Unit 3

Multi - Variable CALCULUS :-

Introduction :-

In many cases, 1 Quantity may be dependent on the Variation of 2 (or) more variables

ex:- ① The area of a Δ^{le} is determined when its base and altitudes are given. so, area is a function of its base & Altitude

② The charles law for a given quantity of a gas is $\frac{PV}{t} = c$, where c is a constant

P for pressure, V for volume, T = absolute temp.

$$\therefore P = \frac{CT}{V}$$

so, the pressure Varries with two Other Variables i.e., Volume & Temp.

Hence, If three Variables u, x, y are so related that u is determined when x & y are known, u is said to be a function of the two independent Variables (x, y) & This is denoted by $u = f(x, y)$

lly, if u is a function of three (or) more Variables $x_1, x_2, x_3 \ldots x_n$ then u can be symbolically denoted as $u = f(x_1, x_2, x_3 \ldots x_n)$

Partial differentiation :-

let (det) $z = f(x, y)$ be a function of two Variables x & y. Then

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

If it exist, is said to be partial derivative of 'z' (or) $f(x, y)$ w.r.t x. It is denoted by

by symbol $\frac{\partial z}{\partial x}$ (or) $\frac{\partial f}{\partial x}$ (or) f_x

Thus, we see that for the partial derivative of $z = f(x,y)$ w.r.t x; y is kept constant.

Ily;

The partial derivative of $z = f(x,y)$ w.r.t y keeping x as constant is defined as

$$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$ & it is denoted by $\frac{\partial z}{\partial y}$ (or) $\frac{\partial f}{\partial y}$ (or) f_y

In the same way, the partial derivatives of the function $z = f(x_1, x_2, x_3, \ldots x_n)$ w.r.t x_i keeping other Variables as constant. can be defined by

$$\frac{\partial z}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots x_i + \Delta x_i, \ldots x_n) - f(x_1, x_2, \ldots x_n)}{\Delta x_i}$$

$$i = 1, 2, 3, \ldots n$$

Higher order partial (differential) derivatives :-

In general the 1st order partial derivatives $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are also functions of x & y & they can be differentiated repeatedly to get higher order partial derivatives

so $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$,

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$, $\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3}$ ....

Sometimes, we also use the symbols f_{xx} for $\frac{\partial^2 f}{\partial x^2}$, f_{yy} for $\frac{\partial^2 f}{\partial y^2}$, f_{xy} for $\frac{\partial^2 f}{\partial x \partial y}$ etc.... for functions having continuous partial derivatives, we have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial y \partial x^2} \text{ etc....}$$

Problems :-

① If $w = (y-z)(z-x)(x-y)$, find value of $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}$

Sol: Given that

$$w = (y-z)(z-x)(x-y)$$

$$= (yz - z^2 - yx + zx)(x-y)$$

$$= \cancel{xyz} - x^2y - xz^2 + x^2z - y^2z + y^2x + yz^2 - \cancel{xyz} + x^2z$$

$$w = yz^2 - x^2y - xz^2 - y^2z + y^2x \quad \text{①}$$

Diff ① w.r.to x partially

$$\frac{\partial w}{\partial x} = 0 - 2yx - z^2 - 0 + y^2 + 2xz$$

$$\frac{\partial w}{\partial x} = y^2 - 2yx - z^2 + 2xz$$

Diff ① w.r.to y partially

$$\frac{\partial w}{\partial y} = z^2 - x^2 - 0 - 2yz + 2yx + 0$$

$$= 2yx - 2yz - x^2 + z^2$$

Diff ① w.r.to z partially

$$\frac{\partial w}{\partial z} = 2zy - 0 - 2zx - y^2 + 0 + x^2$$

$$\frac{\partial w}{\partial z} = 2zy - 2zx - y^2 + x^2$$

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = \cancel{y^2} - \cancel{2yx} - \cancel{z^2} + \cancel{2xz} + \cancel{2yx} - \cancel{2yz} - \cancel{x^2} + \cancel{z^2} + \cancel{2zy} - \cancel{2zx} - \cancel{y^2} + \cancel{x^2}$$

$$= 0.$$

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

$\therefore$ Hence, proved

② ~~If $u = \log(x^3+y^3+z^3)$~~

② If $u = \log(x^3+y^3+z^3-3xyz)$, p.T $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$

$= \frac{-9}{(x+y+z)^2}$

Sol: Given $u = \log(x^3+y^3+z^3-3xyz) \longrightarrow (1)$

Diff eq (1) w.r.t x partially

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \cdot 3x^2-3yz$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

lly

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-xz-yz)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z} \quad \text{—} (2)$$

$a^3+b^3+c^3-3abc$
$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

$\boxed{\because \frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}}$

$\therefore$ Hence proved.

③ Verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$ for the function $u = \tan^{-1}\left(\frac{x}{y}\right)$

Sol: Given $u = \tan^{-1}\left(\frac{x}{y}\right) \longrightarrow (1)$

Diff eq (1) w.r.t y partially

LHS: $\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{-x}{y^2}\right)$

$$= \frac{-x/y^2}{\frac{y^2+x^2}{y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y^2+x^2}$$

Again diff. w.r.t x partially

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{y^2+x^2}\right)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = -\left(\frac{(x^2+y^2)(1) - x(0+2x)}{(y^2+x^2)^2}\right)$$

$$= -\left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right)$$

$$= -\left(\frac{-x^2+y^2}{(x^2+y^2)^2}\right)$$

$$\boxed{\frac{\partial^2 u}{\partial x\,\partial y}} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

RHS:- Diff eq (1) w.r.t x partially

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{1}{y}\right)$$

$$= \frac{\frac{1}{y}}{\frac{y^2+x^2}{y^2}}$$

$$\frac{\partial u}{\partial x} = \frac{y}{y^2+x^2}$$

Again diff w.r.t y partially =

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{(x^2+y^2)\cdot(1) - y(2y)}{(y^2+x^2)^2}$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$\therefore$ LHS = RHS.

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}}$$

$\therefore$ Hence Verified.

(4) If $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$. P.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol: Given $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right) \longrightarrow (1)$

Differentiate (1) w.r.t x partially.

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left[\frac{2xy}{x^2-y^2}\right]^2} \cdot \frac{(x^2-y^2)\,2y - 2xy\,(2x-0)}{(x^2-y^2)^2}$$

$$= \frac{\frac{2x^2y - 2y^3 - 4x^2y}{(x^2-y^2)^2}}{1+\frac{4x^2y^2}{(x^2-y^2)^2}}$$

$$= \frac{\frac{-2x^2y - 2y^3}{\cancel{(x^2-y^2)^2}}}{\frac{(x^2-y^2)^2 + 4x^2y^2}{\cancel{(x^2-y^2)^2}}}$$

$$= \frac{-2x^2y - 2y^3}{x^4+y^4-2x^2y^2+4x^2y^2}$$

$$= \frac{-2x^2y - 2y^3}{x^4+y^4+2x^2y^2}$$

$$= \frac{-2y\,\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{2}}}$$

$$\frac{\partial v}{\partial x} = \frac{-2y}{x^2+y^2}$$

Again diff w.r.t 'x' partially

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = -2y\,\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$\frac{\partial^2 v}{\partial x^2} = -2y\left(\frac{-1}{(x^2+y^2)^2}\cdot 2x\right)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{4xy}{(x^2+y^2)^2}$$

Differentiate eq ① ~~w.r~~ w.r.to 'y' partially

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\left[\frac{(x^2-y^2)2x - 2xy(-2y)}{(x^2-y^2)^2}\right]$$

$$= \frac{\dfrac{2x^3-2xy^2+4xy^2}{\cancel{(x^2-y^2)^2}\,(x^2-y^2)^2}}{1+\dfrac{4x^2y^2}{\cancel{(x^2-y^2)^2}\,(x^2-y^2)}}$$

$$= \frac{2x^3+2xy^2/\cancel{(x^2-y^2)^2}}{\dfrac{(x^2-y^2)^2+4x^2y^2}{\cancel{(x^2-y^2)^2}}}$$

$$= \frac{2x(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2x}{x^2+y^2}$$

Again Diff w.r.to 'y' partially

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = 2x \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = 2x\left(\frac{-1}{(x^2+y^2)^2}\right)2y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-4xy}{(x^2+y^2)^2}$$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4xy}{(x^2+y^2)^2} - \frac{4xy}{(x^2+y^2)^2}$

$$= 0$$

(5) If $z = \log(e^x + e^y)$ s.T $rt - s^2 = 0$

Sol: where $r = \frac{\partial^2 z}{\partial x^2}$; $s = \frac{\partial^2 z}{\partial x \partial y}$; $t = \frac{\partial^2 z}{\partial y^2}$

