UNIT-I

Mathematical logic

Introduction:

Logic is a science dealing with the methods of resining logic which uses a symbolic language to express it's principles in presins unambiguous term is known as mathematical logic.

Mathematical logic has become an acceptance part of discrete mathematics. one season for that all efforts at the verification of algorithms in eritely the notation and methods of logic.

propositions:

A number of words making compleatic grammetical stucture having a sence and meaning also meant and assertion in logic (or) mathematics is called a sentence. This assertion may be in two Types.

1. Declarative
2. Non Declarative

Def:

A Propotion (or) statement is declarative sentence that is true; (or) false.

Ex: $3+3=6 \longrightarrow$ True

$3+3=7 \longrightarrow$ False

⇒ similarly $x+y>1$ is not a statement because for some values of x & y the sentence is true, where as for others it is false.

⇒ For instance $x=1$ & $y=2$. The sentence is True.

⇒ If $x=-2$, $y=-1$ then the sentence is false.

⇒ Question, exclamations (or) commands are not statement.

Eg: The sun rises at west → False

The sun rises at East → True

$5+6>1$ → True $(5,6) \subset (5,6,7)$ → True

$-2-1>1$ → False $(2,3) \underset{\text{sub set}}{\subset} (7,8,9)$ → False.

Do you speak English? is a question was not declarative sentence is not a statement.

close the door → command $y-x=5$

Is not a declarative not a statement since it is true (or) false depending value of x.

NOTE:

→ Simple statements are represented by letters p, q, r called as propositional constants T & F

→ That represents true and false respectively.

compound proposition:

* A proposition consisting of only a single propositional variable are a single propositional constant is called "atomic (primitive, primary) propositional constant is called (or) simply proposition that is they cannot be further subdivided.

* A proposition obtained from the combination two (or) more proposition by means of logical apparators (or) connectives of two (or) more prepositions (or) by negating a single proposition is reflected to molecular (or) compound proposition.

Connectives :

The words and symbols are used to form compound sentence propositions are called connectives

* There are five connectives ; they are

* Negation
* conjunction
* Disjunction
* conditional (or) Implication
* Bi - conditional (or) Equivalence.

Negation :

The Negation of a statement is generally formed by introducing the word 'Not' at a proper place in the statement an by free firing the statement with the phare.

=> It is not in the case that

=> If 'p' denotes a statement then negation of p can be written as '¬p' and read as not p.

=> If the Truth Value of 'P' is true, then the Truth value of '¬p' is false. Also if truth, then of 'p' is false, Then the Truth value of '¬p' is True.

Eg: If P: London is a city

Then ¬P : It is not in case that London is a city.

In General form : London is not a city.

TRUTH TABLE

P	¬P
T	F
F	T

Note: Negation is also denoted as $\sim P$, $\bar{P}$, Not P.

* Negation is a unary operation which operates on a single statement (or) single variable.

Conjunction:

The conjunction of two statements P, Q are the statements P & Q which is read as P and Q. $(P \wedge Q)$

* The statement $P \wedge Q$ has the Truth value 'T' whenever both P & Q have Truth value 'T'. Otherwise it has truth value 'F'.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Eg:
Form the conjunction of

P: ⇒ It is raining today.
Q: ⇒ There are 20 tables in this room

$P \wedge Q$: It is raining today and there are 20 tables in this class

Disjunction (∨)

* The disjunction of two statements P and Q is the statement $P \vee Q$ which is read as "P" or "Q". This statement $P \vee Q$ has the truth value F only when both P and Q have the Truth value F. Otherwise it is T.

TRUTH TABLE

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional (or) Implication ($P \to Q$)

If P and Q are any two statements, then the statement $P \to Q$ which is read "If P, then Q" is called a conditional statement.

* The statement $P \to Q$ has a truth value False when P has the Truth value T and Q has the Truth value F. Otherwise it has the Truth value T.

* In $P \to Q$, P is called antecedent and Q is the consequent.

TRUTH TABLE

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex:-

* Express in on english the statement $P \to Q$ where

P: The sun is shining today

Q: $2+7>4$

sol Given

P: The sun is shining today

Q: $2+7>4$

$P \to Q$: If the sun is shining today, then $2+7>4$.

* write the following statement in symbolic form.

If either Jerry takes calculs or Kari takes socialajy, then Larry will take english.

sol Denotating the statements as

J : Jerry takes calculs

K : Kari takes socialajy

L : Larry takes english

The given statement can be symbollshed as

$(J \vee K) \rightarrow L$.

* write a symbolic form the statement

* The crop will be destroyed If there is a flood.

sol Denotating the statements as

c : The crop will be destroyed

f : ~~If~~ there is a flod.

Rewrite the given statement as

If there is a flod, then the crop will be destroyed.

above statement in symbolic form as

$f \rightarrow c$.

Bi-conditional

If P and Q are any two statements, Then the statement $P \rightleftarrows Q$. which is read as P if and only if Q (or) P iff Q → Abbrivated. is called a Bi-conditional statement.

* The statement $P \rightleftarrows Q$ has the truth value True when both statements P and Q have ideantical Truth values.

P	Q	$P \rightleftarrows Q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex:

* construct the Truth table for the statement formula $P \vee \neg Q$

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

* $(P \vee Q) \vee \neg P$

P	Q	$P \vee Q$	$\neg P$	$(P \vee Q) \vee \neg P$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

* $\neg(\neg P \vee \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\sim(\neg P \vee \neg Q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

* $P \wedge \neg P$

~~P ¬P P∧¬P~~

~~T F~~
~~T F~~
~~F F~~
F F

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

* $P \wedge (P \vee Q)$

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

* $\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R)$

P Q R $\neg P$ $\neg Q$ $\neg Q \wedge R$ $\neg P \wedge (\neg Q \wedge R)$

$Q \wedge R$ $P \wedge R$ $\neg P \wedge (\neg Q \wedge R) \vee$

P	Q	R	$\neg P$	$\neg Q$	$\neg Q \wedge R$	$\neg P \wedge (\neg Q \wedge R)$	$Q \wedge R$	$P \wedge R$	$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R)$
T	T	T	F	F	F	F	T	T	T
T	T	F	F	F	F	F	F	F	F
T	F	T	F	T	T	F	F	T	T
T	F	F	F	T	F	F	F	F	F
F	T	T	T	F	F	F	T	F	T
F	T	F	T	F	F	F	F	F	F
F	F	T	T	T	T	T	F	F	F
F	F	F	T	T	F	F	F	F	F

Question

T	T
F	F
T	F
F	F

* $(P \to Q) \land (Q \to P)$

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \land (Q \to P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

* $\neg(P \land Q) \rightleftarrows (\neg P \lor \neg Q)$

P	Q	$P \land Q$	$\sim(P \land Q)$	$\neg P$	$\neg Q$	$\neg P \lor \neg Q$	Question
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

* $\neg(P \lor (Q \land R)) \rightleftarrows ((P \lor Q) \land (P \lor R))$

P	Q	R	$Q \wedge R$	$P \vee Q$	$P \vee R$	$P \vee (Q \wedge R)$	$\cancel{\neg(P \vee Q)}$ $\neg(P \vee (Q \wedge R))$	$(P \vee Q) \wedge (P \vee R)$	Question
T	T	T	T	T	T	T	F	T	F
T	T	F	F	T	T	T	F	T	F
T	F	T	F	T	T	T	F	T	F
T	F	F	F	T	T	T	F	T	F
F	T	T	T	T	T	T	F	T	F
F	T	F	F	F	F	F	T	F	F
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	F	F	T	F	F

Well formed Formulas (w.f.f)

* A well formed formula can be generated by the following rules.

* A statement variable standing alone is a well formed formula.

* If 'A' is a wff, Then its negation is also wff.

* If 'A' and 'B' are w.f.f, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \rightleftarrows B)$ are w.f.f.

* A string of symbols containing the statement variables, conentives and parenthesis is a wff

* The following are well formed formulas.

$$\neg(P \wedge Q),\ \neg(P \vee Q),\ (P \to (P \vee Q))$$

* The following are not wellformed formulas

$P \wedge Q$, $P \vee Q$, obevisoly P and Q are w.f.f a well formed formula would be either $(\neg P \wedge Q)$ or $\neg(P \wedge Q)$.

* without Parenthesis it is a well formed formula.

* $(P \to Q) \to (\wedge Q)$ is not a w.f.f

Tautology

* A compound proposition i.e always True for all possible Truth values of its variables.

(or)

* In otherwords contain only True in the last column of its Truth Table is called a Tautology

contradiction

* A compound proposition i.e always False for all possible Truth Tables of its variables. (or) In otherwords contain only False in the last column of its Truth table is called contradicion.

contingency

* A proposition i.e neither a tautology nor a contradicion is called a contingency.

Equivalency of formulas

* Let A, B are two statement formulas and let $P_1, P_2, \ldots P_n$ denotes all the variables occured in both A and B. consider an assignment of truth values to $P_1, P_2, \ldots P_n$ and the resultanting truth values of A and B.

* If the Truth value of A is equal to the truth value of B for every one of 2^n possible sets of truth values assigned to $P_1, P_2, \cdots P_n$, Then A and B are said to be "equivalent".

* Assuming that the variables and the assignment of truth values, to the variables appeared in the same order in Truth tables of A and B, Then the final columns in the Truth tables for A and B are identical If A and B are equivalent.

Ex:-

* $\sim\sim P \Leftrightarrow P$
* $P \vee P \Leftrightarrow P$
* $P \vee \neg P \Leftrightarrow Q \vee \neg Q$

* We shall represent the equivalence of two formulas are same A and B by writing $A \Leftrightarrow B$.

which is read as A is equivalent to B.

