SOLUTION FOR HOMEWORK 7, STAT 6331

1. Exerc.7.33. Here we just recall that

$$\text{MSE}(\hat{p}_B) = \frac{np(1-p)}{(\alpha+\beta+n)^2} + \Big(\frac{np+\alpha}{\alpha+\beta+n} - p\Big)^2.$$

Then you plug in $\alpha = \beta = (n/4)^{1/2}$. After simplifications

$$\text{MSE}(\hat{p}_B) = \frac{n}{4(n^{1/2}+n)2}$$

and this is constant in p.

How do we get such α and β? We take a partial derivative in p and then set it equal to zero.

2. Exerc. 7.37. Let $X_1, \ldots, X_n$ be iid according to the pdf

$$f(x|\theta) = (2\theta)^{-1} I(|x| < \theta), \;\; \theta \in \Omega = (0, \infty).$$

We discussed in class that $|X|_{(n)}$ is the CSS. At the same time, $(X_{(1)}, X_{(n)})$ is not CSS because $-E_\theta(X_{(1)} = E_\theta(X_{(n)}$. (What do you think about the SS statistic $(X_{(1)}, X_{(n)})$ for the cases of $Unif(\theta, 2\theta)$ or $Unif(\theta - 1/2, \theta + 1/2)$?)

Now, $Y := |X|_{(n)}$ is SS by Factorization Theorem because

$$f(\mathbf{x}|\theta) = (2\theta)^{-n} I(|x|_{(n)} \le \theta).$$

Further, let us find the pdf of Y. Write,

$$F_Y(y) = \text{P}(Y \le y) = \text{P}(|X|_{(n)} \le y) = \prod_{l=1}^{n} |X_l| \le y) = \theta^{-n} y^n I(y \in (0, \theta)).$$

Thus, the pdf of Y is

$$f_Y(y) = \frac{d}{dz} F_Y(z)\Big|_{z=y} = n\theta^{-n} y^{n-1} I(o < y < \theta). \tag{1}$$

Let us show that Y is CSS. Suppose that

$$E_\theta(g(Y)) = \int_0^\infty g(y) n\theta^{-n} y^{n-1} dy \equiv 0, \quad \text{for all } \theta > 0. \tag{2}$$

Then $dE_\theta(g(Y))/d\theta \equiv 0$ for all $\theta > 0$. This yields (remember the Leibnitz rule on p.69 of how to take the derivative)

$$-n\theta^{-n-1} \int_0^\theta g(y) n y^{n-1} dy + g(\theta) n\theta^{-n} \theta^{n-1} \equiv 0, \;\; \theta > 0.$$

But the integral in the above-written identity is zero due to (2). This implies that $g(\theta)\theta^{-1} \equiv 0$ for all $\theta > 0$, and this $g(\theta \equiv 0$ for all $\theta > 0$. We proved (directly) that Y is complete.

Our next step is to find an unbiased estimator $\delta(Y)$ which is also the UMVUE. Let us check, using (1), that

$$E_\theta(Y) = \int_0^\theta yn\theta^{-n}y^{n-1}dy\theta^{-n}n\int_0^\theta y^n dy$$
$$= \theta^{-n}n(n+1)^{-1}\theta^{n+1} = \frac{n}{n+1}\theta.$$

Thus, the UMVUE is

$$\delta^*(Y) = \frac{n+1}{n}\max_l |X_l|.$$

3. Exers. 7.40. Let $X_1, \ldots, X_n$ be a sample from $Bernoulli(p)$. We calculate the Fisher information for a single observation:

$$I(p) := -E_p\{\frac{\partial^2}{\partial p^2}\ln(p^X(1-p)^{1-X})\} = -E_p\{\frac{\partial^2}{\partial p^2}[X\ln(p) + (1-X)\ln(1-p)]\}$$

$$= -E_p\{\frac{\partial}{\partial p}[\frac{x}{p} - \frac{1-X}{1-p}]\} = -E_p\{-\frac{x}{p^2} - \frac{1-X}{(1-p)^2}\}$$
$$= \frac{(1-p)^2p + p^2(1-p)}{p^2(1-p)^2} = \frac{1-p+p}{p(1-p)} = \frac{1}{p(1-p)}.$$

The Cramér-Rao lower bound tells us that

$$\mathrm{Var}_p(\delta(\mathbf{X})) \geq \frac{[\partial E_p(\delta(\mathbf{X}))/\partial p]^2}{nI(p)}.$$

If $\delta^*(\mathbf{X})$ is unbiased, the numerator in the lower bound is 1, and this yields that

$$\mathrm{Var}_p(\delta^*(\mathbf{X})) \geq \frac{p(1-p)}{n},$$

and because $\mathrm{Var}_p(\bar{X}) = p(1-p)/n$, the sample mean is the best unbiased estimator of p.

4. Exerc. 7.41. A sequence of iid RVs $X_1, \ldots, X_n$ is observed. It is known that $E_{\mu,\sigma^2}(X) = \mu$, $\mathrm{Var}_{\mu,\sigma^2} = \sigma^2$.

