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UNIT - I

Matrices

* Matrix is an inevitable tool in the study of many subjects like physics, Mechanics, statistics, electronic circuits & Computers

Defination of Matrix :

A system of m.n numbers (real (or) complex) arranged in the form of an ordered set of m rows, each row consisting of an ordered set of n numbers between [ ] (or) ( ) (or) | | is called a matrix of order m×n

ex :-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = (a_{ij})_{m\times n}$$

where $1 \leq i \leq m$

$1 \leq j \leq n$

Order of matrix :

Order of the matrix is Obtained by Counting no. of rows & no. of coloumns in the matrix

$\therefore$ Order of matrix = no. of rows × no. of columns

ex: $\begin{pmatrix} 3 & 5 \\ -2 & 1 \end{pmatrix}$ is 2×2 order matrix

Types of Matrix :

Square matrix: In a matrix, If no. of rows = no. of columns then, that matrix is called sq. matrix

ex: $\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}$

Rectangular Matrix :

In a matrix, if no. of rows $\neq$ no. of columns then it is called a Rectangular matrix

ex : $\begin{bmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \end{bmatrix}_{2\times 3}$

Row matrix : A matrix with only one row is known as Row matrix

$\begin{bmatrix} 1 & 4 & 3 & 2 \end{bmatrix}_{1\times 4}$

Column matrix

A matrix with only one column

$\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}_{3\times 1}$

Zero matrix (or) Null matrix :

A matrix in which, all the elements are zero is called Null matrix

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} = [0]_{3\times 3}$

principle diagonal of sq. matrix :

In a Sq. matrix starting from 1^{st} row of 1^{st} element, 2^{nd} row of 2^{nd} element upto last row of last element is called "diagonal elements". The line along the diagonal elements is called principle diagonal

ex : $\begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 3 \\ 3 & 4 & 1 \end{bmatrix}$ 1, 4, 1 are diagonal elements

Diagonal Matrix :

In a sq. matrix except principle diagonal, remaining elements are zero then that matrix is called Diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ can be represented as diag $[1\ 4\ 3]$

Scalar matrix :

A diagonal matrix in which all the diagonal elements are equal is called Scalar matrix

ex: $$\begin{bmatrix} 0 & 9 & 9 \\ 9 & 0 & 9 \\ 9 & 9 & 0 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

Identity matrix (or) Unit matrix :

In a sq. matrix, if the principle diagonal elements are equal to 'one' & Remaining elements are Zero is called Identity matrix (or) unit matrix. denoted by I.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Upper triangular matrix :

In a sq. matrix, if the elements below principle diagonal are zero is called U.T.M

ex: $$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Lower Triangular matrix :

In a sq. matrix, if the elements upper principle diagonal are zero is called L.T.M

ex: $$\cancel{\begin{bmatrix} K & G & F \\ 0 & R & R \\ 0 & 0 & M \end{bmatrix}} \begin{pmatrix} K & 0 & 0 \\ R & G & 0 \\ R & M & F \end{pmatrix}$$

Transpose of a matrix: (A^T)

Matrix Obtained by interchanging rows as columns (or) columns as rows is called Transpose of a matrix (A^T)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Symmetric matrix:

If the Transpose of a matrix is equal to the same matrix then that matrix is called symmetric matrix

ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$\therefore A = A^T$

Skew-Symmetric matrix:

If the transpose of a matrix is equal to its additive inverse ($A^T = -A$), then the matrix is called Skew-Symmetric matrix.

ex: $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$
$= A$

Orthogonal matrix:

A Sq. matrix is said to be orthogonal, if

$A \cdot A^T = A^T \cdot A = I$

Complex matrices:

Matrices with complex elements are called complex matrices.

ex:- $A = \begin{bmatrix} 7 & 8 & 9+3i \\ 6 & 5 & 2 \\ -2i & 4i & 5-4i \end{bmatrix}_{3\times 3}$

Elementary Transformation [(or) Operations] on A Matrix :

(i) Interchanging of two rows : If i^{th} row & j^{th} row are interchanged, it is denoted by "$R_i \longleftrightarrow R_j$"

(ii) Multiplication of each element of a row with a non zero Scalar. If i^{th} row is multiplied with k then it is denoted by "$R_i \longrightarrow kR_i$"

(iii) Multiplying every element of a row with a non-zero scalar & adding to the corresponding elements of another row.

If all the elements of i^{th} row are multiplied with 'k' & added to the corresponding elements of j^{th} Row then it is denoted by

$$R_j \longrightarrow R_j \pm kR_i$$

The corresponding column transformations will be denoted by writting C, instead of R

i.e, $C_i \longleftrightarrow C_j$

$C_i \longrightarrow kC_i$

$C_j \longrightarrow C_j + kC_i$ respectively

* An elementary transformation is called a row transformation (or) a Column transformation according as it applied to rows (or) columns.

Equivalance of Matrices :

If B is obtained from A after a finite chain of elemantary transformations. Then B is said to be Equivalent to A. Symbolically it is denoted as $B \sim A$

Results : 1. If A & B are two equivalent matrices, Then rank A = Rank B

2. If 2 matrices A & B have the Same Size & the Same Rank, then the two matrices A & B are equivalent

Sub Matrix :

A Matrix Obtained by deleting some rows or columns (or) both of a given matrix is called it's Sub Matrix.

Minor of a Matrix :

Let A be an $m \times n$ matrix that determinant of a Sq. Sub matrix of A is called a Minor of the Matrix.

If the oder of the Sq Sub matrix 't'. Then its determenent is called a Minor of order 't'.

ex:- Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$ be a matrix

we have $B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is Sub matrix of order '2'

det $B = 2-3 = -1$, is a minor of Order 2.

Rank of a matrix :

Let A be an $m \times n$ matrix. If A is a null matrix, we define its rank to be "0" (zero)

If A is a non zero matrix, we say that r is the rank of "A"

If

(i) Every $(r+1)^{th}$ order minor of 'A' is '0'. &

(ii) There exists at least one r^{th} order minor of 'A' Which is not zero Rank of "A" is denoted by $\rho(A)$

Note: It can be noted that the Rank of a non-zero matrix is the order of highest order non-zero minor of A

Note:

1. Every matrix will have a Rank
2. Rank of a matrix is unique
3. $\rho(A) \geq 1$, when A is a non-zero matrix
4. If A is a matrix of order $m \times n$, Rank $(A) = \rho(A) \leq \min(m,n)$
5. Rank of Identity matrix I_n is 'n'
6. If A is a matrix of order n & A is non-singular (i.e, $\det A \neq 0$). then Rank of $A = n$

Prb:- 1) Find the Rank of the matrices

(i) $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

iii Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}_{3 \times 4}$

W.K.T $\rho(A) \leq \min(m,n)$

$\leq \min(3,4)$

$\rho(A) \leq 3$

Consider a 3×3 matrix, Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

$= 1(0-49) - 2(0-56) + 3(35-48)$

$= 24 \neq 0$ $\therefore$ Rank of $A = 3$

(i) Sol: Given $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}_{3 \times 3}$ $= -1(18-1) - 0(9+5) + 6(3+30)$

$= -1(17) + 6(33)$

$= -17 + 198$

$= 181 \neq 0$ $\therefore$ Rank of $A = 3$

(ii) Sol: Given $A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix}_{3 \times 3}$ $= 1(24-25) - 2(18-20) + 3(15-16)$

$= 1(-1) - 2(-2) + 3(-1)$

$= -1 + 4 - 3 = -4 + 4 = 0.$

Consider 2X2 matrix

let $B = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$

$\therefore \rho(A) = 2.$

Zero row & non zero row :

If all the elements in a row of a matrix are zero row & if there is at least one non-zero element in a row, then it is called non-zero row.

* Echelon form of a matrix :

A matrix is said to be in Echelon form if it has the following properties

(i). Zero rows, if any, are below any non-zero row

(ii) The 1st non-zero entry in each non-zero row is equal to 1

(iii) The no. of zeros before the 1st non-zero element in a row is less than the no. of such zero in the next row.

Note : condition (ii) is Optional.

Result : The no. of non-zero rows in a row Echlon form of 'A' is the rank of A.

Note : For a Echlon form we use only rows transformations.

ex:

(1) $\begin{pmatrix} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is row echlon form

(ii) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echlon form

* **Working form of Echlon form :**

Consider a given general matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Step 1 : Get zeros below a_{11} by using row Transfor-mation & by using a_{11}

Step : 2 Get zero below a_{22} by using row transfor-mation by using a_{22}. repeat the process Untill we get Echlon form.

Problems :

① Reduce the matrix to echelon ~~form~~ form & find its rank

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

sol: Given
$$A = \begin{array}{c} \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$R_2 \longrightarrow R_2 + R_1$

$R_3 \longrightarrow R_3 + 2R_1$

$R_4 \longrightarrow R_4 - R_1$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$R_3 \rightarrow 2R_3 - 11R_2$

$R_4 \rightarrow R_4 + 2R_2$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$R_4 \rightarrow 6R_4 + R_3$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is a row Echelon form

Here no. of non-zero rows = 4 $\therefore$ Rank of $A = \rho(A) = 4$

(2) Find rank of the matrix $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

Sol: Given.

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

which is a row echlon form

Here no. of non zero rows $= 3$

Rank of $A = \rho(A) = 3$.

