UNIT - III

Elementary combinatorics

Introduction:-

In daily life many a times one needs to findout the no: of all Possible outcomes for a series of events for instance, in how many ways can a panel of the judges comprising of 6 men and four women be choosen from among 15 men and 38 women.

How many different 10 Lettered PAN number can be generated such that the five Letters are capital alphabets the next four are digits and last again a capital Letter. For solving these problems mathematical theory are used counting mainly encompasses fundamental counting rule, the Perimetation rule and the combinational rule.

Basics of counting:-

* If 'X' is a set, Let us use |X| to denote the number of elements in X.

The rules of sum and product

The rule of sum and rule of product are used to decompose difficult counting problems into simple problems.

The rule of sum

If a sequence of tasks $T_1, T_2, T_3 \ldots T_n$ can be done in $w_1, w_2, w_3 \ldots w_n$ ways respectively. [The condition is That no tasks can be Perfomed simultaneously] Then the no: of ways to do one of these tasks is $w_1 + w_2 + w_3 + \dots$

$\dots + w_n$

* If we consider two tasks A and B which are distant ($A \cap B = \phi$) then the mathematically

$$|A \cup B| = |A| + |B|$$

1. In How many ways can will draw a heart or a speade from an ordinary deck of playing cards. i. A heart or an ace ii. An ace or king iii. A card numbered to through 10. iv. A numbered card or a king

Sol: Since There are 13 hearts and 13 spades we may draw a heart or spade in

$13 + 13 = 26$ ways

i. we may draw a heart or an ace in

$13 + 3 = 16$ ways

already one ace in that.

* Since there are only 3 Aces that are not hearts

ii. we may draw an ace or king in

$4 + 4 = 8$ ways

iii. There are 9 numbered cards 2 through 10 in each of 4 suits, (head, diamond, spade, clubs)

we may choose a numbered card in $4 \times 9 = 36$ ways

iv. we may choose a numbered cards or a king in $36 + 4 = 40$ ways.

2. How many ways can we get a sum of four or of 8 when two distigusiable dies (one is red and another is white) are rolled? How many ways can we get even sum.

Sol: Let us label the outcome of a one (1) on the red die and a 3 on the white die as the ordered pair $\langle 1,3 \rangle$.

Then the outcomes $\langle 1,3 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle$ are the only one's whose sum is 4.

There are 3 ways to obtain the sum four.

Similarly we obtain the sum 8 from the outcomes $\langle 2,6 \rangle, \langle 3,5 \rangle, \langle 4,4 \rangle, \langle 5,3 \rangle, \langle 6,2 \rangle$ - There are 5 ways to obtain sum 8.

There are $3+5=8$ outcomes whose sum is 4 or 8

$$3+5=8.$$

The no: of ways to obtain an even sum is the same the no: of ways to obtain either the sum is 2, 4, 6, 8, 10, 12.

For '2' we have one way

i.e $\langle 1,1 \rangle$

For '4' we have 3 ways

i.e $\langle 1,3 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle$ –

For '6' we have 5 ways

i.e $\langle 1,5 \rangle, \langle 2,4 \rangle, \langle 3,3 \rangle, \langle 4,2 \rangle, \langle 5,1 \rangle$

For '8' we have 5 ways

i.e $\langle 2,6 \rangle, \langle 3,5 \rangle, \langle 4,4 \rangle, \langle 5,3 \rangle$

For '10' we have 3 ways

i.e $\langle 4,6\rangle$, $\langle 5,5\rangle$, $\langle 6,4\rangle$

For '12' we have 1 way

$\langle 6,6\rangle$.

There are $1+3+5+5+3+1 = 18$ ways to obtain an even sum.

3. How many ways can we get a sum of 8 when 2 undistinguishable dies are rolled? An even sum is.

Sol Had the dice be indistinguisable, we would obtain a sum of 8 by outcomes $\langle 2,6\rangle$, $\langle 3,5\rangle$, $\langle 4,4\rangle$, $\langle 5,3\rangle$, $\langle 6,2\rangle$

Since the dices are similar $\langle 2,6\rangle$ and $\langle 6,2\rangle$, and as well $\langle 3,5\rangle$ as $\langle 5,3\rangle$ cannot be differentgated and we obtain sum of 8 with the role of two similar dices in only 3 ways i.e $\langle 2,6\rangle$, $\langle 3,5\rangle$, $\langle 4,4\rangle$.

The no: of ways to obtain an even sum is the same as the no: of ways to obtain either the sum is 2, 4, 6, 8, 10, 12

For 2 we have 1 ways, i.e $\langle 1,1\rangle$.

