UNIT - IV

Multivariable Distribution Theory

In many situations, a scientist has to measure more than one factor (or) character at a time on the same item (or) some adjutant items. These measurements may have different units of measurements and more over the range (or) the domain of the variables may also be different. If two variables are studied simultaneously in respect of their distribution, then they are known as bivariate random variable. Two (or) more variables studied concurrently with regard to their distributions known in general multi-variate studied. The variables may be discrete (or) continuous (or) the mixture of two.

The variables should be defined over the same sample space. In this situation the variables can be jointly discrete (or) jointly continuous. We are dealing with 2 dimensional spaces (or) planes.

Bivariate discrete random variable:-

A R.V (x,y) is said to be a two dimensional discrete random variable if it can take only a countable no. of points (x,y) in a 2D space. The random variable (x,y) is also said to be joint discrete random variable.

Ex:- Let us consider an experiment of tossing a coin 3 times and take the variable 'x' as the no. of tail in 1st tossing and 'y' the no. of heads in 3 tosses. Here, our bivariate random variable is the ordered pair (X,Y). Also both X and Y are discrete. The possible outcomes are HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

$X=\{0,1\}$

The pairs of variate values (x,y) are $Y=\{0,1,2,3\}$

(x,y): (0,3) (0,2) (0,2) (0,1) (1,2) (1,1) (1,1) (1,0)

Bi-variate Continuous Random variable:-

A 2D R.V (X, Y) is called a bivariate continuous random variable if there exists a function $f(x,y) \geq 0$ such that for $-\infty < x, y < \infty$, the distribution function $F(x,y)$ of (X,Y) is given as

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

The function $f(x,y)$ is called the joint probability density function of (X,Y).

Ex: The age X of husband and age Y of wife at marriage when treated jointly, they represent the bivariate continous random variable.

Joint distribution function:-

Let X and Y be two random variables defined over the same probability space. The joint cummulative distribution function (c.d.f) of X and Y is denoted by $F_{XY}(x,y)$ and is defined as $P(X \leq x, Y \leq y)$ for all (x,y) in the xy-plane. Its domain is just the xy-plane. Cummulative distribution function is often is known as cummulat. distribution function.

Properties:-

i) $\lim_{\substack{x\to\infty \\ y\to\infty}} F(x,y) = F(\infty,\infty) = 1$

ii) $\lim_{x\to-\infty} F(x,y) = F(-\infty,y) = 0$

iii) $\lim_{y\to-\infty} F(x,y) = F(x,-\infty) = 0$

iv) $F(x,y)$ is right continuous in each argument

i.e., $\lim_{h\to 0} F(x+h,y) = \lim_{h\to 0} F(x,y+h) = F(x,y)$

v) $P(x_1 < x \leq x_2, y_1 \leq y \leq y_2) = F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) + F(x_1,y_1)$

vi) $0 \leq F(x,y) \leq 1$

* Joint discrete density (probability mass) function:-

The joint density function of X and Y is said to be discrete if there exists a non negative function such that it vanishes every where except at a finite (or) countably infinite no of mass points.

The joint probability distribution describes the relationships b/w x and y and assigns probabilities to all possible outcomes (x,y).

Thus $p(x,y) = P(X=x, Y=y)$ for all x,y in xy-plane. $p(x,y)$ is called the joint probability mass function.

Properties:-

i) $p(x_i, y_j) \geq 0$ for all i,j

(ii) $F(x,y) = \sum_{x_i \leq x} \sum_{y_j \leq y} p(x_i, y_j)$

(iii) $\sum_i \sum_j p(x_i, y_j) = 1$ for $i,j = 1,2,3 \ldots$

Joint probability density function of the C.R.Vs-

A function $f_{XY}(x,y)$ is a continuous function of x,y which gives the joint probability distribution of (X and Y) such that the probability of the variates falling within the infinitesimal rectangular region bounded by the lines $x \pm \frac{1}{2}dx$ and $y \pm \frac{1}{2}dy$ is expressed as $f_{xy}(x,y)dxdy$.

This function is called the joint probability density function (p.d.f)

Symbolically;

$P(x - \frac{1}{2}dx \leq X \leq x + \frac{1}{2}dx,\ y - \frac{1}{2}dy \leq X \leq y + \frac{1}{2}dy)$

$= f_{xy}(xy)\,dx\,dy$ Often the suffix X,Y to 'f' is omitted.

Marginal Probability distribution functions:-

If X and y are joint joint descrete R.V's with probability function $p(x,y)$, the individual distribution of either X (or) Y is called the marginal distributions of X (or) Y. Denoted as $P_X(x)$ or $P_Y(y)$ respectively.

$$P_X(x) = P(X=x) = \sum_y p(x,y) = \sum_y p(X=x, Y=y)$$

Similarly;

$$P_Y(y) = P(Y=y) = \sum_x p(x,y) = \sum_x p(X=x, Y=y)$$

Marginal probability density function:-

If X and Y are jointly C.R.V and their joint p.d.f is $f_{XY}(x,y)$, the marginal probability density function of the variable X is given as

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

Ily the marginal probability density function of the variable Y is given as

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$$

Conditional Random Variables:-

If (X,Y) is a bivariate random variable, the consideration of the variable Y for given value of X is known as the conditional variable Y for given X and is denoted as $Y/_{X=x}$ (or) Y/x. Ily the conditional X for given Y can be defined and denoted as $X/_{Y=y}$ (or) X/y

Conditional probability Mass function:-(PMF)

Let X and Y be two discrete random variables with their p.m.f $P_{XY}(x,y)$. The conditional probability Mass function of Y given X usually denoted as $P_{Y/X}(y/x)$ or $P(y/x)$ is defined as the ratio of the joint probability mass function to the marginal probability of X.

i.e; $\boxed{P_{Y/X}(y/x) = \dfrac{P_{XY}(x,y)}{P_X(x)}}$ (1.01)

IIly the conditional probability mass function of X given Y can be denoted as

$$P_{X/Y}(x/y) = \frac{P_{XY}(x,y)}{P_Y(y)}$$

Also the conditional descrete distribution function

$$F_{Y/x}(y/x) = \sum_{y_j \le y} P_{Y/x}(y_j/x) = P(Y \le y/X=x)$$

and $F_{X/y}(x/y) = \sum_{x_i \le x} P_{X/y}(x_i/y) = P(X \le x/Y=y)$

[Note: The suffix X, Y or Y/x or X/y with P or f is often not indicated]

Conditional probability density function:-

If X and Y are two bi-variate continuous random variable and joint p.d.f is $f_{X,Y}(x,y)$ the conditional probability density function of Y given X=x is denoted as

$$f_{Y/X=x}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} \text{ for } f_X(x) > 0 \text{; undefined at } f_X(x) = 0$$

IIly $$f_{X/y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} \text{ for } f_Y(y) > 0 \text{; undefined at } f_Y(y) = 0$$

Independency of Random variables:-

If X and Y are two R.V's and their joint p.d.f is $f_{X,Y}(x,y)$. The variables X and Y are independent if and only if

$$f_{XY}(x,y) = f_Y(y) \cdot f_X(x) \ \forall \ x, y \text{ of } X, Y$$

In terms of conditional distributions,

$$f_{Y/x}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_X(x) \cdot f_Y(y)}{f_X(x)} = f_Y(y)$$

IIly $f_{X/Y}(x/y) = f_X(x)$

Also $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ - discrete.