(3). $\begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{pmatrix} = A$

Sol:

Applying $R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & 8 \end{bmatrix}$$

$R_3 \to 4R_3 + R_2$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & -32 \end{pmatrix}$$

$\rho(A) = 3$.

(4) Define the rank of matrix & find the rank of the following matrix.

$$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$2-2(1)$
$2-2$
$1-2(3)$
$1-6$
$-3-4(3)$
-15
$-1-4(5)$
-21

Applying $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - 4R_1$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -15 & -21 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$

Practice

(1). $\begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ Ans: $\rho(A) = 4$

(2) $\begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$

Sol: Given Matrix $A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$

Applying $R_2 \to 3R_2 - 2R_3$

$R_3 \to 2R_3 - 3R_2$

$3(R_2) - 2(R_3)$

$6 - 2$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + 3R_1$

$\sim \begin{bmatrix} -1 & 2 & 1 & 8 \\ 0 & 5 & 1 & 8 \\ 0 & 8 & 4 & 31 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 1 & 8 \\ 0 & -1 & -5 & -14 \\ 0 & 1 & 5 & 14 \end{bmatrix}$

$R_3 \to 5R_3 - 8R_2$ $\qquad R_3 \to R_3 + R_2$

$\sim \begin{bmatrix} -1 & 2 & 1 & 8 \\ 0 & 5 & 1 & 8 \\ 0 & 0 & 12 & 91 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 & 1 & 8 \\ 0 & -1 & -5 & -14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Which is a Row Echolon form

$\therefore$ Here no. of non-zero rows = 3

$\therefore \rho(A) = 3$.

(7) for what value of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has Rank. 3.

Sol:

Sol: Given

$$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix}$$

Applying $R_2 \rightarrow 4R_2 - R_1$

$R_3 \rightarrow 4R_3 - kR_1$

$R_4 \rightarrow 4R_4 - 9R_1$

$4\times1 - 4$

$4-4=0$

$4k-4k=0$

$4\times9 = 36 - 36 = 0$

$$\sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$$

given that the rank of A is 3

so, det A = 0

$$4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

$0 + 1(24-12k-0) - 1(32k+216-16k^2-108k) = 0$

$24-12k-32k-216+16k^2+108k = 0$

$(16k^2+64k-192 = 0) \div 16$

$k^2+4k-12=0$

$k^2+6k-2k-12=0$

$k(k+6)-2(k+6)=0$

$\boxed{k=-6 \ (\text{or}) \ k=2.}$

⇒ Find the Value of k. Such that the Rank of A is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{pmatrix}$$ is 2

Sol: Given A =

det = 0

$1(10k-42) - 2(20-21) + 3(12-3k) = 0$

$10k-42-40+42+36-9k=0$

$k + \cancel{1/2} + 36 - 40 - \cancel{1/2} = 0$

$$\boxed{k = 4}$$

$\Rightarrow$ find the value of 'k' such that the rank of

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \text{ is } 2$$

<u>sol</u>: Given $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & 0 & 2-k & 0 \end{bmatrix}$$

Which is in echolon form

Given that $\rho(A) = 2$

Here should be 2 no. of non-zero rows

third row should be zero

$2 - k = 0$

$$\boxed{k = 2}$$

** Reduction to Normal form :

We use normal form to find the Rank of a matrix

~~Theorem:~~

Theorem: Every $m \times n$ matrix of rank 'r' can be reduced to the form I_r, $[I_r, 0]$ (or) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite ~~chin~~ chain of elementry row (or) column Operations, where ~~X~~ I_r is the r-rowed unit matrix.

The above form is called "Normal form".

(or) 1st Cannonical form.

Corollary: The rank of a $m \times n$ matrix 'A' is r. Iff it can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementry row (or) column Operations.

Procedure to obtain a normal form :-

- Consider a General matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Step :- 1 :- Get Zeros below a_{11} by using row transforma-tion by using a_{11}

Step :- 2 :- Get zeros right sides to a_{11} by using column transformation by using a_{11}

Step :- 3 :- Get zeros below a_{22} by using row transforma-tion by using

Step :- 4 :- Get zeros right sides to a by using column transformation by using a_{22}

~~Reape~~ Repeat the Same untill we get Normal form.

① Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to Canonical form. & hence find its Rank.

Sol: Given $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

Applying $R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

Applying $C_2 \to C_2 - 2C_1$
$C_3 \to C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

Applying $R_3 \to 4R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$$

Applying $C_3 \to 8C_3 - 5C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 72 & -32 \end{bmatrix}$$

Applying $C_4 \to 72C_4 + 32C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 72 & 0 \end{bmatrix}$$

Applying $R_2 \to R_2/8$
$R_3 \to R_3/72$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim [I_3, 0]$$

Which is a normal form

Hence the rank of A $\rho(A) = 3$.

(2) Reduce the matrix $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ to normal form & hence find the Rank ?

Sol: Given $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

$C_1 \leftrightarrow C_2$

$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_1$

$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & +1 & 3 \end{bmatrix}$

$C_3 \rightarrow C_3 - 2C_1$

$C_4 \rightarrow C_4 + 2C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$R_3 \rightarrow 2R_3 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \rightarrow R_2/2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \rightarrow 2C_3 - C_2$

$C_4 \rightarrow 2C_4 - 3C_2$

$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_2 \rightarrow R_2 \div 2$

$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

which is a normal form

Hence, the rank of A

$= \rho(A) = 2.$

(3) Reduce the matrix $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ to normal form & Hence find the Rank?

Sol: Given $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to 2R_3 - R_1$

$R_4 \to 2R_4 - R_1$

$\sim \begin{bmatrix} 2 & -2 & 0 & 6 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & -2 \end{bmatrix}$

$R_3 \leftrightarrow R_4$

$\sim \begin{bmatrix} 2 & -2 & 0 & 6 \\ 0 & 6 & 0 & -10 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 \to C_2 + C_1$

$C_4 \to C_4 - 3C_1$

$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \to 3R_3 + R_2$

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 6 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \to 10R_3 - 16R_2$, $R_2 \to 16R_2 - 10R_3$

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 96 & 0 & 0 \\ 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 \to \frac{R_1}{2}$, $R_2 \to \frac{R_2}{96}$, $R_3 \to \frac{R_3}{60}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\sim [I_3, 0]$

which is a normal form

Hence the Rank of A. $\rho(A) = 3$

(4). $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ Reduce A to normal form and also find its Rank ?

Sol: Given matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_2 \to 2R_2 - R_1$,

$R_3 \to 2R_3 - 3R_1$

$R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -5 & -3 & -7 \\ 0 & -7 & 9 & -1 \\ 0 & -6 & 3 & -4 \end{bmatrix}$$

$C_2 \to 2C_2 - 3C_1$

$C_3 \to 2C_3 + C_1$

$C_4 \to 2C_4 + C_1$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -10 & -6 & -14 \\ 0 & -14 & 18 & -2 \\ 0 & -12 & 6 & -8 \end{bmatrix}$$

$R_3 \to 5R_3 - 7R_2$

$R_4 \to 5R_4 - 6R_2$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -10 & -6 & -14 \\ 0 & 0 & 132 & 88 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$C_3 \to 5C_3 - 3C_2$

$C_4 \to 5C_4 - 7C_2$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & 660 & 440 \\ 0 & 0 & 330 & 220 \end{bmatrix}$$

$R_4 \to 2R_4 - R_3$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & 660 & 440 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to 3C_4 - 2C_3$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & 660 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{R_1}{2}$, $R_2 \to \frac{R_2}{-10}$, $R_3 \to R_3/660$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\left[I_3 \,\vdots\, 0 \right]$

∴ which is in normal form

∴ Here, non zero rows is 3

∴ $\rho(A) = 3$.

Linear Equations :

An eq. of the form $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b \longrightarrow (1)$

Where $x_1, x_2 \dots x_n$ are unknowns and $a_1, a_2, \dots a_n, b$ are constants is called a linear equation in 'n' unknowns

Consider the system of 'm' linear eq.'s in n unknowns "$x_1, x_2, x_3 \dots x_n$" as given below

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n &= b_i \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \longrightarrow (2)$$

where a_{ij} be is & $b_1, b_2, b_3 \dots b_m$ are constant

The System of eq.'s in (2) can be written in matrix form as $AX = B \longrightarrow (3)$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow (4)$

The matrix $[A/B]$ is called the Augmented matrix of the system (2) by reducing $[A/B]$ to its row echlon form, all the Questions related to the existence & uniquenem of the Solutions of ~~ax~~ $AX = B$

Defination : If $B = 0$ in $AX = B$, the system is said to be homogenous, Other wise the system is said to non-homogeneou

Augmented Matrix :

A matrix is Obtained by attaching the elements of 'B' as last column in The coeff. matrix 'A'. It is denoted by $[A/B]$ (or) $[A'B]$.

ex: $AX = B$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$A'B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

Consistancy :- By reducing $[A'B]$ into echolon form, we get $\rho(A)$ & $\rho(A'B)$

If $\rho(A) = \rho(A'B) \longrightarrow$ the given system of eqs are said to be consistence. It has 2 types of solution

1. Unique solution :

   If $\rho(A) = \rho(A'B) = n$ (no. of unknowns)