$p = \frac{\partial z}{\partial x}$; $q = \frac{\partial z}{\partial y}$

Given $z = \log(e^x + e^y) \longrightarrow (1)$.

$$\frac{\partial z}{\partial x} = \frac{1}{e^x+e^y} \cdot e^x = \frac{e^x}{e^x+e^y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{e^x}{e^x+e^y}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{(e^x+e^y)\,e^x - e^x(e^x)}{(e^x+e^y)^2}$$

$$r = \frac{\cancel{e^{2x}} + e^{x+y} - \cancel{e^{2x}}}{(e^x+e^y)^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{e^x}{e^x+e^y}\right)$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{-e^x}{(e^x+e^y)^2}\cdot e^y$$

$$s = \frac{-e^{x+y}}{(e^x+e^y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\log(e^x+e^y)\right) = \frac{1}{e^x+e^y}\,e^y$$

$$\frac{\partial z}{\partial y} = \frac{e^y}{e^x+e^y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{e^y}{e^x+e^y}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(e^x+e^y)e^y - e^y(e^y)}{(e^x+e^y)^2}$$

$$= \frac{e^{x+y} + \cancel{e^{2y}} - \cancel{e^{2y}}}{(e^x+e^y)^2}$$

$$t = \frac{e^{x+y}}{(e^x+e^y)^2}$$

$$rt - s^2 = \frac{e^{x+y}}{(e^x+e^y)^2}\cdot\frac{e^{x+y}}{(e^x+e^y)^2} - \left(\frac{-e^{x+y}}{(e^x+e^y)^2}\right)^2 = 0$$

(6) If $U = e^{xyz}$, S.T $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$.

Sol: Given $U = e^{xyz}$

Differentiable w.r.to z partially

$$\frac{\partial U}{\partial z} = e^{xyz}\cdot xy$$

Diff w.r.t y partially

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = x\frac{\partial}{\partial y}\left(e^{xyz}\cdot y\right)$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = x\left(e^{xyz} + ye^{xyz}\cdot xz\right)$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = xe^{xyz} + x^2yze^{xyz}$$

Diff w.r.t x partially

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\,\partial z}\right) = \frac{\partial}{\partial x}\left(xe^{xyz}\right) + yz\frac{\partial}{\partial x}\left(x^2e^{xyz}\right)$$

$$= xe^{xyz}\cdot yz + e^{xyz} + yz\left[x^2e^{xyz}\cdot yz + e^{xyz}\cdot 2x\right]$$

$$= xyz\,e^{xyz} + e^{xyz} + yz\left(x^2yz\,e^{xyz} + 2xe^{xyz}\right)$$

$$= xyz\,e^{xyz} + e^{xyz} + x^2y^2z^2e^{xyz} + 2xyz\,e^{xyz}$$

$$= \left(xyz + 1 + x^2y^2z^2 + 2xyz\right)e^{xyz}$$

$$= \left(1 + 3xyz + x^2y^2z^2\right)e^{xyz}$$

(7) If $z = f(x+ay) + \phi(x-ay)$, P.T $\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}$

Sol: Given $z = f(x+ay) + \phi(x-ay) \longrightarrow (1)$

diff eq (1) w.r.to y pratially

$$\frac{\partial z}{\partial y} = f'(x+ay)\,a + \phi(x-ay)(-a)$$

again diff w.r.to y partially.

$$\frac{\partial^2 z}{\partial y^2} = f''(x+ay)\,a^2 + \phi''(x-ay)\,a^2$$

$$\frac{\partial^2 z}{\partial y^2} = a^2\left[f''(x+ay) + \phi''(x-ay)\right] \longrightarrow (2)$$

diff eq (1) w.r.to x partially

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay)$$

Again diff w.r.to x partially

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \longrightarrow (3)$$

Sub. eq (3) in (2)

$$\frac{\partial^2 z}{\partial y^2} = a^2\left(\frac{\partial^2 z}{\partial x^2}\right)$$

$\therefore$ Hence proved.

The chain rule of partial differentiation :-

(i) If $z = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$ then z is called a composite ~~co~~ function of a Variable 't'.

(ii) If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$ then z is called as composite function of two Variables 'u' & 'v'

Theorem :-

let $z = f(u,v)$ where $u = \phi(x,y)$ & $v = \psi(x,y)$ then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \; ; \; \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

these are reffered to as the chain-rule of partial differentiation. the above rule can be extended to functions of morethan two independent Variables

Total differential co-efficient :-

Let $z = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$

Substuting x & y in $z = f(x,y)$, z becomes a function of single Variable 't'

then the derivative of z w.r.to t

i.e., $\frac{dz}{dt}$ called the total differential co-efficient (or) Total derivative of 'z'.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

In the differential form, this result can be written as

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$$

du is called the total differential of 'u'.

Problems :-

(1). If $u = f(y-z, z-x, x-y)$. Then find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

Sol: Given that $u = f(y-z, z-x, x-y)$

let

$$r = y-z, \quad s = z-x, \quad t = x-y$$

$$u = f(r, s, t)$$

$$\frac{\partial r}{\partial x} = 0 \qquad \frac{\partial s}{\partial x} = -1 \qquad \frac{\partial t}{\partial x} = 1$$

$$\frac{\partial r}{\partial y} = 1 \qquad \frac{\partial s}{\partial y} = 0 \qquad \frac{\partial t}{\partial y} = -1$$

$$\frac{\partial r}{\partial z} = -1 \qquad \frac{\partial s}{\partial z} = 1 \qquad \frac{\partial t}{\partial z} = 0$$

By the chain rule of differentiation we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r} \cdot (1) + \frac{\partial u}{\partial s} \cdot (0) + \frac{\partial u}{\partial t} \cdot (-1)$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}\cdot(-1) + \frac{\partial u}{\partial s}(1) + 0$$

$$= -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}}$$

$$= 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

② If $u = f(r,s,t)$ where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ S.T.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Sol: Given that $u = f(r,s,t)$

let $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$

$$\frac{\partial r}{\partial x} = 1/y \qquad \frac{\partial s}{\partial x} = 0 \qquad \frac{\partial t}{\partial x} = \frac{-z}{x^2}$$

$$\frac{\partial r}{\partial y} = \frac{-x}{y^2} \qquad \frac{\partial s}{\partial y} = 1/z \qquad \frac{\partial t}{\partial y} = 0$$

$$\frac{\partial r}{\partial z} = 0 \qquad \frac{\partial s}{\partial z} = \frac{-y}{z^2} \qquad \frac{\partial t}{\partial z} = 1/x$$

By the chain rule of differentiation we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}\cdot(0) + \frac{\partial u}{\partial t}\left(\frac{-z}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}\left(\frac{\partial u}{\partial r}\right) - \frac{z}{x^2}\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{z}\right)$$

$$= \frac{-y}{z^2}\cdot\frac{\partial u}{\partial s} + \frac{1}{z}\frac{\partial u}{\partial t}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} - \frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} - \frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t} = 0$$

(3) If (i) $u = x^2+y^2$, $x = at^2$, $y = 2at$

(ii) $u = x^2+y^2+z^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$

then find $\frac{du}{dt}$.