For 4, we have 2 ways $\langle 1,3\rangle$, $\langle 2,2\rangle$

For '6' we have 3 ways.

$\langle 1,5\rangle$, $\langle 2,4\rangle$, $\langle 3,3\rangle$.

For '8' we have 3 ways

i.e <2,6>, <3,5>, <4,4>.

For '10' we have 2 ways

i.e <4,6>, <5,5>

For '12' we have → 1 way

i.e <6,6>.

There are 1+2+3+3+2+1 = 12 ways to obtain an even sum in indistiguisable.

Product Rule

If a sequence of task $T_1, T_2, T_3, \ldots, T_n$ are can be done in $w_1, w_2, w_3, \ldots, w_n$ ways respectively and every task arives after occurence of the previous task, Then there are $w_1 \times w_2 \times w_3 \ldots \times w_n$ ways to perform the task.

Mathematically if a task 'B' arrives 'A', then $|A \times B|$ (no: of ways to perform A and B tasks)

$|A \times B| = |A| \times |B|$.

1. If 2 distinguisable dies are rolled, in how many ways can they fall? If 5 distinguisable dies are rolled, how many possible out comes. How many ways if 100 distinguisable dices are tossed?

Sol The first die can fall in 6 ways

The second die can fall in 6 ways

If two dies are rolled, then the possible no: of out comes are $6^2 = 6 \times 6 = 36$ ways.

Also 3rd, 4th, 5th die each have 6 possible outcomes. There are 6^5 possible outcomes when all 5 dies are rolled.

Simillarly we have 6^{100} possible out comes when all 100 dies are tossed.

2. Suppose that the license plates of a certain state requires 3- English letter followed by 4-digits. i. How many different plates can be manufactured if repetition of Letters and digits are allowed.

ii. How many plates are possible if only the letters can be repeated.

iii. How many are possible if only the digits can be repeated.

iv. How many are possible if no repetitions ~~are~~ allowed at all

Sol: We have 26-English Letters and 10-digits

i. No: of plates if repetitions of digits and Letters are allowed.

$\Rightarrow 26 \times 26 \times 26 = 26^3$ (Letters)

$\Rightarrow 10 \times 10 \times 10 \times 10 = 10^4$ (digits)

∴ Required solution is $= 26^3 \times 10^4$

ii. The No: of plates are possibles with repetition of Letters is allowed,

$26 \times 26 \times 26 \longrightarrow$ (Letters)

$10 \times 9 \times 8 \times 7$ (digits)

∴ required solutions is $\cancel{26 \times 26 \times 26 \times 10^4} =$
$26^3 \times 10 \times 9 \times 8 \times 7$

iii. The no: of plates are possible with repetition of digits are allowed.

$\cancel{10 \times 10 \times 10 \times 10}$

$10 \times 10 \times 10 \times 10 = 10^4$

Required solution $= 26 \times 10 \times 9 \times 8 \times 7$

$10^4 \times 26 \times 25 \times 24 =$

iv. The no: of plates are possible without repetitions of digits and letters is $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 =$

3. i. How many 3-digit numbers can be formed using digits 1, 3, 4, 5, 6, 8 & 9?

ii. How many can be formed if no digit can be repeated.

Sol i. The Required number have 3-digits

The digits are 1, 3, 4, 5, 6, 8, 9

i. Since repetition of number is allowed, we have $7 \times 7 \times 7 = 7^3 = 343$ possible numbers

ii. The no: of possible ways without repetition, is $7 \times 6 \times 5 = 210$.

4. How many different license plates are there that involve 1, 2 and 3 letters followed by by 4-digits.

Sol The no: of plates with '1' letter followed by 4-digits is 26×10^4

The no: of plates with '2' Letter followed by 4-digits is $26^2 \times 10^4$

The no: of plates with '3' Letter followed by 4-digit is $26^3 \times 10^4$

These events are mutually exclusive or disjoint. so we can apply the sum rule

$(26 \times 10^4) + (26^2 \times 10^4) + (26^3 \times 10^4)$

$\doteq (26 + 26^2 + 26^3) 10^4$

5. How many 3-digit numbers are there which are even and have no repeated digits.

Sol We know that an even number ends with 0, 2, 4, 6, 8

Here we can take 2 cases

i. number ends with zero

ii. Numbers not ends with zero.

i. There are 9 possibility for the first digit and 8 possibilities for the second digit since repetitions are not allowed

The number of 3 digit number ends with zero is $\underline{9} \times \underline{8} \times \underline{1} \rightarrow$ one zero value.

