7/01/2022

UNIT : II. - Mean Value Theorems:

Continous and differentiable functions have many intresting properties, Some of which are studied through rolle's Theorem and Other mean value theorems.

Taylor's and Maclaurins series, which are generalisations of the mean value theorems are useful in approximating transcendental functions.

Continuity:

A function $f:[a,b] \to R$ is said to be continous at a point $c \in [a,b]$ if $\lim_{x \to c} f(x) = f(c)$

f is continous on the Open interval (a,b) if it is continous at every point 'c' of (a,b)

f is continous on the closed interval $[a,b]$ if

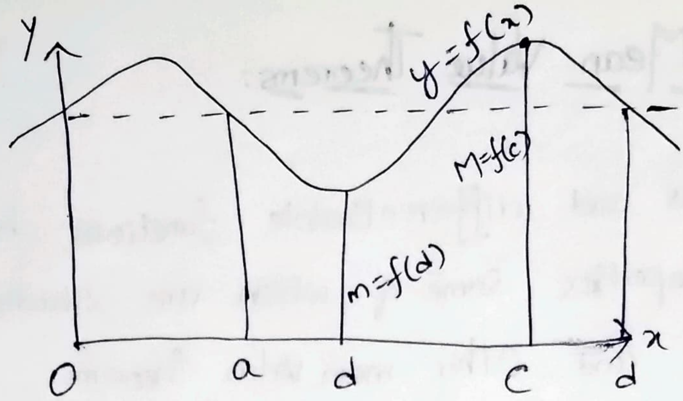
1. f is continous on the Open interval (a,b)
2. f is continous from the right at the left end point 'a' i.e., $f(a^+) = \lim_{x \to a^+} f(x) = f(a)$
3. f is continous from the left at the right end i.e., $f(b^-) = \lim_{x \to b^-} f(x) = f(b)$

* Properties of Continous function:

1. If f is continous in a closed interval $[a,b]$ then f is bounded in that interval and further it attains its bounds atleast once in $[a,b]$

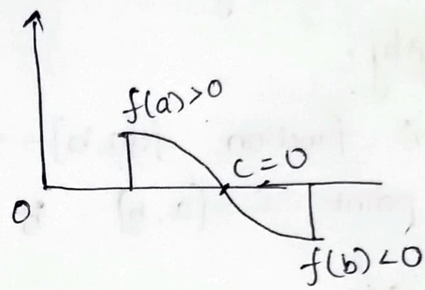
i.e., there exist $c, d \in [a,b]$ such that

$$f(c) = \operatorname{Sup} f = M$$
$$f(d) = \operatorname{Inf} f = m$$



② $f(x)$ attains every value between $f(a)$ and $f(b)$

③ If $f(a)>0$, $f(b)<0$, then there exists $c \in (a,b)$ such that $f(c)=0$



* Differentiability :

A function $f:[a,b] \to R$ is said to be differentiable (derivable) at a point $c \in (a,b)$ if $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists and is finite. The limit is called the derivative of 'f' at 'c' is denoted by $f'(c)$ (or) $\left(\frac{df}{dx}\right)_{x=c}$

f is derivable in (a,b) if 'f' is derivable at each point of (a,b). The function defined by those derived values is called the derivate $\frac{df}{dx}$ of f on (a,b)

Note :-

① let $x_1, x_2 \in R$ we define the distance b/w x_1 & x_2 by $d(x_1, x_2) = |x_1 - x_2|$ and call it the distance function

Clearly $d(x_1, x_2) = d(x_2, x_1)$

(2) let a be any real number. then the Open interval $(a-\delta, a+\delta)$ where $\delta>0$ is Called a δ-neighbourhood of a (nbd of a). If a is exclude from the nbd of a then it is Called a deleted nbd of a

* **limit of a function :**

Let f be a real valued function on a set 'S' and 'l' be a real number. The function 'f' is Said to tend to the limit 'l' as x tends to $a \in S$.
If, for a given real number $\varepsilon>0$, there exists a $\delta>0$ Such that

$$|f(x)-l| < \varepsilon \text{ whenever } 0<|x-a|<\delta$$

Symbolically, we write $\lim\limits_{x\to a} f(x) = l$

Right-handed limit : (l^+)