Sol: By total derivative of u, we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \longrightarrow (1)$$

Given that:

$u = x^2+y^2$ $\quad x = at^2$ $\quad y = 2at$

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{dx}{dt} = 2at$ $\quad \frac{dy}{dt} = 2a$

$\frac{\partial u}{\partial y} = 2y$

$$\frac{du}{dt} = 2x\cdot 2at + 2y\cdot 2a$$

$$= 2(at^2)(2at) + 2(2at)(2a)$$

$= 4a^2t^3 + 8a^2t \Rightarrow 12a^2t^2 + 8a^2$

(ii) By total derivative of u, we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \longrightarrow (1)$$

$\frac{\partial u}{\partial x} = 2x$ $\quad x = e^t$ $\quad z = e^t \cos t$

$\frac{dx}{dt} = e^t$ $\quad = -e^t \sin t + e^t \cos t$

$\frac{\partial u}{\partial y} = 2y$ $\quad \frac{dz}{dt} = e^t(\cos t - \sin t)$

$y = e^t \sin t$

$\frac{\partial u}{\partial z} = 2z$ $\quad \frac{dy}{dt} = e^t \cos t + e^t \sin t$

$= e^t(\cos t + \sin t)$

$(1) \Rightarrow \frac{du}{dt} = 2x\, e^t + 2y\, e^t(\cos t + \sin t) + 2z\, e^t(\cos t - \sin t)$

$= 2e^t e^t + 2(e^t \sin t)\, e^t(\cos t + \sin t) + 2(e^t \cos t)\, e^t(\cos t - \sin t)$

$= 2e^{2t} + 2e^{2t}\sin t\cos t + 2e^{2t}\sin^2 t + 2e^{2t}\cos^2 t - 2e^{2t}\cos t \sin t$

$= 2e^{2t} + 2e^{2t}(\sin^2 t + \cos^2 t)$

$= 4e^{2t}(1) = 4e^{2t}$

(4). If $u = f(2x-3y, 3y-4z, 4z-2x)$ then P.T $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.

(5) If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$ P.T. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''r + \frac{1}{r}f'(r)$

Sol:

$u = f(r) \longrightarrow (1)$

Diff w.r.to 'x' partially

$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}$

Again diff w.r.to 'x' partially

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = f'(r)\frac{\partial^2 r}{\partial x^2} + f''(r)\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial x}\right)$

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\frac{\partial^2 r}{\partial x^2} + \left(\frac{\partial r}{\partial x}\right)^2 f''(r) \longrightarrow (2)$$

lly

$$\frac{\partial^2 u}{\partial y^2} = f'(r)\frac{\partial^2 r}{\partial y^2} + \left(\frac{\partial r}{\partial y}\right)^2 f''(r) \longrightarrow (3)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f'(r)\frac{\partial^2 r}{\partial x^2} + \left(\frac{\partial r}{\partial x}\right)^2 f''(r) + f'(r)\frac{\partial^2 r}{\partial y^2} + \left(\frac{\partial r}{\partial y}\right)^2 f''(r)$$

$$= f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\left(\frac{\partial^2 r}{\partial x^2}\right) + \left(\frac{\partial^2 r}{\partial y^2}\right)\right] \longrightarrow (4)$$

Now, we find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial^2 r}{\partial x^2}, \frac{\partial^2 r}{\partial y^2}$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2(1)$$

$$x^2 + y^2 = r^2 \longrightarrow (5)$$

Diff. eq (5) w.r.to x ~~part~~ partially

$$2r.\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{lly} \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Again diff w.r.to x partially

$$\frac{\partial^2 r}{\partial x^2} = \frac{r.1 - x.\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x.\frac{x}{r}}{r^2}$$

$$= \frac{r - \frac{x^2}{r}}{r^2} = \frac{\frac{r^2 - x^2}{r}}{\frac{r^2}{1}}$$

$$= \frac{r^2 - x^2}{r^3} \quad \text{Put } r^2 = x^2 + y^2$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\cancel{x^2} + y^2 - \cancel{x^2}}{r^3} = \frac{y^2}{r^3}$$

lly $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$

Substuting above values in (4)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left(\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right) + f'(r)\left(\frac{y^2}{r^3} + \frac{x^2}{r^3}\right)$$

$$= f''(r)\left(\frac{x^2+y^2}{r^2}\right) + f'(r)\left(\frac{x^2+y^2}{r^3}\right)$$

$\because$ put $x^2+y^2=r^2$

$$= f''(r)\left(\frac{\cancel{r^2}}{\cancel{r^2}}\right) + f'(r)\left(\frac{\cancel{r^2}}{r^{\cancel{3}}}\right)$$

$$= f''(r) + \frac{1}{r} f'(r)$$

Hence proved

(6) If $u = x\log xy$ where $x^3+y^3+3xy=1$ find $\frac{du}{dx}$

Sol: By the total derivatives of u w.r.to x we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \rightarrow (1)$$

Given that $u = \cancel{\log}\, x \log xy$.

$$= x(\log x + \log y)$$

$$u = x\log x + x\log y \rightarrow (2)$$

Diff eq (2) w.r.to x partially

$$\cancel{\frac{\partial u}{\partial x} = x\cdot\frac{1}{x} + \log x}$$

$$\frac{\partial u}{\partial x} = x\cdot\frac{1}{x} + \log x\cdot 1 + \log y$$

$$\frac{\partial u}{\partial x} = 1 + \log x + \log y$$

Diff eq (2) w.r.to y partially

$$\frac{\partial u}{\partial y} = 0 + x\cdot\frac{1}{y}$$

$$\frac{\partial u}{\partial y} = \frac{x}{y}$$

also given $x^3+y^3+3xy=1$

Diff x^3+y^3+3yx w.r.to x

$$3x^2+3y^2\frac{dy}{dx}+3\left[x\frac{dy}{dx}+y.1\right]=0$$

$$3x^2+3y^2\frac{dy}{dx}+3x\frac{dy}{dx}+3y=0$$

$$x^2+y+\frac{dy}{dx}(y^2+x)=0$$

$$\frac{dy}{dx}(y^2+x)=-(x^2+y)$$

$$\frac{dy}{dx}=\frac{-(x^2+y)}{y^2+x}$$

$$① \Rightarrow \frac{du}{dx}=1+\log x+\log y+\frac{x}{y}\left(\frac{-(x^2+y)}{y^2+x}\right)$$

$$=1+\log x+\log y-\frac{x^3+xy}{y^3+xy}$$ //.

★★★

Jacobians :

If u & v are functions of 2 independent Variables x & y then, the determinent $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the jacobian of u,v w.r.to x,y &

is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ (or) $J\left(\frac{u,v}{x,y}\right)$

lly. The Jacobian of u,v,w w.r.to x,y,z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

The jacobian of n functions of n independent variable can be defined in similar way

functional dependence :-

let $u(x,y)$ and $v(x,y)$ be 2 functions of x and y then, the functions u & v are said to be functionally dependent if they are functions of each other.

Suppose that u & v are functionally dependent then, there exist a relation of the type $f(u,v)=0$ → ①

$$f(u,v)=0 \quad \text{—①}$$

Diff eq ① w.r.to x partially

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=0 \longrightarrow ②$$

Diff eq ① w.r.to y partially

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=0 \longrightarrow ③$$

Eliminating 'f' from ② & ③ we obtain

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}=0$$

$$\frac{\partial(u,v)}{\partial(x,y)}=0$$

Hence, the functions u & v are functionally dependent if the jacobian of u & v is zero

Note:- If u,v,w are functions of x,y,z then u,v,w are functionally dependent provided

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=0$$

Properties :

1. let U and V are functions of x and y and $J = \frac{\partial(u,v)}{\partial(x,y)}$

$J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $J \cdot J' = 1$

2. If U, V are functions of r & s, and r, s are functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(U,V)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$ which is known as chain rule.

Problems : (5 marks)

(1). If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, S.T. $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

(OR)

$y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$ then S.T

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4.$$

Sol: Given that

$u = \frac{yz}{x} \Rightarrow \frac{\partial u}{\partial x} = \frac{-yz}{x^2}$, $\frac{\partial u}{\partial y} = \frac{z}{x}$, $\frac{\partial u}{\partial z} = \frac{y}{x}$

$v = \frac{zx}{y} \Rightarrow \frac{\partial v}{\partial x} = \frac{z}{y}$, $\frac{\partial v}{\partial y} = \frac{-zx}{y^2}$, $\frac{\partial v}{\partial w} = \frac{x}{y}$

$w = \frac{xy}{z} \Rightarrow \frac{\partial w}{\partial x} = \frac{y}{z}$, $\frac{\partial w}{\partial y} = \frac{x}{z}$, $\frac{\partial w}{\partial z} = \frac{-xy}{z^2}$

Now,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ y/z & x/z & -xy/z^2 \end{vmatrix}$$

Taking common $\frac{1}{x^2}$ from 1st row, $\frac{1}{y^2}$ from 2nd & $1/z^2$ from 3rd row

$$= \frac{1}{x^2y^2z^2}\begin{vmatrix} -yz & zx & xy \\ yz & -zx & xy \\ yz & zx & -xy \end{vmatrix}$$

