SOLUTION FOR HOMEWORK 8, STAT 6331

1. Exerc.8.2 Here $X \sim Poisson(\lambda)$ with $\lambda = 15$. We observe $X = x = 10$ while a historical "opinion" is that $\lambda = 15$. Let us calculate the probability $X \le x$,

$$\mathrm{P}_{\lambda=10}(X \le x) = \sum_{i=1}^{10} \frac{e^{-15}(15)^i}{i!} = 0.12.$$

This is the probability that the drop in accidents occured due to stochastic fluctuations. Then it is up to you to make a conclusion...

Note that here $H_0 : \lambda = 15$ and there is an alternative hypothesis $H_a : \lambda < 15$ which is supported by the data.

Remark: You may use a normal approximation as well - it impliesabout the same probability.

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2. Exerc. 8.3. Let $Y_1, \ldots, X_m$ be iid according to $Bernoulli(\theta)$. Test $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$ using the LRT.

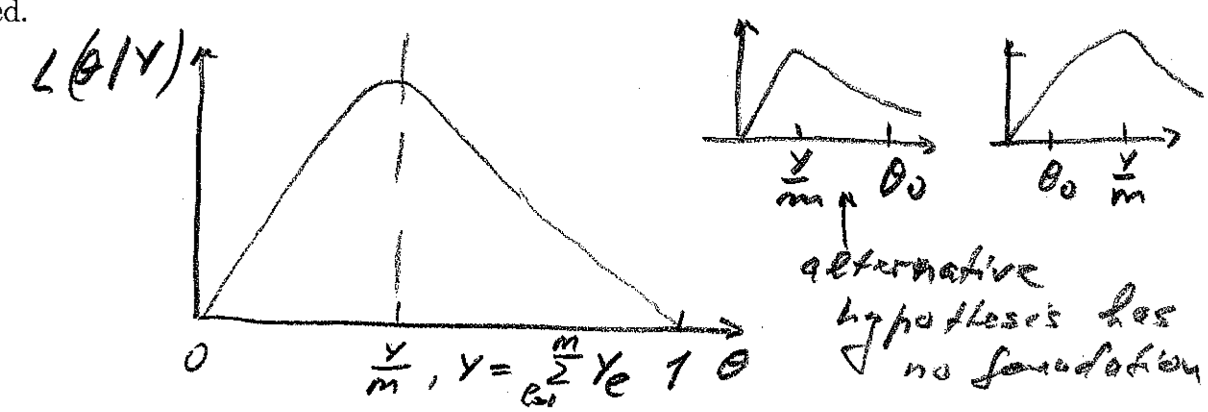
Solution: Let us look at the likelihood ratio test statistic

$$\lambda(\mathbf{Y}) = \frac{\sup_{\theta \in [\theta_0, 1]} L(\theta|\mathbf{Y})}{\sup_{\theta \in [0,1]} L(\theta|\mathbf{Y})}$$

$$= \frac{\sup_{\theta \in [\theta_0, 1]} \prod_{l=1}^m \theta^{Y_l}(1-\theta)^{1-Y_l}}{\sup_{\theta \in [0,1]} \prod_{l=1}^m \theta^{Y_l}(1-\theta)^{1-Y_l}}$$

$$= \frac{\sup_{\theta \in [\theta_0,1]} \exp\{(\sum_{l=1}^m Y_l)\ln(\theta/(1-\theta)) + m\ln(1-\theta)\}}{\sup_{\theta \in [0,1]} \exp\{(\sum_{l=1}^m Y_l)\ln(\theta/(1-\theta)) + m\ln(1-\theta)\}}.$$

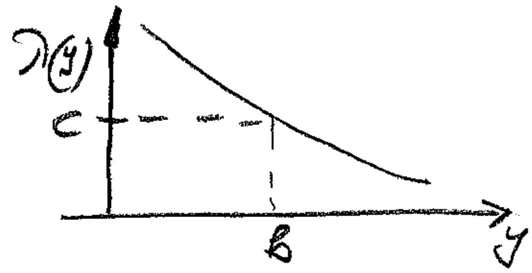
Here in the denominator we have a classical problem of seraching after the MLE which we did many times. In any case, the likelihood function looks like shown below (check this via derivatives) and note that $\bar{Y} > \theta$ because otherwise the alternative hypothesis could not be proposed.



