SOLUTION FOR HOMEWORK 9, STAT 6331

1. Exerc.8.20 (with $\alpha = 0.037$). Solution: Remember that the UMP test rejects H_0 if the likelihood ratio is large:

$$r(x) = \frac{f(x|H_1)}{f(x|H_2)} > k$$

and utilizes randomization if

$$r(x) = k$$

where k is chosen to yield the given zise.

Let us look at $r(x)$,

x	1	2	3	4	5	6	7
r(x)	6	5	4	3	2	1	.84

Thus, we reject if $x < c$ and randomize if $x = c$. To get $\alpha = 0,037$ we set $c = 4$ (using trials and errors approach) and if $x = 4$ then reject with probability 0.7. In this case the critical function is

$$\phi(x) = \begin{cases} 1 & \text{if } x < 4 \\ .7 & \text{if } x = 4 \\ 0 & \text{if } x > 4. \end{cases}$$

Now we check that

$$\alpha = \mathrm{P}(\text{Reject}|H_0) = E(\phi(X)|H_0)$$

$$= \sum_{i=1}^{3} \mathrm{P}(X = i|H_0) + (0.7)\mathrm{P}(X = 4|H_0) = 0.01 + 0.01 + 0.01 + (0.7)(0.01) = 0.037.$$

Further,

$$\text{Type II Error} = \alpha_1 - \mathrm{P}(\text{Accept}|H_1) = 1 - \mathrm{P}(\text{Reject}|H_1) = 1 - E(\phi(X)|H_1)$$

$$= 1 - [.06 + .05 + .04 + (.7)(.03)] = 1 - [.15 + .021] = 1 - .171 = .829.$$

Please note that .171 is the *power* of the test.

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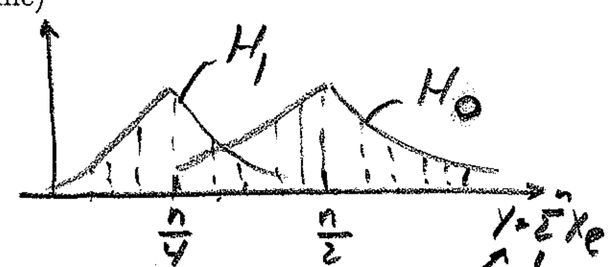
2. Exerc. 8.22. Given: $X_1, \ldots, X_n$ are iid *Bernoulli*(p).

(a) Test $H_0: \; p = 1/2$ versus $p = 1/4$ with $\alpha = .0547$.

This is a distribution with the MSS $Y = \sum_{l=1}^{n} X_l$ (check this!), which is *Binomial*(p, n), and the critical function of the UMP test is (see the graphic)

$$\phi(Y) = \begin{cases} 1 & \text{if } Y < c \\ \gamma & \text{if } Y = c \\ 0 & \text{if } Y > c. \end{cases}$$



MSS

so test depends on $Y \sim Bin(p, n)$

Then, for $n = 10$ using $Binomial(p = 1/2, n = 10)$ Table, we get $c = 3$, $\gamma = 0$. Check this:

$$\mathrm{P}(Y < 3|p = 1/2, n = 10) = .0547.$$

This critical function (the UMP test) also yields the power

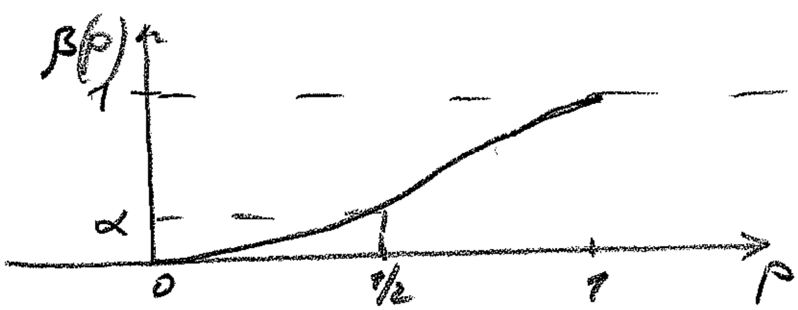
$$\text{Power} = \mathrm{P}(Y < 3|p = 1/4, n = 10) = 0.526$$

b) Given that $H_0: \; p \leq 1/2$ and $H_1: \; p > 1/2$, and also that the critical function is

$$\phi(x) = \begin{cases} 1 & \text{if } Y \geq 6 \\ 0 & \text{if } Y \leq 5. \end{cases}$$

