UNIT - IV

Recurrence Relations

Generating functions of sequences

Consider a sequence of real numbers $a_0, a_1, a_2, \ldots$

Let us denote the sequence by $\langle a_r \rangle$, $r = 0, 1, 2, 3, \ldots$

given this sequence, suppose there exist a function $f(x)$ whose expansion in a series of power of x is as given below.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_r x^r + \cdots + a_{n-1} x^{n-1} + \cdots$$

$$= \sum_{r=0}^{\infty} a_r x^r$$

Then $f(x)$ is called a generating function for the sequence $a_0, a_1, a_2, \cdots = \langle a_r \rangle$

Example

1. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\cdots = \sum_{r=0}^{\infty} x^r$

Here $f(x) = (1-x)^{-1}$ is a generating function for the sequence $1, 1, 1, 1, \ldots\ldots$

Similarly

2. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\cdots = \sum_{r=0}^{\infty} (-x)^r$

Here $f(x) = (1+x)^{-1}$ is a generating function for the sequence $1, -1, 1, -1, \ldots\ldots$

NOTE

* $e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$
* $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$
* $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$
* $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

Binomial Expansions

* $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

$$= \sum_{r=0}^{\infty} \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{r!} x^r$$

Also

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

* $(1+x)^n$ is a generating function for the sequence

$\binom{n}{0}, \binom{n}{1} \binom{n}{2} \cdots \binom{n}{r} \cdots$

Properties

* If $f(x)$ is a generating function for a sequence $\langle a_r \rangle$ and $g(x)$ is generating function for a sequence $\langle b_r \rangle$ then $pf(x) + qg(x)$ is generating function for the sequence $\langle pa_r + qb_r \rangle$ where p and q are any real numbers.
* If $f(x)$ is generating function for a sequence $\langle a_r \rangle$ then $xf'(x)$ is a generating function for a sequence $\langle r a_r \rangle$.

1. Find the sequence generated by the following functions

i. $(3+x)^3$

Sol Given $f(x) = (3+x)^3$

$$= 3^3\left(1+\frac{x}{3}\right)^3$$

$$= 27\left(1+\frac{x}{3}\right)^3 \quad \text{---} (1)$$

consider $\left(1+\frac{x}{3}\right)^3$

by using binomial expansion

$$\left(1+\frac{x}{3}\right)^3 = \sum_{r=0}^{3} \binom{3}{r} \left(\frac{x}{3}\right)^r$$

$$= \binom{3}{0}\left(\frac{x}{3}\right)^0 + \binom{3}{1}\left(\frac{x}{3}\right)^1 + \binom{3}{2}\left(x/3\right)^2 + \binom{3}{3}\left(\frac{x}{3}\right)^3$$

$$= (1)(1) + \cancel{3}\left(\frac{x}{\cancel{3}}\right) + \cancel{3}\left(\frac{x^2}{\cancel{9}_3}\right) + 1\left(\frac{x^3}{27}\right)$$

$$= 1 + x + \frac{x^2}{3} + \frac{x^3}{27}$$

from (1)

$$f(x) = 27\left[1 + x + \frac{x^2}{3} + \frac{x^3}{27}\right]$$

$$= 27 + 27x + 9x^2 + x^3$$

The sequences are 27, 27, 9, 1.

ii. $2x^2(1-x)^{-1}$

Sol Given $f(x) = 2x^2(1-x)^{-1}$

$$f(x) = 2x^2[1 + x + x^2 + x^3 + \dots]$$

$$= 2x^2 + 2x^3 + 2x^4 + 2x^5 + \dots$$

The sequences are 0, 0, 2, 2, 2, 2, . . .

iii. $\frac{1}{1+x} + 2x^3$

Sol Given $f(x) = \frac{1}{1+x} + 2x^3$

$$= (1+x)^{-1} + 2x^3$$

$$= [1 - x + x^2 - x^3 + \dots] + 2x^3$$

$$= [1 - x + x^2 + x^3 + \dots]$$

The sequences are 1, -1, 1, 1, . . .

iv. $(1+3x)^{-1/3}$

Sol Given $f(x) = (1+3x)^{-1/3}$

By using Binomial expansion

$$(1+3x)^{-1/3} = 1+\frac{(-1/3)(3x)}{1!}+\frac{(-1/3)(-1/3-1)}{2!}(3x)^2+\frac{(-1/3)(-1/3-1)(-1/3-2)}{3!}(3x)^3+$$

$$= 1+\frac{(-x)}{1!}+\frac{\frac{4}{9}}{2!}(3)^2x^2+\frac{(-1/3)(-4/3)(-7/3)}{3!}3^3x^3+\dots$$

$$= 1-\frac{x}{1!}+\frac{4}{9\cdot 2!}9x^2-\frac{28}{27\cdot 3!}27x^3+\dots$$

$$= 1-\frac{x}{1!}+2x^2-\frac{14}{3}x^3+\dots$$

$\therefore$ The sequences are $1, -1, 2, -14/3$

v. $f(x) = 3x^3+e^{2x}$

$$= 3x^3+1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\dots$$

$$= 3x^3+1+\frac{2x}{1!}+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\dots$$

$$= 1+\frac{2x}{1!}+2x^2+\frac{26x^3}{3!}+\dots$$

$$= 1+\frac{2x}{1!}+2x^2+\frac{13}{3}x^3+\dots$$

The sequences are $1, 2, 2, \frac{13}{3}$

2. Find the generating functions from the following sequences.

i. $1, 1, 0, 1, 1, 1, \dots$

Sol Given sequences are $1, 1, 0, 1, 1, 1, \dots$

Here $a_0 = 1, a_1 = 1, a_2 = 0, a_3 = 1, a_4 = 1, a_5 = 1 \dots$

We know that

$$f(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\dots$$

$$= 1+x+0+x^3+x^4+x^5+\dots$$

$$= 1+x+x^3+x^4+x^5+\dots$$

$$= (1+x+x^2+x^3+x^4+x^5+\dots)-x^2$$

$$= (1-x)^{-1}-x^2$$

ii. 1, 2, 3, 4

sol Given sequences are

1, 2, 3, 4.

Here $a_0 = 1, a_2 = 2, a_3 = 3, a_4 = 4$.

we know that

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= 1 + 2x + 3x^3 + 4x^3 + \dots$$

$$= (1-x)^{-2}$$

iii. 1, -2, 3, -4

sol Given sequences are

1, -2, 3, -4

Here $a_0 = 1, a_1 = -2, a_2 = 3, a_3 = -4$

WKT $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$$= 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$= (1+x)^{-2}$$

iv. 0, 1, 2, 3, 4, ...

sol Given sequences are 0, 1, 2, 3, 4, ...

$a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$

WKT

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$= 0 + 1.x + 2.x^2 + 3.x^3 + 4.x^4 + \dots$$

$$= x(1 + 2x + 3x^2 + 4x^3 + \dots)$$

$$= x(1-x)^{-2}.$$

v. 0, 1, -2, 3, -4, ...

sol Given sequences are 0, 1, -2, 3, -4

Here $a_0 = 0, a_1 = 1, a_2 = -2, a_3 = 3, a_4 = -4$

WKT $f(x) = a_0 + a_1 x + a_1 x^2 + a_3 x^3 + a_4 x^4 + \dots$

$$= 0 + 1.x - 2x^2 + 3x^3 - 4x^4 + \dots$$

$$= x(1 - 2x + 3x^2 - 4x^3 + \dots) = x(1+x)^{-2}$$

Note:

* If 'n' is a +ve integer then $\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$

$$= (-1)^r \binom{n+r-1}{r-1}$$

* $$(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r.$$

1. Let $f(x) = (1+x+x^2)(1+x)^n$ where ~~for~~ n is a +ve integer. Find the coefficients of x^7, x^8, x^k for $0 \le k \le n+2$ in $f(x)$.

Sol Let $f(x) = (1+x+x^2)(1+x)^n$

$$= (1+x+x^2) \sum_{r=0}^{n} \binom{n}{r} x^r$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^{r} + \sum_{r=0}^{n} \binom{n}{r} x^{r+1} + \sum_{r=0}^{n} \binom{n}{r} x^{r+2}$$

The coefficients of x^7 is

$$\binom{n}{7} + \binom{n}{6} + \binom{n}{5}$$

The coefficient of x^8 is

$$\binom{n}{8} + \binom{n}{7} + \binom{n}{6}$$

The coefficient of x^k is

$$\binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-2}$$

2. Find the coefficients of x^{27} in the following functions

(i) $(x^4+x^5+x^6+\cdots)^5$

(ii) $(x^4+2x^5+3x^6+\cdots)^5$

i. (a). Given $f(x) = (x^4+x^5+x^6+\dots)^5$

$$= (x^4)^5[1+x+x^2+\dots]^5$$

$$= x^{20}[(1-x)^{-1}]^5$$

$$= x^{20}(1-x)^{-5}$$

$$= x^{20}\sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$$

$$= x^{20}\sum_{r=0}^{\infty}\binom{5+r-1}{r}x^r$$

$$= x^{20}\sum_{r=0}^{\infty}\binom{4+r}{r}x^r$$

$$= \sum_{r=0}^{\infty}\binom{4+r}{r} x^{20+r}$$

put $r=7$

$$f(x) = \binom{4+7}{7}x^{27}$$

$$= \binom{11}{7}x^{27}$$

The coefficient of x^{27} is 330

ii. Given $f(x) = (x^4+2x^5+3x^6+\dots)^5$

$$= (x^4)^5[1+2x^2+3x^3+\dots]^5$$

$$= x^{20}[(1-x)^{-2}]^5$$

$$= x^{20}[(1-x)^{-10}]$$

$$= x^{20}\sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$$

$$= x^{20}\sum_{r=0}^{\infty}\binom{10+r-1}{r}x^r$$

$$= x^{20}\sum_{r=0}^{\infty}\binom{9+r}{r}x^r$$

$$= \sum_{r=0}^{\infty} \binom{9+r}{r} x^{20+r}$$

put $r=7$

$$f(x) = \binom{9+7}{7} x^{27}$$

$$= \binom{16}{7} x^{27}$$

$\therefore$ The coefficient of x^{27} is $\binom{16}{7} = 11440$

3. Find the coefficient of x^{18} in the following products.

$(x+x^2+x^3+x^4+x^5)(x^2+x^3+x^4+\cdots)^5$

<u>Sol</u> Given $f(x) = (x+x^2+x^3+x^4+x^5)(x^2+x^3+x^4+\cdots)^5$

$$= x(1+x+x^2+x^3+x^4)(x^2(1+x+x^2+\cdots))^5$$

$$= x^{11}(1+x+x^2+x^3+x^4)((1-x)^{-1})^5$$

$$= x^{11}(1+x+x^2+x^3+x^4)(1-x)^{-5}$$

$$= x^{11}(1+x+x^2+x^3+x^4)\sum_{r=0}^{\infty}\binom{5+r-1}{r}x^r$$

$$= x^{11}(1+x+x^2+x^3+x^4)\sum_{r=0}^{\infty}\binom{4+r}{r}x^r$$

$$= (x^{11}+x^{12}+x^{13}+x^{14}+x^{15})\sum_{r=0}^{\infty}\binom{4+r}{r}x^r$$

$$= \sum_{r=0}^{\infty}\binom{4+r}{r}x^{r+11} + \sum_{r=0}^{\infty}\binom{4+r}{r}x^{r+12} +$$

$$\sum_{r=0}^{\infty}\binom{4+r}{r}x^{r+13} + \sum_{r=0}^{\infty}\binom{4+r}{r}x^{r+14} +$$

$$\sum_{r=0}^{\infty}\binom{4+r}{r}x^{r+15}$$

The coefficient of x^{18} is $\binom{4+7}{7} + \binom{4+6}{6} + \binom{4+5}{5} +$

$\binom{4+4}{4} + \binom{4+3}{4}$

$= \binom{11}{7} + \binom{10}{6} + \binom{9}{5} + \binom{8}{4} + \binom{7}{4}$

$= 330 + 210 + 126 + 70 + 35$

$= 771$

4. Determine the coefficient of x^{12}

i. $x^3(1-2x)^{10}$

Sol Given $f(x) = x^3(1-2x)^{10}$

$$= x^3 \sum_{r=0}^{10} \binom{n}{r} (-2x)^r$$

$$= x^3 \sum_{r=0}^{10} \binom{10}{r} (-2)^r x^r$$

$$= \sum_{r=0}^{10} \binom{10}{r} (-2)^r x^{3+r}$$

put $r=9$

$$f(x) = \binom{10}{9} (-2)^9 x^{3+9}$$

$$= \binom{10}{9} (-2)^9 x^{12}$$

$$= 10 \times -512 \, x^{12}$$

$$= -5120 \, x^{12}$$

$\therefore$ The coefficient of x^{12} is -5120.