With some straightforward algebra (based on the following: if $\psi(\theta) = a\ln(\theta) + (1 - a)\ln(1-\theta)$ then $\psi'(\theta) = a/\theta - (1-a)/(1-\theta) = 0$ yields the solution $\theta^* = a$)

$$\lambda(\mathbf{Y}) = \lambda(\bar{\mathbf{Y}}) = \begin{cases} 1 \text{ if } \bar{Y} \le \theta_0 \\ \frac{\theta_0^{m\bar{Y}}(1-\theta_0)^{1-m\bar{Y}}}{\bar{Y}^{m\bar{Y}}(1-\bar{Y}^{m-m\bar{Y}})} \text{ if } \bar{Y} > \theta_0 \end{cases}$$

This we reject H_0 if (note that $\lambda(\bar{Y}) \leq 1$)



$$\lambda(\bar{Y}) < c$$

which is equivalent to

$$\frac{\theta_0^{m\bar{Y}}(1-\theta_0)^{m(1-\bar{Y})}}{\bar{Y}^{m\bar{Y}}(1-\bar{Y})^{m(1-\bar{Y})}} < c.$$

To finish the problem we need to show that $\lambda(y)$ is decreasing in y. To see this consider

$$\mu(y) := \ln(\lambda(y)) = my\ln(\theta_0) + m(1-y)\ln(1-\theta_0) - my\ln(y) - m(1-y)\ln(1-y)$$

$$= my\ln(\theta_0/(1-\theta_0)) - my\ln(y/(1-y)) + \ln(1-\theta_0) - m\ln(1-y).$$

Now we are taking the derivative,

$$d\mu(y)/dy = \ln(\theta_0/(1-\theta_0)) - \ln(y/(1-y)) - y\frac{m^{-1}(1-y)+y/m}{(1-y)^2}\frac{1-y}{y} + m\frac{y}{1-y} = \ln(\frac{\theta_0(1-y)}{y(1-\theta_0)}).$$

We consider the case $\theta_0 < y$ and this yields that the derivative is negative. Thus $\lambda(y) < c$ is equivalent to $y > c'$. What was wished to show.

3. Exers. 8.5 Let $X_1, \ldots, X_n$ are from $Pareto(\theta, \nu)$ with the pdf

$$f(x|\theta,\nu) = \theta\nu^\theta x^{-(\theta+1)} I(x \geq \nu), \quad \theta > 0, \nu > 0.$$

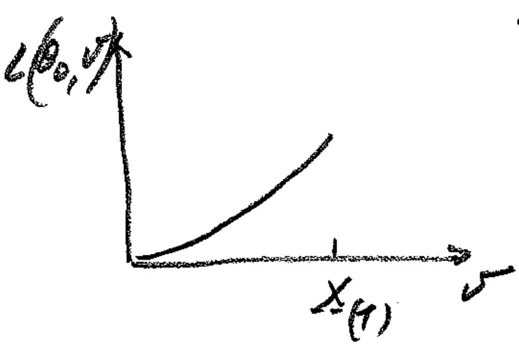
Note that ν is the scale parameter.

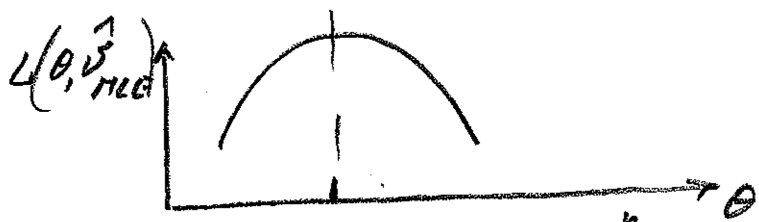
(a) Find the MLEs of θ and ν.

Solution: Write

$$L(\theta,\nu) = \frac{\theta^n \nu^{n\theta}}{\prod_{l=1}^n X_l^{\theta+1}} I(X_{(1)} \geq \nu) = \theta^n \nu^{n\theta} e^{-(\theta+1)\sum_{l=1}^n \ln(X_l)} I(X_{(1)} \geq \nu).$$

Let us look at graphics:





$$\frac{n}{\ln(\prod X_l / X_{(1)})} = \frac{n}{Y},$$

$$Y = \sum_{l=1}^{n} \ln\left(\frac{X_l}{X_{(1)}}\right)$$

We easily see that $\hat{\nu}_{MLE} = X_{(1)}$. To find the MLE for θ we take derivative (note that I set $\nu = \hat{\nu}_{MLE}$

$$d\ln(L(\theta,\hat{\nu}_{MLE}))/d\theta = n\theta^{-1} + n\ln(X_{(1)}) - \sum_{l=1}^{n}\ln(X_l)$$

and the extreme point (check via second derivative that it is also the maximum point) is $\hat{\theta}_{MLE} = n/\sum_{l=1}^{n} \ln(X_l/X_{(1)})$.

(b) Let us find the LRT for $H_0 : \theta = 0$ and ν is unknown versus $H_1 : \theta \neq 1$ and ν is unknown.

