d) at none of these points

19. If $f(z) = \dfrac{\cos 2\theta}{r^2} - i\dfrac{\sin 2\theta}{r^2}$, then $f'(z)$ is _____.

20. The C-R equations in polar form for a function $f(z) = u(r, \theta) + iv(r, \theta)$ are given by _____.

21. $f(x, y) = e^y(\cos x + i \sin x)$ is

(a) an entire function (b) analytic in $x^2 + 4y^2 < 24$
(c) nowhere analytic (d) none of these

22. If f is analytic in D: $|z| < 2$ and $|f|$ is constant in D with $f(0) = 3$, then $f(i)$ is

(a) $3 + i$ (b) $3 - i$
(c) 3 (d) cannot be determined

23. The function $w = e^z$ is

(a) entire periodic (b) entire non-periodic
(c) neither entire nor periodic (d) periodic but not entire

24. The function $w = \cos z$ is

(a) unbounded and entire (b) bounded and entire
(c) bounded but nowhere analytic (d) unbounded but nowhere analytic

25. $e^{\alpha x} \cos \beta y$ is a harmonic function

(a) for all α, β (b) for $\alpha^2 + \beta^2 = 0$
(c) for $\alpha^2 - \beta^2 = 0$ (d) for none of these

26. The harmonic conjugate of the function $4xy + 3$ is _____.

27. The corresponding analytic function in Question 26 is _____.

28. The $u(x, y) = \sinh x \sin y$ is harmonic in a domain D, then its harmonic conjugate in D is _____.

29. Answer 'yes' or 'no' only:

(a) Is $\text{Arg } \bar{z} = -\text{Arg } z$ for all z?

(b) $f(z) = u(x, y) + iv(x, y)$ is such that u and v are differentiable at every point of domain D. Is f analytic in D?

(c) Is the function $f(z) = |z|^2$ analytic at 0?

(d) Is the function $f(z) = e^{|z|}$ analytic at $z = 0$?

(e) Does the limit e^{iz} when $x \to \infty$ exist?

(f) Does the limit e^{iz} when $y \to \infty$ exist?

(g) Does the $\lim_{z \to \infty} e^{-z^2}$ exist?

(h) Is it possible to construct an analytic function with real part as $x^2 + 2xy$?

(i) Does the $\lim_{z \to \infty} \tanh^{-1} z$ exist?

(j) $(z_1 z_2)^s = z_1^s z_2^s, z_1 z_2 \neq 0$?

30. Let $n \in \mathbb{I}$. Then $e^{i\bar{z}} = e^{\overline{iz}}$

(a) for all z (b) only for $z = 2n\pi$

(c) for $z = n\pi$ (d) only for $z = (2n + 1)\pi$

31. $e^{i\bar{z}} = e^{\overline{iz}}$ if and only if _____.

32. If $e^{iz} = -1$, then $z =$ _____.

33. Let $n \in \mathbb{I}$. Then $\sin(i\bar{z}) = \overline{\sin iz}$

(a) for all z (b) only for $z = 2n\pi$

(c) for $z = n\pi i$ (d) only for $z = (2n + 1)\pi i$

34. Let $n \in \mathbb{I}$. Then $\cos(i\bar{z}) = \overline{\cos iz}$

(a) for all z (b) only for $z = 2n\pi$

(c) only for $z = n\pi i$ (d) only for $z = (2n + 1)\pi i$

35. $\text{Log}(-2 + 2i)^2$ is

(a) $2[\text{Log}(-1 + i) + \text{Log } 2]$ (b) $\ln 8 - i\pi/2$

(c) $2\text{Log}(-2 + 2i)$ (d) none of these

36. Consider the function

$$f(z) = \frac{\text{Log}(z - i)}{z^2 + 4}.$$

Its singularities are _____.

37. Let

$$f(z) = \frac{z^2 + 1}{(z^3 + 1)(z - 1)}.$$

The singular points of this function are _____.

38. The singular points of cosech z Log z are _____.

28. The $u(x, y) = \sinh x \sin y$ is harmonic in a domain D, then its harmonic conjugate in D is _____.

29. Answer 'yes' or 'no' only:

(a) Is $\text{Arg } \bar{z} = -\text{Arg } z$ for all z?

(b) $f(z) = u(x, y) + iv(x, y)$ is such that u and v are differentiable at every point of domain D. Is f analytic in D?

(c) Is the function $f(z) = |z|^2$ analytic at 0?

(d) Is the function $f(z) = e^{|z|}$ analytic at $z = 0$?

(e) Does the limit e^{iz} when $x \to \infty$ exist?

(f) Does the limit e^{iz} when $y \to \infty$ exist?

(g) Does the $\lim_{z \to \infty} e^{-z^2}$ exist?

(h) Is it possible to construct an analytic function with real part as $x^2 + 2xy$?

(i) Does the $\lim_{z \to \infty} \tanh^{-1} z$ exist?

(j) $(z_1 z_2)^s = z_1^s z_2^s, z_1 z_2 \neq 0$?

30. Let $n \in \mathbb{I}$. Then $e^{i\bar{z}} = e^{\overline{iz}}$

(a) for all z (b) only for $z = 2n\pi$

(c) for $z = n\pi$ (d) only for $z = (2n + 1)\pi$

31. $e^{i\bar{z}} = e^{\overline{iz}}$ if and only if _____.

32. If $e^{iz} = -1$, then $z =$ _____.

33. Let $n \in \mathbb{I}$. Then $\sin(i\bar{z}) = \overline{\sin iz}$

(a) for all z (b) only for $z = 2n\pi$

(c) for $z = n\pi i$ (d) only for $z = (2n + 1)\pi i$

34. Let $n \in \mathbb{I}$. Then $\cos(i\bar{z}) = \overline{\cos iz}$

(a) for all z (b) only for $z = 2n\pi$

(c) only for $z = n\pi i$ (d) only for $z = (2n + 1)\pi i$

35. $\text{Log}(-2 + 2i)^2$ is

(a) $2[\text{Log}(-1 + i) + \text{Log } 2]$ (b) $\ln 8 - i\pi/2$

(c) $2\text{Log}(-2 + 2i)$ (d) none of these

36. Consider the function

$$f(z) = \frac{\text{Log}(z - i)}{z^2 + 4}.$$

Its singularities are _____.

37. Let

$$f(z) = \frac{z^2 + 1}{(z^3 + 1)(z - 1)}.$$

The singular points of this function are _____.

38. The singular points of cosech z Log z are _____.

39. The number of singularities of $\tanh z$ in the circle $|z| = 1$ is

(a) 1 (b) 2 (c) 4 (d) none of these

40. The set of all singularities of the function

$$\frac{\text{Log}(2z+3)}{(2z+3)(z+2)}$$

is given by _____.

41. The number of singularities of $\coth(z/2)$ in the circle $|z| = 12$ is

(a) zero (b) one (c) three (d) four

42. The periodicity of the function $\sin(13iz)$ is _____.

43. The function $\cos(kz + 23)$ is 1/2 periodic. Then k is equal to

(a) 1 (b) 2 (c) 4π (d) $4\pi i$

44. The period of $\tan 2\pi z$ is

(a) π (b) 2π (c) 2 (d) none of these

45. The period of $\sin(iz + 67)$ is

(a) π (b) $2\pi i$ (c) $-\pi$ (d) none of these

46. The period of $\tan(kz + \alpha)$ is

(a) $\pi/2$ (b) π/k (c) $2\pi/k$ (d) none of these

47. The period of the function $\sin 2z/\cos 3z$ is

(a) π (b) 2π (c) $3\pi i$ (d) 4π

48. The period of the function $\sin 3z/\cos 5z$ is

(a) π (b) 2π (c) $3\pi i$ (d) 4π

49. Let $n \in \mathbb{I}$. Then $\cos z_1 = \cos z_2 \Leftrightarrow z_1 + z_2$ or $z_1 - z_2$ is

(a) an integral multiple of π (b) an integral multiple of 2π
(c) an integral multiple of πi (d) none of these

50. Let $n \in \mathbb{I}$. Then $\cos z_1 + \cos z_2 = 0$ if and only if $z_1 + z_2$ or $z_1 - z_2$ is

(a) an integral multiple of π (b) an even integral multiple of π
(c) an odd integral multiple of π (d) none of these

51. Let $n \in \mathbb{I}$. Then $\sin z_1 = \sin z_2 \Leftrightarrow z_1 + z_2$ or $z_1 - z_2$ is

(a) $z_1 + z_2 = n\pi$ or $z_1 - z_2 = 2n\pi$
(b) $z_1 + z_2 = 2n\pi$ or $z_1 - z_2 = (2n+1)\pi$
(c) $z_1 + z_2 = (2n+1)\pi$ or $z_1 - z_2 = 2n\pi$
(d) none of these

52. Let $n \in \mathbb{I}$. Then $\sin z_1 + \sin z_2 = 0$ if and only if

$z_1 + z_2 =$ _____ or $z_1 - z_2 =$ _____.

53. Let $n \in \mathbb{I}$. Then $\cosh z_1 = \cosh z_2 \Leftrightarrow z_1 + z_2$ or $z_1 - z_2$ is

(a) an integral multiple of πi (b) an integral multiple of $2\pi i$
(c) only $z_1 \pm z_2 = 0$ (d) none of these

54. Let $n \in \mathbb{I}$. Then $\cosh z_1 + \cosh z_2 = 0$ if and only if $z_1 + z_2$ or $z_1 - z_2$ is
(a) an integral multiple of π (b) an integral multiple of $2\pi i$
(c) an odd integral multiple of πi (d) none of these

55. Let $n \in \mathbb{I}$. Then $\sinh z_1 = \sinh z_2$ if and only if
(a) $z_1 + z_2 = n\pi i$ or $z_1 - z_2 = 2n\pi i$
(b) $z_1 + z_2 = 2n\pi i$ or $z_1 - z_2 = (2n + 1)\pi i$
(c) $z_1 + z_2 = (2n + 1)\pi i$ or $z_1 - z_2 = 2n\pi i$
(d) none of these

56. Let $n \in \mathbb{I}$. Then $\sinh z_1 + \sinh z_2 = 0$ if and only if

$z_1 + z_2 =$ _____ or $z_1 - z_2 =$ _____.

57. The value of $\tan^{-1}(2i) - \cot^{-1}(2i)$ is _____.

58. The value of $\tanh^{-1}z - \coth^{-1}z$ is _____.

59. The value of $\tanh^{-1}2 + \coth^{-1}2$ is _____.

60. The principal value of i^i is
(a) $i \operatorname{Log} i$ (b) $e^{-\pi/2}$ (c) $e^{\pi/2}$ (d) none of these

61. The general value of i^i is _____.

62. The principal value of $i^{\sqrt{2}}$ is _____.

63. the principal value of $(1 - i)^{1+i}$ is _____.

