UNIT-V

Graphs

Introduction

* A Graph 'G' be an ordered triple where

1. vertex set: $V=\{v_1, v_2, v_3, \ldots v_n\}$

where V = vertices, node, points

2. Edgeset: $E=\{e_1, e_2, e_3, \ldots e_n\}$

where E = edges, links, lines

* A Graph G is defined by the incidence function:

3. Incidence function: $\psi_G : V \rightarrow E$

Ex:-

$\psi(e_1) = v_1, v_2$
$\psi(e_2) = v_1, v_3$
$\psi(e_3) = v_2, v_3$

⇒ order of a Graph = No: of vertices in a Graph. (N)

⇒ Size of a Graph = No: of edges in a Graph i.e $|E|$

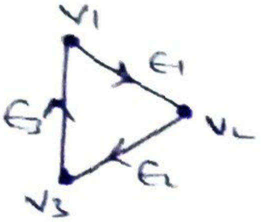
Graph:

* A Graph $G = \langle V, E, \psi \rangle$ consists of a non-empty set V. called set of vertices of the graph, E is said to be the set of Edges of the graph and ψ is a mapping from the set of edges 'E' to a set of ordered or un ordered pair of elements of 'V'.

Directed Graph:

* A Graph in which every edge is directed is called a directed Graph. (or) Digraph

EX:-



Undirected Graph:

* A Graph in which every edge is undirected is called a undirected Graph.

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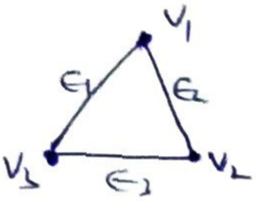
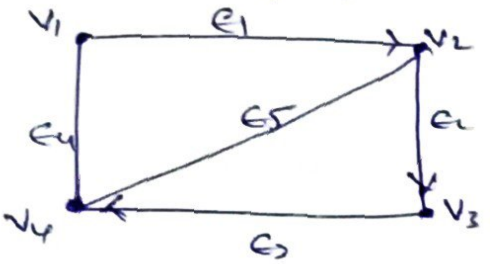


Mixed Graph:

* If some edges directed and some edges are undirected in a graph then the graph is called mixed graph

EX:



Loop: (Sting)

* An edge of a graph which joins a node or vertex to it self is called a loop. The direction of a loop is of no significance. Hence it can be consider eith a edge directed or undirected edge.

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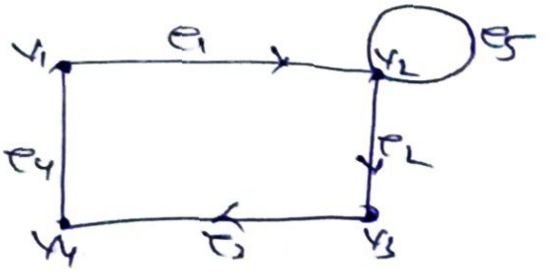

parallel Edges

* In some directed as well as undirected graphs we may have certain pairs of vertices joined by more than one edge. such edges are called parallel.

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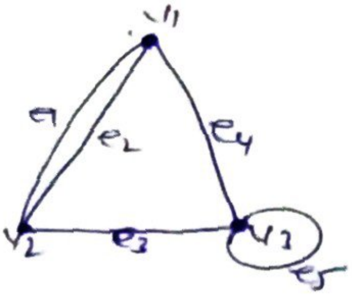
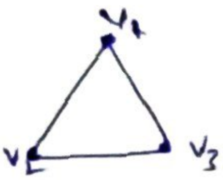


$e_1 = \{v_1, v_2\}$ ⎫
$e_2 = \{v_1, v_2\}$ ⎭ parallel edges
$e_3 = \{v_2, v_3\}$
$e_4 = \{v_1, v_3\}$
$e_5 = \{v_3\}$

Simple Graph:

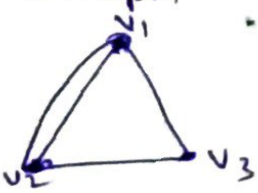
* A Graph which does not contain loops and no multiple edges is called a simple Graph.

Ex:-



Multi Graph

* A Graph which contain multiple edges but no loop is called a multiGraph



General Graph

* A Graph which contains multiple edges or loops or both is called a General Graph.

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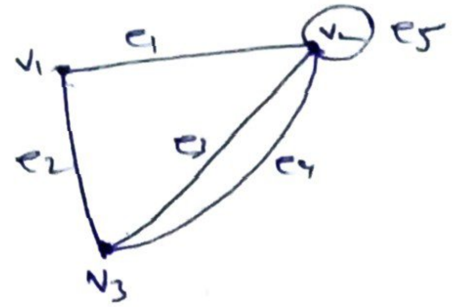


Incidence:

* If a vertex 'v' is an end vertex of some edge 'e' than 'v' and 'E' are said to be incidence with each other.

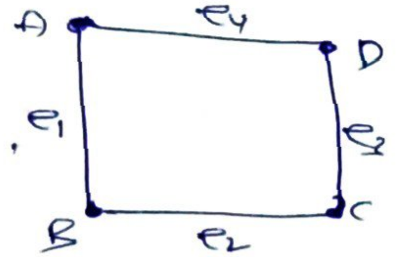
Adjacent edges:

* Two non-parallel edges are said to be adjacent edges if they are incident on a common vertex.

Adjacent vertices

* Two vertices are said to be adjacent vertices if there is an edge joining them.

Ex:-



* A, B are adjacent vertices and 'e_1' and 'e_2' adjacent edges.

* A, C are not adjacent vertices and 'e_1' and 'e_3' are not adjacent edges.

Isolated vertex:

* A vertex in a Graph which not an end vertex of any edge of the graph is called an isolated vertex.

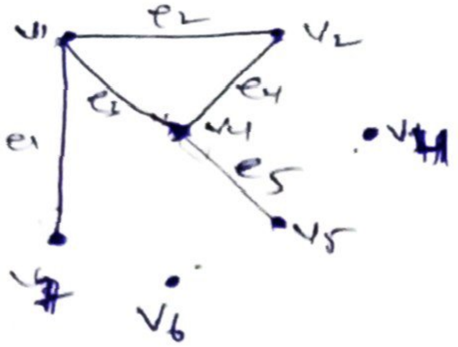
pendent vertex:

* A vertex of degree '1' is called a pendent vertex

pendent edge

* An edge incident on a pendent vertex is called a pendent edge.

Ex:-



* The vertices v_4 and v_6 are isolated vertices.
* v_5 and v_7 are pendent vertices
* The edges e_1 and e_5 are pendent edges.

Null Graph

* A Graph containing only isolated vertices is called Null Graph

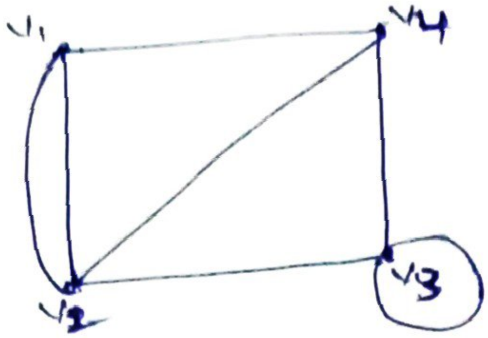
i.e the set of edges in an null graph is empty.

Ex:-

Degree of the vertex:

* Let 'G' be a graph and 'v' is a vertex of 'G' then the no: of edges of 'G' that are incident on 'v':

with the loops counted twice is called Degree of the vertex 'v' and is denoted by deg(v) or d(v).

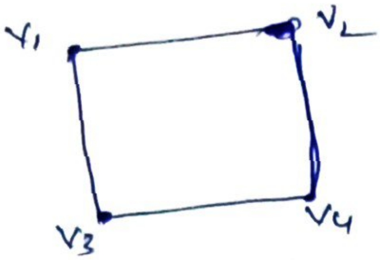


deg(v1) = 3
deg(v2) = 4
deg(v3) = 4
deg(v4) = 3

Regular Graphs

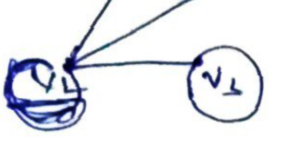
* A Graph in which all the vertices of the same degree 'k' is called a regular Graph of degree 'k' or k-regular Graph. In particular 3 regular graphs are called cubic Graphs.

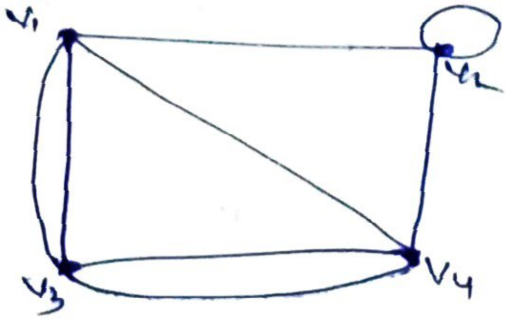
Ex:



2-regular Graph.

deg(v1) = 2
deg(v2) = 2
deg(v3) = 2
deg(v4) = 2





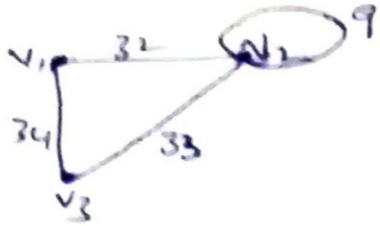
4-regular Graph

deg(v1) = 4
deg(v2) = 4
deg(v3) = 4
deg(v4) = 4

Weighted Graph

* A Graph in which weights are assigned to every edges is called a weighted Graph.

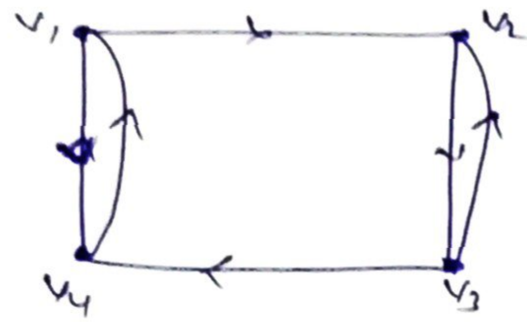
Ex:-



Indegree and outdegree

* The no: of edges directed towards to 'v' is called "Indegree of 'v'".
* The no: of edges directed outwards to 'v' is called "outdegree of 'v'".

Ex:-

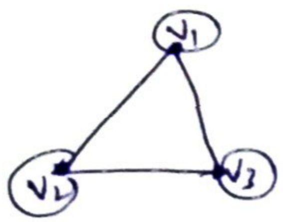


Indegree(v_1) = 1
Indegree(v_2) = 2
Indegree(v_3) = 1
Indegree(v_4) = 2

out degree(v_1) = 2
out degree(v_2) = 1
out degree(v_3) = 2
out degree(v_4) = 1

pseudo Graph:

* A Graph having loops but no multiple edges is called a pseudo Graph.



complete Graph:

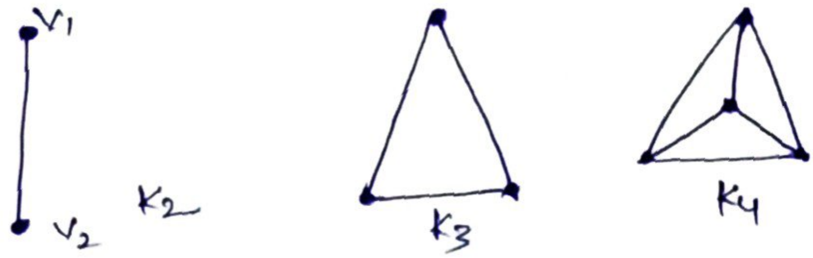
* A simple Graph of order ≥ 2 in which there is an edge between every pair of vertices is called a

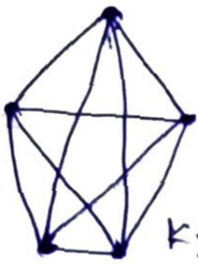
complete Graph or full Graph.