ii x^5 in $(1-2x)^{-7}$

Sol Given $f(x) = (1-2x)^{-7}$

$$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{7+r-1}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{6+r}{r} x^r$$

put $r=5$

$$= \binom{6+5}{5} x^5$$

$$= \binom{11}{5} x^5$$

The coefficient of x^5 is $\binom{11}{5} = 462$

5. Determine the coefficient of x^{10} in $\dfrac{(x^3-5x)}{(1-x)^3}$

Sol Given $f(x) = \dfrac{x^3-5x}{(1-x)^3}$

$$= (x^3-5x)(1-x)^{-3}$$

$$= (x^3-5x) \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

$$= (x^3-5x) \sum_{r=0}^{\infty} \binom{3+r-1}{r} x^r$$

$$= (x^3-5x) \sum_{r=0}^{\infty} \binom{2+r}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{2+r}{r} x^{r+3} - 5 \sum_{r=0}^{\infty} \binom{2+r}{r} x^{r+1}$$

The coefficients of x^{10} is

$$\binom{2+7}{7} - 5\binom{2+9}{9}$$

$$= \binom{9}{7} - 5\binom{11}{9}$$

$$= 36 - 5(55)$$

$$= -239$$

6. Determine the coefficient of x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$

Sol. Given $f(x) = (x^2+x^3+x^4+x^5+x^6)^5$

$$= (x^2)^5[1+x+x^2+x^3+x^4]^5$$

$$= x^{10}[1+x+x^2+x^3+x^4]^5$$

[$\because$ $a_r=1$, for $0\le r\le n$ and $a_r=0$ for $r\ge n+1$]

[$\because$ for the sequence $\langle a_r\rangle$

$$= \sum_{r\ge0} a_r x^r$$

$$= 1+x+x^2+x^3+\dots+x^n$$

$$= \frac{1-x^{n+1}}{1-x}\ \because]$$

$$f(x) \Rightarrow x^{10}\left[\frac{1-x^{4+1}}{1-x}\right]^5$$

$$= x^{10}\left[\frac{1-x^5}{1-x}\right]^5$$

$$= x^{10}(1-x^5)^5(1-x)^{-5}$$

$$= x^{10}(1-x^5)^5(1-x)^{-5}$$

$$= x^{10}\sum_{r=0}^{5}\binom{5}{r}(-x^5)^r \times \sum_{s=0}^{\infty}\binom{5+s-1}{s}x^s$$

$$= x^{10}\sum_{r=0}^{5}\binom{5}{r}(-x^5)^r \times \sum_{s=0}^{\infty}\binom{4+s}{s}x^s$$

$\because$ $r=0$, $s=10$

$r=1$, $s=5$

$r=2$, $s=0$

The coefficients of x^{20} is

$$\binom{5}{0}\binom{5+10-1}{10}+\binom{5}{1}\binom{5+5-1}{5}+\binom{5}{2}\binom{5+0-1}{0}$$

$= 1\times1001 + 5\times126 + 10\times1$

$= 1001 + 630 + 10$

$= 1641$

A counting Techinque

* suppose we wish to determine the no: of integer solutions of the equation $x_1+x_2+x_3+\dots+x_n=r$. where $0\le r\le n$.

* under the constraints that x_1 can take the integer values $P_{11}, P_{12}, P_{13}, \dots$

x_2 can take the integer values $P_{21}, P_{22}, P_{23}, \dots$

$\vdots$

x_n can take the integer values $P_{n1}, P_{n2}, P_{n3}, \dots$

* To solve this problem we first define the functions $f_1(x), f_2(x), f_3(x), \dots f_n(x)$ as follows

$f_1(x) = x^{P_{11}} + x^{P_{12}} + x^{P_{13}} + \dots$

$f_2(x) = x^{P_{21}} + x^{P_{22}} + x^{P_{23}} + \dots$

$\vdots$

$f_n(x) = x^{P_{n1}} + x^{P_{n2}} + x^{P_{n3}} + \dots$

* Then we consider the function $f(x)$ defined by $f(x) = f_1(x)\cdot f_2(x)\cdot f_3(x)\dots f_n(x)$ and determine the coefficient of x^r in this function.

* This coefficient happens to be equal to the no: of solutions that we desired to find the function $f(x)$ is called the generating function.

1. Find the generating function that determines the no: of non-negative integer solutions of the equation $x_1+x_2+x_3+x_4+x_5=20$ under the constraints $x_1 \leq 3$, $x_2 \leq 4$, $2 \leq x_3 \leq 6$, $2 \leq x_4 \leq 5$, x_5 is odd with $x_5 \leq 9$.

Sol) Given equation

$$x_1+x_2+x_3+x_4+x_5=20$$

Given constraints

$$x_1 \leq 3,\ x_2 \leq 4,\ 2 \leq x_3 \leq 6,\ 2 \leq x_4 \leq 5,\ x_5$$

The constraints for x_1 is that x_1 is a non-negative integer and $x_1 \leq 3$.

Hence 'x_1' can take the values 0, 1, 2, 3.

$$f_1(x) = x^0+x^1+x^2+x^3$$
$$= 1+x+x^2+x^3$$

Similarly x_2 can take the values 0, 1, 2, 3, 4

$$f_2(x) = x^0+x^1+x^2+x^3+x^4$$
$$= 1+x+x^2+x^3+x^4$$

x_3 can take the values 2, 3, 4, 5, 6

$$f_3(x) = x^2+x^3+x^4+x^5+x^6$$
$$= x^2(1+x+x^2+x^3+x^4)$$

x_4 can take the values 2, 3, 4, 5.

$$f_4(x) = x^2+x^3+x^4+x^5$$
$$= x^2(1+x+x^2+x^3)$$

x_5 can take values 1, 3, 5, 7, 9

$$f_5(x) = x^1+x^3+x^5+x^7+x^9$$
$$= x(1+x^2+x^4+x^6+x^8)$$

Generating function

$f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x)$

$= 1+x+x^2+x^3 + 1+x^1+x^2+x^3+x^4 + x^2(1+x+x^2+x^3+x^4)$
$+ x^2(1+x+x^2+x^3) + x(1+x^2+x^4+x^6+x^8)$

$f(x) = x^5 (1+x+x^2+x^3)^2 \cdot (1+x+x^2+x^3+x^4)^2 \cdot$

$(1+x^2+x^4+x^6+x^8)$

2. Using generating function, find the no: of

i. Non-negative solutions and

ii. positive integer solutions, of the equation

$x_1+x_2+x_3+x_4 = 25$

Sol Given $x_1+x_2+x_3+x_4=25$

i. x_1 can takes the non-negative solutions.

$0, 1, 2, 3, 4, \ldots$

$f_1(x) = x^0 + x^1 + x^2 + x^3 + x^4 + \ldots\ldots$

$= 1+x+x^2+x^3+x^4+\ldots$

Similarly x_2, x_3 and x_4 can takes the non-negative solutions $0, 1, 2, 3, \ldots$

$f_2(x) = x^0 + x^1 + x^2 + x^3 + \ldots$

$= 1+x+x^2+x^3+x^4+\ldots$

$f_3(x) = 1+x+x^2+x^3+x^4+\ldots$

$f_4(x) = 1+x+x^2+x^3+x^4+\ldots$

The Generation function is

$$f(x) = f_1(x)\, f_2(x) \cdot f_3(x)\, f_4(x)$$

$$= \left(1+x+x^3+x^4+\cdots\right)^4$$

$$= \left((1-x)^{-1}\right)^4$$

$$= (1-x)^{-4}$$

To find the non-negative solutions.
it is enough to find that coefficient of x^{25}.

$$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{4+r-1}{r} x^r$$

~~put r=25~~

$$= \sum_{r=0}^{\infty} \binom{3+r}{r} x^r$$

put $r = 25$

$$= \binom{3+25}{25} x^{25}$$

$$= \binom{28}{25} x^{25}$$

$$= 3276 x^{25}$$

$\therefore$ The coefficient of x^{25} is 3276

$\therefore$ The non-negative solutios is 3276.

ii. x_1 can take the +ve integer solutions is 1, 2, 3, 4, ...

$$f_1(x) = x^1+x^2+x^3+x^4+\cdots$$

$$= x(1+x+x^2+\cdots)$$

Similarly x_2, x_3, x_4 can takes the +ve integer solutions are 1, 2, ~~1~~, 3, 4 ...

$f_2(x) = x^1 + x^2 + x^3 + x^4 + \cdots$

$= x(x + x + x^2 + \cdots)$

$f_3(x) = x(1 + x + x^2 + x^3 + \cdots)$

$f_4(x) = x(1 + x + x^2 + x^3 + \cdots)$

The Generating function is

$$f(x) = f_1(x)\, f_2(x) \cdot f_3(x) \cdot f_4(x)$$

$$= x^4 (1 + x + x^2 + x^3 + \cdots)^4$$

$$= x^4((1-x)^{-1})^4$$

$$= x^4 (1-x)^{-4}$$

To find the +ve integer solutions is

It is enough to find the coefficient of x^{25}.

$$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

$$= x^4 \sum_{r=0}^{\infty} \binom{3+r}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{3+r}{r} x^{r+4}$$

put $r = 21$

$$= \binom{3+21}{21} x^{25}$$

$$= 2024\, x^{25}$$

The coefficient of x^{25} is 2024.

The +ve integer solutions is 2024

3. Find the no: of integer solutions of the equation $x_1+x_2+x_3+x_4+x_5=30$ under ~~and~~ the constraints $x_i \geq 0$ for $i=1,2,3,4,5$ and further x_2 is even and x_3 is odd.

Sol Given equations

$$x_1+x_2+x_3+x_4+x_5=30$$

Given constraints

$x_i \geq 0$ for $i=1,2,3,4,5$ and further x_2 is even and x_3 is odd

x_i can takes the non-negative solutions is $0,1,2,3,\ldots$

$$f_1(x) = x^0+x^1+x^2+x^3+\ldots$$
$$= 1+x+x^2+x^3+\ldots$$
$$=(1-x)^{-1}$$

x_2 can takes the values $0,2,4,6,8,\ldots$

$$f_2(x) = x^0+x^2+x^4+x^6+x^8+\ldots$$
$$= 1+x^2+x^4+x^6+x^8+\ldots$$
$$= 1+x^2+(x^2)^2+(x^2)^3+(x^2)^4+\ldots$$
$$= (1-x^2)^{-1}$$

x_3 can takes the values $1,3,5,7,\ldots$

$$f_3(x) = x^1+x^3+x^5+x^7+\ldots$$
$$= x(1+x^2+x^4+x^6+\ldots)$$
$$= x(1+x^2+(x^2)^2+(x^3)^2+\ldots)$$
$$= x(1-x^2)^{-1}$$

x_4, x_5 can take the values $0,1,2,3,4,\ldots$

$$f_4(x) = x^0+x^1+x^2+x^3+\ldots = 1+x+x^2+x^3+\ldots = (1-x)^{-1}$$

$$f_5(x) = (1-x)^{-1}$$

The generating function

$f(x) = f_1(x)\, f_1(x)\, f_3(x)\, f_4(x)\, f_5(x)$

$= (1-x)^{-1}(1-x^2)^{-1} x (1-x^2)^{-1} (1-x)^{-1} (1-x)^{-1}$

$= [(1-x)^{-1}]^3 x ((1-x^2)^{-1})^2$

$= x (1-x)^{-3} (1-x^2)^{-2}$

To find the non-negative integer solutions, It is enough to find the coefficient of x^{30}.

$$= x\left[\sum_{r=0}^{\infty} \binom{3+r-1}{r} x^r \left(\sum_{s=0}^{\infty} \binom{2+s-1}{s} (x^2)^s \right)\right]$$

$$= \sum_{r=0}^{\infty} \binom{2+r}{r} x^{r+1} \sum_{s=0}^{\infty} \binom{1+s}{s} x^{2s}$$

put $r = 29$, $s = 0$

$r = 27$, $s = 1$

$r = 25$, $s = 2$

$r = 23$, $s = 3$

$\vdots$

$r = 1$, $s = 14$.

The coefficients, x^{30} is $\binom{31}{29}\binom{1}{0} + \binom{29}{27}\binom{2}{1} + \binom{27}{25}\binom{3}{2} + \cdots + \binom{3}{1}\binom{15}{14}$

$= 465 + 406(2) + 351(3) + \cdots + 3(15)$

$= 465 + 812 + 1053 + \cdots + 45$

4. Find the generating function for the number of integer solution to the equation $c_1+c_2+c_3+c_4=20$ where $-3 \le c_1$, $-3 \le c_2$, $-5 \le c_3 \le 5$ and $0 \le c_4$. Hence find the no. of such solutions.

