UNIT - II

Set Theory (George cantor)

* set theory was developed by George canter.

Set: A well defined collection of objects (or) an ideas is known as set

* well defined means that

i. All the objects in the set should have a common feature in the property.

ii. It should be possible to decide whether any given object belongs to the set or not.

Note:

Given a set 's' and object 'p' exactly one of the statement following statements should be true (i) $p \in s$

(ii) $p \notin s$ (does not belongs to)

Representation of a set:

A set can be representation by 2 methods.

① Rule method (or) set Builder method.

② Roster method (or) Tabulation.

① Rule method

In this method the set is represented by a statement (or) a rule (or) a common property which all the members of the set posses that is if the common property possessed by all the elements in 'p', the set is represented in the rule method by $S = \{ x : x$ has the property p. this is read as "the set of all elements x such that x has property p" }

Ex:-

i. If P is the set of all prime numbers, then

$P = \{x;\ x \text{ is a prime number}\}$

ii. $A = \{1, 2, 5, 10, 25\}$ can be written as $A = \{x;\ x$ is a positive diviser of 50 which is less than $50\}$

iii. If A is the set of all natural numbers b/w 10 and 100, then $A = \{x;\ x \in N \text{ and } 10 < x < 100\}$

② **Roster Method**

In this method the elements of the given set are enlisted and enclosed in brackets (curly brackets).

Ex:

i. If $A = \{a, b, c, 1, 2, 3\}$ the where

$A = \{$ set of letters a, b, c and numbers $1, 2, 3\}$

ii. If A is the set of letters of the word "MATHEMATICS", then in roaster form

$A = \{M, A, T, H, E, M, A, T, I, C, S\}$

Null set (or) Empty set

A set which does not contain any element is called an empty set (or) a null set. An Empty set will be denoted by ϕ (or) $\{\}$

Ex:- A set of all even numbers b/w 2, 4

Finite & Infinite Set:

A set is said to be finite if it has a finite number of element otherwise it is said to be infinite.

Ex:- ① The set of vowels in english alphabet is a finite set

ii. The set of natural numbers is an infinite set.

Singleton set:

A set containing a single elements is called a singleton set.

Ex:- $\{1\}, \{a\}, \{\phi\}, \{\pi\}$

Cardinality of a finite set

The number of elements in a finite set 'A' is called its cardinal number and is denoted by $n(A)$ (or) $|A|$

* If $A = \{1, 2, 3, 4\}$

$$n(A) = 4$$

Complement of a set:

Let 'U' be the universal set and $A \subset U$. Then The complement of A is denoted by A^c symbolically $A^c = U - A$

* $A^c = \{x : x \in U \ \& \ x \notin A\}$

$$A^c = \{x : x \notin A\}$$

Ex:- If U is a set of natural numbers and A is the set of even numbers then $A^c = U - A =$ = the set of natural numbers which are not even.

* A^c = The set of odd natural numbers.

Equal sets:

Two sets A & B is said to be equal if they have the same elements, then we write $A = B$.

Note: Two sets A & B are equal $A \subseteq B$ and $B \subseteq A$.

we write $A = B$

Ex: $A = \{1, 4, 5, 7\}$ $B = \{1, 4, 5, 7\}$

$\boxed{A = B}$

Sub Sets:

A set A is a subset of the set B, if and only if every element of A is also an element of B. we also say that A is contained B and use the notation $A \subseteq B$.

symbolically $x \in A \Rightarrow x \in B$ then $A \subseteq B$ otherwise we write $A \not\subseteq B$.

proper set:

* A set A is called proper set of the set B.

If (i) A is subset of B

(ii) B is not a subset of A

i.e. A is said to be proper set of B is every element of $A \in B$ (belongs to set B) but their is atleast one element of B which is not on A.

denoted $\boxed{A \subset B}$

Super set:

If A is a subset of B, then B is called a super set of A.

Examples:

Subset

$A = \{1,2,3\}$ $B = \{0,2,1,3,7\}$

proper set

$A = \{1,2,3\}$ $B = \{1,2,3,4,5,6\}$

super set

$A = \{1,2,3\}$ $B = \{1,2,3,4,5,6\}$, A is a proper subset of B, and B is a super subset of A.

Power set:

For a set A, a collection (or) family of all subsets of A is called powerset of A.

* The powerset of A is denoted by $P(A)$ (or) 2^A

* $P(A)$ (or) $2^A = \{x : x \subseteq A\}$

Ex:- For a set $S = \{a\}$; $P(S) = \{\phi, \{a\}\} = \{\phi, \{S\}\}$

$S = \{a, b\}$, $P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\} = \{\phi, \{a\}, \{b\}, \{S\}\}$

Universal set:

A set 'U' which contains all the sets under consideration as subsets is called a universal sets.

Ex:- In plane geometry the universal set consist of all the points in the plane.

Disjoint set

Two sets are said to be disjoint, if they have no element in common.

Ex:- $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$ then A & B are disjoint $A \cap B = \phi$

Ordered set:

A set which an ordering imposed on it is said to be an ordered set.

Ex:- $S = \{2, 3, 6, 8, \ldots\}$

Same operations on set

Intersection:

The intersection of any two sets A and B written as $A \cap B$, is the set consisting of all the elements which belongs to both A and B

* Symbollically $A \cap B = \{x : x \in A \wedge x \in B\}$

* For any sets A and B, $A \cap B = B \cap A$

$A \cap A = A$

$A \cap \phi = \phi$, $(A \cap B) \cap C = A \cap (B \cap C)$.

① If $A_1 = \{\{1,2\},\{3\}\}$, $A_2 = \{\{1\},\{2,3\}\}$ & $A_3 = \{\{1,2,3\}\}$ Then show that A_1, A_2 and A_3 are mutually disjoint.

Sol

$A_1 \cap A_2 = \phi$

$A_1 \cap A_3 = \phi$

$A_2 \cap A_3 = \phi$ the given sets are disjoint.

② Show that $A \subseteq B \Leftrightarrow A \cap B = A$

Sol For any x, $\boxed{P \rightarrow Q \Leftrightarrow ((P \wedge Q) \rightleftarrows P))}$

$x \in A \rightarrow x \in B \Leftrightarrow (x \in A \wedge x \in B \rightleftarrows x \in A)$

$A \subseteq B \Leftrightarrow (x)(x \in A \rightarrow x \in B) [\because x : x \in A \rightarrow x \in B]$

$A \cap B = A \Leftrightarrow (x)(x \in A \wedge x \in B) \rightleftarrows x \in A)$

Union:

* For any two sets A and B, the union of A and B is written as "$A \cup B$" is the set of all elements which are members are the set A (or) the set B (or) both.

* symbolically $A \cup B = \{x : x \in A \vee x \in B\}$

$A \cup B = B \cup A$ (Commutative)

$A \vee \phi = A$

$A \cup A = A$

$(A \cup B) \cup C = A \cup (B \cup C)$

① Show that $A \cup A = A$

Sol For any x

$x \in (A \cup A) \Leftrightarrow x \in \{x : x \in A \cup x \in A\}$

$\Leftrightarrow x \in A \cup x \in A$

$\Leftrightarrow x \in A$

$\Leftrightarrow x \in \{x : x \in A\}$

$\therefore A \cup A = A$. $\Leftrightarrow x \in A$

(2) If $A_1 = \{1,2\}$, $A_2 = \{2,3\}$ & $A_3 = \{1,2,3,6\}$ what are $\bigcup_{i=1}^{3} A_i$ & $\bigcap_{i=1}^{3} A_i$

Sol Given sets $A_1 = \{1,2\}$, $A_2 = \{2,3\}$, $A_3 = \{1,2,3,6\}$

$$\bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3$$
$$= \{1,2,3,6\}$$

$$\bigcap_{i=1}^{3} A_i = A_1 \cap A_2 \cap A_3$$

$$\bigcap_{i=1}^{3} A_i = \{2\}$$

Relative complement (or) Difference

Let A and B be any two sets the relative complement of B in A is written as A-B, is the set consisting all elements of A which are not elements of B that is $A-B = \{x : x \in A \wedge x \notin B\}$ (or) $\{x : x \in A \wedge \neg(x \in B)\}$

1. If $A = \{2,5,6\}$, $B = \{3,4,2\}$, $C = \{1,3,4\}$ find A-B, B-A show that $A-B \neq B-A$ and $A-C = A$

Sol Given $A = \{2,5,6\}$, $B = \{3,4,2\}$, $C = \{1,3,4\}$

$$A-B = \{2,5,6\} - \{3,4,2\}$$
$$= \{5,6\}$$

$$B-A = \{3,4,2\} - \{2,5,6\}$$
$$= \{3,4\}$$

$$A-B \neq B-A$$

$$A-C = \{2,5,6\} - \{1,3,4\}$$
$$= \{2,5,6\}$$

$$\boxed{A-C=A}$$

② show that (i) $A-B = A\cap B^c$ (ii) $A \subseteq B \Leftrightarrow B^c \subseteq A^c$.

sol. For any x,

$x \in (A-B) \Leftrightarrow x \in \{x : x\in A \cap x \notin B\}$

$\Leftrightarrow x\in A \cap x \notin B$

$\Leftrightarrow x \in A \cap x \in B^c$

$\Leftrightarrow x \in (A\cap B^c)$

$\therefore$ i.e $A-B = A\cap B^c$

ii. for any x,

$(A \subseteq B) \Leftrightarrow (x)\ (x\in A \rightarrow x\in B)$

$\boxed{P\rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P}$

$\Leftrightarrow (x)\ (\neg(x\in A) \rightarrow \neg(x\in A))$

$\Leftrightarrow (x)\ (x\notin B \rightarrow x\notin A)$

$\Leftrightarrow (x)\ (x\in B^c \rightarrow x\in A^c)$

$\boxed{(A \subseteq B) \Leftrightarrow B^c \subseteq A^c}$

3. show that for any two sets A and B

$A-(A\cap B) = A-B$.

sol For any x

$x \in (A-(A\cap B)) \Leftrightarrow x \in \{x: x\in A \cap x \notin (A\cap B)\}$

$\Leftrightarrow x\in A \cap x \notin (A\cap B)$

$\Leftrightarrow x\in A \cap \neg(x\in (A\cap B))$

$\Leftrightarrow x\in A \cap \neg(x\in A \cap x\in B)$

$\Leftrightarrow (x\in A) \cap (x\notin A \vee x \notin B)$

using distributive law.

$\Leftrightarrow (x\in A \cap x\in A) \vee (x\in A \cap x\notin B)$

$\Leftrightarrow F \cap (x\in A \cap x\notin B)$ $\boxed{P\vee F \Leftrightarrow P}$

$\Leftrightarrow x\in A \quad x\notin B$

$\Leftrightarrow x\in\{x : x\in A \cap x\notin B\}$

$\Leftrightarrow \quad x \in A \cap x \notin B$

$\Leftrightarrow \quad x \in A - B$

$A - (A \cap B) = A - B$

Symmetric Difference (or) Boolean sum

Let A and B be any two sets. The symmetric difference of A and B is the set $A + B$ defined by $\boxed{A+B = (A-B) \cup (B-A)}$

(or.) $(A \cap B^c \cup B \cap A^c)$

* symbolically $x \in A+B \Leftrightarrow x \in \{x : x \in A \,\bar{\vee}\, x \in B\}$
where $\bar{\vee}$ is the exclusive disjunction.

$A+B = B+A$

$(A+B)+C = A+(B+C)$

$A+\phi = A \quad , \quad A+A = \phi$

1. prove that $A+\phi = A$

Sol For any x

$x \in (A+\phi) \Leftrightarrow x \in \{x : (x \in A \wedge x \notin \phi) \vee (x \in \phi \wedge x \notin A)\}$

$\Leftrightarrow (x \in A \cap x \in \phi) \vee (x \in \phi \cap x \notin A)$ $\quad \boxed{x \in \phi = F}$

$\Leftrightarrow (x \in A \cap T) \vee (F \cap x \notin A)$ $\quad \boxed{x \notin \phi = T}$

$\Leftrightarrow (x \in A) \vee F$

$\Leftrightarrow x \in A$ $\quad \boxed{P \wedge T = P}$

$\Leftrightarrow x \in \{x : x \in A\}$ $\quad \boxed{P \vee T = T}$

$\Leftrightarrow x \in A$

$\boxed{P \cap F = F}$

$\therefore A+\phi = A.$ $\quad \boxed{P \vee F = P}$

2. Given $A = \{x : x \text{ is an integer and } 1 \leq x \leq 5\}$, $B = \{3, 4, 5, 17\}$ & $C = \{1, 2, 3, \ldots\}$ find $A \cap B$, $A \cap C$, $A \cup B$ & $A \cup C$.

Sol Given $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 17\}$

$C = \{1, 2, 3, \ldots\}$

$A \cap B = \{3,4,5\}$

$A \cap C = \{1,2,3,4,5\} = A$

$A \cup B = \{1,2,3,4,5,17\}$

$A \cup C = \{1,2,3,4,5, \cdots\} = C$

3. Given $A=\{2,3,4\}$, $B=\{1,2\}$ & $C=\{4,5,6\}$, find $A+B$, $B+C$, $A+B+C$ & $(A+B)+B+C$.

Sol Given $A=\{2,3,4\}$, $B=\{1,2\}$; $C=\{4,5,6\}$

$A+B = (A-B) \cup (B-A)$

$= (\{2,3,4\}-\{1,2\}) \cup (\{1,2\}-\{2,3,4\})$

$= \{3,4\} \cup \{1\}$

$\boxed{A+B = \{1,3,4\}}$

$B+C = (B-C) \cup (C-B)$

$= (\{1,2\}-\{4,5,6\}) \cup (\{4,5,6\}-\{1,2\})$

$= \{1,2\} \cup \{4,5,6\}$

$\boxed{B+C = \{1,2,4,5,6\}}$

$(A+B)+C = ((A+B)-C) \cup (C-(A+B))$

$= (\{1,3,4\}-\{4,5,6\}) \cup (\{4,5,6\}-\{1,3,4\})$

$= \{1,3\} \cup \{5,6\}$

$\boxed{A+B+C = \{1,3,5,6\}}$

$(A+B)+(B+C) = \{(A+B)-(B+C)\} \cup \{(B+C)-(A+B)\}$

$= (\{\cancel{1},3,\cancel{4}\}-\{\cancel{1},\cancel{2},\cancel{4},5,6\}) \cup (\{1,2,4,5,6\}-\{\cancel{1},3,\cancel{4}\})$

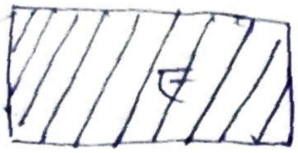
$= \{3\} \cup \{2,5,6\}$

$\boxed{(A+B)+(B+C) = \{3,2,5,6\}}$

Veen Diagrams

The universal set permits the use of a pictorial device to study the connections b/w the subsets of a universal set and their intersection union, difference and other operation

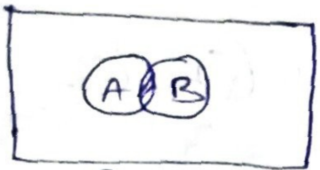
* A veen diagram is a schematic representation of a set by a set of points. The universal set E is represented by a set of points in a rectangle and a subset say 'A' of 'E' is represented by the interior of a circle (or) some other simple closed curve inside the rectangle.