(or)

In otherwords a complete graph is a simple graph in which every pair distinct vertices are adjacent.

* A complete graph with 'n' verties is denoted by K_n.





K_5 vertices

~~Bg~~ parted

Bipartite graph:

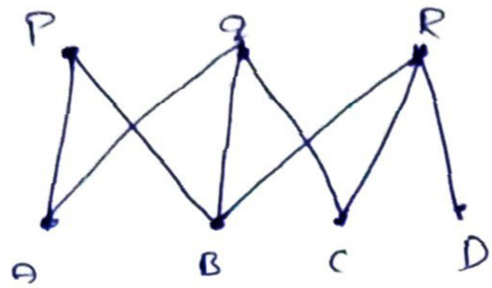
* suppose a simple graph 'G' is such that it's vertex set 'V' is The union of two mutually disjoint non-empty sets V_1 and V_2 which are such that every edge in joins a vertex V_1 and a vertex V_2. Then 'G' is called a Bipartite graph.

* If 'E' is the edge set of this graph, the graph denoted by $G = (V_1, V_2 : E)$ (or) $G = G(V_1, V_2 : E)$. The sets V_1 and V_2 are called bipartites (partitions of the vertexset V)

* A Bipartite graph doesnot contain any self loops.
* A Bipartite graph can be denoted by $K_{r,s}$

where r contains the vertices in sub-set V_1 and 's' contains the vertices in sub-set V_2.

Ex:-



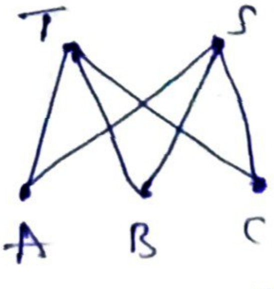
$V_1 = \{P, Q, R\}$
$V_2 = \{A, B, C, D\}$

$K_{3,4}$ - bipartitegraph.

Complete Bipartite Graph

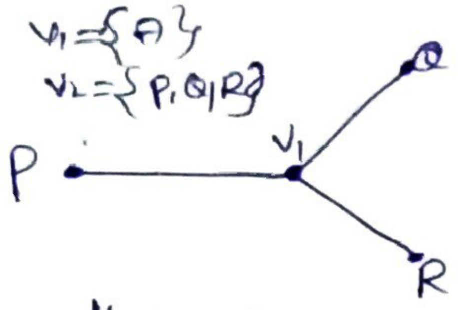
* A Bipartite Graph $G = (V_1, V_2; E)$ is called a complete Bipartite graph, if there is an edge between every vertex in V_1 and every vertex V_2.
* A complete Bipartite Graph $G = (V_1, V_2; E)$ in which the ~~by~~ Bipartites V_1 and V_2 contain r, s vertices respectively with $r \leq s$ is denoted by $K_{r,s}$.
* In this Graph each of 'r' vertices in 'V_1' is joined to each of 'S' vertices in 'V_2'. Thus $K_{r,s}$ has r+s vertices and rs edges i.e $K_{r,s}$ is of order r+s and size rs. Therefore it is a $(r+s, rs)$ graph.

Ex:-



$V_1 = \{T, S\}$
$V_2 = \{A, B, C\}$

$K_{2,3}$ - complete Bipartite graph.



$K_{1,3}$ - complete Bipartite graph

* How many vertices and how many edges are there in complete bipartite graph (i) $K_{4,7}$ and (ii) $K_{7,11}$. If the graph $K_{r,12}$ has 72 edges what is r.

Sol we have complete bipartite graph $K_{r,s}$ has $r+s$ vertices and $r.s$ edges.

i. $K_{4,7}$

vertices $= r+s = 4+7 = 11$

edges $= rs = 4.7 = 28$

ii. $K_{7,11}$

vertices $= r+s = 7+11 = 18$

edges $= rs = 7.11 = 77$.

The Graph $K_{r,12}$ has 72 edges

we have

$12r = 72$

$\boxed{r = 6}$

* prove that the no: of edges in a simple graph of order 'n' (≥ 2) cannot exceed $\frac{n(n-1)}{2}$.

Sol Each edge of a graph is determined by a pair of vertices.

* In a simple graph there no multiple edges has such in a simple graph the no: of edges cannot exceed

the no: of pairs of vertices.

* The no: of pairs of vertices that can be choosen from 'n' vertices is $nC_2 = C(n,2)$

$$= \frac{n!}{(n-2)!\,2!}$$

$$= \frac{n(n-1)\cancel{(n-2)}!}{\cancel{(n-2)}!\,2!}$$

$$= \frac{n(n-1)}{2}$$

Thus in a simple graph of order n (≥ 2), the no: of edges cannot exceed $\frac{n(n-1)}{2}$.

Handshacking Theorem

Let 'G' be an undirected graph with $|E|$ and $|V| = n$ vertices. Then show that

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

Proof:

Let 'G' be a graph with $|E|$ edges and $|V| = $ 'n' vertices $v_1, v_2, v_3, \ldots v_n$

we have to show that $\sum_{i=1}^{n} \deg(v_i) = 2|E|$

when we sum over the degree of all vertices, which we count each edge (v_i, v_j) is twice.

once we count it has (v_i, v_j) in the degree $\deg(v_i)$ and again when we count it has (v_j, v_i) in the degree $\deg(v_j)$

Hence The conclusion follows that

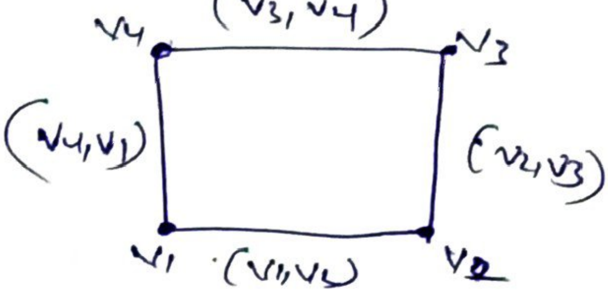
$$\sum_{i=1}^{n} deg(v_i) = 2|e|$$

$deg(v_1) = 2$

$deg(v_2) = 2$

$deg(v_3) = 2$

$deg(v_4) = 2$



$|e| = 4$

$\therefore \quad deg(v_1) + deg(v_2) + deg(v_3) + deg(v_4) = 2|e|$

$2 + 2 + 2 + 2 = 2(4)$

$\boxed{8 = 8}$

$\therefore$ Hence Hand Shaking theorem is satisfied.

Theorem:- In any Graph 'G' prove that total no: of vertices of odd degree is even.

Proof:- Let 'G' be any given graph.

Let V_1, V_2 be the set of vertices of odd and even degrees in 'G'

$V = V_1 + V_2 \quad , V_1 \cap V_2 = \phi$

Where 'V' is the set of vertices in the given graph 'G'

We know that $\sum_{i=1}^{n} deg(v_i) = 2|e|$, Where |e| is the no: of edges of 'G'.

$$\sum_{v \in V_1 \cup V_2} \deg(v_i) = 2|e|$$

$$\sum_{v \in V_1} \deg(v_i) + \sum_{v \in V_2} \deg(v_i) = \text{Even number}$$

$$\sum_{v \in V_1} \deg(v_i) + \text{even number} = \text{Even number}$$

$$\sum_{v \in V_1} \deg(v_i) = \text{Even number}.$$

'V_1' must contain an even number of vertices

∴ Hence The no: of vertices of odd degree is even.

1. Determine the order |V| of the Graph G = (V, E).

(a) G is a cubic graph with 9 edges.

(b) G is regular with 15 edges.

(c) G has 10 edges with 2 vertices of degree 4 & all others of degree 3.

Sol a. Suppose the order of 'G' is n.

Since G is a cubic graph, all the vertices of 'G' have degree '3'.

The sum of the degrees of vertices is 3n.

Since 'G' has 9 edges

By hand shaking property

$$3n = 2(9)$$

$$\boxed{n = 6}$$

b. Since 'G' is Regular all the vertices of 'G'

must be of the same degree 'k' (say)

consider order of 'G' is 'n'

Then the sum of the degree of the vertices is

kn

'G' has '15' edges

By handshaking property

$kn = 2(15)$

$kn = 30$

$k = \frac{30}{n}$

Since k has to be a +ve integer.

We follows that 'n' must be a devices of the 30'

The possible orders of 'G' is 1,2,3, 5,6,10,15,30

c. Suppose the order of 'G' is n Since two vertices of degree 4 & all other (n-2) vertices of degree is '3'.

Since 'G' has 10 edges.

By handshaking property

$2 \times 4 + (n-2)3 = 2(10)$

$8 + 3n - 6 = 20$

$3n + 2 = 20$

$\boxed{n = 6}$

2. Show that there is no Graph with '12' vertices and '28' edges in the following cases.

i. The degree of a vertex is either 3 or 4

ii. The degree of a vertex is either 2 or 6

Sol. Suppose there is a graph with 28 edges with 12 vertices of which k.

$0 \leq k \leq 12$

vertices are of degree '3' each

i. If 'k' no: of vertices of degree '3' and remaining (12-k) degree 4 by hand shaking property.

$3k + (12-k)4 = 2 \times 28$

$3k + 48 - 4k = 56$

$-k = 8$

$\boxed{k = -8}$ which cannot hold because k has to be non-negative.

ii. If 'k' no: of vertices of degree '3' and remaining (12-k) degree '6' by hand shaking property

$3k + (12-k)6 = 2 \times 28$

$3k + 72 - 6k = 56$

$-3k = -16$

$\boxed{k = 16/3}$ which cannot hold because k has to be non-negative (fraction)

Isomorphic Graph

consider two Graphs $G = (V, E)$ & $G' = (V', E')$

suppose there exists a function $f: V \rightarrow V'$ such that

i. f is one to one and onto.

ii. For all vertices A, B of G, the edge $\{A, B\} \in E \iff$ the edge $\{f(A), f(B)\} \in E'$. Then f is called an isomorphism between G and G' and we say that G and G' are isomorphic Graphs.

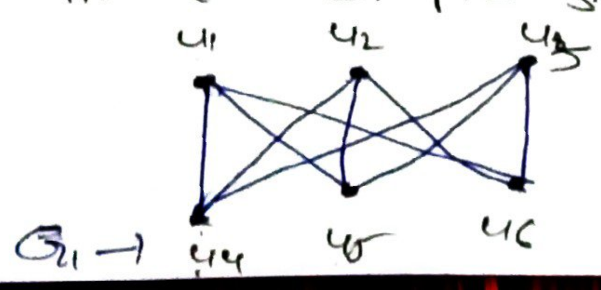
* when G and G' are isomorphic, we write $G \cong G'$

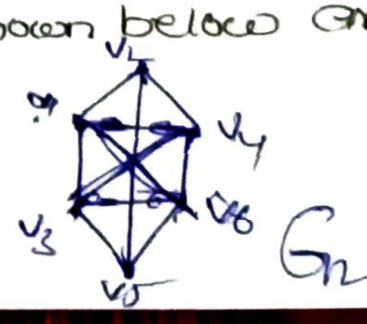
procedure

steps to be follow for checking Given two Graphs are isomorphic or not.

* Both Graphs have the same no: of vertices.
* Both Graphs have the same no: of edges.
* Both Graphs have an equal no: of vertices with Given degree.
* one to one correspondance between the vertices of two Graphs.
* Edge preserving is also satisfying between the two Graphs.

