Unit - IV

Multiple integrals

A double or triple integral is known as Multiple integral is an extension of a definite integral of a function are single variable to a function of 2 or 3 variables. The multiple integral is usually called as improper the region of integration extends to '∞'. (or) when the ~~while~~ integral becomes infinite at a point in side the region (or) on the boundary of the region.

These are Useful in evaluating area, volume, mass, Centroid, & moments of inertia in plane & solid region

Double integrals:

If $f(x)$ is definite over a region 'R' then the double integral of $f(x,y)$ is denoted by

$$\iint_R f(x,y)\,dx\,dy \quad \text{(or)}$$

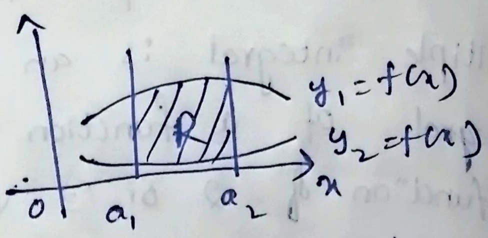
$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy$$, where R is Region of Integration

Evaluation of double integrals:

It involves 3 cases;

Case(1): when y_1 & y_2 are functions of x_1 & x_2 are constants then $f(x,y)$ is 1st integrated w.r.to 'y' keeping x- fixed (or) Constants b/w the limits y_1 & y_2 the resulting expression is integrated w.r.to x b/w the limits x_1 & x_2

i.e., $\int_{a_1}^{a_2}\left(\int_{y_1=f(x)}^{y_2=f(x)} f(x,y)\, d^2y\right) dx$



Graphical representation

Case 2: When x_1 & x_2 are functions of y and y_1 & y_2 are constants, the $f(x,y)$ is first integerated w.r.to x, keeping y fixed (or) constants b/w the limits x_1 & x_2 the resulting expression is integrated w.r.t y b/w the limits y_1 & y_2

$$\int_{y_1}^{y_2}\left(\int_{x_1=f(y)}^{x_2=f(y)} f(x,y)\, d^2x\right) dy$$

Case 3: If all the 4 limits of an integration are constants then the double integral can be calculated in either way i.e., we first integrate w.r.to x then w.r.to y (or) we 1st integrate w.r.to y later w.r.to x.

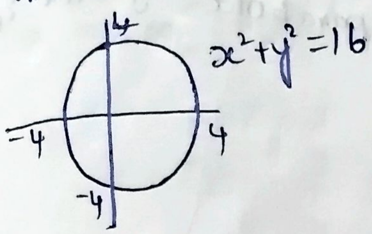
1. Evaluate $\int_0^1 \int_0^x e^{x+y}\, dy\, dx$

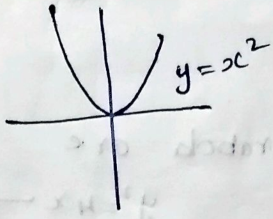
Sol: given expression is

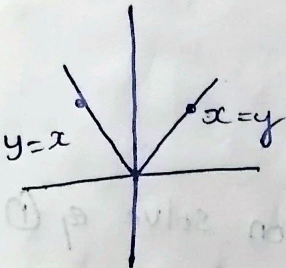
$$\int_{x=0}^{1}\int_{y=0}^{x} e^{x+y}\, dy\, dx$$

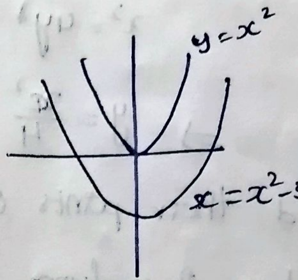
The limits of the inner integral are functions of x these are limits for y, the outer integral as constant limits for x so, we 1^{st} integrate w.r.to y

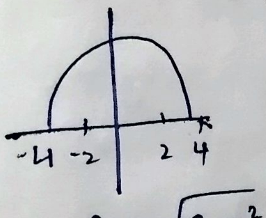
graphs:

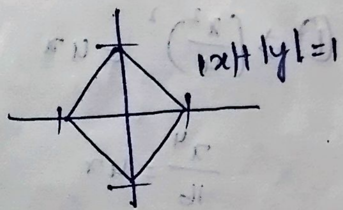


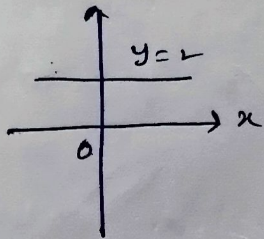


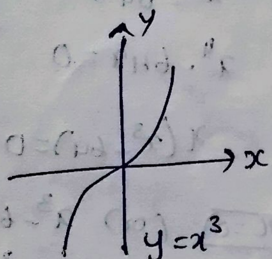


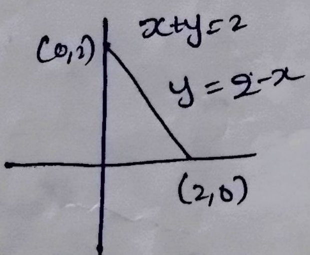


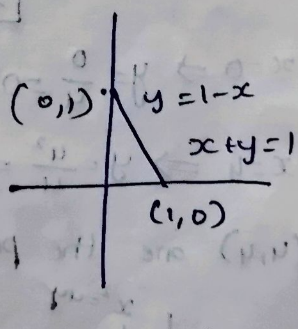


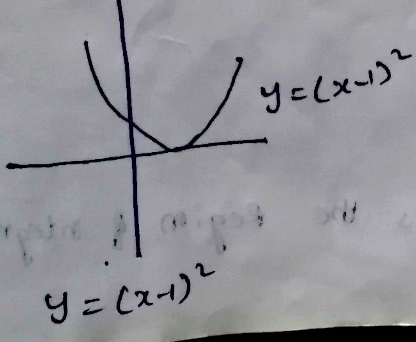












(5) ~~Sol~~ Evaluate $\iint_R y\,dx\,dy$ where R is the region of integration bounded by the parabola $y^2=4x$ & $x^2=4y$

Sol: Given parabola are

$$y^2=4x \longrightarrow (1)$$

$$x^2=4y \longrightarrow (2)$$

$$\Rightarrow y=\frac{x^2}{4}$$

To find their points of intersection solve eq (1) & (2) sub. the values of y from eq (2) in eq (1) then,

$$(1) \Rightarrow \left(\frac{x^2}{4}\right)^2=4x$$

$$\frac{x^4}{16}=4x$$

$$x^4=64x$$

$$x^4-64x=0$$

$$x(x^3-64)=0$$

$$\boxed{x=0} \text{ (or) } x^3-64=0$$

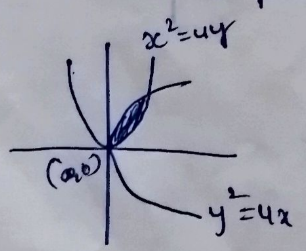
$$x^3=64$$

$$\boxed{x=4}$$

if $x=0 \Rightarrow y=\frac{0}{4}=0$

if $x=4 \Rightarrow y=\frac{4^2}{4}=4$

$(0,0)$ & $(4,4)$ are the points of intersection



The shaded area is the Region of integration from the figure, limits are

y rasies from $y=\frac{x^2}{4}$ to $y=2\sqrt{x}$

x rasies from $x=0$ to $x=4$

$$\iint_R y\,dx\,dy = \int_{x=0}^{4}\left(\int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\right)dx$$

$$= \int_{x=0}^{4}\left(\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{2}\int_{x=0}^{4}\left[(2\sqrt{x})^2 - \left(\frac{x^2}{4}\right)^2\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{4}\left(4x - \frac{x^4}{16}\right)dx$$

$$= \frac{1}{2}\left(\frac{\cancel{4}^{2}x^2}{\cancel{2}} - \frac{x^5}{16\times 5}\right)_0^4$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4$$

$$= \frac{1}{2}\left[2(4)^2 - \frac{4^5}{80} - 0\right]$$

$$= \frac{1}{2}\left[30 - \frac{4^{\cancel{8}\,5}}{80}\right]$$

$$= \frac{1}{2}\left(\frac{96}{5}\right)$$

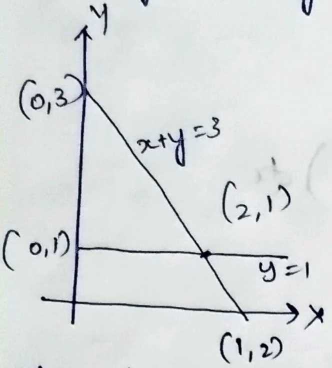
$$= \frac{48}{5} \text{ sq. units}$$

(6) $\iint_R (x+y)\,dA$, where R is the region b/w the lines

$x=0$, $y=1$ & $x+y=3$.