If, for a given real number $\varepsilon>0$ there exists a $\delta>0$ Such that $|f(x)-l^+|<\varepsilon$ whenever $a<x<a+\delta$; we Say that right handed limit l^+ exists for f at $x=a$ and write

$$\lim_{x\to a^+} f(x) = l^+$$

Left-handed limit : (l^-)

If, for a given real number $\varepsilon>0$ there exists a $\delta>0$ Such that $|f(x)-l^-|<\varepsilon$ whenever $a-\delta<x<a$, we Say that left-handed limit l^- exists for f at $x=a$ and write

$$\lim_{x\to a^-} f(x) = l^-$$

Note that $\lim\limits_{x\to a} f(x) = l \iff \begin{cases} 1.\ \lim\limits_{x\to a^+} f(x) = l^+ \\ 2.\ \lim\limits_{x\to a^-} f(x) = l^- \\ 3.\ l^+ = l^- = l \end{cases}$

Properties of continous and differentiable functions :

1. All polynomials always continous & differentiable.
2. Trigonometric function sin, cos are always continous and differentiable.
3. Exponential functions (e^x (or) e^{-x}) are always continous and differentiable.
4. functions defined in the interval (a, b) are always continous and differentiable.
5. Every differentiable function is continous.

Rolle's Theorem :

Statement: If $f:[a,b] \longrightarrow R$:s

(i) Continous in the closed interval $[a,b]$

(ii) Derivable in the open interval (a,b) and

(iii) $f(a) = f(b)$ then there exists atleast one point 'c' in (a,b) such that $f'(c) = 0$

Problms

① Verify Rolle's theorem for $f(x) = e^x \sin x$ in $[0, \pi]$

Sol: Given that $f(x) = e^x \sin x$

$$a = 0 , \quad b = \pi$$

(i) Since $\sin x$, e^x are both continous functions in $[0,\pi]$ so, $f(x) = e^x \sin x$ is also continuous

(ii) $f(x) = e^x \sin x$

$$f'(x) = e^x \cos x + \sin x \, e^x$$

$$= e^x (\cos x + \sin x)$$

$\therefore f(x)$ is derivable in $[0,\pi]$

(iii) $f(x) = e^x \sin x$

$$f(a) = f(0) = e^0 \sin 0 = 0$$

$$f(b) = f(\pi) = e^\pi \sin \pi = 0$$

$$\therefore f(a) = f(b)$$

for satisfies all the three conditions of rolle's theorem in $[0,\pi]$ then there exist atleast one value of 'c' of x in $[0,\pi]$ such that $f(c)=0$

i.e., $e^c(\cos c+\sin c)=0$

Since, $e^c \neq 0$, we have $\cos c+\sin c=0$

$$-\cos c = \sin c$$

$$\frac{\sin c}{\cos c} = -1$$

$$\tan c = -1$$

$$\tan c = -\tan \pi/4$$

$$\cancel{\theta} = \cancel{\pi/4} + \tan(\pi - \pi/4)$$

$$\therefore\ c = 3\pi/4 \in (0,\pi)$$

Hence, Rolle's theorem Verified.

③. Verify Rolle's theorem for the function $f(x)=\frac{\sin x}{e^x}$ (or) $e^{-x}\sin x$ in $[0,\pi]$

Sol: Given $f(x)=e^{-x}\sin x$

$a=0 \quad b=\pi$

(i) here e^{-x}, $\sin x$ both are continuous, so $f(x)=e^{-x}\sin x$ is also continuous

(ii) $f(x)=e^{-x}\sin x$

$$f'(x)=e^{-x}\cos x+\sin x\, e^{-x}(-1)$$

$$= \cancel{\cos x+\sin x}\ e^{-x}(\cos x-\sin x)$$ exists every value of x in $(0,\pi)$

$\therefore f(x)$ is derivable

(iii) $f(x)=e^{-x}\sin x$

$$f(a)=f(0)=e^{-0}\sin 0=0$$

$$f(b)=f(\pi)=e^{-\pi}\sin\pi=0 \qquad \therefore f(a)=f(b)$$

$f(x)$ satisfies all the three conditions of rolle's theorem

So, there exist a $c\in[0,\pi]$ such that $f'(c)=0$

i.e., $f'(c)=e^{-c}(\cos c-\sin c)=0$

Since $e^{-c} \neq 0$, $\cos c - \sin c = 0$

$$\cos c = \sin c$$

$$1 = \frac{\sin c}{\cos c}$$

$$\tan c = 1$$

$$\tan c = \tan \pi/4$$

$$c = \pi/4 \in (0, \pi)$$

$\therefore$ Hence Rolle's theorem Verified.