Conditional Expectation:-

Let X and Y be two R.V's having joint p.d.f $f_{X,Y}(x,y)$ and their Marginal p.d.f's be $f_X(x)$ and $f_Y(y)$. The conditional expectation of $Y/X=x$ is given as

$$E[Y/x] = \int_{-\infty}^{\infty} y \cdot \frac{f_{X,Y}(x,y)}{f_X(x)} dy \quad \text{where } f_X(x) > 0$$

lly $$E(X/y) = \int_{-\infty}^{\infty} x \cdot \frac{f_{X,Y}(x,y)}{f_Y(y)} dx \quad \text{where } f_Y(y) > 0$$

for discrete variables

$$E[Y/x] = \sum_j y_j \frac{P_{ij}}{P_i} \text{ for } P_i > 0$$

$$= \sum y_j \, P(j/i)$$

lly $$E[X/y] = \sum_i x_i \frac{P_{ij}}{P_j} \text{ for } P_j > 0$$

where $P_{ij} = P(X=x_i, Y=y_j)$

$P_i = P(X=x_i)$, $P_j = P(Y=y_j)$

* Properties:-

(i) $E\{E(Y/x)\} = E(Y)$

(ii) $E[E \cdot Y/x] = X \, E[Y/x]$

(3) $E(XY) = E\{X \, E(Y/x)\}$

(4) If X & Y are independent then;
$E(XY) = E(X) \cdot E(Y)$ but the converse is not true

(5) $E(Y/X=x)$ is called the regression curve of Y on X

(6) $E(X/Y=y)$ is called the regression curve of X on Y

7) The conditional expectation of a function $g(x,y)$ of X, Y is $E[g(x,y)/X=x] = \int_{-\infty}^{\infty} g(x,y) \cdot f_{Y/X}(y/x) \, dy$ when X & Y are continuous.

lly $E[g(x,y)/Y=y] = \int_{-\infty}^{\infty} g(x,y) \cdot f_{X/Y}(x/y) \, dx$

Also $E[g(x,y)/X=x] = \sum_j g(x_i, y_j) \frac{P_{ij}}{P_i}$ when X, Y are discrete

lly $E[g(x,y)/Y=y] = \sum_i g(x_i, y_j) \frac{P_{ij}}{P_j}$

1.01

8) If X & Y are independent, then

$$E[\phi_1(x)\cdot\phi_2(y)] = E[\phi_1(x)]\cdot E[\phi_2(y)]$$

Conditional variance:-

The conditional variance of a variable Y given $X=x$ is denoted by $V(Y/X=x)$ and is defined as $V(Y/X=x) = E[Y^2/X=x] - \{E[Y/X=x]\}^2$

Covariance b/w two variables:-

Covariance b/w two variables x and y is defined as

$$\text{cov}(x,y) = E[\{x-E(x)\}\{y-E(y)\}]$$

$$= E[\{x-\mu_x\}\{y-\mu_y\}]$$

$$= E(xy) - \mu_x\mu_y$$

$$\text{cov}(x,y) = \sigma_{xy}$$

Properties:-

1) Covariance b/w two variables is a measure of strength and direction of association b/w two variables

2) Covariance tends to measure the linear relationship of x and y

3) If the variates x and y are independent then $\text{cov}(x,y)=0$

4) If $\text{cov}(x,y)=0$, it means that the variables x and y are uncorrelated but not necessarily independent

Conditional Covariance:-

If x, y and z are 3 random variables, the covariance b/w x & y given $z=z$ is

$$\text{cov}(x,y/z) = E[\{x-E(x/z)\}\{y-E(y/z)\}/z]$$

Joint Moment generating function for bivariate R.V's:-

If X and Y are two random variables having the joint pdf $f(x,y)$, then their joint moment generating function (MGF) $M_{X,Y}(t_1,t_2)$ is defined as

$$M_{X,Y}(t_1,t_2) = E\left(e^{t_1 x + t_2 y}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x,y)\,dx\,dy$$

Joint raw moments for bivariate distribution:-

The $(r,s)^{th}$ moment of the variables X & Y about the origin is called the raw moment of bivariate distribution and is given as

$$\mu'_{r,s} = E\left[X^r Y^s\right]$$

also $\mu'_{r,0} = E\left[X^r\right]$ and $\mu'_{0,s} = E\left[Y^s\right]$

for $r,s = 0,1,2,3\ldots$

Joint Central moments for the bivariate distribution:

The $(r,s)^{th}$ moment about the means μ_x, μ_y of (X,Y) is called the central moment of bivariate distribution and is obtained as

$$\mu_{r,s} = E\left[(X-\mu_x)^r (X-\mu_y)^s\right]$$

for $r,s = 0,1,2\ldots$

also $\mu_{1,0} = E[(X-\mu_x)] = 0$

linear operator.
$E[X-\mu_x] = E(X) - \mu_x$
$= \mu_x - \mu_x$
$= 0$

$\mu_{0,1} = E[(Y-\mu_y)] = 0$

$\mu_{2,0} = E\left[(X-\mu_x)^2\right] = \sigma_x^2$

$\mu_{0,2} = E\left[(Y-\mu_y)^2\right] = \sigma_y^2$

$\mu_{1,1} = E\left[(x-\mu_x)(y-\mu_y)\right] = \text{cov}(XY) = \sigma_{xy}$

1) Let x, y, z denotes 3 jointly distributed random variable with joint density function then

$$f(x,y,z) = \begin{cases} k(x^2+yz); & 0\le x\le 1, 0\le y\le 1, 0\le z\le 1 \\ 0 & \text{otherwise} \end{cases}$$

i) find the value of k

ii) Determine the marginal distributions of x, y, z

iii) Determine the joint marginal distributions of x, y
x, z
y, z

Sol:- Given probability function

$$f(x,y,z) = \begin{cases} k(x^2+yz) & 0\le x\le 1, 0\le y\le 1, 0\le z\le 1 \\ 0 & \text{otherwise} \end{cases}$$

i) To find the value of k:-

Since the probability is unity, we have

$$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz = 1$$

$$\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} k(x^2+yz)\,dx\,dy\,dz = 1$$

$$k\int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^3}{3}+yzx\right]_0^1 dy\,dz = 1$$

$$k\int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{1}{3}+yz\right]dy\,dz = 1$$

$$k\int_{z=0}^{1}\left[\frac{y}{3}+\frac{y^2z}{2}\right]_0^1 = 1$$

$$k\int^{1}\left[\frac{1}{3}+\frac{z}{2}\right] = 1$$

$$\Rightarrow k\left[\frac{z}{3}+\frac{z^2}{4}\right]_0^1 = 1$$

$$k\left[\frac{1}{3}+\frac{1}{4}\right] = 1$$

$$k\left[\frac{7}{12}\right] = 1$$

$$k = \frac{12}{7}$$

∴ The p.d.f becomes

$\therefore$ The p.d.f becomes

$$f(x,y,z) = \begin{cases} \frac{12}{7}(x^2+yz) ; & 0\le x\le 1 \quad 0\le y\le 1 \quad 0\le z\le 1 \\ 0 ; & \text{otherwise} \end{cases}$$

(ii) Marginal distribution of x:-

$$f_X(x) = \int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} f(x,y,z)\,dy\,dz$$

$$f_X(x) = \int_{y=0}^{1}\int_{z=0}^{1} \frac{12}{7}(x^2+yz)\,dz.dy$$

$$= \frac{12}{7}\int_{y=0}^{1}\left[x^2 z + \frac{yz^2}{2}\right]_0^1 dy$$

$$= \frac{12}{7}\int_{y=0}^{1}\left[x^2 + \frac{y}{2}\right]dy$$

$$= \frac{12}{7}\left(x^2 y + \frac{y^2}{4}\right)_0^1$$

$$f_X(x) = \frac{12}{7}\left[x^2 + \frac{1}{4}\right]$$

The marginal distribution of y:-

$$f_Y(y) = \int_{x=-\infty}^{\infty}\int_{z=-\infty}^{\infty} f(x,y,z)\,dx\,dz$$

$$f_Y(y) = \int_{x=0}^{1}\int_{z=0}^{1} \frac{12}{7}(x^2+yz)\,dz\,dx$$

$$= \frac{12}{7}\int_{x=0}^{1}\left[x^2 z + \frac{yz^2}{2}\right]_0^1 dx$$

$$= \frac{12}{7}\int_{x=0}^{1}\left[x^2 + \frac{y}{2}\right]_0 dx$$

$$= \frac{12}{7}\left[\frac{x^3}{3} + \frac{yx}{2}\right]_0^1$$

$$= \frac{12}{7}\left(\frac{1}{3} + \frac{y}{2}\right)$$

The marginal distribution of z:-

$$f_z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f(x,y,z)\,dy\,dx$$

$$f_z(z) = \int_{x=0}^{1} \int_{y=0}^{1} \frac{12}{7}(x^2+yz)\,dy\,dx$$

$$\frac{12}{7}\int_{x=0}^{1} \left(x^2y + \frac{y^2z}{2}\right)_0^1 dx$$

$$= \frac{12}{7}\int_{x=0}^{1}\left(x^2 + \frac{z}{2}\right)dx$$

$$= \frac{12}{7}\left[\frac{x^3}{3} + \frac{zx}{2}\right]_0^1$$

$$f_z(z) = \frac{12}{7}\left[\frac{1}{3} + \frac{z}{2}\right]$$

iii) The joint marginal distribution of x,y:-

$$f_{x,y}(x,y) = \int_{z=-\infty}^{\infty} f(x,y,z)\,dz$$

$$= \int_{z=0}^{1} \frac{12}{7}(x^2+yz)\,dz$$

$$= \frac{12}{7}\left[x^2z + \frac{yz^2}{2}\right]_0^1$$

$$= \frac{12}{7}\left(x^2 + \frac{y}{2}\right)$$

The joint marginal distribution of y,z:-

$$f_{y,z}(y,z) = \int_{x=a}^{a} f(x,y,z)\,dx$$

$$= \int_{x=0}^{1} \frac{12}{7}(x^2+yz)\,dx$$

$$= \frac{12}{7}\left(\frac{x^3}{3} + yzx\right)_0^1$$

$$f_{y,z}(y,z) = \frac{12}{7}\left(\frac{1}{3} + yz\right)$$

The joint marginal distribution of X, Z :-

$$f_{XZ}(x,z) = \int_{y=-\infty}^{\infty} f(x,y,z)\,dy$$

$$= \int_{y=0}^{1} \frac{12}{7}(x^2+yz)\,dy$$

$$= \frac{12}{7}\left(x^2y + \frac{y^2z}{2}\right)_0^1$$

$$f_{XZ}(x,z) = \frac{12}{7}\left(x^2+\frac{z}{2}\right)$$

1) The conditional distribution of z given $X=x, Y=y$:

The marginal distribution of X, Y is

$$f_{XY}(x,y) = \frac{12}{7}\left(x^2+\frac{1}{2}y\right) \text{ for } 0\le x\le 1,\ 0\le y\le 1$$

∴ The conditional distribution of z given $X=x, Y=y$ is

$$\frac{f(x,y,z)}{f_{XY}(x,y)} = \frac{\frac{12}{7}(x^2+yz)}{\frac{12}{7}\left(x^2+\frac{y}{2}\right)}$$

$$= \frac{x^2+y^2}{x^2+\frac{y}{2}} \text{ when } 0\le z\le 1$$

4) The conditional distribution of Y, Z given $X=x$

The marginal distribution of X is $f_X(x) = \frac{12}{7}\left(x^2+\frac{1}{4}\right)$

The conditional distribution of Y, Z given $X=x$ is

$$\frac{f(x,y,z)}{f_X(x)} = \frac{\frac{12}{7}(x^2+yz)}{\frac{12}{7}\left(x^2+\frac{1}{4}\right)}$$

$$= \frac{x^2+yz}{x^2+\frac{1}{4}} ; \text{ for } 0\le y\le 1,\ 0\le z\le 1$$

7) The conditional distribution of Y given $X=x$

The marginal function of X is

$$f_X(x) = \frac{12}{7}\left(x^2+\frac{1}{4}\right)$$

Q:2!
2
x
22

$\frac{4.2}{1.01}$

∴ The conditional distribution of Y given $X = x$ is

$$f_{xy}(x,y) = \frac{\frac{12}{7}\left(x^2 + \frac{y}{2}\right)}{\frac{12}{7}\left(x^2 + \frac{1}{4}\right)}$$
$$f_x(x)$$

$$= \frac{x^2 + \frac{y}{2}}{x^2 + \frac{1}{4}} \quad 0 \le y \le 1$$

$$Y/X = x \quad \frac{f_{xy}(x,y)}{f_x(x)}$$

2) Let X, Y, Z denote 3 jointly distributede random variable with joint density function then

$$f(x,y,z) = \begin{cases} \frac{12}{7}(x^2 + yz) & 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine Mean $E(XYZ)$

Sol:- Given probability density function

$$f(x,y,z) = \begin{cases} \frac{12}{7}(x^2 + yz) & 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[XYZ] = \int_{z=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} xyz\, f(x,y,z)\, dx\, dy\, dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} xyz \cdot \frac{12}{7}(x^2 + yz)\, dx\, dy\, dz$$

$$= \frac{12}{7}\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (x^3yz + xy^2z^2)\, dx\, dy\, dz$$

$$= \frac{12}{7}\int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^4yz}{4} + \frac{x^2y^2z^2}{2}\right]_{x=0}^{1} dy\, dz$$

$$= \frac{12}{7}\int_{z=0}^{1}\int_{y=0}^{1} \frac{yz}{4} + \frac{y^2z^2}{2}\, dy\, dz$$

$$= \frac{12}{7}\int_{z=0}^{1}\left[\frac{y^2z}{8} + \frac{y^3z^2}{6}\right]_{y=0}^{1} dz$$

$$\int f(x)\int f(x,y,z)\, dx$$

$= \frac{12}{7} \int_{z=0}^{1} \frac{z}{8} + \frac{z^2}{6} \, dz$

$\frac{72 \times 7}{154}$ $\frac{12 \times 6}{102}$

$= \frac{12}{7} \left[\frac{z^2}{16} + \frac{z^3}{18} \right]_{z=0}^{1}$

$\frac{9+8}{72}$ $\frac{17}{72} \times \frac{6}{7}$ $\frac{102}{504}$

$= \frac{12}{7} \left(\frac{1}{16} + \frac{1}{18} \right)$

$9+8$

$= \frac{6}{7} \left(\frac{1}{8} + \frac{1}{9} \right)$ $\frac{17}{72} \times$

$= \frac{6}{7} \left(\frac{17}{72} \right)$

$= \frac{102}{504}$

$= \frac{17}{84}$

3) Let X, Y, Z denote 3 jointly distributed RV's with joint density function they

$f(x,y,z) = \begin{cases} \frac{12}{7}(x^2+yz) & 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$

Determine the conditional expectation of $U = X^2 + Y + Z$ given $X = x, Y = y$

Sol: Integration over z gives us the marginal distribution of X, Y:

$f_{xy}(x,y) = \frac{12}{7}\left(x^2 + \frac{1}{2}y\right)$ for $0 \le x \le 1, 0 \le y \le 1$

Given probability density function

~~f_{xy}~~ $f(x,y,z) = \begin{cases} \frac{12}{7}(x^2+yz) & 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1 \\ 0; & \text{otherwise} \end{cases}$

Joint Marginal distribution function of x, y

$f_{xy}(x,y) = \int_{z=-\infty}^{\infty} f(x,y,z)\, dz$

$f_{xy}(x,y) = \frac{12}{7}\left(x^2 + \frac{y}{2}\right)$

$= \int_{z=0}^{1} \frac{12}{7}(x^2+yz)\, dz$ for $0 \le x, y \le 1$

$= \frac{12}{7}\left(x^2 z + \frac{yz^2}{2}\right)_{z=0}^{1}$

$\frac{4}{1.01}$

The marginal distribution of X, Y

$f_{xy}(x,y) = \frac{12}{7}\left(x^2 + \frac{y}{2}\right)$ for $0 \le x \le 1,\ 0 \le y \le 1$

The conditional distribution of Z given X=x, Y=y is

$$\frac{f(x,y,z)}{f_{xy}(x,y)} = \frac{\cancel{\frac{12}{7}}(x^2+yz)}{\cancel{\frac{12}{7}}\left(x^2+\frac{y}{2}\right)}$$

$$= \frac{x^2+yz}{x^2+\frac{y}{2}} \quad \text{for } 0 \le z \le 1$$

The conditional expectation of $U = X^2+Y+Z$ given $X=x,\ Y=y$

$$E[U/x,y] = \int_{z=0}^{1} (x^2+y+z)\, f(x,y,z)\, dz$$

$$= \int_{z=0}^{1} (x^2+y+z)\, \frac{x^2+yz}{x^2+\frac{y}{2}}\, dz$$

$$= \frac{1}{x^2+\frac{y}{2}} \int_{z=0}^{1} (x^2+y+z)(x^2+yz)\, dz$$

$$= \frac{1}{x^2+\frac{y}{2}} \int_{z=0}^{1} x^4 + x^2y + x^2z + x^2yz + y^2z + yz^2\, dz$$

$$= \frac{1}{x^2+\frac{y}{2}} \left[x^4z + x^2yz + \frac{x^2z^2}{2} + \frac{x^2yz^2}{2} + \frac{y^2z^2}{2} + \frac{yz^3}{3}\right]_{z=0}^{1}$$

$$= \frac{1}{x^2+\frac{y}{2}} \left[x^4 + x^2y + \frac{x^2}{2} + \frac{x^2y}{2} + \frac{y^2}{2} + \frac{y}{3}\right]$$

$$= \frac{1}{\cancel{x^2+\frac{y}{2}}} \left(\cancel{x^2\left[x^2+\frac{1}{2}\right]} + x^2y\right.$$

$$= \frac{1}{x^2+\frac{y}{2}} \left[x^2(x^2+y) + \frac{y}{2}(x^2+y) + \frac{x^2}{2} + \frac{y}{3}\right]$$

$$= \frac{1}{x^2+\frac{y}{2}}\left[(x^2+y)\left(x^2+\frac{y}{2}\right)+\left(\frac{x^2}{2}+\frac{y}{3}\right)\right]$$

$$= \frac{(x^2+y)\cancel{\left(x^2+\frac{y}{2}\right)}}{\cancel{x^2+\frac{y}{2}}} + \frac{\frac{x^2}{2}+\frac{y}{3}}{x^2+\frac{y}{2}}$$

$$= (x^2+y) + \frac{\frac{x^2}{2}+\frac{y}{3}}{x^2+\frac{y}{2}}$$

4) Suppose that a rectangle is constructed by first choosing its length, x and then choosing its width y, its length x is selected from an exponential distribution with mean $\mu = 1/\lambda = 5$. Once the length has been choosen its width, y, is selected from a uniform distribution from 0 to half its length. find the mean and variance of the area of the rectangle $A = XY$.