The power function is

$$\beta(p) = E(\phi(Y)|p) = \mathrm{P}(Y \geq 6|p) = \sum_{k=6}^{10} \frac{10!}{k!(10-k)!} (\frac{p}{1-p})^k (1-p)^{10}.$$



Please note how the graphic also follows from the fact that the test is UMP.
Size is

$$\text{Size} = \beta(1/2) = \mathrm{P}(Y \geq 6|p = 1/2, n = 10) \approx .377$$

(c) A nonrandomized test exists for α equal to $P(Y \leq j|p = 1/2, n)$. For $n = 10$ these are (from the Table)

$$\alpha = 0, \; \frac{1}{1024}, \frac{11}{1024}, \frac{56}{1024}, \ldots, 1.$$

3. Exers. 8.23 Given that $X \sim Beta(\theta, 1)$, that is

$$f_X(x|\theta) = \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma^1} x^{\theta-1}(1-x)^{1-1} I(x \in [0,1]), \quad \theta > 0.$$

Also remember (p.99 in the text) that $\Gamma(\theta + 1) = \theta\Gamma(\theta)$ and $\Gamma(1) = 1$.

(a) We test $H_0: \; \theta \leq 1$ versus $H_1: \; \theta > 1$ and the given critical function is $\phi(x) = 1$ whenever $X > 1/2$ and it is zero otherwise.

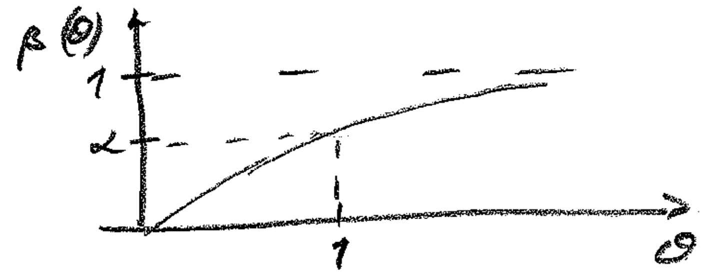
The power function is

$$\beta(\theta) = E_\theta\{\phi(X)\} = \int_{1/2}^{1} \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} x^{\theta-1} dx$$

$$= \frac{1}{\theta}\frac{\Gamma(\theta+1)}{\Gamma(\theta)}x^{\theta}|_{x=1/2}^{1} = (\theta/\theta)x^{\theta}|_{x=1/2}^{1} = 1 - 2^{-\theta}.$$

I used $\Gamma(\theta+1) = \theta\Gamma(\theta)$, $\Gamma(1) = 1$ (see p. 29 in the text)

The graphic is



Further,

$$\text{Size} = \max_{\theta \in (0,1]} \beta(\theta) = \beta(1) = 1/2.$$

(b) Here we test $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ with a given α. Then the UMP test rejects whenever

$$r(x) = \frac{f_X(x|\theta = 2)}{f_X(x|\theta = 1)} > k.$$

This yields

$$r(x) = \frac{\Gamma(3)x^{2-1}\Gamma(1)\Gamma(1)}{\Gamma(2)\Gamma(1)\Gamma(2)x^{1-1}} = \frac{\Gamma(3)}{(\Gamma(2))^2}x > k.$$

Remember that $\Gamma(n) = (n-1)!$ and then $r(x) = 2x > k$ yields the rejection region $x > c$. Because α is given we write

$$\alpha = \mathrm{P}(X > c|\theta = 1) = \int_c^1 \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}x^0 dx = 1 - c.$$

We conclude that $c = 1 - \alpha$ and the UMP rejects whenever $X > 1 - \alpha$.

(c) Here we test $H_0 : \theta \leq 1$ versus $H_a : \theta > 1$. The issue is about the UMP. For any pair $\theta_2 > \theta_1$ of two values of the parameter, we notice that the fuunction

$$r(x|\theta_1, \theta_2) = \frac{f(x|\theta_2)}{f(x|\theta_1)} = [\theta_2/\theta_1]x^{\theta_2 - \theta_1}$$

is monotonic in x. This yields MLR and concequently the UMP test exists, and it is $X > c_\alpha$.