Denote

$$T := \ln(\prod_{l=1}^{n} X_l/(X_{(1)})^n).$$

Well, we know that $(X_{(1)}, \ln(\prod_{l=1}^{n} X_l)$ is sufficient (actually it is MSS) so a test should depend on this statistic. Also, from (a) we have that $\hat{\theta}_{MLE} = n/T$.

Under H_0 we have $\hat{\theta}_{MLE=1}$ because (do you recall it?)

$$\hat{\theta}_{MLE} = \text{argmax}_{\theta \in \{1\}} L(\theta|\mathbf{X}) = 1.$$

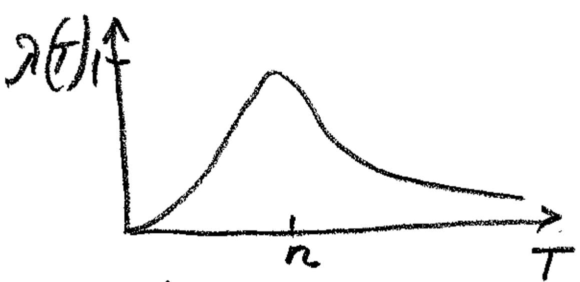
And we know that $\hat{\nu}_{MLE} = X_{(1)}$.

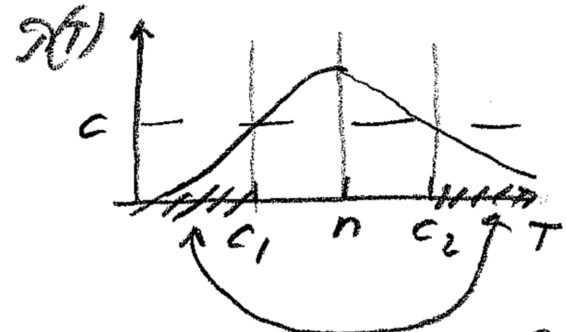
Thus the likelihood ratio statistics is

$$\lambda(\mathbf{X}) = \frac{\sup_{\theta \in \{1\}, \nu > 0} f(\mathbf{X}|\theta, \nu)}{\sup_{\theta > 0, \nu > 0} f(\mathbf{X}|\theta, \nu)} = \frac{f(\mathbf{X}|\theta = 1, \nu = X_{(1)})}{f(\mathbf{X}|\theta = n/T, \nu = X_{(1)})}$$

$$= \frac{X_{(1)}^n / \prod_{l=1}^{n} X_l^2}{(n/T)^n X_{(1)}^{n^2/T} / \prod_{l=1}^{n} X_l^{n/T+1}} = (T/n)^n \frac{e^{-T}}{(e^{-T})^n T} = (T/n)^n e^{-T+n}.$$

Let us look at graphics. Note that $d \ln(\lambda(t)/dt = n/t - 1$ and the derivative is nonnegative if $T \leq n$ and negative otherwise.





because

$$\frac{\partial \ln \lambda(t)}{\partial T} = \frac{n}{T} - 1 = \begin{cases} \geq 0, & T \leq n \\ < 0 & T > n \end{cases}$$

reject for LRT

4. Exerc. 8.6 Here $X_1, \ldots, X_n$ are from $Expon(\theta)$ and $Y_1, \ldots, Y_m$ are from $Expon(\mu)$. Remember that $f(x|\theta) = \theta^{-1} e^{-x/\theta}$, and $E(X) = \theta$, $\theta > 0$.

(a) We test $H_0 : \theta = \mu$ versus $H_a : \theta \neq \mu$. The likelihood ratio statistic is (remember that $z^k := (z_1, \ldots, z_k)$)

$$\lambda(x^n, y^m) = \frac{\sup_{\theta > 0} f(x^n|\theta) f(y^m|\theta)}{\sup_{\theta > 0} f(x^n|\theta) \sup_{\mu > 0} f(y^m|\mu)}$$

$$= \frac{\sup_{\theta>0} \theta^{-n-m} e^{-(sum_{l=1}^{n} x_l + \sum_{i=1}^{m} y_i)/\theta}}{\sup_{\theta>0} \theta^{-n} e^{-sum_{l=1}^{n} x_l/\theta} \sup_{\mu>0} \mu^{-m} e^{-sum_{l=1}^{n} y_l/\mu}}$$

To find the sup's, we take derivatives of correspond log-likelihoods (and then check the negativity of second derivatives). Note that

$$\frac{\partial}{\partial\theta} \ln(\theta^{-a} e^{-t/\theta}) = -a/\theta + t/\theta^2.$$