$= 72$

ii. We have four choices for the last digit (2, 4, 6, 8) when this digit is specified, we have 8 possibilities for the 1st digit why because no number starts with zero.

Also there are '8' choices for the second digit (include zero)

The no: of three digit number not end's with zero is $8 \times 8 \times 4 = 256$

These two cases are mutually exclusive

so we can apply sum rule.

⇒ The no. of three digit even number with out repetitions is $72+256$

$= 328$

Permutations and combinations

Permutations

* An ordered selection or arrangements of objects from a set of 'n' objects is called a permutation. It is denoted by nP_r (or) $P(n,r)$

* If all the elements are distinct and repetition is not allowed, then by applying product rule.

* $P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

* This is the problem of counting the no. of ways of arranging 'r' objects from a set of 'n' objects when the repetition is not allowed and they ordered matters

* If $r=n$, we have $P(n,n) = n!$

combinations

* An unordered selection of 'r' objects from a set of 'n' objects is called a combination. It is denoted by nC_r (or) $C(n,r)$.

* If all the elements are distinct and repetition is not allowed by sum rule.

* $C(n,r) = \frac{n!}{(n-r)!\,r!}$ ∵ $[C(n,n) = 1]$

$\boxed{nCn = 1}$

* This is the no: of ways of selecting 'r' elements from a set of 'n' objects when repetition is not allowed and ordered does not matter

1. How many different strings or sequences of length '4' can be found using letters of the word FLOWER.

Sol A Given word has 6 Letters all of which are distinct

The required no: of strings is

$$P(6,4) = {}^{6}P_{4} = \frac{6!}{(6-4)!}$$

$$= \frac{6!}{2!} = 360$$

2 Find the no: of permitations of the letters of the word SUCCESS.

Sol The Given word has 7 letters of which 3 are 'S', 2 are 'C' and 1 each are 'U' and 'E'

∴ The required no: of permitations is

$$P(SUCCESS) = \frac{7!}{3!\,2!\,1!\,1!}$$

$$= \frac{7\times\cancel{6}\times5\times4\times3\times\cancel{2}\times1}{\cancel{3}\times\cancel{2}\times1\times\cancel{1}\times1}$$

$$= 420$$

3 Find the no: of Permitations of the Letters of the word MASSASAUGA. In how many of these all fours 'A's are together? how many of them begin with 'S'?

Sol The Given word has 10 Letters of which 3 are 'S' and 4 are 'A' and 1 each are 'M', 'U', 'G'

i. The required no. of permutations is $\frac{10!}{4!3!1!1!1!} = 25200$

ii. In a permutations all A's to be together, we treat all of A's are as one single letter. Then the letter to be permuted read (AAAA) S, S, S, M, U, G and

The required no. of permutations $= \frac{7!}{1!3!1!1!1!} = 840$

iii. For permutations begining with 'S', then occurs 9 open positions to fill, where 2 are 'S', 4 are 'A', and 1 each are M, U, G.

The required no. of permutations is $= \frac{9!}{4!2!1!1!1!} = 7560$

4. In how many ways can the letters of the word 'MONDAY' be arranged. How many of them begin with 'M' and end 'Y'? How many of them doesn't begin with 'M' but end 'Y'.

Sol: The word 'MONDAY' consists of '6' distinct letters.

The no. of permutations possible with this letter is $P(6,6) = {}^6P_6 = 6! = 720$

ii. If 'M' occupies the 1st place and 'Y' occupies last place, Then there are 4 letter (O,N,D,A) left to be arranged in 4 places. M - - - - Y between M and Y.

The no. of possible arrangements $= 4! = 24$

iii. If 'M' doesnot occupie the first place but 'Y' occupies the last place, the 1st place can be occupied in '4' ways (By any one of the letter (O, N, D, A).

From the 2nd place again four letter available including M, The third, fourth, fifth places can be filled by 3, 2, 1 ways.

Hence by product Rule the required no: of arrangement is $= 4 \times 4 \times 3 \times 2 \times 1$

$= 96.$

5. Consider the set $\{a, b, c, d\}$. In how many ways can we select 2 of these letters when

i. order matters

ii. order doesnot matters

Sol If i. order matters but repetitions is not allowed $n=4, r=2$

The no: of ways of selecting 2 from 4 is

$$^nP_r = {}^4P_2 = \frac{4!}{(4-2)!} = 12$$

ii. Order doesnot matters but repetitions are not allowed $n=4, r=2$

The no: of ways of selecting 2 from 4 is

$$^nC_r = {}^4C_2 = \frac{4!}{(4-2)!\,2!} = \frac{4!}{4} = 6$$

6. A man has 7 relatives 4 of them are ladies and 3 gentle mens and its wife has 7 relatives and 3 of them are ladies and 4 gentle man. In How many ways can they invite a dinner party of 3 ladies, 3 gentle man's, so that there are 3 of man's relatives and 3 of wife's relatives.

