UNIT - III

Stochastic process & Markov chains

Introduction:-

Markov analysis is a way of analyzing the current movement of some variable in an effort to forecast its future movement.

As a management tool, Markov analysis has been used during the last several years, mainly as a marketing aid for examining and predicting the behaviour of customers from the stand point of the loyalty to one brand and their switching patterns to other brands. In the field of accounting it can be applied to the behaviour of accounts receivable.

In production, it is helpful in evaluating alternative maintainance policies, queueing systems and work assignments. The technique is also useful to the personal department in determining future manpower requirements of an organizations.

Stochastic process:-

Stochastic process is a set of random variables depending on some real parameter like time (t)

Markov process:-

A random process in which the occurrence of each future state depends on the immediately preceding state and only on it. is known as markov process or Markov chain.

Markov process helps us to identify

i) A specific state of the system being studied (two states working or not working)

ii) The state - transition relationship (behaviour of customers in terms of their brand loyality and Switching pattern)

State :-
Is a condition or location of an object in the system at particular time.

Assumptions:-
1) The process consists of a finite number of states.
2) The probability of moving from one state to another depends only on the immediately preceding state.
3) Transition probabilities are constant.
4) The process has a set of initial probabilities which may be given or determined.

Transition probability:-
The probability of moving from one state to another state ~~are~~ or remaining in the same state during a single time period is called the transition probability.

Mathematically

$$P_{ij} = P(\text{next state } S_j \text{ at } t=1 / \text{Initial state } S_i \text{ at } t=0)$$

i — initial state, j — next state

Whenever a new result ~~are~~ or outcome occurs, process is said to have incremented one step. Each step represents a time period or any other condition which would result in another state.

If $n=0$ then it represents the initial state.

Transition probability Matrix (TPM):-
With the TPM we predict the movement of system from one state to the next state

$$P = \text{Initial state}[i] \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \text{ next state}[j] \\ \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \end{array}$$

$P_{11} = P[\text{In state } S_1 \text{ next state at } t=1 / \text{In state } S_1 \text{ initial state at } t=0]$

(or)

$P[S_1 \text{ at time } t=1 / S_1 \text{ at time } t=0]$

$P_{12} = P[S_2 \text{ at time } t=1 \mid S_1 \text{ at time } t=0]$

$P_{21} = P[S_1 \text{ at time } t=1 / S_2 \text{ at time } t=0]$

one-step transition process

2-step TPM:-

$$P^2 = \text{Initial state}[i] \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \text{next state}[j] \\ \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} P_{11}^2 & P_{12}^2 & P_{13}^2 \\ P_{21}^2 & P_{22}^2 & P_{23}^2 \\ P_{31}^2 & P_{32}^2 & P_{33}^2 \end{bmatrix} \end{array}$$

$P_{11}^2 = P[S_1 \text{ at time } t=2 \mid S_1 \text{ at time } t=0]$

n-step TPM :-

$$P^n = \text{state}[i] \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \text{next state}[j] \\ \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} P_{11}^n & P_{12}^n & P_{13}^n \\ P_{21}^n & P_{22}^n & P_{23}^n \\ P_{31}^n & P_{32}^n & P_{33}^n \end{bmatrix} \end{array}$$

$P_{11}^n = P[S_1 \text{ at time } t=n \mid S_1 \text{ at time } t=0]$

Properties of TPM :-

1) It is a Square matrix
2) All entries are b/w 0 & 1
3) The Sum of entries in any row must be 1.

Definition:-

A Stochastic matrix P is Said to be regular if all the entries of sum power P^m are positive

Definition :-
A stochastic matrix 'P' is not-regular if '1' occurs in the principle main diagonal

Ex:- The given matrix is regular or not ?

$$A = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5625 & 0.3125 & 0.125 \\ 0.3 & 0.45 & 0.25 \\ 0.45 & 0.35 & 0.2 \end{bmatrix}$$

A^2 having all positive & there is no '1' in diagonal

$\therefore$ The given matrix is regular.

2) Is the given matrix $B = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is regular or not ?

Sol

$$B^2 = B \times B = \begin{bmatrix} 0.25 & 0 & 0.75 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^3 = B \times B^2 = \begin{bmatrix} 0.125 & 0 & 0.875 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^4 = B^3 \times B = \begin{bmatrix} 0.0625 & 0 & 0.9375 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ there is no positive elements in B^4 and diagonal elements having is $\therefore$ B is not regular

3) Find the value of x, y, z if $\begin{bmatrix} 0 & x & 1/3 \\ 0 & 0 & y \\ 1/3 & 1/4 & z \end{bmatrix}$ is a Transition probability Matrix?

Sol We know that, in a TPM the sum of entries in each row is equal to '1'.

So

$$0 + x + 1/3 = 1 \longrightarrow ①$$

$$0 + 0 + y = 1 \longrightarrow ②$$

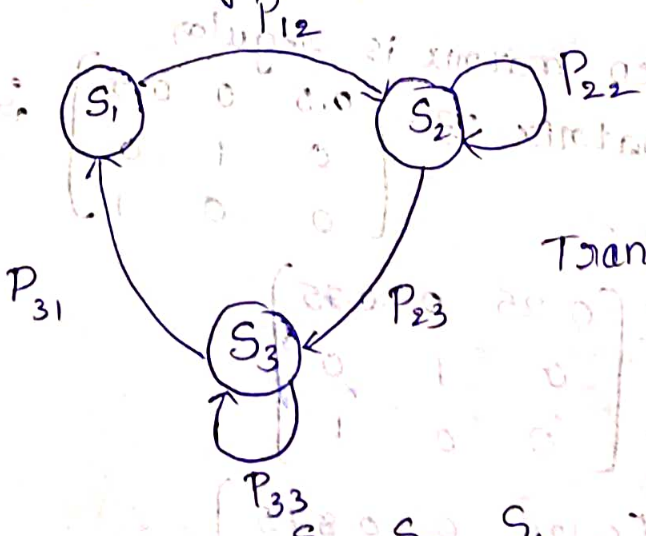
$$1/3 + 1/4 + z = 1 \longrightarrow ③$$

$$\therefore x = 2/3,\ y = 1,\ z = 5/12$$

Transition diagram:-

A transition diagram shows the transition probabilities that can occur in any situation. The arrows from each state indicate the possible states to which a process can move from the given state.