NOTE :-

* If A and B are equivalent, then $A \Leftrightarrow B$ is a Tautology

* prove that $(P \to Q) \Leftrightarrow (\neg P \vee Q)$

P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$	$P \to Q \Leftrightarrow (\neg P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

* The truths values in the column for $P \to Q$ and $\neg P \vee Q$ are equivalent (or) Identical. so the bi-conditional will have the truth value True. Therefore $(P \to Q) \Leftrightarrow (\neg P \vee Q)$.

Equivalent formulas

1. $P \vee P \Leftrightarrow P$, $P \wedge P \Leftrightarrow P$ } Idempotent laws.

2. $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$, $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ } Associative law.

3. $P \vee Q \Leftrightarrow Q \vee P$, $P \wedge Q \Leftrightarrow Q \wedge P$ } commutative laws.

4. $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$, $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ } Distributive laws.

5. $P \vee F \Leftrightarrow P$, $P \wedge T \Leftrightarrow P$

6. $P \vee T \Leftrightarrow T$, $P \wedge F \Leftrightarrow F$

7. $P \vee \neg P \Leftrightarrow T$, $P \wedge \neg P \Leftrightarrow F$

8. $P \vee (P \wedge Q) \Leftrightarrow P$
$P \wedge (P \vee Q) \Leftrightarrow P$ } Absorption law.

9. $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ } Demorgan's laws

* show that $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \to R$.

sol Given $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R)$

$(\because P \to Q \Leftrightarrow \neg P \vee Q)$

$\Leftrightarrow \neg P \vee (\neg Q \vee R)$

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ (By associative law)

$\Leftrightarrow \neg(P \wedge Q) \vee R$ (By using Demorgan's law)

$\Leftrightarrow (P \wedge Q) \to R$

* show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

sol Given

$\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow (\neg P \wedge (\neg Q \wedge R) \vee ((Q \vee P) \wedge R)$ (Distributive law)

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ (Associative law)

$\Leftrightarrow \neg(P \vee Q) \wedge R) \vee ((Q \vee P) \wedge R)$ (Demorgan's law)

$\Leftrightarrow \neg(P \vee Q) \wedge R) \vee ((P \vee Q) \wedge R)$ (commulative law)

$\Leftrightarrow \underline{\neg((P \vee Q) \vee (P \vee Q))} \wedge R$ (Distributive law)

$\Leftrightarrow T \wedge R$. $(P \vee \neg P \Leftrightarrow T)$

$\Leftrightarrow R$

Duality Law

* Two formulas A and A^* are said to be dual of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives and $\wedge$ and $\vee$ are also called duals of each other.

* If the formula A contains the special variables T and F, Then A^* its duals is obtained by replacing T by F and F by T.

EX:-

* write duals of

i. $(P \vee Q) \wedge R$

Here $A = (P \vee Q) \wedge R$

$A^* = (P \wedge Q) \vee R$

ii. $(P \wedge Q) \vee T$

Here $A = (P \wedge Q) \vee T$

$A^* = (P \vee Q) \wedge T$

iii. $\neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg S))$

NOTE:

* dual of the dual is primal.

* If any two formulaes are equivalent then the duals also equivalent to each other.

i.e If $A \Leftrightarrow B$, then $A^* \Leftrightarrow B^*$

* write an equivalent formula for $P \wedge (Q \rightleftarrows R) \vee (R \rightleftarrows P)$. which does not contain the biconditional.

$P \rightleftarrows Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

Sol Given

$P \wedge (Q \rightleftarrows R) \vee (R \rightleftarrows P) \Leftrightarrow$

$P \wedge ((Q \to R) \wedge (R \to Q)) \vee ((R \to P) \wedge (P \to R))$

* write an equivalent formula for $P \wedge (Q \rightleftarrows R)$ which contains neither biconditional nor conditionals.

Sol Given $P \wedge (Q \rightleftarrows R) \Leftrightarrow P \wedge ((Q \to R) \wedge (R \to Q))$

$\Leftrightarrow P \wedge ((\neg Q \vee R) \wedge (\neg R \vee Q))$

$[\because P \to Q \Leftrightarrow \neg P \vee Q]$

Other connectives

Exclusive OR :-

* Let P and Q be any two formulas then the formula $P \bar{\vee} Q$, in which the connective $\bar{\vee}$ is called an "Exclusive OR". which is T whenever either P or Q is T but not both. The Exclusive OR is called Exclusive disjunction.

Properities

i. $P \bar{\vee} Q \Leftrightarrow Q \vee P$ (commutative law or symmetry)

ii. $(P \bar{\vee} Q) \bar{\vee} R \Leftrightarrow P \bar{\vee} (Q \bar{\vee} R)$ (Associative)

iii. $P \wedge (Q \bar{\vee} R) \Leftrightarrow (P \wedge Q) \bar{\vee} (P \wedge R)$ (Distributive)

iv. $P \bar{\vee} Q \Leftrightarrow \neg (P \rightleftarrows Q)$

TRUTH TABLE

P	Q	$P \bar{\vee} Q$
T	T	F
T	F	T
F	T	T
F	F	F

Note:

* The exclusive OR is ~~equal~~ to negation of biconditional.

NAND

* The word NAND is a combination of "Not" and "AND" where "NOT" standars for negation and "AND" standards for conjunction. The connective "NAND" is denoted by symbol "↑". For any two formulas P and Q is $P \uparrow Q \Leftrightarrow \neg(P \wedge Q)$

TRUTH Table

P	Q	$P \wedge Q$	$P \uparrow Q$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

NOR

* The word "NOR" is a combination "NOT" and "OR" where NOT standars for negation and "OR" standard for disjunction. The Connective is "NOR" is denoted by symbol (↓). For any two formulas P and Q is

$$P \downarrow Q \Leftrightarrow \neg(P \vee Q)$$

TRUTH TABLE

P	Q	$P \vee Q$	$P \downarrow Q$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Tautological Implication:

* The connectives $\land$, $\lor$ and $\rightleftarrows$ are symmetric in the scence that

$$P \land Q \Leftrightarrow Q \land P$$
$$P \lor Q \Leftrightarrow Q \lor P$$
$$P \rightleftarrows Q \Leftrightarrow Q \rightleftarrows P$$

* On the otherhand $P \to Q$ is not equivalent to $Q \to P$.
* For any statement formula $P \to Q$ that statement formula $Q \to P$ is called its converse, $\neg P \to \neg Q$ is its inverse and $\neg Q \to \neg P$ is called its contra-positive

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

From the Truth Table it is clear that

$$P \to Q \Leftrightarrow \neg Q \to \neg P$$
$$Q \to P \Leftrightarrow \neg P \to \neg Q$$

Normal Forms

* Elementary Product
* Let P and Q be any two atomic variables, a conjunction (product) of the variables and their negation in a formula is called an elementary product.

Examples: $\neg P \wedge Q$, $P \wedge Q$, $P \wedge \neg Q$, $\neg Q \wedge \neg P \wedge \neg P$.

Elementary sum (Disjunction(sum)).

* Disjunction of the variables and their negation is called elementary sum.

examples: $\neg P \vee Q$, $P \vee Q$, $P \vee \neg Q$,

Factors:-

* Any part of an elementary sum or product which is itself an elementary sum or product is called a factor of the original elementary sum (or) product.

ex:-

* $\neg Q$, $P \wedge \neg P$ and $\neg Q \wedge P$ are sum of factors $\neg Q \wedge \neg P \wedge \neg P$

NOTE:-

* A neccessary and sufficient condition for an elementary product to be identically false is that it contains atleast one pair of factors in which one is the negation of the other.

* A neccessary and sufficient condition for an elementary sum to be identically TRUE is that it contains atleast one pair of factors in which one is the negation of the other.

* We know that for any variable P, $P \wedge \neg P$ is identically False, Hence if $P \wedge \neg P$ appears in the elementary Product, Then the product is identically False.

Disjunctive Normal Form

* A Formula which is equivalent to a given formula and which consists of a "sum of elementary products" is called a disjunctive normal form of a given formula.

* obtain disjunctive normal forms of

a. $P \wedge (P \rightarrow Q)$

b. $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

a. $P \wedge (P \rightarrow Q)$

Sol Given $P \wedge (P \rightarrow Q) \Leftrightarrow$ $[\because P \rightarrow Q \Leftrightarrow \neg P \vee Q]$

$\Leftrightarrow P \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q) \longrightarrow$ D.N.F

b. $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$\because R \rightleftarrows S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)$

Sol Given $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$\because R \rightleftarrows S \Leftrightarrow (R \rightarrow S) \wedge (S \rightarrow R)$

$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q))$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q))$

which the required D.N.F.

Conjunctive Normal form

* A Formula which is equivalent to a given formula and which consists of "product of elementary sum" is called conjunctive normal forms of a given formula.

* obtain the conjunctive normal form :-

① $P \wedge (P \rightarrow Q)$

Sol Given $P \wedge (P \rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q) \rightarrow$ C.N.F

② $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$ $[\because R \rightleftarrows S \Leftrightarrow (R \rightarrow S) \wedge (S \rightarrow R)]$

Sol Given $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$\Leftrightarrow (\neg(P \vee Q) \rightarrow P \wedge Q)) \wedge (P \wedge Q \rightarrow \neg(P \vee Q))$

$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee \neg(P \vee Q))$

$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q))$

$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge ((\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q))$

which is the required CNF.

* show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a Tautology.