③ Verify Rolle's theorem for the function $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ in $[a,b]$

Sol: Given that $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$

$$a = a \quad , b = b$$

$$f(x) = \log(x^2+ab) - \log(x(a+b))$$

$$= \log(x^2+ab) - \log x - \log(a+b)$$

(i) since $f(x)$ is composite continuous functions in $[a,b]$, so $f(x)$ is continuous in $[a,b]$.

(ii) $f'(x) = \frac{1}{x^2+ab} \cdot 2x - \frac{1}{x} - 0$

$$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} = \frac{2x^2 - x^2 - ab}{x(x^2+ab)}$$

$f'(x) = \frac{x^2-ab}{x(x^2+ab)}$ exists for every value of x in (a,b)

(iii) $f(a) = \log\left(\frac{a^2+ab}{a(a+b)}\right) = \log\left(\frac{(a^2+ab)}{(a^2+ab)}\right) = \log 1 = 0$

$f(b) = \log\left(\frac{b^2+ab}{b(a+b)}\right) = \log\left(\frac{b^2+ab}{b^2+ab}\right) = \log 1 = 0$

$\therefore f(a) = f(b)$

$f(x)$ Satisfies all three conditions of rolle's theorem then there exist atleast one value of c of x in (a,b)

Such that $f'(c) = 0$

$$\frac{c^2-ab}{c(c^2+ab)} = 0$$

$$c^2 = ab$$

$$c = \sqrt{ab}$$

$\therefore c = \sqrt{ab} \in (a,b)$

$\therefore$ Hence, Rolle's theorem Verified

(4). Verify Rolle's theorem for $f(x) = x^2 - 2x - 3$ in $[-1, 3]$

Sol: Given that $f(x) = x^2 - 2x - 3$

$a = -1$, $b = 3$

(i) $f(x)$ is continuous because all Polynomials ~~exponentional~~ are Continuous

(ii) $f(x) = x^2 - 2x - 3$

$f'(x) = 2x - 2$ exists for every value of x in $(-1, 3)$

$\therefore f'(x)$ is derivable in $(-1, 3)$

(iii) $f(a) = f(-1) = (-1)^2 - 2(-1) - 3 = 0$

$f(b) = f(3) = 3^2 - 2(3) - 3 = 0$

$f(a) = f(b)$

$\therefore f(x)$ Satisfies all three conditions of Rolle's theorem then there exist atleast one value of c of x in $(-1, 3)$ such that $f'(c) = 0$

$2c - 2 = 0$

$2c = 2$

$\boxed{c = 1}$

$c = 1 \in (-1, 3)$, Hence Rolle's theorem Verified.

h/w

(5) (i) Verify Rolle's theorem for $f(x) = x^2 - 5x + 6$ in $(2, 3)$

Sol: Given that $f(x) = x^2 - 5x + 6$

$a = 2$, $b = 3$

(i) $f(x)$ is continuous, because all polynomials are continuous

(ii) $f(x) = x^2 - 5x + 6$

$f'(x) = 2x - 5$ exists for every value of x in $(2, 3)$

$\therefore f(x)$ is derivable.

(iii) $f(a) = f(2) = (2)^2 - 5(2) + 6 = 4 - 10 + 6 = 0$

$f(b) = f(3) = (3)^2 - 5(3) + 6 = 9 - 15 + 6 = 0$

$\therefore f(a) = f(b)$.

$f(x)$ satisfies all three conditions of Rolle's theorem

Then there exist atleast one value for c of x in $(2,3)$

Such that $f'(c) = 0$

$$2c - 5 = 0$$

$$2c = 5$$

$$\boxed{c = \frac{5}{2}} = 2.5$$

$\therefore$ Hence, Rolle's theorem verified.

h/w

(ii) Verify Rolle's theorem for $f(x) = 9x^3 - 4x$ in $[-\frac{2}{3}, \frac{2}{3}]$