2. Infinite solution :

   If $\rho(A) = \rho(A'B) < n$

   the given Systems of eqs has an infinite no. of sol.s.
   let us introduce $(n-r)$ Orbitary linearly independent varaible ~~sol.n~~

   where r = rank of A

Inconsistancy :

If $\rho(A) \neq \rho(A'B) \rightarrow$ It has no Sol.n

Q.1 Show that the eq.s

(i) $x+y+z=4$;
$2x+5y-2z=3$; are not consistant
$x+7y-7z=5$

Sol: We write the given eq.s in the form $AX=B$, where

$$A=\begin{bmatrix}1 & 1 & 1\\ 2 & 5 & -2\\ 1 & 7 & -7\end{bmatrix};\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\quad B=\begin{bmatrix}4\\ 3\\ 5\end{bmatrix}$$

Consider an Augmented matrix

$$(A{:}B)=\left[\begin{array}{ccc|c}1 & 1 & 1 & 4\\ 2 & 5 & -2 & 3\\ 1 & 7 & -7 & 5\end{array}\right]$$

Applying $R_2 \to R_2-2R_1$
$R_3 \to R_3-R_1$

$$\sim\left[\begin{array}{ccc|c}1 & 1 & 1 & 4\\ 0 & 3 & -4 & -5\\ 0 & 6 & -8 & 1\end{array}\right]$$

Applying $R_3 \to R_3-2R_2$

$$\sim\left[\begin{array}{ccc|c}1 & 1 & 1 & 4\\ 0 & 3 & -4 & -5\\ 0 & 0 & 0 & 11\end{array}\right]$$

Which is in Echolon form

$\therefore$ Here $\rho(A)=2$ $\quad\rho(A{:}B)=3$

$\because \rho(A)\neq\rho(A{:}B) \Rightarrow$ The system is not consistent

$\therefore$ It has no solution.

(2). Find the values of a & b for which the eq.s $x+y+z=3$; $x+2y+2z=6$; $x+ay+3z=b$, have
(i) No solution (ii) Unique solution (iii) infinite no. of sol.s

Sol: we write the given eq.s in the form of $AX=B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & 3 \end{bmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix}$$

consider an Augmented matrix

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & a & 3 & b \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & 2 & b-3 \end{array}\right]$$

$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-3 & 0 & b-9 \end{array}\right]$$

which is in echolon form

(i) No Solution:

for $a=3$ & $b \neq a$

$\rho(A) = 2$ $\rho(A:B) = 3$

$\because \rho(A) \neq \rho(A:B) \Rightarrow$ system has no sol.

(ii) Unique sol.:

~~for~~ no. of unknowns $n=3$

for $a \neq 3$ & for any value of b,

$\rho(A) = 3$ $\rho(A'B) = 3$

$\rho(A) = \rho(A'B) = n = 3 \Rightarrow$ the system has unique sol.

(iii) Infinite sol:

for $a = 3$ & $b = 9$

$\rho(A) = 2$ $\rho(A'B) = 2$ $n = 3$

$\rho(A) = \rho(A'B) < n (=3) \Rightarrow$ The system has infinite sol.

Let us introduce $n - r = 3 - 2 = 1$ L.I.S.

(3). Discuss for what values of λ, μ the simultaneous eq:

$x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have

(1) no sol. (ii) A unique sol. (iii) An infinte no. of sol.

sol: Given.. Consider an Augmented matrix.

$$(A'B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

Which is echolon form

(i) No sol.:

for $\lambda = 3$ $\mu \neq 10$

$\rho(A) = 2$ $\rho(A'B) = 3$

$\rho(A) \neq \rho(A'B) \Rightarrow$ It has no sol.

(ii) Unique sol:

no. of unknowns $n=3$

for $\lambda \neq 3$ & for any value of μ,

$\rho(A)=3$ $\quad$ $\rho(A'B)=3$

$\rho(A)=\rho(A'B)=n=3 \Rightarrow$ It has unique sol.

(iii) Infinite sol:

for $\lambda=3$ & $\mu=10$

$\rho(A)=2$ $\quad$ $\rho(A'B)=2$ $\quad$ $n=3$

$\rho(A)=\rho(A'B) < n(=3) \Rightarrow$ It has infinite sol.

Let us introduce $n-r=3-2=1$ LIs

(4). Find ~~whete~~ wether the following eq.s are consistents if so solve them.

$x+y+2z=4$, $2x-y+3z=9$, $3x-y-z=2$.

sol: We write the given in the form $AX=B$, where

$$A=\begin{bmatrix}1 & 1 & 2\\ 2 & -1 & 3\\ 3 & -1 & -1\end{bmatrix} \quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \quad B=\begin{bmatrix}4\\ 9\\ 2\end{bmatrix}$$

Consider Augmented matrix

$$(A'B)=\left[\begin{array}{ccc|c}1 & 1 & 2 & 4\\ 2 & -1 & 3 & 9\\ 3 & -1 & -1 & 2\end{array}\right]$$

Applying $R_2 \to R_2-2R_1$
$R_3 \to R_3-3R_1$

$$\sim\left[\begin{array}{ccc|c}1 & 1 & 2 & 4\\ 0 & -3 & -1 & 1\\ 0 & -4 & -7 & -10\end{array}\right]$$

$R_3 \to 3R_3-4R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{array}\right]$$

Which is in echelon form

$\rho(A) = 3$. $\rho(A'B) = 3$

$\rho(A) = \rho(A'B) \rightarrow$ System has consistent

no. of unknowns $(n) = 3$

$\rho(A) = \rho(A'B) = n = 3 \rightarrow$ It has unique sol.

rewrite in $AX = B$ form

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$x + y + 2z = 4 \quad -(1)$

$-3y - z = 1 \quad -(2)$

$\cancel{-}17z = \cancel{-}34 \quad -(3)$

$\boxed{z = 2}$

$(2) \Rightarrow -3y - 2 = 1$

$-3y = 3$

$\boxed{y = -1}$

$(1) \Rightarrow x + (-1) + 2(2) = 4$

$x = 4 - 3$

$\boxed{x = 1}$

$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ is the req. sol.

practice

(5). Solve the system of eq. if its is consistent

$2x - y + 4z = 12$, $3x + 2y + z = 10$, $x + y + z = 6$

(6) $x + y + z + t = 4$, $x - z + 2t = 2$, $y + z - 3t = -1$, $x + 2y - z + t = 3$.

Consistancy of System of Homogneous linear eq.s

Consider a system of 'm' homogneous linear eqs in 'n' unknowns, namely

let
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots \quad a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots \quad a_{mn}x_n = 0$$

$$AX = 0$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Trivial Solution :

If each & every unknown should be 'zero'.

i.e., $x_1 = 0$, $x_2 = 0$, ... $x_n = 0$ are called Zero sol. (or) Trivial sol.

Non-Trivial sol. :

If atleast one unknown should not be 'zero' are called Non-zero sol. (or) Non-Trivial sol.

*** Working rule to solve H.L.E :-** (homogen linear eq.)

1. $\rho(A) = n \Rightarrow$ The given system of eq.s has a zero sol. (or) Trivial sol.

2. $\rho(A) < n \Rightarrow$ The given system of eq.s has ~~an non-zero sol~~ infinite no. of non-Trivial sol.

Let us introduce $(n-r)$ L.I.s

(3). If the no. of eq.s is less than no. of Varaibles, The sol. is always non-Trivial

(4). If the no. of eq.s = no. of Variables, the necessary and Sufficient condition for sol. a non-Trivial sol. is that determinent of the coeff. matrix is zero ($|A| = 0$)

Note :

1. $AX = 0$ is always consistent, i.e., it has a sol.
2. If A is a non-singular matrix ($|A| \neq 0$) then the linear System $AX = 0$ has only the "Zero" sol.

(practice problms).

⇒ (5).

sol: We write the given in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 12 \\ 10 \\ 6 \end{bmatrix}$$

consider Augmented matrix

$$(A'B) = \left[\begin{array}{ccc|c} 2 & -1 & 4 & 12 \\ 3 & 2 & 1 & 10 \\ 1 & 1 & 1 & 6 \end{array}\right]$$

Applying $R_2 \to 2R_2 - 3R_1$,
$R_3 \to 2R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 4 & 12 \\ 0 & 7 & -10 & -16 \\ 0 & 3 & -2 & 0 \end{array}\right]$$

$R_3 \to 7R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 4 & 12 \\ 0 & 7 & -10 & -16 \\ 0 & 0 & 16 & +48 \end{array}\right]$$

... 36
4-(-3)
4+3
2-12
2-(-1)

-14-(-30)
-14+30

-48-0

Where is in the echolon form

$\rho(A) = 3$ $\quad \rho(A'B) = 3$

$\rho(A) = \rho(A'B) \longrightarrow$ System is consistent

no. of unknowns $(n) = 3$

$\therefore \rho(A) = \rho(A'B) = n = 3 \longrightarrow$ It has uniquesol.

we write in $AX=B$ form

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 7 & -10 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -16 \\ 48 \end{bmatrix}$$

$2x - y + 4z = 12$ — ①

$7y - 10z = -16$ — ②

$16z = 48$ — ③

$\boxed{z = 3}$

② $\Rightarrow 7y - 10(3) = -16$

$7y - 30 = -16$

$7y = 14$

$\boxed{y = 2}$

① $\Rightarrow 2x - 2 + 4(3) = 12$

$2x + 10 = 12$

$2x = 2$

$\boxed{x = 1}$

$\therefore X \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the req. sol.