Sol They can invite '4' possible ways.

i. A man invities '3' gentlemens and his wife invites '3' ladies

$$^3C_3 \times {}^3C_3 = 1 \times 1 = 1$$

ii. A man invities '3' ladies and his wife invites '3' gentlemen ~~ladies~~.

$$(\because {}^nC_r = {}^nC_{n-r})$$

$$^4C_3 \times {}^4C_3 = 4 \times 4 = 16$$

iii. A man invities '2' gentlemans, 1 ladie and his wife invities '2' ladies, 1 gentleman.

$$({}^3C_2 \times {}^4C_1) \times ({}^3C_2 \times {}^4C_1)$$

$$= (3 \times 4) \times (3 \times 4)$$

$$= 12 \times 12$$

$$= 144$$

iv. A man invities '1' gentleman, 2 ladies, and his wife invities '1' ladie, '2' gentleman.

$$({}^3C_1 \times {}^4C_2) \times ({}^3C_1 \times {}^4C_2)$$

$$= (3 \times 6) \times (3 \times 6)$$

$$= 18 \times 18$$

$$= 324$$

$\therefore$ The total no: of possible ways $= 1 + 16 + 144 + 324 = 485$

7. How many committies of 5 with a given Chair person can be selected from 12 persons.

Sol The chair person can be choosen in $12C_1 - 12$ ways and the other '4' on the committe can be choosen in $^{11}C_4$ ways.

$\therefore$ The possible no: of such committies $= {}^{12}C_1 \times {}^{11}C_4 = 12 \times 330 = 3960$

8. Find the no: of commities of '5' that can be selected from 7 men and 5 women if the commite is to consist of atleast one men and atleast '1' women

Sol/ From the given '12' persons the no' of commities of '5' that can be selected is $12C_5$

Among these possible commities there are $7C_5$ commities consisting of 5 men and $5C_5$ commities consisting of 5 women.

The no: of commities containing atleast '1' men and atleast '1' women is

$$12C_5 - 7C_5 - 5C_5$$

$$= 792 - 21 - 1$$

$$= 770.$$

Restricted Permitations

i. The number of permitations of 'n' different objects taken 'r' at a time in which k particular objects donot occur is

$$\boxed{P(n-k, r)}$$

ii. The number of permitations of different objects taken 'r' at time in which 'k' particular objects are always present is

$$\boxed{P(n-k, r-k) \times P(r, k)}$$

Restricted combinations

i. The number of combinations of 'n' different objects taken 'r' at a time when 'k' particular objects never occur is given by $\boxed{C(n-k, r)}$

ii. The number of combinations of 'n' different objects taken 'r' at a time when 'k' particular objects always occur is given by

$$\boxed{C(n-k, r-k)}$$

① In how many ways can a cricket team of '11' be choosen out of batch of 15 players? i. How many of them in each build include a particular player. ii. Exclude a particular player.

SOL. The no: of ways of selecting '11' players out of 15 is $^{15}C_{11}$ ways → 1365

i.
The no: of ways in which a particular include is $C(n-k, r-k) = C(15-1, 11-1)$

$= C(14, 10)$

$= {}^{14}C_{10} = 1001$

ii. The no: of ways in which a particular exclude is $C(n-k, r) = C(15-1, 11)$

$= C(14, 11)$

$= {}^{14}C_{11} = 364$

• Perimitations with repetitions allowed

When there is no restriction on the number of times a particular element may occur in the 'r' permitations of a set of 'n' elements their are 'n' ways to select an element of the set per each of 'r' positions,

* Hence by the multiplication rule the no: of posibilities $n \times n \times n \times \cdots$ to r factors $= n^r$. Thus.

* Thus the no: of r permitations of a set of n objects with repetition allowed is n^r.

1. A bit is either 0 or 1 : a bit is a sequence of 8 bits. Find i. The no: of bytes That can be formed from 8 bits.

ii. The no: of bytes that begin with 11 and end 11

iii. The no: of bytes that begin with 11 and does not end with 11.

iv. The no: of bytes that begin with 11 or end with 11.

Sol

i. since the bits 0 or 1 can repeat the 8 position can be filled up either by 0 or 1 in 2^8 ways

* Hence the no: of bytes that can be formed from 8 bits is $2^8 = 256$ ways,

ii. Keeping two positions at the begining by 11 and the last two position by 11, there are 4 open positions which can be filled up in 2^4 ways.