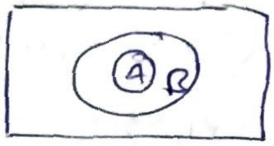
E

A

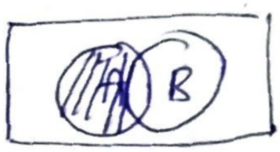
$A \cup B$



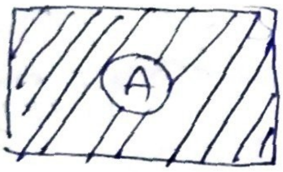
$A \cap B$



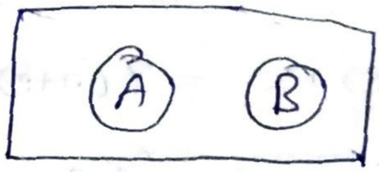
$A \subseteq B$



$A - B$



$A^c = U - A$



$A \cap B = \phi$

ordered pairs and n-tuples

An ordered pair consists of two objects in a given fixed order. Note that an ordered pair is not a set consisting two elements the ordering of the two objects is important the two objects need to be distinct. we shall denote an order pair by $\langle x, y \rangle$.

* A familiar example of an ordered pair is the representation of a point in a two dimensional plane in cartisian coordinates. Accordingly the ordered pairs $\langle 1,2\rangle$, $\langle 2,4\rangle$, $\langle 1,3\rangle$, $\langle 4,5\rangle$ represent different points in a plane.

* The equality of two ordered pairs $\langle x,y\rangle$ and $\langle u,v\rangle$ is defined by $\langle x,y\rangle = \langle u,v\rangle \Leftrightarrow ((x=u) \wedge (y=v))$.

* so that $\langle 1,2\rangle \neq \langle 2,1\rangle$ and $\langle 1,1\rangle \neq \langle 2,2\rangle$.

* order thriple $\langle\langle x,y\rangle, z\rangle = \langle\langle u,v\rangle, w\rangle \Leftrightarrow$
$$((x=u) \wedge (y=v) \wedge (z=w))$$

* order 'n' tuple $\langle\langle x, x_2, \ldots x_{n-1}\rangle, x_n\rangle =$
$$\langle\langle u_1, u_2 \ldots u_n\rangle, u_n\rangle \Leftrightarrow$$
$$((x_1=u_1) \wedge (x_2=u_2) \wedge (x_3=u_3) \wedge \ldots \wedge (x_n=u_n))$$

cartisian products

Let 'A' and 'B' be any two sets. The set of all ordered pairs such that the first member of the ordered pair is an element of 'A' and the second members is an element of B is called the cartisian product of A & B is written as $A \times B$

$$A \times B = \{\langle x,y\rangle : (x \in A) \wedge (y \in B)\}$$

1. If $A = \{\alpha, \beta\}$ & $B = \{1,2,3\}$ what are $A\times B$, $B\times A$, $A\times A$, $B\times B$, $(A\times B) \cap (B\times A)$.

sol Given $A = \{\alpha, \beta\}$, $B = \{1,2,3\}$

$A \times B = \{\alpha, \beta\} \times \{1,2,3\}$

$= \{\langle\alpha,1\rangle, \langle\alpha,2\rangle, \langle\alpha,3\rangle, \langle\beta,1\rangle, \langle\beta,2\rangle, \langle\beta,3\rangle\}$

$B \times A = \{1,2,3\} \times \{\alpha, \beta\}$

$= \{\langle 1,\alpha\rangle, \langle 1,\beta\rangle, \langle 2,\alpha\rangle, \langle 2,\beta\rangle, \langle 3,\alpha\rangle, \langle 3,\beta\rangle\}$

$B\times B = \{1,2,3\} \times \{1,2,3\}$

$= \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle,$
$\langle 3,2\rangle, \langle 3,3\rangle\}$

$A\times A = \{\alpha,\beta\} \times \{\alpha,\beta\}$

$= \{\langle\alpha,\alpha\rangle, \langle\alpha,\beta\rangle, \langle\beta,\alpha\rangle, \langle\beta,\beta\rangle\}$

$(A\times B) \cap (B\times A) = \{\}$

2 If $A = \phi$ & $B = \{1,2,3\}$ what are $A\times B$ & $B\times A$?

sol Given $A = \phi$
$B = \{1,2,3\}$

$A\times B = \{\phi\} \times \{1,2,3\}$

$A\times B = \phi$

$B\times A = \phi$

$A\times B = B\times A = \phi$

cartisian product more than two sets

$(A\times B)\times C = \{\langle\langle a,b\rangle, c\rangle : \{(\langle a,b\rangle \in A\times B) \wedge (c\in C)\}$

$= \{\langle a,b,c\rangle : a\in A \wedge b\in B \wedge c\in C\}$

* write $A\times B$, $A\times B\times C$, B^2, A^3 & $B^2\times A$ where $A = \{1\}$, $B = \{a,b\}$ & $C = \{2,3\}$

sol Given $A = \{1\}$
$B = \{a,b\}$
$C = \{2,3\}$

$A\times B = \{1\} \times \{a,b\}$

$= \{\langle 1,a\rangle, \langle 1,b\rangle\}$

$(A \times B) \times C = \{1\} \times \{a,b\} \times \{2,3\}$ (or) $\{\langle 1,a\rangle, \langle 1,b\rangle\} \times \{2,3\}$

$= \{\langle 1,a,2\rangle, \langle 1,a,3\rangle, \langle 1,b,2\rangle, \langle 1,b,3\rangle\}$

$B^2 = B \times B$

$= \{a,b\} \times \{a,b\}$

$= \{\langle a,a\rangle, \langle a,b\rangle, \langle b,a\rangle, \langle b,b\rangle\}$

$A^3 = A \times A^2$

$A^2 = \{1\} \times \{1\}$

$= \{\langle 1,1\rangle\}$

$= \{1\} \times \{1\} \times \{1\}$

$= \{\langle 1,1,1\rangle\}$

$B^2 \times A = \{\langle a,a\rangle, \langle a,b\rangle, \langle b,a\rangle, \langle b,b\rangle\} \times \{1\}$

$= \{\langle a,a,1\rangle, \langle a,b,1\rangle, \langle b,a,1\rangle, \langle b,b,1\rangle\}$

Relation and ordering

Relations

The word relation suggests some familiar examples of relations such as the relation of father to son, mother to son, brother to sister etc. familiar examples in arithmetic are relations such as greater than, less than (or) that of equality b/w two numbers

We also know the relation b/w the area of the circle and its radius, and b/w the area of square and its sides. This examples suggest relations b/w two objects.

* The relation b/w parent & child the coincidence of three lines and that a point lying b/w two points are examples of relations among three objects. Similar example exist for relations among 4 (or) more objects.

Binary Relation

Any set of ordered pairs defines an binary relations.

* We shell call a binary relation simply a relation. It is some times convineant to express the fact that a particular order pair say $\langle x, y\rangle \in R$.
where 'R' is a relation by written xRy - which may be read as "x is in Relation R to y".

* In mathematics relations are often denoted by special symbols rather than by capital letters. A familiar example is the relation greater than for real numbers. This relation is denoted by '>'.

* In 'a & b' are two real numbers such that 'a > b' then we say that $\langle a, b\rangle \in >$ (or) $a > b$.

* More preciously relation greater than is

$> = \{\langle x, y\rangle ; x, y$ are real numbers and $x > y\}$.

* similarly

$F = \{\langle x, y\rangle ;$ x is the father of $y\}$

$S = \{\langle 2, 4\rangle, \langle 1, 3\rangle, \langle 7, 6\rangle, \langle \text{Joan}, 4\rangle\}$

where F & S are relations.

Domain & Ranges

Let 'S' be a binary relation the set $D(S)$ of all objects 'x' such that for some 'y', $\langle x, y\rangle \in S$ is called the domain of 'S' i.e $D(S) = \{x : (\exists y)(\langle x, y\rangle \in S)\}$
Similarly the set $R(S)$ of all objects 'y' such that for some 'x', $\langle x, y\rangle \in S$ is called the range of 'S' $R(S) = \{y : (\exists x)(\langle x, y\rangle \in S)\}$

Ex:- $S = \{\langle 2,4\rangle, \langle 5,6\rangle, \langle 7,8\rangle\}$

$D(S) = \{2,5,7\}$

$R(S) = \{4,6,8\}$

1. Let $X = \{1,2,3,4\}$ if $R = \{\langle x,y\rangle;\ x \in X \cap y \in X \wedge (x-y)$ is an internal non-zero multiple of $2)\}$

$S = \{\langle x,y\rangle / x \in X \cap y \in X \cap (x-y)$ is an integral non-zero multiple of $3)\}$.

Find $R \cup S$ & $R \cap S$.

Sol Given $X = \{1,2,3,4\}$

$R = \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$

$S = \{\langle 1,4\rangle, \langle 4,1\rangle\}$

$R \cup S = \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle, \langle 1,4\rangle, \langle 4,1\rangle\}$

$R \cap S = \phi$

2. Let 'L' denote the relation "less than or equal to" and 'D' denote the relation "divides", $x\,D\,y$ means "x divides y". Both 'L' and 'D' are define on the set $\{1,2,3,6\}$. write "L" and "D" as sets and find "$L \cap D$".

Sol Given set is $\{1,2,3,6\}$

$L = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,6\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,6\rangle,$
$\langle 3,6\rangle, \langle 3,3\rangle, \langle 6,6\rangle\}$

$D = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,6\rangle, \langle 2,6\rangle, \langle 3,6\rangle,$
$\langle 2,2\rangle, \langle 3,3\rangle, \langle 6,6\rangle\}$

$L \cap D = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,6\rangle, \langle 2,6\rangle, \langle 3,6\rangle,$
$\langle 2,2\rangle, \langle 3,3\rangle, \langle 6,6\rangle\} = D$

3. Let $P = \{<1,2>, <2,4>, <3,3>\}$ and $Q = \{<1,3>, <2,4>, <4,2>\}$ find $P \cup Q$, $D(P)$, $D(Q)$, $D(P \cup Q)$, $R(P)$, $R(Q)$ & $R(P \cap Q)$ and also show that: $D(P \cup Q) = D(P) \cup D(Q)$, $R(P \cap Q) \subseteq R(P) \cap R(Q)$

Sol Given $P = \{<1,2>, <2,4>, <3,3>\}$,

$Q = \{<1,3>, <2,4>, <4,2>\}$

$P \cup Q = \{<1,2>, <2,4>, <3,3>, <1,3>, <4,2>\}$

$D(P) = \{1,2,3\}$

$D(Q) = \{1,2,4\}$

$D(P \cup Q) = \{1,2,3,4\}$

$R(P) = \{2, 4, 3\}$

$R(Q) = \{3, 4, 2\}$

$P \cap Q = \{<2,4>\}$

$D(P \cap Q) = \{2\}$

$D(P \cup Q) = \{1, 2, 3, 4\}$

$D(P) \cup D(Q) = \{1,2,3,4\}$

$\boxed{D(P \cup Q) = D(P) \cup D(Q)}$

$R(P \cap Q) = \{4\}$

$R(P) \cap R(Q) = \{4,2,3\}$

$R(P \cap Q) \subseteq R(P) \cap R(Q)$

$\{4\} \subseteq \{2, 3, 4\}$ ($\subseteq$ = subset)

Properties of Binary Relations in a sets:

Reflexive: A binary relation a ref in a set 'x' is reflexive if for every $x \in X$, xRx, $<x,x> \in R$. (or) R is reflexive in x

$\Leftrightarrow (x)(x \in X \rightarrow xRx)$

* The Relations $\leq$ its reflexive in the set of real numbers.
* since for any 'x' we have $x \leq x$.
* similarly the relation of inclusion is reflexive in the family of all subsets of universal sets. The relation of equality of sets is also reflexive.
* The relation '<' is no reflexive in the set of real number,

Symmetric:

A Relation 'R' in a set 'x' is symmetric for every x and y in 'x' whenever xRy then yRx i.e R is symmetric in $x \Leftrightarrow (x)(y)$ $(x \in x \cap y \in x \cap xRy \to yRx)$

* The Relations '$\leq$' and '<' are not symmetric in the set of real numbers, while the relation of equality is the relation of symilarity in the set of triangles in a plane is both reflexive and symmetric.

Transitive: The Relation 'R' in a set 'x' is transitive if for every x, y and z in 'x': whenever xRy, yRz, then xRz.

* i.e 'R' is transitive in 'x', wherever $\Leftrightarrow (x)(y)(z)$ $(x \in x \cap y \in x \cap z \in x \cap xRy \cap yRz \to xRz)$
* The relations '$\leq$', '<', '=' are transitive in the set of real numbers. The relations '$\subseteq$, $\subset$' and equality are also transitive in the family of subsets of a universal set.

Irreflexive: A relation 'R' in a set 'x' is irreflexive for every $x \in x$, $\langle x, x \rangle \notin R$.

* Note that any relation which is not reflexive is not necessarily irreflexive & viceversa.

* The relation '<' in the set of real numbers is irreflexive.

Anti symmetric:

A Relation 'R' in a set 'X' is anti symmetric if for every x & y in 'X', whenever xRy, yRx then x = y.

* symbolically 'R' is anti symmetric in 'X' if and only if ($\Leftrightarrow$) $(x)(y)(x \in X \cap y \in X \cap xRy \cap yRx \rightarrow x = y)$

Equivalence Relation:

A Relation 'R' on a set 'A' is an equivalance relation if and only if ($\Leftrightarrow$) (i) R is reflexive

i. for every $a \in A$, $\langle a, a \rangle \in R$.

ii. R is symmetric

For every $a, b \in A$, $\langle a, b \rangle \in R \Rightarrow \langle b, a \rangle \in R$

iii. R is transitive

$\langle a, b \rangle \in R$, $\langle b, c \rangle \in R \Rightarrow \langle a, c \rangle \in R$

Ex:- $A = \{a, b, c\}$, $R = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle, \langle c, b \rangle, \langle b, a \rangle, \langle c, a \rangle\}$ is an equivalance relation.

1. Let $A = \{1, 2, 3, 4\}$ and $R = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 1 \rangle, \langle 3, 3 \rangle, \langle 1, 3 \rangle, \langle 4, 1 \rangle, \langle 4, 4 \rangle\}$ is and R equivalance relation? why?

Sol Given $A = \{1, 2, 3, 4\}$

$R = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 1 \rangle, \langle 3, 3 \rangle, \langle 1, 3 \rangle, \langle 4, 1 \rangle, \langle 4, 4 \rangle\}$

i. Reflexive: For every $a \in A$, $\langle a, a \rangle \in R$

$1 \in A, \langle 1,1\rangle \in R, 2 \in A, \langle 2,2\rangle \in R$

$3 \in A, \langle 3,3\rangle \in R,$

$4 \in A, \langle 4,4\rangle \in R.$

Symmetric:

for every $a, b \in A$, $\langle a,b\rangle \in R$ then $\langle b,a\rangle \in R$.

If $\langle 1,2\rangle \in R \Rightarrow \langle 2,1\rangle \in R$.

If $\langle 2,1\rangle \in R \Rightarrow \langle 1,2\rangle \in R$

If $\langle 3,1\rangle \in R \Rightarrow \langle 1,3\rangle \in R$

If $\langle 1,3\rangle \in R \Rightarrow \langle 3,1\rangle \in R$

If $\langle 4,1\rangle \in R \Rightarrow \langle 4,1\rangle \notin R$

R is not symmetric.

∴ R is not an equivallence relation there is no need to check after transitive.

2. If $A = \{1, 2, 3, 4\}$, $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$. Check R is an equivalence (or) not?

Sol Given $A = \{1, 2, 3, 4\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$

Reflexive:

for every $a \in A$, $\langle a,a\rangle \in R$

$1 \in A, \langle 1,1\rangle \in R$

$2 \in A, \langle 2,2\rangle \in R$

$3 \in A, \langle 3,3\rangle \in R$

$4 \in A, \langle 4,4\rangle \in R$

Symmetric:

For every $a, b \in A$, $\langle a,b\rangle \in R$, then $\langle b,a\rangle \in R$.