This the extreme point is $\theta = t/a$ and it is plainly the maximum point. Using this result we conclude that

$$\lambda(x^n, y^m) = \frac{(\frac{\sum_{l=1}^{n} x_l + \sum_{l=1}^{m} y_l}{n+m})^{-n-m} e^{-(\sum_{l=1}^{n} x_l + \sum_{l=1}^{m} y_l)/[(\sum_{l=1}^{n} x_l + \sum_{l=1}^{m} y_l)/(n+m)]}}{(\sum_{l=1}^{n} x_l/n)^{-n} e^{-\sum_{l=1}^{n} x_l/[\sum_{j=1}^{n} x_j/n]} (\sum_{l=1}^{n} y_l/n)^{-n} e^{-\sum_{l=1}^{m} y_l/[\sum_{j=1}^{n} y_j/m]}}$$

$$= \frac{(n+m)^{n+m}}{n^n m^m} \frac{(\sum_{l=1}^{n} x_l)^n (\sum_{l=1}^{m} y_l)^m}{(\sum_{l=1}^{n} x_l + \sum_{l=1}^{m} y_l)^{n+m}}.$$

And the LRT should reject H_0 iff $\lambda(x^n, y^m) \le c$.

(b) Write,

$$\lambda(x^n, y^m) = \frac{(n+m)^{n+m}}{n^n m^m} \frac{(\sum_{l=1}^{n} x_l)^n (\sum_{l=1}^{m} y_l)^m}{(\sum_{l=1}^{n} x_l + \sum_{l=1}^{m} y_l)^{n+m}}$$

$$= \frac{(n+m)^{n+m}}{n^n m^m} (\frac{\sum_{l=1}^{n} x_l}{\sum_{l=1}^{n} x_l + \sum_{l=1}^{m} y_l})^n (1 - \frac{\sum_{l=1}^{n} x_l}{\sum_{l=1}^{n} x_l + \sum_{l=1}^{m} y_l})^m.$$

(c) Note that $\sum_{l=1}^{n} X_l \sim Gamma(n, \theta)$, $\sum_{l=1}^{m} Y_l \sim Gamma(m, \theta)$, and we know that

$$\frac{Gamma(n, \theta)}{Gamma(n, \theta) + \Gamma(m, \mu)} \sim Beta(n, m).$$

Here the two Gamma's should be independent!

5. Exerc. 8.10 Given that X^n is a sample from $Poisson(\lambda)$, and under a Bayesian approach we have $\Lambda \sim Gamma(\alpha, \beta)$. Consider a bauesian test for $H_0 : \lambda < \lambda_0$ versus $H_a : \lambda > \lambda_0$.

Solution: As we know, a Bayesian test rejects H_0 iff

$$\mathrm{P}(\Lambda \le \lambda_0 | X^n) < \mathrm{P}(\Lambda > \lambda_0 | X^n)$$

which is equivalent to

$$\mathrm{P}(\Lambda > \lambda_0 | X^n) > 1/2.$$

Remember that if $Y := \sum_{l=1}^{n} X_l$, then according to Exerc 7.24 from our previous homework,

$$f_{\Lambda|X^n}(\lambda|x^n) \sim Gamma(y + \alpha, \beta/(n\beta + 1)).$$

Now we can solve requested problems.

(a) Write,

$$\mathrm{P}(\Lambda > \lambda_0|y) = [(b+1)/\beta]^{y+\alpha}[1/\Gamma(y+\alpha)] \int_{\alpha_0}^{\infty} u^{y+\alpha-1} e^{-u(n\beta+1)/\beta} du$$

and $\mathrm{P}(\Lambda \leq \lambda_0|y) = 1 - \mathrm{P}(\Lambda > \lambda_0|y)$.

(b) Because in the considered Gamma distribution $\beta/(n\beta+1)$ is the scale parameter, $Z := (2(n\beta+1)/\beta)\Lambda$ given $Y = y$ has $Gamma(y+\alpha, 2)$ distribution. Further, if 2α is integer then $Z \sim \chi^2_{2y+2\alpha}$ because y is integer. Thius, if $\alpha = 5/2$ and $\beta = 2$ then

$$\mathrm{P}_{\beta=2,\alpha=5/2}(\Lambda > \lambda_0|y) = \mathrm{P}(\frac{2(n\beta+1)}{\beta}\Lambda > \frac{2(n\beta+1)}{\beta}\lambda_0|y) = \mathrm{P}(\chi^2_{2y+5} > (2n+1)\lambda_0).$$

If the last probability is larger 0.5 then we reject H_0.

6. Exerc. 8.11 Here we discuss Bayes testing σ^2 for a normal RV with conjugate prior. We test $H_0 : \sigma \leq 1$ versus $H_1 : \sigma > 1$.

(a) The posterior of Σ^2 given S^2 is, according to our HW (Exerc. 7.23) is inverted Gamma $IG(\gamma, \delta)$ where $\gamma = \alpha + (n-1)/2$ and $\delta = [(n-1)S^2/2 + 1/\beta]^{-1}$. Set $Y := 2/(\Sigma^2\delta)$. Then conditional $Y|S^2$ has $Gamma(\gamma, 2)$ distribution.