1. Show that the two Graphs shown below are isomorphic





Sol. The no: of vertices in $G_1 = 6 =$ The no: of vertices in G_2

The no: of edges in $G_1 = 9 =$ The no: of edges in G_2

Degree sequence in G_1 & G_2

G_1	G_2
$\deg(u_1) = 3$	$\deg(v_1) = 3$
$\deg(u_2) = 3$	$\deg(v_2) = 3$
$\deg(u_3) = 3$	$\deg(v_3) = 3$
$\deg(u_4) = 3$	$\deg(v_4) = 3$
$\deg(u_5) = 3$	$\deg(v_5) = 3$
	$\deg(v_6) = 3$

one to one correspondance between the Given two graphs is

$u_1 \longleftrightarrow v_1$

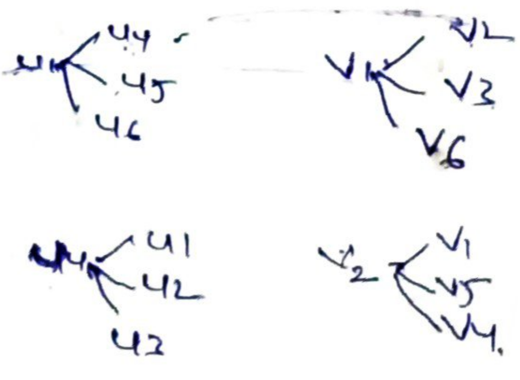
$u_4 \longleftrightarrow v_2$

$u_5 \longleftrightarrow v_3$

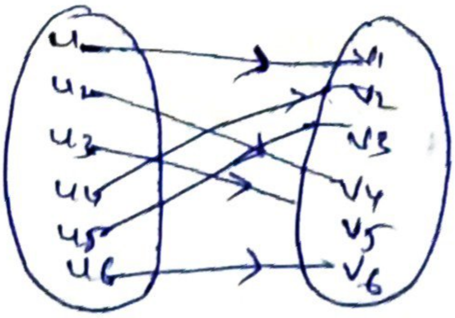
$u_6 \longleftrightarrow v_6$

$u_2 \longleftrightarrow v_4$

$u_3 \longleftrightarrow v_5$



we define $\phi : V(G_1) \to V(G_2)$



Edge preserving

$\{u_1, u_4\} \leftrightarrow \{v_1, v_2\}$, $\{u_1, u_5\} \leftrightarrow \{v_1, v_3\}$

$\{u_1, u_6\} \leftrightarrow \{v_1, v_6\}$, $\{u_2, u_4\} \leftrightarrow \{v_4, v_2\}$

$\{u_2, u_5\} \leftrightarrow \{v_4, v_3\}$, $\{u_2, u_6\} \leftrightarrow \{v_4, v_6\}$

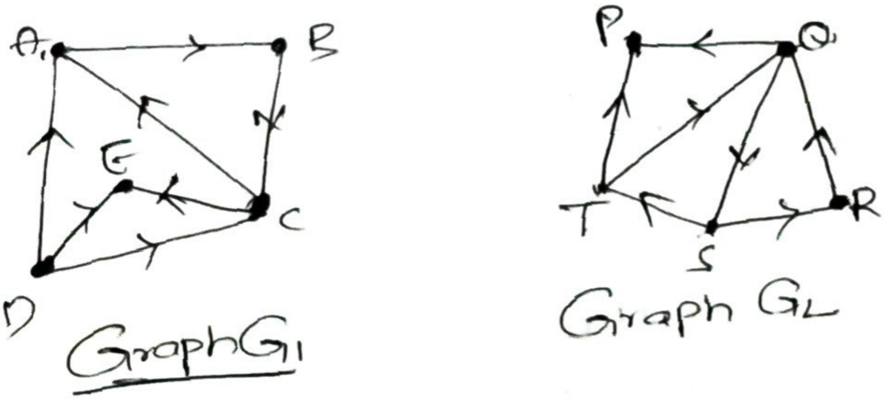
$\{u_3, u_4\} \leftrightarrow \{v_5, v_2\}$, $\{u_3, u_5\} \leftrightarrow \{v_5, v_3\}$

$\{u_3, u_6\} \leftrightarrow \{v_5, v_6\}$

This represents one to one correspondance between the edges of the two Graphs under which the adjacent vertices in the first Graph corresponse to adjacent vertices in second Graph and vice versa.

$\therefore$ The two Graphs are isomorphic

2. Show that the following digraphs are not isomorphic.



Sol The number of vertices in $G_1 = 5 =$ The number of vertices in G_2.

The number of edges in $G_1 = 7 =$ The number of edges in G_2.

Degree sequence G_1 & G_2

Indegree + outdegree = Total degree

G_1	G_2
deg(A) = 2+1 = 3	deg(P) = 2+0 = 2
deg(B) = 1+1 = 2	deg(Q) = 2+2 = 4
deg(C) = 2+2 = 4	deg(R) = 1+1 = 2
deg(D) = 0+3 = 3	deg(S) = 1+2 = 3
deg(E) = 2+0 = 2	deg(T) = 1+2 = 3

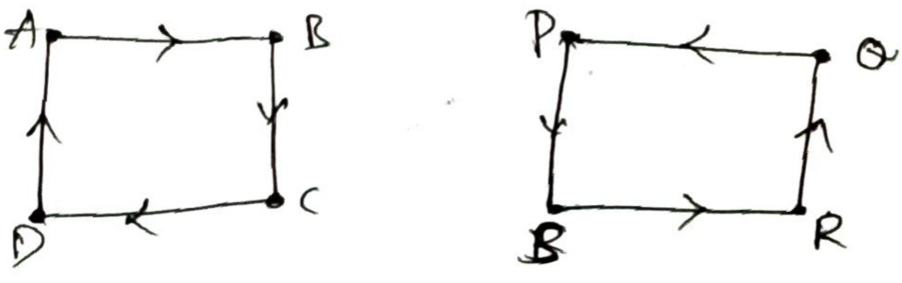
We observe that the vertex 'A' of the first digraph has 2 as indegree and '1' as its out degree.

There is no such vertex in the second diagraph.

∴ There cannot be any one to one correspondance b/w the vertices of the two diagraphs which preserves the direction of edges.

∴ G_1 & G_2 are not isomorphic.

3. Show that following digraphs are isomorphic.



Sol. The number of vertices in G_1 = 4 = The number of vertices in G_2.

The no: of edges in G_1 = 4 = the number of edges in G_2.

Degree sequence G_1 & G_2

Indegree + out degree = Total degree

G_1	G_2
$\deg(A) = 1+1 = 2$	$\deg(P) = 1+1 = 2$
$\deg(B) = 1+1 = 2$	$\deg(Q) = 1+1 = 2$
$\deg(C) = 1+1 = 2$	$\deg(R) = 1+1 = 2$
$\deg(D) = 1+1 = 2$	$\deg(S) = 1+1 = 2$

The one to one correspondance b/w the vertices of the given two ~~diagr~~ digraphs is

G_1	G_2
$A < \begin{matrix} B \\ D \end{matrix}$	$Q < \begin{matrix} P \\ R \end{matrix}$

$A \leftrightarrow Q$

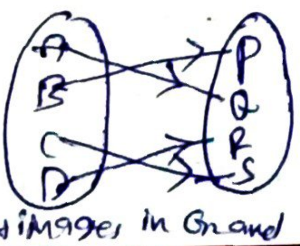
$B \leftrightarrow P$

$D \leftrightarrow R$

G_1	G_2
$B < \begin{matrix} A \\ C \end{matrix}$	$P < \begin{matrix} Q \\ S \end{matrix}$

$C \leftrightarrow S$

We define $\phi : V(G_1) \rightarrow V(G_2)$



ϕ is a bijective because of G_1 has different images in G_2 and G_2 has preimages in G_1.

Edge preserving

$\{A, B\} = \{Q, P\}$

$\{B, C\} = \{R, S\}$

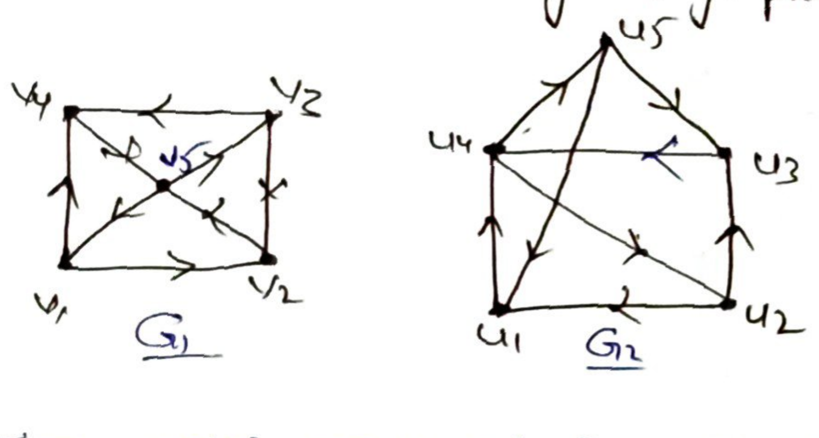
$\{C, D\} = \{S, R\}$

$\{D, A\} = \{R, Q\}$

This represents one to one correspondance between the edges of the Graphs under which the adjacent vertices in the first Graph corresponse to adjacent vertices in second Graph and viceversa.

$\therefore$ The two Graphs are isomorphic

4. show that the following digraphs are isomorophic



Sol

The no: of vertices in $G_1 = 5 =$ The no: of vertices in G_2

The no: of edges in $G_1 = 8 =$ The no: of edges in G_2

Degree sequence in G_1 & G_2

Total degree = indegree + outdegree.

G_1	G_2
$\deg(v_1) = 1+2 = 3$	$\deg(u_1) = 2+1 = 3$
$\deg(v_2) = 2+1 = 3$	$\deg(u_2) = 1+2 = 3$
$\deg(v_3) = 1+2 = 3$	$\deg(u_3) = 2+1 = 3$
$\deg(v_4) = 2+1 = 3$	$\deg(u_4) = 2+2 = 4$
$\deg(v_5) = 2+2 = 4$	$\deg(u_5) = 1+2 = 3$

The one to one correspondance b/w the vertices of the given two digraphs is

$v_5 \leftrightarrow u_4$

$v_1 \leftrightarrow u_2$

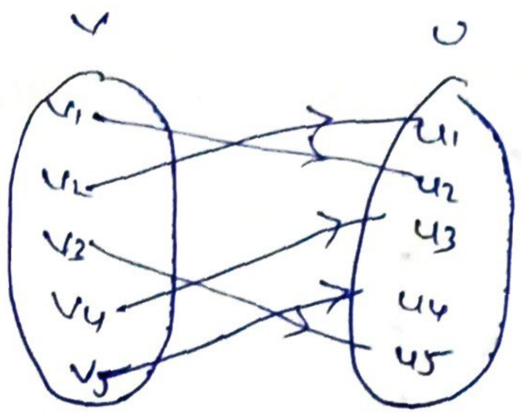
$v_2 \leftrightarrow u_1$

$v_3 \leftrightarrow u_5$

$v_4 \leftrightarrow u_3$

We define the mapping

$\phi : V(G_1) \to U(G_2)$



$\therefore$ ϕ is a bijective.