Sol: Given eq. of lines are

$x=0,\ y=1,\ x+y=3$



The shaded area is the region of integration from the graph. limits are

y varies from $y=1$ to $y=3-x$

x varies from $x=0$ to $x=2$

Given that

$$\iint_R (x+y)\,dA = \int_{x=0}^{2}\left(\int_{y=1}^{3-x}(x+y)\,dy\right)dx$$

$$= \int_{x=0}^{2}\left[xy+\frac{y^2}{2}\right]_{y=1}^{3-x} dx$$

$$= \int_0^2 \left[x(3-x)+\left(\frac{3-x}{2}\right)^2 - x(1)-\frac{(1)^2}{2}\right]dx$$

$$= \int_{x=0}^{2}\left(3x-x^2+\frac{9+x^2-6x}{2}-x-\frac{1}{2}\right)dx$$

$$= \frac{1}{2}\int_0^2 (6x-2x^2+9+x^2-6x-2x-1)\,dx$$

$$= \frac{1}{2}\int_0^2 (-x^2-2x+8)\,dx$$

$$= \frac{1}{2}\left[\frac{-x^3}{3}-\frac{\cancel{2}x^2}{\cancel{2}}+8x\right]_0^2$$

$$= \frac{1}{2}\left[-\frac{2^3}{3}-2^2+8(2)-0\right]$$

$$= \frac{1}{2}\left[\frac{-8-12+48}{3}\right]$$

$$= \frac{1}{2}\left[\frac{28}{3}\right]$$

$$= \frac{28}{6}$$

$$= \frac{14}{3} \text{ sq. units.}$$

(7) Evaluate $\iint_R xy\,dA$, the region 'R' of the integration is enclosed by the curve $(x-1)^2=2y$ and $y=2$

Sol: Given curves are

$(x-1)^2 = 2y \rightarrow (1)$

$y=2 \rightarrow (2)$

To find their points of ~~insertion~~ intersection solve (1) & (2)

put $y=2$ in eq (1)

$(x-1)^2 = 2(2)$

$x^2+1-2x-4=0$

$x^2-2x-3=0$

$x^2-3x+x-3=0$

$x(x-3)+1(x-3)=0$

$(x-3)(x+1)=0$

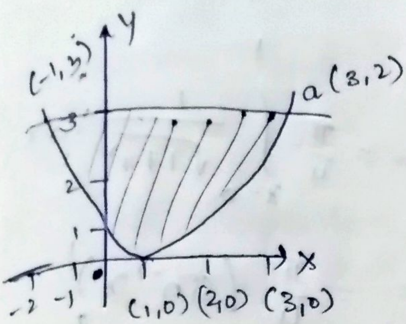
$x=3,-1$

$(1) \Rightarrow y=\frac{(x-1)^2}{2}$

if $x=3 \Rightarrow \frac{(3-1)^2}{2}=2$

if $x=-1 \Rightarrow \frac{(-1-1)^2}{2}=2$

$\therefore (3,2)$ & $(-1,2)$ are points of intersection



The shaded area is the region of integration from the graph limit are

y varies from $y=0$ to $y=2$

x varies from $x=1+\sqrt{2y}$ to $x=1-\sqrt{2y}$

Given that $\iint_R xy\,dA =$

$$\int_{y=0}^{2}\left(\int_{x=1+\sqrt{2y}}^{1-\sqrt{2y}} xy\,dx\right)dy$$

$$= \int_{y=0}^{2}\left(\frac{4x^2}{2}\right)_{x=1+\sqrt{2y}}^{1-\sqrt{2y}} dy$$

$$= \frac{1}{2}\int_{y=0}^{2}\left(y\left[1-\sqrt{2y}\right]^2 - y\left[1+\sqrt{2y}\right]^2\right)dy$$

$$= \frac{1}{2}\int_{y=0}^{2} y\left(1+2y-2\sqrt{2y}\right) - y\left(1+2y+2\sqrt{2y}\right)dy$$

$$= \frac{1}{2}\int_{y=0}^{2}\left(y+\cancel{2y^2}-2y\sqrt{2y}-y-\cancel{2y^2}-2y\sqrt{2y}\right)dy$$

$$= \frac{1}{2}\int_{y=0}^{2}\left(-4y\sqrt{2}\sqrt{y}\right)dy$$

$$= \frac{-\cancel{4}^{2}\sqrt{2}}{\cancel{2}}\int_0^2 y\sqrt{y}]\,dy$$

$$= -2\sqrt{2}\int_0^2 \left(y^{3/2}\right)dy$$

$$= -2\sqrt{2}\;\frac{y^{3/2+1}}{3/2+1}\Bigg]_0^2$$

$$= -2\sqrt{2} + \frac{2}{5}\left(y^{5/2}\right)_0^2$$

$$= \frac{-4\sqrt{2}}{5}\left(2^{5/2}\right)$$

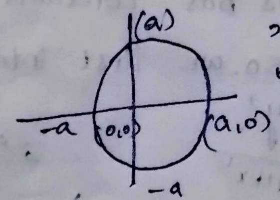
$$= \frac{-32}{5}$$

Area should not be negative. so.

$$\iint_R xy\,dx = \frac{32}{5} \text{ sq. units.}$$

(8). Evaluate $\iint xy\,dx\,dy$ over the positive quadrant $x^2+y^2=a^2$

Sol: Given eq. is $x^2+y^2=a^2$



$x^2+y^2=a^2$

$y^2=a^2-x^2$

$y=\sqrt{a^2-x^2}$

The shaded area is the ~~integration~~ region of integration from the graph limits are

x varies from $x=0$ to $x=a$

y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

Given that $\iint xy\,dx\,dy$

$$= \int_{x=0}^{a}\left(\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\right)dx$$

$$= \int_0^a\left(\frac{xy^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\left[x\left(\sqrt{a^2-x^2}\right)^2 - x(0)\right]dx$$

$$= \frac{1}{2}\int_0^a (a^2x - x^3 - 0)\,dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left(\frac{a^2a^2}{2} - \frac{a^4}{4}\right)$$

$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{1}{2}\left[\frac{4a^2-2a^4}{8}\right]$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}a^4}{8}\right) = \frac{a^4}{8} \text{ sq. units}$$

(9) Evaluate $\int_{x=0}^{1}\int_{y=0}^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol: Since the inner integrals has variable limits for y and outer integrals has constants limits for x. so, we first integrate w.r.to 'y'.

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$\int_{x=0}^{1}\left(\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}dy\right)dx =$$

$$= \int_{x=0}^{1}\left(\int_{y=0}^{\sqrt{1+x^2}}\left(\frac{1}{\sqrt{1+x^2}^2+y^2}\right)dy\right]dx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}}\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right)_{y=0}^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}}\tan^{-1}\frac{\cancel{\sqrt{1+x^2}}}{\cancel{\sqrt{1+x^2}}} - \frac{1}{\sqrt{1+x^2}}\tan^{-1}(0)\right)dx$$

$$= \int_{x=0}^{1}\left(\frac{1}{\sqrt{1+x^2}}\right)\cancel{\tan^{-1}}\left(\cancel{\tan}\frac{\pi}{4}\right)\Big]dx$$

$$= \frac{\pi}{4}\int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}dx$$

$$= \frac{\pi}{4}\left(\sin^{-1}x\right)_0^1$$

$$= \frac{\pi}{4}\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$$

$$= \pi/4\left[\cancel{\sin^{-1}}(\cancel{\sin}\,\pi/2) - \cancel{\sin^{-1}}(\cancel{\sin}\,0)\right]$$

$$= \frac{\pi}{4}\left[\pi/2 - 0\right]$$

$$= \pi^2/8 \text{ sq. units}$$

(10) Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$

Sol: Given equations are

$$y=x^2 \longrightarrow (1)$$

$$y=x \longrightarrow (2)$$

To find their point of intersection solve eq (1) & (2)

substitute the value of 'y' in eq (1)

$$x=x^2$$

$$x^2-x=0$$

$$x(x-1)=0$$

$$\boxed{x=0}, \boxed{x=+1}$$

If $x=0$ then $y=0$

If $x=1$ then $y=1$

The shaded area is the region of intersection from the graph limits are

'y' varies from $y=x^2$ to $y=x$

'x' varies from $x=0$ to $x=1$

Given that

$$\iint xy(x+y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x^2y + xy^2)\,dy\,dx$$

$$= \int_{x=0}^{1}\left(\int_{y=x^2}^{x} (xy+y^2)\,dy\right)dx$$

$$= \int_{0}^{1}\left(x\left(\frac{xy^2}{2}+\frac{y^3}{3}\right)_{x^2}^{x}\right)dx$$

$$= \int_{0}^{1} x\left(\frac{x\,x^2}{2}+\frac{x^3}{3}-\frac{x\,x^4}{2}-\frac{x^6}{3}\right)dx$$

$$= \int_{x=0}^{1}\left(\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$= \int_{0}^{1}\left(\frac{3x^4+2x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$= \int_{0}^{1}\left(\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$= \left(\frac{5}{6}\cdot\frac{x^5}{5}-\frac{1}{2}\frac{x^7}{7}-\frac{1}{3}\cdot\frac{x^8}{8}\right)_0^1$$

$$= \left(\frac{\cancel{5}}{6}\times\frac{1}{\cancel{5}}-\frac{1}{2}\times\frac{1}{7}-\frac{1}{3}\times\frac{1}{8}-0\right)$$

$$= \frac{1}{6}-\frac{1}{14}-\frac{1}{24}$$

$$= \frac{84-36-21}{504}$$

$$= \frac{27}{504} \text{ sq. units}$$

Changing the Order of integration:

There are two ways of evaluation of double integrals, depending on the order of integration, ~~depending~~

$$\int_{x=a}^{b}\left(\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy\right)dx$$

$$\int_{y=c}^{d}\left(\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx\right)dy.$$

In some cases, evaluation of the integral in the given order is impossible. Change of the order of integration makes it possible (or) easy for evaluation

Procedure:

Identify the boundaries of the region of integration by drawing the curves

$$y=f_1(x),\quad y=f_2(x),\quad x=a,\quad x=b$$

While changing the order, we have to fix the limits for x in terms of y and then find the constant limits for y

Note: 'x' may have two functions in y as 2 limits. Then we have to sub divide the region R into subregions R_1 & R_2

Problems:-

1. Change the order of integration and evaluate

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$$

Sol: Given expression is $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$

the inner integral has functions of x as limits. so, it has to be evaluated w.r.to y.