Sol Given $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

$\Leftrightarrow Q \vee ((P \vee \neg P) \wedge \neg Q))$

$\Leftrightarrow (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)$

$\Leftrightarrow T \wedge T$

$\Leftrightarrow T$

Since each of the elementary is a Tautology, The given formula is Tautology.

Minterms:-

Let P and Q be two statement variables. Then $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$ are called minterms of P and Q.

* It may be noted that none of the minterms should contain both a variables and its negation.

* For given 2 variables there are $2^2 = 4$ minterms

The no. of minterms in n variables is $\underline{2^n}$.

P	Q	minterms
T	T	$P \wedge Q$
T	F	$P \wedge \neg Q$
F	T	$\neg P \wedge Q$
F	F	$\neg P \wedge \neg Q$

* For 3 variables $\rightarrow 2^3 = 8$

P	Q	R	minterms
T	T	T	$P \wedge Q \wedge R$
T	T	F	$P \wedge Q \wedge \neg R$
T	F	T	$P \wedge \neg Q \wedge R$
T	F	F	$P \wedge \neg Q \wedge \neg R$
F	T	T	$\neg P \wedge Q \wedge R$
F	T	F	$\neg P \wedge Q \wedge \neg R$
F	F	T	$\neg P \wedge \neg Q \wedge R$
F	F	F	$\neg P \wedge \neg Q \wedge \neg R$

Principle Disjunctive Normal Forms (PDNF)

* A Formula which is equivalent to a given formula and which consisting of disjunction of minterms only, This is also called "sum of the products canonical form".

Process for obtaining PDNF

① By using Truth Table

* construct the Truth Table of the given compound propositions.

* For every Truth value T for (given propositions in the final column) collect minterms which also have the values 'T' for the same combination of the Truth value of the statement variables.

* If p has the Truth value T, we get take p only

* If p has the Truth value F, we take negation p.

* The disjunction of the minterms selected in step-2 is the required PDNF.

② Without constructing Truth Tables

* obtain the DNF

* Drop elementary products which are contradictions, such as $p \wedge \neg p$.

* If P_i and $\sim P_i$ are missing in a elementary product 'A', Replace A by $(A \wedge P_i) \vee (A \wedge \neg P_i)$

* Repeat set-3 until all elementary product are reduced to sum of minterms. Identical minterms appearing in the disjunction are deleted.

Advantages of PDNF

* The PDNF of a given formula is unique.
* Two formulas are equivalent if and only if the PDNF coincides.
* If the given compound proposition is a tautology then its PDNF will contain all possible minterms of its components.

* obtain the PDNF of the given formulas.

① $P \to Q$

sol

P	Q	$P \to Q$	minterms
T	T	T	$P \wedge Q$
T	F	F	$P \wedge \neg Q$
F	T	T	$\neg P \wedge Q$
F	F	T	$\neg P \wedge \neg Q$

The PDNF consists of the minterms of $P \to Q$ are $P \wedge Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$

Then condition of PDNF

$(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

② $\neg(P \wedge Q)$

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	minterms
T	T	T	F	$P \wedge Q$
T	F	F	T	$P \wedge \neg Q$
F	T	F	T	$\neg P \wedge Q$
F	F	F	T	$\neg P \wedge \neg Q$

$\neg(P \wedge Q)$ has minterms are $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$

Here we have '3' of T's have.

The condition for PDN of given condition

$\sim(P \wedge Q)$ is $(P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

iii. <u>$P \vee Q$</u>

P	Q	$P \vee Q$	minterms
T	T	T	$P \wedge Q$
T	F	T	$P \wedge \neg Q$
F	T	T	$\neg P \wedge Q$
F	F	F	$\neg P \wedge \neg Q$

minterms are $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$

The condition for PDNF $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

* obtain the PDNF of

(i) $\neg P \vee Q$

sol Given $\neg P \vee Q \Leftrightarrow (\neg P \wedge T) \vee (Q \wedge T)$

$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

take single

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \vee (\neg P \wedge Q)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

(ii). $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

sol Given $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\Leftrightarrow ((P \wedge Q) \wedge T) \vee ((\neg P \wedge R) \wedge T) \vee ((Q \wedge R) \wedge T)$

$\Leftrightarrow ((P \wedge Q) \wedge (R \vee \neg R)) \vee ((\neg P \wedge R) \wedge (Q \vee \neg Q)) \vee ((Q \wedge R) \wedge (P \vee \neg P))$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q) \vee$

$(Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$

$\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge Q\wedge \neg R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge R)\vee$

$(P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge R)$

$\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge R)$ $\longrightarrow$ PDNF

* show that the following are equivalent formulas

ⓐ $P\vee(P\wedge Q)\Leftrightarrow P$

sol Given $\overset{LHS}{P\vee(P\wedge Q)}$

$\Leftrightarrow (P\wedge T)\vee(P\wedge Q)$

$\Leftrightarrow P\wedge(Q\vee\neg Q))\vee(P\wedge Q)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)\vee(P\wedge Q)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)$

R.H.S. $P\wedge T$

$\Leftrightarrow P\wedge(Q\vee\neg Q)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)$

$\boxed{LHS = RHS}$

ⓑ $P\vee(\neg P\wedge Q)\Leftrightarrow P\vee Q$

sol Given LHS $= P\vee(\neg P\wedge Q)$

$\Leftrightarrow (P\wedge T)\vee(\neg P\wedge Q)$

$\Leftrightarrow P\wedge(Q\vee\neg Q)\vee(\neg P\wedge Q)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)\vee(\neg P\wedge Q)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)\vee(\neg P\wedge Q)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)\vee(\neg P\wedge Q)$

RHS $= P\vee Q$

$\Leftrightarrow (P\wedge T)\vee(Q\wedge T)$

$\Leftrightarrow P\wedge(Q\vee\neg Q)\vee(Q\wedge(P\vee\neg P))$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)\vee(Q\wedge P)\vee(Q\wedge\neg P)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)\vee(P\wedge Q)\vee(\neg P\wedge Q)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)\vee(\neg P\wedge Q)$

LHS = RHS

* obtain the P.D.N.F of

$$P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$$

sol Given

$$P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$$

$\Leftrightarrow P \to ((\neg P \vee Q) \wedge (Q \wedge P))$

$\Leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P))$

$\Leftrightarrow \neg P \vee ((\neg P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P))$

$\Leftrightarrow \neg P \vee F \vee (Q \wedge P)$

$\Leftrightarrow \neg P \vee (Q \wedge P)$ $\longrightarrow$ P.N.F $[\because P \vee F \Leftrightarrow P]$

$\Leftrightarrow (\neg P \wedge T) \vee (Q \wedge P)$

$\Leftrightarrow \neg P \wedge (Q \vee \neg Q) \vee (Q \wedge P)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P)$ $\longrightarrow$ P.D.N.F

Max terms

* Let P and Q be two statement variables, then $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$, $\neg P \vee \neg Q$ are called max terms of P and Q. It may be noted that none of the max terms should contain both a variable and its negation.
* For given two variables, there are $2^2 = 4$ max-terms. The no. of max terms in n variables $= 2^n$.
* For 3 variable P, Q, R, we have $2^3 = 8$ max terms. There are $P \vee Q \vee R$, $P \vee Q \vee \neg R$, $P \vee \neg Q \vee R$, $P \vee \neg Q \vee \neg R$, $\neg P \vee Q \vee R$, $\neg P \vee Q \vee \neg R$, $\neg P \vee \neg Q \vee R$, $\neg P \vee \neg Q \vee \neg R$

Principle Conjunctive Normal forms (PCNF)

* A formula which is equivalent to a given formula and which consisting of conjunction of maxterms only. This is also called the product of sum canincal form.

Process of obtaining PCNF

* By using Truth table

* construct a truth table for a given compound proposition.

* For every Truth value 'F' of the given proposition select the max term, which also has the Truth value 'F' for the same combination of the Truth value of the statement variables.

* If p has the Truth value T, we take $\neg p$

* If p has the Truth value F, we take p

$$P \to T \Rightarrow \neg P$$
$$P \to F \Rightarrow P$$

* The conjunction of the maxterms selected in step-2 is the required PCNF.

1. obtain the PCNF for the formulas

(i) $P \vee Q$

P	Q	$P \vee Q$	maxterms
T	T	T	$\neg P \vee \neg Q$
T	F	T	$\neg P \vee Q$
F	T	T	$P \vee \neg Q$
F	F	F	$P \vee Q$

The PCNF of $P \vee Q$ is $P \vee Q$

(ii) $P \to Q$

P	Q	$P \to Q$	maxterms
T	T	T	$\neg P \vee \neg Q$
T	F	F	$\neg P \vee Q$
F	T	T	$P \vee \neg Q$
F	F	T	$P \vee Q$

The PCNF of $P \to Q$ is $\neg P \vee Q$

(iii) $\neg(P \wedge Q)$

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	max term,
T	T	T	F	$\neg P \vee \neg Q$
T	F	F	T	$\neg P \vee Q$
F	T	F	T	$P \vee \neg Q$
F	F	F	T	$P \vee Q$

The PCNF of $\neg(P \wedge Q)$ is $\neg P \vee \neg Q$

* Find PCNF of $(\neg P \to R) \wedge (Q \rightleftarrows P)$

Sol Given $(\neg P \to R) \wedge (Q \rightleftarrows P)$

$\Leftrightarrow (P \vee R) \wedge ((P \to Q) \wedge (Q \to P))$

$\Leftrightarrow (P \vee R) \wedge ((\neg P \vee Q) \wedge (\neg Q \vee P))$

$\Leftrightarrow (P \vee R) \wedge (\neg P \vee Q) \wedge (\neg Q \vee P) \to$ C.N.F

$\Leftrightarrow ((P \vee R) \vee F) \wedge ((\neg P \vee Q) \vee F) \wedge (\neg Q \vee P \vee F)$

$\Leftrightarrow ((P \vee R) \vee (Q \wedge \neg Q)) \wedge ((\neg P \vee Q) \vee (R \wedge \neg R)) \wedge$
$(\neg Q \vee P \vee (R \wedge \neg R))$

$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg P \vee Q \vee R) \wedge$
$(\neg P \vee Q \vee \neg R) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R)$

$\Leftrightarrow (P \vee Q \vee R) \wedge \underline{(P \vee \neg Q \vee R)} \wedge (\neg P \vee Q \vee R) \wedge$
$(\neg P \vee Q \vee \neg R) \wedge \underline{(P \vee \neg Q \vee R)} \wedge (\neg P \vee Q \vee R)$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \wedge \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge$
$(\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$ of max term PCNF.