Sol:- Since length (x) follows exponential distribution with $\mu = \frac{1}{\lambda} = 5$, So marginal function of

$\lambda = \frac{1}{5}$

$P(x,\lambda) = \lambda e^{-\lambda x}$

$= \frac{1}{5} e^{-\frac{1}{5}x}$

x is $f_X(x) = \frac{1}{5} e^{-1/5 x}, x \geq 0$

width y follows uniform distribution and y varies from 0 to $x/2$

uniform distribution $[a,b]$

$= \frac{1}{b-a}$

$= (0, x/2) = \frac{1}{\frac{x}{2}-0} = \frac{1}{x/2}$

choosing y after x is

$f_{Y/X}(y/x) = f_X(x) \cdot f_{Y/X}(y/x)$

$= \frac{1}{5} e^{-1/5 x} \cdot \frac{1}{x/2}$

$\frac{1}{5} e^{-1/5 x} \cdot \frac{2}{x} = \frac{2}{5x} e^{-\frac{1}{5}x}$

$= \boxed{\frac{2}{5x} e^{-\frac{1}{5}x}}$ where $x \geq 0$

$0 \leq y \leq \frac{x}{2}$

Now we will find mean and variance of $A = XY$

1.01

$E[A] = E[XY]$

$$= \int_{x=0}^{\infty}\int_{y=0}^{x/2} xy.\, f(x,y)\, dy\, dx$$

$$= \int_{x=0}^{\infty}\int_{y=0}^{x/2} \cancel{x}y.\, \frac{2}{5\cancel{x}}\, e^{1/5\, x}\, dy\, dx$$

$$= \frac{2}{5}\int_{x=0}^{\infty}\left[\int_{y=0}^{x/2} y\,dy\right] e^{-1/5\,x}\, dx$$

$$= \frac{2}{5}\int_{x=0}^{\infty}\left[\frac{y^2}{2}\right]_{y=0}^{x/2} e^{-1/5\,x}\, dx$$

$$= \frac{1}{5}\int_{x=0}^{\infty}\left(\frac{x}{2}\right)^2 e^{-1/5\,x}\, dx$$

$$= \frac{1}{20}\int_{x=0}^{\infty} x^2 e^{-\frac{1}{5}x}\, dx$$

Divide & Multiply with $\frac{\Gamma[3]}{(\frac{1}{5})^3}$

$$= \frac{1}{20}\cdot\frac{\Gamma(3)}{(\frac{1}{5})^3}\int_{x=0}^{\infty}\frac{(\frac{1}{5})^3}{\Gamma[3]}\, x^2 e^{-\frac{1}{5}x}\, dx$$

$$= \frac{1}{20} \times \frac{2}{\frac{1}{125}}$$

$$= \frac{1}{20} \times 2 \times 125$$

$E[XY] = 12.5$

$\frac{1}{5}x = t$

$x = 5t$

$dx = 5\,dt$

$$= \frac{1}{20}\int_{x=0}^{\infty}(5t)^2 e^{-t}\, 5\,dt$$

$$= \frac{1}{20}\cdot 125\int t^2 e^{-t}\, dt$$

$$= \frac{1}{20}\cdot 125\int e^{-t}\, t^{3-1}\, dt$$

$$= \frac{1}{20}\cdot 125\; \Gamma[3]$$

$$= \frac{1}{20}\times 125 \times 2$$

$\Gamma(n) = e^{-x} x^{n-1} dt$

$\Gamma[n+1] = n!$

$\Gamma[3] = \Gamma[2$

$E[A^2] = E[X^2Y^2]$

$$= \int_{x=0}^{\infty}\int_{y=0}^{x/2} x^2 y^2 f(x,y)\, dy\, dx$$

$$= \int_{x=0}^{\infty}\int_{y=0}^{x/2} x^{\cancel{2}} y^2 \cdot \frac{2}{5\cancel{x}}\, e^{-\frac{1}{5}x}\, dx$$

$$= \frac{2}{5}\int_{x=0}^{\infty}\left(\int_{y=0}^{x/2} y^2\,dy\right) x \cdot e^{-\frac{1}{5}x}\, dx$$

$$= \frac{2}{5} \int_{x=0}^{\infty} \left[\frac{y^3}{3}\right]_{y=0}^{x/2} x \cdot e^{-\frac{1}{5}x} dx$$

Put $\frac{1}{5}x = t$
$x = 5t$
$dx = 5dt$
if $x = 0 \Rightarrow t = 0$
$x = \infty \Rightarrow t = \infty$

$$= \frac{2}{5} \cdot \frac{1}{3} \int_{x=0}^{\infty} \left(\frac{x}{2}\right)^3 \cdot x \cdot e^{-\frac{1}{5}x} dx$$

$$= \frac{2}{15} \int_{x=0}^{\infty} \frac{x^3}{8} \cdot x \cdot e^{-\frac{1}{5}x} dx$$

$\frac{1}{60} \times \int_{t=0}^{\infty} (5t)^4 \cdot e^{-t} \, 5dt$

$$= \frac{1}{60} \int_{x=0}^{\infty} x^4 \cdot e^{-\frac{1}{5}x} dx$$

$\frac{1}{60} \times 5^5 \int_{t=0}^{\infty} e^{-t} t^4 dt$

$\frac{1}{60} \times 5^5 \int_{t=0}^{\infty} e^{-t} t^{5-1} dt$

Multiple and divide with $\frac{\Gamma[5]}{\left(\frac{1}{5}\right)^5}$

$\frac{1}{60} \times 5^5 \times \Gamma[5]$

$= \frac{1}{60} \times 5^5 \times 4! = 1250$

$$= \frac{1}{60} \times \frac{\Gamma[5]}{\left[\frac{1}{5}\right]^5} \int_{x=0}^{\infty} \frac{\left[\frac{1}{5}\right]^5}{\Gamma[5]} \cdot x^4 \cdot e^{-\frac{1}{5}x} dx$$

$$= \frac{1}{60} \cdot \frac{4!}{\frac{1}{5^5}} \Rightarrow \frac{1}{60} \times 24 \times 5^5$$

$$= \frac{1}{60} \times 3125 \times 24$$

$E[A^2] = 1250$

$V[A] = E[A^2] - [E(A)]^2$

$= 1250 - (12.5)^2$

$= 1093.75$

5) Suppose that x_1, x_2 are independent with density functions $f_1(x_1)$ and $f_2(x_2)$. find the distribution of $u_1 = x_1 + x_2$ & $u_2 = x_1 - x_2$

Sol:- Solving for x_1 and x_2 we get the inverse transformation: $x_1 = \frac{u_1+u_2}{2}$, $x_2 = \frac{u_1-u_2}{2}$

The Jacobian of the transformation

$$J = \frac{d(x_1, x_2)}{d(u_1, u_2)} = \det \begin{vmatrix} \frac{dx_1}{du_1} & \frac{dx_1}{du_2} \\ \frac{dx_2}{du_2} & \frac{dx_2}{du_2} \end{vmatrix}$$

$$J = \frac{d(x_1, x_2)}{d(u_1, u_2)} = \det\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{-1}{2}$$

The joint density of x_1, x_2 is

$$f(x_1, x_2) = f_1(x_1) f_2(x_2)$$

Hence the joint density of u_1 and u_2 is:

$$g(u_1, u_2) = f(x_1, x_2)\,|J|$$
$$= f\left(\frac{u_1+u_2}{2}\right) f_2\left(\frac{u_1-u_2}{2}\right) \frac{1}{2}$$

from $g(u_1, u_2) = f_1\left(\frac{u_1+u_2}{2}\right) f_2\left(\frac{u_1-u_2}{2}\right) \frac{1}{2}$

we can determine the distribution of $u_1 = x_1 + x_2$

$$g_1(u_1) = \int_{-\infty}^{\infty} g(u_1, u_2)\, du_2$$
$$= \int_{-\infty}^{\infty} f_1\left(\frac{u_1+u_2}{2}\right) f_2\left(\frac{u_1-u_2}{2}\right) \frac{1}{2}\, du_2.$$

put $v = \frac{u_1+u_2}{2}$ then $\frac{u_1-u_2}{2} = u_1 - v$. $\frac{dv}{du_2} = \frac{1}{2}$

Hence

$$g_1(u_1) = \int_{-\infty}^{\infty} f_1\left(\frac{u_1+u_2}{2}\right) f_2\left(\frac{u_1-u_2}{2}\right) \frac{1}{2}\, du_2$$
$$= \int_{-\infty}^{\infty} f_1(v) f_2(u_1 - v)\, dv$$

This is called the convolution of the two densities f_1 and f_2.