If $\langle 1,2\rangle \in R \Rightarrow \langle 2,1\rangle \in R$

If $\langle 2,1\rangle \in R \Rightarrow \langle 1,2\rangle \in R$

If $\langle 3,4\rangle \in R \Rightarrow \langle 4,3\rangle \in R$

If $\langle 4,3\rangle \in R \Rightarrow \langle 3,4\rangle \in R$

$\therefore$ R is symmetric.

Transitive

For every $a,b,c \in A$

If $\langle a,b\rangle \in R$ & $\langle b,c\rangle \in R$, then $\langle a,c\rangle \in R$

If $\langle 1,2\rangle \in R$, $\langle 2,1\rangle \in R \Rightarrow \langle 1,1\rangle \in R$

If $\langle 3,4\rangle \in R$, $\langle 4,3\rangle \in R \Rightarrow \langle 3,3\rangle \in R$

If $\langle 2,1\rangle \in R$, $\langle 1,2\rangle \in R \Rightarrow \langle 2,2\rangle \in R$

If $\langle 4,3\rangle \in R$, $\langle 3,4\rangle \in R \Rightarrow \langle 4,4\rangle \in R$

$\therefore$ R is Transitive.

$\therefore$ 'R' is an equivalence relation.

IMP * composition of binary relation

Let 'R' be a relation from 'x' to 'y' and 's' be a relation written as "ROS" is called a composite relation of "R & S" where ROS

$= \{\langle x,z\rangle : x \in X \wedge z \in Z \wedge (\exists y)(y \in Y \wedge \langle x,y\rangle \in R \wedge \langle y,z\rangle \in S\}$. The operation of obtaining ROS from R&S is called composition of relation.

$R \to X$ to X

$S \to Y$ to Z

$X \xrightarrow{R} Y \xrightarrow{S} Z \qquad X \xrightarrow{ROS} Z$

Note

$R\,o(S\,o\,P) = (R\,o\,S)\,o\,P$

$R\,o(S\,o\,P) = (R\,o\,S)\,o\,P = R\,o\,S\,o\,P$

1. Find SOR and ROS, $R = \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$.

Sol Given

$R = \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle$

$S = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$

(∵ repetation is neglect)

ROS

$\langle 1,1\rangle \in R$, $\langle 1,1\rangle \in S \Rightarrow \langle 1,1\rangle \in ROS$

$\langle 1,3\rangle \in R$, $\langle 3,1\rangle \in S \Rightarrow \langle 1,1\rangle \in ROS$

$\langle 3,3\rangle \in S \Rightarrow \langle 1,3\rangle$ ROS

$\langle 2,1\rangle \in R$, $\langle 1,1\rangle \in S \Rightarrow \langle 2,1\rangle \in ROS$

$\langle 2,2\rangle \in R$, $\langle 2,2\rangle \in S \Rightarrow \langle 2,2\rangle \in ROS$

$\langle 2,3\rangle \in S \Rightarrow \langle 2,3\rangle \in ROS$

$\langle 2,3\rangle \in R$, $\langle 3,1\rangle \in S \Rightarrow \langle 2,1\rangle \in ROS$

$\langle 3,3\rangle \in S \Rightarrow \langle 2,3\rangle \in ROS$

$\langle 3,2\rangle \in R$, $\langle 2,2\rangle \in S \Rightarrow \langle 3,2\rangle \in ROS$

$\langle 2,3\rangle \in S \Rightarrow \langle 3,3\rangle \in ROS$.

$ROS = \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$.

Find

$SOR = \{\langle 1,1\rangle \in S$, $\langle 1,1\rangle \in R \Rightarrow \langle 1,1\rangle \in SOR$

$\langle 1,3\rangle \in R \Rightarrow \langle 1,3\rangle \in SOR$

$\langle 2,2\rangle \in S$, $\langle 2,1\rangle \in R \Rightarrow \langle 2,1\rangle \in SOR$

$\langle 2,2\rangle \in R \Rightarrow \langle 2,2\rangle \in SOR$

$\langle 2,3\rangle \in R \Rightarrow \langle 2,5\rangle \in SOR$

$\langle 2,3\rangle \in S$, $\langle 3,2\rangle \in R \Rightarrow \langle 2,2\rangle \in SOR$.

$\langle 3,1\rangle \in S$, $\langle 1,1\rangle \in R$, $\langle 1,3\rangle \in R \Rightarrow \langle 3,1\rangle \in SOR$

$\langle 3,3\rangle \in SOR$

$\langle 3,3\rangle \in S$, $\langle 3,2\rangle \in R \Rightarrow \langle 3,2\rangle \in SOR$.

$SOR = \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle, \langle 3,2\rangle\}$.

2. Let $R = \{\langle 4,2\rangle, \langle 3,4\rangle, \langle 2,2\rangle\}$ & $S = \{\langle 4,2\rangle, \langle 4,5\rangle, \langle 3,1\rangle, \langle 1,3\rangle\}$ find ROS, SOR, Ro(SOR), (ROS)OR, ROR, SOS, ROROR.

Sol Given $R = \{\langle 1,2\rangle, \langle 3,4\rangle, \langle 2,2\rangle\}$

$S = \{\langle 4,2\rangle, \langle 2,5\rangle, \langle 3,1\rangle, \langle 1,3\rangle\}$

$R \circ S = \{\langle 1,5\rangle, \langle 3,2\rangle, \langle 2,5\rangle\}$

$S \circ R = \{\langle 4,2\rangle, \langle 3,2\rangle, \langle 1,4\rangle\}$

$R \circ (S \circ R) = \{\langle 3,2\rangle\}$

$(R \circ S) \circ R = \{\langle 3,2\rangle\}$

$R \circ R = \{\langle 1,2\rangle, \langle 2,2\rangle\}$

$S \circ S = \{\langle 4,5\rangle, \langle 3,3\rangle, \langle 1,1\rangle\}$

$(R \circ R) \circ R = \{\langle 1,2\rangle, \langle 2,2\rangle\} \circ \{\langle 1,2\rangle, \langle 3,4\rangle, \langle 2,2\rangle\}$

$= \{\langle 1,2\rangle, \langle 2,2\rangle\}$

3. Let R & S be two relations on a set of positive integers "I" are

$R = \{\langle x, 2x\rangle : x \in I\}$, $S = \{\langle x, 7x\rangle : x \in I\}$

Find $R \circ S$, $S \circ R$, $R \circ R \circ R$, $S \circ R \circ R$.

Sol Given $R = \{\langle x, 2x\rangle : x \in I\}$

$S = \{\langle x, 7x\rangle : x \in I\}$ $\quad I = 1, 2, \ldots$

$R = \{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, \langle 4,8\rangle, \langle 5,10\rangle, \ldots\}$

$S = \{\langle 1,7\rangle, \langle 2,14\rangle, \langle 3,21\rangle, \langle 4,28\rangle, \langle 5,35\rangle, \langle 6,42\rangle \ldots\}$

<u>$R \circ S$</u>

If $\langle 1,2\rangle \in R$, $\langle 2,14\rangle \in S \Rightarrow \langle 1,14\rangle \in R \circ S$

If $\langle 2,4\rangle \in R$, $\langle 4,28\rangle \in S \Rightarrow \langle 2,28\rangle \in R \circ S$

If $\langle 3,6\rangle \in R$, $\langle 6,42\rangle \in S \Rightarrow \langle 3,42\rangle \in R \circ S$

$R \circ S = \{\langle 1,14\rangle, \langle 2,28\rangle, \langle 3,42\rangle \ldots\}$

$\boxed{R \circ S = \{\langle x, 14x\rangle / x \in I\}}$

$\underline{SoR}$: $\langle 1,7\rangle \in S$, $\langle 7,14\rangle \in R \Rightarrow \langle 1,14\rangle \in SoR$

If $\langle 2,14\rangle \in S$, $\langle 14,28\rangle \in R \Rightarrow \langle 2,28\rangle \in SoR$

$SoR = \{\langle 1,14\rangle, \langle 2,28\rangle, \ldots\}$

$SoR = \{\langle x, 14x\rangle : x \in I\}$

$\underline{RoR}$

If $\langle 1,2\rangle \in R$, $\langle 2,4\rangle \in R \Rightarrow \langle 1,4\rangle \in RoR$

If $\langle 2,4\rangle \in R$, $\langle 4,8\rangle \in R \Rightarrow \langle 2,8\rangle \in RoR$

If $\langle 3,6\rangle \in R$, $\langle 6,12\rangle \in R \Rightarrow \langle 3,12\rangle \in RoR$.

$RoR = \{\langle 1,4\rangle, \langle 2,8\rangle, \langle 3,12\rangle, \ldots\}$

$= \{\langle x, 4x\rangle : x \in I\}$

$(RoR)oR = \{\langle 1,4\rangle, \langle 2,8\rangle, \langle 3,12\rangle, \ldots\}\, o$

$\{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle \ldots\}$

$= \{\langle 1,8\rangle, \langle 2,16\rangle, \langle 3,24\rangle \ldots\}$

$RoRoR = \{\langle x, 8x\rangle : x \in I\}$

$(SoR)oR = \{\langle 1,14\rangle, \langle 2,28\rangle \ldots\}\, o\, \{\langle 1,2\rangle, \langle 2,4\rangle,$
$\langle 3,6\rangle \ldots\}$

$= \{\langle 1,28\rangle, \langle 2,56\rangle, \ldots\}$

$= \{\langle x, 28x\rangle : x \in I\}$

* Representation of discrete structures:

i. Relation Matrix:

A Relation 'R' from a finite set 'x' to a finite set 'y' can also be represented by a matrix is called the relation matrix of 'R'

* $x = \{x_1, x_2, x_3, \ldots x_m\}$ – m rows

$y = \{y_1, y_2, y_3 \ldots y_n\}$ – n column.

* R be a relation from 'x' to 'y'

* The relation matrix 'R' can be obtained by first constructing a table whose coloumns are preseded by a coloumn consisting of successive elements of 'x' and whose rows are headed by a row consisting of the successive elements of y.

* If $x_i R y_i$ then we enter a '1' in the ith row and jth coloumn. If $x_k \not R y_k$ then we enter a '0' in the kth row and jth coloumn.

Ex:- consider m=3, n=2 and 'R' is given by
$R = \{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \langle x_3, y_3\rangle, \langle x_2, y_1\rangle\}$.
the required coloumns is

	y_1	y_2
x_1	1	0
x_2	1	1
x_3	0	1

The relation matrix can be constructed in the following matrix.

$$r_{ij} = \begin{cases} 1 & \text{if } x_i R y_j \\ 0 & \text{if } x_i \not R y_j \end{cases}$$

where 'r_{ij}' is the element in the ith row and jth coloum.

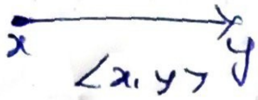
* If 'x' has 'm' elements and 'y' has 'n' elements then the relation matrix is an m×n matrix. For the relation R given in eq(1) the relation matrix is

$$M_R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Graph of a relation (Di Graph):
From the Graph of relation it is possible to observe some of its properties. The relation is reflexive, then their must be

a loop at each node (vertex).

* If the relation is irreflexive, then their is no loop at any node.



$\langle x, y \rangle$

xRy

xRx

$xRy \wedge yRx$

$xRy \wedge yR$
$z \wedge zRx.$

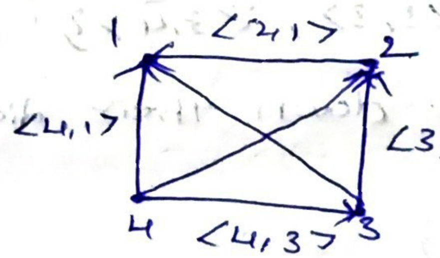
$xRy \wedge yRy$

1. Let $x = \{1,2,3,4\}$ and $R = \{\langle x, y \rangle ; x > y\}$ draw graph of 'R' and also give its matrix.

Sol Given $x = \{1,2,3,4\}$

$R = \{\langle 4,1 \rangle, \langle 4,2 \rangle, \langle 4,3 \rangle, \langle 3,1 \rangle, \langle 3,2 \rangle, \langle 2,1 \rangle\}$

Graph of Relations.



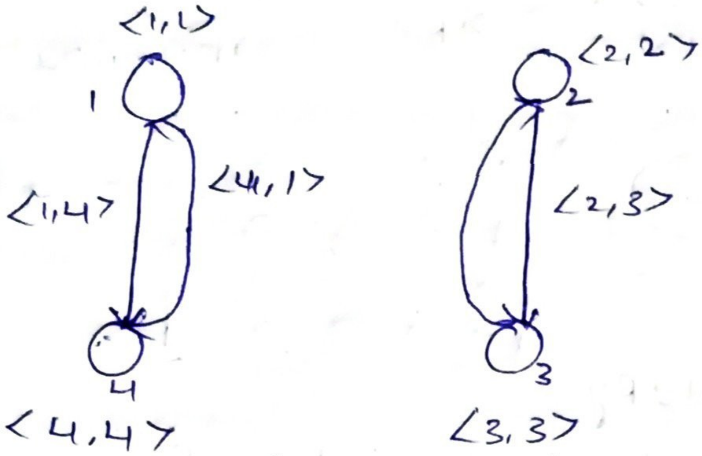
	1	2	3	4
1	0	0	0	0
2	1	0	0	0
3	1	1	0	0
4	1	1	1	0

matrix $M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

2. Let $x = \{1,2,3,4\}$ & $R = \{\langle 1,1 \rangle, \langle 1,4 \rangle, \langle 4,1 \rangle, \langle 4,4 \rangle, \langle 2,2 \rangle, \langle 2,3 \rangle, \langle 3,2 \rangle, \langle 3,3 \rangle\}$ write matrix of 'R' and sketch its graph.

Sol Given $X = \{1,2,3,4\}$

$R = \{\langle 1,1\rangle, \langle 1,4\rangle, \langle 4,1\rangle, \langle 4,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$



	1	2	3	4
1	1	0	0	1
2	0	1	1	0
3	0	1	1	0
4	1	0	0	1

matrix $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

3. If $A = \{1,2,3,4\}$ and R is a relation on A defined by $R = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle\}$ Find R^2 and R^3. write down their diagrams.
$R^2 oR$

Sol Given $A = \{1,2,3,4\}$

$R = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle\}$

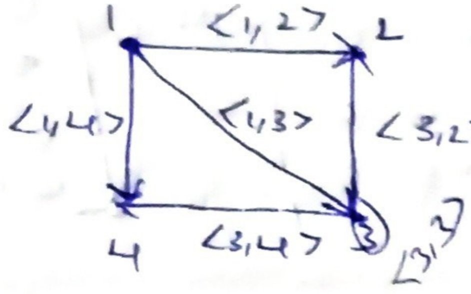
$R^2 = RoR$

$= \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle\} o$
$\{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle\}$

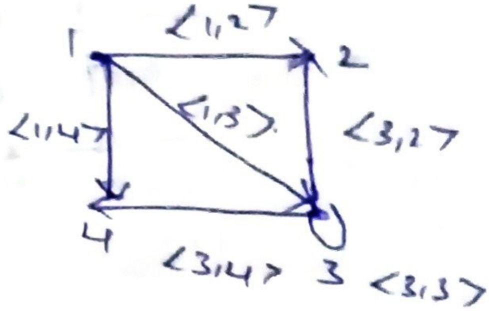
$= \{\langle 1,4\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 1,3\rangle, \langle 3,4\rangle, \langle 3,2\rangle\}$

$R^2 = \{\langle 1,4\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 3,4\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

R^2 diagram



R^3



4. If $A = \{1,2,3,4\}$ & R, S are relations on 'A' defined by $R = \{<1,2>, <1,3>, <2,4>, <4,4>\}$ & $S = \{<1,1>, <1,2>, <1,3>, <1,4>, <2,3>, <2,4>\}$ find RoS, SoR, R^2, S^2, write down their matrices.