The following transition matrix corresponds to the adjoining diagram.



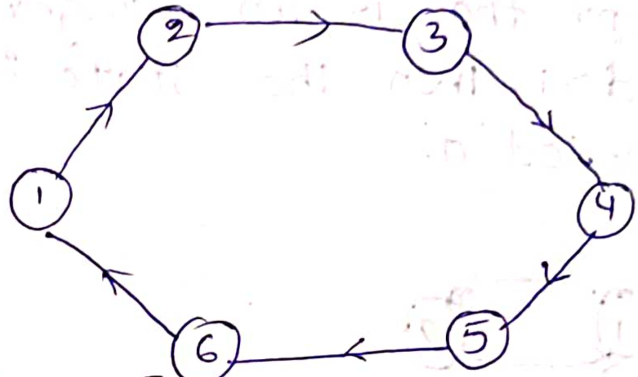
Transition diagram

TP Matrix =
$$\begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0 & P_{12} & 0 \\ 0 & P_{22} & P_{23} \\ P_{31} & 0 & P_{33} \end{bmatrix} \end{array}$$

A zero element in the matrix indicates that the transition is impossible.

Irreducible:-

A Markov chain is said to be irreducible, If all states communicate with each other.



1) If $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ It is irreducable or not?

let $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{array}$

Transition diagram

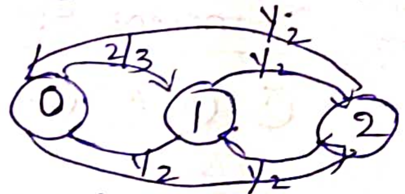
* class of state

$c(0) = \{1,2,0\}$
$c(1) = \{1,0,2\}$
$c(2) = \{1,0,2\}$

∴ Every State is communicate with each other states So, the given markov chain is irreducable.

2) $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

let $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 2/3 & 1/3 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 1/2 & 1/2 & 0 \end{array}$

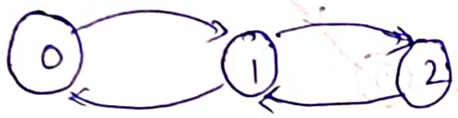


class of state $c(0) = \{0,1,2\}$
$c(1) = \{0,1,2\}$
$c(2) = \{0,1,2\}$

∴ Every state is communicate with each other States, so The given markov chain is irreducable

* Periodic and Aperiodic states:-

Suppose that the structure of Markov chain is such that state i is visited after a number of steps that is an integer multiple of an integer $d>1$. Then the state is called periodic with period 'd'.



periodic state $d=2$

* If no such integer exists (i.e. $d=1$) then the state is called Aperiodic.

periodic if $d(i) = \gcd\{n>0;\ P_{ij}^{n}>0\}$

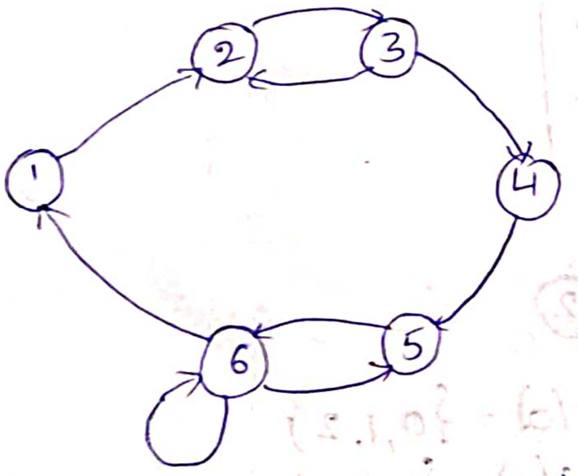
Aperiodic if $d(i)=1$

* Recurrent:-

In general a state is said to be recurrent, if any time that we leave that state and we will return to that state with probability 1.

* Transient:-

If the probability of returning is less than '1'. (<1), the state is called transient.



Recurrent states (come back)

2 3

Transient states (Not come back)

1 4 5 6

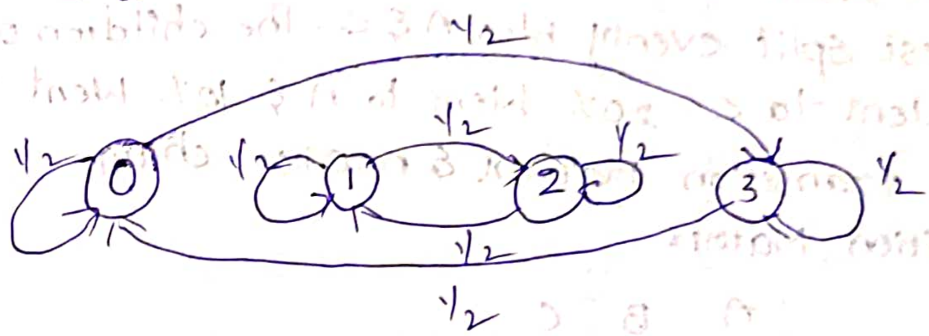
Ergodic:-

A Markov chain is Ergodic if and only if it has atmost one recurrent class and is aperiodic (All the states are aperiodic)

{All the elements of a class is recurrent & aperiodic}

Ex:-

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \end{array}$$



$d(i) = 1$

$d(0) = \gcd\{1,2,3\} = 1$

$d(1) = \gcd\{1,2,3\} = 1$

$d(2) = 1 \quad d(3) = 1$

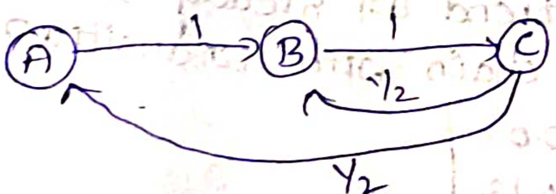
All states are Aperiodic So it is erogodic

1) 3 boys A, B, C are throwing a ball to each other. A always throws the ball to B and B always throws the ball To C but C is just as likely to throw the ball to B as To A. Find The transition Matrix and classify the states. Do all the states are ergodic?

let us construct a transition Matrix for given data.