The Outer integral has constant limits for x

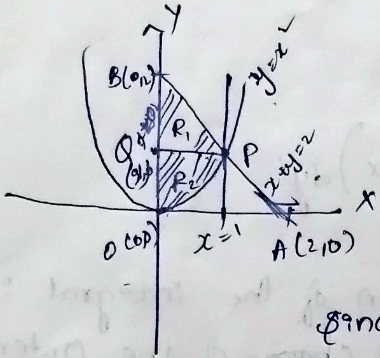
$y = x^2$ (parabola)

$y = 2 - x$

$x + y = 2$ (str. line) $\longrightarrow$ (1)

$x = 0$ (y-axis)

$x = 1$ (line) $\longrightarrow$ (2)



The region of intersection is ~~formed~~ bounded by y-axis, the parabola & straight line $x+y=2$.

Since there are two different functions of x, defining the limits for 'y' in eq. (1) We have to divide R into two regions R_1 (QPBQ) and R_2 (OPQO)

After changing the limits, limits are;

R_1(QPBQ)

x Varies from $x = 0$ to $x = 2 - y$

y Varies from $y = 1$ to $y = 2$

R_2 (OPQO)

x varies from $x = 0$ to $x = \sqrt{y}$

y varies from $y = 0$ to $y = 1$

$R = R_1 + R_2$

$$= \int_{y=1}^{2} \left(\int_{x=0}^{2-y} xy \, dx \right) dy + \int_{y=0}^{1} \left(\int_{x=0}^{\sqrt{y}} xy \, dx \right) dy$$

$$= \int_{y=1}^{2} \left[\frac{yx^2}{2} \right]_0^{2-y} dy + \int_{y=0}^{1} \left(\frac{yx^2}{2} \right)_0^{\sqrt{y}} dy$$

$$= \frac{1}{2} \int_{y=1}^{2} \left[y(2-y)^2 - y(0) \right] dy + \frac{1}{2} \int_{y=0}^{1} \left[y(\sqrt{y})^2 - y(0) \right] dy$$

$$= \frac{1}{2}\int_{y=1}^{2}\left[y(4+y^2-4y)\right]dy + \frac{1}{2}\int_{y=0}^{1} y^2\, dy$$

$$= \frac{1}{2}\int_{y=1}^{2}(4y+y^3-4y^2)\, dy + \cancel{\frac{1}{2}\int_{y=0}\left(\frac{y^3}{3}\right)dy}\ \frac{1}{2}\left[\frac{y^3}{3}\right]_{y=0}^{1}$$

$$= \frac{1}{2}\left[\frac{4y^2}{2}+\frac{y^4}{4}-\frac{4y^3}{3}\right]_{y=1}^{2} + \frac{1}{2}\left(\frac{1}{3}-0\right)$$

$$= \frac{1}{2}\left[\left(\frac{4(4)}{2}+\frac{16}{4}-\frac{4(8)}{3}\right)-\left(\frac{4}{2}+\frac{1}{4}-\frac{4}{3}\right)\right]+\frac{1}{2}\left(\frac{1}{3}\right)$$

$$\cancel{= \frac{1}{2}\left[\left(\frac{16}{2}^{8}+\frac{16}{4}^{4}-\frac{32}{3}\right)-\left(\frac{4}{2}^{2}+\frac{1}{4}-\frac{4}{3}\right)\right]+\frac{1}{6}}$$

$$\cancel{= \frac{1}{2}\left(8+4-\frac{32}{3}\right)-\left(2+\frac{1}{4}-\frac{4}{3}\right)+\frac{1}{6}}$$

$$= \frac{1}{2}\left(\frac{16}{2}+\frac{16}{4}-\frac{32}{3}-\frac{4}{2}-\frac{1}{4}+\frac{4}{3}\right]+\frac{1}{6}$$

$$= \frac{1}{2}\left(\frac{16}{2}-\frac{4}{2}+\frac{16}{4}-\frac{1}{4}-\frac{32}{3}+\frac{4}{3}\right]+\frac{1}{6}$$

$$= \frac{1}{2}\left(\frac{12}{2}+\frac{15}{4}-\frac{28}{3}\right]+\frac{1}{6}$$

$$= \frac{1}{2}\left(\frac{72+45-112}{12}\right]+\frac{1}{6} = \frac{1}{2}\left[\frac{5}{12}\right]+\frac{1}{6}$$

$$= \frac{5}{24}+\frac{1}{6}$$

$$\frac{5+4}{24} = \frac{9^{3}}{\cancel{24}_{8}} = \frac{3}{8} \text{ sq. units}$$

(2) Evaluate $\int_{0}^{\infty}\int_{y=x}^{\infty}\frac{e^{-y}}{y}\,dy\,dx$ by changing order of integration

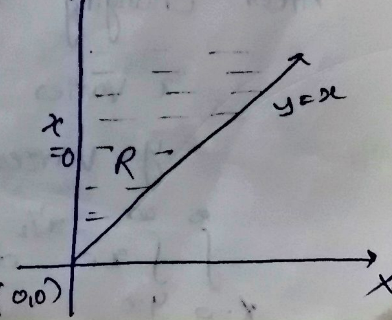
Sol: Given expression is $\int_{x=0}^{\infty}\int_{y=x}^{\infty}\frac{e^{-y}}{y}\,dy\,dx$

The inner integral has functions of x has limits. so, It has to be evaluated with respect to y.

The outer integral has constant limits for x

$y = x \qquad y = \infty$ —— (1)

$x = 0$ (y-axis) $\qquad x = \infty$ —— (2)



The region of integration is bounded by Y-axis & $y = x$ line

After changing the limits,

x varies from $x=0$ to $x=y$

y varies from $y=0$ to $y=\infty$

$$\int_0^\infty \int_{y=x}^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty} \left(\int_{x=0}^{y} dx \right) \frac{e^{-y}}{y}\, dy$$

$$= \int_{y=0}^{\infty} [x]_{x=0}^{y} \frac{e^{-y}}{y}\, dy$$

$$= \int_{y=0}^{\infty} (y-0) \frac{e^{-y}}{y}\, dy$$

$$= \int_{y=0}^{\infty} \cancel{y} \frac{e^{-y}}{\cancel{y}}\, dy$$

$$= \left[-e^{-y}\right]_{y=0}^{\infty}$$

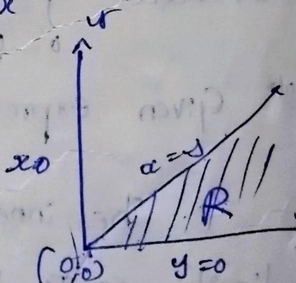
$$= \left[-e^{-\infty} - (-e^{0})\right] = \left[0-(-1)\right] = 1 \text{ sq. unit}$$

(3) Evaluate $\int_0^\infty \int_0^\infty x e^{-x^2/y}\, dy\, dx$ by C.T.O.I.

Sol: Given expression $\int_0^\infty \int_0^\infty x e^{-x^2/y}\, dy\, dx$

$y=0 \qquad y=\infty \longrightarrow (1)$

$x=0 \qquad x=\infty \longrightarrow (2)$



The region of integration is bounded by $y=0$ & $x=y$

After changing the order, the limits are

x varies from $x=y$ to $x=\infty$

y varies from $y=0$ to $y=\infty$

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} x e^{-x^2/2}\, dy\, dx = \int_{y=0}^{\infty} \left(\int_{x=y}^{\infty} x e^{-x^2/y}\, dx \right) dy \longrightarrow (1)$$

let $\frac{-x^2}{y} = t$

$$\frac{-2x\,dx}{y} = dt$$

$$x\,dx = \frac{-1}{2} y\,dt$$

limits are $x = y \Rightarrow t = \frac{-y^2}{y} = -y$

$x = \infty \Rightarrow t = \infty$

from (1)

$$\int_{y=0}^{\infty}\left(\int_{t=-y}^{\infty} \frac{1}{2} y e^{t}\, dt\right) dy = \frac{-1}{2}\int_{y=0}^{\infty} y\left(\int_{t=-y}^{\infty} e^{t}\, dt\right) dy$$

$$= \frac{-1}{2}\int_{0}^{\infty} y\left(e^{t}\right)_{t=-y}^{\infty} dy$$

$$= \frac{-1}{2}\int_{0}^{\infty} y\left(e^{t}\right)_{t=-y}^{\infty} dy$$

$$= \frac{-1}{2}\int_{y=0}^{\infty} y\left[e^{-\infty} - e^{-y}\right] dy$$

$$= \frac{-1}{2}\int_{y=0}^{\infty} -ye^{-y}\, dy$$

$$= \frac{1}{2}\int_{0}^{\infty} ye^{-y}\, dy.$$

$$= \frac{1}{2}\left[y(+e^{-y}) - 1(+e^{-y})\right]_{0}^{\infty}$$

$$= \frac{1}{2}\left(-ye^{-y} - e^{-y}\right)_{0}^{\infty} = \frac{1}{2}\left[0 + 0 + e^{0}\right]$$

$$= \frac{1}{2}(1)$$

$= \frac{1}{2}$ sq. units.