* The no: of bytes that begin with 11 and end 11 is $2^4 = 16$

iii. Keeping two positions at the begining by 11, The remaining 6 open positions can be filled up by 2^6 ways = 64 ways

From these we subtract the no: of bytes that end with 11.

* The required number is $2^6 - 2^4$

$= 64 - 16$

$= 48$

iv. 64 bytes begin with 11, Similarly 64 bytes end with 11

16 bytes begin and end with 11

* The required number is $64+64-16$
$= 128-16$
$= 112$

~~permutation~~

Combinations with repetitions

* suppose we wish to select with repetition the combination of 'r' objects from a set of 'n' distinct objects. The number of such selections is given by

$$C(n+r-1, r) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{(n-1)!\,r!}$$

$$= \binom{r+n-1}{r} = C(r+n-1, n-1)$$

* In other words $C(n+r-1, r) = C(r+n-1, n-1)$ represents the no: of combinations of n' distinct objects taken 'r' at a time with repetitions allowed.

* The following are other interpretations of this number.

i. $C(n+r-1, r) = C(r+n-1, n-1)$ represents the no: of ways in which 'r' identical objects can be distributed among n' distinct containers.

ii. $C(n+r-1, r) = C(r+n-1, n-1)$ represents the no: of non-negative integer solutions of the equation $x_1+x_2+x_3+\dots+x_n = r$

1. A bag contains coins of seven different denominations with atleast one dozen coins in each denomination. In how many ways can we select a dozen coins from the bag.

Sol The selection consists in choosing with repetitions $r=12$ points of 7 different dimensions.

The no: of ways of making the selection is $C(n+r-1, r)$

$= C(7+12-1, 12)$

$= C(18, 12)$

$= {}^{18}C_{12} = 18564$

2. In How many ways can we distribute 10 identical marbles among 6 distinct containers

Sol No: of identical marbles $r=10$

No: of distinct containers $n=6$

∴ The no: of ways of making selection is $C(n+r-1, r)$

$= C(6+10-1, 10)$

$= C(15, 10)$

$= {}^{15}C_{10}$

$= 3003$

3. Find the no: of non-negative integer solutions of the equation $x_1+x_2+x_3+x_4+x_5=8$

Sol Given equation $x_1+x_2+x_3+x_4+x_5=8$

Comparing with $x_1+x_2+\cdots\cdots+x_n=r$

Here $n=5$, and $r=8$

The no: of non-negative integer solution is $C(n+r-1, r)$

$= C(5+8-1, 8) = C(12, 8) = {}^{12}C_8 = 495$

4. Find the no: of integer solutions of $x_1+x_2+x_3+x_4+x_5=30$, where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2; x_5 \geq 0$

Sol Given $x_1+x_2+x_3+x_5=30$, where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0$. Then y_1, y_2, y_3, y_4, y_5 are all the non-negative integers. when written integers 4 y' the eq $x_1+x_2+x_3+x_4+x_5=30$

Let $y_1=x_1-2$, $y_2=x_2-3$, $y_3=x_3-4$, $y_4=x_4-2$, $y_5=x_5$.

$$x_1+x_2+x_3+x_4+x_5=30$$

$$y_1+2+y_2+3+y_3+4+y_4+2+y_5=30$$

$$y_1+y_2+y_3+y_4+y_5=30-11$$

$$y_1+y_2+y_3+y_4+y_5=19$$

comparing with $x_1+x_2+\cdots\cdots+x_n=r$.

Here $n=5, r=19$

The no: of non-negative integer solution,

$$C(n+r-1, r) = C(5+19-1, 19)$$
$$= C(23, 19)$$
$$= {}^{23}C_{19} = 8855$$

5. Find the no: of +ve integer solutions of the equation $x_1+x_2+x_3=17$.

Sol Given $x_1+x_2+x_3=17$ where $x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$

Then y_1, y_2, y_3 are the non-negative integers when written integer of 'y' the eq $x_1+x_2+x_3=17$

Let $y_1=x_1-1$, $y_2=x_2-1$, $y_3=x_3-1$

$$x_1+x_2+x_3=17$$

$$y_1+1+y_2+1+y_3+1=17$$

$$y_1+y_2+y_3=14$$

comparing with $x_1+x_2+\cdots+x_n=r$

Here $n=3$, $r=14$

The no: of non-negative integers is

$$C(n+r-1, r) = C(3+14-1, 14)$$

$$= C(16, 14)$$

$$= {}^{16}C_{14}$$

$$= 120.$$

6. In how many ways can we distribute 12 ideantical pencils to 5 children so that Every child gets atleast one pencil.

Sol) First we distribute '1' pencil to each children. Then remaining 7 pencils to be distributed.