Sol: Given that $f(x) = 9x^3 - 4x$

$$a = -\frac{2}{3}, \quad b = \frac{2}{3}$$

(i) $f(x)$ is continuous, because all polynomials are continuous

(ii) $f(x) = 9x^3 - 4x$

$f'(x) = 27x^2 - 4$ exist for every value of x in $(-\frac{2}{3}, \frac{2}{3})$

$\therefore f(x)$ is derivable

(iii) $f(a) = f(-2/3) = 9(-\frac{2}{3})^3 - 4(-2/3)$

$$= 9\left(\frac{-8}{27}\right) - 4(-2/3)$$

$$= -\frac{8}{3} + \frac{8}{3} = 0$$

$f(b) = f(2/3) = 9(2/3)^3 - 4(2/3) = \frac{8}{3} - \frac{8}{3} = 0$

$f(x)$ satisfies all three conditions of rolles theorem then there exist atleast one value of c such that $f'(c) = 0$

$$27c^2 - 4 = 0$$

$$c^2 = \frac{4}{27}$$

$$c = \sqrt{4/27}$$

$$\boxed{c = \frac{2}{\sqrt{27}}}$$

h/w

6) Using Rolle's theorem, S.T $g(x) = 8x^3 - 6x^2 - 2x + 1$ has a zero b/w 0 and 1.

Sol: Given that $g(x) = 8x^3 - 6x^2 - 2x + 1$

$$a = 0, \quad b = 1$$

(i) $g(x)$ is continuous

(ii) $g(x) = 8x^3 - 6x^2 - 2x + 1$

$g'(x) = 24x^2 - 12x - 2$ exist for every value of x in $(0,1)$

(iii) $g(a) = g(0) = 8(0)^3 - 6(0)^2 - 2(0) + 1 = 1$

$g(b) = g(1) = 8 - 6 - 2 + 1 = 1$

$\therefore g(a) = g(b)$

$g(x)$ satisfies all three conditions of rolle's theorem, then there exists atleast one value of c of x in $(0,1)$ such that $g'(c)=0$

$$24c^2 - 12c - 2 = 0$$
$$\Rightarrow 12c^2 - 6c - 1 = 0$$
$$c = \frac{3 \pm \sqrt{21}}{12} \Rightarrow c = \frac{3+\sqrt{21}}{12}$$
$$\Rightarrow \boxed{c = 0.6} \Rightarrow \boxed{c = \frac{3+\sqrt{21}}{12}}$$

$\therefore$ Rolle's theorem verified.

(7). Verify Rolles theorem for the function $f(x) = (x-a)^m (x-b)^n$, where m, n are +ve integers in $[a,b]$

Sol: Given that $f(x) = (x-a)^m (x-b)^n$

(i) $f(x)$ is continuous

(ii) $f(x) = (x-a)^m (x-b)^n$

$$f'(x) = (x-a)^m n(x-b)^{n-1} + (x-b)^n m(x-a)^{m-1}$$
$$= \frac{n(x-a)^m (x-b)^n}{x-b} + \frac{m(x-a)^m (x-b)^n}{x-a}$$
$$= (x-a)^m (x-b)^n \left(\frac{n}{x-b} + \frac{m}{x-a} \right)$$
$$= (x-a)^m (x-b)^n \left[\frac{n(x-a) + m(x-b)}{(x-a)(x-b)} \right]$$
$$= (x-a)^{m-1} (x-b)^{n-1} (nx - na + mx - mb)$$

$f'(x) = (x-a)^{m-1}(x-b)^{n-1}\left[x(m+n) - (na+mb)\right]$ exists for every value of x in (a,b).

$\therefore f(x)$ is derivable in (a,b).

(iii) $f(x) = (x-a)^m (x-b)^n$

$f(a) = 0.(a-b)^n = 0$

$f(b) = (b-a)^m.0 = 0$

$f(a) = f(b)$

$\therefore f(x)$ satisfies all three conditions of rolle's theorem

~~in~~ then there exist, atleast one value of c of x in (a,b) such that $f'(c) = 0$

$$(c-a)^{m-1}(c-b)^{n-1}(c(m+n) - (na+mb)) = 0$$

Since $(c-a)^{m-1}(c-b)^{n-1} \neq 0$, $c(m+n) - (na+mb) = 0$

$$c(m+n) = na + mb$$

$$c = \frac{na+mb}{m+n} \in (a,b)$$

Hence, Rolle's theorem is Verified.