Sol Given $A = \{1,2,3,4\}$

$R = \{<1,2>, <1,3>, <2,4>, <4,4>\}$

$S = \{<1,1>, <1,2>, <1,3>, <1,4>, <2,3>, <2,4>\}$

$RoS = \{<1,3>, <1,4>\}$

$SoR = \{<1,2>, <1,3>, <1,4>, <2,4>\}$

$R^2 = RoR = \{<1,4>, <2,4>, <4,4>\}$

$S^2 = SoS = \{<1,2>, <1,3>, <1,4>\}$

$R^2 =$

	1	2	3	4
1	0	1	0	1
2	0	0	0	1
3	0	0	0	0
4	0	0	0	1

$S^2 =$

	1	2	3	4
1	0	1	1	1
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0

matrices of above composite relations.

$M_{RoS} =$

	1	2	3	4
1	0	0	1	1
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0

$SoR =$

	1	2	3	4
1	0	1	1	1
2	0	0	0	1
3	0	0	0	0
4	0	0	0	0

$$M_{RoS} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_{SoR} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_{S^2} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

complementation of a Relation

Given a relation R from a set A to set B, the complement of R is denoted by $\overline{R}$ is defined as a relation from A to B with the property ordered pair $\langle a, b \rangle \in \overline{R}$ iff $\langle a, b \rangle \notin R$

* In other words $\overline{R}$ is complement of the set R in the universal set $A \times B$.

$$\overline{R} = (A \times B) - R$$

converse of a Relation

Given a Relation R from a set A to set B, the converse of R is denoted by R^c is defined as a relation from B to A with the property order pair $\langle a, b \rangle \in R^c$ iff $\langle a, b \rangle \in R$

* If M_R matrix R is then $(M_R)'$, The Transpose M_R is the matrix of R^c and

* $(R^c)^c = R$

1. consider set $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ and the Relations $R = \{\langle a,1\rangle, \cancel{\langle a,2\rangle}, \langle b,1\rangle, \langle \cancel{b}c,2\rangle$ $\langle c,3\rangle\}$ and $S = \{\langle a,1\rangle, \langle a,2\rangle, \langle b,1\rangle, \langle b,2\rangle\}$ from A to B. Determine $\bar{R}$, $\bar{S}$, $R \cup S$, $R \cap S$, R^c, S^c.

sol Given $A = \{a, b, c\}$

$B = \{1, 2, 3\}$

$R = \{\langle a,1\rangle, \langle b,1\rangle, \langle c,2\rangle, \langle c,3\rangle\}$

$S = \{\langle a,1\rangle, \langle a,2\rangle, \langle b,1\rangle, \langle b,2\rangle\}$

$A \times B = \{a, b, c\} \times \{1, 2, 3\}$

$= \{\langle a,1\rangle, \langle a,2\rangle, \langle a,3\rangle, \langle b,1\rangle, \langle b,2\rangle, \langle b,3\rangle, \langle c,1\rangle, \langle c,2\rangle, \langle c,3\rangle\}$

$\bar{R} =$ complement R in $A \times B$

$= (A \times B) - R$

$= \{\langle a,1\rangle, \langle a,2\rangle, \langle a,3\rangle, \langle b,1\rangle, \langle b,2\rangle, \langle b,3\rangle, \langle c,1\rangle, \langle c,2\rangle, \langle c,3\rangle\} - \{\langle a,1\rangle, \langle b,1\rangle, \langle c,2\rangle, \langle c,3\rangle\}$

$= \{\langle a,2\rangle, \langle a,3\rangle, \langle b,2\rangle, \langle b,3\rangle, \langle c,1\rangle\}$

$\bar{S} =$ complement S in $A \times B$

$= (A \times B) - S$

$= \{\langle a,1\rangle, \langle a,2\rangle, \langle a,3\rangle, \langle b,1\rangle, \langle b,2\rangle, \langle b,3\rangle, \langle c,1\rangle, \langle c,2\rangle, \langle c,3\rangle\} - \{\langle a,1\rangle, \langle a,2\rangle, \langle b,1\rangle, \langle b,2\rangle\}$

$= \{\langle a,3\rangle, \langle b,3\rangle, \langle c,1\rangle, \langle c,2\rangle, \langle c,3\rangle\}$

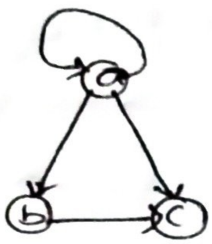
$R \cup S = \{\langle a,1\rangle, \langle a,2\rangle, \langle b,1\rangle, \langle b,2\rangle, \langle c,2\rangle, \langle c,3\rangle\}$

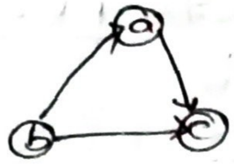
$R \cap S = \{\langle a,1\rangle, \langle b,1\rangle\}$

$R^c = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,c\rangle, \langle 3,c\rangle\}$

$S^c = \{\langle 1,a\rangle, \langle 2,a\rangle, \langle 1,b\rangle, \langle 2,b\rangle\}$

2. The diagraphs of two relations R and S on the set $A = \{a, b, c\}$ are given below. Draw the digraphs of $\bar{R}$, $R \cup S$, $R \cap S$ and R^c.





S

R

Sol Let $R = \{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,c\rangle\}$

$S = \{\langle a,c\rangle, \langle b,a\rangle, \langle b,c\rangle\}$

$A \times A = \{a, b, c\} \times \{a, b, c\}$

$= \{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,a\rangle, \langle b,c\rangle, \langle b,b\rangle, \langle c,a\rangle, \langle c,b\rangle, \langle c,c\rangle\}$

$\bar{R}$ = complement of R in $A \times A$

$= A \times A - R$

$= \{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,a\rangle, \langle b,b\rangle, \langle b,c\rangle, \langle c,a\rangle, \langle c,b\rangle, \langle c,c\rangle\} - \{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,c\rangle\}$

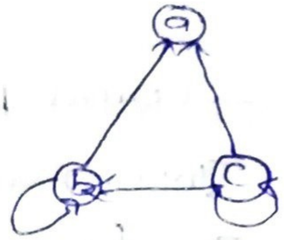
$= \{\langle b,a\rangle, \langle b,b\rangle, \langle c,a\rangle, \langle c,b\rangle, \langle c,c\rangle\}$

$R \cup S = \{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,a\rangle, \langle b,c\rangle\}$

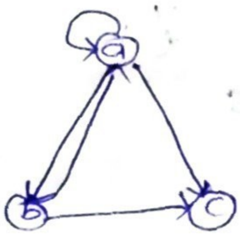
$R \cap S = \{\langle a,c\rangle, \langle b,c\rangle\}$

$R^c = \{\langle a,a\rangle, \langle b,a\rangle, \langle c,a\rangle, \langle c,b\rangle\}$

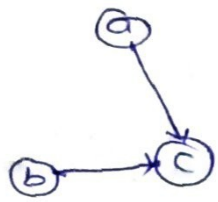
The diagraphs of $\overline{R}$



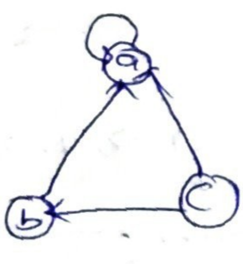
The diagraph of $R \cup S$



The diagraph of $R \cap S$



The diagraph of R^c



Covering of a set

Let 'S' be a given set and $A = \{A_1, A_2, A_3 \ldots A_m\}$ where each A_i is subset of S and union $\bigcup_{i=1}^{m} A_i = S$. Then set A is called a covering of set S and the subsets $A_1, A_2, A_3, \ldots A_m$ are said to cover set 'S'.

Ex:-

Let $S = \{a, b, c\}$ and consider the following collection of subsets of 'S' $A = \{\{a,b\}, \{b,c\}\}$, $B = \{\{a\}, \{a,c\}\}$, $C = \{\{a\}, \{b,c\}\}$, $D = \{\{a,b,c\}\}$, $E = \{\{a\}, \{b\}, \{c\}\}$, $F = \{\{a\}, \{a,b\}, \{a,c\}\}$.

Among them which which subsets are covering sets?

Sol: Except 'B' all subsets are covering sets of 'S'.

Partision of a set

Let 'S' be a non-empty set, $S_1, S_2, S_3 \ldots S_m$ are the subsets of S, the collection of subsets S_i is a partision of S iff and only if

i. $S_i \neq \phi$, for each i

ii. $S_i \cap S_j = \phi$ for $i \neq j$

iii. $\bigcup_{i=1}^{m} S_i = S$.

Where $\bigcup_{i=1}^{m}$ represents the union of the subsets S_i, for all i and $S_1, S_2, S_3, \ldots S_m$ are called blocks or cells of partisions.

Ex

* Let $S = \{1, 2, 3, 4, 5, 6\}$ and the subsets $S_1 = \{1, 3, 5\}$, $S_2 = \{2, 4\}$, $S_3 = \{1, 3\}$, $S_4 = \{3, 5\}$, $S_5 = \{2, 4, 6\}$, $S_6 = \{1, 2, 3\}$, $S_7 = \{4, 5\}$, $S_8 = \{6\}$ Check whether the following partision are valid or not?

$\pi_1 = \{S_6, S_7, S_8\}$.

Sol: $S_6 \neq \phi$, $S_7 \neq \phi$, $S_8 = \phi$

The first condition is satisfied.

$$\{S_6, S_7\} = \{S_6\} \cap \{S_7\}$$
$$= \{1, 2, 3\} \cap \{4, 5\}$$
$$= \phi$$

$\{S_7, S_8\} = \{S_7\} \cap \{S_8\}$

$= \phi$

$\{S_6 S_8\} = \{S_6\} \cap \{S_8\}$

$= \phi$

The second condition is satisfied.

iii. $\{S_6 S_7 S_8\} = \{S_6\} \cup \{S_7\} \cup \{S_8\}$

$= \{1,2,3\} \cup \{4,5\} \cup \{6\}$

$= \{1,2,3,4,5,6\}$

$= S$

∴ The Third condition is satisfied

$\Pi_1 = \{S_6, S_7, S_8\}$ is a valid Partision.

Partial ordering

* A binary relation R in a set 'P' is called a partial order relation. (or) A partial ordering in 'P' ⟺ R is reflexive, Anti symmetric and Transitive.

* We'll denote a Partial ordering by the symbol "$\leq$"

* If $\leq$ is a partial ordering on P, then the order pair $\langle P, \leq \rangle$ is called Partial ordering set or "poset".

Totally ordered set

Let $\langle P, \leq \rangle$ be a partially ordered set. If for every $x, y \in P$ we have either $x \leq y \vee y \leq x$

Then $\leq$ is called simple ordering or Linear ordering on 'P', and $<P, \leq>$ is called a Totally ordered set or Simply ordered or a chain.

Compatibility Relation

* A Relation 'R' in 'x' is said to be a compatability relation if it is reflexive and symmetric.
* The Relation 'R' is given by

$$R = \{<x, y> \; x, y \in X \wedge x R y \; , \text{ if } x \; \& \; y$$

Contain some common letter}.

Hasse Diagram

* A partial ordering '$\leq$' on a set 'P' can be represented by means of a diagram is known as Hasse diagram of $<P, \leq>$.

* In Hasse diagram each element is represented by small circle or dots.
* In Hasse diagram we represent their vertices by dots or circles, we doesnot put arrows on edges and we doesnot draw selfloops at vertices.
* In diagraph of partial order, there is an edge from vertex A to vertex B and there is an edge from vertex B to vertex C, There should be an edge from vertex A to vertex C as such we need not exhibit an edge from A to C explicitly.

1. Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$. If x divides y. Draw the Hasse diagram of $<X, \leq>$

Sol Given $X = \{2, 3, 6, 12, 24, 36\}$

$2 \rightarrow 2, 6, 12, 24, 36$

ordered pairs: $\langle 2,2\rangle, \langle 2,6\rangle, \langle 2,12\rangle, \langle 2,24\rangle, \langle 2,36\rangle$.

$3 \rightarrow 3, 6, 12, 24, 36$

ordered pairs: $\langle 3,3\rangle, \langle 3,6\rangle, \langle 3,12\rangle, \langle 3,24\rangle, \langle 3,36\rangle$.

$6 \rightarrow 6, 12, 24, 36$

ordered pairs: $\langle 6,6\rangle, \langle 6,12\rangle, \langle 6,24\rangle, \langle 6,36\rangle$.

$12 \rightarrow 12, 24, 36$

Ordered pairs: $\langle 12,12\rangle, \langle 12,24\rangle, \langle 12,36\rangle\}$

$24 \rightarrow \langle 24,24\rangle$

$36 \rightarrow \langle 3636\rangle$

$R = \{\langle 2,2\rangle, \langle 2,6\rangle, \langle 2,12\rangle, \langle 2,24\rangle, \langle 2,36\rangle, \langle 3,3\rangle, \langle 3,6\rangle, \langle 3,12\rangle, \langle 3,24\rangle, \langle 3,36\rangle, \langle 6,6\rangle, \langle 6,12\rangle, \langle 6,24\rangle, \langle 6,36\rangle, \langle 12,12\rangle, \langle 12,24\rangle, \langle 12,36\rangle, \langle 24,24\rangle, \langle 36,36\rangle\}$

Reflexive: For all $a \in X$, $\langle a,a\rangle \in R$

R is a Reflexive

Antisymmetric: For all $a,b \in X$, if $\langle a,b\rangle \in R$ & $\langle b,a\rangle \in R$ then $a=b$.

R is anti symmetric

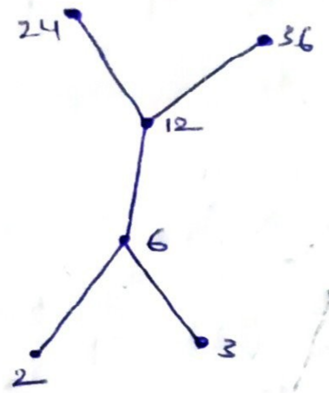
Transitive:

For all $a,b,c \in X$, if $\langle a,b\rangle \in R$, $\langle b,c\rangle \in R$ and $\langle a,c\rangle \in R$

$\therefore$ R is a partially ordered relation.

Hasse Diagram

$X = \{2, 3, 6, 12, 24, 36\}$



2. Let $A = \{1, 2, 3, 4, 6, 12\}$ on A, defined a relation R by aRb iff and only if a divides b. prove that R is a partial order on A. Draw the Hasse diagram for this Relation

sol Let $A = \{1, 2, 3, 4, 6, 12\}$

$R = \{\langle a, b \rangle / a, b \in A$ and a divides b$\}$

$R = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 1,3 \rangle, \langle 1,4 \rangle, \langle 1,6 \rangle, \langle 1,12 \rangle, \langle 2,2 \rangle,$
$\langle 2,4 \rangle, \langle 2,6 \rangle, \langle 2,12 \rangle, \langle 3,3 \rangle, \langle 3,6 \rangle, \langle 3,12 \rangle,$
$\langle 4,4 \rangle, \langle 4,12 \rangle, \langle 6,6 \rangle, \langle 6,12 \rangle, \langle 12,12 \rangle\}$

Reflexive: For all $a \in A$, $\langle a, a \rangle \in R$

$\therefore$ R is Reflexive.

Anti symmetric:

For all $a, b \in A$, if a divides b and b divides a then $a = b$

$\therefore$ R is antisymmetric.

Transitive

For all $a, b, c \in R$ if $\langle a, b \rangle \in R$, $\langle b, c \rangle \in R$

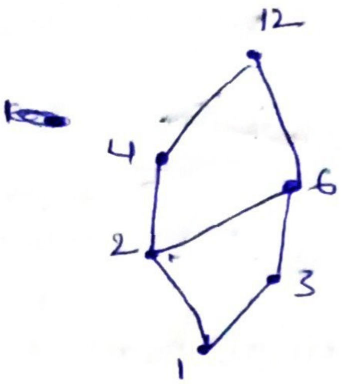
then $\langle a, c \rangle \in R$

$\therefore$ R is Transitive

∴ R is poset.