(a) If $\delta := \sum_{i=1}^n a_iX_i$ then

$$E_{\mu,\sigma^2}(\delta) = \sum_{i=1}^m a_i\mu = \ mu\sum i = 1^n a_i = \mu i f f \sum_{i=1}^n a_i = 1.$$

(b) Let us find $\{a_i\}$ that minimize the variance.

$$\mathrm{Var}_{\mu,\sigma^2}(\delta) = E_{\mu,\sigma^2}(\sum_{i=1}^n (a_iX_i - \mu)^2) = \mathrm{Var}_{\mu,\sigma^2}(\sum_{i=1}^n a_i(X_i - \mu))$$

[because observations are iid we continue]

$$= \sum_{i=1}^{n} a_i^2 \mathrm{Var}_{\mu,\sigma^2}(X_i) = \sigma^2 \sum_{i=1}^{n} a_i^2.$$

Now we should find $\{a_i\}$ that minimize $\sum_{i=1}^n a_i^2$ given $\sum_{i=1}^n a_i = 1$. Let us check that $a_i \equiv 1/n$ are the extreme (what else can we try?). Write (in what follows the summation is over $i \in \{1, 2, \ldots, n\}$),

$$\sum a_i^2 = \sum (a_i - n^{-1} + n^{-1})^2 = \sum (a_i - n^{-1})^2 + 2n^{-1} \sum (a_i - n^{-1}) + n^{-1}.$$

The last sum is zero because $\sum a_i = 1$. As a result, we get that

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{m} (a_i - n^{-1})^2 + n^{-1} \geq n^{-1} \text{ with the equality iff } a_i \equiv n^{-1}.$$

5. Exerc. 7.47 We have $X = \mu + \epsilon$ where $\epsilon \sim N(0, \sigma^2)$ and μ is an underlying radius. A sample of size n is observed. What is the UMVUE of $a = \pi\mu^2$?

Here $\bar{X}$ is CSS (due to the exponential family) so I just note that $E_{\mu,\sigma^2}(\bar{X}^2) - n^{-1}\sigma^2 = \mu^2$, so

$$\hat{a}_{unb} = \pi(\bar{X}^2 - n^{-1}\sigma^2).$$

Because it is a function of the CSS, it is UMVUE.

Here σ^2 is known, but what if it is also unknown? Consider the following estimator:

$$\tilde{a} := \pi(\bar{X} - n^{-1} S_n^2)?$$

What do you think about its properties?

6. Exerc. 7.48 Here $X_1, \ldots, X_n$ are iid $Bernoulli(p)$.

(a). We found in Exerc. 7.40 that $I(p) = 1/[p(1-p)]$, so $\mathrm{Var}_p(\bar{X}) = n^{-1}p(10p)$ attains the lower bound $[nI(p)]^{-1} = n^{-1}p(1-p)$.

(b) Well, we can write using iid,

$$E_p\{X_1 X_2 X_3 X - 4\} = \prod_{l=1}^{4} E_p\{X_l\} = p^4.$$

Then, because $\sum_{l=1}^n X_l$ is CSS (remember that we are dealing with an exponential class), the statistic

$$\delta(\sum_{l=1}^{n} X_l) := E_p\{X_1 X_2 X_3 X_4 | \sum_{l=1}^{n} X_l\}$$

is the UMVUE of p^4. Can you calculate it? Try by yourself and then look at this:

$$\delta(t) = E_p\{X_1 X_2 X_3 X_4 | \sum_{l=1}^{n} X_l = t\} = \mathrm{P}_p(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1 | \sum_{l=1}^{n} X_l = t)$$

$$= \frac{\mathrm{P}_p(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1, \sum_{l=1}^n X_l = t)}{\mathrm{P}_p(\sum_{l=1}^n X_l = t)}$$

$$= \frac{p^4[(n-4)!/(t-4)!(n-t)!]p^{t-4}(1-p)^{n-t}}{[n!/t!(n-t)!]p^t(1-p)^{n-t}]} = \frac{(n-4)!t!}{n!(t-4)!}.$$

7. Exerc. 7.49 Let $X_1, \ldots, X_n$ be a sample from $Expon(\lambda)$.

(a) Find an UE of λ based on $X_{(1)}$.

Well, because $f_X(x|\lambda) = \lambda^{-1}e^{-x/\lambda}I(x > 0)$ we use our technique to find the density of the first ordered observation. Remember how we do this:

$$F_{X_{(1)}}(x|\lambda) = \mathrm{P}_\lambda(X_{(1)} \le x) = 1 - \mathrm{P}(X_{(1)} > x)$$

$$= 1 - \mathrm{P}_\lambda(X_1 > x, \ldots, X_n > x) = 1 - [\lambda^{-1}\int_x^\infty e^{-z/\lambda}dz]^n = 1 - e^{-nx/\lambda}.$$

Take derivative and get $f_{X_{(1)}}(x|\lambda) = n\lambda^{-1}e^{-nx/\lambda}$.