Edge preserving

$\{v_1, v_2\} \leftrightarrow \{u_2, u_1\}$, $\{v_2, v_3\} \leftrightarrow \{u_1, u_5\}$

$\{v_3, v_4\} \leftrightarrow \{u_5, u_3\}$, $\{v_4, v_1\} \leftrightarrow \{u_3, u_2\}$

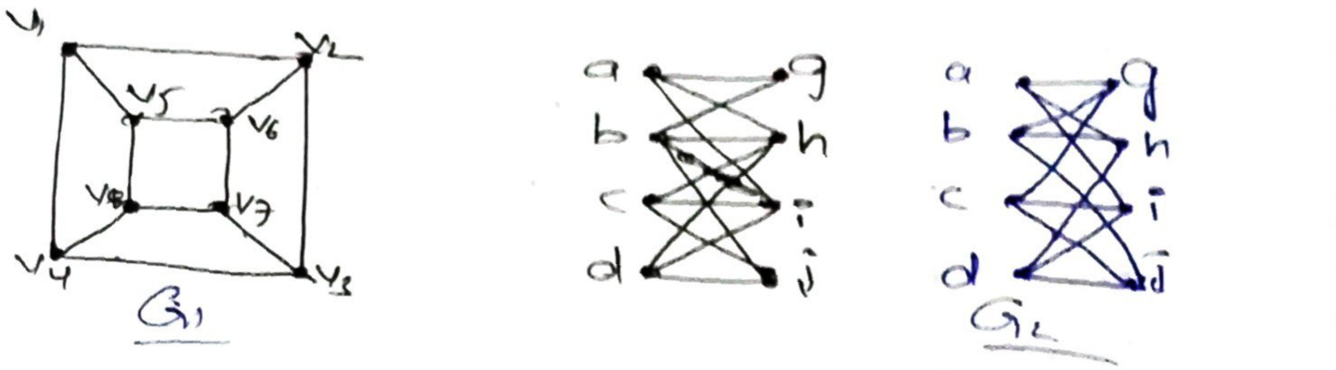
$\{v_1, v_5\} \leftrightarrow \{u_2, u_4\}$, $\{v_2, v_5\} \leftrightarrow \{u_1, u_4\}$

$\{v_3, v_5\} \leftrightarrow \{u_5, u_4\}$, $\{v_4, v_5\} \leftrightarrow \{u_3, u_4\}$

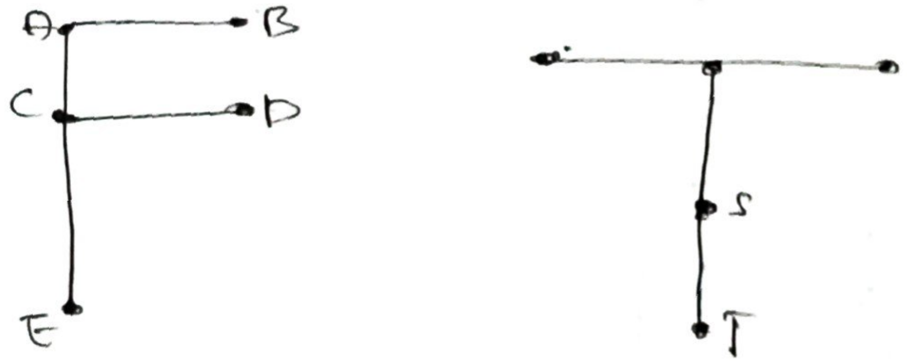
This represents one to one correspondance b/w the edges of the two graphs under which the adjacent vertices in the first Graph correspondence to adjacent vertices in second Graph and vice versa.

∴ The two Graphs isomorphic.

5. When we say that two Graphs G_1 and G_2 are isomorphic. Are the following two digraphs are isomorphic or not.



6. check whether the following graphs are isomorphic.



5. Sol

The no: of vertices in $G_1 = 8 =$ The no: of vertices in G_2

The no: of edges in $G_2 = 12 =$ the no: of edges in G_2

Degree of Sequence in G_1 & G_2

G_1	G_2
$deg(v_1) = 3$	$deg(a) = 3$
$deg(v_2) = 3$	$deg(b) = 3$
$deg(v_3) = 3$	$deg(c) = 3$

$deg(v_4) = 3$ $deg(d) = 3$

$deg(v_5) = 3$ $deg(g) = 3$

$deg(v_6) = 3$ $deg(h) = 3$

$deg(v_7) = 3$ $deg(i) = 3$

$deg(v_8) = 3$ $deg(j) = 3$

The one to one correspondance b/w the vertices of the given two digraphs is

$v_1 \longleftrightarrow a$ $v_6 \longleftrightarrow h$

$v_2 \longleftrightarrow b$ $v_7 \longleftrightarrow i$

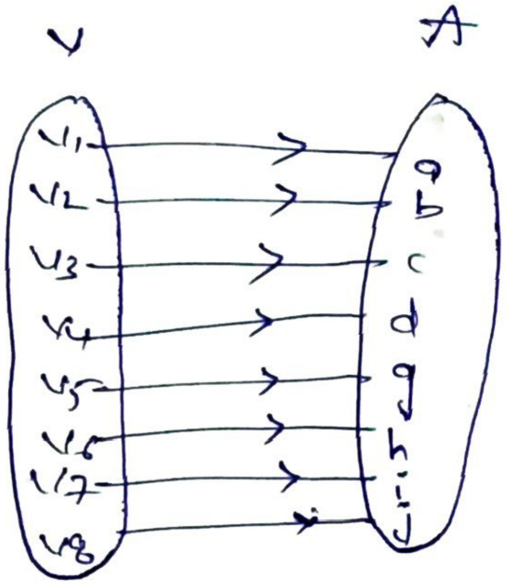
$v_3 \longleftrightarrow c$ $v_8 \longleftrightarrow j$.

$v_4 \longleftrightarrow d$

$v_5 \longleftrightarrow g$

we define the mapping

$$\phi : V(G_1) \longrightarrow A(G_2)$$



Edge preserving

$(v_1, v_2) \longleftrightarrow (a, g)$

$(v_1, v_4) \longleftrightarrow (a, i)$

$(v_1, v_5) \longleftrightarrow (a, h)$

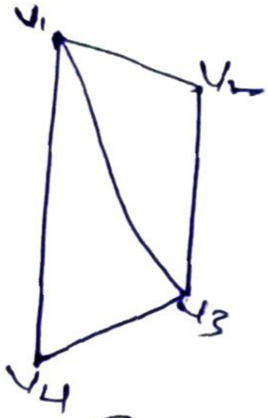
Sub Graph

Given two Graphs G and G_1 we say that G_1 is a sub Graph of G if the following conditions hold:

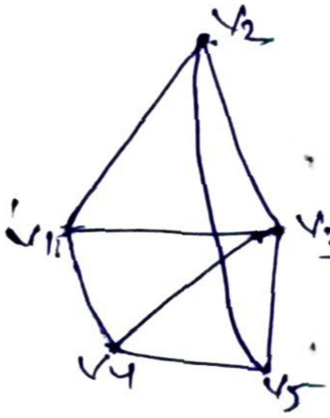
i. All the vertices and all the edges of G_1 are in G.

ii. Each edge of G_1 has the same end vertices in G as in G_1

Ex:-



G_1

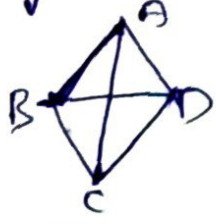


G_2

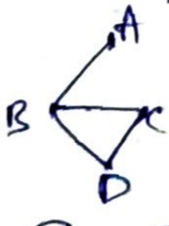
Spanning sub Graph:

Given a Graph $G=(V,E)$ if there is a subgraph $G_1=(V_1,E_1)$ of G such that $V_1=V$, then G_1 is called Spanning sub Graph of G.

EX:-



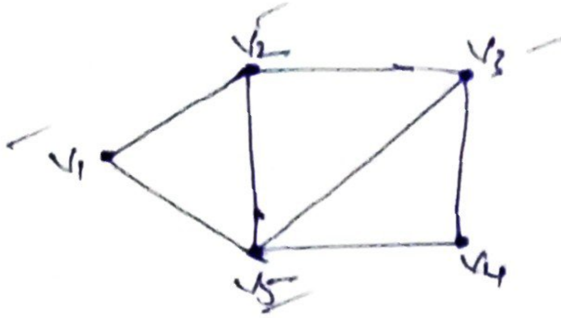
$G_1=(V,E)$



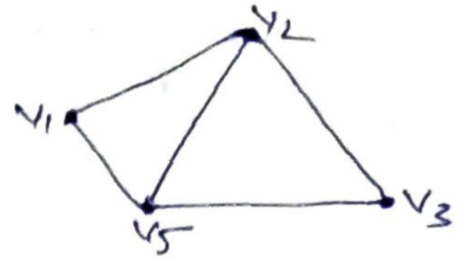
$G_2=(V_1,E_1)$
Spanning subgraph of G_1

Induced Sub Graph:

Given a Graph $G=(V,E)$, suppose there is a subgraph $G_1=(V_1,E_1)$ of G such that every edge $\{A,B\}$ of G where $A,B \in V_1$ is an edge of G_1 also. Then G_1 is called a subGraph of G induced by V_1 and is denoted by $\langle V_1 \rangle$.

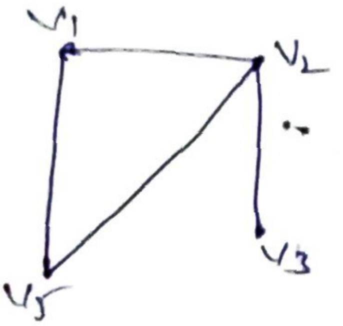


Graph $G=(V,E)$



$G_1=(V_1,E_1)$

Induced sub graph.



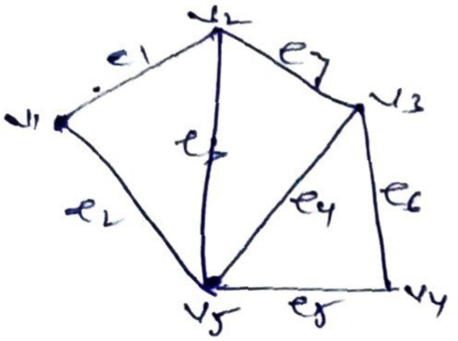
$G_2=(V_2,E_1)$

is not a Induced sub Graph.

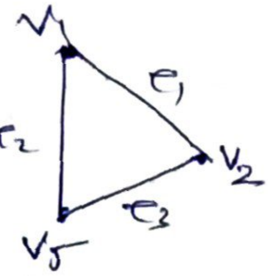
Edge disjoint sub Graph

* Let G be Graph and G_1 and G_2 be two Sub graphs G then G_1 and G_2 are said to be edge disjoint if they donot have any common edge

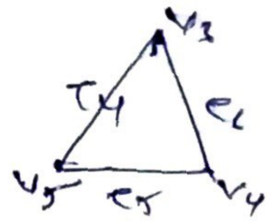
$$\boxed{E_1(G_1)\cap E(G_2)=\phi}$$



G = (V, E)



$G_1 = (V_1, E_1)$

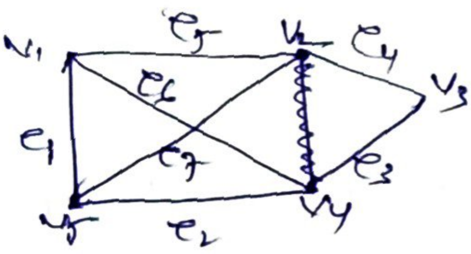


$G_2 = (V_2, E_2)$

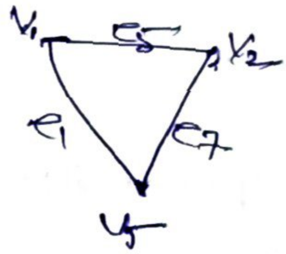
vertex disjoint subGraph

* Let 'G' be a Graph and G_1 and G_2 be two sub graphs of 'G' then G_1 and G_2 are said to be vertex disjoint if they donot have any common edge and any common vertex..