Hasse Diagram

$A = \{1, 2, 3, 4, 6, 12\}$



3. Draw the Hasse diagram for $[\{1, 2, 3, 5, 6, 10, 15, 30\}$ /]

Sol Given $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$R = \{\langle a,b\rangle / a,b \in A$ and a divides $b\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle \langle 1,5\rangle, \langle 1,6\rangle, \langle 1,10\rangle, \langle 1,15\rangle, \langle 1,30\rangle,$
$\langle 2,2\rangle, \langle 2,6\rangle, \langle 2,10\rangle, \langle 2,30\rangle, \langle 3,3\rangle, \langle 3,6\rangle, \langle 3,15\rangle,$
$\langle 3,30\rangle, \langle 5,5\rangle, \langle 5,10\rangle, \langle 5,15\rangle, \langle 5,30\rangle, \langle 6,6\rangle, \langle 6,30\rangle$
$\langle 10,10\rangle, \langle 10,30\rangle, \langle 15,15\rangle, \langle 15,30\rangle, \langle 30,30\rangle\}$

Reflexive : For all $a \in A$, $\langle a,a\rangle \in R$.

∴ R is Reflexive.

Anti symmetric

For all $a,b \in A$, if a divides b and b divides a then $a = b$.

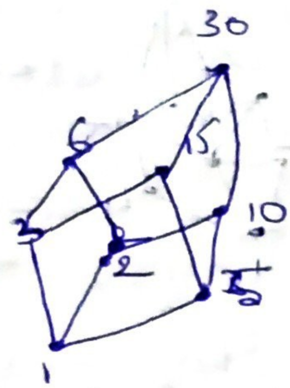
∴ R is Antisymmetric

Transitive : For all $a,b,c \in A$ if $\langle a,b\rangle \in R$ and $\langle b,c\rangle \in R$ then $\langle a,c\rangle \in R$. ∴ R is Transitive

R is a Poset.

Hasse Diagram

$A = \{1, 2, 3, 5, 6, 10, 15, 30\}$



4. Draw the Hesse diagram representing the positive divises of 36

Sol The set of all positive divises of 36 is

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

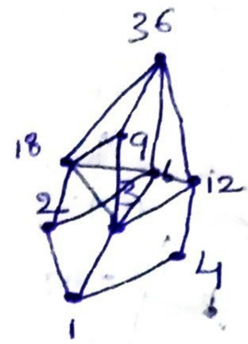
$R = \{\langle a, b\rangle / a, b \in A$ and a divides b$\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,6\rangle, \langle 1,9\rangle, \langle 1,12\rangle,$
$\langle 1,18\rangle, \langle 1,36\rangle, \langle 2,2\rangle, \langle 2,4\rangle, \langle 2,6\rangle, \langle 2,12\rangle, \langle 2,18\rangle \langle 2,36\rangle$
$\langle 3,3\rangle, \langle 3,6\rangle, \langle 3,9\rangle, \langle 3,12\rangle, \langle 3,18\rangle, \langle 3,36\rangle, \langle 4,4\rangle, \langle 4,12\rangle$
$\langle 4,36\rangle, \langle 6,6\rangle, \langle 6,18\rangle, \langle 6,36\rangle, \langle 9,9\rangle, \langle 9,36\rangle, \langle 12,12\rangle,$
$\langle 12,36\rangle, \langle 18,18\rangle, \langle 18,36\rangle, \langle 36,36\rangle$

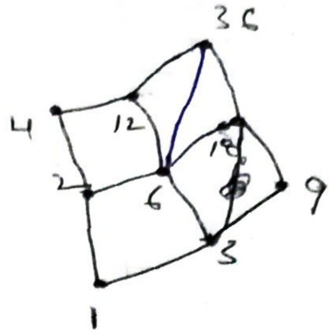
$\therefore$ R is a Reflexive, Antisymmetric, Transitive then R is a Poset.

Hasse Diagram

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



(or)



5. Let A be a given finite set and $P(A)$ it's power set. Let $\subseteq$ be the inclusion relation on the elements of $P(A)$. Draw the Hasse diagram of $\langle P(A), \subseteq \rangle$ for i. $A = \{a\}$ ii. $A = \{a, b\}$, iii. $A = \{a, b, c\}$

Sol: i. Given $A = \{a\}$

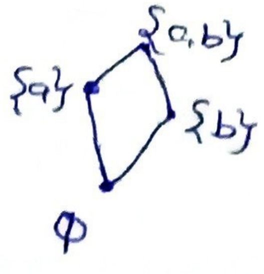
$P(A) = \{\phi, \{a\}\}$

Hasse Diagram for $\langle P(A), \subseteq \rangle$

ii. Given $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

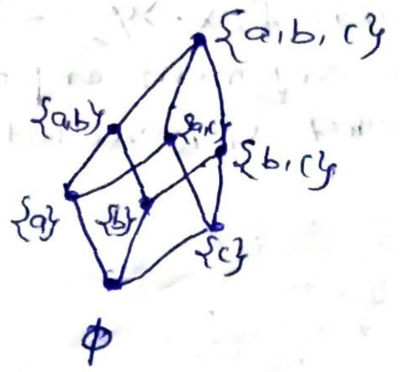
Hasse Diagram for $\langle P(A), \subseteq \rangle$



iii. Given $A = \{a, b, c\}$

$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$

Hasse diagram for $< P(A), \subseteq >$



6. Draw the Hasse Diagram for $[\{1, 2, 3, 4, 6, 9\} /]$.

Sol Let $A = \{1, 2, 3, 4, 6, 9\}$

$R = \{ <a, b> / a, b \in A$ and a divides $b \}$

$R = \{ <1,1>, <1,2> <1,3>, <1,4>, <1,6> <1,9>,$

Functions

* Let X and Y be any two sets. A Relation f from X to Y ($f: X \to Y$) is called function. If for every $x \in X$, there is a unique $y \in Y$ such that order pair $< x, y > \in f$. It is denoted as $f: X \to Y$.

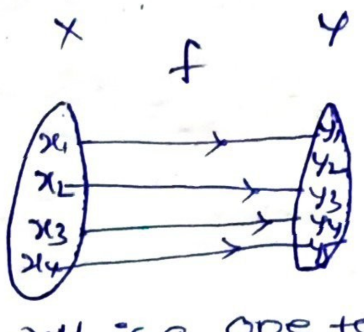
* For a function $f: X \to Y$, $< x, y > \in f$, then x is called an argument or preimage of y and the corresponding y is called image of x ~~correlary~~ under f.

Types of functions

1. One to one function (or) Injective or 1-1

A function $f: X \to Y$ is said to be a one to one function. If different elements of X have different images in Y under f.

* That is if whenever $x_1, x_2 \in X$ with $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ (or) Equivalently if whenever $f(x_1) = f(x_2)$ for $x_1 x_2 \in X$ then $x_1 = x_2$

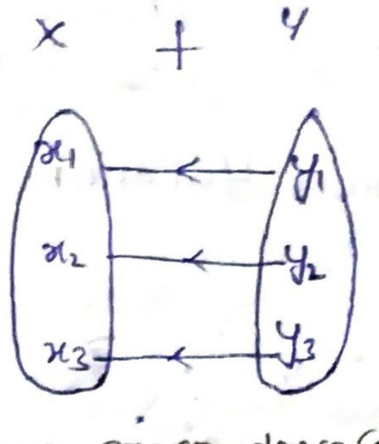


* If $f: X \to Y$ is a one to one function, then Every element of X has a unique image in Y and Every element of $f(x) = Y$ has a unique preimage in X.

2. On to function (or) surjection

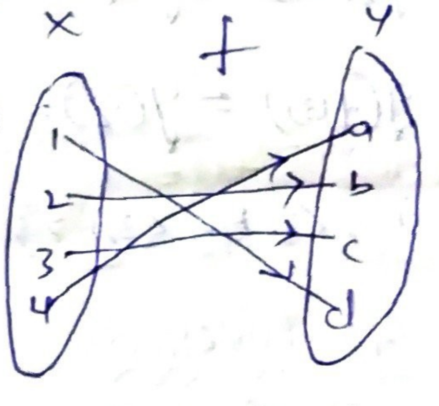
A function $f: X \to Y$ is said to be onto function if every element of Y has a preimage in X under f.

* In other words f is an onto-function from X to Y if the Range of f is equal to Y ($R_f = Y$). i.e ($y = f(x)$).



3. one to one corrspondense (or) Bijection (or) one to one onto.

* A function which is both one to one and on to is called a one to one corrspondense. If $f: X \to Y$ is such a function, then every element of X has a unique image in Y and Every element of Y has a unique preimage in X.

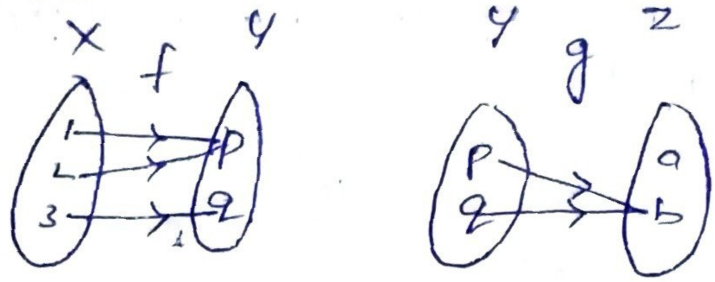


1. Let $X = \{1, 2, 3\}$, $Y = \{P, Q\}$, $Z = \{a, b\}$ also let $f: X \to Y$ be $f = \{\langle 1, P\rangle, \langle 2, P\rangle, \langle 3, Q\rangle\}$ and $g: Y \to Z$ be given by $g = \{\langle P, b\rangle \langle Q, b\rangle\}$ Find $g \circ f$.

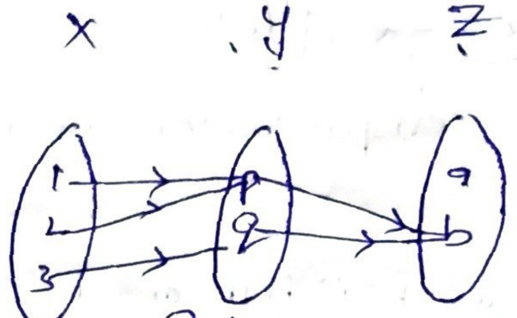
Sol Given $X = \{1, 2, 3\}$, $Y = \{P, Q\}$, $Z = \{a, b\}$

$f: X \to Y$, $f = \{\langle 1, P\rangle, \langle 2, P\rangle, \langle 3, Q\rangle\}$

$g: Y \to Z$, $g = \{\langle P, Q\rangle, \langle Q, b\rangle\}$



The composition function of gof.



gof

<u>Finding gof</u> $g\{f(1)\}$ or

$g \circ f(1) = g(f(1)) = g(p) = b$

$g \circ f(2) = g(f(2)) = g(p) = b$

$g \circ f(3) = g(f(3)) = g(q) = b$

$\therefore g \circ f = \{\langle 1,b \rangle, \langle 2,b \rangle, \langle 3,b \rangle\}$

Composition of function:-

* Let $f: X \to Y$ and $g: Y \to Z$ be two functions the composit relation gof such that.

$$g \circ f = \{\langle x,z \rangle / (x \in X) \,\&\, (z \in Z) \wedge (\exists y)(y \in Y \wedge y = f(x) \wedge z = g(y)\}$$

is called composition of function (or) relative product of functions f and g.

2. Let $X = \{1,2,3\}$ and f, g, h, s be functions from X to X given by $f = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle\}$,

$g = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$, $h = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle\}$

$s = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle\}$. Find fog, gof, $fohog$, sog, gos, sos & fos.

Sol/ Given $X = \{1,2,3\}$,

$f = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle\}$

$g = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$

$h = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle\}$

$s = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle\}$

<u>fog</u>

$fog(1) = f(g(1)) = f(2) = 3$

$fog(2) = f(g(2)) = f(1) = 2$

$fog(3) = f(g(3)) = f(3) = 1$

$\therefore fog = \{\langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle\}$

<u>gof</u>

$gof(1) = g(f(1)) = g(2) = 1$

$gof(2) = g(f(2)) = g(3) = 3$

$gof(3) = g(f(3)) = g(1) = 2$

$\therefore gof = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$

<u>fohog</u>

$fohog(1) = foh(g(1)) = f(h(2)) = f(2) = 3$

$fohog(2) = foh(g(2)) = f(h(1)) = f(1) = 2$

$fohog(3) = foh(g(3)) = f(h(3)) = f(1) = 2$

$\therefore fohog = \{\langle 1,3\rangle, \langle 2,2\rangle, \langle 3,2\rangle\}$

sog

$sog(1) = s(g(1)) = s(2) = 2$

$sog(2) = s(g(2)) = s(1) = 1$

$sog(3) = s(g(3)) = s(3) = 3$

$\therefore sog = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$

gos

$gos(1) = g(s(1)) = g(1) = 2$

$gos(2) = g(s(2)) = g(2) = 1$

$gos(3) = g(s(3)) = g(3) = 3$

$\therefore gos = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$

sos

$sos(1) = s(s(1)) = s(1) = 1$

$sos(2) = s(s(2)) = s(2) = 2$

$sos(3) = s(s(3)) = s(3) = 3$

$\therefore sos = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle\} = s$

fos

$fos(1) = f(s(1)) = f(1) = 2$

$fos(2) = f(s(2)) = f(2) = 3$

$fos(3) = f(s(3)) = f(3) = 1$

$\therefore fos = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle\} = f$

3. Let $A=\{1,2,3,4\}$, $B=\{a,b,c\}$ & $C=\{w,x,y,z\}$ with $f: A \to B$, $g: B \to C$ given by $f=\{\langle 1,a\rangle, \langle 2,a\rangle, \langle 3,b\rangle, \langle 4,c\rangle\}$ and $g=\{\langle a,x\rangle, \langle b,y\rangle, \langle c,z\rangle\}$, find gof

Sol Given $A=\{1,2,3,4\}$

$B=\{a,b,c\}$

$C=\{w,x,y,z\}$

$f: A \to B$, $g: B \to C$.

$f=\{\langle 1,a\rangle, \langle 2,a\rangle, \langle 3,b\rangle, \langle 4,c\rangle\}$

$g=\{\langle a,x\rangle, \langle b,y\rangle, \langle c,z\rangle\}$

gof

$gof(1) = g(f(1)) = g(a) = x$

$gof(2) = g(f(2)) = g(a) = x$.

$gof(3) = g(f(3)) = g(b) = y$

$gof(4) = g(f(4)) = g(c) = z$

$\therefore gof = \{\langle 1,x\rangle, \langle 2,x\rangle, \langle 3,y\rangle, \langle 4,z\rangle\}$

4. Let $f(x) = x+2$, $g(x) = x-2$ and $h(x) = 3x$ for $x \in R$ where R is the set of real numbers find gof, fog, fof, gog, foh, hog, & $fohog$.