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

State transition diagram.



from The transition diagram it is clear that it is a recurrent

$d(A) = \gcd\{3,5\} = 1$

$d(B) = \gcd\{2,3,5\} = 1$

$d(C) = \gcd\{2,3,5\} = 1$

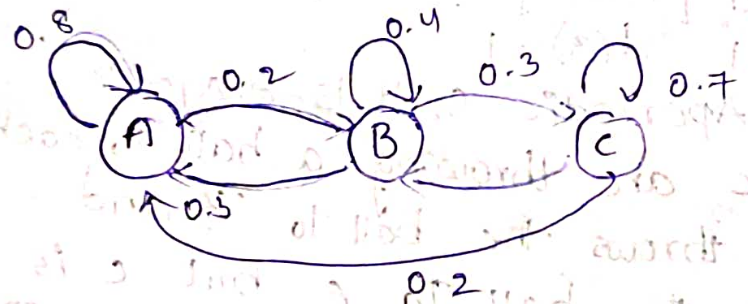
$\therefore d(i) = 1 \Rightarrow$ The Markov chain is aperiodic

$\therefore$ The problem is Ergodic.

2) 3 universities A, B, C are admitting students. It is given that 80% of the children of A went to A and rest went to B. 40% of the children of B went B. and rest split evenly b/w A & C. The children of C. 70% went to C. 20% went to A & 10% went to B. from the transition Matrix & markov chain.

Transition Matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$



Steady state vector (or) fixed vector (or) Equilibrium vector of a Markov chain (or) long range trend of the Markov chain:-

If a markov chain with transition matrix P is regular, then there is a unique vector v such that for any probability vector v & for any large value of n $\boxed{vP^n = v}$ vector V is called the equilibrium (or) steady State vector

problem ①:-

Find the long range trend or steady state vector for the markov chain with transition matrix $\begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$

Given matrix $P = \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$

∴ All the entries in the matrix are positive.
so that Given matrix is regular.
let v is a probability vector $[v_1 \; v_2 \; v_3]$

$$vP = v \quad \{n=1\}$$

$$[v_1 \; v_2 \; v_3]\begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix} = [v_1 \; v_2 \; v_3]$$

$$[0.65v_1+0.15v_2+0.12v_3 \quad 0.28v_1+0.67v_2+0.36v_3 \quad 0.07v_1+0.18v_2+0.52v_3] = [v_1 \; v_2 \; v_3]$$

$$0.65v_1+0.15v_2+0.12v_3 = v_1$$

$$-0.35v_1+0.15v_2+0.12v_3 = 0 \quad \text{--- (1)}$$

$$0.28v_1+0.67v_2+0.36v_3 = v_2$$

$$0.28v_1-0.33v_2+0.36v_3 = 0 \quad \text{--- (2)}$$

$$0.07v_1+0.18v_2+0.52v_3 = v_3$$

$$0.07v_1+0.18v_2+(-0.48)v_3 = 0 \quad \text{--- (3)}$$

Since v is a probability vector so we have

$$v_1+v_2+v_3 = 1 \quad \text{--- (4)}$$

Solving (1) (2) & (4)
Which is in $AV = B$
Agumented matrix

$$A'B = \left[\begin{array}{ccc|c} -0.35 & 0.15 & 0.12 & 0 \\ 0.28 & -0.33 & 0.36 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.28 & -0.33 & 0.36 & 0 \\ -0.35 & 0.15 & 0.12 & 0 \end{bmatrix}$$

Applying $R_2 \leftrightarrow R_2 - 0.28R_1$
$R_3 \leftrightarrow R_3 + 0.35R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -0.61 & 0.08 & -0.28 \\ 0 & 0.5 & 0.47 & 0.35 \end{bmatrix}$$

$$R_3 \leftrightarrow R_3 0.61 + 0.5R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -0.61 & 0.08 & -0.28 \\ 0 & 0 & 0.33 & \end{bmatrix}$$

Re write in AV=B for

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -0.61 & 0.08 \\ 0 & 0 & 0.33 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.28 \\ 0.19 \end{bmatrix}$$

$V_1 + V_2 + V_3 = 1 \rightarrow$ (i)

$-0.61 V_2 + 0.08 V_3 = -0.28 \rightarrow$ (ii)

$0.33 V_3 = 0.07$

$\boxed{V_3 = 0.21}$

$-0.61 V_2 + 0.08 (0.21) = -0.28$

$\cancel{+}0.61 V_2 = \cancel{+}0.2968$

$\boxed{V_2 = 0.49}$

$V_1 + 0.49 + 0.21 = 1$

$\boxed{V_1 = 0.30}$

$\therefore$ The required equilibrium vector

$V = [0.30 \;\; 0.49 \;\; 0.21]$

2) $P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$ 3) $P = \begin{pmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{pmatrix}$

Given matrix $P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$

All the entries in The matrix are positive.
So the given matrix is regular
let v is the probability vector $[v_1 \; v_2 \; v_3]$

$VP = V \quad \{n=1\}$

$$[v_1 \; v_2 \; v_3] \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} = [v_1 \; v_2 \; v_3]$$

$$[V_1\,0.5+0.1V_2+0.2V_3 \quad 0.2V_1+0.4V_2+0.2V_3 \quad 0.3V_1+0.5V_2+0.6V_3] = [V_1 \quad V_2 \quad V_3]$$

$0.5V_1+0.1V_2+0.2V_3=V_1$

$-0.5V_1+0.1V_2+0.2V_3=0 \rightarrow \text{I}$

$0.2V_1+0.4V_2+0.2V_3=V_2$

$0.2V_1-0.6V_2+0.2V_3=0 \rightarrow \text{II}$

$0.3V_1+0.5V_2+0.6V_3=V_3$

$0.3V_1+0.5V_2-0.4V_3=0 \rightarrow \text{III}$

Since v is the probability vector, then

$V_1+V_2+V_3=1 \rightarrow \text{IV}$

Solving I, II & IV

which is in $AV=B$

Agumented matrix

$$A|B = \begin{bmatrix} -0.5 & 0.1 & 0.2 & 0 \\ 0.2 & -0.6 & 0.2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & -0.6 & 0.2 & 0 \\ -0.5 & 0.1 & 0.2 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - R_1 0.2, \quad R_3 \leftrightarrow R_3 + 0.5R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -0.8 & 0 & -0.2 \\ 0 & 0.6 & 0.7 & 0.5 \end{bmatrix}$$