Ex:-



G = (V, E)



$G_1 = (V_1, E_1)$

$G_2 = (V_2, E_2)$

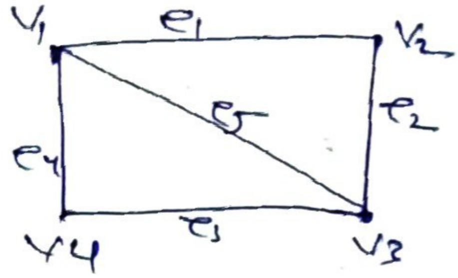
Walk (traverse)

* A walk is defined as a finite altomative sequence of vertices and edges begining and ending with vertices such that edge is incident with vertices.

* An edge appears more than one's in walk.
* The vertex may appears more than once.
* A walk in a Graph 'G' is a subgraph of G.

* A vertex with which a walk ends is called terminal vertex.

EX:-



$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1 e_5 v_3$ → open walk

→ closed

$v_1 e_1 v_2 e_2 v_3 e_5 v_1$

open walk

* A walk which begins and ends with the different vertices is a open walk.

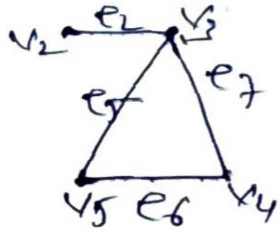
closed walk

* A walk which begins and ends with the same vertex is called a closed walk.

Trail:-

* A Trail is an openwalk in which no edge appears more than once.

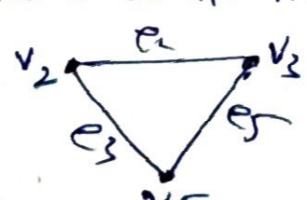
EX:-



walk sequence: $v_2 e_2 v_3 e_5 v_5 e_6 v_4 e_7 v_3$

circuit:-

* A circuit is an closed walk in which no edge appears more than one,

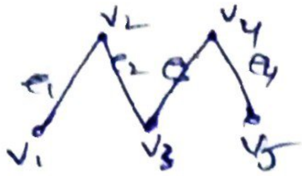


walk sequence: $v_2 e_2 v_3 e_5 e_3 v_2$

path:-

* A trail in which no vertex appears more than once

Ex:

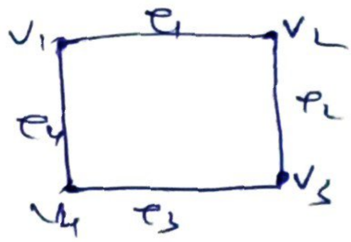


walk sequence: $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$.

Cycle :-

* A circuit in which no vertex appears more than once except starting and ending vertices.

Ex:-

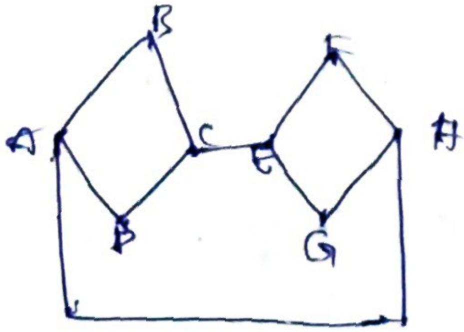


walk sequence: $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$

Connected Graph

* A Graph is said to be connected Graph, if there exist a Path or atleast one edge between every Pair of vertices in the Given Graph.

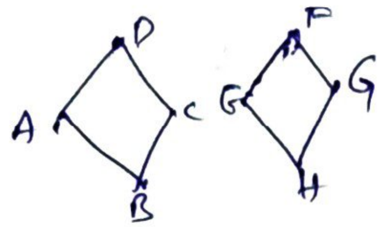
Ex:



$G = (V, E)$

Disconnected Graph:

* A Graph is said to be disconnected Graph, in which any path doesn't exist between every pair of vertices in the given Graph.



G

Length of the path

* The no: of edges appearing in the sequence of a path is called the Length of the path.

* How many paths of length 'M' are there in the complete Graph K_n with $m,n \in Z^+$. How many paths of length '4' are there in the complete Graph K_7.

Sol

For $0<m<n$, the no: of paths of length 'M' in K_n is $\frac{1}{2}n(n-1)(n-2)(n-3)\ldots\ldots(n-m)$

In K_7, the no: of paths of length '4' is

$\frac{1}{2}n(n-1)(n-2)(n-3)(n-4)$

$\frac{1}{2}7(7-1)(7-2)(7-3)(7-4)$

$=\frac{1}{2}\times 7\times 6\times 5\times 4\times 3$

$=1260$

Theorem:

Let $G=(V,E)$ be an undirected Graph with $a,b \in V$, $a \neq b$. If there exist a trail (in G) from a to b, then there is a path (in G) from a to b.

proof:-

Given that

$G=(V,E)$ be an undirected Graph.

Since there is a trail from a to b.

We select one of the shortest path from a to b.

$\{a,x_1\}, \{x_1,x_2\}, \{x_2,x_3\}. \cdots \{x_n,b\}$

In this trail is not a path we have

$\{a,x_1\}, \{x_1,x_2\}, \{x_2,x_3\} \cdots \{x_{k-1},x_k\}, \{x_k,x_{k+1}\},$
$\{x_{k+1},x_{k+2}\}, \cdots \{x_{m-1},x_m\}, \{x_m,x_{m+1}\} \cdots \{x_n,b\}$

Where $k<m$ and $x_k=x_m$

possibily with $k=0$ and $a=(x_0)=x_m$.

(or)

$m=n+1$ and $b=(x_{n+1})=x_k$

But then we have a contradiction because

$\{a,x_1\}, \{x_1,x_2\}, \{x_2,x_3\}. \cdots \{x_{k-1},x_k\}, \{x_m,x_{m+1}\} \cdots$
$\{x_n,b\}$

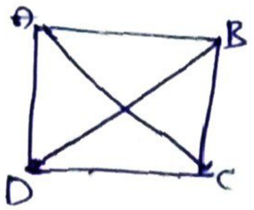
is a shorter trail from a to b.

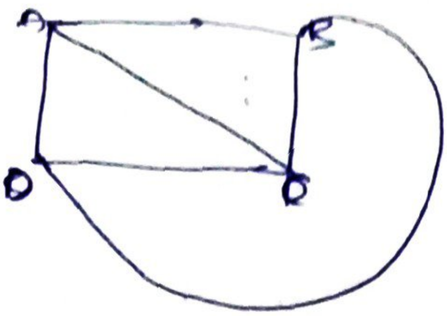
Hence if there exist a trail from a to b then there is a path from a to b.

plannar Graphs:

* A Graph which can be represented by atleast one plane drawing in which the edges meet only at the vertices is called a plannar Graph.

Ex:-



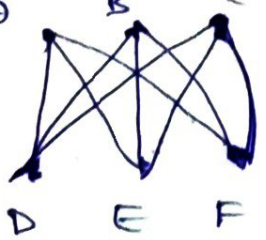


plannar Graph.

Non-planner Graph:

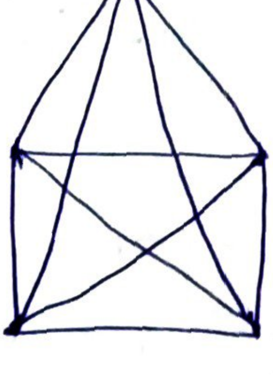
* A Graph which cannot be represented by a plane drawing in which the edges meet only at the vertices is called a Non-planner Graphs.

Ex:- complete biparted $K_{3,3}$



Non-plannar Graph.

Ex:- complete biparted $K_{5,5}$



Non-plannar Graph

Euler's Formula:-

* A connected planner Graph 'G' with |V| vertices and |E| edges has exactly |E|-|V|+2 regions in all of its diagrams.

$$\boxed{|R| = |E| - |V| + 2}$$

(or)

$$\boxed{|V| - |E| + |R| = 2}$$

* |R| = No: of Regions
  |E| = No: of Edges
  |V| = No: of vertices

* A connected planner Graph has '9' vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4, 5. How many edges are there. How many phases (or) Regions.

Given '9' vertices and degree's 2, 2, 2, 3, 3, 3, 4, 4, 5

By handshacking property

$$\sum d(v) = 2|E|$$

$$2+2+2+3+3+3+4+4+5 = 2|E|$$

$$28 = 2|E|$$

$$\boxed{|E| = 14}$$

Given |V| = 9

By using Euler's Formula

$$|R| = |E| - |V| + 2$$

$$= 14 - 9 + 2$$

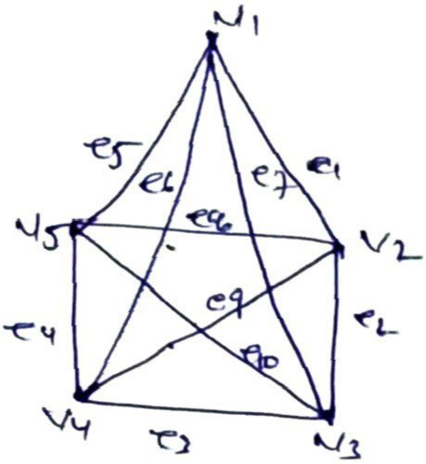
$$= 7$$

NOTE:- If $G=(V,E)$ is a connected plannar simple Graph of order '3' or more, than $|E| \leq 3|V|-6$

* If $G=(V,E)$ is a connected plannar simple graph of order '3' or more and no '3' cycles, than $|E| \leq 2|V|-4$

* Show that kuratowski's first Graph, K_{5}, is non-plannar.

Sol



The Graph '$K_{5,5}$' simple, connected and $|V|=5$ and $|E|=10$

* If this Graph is plannar, then we have

$$|E| \leq 3|V|-6$$

$$10 \leq 3(5)-6.$$

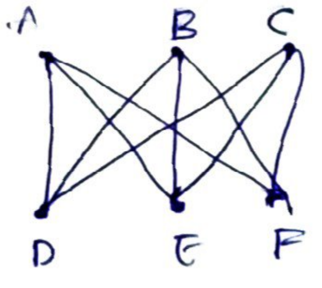
$$10 \leq 9$$

* which is not True.

$\therefore$ $K_{5,5}$ is non-plannar.

* Show that kuratowski's second graph K_3 is non-plannar.

Sol



The Graph '$K_{3,3}$' simple connected and $|V| = 6$ and $|E| = 9$

* If the Graph is plannar, then we have

$|E|$

* In this Graph '$K_{3,3}$' has no Triangles.

$|E| \leq 2|V| - 4$

$|9| \leq 2|6| - 4$

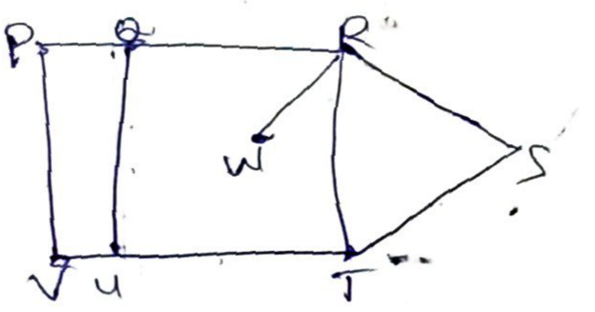
$|9| \leq 8$

$9 \leq 8$

* which is not True

$\therefore$ '$K_{3,3}$' is a non-plannar.

* show that the following Graph is verified euler's formula.



Sol Given $|V| = 8, |E| = 10$

Euler's formula

$|R| = |E| - |V| + 2$

$\Rightarrow 10 - 8 + 2$

$|R| = 4$

Now

$|V| - |E| + |R| = 2$

$8 - 10 + 4 = 2$

$\boxed{2 = 2}$

$\therefore$ The Graph satisfies the Euler's formula.

