Simplex Method

Q. Solve the following LPP using the Simplex method.

Maximize $z = 12x_1 + 16x_2$

s.t

$10x_1 + 20x_2 \leq 120$

$8x_1 + 8x_2 \leq 80$

$x_1 \& x_2 \geq 0$

Solⁿ. →

Max $Z = 12x_1 + 16x_2 + 0S_1 + 0S_2$

subject to

$10x_1 + 20x_2 + S_1 = 120$ —(1)

$8x_1 + 8x_2 + S_2 = 80$ —(2)

(we add slack variable S_1 & S_2 in both the eqns (constraints) to balance the eqn (constraint) and also add in the objective fn with coeff. 0)

x_1, x_2, S_1 and $S_2 \geq 0$

Initial Simplex Table

C_{B_i}	C_j	12	16	0	0	Solution	Ratio
	Basic variable	x_1	x_2	S_1	S_2		
		Coeff. of x_1	Coeff. of x_2	Coeff. of S_1	Coeff. of S_2		
0	S_1	10	20	1	0	120	$\frac{120}{20}=6$ → from eqn (1)
0	S_2	8	8	0	1	80	$\frac{80}{8}=10$ → from eqn (2)
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	12	16	0	0		

C_j → Coeff. of objective fn

C_{B_i} → Cost of objective fn i.e., coeff. of basic variable in z.

Basic variable → no. of variables in obj. fn are 4 i.e., x_1, x_2, S_1 & S_2

$Z_j = \sum_{i=1}^{2} (C_{B_i})(a_{ij})$

$(0\times10)+(0\times8) = 0+0 = 0$

$(0\times20)+(0\times8) = 0+0 = 0$

$(0\times1)+(0\times0) = 0+0 = 0$

$(0\times0)+(0\times1) = 0+0 = 0$

and $(0\times120)+(0\times80) = 0+0 = 0$

$c_j - z_j = 12 - 0 = 12$, $16 - 0 = 16$, $0 - 0 = 0$, $0 - 0 = 0$,

optimality Condition :

For Max :

all $c_j - z_j \leq 0$ ie, all -ve or 0 should be $c_j - z_j$

$z_j - c_j \geq 0$ ie, all +ve or 0, should be $z_j - c_j$

For Min :

all $c_j - z_j \geq 0$ ie, all +ve or 0 should be $c_j - z_j$

$z_j - c_j \leq 0$ ie, all -ve or 0 should be $z_j - c_j$

Here we have got all +ve or 0 so we did not reach optimality.

we should proceed further for optimality.

In order to proceed further

1st step ⇒ to find the largest no. in $c_j - z_j$ or, most negative in $z_j - c_j$

Here 16 is the largest no. so x_2 is the key column

or -16 is most negative,

then ~~for~~ select the column as key column

and find Min Ratio to find key row.

Ratio = Solution / key column ie, $120/20 = 6$

$80/8 = 10$

key element

C_{Bi}	c_j Basic variable	x_1	x_2	S_1	S_2	Solution	Ratio
0	S_1 ←	10	20	1	0	120	$\frac{120}{20} = 6$ ← min value
0	S_2	8	8	0	1	80	$\frac{80}{8} = 10$
	z_j	0	0 ↑	0	0	0	
	$c_j - z_j$	12	16	0	0		

← max^m value

∵ 16 is the max^m value in $c_j - z_j$ so x_2 is the key column and since 6 is the min Ratio so S_1 is the key row and 20 is the intersection point of column and row so 20 is the key element. So ~~the~~ x_2 will enter as basic variable and the basic variable S_1 will leave. we shall form new Iteration table.

Iteration I

C_{B_i}	C_j / BV	12 / x_1	16 / x_2	0 / S_1	0 / S_2	Solution	Ratio
16	x_2	1/2	1	1/20	0	6	$\frac{6}{(1/2)} = 12$
0	S_2	4	0	$-2/5$	1	32	$\frac{32}{4} = 8$
	Z_j	8	16	4/5	0		
	$C_j - Z_j$	4	0	$-\frac{4}{5}$	0		

There is no change in C_j & BV from initial table so copy the values from initial table.

→ key row

key element

key column

max^m value so x_1 is key row, x_1 will enter and S_2 will leave

For C_{B_i} → coeff. of x_2 in Z is 16

for BV we write x_2 in place of S_1 as S_1 will leave.

So row value of x_2 is old value / key element

new value = $10/20 = \frac{1}{2}$, $20/20 = 1$, $1/20$, $0/20 = 0$, $120/20 = 6$

New value for S_2 row →

$$\text{New value} = \text{old value} - \frac{\text{Corr. key column value} \times \text{Corr. key row value}}{\text{key element}}$$

$$8 - \frac{8 \times 10}{20} = 8 - \frac{80}{20} = 8 - 4 = 4$$

$$8 - \frac{8 \times 20}{20} = 8 - \frac{8 \times 20}{20} = 8 - \frac{160}{20} = 8 - 8 = 0$$

$$0 - \frac{8 \times 1}{20} = 0 - \frac{8}{20} = 0 - \frac{2}{5} = -2/5$$

$$1 - \frac{8 \times 0}{20} = 1 - \frac{0}{20} = 1$$

$$80 - \frac{8 \times 120}{20} = 80 - \frac{960}{20} = 80 - 48 = 32$$

$$Z_j = 16 \times \frac{1}{2} + 0 \times 4 = 8, \quad 16 \times 1 + 0 \times 0 = 16, \quad 16 \times \frac{1}{20} + 0 \times -\frac{2}{5} = \frac{4}{5},$$

$$16 \times 0 + 0 \times 1 = 0$$

Again we found that $c_j - z_j$ are +ve value but we need all -ve values or zero. So we have to proceed further and continue the same process. We need to find key column, key row, key elt and min ratio for next iteration. Iteration II.

C_{Bi}	c_j	12	16	0	0	Solution	Ratio
	BV	x_1	x_2	S_1	S_2		
16	x_2	0	1	1/10	-1/8	2	
12	x_1	1	0	-1/10	1/4	8	
	Z_j	12	16	2/5	1	128	
	$c_j - z_j$	0	0	-2/5	-1		

For x_1 row → Divide S_2 row of Iteration 1 by key element we get

$$4/4 = 1,\ 0/4 = 0,\ \frac{-2/5}{4} = -\frac{1}{10},\ \frac{1}{4},\ 32/4 = 8$$

New value of row x_2 =

$$\frac{1}{2} - \left(\frac{\frac{1}{2} \times 4}{4}\right) = 0$$

$$1 - \left(\frac{\frac{1}{2} \times 0}{4}\right) = 1$$

$$1/20 - \left(\frac{\frac{1}{2} \times -2/5}{4}\right) = 1/10$$

$$0 - \left(\frac{1/2 \times 1}{4}\right) = -\frac{1}{8}$$

$$6 - \frac{\frac{1}{2} \times 32}{4} = 2$$

For maximization problem we should get values $c_j - z_j$ less than or equal to 0.

Here $c_j - z_j$ is -ve or 0.

So optimal sol^n is $x_1 = 12$, $x_2 = 16$, $z_{max} = 128$

So optimality is reached.

Q. Use the Simplex method to solve the following LPP and identify the movements graphically.

$$\text{Max } z = 4x_1 + 3x_2$$

subject to

$$x_1 + x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

Sol^n. →

$$\text{Max } z = 4x_1 + 3x_2 + 0S_1 + 0S_2$$

s.t

$x_1 + x_2 + S_1 = 8$ —(1)

$2x_1 + x_2 + S_2 = 10$ —(2)

x_1, x_2, S_1 and $S_2 \geq 0$

(we add slack variable S_1 & S_2 in both the eqns (constraints) to balance the eqn (constraint) and also add in the objective fn with coeff. zero)

Initial Simplex Table

C_{B_i}	C_j	4	3	0	0	Solution	Ratio
	Basic variable	x_1	x_2	S_1	S_2		
0	S_1	1	1	1	0	8	$\frac{8}{1} = 8$
0 ←	S_2	2	1	0	1	10	$\frac{10}{2} = 5$ → min value
	Z_j	0	0	0	0	0	
	$Z_j - C_j$	−4 ↑	−3	0	0		

Put $x_1 = 0$ and $x_2 = 0$ in (1) & (2) we get $S_1 = 8$, $S_2 = 10$

C_j → Coeff. of objective fn

C_{B_i} → cost of objective fn i.e, coeff. of basic variable

Basic variable → variable which form identity matrix (Column of Basic variable) i.e, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Here S_1 & S_2 form identity matrix so S_1 & S_2 are basic variable.

Basic variable (row) → no. of variables in obj. fn i.e, x_1, x_2, S_1 & S_2

$Z_j = \Sigma C_{B_i} a_{ij}$ | $(0 \times 1) + (0 \times 2) = 0$ and $0 \times 0 + 0 \times 1 = 0$

$0 \times 1 + 0 \times 1 = 0$

$0 \times 1 + 0 \times 0 = 0$

$0 \times 8 + 0 \times 10 = 0$

optimality condition

For max

all $z_j - c_j \geq 0$ ie, all +ve or 0, should be $z_j - c_j$

For min

all $z_j - c_j \leq 0$ ie, all -ve or 0, should be $z_j - c_j$

Here we have got all -ve or 0, so we did not reach optimality. we should proceed further for optimality. In order to proceed further

1st step → To find the most negative no in $z_j - c_j$ for key column

Here -4 is the most negative no so x_1 is the key column.

2nd step → To find key row, we have to find

ratio = $\frac{\text{Solution}}{\text{Key column}}$ and find the minimum value

Here 5 is the minimum ratio.

So row S_2 will leave and column x_1 will enter as new as basic variable in place of S_2. 2 is the key value as it is the intersection point of key row and key column.

we shall form new Iteration table.

Iteration I.