As we see, $X_{(1)} \sim Expon(\lambda/n)$ so $E(X_{(1)}) = \lambda/n$ and

$$\tilde{\lambda} := nX_{(1)}$$

is UE.

(b) Let us find UMVUE. Here $Y := \sum_{l=1}^n X_l$ is CSS (again due to the exponential family). Because $E_\lambda(Y) = n\lambda$ we get

$$\hat{\lambda}_{UMVU} = \bar{X}.$$

8. Exerc. 7.52 Here $X_1, \ldots, X_n$ are iid from $Poisson(\lambda)$.

(a) Write

$$f_{\mathbf{X}}(\mathbf{x}|\lambda) = \prod_{l=1}^n \frac{e^{-\lambda}\lambda^{X_l}}{x_l!} = \frac{e^{-n\lambda}\lambda^{\sum_{l=1}^n x_l}}{\prod_{l=1}^n x_l!}.$$

As we see, this is an exponential family with $Y := \sum_{l=1}^n X_l$ being the CSS.

We conclude, using our theory, that $\bar{X} = Y/n$ is the UMVUE of λ.

(b) To analyze directly

$$E_\lambda(S^2|\bar{X}) = E_\lambda\{(n-1)^{-1}\sum_{l=1}^n (X_l - \bar{X})^| \bar{X}\}$$

is possible but rather complicated.

Let be smart and use the theory. We know that $E_\lambda(S^2) = \lambda$ because λ is the variance and S^2 is UE of the variance. But λ is also the mean for poisson distribution, so $E_\lambda(S^2|\bar{X})$ is the UMVUE of the mean. Further, $\bar{X}$ is also UMVUE of λ and $\bar{X}$ is the CSS, so by uniqueness of the UMVUE we have

$$E_\lambda(S^2|\bar{X}) = \bar{X}$$

for Poisson distribution!

Then we also can write that

$$\mathrm{Var}_\lambda(S^2) = \mathrm{Var}_\lambda(E_\lambda(S^2|\bar{X})) + E_\lambda\{\mathrm{Var}_\lambda(S^2|\bar{X})\} > \mathrm{Var}_\lambda(E_\lambda(S^2|\bar{X})) = \mathrm{Var}_\lambda(\bar{X}).$$

(c) If Y is CSS and Z is any other statistic such that $E_\theta(Y) \equiv E_\theta(Z)$ for all $\theta \in \Omega$ then $E_\theta(Z|Y) = Y$ for all $\theta \in \Omega$. Indeed, let $E_\theta(Z|Y) =: g(Y)$, then $E_\theta(g(Y) - Y) \equiv 0$ for all $\theta \in \Omega$ because $E_\theta g(Y) = E_\theta(Z) = E_\theta(Y)$ for all $\theta \in \Omega$. Because Y is complete this yields that $g(Y) = Y$ a.s.

Finally, we know that conditioning on a CSS reduces the variance, so

$$\mathrm{Var}_\theta(Z) > \mathrm{Var}_\theta(E_\theta(Z|Y)) = \mathrm{Var}_\theta(Y).$$

9. Exerc. 7.55

(a) Given that the pdf is $f(x|\theta) = \theta^{-1}I(0 < x < \theta)$.

Here $Y := X_{(n)}$ is the CSS and $f_Y(y|\theta) = n\theta^{-n}y^{n-1}$. Then

$$E_\theta(Y^r) = \int_0^\theta y^r n\theta^{-n}y^{n-1}dy = n\theta^{-n}\int_0^\theta y^{n+r-1}dy = \frac{n}{n+r}\theta^{-n}\theta^{n+r} = \frac{n}{n+r}\theta^r.$$

As a result,

$$\hat{\theta}_{UMVU} := \frac{n+r}{n}X^r_{(n)}.$$

10. Exerc 7.59. Here $X_1, \ldots, X_n$ are iid from $N(\mu, \sigma^2)$. Find UMVUE for σ^p, $p > 0$.

Well, it is reasonable to try to analyze $(S^2)^{p/2}$ because S^2 is a good estimate of σ^2. We know that $S^2 \stackrel{D}{=} \sigma^2(n-1)\chi^2_{n-1}$, so

$$E_{\mu,\sigma^2}(S^2)^{p/2} = \sigma^p[(n-1)^{-p/2}E\{(\chi^2_{n-1})^{p/2}\}].$$

Denote $K_p := (n-1)^{-p/2}E\{(\chi^2_{n-1})^{p/2}\}$. This is a function in p which can be calculated for all p (I skip its calculation but you can think about moment generating function for chi-squared RV), and then

$$\hat{\delta}_p := \frac{(S^2)^{p/2}}{K_p} \tag{3}$$

is the UMVUE. Indeed, $(\bar{X}, S^2)$ is the CSS for the normal distribution and the mean of the estimator is σ^p.