SOLUTION FOR HOMEWORK 1, STAT 6331

Welcome to your second homework. Reminder: if you find a mistake/misprint, do not e-mail or call me. Write it down on the first page of your solutions and you may give yourself a partial credit — but keep in mind that the total for your homeworks cannot exceed 20 points.

Now let us look at your problems.

1. Problem 5.32. (page 261). Given: $X_n \xrightarrow{P} a$ and $P(X_i > 0) = 1$. Note that this implies $a > 0$. Solution:

a) For $Y_i := X_i^{1/2}$ we can write

$$P(|Y_i - a^{1/2}| > \epsilon) = P(|X_i^{1/2} - a^{1/2}| > \epsilon) = P(|X_i^{1/2} - a^{1/2}|(X_i^{1/2} + a^{1/2}) > \epsilon(X_i^{1/2} + a^{1/2}))$$

$$= P(|X_i - a| > \epsilon(X_i^{1/2} + a^{1/2})) \leq P(|X_i - a| > \epsilon a^{1/2}) \to 0 \text{ as } i \to \infty.$$

Now for $Y_i' := a/X_i$ and any $\epsilon \in (0,1)$

$$P(|Y_i' - 1| \leq \epsilon) = P(|aX_i^{-1} - 1| \leq \epsilon) = P(\frac{a}{1+\epsilon} \leq X_i \leq \frac{a}{1-\epsilon})$$

$$= P(a - \frac{a\epsilon}{1+\epsilon} \leq X_i \leq a + \frac{a\epsilon}{1-\epsilon}).$$

Now I use $(1+\epsilon)^{-1} < (1-\epsilon)^{-1}$ and continue

$$P(|Y_i - a^{1/2}| > \epsilon) \geq P(a - \frac{a\epsilon}{1+\epsilon} \leq X_i \leq a + \frac{a\epsilon}{1+\epsilon}) = P(|X_i - a| \leq \epsilon\frac{a}{1+\epsilon}) \to 1 \text{ as } i \to \infty.$$

What was wished to show.

Now part b). Using the above-established result, and that $S_n \xrightarrow{P} \sigma$, we established the wished result.

2. Problem 5.33. Given: $X_n \xrightarrow{D} X$, $\lim_{n\to\infty} P(Y_n > c) = 1$ for any finite c. Show that $P(X_n + Y_n > c) \to 1$.

In what follows I may use notation $P(A, B) := P(A \cap B)$.

Solution: Choose an arbitrary $a > 0$ and write,

$$P(X_n + Y_n > c) = P(X_n > -a, X_n + Y_n > c) + P(X_n \leq -a, X_n + Y_n > c)$$

$$\geq P(X_n > -a, Y_n > c + a) = P(Y_n > c + a) - P(X_n \leq -a, Y_n > c + a)$$

$$\geq P(Y_n > c + a) - P(X_n \leq -a) \to 1 - P(X \leq -a) \text{ as } n \to \infty.$$

Note that $P(X \leq -a) = F_X(-a) \to 0$ as $a \to \infty$. This remark finishes the proof.

3. Problem 5.34. Write,

$$E\{\frac{n^{1/2}(\bar{X}_n - \mu)}{\sigma}\} = \frac{n^{1/2}(E\{\bar{X}_n\} - \mu)}{\sigma} = 0.$$

Further,

$$\mathrm{Var}(\frac{n^{1/2}(\bar{X}_n-\mu)}{\sigma}) = [n/\sigma^2]\mathrm{Var}(\bar{X}_n-\mu) = [n\sigma^2][n^{-2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = [n/\sigma^2][n\sigma^2/n^2] = 1.$$

What was wished to show.

4. Problem 5.39. This is a nice problem because here we need to use the definition of a random variable as a measurable function of $w \in S$ from an underlying probability space $(S, \mathcal{F}, P)$.

Given: $X_n(w) \xrightarrow{P} X(w)$, that is for any $\epsilon > 0$ we have $P(w: |X_n(w) - X(w)| > \epsilon) \to 0$ as $n \to 0$, and $h(z)$ is continuous on R. Show that $h(X_n(w)) \xrightarrow{P} h(X(w))$.

Solution: Unfortunately, the hint in the text is confusing. This is the situation. Suppose that $h(z)$ is *uniformly* continuous on R, that is, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|h(z) - h(y)| < \epsilon$ for any $(z, y) \in R^2$ whenever $|z - y| < \delta$. An example is $h(z) = sin(z)$. At the same time, $h(z) = z^2$ is continuous but not uniformly continuous on R.

If this is the case then (think about why the next inequality holds; you can use the two examples as the pivot)

$$P(w: |h(X_n(w)) - h(X(w))| < \epsilon) \geq P(w: |X_n(w) - X(w)| < \delta) \to 1$$

due to the given property of X_n.

In the general case of a continuous $h(z)$, we need to remember the following calculus' result: $h(z)$ is uniformly continuous on any finite interval $[-C, C]$. As a corollary, it is also uniformly continuous on some sequence of increasing intervals $[-C_n, C_n]$ where $C_n \to \infty$ as slowly as necessary. This is the main trick. Also, you need to use the facts that $P(w: |X(w)| > C) \to 0$ as $C \to \infty$ and that there exists a sequence C'_n increasing to infinity such that $P(w: |X_n(w)| > C'_n) \to 0$ as $n \to \infty$. Using these results finishes the proof.

b) Look how I mathematically write down the problem. Let U be $Unif([0, 1])$ and

$$Y_{kl} := U + I(U \in [l/k, (l+1)/k]), \quad k = 1, 2, \ldots; \quad l = 0, \ldots, k-1.$$

Set $n = n(k, l) = \sum_{s=0}^{k} 2^s + l$ and note that if we know n then we know the pair (k, l) and vise versa. As a result, we can denote $X_n := Y_{kl}$.