Taking common yz from 1st column, zx from 2nd & xy from 3rd column

$$= \frac{(yz)(zx)(xy)}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{x^2y^2z^2}{x^2y^2z^2}\left[-1(1-1)-1(-1-1)+1(1+1)\right]$$

$$= 1\,(2+2) = 4$$

Hence proved

(Spherical polar Cordinates)

② If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$; $z = r\cos\theta$ then s.T. $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Sol: Given

$$x = r\sin\theta\cos\phi$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi, \quad \frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi, \quad \frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi$$

$$y = r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi, \quad \frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi$$

$$z = r\cos\theta$$

$$\frac{\partial z}{\partial r} = \cos\theta, \quad \frac{\partial z}{\partial\theta} = -r\sin\theta, \quad \frac{\partial z}{\partial\phi} = 0$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Taking common r from 2nd and 3rd columns, $\sin\theta$ from 3rd column

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & -0 \end{vmatrix}$$

(finding det with 3rd row)

$$= r^2\sin\theta\left[\cos\theta(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) + \sin\theta(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)\right]$$

$$= r^2\sin\theta\left[\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta(\cos^2\phi + \sin^2\phi)\right]$$

$$= r^2\sin\theta\,[1]$$

$$= r^2\sin\theta$$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

Hence proved.

(3) If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$ find $\frac{\partial(u,v)}{\partial(x,y)}$ are u & v functionally dependent? If so, what is relation b/w them?

Sol: Given $u = \frac{x+y}{1-xy}$ $\quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (x+y)(0-y)}{(1-xy)^2}$$

$$= \frac{(1-xy) + y(x+y)}{(1-xy)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1 - \cancel{xy} + \cancel{xy} + y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1) - (x+y)(0-x)}{(1-xy)^2}$$

$$= \frac{1 - \cancel{xy} + \cancel{xy} + x^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$V = \tan^{-1} x + \tan^{-1} y$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{\cancel{1+y^2}}{(1-xy)^2} \cdot \frac{1}{\cancel{1+y^2}} - \frac{\cancel{1+x^2}}{(1-xy)^2} \cdot \frac{1}{\cancel{1+x^2}}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

$\Rightarrow$ u and v are functionally independent

W.K.T, $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$V = \tan^{-1} u$

$u = \tan V$

$\therefore$ is relation b/w u & V

(4) If $u = x+y+z$, $uv = y+z$, $uvw = z$ then P.T

$$\frac{\partial(xyz)}{\partial(uvw)} = u^2 v$$

Sol: Given

$$u = x+y+z \quad \text{—} (1)$$
$$uv = y+z \quad \text{—} (2)$$
$$uvw = z \quad \text{—} (3)$$

$(1) \Rightarrow x = u - y - z$

$x = u - uv + \cancel{uvw} - \cancel{uvw}$

$x = u - uv$

$(3) \Rightarrow z = uvw$

$(2) \Rightarrow y = uv - z$

$y = uv - uvw$

$$\frac{\partial x}{\partial u} = 1 - v \qquad \frac{\partial x}{\partial v} = 0 - u \qquad \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = v - vw \qquad \frac{\partial y}{\partial v} = u - uw \qquad \frac{\partial y}{\partial w} = 0 - uv$$

$$\frac{\partial z}{\partial u} = vw \qquad \frac{\partial z}{\partial v} = uw \qquad \frac{\partial z}{\partial w} = uv$$

$$\frac{\partial(xyz)}{\partial(uvw)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= u^2 v \begin{vmatrix} 1-v & -1 & 0 \\ v(1-w) & 1-w & -1 \\ vw & w & 1 \end{vmatrix}$$

Take common u^2 & v in 2nd & 3rd column respectively

$$= u^2v\left(1 - v(1 - \cancel{v} + \cancel{v}) + 1(v - \cancel{v} + \cancel{v}) + 0\right)$$

$$= u^2v(1 + \cancel{v} + \cancel{v})$$

$$= u^2v(1)$$

$$= u^2v$$

$\therefore$ Hence proved

(5) If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$

find $\dfrac{\partial(FGH)}{\partial(uvw)} = ?$

Sol: Given $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$

$$\frac{\partial F}{\partial u} = x \quad \frac{\partial F}{\partial v} = 1 \quad \frac{\partial F}{\partial w} = 0$$

$$\frac{\partial G}{\partial u} = 2u \quad \frac{\partial G}{\partial v} = y \quad \frac{\partial G}{\partial w} = 1$$

$$\frac{\partial H}{\partial u} = z \quad \frac{\partial H}{\partial v} = -1 + w \quad \frac{\partial H}{\partial w} = v$$

$$\frac{\partial(FGH)}{\partial(uvw)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} & \frac{\partial G}{\partial w} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} & \frac{\partial H}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} x & 1 & 0 \\ 2u & y & 1 \\ z & -1+w & v \end{vmatrix}$$

$$= x(yv + 1 - w) - 1(2uv - z)$$

$$= x(vy + 1 - w) - 2uv + z$$

$$= x(yv + 1 - w) + z - 2uv$$

Ans $= \left(x(yv + 1 - w) + z - 2uv\right)$

Prob: 6

If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$

Sol: Given that $u = 2xy$ $v = x^2 - y^2$

put $x = r\cos\theta$, $y = r\sin\theta$

$u = 2r\cos\theta\, r\sin\theta$ $\quad v = r^2\cos^2\theta - r^2\sin^2\theta$

$= 2r^2\cos\theta\sin\theta$ $\quad = r^2(\cos^2\theta - \sin^2\theta)$

$u = r^2\sin 2\theta$ $\quad v = r^2$

$u = r^2\sin 2\theta$ $\quad v = r^2\cos 2\theta$

$\frac{\partial u}{\partial r} = 2r\sin 2\theta$ $\quad \frac{\partial v}{\partial r} = 2r\cos 2\theta$

$\frac{\partial u}{\partial \theta} = r^2\cos 2\theta . 2$ $\quad \frac{\partial v}{\partial \theta} = r^2(-\sin 2\theta)2$

$= 2r^2\cos 2\theta$ $\quad = -2r^2\sin 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\sin 2\theta & 2r^2\cos 2\theta \\ 2r\cos 2\theta & -2r^2\sin 2\theta \end{vmatrix}$$

$= -4r^3\sin^2 2\theta - 4r^3\cos^2 2\theta$

$= -4r^3(\sin^2 2\theta + \cos^2 2\theta)$

$= -4r^3$

(7) If $x = a\cosh u\cos v$ and $y = a\sinh u\sin v$ then show that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2}{2}(\cosh 2u - \cos 2v)$

Sol: $\frac{\partial x}{\partial u} = a\cos\theta\sinh u$ $\quad \frac{\partial v}{\partial u} = a\cosh u(-\sin v)$

$= -a\cosh u\sin v$

$\frac{\partial y}{\partial u} = a\sin v\cosh u$ $\quad \frac{\partial y}{\partial v} = a\sinh u\cos v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a\cos v \sinh u & -a\cosh u \sin v \\ a\sin v \cosh u & a\sinh u \cos v \end{vmatrix}$$

$$= a^2\cos^2 v \sinh^2 u + a^2 \sin^2 v \cosh^2 u$$

$\sinh^2 u - \cosh^2 u = -1$

$$= a^2\left[\cos^2 v \sinh^2 u + \sin^2 v \cosh^2 u\right]$$

$\sinh^2 u = \cosh^2 u - 1$

$\sin^2 v + \cos^2 u = 1$

$$= a^2\left[\cos^2 v(\cosh^2 u - 1) + (1-\cos^2 v)\cosh^2 u\right]$$

$$= a^2\left[\cancel{\cos^2 v \cosh^2 u} - \cos^2 v + \cosh^2 u - \cancel{\cos^2 v \cosh^2 u}\right]$$

$$= a^2\left[(\cosh^2 u) - (\cos^2 v)\right]$$

$$= a^2\left(\frac{1+\cosh 2u}{2} - \frac{1+\cos 2v}{2}\right)$$

$$= \frac{a^2}{2}(\cancel{1}+\cosh 2u - \cancel{1} - \cos 2v)$$

$$= \frac{a^2}{2}(\cosh 2u - \cos 2v)$$

$\therefore$ Hence proved.