Sol: Given

$$c_1+c_2+c_3+c_4=20$$

Given constraints

$-3 \le c_1$, $-3 \le c_2$, $-5 \le c \le 5$ and $0 \le c_4$

The Given constraints can be written as

$x_1 = c_1+3$, $x_2 = c_2+3$, $x_3 = c_3+5$, $x_4 = c_4$

The given equation can written a.

$$c_1+c_2+c_3+c_4=20$$

$$x_1-3+x_2-3+x_3-5+x_4=20$$

$$x_1+x_2+x_3+x_4=20+11$$

$$\boxed{x_1+x_2+x_3+x_4=31}$$

The constraints are

$x_1 \ge 0$, $x_2 \ge 0$, $0 \le x_3 \le 10$, $x_4 \ge 0$

Now

$$f_1(x) = x^0+x^1+x^2+x^3+\cdots$$
$$= 1+x+x^2+x^3+\cdots$$
$$= (1-x)^{-1}$$

$$f_2(x) = (1-x)^{-1}$$

$$f_3(x) = x^0+x^1+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}$$

$$= 1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}$$

$$f_4(x) = (1-x)^{-1}$$

The generating function

$$f(x) = f_1(x) f_2(x) f_3(x) f_4(x).$$

$$f(x) = (1-x)^{-1}(1-x)^{-1}(1+x+x^2+x^3+\cdots+x^{10})(1-x)^{-1}$$

$$= ((1-x)^{-1})^3\left[1+x+x^2+x^3+\cdots+x^{10}\right]$$

$$= (1-x)^{-3}\left[1+x+x^2+x^3+\cdots+x^{10}\right]$$

To ~~to~~ find the no: of integer solutions it is enough to find x^{31} coefficient

$$= \sum_{r=0}^{\infty}\binom{3+r-1}{r} x^r (1+x+x^2+\cdots+x^{10})$$

$$= \sum_{r=0}^{\infty}\binom{2+r}{r} x^r (1+x+x^2+\cdots+x^{10})$$

$$= \sum_{r=0}^{\infty}\binom{2+r}{r}x^r + \sum_{r=0}^{\infty}\binom{2+r}{r}x^{r+1} + \sum_{r=0}^{\infty}\binom{2+r}{r}x^{r+2} + \cdots$$

$$\cdots\cdots + \sum_{r=0}^{\infty}\binom{2+r}{r}x^{r+10}$$

The coefficient of x^{31} is

$$= \binom{2+31}{31} + \binom{2+30}{30} + \binom{2+29}{29} + \cdots + \binom{2+21}{21}$$

$$= \binom{33}{31} + \binom{32}{30} + \binom{31}{29} + \cdots + \binom{23}{21}$$

$$= 528 + 496 + 465 + \cdots + 253$$

5. In how many ways can 12 organes can be distributed among three children A, B, C, so that A gets atleast 4, B and C get atleast 2, but C gets no more than 5.

Sol Let x_1 be the no: of oranges which A can get

Let x_2 be the no: of oranges which B can get

Let x_3 be the no: of oranges which C can get.

Total No: of oranges is $x_1+x_2+x_3=12$

The Given constraints are

$$x_1 \ge 4,\ x_2 \ge 2,\ 2 < x_3 \le 5.$$

Now $f_1(x) = x^4+x^5+x^6+\ldots\ldots$

$$= x^4(1+x+x^2+x^3+\ldots\ldots)$$

$$= x^4(1-x)^{-1}$$

$$f_2(x) = x^2+x^3+x^4+x^5+\ldots$$

$$= x^2(1+x+x^2+x^3+\ldots\ldots)$$

$$= x^2(1-x)^{-1}$$

$$f_3(x) = x^2+x^3+x^4+x^5$$

$$= x^2(1+x+x^2+x^3)$$

The Generating function is

$$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x)$$

$$= x^4(1-x)^{-1}(x^2)(1-x)^{-1}x^2(1+x+x^2+x^3)$$

$$= x^8(1-x)^{-2}(1+x+x^2+x^3)$$

$$= x^8 \sum_{r=0}^{\infty} \binom{2+r-1}{r} x^r (1+x+x^2+x^3)$$

$$= \sum_{r=0}^{\infty} \binom{1+r}{r} x^{r+8} (1+x+x^2+x^3)$$

$$= \sum_{r=0}^{\infty} \binom{1+r}{r} x^{8+r} + \sum_{r=0}^{\infty} \binom{1+r}{r} x^{r+9} + \sum_{r=0}^{\infty} \binom{1+r}{r} x^{r+10} + \sum_{r=0}^{\infty} \binom{1+r}{r} x^{r+11}$$

The coefficient of x^{12} is

$$= \binom{1+4}{4} + \binom{3+1}{3} + \binom{2+1}{1} + \binom{1+0}{0}$$

$$= \binom{5}{4} + \binom{4}{3} + \binom{3}{1} + \binom{2}{0}$$

$$= 5+4+3+2$$

$$= 14$$

The no: of ways of n· oranges distributed is 14

6. In how many ways can we distribute 24 pencils to 4 children so that each child gets atleast 3 pencils but not more than 8.

Sol Given that

Total no. of pencils $= x_1+x_2+x_3+x_4 = 24$

and constraints $3 \le x_i \le 8$ for $i = 1, 2, 3, 4$

Now $f_1(x) = x^3+x^4+x^5+x^6+x^7+x^8$

$= x^3(1+x+x^2+x^3+x^4+x^5)$

$f_2(x) = x^3+x^4+x^5+x^6+x^7+x^8$

$= x^3(1+x+x^2+x^3+x^4+x^5)$

$f_3(3) = x^3(1+x+x^2+x^3+x^4+x^5)$

$f_4(x) = x^3(1+x+x^2+x^3+x^4+x^5)$

The Generating function is

$f(x) = f_1(x)\, f_2(x)\, f_3(3)\, f_4(x)$

$= (x^3)^4 (1+x+x^2+x^3+x^4+x^5)^4$

$= x^{12}(1+x+x^2+x^3+x^4+x^5)^4$

$= x^{12}\left(\frac{1-x^{5+1}}{1-x}\right)^4 \quad \left[\because \frac{1-x^{n+1}}{1-x}\right]$

$= x^{12}\left[(1-x)^6 (1-x)^{-1}\right]^4$

$= x^{12}\left((1-x)^6\right)^4 (1-x)^{-4} \rightarrow (1+x)^n$

$= x^{12} \sum_{r=0}^{4} \binom{4}{r}(-x^6)^r \times \sum_{s=0}^{\infty} \binom{4+s-1}{s} x^s$

$= x^{12} \sum_{r=0}^{4} \binom{4}{r}(-1)^r (x^{6r}) \cdot \sum_{s=0}^{\infty} \binom{3+s}{s} x^s$

put $r = 12$ | $s = 0$ | $r = 0$ | $s = 12$

$r = 11$ | $s = 1$ | $r = 1$ | $s = 6$

$r = 10$ | $s = 2$ | $r = 2$ | $s = 0$

$s = 3$

The coefficient of x^{24} is

$$\binom{4}{0}\binom{3+12}{12} + (-1)\binom{4}{1}\binom{3+6}{6} + \binom{4}{2}\binom{3+0}{0}$$

$$= \binom{4}{0}\binom{15}{12} + (-1)\binom{4}{1}\binom{9}{6} + \binom{4}{2}\binom{3}{0}$$

$$= 1\times455 - 4\times84 + 6\times1$$

$$= 455 - 336 + 6$$

$$= 461 - 336$$

$$= 125.$$

7. A bag contain a large no: of red, green, white, and black marbles, with atleast 24 of each color. In how many ways can one select 24 of these marbles, so that there are even no: of white marbles and atleast 6 black marbles.

Let x_1, x_2, x_3 and x_4 represent the no: of red, green, white and black marbles that can be selected.

we have $x_1+x_2+x_3+x_4 = 24$

Given constraints are

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 = 0, 2, 4, \ldots,\ x_4 \geq 6$$

Now $f_1(x) = x^0 + x^1 + x^2 + x^3 + \ldots$

$= 1 + x^1 + x^2 + x^3 + \ldots$

$= (1-x)^{-1}$

$f_2(x) = (1-x)^{-1}$

$f_3(x) = x^0 + x^2 + x^4 + x^6 + \ldots$

$= \cancel{x^2(1+}$

$= 1 + x^2 + (x^2)^2 + (x^2)^3 + \ldots$

$= (1-x^2)^{-1}$

$f_4(x) = x^6 + x^7 + x^8 + x^9 + \cdots$

$= x^6(1 + x + x^2 + x^3 + \cdots)$

$= x^6(1-x)^{-1}$

The Generating function is

$f(x) = f_1(x)\, f_2(x)\, f_3(x)\, f_4(x)$

$= (1-x)^{-1}(1-x)^{-1}(1-x^2)^{-1}x^6(1-x)^{-1}$

$= x^6(1-x)^{-3}(1-x^2)^{-1}$

$= x^6 \sum_{r=0}^{\infty}\binom{3+r-1}{r}x^r \sum_{s=0}^{\infty}\binom{1+s-1}{s}x^{2s}$

$= x^6 \sum_{r=0}^{\infty}\binom{2+r}{r}x^r \sum_{s=0}^{\infty}\binom{s}{s}x^{2s}$

put

$r = 18 \quad s = 0$

$r = 16 \quad s = 1$

$r = 14 \quad s = 2$

$r = 12 \quad s = 3$

$r = 10 \quad s = 4$

$r = 8 \quad s = 5$

$r = 6 \quad s = 6$

$r = 4 \quad s = 7$

$r = 2 \quad s = 8$

$r = 0 \quad s = 9$

The coefficients of x^{24} is

$\binom{20}{18} + \binom{18}{16} + \binom{16}{14} + \binom{14}{12} + \binom{12}{10} + \binom{10}{8} +$

$\binom{8}{6} + \binom{6}{4} + \binom{4}{2} + \binom{2}{0}$

$= 190 + 153 + 120 + 91 + 66 + 45 + 28 + 15 + 6 + 1$

$= 715$

8. Use generating function to determine the no. of 4 elements subsets of $S = \{1, 2, 3, 4, \dots 15\}$ that contain no consecutive integers

Sol Every subset S of the required type is of the form $A = \{a_1, a_2, a_3, a_4\}$

where $1 \le a_1 < a_2 < a_3 < a_4 \le 15$ and

$a_2 - a_1 \ge 2, a_3 - a_2 \ge 2, a_4 - a_3 \ge 2$

Let $x_1 = a_1 - 1$

$x_2 = a_2 - a_1$

$x_3 = a_3 - a_2$

$x_4 = a_4 - a_3$

$x_5 = 15 - a_4$

Now $x_1 + x_2 + x_3 + x_4 + x_5 = a_1 - 1 + a_2 - a_1 + a_3 - a_2 + a_4 - a_3 + 15 - a_4$

$= -1 + 15$

$x_1 + x_2 + x_3 + x_4 + x_5 = 14$

the constraints are

$x_1 \ge 0, \quad x_2 \ge 2, \; x_3 \ge 2, \; x_4 \ge 2, x_5 \ge 0$

Now

$f_1(x) = x^0 + x^1 + x^2 + \dots$

$= 1 + x + x^2 + x^3 + \dots$

$= (1-x)^{-1}$

$f_2(x) = x^2 + x^3 + x^4 + x^5 + x^6 + \dots$

$= x^2(1 + x + x^2 + x^3 + \dots)$

$= x^2(1-x)^{-1}$

$f_3(x) = x^2(1-x)^{-1}$

$f_4(x) = x^2(1-x)^{-1}$

$f_5(x) = (1-x)^{-1}$

The Generating function is

$$f(x) = f_1(x)\, f_2(x)\, f_3(x)\, f_4(x)\, f_5(x)$$

$$= (1-x)^{-1}\, x^2(1-x)^{-1}\, x^2(1-x)^{-1}\, x^2(1-x)^{-1}\, (1-x)^{-1}$$

$$= x^6 (1-x)^{-5}$$

$$= x^6 \sum_{r=0}^{\infty} \binom{5+r-1}{r} x^r$$

$$= x^6 \sum_{r=0} \binom{4+r}{r} x^r$$

$$= \sum_{r=0} \binom{4+r}{r} x^{6+r}$$

The coefficient of x^{14}

$$= \binom{8+4}{8}$$

$$= \binom{12}{8}$$

$$= \cancel{495}$$

$$= 495$$

Recurrence Relations

* A Recurrence Relation of the sequence $\{a_n\}$ is an equation that express a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \dots a_n$ for all integers 'n' with $n \geq n_0$ where n_0 is a non-negative integer.

-A Recurrence Relation is also called difference Relation.

1. Let sequence of a_n be a sequence that Satisfies the Recurrence Relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, \dots$ and suppose that $a_0 = 3$, $a_1 = 5$.

What are a_2 & a_3?