Remark: remember that Z is $Gamma(\alpha, \beta)$ iff

$$f_Z(z|\alpha, \beta) := \frac{1}{\Gamma(\alpha)\beta^\alpha} z^{\alpha-1} e^{-z/\beta} I(z > 0), \quad \alpha > 0, \beta > 0.$$

Further, $W := 1/Z$ is the inverted Gamma. Further, β is the scale parameter. Further, if 2γ is integer then $Y|S^2$ has a $\chi^2_{2\gamma}$ distribution.

Let M denote the median of a $Gamma(\gamma, 2)$ distribution. Clearly M depends on α and n but not on S^2 or β. Write (note that $\sigma \leq 1$ iff $Y \geq 2/\delta$)

$$\mathrm{P}Y \geq 2/\delta|S^2) = \mathrm{P}(\sigma^2 \leq 1|S^2) > 1/2$$

iff

$$M > 2/\delta = (n-1)S^2 + 2/\beta$$

which is equivalent to

$$S^2 < \frac{M - 2/\beta}{n-1}.$$

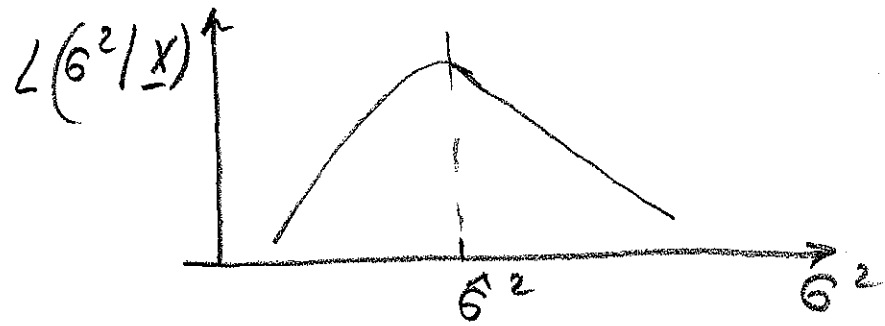
(b) The unrestricted MLEs are

$$\hat{\mu} := \bar{X}, \qquad \hat{\sigma}^2 := [(n-1)/n]S^2.$$

Under H_0 (which is the variance is at most 1), the sample mean $\bar{X}$ is still the MLE, but there is a new restricted MLE for the variance

$$\hat{\sigma}^2_{RMLE} := \min(1, \hat{\sigma}^2).$$

See the figure below that explains this result:



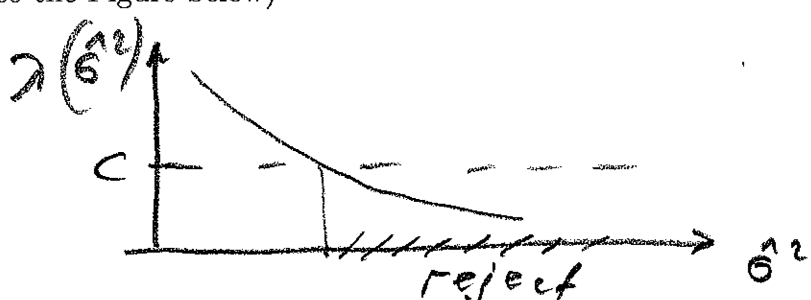
We conclude that

$$\lambda(\mathbf{X}) = \begin{cases} 1 \text{ if } \hat{\sigma}^2 \leq 1 \\ (\hat{\sigma}^2)^{n/2} e^{-n(\hat{\sigma}^2-1)/2} \text{ if } \hat{\sigma}^2 > 1. \end{cases}$$

Now we need to understand where to reject in terms of the statistic. Note that for $\hat{\sigma} > 1$ we have

$$\partial \ln(\lambda(\mathbf{X}))/\partial\hat{\sigma}^2 = (n/2)(\frac{1}{\hat{\sigma}^2} - 1) < 0$$

(see also the Figure below)



We conclude that the rejection region is where $\hat{\sigma}^2$, that is §^{2}, is large.
Then you can match the regions —- I skip this part...