(9) $\frac{\partial(u,v)}{\partial(x,y)}$ and $\frac{\partial(x,y)}{\partial(u,v)}$ and verify that $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

(i) $u = e^x \quad v = e^y$

(ii) $u = e^{x+y} \quad v = e^{-x+y}$

(ii) Sol: ~~Given~~ $\frac{\partial u}{\partial x} = e^{x+y} \quad \frac{\partial u}{\partial y} = e^{x+y}$

$\frac{\partial v}{\partial x} = e^{-x+y}(-1) \quad \frac{\partial v}{\partial y} = e^{-x+y}$

$= -e^{-x+y}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} e^{x+y} & e^{x+y} \\ e^{-x+y} & e^{-x+y} \end{vmatrix}$$

$$= e^{x+y} \cdot e^{-x+y} + e^{-x+y} \cdot e^{x+y}$$

$$= e^{\cancel{x}+y-\cancel{x}+y} + e^{-x+y+x+y}$$

$$= e^{2y} + e^{2y} = 2e^{2y}.$$

consider, $\frac{u}{v} = \frac{e^{x+y}}{e^{-x+y}}$

$$= e^{x+y} \cdot e^{x-y}$$

$$= e^{x+y+x+y}$$

$$\frac{u}{v} = e^{2x}$$

take log on O.B.S

$$\log\left(\frac{u}{v}\right) = 2x$$

$$x = \frac{1}{2}\log\left(\frac{u}{v}\right)$$

$$x = \frac{1}{2}(\log u - \log v) = \frac{1}{2}\log u - \frac{1}{2}\log v$$

$$uv = e^{x+y} \cdot e^{-x+y} = e^{2y}$$

$$\log uv = 2y$$

$$y = \frac{1}{2}\log uv$$

$$\cancel{\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2e^{2y}}}$$

$$y = \frac{1}{2}(\log u + \log v)$$

$$y = \frac{1}{2}\log u + \frac{1}{2}\log v$$

$$\frac{\partial x}{\partial u} = \frac{1}{2u} \qquad \frac{\partial x}{\partial v} = \frac{-1}{2v}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2u} \qquad \frac{\partial y}{\partial v} = \frac{1}{2v}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2u} & \frac{-1}{2v} \\ \frac{1}{2u} & \frac{1}{2v} \end{vmatrix} = \frac{1}{4uv} + \frac{1}{4uv}$$

$$= \frac{1}{2uv}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2e^{2y}} \qquad \text{put } uv = e^{2y}$$

Maxima & Minima of functions having two (or) more Variables:

Working Rule:

Step :- 1 :

Solve $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ to get the values $x = a, y = b$

Step :- 2

Calculate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$ & $t = \frac{\partial^2 f}{\partial y^2}$

Step :- 3 :-

- If $rt - s^2 > 0$ and $r < 0$, then the function $f(x,y)$ has a maximum values at $x = a$, $y = b$.
- If $rt - s^2 > 0$ and $r > 0$ then the function $f(x,y)$ has a minimum values at $x = a$, $y = b$

(extrem points)

Note:

1. If $rt - s^2 < 0$, there is no maximum or minimum at $x = a$, $y = b$

   i.e., the point (a, b) is a saddle point

2. If $rt - s^2 = 0$, ~~for~~ further investigation required to find whether there is a maximum or minimum at $x = a, y = b$.

3. $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $\frac{\partial f(a,b)}{\partial x} = 0$, $\frac{\partial f(a,b)}{\partial y} = 0$.

   i.e., The function is stationary at (a,b)

Thus, every extreme value is stationary value but the Converse may not be true.

problems:

① Find the maximum or mininum of $f(x,y) = x^3+y^3-3x-12y+20$

Sol: Given that $f = x^3+y^3-3x-12y+20$

$$\frac{\partial f}{\partial x} = 3x^2-3$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^3-3=0 \longrightarrow ①$$

$$\frac{\partial f}{\partial y} = 3y^2-12$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2-12=0 \longrightarrow ②$$

from ① $3x^2=3$, $x^2=1$, $x=\pm 1$

from ② $3y^2=12$, $y^2=4$, $y=\pm 2$

$A(1,2)$, $B(1,-2)$, $C(-1,2)$, $D(-1,-2)$ are stationary point

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(3x^2-3)$$

$$r = 6x$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(3y^2-12)$$

$$s = 0,$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(3y^2-12)$$

$$t = 6y$$

At $A(1,2)$

$x=1$, $y=2$

$r = 6x = 6(1) = 6 > 0$

So $s = 0$

$t = 6y = 6(2) = 12$

Now $rt - s^2 = (6)(12) - 0 = 72 > 0$

Since $rt - s^2 > 0$ & $r > 0$ so the function f has minima at $(1,2)$

$f_{min}(1,2) = 1^3 + 2^3 - 3(1) - 12(2) + 20$

$= 1 + 8 - 3 - 24 + 20$

$f_{min} = 2$

At $B(1,-2)$

$x = 1, y = -2$

$r = (6)(1) = 6 > 0$

$s = 0$

$t = 6(-2) = -12$

$rt - s^2 = 6(-12) - 0 = -72 < 0$

Since $rt - s^2 < 0$, the point $(1,-2)$ is on saddle point

At $C(-1,2)$:-

$r = 6(-1) = -6 < 0$

$s = 0$

$t = 6(2) = 12$

$rt - s^2 = (-6)(12) - 0$

Since $rt - s^2 < 0$, the point $(-1,2)$ in a saddle point

At $D(-1,-2)$

$r = 6(-1) = -6 < 0$

$s = 0$

$t = 6(-2) = -12$

$rt - s^2 = (-6)(-12) - 0 = 72 > 0$

$rt - s^2 > 0$ & $r < 0$, so the function f has maximum at $(-1,-12)$

$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$

$= -1 - 8 + 3 + 24 + 20$

$f(-1,-2) = 38$

(2) find the extrema of $f(x,y)$ if

(i) $f(x,y) = a^2 - x^2 - y^2$

(ii) $f(x,y) = x^3 y^2 - xy$

Sol: (i) Let $f = a^2 - x^2 - y^2$

$$\frac{\partial f}{\partial x} = -2x$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow -2x$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow -2x = 0$$

$$\boxed{x = 0}$$

$$\frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow -2y = 0$$

$$\boxed{y = 0}$$

$(0,0)$ is the stationary point of f

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(-2x)$$

$$r = -2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(-2y)$$

$$s = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(-2y)$$

$$t = -2$$

At $(0,0)$

$r = -2 < 0$

$s = 0$

$t = -2$

Now $rt - s^2 = (-2)(-2) - 0 = 4 > 0$

Since $rt - s^2 > 0$, $r < 0$ so the function f has maxima at $(0,0)$

$f_{max}(0,0) = a^2 - 0 - 0 = a^2$

(ii) Let $f = x^3y^2 - xy$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - y$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2y^2 - y = 0$$

$$y(3x^2y - 1) = 0$$

$$\boxed{y=0} \qquad 3x^2y - 1 = 0 \text{ — (1)}$$

$$\frac{\partial f}{\partial y} = 2x^3y - x$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2x^3y - x = 0$$

$$x(2x^2y - 1) = 0$$

$$\boxed{x=0} \qquad 2x^2y - 1 = 0 \text{ — (2)}$$

(0,0) is only stationary point at (0,0).

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(3x^2y^2 - y)$$

$$r = 6xy^2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(2x^3y - x)$$

$$s = 6x^2y - 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(2x^3y - x)$$

$$t = 2x^3$$

$$r = 6(0)(0) = 0$$

$$s = 6(0)(0) - 1 = -1$$

$$t = 2(0) = 0$$

$$rt - s^2 = 0 - (-1)^2 = -1 < 0$$

Since $rt - s^2 < 0$, so the point (0,0) is a saddle point.