Sol Given Recurrence Relation

$$a_n = a_{n-1} - a_{n-2} \quad (1)$$

$a_0 = 3,\ a_1 = 5$

put $n=2$ in (1)

$a_2 = a_{2-1} - a_{2-2}$

$= a_1 - a_0$

$= 5 - 3$

$= 2$

put $n=3$ in eq (1)

$a_3 = a_{3-1} - a_{3-2}$

$= a_2 - a_1$

$= 2 - 5$

$= -3$

2. Find out the sequence generated by the Recurrence Relations below.

$$T_n = 2T_{n-1} \text{ with } T_1 = 4$$

Sol Given $T_n = 2T_{n-1}$ with $T_1 = 4$

Initial condition

For $n=2$

$T_2 = 2T_{2-1}$

$= 2T_1$

$= 2(4)$

$= 8$

For $n=3$

$T_3 = 2T_{3-1}$

$= 2T_2$

$= 2(8)$

$= 16$

For $n=4$

$T_4 = 2T_{4-1}$

$= 2T_3$

$= 2(16)$

$= 32$

For $n=5$

$T_5 = 2T_{5-1}$

$= 2T_4$

$= 2(32)$

$= 64$

$\therefore$ The sequence is

$4, 8, 16, 32, 64, \ldots\ldots\ldots$

First order Linear or homogenous recurrence Relation

* First we consider for solution of recurrence relation of the form $a_n = c\,a_{n-1} + f(n)$, for $n \geq 1$ — (1)

where 'c' is a known constant and $f(n)$ is a known function such a relation is called a linear recurrence relation of first ordered with constant coefficient. If $f(n) = 0$, the relation is called homogeneous. Otherwise it is called non-homogeneous (or) inhomogeneous

* The Relation (eq) (1) can be solved in ~~[illegible]~~ Trivial way.

* substituting $n = n+1$, in eq (1)

$$a_{n+1} = c\,a_{n+1-1} + f(n+1), \text{ for } n+1 \geq 1$$

$$\Rightarrow n+1-1 \geq 0$$

$$n \geq 0 \text{ — (2)}$$

For $n = 0, 1, 2, 3, \ldots\ldots$ in eq (2)

$$a_1 = c\,a_0 + f(1)$$

$$a_2 = c\,a_1 + f(2)$$

$$= c(c\,a_0 + f(1)) + f(2)$$

$$= c^2 a_0 + c f(1) + f(2)$$

* By induction

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k) \text{ for } n \geq 1 \text{ — (3)}$$

this is the General solution of the Recurrence relation (2) which equivalent to the eq(1).

* If $f(n)=0$, The Recurrence Relation is Homogeneous the solution of eq(3) becomes

$$\boxed{a_n = c^n a_0} \quad (4) \quad \text{for } n \geq 1$$

The solutions (3) and (4) yields particular solutions if a_0 is specified. The specified value of 'a_0' is called the initial condition.

(1) Solve the recurrence Relation $a_{n+1} = 4a_n$, for $n \geq 0$ given that $a_0 = 3$.

sol Given Relation

$$a_{n+1} = 4a_n \text{ for } n \geq 0 \quad (1)$$

Initial condition $a_0 = 3$

This is the First order Linear recurrence relation.

- The General solution of the First order Linear Recurrence relation is

$$a_n = c^n a_0 \quad (2)$$

If $a_n = c^n$ Then $a_{n+1} = c^{n+1}$

$$4a_n = c^n \cdot c^1$$

$$4\cancel{c^n} = \cancel{c^n} \cdot c^1$$

$$\boxed{c = 4}$$

from (2) $a_n = 4^n a_0$

$$= 4^n (3)$$

$$a_n = 3 \cdot 4^n$$

2. Solve the Recurrence Relation $a_n = 7a_{n-1}$ where $n \geq 1$, given that $a_2 = 98$.

Sol Given Relation

$$a_n = 7a_{n-1} \quad \text{--- (1)}$$

Initial condition is $a_2 = 98$.

This is a First order Linear Recurrence Relation

The General solution of the First order Linear Recurrence Relation.

$$a_n = c^n a_0 \quad \text{--- (2)}$$

put $n = n+1$ in eq (1)

$$a_{n+1} = 7a_{n+1-1}$$

$$a_{n+1} = 7a_n, \ n \geq 0 \quad \text{--- (3)}$$

If $a_n = c^n$, then $a_{n+1} = c^{n+1}$

$$7a_n = c^n \cdot c^1$$

$$7\cancel{c^n} = \cancel{c^n} \cdot c^1$$

$$\boxed{c = 7}$$

(2) $\rightarrow$ $a_n = 7^n a_0$ --- (4)

put $n = 2$

$$a_2 = 7^2 a_0$$

$$98 = 49 a_0$$

$$a_0 = 2$$

(4) $a_n = 7^n 2$

$$\boxed{a_n = 2 \cdot 7^n}$$

3. Solve the Recurrence Relation $3a_{n+1} - 4a_n = 0$ where $n \geq 0$, given that $a_1 = 5$.

Sol Given Relation

$$3a_{n+1} - 4a_n = 0 \Rightarrow 3a_{n+1} = 4a_n$$

Initial condition $a_1 = 5$ $\Rightarrow a_{n+1} = \frac{4}{3} a_n$ ← (1)

The General solution of First order linear recurrence relation.

$$a_n = c^n a_0 \quad \text{(2)}$$

If $a_n = c^n$, then $a_{n+1} = c^{n+1}$

$$\frac{4}{3} a_n = c^n . c^1$$

$$\frac{4}{3} \cancel{c^n} = \cancel{c^n} . c^1$$

$$\boxed{c = 4/3}$$

(2) $a_n = \left(\frac{4}{3}\right)^n a_0$ — (3)

put $n = 1$

$$a_1 = \left(\frac{4}{3}\right)^n a_0$$

$$5 = \left(\frac{4}{3}\right) a_0$$

$$a_0 = \frac{15}{4}$$

from (3) $a_n = \left(\frac{4}{3}\right)^n \left(\frac{15}{4}\right)$

4. Solve the Recurrence Relation $4a_n - 5a_{n-1} = 0$, $n \geq 1$, $a_0 = 1$

Linear Recurrence Relation with constant values

A Recurrence Relation of the form

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \cdots + c_k a_{r-k} = f(r) \quad (1)$$

where c_i's are constant is called a Linear Reccurence Relation with constant coefficient.

The Reccurence Relation in eq(1) is known as a k^{th} ordered Relation, provided that both $c_0 \neq 0$ and $c_k \neq 0$.

Second ordered Reccurence Relation

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} = f(r).$$

Third ordered Recurrence Relation

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + c_3 a_{r-3} = f(r)$$

* The solution of eq(1) is $a_r = a_r^{(h)} + a_r^{(p)}$

where $a_r^{(h)}$ = Homogeneous solution

$a_r^{(p)}$ = particular solution

Homogeneous Recurrence Relation

* If $f(r)=0$ in eq(1), Then it is Homogeneous Recurrence Relation.

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \cdots + c_k a_{r-k} = f(r) = 0$$

Non-Homogeneous Recurrence Relation.

* If $f(r) \neq 0$ in eq(1), Then it is called Non-Homogeneous Recurrence Relation.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} \neq 0$$

Homogeneous Linear Recurrence Relation with constant coefficients

* Suppose the second ordered Homogeneous Linear Recurrence relation is $C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = 0$

* The characteristic equation or Auxiliary equation is

$$C_0 m^2 + C_1 m + C_2 = 0$$

* It has 2 roots m_1 and m_2 are called characteristic roots.

Case-1

* If the roots Auxiliary equation are real and unequal.

i.e $m_1 \neq m_2$

* The General solution is

$$a_r = C_1 m_1^r + C_2 m_2^r$$

Case-2

* If the roots of auxiliary equation are real and equal.

i.e $m_1 = m_2 = m$

* The General solution is

$$a_r = (C_1 + C_2 r) m^r$$

Case-3

* If the roots of Auxiliary equation are in complex number

$m = \alpha \pm i\beta$

* The General solution is

$$a_r = [(C_1 \cos r\theta + C_2 \sin r\theta) R^r]$$

where $R = \sqrt{\alpha^2+\beta^2}$

$\theta = \tan^{-1}(\beta/\alpha)$

1. Solve the Recurrence Relation $a_r + 5a_{r-1} + 6a_{r-2} = 0$

Sol Given Recurrence Relation

$$a_r + 5a_{r-1} + 6a_{r-2} = 0$$

This is a second ordered Linear Recurrence Relation.

The characteristic Equation is

$$m^2 + 5m + 6 = 0$$

$$m^2 + 3m + 2m + 6 = 0$$

$$m(m+3) + 2(m+3) = 0$$

$$(m+2)(m+3) = 0$$

$$m = -2, \quad m = -3$$

we consider $m_1 = -2$, $m_2 = -3$

The characteristic roots are real and unequal

The General solution is

$$a_r = c_1 m_1^r + c_2 m_2^r$$

$$= c_1(-2)^r + c_2(-3)^r$$

2. Solve the Recurrence Relation $a_r - 7a_{r-1} + 10a_{r-2} = 0$, for $r \geq 2$

Given that $a_0 = 0$, $a_1 = 3$

Sol Given Recurrence Relation

$$a_r - 7a_{r-1} + 10a_{r-2} = 0$$

This is a second order Linear Recurrence Relation

The Characteristic equation is

$$m^2 - 7m + 10 = 0$$

$$m^2 - 5m - 2m + 10 = 0$$

$$m(m-5) - 2(m-5) = 0$$

$$(m-2)(m-5)=0$$

$$m=2,\ m=5$$

$$m_1=2,\ m_2=5$$

The characteristic roots are real and unequal.

The General solution is

$$a_r = C_1 m_1^r + C_2 m_2^r$$

$$= C_1(2)^r + C_2(5)^r \quad \text{— (1)}$$

and also given that

$$a_0=0,\ a_1=3$$

put $r=0$ in eq (1)

$$a_0 = C_1 2^0 + C_2 5^0$$

$$a_0 = C_1 + C_2$$

$$\Rightarrow C_1+C_2=0$$

$$\Rightarrow C_1=-C_2 \quad \text{— (2)}$$

put $r=1$ in eq (1)

$$a_1 = C_1 2^1 + C_2 5^1$$

$$a_1 = 2C_1 + 5C_2$$

$$3 = 2C_1+5C_2$$

$$\Rightarrow 2(-C_2)+5C_2=3$$

$$\Rightarrow -2C_2+5C_2=3$$

$$3C_2=3$$

$$\boxed{C_2=1}$$

from (2)

$$\boxed{C_1=-1}$$

C_1, C_2 substitute in eq (1)

$$a_r = (-1)2^r + (+1)5^r$$

$$\boxed{a_r = -2^r + 5^r}$$

3. Solve the Recurrence Relation $a_r - 3a_{r-1} + 3a_{r-2} - a_{r-3} = 0$, $r \geq 3$, given that $a_0 = 1$, $a_1 = -2$, $a_2 = -1$

Sol Given Recurrence Relation

$$a_r - 3a_{r-1} + 3a_{r-2} - a_{r-3} = 0$$

This is a Third ordered Linear Recurrence equation.

The characteristic equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$

	1	-3	3	-1
$m=1$	0	1	-2	1
	1	-2	1	0

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$(m-1)(m-1) = 0$$

$$m = 1, \; m = 1$$

The characteristic roots are equal

The General solution is

$$a_r = (c_1 + c_2 r + c_3 r^2) m^r$$

$$a_r = (c_1 + c_2 r + c_3 r^2)(1)^r \quad \text{--- (1)}$$

Given $a_0 = 1$, $a_1 = -2$, $a_2 = -1$

Put $r = 0$ in eq (1)

$$a_0 = (c_1 + 0 + 0)(1)^0$$

$$a_0 = c_1$$

$$\Rightarrow \boxed{c_1 = 1}$$

put $r = 1$ in eq (1)

$$a_1 = (c_1 + c_2 + c_3)(1)^1$$

$$-2 = 1 + c_2 + c_3$$

$$\Rightarrow c_2 + c_3 = -3 \quad \text{--- (2)}$$

put $r = 2$

$$a_2 = (c_1 + 2c_2 + 4c_3)$$

$$-1 = 1 + 2c_2 + 4c_3$$

$$\Rightarrow 2c_2 + 4c_3 = -2$$

$$\Rightarrow 2(c_2 + 2c_3) = -2$$

$$c_2 + 2c_3 = -1 \quad \text{--- (3)}$$

[∵ solve (2) and (3)

$$c_2 + c_3 = -3$$

$$c_2 + 2c_3 = -1$$

$$-c_3 = -4 \Rightarrow \boxed{c_3 = 4}$$

c_3 in eq (2)

$$c_2 + 4 = -3$$

$$\boxed{c_2 = -7}$$]

Solving (2) & (3)

$$(2)\times 2 \Rightarrow 2C_2 + 2C_3 = -6$$
$$(3)\times 1 \Rightarrow 2C_2 + 4C_3 = -2$$
$$-2C_3 = -4$$
$$\boxed{C_3 = 2}$$

in eq (3)

$$C_2 + C_3 = -3$$
$$\boxed{C_2 = -5}$$

from (1)

$$a_r = (1 + (-5)r + 2r^2)(1)^r$$
$$= (1 - 5r + 2r^2)(1)^r.$$

4 Solve the Recurrence Relation

$a_r + 2a_{r-1} + 2a_{r-2} = 0$, for $r \geq 2$, given that $a_0 = 0, a_1 = 1$

Sol Given Recurrence Relation

$$a_r + 2a_{r-1} + 2a_{r-2} = 0$$

This is a second order Recurrence Relation.