The no: of ways of distributing these 7 pencils to 5 childrens is the required number. Here $n=5$, $r=7$

The required number is $C(n+r-1, r)$

$$= C(5+7-1, 7)$$

$$= C(11, 7)$$

$$= {}^{11}C_7$$

$$= 330.$$

7. In how many ways can we distributed 7 apples and 6 organes among 4 children, so That each child gets atleast one apple.

Sol) First we distibute one apple to each child. The remaining 3 apples can we distributed among the 4 children is $C(n+r-1, r) = C(4+3-1, 3) = 20$ ($n=4, r=3$)

Also 6 oranges can we distributed among the 4 children is $n=4, r=6$

$$C(n+r-1, r)$$
$$= C(4+6-1, 6)$$
$$= C(9,6)$$
$$= {}^{9}C_{6}$$
$$= 84.$$

By using product Rule the no. of ways of distributing the given fruits is

$$= 20 \times 84$$
$$= 1680$$

8. Find the no. of unordered sample up size 5 (Repetition allowed) from the set {a, b, c, d, e, f}
i. No further restrictions.
ii. 'a' occurs atleast twice.
iii. 'a' occurs exactly twice.

Sol i. $r=5, n=6$

The required number is $C(n+r-1, r)$
$$= C(6+5-1, 5) = {}^{10}C_5 = 252$$

ii. Since 'a' occurs atleast twice we have to find the unordered samples of size 3 from the 6 element set

$\underline{a}\ \underline{a}$ (- - -)

$n=6, r=3$

The required number is $C(n+r-1, r)$
$$= C(6+3-1, 3)$$
$$= C(8,3)$$
$$= {}^{8}C_3 = 56$$

iii. since 'a' occurs ~~atleast~~ exactly twice we have to find the unordered sample of size 3 from the 5 element set {b, c, d, e, f}

$r=3, n=5$

The required number is $C(n+r-1, r)$
$$= C(5+3-1, 3) = {}^{7}C_3 = 35$$

9. Find the number of distinct terms in the expansion of $(x_1+x_2+x_3+x_4+x_5)^{16}$.

By using Multinomial Theorem

$$(x_1+x_2+x_3+\cdots\cdots+x_t)^n = \binom{n}{n_1, n_2, n_3, \ldots n_t} x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$$

(or)

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_t!}$$

Now $(x_1+x_2+x_3+x_4+x_5)^{16} = \binom{16}{n_1, n_2, n_3, n_4, n_5} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5}$

where each n_i is a non-negative integer and sum of n_i is 16.

i.e $n_1+n_2+n_3+n_4+n_5 = 16$

Comparing with $x_1+x_2+x_3+x_4+x_5+\cdots\cdots+x_n = r$

$n=5,\ r=16$

The no: of distinct terms in this expansion

is $C(n+r-1, r)$

$C(5+16-1, 16)$

$= {}^{20}C_{16}$

$= 4845$

Circular permitations

There are $(n-1)!$ permitations of 'n' distinct objects in a circle.

Ex: In how many ways can 5 children arrange them self in a aing.

Sol There $n=5$

The no: of circular permitations $(n-1)! = (5-1)! = 4! = 24$

Binomial Theorem

* one of the basic properties of $c(n,r) = {}^nC_r$ is that it is the coefficient of $x^r y^{n-r}$ is the expansion of the expression $(x+y)^n$, where x and y are any real numbers

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

for $\binom{n}{r}$

* The numbers $r = 0, 1, 2, \ldots n$.

In the above result are known as the binomial coefficients.

Multinomial Theorem

* For positive integers n, t, the coefficients of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_t^{n_t}$ in the expansion of $(x_1+x_2+x_3+\ldots \ldots + x_t)^n$ is $\binom{n}{n_1, n_2, n_3, \ldots n_t}$, where each n_i is a non-negative integer $\leq n$ and $n = n_1+n_2+n_3+\ldots\ldots+n_t$.