Sol Given $f(x) = x+2$, $g(x) = x-2$, $h(x) = 3x$

gof

$gof(x) = g(f(x)) = g(x+2) = x+2-2 = x$

$gof = \{\langle x,x\rangle / x \in R\}$

$$fog(x) = f(g(x)) = f(x-2) = x-2+2 = x$$

$\therefore fog = \{<x, x> / x \in R\}$

$$fof(x) = f(f(x)) = f(x+2) = x+2+2 = x+4$$

$\therefore fof = \{<x, x+4 / x \in R\}$

$$gog(x) = g(g(x)) = g(x-2) = x-2-2 = x-4$$

$\therefore gog = \{<x, x-4 / x \in R\}$

$$foh(x) = f(h(x)) = f(3x) = 3x+2$$

$foh = \{<x, 3x+2 / x \in R\}$

$$hog(x) = h(g(x)) = h(x-2) = 3(x-2)$$
$$= 3x-6$$

$hog = \{<x, 3x-6> / x \in R\}$

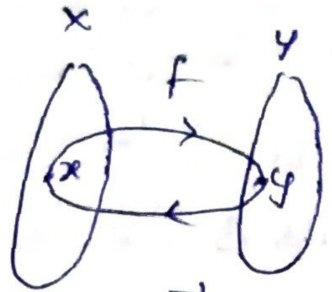
$$fohog(x) = foh(g(x)) = f(h(x-2)) = f(3x-6)$$
$$= 3x-6+2$$
$$= 3x-4$$

$fohog = \{<x, 3x-4> / x \in R\}$

Inverse Functions (or) Invertiable Functions

* A function $f: X \to Y$ is said to be invertiable if there exist a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$, where I_X is the Identity function on X and I_Y is the Identity function on Y.

* Then g is called an inverse of f and we write $g = f^{-1}$



$g = f^{-1}$

NOTE:

Let $X = \{1,2,3\}$, $Y = \{p,q,r\}$ and $f: X \to Y$ be given by $f = \{<1,p>, <2,q>, <3,q>\}$ then $f^{-1} = \{<p,1>, <q,2>, <q,3>\}$ and f^{-1} is not a function

Identity Maping:-

A maping $I_x : X \to X$ is called an Identity Maping if $I_x = \{<x,x> / x \in X\}$

1. Show that the functions $f(x) = x^3$ and $g(x) = x^{1/3}$ for $x \in R$ are inverse of one another.

Sol. Given $f(x) = x^3$

$g(x) = x^{1/3}$

fog

$fog(x) = f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x = I_x$ (or) $I_R(x)$

$\rightarrow fog = \{<x,x> / x \in R\}$

gof

$gof(x) = g(f(x)) = g(x^3) = (x^3)^{1/3} = x = I_x$ (or) $I_R(x)$

$gof = \{<x,x> / x \in R\}$

$\therefore fog = gof$

$\therefore f = g^{-1}$ and $g = f^{-1}$

$\therefore f$ and g are inverse of each other.

2. Let $A=\{1,2,3,4\}$ and f and g the functions from A to A given by $f=\{\langle 1,4\rangle, \langle 2,1\rangle, \langle 3,2\rangle, \langle 4,3\rangle\}$ and $g=\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle\}$. Prove that f and g are inverse of each other

Sol Given $A=\{1,2,3,4\}$

$$f=\{\langle 1,4\rangle, \langle 2,1\rangle, \langle 3,2\rangle, \langle 4,3\rangle\}$$

$$g=\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle\}$$

<u>fog</u>

$fog(1) = f(g(1)) = f(2) = 1 = I_A(1)$

$fog(2) = f(g(2)) = f(3) = 2 = I_A(2)$

$fog(3) = f(g(3)) = f(4) = 3 = I_A(3)$

$fog(4) = f(g(4)) = f(1) = 4 = I_A(4)$

$fog = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\}$

<u>gof</u>

$gof(1) = g(f(1)) = g(4) = 1 = I_A(1)$

$gof(2) = g(f(2)) = g(1) = 2 = I_A(2)$

$gof(3) = g(f(3)) = g(2) = 3 = I_A(3)$

$gof(4) = g(f(4)) = g(3) = 4 = I_A(4)$

$gof = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle$

$\therefore$ $fog = gof$

$\therefore$ $f = g^{-1}$ and $g = f^{-1}$

$\therefore$ f and g are inverse of each other.

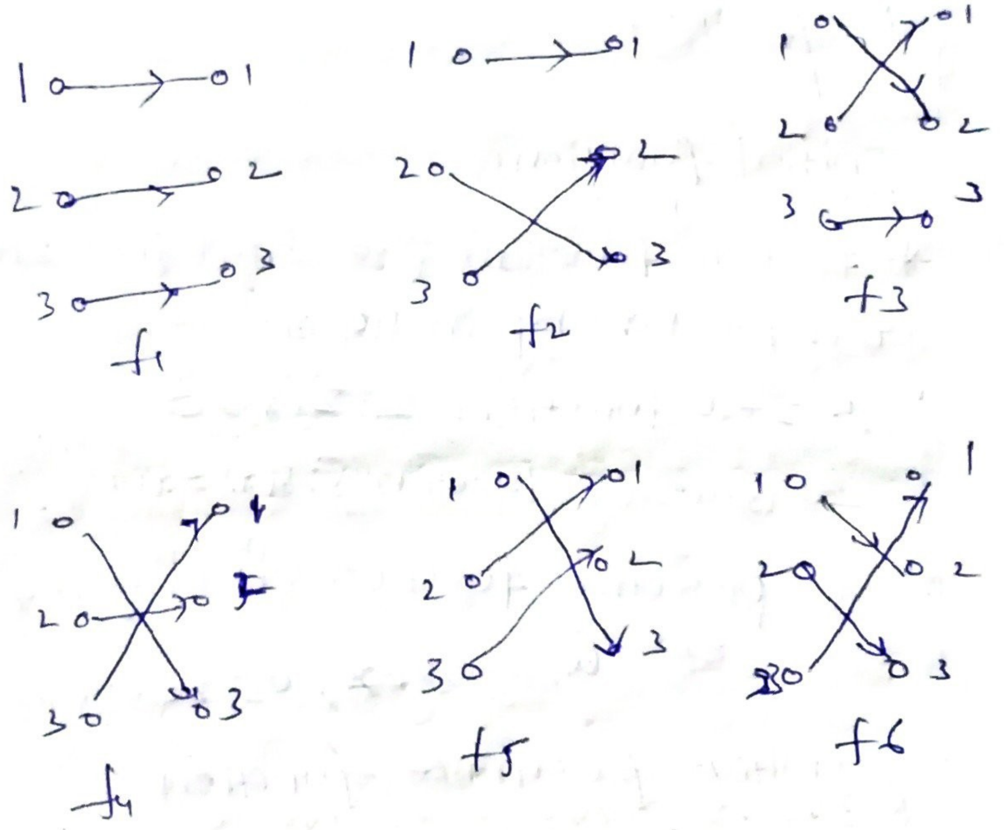
3. Let F_x be the set of all one to one mapping from x on to X. where $X=\{1,2,3\}$. Find all the elements of F_x and find the inverse of each element.

Sol NOTE: If a set x has n elements, then there are $n!$ functions from x to x which are bijective.

Given $x=\{1,2,3\}$

Since x consists of 3 elements then a set of bijective function from x to x consists of $3!=6$ elements.

$$F_x=\{f_1, f_2, f_3, f_4, f_5, f_6\}$$



$f_1^{-1}=f_1$, $f_2^{-1}=f_2=f_3^{-1}=f_3$, $f_4^{-1}=f_4$ & $f_5^{-1}=f_6$

Recursive function

* Any function $f: N^n \to N$ is called total because it is defined for every n-tuple in N^n. on the otherhand $f: D \to N$ where $D \subseteq N^n$ f is called partial.

Ex:

1. $f\langle x, y\rangle = x+y$ which is defined for all $x, y \in N$ and hence is a total function.
2. $g\langle x, y\rangle = x-y$ which is defined for only these $x, y \in N$ which satisfies $x \geq y$. hence $g\langle x, y\rangle$ is partial.

Initial functions

* Initial functions are used in defining other functions by induction

1. zero function $Z: Z(x) = 0$
2. Successor function $S: S(x) = x+1$
3. projection function U_i^n: $U_i^n \langle x_1, x_2, \ldots x_n\rangle = x_i$

   ex:- $U_1^2 \langle x, y\rangle = x$, $U_2^3 \langle 2, 4, 6\rangle = 4$

Premitive Recursive function

* A function f is called premitive recursive iff it can be obtain from the initial functions by a finite number of operations of composition and recursion.

1. Show that the function $f\langle x,y\rangle = x+y$ is premitive recursive.

Sol Given $f\langle x,y\rangle = x+y$

Zero function:

$f\langle x,0\rangle = x+0 = x$

Successor function

$f\langle x, y+1\rangle = x+(y+1) = (x+y)+1$ $\quad \because S: S(x) = x+1$

$= f\langle x,y\rangle+1$

$= S(f\langle x,y\rangle)$

Projection function

$f\langle x,0\rangle = x = U_1^1(x) \rightarrow$ base function.

$f\langle x,y+1\rangle = S(f(\langle x,y\rangle)) = S(U_3^3\langle x,y, f\langle x,y\rangle\rangle)$

Induction Step function.

2. Using Recursion, defined the multiplication function * given by $g\langle x,y\rangle = x*y$.

Sol Given $g\langle x,y\rangle = x*y$

Zero function

$g\langle x,0\rangle = x*0 = 0$

Successor function

$g\langle x,y+1\rangle = x*(y+1)$

$= (x*y)+x$

$= g\langle x,y\rangle + x$

$= S(g\langle x,y\rangle)$

Projection function

$g\langle x,y+1\rangle = g\langle x,y\rangle + x = f\langle u_3^3\langle x,y,g\langle x,y\rangle\rangle,$

$u_1^3\langle x,y,g\langle x,y\rangle\rangle\rangle$

premitive Recursive function:

* sign function or Non-Test function (sg)

$sg(0)=0,\ sg(y+1)=1$

* zero Test function ($\overline{sg}$)

$\overline{sg}(0)=1,\ \overline{sg}(y+1)=0$

* ~~pratiser~~ function
predecessor

$p(0)=0,\ p(y+1)=y=u_1^2\langle y,p(y)\rangle$

Ex:- $p(8)=7,\ p(9)=8.$

* Proper subtraction ($\dot{-}$)

$x\dot{-}0=x,\quad x\dot{-}(y+1)=p(x\dot{-}y)$

NOTE

* $x\dot{-}y=0$ for $x<y$
* $x\dot{-}y=x-y$ for $x\geq y$

3. Show that $f\langle x,y\rangle = x^y$ is a premitive recursive function.

Sol Given $f\langle x,y\rangle = x^y$

zero function

$f\langle x,0\rangle = x^0 = 1$

Successor function

$$f\langle x, y+1\rangle = x^{y+1}$$
$$= x^y \cdot x$$
$$= x^y * x$$
$$= f\langle x,y\rangle * x$$

Projection function : $x^0 = 1$, for $(x \neq 0)$

$f\langle x, 0\rangle = sg(x)$ | put $x^0 = 0$, for $x=0$.

$$f\langle x, y+1\rangle = f\langle x, y\rangle * x$$

$$= u_3^3\langle x, y, f\langle x,y\rangle\rangle * u_1^3\langle x, y, f\langle x, y\rangle\rangle.$$

4. Let $\lfloor\sqrt{x}\rfloor$ be the greatest integer $\leq \sqrt{x}$.
Show that $\lfloor\sqrt{x}\rfloor$ is premitive recursive.

Sol Let $\lfloor\sqrt{x}\rfloor$ be the greatest integer $\leq \sqrt{x}$.
We observe that $(y+1)^2 \dot{-} x$ is zero for $(y+1)^2 \leq x$
and non-zero for $(y+1)^2 > x$

* Therefore $\overline{sg}((y+1)^2 \dot{-} x)$ is 1 for $(y+1)^2 \leq x$
and can't be equals to zero.

* The smallest value of y for which
$(y+1)^2 > x$ is the required number $\lfloor x \rfloor$

* Hence $\lfloor\sqrt{x}\rfloor = \mu y \langle \overline{sg}((y+1)^2 \dot{-} x)\rangle = 0$

Algebraic Structures

Definations and Examples

* Let $f: X^n \to X$, for $n=1$, Such an operation is called
unary operation.

* Similarly $n=2$, is called a Binary operation. any
* distinguished element of X such as an identity

element or zero element with respect to a Binary operation is considered as a 0-ary operation

* A Binary operation will be denoted by means of symbols as $*$, Δ, $+$, $\oplus$ and the result of Binary operation on the elements say $x_1, x_2 \in X$ is expressed by writting $x_1 * x_2$.

* If f denotes an n-ary operation when $f\langle x_1, x_2, \ldots x_n\rangle$ operation is the image in X of the n-tuple $\langle x_1, x_2, \ldots x_n\rangle \in X^n$?

* We shall denote an algebraic system $\langle S, f_1, f_2, \ldots\rangle$ where S is a non-empty set $f_1, f_2 \ldots\ldots$ are the operations on S.

* Since the operations and relations on the set S define a structure on the element of S an algebraic system is called a algebraic structure.

Examples:

* Let I be set of integers consider the algebraic system $\langle I, +, * \rangle$ where $+$ and $*$ are the operations of addition and multiplication on I.

* (A-1) for any $a, b \in I$.

* $a+(b+c) = (a+b)+c$ → Associative.

* (A-2) for any $a, b \in I$

* $a+b = b+a$ → commutative.

* (A-3) ~~there~~ there exist a distinguished element $0 \in I$ such that for any $a \in I$

* $a+0 = 0+a = a$ (Identity law)

* $0 \in I$ is the Identity element under addition

* (A-4) for each $a \in I$ there exist an element in I denoted by $-a$ and called the negative of a such that $a+(-a) = a-a = 0$ (Inverse Law).

* (M-1) for any $a, b, c \in I$

$a*(b*c) = (a*b)*c \rightarrow$ Associative

* (M-2) for any $a, b, \in I$.

$a*b = b*a$ (commutative)

* (M-3) there exist a distinguished element $1 \in I$ such that for any $a \in I$.

$a*1 = 1*a = a$ (Identity law)

* (D), For any $a, b, c \in I$, the

$a*(b+c) = (a*b)+(a*c)$ (Distributive)

* (C) For any $a, b, c \in I$ and $a \neq 0$

$a*b = a*c \Rightarrow b = c$ (Left Cancelation Law)

* Consider a set $B = \{0, 1\}$ and the operations $+_2$ & $*_2$ on B given by the following tables

$+_2$	0	1
0	0	1
1	1	0

$2)\,2\,(1$
$\underline{2}$
0

$*_2$	0	1
0	0	0
1	0	1

1. which of the following system satisfies the properties of $\langle I, +, * \rangle$. i. $\langle Z_6, +_6, *_6 \rangle$

Sol Given $Z_6 = \{0, 1, 2, 3, 4, 5\}$

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

6)7(1
6
(1)

$*_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

* $\langle Z_6, +_6 \rangle$ satisfies closer and associative laws
* $\langle Z_6, +_6, *_6 \rangle$ satisfies Identity law
* $\langle Z_6, +_6 \rangle$ satisfies Inverse law.
* i.e $0^{-1} = 0$

$1^{-1} = 5$

$2^{-1} = 4$

$3^{-1} = 3$

$4^{-1} = 2$

$5^{-1} = 1$

* $\langle Z_6, *_6 \rangle$ are not satisfies the Inverse law.

Sub-algebra :-

* Let $\langle X, \circ \rangle$ be an algebraic system and $Y \subseteq X$ which is closed under the operation 'o' then $\langle Y, o \rangle$ is called a sub-algebra of $\langle X, o \rangle$

Homomorphism :-

* Let $\langle X, o \rangle$ and $\langle Y, * \rangle$ be two algebraic system of the same type in the sence that both o and * are binary operation (~~n-ary~~) (n-ary).
* A mapping $g: X \to Y$ is called Homomorphism (or) simply morphism from $\langle X, o \rangle$ to $\langle Y, * \rangle$ if x_1, x_2

Ex $$g(x_1 o x_2) = g(x_1) * g(x_2)$$

Defination

Let 'g' be a homomorphism from $\langle X, o \rangle$ to $\langle Y, * \rangle$.