Bivariate Normal Distribution:-

The Bivariate continuous RV's (X, Y) with population means μ_x and μ_y, population variances σ_x^2 and σ_y^2 respectively and constant "ρ" which is known as the population correlation coefficient b/w X and Y are said to follow bivariate normal distribution if there probability density function is

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot e^{\frac{-1}{2(1-\rho)^2}}$$

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot e^{\frac{-1}{2(1-\rho)^2}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}}$$

for $-\infty < \mu_x, \mu_y < \infty$ $\sigma_x^2, \sigma_y^2 > 0$ and $0 \le \rho \le 1$

Bivariate Normal distribution is sometimes named after its inventors as Gaussia and Laplace Gauss

Bivariate Normal distribution has 5 parameters namely $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho$

If we put $\frac{x-\mu_x}{\sigma_x} = u$ and $\frac{y-\mu_y}{\sigma_y} = v$, then

$$g(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}(u^2-2\rho uv+v^2)}$$

This is known as the standardised bivariate normal distribution with parameters $(0,0,1,1,\rho)$

Latest definition:-

A random vector (X,Y) is said to possess bivariate normal distribution if for any 2 constants a & b, $Z = aX+bY$ is a normal variate.

Bivariate normal distribution is denoted as $BVN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

Moment generating function of Bivariate Normal Distribution:-

Let $(X,Y) \sim BVN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

By the defn of M.g.f

$$M_{XY}(t_1,t_2) = E[e^{t_1X+t_2Y}]$$

$$= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\,dx\,dy$$

put $\frac{x-\mu_x}{\sigma_x} = u$, $\frac{y-\mu_y}{\sigma_y} = v$ $(-\infty < (u,v) < \infty)$

$x = \sigma_x u + \sigma_x$ $\quad y = \sigma_y v + \mu_y$

$|J| = \sigma_x \sigma_y$

$$M_{XY}(t_1,t_2) = \int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty} e^{t_1(u\sigma_x+\mu_x)+t_2(v\sigma_y+\mu_y)}$$

$$\frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}(u^2-2\rho uv+v^2)}$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty} e^{t_1 u\sigma_x + t_1\mu_y + t_2 v\sigma_y + t_2\mu_y + \left(\frac{-u^2+2\rho uv - v^2}{2(1-\rho^2)}\right)} du\,dv$$

$$= \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty} e^{t_1 u\sigma_x + t_2 v\sigma_y + \left(\frac{-u^2+2\rho uv - v^2}{2(1-\rho^2)}\right)} du\,dv$$

$$M_{XY}(t_1,t_2) = \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \int_u\int_v e^{\frac{-1}{2(1-\rho^2)}\left[(u^2-2\rho uv+v^2) - 2(1-\rho^2)(t_1 u\sigma_x + t_2\sigma_y v)\right]} du\,dV \longrightarrow (1)$$

Consider ;

$$\frac{-1}{2(1-\rho^2)}\left[(u^2-2\rho uv+v^2) - 2(1-\rho^2)(t_1 u\sigma_x + t_2 v\sigma_y)\right]$$

$$\frac{-1}{2(1-\rho^2)}\left[u^2 - 2\rho uv + v^2 - 2(1-\rho^2)t_1 u\sigma_x - 2(1-\rho^2)t_2 v\sigma_y\right]$$

$$= \frac{-1}{2(1-\rho^2)}\left[u^2 - 2u\{\rho v + (1-\rho^2)t_1\sigma_x\} + v^2 - 2(1-\rho^2)t_2 v\sigma_y\right]$$

$$= \frac{-1}{2(1-\rho^2)}\left[\{\underbrace{u^2}_{a^2}\underbrace{-2u\{\rho v+(1-\rho^2)t_1\sigma_x\}}_{-2ab}\} + v^2 - 2(1-\rho^2)t_2 v\sigma_y + \{\rho v + (1-\rho^2)t_1\sigma_x\}^2_{b^2} - \underbrace{\{\rho v + (1-\rho^2)t_1\sigma_x\}^2}_{(a+b)^2}\right]$$

$$\frac{-1}{2(1-\rho^2)}\left[\{u - (\rho v + (1-\rho^2)t_1\sigma_x)\}^2 - \rho^2 v^2 - (1-\rho^2)^2 t_1^2\sigma_x^2 - 2\rho v(1-\rho^2)t_1\sigma_x + v^2 - 2(1-\rho^2)t_2 v\sigma_y\right]$$

$$\frac{-1}{2(1-\rho^2)}\left[\{u - \rho v - (1-\rho^2)t_1\sigma_x\}^2 + v^2 - \rho^2 v^2 - 2\rho v(1-\rho^2)t_1\sigma_x - 2(1-\rho^2)t_2\sigma_y v - (1-\rho^2)^2 t_1^2\sigma_x^2\right]$$

$$= \frac{-1}{2(1-\rho^2)}\left[\{u-\rho v-(1-\rho^2)t_1\sigma_x\}^2 + v^2(1-\rho^2) - 2v(1-\rho^2)[\rho t_1\sigma_x + t_2\sigma_y] - (1-\rho^2)^2 t_1^2\sigma_x^2\right]$$

$$= \frac{-1}{2(1-e^2)}\Big[\{u - ev - (1-e^2)t_1\sigma_x\}^2 + (1-e^2)\{v^2 - 2v(et_1\sigma_x + t_2\sigma_y)^2 - (1-p^2)t_1^2\sigma_x^2$$

$$(1-e^2)t_1^2\sigma_x\Big] + (pt_1\sigma_x + t_2\sigma_y\} - (pt_1\sigma_x + t_2\sigma_y)^2$$

$$= \Big[\frac{-1}{2}\Big\{\frac{u - ev - (1-e^2)t_1\sigma_x}{\sqrt{1-e^2}}\Big\}^2 - \frac{1}{2}\{v - pt_1\sigma_y + t_2\sigma_y\}$$

$$+ \frac{1}{2}\{t_1^2\sigma_x^2 + 2pt_1t_2\sigma_x\sigma_y + t_2^2\sigma_y^2\Big]$$

If we substitute

$$w = \frac{u - ev - (1-e^2)t_1\sigma_x}{\sqrt{1-e^2}}$$

$$z = v - pt_1\sigma_x - t_2\sigma_y$$

$$J = \frac{\partial u\,\partial v}{\partial w\,\partial z}$$

$$\frac{1}{J} = \frac{\partial w\,\partial z}{\partial u\,\partial v} = \begin{vmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{\sqrt{1-e^2}} & \frac{-p}{\sqrt{1-e^2}} \\ 0 & 1 \end{vmatrix}$$

$$\frac{1}{J} = \frac{1}{\sqrt{1-e^2}}$$

$$J = \sqrt{1-e^2}$$

$$\partial u\,\partial v = \sqrt{1-e^2}\,\partial w\,\partial z$$

eqn ① becomes

$$M_{X,Y}(t_1,t_2) = \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{(1-\rho^2)}} \int_w \int_z e^{-1/2(\omega)^2 - \frac{1}{2}(z)^2 + \frac{1}{2}(z)^2 + \frac{1}{2}(t_1^2\sigma_x^2 + t_2^2\sigma_y^2 + 2\rho t_1 t_2 \sigma_x\sigma_y))} \frac{1}{\sqrt{(1-\rho^2)}} \partial\omega\, \partial z$$

$$= e^{t_1\mu_x + t_2\mu_y + \frac{1}{2}(t_1^2\sigma_x^2 + t_2^2\sigma_y^2 + 2\rho t_1 t_2\sigma_x\sigma_y)} * \left[\frac{1}{\sqrt{2\pi}}\int_{w=-\infty}^{\infty} e^{-\frac{1}{2}w^2} dw\right]\left[\frac{1}{\sqrt{2\pi}}\int_{z=-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz\right]$$

$$= e^{t_1\mu_x + t_2\mu_y + \frac{1}{2}(t_1^2\sigma_x^2 + t_2^2\sigma_y^2 + 2\rho t_1 t_2\sigma_x\sigma_y)}$$