$\Rightarrow$ practice

⑥ sol: We write the given matrix in the form of $AX=B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

consider Augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 1 & 2 & -1 & 1 & 3 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$,
$R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & 1 & -2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & -2 & 0 & -1 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$R_4 \to R_4 + R_2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & 1 & -2 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & -4 & 1 & -3 \end{array}\right]$$

$-4-(-4)$

$-1-(-8)$

$-1+8$

$-12-(-3)$

$-12+3$

$-3-(-12)$

$-3+12$

$R_4 \to R_4 - 4R_3$

$-1-(-8)$

$-1+8$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & 1 & -2 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 9 & 9 \end{array}\right]$$

which is in echolon form

$\rho(A) = 4 \quad \rho(A'B) = 4$

$\rho(A) = \rho(A'B) \Rightarrow$ It is consitent

- no. of unknowns $(n) = 4$

$\rho(A) = \rho(A'B) = n = 4 \to$ It has unique sol.

re write in $AX = B$ form

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \\ 9 \end{bmatrix}$$

$$x+y+z+t = 4 \quad — (1)$$
$$-y - 2z + t = -2 \quad — (2)$$
$$-z - 2t = -3 \quad — (3)$$
$$9t = 9 \quad — (4)$$

$$\boxed{t = 1}$$

$(3) \Rightarrow -z - 2\left(\frac{9}{7}\right) = -3$

$-z - \left(\frac{18}{7}\right) = -3$

$-z = -3 + \frac{18}{7}$

$+z = +3$

$\boxed{z = 3}$

$(2) \Rightarrow -y + 2(3) + \left(\frac{9}{7}\right) = -2$

$-y + 6 = -2 + \frac{9}{7}$

$-y = -5 + 6$

$\boxed{y = -1}$

$21 \times$
18
9
$28 -$
9
37

$(1) \Rightarrow x + (-1) + (3) + \left(\frac{9}{7}\right) = 4$

$x + 2 = 4 + \frac{9}{7}$

$x = 37 - 2$

$\boxed{x = 35}$

$$X \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 35 \\ -1 \\ 3 \\ \left(\frac{9}{7}\right) \end{bmatrix}$$ ~~is the req. sol.~~

$(3) \Rightarrow -z - 2(1) = -3$

$-z - 2 = -3$

$-z = -1$

$\boxed{z = 1}$

$(2) \Rightarrow -y - 2(1) + (1) = -2$

~~$-y - 3 = -2$~~

$-y - 1 = -2$

$+y = +1$

$\boxed{y = 1}$

$(1) \Rightarrow x + y + z + t = 4$

$x + 1 + 1 + 1 = 4$

$\boxed{x = 1}$

$$X \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$ is the req. sol.

homogeneous
Prb:- 1

1. Solve completely the system of eq.s

$x+y-2z+3w=0$, $x-2y+z-w=0$, $4x+y-5z+8w=0$

$5x-7y+2z-w=0$.

Sol: Given system of eq.s can be written in matrix form $AX=0$

$$AX \Rightarrow \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - 5R_1$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$

$R_4 \to R_4 - 4R_2$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ which is in echolon form

$\therefore$ The Rank of A $\therefore \rho(A) = 2$

No. of unknowns, $n = 4$

$\therefore \rho(A) < n \to$ The system has non-Trivial sol.

Let us introduce $n-r = 4-2 = 2$ L.I.S

Let, $w = \lambda$, $z = \mu$

rewrite in $AX = 0$ form

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y - 2z + 3w = 0 \longrightarrow (1)$$

$$-3y + 3z - 4w = 0 \longrightarrow (2)$$

Sub. $w = \lambda$ and $z = \mu$ in (2)

$$-3y + 3(\mu) - 4(\lambda) = 0$$

$$-3y = -3\mu + 4\lambda$$

$$y = \frac{3\mu - 4\lambda}{3}$$

$$y = \frac{\cancel{3}\mu}{\cancel{3}} - \frac{4\lambda}{3}$$

$$\boxed{y = \mu - \frac{4\lambda}{3}}$$

Sub. $y = \mu - \frac{4\lambda}{3}$ in (1)

$$x + \mu - \frac{4\lambda}{3} - 2(\mu) + 3\lambda = 0$$

$$x - \mu - \frac{5\lambda}{3} = 0$$

$$\boxed{x = \mu - \frac{5\lambda}{3}}$$

$$\therefore X \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \mu - \frac{5\lambda}{3} \\ \mu - \frac{4\lambda}{3} \\ \mu \\ \lambda \end{bmatrix}$$ is the req. sol.

②. Solve the system of eq.s $x+y-3z+2w=0$, $2x-y+2z-3w=0$, $3x-2y+z-4w=0$, $-4x+y-3z+w=0$.

Sol: Given system of eq.s can be written as $AX=0$

$$AX \Rightarrow \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = O$$

Applying $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 + 4R_1$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

Applying $R_3 \to 3R_3 - 5R_2$
$R_4 \to 3R_4 + 5R_2$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & -5 & -8 \end{bmatrix}$$

$R_4 \to 2R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 0 & -21 \end{bmatrix}$$

∴ which is in the echolon form
$\rho(A) = 4$

no. of unknown $= 4$ $\quad \rho(A) = 4$

$\therefore$ since $\rho(A) = n = 4$ $\rightarrow$ the system has Trivial (zero) Sol.

$\Rightarrow x = 0, y = 0, z = 0$ and $w = 0$ is the req. sol.

Practice:
Solve ① $x + 3y - 2z = 0$
$2x - y + 4z = 0$
$x - 11y + 14z = 0$

② $x + y + w = 0$
$y + z = 0$
$x + y + z + w = 0$
$x + y + 2z = 0$.

① Sol: Given eqs we can written in the form of $AX = 0$

$$AX \Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$16 - 16$

$R_3 \rightarrow R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ which is in echolon form

$\therefore \rho(A) = 2$, no. of unknowns $= 3$

$\therefore \rho(A) < n \rightarrow$ system has nonTrivial sol.

let us introuse $n - r = 3 - 2 = 1$ L.I.S

let, $z = \lambda$

rewrite ~~the~~ in $AX = 0$ form

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 3y - 2z = 0 \quad — (1)$

$-7y + 8z = 0 \quad — (2)$

Sub $z = \lambda$ in (2)

$-7y + 8(\lambda) = 0$

$+7y = +8\lambda$

$y = \boxed{+\frac{8}{7}\lambda}$

Sub y & z in (1)

$x + 3\left(+\frac{8}{7}\lambda\right) - 2\lambda = 0$

~~$x + \frac{24}{7}\lambda - 2\lambda = 0$~~

$x + \frac{24}{7}\lambda - 2\lambda = 0$

$x = \frac{10}{7}\lambda.$

~~$x + 3\lambda - 2\lambda = 0$~~

~~$x - 5\lambda = 0$~~

~~$x = 5\lambda$~~

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10/7\,\lambda \\ 8/7\,\lambda \\ \lambda \end{bmatrix}$ is req. sol.

(2). Given eq.s can be written as in the form of $AX = 0$

Sol:

$$AX \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

which is in echolon form

$\therefore \rho(A)=4$ no. of unknowns $=4$

$\rho(A)=n=4 \rightarrow$ It has Trivial sol.

$x=0,\ y=0,\ z=0,\ w=0,$ is req. sol.

Determine the ~~Vow~~ values of 'λ' for which the following set of eq. posses non-trivial sol.
$3x_1+x_2-\lambda x_3=0,\ 4x_1-2x_2-3x_3=0,\ 2\lambda x_1+4x_2+\lambda x_3=0$
for each permisable value of 'λ', Determine the general sol.