(4) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dx \, dy$

Sol: Given expression $\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y^2 \, dx \, dy$

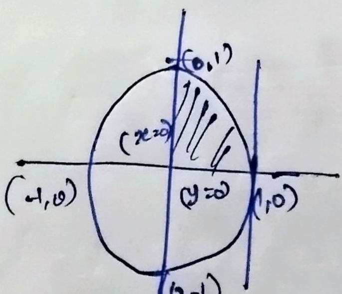
limits are

$x = 0$ (y-axis) $\quad x = 1 \longrightarrow (1)$

$y = 0$ (x-axis) $\quad y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$ (circle with radius 1) $\longrightarrow (2)$



the region of integration is bounded by y-axis, x-axis & curve $x^2+y^2=1$

After changing the order, the limits are

y values from $y=0$ to $y=1$

x values from $x=0$ to $x=\sqrt{1-y^2}$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dx \, dy = \int_0^1 \left(\int_{x=0}^{\sqrt{1-y^2}} dx \right) y^2 \, dy$$

$$= \int_{y=0}^{1} (x)_{x=0}^{x=\sqrt{1-y^2}} \, y^2 \, dy$$

$$= \int_{y=0}^{1} \left(\sqrt{1-y^2} - 0\right) y^2 \, dy$$

$$= \int_{y=0}^{1} y^2 \sqrt{1-y^2} \, dy \longrightarrow (3)$$

put $y = \sin\theta$

$dy = \cos\theta \cdot d\theta$

limits if $y = 0 \Rightarrow 0 = \sin\theta$

$\boxed{\theta = 0}$

if $y = 1 \Rightarrow 1 = \sin\theta$

$\boxed{\theta = \pi/2}$

(3) $\Rightarrow$ $$\int_{\theta=0}^{\pi/2} \sin^2\theta \sqrt{1-\sin^2\theta} \cdot \cos\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin^2\theta \sqrt{\cos^2\theta} \, \cos\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi/2} (1^2 - \cos^2 2\theta) \, d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi/2} 1 \, d\theta - \frac{1}{4}\int_{\theta=0}^{\pi/2} \cos^2 2\theta \cdot d\theta$$

$$= \frac{1}{4}(\theta)_{\theta=0}^{\pi/2} - \frac{1}{4}\left[\frac{(2-1)}{2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{1}{4}\left(\pi/2 - 0\right) - \frac{1}{4}\left(\pi/4\right)$$

$$= \pi/8 - \frac{\pi}{16} = \frac{2\pi - \pi}{16} = \frac{\pi}{16}$$

(5) Evaluate $$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2) \, dy . dx.$$

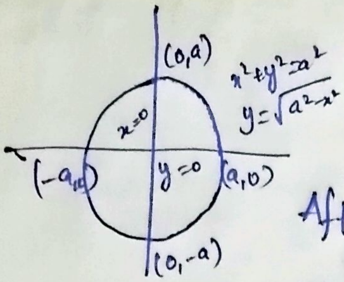
sol: given expression $= \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2) \, dy \, dx$

Limits are

$y = 0$ (x-axis) $\quad y = a \rightarrow$ (1)

$x = 0$ (y-axis) $\quad x = \sqrt{a^2-y^2}$

$x^2+y^2 = a^2$ (circle radius a) $\rightarrow$ (2)



The region of integration is bounded by $y=0$, $x=0$ and curve $x^2+y^2=a^2$

After changing the order the limits are

x varies from $x=0$ to $x=a$

y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy = \int_{x=0}^{a}\left(\int_{y=0}^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\right)dx$$

if $x=a \Rightarrow a = a\cos\theta$

$\boxed{\theta=0}$

$$= \int_{x=0}^{a}\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left((x^2\sqrt{a^2-x^2}) + \frac{(\sqrt{a^2-x^2})^3}{3} - 0\right)dx$$

$$= \int_0^a \left(x^2(a^2-x^2)^{1/2} + \frac{(a^2-x^2)^{3/2}}{3}\right)dx$$

$$= \int_0^a \left(x^2(a^2-x^2)^{1/2} + \frac{(a^2-x^2)(a^2-x^2)^{1/2}}{3}\right)dx$$

$$= \int_0^a (a^2-x^2)^{1/2}\left(x^2+\frac{a^2-x^2}{3}\right)dx$$

$$= \int_0^a (a^2-x^2)^{1/2}\left(\frac{3x^2+a^2-x^2}{3}\right)dx$$

$$= \int_0^a (a^2-x^2)^{1/2}\left(\frac{2x^2+a^2}{3}\right)dx$$

$$= \frac{1}{3}\int_0^a (a^2-x^2)^{1/2}(2x^2+a^2)\,dx$$

$$= \frac{1}{3}\int_0^a 2x^2(a^2-x^2)^{1/2}dx + \frac{1}{3}\int_0^a a^2(a^2-x^2)\,dx$$

$x = a\cos\theta$

$dx = -a\sin\theta$

if $x=0 \Rightarrow 0 = a\cos\theta$

$\cos\theta = \cos\pi/2$

$\boxed{\theta=\pi/2}$

$$= \frac{2}{3}\int_{\theta=\pi/2}^{0} a^2\cos^2\theta\,(a^2-a^2\cos^2\theta)^{1/2}(-a\sin\theta)\,d\theta + \frac{a^2}{3}\int_0^a\sqrt{a^2-x^2}\,dx$$

$$= \frac{2}{3}\int_{\theta=0}^{\pi/2} a^2\cos^2\theta\,(a^2\sin^2\theta)^{1/2}(a\sin\theta)\,d\theta + \frac{a^2}{3}\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{a}$$

$$= \frac{2}{3}\int_{\theta=0}^{\pi/2} a^2\cos^2\theta\, a^2\sin^2\theta\, d\theta + \frac{a^2}{3}\left(0+\frac{a^2}{3}(\sin^{-1}(a/a)) - 0 - \frac{a^2}{2}(\sin^{-1}(0))\right)$$

$$= \frac{2a^4}{3}\int_{\theta=0}^{\pi/2}\left(\frac{1+\cos2\theta}{2}\right)\left(\frac{1-\cos2\theta}{2}\right)d\theta + \frac{a^2}{3}\left(\frac{a^2}{2}\sin^{-1}(\sin\pi/2) - \frac{a^2}{2}(0)\right)$$

$$= \frac{\cancel{2}a^4}{\cancel{12}\,6}\int_{\theta=0}^{\pi/2}(1-\cos^2 2\theta)\,d\theta + \frac{a^4}{6}(\pi/2)$$

$$= \frac{a^4}{6}\int_{\theta=0}^{\pi/2} 1\,d\theta - \frac{a^4}{6}\int_0^{\pi/2}\cos^2\theta\,d\theta + \frac{\pi a^4}{12}$$

$$= \frac{a^4}{6}(\theta]_0^{\pi/2} - \frac{a^4}{6}\left(\left(\frac{2-1}{2}\right)\cdot\frac{\pi}{2}\right) + \frac{\pi a^4}{12}$$

$$\frac{a^4}{6}\left(\frac{\pi}{2}\right) - \frac{a^4}{6}\left(\frac{\pi}{4}\right) + \frac{\pi a^4}{12}$$

$$= \frac{\pi a^4}{12} - \frac{\pi a^4}{24} + \frac{\pi a^4}{12}$$

$$= \frac{2\pi a^4 - \pi a^4 + 2\pi a^4}{24} = \frac{3\pi a^2}{24}$$

$$= \frac{\pi a^4}{6}$$

Transform of co-ordinate:

If a region 'G' in the uv plane is transformed into the region R in the xy plane by differentiable eq.s of the form $x = f(u,v)$ and $y = g(u,v)$ then, a function $F(x,y)$ defined on R can be thought of ~~part~~ of

$F(f(u,v), g(u,v))$

polar co-ordinates:

In the case of polar co-ordinates, we have $u = r, v = \theta$, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2 + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

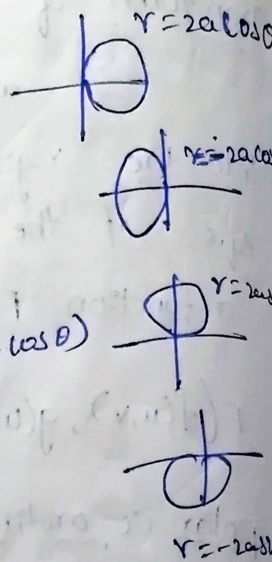
$$\therefore \iint_R f(x,y)\,dx\,dy = \iint_G F(r\cos\theta, r\sin\theta) r.\,dr.\,d\theta$$

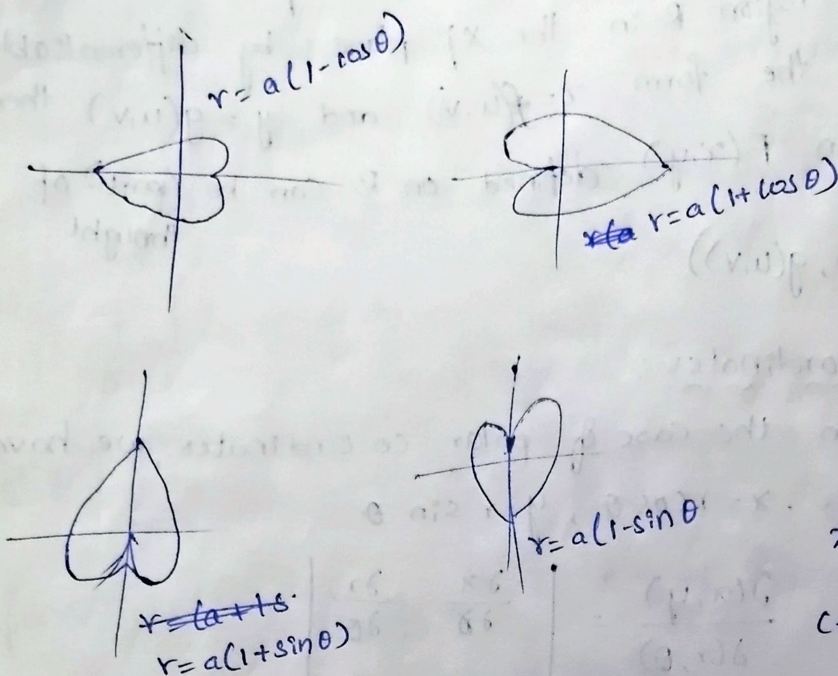
this corresponds to

$$\iint F(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)\, r\,dr\,d\theta$$

Cardioid: Vertical for $\sin\theta$
Horizontal for $\cos\theta$

Circle





Circle eq:-

$x^2+y^2\pm 2gx+2fy+c$

centre $=(-g,-f)$

Double integrals in polar coordinates.