In particular if $(X,Y) \sim BVN(\overset{\mu_x}{0}, \overset{\mu_y}{0}, \overset{\sigma_x}{1}, \overset{\sigma_y}{1}, \rho)$

then $\frac{1}{2}(t_1^2 + t_2^2 + 2\rho t_1 t_2)$

$$M_{X,Y}(t_1,t_2) = e^{\frac{1}{2}(t_1^2 + t_2^2 + 2\rho t_1 t_2)}$$

Marginal distribution of Bivariate Normal

Marginal Distribution:-

The marginal distribution of Random variable 'X' is given by

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f_X(x) = \int_{y=-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot e^{\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}} dy$$

Put $\frac{y-\mu_x}{\sigma_y} = u$

$y = u\sigma_y + \mu_x$

$dy = \sigma_y du$

Limits:-

$y = \infty \Rightarrow u = \infty$

$y = -\infty \Rightarrow u = -\infty$

$$f_X(x) = \frac{1}{2\pi\sigma_x\cancel{\sigma_y}\sqrt{1-\rho^2}} \int_{u=-\infty}^{+\infty} e^{\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho u\left(\frac{x-\mu_x}{\sigma_x}\right) + u^2\right\}} \cancel{\sigma_y} du$$

Add & Subtract $e^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2$

$$f_X(x) = \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{u=-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho u\left(\frac{x-\mu_X}{\sigma_X}\right) + u^2\right]} du$$

$$= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{u=-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2(1-\rho^2) + \left\{u-\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right\}^2\right]} du$$

$$= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{u=-\infty}^{\infty} e^{\frac{-1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} e^{\frac{-1}{2(1-\rho^2)}\left(u-\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right)^2} du$$

$$= \frac{e^{\frac{-1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{u=-\infty}^{\infty} e^{\frac{-1}{2}\left(\frac{u-\rho\left(\frac{x-\mu_X}{\sigma_X}\right)}{\sqrt{1-\rho^2}}\right)^2} du$$

$\Rightarrow$ Let $z = \dfrac{u-\rho\left(\frac{x-\mu_X}{\sigma_X}\right)}{\sqrt{1-\rho^2}}$

$$f_X(x) = \frac{e^{\frac{-1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{u=-\infty}^{\infty} e^{\frac{-1}{2}z^2} du$$

$$f_X(x) = \frac{e^{\frac{-1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}{2\pi\sigma_X\sqrt{1-\rho^2}} \Rightarrow$$

$$= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} e^{\frac{-1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

$|||^{ly}$ $f_Y(y) = \dfrac{1}{2\pi\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}$

Conditional Distribution

Conditional Distribution of Y for given X for the BVN $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$ distribution is

$$f_{Y/X}(y/x) = \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)\sigma_Y^2}\left\{(y-\mu_Y) - \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right\}^2}$$

for $-\infty \le x \le \infty$

The right hand expression is the probability density function of Uni-variate Normal distribution of y with $\boxed{\text{mean} = \mu_y + \rho \frac{\sigma_y}{\sigma_x}(x-\mu_x)}$

$\boxed{\text{Variance} = \sigma_y^2(1-\rho^2)}$

$|||^{ly}$ The conditional distribution x given y is $f_{x/y}(x/y) = \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \cdot e^{\frac{-1}{2(1-\rho^2)\sigma_x^2}\left\{(x-\mu_x)-\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right\}^2}$

for $-\infty \le y \le \infty$

with $\boxed{\text{mean} = E[X/y] = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)}$

Variance $= V(X/y) = \sigma_x^2(1-\rho^2)$

1) Given the joint distribution of X, Y as BVN $(3, 4, 16, 25, 0.8)$ find (i) $P(5 < X < 9/_{y=6})$

(ii) $P(-3 < Y < 3/_{X=5})$

Sol:- Given that

$\mu_x = 3$ $\mu_y = 4$

$\sigma_x^2 = 16$ $\sigma_y^2 = 25$

$\rho = 0.5$

i) The conditional distribution of X given y has

Mean, $\mu = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$

Put $y = 6$

$\mu = 3 + 0.8\,\frac{4}{5}(6-4)$

$\mu = 4.28\ E(X/y)$

Variance, $\sigma^2 = \sigma_x^2(1-\rho^2)$

$= 16(1-0.64)$

$\sigma^2 = 5.76$

$\sigma = 2.4\ V(X/y)$

By using the conditional distribution $z = \frac{x-y}{\sigma}$

$P(5 < X < 9 / y=6) = P\left(\frac{5-4.28}{2.4} < \frac{x-4.28}{2.4} < \frac{9-4.28}{2.4}\right)$

$= P(0.3 < z < 1.96)$

$= A(1.96) - A(0.3)$

$= 0.0584 - 0.1179$

$= 0.0595$

ii) We know That the conditional distribution Y given X has the ~~mea~~

Mean $= E[Y/x] = \mu_y + e \frac{\sigma y}{\sigma x}(x - \mu x)$

for $x = 5$

$= 4 + 0.8 \cdot \frac{5}{4}(5-3)$

$= 4 + 0.8 \times \frac{5}{4} \times 2$

$\mu = 6$

Variance(σ) $V(Y/x) = \sigma_y^2(1-e^2)$

$\sigma^2 = 25(1-0.64)$

$\sigma^2 = 9$

$\sigma = 3$

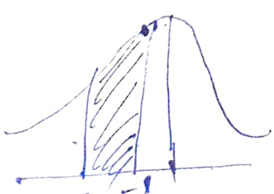
64
36
10

25×0.36

By using the conditional distribution $z = \frac{x-y}{\sigma}$

$P(-3 < y < 3 / x=5) = \left(\frac{-3-6}{3} < \frac{y-6}{3} < \frac{3-6}{3}\right)$

$= \left(\frac{-9}{3} < \frac{y-6}{3} < \frac{-3}{3}\right)$



$= \left(-3 < \frac{y-6}{3} < 1\right)$

$= A(-3) - A(-1)$

$= 0.4987 - 0.3413$

$= 0.1574$

8) If (X, Y) follows BVN $(3, 6, 16, 25, \rho)$ and $P(1 < X < 5 / Y=6) = 0.724$. find the value of ρ?

Given that $\mu_x = 3$ $\mu_y = 6$

$\sigma_x^2 = 16$ $\sigma_y^2 = 25$

$\rho = ?$

we know that the conditional distribution of X given y has

Mean $E(X/y) = \mu_x + \rho \frac{\sigma_x}{\sigma_y}(y - \mu_y)$

$y = 6$

$= 3 + \rho \frac{4}{25}(6-6)$

$E(X/y) = 3 \Rightarrow \mu$

Variance $\sigma^2_{x/y}[X/y] = \sigma_x^2(1-\rho^2)$

$\sigma^2 = 16(1-\rho^2)$

$\sigma = 4\sqrt{1-\rho^2}$

Also Given that $P(1 < X < 5/y=6) = 0.724$

$$P\left(\frac{1-3}{4\sqrt{1-\rho^2}} < \frac{X-3}{4\sqrt{1-\rho^2}} < \frac{5-3}{4\sqrt{1-\rho^2}}\right) = 0.724$$

$$P\left(\frac{-2}{4\sqrt{1-\rho^2}} < Z < \frac{2}{4\sqrt{1-\rho^2}}\right) = 0.724$$

$$P\left(\frac{-1}{2\sqrt{1-\rho^2}} < Z < \frac{1}{2\sqrt{1-\rho^2}}\right) = 0.724$$

$$\frac{1}{\sqrt{2\pi}} \int_{z=\frac{-1}{2\sqrt{1-\rho^2}}}^{\frac{1}{2\sqrt{1-\rho^2}}} e^{-\frac{1}{2}z^2} dz = 0.724$$

$$\frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Since integration is event

so $$\frac{2}{\sqrt{2\pi}} \int_{z=0}^{\frac{1}{2\sqrt{1-\rho^2}}} e^{-\frac{1}{2}z^2} dz = 0.724$$

$$\frac{1}{\sqrt{2\pi}} \int_{z=0}^{\frac{1}{2\sqrt{1-e^2}}} e^{-\frac{1}{2}z^2} \, dz = 0.362$$

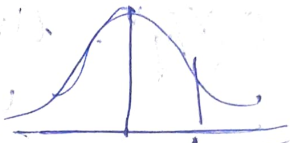
By using the Area under normal curve

$d(1.09) = 0.362$

$$\frac{1}{2\sqrt{1-e^2}} = 1.09$$

$$\frac{1}{\sqrt{1-e^2}} = 2.18$$

$$\sqrt{1-e^2} = \frac{1}{2.18}$$



$$\sqrt{1-e^2} = 0.458$$

$$1-e^2 = 0.2097$$

$$e^2 = 1 - 0.2097$$

$$e^2 = 0.7903$$

$$e = 0.88$$

3) Let (X, Y) have BVN $(5, 10, 1, 25, e)$ and $\operatorname{cor}(X,Y) = e$

i) If $e > 0$, find e when $P(4 < Y < 16 / X=5) = 0.954$

ii) If $e = 0$, find $(X + Y \leq 16)$

Given That $\mu_x = 5$, $\mu_y = 10$, $\sigma_x^2 = 1$, $\sigma_y^2 = 25$, $e = ?$