(3) Discuss the minima & maxima of $f(x,y) = x^3y^2(1-x-y)$

(4) Discuss the maximam (or) minimum values of u, if

$u = ax^3y^2 - x^4y^2 - x^3y^3$

(5) A rectangular box open at the top is to have a Volume of 32 cft find the dimension of the box requiring least material for its construction

Sol: let x, y, z be the length, breadth, height of the box respectively., V its vol. & S is its Surface area

Given that $V = xyz = 32$ cft

$$\boxed{z = \frac{32}{xy}}$$

$$S = xy + 2yz + 2zx$$

put $z = \frac{32}{xy}$

$$S = xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{y}\right)x$$

$$S = xy + \frac{64}{x} + \frac{64}{y}$$

for ~~stab~~ S to be minimum $\frac{\partial S}{\partial x} = 0$; $\frac{\partial S}{\partial y} = 0$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}$$

$$\frac{\partial S}{\partial x} = 0 \Rightarrow y = \frac{64}{x^2}$$

$$\frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

$$\frac{\partial S}{\partial y} = 0 \Rightarrow x = \frac{64}{y^2}$$

put $y = \frac{64}{x^2}$

$$x = \frac{64}{\left(\frac{64}{x^2}\right)^2}$$

$$x = \cancel{64} \times \frac{x^4}{(64)^{\cancel{2}}}$$

$$x^4 = 64x$$

$x^4 - 64x = 0$,

$x(x^3 - 64) = 0$

$x \neq 0 \qquad x^3 - 64 = 0$

$x^3 = 64$

$\boxed{x = 4}$

$y = \frac{64}{(4)^2} = \frac{\cancel{64}^{4}}{\cancel{16}_{1}} = 4$,

$\boxed{y = 4}$

$z = \frac{32}{(4)(4)} = \frac{32}{16} = 2$

length $= 4$, breadth$(y) = 4$,

height $(z) = 2$

$r = \frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial s}{\partial x}\right)$

$= \frac{\partial}{\partial x}\left(y - \frac{64}{x^2}\right)$

$r = \frac{128}{x^3}$

$s = \frac{\partial^2 s}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial s}{\partial y}\right)$

$= \frac{\partial}{\partial x}\left(x - \frac{64}{y^2}\right)$

$s = 1$

$t = \frac{\partial^2 s}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial s}{\partial y}\right)$

$= \frac{\partial}{\partial y}\left(x - \frac{64}{y^2}\right)$

$t = \frac{128}{y^3}$

At $(x, y) = (4, 4)$

$r = \frac{128}{4^3} = \frac{128}{64} = 2 > 0$

$s = 1$

$t = \frac{128}{4^3} = 2$

$rt - s^2 = 2 \times 2 - 1^2 = 3 > 0$

$\therefore rt - s^2 > 0$ & $r > 0$

so the function surface area of a rectangular box is minimum of $x = 4$, $\boxed{y = 4}$

⑥ Discuss the maxima (or) minima of $f(x,y) = \sin x + \sin y + \sin(x+y)$

Sol: given $f = \sin x + \sin y + \sin(x+y)$

$$\frac{\partial f}{\partial x} = \cos x + \cos(x+y)$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow \cos x + \cos(x+y) = 0 \rightarrow (1)$$

$$\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$$

$$\frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0 \quad -(2)$$

$$(1) \Rightarrow \cos x + \cos(x+y) = 0$$

$$(2) \Rightarrow \cos x + \cos(x+y) = 0$$

$$\cos x - \cos y = 0$$

$$\cos x = \cos y$$

$$\boxed{x = y}$$

put $x = y$ in (1)

$$\cos y + \cos(y+y) = 0$$

$$\cos y + \cos 2y = 0$$

$$2\cos\left(\frac{y+2y}{2}\right)\cos\left(\frac{y-2y}{2}\right) = 0$$

$$\cos\frac{3y}{2} \cdot \cos\frac{y}{2} = 0$$

$$\cos\frac{3y}{2} = 0 \text{ (or) } \cos\frac{y}{2} = 0$$

$$\cos\frac{3y}{2} = \cos\frac{\pi}{2} \qquad \cos\frac{y}{2} = \cos\frac{\pi}{2}$$

$$\frac{3y}{\cancel{2}} = \frac{\pi}{\cancel{2}} \qquad \frac{y}{\cancel{2}} = \frac{\pi}{\cancel{2}}$$

$$y = \frac{\pi}{3} \qquad y = \pi$$

$\therefore x = y \Rightarrow$ if $y = \frac{\pi}{3}$ then $x = \frac{\pi}{3}$

if $y = \pi$ then $x = \pi$

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ & (π, π) are stationary points

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(\cos x + \cos(x+y))$$

$$\Rightarrow r = -\sin x - \sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(\cos x + \cos(x+y))$$

$$\Rightarrow s = -\sin(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(\cos y + \cos(x+y))$$

$$\Rightarrow t = -\sin y - \sin(x+y)$$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ $\left(\sin \pi/3 = \frac{\sqrt{3}}{2}, \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\right)$

$r = -\sin\frac{\pi}{3} - \sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2} - \sin\left(\frac{2\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{2\sqrt{3}}{2} = -\sqrt{3} < 0$

$s = -\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = -\sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$

$t = -\sin\frac{\pi}{3} - \sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

Now $rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2$

$= 3 - \frac{3}{4} = \frac{9}{4} > 0$

$rt - s^2 > 0$ & $r < 0$, so the function f has maximum value at $x = \frac{\pi}{3}, y = \frac{\pi}{3}$

$f_{max}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3}$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$f_{max} = \frac{3\sqrt{3}}{2}$

At (π, π)

$r = 0$

$s = 0$

$t = 0$

$rt - s^2 = 0 \Rightarrow$ No conclusion

(7) Discuss the maxima or minima of $f(x,y) = \sin x \sin y - \sin(x+y)$

Sol: $f = \sin x \sin y \sin(x+y)$

$\frac{\partial f}{\partial x} = \sin y \frac{\partial}{\partial x}\left(\sin x . \sin(x+y)\right)$

$= \sin y\left(\sin x \cos(x+y) + \sin(x+y)\cos x\right)$

$= \sin y\left[\sin(x+x+y)\right]$

$\frac{\partial f}{\partial x} = \sin y \sin(2x+y)$

$\sin n\pi = 0$

$\sin n\pi = 0$

$\sin y = 0$ $\quad$ $\sin(2x+y) = 0$

$\sin y = \sin 0$ $\quad$ $\sin(2x+y) = \sin \pi$

$y = 0$ $\quad$ $2x+y = \pi \rightarrow (1)$

$$\frac{\partial f}{\partial y} = \sin x \frac{\partial}{\partial y}\left[\sin y . \sin(x+y)\right]$$

$$\frac{\partial f}{\partial y} = \sin\left[\sin y \cos(x+y) . + \sin(x+y) . \cos y\right]$$

$$= \sin x \left(\sin(y+x+y)\right]$$

$$\frac{\partial f}{\partial y} = \sin x . \sin(x+2y)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \sin x \sin(x+2y) = 0$$

$\sin x = 0$ $\quad$ $\sin(x+2y) = 0$

$\sin x = \sin 0$ $\quad$ $\sin(x+2y) = \sin \pi$

$\boxed{x = 0}$ $\quad$ $x+2y = \pi \rightarrow (2)$

solving (1) & (2)

$(1) \times 2 \Rightarrow 4x + 2y = 2\pi$

$x + 2y = \pi$

$3x = \pi$

$$\boxed{x = \frac{\pi}{3}}$$

$(2) \Rightarrow \frac{\pi}{3} + 2y = \pi$

$2y = \pi - \frac{\pi}{3}$

$$\boxed{y = \pi/3}$$

$\therefore (0,0)\ (\pi/3, \pi/3)$ and stationary points

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\sin y \sin(2x+y)\right)$$

$$= \sin y \cos(2x+y) . 2$$

$$\rightarrow r = 2 \sin y \cos(2x+y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\sin x . \sin(x+2y) + \sin(x+2y) . \cos x\right)$$

$$= \sin x \cos(x+2y) + \sin(x+2y) . \cos x$$

$$= \sin(x+x+2y)$$

$$\Rightarrow s = \sin(2x+2y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\sin x . \sin(x+2y)\right)$$

$$t = \sin x . \cos(x+2y) . 2$$

$$t = 2 \sin x . \cos(x+2y)$$

At $(\pi/3, \pi/3)$

$$r = 2 \sin \pi/3 \cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right) = \cancel{2}\frac{\sqrt{3}}{\cancel{2}}(-1) = -\sqrt{3}$$

$$s = \sin\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \sin\left(\frac{4\pi}{3}\right)$$

$$= \sin(180 + \pi/3)$$

$$= -\sin \pi/3 = -\frac{\sqrt{3}}{2}$$

$$t = 2 \sin \pi/3 \cos\left(\pi/3 + \frac{2\pi}{3}\right)$$

$$= \cancel{2}\frac{\sqrt{3}}{\cancel{2}} \cos \pi = -\sqrt{3}$$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