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the lines $\theta=\theta_1, \theta=\theta_2$ and the curves $r=r_1, r=r_2$, we first integrate w.r.to 'r' b/w the limits r_1 & r_2 keeping θ-fixed the resulting expression integrated w.r.to θ from θ_1 to θ_2. In this integral r_1, r_2 are functions of θ & θ_1, θ_2 are constant.

$\Rightarrow$ Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r=2\sin\theta$, $r=4\sin\theta$

sol: Given circles,

$r=2\sin\theta$ $\qquad$ $x=r\cos\theta$

Multiplying w.r.to r $\qquad$ $y=r\sin\theta$

$r^2=2r\sin\theta$ $\qquad$ $x^2+y^2=r^2$

$x^2+y^2=2y$

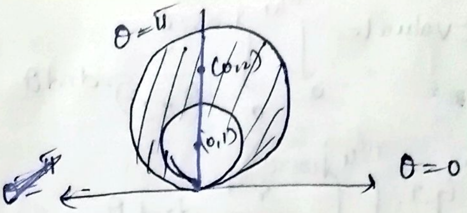
$x^2+y^2-2y=0$ represents a circle having centre (0,1)

$r = 4\sin\theta$

$r^2 = 4r\sin\theta$

$x^2+y^2 = 4y$

$x^2+y^2-4y=0$ is a circle having center $(0,2)$



For the region of integration, limits are r varies from $r = 2\sin\theta$ to $r = 4\sin\theta$, θ varies from $\theta=0$ to $\theta=\pi$

$$\iint r^3\,dr\,d\theta = \int_{\theta=0}^{\pi}\left(\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\right)d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi}\left[r^4\right]_{r=2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi}\left(4^4\sin^4\theta - 2^4\sin^4\theta\right)d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi}\left(256\sin^4\theta - 16\sin^4\theta\right)d\theta$$

$$= \frac{240}{4}\int_{\theta=0}^{\pi}\sin^4\theta\,d\theta$$

$$= \frac{240}{\cancel{4}\,2}\times\cancel{2}\int_{\theta=0}^{\pi}\sin^4\theta\,d\theta$$

$$= 120\left[\frac{3.1}{4.2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{360}{6}\pi$$

$$= \frac{90\pi}{4}$$

$$= \frac{45\pi}{2}$$

② Evaluate $\int_{\theta=0}^{\pi}\int_{0}^{a\sin\theta} r\,dr\,d\theta$

Sol: Given expression $\int_{0}^{\pi}\int_{0}^{a\sin\theta} r\,dr\,d\theta$

$$= \frac{1}{2}\int_{0}^{\pi}\left[r^2\right]_{r=0}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi}\left(a^2\sin^2\theta\right)d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi}\sin^2\theta\,d\theta$$

$$= \frac{a^2}{\cancel{2}}\times\cancel{2}\int_{\theta=0}^{\pi/2}\sin^2\theta\,d\theta$$

$$= a^2\left[\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{a^2\pi}{4}\text{ sq. units}$$

③ Evaluate $\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta$

Sol: G.T. $\int_{\theta=0}^{\pi/4}\int_{r=0}^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta$

divide & multiply with '2'

$$= \frac{1}{2}\int_{\theta=0}^{\pi/4}\int_{r=0}^{\sqrt{\cos 2\theta}} \frac{2r}{(1+r^2)^2}\,dr\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/4}\left(\int_{r=0}^{\sqrt{\cos 2\theta}} \frac{d}{dr}\left(\frac{-1}{1+r^2}\right)dr\right)d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/4}\left[\frac{-1}{1+r^2}\right]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/4}\left(\frac{-1}{1+\cos 2\theta}+\frac{1}{1+0}\right)d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/4}\left(1-\frac{1}{1+\cos 2\theta}\right)d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/4}\left(1-\frac{1}{2\cos^2\theta}\right)d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/4}\left(1-\frac{\sec^2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} d\theta - \frac{1}{4}\int_{\theta=0}^{\pi/4}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}[\theta]_{\theta=0}^{\pi/4} - \frac{1}{4}[\tan\theta]_{\theta=0}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{4}\left[\tan\frac{\pi}{4} - \tan 0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4}\right] - \frac{1}{4}(1)$$

$$= \left(\frac{\pi}{8} - \frac{1}{4}\right) \text{ sq. units.}$$

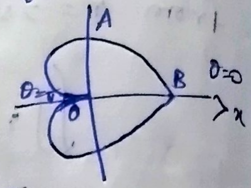
④ find the area of cardioid ~~r = a(1+~~ $r = a(1+\cos\theta)$

Sol: Given eq. of cardioid

$r = a(1+\cos\theta) \rightarrow ①$

The curve is symmetrical about x-axis the eq. of the loop

2 OABO



~~Over~~ θ varies from

$\theta = 0$ to $\theta = \pi$

$$= 2\int_{\theta=0}^{\pi} \frac{r^2}{2}\,d\theta$$

put $r = a(1+\cos\theta)$

$1+\cos 2\theta = 2\cos^2\theta$

$1+\cos\theta = 2\cos^2\theta/2$

$$= \int_{\theta=0}^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(2\cos^2\theta/2)^2\,\frac{d\theta}{2}$$

$$= 4a^2\int_0^{\pi}\cos^4\theta/2\; d\theta/2$$

$$= 4a^2 \times 2\int_0^{\pi/2}\cos^4\theta/2\; d\theta/2$$

$$= 8a^2\left[\frac{(4-1)(4-3)}{4(4-2)}\cdot\frac{\pi}{2}\right]$$

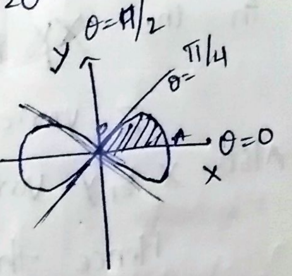
$$= \cancel{8}a^2\left(\frac{3}{\cancel{8}}\cdot\frac{\pi}{2}\right)$$

$$= \frac{3\pi a^2}{2} \text{ sq. units.}$$

1) Find the whole area of the lemniscale $r^2 = a^2\cos 2\theta$.

Sol: Given eq. of lemniscale $r^2 = a^2 \cos 2\theta$

The curve is symmetrical about X-axis



The area of the loop = 4 OAO

θ Varies from $\theta=0$ to $\theta = \pi/4$

$$= 4\int_{\theta=0}^{\pi/4} \frac{r^2}{2}\, d\theta$$

put $r^2 = a^2\cos 2\theta$

$$= \frac{4}{2}\int_{\theta=0}^{\pi/4} a^2 \cos 2\theta \, d\theta$$

$$= 2a^2 \int_{\theta=0}^{\pi/4} \cos 2\theta\, d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \frac{2a^2}{2}\left(\sin \cancel{2}\frac{\pi}{\cancel{4}_2} - \sin 0\right)$$

$$= a^2\left[\sin \pi/2 - \sin 0\right]$$

$= a^2$ sq. units.

* Double integral volume enclosed by a cylindrical surface:

If $z = f(x,y)$, then the total volume of the cylinder with 'S' as base is $V = \iint_S z\,dx\,dy = \iint_S f(x,y)\,dx\,dy$

in Cartesian Cordinates and

$$V = \iint_S F(r,\theta)\, r\, dr\, d\theta$$ in polar coordi-nates

Problem :- 1

1. Find the volume of the cylinder $x^2+y^2=1$ bounded above by the plane $x+y+z=4$ and below by the XY plane

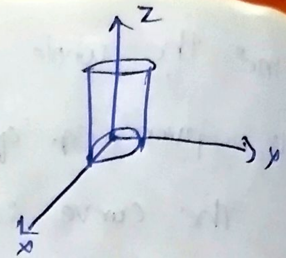
Sol: The given eq. of the plane is $x+y+z=4$

$$z = 4-x-y \longrightarrow (1)$$

eq. of the base of the cylinder

$$x^2+y^2=1 \;,\; z=0 \;\text{———}\; (2).$$

For any fixed values of x & y in the xy plane



$\therefore z$ varies from $z=0$ to $z=4-x-y$

Also x & y vary within $x^2+y^2 \le 1$.