Sol: The given system of eq.s is equivalent to the matrix eq.

$$AX=\begin{pmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

The given system posses non trivial sol., if $\rho(A)<n$ i.e., $\rho(A)<3$

$\therefore \det A=0$

$$\begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix}=0$$

$$3(-2\lambda+12)-1(4\lambda+6\lambda)-\lambda(16+4\lambda)=0$$

$$(-4\lambda^2-32\lambda+36=0)\div -4$$

$$\lambda^2+8\lambda-9=0$$

$$\lambda^2+9\lambda-\lambda-9=0$$

$$\lambda=-9 \text{ (or) } \lambda=1$$

-9

$9 \quad -1$

Case : 1

put $\lambda = -9$

the given system becomes

$$3x_1 + x_2 + 9x_3 = 0$$

$$4x_1 - 2x_2 - 3x_3 = 0$$

$$-18x_1 + 4x_2 - 9x_3 = 0$$

consider a minor of order '2'

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = -6 - 4 = -10 \neq 0$$

$\therefore \rho(A) = 2$, $n = 3$

let us introduce $n - r = 3 - 2 = 1$ L.I.s

let $x_3 = k$

the above given eq.s

$$3x_1 + x_2 + 9k = 0 \rightarrow (1)$$

$$4x_1 - 2x_2 - 3k = 0 \rightarrow (2)$$

solving (1) & (2)

$$\begin{array}{cccc} x_1 & x_2 & k & \\ 1 & 9 & 3 & 1 \\ -2 & -3 & 4 & -2 \end{array}$$

$$\frac{x_1}{-3+18} = \frac{x_2}{36+9} = \frac{k}{-6-4}$$

$$\frac{x_1}{\cancel{15}\,3} = \frac{x_2}{\cancel{45}\,9} = \frac{k}{-\cancel{10}\,2}$$

$$\frac{x_1}{3} = \frac{-k}{2} \qquad \frac{x_2}{9} = \frac{-k}{2}$$

$$\boxed{x_1 = \frac{-3k}{2}} \qquad \boxed{x_2 = \frac{-9k}{2}}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} -3k/2 \\ -9k/2 \\ k \end{pmatrix}$$

Case: 2

put $\lambda = 1$

The given system becomes

$$3x_1 + x_2 - x_3 = 0$$

$$4x_1 - 2x_2 - 3x_3 = 0$$

$$-18x_1 + 4x_2 + [illegible]\, x_3 = 0$$

consider a minor of order 2

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = -10 \neq 0$$

$\therefore \rho(A) = 2$, $n = 3$

let us ~~cons~~ introduce $n - r = 3 - 2 = 1$ L.I.s

Let $x_3 = k$

The above given eq.s

$$3x_1 + x_2 - 1k = 0 \longrightarrow (3)$$

$$4x_1 - 2x_2 - 3k = 0 \longrightarrow (4)$$

solving (3) & (4)

$$\begin{array}{cccc} x_1 & x_2 & k & \\ 1 & -1 & 3 & 1 \\ -2 & -3 & 4 & -2 \end{array}$$

$$\frac{x}{-3-2} = \frac{y}{-4+9} = \frac{k}{-6-4}$$

$$\frac{x}{-\cancel{5}\,1} = \frac{y}{\cancel{5}\,1} = \frac{k}{-\cancel{10}\,2}$$

$$\frac{x}{-1} = \frac{y}{1} = \frac{k}{-2}$$

$$\frac{x}{1} = \frac{k}{2} \qquad \frac{y}{1} = \frac{-k}{2}$$

$2x = k$ $\qquad$ $2y = -k$

$$\boxed{x = k/2} \qquad \boxed{y = -\frac{k}{2}}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} k/2 \\ -k/2 \\ k \end{pmatrix}$$

* Show that the only real no. 'λ' for which the System

$x+2y+3z=\lambda x$, $3x+y+2z=\lambda y$, $2x+3y+z=\lambda z$

has non zero sol. is '6' & solve them, when $\lambda=6$

Sol: The given system can be written as

$(1-\lambda)x+2y+3z=0$, $3x+(1-\lambda)y+2z=0$

$2x+3y+(1-\lambda)z=0$.

The matrix form $AX=0$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

Given that the system posses non-trivial sol.

$\therefore \det A=0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}=0$$

$R_1 \to R_1+R_2+R_3$

$$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}=0$$

$$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}=0$$

$C_2 \to C_2-C_1$

$C_3 \to C_3-C_1$

$$(6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -\lambda-2 & -1 \\ 2 & 1 & -\lambda-1 \end{vmatrix}=0$$

$(6-\lambda)\,1(\lambda^2+\lambda+2\lambda+2+1)=0$

$(6-\lambda)$ $\quad \lambda^2+3\lambda+3=0$

$-\lambda=-6$ $\quad \therefore \lambda=6$ is the only real root & ~~im~~

$\boxed{\lambda=6}$ ~~imaginary~~ remaining are imaginary root

for $\lambda=6$, the given system becomes

$$-5x+2y+3z=0$$
$$3x-5y+2z=0$$
$$2x+3y-5z=0$$

$\therefore$ det $A=0$, consider a minor of order 2

$$\begin{vmatrix} -5 & 2 \\ 3 & -5 \end{vmatrix} = 25-6=19 \neq 0$$

$\therefore$ Rank of $A=2$, $n=3$

let us introduce $n-r=3-2=1$ LIS
let $z=k$.

The above eq.s

$$-5x+2y+3k=0 \longrightarrow (1)$$
$$3x-5y+2k=0 \longrightarrow (2)$$

Solving (1) & (2).

$$\begin{array}{cccc} & x & y & k \\ 2 & 3 & -5 & 2 \\ -5 & 2 & 3 & -5 \end{array}$$

$$\frac{x}{4+15} = \frac{y}{9+10} = \frac{k}{+25-6}$$

$$\frac{x}{\cancel{19}} = \frac{y}{\cancel{19}} = \frac{k}{\cancel{19}}$$

$x=k$

$y=k$ $\quad \therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix}$ 1.

Part - II

Eigen Values & Eigen Vectors of a Matrix :

Introduction :

Eigen Value problems are Some times called as characterstic Value problems and are among the most important class of physical problems for engineering use. They occur in stability problems, dynamics and vibrations, large displacement analysis and several other application areas. They appear when ever a problem has Valid sol. only for certain Specific Values. Such Special Values of physical constants are Called as Characteristic Values. (or) Eigen Values.

Eigen Vector and Eigen values :

let $A = [a_{ij}]_{n \times n}$ Consider $AX = \lambda X$.

- A non-Zero Vector X is Said to be a Eigen Vector (or) Characteristic Vector (or) latent Vector of "A", if there exist a Scalar 'λ' such that '$AX = \lambda X$'

- Here λ is called Eigen Value (or) Characteristic Value (or) latent Value of 'A'.

Characteristic eq. :-

Let 'A' be a square matrix of order 'n' whose eigen Vector is 'X', Corresponding eigen Value 'λ'. Then '$AX = \lambda X$' i.e., $AX = I\lambda X$

$$AX - I\lambda X = 0$$

$$(A - I\lambda)X = 0$$

$$\therefore (A - \lambda I)X = 0.$$

$\therefore (A - \lambda I)$ is Called characteristic matrix of 'A'.

$|A - \lambda I| = 0$ is Called Characteristic eq. of 'A'.

- **Note:** ①. The set of eigen values of a matrix 'A', is known as Spectrum.
- A matrix 'A' of order 'n' has max. 'n' and min. one eigen value.
- Eigen vector is always non-zero vector
- Eigen values of a matrix are unique
- Eigen vector corresponding to a eigen value is not unique.
- Two vectors X_1 & X_2 are said to be Orthogonal, if $X_1 \cdot X_2' = X_2 \cdot X_1^T = 0$.

Properties of Eigen Values & Eigen Vectors:

Theorem ①.

Statement: The sum of the Eigen Values of a matrix 'A' is equal to the trace of 'A'.

Proof: We shall prove this property by considering a sq. matrix of order '3'.

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

let 'λ' be an Eigen Value of A

Characteristic ~~Vat~~ eq. of 'A' is $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} \right| = 0$$

$$|A-\lambda I| = -\lambda^3 + \lambda^2 (a_{11}+a_{22}+a_{23}) - \lambda(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31}) + (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}) \longrightarrow ①$$

If $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of A, then

$$|A-\lambda I| = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) \rightarrow ②$$

from eq. ① & ② equating RHS

$$\lambda_1+\lambda_2+\lambda_3 = a_{11}+a_{22}+a_{33}$$

i.e., Sum of Eigen Values is equal to the trace of A.

$\therefore$ Hence, it is proved.

Theorem :- ②

Statement: The pdt of the Eigen Values of a matrix is equal to its det.

Proof: Let $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ be the Eigen Values of a matrix 'A'. then

$$|A-\lambda I| = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)\ldots(\lambda-\lambda_n)$$

put $\lambda = 0$

$$|A| = (-1)^n(-\lambda_1)(-\lambda_2)(-\lambda_3)\ldots(-\lambda_n)$$

$$= (-1)^n(-1)^n \lambda_1\lambda_2\lambda_3\ldots\lambda_n$$

$$= -1^{2n} \lambda_1\lambda_2\lambda_3\ldots\lambda_n$$

$$|A| = \lambda_1\lambda_2\lambda_3\ldots\lambda_n$$

i.e., pdt of Eigen Values is equal to det of $|A|$

$\therefore$ Hence, proved.

Note: If one of the eigen values of a matrix 'A' is zero, then $|A| = 0$

Hence $|A|$ is Singular Matrix.

Theorem :- 3

Statement : If λ is an Eigen Value of 'A' corresponding to the Eigen vector 'X', then λ^n is Eigen Value of A^n corresponding to the Eigen vector 'X'.

proof : Given that λ is an Eigen Value of 'A' corresponding to the Eigen Vector 'X'.

$$AX = \lambda X \longrightarrow ①$$

pre multiply with 'A' on both sides

$$A(AX) = A(\lambda X)$$

$$A^2 X = \lambda (AX)$$

$$A^2 X = \lambda(\lambda X) \text{ from } ①$$

$$A^2 X = \lambda^2 X$$

i.e., λ^2 is an Eigen Value of A^2 corresponding to Eigen vector X,

lly, $A^n X = \lambda^n X$

i.e., λ^n is an Eigen Value of A^n corresponding to Eigen Vector X,

∴ Hence it is proved.