WKT, The conditional probability of Y given X ay

Mean $\mu = \mu_y + e \frac{\sigma_y}{\sigma_x}(x - \mu_x)$

$$= 10 + e\left(\frac{5}{1}\right)(5-5) = 10$$

$$e(Y/x) = 10 = \mu$$

Variance $V(Y/x) = \sigma_y^2 (1-e^2)$

$$\sigma^2 = 25(1-e^2)$$

$$\sigma = 5\sqrt{1-e^2}$$

Also given that $P(4<y<16/x=5) = 0.954$

$$P\left(\frac{4-10}{5\sqrt{1-e^2}} < \frac{x-10}{5\sqrt{1-e^2}} < \frac{16-10}{5\sqrt{1-e^2}}\right) = 0.954$$

$$P\left(\frac{-6}{5\sqrt{1-e^2}} < z < \frac{6}{5\sqrt{1-e^2}}\right) = 0.954$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{-6}{5\sqrt{1-e^2}}}^{\frac{6}{5\sqrt{1-e^2}}} e^{-\frac{1}{2}z^2} dz = 0.954$$

since integration is even

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\frac{6}{5\sqrt{1-e^2}}} e^{-\frac{1}{2}z^2} dz = 0.954$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{6/5\sqrt{1-e^2}} e^{-\frac{1}{2}z^2} dz = 0.477$$

By using the area under normal curve

$$\phi(\ ) = 0.477$$

$$\frac{6}{5\sqrt{1-e^2}} = 2$$

$$6 = 10\sqrt{1-e^2}$$

$$3 = 5\sqrt{1-e^2}$$

$$\frac{3}{5} = \sqrt{1-e^2}$$

$$0.6 = \sqrt{1-e^2}$$

$$0.36 = 1-e^2$$

$$e^2 = 0.64$$

$$\boxed{e = 0.8}$$

Since x, y have Bivariate Normal distribution

$e=0 \Rightarrow$ X and Y are independent RV's and

$$X+Y \sim N(\mu_x+\mu_y, \sigma_x^2+\sigma_y^2)$$

$$= N(5+10, 1+25)$$

$$= N(15, 26)$$

By using the conditional distribution

$$P(X+Y \le 16) = P\left(\frac{(X+Y)-15}{\sqrt{26}} < \frac{16-15}{\sqrt{26}}\right)$$

$$= P\left(Z \le \frac{1}{\sqrt{26}}\right)$$

$$= P(Z \le 0.196)$$

$$= 0.5 + \phi(0.196)$$

$$= 0.5 + 0.754$$

$$= 1.254$$

* Multiple & partial correlation

When the values of one variable are associated with or influenced by other variable, Ex, the age of husband and wife, the height of father and son, the supply and demand of a commodity and so on. Karl Pearsons coefficient of correlation as a measure of linear relationship b/w them.

But some times there is inter relation b/w many variables and the value of one variable may be influenced by many others. Ex:- The yield of crop per acre (X_1) depends upon quality of seed (X_2), fertility of soil (X_3), fertilizers used (X_4), irrigation facilities (X_5), weather condition (X_6) and so on.

Whenever we are intrested in studying the joint effect of a group of variables upon a variable not included in that group. our study is that of multiple correlation and multiple regression

Suppose in a trivariate (or) multivariate distribution we are intrested in the relationship b/w two variables only. There are 2 alternatives :(i) we consider only those two members of the observed data in which the other members have specified value.

(or) we may eliminate mathematically the effect of other variates on two variates.

The first method has the disadvantage thus its limits the size of the data and also it will be applicable to only the data in which the other variates have assigned value.

In the second method it may not be possible to eliminate the entire influence of the variates but the linear effect can be easily eliminated.

The correlation and regression b/w only two variates eliminating the linear effect of other variates in them is called the partial correlation and partial regression.

Coefficient of Multiple correlations:-

In a trivariate distribution in which each of the variables X_1, X_2 and X_3 has 'n' observations, the multiple correlation coefficient of X_1 on X_2 and X_3, usually denoted by $R_{1\cdot 23}$ is the simple correlation coefficient b/w X_1 and the joint effect of X_2 and X_3 on X_1.

In other words $R_{1\cdot 23}$ is the correlation coefficient b/w X_1 and its estimated value as given by the plane of regression of X_1 on X_2 and X_3

$$e_{1\cdot 23} = b_{12\cdot 3}X_2 + b_{13\cdot 2}X_3$$

we have

$$X_{1\cdot 23} = X_1 - b_{12}X_2 - b_{13\cdot 2}X_3 = X_1 - e_{1\cdot 23}$$

$$\therefore e_{1\cdot 23} = X_1 - X_{1\cdot 23}$$

$$E[X_{1\cdot 23}] = 0 \text{ and } E(e_{1\cdot 23}) = 0 \; (\because E(X_i) = 0;\ i = 1,2,3\cdots)$$

$$\text{By def } R_{1\cdot 23} = \frac{\mathrm{Cov}(X_1, e_{1\cdot 23})}{\sqrt{V(X_1)\,V(e_{1\cdot 23})}}$$

$$R_{1\cdot 23} = \frac{\sigma_1^2 - \sigma_{1\cdot 23}^2}{\sqrt{\sigma_1^2(\sigma_1^2 - \sigma_{1\cdot 23}^2)}}$$

$$R_{1\cdot 23}^2 = 1 - \frac{\sigma_{1\cdot 23}^2}{\sigma_1^2}$$

$$\boxed{R_{1\cdot 23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}}$$

Coefficient of partial correlation:-

Sometimes the correlation b/w two variables X_1 and X_2 may be partly due to the correlation of a 3rd variable, 'X_3' with both X_1 & X_2. In such a situation, one may want to know what the correlation X_1 & X_2 would be if the effect of X_3 on each of X_1 & X_2 were eliminated. This correlation is called the partial correlation and the correlation coefficient b/w X_1 & X_2 after the linear effect of X_3 on each of them has been eliminated is called the partial correlation coefficient.

The residual $X_{1\cdot 3} = X_1 - b_{13}X_3$ may be regarded as that part of the variable X_1 which remains after the linear effect of X_3 has been eliminated. Similarly the residual $X_{2\cdot 3} = X_2 - b_{23}X_3$ may be interpreted as the part of the variable X_2 obtained after eliminating the linear effect of X_3. Thus the partial correlation coefficient b/w X_1 & X_2, usually denoted by $r_{12\cdot 3}$ is given by

$$\boxed{r_{12\cdot 3} = \frac{\operatorname{cov}(X_{1\cdot 3}, X_{2\cdot 3})}{\sqrt{V(X_{1\cdot 3})V(X_{2\cdot 3})}}}$$

we have $cov(X_{1.3}, X_{2.3}) = \frac{1}{N} \Sigma X_{1.3} X_{2.3}$

$= \sigma_1 \sigma_2 (r_{12} - r_{13} r_{23})$

$V(X_{1.3}) = \frac{1}{N} \Sigma X_{1.3}^2$

$= \sigma_1^2 (1 - r_{13}^2)$

$V(X_{2.3}) = \sigma_2^2 (1 - r_{23}^2)$

Hence

$$r_{12.3} = \frac{(r_{12} - r_{13} r_{23})}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$r_{13.2} = \frac{(r_{13} - r_{12} \cdot r_{32})}{\sqrt{(1-r_{12}^2)(1-r_{32})^2}}$$

$$r_{23.1} = \frac{(r_{23} - r_{21} \cdot r_{31})}{\sqrt{(1-r_{21}^2)(1-r_{31}^2)}}$$

1) from the data relating to the yield of dry bark (X_1), height (X_2) and girth (X_3) for eighteen (18) cinchona plants. The following correlation coefficients were obtained.