$$m^2 + 2m + 2 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm \sqrt{4i^2}}{2} = \frac{-2 \pm 2i}{2}$$

$$= 2(-1 \pm i)/2 = -1 \pm i$$
$$= \alpha \pm i\beta$$

The characteristic roots are in complex number

The General solution is

$$a_r = (c_1 \cos r\theta + c_2 \sin r\theta) R^r \quad (1)$$

where $R = \sqrt{\alpha^2+\beta^2}$

$= \sqrt{(-1)^2+(1)^2}$

$= \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}(\beta/\alpha) = \tan^{-1}\left(\frac{1}{-1}\right)$

$= \tan^{-1}(-1)$

$= \tan^{-1}(\tan(\pi - \pi/4))$

$= \tan^{-1}(\tan(3\pi/4))$

$\theta = +\frac{3\pi}{4}$

$(1) \Rightarrow a_r = \left[c_1 \cos r(3\pi/4) + c_2 \sin r\left(\frac{3\pi}{4}\right)\right] (\sqrt{2})^r \quad (2)$

Given $a_0 = 0,\ a_1 = -1$

put $r=0$ in (2)

$a_0 = [c_1 \{\cos(0)(3\pi/4) + c_2 \sin(0)(3\pi/4)] (\sqrt{2})^0$

$= [k\, d(0)(3\pi/4) + (0)]\, 1$

$0 = [c_1 \cos(0) + c_2 \sin(0)]\, 1$

$0 = c_1 + 0$

$\Rightarrow \boxed{c_1 = 0}$

put $r=1$ in eq (2)

$a_1 = [c_1 \cos(1)(3\pi/4) + c_2 \sin(1)(3\pi/4)] (\sqrt{2})^1$

$= [c_1 \cos(3\pi/4) + c_2 \sin(3\pi/4)] \sqrt{2}$

$= [c_1 \cos(\pi - \pi/4) + c_2 \sin(\pi - \pi/4)] \sqrt{2}$

$= [c_1 - \cos\pi/4 + c_2 \sin\pi/4] \sqrt{2}$

$$-1 = \left[c_1\left(\frac{-1}{\sqrt{2}}\right) + c_2\left(\frac{1}{\sqrt{2}}\right)\right]\sqrt{2}$$

$$\Rightarrow \quad \frac{-c_1 + c_2}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$\Rightarrow \quad -c_1 + c_2 = -1$$

$$\boxed{c_1 = 0}$$

$$\boxed{c_2 = -1}$$

substitute c_1, c_2 in eq (2)

$$a_r = \left[0 \cos r(3\pi/4) + (-1) \sin r(3\pi/4)\right](\sqrt{2})^r$$

$$a_r = \left[-\sin r(3\pi/4)\right](\sqrt{2})^r$$

5. Solve the Recurrence Relation $a_r + a_{r-1} - 6a_{r-2} = 0$ for $r \geq 2$ given that $a_0 = -1, a_1 = 8$

6. $a_r = 2(a_{r-1} - a_{r-2})$ for $r \geq 2$, given that $a_0 = 1, a_1 = 2$

5.
Sol Given Recurrence Relation

$$a_r + a_{r-1} - 6a_{r-2} = 0, \text{ for } r \geq 2$$

This is a second ordered Recurrence Relation

$$m^2 + m - 6 = 0$$

$$m^2 - 2m + 3m - 6 = 0$$

$$m(m-2) + 3(m-2) = 0$$

$$(m+3)(m-2) = 0$$

$$m = -3, m = 2$$

The ~~carn~~ characteristic roots are real and unequal.

The General solution is

$$a_r = c_1 m_1^r + c_2 m_2^r$$

$$= c_1(-3)^r + c_2(2)^r \quad \text{--- (1)}$$

also given that

$a_0 = -1, a_1 = 8$

put $r=0$ in eq (1)

$a_0 = c_1(-3)^0 + c_2(2)^0$

$\boxed{-1 = c_1 + c_2}$

$c_1 + c_2 + 1 = 0$ —— (3)

put $r=1$ in eq (1)

$a_1 = c_1(-3)^1 + c_2(2)^1$

$\Rightarrow -3c_1 + 2c_2 = 8$ —— (4)

$\Rightarrow -3c_1 + 2c_2 - 8 = 0$

solve (3) and (4)

(3) & (4) by c_1 & c_2

4 c x y → method

1 1 1 1

2 -8 -3 2

$$\frac{c_1}{-8-2} = \frac{c_2}{-3+8} = \frac{1}{2+3}$$

$$\frac{c_1}{-10} = \frac{1}{5}, \quad \frac{c_2}{5} = \frac{1}{5}$$

$c_1 = \frac{-10}{5}$ | $c_2 = 5/5$

$\boxed{c_1 = -2}$ | $\boxed{c_2 = 1}$

c_1 & c_2 substitute in eq (1)

$a_r = (-2)(-3)^r + (1)(2)^r$

$a_r = (-2)(-3)^r + 2^r$

6.

sol Given recurrence relation

$a_r = 2(a_{r-1} - a_{r-2})$

$a_r = 2a_{r-1} - 2a_{r-2}$

$a_r - 2a_{r-1} + 2a_{r-2} = 0$

This is a second ordered recurrence relation.

$$m^2 - 2m + 2 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm \sqrt{4i^2}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$$= 1 \pm i$$

$$= \alpha \pm i\beta$$

The characteristic roots are in complex number.

The General solution is

$$a_r = (c_1 \cos r\theta + c_2 \sin r\theta) R^r \quad \text{--- (1)}$$

where $R = \sqrt{\alpha^2 + \beta^2}$

$$= \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(\tan \pi/4)$$

$$= \pi/4$$

R, θ is substitute in eq (1)

$$a_r = [c_1 \cos r(\pi/4) + c_2 \sin r(\pi/4)](\sqrt{2})^r$$

also given that — (2)

$$a_0 = 1, \ a_1 = 2$$

put $r = 0$ in eq (2)

$$a_0 = [C_1 \cos(0)(\pi/4) + C_2 \sin(0)(\pi/4)](\sqrt{2})^0$$

$$= [C_1 \cos 0 + C_2 \sin 0](1)$$

$$1 = C_1(1) + 0$$

$$\boxed{C_1 = 1}$$

put $r = 1$ in eq (2)

$$a_1 = [C_1 \cos(1)(\pi/4) + C_2 \sin(1)(\pi/4)](\sqrt{2})^1$$

$$2 = [C_1(1/\sqrt{2}) + C_2(1/\sqrt{2})]\sqrt{2}$$

$$\Rightarrow \frac{2}{\sqrt{2}} = \frac{C_1 + C_2}{\sqrt{2}}$$

$$\Rightarrow C_1 + C_2 = 2$$

$$1 + C_2 = 2$$

$$\boxed{C_2 = 1}$$

$\therefore$ C_1 & C_2 substitute in eq (2)

$$a_r = [(1) \cos r(\pi/4) + (1) \sin(r)(\pi/4)](\sqrt{2})^r$$

$$a_r = [\cos r(\pi/4) + \sin r(\pi/4)](\sqrt{2})^r.$$

Non-Homogeneous Linear Recurrence Relation of second and higher orders

* suppose the k^{th} order non-homogeneous linear Recurrence relation with constant coefficient is

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r) \quad (1)$$

* The General solution of eq (1) is

$$a_r = a_r^{(h)} + a_r^{(p)} \quad (2)$$

where $a_r^{(h)}$ = Homogeneous Solution

$a_r^{(P)}$ = particular solution

where $a_r^{(h)}$ is the General solution of Homogeneous part of eq(1)

i.e $f(r)=0$

and $a_r^{(P)}$ is any particular solution of eq(1).

To find the particular solution, when $f(r) \neq 0$

The following are some of the special cases

Case-i: Suppose $f(r)$ is a polynomial of degree 'q' and '1' is not a root of the characteristic equation of the homogeneous part of the eq(1).

i.e $a_r(P) = A_0 + A_1 r + A_2 r^2 + \ldots\ldots + A_q r^q$

Where $A_0, A_1, A_2, \ldots\ldots A_q$ are constants.

Case-ii

* Suppose $f(r)$ is a polynomial of degree 'q' and '1' is a root of multiplicity 'm' of the characteristic equation of the homogeneous part of eq(1)

* $a_r(P) = r^m [A_0 + A_1 r + A_2 r^2 + \ldots + A_q r^q]$

Case-iii

* Suppose $f(r) = \alpha . b^r$ where 'α' is a constant and 'b' is not a root of the characteristic equation of their Homogeneous part of the eq(1)

$a_r^{(P)} = A_0 b^r$

* Case-IV

Suppose $f(r) = \alpha . b^r$, where 'α' is a constant and

'b' is a root of the characteristic equation of multiplicity 'm'
homogeneous part of eq(1).

$$a_r^{(p)} = A_0 r^m b^r$$

1. Solve the Recurrence Relation $a_r - 7a_{r-1} + 10a_{r-2} = 3^r$, given that $a_0 = 0, a_1 = 1$

Sol Given Recurrence Relation

$$a_r - 7a_{r-1} + 10a_{r-2} = 3^r \quad \text{--- (1)}$$

This is a second ordered Linear Non-homogeneous recurrence Relation.

The General solution of eq(1)

$$a_r = a_r^{(h)} + a_r^{(p)} \quad \text{--- (2)}$$

First we have to find $a_r^{(h)}$

The homogeneous Recurrence Relation solution of eq(1) is

$$a_r - 7a_{r-1} + 10a_{r-2} = 0$$

The characteristic Equation is

$$m^2 - 7m + 10 = 0$$

$$m^2 - 5m - 2m + 10 = 0$$

$$m(m-5) - 2(m-5) = 0$$

$$m = 2, \; m = 5$$

$$m_1 = 2, \; m_2 = 5$$

The characteristic roots are real and unequal

The homogeneous solution is

$$a_r^{(h)} = C_1 m_1^r + C_2 m_2^r$$

$$a_r^{(h)} = C_1 2^r + C_2 5^r \quad \text{--- (3)}$$

To find $a_r^{(p)}$

The Non-homogeneous recurrence equation is

$$a_r - 7a_{r-1} + 10a_{r-2} = 1.3^r$$
$$= d.b^r$$

Here $f(r) = 3^r$, '3' is not a root of the characteristic equation

$$a_r^{(p)} = A_0 b^r$$

$$a_r^{(p)} = A_0 3^r \quad \text{--- (4)}$$

(1) $\Rightarrow$ $a_r - 7a_{r-1} + 10a_{r-2} = 3^r$

$$A_0 3^r - 7A_0 3^{r-1} + 10A_0 3^{r-2} = 3^r$$

$$A_0 \cancel{3^r}\left[1 - 7\cdot\frac{1}{3} + 10\cdot\frac{1}{3^2}\right] = \cancel{3^r}$$

$$A_0\left[1 - \frac{7}{3} + \frac{10}{9}\right] = 1$$

$$A_0\left[\frac{9-21+10}{9}\right] = 1$$

$$A_0\left[\frac{19-21}{9}\right] = 1$$

$$A_0\left[\frac{-2}{9}\right] = 1$$

$$\boxed{A_0 = -9/2}$$

'A_0' substitute in eq (4)

$$a_r^{(p)} = \left(\frac{-9}{2}\right)(3^r) \quad \text{--- (5)}$$

The General solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = C_1 2^r + C_2 5^r + (-9/2) 3^r \quad \text{— (6)}$$

Given initial conditions

$$a_0 = 0, \; a_1 = 1$$

put $r=0$ in eq (6)

$$a_0 = C_1 2^0 + C_2 5^0 + (-9/2) 3^0$$

$$0 = C_1 + C_2 - \frac{9}{2}$$

$$\Rightarrow C_1 + C_2 = \frac{9}{2} \quad \text{— (7)}$$

put $r=1$ in eq (6)

$$a_1 = C_1 2^1 + C_2 5^1 + \left(\frac{-9}{2}\right) 3^1$$

$$1 = 2C_1 + 5C_2 - \frac{27}{2}$$

$$\Rightarrow 2C_1 + 5C_2 = 1 + \frac{27}{2}$$

$$2C_1 + 5C_2 = \frac{2+27}{2}$$

$$2C_1 + 5C_2 = \frac{29}{2} \quad \text{— (8)}$$

solving (7) & (8)

$$(7) \times 2 \Rightarrow \cancel{2C_1} + 2C_2 = \frac{18}{2} = 9$$

$$(8) \times 1 \Rightarrow \cancel{2C_1} + 5C_2 = \frac{29}{2}$$

$$(-) \quad (-) \quad (-)$$

$$-3C_2 = 9 - \frac{29}{2}$$

$$-3C_2 = \frac{18-29}{2}$$

$$-3C_2 = \frac{-11}{2} \Rightarrow \boxed{C_2 = 11/6}$$

$7 \Rightarrow C_1 + C_2 = 9/2$

$C_1 + 11/6 = 9/2$

$C_1 = \frac{9}{2} - \frac{11}{6}$

$C_1 = \frac{27-11}{6}$

$\boxed{C_1 = \frac{16}{6}} = 8/3$

$C_1 \& C_2$ substitute in eq 6

$a_r = \frac{8}{3} 2^r + \frac{11}{6} 5^r + (-9/2) 3^r$

$= \frac{8}{3} 2^r + 11/6\, 5^r - \frac{9}{2} 3^r$

2. Solve the Recurrence Relation $a_r - 5a_{r-1} + 6a_{r-2} = 5^r$.

Sol Given the Recurrence Relation

$a_r - 5a_{r-1} + 6a_{r-2} = 5^r$ — 1

This is a second ordered Linear non-homogeneous recurrence Relation

The General solution of eq 1

is $a_r = a_r(h) + a_r(P)$ — 2

Find $a_r(h)$

The homogeneous recurrence relation is

$a_r - 5a_{r-1} + 6a_{r-2} = 0$

The characteristic equation is

$m^2 - 5m + 6 = 0$

$m^2 - 2m - 3m + 6 = 0$

$m(m-2) - 3(m-2) = 0$

$(m-3)(m-2)=0$

$m=3,\ m=2$

$m_1=3,\ m_2=2$

The characteristic roots are real and unequal.