$R_3 \leftrightarrow 0.8R_3 + 0.6R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -0.8 & 0 & -0.2 \\ 0 & 0 & 0.56 & -0.08 \end{bmatrix}$$

Re write in $AV=B$ then

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -0.8 & 0 \\ 0 & 0 & 0.56 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.2 \\ -0.08 \end{bmatrix}$$

$V_1 + V_2 + V_3 = 1$

$-0.8V_2 = -0.2$

$V_2 = \frac{0.2}{0.8} = 1/4 = 0.25$

$\boxed{V_2 = 0.25}$

$0.56V_3 = -0.08$

$V_3 = \frac{-0.08}{0.56} = -1/7 = -0.142$

$\boxed{V_3 = -0.142}$

$V_1 + 0.25 - 0.14 = 1$

$V_1 = 1.14 - 0.25$

$\boxed{V_1 = 0.89}$

∴ The required equlibrium vector

$V = [0.89 \quad 0.25 \quad -0.14]$

3) Given matrix $P = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$

All the entries in the matrix are positive. so the given matrix is regular. let v is the probability vector $[v_1 \quad v_2 \quad v_3]$

$vp = v \, (n=1)$

$[v_1 \quad v_2] \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix} = [v_1 \quad v_2]$

$0.25V_1 + 0.5V_2 = V_1$

$-0.75V_1 + 0.5V_2 = 0 \rightarrow (1)$

$0.75V_1 + 0.5V_2 = V_2$

$0.75V_1 - 0.5V_2 = 0 \rightarrow (2)$

Since v is the probability vector then

$V_1 + V_2 = 1 \rightarrow (3)$

Solving (1) & (3)

$-0.75V_1 + 0.5V_2 = 0$

$V_1 + V_2 = 1$

$-0.75V_1 + 0.5V_2 = 0$

$0.75V_1 + 0.75V_2 = 0.75$

$1.25V_2 = 0.75$

0.75
0.5
1.25

$$V_2 = \frac{0.75}{1.25} = \frac{75}{125}$$
$$= 0.6$$

$5)\ 30(0.6$
30
(0)

from ①

$$-0.75V_1 + 0.5V_2 = 0$$
$$-0.75V_1 + 0.5 \times 0.6 = 0$$
$$-0.75V_1 + 0.30 = 0$$
$$+0.75V_1 = +0.30$$
$$V_1 = \frac{0.30}{0.75} = \frac{30}{75}$$
$$= 0.4$$

0.5×0.6
30
00
0.30

$75)30$

$13)\ 60(0.4$
52
80

$$V_1 = 0.4$$

$\therefore$ The required equilibrium vector

$$v = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$$

P = Initial state
(i) [n=0]

$$\begin{array}{cccc} & S_1 & S_2 & S_3 \\ S_1 & \multicolumn{3}{c}{\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}} \end{array}$$

↓ Retention & gain

→ Retention

→ Retention & loss

1) In a certain market, only two brands of cold drinks A & B are sold. Given that a man last purchased brand A, there is 80% chance that he would buy the same brand in the next purchase, while if a man purchased brand B, there is 90% chance that his next purchase would be brand B. Using this information

a) Develop transition probability matrix

b) Interpret the state transition matrix in terms of i) Retention and loss ii) Retention & gain.

c) Draw transition diagram.

Sol a)

$$P = \begin{array}{c} \text{present} \\ \text{purchase} \\ (n=0) \end{array} \begin{array}{c} \\ A \\ B \end{array} \overset{\text{next purchase } (n=1)}{\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix} \end{array}}$$

↓ Retention & gain

Retetion & loss →

b) i) Retention & loss

P_{11} (or) $P_{AA} = P[A$ next at $t=1 \mid A$ at time $t=0]$

$P_{11} = 0.8$ Retention of A

P_{12} (or) $P_{AB} = P[B$ next at $t=1 \mid A$ at time $t=0]$

$P_{12} = 0.2$ loss to A

ii) Retention & gain

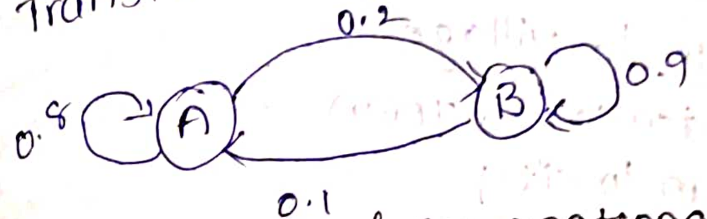
P_{21} (or) $P_{BA} = P[A$ next at $t=1 \mid B$ at time $t=0]$

$P_{21} = 0.1$ gain to A

P_{22} (or) P_{BB} = P [B next at t=1 / B at time t=0]

$P_{22} = 0.9$ Retention of B.

(c) Transition diagram



2) The school of international studies for population found out by its survey that the mobility of a population of a state to the village, town and city is in the following percentages.

		To		
		village	Town	city
	village	50%	30%	20%
from	Town	10%	70%	20%
	city	10%	40%	50%

a) Interpret the state transition matrix in terms of
i) Retention & loss ii) Retention & gain.

b) Draw transition Diagram

a) Transition matrix is

$$P = \text{from} \quad \begin{array}{c|c} & \begin{matrix} V & T & C \end{matrix} \\ \begin{matrix} V \\ T \\ C \end{matrix} & \begin{bmatrix} 0.50 & 0.30 & 0.20 \\ 0.10 & 0.70 & 0.20 \\ 0.10 & 0.40 & 0.50 \end{bmatrix} \end{array}$$