C_{B_i}	C_j	4	3	0	0	Solution	Ratio
	Basic variable	x_1	x_2	S_1	S_2		
0	S_1	0	1/2	1	-1/2	3	$\frac{3}{1/2} = 6$
4	x_1	1	1/2	0	1/2	5	$5/\frac{1}{2} = 10$
	z_j	4	2	0	2	20	
	$z_j - c_j$	0	-1	0	2		

C_{B_i} → coeff. of S_1 and x_2 in z.

Basic variable → x_1 as S_2 will leave

Row value of x_1 → old value of S_2 row / key element

ie, $2/2 = 1$, $1/2$, $0/2 = 0$, $1/2$, $10/2 = 5$

New value of S_1 row $\rightarrow$ old value $-$ $\dfrac{\text{Corr. key column value} \times \text{Corr. key row value}}{\text{key element}}$

$1 - \frac{1\times 2}{2} = 1 - \frac{2}{2} = 1 - 1 = 0$

$1 - \frac{1\times 2}{2} = 1 - \frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$

$1 - \frac{1\times 0}{2} = 1 - 0 = 1$

$0 - \frac{1\times 1}{2} = -\frac{1}{2}$

$8 - \frac{1\times 10}{2} = 8 - 5 = 3$

Again we found that $z_j - c_j$ has -ve value but we need all +ve value or 0. So we have to proceed further and continue the same process. we need to find key column, key row and key element and min ratio for next iteration.

Iteration II

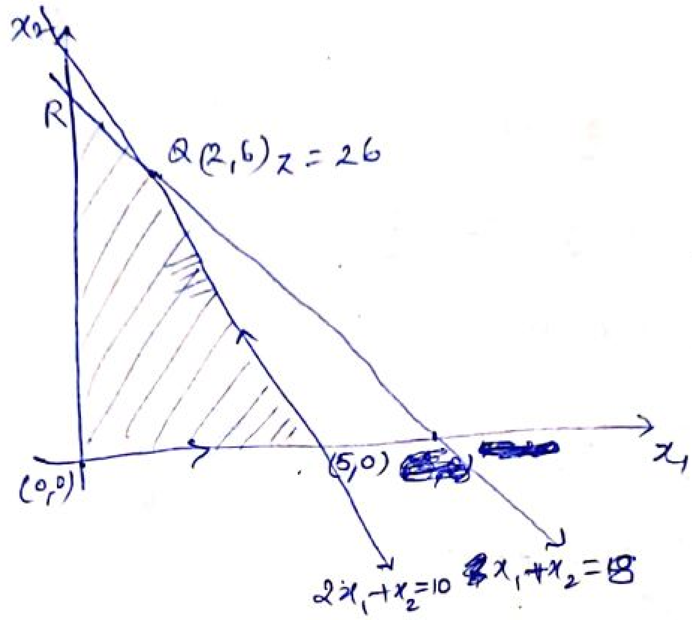
C_{B_i}	c_j / Basic variable	4 x_1	3 x_2	0 s_1	0 s_2	solution	Ratio
3	x_2	0	1	2	-1	6	
4	x_1	1	0	-1	1	2	
	Z_j	4	3	2	1	26	
	$Z_j - C_j$	0	0	2	1		

Row $x_2 \rightarrow 0/\tfrac{1}{2} = 0$, $\tfrac{1}{2}/\tfrac{1}{2} = 1$, $1/\tfrac{1}{2} = 2$, $-\tfrac{1}{2}/\tfrac{1}{2} = -1$, $3/\tfrac{1}{2} = 6$ [old row S_1/key elt]

New value for row $x_1 \rightarrow 1 - \frac{\frac{1}{2}\times 0}{\frac{1}{2}} = 1$, $\frac{1}{2} - \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2} - \frac{1}{2} = 0$

$0 - \frac{\frac{1}{2}\times 1}{\frac{1}{2}} = -1$, $\frac{1}{2} - \frac{\frac{1}{2}\times -\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$

$5 - \frac{\frac{1}{2}\times 3}{\frac{1}{2}} = 5 - 3 = 2$. Thus we have $z_j - c_j$ all +ve and 0. So optimality is reached. Max $Z = 26$ at $x_1 = 2$ & $x_2 = 6$ which is basic feasible solution.



As the initial basic feasible solution corresponds to the corner point $(0,0)$, the above simplex tableaus tell that we move from corner point $(0,0)$ to $(5,0)$ and then to $(2,6)$ which is the optimal corner point.

The Two Phase Method

Consider two LPPs

(1)

$$\text{Min } z = x_1 - 3x_2 + 2x_3$$
s.t.
$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

(2)

$$\text{Maximize } z = 5x_1 - 3x_2 + x_3$$
s.t.
$$3x_1 - x_2 - x_3 \ge 3$$
$$x_1 + 2x_2 = 4$$
$$x_1 - x_2 + 2x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

Simplex Method

Type 1:
when all constraints $\le$ form
↓
we use simplex method

Type 2:
when at least one constraint is either $\ge$ or $=$ form
↓
we use Two-Phase or Big M method

Here (1) is of type 1 and (2) is of type 2.

Two Phase Method: when we get $\ge$ sign we are unable to find identity matrix for basic feasible soln or basic variable. So we introduce artificial variable so that we get identity matrix.

The artificial variable technique is merely a device to get the starting basic feasible solution (BFS), so that the simplex procedure maybe adopted as usual until the optimal solution is obtained.

Goal of simplex method → Feasibility → optimality

Two Phase Method

Phase-I → The task of the Phase I is to eliminate the artificial variable (ie, to check the feasibility of the LPP

Phase II → while Phase-II is used to get the optimal solution. when we complete Phase I then only we will be able to enter Phase II.

Since the solution of the LPP is completed in two phases, it is called Two-Phase Simplex Method.

Eg- Use Two Phase Method, to

Minimize $z = 7.5x_1 - 3x_2$

s.t $3x_1 - x_2 - x_3 \geq 3,$

$x_1 - x_2 + x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Sol^n.

→ To convert $\geq$ to $\leq$ add slack variable

The standard form of LPP is

Max $z = -7.5x_1 + 3x_2$

s.t $3x_1 - x_2 - x_3 - s_1 = 3$

$x_1 - x_2 + x_3 - s_2 = 2$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

C_{Bi}	c_j	-7.5	3	0	0	0	Soln
	BV	x_1	x_2	x_3	s_1	s_2	
?		3	-1	-1	-1	0	3
?		1	-1	1	0	-1	2
	z_j						
	$z_j - c_j$						

There does not exist $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ in this table for BFS a BV

To handle it, we added an artificial variable denoted by A in the given LPP (for all those constraints which are either $\geq$, or $=$ form) which gives

Max $z = -7.5x_1 + 3x_2 - A_1 - A_2$

s t $3x_1 - x_2 - x_3 - s_1 + A_1 = 3$

$x_1 - x_2 + x_3 - s_2 + A_2 = 2$

$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$

C_{Bi} ↓	c_j →	-7.5	3	0	0	0	-1	-1	Soln	Ratio
	BV ↓	x_1	x_2	x_3	s_1	s_2	A_1	A_2		
-1	A_1	3	-1	-1	-1	0	1	0	3	
-1	A_2	1	-1	1	0	-1	0	1	2	
	z_j									
	$z_j - c_j$									

∵ A_1 & A_2 form identity matrix $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ so basic variables are A_1 & A_2

Our aim is get $A_1, A_2 = 0$ i.e, to eliminate A_1 & A_2 to get the feasible sol^n.

To start Phase I we ~~also~~ delete $-7.5x_1 + 3x_2$ and take only artificial variable i.e, $-A_1 - A_2$ in z

Q.

C_{Bi}	C_j / BV	0 x_1	0 x_2	0 x_3	0 s_1	0 s_2	0 A_1	-1 A_2	Solution
0	x_1	1	-1/2	0	-1/4	-1/4	1/4	1/4	5/4
0	x_3	0	-1/2	1	1/4	-3/4	-1/4	3/4	3/4
	z_j	0	0	0	0	0	0	0	0
	$z_j - c_j$	0	0	0	0	0	1	1	0

$\because z_j - c_j \geq 0$ for all j. Thus Phase I. terminates and BFS is feasible ie, A_1 & A_2 are eliminated.

Note : Since no artificial variable appears in basic variables and $z=0$, so we proceed to Phase II.

In Phase II we delete A_1 & A_2 from the eqn.

ie, $\text{Max } z = -7.5x_1 + 3x_2$

s.t $3x_1 - x_2 - x_3 - s_1 + A_1 = 3$.

$x_1 - x_2 + x_3 - s_2 + A_2 = 2$

$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$

C_{Bi}	C_j / BV	-7.5 x_1	3 x_2	0 x_3	0 s_1	0 s_2	Solution
-7.5	x_1	1	-1/2	0	-1/4	-1/4	5/4
0	x_3	0	-1/2	1	+1/4	-3/4	3/4
	z_j	-7.5	.75	0	75/40	75/40	
	$z_j - c_j$	0	3/4	0	15/8	15/8	-75/8

$.75 = \frac{75}{100} = \frac{3}{4}$

$\frac{7.5}{4} = \frac{75}{40} = \frac{15}{8}$

Since $z_j - c_j \geq 0$ for all j

Thus the current BFS is optimal

$x_1 = 5/4, \; x_2 = 0, \; x_2 = 3/4, \; \text{Max } z = \frac{-75}{8}$

Thus $\text{Min } z = \frac{75}{8}$

Phase I

$\text{Max } Z = -A_1 - A_2$

s.t. $3x_1 - x_2 - x_3 - s_1 + A_1 = 3$

$x_1 - x_2 + x_3 - s_2 + A_2 = 2$

$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$

key element ←

C_{B_i}	C_j / BV	0 / x_1	0 / x_2	0 / x_3	0 / s_1	0 / s_2	-1 / A_1	-1 / A_2	Soln.	Ratio
-1	A_1	$\boxed{3}$	-1	-1	-1	0	1	0	3	$3/3 = 1$ ← min ratio key column
-1	A_2	1	-1	1	0	-1	0	1	2	$\frac{2}{1} = 2$
	Z_j	-4	2	0	1	1	-1	-1	-5	
	$Z_j - C_j$	-4	2	0	1	1	0	0		