(or)

The General Term in the expansion of

$$(x_1+x_2+x_3+\ldots\ldots+x_t)^n = \binom{n}{n_1, n_2, n_3, n_4 \ldots n_t} (x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_n^{n_t})$$

(or)

$$\frac{n!}{n_1! n_2! n_3! n_4! \ldots n_t!}$$

1. prove the following identities for a +ve integer n

i. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots\ldots + \binom{n}{n} = 2^n$

ii. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots\ldots \binom{n}{n}(-1)^n = 0$

Sol By using Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \quad \text{——} \ (1)$$

put $x=1, y=1$

$$(1+1)^n = \sum_{r=0}^{n} \binom{n}{r} (1)^r (1)^{n-r}$$

$$\Rightarrow \sum_{r=0}^{n} \binom{n}{r} = 2^n$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots\cdots + \binom{n}{n} = 2^n$$

put $x=-1, y=1$ in eq (1)

$$(-1+1)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r (1)^{n-r}$$

$$\Rightarrow \sum_{r=0}^{n} (-1)^r = 0$$

$$\Rightarrow \binom{n}{0} + \binom{n}{1}(-1) + \binom{n}{2}(+) + \binom{n}{3}(-1) + \cdots\cdots + \binom{n}{n}(-1)^n = 0$$

$$\Rightarrow \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots\cdots + \binom{n}{n}(-1)^n = 0$$

2. Evalulate $\binom{12}{5,3,2,2} = \dfrac{12!}{5!\,3!\,2!\,2!}$

$$= \frac{479001600}{2880}$$

$$= 166320$$

3. Find the coefficient of x^9y^3 in the expansion of $(2x-3y)^{12}$.

Sol By using Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Now

$$(2x-3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} (2x)^r (-3y)^{12-r}$$

$$(2x-3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} 2^r (-3)^{12-r} x^r y^{12-r}$$

put $r=9$.

$$(2x-3y)^{12} = \binom{12}{9} 2^9 (-3)^{12-9} x^9 y^{12-9}$$

$$\Rightarrow \binom{12}{9} 2^9 (-3)^3 x^9 y^3$$

$$= 220 \times 512 \times (-27) x^9 y^3$$

$$= -3041280 \, x^9 y^3$$

The coefficient of x^9y^3 is -3041280

7. Determine the coefficients of xyz^2 in the expansion of $(2x-y-z)^4$

Sol By using multinomial theorem.

$$(2x-y-z)^4 = \binom{4}{n_1, n_2, n_3} (2x)^{n_1} (-y)^{n_2} (-z)^{n_3}$$

$$= \binom{4}{n_1 n_2 n_3} 2^{n_1} (-1)^{n_2} (-1)^{n_3} (x)^{n_1} (y)^{n_2} (z)^{n_3}$$

put $n_1=1$, $n_2=1$, $n_3=2$

$$(2x-y-z)^4 = \binom{4}{1\ 1\ 2} 2^1 (-1)^1 (-1)^2 \, x^1 y^1 z^2$$

$$= \frac{4!}{1!1!2!} 2 \times -1 \times 1 \; x^1 y^1 z^2$$

$$= 12 \times (-2) \, x y z^2$$

$$= -24 \, xyz^2$$

The coefficient of xyz^2 is -24

5. Determine the coefficients of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$.

Sol By using multinomial theorem

$$(a+2b-3c+2d+5)^{16} = \binom{16}{n_1,n_2,n_3,n_4,n_5} a^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}$$

put $n_1 = 2,\ n_2 = 3,\ n_3 = 2,\ n_4 = 5$

since $n_1+n_2+n_3+n_4+n_5 = n$

$2+3+2+5+n_5 = 16$

$n_5 = 16-12$

$n_5 = 4$

$$= \frac{16!}{2!\,3!\,2!\,5!\,4!} a^2 (2b)^3 \cdot (-3c)^2 (2d)^5 (5)^4$$

$$= \frac{16!}{2!3!2!5!4!} \times 2^3 \times 3^2 \times 2^5 \times 5^4 (a^2b^3c^2d^5)$$

$$= 302702400 \times 2^3 \times 3^2 \times 2^5 \times 5^4 (a^2b^3c^2d^5)$$

$$= 302702400 \times 1440000 (a^2b^3c^2d^5)$$

$$= 4.42371456 \times 10^{14} (a^2b^3c^2d^5)$$

$$= \frac{16!}{(4!)^2} \times 3 \times 2^5 \times 5^3 \times (a^2b^3c^2d^5)$$

The coefficient of $a^2b^3c^2d^5$ is $\frac{16!}{(4!)^2} \times 3 \times 2^5 \times 5^3$

6. Find the term which contains $x^{11}y^4$ in the expansion of $(2x^3 - 3xy^2 + z^2)^6$.

Sol By using multinomial Theorem

$$(2x^3-3xy^2+z^2)^6 = \binom{6}{n_1,n_2,n_3} (2x^3)^{n_1} (-3xy^2)^{n_2} (z^2)^{n_3}$$

$$= \binom{6}{n_1,n_2,n_3} 2^{n_1} (-3)^{n_2} (1)^{n_3} x^{3n_1} x^{n_2} y^{2n_2} z^{2n_3}$$

$$= \binom{6}{n_1, n_2, n_3} 2^{n_1} (-3)^{n_2} x^{3n_1+n_2} y^{2n_2} z^{2n_3}.$$