The theory of Inference for statement calculus

Inference theory

The main aim of logic is to provide rules of inference to infer a conclusion from certain premises.

* The theory associated with the rules of inference is known as Inference theory

valid argument (or) valid conclusion

* If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called "dedection" (or) "Formal proof" and the argument or conclusion is called valid argument (or) valid conclusion.

Note:

* premises are also called as assumption (or) axioms (or) hypothesis

Methods used to determine the conclusion

1. using Truth table method

Let $P_1, P_2, \ldots P_n$ be the variables appearing in the premises $[H_1, H_2, \ldots H_m]$ and the conclusion C

* Look for the rows in which all $H_1, H_2, H_3 \ldots H_m$ have the value 'T', if every such row 'C' also has the value 'T'. Then $H_1 \wedge H_2 \wedge H_3, \ldots \wedge H_m \Rightarrow C$.

* Look for the rows in which 'C' has the value F, if in every such row atleast one of the values of $H_1, H_2 \ldots H_m$ is F. Then $H_1 \wedge H_2 \wedge H_3 \ldots H_m \Rightarrow C$

* Determine whether the conclusion 'C' follow logically from the premises H_1 and H_2.

i. $H_1: P \to Q$, $H_2: P$ $\quad C: Q$

Sol Given $H_1: P \to Q$, $H_2: P$

P	Q	$H_1: P\to Q$	$H_2: P$	$H_1 \wedge H_2$	$C: Q$	$H_1 \wedge H_2 \to C$
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	F	F	F	T

* $H_1 \wedge H_2 \to C$ is Tautology.

$\therefore H_1 \wedge H_2 \to C$. 'C' is a valid conclusion

ii. $H_1: P \to Q$ $\quad H_2: \neg P$ $\quad C: Q$

P	Q	$H_1: P\to Q$	$H_2: \neg P$	$H_1 \wedge H_2$	$C: Q$	$H_1 \wedge H_2 \to C$
T	T	T	F	F	T	T
T	F	F	F	F	F	T
F	T	T	T	T	T	T
F	F	T	T	T	F	F

C is not a valid conclusion. (or) invalid

iii. $H_1: P \to Q$ $\quad H_2: \neg(P \wedge Q)$ $\quad C: \neg P$

P	Q	$H_1: P\to Q$	$H_2: \neg(P\wedge Q)$	$H_1 \wedge H_2$	$C: \neg P$	$H_1 \wedge H_2 \to C$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	F	T	T	T	T	T

$H_1 \wedge H_2 \to C$ is Tautology,

$H_1 \wedge H_2 \to C$. 'C' is a valid Conclusion

iv. $H_1: \neg P$ $H_2: P \rightleftarrows Q$ $C: \neg(P \wedge Q)$

P	Q	$H_1: \neg P$	$H_2: P \rightleftarrows Q$	$H_1 \wedge H_2$	$P \wedge Q$	$C: \neg(P \wedge Q)$	$H_1 \wedge H_2 \rightarrow C$
T	T	F	T	F	T	F	T
T	F	F	F	F	F	T	T
F	T	T	F	F	F	T	T
F	F	T	T	T	F	T	T

$H_1 \wedge H_2 \rightarrow C$ is Tautology

∴ 'C' is a valid ~~functio~~ conclusion.

V. $H_1: P \rightarrow Q$ $H_2: Q$ $C: P$

P	Q	$H_1: P \rightarrow Q$	$H_2: Q$	$H_1 \wedge H_2$	$C: P$	$H_1 \wedge H_2 \rightarrow C$
T	T	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	F	F	F	T

$H_1 \wedge H_2 \rightarrow C$ 'C' is not a valid conclusion,
(or) invalid

* Determine whether the conclusion C is valid in the following. when $H_1, H_2, H_3, \ldots H_m$ are the premises

(i) $H_1: P \rightarrow (Q \rightarrow R)$ $H_2: P \wedge Q$ $C: R$

(ii) $H_1: P \rightarrow Q$ $H_2: Q \rightarrow R$ $C: P \rightarrow R$

(i) Sol Given

$H_1: P \rightarrow (Q \rightarrow R)$ $H_2: P \wedge Q$ $C: R$

P	Q	R	$Q \to R$	$H_1: P \to (Q \to R)$	$H_2: P \land Q$	$H_1 \land H_2$	$C: R$	$H_1 \land H_2 \to$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	F	T	T
T	F	F	T	T	F	F	F	T
F	T	T	T	T	F	F	T	T
F	T	F	F	T	F	F	F	T
F	F	T	T	T	F	F	T	T
F	F	F	T	T	F	F	F	T

$H_1 \land H_2 \to C$ is a Tautology

$H_1 \land H_2 \to C$ is a valid conclusion.

(ii). $H_1: P \to Q$ $H_2: Q \to R$ $C: P \to R$

P	Q	R	$H_1: P \to Q$	$H_2: Q \to R$	$H_1 \land H_2$	$C: P \to R$	$H_1 \land H_2 \to C$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$H_1 \land H_2 \to C$ is a Tautology

$H_1 \land H_2 \to C$ is a valid conclusion.

iii. $H_1: P \to (Q \to R)$ $H_2: R$ $C: P$

iv. $H_1: P \lor Q$ $H_2: P \to R$ $H_3: Q \leftrightarrow R$ $C: R$

iii. $H_1: P \to (Q \to R)$ $\quad H_2: R$ $\quad C: P$

P	Q	R	$Q \to R$	$H_1: P \to (Q \to R)$	$H_2: R$	$H_1 \wedge H_2$	C	$H_1 \wedge H_2 \to C$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	F	T	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	F	F
F	T	F	F	T	F	F	F	T
F	F	T	T	T	T	T	F	F
F	F	F	T	T	F	F	F	T

$H_1 \wedge H_2 \to C$ is a invalid conclusion.

iv. $H_1: P \vee Q$ $\quad H_2: P \to R$ $\quad H_3: Q \to R$ $\quad C: R$

P	Q	R	$H_1: P \vee Q$	$H_2: P \to R$	$H_3: Q \to R$	$H_1 \wedge H_2 \wedge H_3$	$C: R$	$H_1 \wedge H_2 \wedge H_3 \to C$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F	T
F	F	T	F	T	T	F	T	T
F	F	F	F	T	T	F	F	T

$H_1 \wedge H_2 \wedge H_3 \to C$. is Tautology

$H_1 \wedge H_2 \wedge H_3 \to C$ is a valid conclusion.

Rules of Inference

Rule - P

* A premise may be introduce at any point in the derivation.

Rule - T

* A Formula 's' may be introduced in a derivation if 's' is tautologically incdied by any one or more of the preceding formula in the derivation.

Implications

* I_1: $P \wedge Q \Rightarrow P$ } simplification
* I_2: $P \wedge Q \Rightarrow Q$
* I_3: $P \Rightarrow P \vee Q$ } Addition
* I_4: $Q \Rightarrow P \vee Q$
* I_5: $\neg P \Rightarrow P \rightarrow Q$
* I_6: $Q \Rightarrow P \rightarrow Q$
* I_7: $\neg(P \rightarrow Q) \Rightarrow P$
* I_8: $\neg(P \rightarrow Q) \Rightarrow \neg Q$
* I_9: $P, Q \Rightarrow P \wedge Q$
* I_{10}: $\neg P, P \vee Q \Rightarrow Q$ (Disjunctive syllogism)
* I_{11}: $P, P \rightarrow Q \Rightarrow Q$ (modus ponens)
* I_{12}: $\neg Q, P \rightarrow Q \Rightarrow \neg P$ (modustollers)
* I_{13}: $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (hypothetical
* I_{14}: $P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ → Dilemma
* I_{15}: $(x)\, A(x) \vee (x)\, B(x) \Rightarrow (x)(A(x) \vee B(x))$
* I_{16}: $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)A(x) \wedge (\exists(x)\, B(x)$

Equivalencies

* E1: $\neg\neg P \Leftrightarrow P$
* E2: $P \land Q \Leftrightarrow Q \land P$
* E3: $P \lor Q \Leftrightarrow Q \lor P$

commutative laws

* E4: $(P \land Q) \land R \Leftrightarrow P \land (Q \land R)$
* E5: $(P \lor Q) \lor R \Leftrightarrow P \lor (Q \lor R)$

Associative laws.