↑ key column

most -ve value is -4 so

Rule to find leaving variable

a. Take a ratio of solution column with key column (only those values which are positive)

b. choose the minimum ratio & corresponding row is called key row

x_1 row will enter in place of A_1

Here A_1 row will leave and x_1 row will enter in place of A_1

C_{B_i}	C_j / BV	0 / x_1	0 / x_2	0 / x_3	0 / s_1	0 / s_2	-1 / A_1	-1 / A_2	Sol^n	Ratio
0	x_1	1	$-1/3$	$-1/3$	$-1/3$	0	$1/3$	0	1	-3 (∵ -ve value so do not consider)
-1	A_2	0	$-2/3$	$\boxed{4/3}$	$1/3$	-1	$-1/3$	1	1	$1/\frac{4}{3} = 3/4$ ←
	Z_j	0	$2/3$	$-4/3$	$-1/3$	1	$1/3$	-1	-1	
	$Z_j - C_j$	0	$2/3$	$-4/3$	$-1/3$	1	$4/3$	0		

∵ BV is 2×2 matrix and it will form identity matrix so $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Row A_2 (new value)

$= 1 - \frac{1 \times 3}{3} = 0,$

$-1 - \frac{1 \times -1}{3} = -1 + \frac{1}{3} = \frac{-2}{3}$

$1 - \frac{1 \times -1}{3} = \frac{4}{3}$

$0 - \frac{1 \times -1}{3} = 1/3$

$-1 - \frac{1 \times 0}{3} = -1$

$0 - \frac{1 \times 1}{3} = -\frac{1}{3}$

$2 - \frac{1 \times 3}{3} = 1$

most negative value is $-4/3$ so key column is x_3

So column x_3 will enter and row A_2 will be deleted and $4/3$ is the key element.

Q. Use Two Phase Method to Minimize $z = 2x_1 + x_2$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Sol^n. → After converting it in standard form we get the following

$$\text{Max } z = -2x_1 - x_2 + 0s_1 + 0s_2$$

s.t

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - s_1 = 6 \quad [\text{surplus variable} \because \geq]$$
$$x_1 + 2x_2 + s_2 = 3 \quad [\text{slack variable} \because \leq]$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Here matrix $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ does not have a 3×3 identity matrix

So we need to add artificial variable for = & ≥ sign in constraint. i.e,

$$\text{Max } z = -2x_1 - x_2 - A_1 - A_2$$

s.t

$$3x_1 + x_2 + A_1 = 3$$
$$4x_1 + 3x_2 - s_1 + A_2 = 6$$
$$x_1 + 2x_2 + s_2 = 3$$

Now $A = \cancel{\begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix}}$ $\begin{array}{cccccc} x_1 & x_2 & A_1 & S_1 & A_2 & S_2 \end{array}$ $\begin{bmatrix} 3 & 1 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$

BV → A_1, A_2, S_2 [$\because$ it forms 3×3 Identity matrix]

C_B	C_j	-2	-1	0	0	-1	-1	Soln	Ratio
	BV	x_1	x_2	s_1	s_2	A_1	A_2		
		3	1	0	0	1	0	3	
		4	3	-1	0	0	1	6	
		1	2	0	1	0	0	3	

C_B	C_j	-2	-1	-1	0	-1	0	Soln	Ratio
	BV	x_1	x_2	A_1	s_1	A_2	s_2		
-1	A_1	3	1	1	0	0	0	3	$3/3=1$
-1	A_2	4	3	0	-1	1	0	6	$6/4=\frac{3}{2}$ min Ratio
0	S_2	1	2	0	0	0	1	3	$3/1=3$
	z_j	-7	-4	-1	1	-1	0	-9	
	z_j-c_j	-5	-3	0	1	0	0		

Phase I:
we take

$$\text{Max } Z = -A_1 - A_2$$
s.t
$$3x_1 + x_2 + A_1 = 3$$
$$4x_1 + 3x_2 - s_1 + A_2 = 6$$
$$x_1 + 2x_2 + s_2 = 3$$
$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

C_B	C_j	0	0	-1	0	-1	0	Soln	Ratio
	BV	x_1	x_2	A_1	s_1	A_2	s_2		
-1	A_1	3	1	1	0	0	0	3	$3/3=1$ → min ratio
-1	A_2	4	3	0	-1	1	0	6	$6/4=\frac{3}{2}$ ~~min ratio~~
0	S_2	1	2	0	0	0	1	3	$3/1=3$
	z_j	-7	-4	-1	0	-1	0	-9	
	z_j-c_j	-7	-4	0	1	0	0	-9	

most negative value
key column (entering as row x_1 in the place of A_1 in next Iteration)
Min^m ratio is 1 is A_1 row is key row and A_1 row will be deleted in next Iteration.

Iteration I.

C_{B_i}	C_j	0	0	-1	0	-1	0	Soln	Ratio
	BV	x_1	x_2	A_1	s_1	A_2	s_2		
0	x_1	1	1/3	1/3	0	0	0	1	$1/1/3=3$
-1	A_2	0	5/3	$-\frac{4}{3}$	-1	1	0	2	$2/5/3=\frac{6}{5}$ ← min ratio (∵ we have to eliminate A_2) (we choose A_2 row)
0	S_2	0	5/3	-1/3	0	0	1	2	$2/5/3=\frac{6}{5}$
	z_j	0	-5/3	4/3	1	-1	0	-2	
	z_j-c_j	0	-5/3	7/3	1	0	0		
0	x_1	1	0	3/5	-1/5	$-\frac{1}{5}$	0	3/5	
0	x_2	0	1	-4/5	-3/5	3/5	0	6/5	
0	S_2	0	0	1	1	1	1	0	
	z_j	0	0	0	0	0	0	0	
	z_j-c_j	0	0	1	0	1	0		

$z_j - c_j \geq 0$ so Phase I is feasible and A_1 & A_2 is eliminated.

Phase II

$$\text{Max } Z = -2x_1 - x_2$$
s.t $3x_1 + x_2 = 3$
$4x_1 + 3x_2 - s_1 = 6$
$x_1 + 2x_2 + s_2 = 3$
$x_1, x_2, x_3, s_1, s_2 \geq 0$

C_B	C_j	-2	-1	0	0	Soln
	BV	x_1	x_2	s_1	s_2	
-2	x_1	1	0	-1/5	0	3/5
-1	x_2	0	1	-3/5	0	6/5
0	S_2	0	0	1	1	0
	z_j	-2	-1	1	0	-12/5
	z_j-c_j	0	0	1	0	

∵ $z_j - c_j \geq 0$ and 0 so

$Z_{max} = -\frac{12}{5}$ or $Z_{min} = 12/5$

at $x_1 = 3/5$, x_2 $6/5$, $s_2 = 0$.

∴ Z is optimal at x_1 & x_2

The Big-M Method (Charne's Penalty Method)

Q. Solve the given LPP by Big-M method

Minimize $Z = 2x_1 + x_2$

s.t $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

Sol^n. → Maximize $Z = -2x_1 - x_2$

s.t $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

The standard form of LPP is

Maximize $Z = -2x_1 - x_2 - MA_1 - MA_2$ (Here M is a +ve no which is very very large)

s.t $3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 - s_2 + A_2 = 6$

$x_1 + 2x_2 + s_3 = 3$

$x_1, x_2, s_2, s_3, A_1, A_2 \geq 0$

C_{Bi}	C_j	-2	-1	0	0	$-M$	$-M$	Sol^n	Ratio
	BV	x_1	x_2	s_2	s_3	A_1	A_2		
$-M$	← A_1	3	1	0	0	1	0	3	$3/3 = 1$ min ratio
$-M$	A_2	4	3	-1	0	0	1	6	$6/4 = \frac{3}{2}$
0	s_3	1	2	0	1	0	0	3	$3/1 = 3$
	Z_j	$-7M$	$-4M$	M	0	$\cancel{-M}$	$-M$	-9	
	$Z_j - C_j$	$2-7M$	$1-4M$	M	0	0	0		

↑ most -ve value so key column is x_1

Row x_1 will enter and row A_1 will leave

optimality condition is $Z_j - C_j \geq 0$

Here we have large M so to check take $M = 100$ we get -ve value so we proceed to next iteration. For key column and key row find min ratio.

Iteration I

C_{B_i}	C_j	-2	-1	0	0	$-M$	$-M$		
	BV	x_1	x_2	s_2	s_3	A_1	A_2	Solⁿ	Ratio
-2	x_1	1	$1/3$	0	0	$1/3$	0	1	3
$-M$	A_2	0	$5/3$	-1	0	$-4/3$	1	2	$6/5$ min ratio
0	S_3	0	$5/3$	0	1	$-1/3$	0	2	$6/5$
	Z_j	-2	$-\frac{2}{3}-\frac{5M}{3}$	M	0	$-\frac{2}{3}+\frac{4M}{3}$	$-M$	$-2-2M$	
	Z_j-C_j	0	$\frac{-5M-1}{3}$	M	0	$\frac{7M-2}{3}$	0		

$\because A_2$ is artificial variable so we select A_2 row as key row.

A_2 row $\rightarrow$

$4-\frac{4\times3}{3}=0$, $3-\frac{4\times1}{3}=\frac{9-4}{3}=5/3$,

$-1-\frac{4\times0}{3}=\cancel{\frac{-3-12}{5}}=\cancel{\frac{-15}{15}}=-1$,

$0-\frac{4\times0}{3}=\cancel{\frac{12}{3}}=0$, $0-\frac{4\times1}{3}=-\frac{4}{3}$,

$1-0\times\frac{4}{3}=1$, $6-4\frac{\times3}{3}=2$

In the same way we can find S_3.