↓ Retention & gain

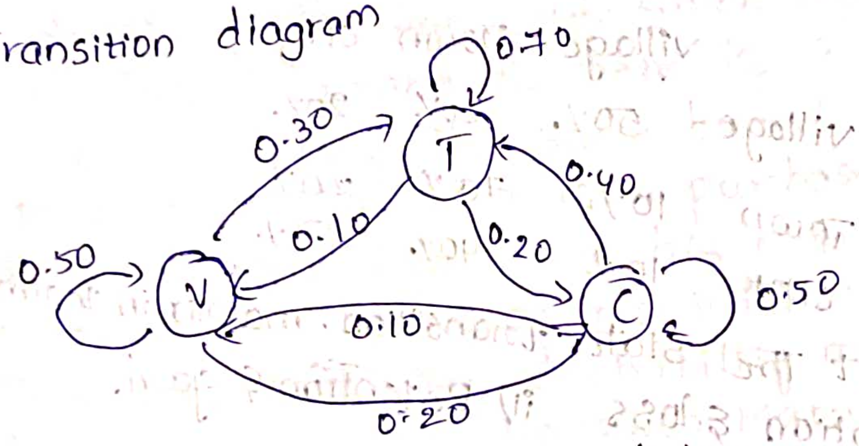
→ Retention

→ Retention & loss

a) i) $P_{11} = 0.50$ Retion to village
$P_{12} = 0.30$ loss to village
$P_{13} = 0.20$ loss to village.
ii) $P_{21} = 0.10$ loss to Town
$P_{22} = 0.70$ ~~loss~~ Retention to town
$P_{23} = 0.20$ loss to town
$P_{31} = 0.10$ loss to city.
$P_{32} = 0.40$ loss to city
$P_{33} = 0.50$ ~~losse to~~ Retention to city.

ii) $P_{11} = 0.50$ Retention to village
$P_{12} = 0.30$ gain to town
$P_{13} = 0.20$ gain to city
$P_{21} = 0.10$ gain to village
$P_{22} = 0.70$ Retention to Town
$P_{23} = 0.20$ gain to city
$P_{31} = 0.10$ gain to village
$P_{32} = 0.40$ gain to town
$P_{33} = 0.50$ Retention to city.

b) Transition diagram



Multi-period Transition probability

R_0 = Initial probability vector of state probability at $n=0$

We have to find.

R_1 = State probability after 1 Execution (or 1 year) (or 1 period)

$R_1 = R_0 \times T.P.M = R_0 \times P$

R_2 = State probability after 2 Executions (or 2 years) (or 2 periods)

$R_2 = R_0 \times P \times P = R_0 \times P^2$ $\boxed{R_2 = R_1 \times P}$

$R_3 = R_0 \times P^3$ $\boxed{R_3 = R_2 \times P}$

||ly R_n = after n years

$R_n = R_0 \times P^n$ (or) $R_n = R_{n-1} \times P$

In a certain Market, only two brands of cold drinks. A & B are sold. Given that a man last purchased brand A, there is 80% chance that he would buy the same brand in the next purchase, while if a man purchased brand B, there is 90% chance that his next purchase would be brand B. Using this information Calculate

(1) The probability that, if a customer is currently a brand A purchaser, he will purchase brand B after two purchases from now.

(2) The probability that if a customer is currently a brand B purchaser, he will purchase brand A after three purchases from now.

Sol/ The transition probability matrix for the given data is

$$P = \text{present } (n=0) \quad \begin{array}{c c} & \text{next}(n=1) \\ & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix} \end{array}$$

1) If customer is currently a brand A purchaser in present purchase at $n=0$ then

$P_{11} = 1$ and $P_{12} = 0$

So Initial vector probability is $R_0 = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}$

after first purchase, the state probability R_1 at $n=1$ [probability after one step]

$R_1 = R_0 \times P$

$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$

$= \begin{bmatrix} 0.8+0 & 0.2+0 \end{bmatrix}$

$R_1 = \begin{array}{c} \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \\ \begin{array}{cc} A & B \end{array} \end{array}$

After second purchase, the state probability R_2 at $n=2$. (probability after two steps)

$R_2 = R_1 \times P$

$$= [0.8 \quad 0.2] \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

$$= [0.64 + 0.02 \quad 0.16 + 0.18]$$

$$R_2 = \begin{array}{cc} A & B \\ \end{array} [0.66 \quad 0.34]$$

Brand A share = 66%

Brand B share = 34%

$\therefore$ The required probability = 0.34

2) If customer is currently a Brand B purchaser in present purchase at $n=0$ then

$$P_{21} = 0 \quad P_{22} = 1 \quad R_0 = [0 \quad 1]$$

- After first purchase the state probability R_1 at $n=1$

$$R_1 = R_0 \times P$$

$$= [0 \quad 1] \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

$$= [0 + 0.1 \quad 0 + 0.9]$$

$$R_1 = \begin{array}{cc} A & B \\ \end{array} [0.1 \quad 0.9]$$

- After second purchase, the state probability R_2 at $n=2$

$$R_2 = R_1 \times P$$

$$= [0.1 \quad 0.9] \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

$$= [0.08 + 0.09 \quad 0.02 + 0.81]$$

$$\cancel{= 0.09 \quad 0.9}$$

$$R_2 = [0.17 \quad 0.83]$$

- After third purchase, the state probability R_3 at $n=3$

$$R_3 = R_2 \times P$$

$$= [0.17 \quad 0.83] \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

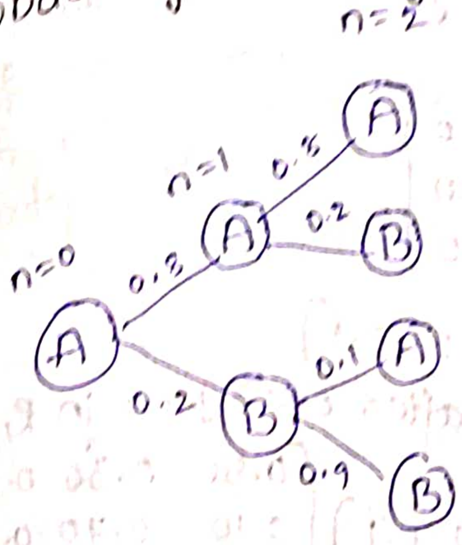
$= [0.136 + 0.083 \quad 0.034 + 0.747]$

$P_3 = [0.219 \quad 0.781]$

$\therefore$ The required probability = 21.9%.

probability tree diagram

1)