64. The principal value of $(1 + \sqrt{3i})^i$ is
(a) $i \operatorname{Log}(1 + \sqrt{3i})$ (b) $e^{i \ln 2 - \pi/3}$ (c) $e^{\ln 2 + i\pi/6}$ (d) none of these

65. The general value of $\operatorname{Re}(\sqrt{2})^i$ is _____.

66. The general value of $\operatorname{Im} i^{\sin i}$ is _____.

67. The set of all value of $\sin^{-1}1$ is given by.

68. For $n \in \mathbb{I}$, all values of $\sin^{-1}z + \cos^{-1}z$ are given by _____.
(a) $2n\pi$ (b) $(2n + 1)\pi$ (c) $(2n + 1/2)\pi$ (d) none of these

69. For $n \in \mathbb{I}$, all value of $\tan^{-1}z + \cot^{-1}z$ are given by
(a) $n\pi$ (b) $(2n + 1)\pi$ (c) $(n - 1/2)\pi$ (d) none of these

70. $\tan^{-1} i$ is
(a) $\dfrac{\sin^{-1} i}{\cos^{-1} i}$ (b) $\log 2$ (c) i (d) none of these

71. $\operatorname{Log}(-3 + 3i)^2$ is
(a) $2 \operatorname{Log}(-3 + 3i)$ (b) $\ln 18 - i\pi/2$
(c) $2[\operatorname{Log} 9 + \operatorname{Log}(-1 + i)]$ (d) none of these

72. The branch cut of the function $f(z) = (z + 2) \operatorname{Log}(z - 1 - i)$ is the set

$$S = \{z = (x, y): \qquad \}$$

73. The branch point of the function in the preceding question is _____.

74. The branch of $w = \text{Log}(z + 4 - i)$ is

(a) $x \le 0, y = 0$ (b) $x \le 0, y = 1$
(c) $x \le -4, y = 0$ (d) $x \le -4, y = 1$

75. The function $f(z) = \text{Log}(z - i)$ is analytic except at $z =$ _____.

76. $\text{Log } z^n = n \text{ Log } z$ is true for

(a) none of the integers n
(b) all z and positive integers n
(c) some z and some integers n
(d) such a relation is never possible

77. The range of cosec z is _____.

78. The range of sec z is _____.

79. The range of tan z is _____.

80. State true or false:

(a) If a function $f = u + iv$ satisfies the C-R equations at a point, then f is differentiable at that point.

(b) Given that $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then u and v are harmonic in D. In this case the converse is also true.

(c) The function $e^{\bar{z}}$ is analytic at $z = 0$.

(d) The function $\sin \bar{z}$ is nowhere analytic.

(e) The relation $-i \sinh iz = \sin z$ holds for all complex numbers.

(f) If $f(z) = z^2$, then the family of curves $x^2 - y^2 = c_1$ and $xy = c_2$ are orthogonal

(g) If $f(z)$ and $\overline{f(z)}$ are analytic in a domain, then $f(z)$ must be constant.

(h) If $f(z)$ is analytic and $|f(z)|$ is constant in a domain D, then $f(z)$ must be constant.

(i) If f is entire and $|f'(z)| \le |z|$ for all z, then $f(z)$ is of the form $a + bz^2$, where a and b are constants.

(j) The equation $|z^2| - 2|z| + 1 = 0$ has a finite number of roots.

81. let $C : z = 2e^{\theta}, 0 \le \theta \le 2\pi$. Then fill up the blank

$$\left| \int_C \frac{e^z}{z^2 + 1} dz \right| \le \text{_____}.$$

82. Let $C : |z - i| = 2$ be in positive sense. Then fill up the blank

$$\int_C \frac{dz}{(z^2 + 4)^3} = \text{_____}.$$

83. Let $C : |z| = 4$. Then fill up the blank

$$\int_C \frac{z^2 e^z \sin z}{z - 8} dz = \text{_____}.$$

84. If C is an ellipse with positive orientation with the centre at the origin, then fill up the blank

$$\int_C \frac{z^3+3}{z}\,dz = \underline{\qquad\qquad}.$$

85. Let $C : |z-1| = 2$, taken in positive sense. Then fill up the blank

$$\int_C \frac{z^3 \sin z}{(z-1)^3}\,dz = \underline{\qquad\qquad}.$$

86. The integral along $|z| = 2$ (positively oriented)

$$\int_{|z|=2} \frac{\cos z}{z(z^2+9)}\,dz$$

has the value

(a) zero (b) $i\pi/2$ (c) $-i\pi/4$ (d) none of these

87. The value of the integral $\int_C z^2 dz$, where C is the line segment joining (0, 0) and (1, 1) is

(a) zero (b) $(1+i)^3/3$ (c) $(1-i)/3$ (d) none of these

88. If C is the boundary of the square $\{-1 \le x \le 1, -1 \le y \le 1\}$ described in anticlockwise direction, then the value of the integral $\int_C \frac{z^2}{4z-1-i}\,dz$ is $\underline{\qquad\qquad}$.

89. C is the curve of the preceding question. Then the value of the integral $\int_C \frac{z^2}{[z-(1+i)/4]^5}\,dz$ is . . .

90. The integral $\int_{|z+i\pi|=1} \frac{z}{e^z-1}\,dz$ has the value

(a) zero (b) πi (c) 2π (d) none of these

91. The value of the integral $\int_C \text{Log}(z-4+3i)dz$, where C is the positively oriented ellipse centred at the origin with semi-major and minor axes as 3 and 2, respectively is.

92. The value of the integral $\int_{|z|=3} \frac{\cos z}{z(z^2+14)}\,dz$ is

(a) zero (b) $2\pi i/7$ (c) $\pi i/7$ (d) none of these

93. The value of the integral $\int\limits_{|z+\pi|=2} \frac{\sin z}{e^z - 1} dz$ is

(a) zero (b) $2\pi i$ (c) 4π (d) none of these

94. The value of the integral $\int_C z^2 dz$, where C is the parabola from (0, 0) to (1, 1), is

(a) zero (b) $(1 + i)/3$ (c) $(1 - i)/3$ (d) none of these

95. Let $f(z)$ be defined in $\{(x, y) : |x| \le 4, |y| \le 3\}$. If $f(z)$ is analytic in D and satisfies $|f(z)| \le 1$ on ∂D, then fill up the blank $|f'(0)| \le$ _____.

96. The radius of convergence of the series $\Sigma_{n=0}^{\infty} z^n/n$ is

(a) 0 (b) 1 (c) ∞ (d) none of these

97. The radii of convergence of the power series $\Sigma a_n z^n$ and $\Sigma b_n z^n$ are R_1 and R_2. The radius of convergence R of the power series $\Sigma_{n=0}^{\infty} (a_n + b_n) z^n$ satisfies

(a) $R = R_1 + R_2$ (b) $R \ge \min\{R_1, R_2\}$
(c) $R \le \min\{R_1, R_2\}$ (d) none of these

98. In view of the preceding question, the radius of convergence R of $\Sigma a_n b_n z^n$ satisfies

(a) $R = R_1 R_2$ (b) $R \le \min\{R_1, R_2\}$
(c) $R \le R_1 R_2$ (d) $R \ge R_1 R_2$

99. The zeros of the function $\sin(3iz + 1)$ are _____.

100. Let f and g be analytic in a domain D, and let each have zeros of order m and n at z_0, respectively. Then the order of zeros of $f + g$ is

(a) $m + n$ (b) $\ge \min\{m, n\}$ (c) $\le \max\{m, n\}$ (d) mn

101. Let f and g be analytic at z_0 and have zeros of order m and n, respectively. Then the order of zeros of $f(z)g(z)$ at $z = z_0$ is

(a) $m + n$ (b) $\le \min\{m, n\}$ (c) $\le \max\{m, n\}$ (d) mn

102. The origin is the zero of $z^3 \sin z$ of order

(a) 1 (b) 3 (c) 4 (d) none of these

103. The sum of the power series $\Sigma_{n=1}^{\infty} z^n/n$ for $|z| < 1$ is

(a) $\log(1 + z)$ (b) $-\log(1 - z)$ (c) $\log(1 - z)$ (d) none of these

104. The sum of the power series $\Sigma_{n=0}^{\infty} z^{2n+1}/(2n + 1)$ for $|z| < 1$ is _____.

105. The sum of the power series $\Sigma_{n=0}^{\infty} (n + 3) z^n$ for $|z| < 1$ is _____.

Suggestion. Multiply by z^2 and integrate.

106. Consider the expansion

$$\sum_{n=0}^{\infty} \frac{z^n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}, \qquad 1 < |z| < 3.$$

Then $z = 0$

(a) is an essential singularity (b) is a pole of order one
(c) is a point of analyticity (d) is none of these

107. Let $f(z)$ be an even analytic function having a pole of order two at $z = a$. Then the residue of $f(z)$ at $z = a$ is

(a) -1 (b) 0 (c) 1 (d) 2

108. The residue of the function

$$f(z) = \frac{\sin z^2}{z^5 \sin z}$$

at $z = 0$ is _____.

109. The residue of $\dfrac{\cos z}{z}$ at $z = 0$ is _____.

110. The poles of $f(z) = (z^3 - 1)/(z^3 + 1)$ are _____.

111. If $f(z)$ has a pole of order three at a, then $\text{Res}_{z=a} f(z)$ is _____.

112. The residue of $\tan z$ at $z = \pi/2$ is _____.

113. The residue of $\tanh z$ at $z = \pi i/2$ is _____.

114. The residue of $\pi \operatorname{cosec}(\pi z)$ at $z = n$, $n \in \mathbb{I}$ is _____.

115. The singularities of the function

$$f(z) = \frac{e^z}{\sin z - \sin a}, \qquad a \neq \left(n + \frac{1}{2}\right)\pi, \quad n \in \mathbb{I}$$

are the simple poles:

(a) $n\pi + a$ (b) $n\pi + (-1)^n a$ (c) $n\pi - a$ (d) none of these

116. State true or false.

(a) $\int_C z^{-1} |dz| = 2\pi$, where C is the circle $|z| = 1$ with positive orientation.

(b) If f is analytic within and on $|z| = 4$, taken in positive sense, then

$$\frac{1}{\pi i} \int_C \frac{2f(z)}{(2z-1)(2z+1)} dz = f\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right).$$

(c) If f and g are analytic in a domain D and continuous on the boundary of D such that $f(z) = g(z)$ on the boundary of D, then $f(z) = g(z)$ for all z in D.

(d) A bounded sequence of complex numbers is always convergent.

(e) A complex sequence of continuous functions may converge to a continuous function, though uniform convergence does not occur.

(f) If two power series have the same finite radius of convergence, then the radius of convergence of their sum power series is always finite.

(g) $\dfrac{1}{2z - z^2} = \displaystyle\sum_{n=0}^{\infty} \frac{z^{n-1}}{2^{n+1}}$ for $0 < z < 2$.

(h) The number of zeros of an analytic function inside a simple closed contour is finite.

(i) If a circle is transformed into a circle under the map $w = 1/z$, the centre of the original circle is never mapped onto centre of the image circle.