Iteration II

C_{B_i}	C_j	-2	-1	0	0	$-M$	$-M$	
	BV	x_1	x_2	s_2	s_3	A_1	A_2	Soln
-2	x_1	1	0	$1/5$	0	$3/5$	$-1/5$	$3/5$
-1	x_2	0	1	$-3/5$	0	$-4/5$	$3/5$	$6/5$
0	s_3	0	0	1	1	1	-1	0
	Z_j	-2	-1	$1/5$	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$-\frac{12}{5}$
	Z_j-C_j	0	0	$1/5$	0	$-\frac{2}{5}+M$	$-\frac{1}{5}+M$	

Since $Z_j-C_j \geq 0$ for all j

Thus, the current BFS is optimal

$x_1=3/5$, $x_2=\frac{6}{5}$

$\text{Max } Z=-\frac{12}{5}$

Thus $\text{Min } Z=12/5$.

HW

Q. Solve the given LPP by the Big-M Method

$$\text{Max } Z=3x_1-x_2-x_3$$

s.t

$$x_1-2x_2+x_3\leq 11$$
$$-4x_1+x_2+2x_3\geq 3$$
$$-2x_1+x_3=1$$
$$x_1,x_2,x_3\geq 0$$

Ans: 3 Iteration

$x_1=4$, $x_2=1$, $x_3=9$

$Z_{max}=2$.

Solⁿ $\rightarrow$

$$\text{Max } Z=3x_1-x_2-x_3+0s_1+0s_2-MA_1-MA_2$$

s.t

$$x_1-2x_2+x_3+s_1=11$$
$$-4x_1+x_2+2x_3-s_2+A_1=3$$
$$-2x_1+x_3+A_2=1$$
$$x_1,x_2,x_3,s_1,s_2,A_1,A_2\geq 0$$

Taking s_1, A_1, A_2 are basic variable and solving by simplex method we obtain the following tableaus–

Solve the foll. problem by the simplex method and verify your answer graphically.

Q. Max $z = 4x_1 + 3x_2$

s.t $x_1 + x_2 \geq 8$

$2x_1 + x_2 \geq 10$

$x_1, x_2 \geq 0$

Sol^n → Here we need two artificial variables A_1 and A_2 and the Phase I problem is

$$\text{Max } z = -A_1 - A_2$$

$$x_1 + x_2 - s_1 + A_1 = 8$$

$$2x_1 + x_2 - s_2 + A_2 = 10$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

Now in the initial basic feasible solution A_1 and A_2 are the basic variables. This gives the simplex tableaus

c_{Bi}	c_j	0	0	0	0	-1	-1	Soln	Ratio
	BV	x_1	x_2	s_1	s_2	A_1	A_2		
-1	A_1	1	1	-1	0	1	0	8	$8/1 = 8$
-1	A_2	2	1	~~0~~	-1	0	1	10	$10/2 = 5$
	z_j	-3	-2	1	1	-1	-1	-18	
	$z_j - c_j$	-3	-2	1	1	0	0		

Iteration I

c_{Bi}	c_j	0	0	0	0	-1	-1	Soln	Ratio
	BV	x_1	x_2	s_1	s_2	A_1	A_2		
-1	A_1	0	1/2	-1	1/2	1	-1/2	3	$3/\frac{1}{2} = 6$
0	x_1	1	1/2	0	-1/2	0	1/2	5	$5/\frac{1}{2} = 10$
	z_j	0	-1/2	1	-1/2	-1	1/2	-3	
	$z_j - c_j$	0	-1/2	1	-1/2	0	3/2		

new value for A_1 row →

$1 - \frac{2 \times 1}{2} = 0$, $1 - \frac{1 \times 1}{2} = \frac{1}{2}$,

$-1 - \frac{1 \times 0}{2} = -1$, $0 - \frac{1 \times -1}{2} = \frac{1}{2}$,

$1 - \frac{1 \times 0}{2} = 1$, $0 - \frac{1 \times 1}{2} = -\frac{1}{2}$

$8 - \frac{1 \times 10}{2} = 3$

Iteration II

c_{Bi}	c_j	0	0	0	0	-1	-1	Soln	Ratio
	BV	x_1	x_2	s_1	s_2	A_1	A_2		
0	x_2	0	1	-2	1	2	-1	6	
0	x_1	1	0	1	-1	-1	1	2	
	z_j	0	0	0	0	0	0	0	
	$z_j - c_j$	0	0	0	0	1	1		

∵ A_1 & A_2 are eliminated and objective fn value for Phase I is zero so the given LPP is feasible. Also both artificial variables are non-basic so drop them and go to Phase II.

The Phase II problem is

$$\text{Max } z = 4x_1 + 3x_2 + 0s_1 + 0s_2$$

s.t

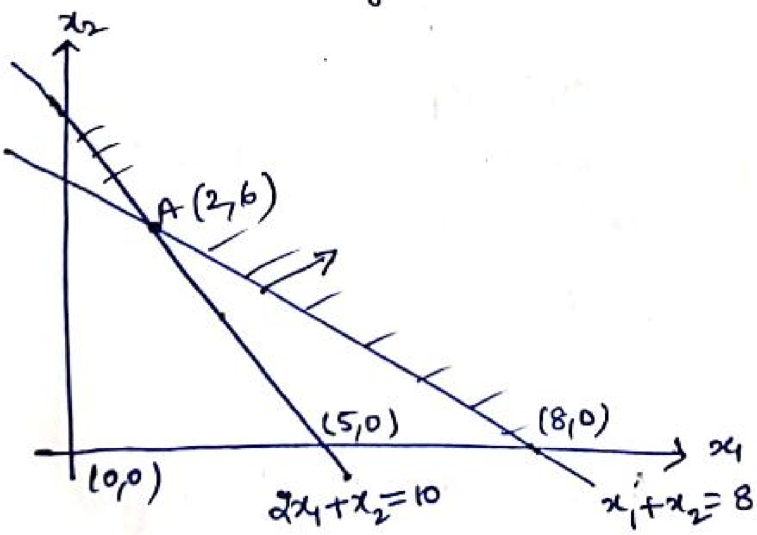
$$x_1 + x_2 - s_1 = 8$$
$$2x_1 + x_2 - s_2 = 10$$
$$x_1, x_2, s_1, s_2 \geq 0$$

The initial tableau for Phase II problem is

C_{Bi}	C_j	4	3	0	0		
	BV	x_1	x_2	s_1	s_2	Soln	Ratio
3	x_2	0	1	-2	1	6	$6/-2 = -3$ → Ignore -ve value
4	x_1	1	0	1	-1	2	$2/1 = 2$
	Z_j	4	3	-2	-1	26	
	$Z_j - C_j$	0	0	-2	-1		

C_{Bi}	C_j	4	3	0	0		
	BV	x_1	x_2	s_1	s_2	Soln	
3	x_2	2	1	0	-1	10	
0	s_1	1	0	1	-1	2	
	Z_j	6	3	0	-3	30	
	$Z_j - C_j$	2	0	0	-3		

∵ $Z_j - C_j$ is not all +ve and zero.
We conclude that the given LPP has unbounded solution.



The arrow indicates the direction of increase of the given objective fn. It clearly indicates that the given LPP has <u>unbounded solution</u>.

Q. Solve the following problem by simplex method and verify your answer graphically.

$$\text{Max } Z = 4x_1 + 3x_2$$

s.t

$$x_1 + x_2 \leq 8$$
$$5x_1 + 6x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

Solⁿ. → Here we need only one artificial variable A_1 and therefore the Phase-I problem is

$$\text{Max } Z = -A_1$$

s.t

$$x_1 + x_2 + s_1 = 8$$
$$5x_1 + 6x_2 - s_2 + A_1 = 60$$
$$x_1, x_2, s_1, s_2, A_1 \geq 0$$

Taking s_1 and A_1 as initial basic variables we have the foll simplex tableau for Phase I problem.

C_{Bi}	C_j	0	0	0	0	-1	Solⁿ	Ratio
	BV	x_1	x_2	s_1	s_2	A_1		
← 0	s_1	1	1	1	0	0	8	$8/1 = 8$
-1	A_1	5	6	0	-1	1	60	$60/6 = 10$
	Z_j	-5	-6	0	1	-1	-60	
	$Z_j - C_j$	-5	-6 ↑	0	1	0		

Iteration I

C_{Bi}	C_j	0	0	0	0	-1	Solⁿ	Ratio
	BV	x_1	x_2	s_1	s_2	A_1		
0	x_2	1	1	1	0	0	8	
-1	A_1	-1	0	-6	-1	1	12	
	Z_j	1	0	6	1	-1	-12	
	$Z_j - C_j$	1	0	6	1	0		

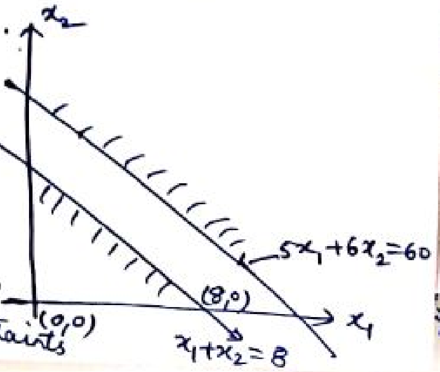
New value for A_1 ~~row~~ row →

$5 - \frac{6\times1}{1} = -1$, $6 - \frac{6\times1}{1} = 0$,

$0 - \frac{6\times1}{1} = -6$, $-1 - \frac{6\times0}{1} = -1$

$1 - \frac{6\times0}{1} = 1$, $60 - \frac{6\times8}{1}$
$= 60 - 48 = 12$

Here ~~$Z_j - C_j$~~ $Z_j - C_j \geq 0$. The Phase I problem has been solved. Here A_1 is not eliminated so Phase I problem is <u>infeasible</u> and $Z_{max} = 12$.



The two lines do not intersect in the first quadrant. There is no pt. in R^2 satisfying all the constraints and so the set of feasible solⁿ is an empty set. Thus the given LPP has infeasible solⁿ and the constraints are inconsistent.