The homogeneous solutions is

$$a_r(h) = C_1 m_1^r + C_2 m_2^r$$
$$= C_1 3^r + C_2 2^r \quad \text{--- (3)}$$

To find $a_r(P)$

The non-homogeneous Recurrence Relation is

$$a_r - 5a_{r-1} + 6a_{r-2} = 5^r$$
$$= 1 \cdot 5^r$$
$$= d \cdot b^r$$

Here $f(r)=5^r$, 5 is not a root of the characteristic equation.

$$a_r(P) = A_0 b^r$$

$$a_r(P) = A_0 5^r \quad \text{--- (4)}$$

from (1)

$$a_r - 5a_{r-1} + 6a_{r-2} = 5^r$$

$$A_0 5^r - 5A_0 5^{r-1} + 6A_0 5^{r-2} = 5^r$$

$$A_0 \cancel{5^r}\left[1 - 5\frac{1}{5} + 6\frac{1}{5^2}\right] = \cancel{5^r}$$

$$A_0\left[1 - 1 + \frac{6}{25}\right] = 1$$

$$A_0\left[\frac{6}{25}\right] = 1$$

$$A_0 = \frac{25}{6}$$

A_0 substitute in eq (4).

$$a_r(P) = A_0 5^r$$

$$a_r(P) = \frac{25}{6} 5^r \quad \text{--- (5)}$$

The General solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$\boxed{a_r = c_1 3^r + c_2 2^r + \frac{25}{6} 5^r}$$

3. Solve the Recurrence Relation $a_r + 4a_{r-1} + 4a_{r-2} = 5(-2)^r$, $r \geq 2$.

Sol Given Recurrence Relation

$$a_r + 4a_{r-1} + 4a_{r-2} = 5(-2)^r \quad \text{--- (1)}$$

The General solution of (1) is

$$a_r = a_r^{(h)} + a_r^{(p)} \quad \text{--- (2)}$$

To find $a_r^{(h)}$

* The homogeneous part of eq (1) is

$$a_r + 4a_{r-1} + 4a_{r-2} = 0$$

The characteristic equation is

$$m^2 + 4m + 4 = 0$$

$$m^2 + 2m + 2m + 4 = 0$$

$$m(m+2) + 2(m+2) = 0$$

$$(m+2)(m+2) = 0$$

$$m = -2, \quad m = -2$$

The characteristic roots are real and equal

$$a_r^{(h)} = (c_1 + c_2 r) m^r$$

$$= (c_1 + c_2 r)(-2)^r \quad \text{--- (3)}$$

To find $a_r^{(p)}$

The non-homogeneous Recurrence Relation.

$$a_r + 4a_{r-1} + 4a_{r-2} = 5(-2)^r = d \cdot \beta^r$$

Here $f(r) = 5(-2)^r$

-2 is a root of the characteristic equation of multiplicity '2'

$$\therefore a_r(P) = A_0 b^r r^m$$
$$a_r(P) = A_0 (-2)^r r^2 \quad \text{—— (4)}$$

from (1)

$$A_0 (-2)^r r^2 + 4[A_0 (-2)^{r-1} (r-1)^2] + 4[A_0 (-2)^{r-2} (r-2)^2] = 5(-2)^r$$

$$\Rightarrow A_0 (-2)^r r^2 + 4A_0 (-2)^{r-1} (r-1)^2 + 4A_0 (-2)^{r-2} (r-2)^2 = 5(-2)^r$$

Dividing on Both sides by $(-2)^{r-2}$

$$\Rightarrow \frac{A_0 (-2)^r r^2}{(-2)^{r-2}} + \frac{4 A_0 (-2)^{r-1} (r-1)^2}{(-2)^{r-2}} + \frac{4 A_0 (-2)^{r-2} (r-2)^2}{(-2)^{r-2}} = \frac{5 \cdot (-2)^r}{(-2)^{r-2}}$$

$$\Rightarrow A_0 r^2 4 + 4 A_0 (r-1)^2 (-2) + 4 A_0 (r-2)^2 = 5(4)$$

$$\Rightarrow 4A_0 r^2 - 8 A_0 (r-1)^2 + 4 A_0 (r-2)^2 = 20$$

$$\Rightarrow 4A_0 r^2 - 8 A_0 (r^2+1-2r) + 4A_0 (r^2+4-4r) = 20$$

$$\Rightarrow \cancel{4A_0 r^2} - \cancel{8A_0 r^2} - 8A_0 + \cancel{16 A_0 r} + \cancel{4A_0 r^2} + 16A_0 - \cancel{16 A_0 r} = 20$$

$$\Rightarrow 8 A_0 = 20$$

$$A_0 = 20/8$$

$$\boxed{A_0 = \frac{5}{2}}$$

from (4)

$$a_r(P) = \frac{5}{2} (-2)^r r^2.$$

from (2).

The General solution is

$$a_r = (C_1 + C_2 r)(-2)^r + \frac{5}{2} (-2)^r r^2 = \left[(C_1 + C_2 r) + \frac{5}{2} r^2\right] (-2)^r$$

4. Solve the recurrence relation $a_{r+2} - 6a_{r+1} + 9a_r = 3.2^r + 7.3^r$ for $r \geq 0$, given that $a_0 = 1$, $a_1 = 4$.

Sol Given the recurrence relation

$$a_{r+2} - 6a_{r+1} + 9a_r = 3.2^r + 7.3^r \quad (1)$$

The General solution of eq (1) is

$$a_r = a_r(h) + a_r(p) \quad (2)$$

To find $a_r(h)$

put $r = r-2$ (1)

The homogeneous part of eq (1) is

$$a_r - 6a_{r-1} + 9a_{r-2} = 3.2^r + 7.3^r = 0$$

The characteristic equation is

$$m^2 - 6m + 9m = 0$$

$$m^2 - 3m - 3m + 9m = 0$$

$$m(m-3) - 3(m-3) = 0$$

$$(m-3)(m-3) = 0$$

$$m = 3, m = 3$$

The characteristic roots are equal and real

$$a_r(h) = (C_1 + C_2 r) m^r$$

$$= (C_1 + C_2 r) 3^r \quad (3)$$

To find $a_r(p)$

The non-homogeneous recurrence relation is

$$a_{r+2} - 6a_{r+1} + 9a_r = 3.2^r + 7.3^r$$

$$= d.b^r + d.b^r$$

$$a_r(P) = A_0 2^r + A_1 3^r . r^2 \quad (4)$$

from (1)

$$A_0 2^{r+2} + A_1 3^{r+2}(r+2)^2 - 6[A_0 2^{r+1} + A_1 3^{r+1}(r+1)^2] + 9[A_0 2^r + A_1 3^r . r^2]$$

$$= 3.2^r + 7.3^r$$

$$\Rightarrow A_0 . 2^r . 2^2 + A_1 3^r 3^2 (r+2)^2 - 6[A_0 2^r . 2 + A_1 3^r . 3 (r+1)^2] + 9A_0 2^r +$$

$$9 A_1 3^r r^2 = 3.2^r + 7.3^r$$

$$(r^2+4+4r) \quad (r^2+1+2r)$$

$$\Rightarrow A_0 2^r [4-12+9] + A_1 3^r [9 (r+2)^2 - 18 (r+1)^2 + 9r^2] = 3.2^r + 7.3^r$$

$$\Rightarrow A_0 2^r . [1] + A_1 3^r [\cancel{9r^2} + 36 + \cancel{36r} - \cancel{18r^2} - 18 - \cancel{36r} + \cancel{9r^2}] = 3.2^r + 7.3^r$$

$$\Rightarrow A_0 2^r + A_1 3^r [18] = 3.2^r + 7.3^r$$

Equating the coefficients.

$$A_0 2^r = 3 2^r \qquad A_1 3^r [18] = 7.3^r$$

$$\boxed{A_0 = 3} \qquad 18 A_1 = 7$$

$$\boxed{A_1 = 7/18}$$

from (4)

$$a_r^{(P)} = 3\, 2^r + \frac{7}{18} 3^r r^2 \quad \text{—— (5)}$$

$\therefore$ from (2)

The General solution is

$$a_r = (C_1 + C_2 r) 3^r + 3.2^r + \frac{7}{18} 3^r r^2$$

$$a_r = [C_1 + C_2 r + \frac{7}{18} r^2] 3^r + 3.2^r \quad \text{—— (6)}$$

Given initial conditions

$a_0 = 1, a_1 = 4$

put $r=0$ in eq (6)

$$a_0 = [C_1 + 0 + \frac{7}{18}(0)] 3^0 + 3.2^0$$

$$1 = C_1 + 3$$

$$\boxed{C_1 = -2}$$

put $r=1$ in eq(6)

$$a_1 = \left[C_1 + C_2 + \frac{7}{18}(1)^2\right]3^1 + 3.2^1$$

$$4 = \left[C_1 + C_2 + \frac{7}{18}\right]3 + 6$$

$$4 = \left[\cancel{+C_1} + C_2 + \frac{7}{18}\right]3 + 6$$

$$4 = \left[\cancel{+3C_1} + 3C_2 + \frac{21}{18}\right] + 6$$

$$4 = \left[+3C_1 + 3C_2 + \frac{7}{6}\right] + 6$$

$$4 = 3C_1 + 3C_2 + \frac{36+7}{6}$$

$$\Rightarrow 3C_1 + 3C_2 = 4 - \frac{43}{6}$$

$$\Rightarrow 3C_1 + 3C_2 = -19/6$$

$$\Rightarrow 3(-2) + 3C_2 = -19/6$$

$$\Rightarrow 3C_2 = \frac{-19}{6} + 6$$

$$3C_2 = \frac{-19+36}{6}$$

$$3C_2 = 17/6$$

$$\boxed{C_2 = \frac{17}{18}}$$

from (6)

$$a_r = \left[-2 + \frac{17}{18}r + \frac{7}{18}r^2\right]3^r + 3.2^r.$$

5. Solve the Recurrence Relation $a_r - 2a_{r-1} + a_{r-2} = 5r$

Sol Given Recurrence Relation

$$a_r - 2a_{r-1} + a_{r-2} = 5r \quad \text{(1)}$$

The General solution of eq(1) is

$$a_r = a_r(h) + a_r(P) \quad \text{(2)}$$

To find $a_r(H)$

The homogeneous part of eq ① is

$$a_r - 2a_{r-1} + a_{r-2} = 0$$

The characteristic equation is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$(m-1)(m-1) = 0$$

$$m = 1, \ m = 1$$

The characteristic roots are real and equal.

$$a_r(h) = (C_1 + C_2 r) m^r$$

$$a_r(h) = (C_1 + C_2 r) 1^r \quad (3)$$

To find $a_r(P)$

The non-homogeneous Reccurence Relation is

$$a_r - 2a_{r-1} + a_{r-2} = 5r$$

* The RHS of eq ① has a polynomial of degree is '1' and '1' is a root of the multiplicity '2' of the characteristic equation.

$$\therefore \quad a_r(P) = [A_0 + A_1 r] r^m$$

$$= [A_0 + A_1 r] r^2 \quad (4)$$

$$a_r(P) = A_0 r^2 + A_1 r^3$$

from ①

$$A_0 r^2 + A_1 r^3 - 2[A_0 (r-1)^2 + A_1 (r-1)^3] + A_0 (r-2)^2 + A_1 (r-2)^3 = 5r$$

$$A_0 r^2 + A_1 r^3 - 2[A_0(r^2+1-2r) + A_1(r^3-1-3r^2\cdot 1+3\cdot r\cdot 1)] +$$
$$[A_0(r^2+4-4r) + A_1[r^3-8-6r^2+12r]] = 5r$$

$\because (a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

$\because (a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$$\Rightarrow \cancel{A_0r^2} + \cancel{A_1r^3} - \cancel{2A_0r^2} - 2A_0 + \cancel{4A_0r} - \cancel{2A_1r^3} + 2A_1 + \cancel{6A_1r^2} - 6A_1r +$$
$$\cancel{A_0r^2} + 4A_0 - \cancel{4A_0r} + \cancel{A_1r^3} - 8A_1 - \cancel{6A_1r^2} + 12A_1r = 5r$$

$$\Rightarrow 6A_1r + 2A_0 - 6A_1 = 5r$$

Equating the corresponding coefficients of r, constants.