Hamiltonian cycle / circuit :-

* In a connected Graph, a closed walk that visities every vertex of graph 'G' exactly once except the starting and ending vertex is called Hamiltonian cycle.

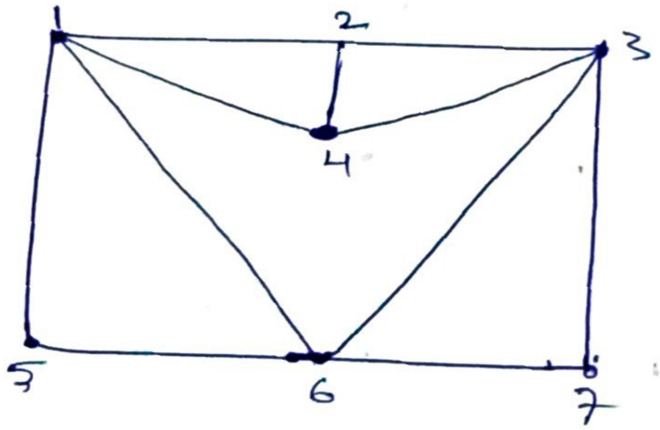
Hamiltonian path

* In a connected Graph an open walk that visities every vertex of graph 'G' exactly once but it contain open walk is called Hamiltonian path,

Hamiltonian Graph

* A Graph 'G' is said to be Hamiltonian Graph, if it contains Hamiltonian cycles.

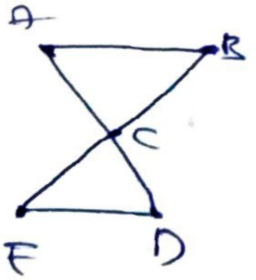
Ex:



Hamiltonian cycle → 1-2-4-3-7-6-5-1

Hamiltonian Graph.

Ex:-



Hamitonian path → A-B-C-D-E

Travelling salesman problem

* A salesman required to visit a no: of cities (Each of city has a road to every other city) during his trip given the distance between cities in what order should the salesman travels so has to visit every city preciously once and return to this home city with minimum Milaze travelled.

Sol

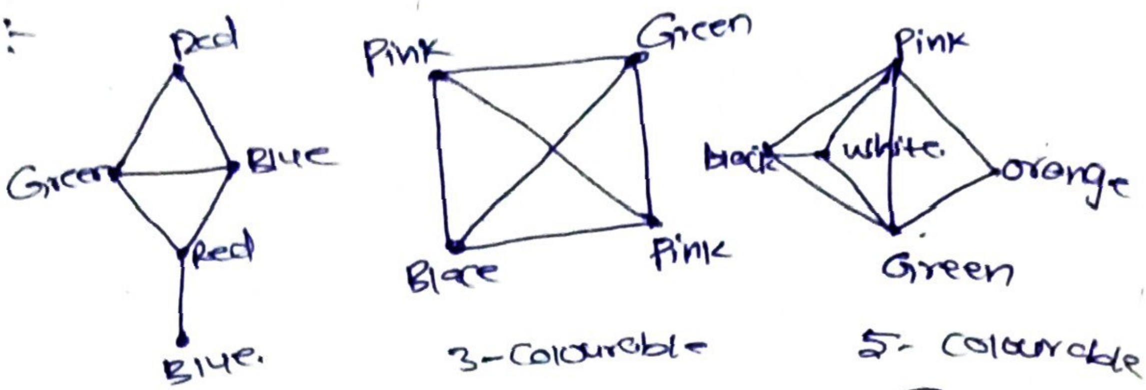
* Represent the cities By vertices and Roads b/w them by Edges. Then we get a Graph.
* In this Graph for every edge E They a corresponds a real number w(e) (Distance in kilometer). Such a Graph is called weighted Graph. Here w(e) is called as the weight of the edge 'e'
* If each of the cities has a road to every other city, we have a complete weighted graph.
* This Graph has numerous hamiltonian circuit and we have to select the hamiltonian circuit, that has the smallest sum of distances.
* The no: of different Hamiltonian circuites in a complete Graph of 'n' vertices is equal to $\frac{(n-1)!}{2}$.
* First we list all the $\frac{(n-1)!}{2}$ Hamiltonian circuites that are possible in the Given Graph.
* Next calculate the distance travelled on each of these Hamiltonian circuites.
* Then select the Hamiltonian circuits with the least distances.
* This provides a solution for the Travelling salesman Problem.

Graph coloring

* Given a planar or non-planar Graph 'G', if we assign colors to its vertices in such a way that no two adjacent vertices have the same color, then we say that the Graph 'G' is properly colored.

* In otherwords proper coloring of a Graph means assigning colors its vertices such that adjacent vertices have different colors.

Ex:-



3 - colourable Graph

3 - Colourable Graph.

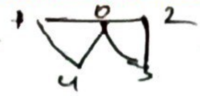
5 - Colourable Graph.

Applications

* Register allocation.
* Map colouring.
* Biparted Graph checking
* Mobile Radio frequency assignment.
* Making Time tables .. etc.

Euler Trails & Euler Circuits.

* consider a connected Graph 'G', if there is a circuit in 'G' that contains all the edges of the 'G', then that circuit is called Euler circuit or Eulerian Line or Euler Tour

in 'G'. Ex:-  ⇒ 1-0-2-3-0-4-1

* If there is a Trail in 'G' that contains all the edges of 'G', then that Trail is called an "Euler Trail" in 'G'.

* A connected Graph that contains an Eulers Circuit is called an Euler Graph. (or) Eulerian Graph.

Theorem: A connected Graph 'G' has an Euler circuit [i.e 'G' is an Euler Graph] iff all the vertices of 'G' are of Even degree.

Proof:

Suppose 'G' is an Euler Graph, Then 'G' contains an Euler circuit.

* So there exist a closed walk running through all the edges of 'G' exactly once.

* Let $v \in V$ be a vertex of 'G'

* Now in tracing the walk it goes through two incident edges on v with one entered 'v' and the other exited.

* This is true not only for all the immediate vertices of the walk, but also true for the terminal vertex. because we exited and entered at the same vertex at the beginning and ending of the walk.

* If 'v' occur 'K' times at Euler circuit, then the degree of 'v' = 2K (Even number)

Thus the 'G' is an Euler Graph then the degree of each vertex is even.

Converse

* Suppose all the vertices of Graph 'G' are even degree.

* Now we have to show that 'G' is an Euler Graph.

* We have to construct a closed walk started at an arbitrary vertex 'v' and running through all edges of 'G' exactly once.

* To find a closed walk, let us start from the vertex 'v' since every vertex is of even degree, we can exit from every vertex we entered.

* The tracing cannot stop at any vertex 'v', since 'v' is also an even degree, we shall eventually reach 'v' when the tracing comes to end.

* If this closed walk includes all the edges of 'G' then 'G' is an Euler Graph.

Chromantic Numbers:

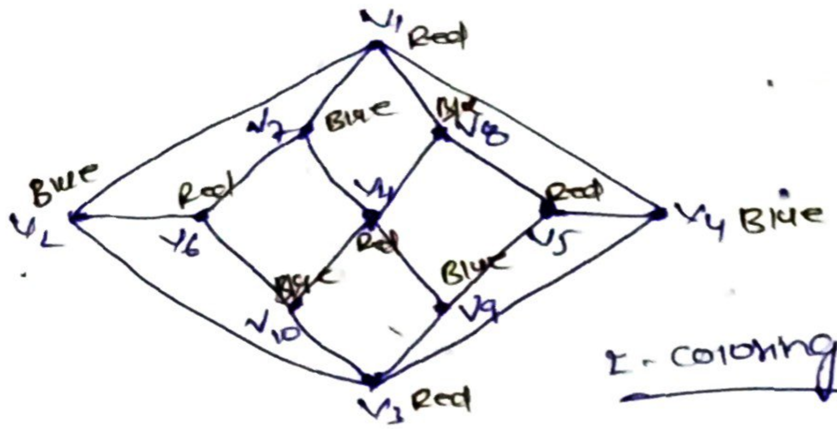
* The minimum numbers of colors required to color all the vertices of a Given Graph is called a chromantic number of a given Graph.

* The chromantic number of a Graph 'G' is usually denoted by $\chi(G)$.

* A Graph 'G' is said to be K - Colourable, if we

or properly color it with k-colors.

* Find the Chromantic Number of the following Graph.



2-coloring Graph

Red — V1, V6, V5, V3, V11

Blue — V2 V4, V9, V8, V10, V7

The Given Graph is called the Herschel Graph

* The chromantic number $\chi(G) = 2$

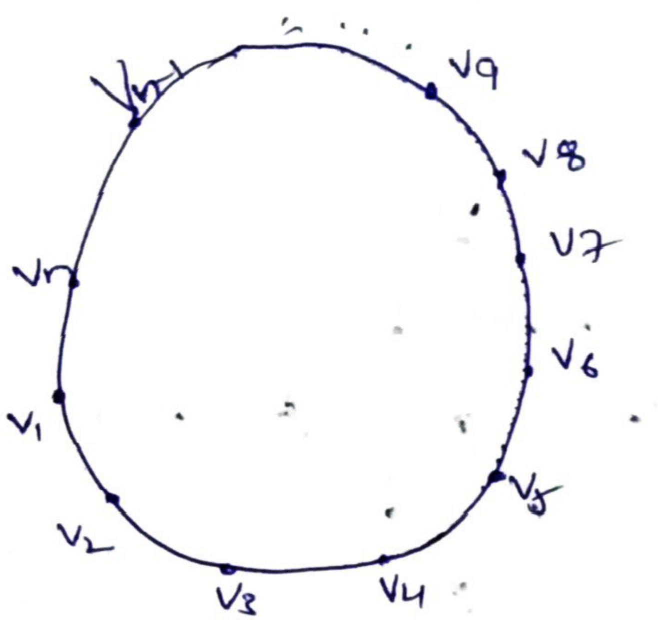
* The vertices $V_1, V_3, V_5, V_6, V_{11}$ can be assigned the same colour Red and all the remaining vertices $V_2, V_4, V_7, V_8, V_9, V_{10}$ can be assigned the same Color Blue.

* Thus two colors are sufficient for proper coloring of G.

* Hence the Chromantic Number $\chi(G) = 2$.

* prove that a Graph of order 'n' (≥ 2) consiting of a single Cycle is 2-chromantic if 'n' is even and 3-chromatic if 'n' is odd.

Sol The Graph being considered is shown below.



* The Graph cannot be properly colored with a single color.
* Assign two colors alternatively to the vertices, starting with 'v_1' then the odd vertices $v_1, v_3, v_5, \ldots$ etc will have color 'α' and the even vertices $v_2, v_4, \ldots$ ... etc will have different color 'β'
* Suppose 'n' is even then the vertex 'v_n' is an even vertex and therefore will have the color 'β' and the Graph gets properly colour. Therefore the Graph is chromatic.
* Suppose 'n' is odd then the vertex 'v_n' is an odd vertex and therefore will have the color 'α' and the Graph is not properly colored.
* To make it properly colored it is enough if 'v_n' is assigned a 3rd color 'γ'. Thus in this case the Graph is '3' chromatic.

4-coloring problem

* The 4-color theorem states that the vertices of every Planar Graph can be coloured with atmost 4 colors. so that no two adjacent vertices receive the same color.

* Every plannar Graph is Four colourable

Trees

* Tree is a discrete structure that represent heiracial relationships b/w individual elements or nodes.
* Every in which a parent has no more than two children is called Binary Tree.

Def: Tree

* A tree is a connected acyclic undirected graph.