Alternate optima

If the given LPP has an optimal soln then either it has unique optimal solution or it has infinitely many optimal solutions.
When there are infinitely many optimal solutions, we say the given LPP has alternative optima. In general, it can be shown that set of all optimal solutions is the convex set spanned by the optimal corner points i.e, it is the convex hull of the optimal corner points.

* In the optimal simplex tableau of the given LPP, if for some non-basic variable x_j, $z_j - c_j = 0$ and for that some $x_{ij} > 0$ then the problem has alternative optima.

Ex.

$$\text{Max } z = x_1 + x_2$$

s.t

$$x_1 + x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

Soln. → Here we need two slack variables to get the following standard form LPP

$$\text{Max } z = x_1 + x_2 + 0s_1 + 0s_2$$

s.t

$$x_1 + x_2 + s_1 = 8$$
$$2x_1 + x_2 + s_2 = 10$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Now in the initial b.f.s, s_1 and s_2 are basic variables.
This gives the foll. simplex tableau

C_{Bi}	c_j	1	1	0	0	Soln	Ratio
	BV	x_1	x_2	s_1	s_2		
0	s_1	1	1	1	0	8	$\frac{8}{1} = 8$
← 0	s_2	2	1	0	1	10	$\frac{10}{2} = 5$
	z_j	0	0	0	0	0	
	$z_j - c_j$	−1 ↑	−1	0	0	0	

2
−1
1

Iteration I

C_{Bi}	C_j	1	1	0	0	Soln	Ratio
	BV	x_1	x_2	s_1	s_2		
0	s_1	0	1/2	1	-1/2	3	$3/1/2 = 6$ min ratio
1	x_1	1	1/2	0	1/2	5	$5/1/2 = 10$
	z_j	1	1/2	0	1/2	5	
	$z_j - c_j$	0	-1/2	0	1/2	5	

Iteration II

C_{Bi}	C_j	1	1	0	0	Soln	Ratio
	BV	x_1	x_2	s_1	s_2		
1	x_2	0	1	2	-1	6	$6/-1 = -6$ (Ignore)
1	x_1	1	0	-1	1	2	$2/1 = 2$
	z_j	1	1	1	0	8	
	$z_j - c_j$	0	0	1	0	8	

So an optimal soln is $z_{max} = 8$, at $x_1 = 2, x_2 = 6$.

As we know $z_j - c_j = 0$ for basic variables but for some non basic variable $z_j - c_j = 0$ and $x_{ij} > 0$, then the problem has alternate optima. Here s_2 is is non-basic variable for which $z_j - c_j = 0$

so s_2 will be basic variable in next table and find another optimal b.f.s. and x_1 will leave the table as x_2 row has -1 so min ratio will be 2.

Iteration III

C_{Bi}	C_j	1	1	0	0	Soln
	BV	x_1	x_2	s_1	s_2	
1	x_2	1	1	1	0	8
0	s_2	1	0	-1	1	2
	z_j	1	1	1	0	8
	$z_j - c_j$	0	0	1	0	8

Thus we obtain another b.f.s given by $x_1 = 0, x_2 = 8, z_{max} = 8$

If the given LPP has has more then one optimal solution then it has infinitely many solutions. The set of all optimal solns. of a LPP is a convex set and has infinitely many solns.

~~Redundancy in LPP.~~ Degeneracy and cycling

Basic Solutions (BS) are of two types

Non-Degenerate B.S ← → Degenerate BS

Non-Degenerate B.S ↓ If none of the basic solution is zero

Degenerate BS ↓ If at least one of the basic solution is zero.

Occurrence of Degeneracy in LPP

$$\boxed{\begin{array}{c} \text{Max/Min } Z = C^T X \\ \text{s.t } AX \leq = \geq b \\ X \geq 0 \end{array}}$$

In LPP, degeneracy may appear in the following two ways :-

1. when some component of b is zero
2. If none of b is zero but the minimum ratio (for finding the leaving variable) is tie (Not unique)

Ex.1. Solve the LPP

$$\text{Max } z = 2x_1 + 3x_2 + 10x_3$$
$$\text{s.t } x_1 + 2x_3 = 0$$
$$x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Sol^n → The standard form of the LPP is

$$\text{Max } z = 2x_1 + 3x_2 + 10x_3$$
$$\text{s.t } x_1 + 2x_3 = 0$$
$$x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

No need to add artificial variable as we get identity matrix. So the table for LPP is

C_{Bi}	C_j	2	3	10	Sol^n	Ratio
	BV	x_1	x_2	x_3		
← 2	x_1	1	0	2	0	0 min ratio
3	x_2	0	1	1	1	1
	Z_j	2	3	7	3	
	$Z_j - C_j$	0	0	-3 ↑		

Row x_1 will leave and column x_3 will enter as basic variable

C_{Bi}	C_j / BV	2 / x_1	3 / x_2	10 / x_3	Soln
10	x_3	1/2	0	1	0
3	x_2	-1/2	1	0	1
	Z_j	7/2	3	10	3
	$Z_j - C_j$	3/2	0	0	

Here at least one of the sol^n for basic variable is zero so degenerate soln occurs. i.e., $x_3 = 0$

Since $Z_j - C_j \geq 0$, so optimal solution is $x_1 = 0, x_2 = 1, x_3 = 0$.

$Z_{max} = 3$

which is degenerate b.f.s.

Ex2. Solve the LPP

$$\text{Max } z = 5x_1 + 3x_2$$
$$\text{s.t } x_1 + x_2 \leq 2$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

Sol^n. The standard form of LPP is

$$\text{Max } z = 5x_1 + 3x_2$$
$$\text{s.t } x_1 + x_2 + s_1 = 2$$
$$5x_1 + 2x_2 + s_2 = 10$$
$$3x_1 + 8x_2 + s_3 = 12$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

C_{Bi}	C_j / BV	5 / x_1	3 / x_2	0 / s_1	0 / s_2	0 / s_3	Soln	Ratio
0	s_1	1	1	1	0	0	2	2 } tie
0	s_2	5	2	0	1	0	10	2
0	s_3	3	8	0	0	1	12	4
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$	-5	-3	0	0	0		

∵ we have same min^m Ratio so we can choose any variable as leaving row between the two i.e, s_1 & s_2. The next table will give degenerate sol^n.

... in R. where $Z_{max} = 0$

sign

Iteration I

C_{Bi}	C_j / BV	5 x_1	3 x_2	0 s_1	0 s_2	0 s_3	Soln	Ratio.
0	s_1	0	3/5	1	-1/5	0	0	0 min ratio
5	x_1	1	2/5	0	1/5	0	2	5
0	s_3	0	34/5	0	-3/5	1	6	30/24
	Z_j	5	2	0	1	0	10	
	$Z_j - C_j$	0	-1	0	1	0		

This table is degenerate but the sol^n is not optimal as $Z_j - C_j \geq 0$ so we shall proceed in next tableau

Iteration II

C_{Bi}	C_j / BV	5 x_1	3 x_2	0 s_1	0 s_2	0 s_3	Soln	Ratio
3	x_2	0	1	5/3	-1/3	0	0	
5	x_1	1	0	-2/3	1/3	0	2	
0	s_3	0	0	-34/3	5/3	1	6	
	Z_j	5	3	5/3	2/3	0	10	
	$Z_j - C_j$	0	0	5/3	2/3	0		

$\because Z_j - C_j \geq 0$, so optimal sol^n is
$x_1 = 2$, $x_2 = 0$, $Z_{max} = 10$ which is degenerate sol^n.

Cycling

cycling occurs due to degeneracy but not everytime degeneracy will lead to cycling. Degenerate b.f.s and basic variables there is no change in the objective fn value. Infact both degenerate b.f.s. and basic soln correspond to the same corner point.

Ex. $\quad \text{Min } z = \frac{3}{4}x_1 - 20x_2 + \frac{1}{2}x_3 - 6x_4$

subject to

$$\frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 \leq 0$$

$$\frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \leq 0$$

$$x_3 \leq 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Soln: Introducing slack variables and writing the problem in the standard form,

$$\text{Max } z = -\frac{3}{4}x_1 + 20x_2 - \frac{1}{2}x_3 + 6x_4$$

s.t

$$\frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 + s_1 = 0$$

$$\frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 + s_2 = 0$$

$$x_3 + s_3 = 1$$

$$x_1, x_2, \cancel{x_2}, x_3, s_1, s_2, s_3 \geq 0$$

If we solve the above problem by the simplex method then the results of various tablaues can be summerized as follows:

Iteration no	Basic variable (b.f.s)	$z_j - c_j = z$	Corner point (Soln)
1	$s_1 = 0, s_2 = 0, s_3 = 1$	0	(0, 0, 0, 0)
2	$s_1 = 0, x_1 = 0, s_3 = 1$	0	(0, 0, 0, 0)
3	$x_2 = 0, x_1 = 0, s_3 = 1$	0	(0, 0, 0, 0)
4	$x_2 = 0, x_3 = 0, s_3 = 1$	0	(0, 0, 0, 0)
5	$x_4 = 0, x_3 = 0, s_3 = 1$	0	(0, 0, 0, 0)
6	$x_4 = 0, x_6 = 0, x_7 = 1$	0	(0, 0, 0, 0)
7	$s_1 = 0, s_2 = 0, s_3 = 1$	0	(0, 0, 0, 0)

Here the b.f.s obtained at the 7th Iteration is the same as obtained at the initial iteration (i.e. Iteration 1). So we have a sequence of 6 degenerate b.f.s. which are going to cycle and they correspond to the same corner point (0, 0, 0, 0) in R^4 where $z_{max} = 0$.

The Simplex Tableau in the Condensed Form

The main advantage of using the tableau in condensed form is that we have to store lesser data and work with a smaller sized tableau. The only change in the earlier working will be with regard to pivoting (key element). The pivoting rules used earlier for the tableau in extended form will need modification if we are working with the tableau in the condensed form.