④. A sq. matrix 'A' and it's transpose A^T have the same Eigen Values

⑤ If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigen Values of a matrix 'A', then $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ are the Eigen Values of a matrix kA where k being any non-zero scalar

⑥ The eigen values of a triangular matrix are just the diagonal element of the matrix

⑦ If A & B are sq. matrix such that 'A' is non-singular, then $A^{-1}B$ & $B.A^{-1}$ have the same Eigen Value.

Working rule to find Eigen Vectors :

Step :-1 Write characteristic eq of 'A'

i.e., $|A-\lambda I|=0$ on solving we get eigen value

$\lambda_1, \lambda_2, \ldots \lambda_n$

Step :-2

let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be a Eigen Vector Corresponding to eigen vector

Step :-3 write the Characteristic matrix of A

i.e, $(A-\lambda I)\lambda = 0$

Step : 4

Reducing the coeff matrix into echolon form by using row transformation (or) Direct Solving.

Problems:

①. Find the Eigen Values and the corresponding Eigen ~~Valu~~ Vectors of a matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol: Given $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Characteristic eq of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$(8-\lambda)(21-7\lambda-3\lambda+\lambda^2-16)+6(-18+6\lambda+8)+2(24-14+2\lambda)=0$

$(8-\lambda)(\lambda^2-10\lambda+5)+6(6\lambda-10)+2(2\lambda+10)=0$

$8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+4\lambda+20=0$

$-\lambda^3+18\lambda^2-45\lambda=0$

$\lambda^3-18\lambda^2+45\lambda=0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda = 0$ (or) $\lambda^2 - 18\lambda + 45 = 0$

$\lambda^2 - 15\lambda - 3\lambda + 45 = 0$

$\lambda = 15 \quad , \quad \lambda = 3$

$\lambda = 0, 3, 15$ ~~and~~ are Eigen Values of A

let $x = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}$ be an Eigen Vectors corresponding to Eigen Value λ. The Characterstic matrix of A is $(A - \lambda I)X = 0$

$$\begin{bmatrix} 8-\lambda & 6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (1)$$

Eigen Vector corresponding to Eigen Value $\lambda = 0$

put $\lambda = 0$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0 \rightarrow (1)$

$-6x_1 + 7x_2 - 4x_3 = 0 \rightarrow (2)$

$2x_1 - 4x_2 + 3x_3 = 0 \rightarrow (3)$

solving (1) & (2)

$$\begin{array}{cccc} \underline{x_1} & & \underline{x_2} & \underline{x_3} \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{array}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k \text{ (let)}$$

$x_1 = k \quad , \quad x_2 = 2k \quad , \quad x_3 = 2k$

let $k = 1$ then $X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Eigen Vector corresponding to Eigen Value $\lambda=3$, put $\lambda=3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \longrightarrow (3)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \longrightarrow (4)$$

solving (3) & (4)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{array}$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k \text{ (let)}$$

$x_1 = 2k$, $x_2 = k$, $x_3 = -2k$

If $k=1$ then

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Eigen Vector corresponding to Eigen Value $\lambda = 15$, put $\lambda=15$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0 \longrightarrow (1)$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \longrightarrow (2)$$

$$2x_1 - 4x_2 - 12x_3 = 0 \longrightarrow (3)$$

solving (1) & (2)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -6 & 2 & -7 & -6 \\ -8 & -4 & -6 & -8 \end{array}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k \text{ (let)}$$

$x_1 = 2k \quad , x_2 = -2k \quad , x_3 = k$

if $k=1$ then

$$X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

problm

(2). Find the Characteristic roots of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Characteristic eq. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(6-3\lambda-2\lambda+\lambda^2-2) - 2(-1-\lambda+2) + 1(2-3+\lambda) = 0$$

$$(2-\lambda)(\lambda^2-5\lambda+4) - 2(1-\lambda) + 1(\lambda-1) = 0$$

$$2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 2 + 2\lambda + \lambda - 1 = 0$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$(\lambda-1)(\lambda^2 - 6\lambda + 5) = 0$$

$$\lambda = 1 \text{ (or) } \lambda^2 - 6\lambda + 5 = 0$$

$$\lambda^2 - 5\lambda - \lambda + 5 = 0$$

$$\lambda = 5 \quad , \lambda = 1$$

$\lambda = 1, 1, 5$ are characteristic roots.

problm

(3) Find the Eigen values and Eigen Vectors of $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Sol: Given $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Characteristic eq. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(9+\lambda^2-6\lambda-1) + 2(-6+2\lambda+2) + 2(2-6+2\lambda) = 0$$

$$(6-\lambda)(\lambda^2-6\lambda+8) + 2(2\lambda-4) + 2(2\lambda-4) = 0$$

$$6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0$$

$$-\lambda^3+12\lambda^2-36\lambda+32=0$$

$$\lambda^3-12\lambda^2+36\lambda-32=0$$

$$(\lambda-2)(\lambda^2-10\lambda+16)=0$$

$$\boxed{\lambda=2} \text{ (or) } \lambda^2-10\lambda+16=0$$

$$\lambda^2-8\lambda-2\lambda+16=0$$

$$\boxed{\lambda=8} \quad \boxed{\lambda=2}$$

$\lambda = 2, 2, 8$ are Eigen Values

let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an Eigen vector corresponding to Eigen Values λ, then characterstic matrix of A is

$$(A-\lambda I)X=0$$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen Vectors corresponding to Eigen Values $\lambda=2$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to 2R_2+R_1$, $R_3 \to 2R_3-R_1$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$$

which is in Echelon form

$$\rho(A)=1 \quad n=3$$

let us ~~red~~ introduce $n-r=3-1=2$ (L.I.S)

let $x_3=l$, $x_2=m$

from the above matrix

$$4x_1-2x_2+2x_3=0$$

put $x_3=l$, $x_2=m$

$$4x_1-2m+2l=0$$

$$4x_1=2m-2l$$

$$x_1=\frac{2m-2l}{4} \Rightarrow x_1=\frac{m}{2}-\frac{l}{2}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} \frac{1}{2}m - \frac{1}{2}l \\ m \\ l \end{bmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}m \\ m \\ 0 \end{pmatrix} + \begin{bmatrix} -\frac{1}{2}l \\ 0 \\ l \end{bmatrix}$$

If $l = 1$; $m = 1$

$$X = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

$$x_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \; \& \; \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} = x_2$$

Eigen Vectors corresponding to Eigen Values $\lambda = 8$, put $\lambda = 8$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = [0]$$

$-2x_1 - 2x_2 + 2x_3 = 0 \rightarrow (1)$

$-2x_1 - 5x_2 - 1x_3 = 0 \rightarrow (2)$

$2x_1 - x_2 - 5x_3 = 0 \rightarrow (3)$

Solving (1) & (2)

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -2 & 2 & -2 & -2 \\ -5 & -1 & -2 & -5 \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k \text{ (let)}$$

$x_1 = 2k$, $x_2 = -k$, $x_3 = k$

If $k = 1$

$$X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(4) Verify that the sum of Eigen Values is equal to trace of the for the matrix $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find the Corresponding Eigen Vectors

Sol: Given $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Charactrise eq. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$$(3-\lambda)(15-5\lambda-5\lambda+\lambda^2-1)+1(-3+\lambda+1)+1(1-5+\lambda)=0$$

$$(3-\lambda)(\lambda^2-8\lambda+14)+1(\lambda-2)+1(\lambda-4)=0$$

$$3\lambda^2-24\lambda+42-\lambda^3+8\lambda^2-14\lambda+\lambda-2+\lambda-4=0$$

$$-\lambda^3+11\lambda^2-36\lambda+36=0$$

$$\lambda^3-11\lambda^2+36\lambda-36=0$$

$$(\lambda-2)(\lambda^2-9\lambda+18)=0$$

$\boxed{\lambda=2}$ (or) $\lambda^2-9\lambda+18=0$

$$\lambda^2-6\lambda-3\lambda+18=0$$

$\boxed{\lambda=6}$ $\boxed{\lambda=3}$

$\lambda = 2, 3, 6$ are Eigen Values

Verification:

trace $= 3+5+3 = 11$

Sum of Eigen Values $= 2+3+6 = 11$

Sum of Eigen Values $=$ trace of n

let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the Eigen Vectors

Eigen Vector corresponding to Eigen Values $\lambda = 2$

Put $\lambda = 2$

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = [0]$$

$x_1 - x_2 + x_3 = 0 \rightarrow (1)$

$-x_1 + 3x_2 - 1x_2 = 0 \rightarrow (2)$

$x_1 - x_2 + x_3 = 0 \rightarrow (3)$

Solving ~~one~~ (1) & (2)

$$\begin{array}{cccccc} x_1 & & x_2 & & x_3 & \\ -1 & 1 & & 1 & & -1 \\ 3 & -1 & & -1 & & 3 \end{array}$$

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1} = k \text{ (let)}$$

$x_1 = -k \quad x_2 = 0 \quad x_3 = k$

If $k = 1$ then

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Eigen vector corresponding to Eigen value $\lambda = 3$, put $\lambda = 3$

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$-x_2 + x_3 = 0 \rightarrow (1)$