* If $g: X \to Y$ is on to then g is called epimorphism.
* If $g: X \to Y$ is one to one, then g is called monomorphism.
* If $g: X \to Y$ is one to one onto or bijective then g is called an isomorphism.

Isomorphic

Let $\langle X, o \rangle$ and $\langle Y, * \rangle$ be two algebraic system of the same type if there exist an isomorphic mapping $g: X \to Y$, then $\langle X, o \rangle$ and $\langle Y, * \rangle$ are said to be Isomorphic.

1. prove that symmetric difference is commutative operation on sets.

Sol Consider any two sets A and B.

$$\therefore A + B = (A - B) \cup (B - A)$$

$A \oplus B = (A-B) \cup (B-A)$

$= \{x / x \in A \cup B \text{ and } x \notin A \cap B\}$

$= \{x / x \in B \cup A \text{ and } x \notin B \cap A\}$

$= B \oplus A$

2. If $f: S \to T$ is a homomorphism $\langle S, * \rangle$ to $\langle T, \Delta \rangle$ and $g: T \to P$ is also a homomorphism $\langle T, \Delta \rangle$ to $\langle T, \nabla \rangle$, Then $g \circ f : S \to P$ is a homomorphism $\langle S, * \rangle$ to $\langle P, \nabla \rangle$.

Sol: Given that

$f: S \to T$ and $g: T \to P$ is a homomorphism.

We have to show that $g \circ f : S \to P$ is a homomorphism.

If f is a homomorphism, Then for any $S_1, S_2 \in S$ we have $f(S_1 * S_2) = f(S_1) \Delta f(S_2)$ But for $f(S_1)$ and $f(S_2) \in T$.

We have $g(f(S_1) \Delta f(S_2)) = g(f(S_1)) \nabla g(f(S_2))$.

For any $S_1, S_2 \in S$ the composition function $g \circ f$ we have

$$g \circ f(S_1 * S_2) = g(f(S_1 * S_2))$$

$$= g(f(S_1) \Delta f(S_2)) \quad [\because f \text{ is a homomorphism}]$$

$$= g(f(S_1)) \nabla g(f(S_2)) \quad [\because g \text{ is a homomorphism}]$$

Hence

$g \circ f : S \to P$ is a homomorphism.

Semi groups and Monoids

Semigroups:

* Let 'S' be a non-empty set, 'o' be a binary operation on S. The algebraic system $\langle S, o \rangle$ is called a semigroup. If the operation circle is associative.

* In another words $\langle S, o \rangle$ is a semi group if for any $x, y, z \in S$, $x o (y o z) = (x o y) o z$

Monoids

* A semi group $\langle M, o \rangle$ with an identity element with respect to the operation 'o' (circle) is called a Monoid.

* In otherwords algebraic system $\langle M, o \rangle$ is a called a monoid if for any $x, y, z \in M$, $x o (y o z) = (x o y) o z$ and there exist an element $e \in M$ such that for any $x \in M$

$$x o e = e o x = x$$

NOTE: A monoid is always a semi group

Examples:

* Let 'N' be the set of natural numbers, then the algebraic system $\langle N, * \rangle$ is a Monoid with identity element "1"., $\langle N, + \rangle$ is not a monoid.

* If 'E' be the set of all +ve even numbers then $\langle E, + \rangle$ and $\langle E, * \rangle$ are semi groups but not monoids.

* Let 'S' be a non-empty set and $P(S)$ be its powerset. The algebraic systems

$\langle P(S), \cup \rangle$ &

$\langle P(S), \cap \rangle$ are monoids with the identities ϕ and S respectively.

Semigroup homomorphism

* Let $\langle S, * \rangle$ and $\langle T, \Delta \rangle$ be any two semi groups. A mapping $g: S \rightarrow T$ such that for any two elements $a, b \in S$.

* $g(a * b) = g(a) \Delta g(b)$ is called a semi group homomorphism.

Monoid homomorphism :-

Let $\langle M, *, e_M \rangle$ and $\langle T, \Delta, e_T \rangle$ be any two monoids. A mapping $g: M \rightarrow T$ such that for any two elements $a, b \in M$.

$$g(a * b) = g(a) \Delta g(b)$$

$$g(e_M) = e_T$$ is called a monoid homomorphism.

Theorem

Let $\langle S, * \rangle$, $\langle T, \Delta \rangle$ and $\langle V, \oplus \rangle$ be semigroups and $g: S \rightarrow V$ and $h: T \rightarrow V$ be semigroup homomorphisms. Then $(h o g): S \rightarrow V$ is a semi group homomorphism from $\langle S, * \rangle$ to $\langle V, \oplus \rangle$.

Proof

Given that

$\langle S, * \rangle$, $\langle T, \Delta \rangle$, $\langle V, \oplus \rangle$ are semigroups.

also given that $g: S \rightarrow V$ and $h: T \rightarrow V$ be semi group homomorphism.

we have to show that $(hog): S \to V$ is a semi group homomorphism.

Let $a, b \in S$

Then $hog(a*b) = h(g(a*b))$

$= h(g(a) \Delta g(b))$

$= h(g(a)) \oplus h(g(b))$

$= hog(a) \oplus hog(b)$

Therefore $\langle S, * \rangle$ to $\langle V, \oplus \rangle$ is a semi group homomorphism.

Theorem-2

Let $\langle S, * \rangle$ and $\langle T, \Delta \rangle$ be to semi groups and g be a semigroup homomorphism from $\langle S, * \rangle$ to $\langle T, \Delta \rangle$ corresponding to the homomorphism g, there exist a congruence relation R on $\langle S, * \rangle$. Defined by $xRy \iff g(x) = g(y)$ for $x, y \in S$.

Proof:-

Given $\langle S, * \rangle$ and $\langle T, \Delta \rangle$ be to semi groups.

$g: S \to T$ is a semi group homomorphism and R is equivalence relation on $\langle S, * \rangle$.

Let x_1, x_2, x_1', x_2' such that $x_1 R x_1'$ and $x_2 R x_2'$

conside $g(x_1 * x_2) = g(x_1) \Delta g(x_2)$

$= g(x_1') \Delta g(x_2')$

$= g(x_1' * x_2')$

* It follows that R is a Congruence relation on $\langle S, * \rangle$

Sub-semi groups and sub monoids

* Let $\langle S, * \rangle$ be a semi group and $T \subseteq S$. If the set T is closed ~~and~~ under the operation $*$, then $T, *$ is said to be a sub-semi group of $\langle S, * \rangle$.

* Let $\langle M, *, e \rangle$ be a monoid and $T \subseteq M$. If the set T is closed under the operation $*$ and $e \in T$, Then $T, *, e$ is said to be a sub-monoid of $\langle M, *, e \rangle$

* For any $a \in S$, the set consisting of all powers of a under the operation $*$ is a sub-semi group i.e $T = \{a, a^2, a^3, \ldots\}$ where $a^2 = a * a$ Then $\langle T, * \rangle$ is a cyclic semi group which is a sub-semi group of $\langle S, * \rangle$ generated by the element a.

Theorem-3

For any commutative monoid $\langle M, * \rangle$, the set Idempotent element of M forms a sub-monoid.

proof

Given that $\langle M, * \rangle$ is a commutative monoid

i.e $a * b = b * a$ $a, b \in M$

Since the identity $e \in M$ is Idempotent.

Let S be set of Idempotent of M ($S \subseteq M$)

i.e Let $a, b \in S$

Then $a * a = a$ and

$b * b = b$

Consider $(a * b) * (a * b) = a * (b * a) * b$

$= a * (a * b) * b$

$= (a * a^{-1}) * (b * b)$

$= a * b$

Hence $a * b \in S$ and $\langle S, * \rangle$ is a sub-monoid.

Groups:-

Let 'G' be a non-empty set in which an operation $*$ is defined then the algebraic structure $\langle G, * \rangle$ is said to be a group. If G satisfies the following axioms.

1. $\forall a, b \in G$, $a * b \in G$ → closure axiom
2. $\forall a, b, c \in G$, $a * (b * c) = (a * b) * c$ → Associative axiom
3. $\forall a \in G$, $\exists e \in G \Rightarrow a * e = e * a = a$ } Identity axiom.

   where e is an Identity element

4. $\forall a \in G$, $\exists a^{-1} \in G \Rightarrow a * a^{-1} = a^{-1} * a = e$

   a^{-1} is called inverse of a.

Abelian group

A Group $\langle G, * \rangle$ is said to an abelian group if $a * b = b * a$, $\forall a, b \in G$ (Commutative property)

Non-abelian group:-

* If the group doesnot satisfies commutative property, then the group 'G' is called non-abelian.

Finite and Infinite group:-

* If the no: of elements in a group 'G' is countable then 'G' is finite group otherwise infinite group.

order of a group

The no: of elements in a finite a group is called order of a group, It is denoted by $O(G)$ or $|G|$

* show that the set of integers Z with respect to addition is an infinite ablian group.

Given that $\langle z, + \rangle$

where $z = \{ \cdots -2, -1, 0, 1, 2, 3, \cdots \}$

and the binary operation '+' is the binary composition in Z.

i. closer property

Let $a, b \in z$, then $a+b \in z$

$\langle z, + \rangle$ satisfies closure property.

ii. Associative property:

Let $a, b, c \in z$ then $a+(b+c) = (a+b)+c$

$\langle z, + \rangle$ satisfies associative property.

iii. Identity property

Let $a \in z$, then there exist an element $0 \in z$ such that

$a+0 = 0+a = a$

'0' is the identity element under addition.

$\langle z, + \rangle$ satisfiers identity property.

iv. Inverse property:-

Let $a \in Z$, there exist an element $-a \in Z$. such that $a+(-a) = a-a = 0$

$-a$ is the additive inverse of a.

$\therefore \langle z, + \rangle$ is a group.

v. commutative property:-

Let $a, b \in z$, then $a+b = b+a$.

$\langle z, + \rangle$ is a abelian group.

The number of integers is infinite.

$\langle Z, + \rangle$ is an infinite abelian group.

Sub-group

Let $\langle G, * \rangle$ be a group and $S \subseteq G$ be such that it satisfies the following conditions.

1. $e \in S$, where e is the identity $\langle G, * \rangle$.
2. For any $a \in S$, $a^{-1} \in S$.
3. For $a, b \in S$, $a * b \in S$

Then $\langle S, * \rangle$ is called a sub-group of $\langle G, * \rangle$

** A subset $S \neq \phi$ of G is a subgroup of $\langle G, * \rangle$ iff for any pair of elements $a, b \in S$, $a * b^{-1} \in S$

Sol Given that $\langle G, * \rangle$ be a group.

assume that S is a sub group of G

Let we know that $a, b \in S$, then $a^{-1}, b^{-1} \in S$

$\therefore a * b^{-1} \in S$

Converse

Let us assume that $a, b \in S$ and $a * b^{-1} \in S$

we have to prove that S is a subgroup of G.

Taking $b = a$, then $a * a^{-1} = 1 = e \in S$

From $a, b, e \in S$, then

$a^{-1} * e = e * a^{-1} = a^{-1} \in S$

Similarly $b^{-1} * e = e * b^{-1} = b^{-1} \in S$

Finally $a, b^{-1} \in S$ we have

$a * b^{-1} \in S$

$\therefore \langle S, * \rangle$ is a sub group of $\langle G, * \rangle$

Group Homomorphism:

* Let $\langle G, * \rangle$ and $\langle H, \Delta \rangle$ be two groups a mapping $g: G \to H$ is called a group homomorphism from $\langle G, * \rangle$ to $\langle H, \Delta \rangle$

* If for any $a, b \in G$

$$g(a*b) = g(a) \Delta g(b)$$

$$g(e_G) = e_H$$

$$g(a^{-1}) = [g(a)]^{-1}$$

Kernel

* Let 'g' be a group homomorphism from $\langle G, * \rangle$ to $\langle H, \Delta \rangle$. The set of elements of 'G' which are mapped into e_H, the identity of H is called the kernel of the homomorphism g. and denoted by ker(g)

or

kernel of g

Theorem:-

The kernel of a homomorphism 'g' from a group $\langle G, * \rangle$ to $\langle H, \Delta \rangle$ is a sub group of $\langle G, * \rangle$

sol Given that $\langle G, * \rangle$ and $\langle H, \Delta \rangle$ be two groups. We have to show that kernel of g is a sub group of $\langle G, * \rangle$.

$g: G \to H$ is a homomorphism from $\langle G, * \rangle$ to $\langle H, \Delta \rangle$.

since $g(e_G) = e_H$, $e_G \in ker(g)$

For $a, b \in$ kernel(g) that is

$$g(a) = g(b) = e_{H'}$$

Then $g(a * b) = g(a) \Delta g(b)$

$$= e_H \Delta e_H$$

$$= e_H$$

$\therefore a * b \in$ Ker(g)

If $a \in$ Ker(g) then $g(a^{-1}) = [g(a)]^{-1} = e_H^{-1}$

$$= e_H$$

$\therefore a^{-1} \in$ ker(g) and kernel of g is a sub group of $\langle G, * \rangle$.

Lattices :-

A lattice is a partially ordered set or poset $\langle L, \leq \rangle$ in which every pair of elements $a, b \in L$ has a greatest lower bound of a subset $\{a, b\} \subseteq L$ and a least upper bound.

* The greatest lower bound of a subset $\{a, b\} \subseteq L$ will be denoted by $a * b$. and the least upper bound by $a \oplus b$.

* It is customary to call the GLB $\{a, b\} = a * b$ the meet or product of a and b and LUB $\{a, b\} = a \oplus b$ the Join or sum of a and b.

* Other symbols such as $\wedge$ and $\vee$ (or) $\cdot$ and $+$ are also used to denote the meet and Join of two elements respectively.

Some Properties of Lattices

* We shall first list sum of the properties of the two binary operations Meet and Join denoted by $*$ and $\oplus$ on a lattice $\langle L, \leq \rangle$ for any $a, b \in L$ we have

1. Idempotent property

$a * a = a$

$a \oplus a = a$

2. $a * b = b * a$
$a \oplus b = b \oplus a$ } Commutative Property.

3. $a * (b * c) = (a * b) * c$
$a \oplus (b \oplus c) = (a \oplus b) \oplus c$ } Associative Property

4. $a * (a \oplus b) = a$
$a \oplus (a * b) = a$ } absorption law

Theorem-1

Let $\langle L, \leq \rangle$ be a Lattice in which $*$ and $\oplus$ denote the operations of meet and Join respectively. For any $a, b, c \in L$, $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$

proof:-

Let $\langle L, \leq \rangle$ be a lattice in which $*$ and $\oplus$ denote the operations of meet and join respectively

We have show that

$a, b, c \in L$, $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$

For this it is enough to show that

i. $a \leq b \Leftrightarrow a * b = a$

ii. $a \leq b \Leftrightarrow a \oplus b = b$

i. First suppose that $a \le b$

To prove that $a * b = a$

By reflexive property $a \le a$ & $b \le b$

For the set $\{a, b\}$, a is the lower bound.

greatest lower bound (g.l.b) of $\{a, b\}$ is $a * b$

$\Rightarrow a \le a * b$ — (1)

Again $a * b$ is the g.l.b of $\{a, b\}$

$\Rightarrow a * b$ is the lower bound

$\Rightarrow a * b \le a, b$

$\Rightarrow a * b \le a$ — (2)

From (1) & (2)

$\therefore a = a * b$

$a \le b \Rightarrow a = a * b.$

<u>Converse</u>

suppose that $a * b = a =$ g.l.b of $\{a, b\}$

$\Rightarrow a \le a, b$

$\Rightarrow a \le b.$

$\therefore a \le b \Leftrightarrow a * b = a.$

$\therefore$ ii. Similarly we can prove that

$a \le b \Leftrightarrow a \oplus b = b$

$\therefore a \le b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b.$

<u>Theorem - 2</u>

Let $\langle L, \le \rangle$ be a lattice. For any $a, b, c \in L$, the following properties called isotonicity hold

$$b \le c \Rightarrow \begin{cases} a * b \le a * c \\ a \oplus b \le a \oplus c \end{cases}$$

Proof

Let $\langle L, \leq \rangle$ be a lattice and $a, b, c \in L$.