(j) If $\{z_n\}$ converges to nonzero number z_0, then $\{\text{Arg } z_n\}$ also converges to Arg z_0.

(k) If $\{z_n\}$ converges to nonzero number z_0, then $\{\arg z_n\}$ also converges to arg z_0.

117. The cross ratio of the four points z_1, z_2, z_3, z_4 is _____.

118. The fixed points of the transformation $w = (1 + z)/(1 - z)$ are _____.

119. The critical points of the transformation $w^2 = (z - a)(z - b)$ are _____.

120. Let $w = (1 - iz)/z$. Then the unit circle $|w| = 1$ is mapped on the line

(a) $2x + 1 = 0$ (b) $2y + 1 = 0$ (c) $2x + 4 = 0$ (d) $y + 1 = 0$

121. The image of the circle $|z| = 2$ under the map $w = 1/z$ is a

(a) circle of radius 2 (b) straight line passing through the origin
(c) circle of radius of 1/2 (d) straight line not passing through the origin

122. The image of $|z + 1| = 1$ under the mapping $w = 1/z$.

123. The image of the right half-plane $x \geq 0$, under the mapping $w = (z - 1)/(z + 1)$ is

(a) the right half-plane $u \geq 0$ (b) the upper-half plane $v \geq 0$
(c) the circle $|w| \leq 1$ (d) none of these

124. The function $f(z) = z^2$ $(0 \leq \text{Arg } z \leq \pi)$ is

(a) even (b) odd
(c) neither even nor odd (d) the one about which nothing can be said

125. If f_E is the even part in $e^z = f_E + f_O$, then f_E is equal to _____.

126. The function $e^z - \sinh z$ is

(a) even (b) odd
(c) neither even nor odd (d) the one about which nothing can be said

127. If f is an even differential function, then f' is

(a) even (b) odd
(c) neither even nor odd (d) the one about which nothing can be decided

128. For the integral

$$\int_0^\infty \frac{\sin mt}{t} dt = \alpha, m > 0$$

(a) $\alpha = 0$ (b) $\alpha = \pi/2$ (c) $\alpha = \pi$ (d) $-\pi$

129. Which one is the Laplace transform of any function of exponential order?

(a) $\log p$ (b) $\sin p$ (c) e^p (d) none of these

130. If $F(p)$ is the Laplace transform of a piecewise continuous function $f(t)$ of exponential order, then $F(p)$ approaches to

(a) zero as $p \to \infty$ (b) ∞ as $p \to \infty$
(c) e as $p \to \infty$ (d) none of these

131. Which of the following results is correct?

(a) $\delta * f = \delta(t)$ (b) $f * 1 = f$
(c) $L[t\mu_1(t)] = e^{-p}/p^2$ (d) none of these

132. If $L[(f(t)] = F(p)$, then $L[f(\alpha t)] = kF(p/\alpha)$, $\alpha \neq 0$, where k is

(a) α (b) $1/\alpha$ (c) α^2 (d) $1/\alpha^2$

133. The outcome of $\delta(t - a) * f(t)(\delta(t - a)$ Dirac-delta function) is _____.

134. State true or false.

(a) If $F(p)$ is the Laplace transform of a function $f(t)$ then always $F(p) \to 0$ as $p \to \infty$.

(b) If $L[f(t)] = F(p)$, then $\lim_{t\to 0} f(t) = \lim_{p\to\infty} pF(p)$.

(c) If $L[f(t)] = F(p)$, then $\lim_{t\to 0} f(t) = \lim_{p\to\infty} pF(p)$.

(d) The convolution of a function f and constant funciton 1 is always f.

(e) The convolution of two functions may be a constant.

(f) Let $F(p)$ be the Laplace transform of a piecewise continuous function $f(t)$ of exponential order. Then $pF(p)$ is a bounded function of p.

135. The vlaue of $1 * 1 * 1 * \cdots * 1$, $(n + 1)$ times is _____.

136. The Laplace transform of the unit impulse function $\delta(t - a)$ is _____.

137. The inverse Laplace transform of $1/(p + 1)$ is

(a) 0 (b) $-x$ (c) e^{-x} (d) none of these

138. Using the Laplace transform, fill up the blank

$$\int_0^\infty e^{-2t} \cos 3t \, dt = _____.$$

139. Use the Laplace transform method to fill up the blank

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt = _____.$$

140. The Laplace transform of y'' in terms of y is _____.

141. The value of $\Gamma'(1)$ is _____.

142. The value of $\Gamma(1/2)$ is _____.

143. The order and type of the entire function $P(z)e^{kz}$, $k > 0$ [$P(z)$ is a polynomial] are

(a) 1, k (b) k, 1 (c) 1, 1 (d) k, k

144. The order of the entire function e^{z^2} is

(a) 1 (b) ∞ (c) $-\infty$ (d) none of these

145. The order of the entire function e^{e^z} is

(a) 1 (b) ∞ (c) $-\infty$ (d) none of these

146. The minimum value of the function $|e^z|$ on the triangle (including the boundry) bounded by the coordinate axes and the line $y - x = 1$ is _____.

147. The point at which the minimum occurs in the preceding question is _____.

Answers to Problems

CHAPTER 1

Problem Set 1.1

1. (a) $x \geq 0$, $y = 0$; (b) $x = 0$ or 1, $y = 0$; (c) $x = 1$, $y = 0$.

6. $-\frac{3}{4} - i$. **7.** $(\text{Im } \beta)^2 = \text{Re } \alpha \text{ Re}(\alpha\bar{\beta})$.

Problem Set 1.2

2. (a) $\frac{10}{13 - 4\sqrt{3}}$; (b) $\frac{1}{3}$.

12. Equality holds $\Leftrightarrow$ the z_k are proportional to the $\overline{w_k}$.

Problem Set 1.3

1. (a) Since $-2\pi < \theta_1 + \theta_2 \leq 2\pi$, $\theta_1 = \text{Arg } z_1$, $\theta_2 = \text{Arg } z_2$, there is an integer n such that $-\pi < \theta_1 + \theta_2 + 2n\pi \leq \pi$. This n is the same as given in the suggestion. For $n = 0$, the region is $\{(x, y), x > 0, y \in \mathbb{R} \text{ or } x = 0 (y \leq 0)\}$.

6. (a) and (b) conclude that z_1 and z_2 lie along the same ray that passes through the origin;

(c) Vectors z_1 and z_2 are perpendicular, $z_1\overline{z_2} + \overline{z_1}z_2 = 0 \Leftrightarrow z_1 = itz_2$, $t \in \mathbb{R}$;

(d) Vectors z_1 and z_2 are parallel, $z_1\overline{z_2} - \overline{z_1}z_2 = 0 \Leftrightarrow z_1 = tz_2$, $t \in \mathbb{R}$.

7. (c) Either $|a| > 1$, $|b| > 1$ or $|a| < 1$, $|b| < 1$.

9. Yes, (a) $1 + \sqrt{3}$, $1 - \sqrt{3}$; (b) 0, 2.

12. Largest for $(2 + \sqrt{1/2})(1 - i)$; Least for $(2 - \sqrt{1/2})(1 - i)$.

13. Use $z^{-1} = \bar{z}\,|z|^2$

17. (a) $(3 + 6i)/5$, (b) $|z + 7 - bi| = \sqrt{48 + b^2}$, $b \in \mathbb{R}$.

19. In this case, the angle between $a - z$ and $b - z$ is constant, and hence the locus of z is an arc $[azb]$ of a circle which passes through a and b such that the segment $a - b$ makes a constant angle k at z.

Problem Set 1.4

1. (a) 2^{72}; (b) $-i2^{-15}(-\sqrt{3}-i)$.

4. (a) $\dfrac{\sin 2n\alpha}{2\sin\alpha}$ $0 < \alpha < 2\pi$. For $\alpha = 0$ or 2π, the sum is n;

(b) $\dfrac{n}{2}+\dfrac{1}{4}-\dfrac{\sin(2n+1)\alpha}{4\sin\alpha}$, $0 < \alpha < 2\pi$. For $\alpha = 0$ or 2π, the sum is zero.

6. $\dfrac{2^{6t}+2}{3}$ for $n = 6t$, $\dfrac{2^{6t+1}+1}{3}$ for $n = 6t+1$, $\dfrac{2^{6t+1}+3}{3}$ for $n = 6t+3$.

8. $1, \omega, \omega^2, \ldots, \omega^{n-1}$, $\omega = e^{2\pi i/n}$, sum: 0, product: $(-1)^{n-1}$, the roots lie on the vertices of a closed regular polygon having n equal sides inscribed in a unit circle.

9. (a) $-\dfrac{n}{1-\omega}$; (b) $\dfrac{n[(1-\omega)n+2]}{3\omega}$

(c) For $m = 1, 3, 5, \ldots$, $S_n = 2/\omega$, $n = 3m$, $S_n = (1-\omega)/(1+\omega)$, $n = 3m+1$, $S_n = -\omega/(1+\omega)$, $n = 3m+2$, and for $m = 0, 2, 4, \ldots$, $S_n = 0$, $n = 3m$, $S_n = 1$, $n = 3m+1$, $S_n = 1-\omega$, $n = 3m+2$.

10. (a) $n\sqrt[n]{r_0}e^{i\theta_0 h/n}$, if h is a multiple of n, and 0 otherwise; (b) 0, if n is even, and $2\sqrt[n]{r_0}e^{i\theta_0 h/n}/(1+w^h)$, if n is odd; (c) 0.

11. (a) $\sqrt{3}-i$, $-\sqrt{3}+i$; (b) $\sqrt{3}-i$, $2i$, $-1+i\sqrt{3}$; (c) $-2+3i$, $2-3i$; (d) $\sqrt{2}(1+i)$, $\sqrt{2}(-1+i)$, $\sqrt{2}(-1-i)$, $\sqrt{2}(1-i)$.

12. (a) $\dfrac{1-\sqrt{2}+i}{\sqrt{2}}, \dfrac{1+\sqrt{2}+i}{\sqrt{2}}$; (b) $\sqrt{3}+i$, $2i$, $-\sqrt{3}+i$, $-\sqrt{3}-i$, $-2i$, $\sqrt{3}-i$;

(c) $\dfrac{1+i}{\sqrt{2}}, \dfrac{-1+i}{\sqrt{2}}, \dfrac{-1-i}{\sqrt{2}}, \dfrac{1-i}{\sqrt{2}}, \dfrac{1+\sqrt{3}i}{2}, \dfrac{-1+\sqrt{3}i}{2}, -i$; (d) $\dfrac{1}{2}, \dfrac{1-i}{2}, \dfrac{1+i}{2}$.

(e) For $n = 1$, z is real. For $n > 1$, $z = e^{2k\pi i/(n+1)}$, $k = 0, 1, \ldots, n$;

(f) $e^{i2\pi/9}$, $e^{i8\pi/9}$, $-e^{i5\pi/9}$, $e^{-i2\pi/9}$, $e^{i4\pi/9}$, $-e^{\pi/9}$.