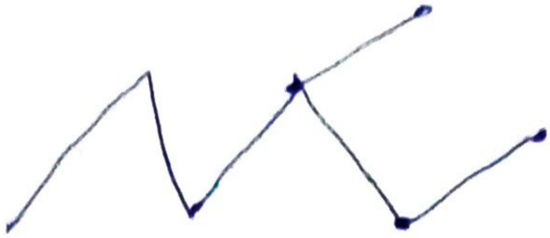
Properities:

* There is a unique path between every pair of vertices in 'G'.

* A Tree with 'n' no. of vertices contain n-1 no. of edges

* The vertex which is of '0' degree is called "Root of the tree".

* The vertex which is of '1' degree is called "Leaf node" and the degree of internal node is atleast '2'.



Theorem:

* If A, B are distinct vertices in a tree $T=(V,E)$ then there is a unique path that contents these vertices.

Proof:-

Given that $T=(V,E)$ be a Tree.

Since 'T' is connected, there is a atleast one path in 'T' that connects from A to B.

If there were more, then from two such that paths some of the edges from a cycle. But 'T' has no cycles.

∴ If A, B are distinct vertices in a tree $T=(V,E)$ then there is a unique path that contents these vertices.

Hence proved.

Theorem 2

* A Tree with 'n' vertices has $(n-1)$ edges.

proof:

Consider a Tree 'T' with 'n' vertices.

Let us reconstruct the Tree from the root vertex. when the first root vertex has been added the no: of edges is '0'

* After the Root vertex, every vertex that is added to the construction of 'T' contributes one edge to 'T'

* Adding the remaining (n-1) vertices to the construction of 'T' after the root vertex will add (n-1) edges

* After the reconstruction of 'T' is complete, 'T' will have 'n' vertices and (n-1) edges

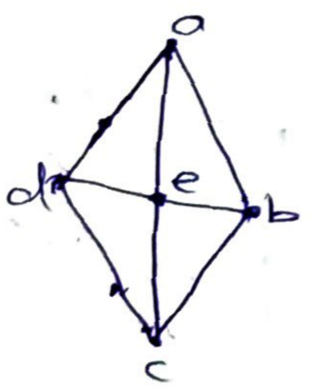
Hence proved

Spanning Tree

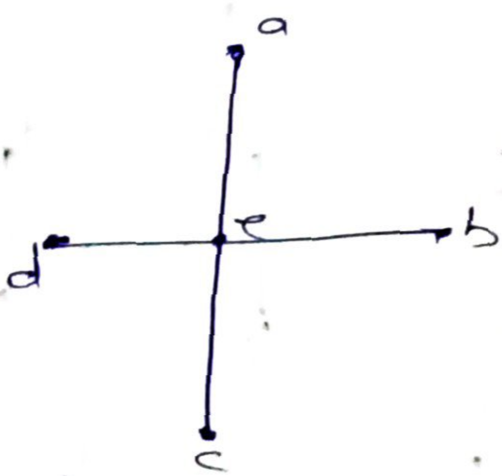
* Def:- A spanning Tree of a connected, undirected Graph 'G' is a tree that minimally includes all the vertices of 'G'.

* A Graph 'G' has many spanning Trees.

Ex:-



Graph G.



Spanning Tree

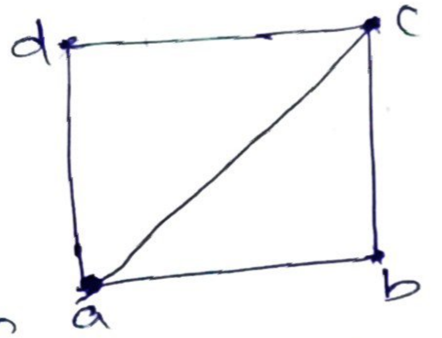
* $m-n+1 \Rightarrow$ No: of edges

m ↓ edges, n ↓ vertices

* How many edges must be removed from connected Graph with 'n' vertices and 'm' edges to produce a spanning Tree.

Sol $\boxed{m-n+1}$

* Draw all the spanning Tree of the Graph.



Given

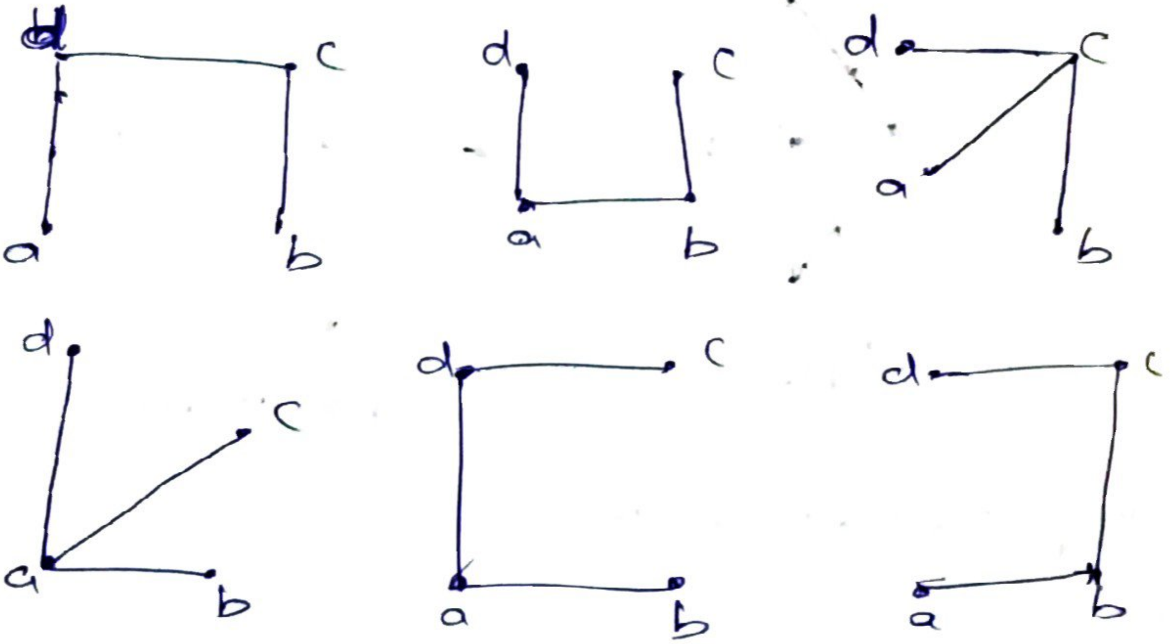
* No: of vertices $n=4$

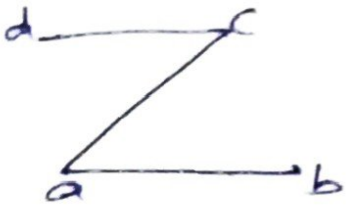
No: of edges $m=5$

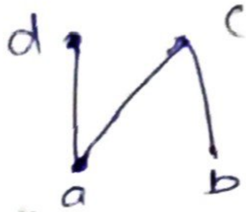
To obtain a spanning Tree we should remove

$m-n+1 = 5-4+1 = 2$ edges.

* The spanning Trees are

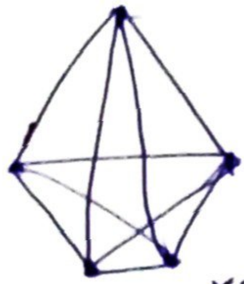






* Find a spanning Tree for the following Graphics K_5 and $K_{4,4}$.

Sol



K_5

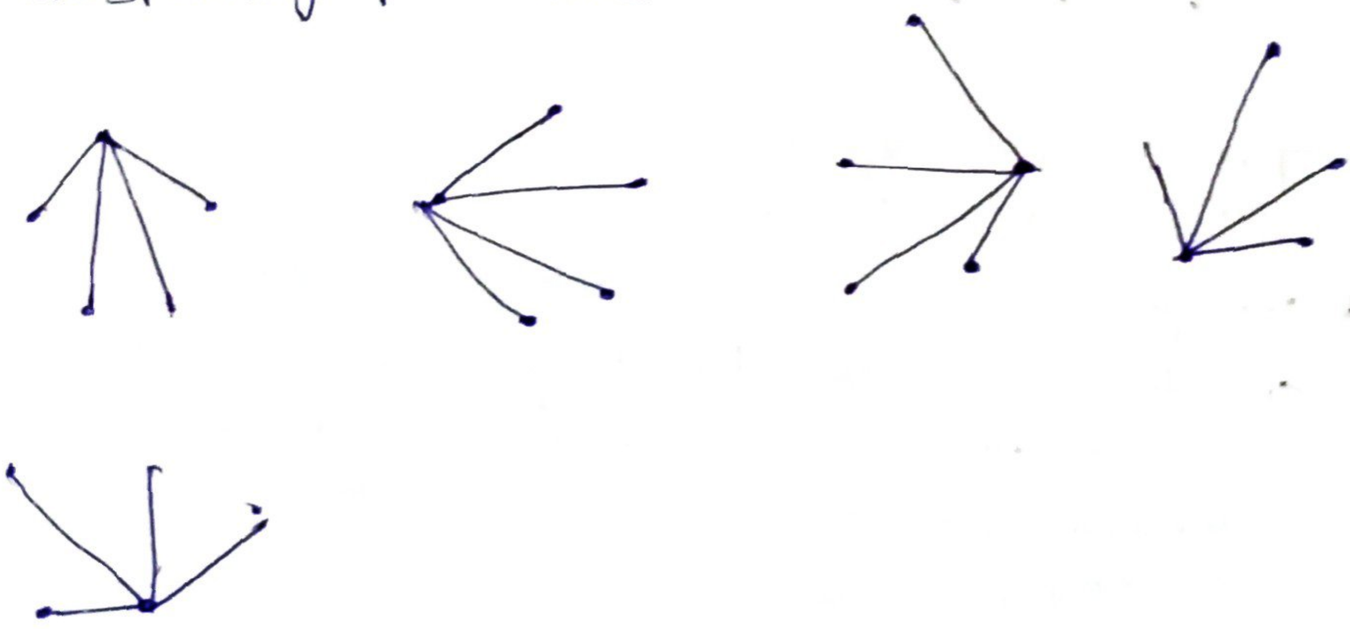
No: of vertices $n = 4$

No: of edges $m = 10$

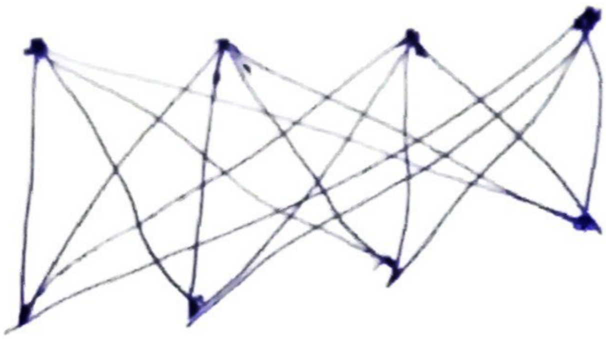
To obtain a spanning Tree we should remove

$m - n + 1 = 10 - \cancel{5} 4 + 1 = \underline{6}$

The Spanning Trees are.



$K_{4,4}$



No: of vertices $n = 8$

No: of edges $m = 16$

To obtain spanning tree

$m - n + 1 = 16 - 8 + 1 = 9$ edges.

Trees and Sorting

* Merge sort is a divide and conquer based on the idea of breaking down a list into several sublists until each sub list consists of a single element and merging those sublists in a manner that results into sorted list.

Merge sort Algorithm

steps

* 1. If $n=1$ then list is already sorted and the process terminates.