Now we have to prove that

$$b \leq c \Rightarrow \begin{cases} a * b \leq a * c \\ a \oplus b \leq a \oplus c \end{cases}$$

we know that

$$a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$$

For $b, c \in L$

we have

$$b \leq c \Leftrightarrow b * c = b \Leftrightarrow b \oplus c = c \quad (1)$$

First suppose that $b \leq c$.

i. To prove that

$$a * b \leq a * c$$

For this it is enough to show that $(a * b) * (a * c)$ $= a * b$

consider $(a * b) * (a * c) = a * (b * a) * c$

$$= a * (a * b) * c$$

$$= (a * a) * (b * c)$$

$$= a * (b * c) \text{ (from (1))}$$

$$= a * b$$

$\because$ $(a * b) * (a * c) = a * b$

$\therefore$ $(a * b) \leq (a * c)$.

$\therefore$ $b \leq c \Rightarrow a * b \leq a * c$

ii. To prove that

$$a \oplus b \leq a \oplus c$$

For this it is enough to show that

$$(a \oplus b) \oplus (a \oplus c) = a \oplus c$$

consider $(a \oplus b) \oplus (a \oplus c) = a \oplus (b \oplus a) \oplus c$

$= a \oplus (a \oplus b) \oplus c$

$= (a \oplus a) \oplus (b \oplus c)$

$= a \oplus (b \oplus c)$ (from ①)

$= a \oplus c$

$\therefore (a \oplus b) \oplus (a \oplus c) = a \oplus c$

$\therefore (a \oplus b) \leq (a \oplus c)$

$\therefore b \leq c \Rightarrow a \oplus b \leq a \oplus c.$

$\therefore b \leq c \Rightarrow \begin{cases} a * b \leq a * c \\ a \oplus b \leq a \oplus c \end{cases}$ Hence proved.

NOTE:-

1. $a \leq b \wedge a \leq c \Rightarrow a \leq (b \oplus c)$
2. $a \leq b \wedge a \leq c \Rightarrow a \leq (b * c)$
3. $a \geq b \wedge a \geq c \Rightarrow a \geq (b * c)$
4. $a \geq b \wedge a \geq c \Rightarrow a \geq (b \oplus c)$

Theorem-3

Let $\langle L, \leq \rangle$ be a Lattice for any $a, b, c \in L$ the following inequalities called the distributive inequalities hold.

i. $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

ii. $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

proof:-

Let $\langle L, \leq \rangle$ be a Lattice for any $a, b, c \in L$.

Now we have to prove that

i. $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

ii. $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

i. To prove that

$a * (b \oplus c) \geq (a*b) \oplus (a*c)$

By the definition of g.l.b

$a*b \leq a, b$ & $a*c \leq a, c$

$\Rightarrow a*b \leq a$ $\quad a*c \leq a$

'a' is the upper bound

The least upper bound of $\{a*b, a*c\}$

$= a*b \oplus a*c$

$\leq a \oplus a$

$\leq a$ — (1)

$a*b \oplus a*c$

we have $a*b \leq b$, $a*c \leq c$

The l.u.b $\{a*b, a*c\} = (a*b) \oplus (a*c)$

$\leq b \oplus c$

$\therefore (a*b) \oplus (a*c) \leq b \oplus c$ — (2)

From (1) & (2)

$(a*b) \oplus (a*c) \leq$ g.l.b$\{a, b \oplus c\}$

$(a*b) \oplus (a*c) \leq a * (b \oplus c)$

$\Rightarrow$ i. $a * (b \oplus c) \geq (a*b) \oplus (a*c)$ — (3).

Dualizing eq (3) [changing $*$ by $\oplus$ and $\oplus$ by $*$ and $\leq$ by $\geq$]

we get.

$\Rightarrow$ ii. $a \oplus (b*c) \leq (a \oplus b) * (a \oplus c)$

$\therefore$ Hence proved.

Theorem-4

Let $\langle L, \leq \rangle$ be a lattice for any $a, b, c \in L$, following inequalities called the modulus inequalities hold $a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$

proof:-

Let $\langle L, \leq \rangle$ be a lattice for any $a, b, c \in L$.

Now we have to prove that

$$a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$$

For $a, c \in L$.

We have $a \leq c \Leftrightarrow a * c = a \Leftrightarrow a \oplus c = c$ — (1)

from distributive inequality

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c) \text{ — (2)}$$

suppose that $a \leq c$

we have to prove that $a \oplus (b * c) \leq (a \oplus b) * c$

from (2) we have

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$
$$\leq (a \oplus b) * c \quad [\text{from}(1)]$$

converse

suppose that $a \oplus (b * c) \leq (a \oplus b) * c$ — (3)

from (2) and (3), we have

$$\cancel{(a \oplus b) *} (a \oplus c) = \cancel{(a \oplus b) *} c$$

By using left cancellation law.

$$a \oplus c = c$$

$$\Rightarrow a \leq c \quad [\text{from}(1)]$$

$\therefore a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$

Theorem-5

Show that in a lattice if $a \leq b \leq c$, then i. $a \oplus b = b * c$

ii. $(a * b) \oplus (b * c) = b = (a \oplus b) * (a \oplus c)$.

Proof:-

Let $\langle L, \leq \rangle$ be a Lattice.

for any $a, b, c \in L$

Such that $a \leq b \leq c$

i. we know that

$a \leq b \Leftrightarrow a * b = a$ — (1)

$\Leftrightarrow a \oplus b = b$ — (2).

$b \leq c \Leftrightarrow b * c = b$ — (3)

$\Leftrightarrow b \oplus c = c$ — (4)

$a \leq c \Leftrightarrow a * c = a$ — (5)

$\Leftrightarrow a \oplus c = c$ — (6)

from eq (2) and (3) we get

$$a \oplus b = b * c$$

ii. L.H.S

$(a * b) \oplus (b * c) = a \oplus b$

$= b$ — (7)

R.HS

$(a \oplus b) * (a \oplus c) = b * c$

$= b$ — (8)

from (7) and (8)

LHS = RHS

$\therefore (a * b) \oplus (b * c) = b = (a \oplus b) * (a \oplus c)$

Sub-Lattice

* Let $\langle L, *, \oplus \rangle$ be a lattice and let $S \subseteq L$ be a subset of L. The algebra $\langle S, *, \oplus \rangle$ is a sub lattice of $\langle L, *, \oplus \rangle \iff S$ is closed under both operations $*$ and $\oplus$.

Some special Lattices:-

1. complete Lattice:

A lattice is called complete if each of its non-empty sub-sets has a least upper bound and Greatest lower bound.

* obviously every final lattice must be complete. Also every complete lattice must have at least element and a greatest element. The least and greatest element of a lattice, if they exist are called the bounce of lattice and are denoted by 0 and 1 respectively.

* $\langle L, *, \oplus \rangle$ satisfies the following identities for any $a \in L$.

Properties:-

$$a \oplus 0 = a, \quad a * 1 = a \quad \text{--- (1)}$$

$$a \oplus 1 = 1, \quad a * 0 = 0 \quad \text{--- (2)}$$

complement of an element

* In a bounded lattice $\langle L, *, \oplus, 0, 1 \rangle$ and element $b \in L$ is called a complement of an element $a \in L$. If $a * b = 0$ and $a \oplus b = 1$

* From the identities eq (1) and eq (2) we have
$0 * 1 = 0$. and $0 \oplus 1 = 1$

which shows that 0 and 1 are complements of each other.

Complemented Lattice:-

* A Lattice $\langle L, *, \oplus, 0, 1 \rangle$ is said to be a complemented Lattice, if every element of L has atleast one complement.

Distributive Lattice

* A Lattice $\langle L, *, \oplus \rangle$ is called a distributive lattice. If for any $a, b, c \in L$.

properties:

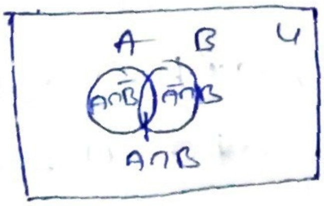
$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

The principle of inclusion - Exclusion:

The principle of inclusion - Exclusion for two sets.

If A and B are subsets of some universal set U, Then $|A \cup B| = |A| + |B| - |A \cap B|$



For three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Demorgan's law

* $|\bar{A}\cap\bar{B}\cap\bar{C}| = |\overline{A\cup B\cup C}| = |U| - |A\cup B\cup C|$

For n-sets

If A_i are finite subsets of a universal set, Then

$$|A_1\cup A_2\cup A_3 \ldots \cup A_n| = \sum_i |A_i|$$
$$= \sum_{i,j} |A_i\cap A_j| + \sum_{i,j,k} |A_i\cap A_j\cap A_k|$$
$$+ \ldots + (-1)^{n-1}|A_1\cap A_2\cap A_3 \ldots A_n|$$

A computer company requires 30 programers to handle systems programming jobs and 40 programers for applications programming. If the company appointes 55 programers to carry out these jobs, how many of these perform jobs of both types? How many handle only system programming jobs? How many handle only applications programing jobs?

Sol Let $|A|$ denote the set of programmers who handle system programming jobs and $|B|$ denotes the set of programmers who handle application programming jobs

$|A\cup B|$ is the set of programmers appointed to carry out these jobs.

we have $|A| = 30$

$|B| = 40$

$|A\cup B| = 55$

$|A\cap B| = ?$

By addition Rule.

i. $|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cap B| = |A| + |B| - |A \cup B|$

$= 30 + 40 - 55$

$= 15$

∴ 15 programers to perform jobs.

ii. The number of programmers who handle only systems programming jobs is

$|A - B| = |A| - |A \cap B|$

$= 30 - 15$

$= 15$

∴ 15 system programers to perform jobs.

iii. The number of programmers who handle only application programming jobs is

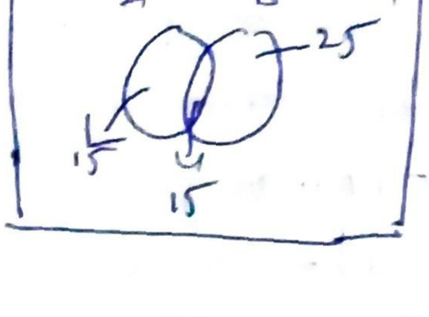
$|B - A| = |B| - |A \cap B|$

$= 40 - 15$

$= 25.$

∴ 25 Application programmes to perform job.

Venn diagrams



2. In a class of 52 students, 30 are studying C++; 28 are studying pascal; and 13 are studying both languages how many in this class are studying atleast one of these languages? How many are studying neither of these languages,

Sol. Let |U| denotes the set of all students in the class.

|A| denote the set of students in the class who are studying C++

|B| denotes the set of students in the class who are studying pascal.

$|A \cap B|$ → Then the set of students in the class who are studying both languages.

The set of students who are studying atleast one ~~these~~ two languages. $|A \cup B|$

The set of students who are studying neither of these languages is $|\overline{A \cup B}|$

We have,

$|U| = 52$

$|A| = 30$

$|B| = 28$

$|A \cap B| = 13$

$|A \cup B| = ?$

$|\overline{A \cup B}| = ?$

By addition Rule,

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 30 + 28 - 13$$
$$= 45$$

ii $|\overline{A \cup B}| = |U| - |A \cup B|$

$= 52 - 45$

$= 7.$

3. In a sample of 100 Logic Chips, 23 have a defect D_1, 26 have a defect D_2, 30 have a defect D_3, 7 have defects D_1 and D_2, 8 have defects D_1 and D_3, 10 have defects D_2 and D_3. and 3 have all the 3 defects. Find the no. of Chips having i. atleast one defect.
ii. No defect.

Sol Let 'U' denotes all logic chips and A, B, C denotes the sets of chips having defects D_1, D_2, D_3 respectively

$|U| = 100$

$|A| = 23$

$|B| = 26$

$|C| = 30$

$|A \cap B| = 7$

$|A \cap C| = 8$

$|B \cap C| = 10.$

$|A \cap B \cap C| = 3$

i. The set of logic chips having atleast one defect is $(A \cup B \cup C)$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 23 + 26 + 30 - 7 - 8 - 10 + 3$

$= 57$

ii. The set of Logic Chips having no Chip deferect $(\overline{A \cup B \cup C})$

$$|\overline{A \cup B \cup C}| = u - |A \cup B \cup C|$$
$$= 100 - 57$$
$$= 43.$$

4. Suppose that 200 faculty memembers can speak French and 50 can speak Russian, while only 20 can speak both French and Russian. Ho many faculty members can speak either french or Russian

Sol Given that

French = 200 => $|F| = 200$

Russian = 50 → $|R| = 50$

$F \cap R = 20$

Faculty members can speak either french or Russian $F \cup R = |F| + |R| - |F \cap R|$

$= 200 + 50 - 20$

$= 230$

Pigean Hole principle

If 'm' pigeans occupy 'n' pigeans holes, and if $m > n$, then 2 or more pigeans occupy the same pigean hole this statement is known as the pigean hole principle.

Generalization of the pigean hole principle:-

If 'm' pigeans occupy 'n' pigean holes, than atleast one pigean hole must contain $(P+1)$ or more pigeans, where $P = \frac{m-1}{n}$

1. Prove that if 30 dictionaries contains total of 61,327 pages then atleast one of the dictionaries must have atleast 2045 pages.

Sol. Treating the pages as pigeons and dictionaries as pigeon holes.

we find by using the Generalized pigeon hole principle that atleast one of the dictionaries must contain $(p+1)$ or more pages.

$$p+1 = \frac{m-1}{n} + 1$$

$$= \frac{61,327-1}{30} + 1$$

$$= 2045.2$$

$$\cong 2045$$

2. prove that in any set of 29 persons atleast 5 persons must have been born on the same day of the week.

Sol Treating the persons as pigeons and Treating the 7 days of week as 7 pigeon holes.

we find by using Generalized pigeon hole principle that atleast one day of the week is assigned to $(p+1)$

$$p+1 = \frac{m-1}{n} + 1$$

$$= \frac{29-1}{7} + 1 = \frac{28}{7} + 1 = 5$$

3. How many persons must be choosen inorder that atleast 5 of them will have birthdays in the same calender month.

Let 'm' be required number of persons.

Since the no: of months over which the birthdays are distributed is 12.

The least no: of who have their birth days in the same month is, by the generalized pigean hole princle

$$\frac{m-1}{n} + 1 = 5$$

$$\frac{m-1}{12} + 1 = 5$$

$$m-1 = 4 \times 12$$

$$\boxed{m = 49}$$

* The required no: of persons is 49.

Cyclic Group:-

A Group 'G' is said to be cyclic if there exist an element $a \in G$, such that every element of 'G' is an integral Power of a.

Here 'a' is called the Generator of 'G' and is denoted by $\langle a \rangle$

i.e $G = \{a^n / n \in z\}$

Theorem:-

Show that Every cyclic Group of order n is isomorphic to the group $\langle z_n, +_n \rangle$

proof:

Let the cyclic group $\langle G, * \rangle$ of order 'n' be Generated by an element $a \in G$.

so that the elements of g are a, $a^2, a^3, \ldots a^n = e$

* Define $g : Z_n \to G$ such that $g([1]) = a$

* so that $[1]$ is the generator of Z_n

Then $(g[j]) = a^j$; for $j = 0, 1, 2, \ldots n-1$

$\therefore$ Hence the isomorphism is established,