For the term containing x^{11} and y^4

we have

$$3n_1+n_2 = 11, \quad 2n_2 = 4$$

$$\boxed{n_2 = 2}$$

$$3n_1 + 2 = 11$$

$$3n_1 = 9$$

$$\boxed{n_1 = 3}$$

Since $n = n_1+n_2+n_3$

$$6 = 3+2+n_3$$

$$\boxed{n_3 = 1}$$

Now

$$= \binom{6}{3, 2, 1} 2^3 (-3)^2 x^{11} y^4 z^{2(1)}$$

$$= \frac{6!}{3!\,2!\,1!} \times 2^3 \times 3^2 \, x^{11} y^4 z^2$$

$$= 60 \times 8 \times 9 \, x^{11} y^4 z^2$$

$(2x^3-3xy^2+z^2)^6 = 4320 \, x^{11} y^4 z^2$

7. Find the coefficient of i. $x^9 y^3$ in the expansion of $(x+2y)^{12}$

ii. $x^5 y^2$ in the expansion of $(2x-3y)^7$.

8. Determine the coefficients i. xyz^5 in the expansion of $(x+y+z)^7$

ii $w^2 x^2 y z^2$ in the expansion of $(2w-x+3y-2z)^8$

7 i. sol By using Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$(x+2y)^{12} = \sum_{r=0}^{12} \binom{12}{r} x^r (2y)^{n-r}$$

$$= \sum_{r=0}^{12} \binom{12}{r} x^r 2^{12-r} y^{12-r}$$

put $r=9$

$$(x+2y)^{12} = \binom{12}{9} x^9 2^{12-9} y^{12-9}$$

$$= \binom{12}{9} x^9 2^3 y^3$$

$$= \binom{12}{9} \times 2^3 x^9 y^3$$

$$= 220 \times 8 \, x^9 y^3$$

$$= 1760 \, x^9 y^3$$

The coefficient of x^9y^3 is 1760.

7 ii sol By using Binomial Theorem

$$(2x-3y)^7 = \sum_{r=0}^{7} \binom{7}{r} (2x)^r (-3y)^{7-r}$$

$$= \sum_{r=0}^{7} \binom{7}{r} 2^r (-3)^{7-r} x^r y^{7-r}$$

put $r=5$

$$(2x-3y)^7 = \binom{7}{5} 2^5 (-3)^{7-5} x^5 y^{7-5}$$

$$= \binom{7}{5} 2^5 (-3)^2 x^5 y^2$$

$$= 21 \times 32 \times 9 \, x^5 y^2$$

$$= 6048 \, x^5 y^2$$

The coefficient of x^5y^2 is 6048.

i. By using multinomial theorem

$$(x+y+z)^7 = \binom{7}{n_1,n_2,n_3} (x)^{n_1} y^{n_2} z^{n_3}$$

put $n_1 = 1, n_2 = 1, n_3 = 5$

$$(x+y+z)^7 = \binom{7}{1,1,5} x^1 y^1 z^5$$

$$= \frac{7!}{1! \times 1! \times 5!} x y z^5$$

$$= 42 \, xyz^5$$

The coefficient of xyz^5 is 42.

ii. By using multinomial Theorem

$$(2w - x + 3y - 2z)^8 = \binom{8}{n_1,n_2,n_3,n_4} (2w)^{n_1} (-x)^{n_2} (3y)^{n_3} (-2z)^{n_4}$$

$$= \binom{8}{n_1,n_2,n_3,n_4} 2^{n_1} (-1)^{n_2} (3)^{n_3} (-2)^{n_4} w^{n_1} x^{n_2} y^{n_3} z^{n_4}$$

put $n_1 = 3, n_2 = 2, n_3 = 1, n_4 = 2$

$$= \binom{8}{3,2,1,2} 2^3 (-1)^2 (3)^1 (-2)^2 w^3 x^2 y^1 z^2$$

$$= \frac{8!}{3! \times 2! \times 1! \times 2!} \times 8 \times 1 \times 3 \times 4 \; w^3 x^2 y^1 z^2$$

$$= 1680 \times 96 \; w^3 x^2 y z^2$$

$$= 161280 \; w^3 x^2 y z^2$$

∴ The coefficient of $w^3 x^2 y z^2$ is 161280