$6A_1 = 5$

$\boxed{A_1 = 5/6}$

$2A_0 - 6A_1 = 0$

$2A_0 = 6A_1$

$2A_0 = 6\frac{5}{6}$

$\boxed{A_0 = 5/2}$

$$④ \Rightarrow a_r(P) = \frac{5}{2}r^2 + \frac{5}{6}r^3 \quad ⑤$$

from ② The General solution is

$$a_r = (C_1 + C_2 r)1^r + \frac{5}{2}r^2 + \frac{5}{6}r^3$$
$$= C_1 + C_2 r + 5r^2\left(\frac{1}{2} + \frac{r}{6}\right)$$
$$= C_1 + C_2 r + 5r^2\left[\frac{3+r}{6}\right]$$
$$= C_1 + C_2 r + 5r^2\left[\frac{3+r}{6}\right]$$
$$a_r = C_1 + C_2 r + 5r^2\left[\frac{3+r}{6}\right]$$

6. Solve the following Recurrence Relation

$$a_{r+2} - 4a_{r+1} + 3a_r = -200, \text{ for } r \geq 0 \text{ and}$$

$a_0 = 3000$, $a_1 = 3300$

Sol Given Recurrence Relation

$$a_{r+2} - 4a_{r+1} + 3a_r = -200 \quad \text{—} (1)$$

Put $r = r-2$

$$a_r - 4a_{r-1} + 3a_{r-2} = -200 \quad \text{—} (2)$$

The General solution is

$$a_r = a_r(h) + a_r(P) \quad \text{—} (3)$$

To find $a_r(h)$

$\because$ The homogeneous part of eq (2) is

$$a_r - 4a_{r-1} + 3a_{r-2} = 0$$

The characteristic equation is

$$m^2 - 4m + 3 = 0$$

$$m^2 - 3m - m + 3 = 0$$

$$m(m-3) - 1(m-3) = 0$$

$$(m-1)(m-3) = 0$$

$$m = 1, m = 3$$

The characteristic roots are unequal and real.

$$\therefore \quad a_r(h) = C_1 m_1^r + C_2 m_2^r$$

$$= C_1 1^r + C_2 3^r \quad \text{—} (4)$$

To find $a_r^{(P)}$

The non-homogeneous Reccurrence Relation is

$$a_{r+2} - 4a_{r+1} + 3a_r = -200$$

The RHS of eq(1) has a Polynomial of degree is '0' and '1' is a root of the multiplicity '1' of the characteristic equation.

$$a_r^{(P)} = A_0 r^m$$

$$a_r^{(P)} = A_0 r^1 \quad (5)$$

from (1)

$$A_0(r+2) - 4[A_0(r+1)] + 3A_0 r = -200$$

$$\cancel{A_0 r} + 2A_0 - 4\cancel{A_0 r} - 4A_0 + 3\cancel{A_0 r} = -200$$

$$-2A_0 = -200$$

$$\boxed{A_0 = 100}$$

(5) $\Rightarrow$

$$a_r^{(P)} = 100r \quad (6)$$

(3) $\rightarrow$

The General solution is

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$a_r = c_1 1^r + c_2 3^r + 100r \quad (7)$$

Given initial conditions $a_0 = 3000$, $a_1 = 3300$

put r=0 in eq (7)

$$a_0 = c_1 1^0 + c_2 3^0 + 100(0)$$

$$3000 = c_1 + c_2$$

$$\Rightarrow c_1 + c_2 = 3000 \quad (8)$$

put r=1 in eq (7)

$$a_1 = c_1 1^1 + c_2 3^1 + 100$$

$$3300 = c_1 + 3c_2 + 100$$

$$\Rightarrow c_1 + 3c_2 = 3200 \quad (9)$$

solving (8) and (9)

$$c_1 + c_2 = 3000$$
$$c_1 + 3c_2 = 3200$$
$$(-)\quad(-)\quad(-)$$
$$-2c_2 = -200$$
$$\boxed{c_2 = 100}$$

from (8)

$$c_1 + c_2 = 3000$$
$$c_1 = 3000 - 100$$
$$\boxed{c_1 = 2900}$$

c_1 and c_2 substitute in eq (7)

$$\boxed{a_r = 2900(1)^r + 100\,3^r + 100.r}$$

7. Solve the Recurrence Relation

$$a_r + 4a_{r-1} + 4a_{r-2} = 8, \text{ for } r \geq 2 \text{ and } a_0 = 1, a_1 = 2$$

8. Solve the Recurrence Relation

$$a_{r+2} - 10a_{r+1} + 21a_r = 3r^2 - 2, r \geq 0$$

7.
Sol Given Recurrence Relation

$$a_r + 4a_{r-1} + 4a_{r-2} = 8 \quad (1)$$

The General solution is

$$a_r = a_r^{(h)} + a_r^{(p)} \quad (2)$$

To find $a_r^{(h)}$

The homogeneous part of eq (1)

$$a_r + 4a_{r-1} + 4a_{r-2} = 0$$

The characteristic equation is

$$m^2+4m+4=0$$
$$m^2+2m+2m+4=0$$
$$m(m+2)+2(m+2)=0$$
$$(m+2)(m+2)=0$$
$$m=-2,\ m=-2$$

The characteristic roots are real and equal

$$a_r^{(h)} = (c_1+c_2 r)m^r$$
$$a_r^{(h)} = (c_1+c_2 r)(-2)^r \quad (3)$$

To find $a_r^{(p)}$

The non-homogeneous Recurrence Relation is

$$a_r+4a_{r-1}+4a_{r-2}=8$$

The RHS of eq(1) has a polynomial of degree is '0' and '1' is not a root of the characteristic equation.

$$a_r^{(P)} = A_0 \quad (4)$$

from (1)
$$A_0+4A_0+4A_0=8$$
$$9A_0=8$$
$$\boxed{A_0=8/9}$$

from (4) $a_r^{(P)}=8/9$

(2) ⇒ The General solution is

$$a_r = a_r^{(h)}+a_r^{(P)}$$
$$a_r = (c_1+c_2 r)(-2)^r+\frac{8}{9} \quad (5)$$

Given initial conditions $a_0=1, a_1=2$

put $r=0$ in eq(5)
$$a_0= c_1+c_2(0)(-2)^0+\frac{8}{9}$$

$$\Rightarrow 1 = c_1 + 0 + \frac{8}{9}$$

$$\Rightarrow c_1 = 1 - \frac{8}{9}$$

$$c_1 = \frac{9-8}{9}$$

$$\boxed{c_1 = 1/9}$$

put $r = 1$

$$a_1 = (c_1 + c_2(1))(-2)^1 + \frac{8}{9}$$

$$2 = -2c_1 - 2c_2 + \frac{8}{9}$$

$$2 = -2(1/9) - 2c_2 + \frac{8}{9}$$

$$2 = -\frac{2}{9} - 2c_2 + \frac{8}{9}$$

$$2c_2 + 2 = \frac{-2}{9} + \frac{8}{9}$$

$$2c_2 = \frac{6}{9} - 2$$

$$2c_2 = \frac{6-18}{9}$$

$$2c_2 = \frac{-12}{9}$$

$$c_2 = \frac{-12}{18}$$

$$\boxed{c_2 = \frac{-6}{9}}$$

c_1 and c_2 substitute in eq 5

$$\boxed{a_r = \left(\frac{1}{9} - \frac{6}{9}r\right)(-2)^r + \frac{8}{9}}$$

To find $a_r(P)$

The non-homogeneous Recurrence Relation is

$$a_{r+2} - 10a_{r+1} + 21a_r = 3r^2-2$$

The RHS of eq(1) has a polynomial degree is '2' and '1' is not a root of the characteristic equation.

$$a_r(P) = A_0 + A_1 r + A_2 r^2 \quad (5)$$

from (1)

$$A_0 - 10A_0 + 21A_0 = 3r^2-2$$

$$22A_0 - 10A_0 = 3r^2-2$$

$$12A_0 = 3r^2-2$$

$$A_0 = \frac{3r^2}{12} - \frac{2}{12}$$

$$\boxed{A_0 = \frac{r^2}{4} - \frac{1}{6}}$$

from 5

$$a_r(P) = \frac{r^2}{4} - \frac{1}{6}$$

(2) ⇒ The General solution is

$$a_r = a_r(h) + a_r(P)$$

$$\boxed{a_r = C_1 3^r + C_2 7^r + \frac{r^2}{4} - 1/6}$$

$$12A_0 = -2 - \frac{5}{2}$$

$$12A_0 = \frac{-4-5}{2}$$

$$12A_0 = \frac{-9}{2}$$

$$\boxed{A_0 = \frac{-9}{24}}$$

$$a_r(P) = \frac{-9}{24} + (-1/2)r + 1/4\, r^2$$

∴ The General solution is

$$a_r = a_r(h) + a_r(P)$$

$$\boxed{a_r = C_1 3^r + C_2 7^r - \frac{9}{24} - \frac{1}{2}r + \frac{1}{4}r^2}$$

(1) ⇒ $[A_0 + A_1(r+2) + A_2(r+2)^2] - 10[A_0 + A_1(r+1) + A_2(r+1)^2] + 21(A_0 + A_1 r + A_2 r^2) = 3r^2-2$

⇒ $A_0 + A_1 r + 2A_1 + A_2 r^2 + 4A_2 + 4A_2 r - 10A_0 - 10A_1 r - 10A_1 - 10A_2 r^2 - 10A_2 - 20A_2 r + 21A_0 + 21A_1 r + 21A_2 r^2 = 3r^2-2$

⇒ $12A_0 + 12A_1 r - 8A_1 + 12A_2 r^2 - 6A_2 + 24A_2 r = 3r^2-2$

comparing the coefficients on Both sides.

$12A_2 = 3$

$\boxed{A_2 = 1/4}$

$12A_1 + 24A_2 = 0$

$12A_1 + 24(1/4) = 0$

$12A_1 + 6 = 0$

$A_1 = \frac{-6}{12} = -1/2$

$12A_0 - 8A_1 - 6A_2 = -2$

$12A_0 - 8(-1/2) - 6(1/4) = -2$

$12A_0 + 4 - \frac{3}{2} = -2$

$12A_0 + \frac{8-3}{2} = -2$

Method of Generating functions

Method of Generating functions for 1st Recurrence Relations

* Suppose the Recurrence Relation to be solved is of the form $a_n = c a_{n-1} + F(n)$; $n \geq 1$

(or) equivalentaly

put $n = n+1$

$$a_{n+1} = c a_{n+1-1} + \phi(n), \quad n \geq 0$$

$$a_{n+1} = c a_n + \phi(n), \quad n \geq 0$$

The Generating function is

$$f(x) = \frac{a_0 + x g(x)}{1 - cx}$$

$$\text{where } g(x) = \sum_{n=0}^{\infty} \phi(n) x^n$$

1. To find a generating function for the Recurrence Relation $a_{n+1} - a_n = 3^n$, $n \geq 0$ and $a_0 = 1$. Hence solve the relation.

Sol Given $a_{n+1} - a_n = 3^n$, $a_0 = 1$

$$a_{n+1} = a_n + 3^n$$

comparing this equation with $a_{n+1} = c a_n + \phi(n)$,

Here $c = 1$, $\phi(n) = 3^n$.