4. Exerc. 8.24 The LRT rejects H_0 when

$$\lambda(x) = \frac{f(x|H_0)}{\max(f(x|H_1), f(x|H_0))} < c.$$

The MP test rejects when

$$r(x) = \frac{f(x|H_1)}{f(x|H_0)} > k.$$

As a result, we can write that

$$\lambda(x) = \begin{cases} 1/r(x) & \text{if } f(x|H_1) \geq f(x|H_0), \text{ that is } \lambda(x) \leq 1 \\ 1 & \text{if } f(x|H_1) \leq f(x|H_0), \text{ that is } \lambda(x) = 1. \end{cases}$$

Thus, in general the two tests are different. At the same time, in practice typically c is smaller than 1 so they are often identical. Also note that if $c = 1$ then the power of the LRT is 1 and the size is 1.

5. Exerc. 8.25 Show that a family has an MLR. (Remember that a family with a monotone likelihood ratio has a UMP test — this is why the MLR is important.)

In what follows $\theta_1 < \theta_2$.

(a) $Normal(\theta, \sigma^2)$ with σ^2 known. Write

$$r(x) = \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{e^{-(x-\theta_2)^2/2\sigma^2}}{e^{-(x-\theta_1)^2/2\sigma^2}} = e^{x(\theta_2-\theta_1)/\sigma^2} e^{(\theta_1^2-\theta_2^2)/2\sigma^2}.$$

The ratio is increasing in x, and this implies the MLR. Please note that the same result holds for a sample of size n.

(b) $Poisson(\theta)$. Write,

$$r(x) = \frac{p(x|\theta_2)}{p(x|\theta_1)} = \frac{e^{-\theta_2}\theta_2^x x!}{x! e^{-\theta_1}\theta_1^x} = e^{\theta_1-\theta_2}[\theta_2/\theta_1]^x.$$

The ratio os increasing in x, and this yields the MLR.

(c) $Binomial(\theta, n)$. Write

$$r(x) = \frac{p(x|\theta_2)}{p(x|\theta_1)} = \frac{\frac{n!}{x!(n-x)!}\theta_2^x(1-\theta_2)^{n-x}}{\frac{n!}{x!(n-x)!}\theta_1^x(1-\theta_1)^{n-x}}$$

$$= [\frac{\theta_2(1-\theta_1)}{\theta_1(1-\theta_2)}]^x[\frac{1-\theta_2}{1-\theta_1}]^n.$$

Again, the ratio is increasing in x, so the family has the MLR.

Remark: Have you noticed that the three distributions are from the exponential family? See also the next problem.

6. Exerc. 8.27 Let $g(t|\theta) = c(\theta)h(t)e^{w(\theta)t}$ be a one-parameter exponential family. Then for $\theta_2 > \theta 1$ we can write

$$r(t|\theta_1, \theta_2) = \frac{g(t|\theta_2)}{g(t|\theta_1)} = e^{(w(\theta_2)-w(\theta_1))t}\frac{c(\theta_2)}{c(\theta_1)}.$$

If $w(\theta)$ is increasing them the likelihood ratio r is also increasing in t, and we establish the MLR. Problem 8.25 gives us 3 examples.

7. Exerc. 8.28. Given: The logistic distribution

$$f(x|\theta) = \frac{e^{x-\theta}}{(1+e^{x-\theta})^2} I(x \in (-\infty, \infty)), \quad \theta \in (-\infty, \infty).$$

(a) Suppose that $\theta_2 > \theta_1$ and write

$$r(x) = \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{e^{x-\theta_2-x+\theta_1}(1+e^{x-\theta_1})^2}{(1+e^{x-\theta_2})^2}.$$

To study the monotonicity, let us look at the derivative

$$dr(x)/dx = e^{\theta_1-\theta_2}[2\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}} \frac{e^{x-\theta_1}(1+e^{x-\theta_2}) - e^{x-\theta_2}(1+e^{x-\theta_1})}{(1+e^{x-\theta_2})^2}]$$

$$= 2e^{\theta_1-\theta_2} \frac{(1+e^{x-\theta_1})[e^{x-\theta_1} - e^{x-\theta_2}]}{(1+e^{x-\theta_2})^3} > 0$$

because $e^{-\theta_1} > e^{-\theta_2}$. The MLR is established.