Ex. Sol^n. → Max $z = 4x_1 + 3x_2 + 0x_3 + 0x_4$

subject to $x_1 + x_2 + x_3 = 8$

$2x_1 + x_2 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Q. Max $z = 4x_1 + 3x_2$

s.t $x_1 + x_2 \leq 8$

$2x_1 + x_2 \leq 10$

$x_1, x_2 \geq 0$

If we are working with the usual tableau (ie, tableau in the extended form) the initial table will look like

C_{Bi}	c_j	4	3	0	0	
	BV	x_1	x_2	x_3	x_4	Soln
0	x_3	1	1	1	0	8
0	x_4	2	1	0	1	10
	z_j	0	0	0	0	
	$z_j - c_j$	−4	−3	0	0	

If we decide to work with the tableau in the condensed form then the initial tableau will look like

C_{Bi}	c_j	4	3	
	BV	x_1	x_2	
0	x_3	1	1	8
0	x_4	2	1	10
	z_j	0	0	0
	$z_j - c_j$	−4	−3	

which we shall normally write as

	1	2		
8	1	1		3
10	2	1		4
0	−4	−3		

x_1 and x_2 are non-basic variables. These indices help us in keeping track of basic and non-basic variables at every iteration.

Here indices in the extreme right column identify the current basic variables and indices in the topmost row identify the current non-basic variables, ie, x_3 is a basic variable whose value is 8 and x_4 is a basic variable whose value is 10.

while finding key element or pivoting we also exchange the indices of leaving and entering variables.

Here in the tableau we have $z_j - c_j$ and -4 is the most $-$ve value which is the entering basic variable i.e, x_1 and x_4 is going to be non-basic variable in next tableau. 2 is the key element.
Modified rules of pivoting for tableau in the condensed form are :-

Step 1. Replace the pivot element by its reciprocal.

Step 2. Divide the remaining entries of the pivot row by the pivot element.

Step 3. Divide the remaining entries of the pivot column by the negative of the pivot element.

Step 4. For remaining entries of the tableau, follow the same update rule as the one for tableau in the extended form.

Step 5. Exchange the indices of the variable to enter and variable to leave.

Here we may note that except for Step 5, nothing (in Step 2, 3 or 4) for indices in the right most column and top most row. As such these are not entries in the tableau, they just identity the basic and non-basic variables at every iteration.
After pivoting we get the next tableau in the condensed form

	4	2	
3	$-1/2$	$1/2$	3
5	$1/2$	$1/2$	1
20	2	-1	

c_j 0 3

$z_j = 2, 2$

$z_j - c_j = 2, -1$

Here x_3 is a basic variable whose value is 3 and x_1 is a basic variable whose value is 5. Also x_4 and x_2 are now non-basic variables. As $z_j - c_j$ is still $-$ve, the current soln is not optimal.

$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

As indicated, now x_2 becomes basic variable and x_3 becomes non-basic variable and therefore the modified pivoting rules gives the next tableau

	4	3	
6	−1	2	2
2	1	−1	1
26	1	2	

$c_j = 0 \quad 0$

$z_j = 26 \quad 1 \quad 2$

$z_j - c_j = 26 \quad 1 \quad 2$

Therefore $x_1 = 2$, $x_2 = 6$ and $z = 26$.
which is an optimal soln.

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Duality in LPP

Primal to Dual Conversion →

Steps

1. First we convert the problem in canonical form

$\text{Min } z \to \text{Min } z \geq$

$\text{Max } z \to \text{Max } z \leq$

2. Change the objective function of maximization in the primal into minimization in the dual and vice-versa. $\text{Max } z \to \text{Min } z \geq$, $\text{Min } z \to \text{Max } z \leq$

3. The no. of variable in the primal will be the no. of constraints in dual and vice-versa.

i.e, Primal $\begin{cases} \text{no. of variable} = n \\ \text{no. of Constraints} = m \end{cases}$, dual $\begin{cases} \text{no. of constraints} = n \\ \text{no. of variable} = m. \end{cases}$

4. Cost coefficient in objective function of the primal will be RHS constant of the constraints in dual and vice-versa.

5. For formulating constraints we consider the transpose of matrix.

6. If restriction in primal is $\geq$ then it will become $\leq$ in dual and vice-versa.

If ~~such~~ constraint is of type '=' in primal then dual form will unrestricted in sign.

Table for construction of the Dual

Primal (Max)	Dual (Min)
i^{th} Constraint $\leq$ type	$w_i \geq 0$
i^{th} constraint $\geq$ "	$w_i \leq 0$
i^{th} " $=$	w_i is unrestricted in sign
$x_j \leq 0$	j^{th} constraint is $\leq$ type
$x_j \geq 0$	j^{th} constraint is $\geq$ type
x_j unrestricted in sign	j^{th} constraint is $=$ type

$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

Q. Write the dual of the foll. LPP

1. Max $-2x_1 - x_2$.

Subject to

$x_1 + x_2 \leq 10$

$x_1 - 2x_2 = -8$

$x_1 + 3x_2 \geq 9$

$x_1 \geq 0$, x_2 unrestricted in sign

Sol^n.

→ For Dual form of LPP we write in Canonical form

Max $Z = -2x_1 - x_2$

s.t $x_1 + x_2 \leq 10$

$x_1 - 2x_2 = -8$

$-x_1 - 3x_2 \leq -9$

$x_1 \geq 0$, x_2 unrestricted in sign

Then dual form is

Min $W = 10w_1 - 8w_2 - 9w_3$

s.t

$w_1 + w_2 - w_3 \geq -2$

$w_1 - 2w_2 - 3w_3 \geq -1$

$w_1 \geq 0$, $w_3 \geq 0$, w_2 unrestricted in sign

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & -3 \end{bmatrix}$$

2. Min $Z = 2x_1 - x_2 + x_3$

s.t $2x_1 + x_2 - x_3 \leq 8$

$-x_1 + x_3 \geq 1$

$x_1 + 2x_2 + 3x_3 = 9$

$x_1 \geq 0$, $x_2 \geq 0$, x_3 unrestricted in sign

Sol^n.

→ Canonical form

Min $Z = 2x_1 - x_2 + x_3$

s.t

$-2x_1 - x_2 + x_3 \geq -8$

$-x_1 + x_3 \geq 1$

$x_1 + 2x_2 + 3x_3 = 9$

$x_1 \geq 0$, $x_2 \geq 0$, x_3 unrestricted in sign

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Dual form

Max $W = -8w_1 + w_2 + 9w_3$

s.t

$-2w_1 - w_2 + w_3 \leq 2$

$-w_1 + w_3 \leq -1$

$w_1 + w_2 + 3w_3 = 1$

$w_1 \geq 0$, $w_2 \geq 0$, w_3 unrestricted in sign

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Note (1) If p^{th} constraint of primal holds strict equality then its dual, the p^{th} dual variable is unrestricted in sign
(2) If k^{th} primal variable is unrestricted in sign then its dual the k^{th} constraint holds the strict equality.

Q. $\text{Min } z = 2x_2 + 5x_3$

subject to $x_1 + x_2 \geq 2$

$$2x_1 + x_2 + 6x_3 \leq 6$$
$$x_1 - x_2 + 3x_3 = 4$$
$$x_1, x_2, x_3 \geq 0$$

Soln. Canonical form,

$$\text{Min } z = 0.x_1 + 2x_2 + 5x_3$$

s.t $x_1 + x_2 \geq 2$

$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$-x_1 + x_2 - 3x_3 \geq -4$$
$$x_1, x_2, x_3 \geq 0$$

[$x_1 - x_2 + 3x_3 = 4$ can be written as $x_1 - x_2 + 3x_3 \geq 4$, $x_1 - x_2 + 3x_3 \leq 4$]

Dual form, let w_1, w_2, w_3, w_4 be dual variables

$$\text{Max } w = 2w_1 - 6w_2 + 4w_3 - 4w_4$$

s.t

$$w_1 - 2w_2 + w_3 - w_4 \leq 0$$
$$w_1 - w_2 - w_3 + w_4 \leq 2$$
$$0w_1 - 6w_2 + 3w_3 - 3w_4 \leq 5$$
$$w_1, w_2, w_3, w_4 \geq 0$$

Rearrange, $w_3 - w_4 = w_3'$

$$\text{Max } w = 2w_1 - 6w_2 + 4w_3'$$

s.t

$$w_1 - 2w_2 + w_3' \leq 0$$
$$w_1 - w_2 - w_3' \leq 2$$
$$-6w_2 + 3w_3' \leq 5$$

$w_1, w_2 \geq 0$ and w_3' is unrestricted in sign.

Weak Duality Theorem:- Let x be feasible for the primal and w be feasible for the dual. Then $c^T x \leq b^T w$

Cor. Let $\bar{x}$ be feasible for the primal and $\bar{w}$ be feasible for the dual. Also let $c^T\bar{w} = b^T\bar{w}$. Then $\bar{x}$ is optimal to the primal and $\bar{w}$ is optimal to the dual.

Strong Duality Theorem:-

(i) Let $\bar{x}$ be an optimal solution of the primal. Then there exists a $\bar{w}$ which is optimal to the dual. Also $c^T\bar{x} = b^T\bar{w}$.

(ii) Let w^* be an optimal solution of the dual. Then there exists a x^* which is optimal to the primal. Also $b^T w^* = c^T x^*$.

Here part (i) is called the direct theorem and (ii) is called the converse duality theorem. As (dual) = primal.