13. $e^{2(k+1)\pi i/7}$, $k = 0, 1, \ldots, 6$ are seven roots of $z^2+z+2 = 0$.

15. $p = 4$, $q = 6$, $r = 6$, $s = 0$.

Problem Set 1.5

1. (a) Straight line $x = 3$, unbounded; (b) $(x-2)^2+(y+1)^2 \le 1$, bounded; (c) $(x+3/2)^2+y^2 > 4$, unbounded; (d) $x^2-y^2 = 1$, unbounded; (e) $\frac{x^2}{4}+\frac{y^2}{3} = 1$, bounded; (f) $(x-1)^2+y^2 > 1$, unbounded; (g) $x^2 > y^2$, unbounded; (h) Hyperbola with foci $(0, \pm i)$, bounded; (i) Circle with centre $(z_1+z_2)/2$, radius $|z_1-z_2|/2$; (j) Straight line passing through z_1 and z_2.

3. (a) Open, domain; (b) Closed; (c) Neither open nor closed; (d) Closed; (e) Neither open nor closed; (f) Neither open nor closed; (g) Open domain; (h) Open.

5. (a) $-\pi \le \arg z \le \pi \cup \{0\}$; (b) $|\text{Re } z| \le |z|$; (c) ϕ;
(d) $i(z+\bar{z})(z-\bar{z}) \ge 4$; (e) $x + iy,\ x \in \mathbb{I},\ y \in \mathbb{R}$; (f) 0;
(g) $0 \le \arg z \le \pi/2 \cup \{0\}$; (h) $\pm(1+i)$; (i) $\pm 1, \pm i$.

9. (a) Connected; (b) Not connected; (c) Connected; (d) Not connected.

12. $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}}\right) \to z_1 = \left(\frac{2}{\sqrt{3}-1}\right) + i\left(\frac{2}{\sqrt{3}-1}\right)$

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right) \to z_2 = \left(\frac{2}{\sqrt{3}+1}\right) + i\left(\frac{2}{\sqrt{3}+1}\right)$

$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right) \to z_3 = \left(\frac{2}{\sqrt{3}+1}\right) - i\left(\frac{2}{\sqrt{3}+1}\right)$

$\left(\frac{1}{-\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right) \to z_4 = -\left(\frac{2}{\sqrt{3}+1}\right) + i\left(\frac{2}{\sqrt{3}+1}\right)$

$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right) \to z_5 = -\left(\frac{2}{\sqrt{3}+1}\right) - i\left(\frac{2}{\sqrt{3}+1}\right)$

$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}}\right) \to z_6 = -\left(\frac{2}{\sqrt{3}-1}\right) + i\left(\frac{2}{\sqrt{3}-1}\right)$

$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}}\right) \to z_7 = \left(\frac{2}{\sqrt{3}-1}\right) - i\left(\frac{2}{\sqrt{3}-1}\right)$

$\left(\frac{1}{-\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}}\right) \to z_8 = -\left(\frac{2}{\sqrt{3}-1}\right) - i\left(\frac{2}{\sqrt{3}-1}\right)$

CHAPTER 2

Problem Set 2.1

1. (a) Domain: $\mathbb{C}\backslash\{y\text{-axis}\}$, range: $\mathbb{C}$; (b) Domain: $\mathbb{C}\backslash\{|z| = 1\}$, range: $\mathbb{C}\backslash\{0\}$, (c) Domain: $\mathbb{C}\backslash\{0\}$, range $\mathbb{C}\backslash\{0\}$; (d) Domain: $\mathbb{C}\backslash\{x\text{-axis}\}$, range $\mathbb{C}\backslash\{0\}$.

2. (a) Not Univalent; (b) Univalent; (c) Univalent; (d) Univalent.

Problem Set 2.2

7. Discontinuous at $z = i$, define $f(i) = 2 + i$, to make it continuous at i.

9. $\{z : |z| = \phi\}$, the spiral of Archimedes.

Problem Set 2.3

1. The function is differentiable only at $z = -i$.

2. (a) $\mathbb{C}\backslash\{0\}$; (b) nowhere differentiable.

7. (a) $1/[2f(z)]$; (b) $if(z)/z$.

Problem Set 2.4

1. (a) $\pm i$, differentiable in $A = \mathbb{C}\backslash\{i, -i\}$, A is open, the function is analytic in A;
(b) 3, $1 + \sqrt{2}i$, $1 - \sqrt{2}i$, differentiable in $B = \mathbb{C}\backslash\{3, 1 + \sqrt{2}i, 1 - \sqrt{2}\}$, B is open, function is analytic in B.
(c) $0, -1$, differentiable in $B = \mathbb{C}\backslash\{0, -1\}$, B is open, function is analytic in B.
(d) $-i$, $2i$; (e) 0; (f) 0.

7. $D = \mathbb{C}\backslash\{\pm 1\}$ and $g(D) \subset \mathbb{C}\backslash\{1\}$.

Problem Set 2.5

1. $a = \pm 1$.

4. When k is any real constant (a) $v = -2x + 3xy - y^3 + k$; (b) $v = \cos x \sinh y + k$; (c) $v = -x + e^x \sin y + k$; (d) $v = (x - y)/(x^2 + y^2) + k$.

8. (c) $z^4 + c$, c: a complex constant; (d) $z^2 + 2zi + c$, c: a complex constant.
(e) $-2iz^2 + z + 1 + 3i$; (f) $i \sinh z + 1$.

9. (a) $-iz^3 + c$, c: a complex constant; (b) $(2 + 3i)z^2 + z + c$, c: a complex constant.

10. $a \ln(\sqrt{x^2 + y^2}) + b$, a and b are real constants.

11. $ax^3 - 3dx^2y - 3axy^2 + dy^3$, $a, d \in \mathbb{C}$, $(a + id)z^3 + c$, c: a complex constant.

13. $f(z) = c\sqrt{z} + k$, $c \in \mathbb{R}$ and $k \in \mathbb{C}$ are fixed.

14. $a = 3i$, $b = -3$, $c = -i$. **16.** No, the real part must be harmonic.

CHAPTER 3

Problem Set 3.1

1. (a) Limit does not exist; (b) 0.

2. Limit does not exist, as $\lim_{z\to\infty} e^{-z^2} = 1$, if $y = x$ and $\lim_{z\to\infty} e^{-z^2} = 0$, if $y = 0$.

3. (a) $(2n + 1)\pi + i(-1 + \ln 2)$, $n \in \mathbb{I}$;
(b) $i(n - 1/2)\pi$, $n \in \mathbb{I}$;
(c) $\{z : z = \ln 2 + i2n\pi\} \cup \{z : z = i(2n - 1/2)\pi\}$, $n \in \mathbb{I}$.

7. (a) 2π, $n \in \mathbb{I}$; (b) πi, $n \in \mathbb{I}$. **10.** $e^z + 2z + c$, c: a complex constant.

11. Derivatives of all orders exist at $x = 0$ but the function is not analytic in the real sense at the origin.

13. Complex numbers are not ordered, so the inequalities concerning real continuous functions referred to in the problem are meaningless for complex quantities.

Problem Set 3.2

2. $\sin z_1 = \sin z_2 \Leftrightarrow z_1 + z_2 = (2n+1)\pi$ or $z_1 - z_2 = 2n\pi,\ n \in \mathbb{I}$.

3. (a) π; (b) π; (c) 2π; (d) 20π.

4. (a) $2n\pi i,\ n\pi - 2,\ n \in \mathbb{I}$; (b) $2\pi,\ (n + 1/2)\pi + 2i,\ n \in \mathbb{I}$;
(c) $\pi i,\ i(n\pi + 3m),\ n \in \mathbb{I}$; (d) $\pi,\ (n + 1/2)\pi,\ n \in \mathbb{I}$.

6. (a) $(2n + 1/2) \pm 4i,\ n \in \mathbb{I}$; (b) $2n\pi + i \cosh^{-1}2,\ n \in \mathbb{I}$;
(c) $\{z : z = \ln 2 + i(2n - 1/2)\pi\} \cup \{n\pi + i \ln (1 + \sqrt{2})\},\ n \in \mathbb{I}$;
(d) $(n + 1/4)\pi,\ n \in \mathbb{I}$.

9. $[1 - \cot (z/2)]/2$.

11. x is indeterminate and y is infinite.

Problem Set 3.3

3. (a) $i(2n \pm 1/3)\pi,\ n \in \mathbb{I}$; (b) $(2n - 1/2)\pi i,\ n \in \mathbb{I}$; (c) $(n + 1/4)\pi i,\ n \in \mathbb{I}$.

4. (a) $\pi i, -\sqrt{3} + (n + 1/2)\pi i,\ n \in \mathbb{I}$; (b) $2\pi,\ n\pi + 2i,\ n \in \mathbb{I}$; (c) $\pi i,\ n\pi i,\ n \in \mathbb{I}$

8. $(n + 1/2)\pi i,\ n \in \mathbb{I}$.

10. $e^z + \sinh z + iz + c$, c: some constant.

11. The difference between z_1 and z_2 is an integral multiple of $2\pi i$.

Problem Set 3.4

2. (a) $z = 0,\ \alpha < \arg z < \alpha + 2\pi,\ \alpha \in \mathbb{R}$; (b) $z = \pm 2$, line segment $-2 \le x \le 2 (y = 0)$;
(c) $z = 0, 1$, line segment $0 \le x \le 1\ (y = 0)$.

4. (a) $\ln 2 + i2(n + 1/3)\pi,\ n \in \mathbb{I}$; (b) $i(2n + 1/2)\pi,\ n \in \mathbb{I}$; (c) $1 + i2n\pi,\ n \in \mathbb{I}$.

11. Branch point: $-b/a$, principal branch: $|az + b| > 0,\ -\pi < \arg (az + b) < \pi$, branch cut: $x \le -b/a (y = 0)$.

12. No.

13. $|z + 4| > 0,\ \alpha < \arg (z + 4) < \alpha + 2\pi,\ \alpha \in (5\pi, 7\pi)$.

Problem Set 3.5

1. (a) $e^{(2n+1)\pi/2},\ n \in \mathbb{I}$; (b) $e^{i(2k+1)\pi/n},\ k = 0, 1, \ldots, n-1$; (c) $e^{i \ln\sqrt{2}} e^{(2n-1/4)\pi},\ n \in \mathbb{I}$;
(d) $(1 + i)\ e^{(2n+1/4)\pi} e^{-i \ln\sqrt{2}},\ n \in \mathbb{I}$; (e) $e^{(2n-1/2)\sinh 1},\ n \in \mathbb{I}$; (f) $e^{2n\pi} \cos(\ln\sqrt{2})$, $n \in \mathbb{I}$; (g) $e^{2n\pi} \sin(\ln \cosh 1),\ n \in \mathbb{I}$; (h) $e^{(n-1/4)\pi^2 i},\ n \in \mathbb{I}$.

2. (a) $e^{2\pi^2} e^{i3\pi \ln 2}$; (b) $e^{i\pi \ln\sqrt{2}} e^{-\pi^2/4}$.