(b) We test $H_0 \theta = 0$ versus $H_a : \theta = 1$. According to part (a) we have $r(x) > k$ iff $x > c$. Then

$$\alpha = \mathrm{P}(X > c|\theta = 0) = \int_c^\infty \frac{e^x}{(1+e^x)^2} dx = 1 - F_0(c)$$

where the logistic cdf $F_\theta(c) := e^{c-\theta}/[1+e^{c-\theta}]$. We conclude that

$$\alpha = 1 - \frac{e^c}{1+e^c} = [1+e^c]^{-1},$$

and the last relation yields

$$c = \ln([1-\alpha]/\alpha).$$

Similarly,

$$\beta = F(c|\theta = 1) = \frac{e^{c-1}}{1+e^{c-1}}$$

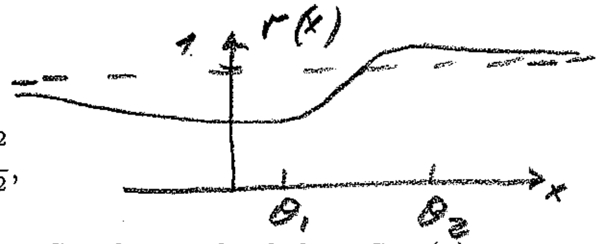
and you can plug-in the c expressed as α (I skip this).

For the given number, if $\alpha = .2$ then $c = 1.39$ and $\beta = .6$.

(c) This follows from the MLR property (Karlin-Rubin Theorem).

8. Exerc. 8.29 Given that $X \sim Cauchy(\theta)$, $\theta_2 > \theta_1$.

(a). This is a very famous result. Write



$$r(x) = \frac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2},$$

and note that $r(x) \uparrow 1$ as $x \to \infty$ and $r(x) \downarrow 1$ as $x \to \infty$. See the graphic below. So $r(x)$ is not MLR.

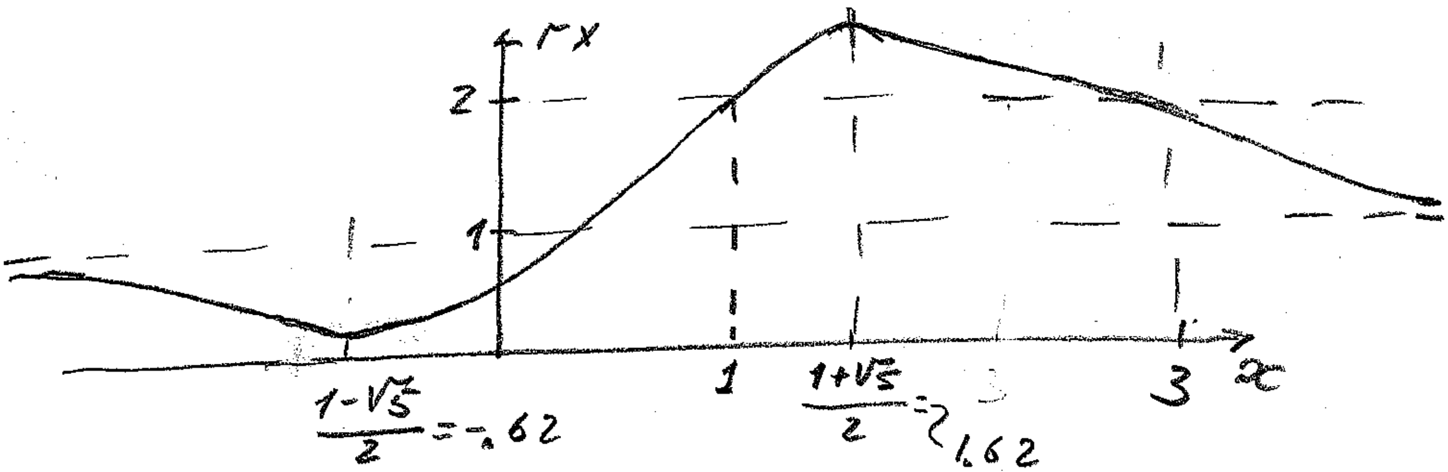
(b) Given $f(x|\theta) = 1/[\pi(1+(x-\theta)^2)]I(x \in (-\infty, \infty))$. Test $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Remember that the MP test rejects when $r(X) > k$. Here

$$r(x) = \frac{1+x^2}{1+(x-1)^2}$$

and

$$dr(x)/dx = \frac{2x(1+(x-1)^2) - 2(x-1(1+x^2)}{(1+(x-1)^2)^2} = \frac{2(1+x-x^2)}{(1+(x-1)^2)^2}.$$

This allows us to plot the graphic. Note that the roots of $1+x-x^2 = 0$ are $x_{12} = [1 \pm 5^{1/2}]/2$.