B share

$P_{12} = (0.8)(0.2) + (0.2)(0.9)$

$= 0.16 + 0.18$

$= 0.34$

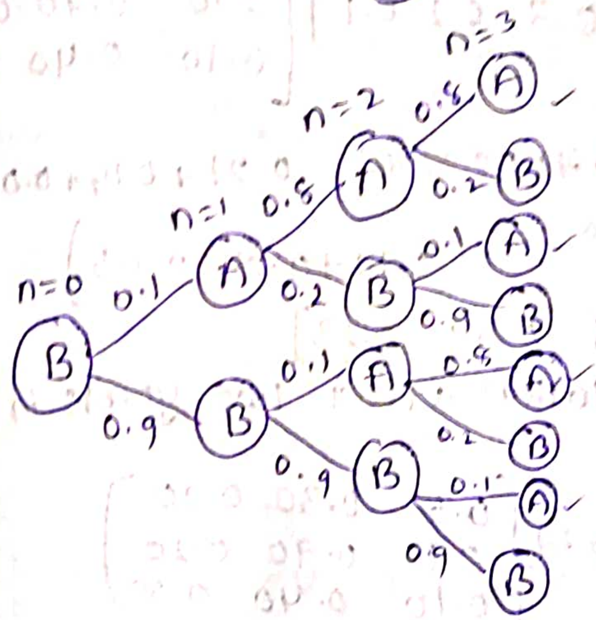
A share

$P_{11} = (0.8)(0.8) + (0.2)(0.1)$

$= 0.64 + 0.02$

$= 0.66$

2)



A share

$P_{21} = (0.1)(0.8)(0.8)$
$+ (0.1)(0.2)(0.1)$
$+ (0.9)(0.1)(0.8)$
$+ (0.9)(0.9)(0.1)$

$= 0.064 + 0.002 + 0.072 + 0.081$

$= 0.219$

B share

$P_{22} = (0.1)(0.8)(0.2) + (0.1)(0.2)(0.9)$
$+ (0.9)(0.1)(0.2) + (0.9)(0.9)(0.9)$

$= 0.781$

2) The School of International studies for population found out by its survey that the mobility of a population of a state to the village, town and city is in the following percentages.

		Village	Town	City (TO)
	Village	50%	30%	20%
from	Town	10%	70%	20%
	City	10%	40%	50%

What will be the proportion of population in village, town and city after two years, given that the present population has proportion of 0.7, 0.2 and 0.1 in the village, town and city respectively.

The transition probability matrix for the given data

$$P = \begin{bmatrix} 0.50 & 0.30 & 0.20 \\ 0.10 & 0.70 & 0.20 \\ 0.10 & 0.40 & 0.50 \end{bmatrix}$$

So initially $R_0 = [0.7, 0.2 \; 0.1]$

The proportion of population after one year in R_1

$$R_1 = R_0 \times P = (0.7 \;\; 0.2 \;\; 0.1) \begin{bmatrix} 0.50 & 0.30 & 0.20 \\ 0.10 & 0.70 & 0.20 \\ 0.10 & 0.40 & 0.50 \end{bmatrix}$$

$$= (0.35 + 0.02 + 0.01 \quad 0.21 + 0.14 + 0.04 \quad 0.14 + 0.04 + 0.05)$$

$$R_1 = [0.38 \quad 0.39 \quad 0.23]$$

The proportion of population after two years in R_2

$$R_2 = R_1 \times P$$

$$= (0.38 \;\; 0.39 \;\; 0.23) \begin{bmatrix} 0.50 & 0.30 & 0.20 \\ 0.10 & 0.70 & 0.20 \\ 0.10 & 0.40 & 0.50 \end{bmatrix}$$

$$= [0.19 + 0.039 + 0.023 \quad 0.114 + 0.273 + 0.092 \quad 0.076 + 0.078 + 0.115]$$

$$R_2 = [\underset{V}{0.252} \quad \underset{T}{0.479} \quad \underset{C}{0.269}]$$

After two years the proportion of population in $V = 25\%$

$T = 48\%$

$C = 27\%$

3) In a town only two brands of LED are availab[le] Syska and Surya. Let each of these two

brands have exactly 50% of the total market
in same period and let market be of a
fixed size. The transition Matrix is given below.

$$\begin{array}{ccc} & & \text{To } [n=1] \\ & & \begin{array}{cc}\text{Syska} & \text{Surya}\end{array} \\ \text{from } n=0 & \begin{array}{c}\text{Syska} \\ \text{Surya}\end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$

If initial market share breakdown is 50% for
each brand, then determine market share in
steady state.

Sol: Given transition probability matrix

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}$$

Since p is a regular matrix, we have

$$VP = V$$

Where v is a probability vector

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$\begin{bmatrix} 0.8v_1 + 0.5v_2 & 0.2v_1 + 0.5v_2 \end{bmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$$

$0.8v_1 + 0.5v_2 = v_1$ $\qquad$ $0.2v_1 + 0.5v_2 = v_2$

$0.5v_2 = 0.2v_1$ $\qquad$ $0.2v_1 = 0.5v_2$

$\cancel{v_2 = \frac{0.2}{0.5}}$ $\qquad$ $0.2v_1 - 0.5v_2 = 0$ $\quad \rightarrow (2)$

$0.2v_1 - 0.5v_2 = 0 \rightarrow (1)$

Since v is a probability vector we have

$$v_1 + v_2 = 1 \rightarrow (3)$$

Solving (1) & (3)

$$(1) \Rightarrow \quad 0.2v_1 - 0.5v_2 = 0$$

$$-0.2v_1 - 0.2v_2 = -0.2$$

$$+0.7v_2 = +0.2$$

$$v_2 = \frac{0.2}{0.7} = \frac{2}{7} = 0.286$$

$(1) \Rightarrow$

$$0.2v_1 - 0.5(0.286) = 0$$

$$0.2v_1 = 0.143$$

$$v_1 = \frac{0.143}{0.2} = 0.715$$

$\therefore$ The steady state vector is $v = \begin{bmatrix} 0.715 & 0.286 \end{bmatrix}$

4) Consider a bike share problem with only 3 stations A, B, C. Suppose that all bikes must be returned to the station at the end of the day, so that all the bikes are at some stations. Each day, the distribution of bikes at each station changes, as the bikes get returned to different stations from where they are borrowed. Of the bikes borrowed from station A, 30% are returned to station A, 50% end up at station B, and 20% end up at station C. Of the bikes borrowed from station B, 10% end up at station A. 60% have been returned to - of the bikes borrowed from station C, 10% end up at station A, 10% end up at station B, and 80% are returned to station C.

i) Express this information as a transition probability matrix and determine the probabilities of bike being at a particular station after two days.

ii) Suppose when we start observing the bike share program, 30% of the bikes are at station A, 45% of the bikes are at station B and 25% are at station C, determine the distribution of bikes at the end of the next day and after two days.