7. Exerc. 8.13. Given: X_1 and X_2 are $Unif(\theta, \theta+1)$, and $H_0: \ \theta = 0$ versus $H_a: \ \theta > 0$. Further, we have two competing tests:

$$\phi_1(X_1): \quad \text{Reject } H_0 \text{ if } X_1 > .95$$

$$\phi_2(X_1, X_2): \quad \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

(a) We are asked to find C such that both tests have the same size. Write (note that the calculation below is elementary — just graph the uniform density and look at the rejection area)

$$\text{Size}(\phi_1) = \alpha_1 := \text{P}(X_1 > .95|\theta = 0) = .05.$$

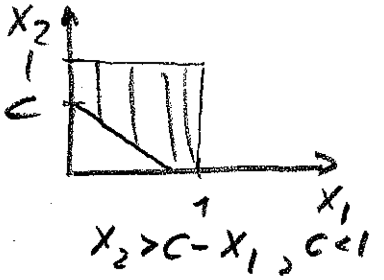
Next,

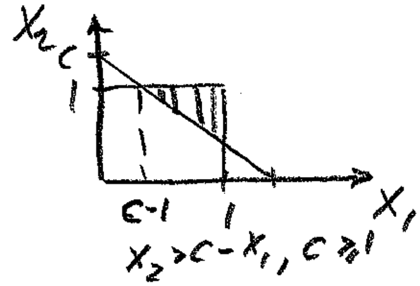
$$\text{Size}(\phi_2) = \alpha_2 = \mathrm{P}(X_1 + X_2 > C|\theta = 0).$$

Note that you need to concider only $C > 1$ because otherwise $\alpha_2 > 1/2$. Then, either using a graphic or integration we continue

$$\alpha_2 = \int_{C-1}^{1} 1dx_1 \int_{C-x_1} 1dx_2 = (2-C)^2/2.$$

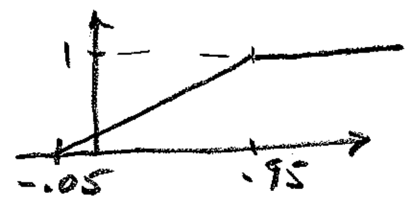
Thus $\alpha_1 = \alpha_2$ implies $.05 = (2-C)^2/2$. This yields $C = 2 - (.1)^{1/2} = 1.7$.





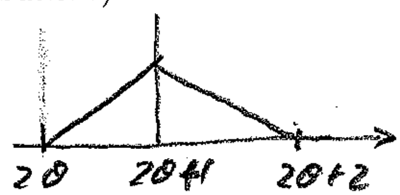
(b) For ϕ_1

$$\beta_1(\theta) = \mathrm{P}(X_1 > .95|\theta) = \begin{cases} 0, \ \theta \leq -.05 \\ \theta + .05, \ -0.05 < \theta \leq .95 \\ 1, \theta > .95. \end{cases}$$



As for $Y = X_1 + X_2$ (do you remember that Y has a triangle distribution?)

$$f_{Y|\theta}(y|\theta) = \begin{cases} y - 2\theta \text{ if } 2\theta \leq y < 2\theta + 1 \\ 2\theta + 2 - y \text{ if } 2\theta + 1 \leq y < 2\theta + 2 \\ 0 \text{ otherwise.} \end{cases}$$



As a result,

$$\beta_2(\theta) = \mathrm{P}_\theta(T > C) = \begin{cases} 0 \text{ if } \theta \leq (C/2) - 1 \\ (2\theta + 2 - C)^2/2 \text{ if } C/2 - 1 < \theta \leq (C-1)/2 \\ 1 - (C - 2\theta)^2/2 \text{ if } (C-1)/2 < \theta \leq C/2 \\ 1 \text{ if } \theta > C/2 \end{cases}$$

(c) Well, it is clear that ϕ_1 is more powerful for θ near 0. On the other hand, ϕ_2 is more powerful for larger θ. We conclude that neither is the UMP.

(d) This is simple. Note that if either $X_1 > 1$ or $X_2 > 1$ then we should reject H_0 (that is $\theta = 0$). Thus the rejection region

$$R := \{x_1 + x_2 > C\} \cup \{x_1 > 1, x_2 > 0\} \cup \{x_2 > 1, x_1 > 0\}$$

has the same size but larger power than ϕ_2.

8. Exerc. 8.14 Given that $X_1, \ldots, X_n$ are iid *Bernoulli*(p). We want to test $H_0: \quad p = .49 = p_0$ versus $H_a: \quad p = .51 = p_1$.

Well, here we reject (Neyman-Pearson, or any other reasonable approach) if

$$\bar{X} > C.$$

Using the CLT $\bar{X} \sim N(p, p(1-p)/n)$.