We have $r(1) = 2/1$ and $r(3) = [1+9]/[1+4] = 2$. As a result, according to these calculations and the graphic, the interval $[1, 3]$ is the MP of its size. Let us make some calculations:

$$\alpha = \mathrm{P}(X \in [1,3]|\theta = 0) = \int_1^3 \frac{1}{\pi(1+x^2)} dx = \pi^{-1} \arctan(x)|_{x=1}^3 \approx .15.$$

Further,

$$\alpha_1 = \text{second type error} = 1 - \mathrm{P}(x \in [1,3]|\theta = 1)$$
$$= 1 - \frac{1}{\pi} \int_1^3 \frac{1}{1+(x-1)^2} dx = 1 - \frac{1}{\pi} \arctan(x-1)|_1^3 \approx .65$$

(c) Clearly this is not the UMP test. Further, note that the rejection region can be a tail, or two tails, etc.

9. Exerc. 8.32 Given $X_1, \ldots, X_n$ are iid $Normal(\theta, 1)$, and θ_0 is given.

(a) Test $H_0 : \theta > \theta_0$ versus $H_1 : \theta < \theta_0$. Since normal family is a MLR family, we know that the UMP test exists and, via a familiar calculation the rejection region is (below $\theta_1 < \theta_0$)

$$r(x) = \frac{f(\mathbf{x}|\theta_1)}{f(\mathbf{x}|\theta_0)} > k \quad \Rightarrow \bar{x} < c.$$

This yields $c = \theta_0 - z_{1-\alpha}\sigma/n^{1/2}$ (check this famous result!).

(b) We discussed this in class (remember that here left and right tails atre the UMP for parameters smaller and larger than θ_0. Due to this fact and importance of the normal distribution, the notion of an unbiased test has been introduced.

10. Exerc 8.33. Given: $X_1, \dots, X_n$ is a sample from the $Uniform(\theta, \theta+1)$. We test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. The proposed test has the critical function $\phi(\mathbf{x}) = 1$ if $X_{(1)} \geq 1$ or $X_{(n)} \geq k$ and it is zero otherwise.

(a) To have the size α, we need (note that under the null hypothesis all observations are at most 1)

$$\alpha = \mathrm{P}(X_{(n)} > k|\theta = 0) = \int_k^1 n(1-y)^{n-1} dy = (1-k)^n.$$

This yields $k = 1 - \alpha^{1/n}$.

(b) Write using $F_{X_{(1)},X_{(n)}}(u, v|\theta) = n(n-1)(v-u)^{n-2} I(\theta < u < v < \theta + 1)$,

$$\beta(\theta) = \mathrm{P}(\{X_{(1)}\} \geq 1\{\cup X_{(n)} > k\}|\theta)$$

$$= \begin{cases} 0 \text{ if } \theta \leq k-1 \\ \int_k^\theta n(1-(y_1-\theta))^{n-1} dy_1 = (1-k+\theta)^n \text{ if } k-1 < \theta \leq 0 \\ \int_k^{\theta+1} n(1-(y_1-\theta))^{n-1} dy_1 + \int_\theta^k \int_1^{\theta+1} n(n-1)(y_n - y_1)^{n-2} dy_n dy_1 = \alpha + 1 - (1-\theta)^n, \text{ if } 0 < \theta < k \\ 1 \text{ if } \theta > k. \end{cases}$$

(c) Note that $X_{(1)}$ and $X_{(n)}$ are MSS. Then for $0 < \theta < 1$ we can write

$$\frac{f_{X_{(1)},X_{(n)}}(y_1, y_n|\theta)}{f_{X_{(1)},X_{(n)}}(y_1, y_n|0)} = \begin{cases} 0 & \text{if } 0 < y_1 \leq \theta, \quad y_1 < y_n < 1 \\ 1 & \text{if } \theta < y_1 < y_n < 1 \\ \infty & \text{if } 1 \leq y_n < \theta + 1, \quad \theta < y_1 < y_n. \end{cases}$$

For $\theta > 1$ we have

$$\frac{f_{X_{(1)},X_{(n)}}(y_1, y_n|\theta)}{f_{X_{(1)},X_{(n)}}(y_1, y_n|0)} = \begin{cases} 0 & \text{if } y_1 < y_n < 1 \\ \infty & \text{if } \theta < y_1 < y_n < \theta + 1. \end{cases}$$

From these results follows the UMP.

(d) According to part (b), $\beta(\theta) = 1$ for all $\theta \geq k = 1 - \alpha^{1/n}$. So these conditions are satisfied for all n.