Sol i) Transition probability matrix for the data

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.30 & 0.50 & 0.20 \\ 0.10 & 0.60 & 0.30 \\ 0.10 & 0.10 & 0.80 \end{bmatrix} \end{array}$$

ii) Given initial probability vector

$$R_0 = \begin{bmatrix} 0.30 & 0.45 & 0.25 \end{bmatrix}$$

The distribution of bikes at the end of the day

$R_1 = R_0 \times P$

$= \begin{bmatrix} 0.30 & 0.45 & 0.25 \end{bmatrix} \begin{bmatrix} 0.30 & 0.50 & 0.20 \\ 0.10 & 0.60 & 0.30 \\ 0.10 & 0.10 & 0.80 \end{bmatrix}$

$= [0.09 + 0.045 + 0.025 \quad 0.15 + 0.27 + 0.025 \quad 0.06 + 0.135 + 0.2]$

$R_1 = [0.16 \quad 0.445 \quad 0.395]$

The distribution of bikes ~~at the~~ after Two days

$R_2 = R_1 \times P$

$= \begin{bmatrix} 0.16 & 0.445 & 0.395 \end{bmatrix} \begin{bmatrix} 0.30 & 0.50 & 0.20 \\ 0.10 & 0.60 & 0.30 \\ 0.10 & 0.10 & 0.80 \end{bmatrix}$

$= [0.048 + 0.0445 + 0.0395 \quad 0.08 + 0.267 + 0.0395 \quad 0.032 + 0.1335 + 0.316]$

$R_2 = [0.132 \quad 0.3865 \quad 0.4815]$

5) A market research team has conducted a Survey of customer buying habits with respect to 3 brands of talcum powder in an area. It estimates at present, 20% of the customers by brand A, 50% of the customers by brand B and 30% of the customers by brand C. In addition the team has analyzed its survey and has determined the following brand switching matrix.

Brand just Bought \ Brand Next Bought	A	B	C
A	0.6	0.3	0.1
B	0.4	0.5	0.1
C	0.2	0.1	0.7

Determine the expected distribution of consumers two time periods later.

Sol Given transition probability matrix is

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}$$

Given Initial probability vector is

$$R_0 = [0.2 \quad 0.5 \quad 0.3]$$

The expected distribution of consumer in one time period is

$$R_1 = R_0 \times P$$

$$= (0.2 \quad 0.5 \quad 0.3) \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}$$

$$= [0.12 + 0.20 + 0.06 \quad 0.06 + 0.25 + 0.03 \quad 0.02 + 0.05 + 0.21]$$

$$= [0.38 \quad 0.34 \quad 0.28]$$

The distribution of consumer in two time period is

$$R_2 = R_1 \times P$$

$$= (0.38 \quad 0.34 \quad 0.28) \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}$$

$$= [0.228 + 0.136 + 0.056 \quad 0.114 + 0.17 + 0.028 \quad 0.038 + 0.034 + 0.196]$$

$$R_2 = [0.42 \quad 0.312 \quad 0.268]$$

Markov Analysis Algorithm:-

Markov Analysis is used to provide management with answer to the following questions

a) What is the rate of gains & loses for the current market

b) For a Given future period, what is the projected market share for each firm in the system.

c) Assuming it could exist what is the equilibrium state

The following algorithm provide a procedure for answering these questions regardless of the area of application

step 1:- Using the data of the problem, determine the probabilities associated with retentions, gain & loses

step 2:- Using the data of step 1, develop the state transition matrix.

step 3:- Using the state transition matrix of step 2 determine the expected future market shares for the next period. The procedure is as follows

$$\begin{bmatrix}\text{market shares begining} \\ \text{of period 1}\end{bmatrix}\begin{bmatrix}\text{state transition} \\ \text{matrix}\end{bmatrix} = \begin{bmatrix}\text{expected market} \\ \text{share begining of} \\ \text{period 2}\end{bmatrix}$$

step 4:- If more than one expected market share calculation is required, apply

$$\begin{bmatrix}\text{expected market shares} \\ \text{begining of period k}\end{bmatrix}\begin{bmatrix}\text{state transition} \\ \text{matrix}\end{bmatrix} = \begin{bmatrix}\text{expected market} \\ \text{share begining of} \\ \text{period k+1}\end{bmatrix}$$

(where k is equals the number of time periods under consideration.

Step 5:- Using the state transition matrix of step 2 determine the equilibrium conditions for the current problems

b) Suppose there are 2 market products of brands A & B respectively. let each of these two brands have exactly 50% of the total market in same period and let the market be of a fixed size. The transition matrix is given below.

$$\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \text{from} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} \end{array} \text{ To}$$

If the initial market share ~~is~~ breakdown is 50% for each brand Then determine their market shares in the steady state.

Given transition probability matrix

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

Since P is a regular matrix, We have

$$VP = V$$

Where V is a probability vector

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$\begin{bmatrix} 0.9v_1 + 0.5v_2 & 0.1v_1 + 0.5v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$0.9v_1 + 0.5v_2 = v_1$ $\qquad$ $0.1v_1 + 0.5v_2 = v_2$

$0.5v_2 = 0.1v_1$ $\qquad$ $0.1v_1 = 0.5v_2$

$0.1v_1 - 0.5v_2 = 0$ — (1) $\qquad$ $0.1v_1 - 0.5v_2 = 0$ — (2)

Since v is the probability vector Then

$$v_1 + v_2 = 1 \quad \text{— (3)}$$

from (1) & (3)

$$(1) \Rightarrow \quad \cancel{0.1v_1} - 0.5v_2 = 0$$
$$\cancel{0.1v_1} + 0.1v_2 = 0.1$$
$$\overline{-0.6v_2 = -0.1}$$

$$v_2 = \frac{0.1}{0.6} = 1/6 = 0.166$$

from (1) $\Rightarrow$ $0.1v_1 - 0.5v_2 = 0$

$$0.1v_1 - 0.5(0.166) = 0$$

$$0.1v_1 - 0.083 = 0$$

$$0.1v_1 = 0.083$$

$$v_1 = 0.834$$

$\therefore$ The steady state vector is $v = \begin{bmatrix} 0.834 & 0.166 \end{bmatrix}$