* E6: $P \land (Q \lor R) \Leftrightarrow (P \land Q) \lor (P \land R)$
* E7: $P \lor (Q \land R) \Leftrightarrow (P \lor Q) \land (P \lor R)$

Distributive laws,

* E8: $\neg(P \land Q) \Leftrightarrow \neg P \lor \neg Q$
* E9: $\neg(P \lor Q) \Leftrightarrow \neg P \land \neg Q$

Demorgan

* E10: $P \lor P \Leftrightarrow P$
* E11: $P \land P \Leftrightarrow P$
* E12: $R \lor (P \land \neg P) \Leftrightarrow R$
* E13: $R \land (P \lor \neg P) \Leftrightarrow R$
* E14: $R \lor (P \lor \neg P) \Leftrightarrow T$
* E15: $R \land (P \land \neg P) \Leftrightarrow F$
* E16: $P \rightarrow Q \Leftrightarrow \neg P \lor Q$
* E17: $\neg(P \rightarrow Q) \Leftrightarrow P \land \neg Q$
* E18: $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
* E19: $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \land Q) \rightarrow R$
* E20: $\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$
* E21: $P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \land (Q \rightarrow P)$
* E22: $(P \rightleftarrows Q) \Leftrightarrow (P \land Q) \; (\neg P \land \neg Q)$

*1. Demonstrate that R is a valid inference from the premises $P \to Q$, $Q \to R$ & P

Sol Given premises

$P \to Q$, $Q \to R$ & P

{1} (1) $P \to Q$ Rule P

{2} (2) $Q \to R$ Rule P

{1,2} (3) $P \to R$ Rule T (1) & (2) from I_{13}.

{4} (4) P Rule P

{1,2,4} (5) R Rule T (3) & (4) from I_{11}.

2* Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ & $(A \wedge \neg B) \to (R \vee S)$

Sol Given premises

$C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$

$(A \wedge \neg B) \to (R \vee S)$

{1} (1) $(C \vee D) \to \neg H$, Rule P

{2} (2) $(\neg H \to (A \wedge \neg B))$ Rule P

{1,2} (3) $C \vee D \to (A \wedge \neg B)$ Rule T, (1), (2) from I_{13}

{4} (4) $(A \wedge \neg B) \to (R \vee S)$ Rule P

{1,2,4} (5) $(C \vee D) \to (R \vee S)$ Rule T (3), (4) from I_{13}

{6} (6) $C \vee D$ Rule P

{1,2,4,6} (7) $R \vee S$ Rule T (5), (6) from I_{11}

*3 show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

a) Given premises

$(P \vee Q)$, $(P \to R)$, $(Q \to S)$

{1}	(1)	$P \vee Q$	Rule P
{1}	(2)	$\neg P \to Q$	Rule T, (1) from E_{16} E_{19}
{3}	(3)	$Q \to S$	Rule P
{1,3}	(4)	$\neg P \to S$	Rule T, (2) (3). from P_{13}
{1,3}	(5)	$\neg S \to P$	Rule T (4) from E_{19} & E_{18}
{6}	(6)	$P \to R$	Rule P
{1,3,6}	(7)	$\neg S \to R$	Rule T (5),(6) from P_{13}
{1,3,6}	(8)	$S \vee R$	Rule T (7) from E_{16} & E_{19}.

4. show that $R \wedge (P \vee Q)$ is a valid conclusion from that premises $P \vee Q$, $Q \to R$, $P \to M$ & $\neg M$

a) Given premises

$P \vee Q$, $Q \to R$, $P \to M$ & $\neg M$

{1}	(1)	$P \to M$	Rule P
{2}	(2)	$\neg M$	Rule P
{1,2}	(3)	$\neg P$	Rule T (1),(2) from P_{12}
{4}	(4)	$P \vee Q$	Rule P
{1,2,4}	(5)	Q	Rule T, (3),(4) from P_{10}
{6}	(6)	$Q \to R$	Rule P
{1,4,4,6}	(7)	R	Rule T, (5),(6) from P_{11}

$\{1,2,4,6\}$ (8) $R \wedge (P \vee Q)$ Rule T (4) (7) from ?

5. Show that I_{12} $\neg Q, P \rightarrow Q \Rightarrow \neg P$

Sol Given premises

$\neg Q, P \rightarrow Q$

$\{1\}$ (1) $P \rightarrow Q$ Rule P

$\{1\}$ (2) $\neg Q \rightarrow \neg P$ Rule T (1) from E_{18}

$\{3\}$ (3) $\neg Q$ Rule P

$\{1,3\}$ (4) $\neg P$ Rule T (2), (3) from I_{11}.

Rule - CP (conditional proof)

* If we can derive S from R and a set of premises, then we can derive $R \rightarrow S$ from the set of premises alone.

* Rule-CP follows from the equivalence E_{19}, which states that $(P \wedge R) \rightarrow S \Leftrightarrow P \rightarrow (R \rightarrow S)$

* Let 'P' denote the conjunction of the set of premises and let 'R' be any formula the above equivalence states that if 'R' is included as an additional premise and 'S' is derived from $P \wedge R$. Let $R \rightarrow S$ can be derived from the premises 'P' alone.

Note

* Rule-CP is also called the deduction theorem and is generally used if the conclusion is of the form $R \rightarrow S$. In such cases 'R' is taken as an additional premise

and 's' is derived from the given premises and R.

1. show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$ & Q.

sol. Given $R \to S$

Given additional premise R

Given premises $P \to (Q \to S)$, $\neg R \vee P$, Q

{1}	(1)	R	Rule P (additional premise)
{2}	(2)	$\neg R \vee P$	Rule P
{2}	(3)	$R \to P$	Rule T (2) from E_{16}
{1,2}	(4)	P	Rule T (1),(3) from P_{11}
{5}	(5)	$P \to (Q \to S)$	Rule P
{1,2,5}	(6)	$Q \to S$	Rule T (4),(5) from P_{11}
{7}	(7)	Q	Rule P
{1,2,5,7}	(8)	S	Rule T, (6),(7) from P_{11}
{1,2,5,7}	(9)	$R \to S$	Rule - CP.

$\therefore$ $R \to S$ is a valid conclusion.

2. show that $P \to S$ can be derived from the premises $\neg P \vee R$, $\neg Q \vee R$, $R \to S$

sol. Given additional premise 'P'

Given premises $\neg P \vee R$, $\neg Q \vee R$, $R \to S$

$\{1\}$	(1)	P	Rule P (additional premise)
$\{2\}$	(2)	$\neg P \vee Q$	Rule P
$\{2\}$	(3)	$P \rightarrow Q$	Rule T (2) from E_{16}
$\{1,3\}$	(4)	Q	Rule T (1), (3) from I_{11}
$\{5\}$	(5)	$\neg Q \vee R$	Rule P
$\{5\}$	(6)	$Q \rightarrow R$	Rule (T) (5) from E_{16}
$\{1,3,5\}$	(7)	R	Rule (T) (4), (6) from I_{11}
$\{8\}$	(8)	$R \rightarrow S$	Rule P
$\{1,3,5,8\}$	(9)	S	Rule T (7), (8) from I_{11}
$\{1,3,5,8\}$	(10)	$P \rightarrow S$	Rule CP

$P \rightarrow S$ is a valid conclusion.

3. Show that $Q \rightarrow S$ can be derived from the premises P, $P \rightarrow (Q \rightarrow (R \wedge S))$

Sol Given Q is additional premise

Given premises P, $P \rightarrow (Q \rightarrow (R \wedge S))$.

$\{1\}$	(1)	P	Rule P
$\{2\}$	(2)	$P \rightarrow (Q \rightarrow (R \wedge S))$	Rule P
$\{1,2\}$	(3)	$Q \rightarrow (R \wedge S)$	Rule T, (1) (2) from I_{11}
$\{4\}$	(4)	Q	Rule P (additional premise)
$\{1,2,4\}$	(5)	$R \wedge S$	Rule T (3), (4) from I_{11}

$\{1,2,4\}$ (5) S Rule T, (5) from B_2

$\{1,2,4\}$ (5) $Q \to S$ Rule CP

$\therefore Q \to S$ is a valid premise.

4. Derive $P \to (Q \to S)$ using Rule - CP if necessary from the premise $P \to (Q \to R)$, $Q \to (R \to S)$

Sol Given additional premise P

Given premises are

$$P \to (Q \to R), \; Q \to (R \to S)$$

$\{1\}$ (1) P Rule P (additional premise)

$\{2\}$ (2) $P \to (Q \to R)$ Rule P

$\{1,2\}$ (3) $Q \to R$ Rule T (1),(2) from I_{11}

$\{1,2\}$ (4) $\neg Q \vee R$ Rule T (3) from E_{16}

$\{5\}$ (5) $Q \to (R \to S)$ Rule P

$\{5\}$ (6) $\neg Q \vee (\neg R \vee S)$ Rule T (5) from E_{16}

$\{1,2,5\}$ (7) $(\neg Q \vee R) \wedge (\neg Q \vee (\neg R \vee S))$ Rule T (4) (6) from I_9

$\neg Q \vee (R \wedge (\neg R \vee S))$

$(\neg Q \vee ((R \wedge \neg R) \vee S))$

$(\neg Q \vee (F \vee S))$

$(\neg Q \vee S)$

$\{1,2,5\}$ (8) $Q \to S$ Rule T (7) from E_{16}

$\{1,2,5\}$ (9) $P \to (Q \to S)$ Rule - CP

$P \to (Q \to S)$ is a valid conclusion.

consistency of premises and indirect method of ~~pr~~ ~~ ~~ proof

A set of formulas $h_1, h_2, \ldots h_m$ is set to be consistent if their conjunction of truth value 'T' for some assignment of the truth values to the atomic variables appearing in $H_1, H_2 \ldots H_m$.