$r_{12} = 0.77$, $r_{13} = 0.72$ and $r_{23} = 0.52$

find the partial correlation coefficient $r_{12.3}$ and multiple correlation coefficient $R_{1.23}$

$$r_{12.3} = \frac{(r_{12} - r_{13} r_{23})}{(1-r_{13}^2)(1-r_{23}^2)}$$

$$= \frac{0.77 - (0.72)(0.52)}{\sqrt{[1-(0.72)^2][1-(0.52)^2]}} \quad \frac{0.77 - 0.3744}{\sqrt{(1-0.5184)(1-0.2704)}}$$

$$= \frac{0.3956}{\sqrt{(0.4816)(0.7296)}} = \frac{0.3956}{\sqrt{0.3513}} = \frac{0.3956}{0.5927}$$

$= 0.6674$

0.73

$$R_{1\cdot23}^2 = \frac{r_{12}^2 + r_{13}^2 - r_{12} r_{13} r_{23}}{1 - r_{23}^2}$$

$$= \frac{(0.77)^2 + (0.72)^2 - (0.77)(0.72)(0.52)}{1 - (0.52)^2}$$

$= \cancel{0.5929 + 0.5184 - 0.5765}$

$$= \frac{0.5929 + 0.5184 - 0.5765}{1 - 0.2704}$$

$$= \frac{0.5348}{0.7296}$$

$R_{1\cdot23}^2 = 0.73$

$R_{1\cdot23} = \sqrt{0.73}$

$$\boxed{R_{1\cdot23} = 0.8544}$$

In a trivariate distribution $\sigma_1 = 2$, $\sigma_2 = \sigma_3 = 3$, $r_{12} = 0.7$, $r_{23} = r_{31} = 0.5$ find (i) $r_{23\cdot1}$ (ii) $R_{1\cdot23}$ (iii) $b_{12\cdot3}$, $b_{13\cdot2}$ and (iv) $\sigma_{1\cdot23}$

(i) $$r_{23\cdot1} = \frac{(r_{23} - r_{21} r_{31})}{\sqrt{(1 - r_{21}^2)(1 - r_{31}^2)}} = \frac{0.5 - (0.7)(0.5)}{\sqrt{(1-(0.7)^2)(1-(0.5)^2)}} = \frac{0.5 - 0.35}{\sqrt{(0.51)(0.75)}}$$

0.3825

$$= \frac{0.15}{0.6184} = 0.2425$$

(ii) $$R_{1\cdot23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2 r_{12} r_{13} r_{23}}{1 - r_{23}^2} = \frac{(0.7)^2 + (0.5)^2 - 2(0.7)(0.5)(0.5)}{1 - (0.5)^2}$$

$R_{1\cdot23}^2 = 0.52$

$R_{1\cdot23} = \sqrt{0.52} = 0.7211$

(iii) $b_{12.3} = r_{12.3} \times \dfrac{\sigma_{1.3}}{\sigma_{2.3}}$ and $b_{13.2} = r_{13.2} \dfrac{\sigma_{1.2}}{\sigma_{3.2}}$

$\cancel{b_{12.3} =}$

$$r_{12.3} = \frac{(r_{12} - r_{13} r_{23})}{\sqrt{(1-r_{13})^2 (1-r_{23})^2}}$$

$$= \frac{0.7 - 0.25}{\sqrt{(1-0.25)^2 (1-0.25)}}$$

$$= \frac{0.45}{\sqrt{(0.75)0.75}}$$

$$= \frac{0.45}{\sqrt{0.5625}}$$

$$= \frac{0.45}{0.75}$$

$$= 0.6$$

$$\cancel{r_{12.3} = r_{12.3} \times \frac{\sigma_{1.3}}{\sigma_{2.3}}}$$

$$\cancel{= 0.6 \times \frac{3}{2}}$$

$$\cancel{= 0.6 \times 0.5}$$

$$b_{12.3} = r_{12.3} \times \frac{\sigma_{1.3}}{\sigma_{2.3}}$$

$$= 0.6 \times \frac{1.732}{2.598}$$

$$= 0.4$$

$$b_{13.2} = r_{13.2} \times \frac{\sigma_{1.2}}{\sigma_{3.2}}$$

$$= 0.2425 \times \frac{1.48}{2.598}$$

$$= 0.1333$$

$$r_{13.2} = \frac{(r_{13} - r_{12} r_{32})}{\sqrt{(1-r_{12}^2)(1-r_{32}^2)}}$$

$$= 0.2425$$

$$\sigma_{1.3} = \sigma_1 \sqrt{(1-r_{13}^2)}$$

$$= 2\sqrt{(0.75)}$$

$$= 2 \times 0.8660$$

$$= 1.732$$

$$\sigma_{2.3} = \sigma_2 \sqrt{(1-r_{23}^2)}$$

$$= 3\sqrt{(0.75)}$$

$$= 3 \times 0.8660$$

$$= 2.598$$

1.00

$$\sigma_{1.2} = \sigma_1 \sqrt{(1-r_{12}^2)}$$

$$= 2\sqrt{0.51}$$

$$= 2 \times 0.7141$$

$$= 1.428$$

$$\sigma_{3.2} = \sigma_3 \sqrt{(1-r_{32}^2)}$$

$$= 3\sqrt{0.75}$$

$$= 3 \times 0.8660$$

$$= 2.598$$

(iv) $\sigma_{1.23} = \sigma_1 \sqrt{\frac{W}{W_{11}}}$

$$W = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix} \begin{vmatrix} 1 & 0.7 & 0.5 \\ 0.7 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{vmatrix}$$

$= 1(1-0.25) - 0.7(0.7-0.25) + 0.5(0.35-0.5)$

$= 1(0.75) - 0.7(0.45) + 0.5(0.15)$

$= 0.75 - 0.315 - 0.075$

$= 0.36$

0.7×0.5, 0.25, 0.45 ; 0.35, 0.50, 0.15

$$W_{11} = \begin{vmatrix} 1 & r_{23} \\ r_{32} & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0.5 \\ 0.5 & 1 \end{vmatrix} = 1 - 0.25 = 0.75$$

$$\sigma_{1.23} = 2\sqrt{\frac{0.36}{0.75}} = 1.385$$

Kull-back Leibloor divergence: * Imp

The KL-divergence measure of how a probability differs from another probability distribution.

Classiclay in bayesian theory, There is some true distribution $P(x)$, we had like to estimate with an approximate distribution $Q(x)$. In this context, the KL-divergence measures the distance from the approximate distribution to the true distribution "P".

$$D_{KL}(P \| Q) = E_{x \sim P(x)}\left[\log\left(\frac{P(x)}{Q(x)}\right)\right]$$

$$= \begin{cases} \sum_x P(x) \log\left(\frac{P(x)}{Q(x)}\right) & \text{if } x \text{ is discrete} \\ \int_x P(x) \log\left(\frac{P(x)}{Q(x)}\right) & \text{if } x \text{ is continuous} \end{cases}$$

Kullback Liebler divergence b/w two normal PDFs:-

Let $P(x) \sim N(\mu, \sigma^2)$, $q(x) \sim N(m, s^2)$. Now

$$KL(P \| q) = \int P(x) \cdot \log \frac{P(x)}{q(x)} \, dx$$

$$= \int P(x) \cdot \log \frac{\frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\frac{1}{s\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-m}{s}\right)^2}}$$

$$= \int P(x) \log \left(\frac{s}{\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 + \frac{1}{2}\left(\frac{x-m}{s}\right)^2} \right) dx$$

$$= \int P(x) \log \left(\frac{s^2}{\sigma^2}\right)^{1/2} dx - \int P(x) \frac{(x-\mu)^2}{2\sigma^2} + \int P(x) \frac{(x-m)^2}{2s^2} dx$$

$$= \frac{1}{2} \log\left(\frac{s^2}{\sigma^2}\right) \int P(x) dx - \frac{1}{2\sigma^2} \int (x-\mu)^2 P(x) dx + \frac{1}{2s^2} \int P(x) (x-\mu+\mu-m)^2 dx$$

$$= \frac{1}{2} \log\left(\frac{s^2}{\sigma^2}\right)(1) - \frac{1}{2\sigma^2} - \int (x-\mu)^2 P(x) dx + \frac{1}{2s^2} \int P(x) (x-\mu+\mu-m)^2 dx$$

$$= \frac{1}{2} \log\left(\frac{s^2}{\sigma^2}\right) - \frac{1}{2\sigma^2} \cdot \sigma^2 + \frac{1}{2s^2} \left[\int P(x)(x-\mu)^2 dx + \int P(x)(\mu-m)^2 dx + \int 2(x-\mu)(\mu-m) P(x) \, dx \right]$$

$$= \frac{1}{2} \log\left(\frac{s^2}{\sigma^2}\right) - \frac{1}{2} + \frac{1}{2s^2} \left[\sigma^2 + (\mu-m)^2 \int P(x) + 2(\mu-m) \int P(x)(x-\mu) dx \right]$$

$$= \frac{1}{2} \log\left(\frac{s^2}{\sigma^2}\right) - \frac{1}{2} + \left[\frac{\sigma^2 + (\mu-m)^2}{2s^2} \right]$$