We have $X_n(U) \xrightarrow{P} U$ because $P(|X_n(U) - U| \geq 0) = 1/k$. At the same time, $P(\cup_{n>n_0}\{|X_n(U) - U| \geq 0\}) = 1$ for any n_0. Therefore, we have convergence in probability but not almost sure (with probability 1).

On the other hand, let us choose a subsequence $n_k := n(k, 0)$. Then $|X_{n(k,o)}(U) - U| > 0$ iff $U \in [0, 1/k]$. As a result,

$$P(\cup_{k=k_0}^{\infty}\{|X_{n_k}(U) - U| > 0) = 1/k_0.$$

We established that the subsequence $\{X_{n_k}, k = 1, 2, \ldots\}$ converges to U almost sure.

5. Problem 5.41. Prove that

$$P(|X_n - \mu| > \epsilon) \to 0 \tag{1}$$

for any $\epsilon > 0$ iff

$$P(X_n \le x) \to I(x \ge \mu). \tag{2}$$

Solution: *Sufficiency* : from (2) follows (1). Write

$$P(|X_n - \mu| > \epsilon) = P(X_n > \mu + \epsilon) + P(X_n < \mu - \epsilon)$$

$$= [1 - P(X_n \le \mu + \epsilon)] + P(X_n < \mu - \epsilon) \to 0.$$

according to the assumption (2).

Necessity: from (1) follows (2). Begin with $x < \mu$ and set $\epsilon := \mu - x$. Write

$$P(X_n \le x) = P(X_n \le \mu - \epsilon) \to 0$$

according to (1). Similarly, for $x > \mu$ set $\epsilon := x - \mu$ and write

$$P(X_n \le x) = P(X_n \le \mu + \epsilon) = 1 - P(X_n > \mu + \epsilon) \to 1$$

according to (1).

6. Problem 5.44. Given: $X_i, i = 1, 2, \ldots$ are iid *Bernoulli*(p), and set $Y_n := \sum_{i=1}^n X_i/n$.

a). We know that $E(X_i) = p$, $\mathrm{Var}(X_i) = p(1-p)$ and then the CLT yields

$$n^{1/2}\frac{Y_n - p}{[p(1-p)]^{1/2}} \xrightarrow{D} N(0,1).$$

We also know, due to Slutsky's Theorem (p.239) that if $Z_n \xrightarrow{D} Z$ then $cZ_n \xrightarrow{D} cZ$. This finishes the verification.

b) Write,

$$n^{1/2}[Y_n(1-Y_n) - p(1-p)] = n^{1/2}[(Y_n - p)(1-Y_n) + p(1-Y_n) - p(1-p)]$$

$$= n^{1/2}(Y_n - p)(1 - Y_n) + n^{1/2}p(p - Y_n) = n^{1/2}(Y_n - p)(1-2p) - n^{1/2}(Y_n - p)^2. \tag{3}$$

Now, if $p \ne 1/2$, repeatedly using Slutsky's Theorem, we get $n^{1/2}(1-2p)(Y_n - p) \xrightarrow{D} N(0, p(1-p)(1-2p)^2)$ while $n^{1/2}(Y_n - p)^2 \xrightarrow{P} 0$. This verifies the assertion.

c) If $p = 1/2$ we get from (3) that

$$n^{1/2}[Y_n(1-Y)n) - 1/4] = -n^{1/2}(Y_n - 1/2)^2.$$

We know that $2n^{1/2}(Y_n - 1/2) \xrightarrow{D} N(0,1)$ so $4n(Y_n - 1/2)^2 \xrightarrow{D} \xi_1^2$. Indeed, if $P(Z_n \le y) \to F_Z(y)$ as $n \to \infty$, then $P(Z_n^2 \le y^2) \to F_Z(|y|) - F_Z(-|y|)$ and the last difference is $F_{\chi_1^2}(y^2)$ if Z is standard normal. What was wished to show.

7. Problem 5.49. Here U is *Uniform*($[0,1]$).

a). $X := -\ln(U)$, and because the logarithmic function is monotone and differentiable, we can use Theorem 2.1.5 Note that $X \in [0, \infty)$ almost sure, $U = e^{-X} =: h(X)$ for $X \in [0, \infty)$, and $h'(x) = -e^{-x}$. We get

$$f_X(x) = |h'(x)| f_U(h(x)) I(x \ge 0) = (e^{-x})(1) I(x \ge 0).$$

Similarly, if $X := -\ln(1-U)$, we have $U = 1 - e^{-X} := h(x)$ for $x \in [0, \infty)$ and $f_X(x) = e^{-x} I(x \geq 0)$.

b) Here $X := \ln[U/(1-U)]$, so $U = e^x(1+e^x)^{-1} =: h(x)$, $x \in R$. The function h is again monotone and differentiable, $h'(x) = e^x[1+e^x]^2$, and we get

$$f_X(x) = e^x[1+e^x]^{-2} I(x \in (-\infty, \infty)) = e^{-x}[1+e^{-x}]^{-2} I(x \in (-\infty, \infty)).$$

What was wished to show.