$-x_1 + 2x_2 - x_3 = 0 \rightarrow (2)$

$x_1 - x_2 = 0 \rightarrow (3)$

Solving (1) & (2)

$$\begin{array}{cccccc} x_1 & & x_2 & & x_3 & \\ -1 & 1 & & 0 & & -1 \\ 2 & -1 & & -1 & & 2 \end{array}$$

$$\frac{x_1}{1-2} = \frac{x_2}{-1+0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-7} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{7} = \frac{x_3}{1} = k \text{ (let)}$$

$x_1 = k,\ x_2 = k,\ x_3 = k$

If $k = 1$ then

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

* Eigen vector corresponding to Eigen value $\lambda = 6$

$$\begin{pmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-3x_1 - x_2 + x_3 = 0$ — (1)

$-x_1 - x_2 - x_3 = 0 \rightarrow (2)$

$x_1 - x_2 - 3x_3 = 0 \rightarrow (3)$

$$\begin{array}{cccccc} x_1 & & x_2 & & x_3 & \\ -1 & 1 & & -3 & & -1 \\ -1 & -1 & & -1 & & -1 \end{array}$$

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} = k \text{ (let)}$$

$x_1 = k,\ x_2 = -2k,\ x_3 = k$

If $k = 1$ then

$$X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

→ Eigen Values & Eigen Vectors of $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

characteristic eq. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix}$$

$$(1-\lambda)\left((4-\lambda)(-3-\lambda)+12\right)+6(0)-4(0)=0$$

$$(1-\lambda)\left(\cancel{-12}-4\lambda+3\lambda+\lambda^2+\cancel{12}\right)=0$$

$$(1-\lambda)\ (\lambda^2-\lambda)=0$$

$$1-\lambda=0 \qquad \lambda(\lambda-1)=0$$

$$\boxed{\lambda=1} \qquad \boxed{\lambda=0},\ \boxed{\lambda=1}$$

$\lambda = 0, 1, 1$ are Eigen Values

let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an Eigen Vector Corresponding to Eigen Values λ, then Characteristic matrix of A is $(A-\lambda I)x = 0$

$$\begin{bmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vectors corresponding to Eigen Values $\lambda = 1$

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to 2R_2 + R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ which is in echolon form $\rho(A) = 1$, $n=3$

let us introduce $n - r = 3 - 1 = 2$

$-6x_2 - 4x_3 = 0$

put $x_2 = m$, $x_3 = l$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ m \\ l \end{bmatrix}$$

put $m=1, n=1$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

put $m=2, n=2$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

*Eigen vector corresponding to Eigen Values λ, put $\lambda = 0$

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$1x_1 - 6x_2 - 4x_3 = 0 \longrightarrow (1)$

$4x_2 + 2x_3 = 0 \longrightarrow (2)$

$-6x_2 - 3x_3 = 0 \longrightarrow (3)$

Solving (1) & (2)

x_1	x_2	x_3	
-6	-4	1	-6
4	2	0	4

$$\frac{x_1}{-12+16} = \frac{x_2}{0-2} = \frac{x_3}{4-0}$$

$$\frac{x_1}{\cancel{4}\,2} = \frac{x_2}{\cancel{2}\,-1} = \frac{x_3}{\cancel{4}\,2}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{2} = k \text{ (let)}$$

$x_1 = 2k$, $x_2 = -k$, $x_3 = 2k$

If $k = 1$ then

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$ //.

* Cayley - Hamilton Theorey :

Statement: Every sq. matrix satisfies its own characteristic eq

prblms :-

1) If $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ verify Cayley-Hamilton theorem & find A^{-1}.

Sol: Given $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

characteristic eq. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & 2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4+\lambda^2-4\lambda+3) + 2(2-\lambda) + 2(-1) = 0$$

$$(1-\lambda)(\lambda^2-4\lambda+7) + 4 - 2\lambda - 2 = 0$$

$$\lambda^2 - 4\lambda + 7 - \lambda^3 + 4\lambda^2 - 7\lambda + 2 - 2\lambda = 0$$

$$-\lambda^3 + 5\lambda^2 - 13\lambda + 9 = 0$$

$$\lambda^3 - 5\lambda^2 + 13\lambda - 9 = 0 \longrightarrow ①$$

Replace λ by A

$$A^3 - 5A^2 + 13A - 9I = 0 \longrightarrow ②$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1-2 & -2-4-2 & 2-6+4 \\ 1+2 & -2+4-3 & 2+6+6 \\ -1 & -2-2 & -3+4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1-8 & 2-16 & -2-24 \\ 3-1 & -6-2-14 & 6-3+28 \\ -1-4 & 2-8-1 & -2-12+2 \end{bmatrix} = \begin{bmatrix} -9 & -14 & -26 \\ 2 & -22 & 31 \\ -5 & -7 & -12 \end{bmatrix}$$

$$5A^2 = \begin{bmatrix} -5 & -40 & 0 \\ 15 & -5 & 70 \\ -5 & -20 & 5 \end{bmatrix} \qquad 13A = \begin{pmatrix} 13 & -26 & 26 \\ 13 & 26 & 39 \\ 0 & -13 & 26 \end{pmatrix}$$

$$9I = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

(2) $\Rightarrow$ $$\begin{bmatrix} -9 & \cancel{4} & -26 \\ 2 & -22 & 31 \\ -5 & -7 & -12 \end{bmatrix} - \begin{bmatrix} -5 & -40 & 0 \\ 15 & -5 & 70 \\ -5 & -20 & 5 \end{bmatrix} + \begin{bmatrix} 13 & -26 & 26 \\ 13 & 26 & 39 \\ 0 & -13 & 26 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Hence Cayley Hamilton theorem is verified.

Multiply eq. (2) with A^{-1} O.B.S

$$A^{-1}(A^3 - 5A^2 + 13A - 9I) = 0$$

$$A^2 - 5A + 13I - 9A^{-1} = 0$$

$$9A^{-1} = A^2 - 5A + 13I$$

$$9A^{-1} = \begin{pmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix}$$

$$= \begin{bmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{bmatrix} + \begin{bmatrix} -5 & 10 & -10 \\ -5 & -10 & -15 \\ 0 & 5 & -10 \end{bmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$$

Practice

(3). Verify Cayley-Hamilton theorem for a matrix $A = \begin{pmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix}$

(1) If $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$ Verify Cayley Hamilton theorem. Find A^4 & A^{-1} using C-H Theorem.

practice.

(3). Given $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

Characteristic eq. $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)\left((-3-\lambda)(1-\lambda) - 8\right) + 8\left(4(1-\lambda) + 6\right) + 2\left(-16 - (-3-\lambda)(3)\right) = 0$$

$$(8-\lambda)\left(\lambda^2 + 2\lambda - 11\right) + 8(-4\lambda + 10) + 2(3\lambda - 7) = 0$$

$$8\lambda^2 + 16\lambda - 88 - \lambda^3 - 2\lambda^2 + 11\lambda + (-32\lambda) + 80 + 6\lambda - 14 = 0$$

$$-\lambda^3 + 6\lambda^2 + \lambda - 22 = 0$$

$$\lambda^3 - 6\lambda^2 - \lambda + 22 = 0$$

Replace λ by A

$$A^3 - 6A^2 - A + 22I = 0 \longrightarrow (1)$$

$$A \cdot A = A^2 = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$A^2 = \begin{pmatrix} 64 - 32 + 6 & -64 + 24 - 8 & 16 + 16 + 2 \\ 32 - 12 - 6 & -32 + 9 + 8 & 8 + 6 - 2 \\ 24 - 16 + 3 & -24 + 12 - 4 & 6 + 8 + 1 \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$$

$$22I = \begin{pmatrix} 22 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 22 \end{pmatrix} \qquad 6A^2 = \begin{pmatrix} 228 & -228 & 208 \\ 84 & -90 & 72 \\ 66 & -96 & 90 \end{pmatrix}$$

(1) $\Rightarrow A^3 - 6A^2 - A + 22I = 0.$

$$\Rightarrow \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix} - \begin{pmatrix} 228 & -288 & 208 \\ 84 & -90 & 72 \\ 66 & -96 & 90 \end{pmatrix} - \begin{pmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} 22 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 22 \end{pmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^3 - 6A^2 - A + 22I$$

$\therefore$ Hence proved.