$f_{max} = 3\sqrt{3}/8$

Since $rt - s^2 > 0$ & $r < 0$ so the function f has maximum at $x = y = \pi/3$

$$f_{max} = \sin\frac{\pi}{3} \sin\frac{\pi}{3} \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$$

(8) The sum of three numbers is constant. P.T. Their product is maximum when they are equal

Sol: let x, y, z be three numbers

given that $x + y + z = c$

$$z = c - x - y$$

let $f = xyz$

$$f(x, y) = xy(c - x - y)$$

$$f = cxy - x^2y - xy^2$$

f to be maximum, $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = cy - 2xy - y^2$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow y(c - 2x - y) = 0$$

$$y \neq 0 \quad c - 2x - y = 0 \longrightarrow (1)$$

$$\frac{\partial f}{\partial y} = cx - x^2 - 2xy$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow x(c - x - 2y) = 0$$

$$x \neq 0, \; c - x - 2y = 0 \rightarrow (2)$$

solving (1) & (2)

$$(1) \times 2 \Rightarrow 2c - 4x - 2y = 0$$

$$(2) \Rightarrow \; c - x - 2y = 0$$

$$- \quad + \quad +$$

$$c - 3x = 0$$

$$\boxed{x = \frac{c}{3}}$$

$$(1) \Rightarrow c - 2(c/3) - y = 0$$

$$y = c - \frac{2c}{3}$$

$$\boxed{y = \frac{c}{3}}$$

$$z = c - \frac{c}{3} - \frac{c}{3} = \frac{3c - c - c}{3} = \frac{c}{3}$$

$$x = y = z = c/3$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(cy - 2xy - y^2)$$

$$r = -2y$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(cx - x^2 - 2y)$$

$$s = c - 2x - 2y$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(cx - x^2 - 2xy)$$

$$t = -2x$$

At $(x,y) = \left(\frac{c}{3}, \frac{c}{3}\right)$

$r = -2(c/3) = \frac{-2c}{3} < 0$

$s = c - \frac{2c}{3} - \frac{2c}{3} = \frac{3c-2c-2c}{3} = \frac{-c}{3}$

$t = \frac{-2c}{3}$

$rt - s^2 = \left(\frac{-2c}{3}\right)\left(\frac{-2c}{3}\right) - \left(\frac{-c}{3}\right)^2$

$= \frac{4c^2}{9} - \frac{c^2}{9}$

$= \frac{3c^2}{9} = \frac{c^2}{3} > 0$

$\therefore rt - s^2 > 0$ & $r < 0 \Rightarrow$ the function product of three numbers is maximum when they are equal.

(9) S.T Rectangular parallelopiped of maximum value that can be incribed in a given sphere is cube.

Sol: let x, y, z be length, breadth, height of the rectangular parallelo pipe.

If v is the volume of the rectangular parallelopipe.

Since each diagonal of ~~each diagonal~~

$V = xyz \rightarrow (1)$

diagonal = diameter of sphere = d ~~let~~ let

diameter of the sphere $d = \sqrt{x^2+y^2+d^2}$

$d^2 = x^2+y^2+z^2$

$z^2 = d^2 - x^2 - y^2$

$z = \sqrt{d^2-x^2-y^2}$

$(1) \Rightarrow V = xy\left(\sqrt{d^2-x^2-y^2}\right)$

$v^2 = x^2y^2(d^2-x^2-y^2)$

$v^2 = d^2x^2y^2 - x^4y^2 - x^2y^4$

let $v^2 = f(x,y) = d^2x^2y^2 = x^4y^2 - x^2y^4$

for v to be maximum $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 2d^2xy^2 - 4x^3y^2 - 2xy^4$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2d^2xy^2 - 4x^3y^2 - 2xy^4 = 0$$

$$xy^2(2d^2 - 4x^2 - 2y^2) = 0$$

since $x \neq 0$, $y \neq 0$ $\quad 2d^2x^2y - 2x^4y - 4x^2y^3$

$$\frac{\partial f}{\partial y} = 2d^2x^2y - 2x^4y - 4x^2y^3$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2d^2x^2y - 2x^4y - 4x^2y^3 = 0$$

$$x^2y(2d^2 - 2x^2 - 4y^2) = 0$$

$$2d^2 - 2x^2 - 4y^2 = 0 \longrightarrow (3)$$

solving (2) & (3)

$$(2) \times 2 \Rightarrow 4d^2 - 8x^2 - \cancel{4y^2} = 0$$

$$(3) \quad \Rightarrow 2d^2 - 2x^2 - \cancel{4y^2} = 0$$

$$- \quad + \quad +$$

$$2d^2 - 6x^2 = 0$$

$$\cancel{2}d^2 = \cancel{6}^{3}x^2$$

$$x^2 = \frac{d^2}{3}$$

$$\boxed{x = \frac{d}{\sqrt{3}}}$$

$$(3) \Rightarrow 2d^2 - 2\left(\frac{d}{\sqrt{3}}\right)^2 = 4y^2$$

$$2d^2 - \frac{2d^2}{3} = 4y^2$$

$$\frac{\cancel{4}d^2}{3} = \cancel{4}y^2$$

$$\boxed{y = \frac{d}{\sqrt{3}}}$$

$$z^2 = d^2 - \frac{d^2}{3} - \frac{d^2}{3}$$

$$= \frac{3d^2 - d^2 - d^2}{3}$$

$$z^2 = \frac{d^2}{3} \Rightarrow z = \frac{d}{\sqrt{3}}$$

since $x = y = z = \frac{d}{\sqrt{3}}$

$\Rightarrow$ The rectangular parallelopiped inscribed in a square is cube.

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(2d^2y^2 - 4x^3y^2 - 2xy^4\right)$$

$$r = 2d^2y^2 - 12x^2y^2 - 2y^2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(2d^2x^2y - 2x^4y - 4x^2y^3\right)$$

$$s = 4d^2xy - 8x^3y - 8xy^3$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(2d^2x^2y - 2x^4y - 4x^2y^3\right)$$

$$t = 2d^2x^2 - 2x^4 - 12x^2y^2$$

At $(x, y), \left(\frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}\right)$

$$r = 2d^2\left(\frac{d}{\sqrt{3}}\right)^2 - 12\left(\frac{d}{\sqrt{3}}\right)^2\left(\frac{d}{\sqrt{3}}\right)^2 - 2\left(\frac{d}{\sqrt{3}}\right)^4$$

$$= \frac{2d^4}{3} - \frac{12d^2}{9} - \frac{2d^4}{9}$$

$$r = \frac{-8d^4}{9} < 0$$

$$s = 4d^2\left(\frac{d}{\sqrt{3}}\right)\left(\frac{d}{\sqrt{3}}\right) - 8\left(\frac{d}{\sqrt{3}}\right)^3\left(\frac{d}{\sqrt{3}}\right) - 8\left(\frac{d}{\sqrt{3}}\right)\left(\frac{d}{\sqrt{3}}\right)^3$$

$$= \frac{4d^4}{3} - \frac{8d^4}{9} - \frac{8d^4}{9}$$

$$= \frac{12d^4 - 8d^4 - 8d^4}{9} = \frac{-4d^4}{9}$$

$$t = 2d^2\left(\frac{d}{\sqrt{3}}\right)^2 - 2\left(\frac{d}{\sqrt{3}}\right)^4 - 12\left(\frac{d}{\sqrt{3}}\right)^2\left(\frac{d}{\sqrt{3}}\right)^2$$

$$= \frac{2d^4}{3} - \frac{2d^4}{9} - \frac{12d^4}{9}$$

$$t = \frac{6d^4 - 2d^4 - 12d^4}{9}$$

$$t = \frac{-8d^4}{9}$$

Now $rt-s^2 = \left(-\frac{8d^4}{9}\right)\left(-\frac{8d^4}{9}\right) - \left(\frac{-4d^4}{9}\right)^2$

$$= \frac{64d^8}{81} - \frac{16d^8}{81} = \frac{48d^8}{81} > 0$$

$rt-s^2>0$ & $r<0$ the function, Volume of Cube is maximum when $x=y=z$.