Hence the required volume is

$$V = \iint_A (4-x-y)\,dx\,dy \longrightarrow (3)$$

put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

r varies from $r=0$ to $r=1$

θ varies from $\theta=0$ to $\theta=2\pi$

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (4 - r\cos\theta - r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \Big(4 - r(\cos\theta+\sin\theta)\Big)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \Big(4r - r^2(\cos\theta+\sin\theta)\Big)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{\cancel{4}^{2} r^2}{\cancel{2}} - \frac{r^3}{3}(\cos\theta+\sin\theta)\right]_0^1 d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(2 - \frac{1}{3}[\cos\theta+\sin\theta] - 0\right] d\theta$$

$$= \int_0^{2\pi} \left(2 - \frac{1}{3}[\cos\theta+\sin\theta]\right] d\theta$$

$$= \int_{\theta=0}^{2\pi} 2\,d\theta - \frac{1}{3}\int_{\theta=0}^{2\pi} \cos\theta - \frac{1}{3}\int_{\theta=0}^{2\pi} \sin\theta \cdot d\theta$$

$$= [2\theta]_0^{2\pi} - \frac{1}{3}[\sin\theta]_0^{2\pi} - \frac{1}{3}(-\cos\theta]_0^{2\pi}$$

$$= [2(2\pi)-0] - \frac{1}{3}(0) + \frac{1}{3}(\cos 2\pi - \cos 0]$$

$$= 4\pi - 0 + \frac{1}{3}(0)$$

$$= 4\pi \text{ Cubic units}$$

$\therefore x+y+z=3$

Triple Integrals:

If $f(x,y,z) = 1$, We get on evaluation of the triple integral, The volume of the solid enclosed between the limits of integration

Triple integrals can be evaluated by repeated integrations w.r.to the Variable x, y, z ... in turn within the limits of variations, as shown below

$$\iiint_R f(x,y,z)\, dR = \int_{x=a}^{b} \left\{ \int_{y=f_1(x)}^{f_2(x)} \left[\int_{z=g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right] dy \right\} dx$$

Problem

1. Evaluate $\int_0^1 dx \int_1^2 dy \int_1^3 xyz\, dz$

Sol: Given that $\int_{x=0}^{1} dx \int_{y=1}^{2} dy \int_{z=1}^{3} xyz\, dz$

consider

$$\int_{z=1}^{3} xyz\, dz = xy \int_1^3 z\, dz$$

$$= xy\left[\frac{z^2}{2}\right]_1^3$$

$$= xy\left(\frac{9}{2} - \frac{1}{2}\right)$$

$$= xy[4]$$

$$= 4xy$$

Now, $\int_{y=1}^{2} dy \int_{z=1}^{3} xyz\, dz = \int_{y=1}^{2} 4xy\, dy$

$$= 4x \int_1^2 y\, dy$$

$$= 4x\left(\frac{y^2}{2}\right]_1^2$$

$$= 4x\left(\frac{4}{2} - \frac{1}{2}\right)$$

$$= \cancel{4}^{2} x\left[\frac{3}{\cancel{2}}\right]$$

$$= 6x.$$

finally $\int_{x=0}^{1} dx \int_{y=1}^{2} dy \int_{z=1}^{3} xyz\, dz = \int_{0}^{1} 6x\, dx$

$$= 6\left[\frac{x^2}{2}\right]_0^1$$

$$= {}^{3}\cancel{6}\left[\frac{1}{\cancel{2}}\right]$$

$$= 3$$

(2). Evaluate $\int_{0}^{3} dx \int_{0}^{2} dy \int_{0}^{1} e^{x+y+z} dz$

Sol: consider $\int_{z=0}^{1} e^{x+y+z} dz = e^{x+y} \int_{z=0}^{1} e^{z} dz$

$$= e^{x+y}\left[e^{z}\right]_0^1$$

$$= e^{x+y}\left[e^{1} - e^{0}\right]$$

$$= e^{x+y}(e-1)$$

Now, $\int_{y=0}^{2} dy \int_{z=0}^{1} e^{x+y+z} dz = \int_{y=0}^{2} e^{x+y}(e-1)\, dy$

$$= (e-1)\, e^{x} \int_{y=0}^{2} e^{y} dy$$

$$= (e-1)\, e^{x} \left(e^{y}\right)_0^2$$

$$= (e-1)\, e^{x}\left(e^{2} - e^{0}\right)$$

$$= (e-1)(e^{2}-1)\, e^{x}$$

finally,

$$\int_{0}^{3} dx \int_{0}^{2} dy \int_{0}^{1} e^{x+y+z} dz = \int_{x=0}^{3} (e-1)(e^{2}-1)\, e^{x} dx$$

$$= (e-1)(e^{2}-1)\left(e^{x}\right)_{x=0}^{3}$$

$$= (e-1)(e^{2}-1)(e^{3}-1)$$

1) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

sol: Given expression $\int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y+z}\, dz\, dy\, dx$

consider $\int_{z=0}^{x+y} e^{x+y+z}\, dz = e^{x+y} \int_{z=0}^{x+y} e^{z}\, dz$

$$= e^{x+y}\left[e^{z}\right]_0^{x+y}$$

$$= e^{x+y}\left[e^{x+y} - e^{0}\right]$$

$$= e^{2x+2y} - e^{x+y}$$

Now, $\int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y+z}\, dz\, dy = \int_{y=0}^{x} \left(e^{2x+2y} - e^{x+y}\right) dy$

$$= e^{2x}\int_{y=0}^{x} e^{2y}\, dy - e^{x}\int_{y=0}^{x} e^{y}\, dy$$

$$= e^{2x}\left(\frac{e^{2y}}{2}\right)_{y=0}^{x} - e^{x}\left(e^{y}\right)_0^{x}$$

$$= \frac{e^{2x}}{2}\left(e^{2x} - e^{0}\right) - e^{x}\left[e^{x} - e^{0}\right]$$

$$= \frac{e^{4x}}{2} - \frac{e^{2y}}{2} - \frac{e^{2x}}{1} + e^{x}$$

finally $\int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y+z}\, dz\, dy\, dx = \int_{x=0}^{a} \left(\frac{e^{4x}}{2} - \frac{e^{2y}}{2} - e^{2x} + e^{x}\right) dx$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2y}}{4} - \frac{e^{2x}}{2} + e^{x}\right]_0^{a}$$

$$= \left[\frac{e^{4a}}{8} - \frac{e^{2a}}{4} - \frac{e^{2a}}{2} + e^{a} - \frac{1}{8} + \frac{1}{4} + \frac{1}{2} - 1\right]$$

$$= \left(\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} + \frac{3}{4} - \frac{9}{8}\right)$$

$$= \left[\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8}\right]$$

(4) evaluate $\int_{y=0}^{1}\int_{x=y}^{1}\int_{z=0}^{1-x} x \, dz \, dx \, dy$ Ans $= \frac{1}{12}$

(5) evaluate $\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} dx \, dy \, dz$ Ans $= 1$

(6) evaluate $\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} e^{x+y+z} \, dx \, dy \, dz$ Ans : $(e-1)^3$

$\frac{d}{dx} \log x = \frac{1}{x}$ $\boxed{\int uv = u\int v - \int u' v}$ $\int \frac{x}{} \log x \, dx = \log x . \frac{x^2}{2} - \int \frac{1}{x} \frac{x^2}{2} dx$

$\int \log x \, dx = x \log x - x$ $= \frac{x^2}{2} \log x - \frac{x^2}{4}$

$\int 1 . \log x \, dx$

$= \log x . \int dx - \int \frac{1}{x} . x \, dx$ $\int \log t = t \log t - t$

$= \log x . x - x$ $\int t \log t = \frac{t^2}{2} \log t - \frac{t^2}{4}$

$= x \log x - x$ //

(7) Evaluate $\int_{1}^{e}\int_{1}^{\log y}\int_{1}^{e^x} \log z \, dz \, dx \, dy$ (imp)

Sol: Given expression $\int_{y=1}^{e}\int_{x=1}^{\log y}\int_{z=1}^{e^x} \log z \, dz \, dx \, dy$

consider $\int_{z=1}^{e^x} \log z \, dz = \left[z \log z - z \right]_1^{e^x}$

$= \left[e^x \log e^x - e^x - \log 1 + 1 \right]$

$= \left[x e^x - e^x + 1 \right]$ //

Now, $\int_{x=1}^{\log y}\int_{z=1}^{e^x} \log z \, dz \, dx$ $\int_{x=1}^{\log y} (x e^x - e^x + 1) \, dx$

$= \int_{x=1}^{\log y} x e^x \, dx - \int_{x=1}^{\log y} e^x dx + \int_{x=1}^{\log y} dx$

$$= \left(xe^x - 1.e^x\right)_{x=1}^{\log y} - \left[e^x\right]_{x=1}^{\log y} + \left[x\right]_{x=1}^{\log y}$$

$$= \log y . e^{\log y} - e^{\log y} - 1.e^1 + e^1 - \left[e^{\log y} - e^1\right] + \left[\log y - 1\right]$$

$$= y \log y - y - y + e + \log y - 1$$

$$= y \log y - 2y + \log y + (e-1)$$

Ans:- $\frac{1}{4}e^2 - 2e + \frac{13}{4}$

$$= \cancel{\int_{y=1}^{e} (y \log y - 2y + \log y + (e-1))\,dy}$$

$$= \int_{y=1}^{e} y \log y \, dy - 2\int_{y=1}^{e} y\,dy + \cancel{\int_{y=1}^{e} \log y\,dy + (e-1)} \int_{y=1}^{e} dy$$

finally. $\displaystyle\int_{y=1}^{e} \int_{x=1}^{\log y} \int_{z=1}^{e^x} \log z \; dz \; dx \; dy$

$$= \int_{y=1}^{e} \left(y \log y - 2y + \log y + (e-1)\right) dy$$

$$= \int_{y=1}^{e} y \log y - 2\int_{y=1}^{e} y + \int_{y=1}^{e} \log y + (e-1) \int_{y=1}^{e} (1)\,dy$$

(8) Evaluate the triple integral $\iiint xy^2z\,dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Sol: The first octant is the octant in which all three of the coordinates are positive.