Ex. Write the dual of the foll. LPP (primal)

$$\left.\begin{array}{l} \text{Max } 4x_1 + 3x_2 \\ \text{Subject to } x_1 + x_2 \leq 8 \\ 2x_1 + x_2 \leq 10 \\ x_1, x_2 \geq 0 \end{array}\right\} (1)$$

Standard form

$$\begin{array}{l} \text{Max } 4x_1 + 3x_2 \\ \text{s.t. } x_1 + x_2 + s_1 = 8 \\ 2x_1 + x_2 + s_2 = 10 \\ x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

Soln. Dual of LPP is

$$\left.\begin{array}{l} \text{Min. } 8w_1 + 10w_2 \\ \text{s.t. } w_1 + 2w_2 \geq 4 \\ w_1 + w_2 \geq 3 \end{array}\right\} (2)$$

$$w_1, w_2 \geq 0$$

we have to find the optimal soln of (2) by solving (1)

C_{B_i}	C_j	4	3	0	0	Soln	Ratio
	BV	x_1	x_2	s_1	s_2		
0	s_1	1	1	1	0	8	$8/1 = 8$
0	s_2	2	1	0	1	10	$10/2 = 5$ min ratio
	z_j	0	0	0	0	0	
	$z_j - c_j$	-4	-3	0	0		
0	s_1	0	$1/2$	1	$-1/2$	3	$3/\frac{1}{2} = 6$
4	x_1	1	$1/2$	0	$1/2$	5	$5/\frac{1}{2} = 10$
	z_j	4	2	0	2	20	
	$z_j - c_j$	0	-1	0	2		

C_{B_i}	c_j	4	3	0	0	Soln
	BV	x_1	x_2	s_1	s_2	
3	x_2	0	1	2	-1	6
4	x_1	1	0	-1	1	2
	z_j	4	3	2	1	26
	z_j-c_j	0	0	2	1	

Thus $\bar{x} = (\bar{x}_1 = 2, \bar{x}_2 = 6, \bar{x}_3 = 0, \bar{x}_4 = 0)$ and we have to find $\bar{\omega}$ i.e., an optimal soln of (2).

So from last tableau of the primal (1) we get basic variable (x_2, x_1) i.e, $(3, 4)$

and $B^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ Hence

$$\bar{\omega} = \text{col}(\bar{\omega}_1, \bar{\omega}_2) = C_B^T B^{-1}$$

$$= (3,4)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \cancel{\begin{pmatrix} 3\times2+4\times-1 \\ 3\times-1+4\times1 \end{pmatrix}} = \cancel{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} = \begin{pmatrix} 3\times2+4\times-1 & 3\times-1+4\times1 \end{pmatrix}$$
$$= (2 \quad 1)$$

i.e, $\bar{\omega}_1 = 2$, and $\bar{\omega}_2 = 1$ is optimal to the dual (2).

By duality theorem optimal value of the dual equals the optimal value of the primal which is 26..

Existence Theorem.

(i) If primal and dual both have feasible solns then both have optimal soln.

(ii) If primal (dual) has unbounded soln then the dual (primal) has no feasible soln.

(iii) If primal (dual) has no feasible soln but the dual (primal) has feasible soln then the dual (primal) has unbounded soln.

(Primal)	Dual: feasible	Dual: Infeasible
feasible	both have optimal soln	Primal has unbounded soln
infeasible	Dual has unbounded soln	This case is possible, i.e primal and dual both could be infeasible

Complementary Slackness Theorem :-

Let $\bar{x}$ and $\bar{w}$ be feasible soln to ... l pair (1)-(2)

Complementary slackness Theorem :-

Statement 1 : If in primal the slack or surplus variable x_{n+i} each positive in the optimal soln. then the i^{th} variable w_i of its dual is zero of its dual i.e, $w_i = 0$ [i.e, $x_{n+i} > 0 \rightarrow w_i = 0$]

Statement 2 : If in the primal j^{th} variable x_j is +ve then j^{th} dual constraint hold as a strict inequality i.e, corresponding dual slack or surplus variable $w_{n+j} = 0$ [i.e, $x_j > 0 \rightarrow w_{n+j} = 0$] (1) and (2)

Primal

$\text{Min } z = x_1 - 2x_2 + x_3$

s.t $x_1 + 2x_2 - 2x_3 \leq 4$

$x_1 - x_3 \geq 3$

$2x_1 - x_2 + 2x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

Decision variables:
$(x_1, x_2, x_3, s_1, s_2, s_3)$

1^{st} and 3^{rd} constraint has opposite sign i.e $\leq$ so in dual form restrictions are also of opposite sign in case of max. i.e, $y_1, y_3 \leq 0$

Dual

$\text{Max } w = 4w_1 + 3w_2 + 2w_3$

s.t $w_1 + w_2 + 2w_3 \geq 1$

$2w_1 - w_3 \leq -2$

$-2w_1 - w_2 + 2w_3 \geq 1$

$y_1, y_3 \leq 0, \; y_2 \geq 0$

Decision variables:
$(w_1, w_2, w_3, s_1', s_2', s_3')$

Primal

$\text{Min } z = x_1 - 2x_2 + x_3$

s.t $x_1 + 2x_2 - 2x_3 \leq 4$

$x_1 - x_3 \leq 3$

$2x_1 - x_2 + 2x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

Minimization case so signs should be $\geq$ but it's $\leq$ so in dual form restrictions are of opposite sign.

Decision variables
$(x_1, x_2, x_3, s_1, s_2, s_3)$

Dual

$\text{Max } w = 4w_1 + 3w_2 + 2w_3$

s.t $w_1 + w_2 + 2w_3 \geq 1$

$2y_1 - y_3 \geq -2$

$-2w_1 - w_2 + 2w_3 \geq 1$

$w_1, w_2, w_3 \leq 0$

Decision variables
$(w_1, w_2, w_3, s_1', s_2', s_3')$

Consider the optimal primal table of the minimization problem of LPP is

BV / C_{Bi}	x_1	x_2	x_3	s_1	s_2	s_3	Soln
x_2	3	1	0	1	0	1	6
s_2	7/2	0	0	1/2	1	1	7
x_3	5/2	0	1	1/2	0	1	4
$z_j - c_j$	-9/2	0	0	-3/2	0	-1	-8

From this table we conclude that decision variables of the Primal Problem:

$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 6, 4, 0, 7, 0)$ basic variable, Non basic variable

Our target is to find the dual variables $(w_1, w_2, w_3, s_1', s_2', s_3') = (?, ?, ?, ?, ?, ?)$

optimal condition for minimization is $z_j - c_j \leq 0 \; \forall j$.

Methods to find the variables/solution to the dual problem from the Primal optimal Table:

1. Read these values from the optimal table of the primal.
2. Compute the values by using $C_B B^{-1}$ where C_B represent the cost values of the Basic variables (B.V) in the optimal table of the primal & B^{-1} represent the Basic Inverse matrix.
3. Using Complementary slackness Conditions.

Complementary Slackness Theorem :-

Let $\bar{x}$ and $\bar{w}$ be feasible solns to the primal-dual pair (1)-(2).

$$\begin{aligned} &\text{Max } c^T x \\ &\text{s.t } Ax \leq b \qquad \text{— (1)} \\ &\quad x \geq 0 \end{aligned}$$

and

$$\begin{aligned} &\text{Min } b^T w \\ &\text{s.t } A^T w \geq c \qquad \text{— (2)} \\ &\quad w \geq 0 \end{aligned}$$

Then $\bar{x}$ and $\bar{w}$ are optimal to the respective problem (1) and (2) if and only if $\bar{w}^T(A\bar{x}-b)=0$ and $\bar{x}^T(c-A^T\bar{w})=0$

The above conditions are called complementary slackness conditions or in short the C.S. conditions.

Complementary Slackness Conditions

1. If, in an optimal table of the primal, the decision variable x_k appears as a basic variable (B.V.) then the k^{th} dual constraint is satisfied as equality as a basic variable (B.V.) then the k^{th} dual constraint, i.e, slack or surplus variable associated with the k^{th} dual constraints assumes zero value

2. If, in an optimal table of the primal, the slack or surplus variable S_k appears as a B.V. then the dual variable w_k associated with the k^{th} primal constraint assumes zero value in the optimal soln of the dual.

That means from table of optimal Table of Primal problem.

① (P) x_3(BV) ② S_2 (slack or surplus) in primal as Basic variable) (P) s_3(BV)

(D) $s'_3 = 0$ (slack or surplus is zero) $(w)_2 = 0$ in dual (D) $w_3 = 0$

To Remember: Easiest Rule

In the Primal optimal Table

Basic variable	In Dual
Rule 1. → If variable ($x_1, x_2, \ldots$) present →	slack or surplus is zero
Rule 2. → If slack/surplus present →	variable ($w_1, w_2, \ldots$) is zero

Ex: Solve the LPP

Minimize $Z = x_1 - 2x_2 + x_3$

s.t $x_1 + 2x_2 - 2x_3 \leq 4$

$x_1 - x_3 \leq 3$

$2x_1 - x_2 + 2x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

and hence find the solution of the dual variables.

1. From the optimal table of the primal problems
2. Using basic inverse matrix method
3. Using Complementary slackness condition.

Soln: standard form of LPP is

$\text{Min } Z = x_1 - 2x_2 + x_3$

Soln: The optimal table of given LPP is

c_j	1	-2	1	0	0	0	
BV	x_1	x_2	x_3	s_1	s_2	s_3	Soln
x_2	3	1	0	1	0	1	6
s_2	7/2	0	0	1/2	1	1	7
x_3	5/2	0	1	1/2	0	1	4
z_j-c_j	-9/2	0	0	-3/2	0	-1	-8

Method 1. Read the dual variables from the optimal primal table

Since (s_1, s_2, s_3) are the initial basic feasible soln for this problem.