* If n>1 then go to step2

2. Divide the array and sort the subarrays performing the following.

i. Assign 'm' the value (n/2)

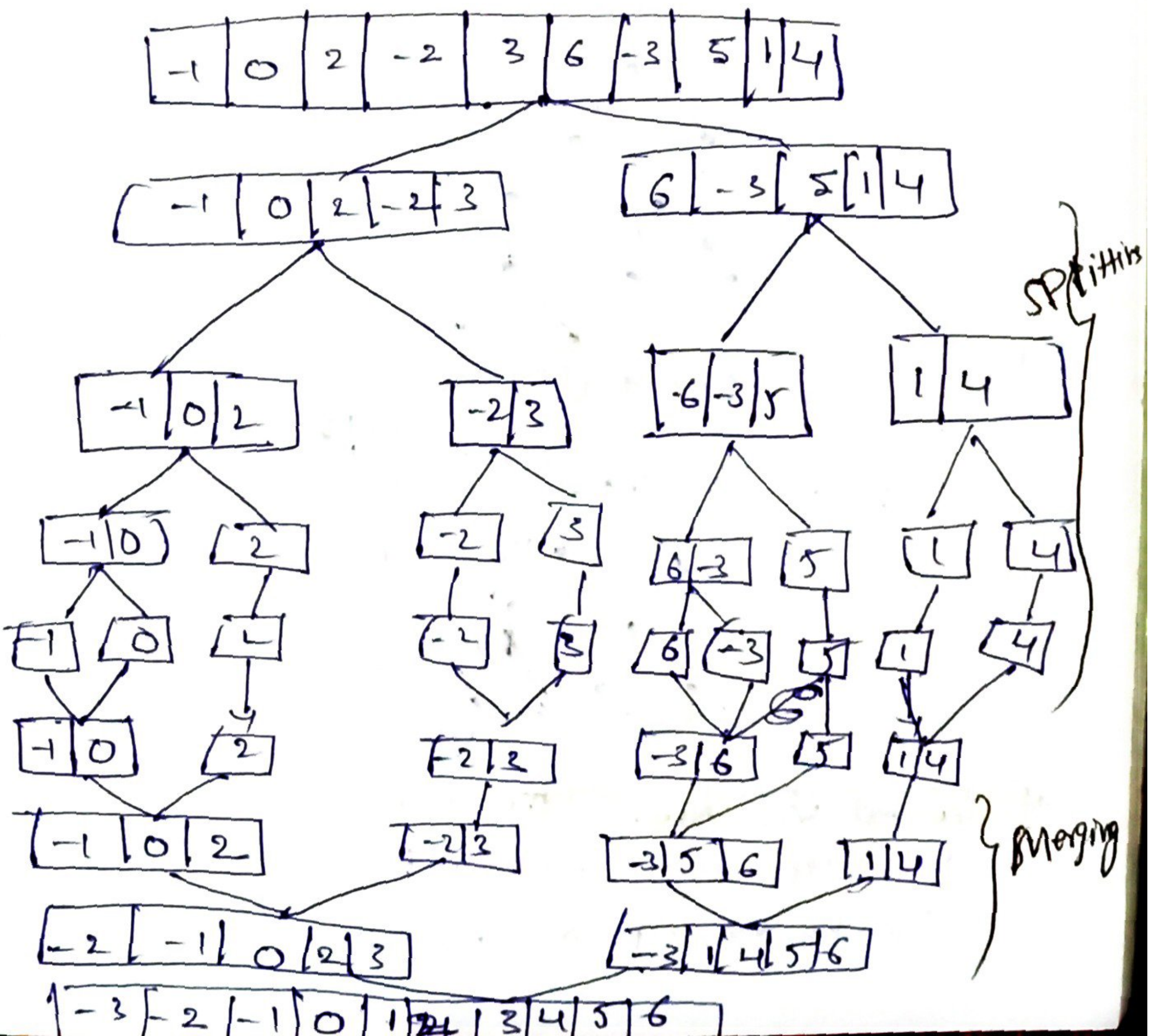
ii. Assign to list, the subarray list (i), list(i) .... list(m).

iii. Assign to list (ii) the subarray list(m+1), list(m+1) .... list(n)

iv. Apply merge sort to list(i) to list(ii).

3. Merge list(i) and list(ii).

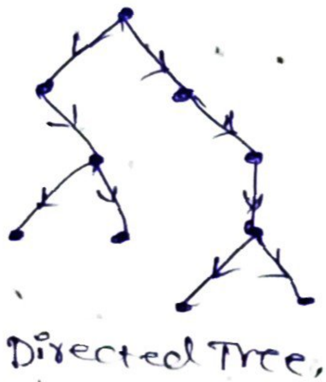
* using the merge sort sort the list -1, 0, 2, -2, 3, 6, -3, 5, 1, 4



Rooted Trees

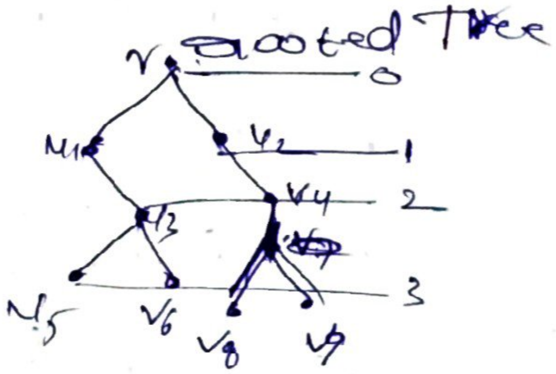
* Let 'D' be a directed Graph and 'G' be its underlying Graph. We say that 'D' is directed Tree whenever 'G' is a Tree. Thus a directed tree is a directed Graph whose underlying graph is a Tree.

* A directed Tree 'T' is called a Rooted Tree, If
  i. 'T' contains a unique vertex called the Root whose indegree is zero
  ii. The indegrees of all other vertices of 'T' are equals to '1'.



Directed Tree.

Ex:



* v_1 and v_2 are first level,
  v_3 and v_4 are second level,
  v_5, v_6, v_8, v_9 are Third level.

* v_1 is the ancesor of v_3, v_5, v_6 (v_3, v_5, v_6 are descends of v_1) and v_2 is the ancesors of v_4, v_8, v_9 (v_4, v_8, v_9 are descends of v_2)

* v_1 is the parent of v_3 and v_3 is a Child of v_1

* siblings are $v_1, v_2, v_5, v_6, v_8, v_9$.

* v_5, v_6, v_8, v_9 are leaf and all other vertices are internal vertices.

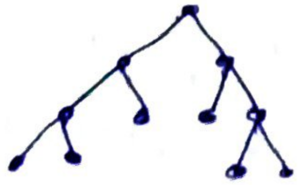
m-ary Tree

* A Rooted Tree 'T' is called an m-ary Tree if every vertex of 'T' is of out degree $\leq m$, i.e if every vertex of 'T' has atmost 'm' children.
* A Rooted Tree 'T' is called a complete "m-ary Tree" if every internal vertex of 'T' is of out degree 'm' only, i.e if every internal vertex of 'T' has exactly 'm' children.

Binary Tree

* A Rooted Tree 'T' is called a Binary rooted Tree or just a Binary Tree, if every internal vertex of 'T' is of out degree 1 or 2 i.e if every vertex has atmost 2 Children.

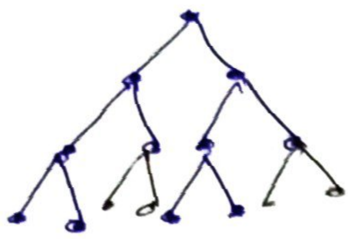
EX:



Complete Binary Tree:

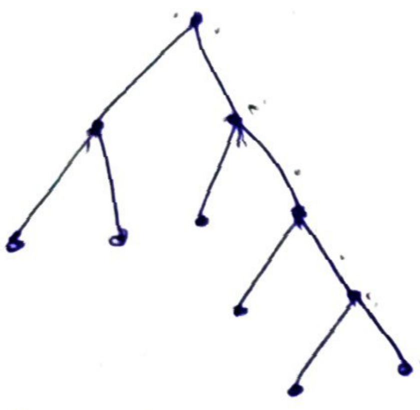
* A rooted Tree 'T' is called a complete binary Tree, if every internal vertex of 'T' is of out degree 2. i.e if every internal vertex two childrens.

Ex:-



Balanced Tree

* If 'T' is a Rooted Tree and 'H' is the largest level number acheived by a leaf of 'T'. Then 'T' is said to have height 'H'.
* A Rooted Tree of Height 'H' is said to be balanced, if the level number of every leaf is 'H' or H-1

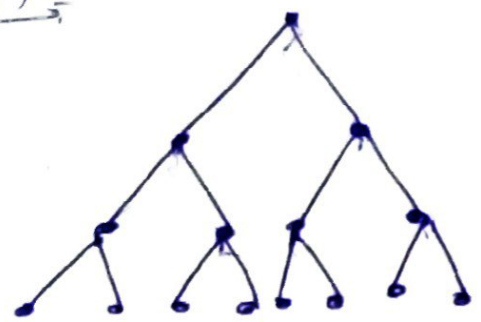


Height h=4 and balanced to.

Full Binary Tree:

* Let 'T' be a complete binary Tree of Height 'h' than 'T' is called a Full binary tree. If all the leaf's in 'T' are at level 'h'

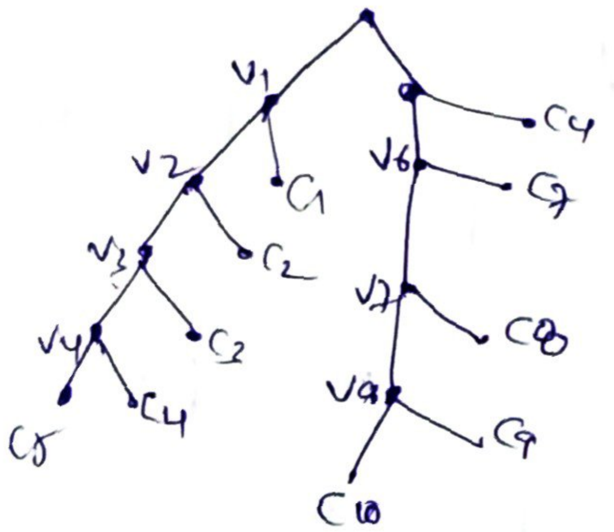
Ex:



A Full Binary tree of height = 3

* ~~The~~ computer Laboratory of a school has '10' computers that are to be connected to a walls socket that has two outlets. Connections are made by using extension cords that have two outlets each. Find the least number of cords needed to get these computers setup for use.

Sol



consider the complete binary Tree having the walls socket has the root, the Computer has the leafs, and the internal vertices, other than the root as extension Cords.

Then The no: of leves in a Tree.

$p = 10$

The no: of internal vertices in the Tree

$q = p-1$

$= 10-1$

$= 9$

The no: of extension cords needed is

(internal vertices − root node)

$9 - 1 = 8$

Weighted Tree and Prefix codes

Weighted Tree

* A Tree to whose nodes or edges labels are assigned.

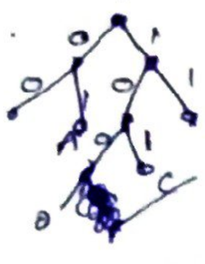
Prefix codes

* A set 'P' of Binary sequences is called a prefix code. If no sequence in 'P' is the prefix of any other sequences in 'P'.

Ex:- i. $P = \{01, 010, 100\}$ → not a prefix code

because 01 is a prefix of 010.

ii ⇒ $P = \{\overset{A}{01}, \overset{B}{100}, \overset{C}{101}\}$ is a prefix code.
because any sequence in P is not prefix of any other code.



* Every prefix code can be represented by a tree.

suppose we have a tree in which every edge left side we label it zero (0) and every edge & rightside by '1'