* If for every assignment of the truth values to the atomic variables, atleast one of the formula $H_1, H_2, \ldots H_3$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, \ldots H_m$ are called inconsistent.

* Alternatively a set of formulas $H_1, H_2, \ldots H_m$ is inconsistent if their conjunction implies a contradaction. i.e $H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow R \wedge \neg R$ where R is any formula note that $R \wedge \neg R$ is a contradaction and it is neccessary and sufficient for the implication that $H_1 \wedge H_2 \wedge H_3, \ldots \wedge H_m$ is be a contradaction.

* The notion of inconsistency is used in a procedure called proof by contradection (or) indirect method of ~~proof~~ True.

1. show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Sol: we introduce $\neg\neg(P \wedge Q)$ as an additional premise and show that this additional premise leads to a contradection.

{1}	(1)	$\neg\neg(P \wedge Q)$	Rule P (additional premise)
{1}	(2)	$P \wedge Q$	Rule T, (1) from E_1

$\{1\}$ (3) P Rule T (2) from P_1

$\{4\}$ (4) $\neg P \wedge \neg Q$ Rule P

$\{4\}$ (5) $\neg P$ Rule T (4) from P_1

$\{1,4\}$ (6) $P \wedge \neg P$ Rule T (3),(5) from P_9
$\Downarrow$
contradiction

$\therefore$ The Given premises are inconsistent

2. Show that the following premises are inconsistent

* i. If Jack misses many class through illness then he fails High school.
ii. If Jack Fails High school, then he is uneducated.
iii. If Jack reads lots of books, then he is not uneducated.
iv. Jack misses many class through illness and reads a lot of books.

Sol Let us form simple statements.

E: Jack misses many class
S: Jack Fails high school
A: Jack reads a lot of books.
H: Jack misses many class through illness
Jack is uneducated.

Given premises
$E \rightarrow S,\ S \rightarrow H,\ A \rightarrow \neg H,\ E \wedge A$

$\{1\}$ (1) $E \rightarrow S$ Rule P

$\{2\}$ (2) $S \rightarrow H$ Rule P

$\{1,2\}$ (3) $E \rightarrow H$ Rule T (1),(2) from P_{13}.

$\{4\}$ (4) $A \rightarrow \neg H$ Rule P

$\{4\}$ (5) $H \to \neg A$ Rule T (4) from E_{18}

$\{1,2,4\}$ (6) $E \to \neg A$ Rule T (3), (5) from I_{13}

$\{1,2,4\}$ (7) $\neg E \vee \neg A$ Rule T (6) from E_{16}

$\{1,2,4\}$ (8) $\neg(E \wedge A)$ Rule T (7) from E_8

$\{9\}$ (9) $E \wedge A$ Rule P

$\{1,2,4,9\}$ (10) $(E \wedge A) \wedge \neg(E \wedge A)$ Rule T (8), (9) from I_9.

$\Downarrow$

contradiction.

$\therefore$ The given premises are inconsistent.

3. Using indirect method of proof derive $P \to \neg S$ from the premises $P \to Q \vee R$, $Q \to \neg P$, $S \to \neg R$ and P.

Sol. Given premises

$P \to Q \vee R$, $Q \to \neg P$, $S \to \neg R$

Given $P \to \neg S \Rightarrow \neg(P \to \neg S)$

$\Rightarrow \neg(\neg P \vee \neg S)$

$\Rightarrow P \wedge S$

$*$ $P \wedge S$ is the additional premise

$\{1\}$ (1) P Rule P

$\{2\}$ (2) $P \to Q \vee R$ Rule P

$\{1,2\}$ (3) $Q \vee R$ Rule T (1), (2) from I_{11}

$\{4\}$ (4) $S \to \neg R$ Rule P

$\{5\}$ (5) $P \wedge S$ Rule P (additional premise)

{5}	(6)	S	Rule T (5) from P_2
{4,5}	(7)	$\neg R$	Rule T (4)(6) from P_{11}
~~{8}~~	~~(8)~~	~~Q~~	
{1,2}	(8)	$R \vee Q$	Rule T (3) from E_3
{1,2,4,5}	(9)	Q	Rule T (7),(8) from P_{10}
{10}	(10)	$Q \to \neg P$	Rule P
{1,2,4,5,10}	(11)	$\neg P$	Rule T (9),(10) from P_{11}
{1,2,4,5,10}	(12)	$P \wedge \neg P$ ↓ contradiction	Rule T (1),(11) from P_9

∴ Given premises are inconsistent.

The predicate calculus

Predicate Logic

* The Logic based upon the analysis of predicates in any statement is called predicate Logic.

predicate

* Let us consider two statements.

i. John is a Bachlor

ii. Smith is a Bachlor

* we introduce some symbols to denote "is a Bachlor" and a method to join it with symbols denoting the names of the indiviuals, Then we have a symbolisim to denote statements about any invisuals being a Bachlor.

* The part "is a Bachlor" is called a predicate. Symbolically Denote the predicate "is a Bachlor"

by the predicate letter "B", John by "j" and Smith by "s".

* The statements i and ii can be written as $B(j)$ and $B(s)$ respectively. In general any statement of the type "p is Q" where "is Q" → predicate and "p" is subject (or) variable can be denoted by $Q(P)$,

* A predicate requires 'm' names is called a m-place predicate. so that statement "John is a Bachlor" is a one-place predicate

* consider the statement "Jack is Taller than Jill". Here "is taller than" is predicate and associated with two objects which is a Two-place predicate. Denoted by $G(j_1, j_2)$.

* The statement "Ravi sits between siva and Hari" is a Three-place predicate.

* In General and 'n-place predicate requires 'n' name of objects to be inserted in fixed positions in order to obtain a statement. If 'S' is an n-place predicate letter and $a_1, a_2, \ldots a_n$ are the names of objects than $S(a_1, a_2, \ldots a_n)$ is a statement.

The statement function, variables, quantifie

* Let "H" be predicate "is a mortal", b the "Mr Brown", c "Canada" and "s A shirt". Then $H(b)$, $H(c)$, $H(s)$ all denotes the statements. These statements have a common form. we can write $H(x)$ for "x is mortal" then $H(b)$, $H(c)$, $H(s)$ and others having the same form can be obtain from $H(x)$ by replacing

x by an appropriate name.

* Note that $H(x)$ is not a statement, but it results in a statement when 'x' is replaced by the name of the object and 'x' is a place holder.

* A statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable; such a statement function becomes a statement when the variable is replaced by the name of any object. The statement resulting from a replacement is called substitution instance of the statement function and is a formula of statement calculs.

Statement calculs

* We can combine two or more simple statement functions by using connectives $\neg, \wedge, \vee, \rightarrow$ & $\rightleftarrows$.

Ex:-

If $M(x)$ be "x is a man"

$H(x)$ be "x is a mortal."

* Then we can from compound statement function such as

i. $M(x) \wedge H(x)$

ii $M(x) \vee H(x)$

iii $M(x) \rightarrow H(x)$

iv $M(x) \rightleftarrows H(x)$ ... etc.

Universal Quantifiers

Let us consider the following statements

i. All men are mortal

ii. Every apple is red

iii. Any integer is either +ve or -ve

Now rewrite the above statement as follows.

i. For all x, if 'x' is man, then 'x' is a mortal
ii For all x, if 'x' is an apple, then x is red
iii For all x, if x is integer, then x is either +ve or −ve

We symbolize "for all x" by the symbol $(\forall x)$ or (x). with an understanding that the symbol be placed before the statement function to which the phrase is applied using

$M(x)$ x is man

$H(x)$: x is a mortal

$A(x)$: x is an apple.

$R(x)$: x is red

$N(x)$: x is an integer

$P(x)$: x is an either +ve or −ve

we write i, ii and iii as

i. $(x)\ (M(x) \rightarrow H(x))$

ii $(x)\ (A(x) \rightarrow R(x))$

iii $(x)\ (N(x) \rightarrow P(x))$

* The symbols $(\forall x)$ or (x) are called universal quantifiers.

* since quantifiers does not change the statement we can write

$$(x)\ (M(x) \rightarrow H(x)) \text{ and } (y)\ (M(y) \rightarrow H(y))$$

are equivalent sentences, it is neccessary to use more than one universal quantifier in a statement.

* For example $G(x, y)$: x is Taller Than y.
We can state that for any x and any y.

if x is Taller than y, Then y is not Taller than x or For any x and y, if x is Taller than y, Then it is not true that y is Taller than x. The statement can be symbolized as

$$(x)(y)\left(G(x,y)\right) \rightarrow \neg G(x,y))$$

Existential Quantifiers

* Another quantifier is introduced to symbolize expressions such as "for some," "Their is atleast one" (or) "There exists some".

* consider the following statements.

  i. There exist a man.

  ii. some men are clever

  iii. some real numbers are rational.

* The first two statements can be expressed in various ways, two such ways being

i. (a) There exist an x such that x is a men

(b) There is exist atleast one x such that x is a men

ii. (a) There exist an x. such that x is a men and x is a clever.

(b) There is exist atleast one x such that x is a men and x is a clever.

iii. (a) There exist an

Ex: Consider The statements

$M(x)$: x is a man

$C(x)$: x is a clever

$R_1(x)$: x is a Real number

$R_2(x)$: x is rational

Using the existential quantifier, we can write the statement i, ii and iii as i $(\exists x)\ (m(x))$

ii $(\exists x)\ (m(x) \land c(x))$

iii $(\exists (x))\ (R_1(x) \land R_2(x))$

Predicate Formulas

* We know that $P(x_1, x_2, \ldots x_n)$ denotes an n-place predicate formula in which the letter p is an n-place predicate and $x_1, x_2, \ldots x_n$ are indiviual variables.