4. (a) $\pm 2\sqrt{2}$; (b) $\pm(1 + i)/\sqrt{2},\ \pm(1 - i)/\sqrt{2}$.

8. Singular points: $(n + 1/2)\pi i \cup x \le 1 (y = -1),\ n \in \mathbb{I}$, Branch cut: $x \le 1 (y = -1)$.

9. $[(\ln 2 + 1) + \pi/2] e^{-\pi} e^{2i \ln 2},\ |z|^x e^{-\theta y} \cos(y \ln|z| + \theta x)\ \theta = \operatorname{Arg} z$;
$[(\pi/2) - i \ln 2] e^{(\ln 2)^2 - (\pi^2/4) + (i\pi \ln 2)/2},\ e^{(\ln|z|^2) - \theta^2},\ \theta = \operatorname{Arg} z$.

Problem Set 3.6

1. (a) $n\pi + i(-1)^{n-1}\ln(1+\sqrt{2})$, $n \in \mathbb{I}$; (b) $i(2n+1)\pi$, $n \in \mathbb{I}$.
3. $\sin^{-1}(\sqrt{\sin\theta}) + i\sinh^{-1}(\sqrt{\sin\theta})$.
5. (a) $z = (2n - 1/2)\pi \pm i\ln(10+\sqrt{99})$, $n \in \mathbb{I}$; (b) $z = (-1)^n \ln(2+\sqrt{5}) + in\pi$, $n \in \mathbb{I}$.
6. (a) $\ln(\sqrt{3}+\sqrt{2}) + i[(2n-1/2)\pi - 1]$, $n \in \mathbb{I}$; (b) $1 + i(2n+1)\pi$, $n \in \mathbb{I}$; (c) $3 - \ln\sqrt{5} + i[(2n+1)\pi - \tan^{-1}2]$, $n \in \mathbb{I}$; (d) $n\pi - i\ln\sqrt{3}$, $n \in \mathbb{I}$.
8. Branch points: ± 1, branch line: $y \le 1(x = 0) \cup y \ge 1(x = 0)$.
9. $-1 \pm i$.

CHAPTER 4

Problem Set 4.1

3. (a) Straight line segment from 0 to $4 + 8i$; (b) Lower semi-circle with radius 2 and centre $1 - i$; (c) Parabola $y = 3x^2$ from $(-1, 1)$ to $(2, 4)$; (d) Ellipse $4x^2 + y^2 = 4$.
4. (a) $3 - 4i + 4e^{it}$, $0 \le t \le 2\pi$; (b) $t + (1/t)i$, $0 \le t \le 3$; (c) $t + it^2$, $0 \le t \le 2$; (d) $(3\cos t + 1) + 2i(\sin t - 1)$, $0 \le t \le 2\pi$.
5. $\gamma(t) = \begin{cases} 2e^{i(t+\pi/2)}, & (-\pi/2) \le t \le 0, \\ -t + (2-t)i, & 0 \le t \le 2. \end{cases}$
7. $2 + 3i$.
8. $-1 + \dfrac{17}{4}i$
9. (a) $1 + \dfrac{1}{3}i$; (b) $a^2[2 - (1+i)\pi]$
11. $\dfrac{\pi^2 e}{36}$.
13. $39\pi/38$.
14. $\dfrac{32}{\pi(2n+1)[1+\sinh(\pi/2)]}$.
18. 8.
19. Tangent makes an angle $\pi/4$ with the positive x-axis at the origin. No direction exists at the point -1.
20. $\gamma(t) = t^2 + t^3 i$, $\gamma'(0) = 0$, tangent at $(0, 0)$ is in the direction of the x-axis.

Problem Set 4.2

4. Yes, $i\alpha^{-1}R^{\alpha}(e^{2\pi\alpha i} - 1)$, $\alpha \ne 0$; $2\pi i$, $\alpha = 0$.
10. (a) Function has singularity inside the path, hence the C-G theorem cannot be applied.

    (b) Function is analytic on $|z| \le 1$, by C-G theorem the integral value is zero.
11. (a) Log $(z + 2)$ is analytic in $\mathbb{C}$, except at $x \le -2(y = 0)$, and hence analytic within and on $|z| \le 1$, by C-G theorem the integral value is zero;

(b) $e^{-z}/(z-\pi/2)$ has singularity at $z = \pi/2$ which lies outside $|z| \le 1$, function is analytic within and on $|z| \le 1$, by C-G theorem the integral value is zero.

12. $2\pi i$, $n = 1$ and 0, $n \neq 1$.

Problem Set 4.3

1. (a) $\pi i \cos 1$; (b) $-\pi i$.

2. $2\pi \cos n$.

3. $2\pi/3$.

5. $g''(p) = 2\int_\gamma \frac{z^2 - z + 1}{(z-p)^3}\,dz$, $g''(2) = 4\pi i$, $g'(5i) = 0$.

12. $-4\pi i$.

17. $4\pi i$.

Problem Set 4.4

8. $-iz + 3i$.

10. $\sin x$ is not analytic anywhere in complex plane and hence the Liouville's theorem cannot be applied to conclude that it is constant.

15. 1/3, −2/3, 5/4.

19. 5.

20. $-\pi \ln 2$.

Problem Set 4.5

3. (a) max $= \sqrt{1+\sinh^2 1}$, min $= 0$; (b) max e, min $= 1/e$; (c) max $= \cosh 2\pi$, min $= 1$; (c) max $= \cosh 1$ on $|z| \le 1$.

11. $f(z) = e^{i\alpha}z$, $\alpha \in \mathbb{R}$.

CHAPTER 5

Problem Set 5.1

2. (a) Limit points ±1; limit does not exist; (b) $1 + i$, both; (c) 0, both; (d) $i/2$, both.

3. $\lim_{n\to\infty} \frac{c_1z_1 + c_2z_2 + \cdots + c_nz_n}{c_1 + c_2 + \cdots + c_n} = z_0$, provided $c_1, c_2, \ldots, c_n$ is any sequence of positive numbers satisfying the $\lim_{n\to\infty}(c_1 + c_2 + \cdots + c_n) = \infty$.

4. $z_n = (-1)^n i$, $n = 1, 2, \ldots$

8. No, see Example 1, Section 5.1.

17. No, for example, $\Sigma_{n=1}^{\infty} 1/n^{3/2}$.

Problem Set 5.2

1. Converges for $|z| > 0$ and converges uniformly for $|z| \geq \delta$, $\delta > 0$.

2. (a) cgs for $|z| \leq 1$, dgs for $|z| > 1$; (b) cgs for $|z| \leq 1/e$, dgs for $|z| > 1/e$; (c) cgs for $|z| < 1/e$, dgs for $|z| \geq 1/e$; (d) cgs for $|z| \leq e$, dgs for $|z| \geq e$; (e) cgs for $|z| < 1$, dgs for $|z| \geq 1$; (f) cgs for $|z| > 1$, dgs for $|z| \leq 1$;

10. $1/z$, except for $z = 0, -1, -2, \ldots$ **11.** $|z - i| \leq 1$.

12. Uniformly convergent but not absolutely convergent.

13. Converges in $\operatorname{Re} z > 1$, converges uniformly in $\operatorname{Re} z \geq \delta$, $\delta > 1$.

19. $\Sigma_{n=0}^{\infty} z^n$ with $|z| = 1$.

20. (a) $\Sigma_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{n} = \ln 2$; $\Sigma_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{2n-1} = \dfrac{\pi}{4}$.

21. Converges for $|\alpha|^{-1} < |z| < |\alpha|$. **22.** (a) Diverges; (b) converges.

Problem Set 5.3

1. (a) Converges; (b) diverges; (c) converges: (d) diverges.

3. (a) Converges absolutely; (b) converges absolutely; (c) converges conditionally (d) converges absolutely in $\operatorname{Im} z \geq 0$, diverges for $\operatorname{Im} z < 0$.

6. (a) $|z| < 1$; (b) $\mathbb{C}$; (c) $\mathbb{C}$; (d) $\operatorname{Re} z > 1$. **7.** $1/(1 - z^2)$.

CHAPTER 6

Problem Set 6.1

1. (a) 2; (b) 4; (c) 1; (d) $4^{-1/3}$. **2.** (a) $\sqrt{2}$; (b) e.

3. $1 + 2/\sqrt{5}$. **7.** (a) R; (b) ∞; (c) R^{α}.

8. (a) $R \geq \min\{R_1, R_2\}$; (b) $R \geq R_1R_2$; (c) $R \geq R_1/R_2$.

9. $f(z) = \displaystyle\sum_{n=1}^{\infty} \frac{1 + n^{n-1}}{2n^n} z^n$, $R = 1$, $g(z) = \displaystyle\sum_{n=1}^{\infty} \frac{1 - n^{n-1}}{2n^n} z^n$,

$$R = 1,\ f(z) + g(z) = \sum_{n=1}^{\infty} \frac{1}{n^n} z^n,\ R = \infty.$$

10. $1/R$ if $a_n \neq 0$ and ∞ if $a_n = 0$.

11. Condition of divergence cannot be dropped, example $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

13. (a) No information from the previous problem; (b) $\{z:|z| \le 1\}\backslash\{1\}$; (c) $|z| \le 1$; (d) $\{z : |z| \le 1\}\backslash\{-1\}$.

14. $b = \pm 1/5$.

Problem Set 6.2

1. (a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{2(2n+1)}}{(2n+1)!}$, $|z| < \infty$; (b) $\displaystyle z + \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} z^{2n+1}$, $|z| < \infty$;

(c) $\displaystyle e^4 \sum_{n=0}^{\infty} \frac{z^{3n}}{n!}$, $|z| < \infty$; (d) $\displaystyle \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} z^{2n}}{(2n)!}$, $|z| < \infty$;

(e) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3(1-9^n)}{4(2n+1)!} z^{2n+1}$, $|z| < \infty$.

2. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} (z - i\pi)^{2n}$, $|z| < \infty$;

(b) $\displaystyle \sin 2 + \frac{\cos 2}{1!}(z-2) + \cdots + \frac{\sin(2 + n\pi/2)}{n!}(z-2)^n + \cdots$, $|z| < \infty$.

3. $\displaystyle\sum_{n=0}^{\infty} \frac{z^{2(2n+1)}}{(2n+1)!}$. **13.** $\displaystyle \frac{1+i}{\sqrt{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(2n-3)!\, i^n}{(n-2)!\, 2^{2n-1}} z^n\right] z^n$, $|z| < 1$.

18. $\dfrac{1}{3+2z}$.

20. (a) $\displaystyle -\frac{z^2}{2} - \frac{z^4}{12} - \frac{z^6}{45} - \cdots$, $|z-1| < 1$;

(b) $\displaystyle -\frac{z^2}{6} - \frac{z^4}{180} - \frac{z^6}{2520} - \cdots$, $|z-1| < 1$;

(c) $\displaystyle 1 + z + \frac{z^2}{2} - \frac{z^4}{8} - \frac{z^5}{15} - \cdots$, $|z| < \infty$. (d) $\displaystyle e\left(1 + z + z^2 + \frac{5}{6}z^3 + \cdots\right)$, $|z| < \infty$.