(a) Under H_0 and the normality assumption we can do the following approximation,

$$\mathrm{P}_{p_0}(\bar{X} > C) = \mathrm{P}_{p_0}(\frac{\bar{X}-\mu}{\sigma_{\bar{X}}} > \frac{C-\mu}{\sigma_{\bar{X}}})$$

$$= \mathrm{P}_{p_0}(Z > \frac{C - p_0}{(p_0(1-p_0)/n)^{1/2}}) = .01.$$

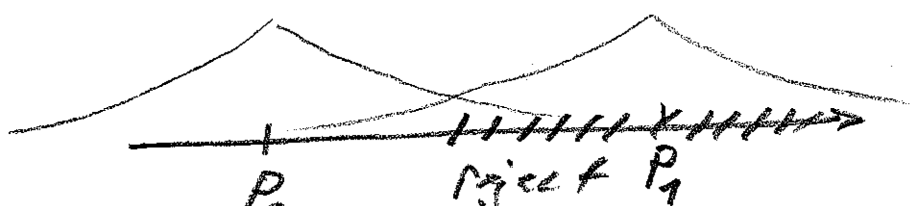
(b) Under H_a, using the same normal approximation (I skip the same elementary steps of z-scoring)

$$\mathrm{P}_{p_1} = \mathrm{P}_{p_1}(Z > \frac{C - p_1}{(p_1(1-p_1)/n)^{1/2}} = .99$$

Then using Norma Table we get

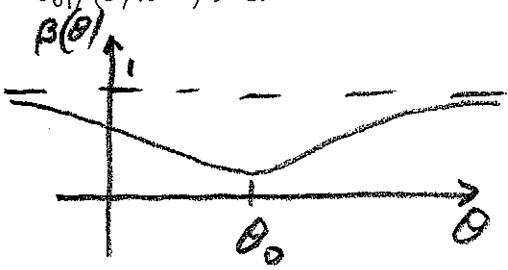
$$\begin{cases} \frac{nC - n(.49)}{(n(.49)(.51))^{1/2}} = 2.33 \\ \frac{nC - n(.51)}{(n(.51)(.49))^{1/2}} = -2.33 \end{cases}$$

Solving this system yields $n = 13567$ and $C = .5$ (as it was expected from the symmetry!).



9. Exerc. 8.18 Given that $X_1, \ldots, X_n$ are iid $Normal(\theta, \sigma^2)$, and σ^2 is known. We test $H_0: \quad \theta = \theta_0$ versus $H_a: \quad \theta \neq \theta_0$. Further, a LRT rejects iff $|\bar{X} - \theta_0|/(\sigma/n^{1/2}) > c$.

(a) For finding the power we write,



$$\beta(\theta) = \mathrm{P}_\theta(|\bar{X} - \theta_0|/(\sigma/n^{1/2} > c)$$

$$= 1 - \mathrm{P}_\theta(\frac{|\bar{X} - \theta_0|}{\sigma/n^{1/2}} \le c)$$

$$= 1 - \mathrm{P}_\theta(-c \le \frac{\bar{X} - \theta_0}{\sigma/n^{1/2}}, c)$$

$$= 1 - \mathrm{P}_\theta(\frac{-c\sigma n^{-1/2} + \theta_0 - \theta}{\sigma n^{-1/2}} < \frac{\bar{X} - \theta}{\sigma / n^{1/2}} < \frac{c\sigma n^{-1/2} + \theta_0 - \theta}{\sigma n^{-1/2}})$$

$$= 1 - \mathrm{P}(-c + \frac{\theta_0 - \theta}{\sigma/n^{1/2}} < Z < c + \frac{\theta_0 - \theta}{\sigma/n^{1/2}})$$

$$= 1 + \Phi(-c + \frac{\theta_0 - \theta}{\sigma/n^{1/2}}) - \Phi(c + \frac{\theta_0 - \theta}{\sigma/n^{1/2}}).$$

Here $Z \sim N(0,1)$ and $\Phi(z)$ is its cdf. Note how symmetric all the functions are!

(b) Type I error (the size) is

$$.05 = \beta(\theta_0 = 1 + \Phi(-c) - \Phi(c),$$

which yields $c = 1.96$. The power for $H_a: \ \theta = \theta_0 + \sigma$ should be at least .75 (it is $1 - \alpha_2$), so

$$.75 \le \beta(\theta_0 + \sigma) = 1 + \Phi(-c - n^{1/2}) - \Phi(c - n^{1/2})$$

$$= 1 + \Phi(-1.96 - n^{1/2}) - \Phi(1.96 - n^{1/2}).$$

The second term is practically zero, so we get $\Phi(-.675) \approx .25$, and this yields $1.96 - n^{1/2} = -.675$ rersulting in $n = 7$.

10. Exerc 8.19. Given: $X \sim f_X(x) = e^{-x} I(x > 0)$ and $Y := X^\theta$. Test $H_0: \ \theta = 1 = \theta_0$ versus $H_a: \ \theta = 2 = \theta_1$.

According to the NP Lemma, the UMP level α test

$$\text{Reject iff } \frac{f_{Y|\theta_1}(y)}{f_{Y|\theta_0}} > c \text{ and } \mathrm{P}_{\theta_0}(\frac{f_{Y|\theta_1}(y)}{f_{Y|\theta_0}} > c) = \alpha.$$

Now let us find the test explicitly.