In a certain ... is sunny, tomorrow will be sunny 80% of the time. If today is cloudy, Tomorrow will be cloudy 60% of the time. Supposing today is sunny, What is the probability it will be cloudy the day after tomorrow?

given
The transition probability matrix is

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

The Initial vector $R_0 = [1 \quad 0]$

The probability it will be cloudy after one day is

$$R_1 = R_0 \times P$$

$$= (1 \quad 0) \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= [0.8+0 \quad 0.2+0]$$

$$R_1 = [0.8 \quad 0.2]$$

The probability it will be cloudy after two days is

$$R_2 = R_1 \times P$$

$$= [0.8 \quad 0.2] \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= [0.64+0.08 \quad 0.16+0.12]$$

$$R_2 = [0.72 \quad 0.28]$$

8) Suppose that new brand of biscuits were introduced in the market by 3 companies at the same time. When they were introduced, each company had an equal share of the market, but during the 1st year the following changes took place.

i) company A retained 90% of its customers, loss 3% to B and 7% to C.

ii) Company B retailed 70% of its customers, loss 10% to A and 20% to C.

iii) Company C retained 80% of its customers, loss 10% to A and loss 10% to B.

Assuming no changes in the buying habits of the consumer occur.

i) What are long run market shares of the 3 companies.

ii) What are the market shares of the 3 companies at the end of first and second years.

Hint:- Initial vector $R_0 = \begin{array}{ccc} & & \\ [0.333 & 0.333 & 0.333] \\ A & B & C \end{array}$

Sol 9) On January 1st (this year), bakery A had 40% of its local market share while the other two bakeries B and C had 40% and 20% respectively. of the market share. Based upon a study by a marketing research firm, the following facts were ~~combined~~ compiled. Bakery A retains 90% of its customers while gaining 5% of competetor B's customers and 10% of C's customers. Bakery B retains 85% of its customers while gaining 5% of A's customers and 7% of C's customers. Bakery C retains 83% of its customers and gains 5% of A's customers and 10% of B's customers.

What will each firm's share be on January 1, next year, and what will each firms share be on January 1, next year and what will each firm's market share be at equilibrium?

Sol Given Transition probability matrix

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0.90 & 0.05 & 0.05 \\ B & 0.05 & 0.85 & 0.10 \\ C & 0.10 & 0.07 & 0.83 \end{array}$$

Given Initial vector $R_0 = [0.4 \quad 0.4 \quad 0.2]$

The probability of each firms share be on January 1 is R_1

$R_1 = R_0 \times P$

$$= \begin{bmatrix} 0.4 & 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 0.90 & 0.05 & 0.05 \\ 0.05 & 0.85 & 0.10 \\ 0.10 & 0.07 & 0.83 \end{bmatrix}$$

$$= [0.36 + 0.020 + 0.02 \quad 0.020 + 0.34 + 0.014 \quad 0.020 + 0.04 + 0.166]$$

$$R_1 = [0.40 \quad 0.374 \quad 0.226]$$

The probability of each firms share be on next year R_2

$R_2 = R_1 \times P$

$$= \begin{bmatrix} 0.40 & 0.374 & 0.226 \end{bmatrix} \begin{bmatrix} 0.90 & 0.05 & 0.05 \\ 0.05 & 0.85 & 0.10 \\ 0.10 & 0.07 & 0.83 \end{bmatrix}$$

$$= [0.045 + 0.0187 + 0.0226 \quad 0.02 + 0.3179 + 0.01582 \quad 0.02 + 0.0374 + 0.18758]$$

$$R_2 = [0.0863 \quad 0.35372 \quad 0.24498]$$

Since p is a regular matrix, We have

$$vp = v$$

Where v is a probability vector

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 0.90 & 0.05 & 0.05 \\ 0.05 & 0.85 & 0.10 \\ 0.10 & 0.07 & 0.83 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$[0.90v_1 + 0.05v_2 + 0.10v_3 \quad 0.05v_1 + 0.85v_2 + 0.07v_3 \quad 0.05v_1 + 0.10v_2 + 0.83v_3] = [v_1 \quad v_2 \quad v_3]$$

$0.90v_1 + 0.05v_2 + 0.10v_3 = v_1$

$-0.10v_1 + 0.05v_2 + 0.10v_3 = 0 \longrightarrow (1)$

$0.05v_1 + 0.85v_2 + 0.07v_3 = v_2$

$0.05v_1 - 0.15v_2 + 0.07v_3 = 0 \longrightarrow (2)$

$0.05v_1 + 0.10v_2 + 0.83v_3 = v_3$

$0.05v_1 + 0.10v_3 - 0.17v_3 = 0 \longrightarrow (3)$

Since v is the probability vector then

$v_1 + v_2 + v_3 = 1$ — (4)

Solving ~~(1)~~ & (2) ~~(3)~~ (4)

$\cancel{-0.10v_1 + 0.05v_2 + 0.10v_3 = 0}$

$0.05v_1 - 0.15v_2 + 0.07v_3 = 0$

$v_1 + v_2 + v_3 = 1$, $0.05v_1 + 0.10v_3 - 0.17v_3 = 0$

Which is in $AV = B$

Augmented matrix $AB = \begin{bmatrix} 0.05 & -0.15 & 0.07 & 0 \\ 0.05 & 0.10 & -0.17 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.05 & -0.15 & 0.07 & 0 \\ 0.05 & 0.10 & -0.17 & 0 \end{bmatrix}$

$R_2 \to R_2 - 0.05R_1$

$R_3 \to R_3 - 0.05R_1$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -0.2 & 0.02 & -1 \\ 0 & 0.05 & -0.22 & -1 \end{bmatrix}$

0.044
0.001

~~$R_3 \to 0.0R_3 \; 0.05$~~

$R_3 \to -0.2R_3 - 0.05R_2$

$= \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & -0.2 & 0.02 & -1 \\ 0 & 0 & 0.043 & -1 \end{matrix}$