* In General $p(x_1, x_2, \ldots x_n)$ will be called an atomic formula of predicate calculs. The following are some examples of atomic formulas.

R (for $n=0$)

$p(x)$ (for $n=1$)

$P(x,y)$ (for $n=2$)

$A(x,y,z)$ (for $n=3$)

Bounded occurance

* Given a formula consist a part of the form $(x)\ p(x)$ or $(\exists x)\ p(x)$ such a part is called an x-Bound part of the formula.

* Any occurance x in an x-Bound part of a formula is called Bounded occurence of x.

Free occurance

* Any occurance of x or any variable that is not bounded occurence is called free occurance

Scope of the quantifiers

The formula $P(x)$ either in $(x)\, P(x)$ or $(\exists x)\, P(x)$ is described as the scope of the quantifiers.

Ex:

i. $(x)\, P(x,y)$ where $P(x,y)$ is the scope of the quantifiers while the occurence of y is the free occurence.

ii. $(x)\, [P(x) \rightarrow Q(x)]$ Here the scope of the universal quantifiers is $P(x) \rightarrow Q(x)$ and all occurences of x are bounded.

iii. $(x)(P(x) \rightarrow (\exists y)\, R(x,y))$ Here the scope of for all x is $P(x) \rightarrow \exists (x)\, R(x,y)$ while the scope of there exist y is $R(x,y)$ and all occurences of both x and y are bounded occurences.

iv. $(x)\, (P(x) \rightarrow R(x)) \vee (x)\, (P(x) \rightarrow Q(x))$ Here the scope of first quantifier for all x is $P(x) \rightarrow R(x)$ and the scope of second quantifier of for all x is $P(x) \rightarrow Q(x)$.

v. $(\exists x)\, P(x) \wedge Q(x)$ Here the scope of there exist is $P(x)$, the occurence of x in $P(x)$ is bounded and occurence of x in $Q(x)$ is free occurence.

Universe of Discourse

* The way in which we substitude the values in an expression within a specified limit. That limit is called the universe of discourse (or) The domain of individuals or simply the universe.

Ex: If the disscussion refers to human beings only, Then the universe of discourse is the class of human beings.

* In elimantary algebra or number theory the universe of discourse could be numbers (real, complex, rational numbers ....etc),

Ex:- $Q(x)$: x is less than 5

a. $x\, Q(x)$ (b) $(\exists x)\, Q(x)$. If the universe of disscourse is given by the sets

(i) $\{-1, 0, 1, 2, 4\}$ (ii) $\{3, -2, 7, 8, 2\}$

(iii) $\{15, 20, 24\}$.

sol: (a) $Q(n)$ is True fo (i) and false for ii and iii. The statement $(\exists x)\, Q(x)$ is True for i and ii and False for iii.

If

1. The universe of discourse of x is $\{a, b, c\}$ Then write $(x)\, p(x)$ and $(\exists x)\, p(x)$

sol: Given universe of discourse is $\{a, b, c\}$

$$(x)\, P(x) \Leftrightarrow (P(a) \wedge P(b) \wedge P(c))$$

$$(\exists x)\, P(x) \Leftrightarrow (P(a) \vee P(b) \vee P(c))$$

2. If the universe of discourse of x is $\{a, b, c\}$ Then write i. $(x)\, R(x) \wedge (x)\, S(x)$

ii. $(x)\, R(x) \wedge (\exists x)\, S(x)$

sol. Given universe of discourse $\{a, b, c\}$

i. $(x)\, R(x) \wedge (x)\, S(x) \Leftrightarrow (R(a) \wedge R(b) \wedge R(c)) \wedge (S(a) \wedge S(b) \wedge S(c))$

ii. $(x)\, R(x) \wedge \exists(x)\, S(x) \Leftrightarrow (R(a) \wedge R(b) \wedge R(c)) \wedge (S(a) \vee S(b) \vee S(c))$

3. Find the truth value of $(\forall x)(P(x) \vee Q(x))$ where $P(x): x=1$, $Q(x): x=2$ and the universe of discourse is $\{1,2\}$.

Sol Given

$P(x): x=1$, $Q(x): x=2$

Given universe of discourse is $\{1,2\}$

$$(\forall x)(P(x) \vee Q(x)) \Leftrightarrow (P(1) \vee Q(1)) \wedge (P(2) \vee Q(2))$$

$$\Leftrightarrow (T \vee F) \wedge (F \vee T)$$

$$\Leftrightarrow T \wedge T$$

$$\Leftrightarrow T$$

4. Find the truth value of $(x)(P \rightarrow Q(x)) \vee R(a)$, where $P: 2>1$, $Q(x): x \leq 3$, $R(x): x>5$ and $a: 5$ and the universe of discourse $\{-2,3,6\}$

Sol Given conditions:

$P: 2>1$, $Q(x): x \leq 3$, $R(x): x>5$, $a: 5$

Given the universe of discourse $\{-2,3,6\}$

$$(x)(P \rightarrow Q(x)) \vee R(a) \Leftrightarrow$$

$$(P \rightarrow Q(-2)) \wedge (P \rightarrow Q(3)) \wedge (P \rightarrow Q(6)) \vee R(5)$$

$$\Leftrightarrow (T \rightarrow T) \wedge (T \rightarrow T) \wedge (T \rightarrow F) \vee F$$

$$\Leftrightarrow T \wedge T \wedge F \vee F$$

$$\Leftrightarrow T \wedge F \vee F$$

$$\Leftrightarrow F \vee F$$

$$\Leftrightarrow F$$

5. Find the Truth values of $(\exists x)(P(x) \rightarrow Q(x)) \wedge T$ where $P(x): x>2$, $Q(x): x=0$ and T is any Tautology with universe of Discourse $\{1\}$

Sol Given condition

$P(x): x>2$, $Q(x): x=0$ and T

Given universe of Discourse is $\{1\}$

$(\exists x)(P(x) \rightarrow Q(x)) \wedge T \Leftrightarrow$

$P(1) \rightarrow Q(1) \wedge T$

$\Leftrightarrow F \rightarrow F \wedge T$

$\Leftrightarrow T \wedge T$

$\Leftrightarrow T$

State and prove Demorgan's laws

statements

(i) $\neg(x)\, P(x) \Leftrightarrow (\exists x) \neg P(x)$

(ii) $\neg(\exists x)\, P(x) \Leftrightarrow (x) \neg P(x)$

proof:

The universe of discourse is $\{x_1, x_2, x_3 \ldots x_n\}$

(i) $\neg(x) P(x) \Leftrightarrow \neg(P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots \wedge P(x_n))$

$\Leftrightarrow \neg P(x_1) \vee \neg P(x_2) \vee \neg P(x_3) \vee \ldots \vee \neg P(x_n)$

$\Leftrightarrow (\exists x) \neg P(x)$

(ii) $\neg(\exists x) P(x) \Leftrightarrow \neg(P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots \vee P(x_n))$

$\Leftrightarrow \neg P(x_1) \wedge \neg P(x_2) \wedge \neg P(x_3) \wedge \ldots \wedge \neg P(x_n)$

$\Leftrightarrow (x) \neg P(x)$

Rule: US (universal specification)

$$\frac{(\forall x)\,P(x)}{P(t) \text{ for all } t}$$

If a statement of the form $(x)P(x)$ is assumed to be true, then the universal quantifiers can be droped to obtain $P(t)$ is true for an orbitery object 't' in universe.

Rule - ES (Existential specification)

If $(\exists x)P(x)$ is assumed to be true, then there is an element 't' in the universe such that $P(t)$ is true.

$$\frac{(\exists x)\,P(x)}{P(t) \text{ for some } t}$$

Rule - UG [Universal Generalization]

If a statement $P(y)$ is true for each element 'y' of universe, then the universal quantifier may be prefixed so obtain $(x)P(x)$, we know that $P(y)$ is true for each element 'y' in the universe.

$$\frac{P(y)}{(x)\,P(x)}$$

Rule - EG [Existential Generalization]

If $P(t)$ is true for some element 't' in the universe, $(\exists x)P(x)$ is true. $\dfrac{P(t)}{(x)\,P(x)}$

Note: Elimination is done by rules of specification called rules US & ES.

① Show that $(x)(H(x) \rightarrow M(x)) \cap H(S) \Rightarrow M(S)$.