(10) If the perimeter of a Δ^{le} is constant. S.T. the Δ^{le} has maximum area when it is an equilateral Δ^{le}.

Sol: let a, b, c the sides of the Δ^{le}

let s be the semi perimeter

$$\therefore s = \frac{a+b+c}{2}$$

$$2s = a+b+c$$

$$s-a = \frac{a}{2} + \frac{b}{2} + \frac{c}{2} - a$$

$$s-a = \frac{-a}{2} + \frac{b}{2} + \frac{c}{2} = \frac{1}{2}(-a+b+c)$$

Now the Ar. of a Δ^{le} to given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Delta^2 = s(s-a)(s-b)(s-c)$$

put $c = 2s-a-b$

let $\Delta^2 = f(a,b) = s(s-a)(s-b)(s-(2s-a-b))$

$$f = s(s-a)(s-b)(a+b-s) \longrightarrow (1)$$

$$\frac{\partial f}{\partial a} = s(s-b)\frac{\partial}{\partial a}((s-a)(a+b-s))$$

$$= s(s-b)\left[(s-a) + (a+b-s)(-1)\right]$$

$= s(s-b)(s-a-a-b+s)$

$\frac{\partial f}{\partial a} = s(s-b)(2s-2a-b)$

$\frac{\partial f}{\partial a} = 0 \Rightarrow 2s-2a-b=0$ — (2)

$\frac{\partial f}{\partial b} = s(s-a)\left[(s-b)1+(a+b-s)(-1)\right]$

$= s(s-a)(s-b-a+b+s)$

$\frac{\partial f}{\partial b} = s(s-a)(2s-a-2b)$

$\frac{\partial f}{\partial b} = 0 \Rightarrow 2s-a-2b=0$ — (3)

from (2) & (3)

(2) × (1) $\Rightarrow 4s-4a-\cancel{2b}=0$

(3) $\Rightarrow 2s-a-\cancel{2b}=0$

$2s-3a=0$

$2s=3a$

$\boxed{a=\frac{2s}{3}}$ $s=\frac{3s}{2}$

(2) $\Rightarrow 2s-2\left(\frac{2s}{3}\right)-b=0$

$b=\frac{6s-4s}{3}$

$b=\frac{2s}{3}$

$c=2s-a-b$

$=2s-\frac{2s}{3}-\frac{2s}{3}$

$=\frac{6s-4s}{3}$

$c=\frac{2s}{3}$

$\therefore a=b=c=\frac{2s}{3}$.

$\therefore$ The Δ^{le} is equilateral Δ^{le}.

$$r = \frac{\partial^2 f}{\partial a^2} = -2s(s-b)$$

$$s = \frac{\partial^2 f}{\partial a \partial b} = -s(3s-2a-2b)$$

$$t = \frac{\partial^2 f}{\partial b^2} = -2s(s-a)$$

At $s = \frac{3a}{2}$, $a = b$

$$r = \frac{-3a^2}{2} < 0$$

$$s = \frac{-3a^2}{4}$$

$$t = \frac{-3a^2}{2}$$

$$rt - s^2 = \frac{27a^4}{16} > 0$$

Hence f is maximum

i.e., $f = a^2$ is maximum.

#6

Lagranges method of Undertermined Multiples:

Suppose it is required to find the extremum (max/min) for the function $f(x,y,z)$, subject to the condition $\phi(x,y,z)=0 \rightarrow (1)$

Working rule

Step:-1

for lagranges function $f(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$ where λ is called the lagrange multiplier, which is determined by following conditions.

Step:2 Obtain, the eqnx

$$\frac{\partial f}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \rightarrow (2)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \rightarrow (3)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \rightarrow (4)$$

Step:-3

solve the eq. (1), (2), (3) & (4)

The values of x, y, z so obtained will gives you stationary point of $f(x,y,z)$

Note: To find the extremum for a function $f(x,y,z)$ subject to the condition (s/c) $\phi(x,y,z)=0$, $\psi(x,y,z)=0$, form the lagrangian functions as

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z) + \mu\psi(x,y,z)$$

where λ & μ over the lagrange multiple & proceed as above

Pending

(4) Discuss the max. (or) min. Values of u, if $u = ax^3y^2 - x^4y^2 - x^3y^3$

Sol: let $u = ax^3y^2 - x^4y^2 - x^3y^3$

$$\frac{\partial u}{\partial x} = 3ax^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow x^2y^2(3a - 4x - 3y) = 0$$

$$3a - 4x - 3y = 0 \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = 2ax^3y - 2x^4y - 3x^3y^2 = 0$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow x^3y(2a - 2x - 3y) = 0$$

$$2a - 2x - 3y = 0 = 0 \longrightarrow (2)$$

Solving (1) & (2)

$$3a - 4x - 3y = 0$$
$$2a - 4x - 3y = 0$$
$$- \quad + \quad$$
$$a - 2x = 0$$
$$\boxed{x = a/2}$$

$$(1) \Rightarrow 3a - 4\left(\frac{a}{2}\right) - 3y = 0$$
$$a - 3y = 0$$
$$\boxed{y = a/3}$$

$\therefore \left(\frac{a}{2}, \frac{a}{3}\right), (0,0)$ are stationary points

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(3ax^2y^2 - 4x^3y^2 - 3x^3y^3\right)$$

$$r = 6axy^2 - 12x^2y^2 - 6xy^3$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(2ax^3y - 2x^4y - 3x^3y^2\right)$$

$$s = \cancel{6ax^2y - 2x^4y - 3x^3}$$
$$6ax^2y - 8x^3y - 9x^2y^2$$

$$t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(2ax^3y - 2x^4y - 3x^3y^2\right)$$

$$t = 2ax^3 - 2x^4 - 6x^3y$$

At $\left(\frac{a}{2}, \frac{a}{3}\right)$

$$r = 6a\left(\frac{a}{2}\right)\left(\frac{a}{3}\right)^2 - 12\left(\frac{a}{2}\right)^2\left(\frac{a}{3}\right)^2 - 6\left(\frac{a}{2}\right)\left(\frac{a}{3}\right)^3$$

$$= \frac{\cancel{6}a^4}{\cancel{18}_{3}} - \frac{\cancel{12}a^4}{\cancel{36}_{3}} - \frac{\cancel{6}a^4}{\cancel{54}_{9}}$$

$$= \frac{a^4}{3} - \frac{a^4}{3} - \frac{a^4}{9}$$

$$r = \frac{-a^4}{9} < 0$$

$$s = 6a\left(\frac{a}{2}\right)^2\left(\frac{a}{3}\right) - 8\left(\frac{a}{2}\right)^3\left(\frac{a}{3}\right) - 9\left(\frac{a}{2}\right)^2\left(\frac{a}{3}\right)^2$$

$$= \frac{6a^4}{12} - \frac{8a^4}{24} - \frac{9a^4}{36}$$

$$= \frac{a^4}{2} - \frac{a^4}{3} - \frac{a^4}{4}$$

$$s = \frac{-a^4}{12}$$

$$t = 2a\left(\frac{a}{2}\right)^3 - 2\left(\frac{a}{2}\right)^4 - 6\left(\frac{a^3}{2}\right)\left(\frac{a}{3}\right)$$

$$= \frac{2a^4}{8} - \frac{2a^4}{16} - \frac{6a^4}{24}$$

$$= \cancel{\frac{a^4}{4}} - \frac{a^4}{8} - \cancel{\frac{a^4}{4}}$$

$$= \frac{-a^4}{8}$$

$$rt - s^2 = \left(-\frac{a^4}{9}\right)\left(-\frac{a^4}{8}\right) - \left(-\frac{a^4}{12}\right)^2$$

$$= \frac{a^8}{72} - \frac{a^8}{144}$$

$$= \frac{a^8}{144} > 0$$

$\therefore rt - s^2 > 0$, $r < 0$ the function has maxima at $(a/2, a/3)$

$$u_{max}\left(\frac{a}{2}, \frac{a}{3}\right) = a\left(\frac{a}{2}\right)^3\left(\frac{a}{3}\right)^2 - \left(\frac{a}{2}\right)^4\left(\frac{a}{3}\right)^2 - \left(\frac{a}{2}\right)^3\left(\frac{a}{3}\right)^3$$

$$= \frac{a^6}{72} - \frac{a^6}{144} - \frac{a^6}{216} = \frac{a^6}{432}$$