Problem Set 6.3

1. Order of zeros of $f + g \ge \min\{m, n\}$.

2. (a) zeros of order 4 at $z = \pm i$; (b) simple zeros at $(1 \pm i)/\sqrt{2}$ and $(-1 \pm i)/\sqrt{2}$, and a zero of order 4 at the origin.

3. (a) 4; (b) 15.

15. $f(z) = \dfrac{3(cz+3)}{1+cz}$, $|c| = 1$

16. $|f(1/7)| \le 1/5$. **18.** No.

22. (i) Such f does not exist as inequality (6.37) is violated; (ii) Inequality (6.37) ensures the existence of such a function f and $f = \dfrac{2cz}{2+cz}$, $|c| = 1$.

Problem Set 6.4

1. $z^{-1} + \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1}$, $0 < |z| < 1$; $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{z^{2n+1}}$, $|z| > 1$.

2. $\sum_{n=0}^{\infty} \dfrac{3^{-n-2} + (-1)^n 2^{-n+2}}{5} z^n - \dfrac{z}{5} z$, $|z| < 2$; $\dfrac{1}{5}\sum_{n=0}^{\infty} \dfrac{z^n}{3^n} + \dfrac{4}{z} + \sum_{n=1}^{\infty} \dfrac{(-1)^n 2^n}{z^{n+1}}$, $2 < |z| < 3$;

$-\dfrac{9}{5z} + \sum_{n=1}^{\infty} \dfrac{3^n + (-1)^n 2^n}{z^{n+1}}$, $|z| > 3$.

3. $\dfrac{1}{a-b}\left(\sum_{n=0}^{\infty} -\dfrac{z^n}{b^{n+1}} + \sum_{n=1}^{\infty} \dfrac{a^{n+1}}{z^n}\right)$, $|a| < |z| < |b|$;

$\dfrac{1}{a-b}\left(\sum_{n=0}^{\infty} \dfrac{a^n}{z^{n+1}} + \sum_{n=1}^{\infty} \dfrac{b^{n+1}}{z^n}\right)$, $|z| > |b|$;

$\dfrac{1}{a-b}\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{1}{a^n} - \dfrac{1}{b^n}\right) z^n$, $|z| < |a|$.

4. (a) $\dfrac{1}{4^{n+2} i^n} \sum_{n=0}^{\infty} (z+2i)^n$, $|z+2i| < 4$; $\sum_{n=1}^{\infty} \dfrac{(4i)^{n-1}}{(z+2i)^n}$, $|z+2i| > 4$;

(b) $-\dfrac{1}{z-\pi/2} + \dfrac{z-\pi}{3} + \dfrac{(z-\pi/2)^3}{45} + \cdots$, $0 < |z-\pi/2] < 1$.

6. $-\dfrac{1}{4} + \dfrac{z-1}{16} - \dfrac{(z-1)^2}{64} - \sum_{n=3}^{\infty} \left(1 + \dfrac{(-1)^n}{4^{n+1}}\right) (z-1)^n$, $|z-1| < 1$;

$\dfrac{3}{4} + \dfrac{33}{16}(z-1) + \dfrac{63}{64}(z-1)^2 - \sum_{n=3}^{\infty} \dfrac{(-1)^n}{4^{n+1}} (z-1)^n + \sum_{n=1}^{\infty} \dfrac{1}{(z-1)^{-n}}$, $1 < |z-1| < 4$;

$$1 + 2(z-1) + (z-1)^2 + \sum_{n=1}^{\infty} [1 - (-1)^{n-1}\, 4^{n-1}](z-1)^{-n},\ |z-1| > 4.$$

10. $|a_k| \le M_R/R^k\ \forall\ k \ge -m.$

14. Taylor series: $2(i-1) + 2(z-i) + \sum_{n=0}^{\infty} (-1)^n (z-i)^n\ (i^{-(n+1)} + (1+i)^{n+1});$

Laurent series: $-1 + z + \sum_{n=2}^{\infty} (-1)^n z^n + \frac{1}{z}.$

16. (a) $z/(z-a)$; (b) $z/(z-1)^2$; (c) $az/(z-a)^2$; (d) $\log(z-1) - \log z$; (e) $e^{1/z}$; (f) $z^2/(z^2-1)$.

Problem Set 6.5

1. $\sum_{k=0}^{\infty} \frac{(24+k)!}{23!k!} z^{k+1},\ |z| < 1,\ \sum_{k=1}^{\infty} \frac{(24+k)!}{25!(k-1)!} z^{k-1},\ |z| < 1.$

2. (a) $-e^i[1 + (1+2i)(z-i) + \frac{-5+4i}{2}(z-i)^2 + \cdots],\ |z-i| < \infty;$

(b) $-\frac{1}{z-1} + \frac{z-1}{2} - \frac{2}{3}(z-1)^2 + \cdots,\ 0 < |z-1| < 1;$

(c) $(z-1) - 2(z-1)^2 + \frac{17}{6}(z-1)^3 - \frac{11}{3}(z-1)^4 + \cdots,\ |z-1| < 1.$

(d) $\frac{1}{6} - \frac{5}{9}(z-2) + \frac{19}{17}(z-2)^2 - \cdots,\ |z-2| < 1.$

4. $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1},\ |z| < 1.$

5. (a) $-\log(1-z)$; (b) $\frac{1}{2}\log\frac{1+z}{1-z}$; (c) $\frac{3-2z}{(1-z)^2}$; (d) $\frac{z(3-5z)}{(1-z)^3}$.

8. $a = i,\ b = 2i,\ c = i.$

9. $\frac{1}{z^3} - \frac{1}{6z} + \frac{7}{360}z + \cdots,\ 0 < |z| < \pi.$

12. (a) $\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{1}{(2k)!}z^n,\ |z| < 1;$

(b) $\sum_{n=0}^{\infty}\sum_{k=0}^{[(n-1)/2]}\frac{1}{(2k+1)!\,(n-2k-1)!}z^n,\ |z| < \infty;$

(c) $\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}z^{2n+1}, |z|<1;$

(d) $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{(2k)![2(n-k)]!}\right)z^{2n}, |z|<\infty.$

13. $-\frac{i}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}[(1+i\sqrt{3})^{n+1}-(1-i\sqrt{3})^{n+1}]\,z^n,\ |z| < 1.$

14. $c_n = \alpha c_{n-3} + \beta c_{n-2} + \gamma c_{n-1}$, $\alpha = -b_3/b_0$, $\beta = -b_2/b_0$, $\gamma = -b_1/b_0$.

15. $E_{2n+1} = 0$, $n = 1, 2, \ldots$, since the function is even.

CHAPTER 7

Problem Set 7.1

1. (a) $\frac{3\pi e(4-3i)}{25}$; (b) $\frac{4\pi i}{25}$; (c) $\frac{9\pi i}{\sqrt{2}}$; (d) $-\pi i/6$; (e) 0.

2. 2/9. **3.** $24\pi i$.

5. (a) $kg(a)$; (b) $-kg(a)$.

6. (a) $z = 1$, simple pole; (b) $z = 0$, pole of order 5;

(c) $z = 0$, pole of order 3, $z = n\pi$, $n \in \mathbb{I}_0$, simple poles;

(d) simple poles: $e^{2\pi ik/5}$, $2ie^{2\pi ik/5}/5\sin(k\pi/5)$, $k = 1, 2, 3, 4$;

(e) simple poles: $z = n\pi$, $(-1)^{n+1}-1$, $n \in \mathbb{I}$, poles of order 2 at: $z = (4n-3)\pi/2$, $n \in \mathbb{I}$, $(4+12i)/3$;

(f) Simple poles: $2n\pi$, $n \in \mathbb{I}$, -2;

(g) $z = 2/(2n+1)\pi$, $n \in \mathbb{I}$ simple poles.

(h) Simple poles: $z = 2n\pi i$, $e^{ai2n\pi}$, $n \in \mathbb{I}$.

8. (a) Pole of order 2, residue: 0; (b) pole of order 3, residue : -3; (c) pole of order 4, residue: 6; (d) removable singularity.

11. $z = \pm 1$, $\operatorname{Res}_{z=1} f(z) = \dfrac{1-2\pi i}{4}$, $\operatorname{Res}_{z=-1} f(z) = \dfrac{1-i\pi}{4}$.

12. $z = (n + 1/2)$, $(-1)^{n/2}/\pi$, $n \in \mathbb{I}$.

13. (a) $-\dfrac{2}{z} + \dfrac{3}{z+4} + \dfrac{1}{z^2}$; (b) $\dfrac{1}{z} - \dfrac{3}{2(z+1)} - \dfrac{2}{(z+1)^2} - \dfrac{5z+3}{2(z^2+1)}$.

16. No exceptional value (except ∞).

Problem Set 7.2

1. (a) Essential sigularity; (b) essential singularity (c) simple pole; (d) pole of order two.

2. (a) 1; (b) 3/2; (c) $a - b$ for all branches; (d) $(\alpha - \beta)^2/8$.

3. $-2a_0a_1$.

5. (a) 1; (b) 0.

7. (a) $32\pi i/63^2$; (b) $2\pi i/(1 - 5^{17})$.

8. Not, counter example $f(z) = z + 1/z$, assumption: $f(z)$ must be analytic at $z = \infty$.

12. f must be a polynomial of degree $\leq n - 1$.

14. $|z| > \max\{|a|, |b|\}$.

Problem Set 7.3

1. $-4\pi i$.

9. (a) No zeros; (b) 3.

Problem Set 7.4

5. (a) $\operatorname{cosech} z = \dfrac{1}{z} + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2z}{z^2 + n^2\pi^2}$; (b) $\tanh z = \displaystyle\sum_{n=1}^{\infty} \frac{2z}{(n-1/2)^2\pi^2 - z^2}$.

8. $\pi \cot \pi z = \dfrac{1}{z} + \displaystyle\sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$.

Problem Set 7.5

3. $2iz \displaystyle\sum_{n=0}^{\infty} i^n[z - (2+i)]^n$, $|z - (2+i)| < 1$, $-2 + 4i$.

8. $f(z) = z + 1/z$, $z \neq 0$.

CHAPTER 8

Problem Set 8.1

4. $\frac{\pi}{4}\coth\pi + \frac{\pi^2}{4}\operatorname{cosech}^2\pi - \frac{1}{2}$.

5. (a) $\frac{\pi\coth(\pi\sqrt{a/b})}{2\sqrt{ab}} + \frac{1}{a}$; (b) $\frac{1}{2}\left[1 + \frac{\pi\sqrt{3}}{3}\tanh\left(\frac{\pi\sqrt{3}}{2}\right)\right]$;

(c) $\frac{\pi}{4}\frac{\sin 2b + \sinh 2b}{\cosh 2b - \cos 2b}$, $b = \frac{a\pi}{4}$; (d) $\frac{\pi^2}{6}$.