The eq's of the sphere in the first octant are

$$x^2+y^2+z^2=0, \quad x \geq 0,\ y \geq 0,\ z \geq 0$$

The limits of integration are

$z^2=a^2-x^2-y^2$

$z=\sqrt{a^2-x^2-y^2}$

z varies from $z=0$ to $z=\sqrt{a^2-x^2-y^2}$

y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

x varies from $x=0$ to $x=\sqrt{a^2}=a$

Given that

$$\iiint xy^2z\,dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy^2z\,dz\,dy\,dx$$

Consider

$$\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy^2z\,dz = xy^2\left[\frac{z^2}{2}\right]_{z=0}^{\sqrt{a^2-x^2-y^2}}$$

$$= \frac{xy^2}{2}\left[a^2-x^2-y^2-0\right]$$

$$= \frac{xy^2a^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{2}$$

$$= \left(\frac{xa^2-x^3}{2}\right)y^2 - \frac{x}{2}y^4$$

Now,

$$\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy^2z\,dz\,dy = \int_{y=0}^{\sqrt{a^2-x^2}}\left(\frac{xa^2-x^3}{2}\right)y^2\,dy - \int_{y=0}^{\sqrt{a^2-x^2}}\frac{x}{2}y^4\,dy$$

$$= \left[\frac{xa^2 - x^3}{2}\right]\left[\frac{y^3}{3}\right]_{y=0}^{\sqrt{a^2-x^2}} - \frac{x}{2}\left[\frac{y^5}{5}\right]_{y=0}^{\sqrt{a^2-x^2}}$$

$$= \frac{x(a^2-x^2)}{6}\left[\left(\sqrt{a^2-x^2}\right)^3 - 0\right] - \frac{x}{10}\left[\left(\sqrt{a^2-x^2}\right)^5 - 0\right]$$

$$= \frac{x(a^2-x^2)(a^2-x^2)^{3/2}}{6} - \frac{x(a^2-x^2)^{5/2}}{10}$$

$$= \frac{x(a^2-x^2)^{5/2}}{6} - \frac{x(a^2-x^2)^{5/2}}{10}$$

$$= \left(\frac{1}{6} - \frac{1}{10}\right) x(a^2-x^2)^{5/2}$$

$$= \frac{1}{15} x(a^2-x^2)^{5/2}$$

finally, $\displaystyle\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy^2z \, dz\, dy\, dx$

$$= \int_{x=0}^{a} \frac{1}{15}\left(x(a^2-x^2)^{5/2}\right) dx$$

$u = a^2 - x^2$

$du = -2x\,dx$

$x\,dx = \frac{du}{-2}$

limits

$x=0 \Rightarrow u = a^2$

$x = a \Rightarrow u = 0$

$$= \frac{1}{15}\int_{a^2}^{0} u^{5/2} \frac{-du}{2}$$

$$= \frac{1}{30}\int_{u=0}^{a^2} u^{5/2} du$$

$$= \frac{1}{30}\left[\frac{u^{5/2+1}}{5/2+1}\right]_{u=0}^{a^2}$$

$$= \frac{1}{30} \times \frac{2}{7}\left[u^{7/2}\right]_0^{a^2} = \frac{1}{105}\left[(a^2)^{7/2} - 0\right]$$

$$= \frac{a^7}{105} \text{ c. units.}$$

Transformation of coordinates - Triple integrals :

Depending upon geometry of the problem under consideration, we may have to use cylindrical (or) spherical polar coordinates in place of rectangular Cartisen coordinates. We now, consider transformation of Cartisen Coordinates to cylindrical ~~to~~ and spherical polar coordinates in 3-D

* Cylindrical coordinates :

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

The jacobian of transformation is :

$$J\left(\frac{x,y,z}{\rho,\phi,z}\right) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \rho\cos^2\phi + \rho\sin^2\phi - 0$$

$$= \rho(\cos^2\phi + \sin^2\phi) = \rho(1)$$

$$= \rho$$

$$\iiint_R f(x,y,z)\,dR = \int_{\rho=a}^{\rho=b}\left\{\int_{\phi=f_1(\rho)}^{\phi=f_2(\rho)}\left(\int_{z=g(\phi,\rho)}^{g_2(\phi,\rho)} f(\rho,\phi,z)\,dz\right]d\phi\right].\rho\,d\rho$$

* Spherical polar co-ordinates

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$J\left(\frac{x,y,z}{r,\theta,\phi}\right) = r^2 \sin\theta$$

→ Evaluate $\displaystyle\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$

Sol: Given expression $\displaystyle\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x^2+y^2)}$

The region of integration is bounded by the curves

$x = 0 \qquad x = \infty$

$y = 0 \qquad y = \infty$

Now, transforming the Cartisen coordinates into polar by $x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

$x^2 + y^2 = r^2$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

r varies from $r=0$ to $r=\infty$

θ varies from $\theta=0$ to $\theta=\pi/2$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

put $r^2 = u$

$2r\,dr = du$

$r\,dr = \frac{du}{2}$

$$= \int_{\theta=0}^{\pi/2}\int_{u=0}^{\infty} e^{-u}\,\frac{du}{2}\,d\theta$$

limits

if $r=0 \Rightarrow u=0$

if $r=\infty \Rightarrow u=\infty$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{e^{-u}}{-1}\right]_{u=0}^{\infty} d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{\pi/2}\left(e^{-\infty} - e^{0}\right) d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

② Evaluate

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$$

Sol:

$$= \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$$

The given region of integration is borded by the area

$y=0 \qquad y=a \longrightarrow ①$

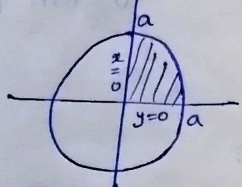
$x=0 \qquad x=\sqrt{a^2-y^2}$

$x^2=a^2-y^2$

$x^2+y^2=a^2 \rightarrow ②$

let $x=r\cos\theta$, $y=r\sin\theta$,

$dx\,dy=r\,dr\,d\theta$, $x^2+y^2=r^2$



from the graph

r varies from $r=0$ to $r=a$

θ varies from $\theta=0$ to $\theta=\pi/2$

$$\int_{y=0}^{0} \int_{x=0}^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy =$$

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_{r=0}^{a} r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^{a} \left(\frac{r^4}{4}\right) d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \left[r^4\right]_0^a d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} a^4\,d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi/2} d\theta$$

$$= \frac{a^4}{4} \left[\theta\right]_0^{\pi/2}$$

$$= \frac{a^4\pi}{8}$$

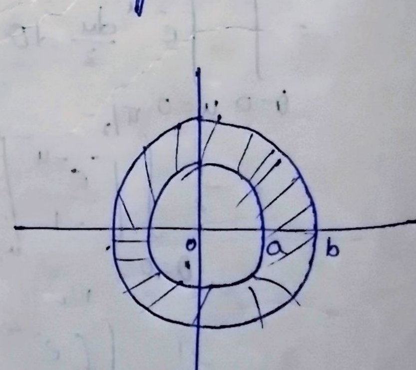
$$= \frac{\pi a^4}{8}.$$

③. Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region enclosed between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$ by changing to polar co-ordinates.

Sol: Given circles are

$x^2+y^2=a^2 \longrightarrow ①$ (small)

$x^2+y^2=b^2 \longrightarrow ②$ (Large)



put

$x=r\cos\theta \quad y=r\sin\theta$

$y^2+x^2=r^2$,

$x^2y^2=r^2\cos^2\theta\, r^2\sin^2\theta$

$=r^4\cos^2\theta\cdot\sin^2\theta$

$dx\,dy=r\,dr\,d\theta$

from the graph

r varies from $r=a$ to $r=b$

θ varies from $\theta=0$ to $\theta=2\pi$

$$\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^4\cos^2\theta\sin^2\theta}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(\int_{r=a}^{b} r^3\,dr\right)\cos^2\theta\sin^2\theta\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_{r=a}^{b}\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=\frac{1}{16}\int_{\theta=0}^{2\pi}(b^4-a^4)(1-\cos^2 2\theta)\,d\theta$$

$$=\frac{b^4-a^4}{16}\int_{\theta=0}^{2\pi}1-\cos^2 2\theta\,d\theta$$

$$=\frac{b^4-a^4}{16}\int_{\theta=0}^{2\pi}1\,d\theta-\frac{b^4-a^4}{16}\int_{\theta=0}^{2\pi}\cos^2 2\theta\,d\theta$$

$$=\frac{b^2-a^4}{16}(\theta)_0^{2\pi}-\frac{b^4-a^4}{\cancel{16}\;8}\cancel{2}\int_{\theta=0}^{\pi}\cos^2 2\theta\,d\theta$$

$$=\frac{b^4-a^4}{16}(2\pi)-\frac{b^4-a^4}{8}\,2\int_{\theta=0}^{\pi/2}\cos^2 2\theta\,d\theta$$

$$=\frac{\pi(b^4-a^4)}{8}-\frac{b^4-a^4}{4}\left[\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$=\frac{\pi}{8}(b^4-a^4)-\frac{(b^4-a^4)}{16}\pi$$

$$=\left(\frac{\pi\, b^4-a^4}{16}\right)$$

$$=\left(\frac{b^4-a^4}{16}\right)\pi \text{ sq.units.}$$

(4) Evaluate $\displaystyle\int_{y=0}^{1}\int_{x=y}^{a}\frac{x\,dx\,dy}{x^2+y^2}$