The Generating function

$$f(x) = \frac{a_0 + x g(x)}{1 - cx}$$

$$f(x) = \frac{1+x\,g(x)}{1-(1)x}$$

$$f(x) = \frac{1+x\,g(x)}{1-x} \quad \text{--- (1)}$$

where $$g(x) = \sum_{n=0}^{\infty} \phi(n)\, x^n$$

$$= \sum_{n=0}^{\infty} 3^n . x^n$$

$$= \sum_{n=0}^{\infty} (3x)^n$$

$$= 1 + (3x)^1 + (3x)^2 + (3x)^3 + \cdots$$

$$= (1-3x)^{-1}$$

from (1) $$f(x) = \frac{1 + x(1-3x)^{-1}}{1-x}$$

$$f(x) = \frac{1 + \frac{x}{1-3x}}{1-x}$$

$$= \frac{\frac{1-3x+x}{1-3x}}{1-x}$$

$$= \frac{1-3x+x}{(1-x)(1-3x)}$$

$$f(x) = \frac{1-2x}{(1-x)(1-3x)} \quad \text{--- (2)}$$

consider $$\frac{1-2x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$\frac{1-2x}{\cancel{(1-x)}\cancel{(1-3x)}} = \frac{A(1-3x) + B(1-x)}{\cancel{(1-x)}\ \cancel{(1-3x)}}$$

$$1-2x = A(1-3x) + B(1-x) \quad \text{--- (3)}$$

put $1-x=0$
$\boxed{x=1}$

from (3) $\Rightarrow$ $1-2 = A(1-3)+0$

$-1 = -2A \quad \Rightarrow \boxed{A = 1/2}$

put $1-3x=0$

$1=3x$

$\Rightarrow \boxed{x=1/3}$

from (3) $1-\frac{2}{3} = A(1-3/3)+B(1-1/3)$

$\frac{3-2}{3} = 0 + B\left(\frac{3-1}{3}\right)$

$\frac{1}{\cancel{3}} = \frac{2B}{\cancel{3}}$

$\boxed{B=1/2}$

[$\because$ A and B values substitute in eq (3)

$1-2x = \frac{1}{2}(1-3x)+\frac{1}{2}(1-x)$

$1-2x = \frac{1}{2}[1-3x+1-2x]$

$1-2x = \frac{1}{2}[2-5x]$ x]

$\frac{1-2x}{(1-x)(1-3x)} = \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1-3x}$

$= \frac{1}{2(1-x)} + \frac{1}{2(1-3x)}$

$\Rightarrow$ (2)

$f(x) = \frac{1}{2}\left[\frac{1}{1-x}+\frac{1}{1-3x}\right]$

$= \frac{1}{2}\left[(1-x)^{-1}+(1-3x)^{-1}\right]$

$= \frac{1}{2}\left[\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (3x)^n\right]$

$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}(1+3^n)x^n$

Since $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$\therefore a_n = \frac{1}{2}(1+3^n)$

2. Using the Generating function method solve the recurrence relation $a_n - 3a_{n-1} = n,\ n \geq 1$, given $a_0 = 1$

Sol. Given

$$a_n - 3a_{n-1} = n,\ a_0 = 1$$

put $n = n+1$

$$a_{n+1} - 3a_{n+1-1} = n+1$$

$$a_{n+1} - 3a_n = n+1$$

$$a_{n+1} = 3a_n + n + 1$$

Comparing this equation with $a_{n+1} = Ca_n + \phi(n)$

Here $C = 3$, $\phi(n) = n+1$

The Generating function

$$f(x) = \frac{a_0 + x\,g(x)}{1 - Cx}$$

$$f(x) = \frac{1 + x\,g(x)}{1 - 3x} \quad \text{— (i)}$$

where $g(x) = \sum_{n=0}^{\infty} \phi(n)\, x^n$

$$= \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 1 + 2x^1 + 3x^2 + 4x^3 + 5x^4 + \dots$$

$$= (1-x)^{-2}$$

from (i) $f(x) = \dfrac{1 + x(1-x)^{-2}}{1 - 3x}$

$$= \frac{1 + \frac{x}{(1-x)^2}}{1 - 3x} = \frac{(1-x)^2 + x}{(1-x)^2(1-3x)}$$

$$= \frac{1+x^2-2(1)(-x)+x}{(1-x)^2(1-3x)}$$

$$= \frac{1+x^2+2x+x}{(1-x)^2(1-3x)}$$

$$f(x) = \frac{1+x^2+3x}{(1-x)^2(1-3x)}$$

$$f(x) = \frac{1+x^2+3x}{(1-x)(1-x)(1-3x)}$$

consider

$$\frac{1+x^2+3x}{(1-x)(1-x)^2(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-3x)}$$

$$\frac{1+x^2+3x}{(1-x)(1-x)(1-3x)} = \frac{A(1-x)(1-3x)+B(1-x)(1-3x)+C(1-x)^2}{(1-x)(1-x)(1-3x)}$$

$$1+x^2+3x = A[1-3x-x+3x^2] + B[1-3x-x+3x^2] + C[1+x^2+2x]$$

$$1+x^2+3x = A(1-x)(1-3x) + B(1-x)(1-3x) + C(1-x)^2 \quad \text{——— (2)}$$

put $x-1=0$

$\boxed{x=1}$

put $x=1$ in eq (2)

$1+1+3 = A$

$\boxed{A=5}$

put $1-3x=0$

$1=3x$

$\boxed{x=1/3}$

$1+\frac{1}{9}+\frac{3}{3} \neq$

put $x=1/3$ in eq (2)

$1+\frac{1}{9}+\frac{3}{3} = C(1-\frac{1}{3})^2$

$1+1+\frac{1}{9} = C(\frac{4}{9})$

$2+\frac{1}{9} = \frac{4C}{9}$

$18+1 = 4C$

$\Rightarrow 4C=19$

$\boxed{C=19/4}$

$$1+x^2+3x = A[3x^2-4x+1] + B[3x^2-4x+1] + C[1+x^2+2x]$$

$$x^2+3x+1 = A[3x^2-4x+1] + B[3x^2-4x+1] + C[x^2+2x-1]$$

comparing the coefficients x^2, x, constants

$1 = 3A+3B+C$ (1), $3 = -4A-4B+2C$, $1 = A+B+C$

$3A+3B+3C = 3$

$3A+3B+C = 1$

$2C = 2$ $\quad \boxed{C=1}$

consider

$$\frac{1+x^2+3x}{(1-x)^2(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{1-3x}$$

$$\frac{1+x^2+3x}{(1-x)^2(1-3x)} = \frac{A(1-x)(1-3x) + B(1-3x) + C(1-x)^2}{(1-x)^2(1-3x)}$$

$$1+x^2+3x = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2$$

$$x^2+3x+1 = A[1-3x-x+3x^2] + B(1-3x) + C(1+x^2-2x)$$

$$x^2+3x+1 = A[3x^2-4x+1] + B(1-3x) + C(x^2-2x+1)$$

Comparing coefficients on Both sides

$1 = 3A + C$ —i $\qquad 3 = -4A - 3B - 2C$ —ii $\qquad 1 = A+B+C$ —iii

solving ii and iii

$-(4A+3B+2C) \neq -B$

$3A+C=1$

$3A = 1-C$

$\boxed{A = \frac{1-C}{3}}$

$A+B+C=1$

$B+C = 1 - \frac{(1-C)}{3}$

$B+C = \frac{3-1+C}{3}$

$B+C = \frac{2+C}{3}$

$B = \frac{2}{3} + \frac{C}{3} - C$

$= \frac{2}{3} - \frac{2C}{3}$

$B = \frac{2}{3}(1-C)$

$3 = -4\frac{(1-C)}{3} - 3\left(\frac{2}{3}(1-C)\right) - 2C$

$\Rightarrow 3 = \frac{-4+4C}{3} - \frac{6}{3} + \frac{6C}{3} - 2C$

$\Rightarrow 3 = \frac{-4+4C-6+6C-6C}{3}$

$\Rightarrow 4C-10 = 9$

$4C = 9+10$

$\boxed{C = 19/4}$

from (i)

$3A+C=1$

$3A = 1-C$

$= 1 - \frac{4}{19}$

$3A = \frac{15}{19}$

$A = \frac{15}{57}$

$\boxed{A = \frac{5}{19}}$

from iii

$B = 1-A-C$

$= 1 - \frac{5}{19} - \frac{19}{4}$

$= \frac{76-20-361}{76}$

$B = -\frac{305}{76} \times \frac{2}{2} = \frac{-610}{152}$

Method of Generating functions for the second order Recurrence Relation

Suppose the Recurrence relation to be solved is of the form $a_n + A a_{n-1} + B a_{n-2} = F(n)$, for $n \geq 2$ or equivalently.

put $n = n+2$

$$a_{n+2} + A a_{n+1} + B a_n = \phi(n), \text{ for } n \geq 0$$

The Generating function is

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x}{1 + Ax + Bx^2}$$

1. Find a generating function for the Recurrence Relation $a_{n+2} - 3a_{n+1} + 2a_n = 0, n \geq 0$. and $a_0 = 1, a_1 = 6$ Hence solve it.

Sol Given Recurrence Relation

$$a_{n+2} - 3a_{n+1} + 2a_n = 0$$

comparing with $a_{n+2} + A a_{n+1} + B a_n = \phi(n), n \geq 0$

Here $A = -3$, $B = 2$ and initial condition are

$a_0 = 1, a_1 = 6$

The Generating function is

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x}{1 + Ax + Bx^2}$$

$$= \frac{1 + (6 + 1(-3))x}{1 + (-3)x + 2x^2} = \frac{1 + 3x}{1 - 3x + 2x^2} = \frac{1 + 3x}{2x^2 - 3x + 1}$$

$$= \frac{1+3x}{(1-x)(1-2x)} \quad \text{(1)}$$

$2x^2-3x+1$

$2x^2-2x-x+1$

$2x(x-1)-1(x-1)$

$(x-1)(2x-1)$

$(1-x)(1-2x)$

consider $\frac{1+3x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$

$$\frac{1+3x}{(1-x)(1-2x)} = \frac{A(1-2x)+B(1-x)}{(1-x)(1-2x)}$$

$$1+3x = A(1-2x)+B(1-x) \quad \text{(2)}$$

put $1-x=0$

$\boxed{x=1}$

$x=1$ in eq (2)

$1+3 = A(1-2)+0$

$\Rightarrow -A = 4$

$\boxed{A=-4}$

put $1-2x=0$

$1=2x$

$\boxed{x=1/2}$ in eq (2)

$1+\frac{3}{2} = A(1-2\cdot\frac{1}{2})+B(1-\frac{1}{2})$

$\frac{2+3}{2} = 0+B(\frac{1}{2})$

$\frac{5}{\cancel{2}} = \frac{B}{\cancel{2}}$

$\boxed{B=5}$

$$\Rightarrow \frac{1+3x}{(1-x)(1-2x)} = \frac{-4}{1-x} + \frac{5}{1-2x}$$

from (1)

$$f(x) = \frac{-4}{1-x} + \frac{5}{1-2x}$$

$$= -4(1-x)^{-1} + 5(1-2x)^{-1}$$

$$= -4\sum_{n=0}^{\infty} x^n + 5\sum_{n=0}^{\infty}(2x)^n$$

$$f(x) = -4\sum_{n=0}^{\infty} x^n + 5\sum_{n=0}^{\infty} 2^n x^n$$

since $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$$a_n = -4+5\cdot 2^n$$

② $a_n + a_{n-1} - 6a_{n-2} = 0$, for $n \geq 2$, given that $a_0 = -1, a_1 = 8$

Sol Given Recurrence Relation

$$a_n + a_{n-1} - 6a_{n-2} = 0 \text{ for } n \geq 2 \quad \text{--- (1)}$$

put $n = n+2$ in eq (1)

$$a_{n+2} + a_{n+1} - 6a_n = 0 \text{ for } n \geq 0$$

comparing with $a_{n+2} + Aa_{n+1} + Ba_n = \phi(n), n \geq 0$

$A = 1$, $B = -6$, initial conditions are

$a_0 = -1, a_1 = 8$

The Generating function is

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x}{1 + Ax + Bx^2}$$

$$= \frac{-1 + (8 + (-1)(1))x}{1 + x - 6x^2}$$

$$= \frac{-1 + 7x}{-6x^2 + x + 1}$$

$$= \frac{+(1 - 7x)}{+(6x^2 - x - 1)}$$

$$= \frac{1 - 7x}{6x^2 - x - 1}$$

$6x^2 - 3x + 2x - 1$

$3x(2x-1) + 1(2x-1)$

$(2x-1)(3x+1)$

$$f(x) = \frac{1 - 7x}{(2x-1)(3x+1)} \quad \text{--- (1)}$$

consider $\dfrac{1-7x}{(2x-1)(3x+1)} = \dfrac{A}{2x-1} + \dfrac{B}{3x+1}$

$$\frac{1-7x}{(2x-1)(3x+1)} = \frac{A(3x+1) + B(2x-1)}{(2x-1)(3x+1)}$$

$$1 - 7x = A(3x+1) + B(2x-1) \quad \text{--- (2)}$$

put $2x-1=0$

$2x=1$

$\boxed{x=1/2}$

x in eq (2)

$1-\frac{7}{2} = A\left(\frac{3}{2}+1\right)+B\left(\frac{2}{2}-1\right)$

$\frac{2-7}{2} = A\left(\frac{3+2}{2}\right)+0$

$\Rightarrow \frac{5A}{\cancel{2}} = \frac{-5}{\cancel{2}}$

$\boxed{A=-1}$

put $3x+1=0$

$3x=-1$

$\boxed{x=-1/3}$

x in eq (2)

$1-7(-1/3) = A(3(-1/3)+1)+ B(2(-1/3)-1)$

$1+\frac{7}{3} = 0+B\left(\frac{-2}{3}-1\right)$

$\frac{10}{\cancel{3}} = -\frac{5}{\cancel{3}}B$

$\Rightarrow -5B=10$

$\boxed{B=-2}$

$$\Rightarrow \frac{1-7x}{(2x-1)(3x+1)} = \frac{(-1)}{2x-1} + \frac{(-2)}{3x+1}$$

from (1)

$$f(x) = \frac{(-1)}{2x-1} + \frac{(-2)}{3x+1}$$

$$= (-1)(2x-1)^{-1} + (-2)(3x+1)^{-1}$$

$$= (1-2x)^{-1} + (1-3x)^{-1}$$

$$= \sum_{n=0}^{\infty} (2x)^n + \sum_{n=0}^{\infty} (3x)^n$$

$$f(x) = \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} 3^n x^n$$

since $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\boxed{a_n = 2^n + 3^n}$$