(a). Let us find the pdf of Y. Write for $y > 0$

$$F_{Y|\theta}(y) = \mathrm{P}(Y \le y|\theta) = P(X^\theta \le y|\theta) = \mathrm{P}(X \le y^{1/\theta})$$

$$= \int_0^{y^{1/\theta}} e^{-x} dx = 1 - e^{-y^{1/\theta}}, \quad y > 0.$$

Take the derivative and get the pdf

$$f_{Y|\theta}(y) = \frac{d}{du} F_{Y|\theta}(u)|_{u=y} = \theta^{-1} y^{\theta^{-1}-1} e^{-y^{\theta^{-1}}} I(y > 0).$$

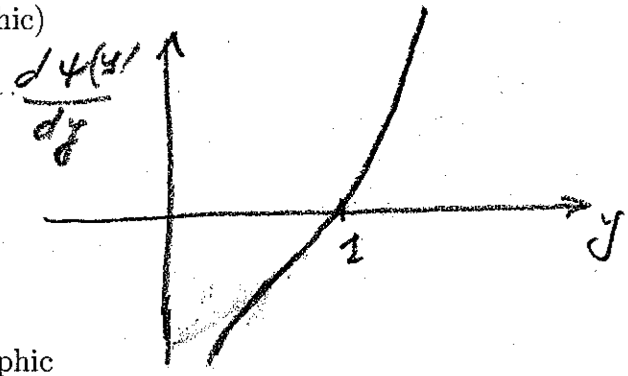
(b) To see the rejection region, let us explore

$$\psi(y) := \frac{f_{Y|\theta_1}(y)}{f_{Y|\theta_0}(y)} = (1/2) y^{-1/2} e^{y - y^{1/2}}.$$

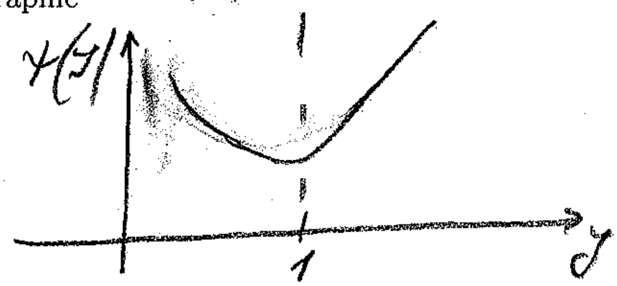
Because

$$\frac{d}{dy}\psi(y) = (1/2) y^{-3/2} e^{y - y^{1/2}} (y - y^{1/2}/2 - 1/2) I(y > 0)$$

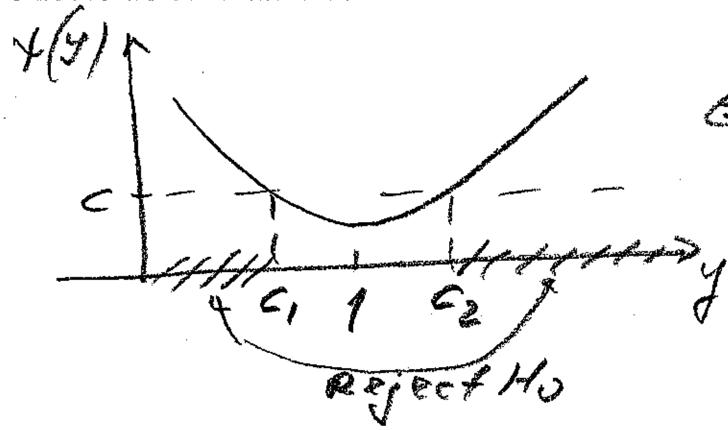
(see also the graphic)



we get the graphic



From the above we conclude that



$\Leftrightarrow \psi(C_1) = \psi(C_2)$

To obtain the given size α we write

$$\begin{cases} \alpha = \mathrm{P}(Y \leq C_1 | \theta = 1) + P(Y \geq C_2 | \theta = 1) = 1 - e^{-C_1} + e^{C_2} \\ \psi(C_1) = \psi(C_2) \end{cases}$$

Solving this system of equations gives us the two constants (I skip this). Finally, type II error is

$$\alpha_2 = 1 - \mathrm{P}(Reject | \theta - 1) = \mathrm{P}(C_1 < Y < C_2 | \theta_2)$$

$$= (1/2) \int_{C_1}^{C_2} y^{-1/2} e^{-y^{1/2}} dy = -e^{-y^{1/2}} |_{C_1}^{C_2} = e^{-C_1^{1/2}} - e^{-C_2^{1/2}}.$$