So $W = (w_1, w_2, w_3)$ are obtained as

Net effect of a_4 = value of $z_j - c_j$ corresponding to s_1 i.e,

$$w_1 - 0 = -3/2 \Rightarrow \boxed{w_1 = -3/2}$$

(w_1 is corresponding to s_1 and c_j value of s_1 is 0) and $z_j - c_j$ corresponding to s_1 is $-3/2$

$$w_2 - 0 = 0 \Rightarrow \boxed{w_2 = 0}$$

$$w_3 - 0 = -1 \Rightarrow \boxed{w_3 = -1}$$

Method 2- compute the values by using $c_B B^{-1}$

B^{-1} = Initial basic feasible soln

$$B^{-1} = \begin{array}{c} \begin{matrix} s_1 & s_2 & s_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 1 & 1 \\ 1/2 & 0 & 1 \end{bmatrix} \end{array}$$

Here $c_B = (x_2, s_2, x_3) = (-2, 0, 1)$

B^{-1} is the column corresponding to (s_1, s_2, s_3)

$$w = c_B B^{-1}$$

$$(w_1, w_2, w_3) = [-2 \quad 0 \quad 1] \begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 1 & 1 \\ 1/2 & 0 & 1 \end{bmatrix} = \left(-\frac{3}{2}, 0, -1\right)$$

Method 3- using complementary slackness conditions:
write the dual of the original LPP

Primal	Dual
Min $z = x_1 - 2x_2 + x_3$	Max $W = 4w_1 + 3w_2 + 2w_3$
s.t $x_1 + 2x_2 - 2x_3 \leq 4$	s.t $w_1 + w_2 + 2w_3 - s_1' = 1$
$x_1 - x_3 \leq 3$	$2w_1 - w_3 - s_2' = -2$
$2x_1 - x_2 + 2x_3 \leq 2$	$-2w_1 - w_2 + 2w_3 - s_3' = 1$
$x_1, x_2, x_3 \geq 0$	$w_1, w_2, w_3 \leq 0$
Decision variables $(x_1, x_2, x_3, s_1, s_2, s_3)$	Decision variables $(w_1, w_2, w_3, s_1', s_2', s_3')$

Take the dual problem form

As x_2 present in BV $\Rightarrow s_2' = 0$

As s_2 present in BV $\Rightarrow w_2 = 0$

As x_3 present in BV $\Rightarrow s_3' = 0$

From 2^{nd} and 3^{rd} constraint of dual we get

$$2w_1 - w_3 = -2 \quad \text{and} \quad -2w_1 - w_2 + 2w_3 = 1$$

After solving, we get $w_1 = -\frac{3}{2}$, $w_3 = -1$

we know the value of $w_1, w_2, w_3, s_2', s_3'$ so by substituting these values in 1^{st} constraint we can find $s_1' = -\frac{9}{2}$

Hence $(w_1, w_2, w_3, s_1', s_2', s_3') = (-\frac{3}{2}, 0, -1, -9/2, 0, 0)$

$\left[-\frac{3}{2} + 0 + 2 - s_1' = 1 \right.$

$\Rightarrow -\frac{7}{2} - s_1' = 1$

$\Rightarrow -\frac{7}{2} - 1 = s_1'$

$\left. = -\frac{9}{2} = s_1' \right]$

Ex. " " "

Dual Simplex Method to Solve the foll. LPP.

Ex. $\text{Min } z = 3x_1 + x_2$

s.t

$x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$

Soln: $\text{Max } z' = -3x_1 - x_2$

s.t

$-x_1 - x_2 \leq -1$

$-2x_1 - 3x_2 \leq -2$

$x_1, x_2 \geq 0$

Standard LPP is

$\text{Max } z' = -3x_1 - x_2$

s.t

$-x_1 - x_2 + s_1 = -1$

$-2x_1 - 3x_2 + s_2 = -2$

$x_1, x_2, s_1, s_2 \geq 0$

Initial Dual Simplex Tableau is

C_{Bi}	C_j	-3	-1	0	0	Soln (X_B)
	BV	x_1	x_2	s_1	s_2	
0	s_1	-1	-1	1	0	-1
← 0	s_2	-2	-3	0	1	-2
	z_j	0	0	0	0	0
	$z_j - c_j$	3	1 ↑	0	0	

* Here -2 is most negative x_{Bi} so s_2 row will leave the tableau

* $\text{Max}\left[\frac{3}{-2}, \frac{1}{-3}, x_r < 0\right]$ for entering column.

Here $-\frac{1}{3}$ is max^m value so

C_{Bi}	C_j	-3	-1	0	0	X_B
	BV	x_1	x_2	s_1	s_2	
← 0	s_1	$-1/3$	0	1	$-1/3$	$-1/3$
-1	x_2	$2/3$	1	0	$-1/3$	$2/3$
	z_j	$-2/3$	-1	0	$1/3$	$-2/3$
	$z_j - c_j$	$7/3$	0	0	↑ $1/3$	

Here $z_j - c_j \geq 0$ and least one $x_B < 0$ so s_1 row will leave the table

$\text{Max}\left(\frac{7/3}{-1/3} = -7, \frac{1/3}{-1/3} = -1, x_r < 0\right]$

→ see on top.

* If all $z_j - c_j \geq 0$ and $x_B \geq 0$, then an optimal basic feasible soln has been obtained.
* If all $z_j - c_j \geq 0$ and at least one ~~basic variable~~ $x_{Bi} < 0$ then
* Select the most negative x_{Bi} for leaving row
* If at least one x_r is negative i.e. $x_r < 0$, then compute $\text{Max}\left[\frac{z_j - c_j}{x_r}, x_r < 0\right]$ for [x_r → leaving row] entering column and intersection of row and column element will be pivot element.

Row $x_2 \rightarrow -2/-3, \frac{-3}{-3}, \frac{0}{-3}, \frac{1}{-3}, \frac{-2}{-3}$

Row $s_1 \rightarrow$ Same rule as simplex table:

→ old value − $\frac{\text{Corr. key row elt} \times \text{Corr. key column}}{\text{key elt}}$.

C_{Bi}	C_j	-3	-1	0	0	X_B
	BV	x_1	x_2	s_1	s_2	
0	s_2	1	0	-3	1	1
-1	x_2	1	1	-1	0	1
	z_j	-1	-1	1	0	-1
	$z_j - c_j$	2	0	1	0	

∵ $z_j - c_j \geq 0$ and $X_B \geq 0$

∴ optimal soln is $\text{Max } z' = -1$

$\text{Min } z = 1$ at $x_1 = 0, x_2 = 1$.

Ex: Use the dual simplex method to solve the foll. LPP

$\text{Max } -2x_1 - x_2$

s.t.

$2x_1 - x_2 - x_3 \geq 3$

$x_1 - x_2 + x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Soln. Standard LPP is

$\text{Max } -2x_1 - x_2$

s.t

$-2x_1 + x_2 + x_3 + s_1 = -3$

$-x_1 + x_2 - x_3 + s_2 = -2$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

Initial dual simplex tableau is

C_{Bi}	c_j	-2	-1	0	0	0	X_B
	BV	x_1	x_2	x_3	s_1	s_2	
← 0	s_1	-2	1	1	1	0	-3
0	s_2	-1	1	-1	0	1	-2
	z_j	0	0	0	0	0	0
	$z_j - c_j$	2 ↑	1	0	0	0	

$\because z_j - c_j \geq 0$ and $X_B < 0$

so most negative value is -3 which is leaving row (s_1)

For entering column, $\text{Max}\left(\frac{z_j - c_j}{x_r}, x_r < 0\right) = \left[\frac{2}{-2}\right] = -1$.

$x_r \to$ key row

$\therefore$ only one $x_r < 0$ so, x_1 column will enter in next table

C_{Bi}	c_j	-2	-1	0	0	0	X_B
	BV	x_1	x_2	x_3	s_1	s_2	
-2	x_1	1	-1/2	-1/2	-1/2	0	3/2
← 0	s_2	0	1/2	-3/2	-1/2	1	-1/2
	z_j	-2	1	1	1	0	-3
	$z_j - c_j$	0	2	1 ↑	1	0	

$\text{Max}\left[\frac{1}{-3/2}, \frac{1}{-1/2}\right] = \left[-\frac{2}{3}, -2\right]$

C_{Bi}	c_j	-2	-1	0	0	0	X_B
	BV	x_1	x_2	x_3	s_1	s_2	
-2	x_1	1	-2/3	0	-2/3	-1/3	5/3
0	x_3	0	-1/3	1	1/3	-2/3	1/3
	z_j	-2	4/3	0	4/3	2/3	-10/3
	$z_j - c_j$	0	7/3	0	4/3	2/3	

$\because z_j - c_j \geq 0$ and $X_B \geq 0$

we obtain optimal soln $x_1 = 5/3$, $x_2 = 0$, $x_3 = 1/3$ and $z_{max} = \frac{-10}{3}$.

$\frac{4}{3}+1$

=

in LPP is

Ex. Use the dual simplex method to check that the foll. LPP is infeasible.

Max $-x_1$

s.t

$x_1 - x_2 \geq 3$

$-x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

Sol^n. SLPP is

Max $-x_1$

s.t

$-x_1 + x_2 + s_1 \leq -3$

$x_1 - x_2 + s_2 \leq -4$

$x_1, x_2, s_1, s_2 \geq 0$

Dual simplex tableau is

C_{Bi}	c_j	-1	0	0	0	
	BV	x_1	x_2	s_1	s_2	x_B
0	s_1	-1	1	1	0	-3
0	s_2	1	-1	0	1	-4
	z_j	0	0	0	0	0
	$z_j - c_j$	1	0	0	0	

$\because z_j - c_j \geq 0$ and $x_B < 0$

So for next table most negative x_B is -4 for leaving row

and for entering column, Max $\left[\frac{z_j - c_j}{x_r}, x_r < 0\right]$ $x_r \rightarrow$ key row = $\{$ here only one $x_r < 0$

$-3 - \frac{1 \times -4}{-1}$

$= -3 - 4 = -7$

so x_2 column will enter.

Next dual tableau is

C_{Bi}	c_j	-1	0	0	0	
	BV	x_1	x_2	s_1	s_2	x_B
0	s_1	0	0	1	1	-7
0	x_2	-1	1	0	-1	4
	z_j	0	0	0	0	0
	$z_j - c_j$	1	0	0	0	

$\because z_j - c_j \geq 0$, $s_1 < 0$, but in that row no $x_r < 0$, hence the given LPP is infeasible.