Problem Set 8.2

1. (a) $\frac{6\pi}{5^{3/2}}$; (b) $\frac{3\pi}{2\sqrt{2}}$; (c) $\frac{\pi}{\sqrt{2}}$; (d) 0. **2.** (a) $\frac{\pi}{\sqrt{1-k^2}}$; (b) $\frac{\pi a^2}{1-a^2}$.

4. $\frac{\pi}{2\sqrt{a(a-1)}}$.

6. For $b = 0$, integral is zero, for $a < 0$, $-2\pi(\sqrt{a^2-b^2} - a)^n/(b^n\sqrt{a^2-b^2})$.

10. (a) $\frac{\pi}{6}$; (b) $\frac{2\pi}{b^2}\left(a - \sqrt{a^2-b^2}\right)$; (c) $\frac{4\pi}{9}$; (d) $\frac{\pi a(a-\sqrt{a^2-b^2})}{b^2\sqrt{a^2-b^2}}$.

Problem Set 8.3

1. (a) $\frac{\sqrt{2}-\sqrt{6}}{4\sqrt{3}}$; (b) π; (c) $\frac{\pi}{3}$; (d) $\frac{\pi}{8a^3}$. **2.** (a) $-\frac{\pi}{5}$; (b) $\frac{2\pi}{ab(a+b)}$.

4. $\frac{\pi(a+2b)}{2ab^3(a+b)^2}$. **6.** (a) $\frac{\pi\cos 2}{e^2}$; (b) $-\pi e^{-1}\sin 2$.

7. $\operatorname{sgn}(m)\pi e^{-|m|a}$, $\operatorname{sgn}(m) = 1, 0 - 1$ if $m > 0$, $m = 0$, $m < 0$, respectively.

8. (a) $\pi e^{-1}(\sin 2 - \cos 2)$; (b) $\frac{\pi}{3e^3}$. **9.** $\frac{\pi}{2a^3}(ma^2 - 1)e^{-ma}$.

Problem Set 8.4

1. (a) $2\pi/5$; (b) $-\pi/\sqrt{3}$. **3.** π; (b) $\pi a^{-1}\sin a$. **7.** $\pi[1-(1+m/2)e^{-m}]$.

Problem Set 8.5

4. $\dfrac{1}{2a} - \dfrac{\pi}{2\sinh \pi a}$. **5.** $\dfrac{2\pi e^{-a}}{4a^2+\pi^2} - \dfrac{1}{1+e^{2a}}$. **8.** $\dfrac{\pi^2}{4m\,|\,m\,|}$.

11. $(\pi/n)\sin(\pi/n)$.

Problem Set 8.6

2. 0. **3.** $\pi \cot a\pi$. **5.** $\dfrac{(1-q)\pi}{4\cos(q\pi/2)}$.

8. (a) $|z| > 0$, $-\pi/2 < \arg z < 3\pi/2$, $\pi^2/6$; (b) $|z| > 0$, $0 < \arg z < 2\pi$, $\dfrac{(\sqrt[3]{3}-1)\pi}{\sqrt{3}}$.

8. $\pi \ln 2$.

CHAPTER 9

Problem Set 9.1

1. $u + v > 1$. **2.** $-1 < u < 0$ $(v > 0)$.

3. (a) $w = (2+i)z + 1 - 3i$; (b) $w = 2(z-1) + 3i$.

4. $\left(u - \dfrac{1}{2}\right)^2 + v^2 > \left(\dfrac{1}{2}\right)^2$, $u > 0$, $v > 0$.

6. (a) Δ' is a triangle with magnification of sides 3 times and translated to vertices $4+i$, $7-5i$, $7+i$;

(b) Δ' is a triangle with rotation $\pi/2$ and translated to vertices $1-i$, 3, 1.

7. Curvilinear triangle with vertices -1, 1, $2i$ and sides as $v = (1-u^2)/2$, $u = 1 - v^2/4$, $u = -1 + v^2/4$.

9. For part (b), when $a = c$, the image curve is a cardioid: $\rho = 2a^2(1 + \cos\phi)$.

11. Parabola $\rho\cos^2(\phi/2) = 1$.

12. The given region is mapped on the annular region $a^2 \le |w| < b^2$ in the upper half-plane.

15. $w = \dfrac{1+i}{\sqrt{2}}z$; semi-major axis $= \sqrt{2}$, semi-minor axis $= 2/\sqrt{3}$.

Problem Set 9.2

1. $2\sqrt{2}$, $-3\pi/4$.

2. Image of $x = 0$ is $u = 0$, image of $y = x - 1$ is $u^2 + v^2 - u + v = 0$. Angles between the curves in the z-plane and image curves in the w-plane are $-\pi/4$.

3. Scale factor: $n|z_0|^{n-1}$, Angle of rotation: $n\theta_0$ at $z_0 = |z_0|e^{i\theta_0}$.

4. $(n + 1/2)\pi$, $n \in \mathbb{I}$.

7. (a) $z = w^{1/2}(|w| > 0, -\pi < \arg w < \pi)$; (b) $z = w^{1/2}(|w| > 0, 0 < \arg w < 2\pi)$; (c) $z = w^{1/2}(|w| > 0, 2\pi < \arg w < 4\pi)$.

Problem Set 9.3

1. $u^2 + v^2 - 2u + v + 1 = 0$, centre $(1, -1/2)$, radius $1/2$.

2. $u \le -1/2$.

6. $\dfrac{az+b}{-bz+a}$, $a^2 + b^2 \neq 0$.

7. $\dfrac{(a-d)z+2b}{2bz-(a-d)}$, $(a-d)^2 - 4b^2 \neq 0$.

8. $\dfrac{z+i}{iz-1}$.

12. (a) i/z, $u = 0$; (b) $z = \dfrac{w-w_1}{w_1-w_2}\,\dfrac{w_2-w_3}{w_3-w}$.

13. $\dfrac{1+iz}{1-iz}$, (a) $\operatorname{Re} w \ge 0$; (b) $\left|w - \dfrac{1+r^2}{1-r^2}\right| = \dfrac{2r}{1-r^2}$.

14. $|a| = |c|$.

20. $w - \alpha = \beta(z-\alpha)$, β: some constant; $w = z + c$, c: some constant.

Problem Set 9.4

2. $w = e^{i\alpha}\dfrac{z^3 - z_0}{z^3 - \bar{z}_0}$, $\operatorname{Im} z_0 > 0$, $\alpha \in \mathbb{R}$.

3. $e^{i\alpha}\dfrac{z^2 + z_0}{z^2 + \bar{z}_0}$, $\operatorname{Im} z_0 > 0$, $\alpha \in \mathbb{R}$.

4. $w = \dfrac{az+b}{cz+d}$; for (a) $ad - bc > 0$; for (b) $ad - bc < 0$.

5. $w = \dfrac{i-z}{i+z}$.

8. $w = \dfrac{z-\beta}{1-\beta z}$, $-1 < \beta < 1$.

9. $w = e^{i\alpha}\dfrac{(z-1-2i)^4 - z_0}{(z-1-2i)^4 - \bar{z}_0}$, $\operatorname{Im} z_0 > 0$, $\alpha \in \mathbb{R}$.

10. $w = i\dfrac{z-2i}{z+2i}$.

12. $w = e^{i\alpha}\dfrac{z^3 - z_0}{1 - \bar{z}_0 z^3}$, $|z_0| < 1$, $\alpha \in \mathbb{R}$.

13. Two; $\pm\alpha/|\alpha|$.

15. $w = 2e^{i\alpha}\dfrac{(1+i)(z-1)-z_0}{(1+i)(z-1)-\bar{z}_0} + 1$, $\operatorname{Im} z_0 > 0$, $\alpha \in \mathbb{R}$, it is not unique as it depends on α. On taking $\alpha = \pi/2, \pi$, we get two such bilinear maps;

$$w = 2e^{i\alpha}\frac{z+z_0}{z+\bar{z}_0} + 1,\ \operatorname{Im} z_0 > 0, \qquad \operatorname{Im} z_0 > 0,\quad \alpha \in \mathbb{R}.$$

Problem Set 9.5

1. Interior and boundary of unit semi-circle above the real axis (except at the origin).
2. Annular region: $e^{-2} < |w| < 1$, cut along the negative real axis.
6. $w = -i(z+1)^3$.
9. $u^2 + v^2 + 2u \cot 2 - 1 = 0$, $u^2 + v^2 - 2v \coth 2 + 1 = 0$.
14. Airfoil having sharp tail at -2 and some portion of the real axis removed.

CHAPTER 10

Problem Set 10.1

2. No, the function is not piecewise continuous.
6. (b) and (d).
7. Converges for $\operatorname{Re} p > 0$.

8. (a) $\dfrac{2}{p(p^2+4)}$; (b) $\dfrac{p(p^2+7)}{(p^2+1)(p^2+4)}$;

(c) $\dfrac{1}{1-e^{-\pi p}}\left[\dfrac{p+e^{-\pi p/2}+(1-p)e^{-\pi p}}{p^2+1}\right]$; (d) $\dfrac{a(p^2+a^2)}{p^4+4a^4}$.

9. (a) $\dfrac{4!}{p^5}+\dfrac{4!}{p^4}+6\dfrac{2!}{p^3}+\dfrac{4}{p^2}+\dfrac{1}{p}$; (b) $\dfrac{36}{(p+1)^4}+\dfrac{4}{(p-2)^2+16}$;

(c) $\dfrac{ap^2}{p^4+4a^4}$; (d) $\dfrac{4(p-2)}{(p^2-4p+8)^2}$.

10. (a) $2e^{-p}\left(\dfrac{1}{p^3}+\dfrac{1}{p^2}+\dfrac{1}{p}\right)$; (b) $\dfrac{1-e^{-2\pi p}}{p^2+1}$;

(c) $\dfrac{4e^{-3p}}{p^2}-\left(\dfrac{2}{p}+\dfrac{3}{p^2}\right)e^{-2p}$; (d) $e^{-3\pi p/2}\left(\dfrac{1}{p}-\dfrac{1}{p^2+1}\right)-\dfrac{e^{-\pi p/2}}{p^2+1}$.

11. (a) $\dfrac{3}{4p^2}\sqrt{\dfrac{\pi}{p}}$; (b) $\dfrac{8p}{(1+p^2)^3} - \dfrac{\pi}{p^2}$; (c) $\dfrac{\sqrt{\pi}}{2p^{3/2}}e^{-1/4p}$; (d) $\sqrt{\dfrac{\pi}{p}}e^{-1/4p}$.

13. $\dfrac{1-e^{-p}}{p(1+e^{-2p})}$.

15. (a) $\pi/2$; (b) $\ln(b/a)$; (c) $\sqrt{\pi}$; (d) 2.

17. (a) $\dfrac{k}{ap^2} - \dfrac{ke^{-ap}}{p(1-e^{-ap})}$; (b) $\dfrac{ke^{-ap}}{p(1-e^{-ap})}$.