Sol: Given expression

$$\int_{y=0}^{1}\int_{x=y}^{a}\frac{x\,dx\,dy}{x^2+y^2}$$

$y=0$ $\quad$ $x=y$

$r\sin\theta=0$ $\quad$ $r\sin\theta=r\cos\theta$

$\sin\theta=\sin 0$ $\quad$ $\tan\theta=1$

$\theta=0$ $\quad$ $\theta=\pi/4$

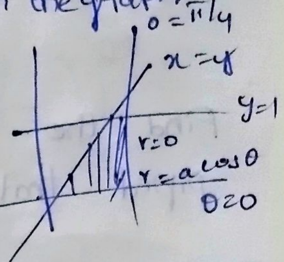
$x=a$

$r\cos\theta=a$

$r=a/\cos\theta$

$r=a\sec\theta$

from the graph



θ varies from $\theta=0$ to $\theta=\pi/4$

r varies from $r=0$ to $r=a\sec\theta$

$$\int_{y=0}^{1}\int_{x=y}^{a}\frac{x\,dx\,dy}{x^2+y^2}=\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sec\theta}\frac{r\cos\theta\,\cancel{r}\,dr\,d\theta}{\cancel{r^2}}$$

$$=\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sec\theta}\cancel{\frac{r}{r}}\cos\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/4} [r]_{r=0}^{a\sec\theta} \, d\theta$$

$$= \int_{\theta=0}^{\pi/4} a \sec\theta \, d\theta$$

$$= a \int_{\theta=0}^{\pi/4} \sec\theta \, d\theta = a \left[\tan\theta\right]_0^{\pi/4}$$

$$= a \left[\pi/4\right]$$

$$= \frac{a\pi}{4} \,//$$

Physical Applications :

① Find the volume of the Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(OR)

Find the volume of the greatest Rectangular parallelopiped that can be incrisbed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol: The Coordinate planes cut the solid given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \longrightarrow (1).$$

into 8 octants. The volume of the solid is equal to 8 times the volume of portion of the solid in the 1^{st} octant bounded by $x=0, y=0, z=0$

for fixed x, y

z varies from $z=0$ to $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

for fixed x

y varies from $y=0$ to $y = b\sqrt{1 - \frac{x^2}{a^2}}$

x varies from $x=0$ to $x=a$

let $z_1 = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ $\quad y_1 = b\sqrt{1 - \frac{x^2}{a^2}}$

$y_1^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

$$y_1^2 = b^2 - \frac{b^2 x^2}{a^2}$$

$$V = 8 \int_{x=0}^{a} \int_{y=0}^{y_1} \int_{z=0}^{z_1} dz\, dy\, dx$$

consider

$$\int_{z=0}^{z_1} dz = [z]_0^{z_1}$$

$$= z_1 = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$= \frac{c}{b}\sqrt{b^2 - \frac{b^2x^2}{a^2} - y^2}$$

$$= \frac{c}{b}\sqrt{y_1^2 - y^2}$$

(from $y_1^2 = b^2 - \frac{b^2x^2}{a^2}$)

Now

$$\int_{y=0}^{y_1}\int_{z=0}^{z_1} dz\, dy = \int_{y=0}^{y_1} \frac{c}{b}\sqrt{y_1^2 - y^2}$$

$$= \frac{c}{b}\left[\frac{y}{2}\sqrt{y_1^2 - y^2} + \frac{y_1^2}{2}\sin^{-1}\left(\frac{y}{y_1}\right)\right]_{y=0}^{y_1}$$

$$\int \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{c}{b}\left[0 + \frac{y_1^2}{2}\sin^{-1}\left(\frac{\cancel{y_1}}{\cancel{y_1}}\right) - 0 - 0\right]$$

$$= \frac{c}{b}\left[\frac{y_1^2}{2}\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi c}{4b} y_1^2$$

$$= \frac{\pi c}{4b}\left(b^2 - \frac{b^2x^2}{a^2}\right)$$

$$= \frac{\pi c b}{4}\left(1 - x^2/a^2\right)$$

$$V = 8\int_{x=0}^{a} \frac{\pi c b}{4}\left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{8\pi c b}{4}\left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= 2\pi c b\left(a - \frac{a^{\cancel{3}}}{3\cancel{a^2}} - 0\right) = 2\pi abc - \frac{2\pi abc}{3}$$

$$= \frac{6\pi abc - 2\pi abc}{3}$$

$$= \frac{4\pi}{3} abc \text{ cub.units}$$

② Find the Volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$

Sol: Given eq. of sphere $x^2+y^2+z^2=a^2 \rightarrow ①$

eq. of cylinder $x^2+y^2=ay \rightarrow ②$

Transforming into cylindrical coordinates,

$$x=\rho\cos\phi, \quad y=\rho\sin\phi \quad z=z$$

$$J\left(\frac{x,y,z}{\rho,\phi,z}\right)=\rho$$

$$① \Rightarrow \rho^2\cos^2\phi+\rho^2\sin^2\phi+z^2=a^2$$

$$\rho^2(\cos^2\phi+\sin^2\phi)+z^2=a^2$$

$$\rho^2+z^2=a^2$$

$$z=\sqrt{a^2-\rho^2}$$

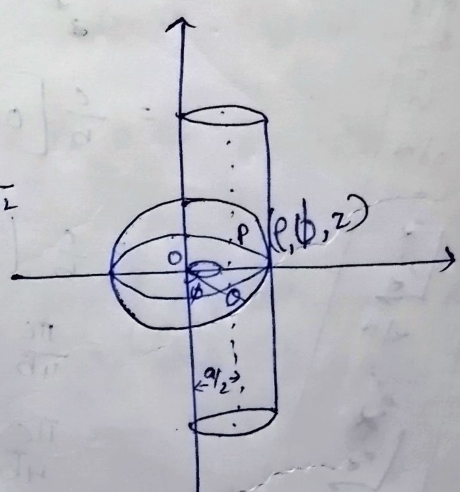
$$② \Rightarrow x^2+y^2=ay$$

$$\rho^{\cancel{2}}=a\cancel{\rho}\sin\phi$$

$$\rho=a\sin\phi$$

eq. of the sphere, $\rho^2+z^2=a^2$

~~cly~~ cylinder $\rho=a\sin\phi$



limits are

z varies from $z=0$ to $z=\sqrt{a^2-\rho^2}$

ρ varies from $\rho=0$ to $\rho=a\sin\phi$

ϕ varies from $\phi=0$ to $\phi=\pi/2$

$$V=4\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a\sin\phi}\int_{z=0}^{\sqrt{a^2-\rho^2}}\rho\,dz\,d\rho\,d\phi$$

$$=4\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a\sin\phi}[z]_{z=0}^{\sqrt{a^2-\rho^2}}\rho\,d\rho\,d\phi$$

$$=4\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a\sin\phi}\sqrt{a^2-\rho^2}\,\rho\,d\rho\,d\phi$$

put $a^2 - \rho^2 = t$

$-2\rho\, d\rho = dt$

$\rho\, d\rho = -\frac{dt}{2}$

limits

if $\rho = 0 \Rightarrow t = a^2 - 0^2$

$\boxed{t = a^2}$

if $\rho = a\sin\phi \Rightarrow t = a^2 - a^2\sin^2\phi$

$\boxed{t = a^2\cos^2\phi}$

$$V = 4\int_{\phi=0}^{\pi/2}\int_{t=a^2}^{a^2\cos^2\phi} t^{1/2}\left(\frac{-dt}{2}\right)d\phi$$

$$= \frac{-4}{2}\int_{\phi=0}^{\pi/2}\left[\frac{t^{1/2+1}}{\frac{3}{2}}\right]_{t=a^2}^{a^2\cos^2\phi} d\phi$$

$$= \frac{-4}{\cancel{2}}\times\frac{\cancel{2}}{3}\int_{\phi=0}^{\pi/2}\left(t^{3/2}\right)_{t=a^2}^{a^2\cos^2\phi} d\phi$$

$$= \frac{-4}{3}\int_{\phi=0}^{\pi/2}\left[(a^2\cos^2\phi)^{3/2} - (a^2)^{3/2}\right]d\phi$$

$$= \frac{-4}{3}\int_{\phi=0}^{\pi/2}\left[(a\cos\phi)^{\cancel{2}\cdot 3/\cancel{2}} - a^3\right]d\phi$$

$$= \frac{-4}{3}\int_{\phi=0}^{\pi/2}(a^3\cos^3\phi - a^3)\,d\phi$$

$$= \frac{-4a^3}{3}\int_{\phi=0}^{\pi/2}\cos^3\phi\, d\phi + \frac{4a^3}{3}\int_{\phi=0}^{\pi/2} d\phi$$

$$= \frac{-4a^3}{3}\left[\frac{(3-1)}{3(3-2)}\,1\right] + \frac{4a^3}{3}\left[\phi\right]_0^{\pi/2}$$

$$= \frac{-8a^3}{9} + \frac{4a^3}{3}\left(\pi/2 - 0\right)$$

$$= \frac{-8a^3}{9} + \frac{\cancel{4}^{2}\pi a^3}{\cancel{6}\,3}$$

$$= a^3\left(\frac{-8}{9} + \frac{2\pi}{3}\right)$$

$$= a^3\left(\frac{-8+6\pi}{9}\right)$$

$$= \frac{a^3}{9}\left(-8+6\pi\right)$$

$$= \frac{2a^3}{9}(3\pi - 4)\ //.$$