* Diagonalization of Matrix :-

If a sq. Matrix 'A' of order n has n Linearly independent Eigen Vectors $(X_1, X_2, X_3 \ldots X_n)$ corresponding to the 'n' Eigen Values $(\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n)$ resp. then a matrix 'P' can be found such that $P^{-1}AP$ is a diagonal matrix

Modal & Spectral Matrix:

let X_1, X_2, X_3 are Eigen Vectors corresponding to the Eigen Values $\lambda_1, \lambda_2, \lambda_3$ of A resp. Then the modal matrix of A is $P = [X_1\ X_2\ X_3]$ and The spectral matrix of A is $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ exist such that $P^{-1}AP = D$

Then P will be an Orthogonal matrix

i.e., $P^T \cdot P = P \cdot P^T = I$

$\Leftrightarrow P^T \cdot P = I \Rightarrow P^T = P^{-1}$

Norm of a Vector :

The norm of a Vector X is the length of that Vector and is denoted $\|X\|$ & defined as

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

Normalised Modal matrix:-

let X_1, X_2, X_3 be the Eigen Vectors of A corresponding to the Eigen Values $\lambda_1, \lambda_2, \lambda_3$. The normalised Modal matrix is

$$P = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|} \right]$$

Calculation of powers of matrix :-

- We can Obtain the powers of a matrix by using diagonalisation

let, A be Sq. matrix, then a non-singular matrix 'p' can be found Such that $D = P^{-1}AP$

$$D^2 = D.D$$

$$D^2 = (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}A(PP^{-1})AP$$

$$D^2 = P^{-1}A^2P$$

$$\text{IIIly } D^3 = P^{-1}A^3P$$

$$\vdots$$

$$D^n = P^{-1}A^nP \longrightarrow (1).$$

to obtain A^n, premultiply eq (1) by P & postmultiply by P^{-1}

$$PD^nP^{-1} = (PP^{-1})A^n(PP^{-1})$$

$$PD^nP^{-1} = IA^n.I$$

$$A^n = PD^nP^{-1}$$

$$\text{Hence } A^n = P\begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}P^{-1}$$

Note:-

$X_1X_2^T = X_2X_1^T = 0$, then X_1 & X_2 are said to be pair wise Orthogonal Vectors. On that times

$$\boxed{P^{-1} = P^T}$$

prblm: Determine the Modal matrix P of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ Verify that $P^{-1}AP$ is a Diagonal matrix

eigen values $\lambda = -3, -3, 5$

sol: Given $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

characteristic eq. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\left[(1-\lambda)(-\lambda) - 12\right) - 2\left[2(-\lambda) - 6\right) - 3\left[-4 - (-1)(1-\lambda)\right] = 0$$

$$2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 + 12 - 3 + 3\lambda = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$(\lambda + 3) \quad (\lambda^2 - 2\lambda - 15) = 0$$

$\lambda = -3 \qquad \lambda^2 + 3\lambda - 5\lambda - 15 = 0$

$\lambda + 3 = 0 \qquad \lambda - 5 = 0$

$\lambda = -3 \qquad \lambda = 5$

$\therefore \lambda = -3, -3, 5$ are Eigen Values

let $X \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ are Eigen ~~Value~~ Vectors corresponding to Eigen Value λ such that

the characteristic matrix of A is

$$(A - \lambda I) X = 0$$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow (1)$$

* Eigen Vectors corresponding to Eigen Value $\lambda = 5$:-

put $\lambda = 5$ in (1).

$$\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

$$\begin{array}{cccc} & x & y & z \\ 2 & -3 & -7 & 2 \\ -4 & -6 & 2 & -4 \end{array}$$

$$\frac{x}{-12-12} = \frac{y}{-6-42} = \frac{z}{28-4}$$

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\quad 1 \qquad 2 \qquad 1$$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

* **Eigen Vectors corresponding to Eigen values $\lambda = -3$:-**

put $\lambda = -3$. in ①

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

here,

$\rho(A) = 1$, no. of unknowns $n = 3$

$\rho(A) \neq n$

let introduce $n = 3-1 = 2$ (L.I.S)

let $x_2 = m$, $x_3 = l$

$x_1 + 2x_2 - 3x_3 = 0$

$x_1 + 2m - 3l = 0$

$x_1 = 3l - 2m$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3l-2m \\ m \\ l \end{pmatrix} = \begin{pmatrix} 3l \\ 0 \\ l \end{pmatrix} + \begin{pmatrix} -2m \\ m \\ 0 \end{pmatrix}$$

If $\lambda=1, m=1$

$$X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad X_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$X_1 X_2^T = (3\ 0\ 1)\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = -3+1 = -2 \neq 0$$

$$X_2 X_3^T = (-2\ 1\ 0)\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = -6 \neq 0$$

Modal matrix $P = \begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|P| = -1(-1) - 3(-1) - 2(-2)$$

$$= 1+3+4 = 8 \neq 0$$

$$P^{-1} = \frac{1}{|P|} \text{Adj } P \longrightarrow ②$$

Adj P = Transpose of co-factors matrix

$$= \begin{pmatrix} -1 & 1 & -2 \\ -2 & 2 & 4 \\ 3 & 5 & 6 \end{pmatrix}^T$$

$$= \begin{pmatrix} -1 & -2 & +3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{pmatrix}$$

$$② \Rightarrow P^{-1} = \frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix}$$

$$P^{-1}A = \frac{1}{8}\begin{pmatrix} -1 & -2 & 3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{pmatrix}\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -5 & -10 & 15 \\ -3 & -6 & -15 \\ 6 & -12 & -18 \end{bmatrix}$$

$$(P^{-1}A)P = \frac{1}{8}\begin{bmatrix} -5 & -10 & 15 \\ -3 & -6 & -15 \\ 6 & -12 & -18 \end{bmatrix}\begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 40 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -24 \end{bmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} = D.$$

A^4

$\Rightarrow A^n = PD^nP^{-1}$

$A^4 = PD^4P^{-1}$

$$= P\begin{pmatrix} 5^4 & 0 & 0 \\ 0 & (-3)^4 & 0 \\ 0 & 0 & (-3)^4 \end{pmatrix} P^{-1}.$$

(2) find A matrix 'p' which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form hence, calculate A^4.

Sol: Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Charastic eq. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\left[6-2\lambda-3\lambda+\lambda^2-2\right) - 1\left(2-4+2\lambda\right) = 0$

$6-2\lambda-3\lambda+\lambda^2-2-6\lambda+2\lambda^2+3\lambda^2-\lambda^3+2\lambda-1(-4+2\lambda) = 0$

$\lambda^2-5\lambda+4-\lambda^3+5\lambda^2-4\lambda+2-2\lambda = 0$

$-\lambda^3+6\lambda^2-11\lambda+6 = 0$

$\lambda^3-6\lambda^2+11\lambda-6 = 0$

$$\begin{array}{c|cccc} \lambda=1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$(\lambda-1)\quad(\lambda^2-5\lambda+6) = 0$

$\lambda = 1 \qquad \lambda-2\lambda-3\lambda+6 = 0$

$\boxed{\lambda = 2}$ $\boxed{\lambda = 3}$ are eigen Values charastic eq. matrix of A is $(A-\lambda I)X=0$

$$\begin{pmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (1)$$

Eigen vectors Corresponding to Eigen values :-

put $\lambda=1$ in (1).

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x_3 = 0 \quad — (1)$

$x_1 + x_2 + x_3 = 0 \rightarrow (2)$

solving (1) & (2)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 0 & & -1 & & 0 & & 0 \\ 1 & & 1 & & 1 & & 1 \end{array}$$

$\dfrac{x_1}{1} = \dfrac{x_2}{-1} = \dfrac{x_3}{0}$; $x_1 = 1$, $x_2 = -1$, $x_3 = 0$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

put $\lambda = 2$ in (1)

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{-x_1 - x_3 = 0}$

$x_1 + x_3 = 0 \rightarrow (1)$

$2x_1 + 2x_2 + x_3 = 0 \rightarrow (2)$

(1) & (2)

$$\begin{array}{cccccc} & x & & x_2 & & x_3 \\ 0 & & 1 & & 1 & & 0 \\ 2 & & 1 & & 2 & & 2 \end{array}$$

$\dfrac{x_1}{0-2} = \dfrac{x_2}{2-1} = \dfrac{x_3}{2-0}$

$\dfrac{x_1}{-2} = \dfrac{x_2}{1} = \dfrac{x_3}{2}$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Eigen vectors corresponding to eigen values put $\lambda = 3$

$$\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2x_1 - x_3 = 0 \quad \text{--- } (1)$

$x_1 - x_2 + x_3 = 0 \quad \longrightarrow (2)$

(1) & (2).

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 0 & & -1 & & -2 & & 0 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{0-1} = \frac{x_2}{-1+2} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$X_1 X_2^T = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} -2 & 1 & 2 \end{pmatrix} = -2-1+0 = -3 \neq 0$$

$$X_2 X_3^T = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 2 \end{pmatrix} = 2+1+4 = 7 \neq 0$$

Model matrix:

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$|P| = 1(2-2) - 2(-2) - 1(-2) = 1(0) - 4 + 2$$

$$= -2 \neq 0.$$

$$P^{-1} = \frac{1}{|P|} \text{Adj } P \longrightarrow (2)$$

Adj P = Transpose of Cofactors matrix.

$$= \begin{pmatrix} 0 & 2 & -2 \\ 2 & 2 & -2 \\ -1 & 0 & -1 \end{pmatrix}$$

$$\text{Adj } P = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$$

$$P^{-1} = \frac{-1}{2}\begin{pmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$$

$$P^{-1}A = \frac{-1}{2}\begin{pmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

$$P^{-1}A = \frac{-1}{2}\begin{pmatrix} 0 & 2 & -1 \\ 4 & 4 & 0 \\ -6 & -6 & -3 \end{pmatrix}$$

$$(P^{-1}A)P = \frac{-1}{2}\begin{pmatrix} 0 & 2 & -1 \\ 4 & 4 & 0 \\ -6 & -6 & -3 \end{pmatrix}\begin{pmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$= \frac{-1}{2}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D_{\prime\prime}$$

$$A^4 = P D^4 P^{-1} = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{pmatrix}P^{-1}.$$

$$A^4 = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{pmatrix} \cdot \frac{-1}{2}\begin{pmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(P.T.O)

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 162 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$