19. (a) $\dfrac{\ln(p+1)}{p}$; (b) $\dfrac{1}{p}\tan^{-1} p$; (c) $\dfrac{\ln(p^2+1)}{2p}$; (d) $\dfrac{(p-1)^n}{p^{n+1}}$.

20. $\dfrac{1}{p^2}\tanh\left(\dfrac{ap}{2}\right)$, $\dfrac{e^{-ap}}{1-e^{-2ap}}\left(\dfrac{e^{ap}}{p^3} - \dfrac{1}{p^2} - \dfrac{a}{p}\right)$.

Problem Set 10.2

1. (a) $-\dfrac{2}{25} + \dfrac{1}{5}t + \dfrac{2}{25}e^{-5t} + \dfrac{1}{5}te^{-5t}$; (b) $\dfrac{e^{2t}}{4}(8\cos 4t + 7\sin 4t)$; (c) $e^{-t} - \cos t + \sin t$; (d) $\cos t - \cos 2t$.

2. (a) $f(t) = 0,\ 0 \le t < 3,\ f(t) = e^{-3(t-3)}\sin(t-3),\ t > 3$;

(b) $f(t) = \dfrac{\sin 2t}{2},\ 0 \le t < \pi,\ f(t) = 0,\ t > \pi$;

(c) $f(t) = 0,\ 0 \le t < 2,\ f(t) = \sqrt{3}\sin\dfrac{t-2}{\sqrt{3}},\ t > 2$;

(d) $f(t) = 0,\ 0 \le t < \pi,\ f(t) = e^{-t(t-\pi)/2}\cosh\dfrac{\sqrt{5}}{2}(t-\pi),\ t > \pi$.

3. (a) $\dfrac{2(e^t - \cos t)}{t}$; (b) $\dfrac{e^{-bt} - e^{-at}}{t}$; (c) $\dfrac{2(1-\cos t)}{t}$; (d) $\dfrac{2(\sinh t - t\cosh t)}{t^2}$.

4. (a) $\dfrac{\sin \omega t}{t}$; (b) $\dfrac{e^{-t}\sin t}{t}$; (c) $\dfrac{\sin at + \sin bt}{t}$; (d) $\dfrac{2\sin t\sinh t}{t}$.

5. $\dfrac{1}{2a^3}(at\cosh at - \sinh at)$.

6. (a) $\sinh 2t - 2t$; (b) $1 + t - \cos t - \sin t$; (c) $\cos\sqrt{2}t + \cosh\sqrt{2}t$.

14. (a) $\dfrac{e^{2t}}{27}(t-1) + \dfrac{e^{-t}}{54}(3t^2 + 4t + 2)$; (b) $\dfrac{1}{4}\sin 2t + \dfrac{1}{2}\sin t$;

(c) For $n = 0, 1, \ldots, f(t) = 0, 2n < t < 2n + 1, f(t) = 1, 2n + 1 \le t \le 2n + 2$;

(d) $e^{t/2}\left(\cos \frac{\sqrt{3}}{2}t + \sqrt{3}\sin\frac{\sqrt{3}}{2}t - e^{-3t/2}\right)$.

CHAPTER 11

Problem Set 11.1

6. (a) $u(r, t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{r^n \sin(2n-1)t}{2n-1}$, $r \in [0, 1]$;

(b) $u(r, t) = -2\sum_{n=1}^{\infty} \frac{r^n \sin nt}{n}$, $r \in [0, 1]$.

Problem Set 11.2

3. $\sin(a - z) = \sin a\, e^{-z\cot a} \prod_{n=-\infty}^{\infty}\left(1 - \frac{z}{a + n\pi}\right) e^{z/(a+n\pi)}$.

10. Simple poles at $z = -n$, $n = 0, -1, \ldots$; $\mathrm{Res}_{z=-n} = -1$.

11. Yes, In view of the Weierstrass factorization theorem.

Problem Set 11.3

1. (a) $\rho = 1$, $\sigma = 3$; (b) $\rho = 2$, $\sigma = \sqrt{5}$; (c) $\rho = 1$, $\sigma = \sqrt{2}$; (d) $\rho = 1/2$, $\sigma = 1$.

2. $\rho = 1/2^m$, $\sigma = 1$.

3. $\rho = 1$, $\sigma = 1$.

4. Order of $(f_1 + f_2) \le \max\{\rho_1, \rho_2\}$, ρ_1 = order of f_1, ρ_2 = order of f_2.

13. $a_n = n^{n^{1+\delta}}$, $\delta < 0$.

14. $a_n = e^{n\delta \ln n}$, $0 < \delta < 1$.

Problem Set 11.4

1. Octant $0 \le \theta \le \pi/4$ with the origin.

7. $\frac{[\Gamma(\alpha/n)]^2}{\Gamma(2\alpha/n)}$.

8. $w = [\Gamma(3/4)/\sqrt{\pi}\Gamma(1/4)\int_0^z x^{-1/2}(1-x)^{-3/4}dx$.

10. $z = \sin\left(\frac{\pi w}{a}\right)$.

11. $w = A\int (z - z_1)^{(\alpha_1/\pi)-1}(z - z_2)^{(\alpha_2/\pi)-1} \cdots (z - z_n)^{(\alpha_n/\pi)-1}dz + B$, A and B are constants.

15. (a) $z = a\cos w + ib\sin w$; (b) $z = 2(\cos w + i\sin w)$.

Answers to Objective-type Questions

1. (c) **2.** (b) **3.** (d) **4.** $i\sqrt{2}$ **5.** $-\pi/4$ **6.** $r_0e^{i\theta_0+2\pi/3}$, $r_0e^{i\theta_0+4\pi/3}$ **7.** zero **8.** 1 **9.** -1 **10.** (d) **11.** ellipse **12.** (d) **13.** $\{0, 1/n, i/n, n \in \mathbb{I}_0\}$ **14.** $0 \le \text{Arg } z \le \pi/2 \cup \{0\}$ **15.** zero **16.** (b) **17.** (b) **18.** (c) **19.** $-2/z^3$ **20.** $ru_r = v_\theta$, $u_\theta = -rv_r$ **21.** (c) **22.** (c) **23.** (a) **24.** (a) **25.** (c) **26.** $2(y^2-x^2) + c$ **27.** $-i(2z^2 + c) + 3$ **28.** $-\cosh x \cos y + c$ **29.** (a) no (b) yes (c) no (d) no (e) no (f) yes (g) yes (h) no (i) no (j) no **30.** (c) **31.** $z = n\pi$, $n \in \mathbb{I}$ **32.** $(2n + 1)\pi i$, $n \in \mathbb{I}$ **33.** (c) **34.** (a) **35.** (b) **36.** $\{2i, -2i, y = 1(x \le 0)\}$ **37.** 1, $(1 + \sqrt{3})/2$, -1, $(1 - \sqrt{3})/2$ **38.** $\{n\pi i, n \in \mathbb{I}\} \cup \{y = 0(x \le 0)\}$ **39.** (d) **40.** $\{-2, -3/2, y = 0(x \le -3/2)\}$ **41.** (c) **42.** $2\pi i/13$ **43.** 4π **44.** (d) **45.** (b) **46.** (b) **47.** (b) **48.** (a) **49.** (b) **50.** (c) **51.** (c) **52.** $z_1 + z_2 = 2n\pi$ or $z_1 - z_2 = (2n + 1)\pi$, $n \in \mathbb{I}$ **53.** (b) **54.** (c) **55.** (c) **56.** $z_1 + z_2 = 2n\pi i$ or $z_1 - z_2 = (2n + 1)\pi i$ **57.** $n\pi i + i \ln 3$, $n \in \mathbb{I}$ **58.** $i(n + 1/2)\pi$, $n \in \mathbb{I}$ **59.** $\ln 3 + i(2n + 1/2)\pi/2$, $n \in \mathbb{I}$ **60.** (b) **61.** $i(n - 1/2)\pi$, $n \in \mathbb{I}$. **62.** $e^{i\sqrt{2}(2n+1/2)\pi}$, $n \in \mathbb{I}$ **63.** $(1 - i)e^{\pi/4}$ **64.** (b) **65.** $e^{2n\pi} \cos(\ln\sqrt{2})$, $n \in \mathbb{I}$ **66.** zero **67.** $(2n - 1/2)\pi$, $n \in \mathbb{I}$ **68.** (c) **69.** (c) **70.** (d) **71.** (b) **72.** $x \le 1(y = 1)$ **73.** $z = 1 + i$ **74.** (d) **75.** $\{z = x + iy : y = 1(x \le 0\}$ **76.** (c) **77.** $\mathbb{C}\backslash\{0\}$ **78.** $\mathbb{C}\backslash\{0\}$ **79.** $\mathbb{C}\backslash\{\pm i\}$ **80.** (a) false (b) false (c) false (d) true (e) true (f) true (g) true (h) true (i) false (j) false **81.** $4\pi e^2/3$ **82.** $3\pi/256$ **83.** zero **84.** $6\pi i$ **85.** $4\pi i(5\sin 1 + 6\cos 1)$ **86.** (d) **87.** (b) **88.** $-\pi$ **89.** zero **90.** (a) **91.** zero **92.** (c) **93.** (a) **94.** (d) **95.** $14/9\pi$ **96.** (b) **97.** (b) **98.** (d) **99.** $i(n\pi + 1)/3$, $n \in \mathbb{I}$ **100.** (b) **101.** (a) **102.** (c) **103.** (b) **104.** $\frac{1}{2}\log\frac{1+z}{1-z}$ **105.** $(3-2z)(1-z)^{-2}$ **106.** (d) **107.** (b) **108.** zero **109.** 1 **110.** $(1-\sqrt{3})/2$, -1, $(1 + \sqrt{3})/2$ **111.** $\frac{1}{2}\frac{d^3}{dz^3}[(z-a)^3 f(z)]$ at $z = z_0$ **112.** -1 **113.** 1 **114.** $(-1)^n$, $n \in \mathbb{I}$ **115.** (b) **116.** (a) false (b) true (c) true (d) false (e) true (f) false (g) true (h) true (i) true (j) false (k) true **117.** $(z_1-z_2)(z_3-z_4)/(z_2-z_3)(z_4-z_1)$ **118.** $\pm i$ **119.** $(a + b)/2$ **120.** (b) **121.** (c) **122.** $2u + 1 = 0$ **123.** (d) **124.** (c) **125.** $\cosh z$ **126.** (a) **127.** (b) **128.** (b) **129.** (a) **130.** (b) **131.** (d) **132.** (b) **133.** constant **134.** (a) false (b) true (c) true (d) false (e) true (f) true **135.** $t^{n-1}/(n-1)!$ **136.** e^{-ap} **137.** (c) **138.** 2/13 **139.** ln 3 **140.** $p^2L[y] - py(0) - y'(0)$ **141.** Euler's constant **142.** $1/\sqrt{2}$ **143.** (a) **144.** (d) **145.** (b) **146.** $1/e$ **147.** $(-1, 0)$.

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