Sol Given premises $(x)(H(x) \rightarrow M(x))$; $H(S)$

{1}	(1)	$(x)(H(x) \rightarrow M(x))$	rule P
{1}	(2)	$H(S) \rightarrow M(S)$	Rule US, (1)
{3}	(3)	$H(S)$	Rule P

$\{1,3\}$ (4) $M(S)$ Rule T, (2) (3) from I_{11}

2. Show that $(x)(P(x) \to Q(x)) \wedge (x)(Q(x) \to R(x)) \Rightarrow (x)(P(x) \to R(x))$

Sol Given premises $(x)(P(x) \to Q(x))$, $(x)(Q(x) \to R(x))$

$\{1\}$	(1)	$(x)(P(x) \to Q(x))$	Rule P.
$\{1\}$	(2)	$P(y) \to Q(y)$	Rule US, (1)
$\{3\}$	(3)	$(x)(Q(x) \to R(x))$	Rule P
$\{3\}$	(4)	$Q(y) \to R(y)$	Rule US, (3)
$\{1,3\}$	(5)	$P(y) \to R(y)$	Rule T, (2),(4) from I_{13}
$\{1,3\}$	(6)	$(x)(P(x) \to R(x))$	Rule UG, (5)

3. Show that $(\exists x) M(x)$ follows logically from the premises $(x)(H(x) \to M(x))$ & $(\exists x) H(x)$)

Sol Given premises $(x)[H(x) \to M(x)]$, $\exists(x) H(x)$
conclusion $\exists(x) M(x)$

$\{1\}$	(1)	$(\exists x) H(x)$	Rule P
$\{1\}$	(2)	$H(y)$	Rule ES, (1)
$\{3\}$	(3)	$(x)(H(x) \to M(x))$	Rule P
$\{3\}$	(4)	$H(y) \to M(y)$	Rule US (3)
$\{1,3\}$	(5)	$M(y)$	Rule T (2) (4), from I_{11}.
$\{1,3\}$	(6)	$\exists(x) M(x)$	Rule EG, (5)

4. prove that $(x)(P(x) \wedge Q(x)) \Rightarrow (\exists(x)) P(x) \wedge (\exists(x) Q(x))$

sol: Given premise $(x)(P(x) \wedge Q(x))$

conclusion $(\exists x) P(x) \wedge (\exists x) Q(x)$

{1}	(1)	$(x)(P(x) \wedge Q(x))$	Rule P
{1}	(2)	$P(y) \wedge Q(y)$	Rule US, (1)
{1}	(3)	$P(y)$	Rule T (2) from I_1
{1}	(4)	$(\exists x) P(x)$	Rule EG (3)
{1}	(5)	$Q(y)$	Rule T (2) from I_2
{1}	(6)	$(\exists x) Q(x)$	Rule EG, (5)
{1}	(7)	$(\exists x) P(x) \wedge (\exists x) Q(x)$	Rule T (4)(6) from I_9

(5). show that $(x)(P(x) \vee Q(x)) \Rightarrow P(x) \vee (\exists x) Q(x)$

sol Given premises

$\neg[(x) P(x) \vee \exists(x) Q(x)]$

{1}	(1)	$\neg[(x) P(x) \vee (\exists x) Q(x)]$	Rule P (additional Premise)
{1}	(2)	$\neg (x) P(x) \wedge \neg(\exists x) Q(x)$	Rule T (1) from E_9
{1}	(3)	$\neg (x) P(x)$	Rule T (2) from I_1
{1}	(4)	$(\exists x) \neg P(x)$	Rule T (3) from E_{26}
{1}	(5)	$\neg P(y)$	Rule ES (4)
{1}	(6)	$\neg((\exists x) Q(x))$	Rule T (2) from I_2
{1}	(7)	$(x) \neg Q(x)$	Rule T (6) from E_{23}
{1}	(8)	$\neg Q(y)$	Rule US (7)
{1}	(9)	$\neg P(y) \wedge \neg Q(y)$	Rule, T, (5) (8) from I_9
{1}	(10)	$\neg(P(y) \vee Q(y))$	Rule T, (9) from E_9
{1}	(11)	$(x)(P(x) \vee Q(x))$	Rule P

$\{1\}$	(12)	$P(y) \vee Q(y)$	Rule US, (11)
$\{1, 11\}$	(13)	$(P(y) \vee Q(y)) \wedge \neg(P(y) \vee Q(y))$	Rule T (12), (10) from I_9

which is a contradiction.

* Demonstrate the following implications.

①. $(x)(\neg P(x) \rightarrow Q(x)), (x)\, \neg(Q(x)) \Rightarrow P(a)$

Sol Given premises $(x)(\neg P(x) \rightarrow Q(x)), (x)\, \neg(Q(x))$

conclusion $P(a)$

$\{1\}$	(1)	$(x)\, \neg(Q(x))$	Rule P
$\{1\}$	(2)	$\neg Q(a)$	Rule US; (1)
$\{3\}$	(3)	$(x)(\neg P(x) \rightarrow Q(x))$	Rule P.
$\{3\}$	(4)	$\neg P(a) \rightarrow Q(a)$	Rule US (3)
$\{1,3\}$	(5)	$P(a)$	Rule T (2) (4) from I_{12}, G

2. $(x)(P(x) \vee Q(x)), (x)\, \neg P(x) \Rightarrow (\exists x)\, Q(x)$.

Sol Given premises $(x)(P(x) \vee Q(x)), (x)\, \neg P(x)$

conclusion $(\exists x)\, Q(x)$.

$\{1\}$	(1)	$(x)\, \neg P(x)$	Rule P
$\{1\}$	(2)	$\neg P(y)$	Rule US, (1)
$\{3\}$	(3)	$(x)(P(x) \vee Q(x))$	Rule P
$\{3\}$	(4)	$P(y) \vee Q(y)$	Rule US (3)
$\{1,3\}$	(5)	$Q(y)$	Rule T (2), (4) from I_{10}
$\{1,3\}$	(6)	$(\exists x)\, Q(x)$	Rule EG (5)

3. $(x)(P(x) \vee Q(x)), (x) \neg P(x) \Rightarrow (x) Q(x)$

Sol Given premises $(x)(P(x) \vee Q(x))$,
$(x) \neg P(x)$

conclusion $(x) Q(x)$

{1}	(1)	$(x) \neg P(x)$	Rule P
{1}	(2)	$\neg P(y)$	Rule US, (1)
{3}	(3)	$(x)(P(x) \vee Q(x))$	Rule P
{3}	(4)	$P(y) \vee Q(y)$	Rule US, (3)
{1,3}	(5)	$Q(y)$	Rule T (2),(4) from I_{10}
{1,3}	(6)	$(x) Q(x)$	Rule UG.

4. $\neg(x)(P(x) \wedge Q(x)), (x) P(x) \Rightarrow \neg(x) Q(x)$

Sol Given premises $\neg(x)(P(x) \wedge Q(x))$; $(x) P(x)$

Conclusion $\neg(x) Q(x)$

{1}	(1)	$(x) P(x)$	Rule P
{1}	(2)	$P(y)$	Rule US (1)
{3}	(3)	$\neg(x)(P(x) \wedge Q(x))$	Rule P
{3}	(4)	$\exists x (\neg (P(x) \wedge Q(x)))$	Rule T (3) from E_{26}
{3}	(5)	$\neg(P(y) \wedge Q(y))$	Rule ES (4)
{3}	(6)	$\neg P(y) \vee \neg Q(y)$	Rule T (5) from E_8.
{1,3}	(7)	$P(y) \wedge (\neg P(y) \vee \neg Q(y))$	Rule T (6)(2) from I_9
{1,3}	(8)	$P(y)$	Rule T (7) from I_1
{1,3}	(9)	$(x) P(x)$	Rule ES (8)

Formulas involving more than one quantifier

① Show that $\neg P(a,b)$ follows logically from $(x)(y)(P(x,y) \rightarrow W(x,y))$ and $\neg W(a,b)$.

Sol Given premises $(x)(y)(P(x,y) \rightarrow W(x,y))$, $\neg W(a,b)$

Conclusion $\neg P(a,b)$

$\{1\}$	(1)	$(x)(y)(P(x,y) \rightarrow W(x,y))$	Rule P
$\{1\}$	(2)	$(y)(P(a,y) \rightarrow W(a,y))$	Rule US(1)
$\{1\}$	(3)	$P(a,b) \rightarrow W(a,b)$	Rule US(2)
$\{4\}$	(4)	$\neg W(a,b)$	Rule P
$\{1,4\}$	(5)	$\neg P(a,b)$	Rule T (3),(4) from I_{12}

② Show that $(\exists y)(x)P(x,y) \rightarrow (x)(\exists y)P(x,y)$

Sol Given premises $(\exists y)(x)P(x,y)$

$\{1\}$	(1)	$(\exists y)(x)P(x,y)$	Rule P
$\{1\}$	(2)	$(\exists y)P(a,y)$	Rule US (1)
$\{1\}$	(3)	$P(a,b)$	Rule ES (2)
$\{1\}$	(4)	$(x)P(x,b)$	Rule UG (3)
$\{1\}$	(5)	$(x)(\exists y)P(x,y)$	Rule EG (4).

3. Write in symbolic form and negate the statement "Every one who is healthy can do all kinds of work".

Sol Given statement is

"Every one who is healthy can do all kinds of work".

$H(x)$: Let x is a healthy person

$W(y)$: y is a kind of work

$D(x,y)$: x can do y

Now the statement can be written as "for all x", if x is healthy and "for all y", if y is a kind of work then "x can do y".

$$[(x)\, H(x) \wedge (y)\, W(y)] \to D(x,y)$$

$$(x)\,(y)\,[H(x) \wedge W(y)] \to D(x,y)$$

It's nagation is given by

$$\neg(x)(y)\,[H(x) \wedge W(y)] \to D(x,y)$$

$$(\exists x)\neg(y)\,[H(x) \wedge W(y)] \to D(x,y)$$

$$(\exists x)\,(\exists y)\,\neg[H(x) \wedge W(y) \to D(x,y)]$$

$$(\exists x)\,(\exists y)\,\neg[\neg(H(x) \wedge W(y)) \vee D(x,y)]$$

$$(\exists x)\,(\exists y)\,[(H(x) \vee W(y)) \wedge \neg D(x,y)]$$

④. write a predicate "x is the Father of the mother of y".

Sol: $P(x)$: x is a person

$F(x,y)$: x is a father of y

$M(x,y)$: x is a mother of y.

In order to symbolize the predicate, we name a person called "z"

obiously we want to saw that "x is the fathe of 'z' and 'z' is the mother of y".

It is assumed that such a person 'z' exist, we symbolize the predicate as

$$(\exists z)\,(P(z) \wedge F(x,z) \wedge M(z,y))$$