8. Verify Rolle's theorem for $f(x) = \frac{x^2-x-6}{x-1}$ in the interval $(-2,3)$

Sol: Given $f(x) = \frac{x^2-x-6}{x-1}$ $a=-2$, $b=3$

(i) $f(x)$ is continuous

(ii) $f(x) = \frac{x^2-x-6}{x-1}$

$$f'(x) = \frac{(x-1)(2x-1)-(x^2-x-6)(1)}{(x-1)^2}$$

$\therefore f'(x)$ becomes ∞ at $x=1$

$\therefore f(x)$ is not derivable

$\therefore$ Rolles theorem is not applicable.

9. Verify whether Rolle's theorem can be applied to the following functions in the intervals cited:

(i) $f(x) = \tan x$ in $[0,\pi]$ (ii) $f(x) = \frac{1}{x^2}$ in $(-1,1)$

(iii) $f(x) = x^3$ in $[1,3]$

Sol: (i) Given that $f(x) = \tan x$

$a=0$ $b=\pi$

$f(x)$ is discontinuous at $x = \pi/2$ as it is not defined there

$\therefore$ Rolle's theorem is not applicable

(ii) $f(x) = \frac{1}{x^2}$ $a=-1$; $b=1$

$f(x)$ is discontinous at $x=0$

$\therefore$ Rolle's theorem is not applicable

(iii) $f(x) = x^3$ $a=1$, $b=3$

Since $f(x)$ being a polynomial, it is continous for every value of 'x' in $[1,3]$

$f(x) = x^3$

$f'(x) = 3x^2$ exist for every value of x in $(1,3)$

$\therefore f(x)$ is derivable in $(1,3)$

~~f(x) is deri~~ $f(1) = 1^3 = 1$

$f(3) = 3^3 = 27$

$\therefore f(1) \neq f(3) \Rightarrow$

Rolle's theorem is not applicable

H/w

(10) Verify Rolles theorem for $f(x) = x(x+3)e^{-x/2}$ in the inter-
-val $[-3, 0]$

Sol: Given that $f(x) = x(x+3)e^{-x/2}$

(i) $f(x)$ is continous.

(ii) $f(x) = x(x+3)e^{-x/2}$

$$f'(x) = (2x+3)e^{-x/2} + (x^2+3x)e^{-x/2}\left(-\frac{1}{2}\right)$$

$$= e^{-x/2}\left[2x+3 - \frac{1}{2}(x^2+3x)\right]$$

$f'(x) = -\frac{1}{2}(x^2-x-6)e^{-x/2}$ exist for every value of x in $(-3,0)$.

(iii) $f(a) = f(-3) = -3((-3)+3)e^{+3/2} = 0$

$f(b) = f(0) = 0(0+3)e^{0/2} = 0$.

$f(a) = f(b)$

$f(x)$ Satisfies all three conditions of Rolles theorem, then there have atleast one value for c of x in $(-3,0)$. Such $f'(c) = 0$

$$-\frac{1}{2}(c^2-c-6)e^{-c/2} = 0$$

$$c^2 - c - 6 = 0$$

$$c = 3, -2$$

here $c = 3$ is not applicable so,

$\boxed{c = -2}$ $c \in (-3, 0)$.

$\therefore$ Hence, Rolles theorem is ~~proved~~ Verified.

(5) $\frac{1}{2}$

(iii) $f(x) = 2x^3 + x^2 - 4x - 2$ in $(-\sqrt{3}, \sqrt{3})$

Sol: Given that $f(x) = 2x^3 + x^2 - 4x - 2$

$a = -\sqrt{3}$, $b = \sqrt{3}$

(i) $f(x)$ is continuous, because all polynomials are continuous

(ii) $f(x) = 2x^3 + x^2 - 4x - 2$

$f'(x) = 6x^2 + 2x - 4$, exist for every value of x in $(-\sqrt{3}, \sqrt{3})$

$\therefore f(x)$ is derivable.

(iii) $f(a) = f(-\sqrt{3}) = 2(-\sqrt{3})^3 + (-\sqrt{3})^2 - 4(-\sqrt{3}) - 2$

$= 2(-3\sqrt{3}) + 3 + 4\sqrt{3} - 2$

$= -6\sqrt{3} + 4\sqrt{3} + 1 = -2\sqrt{3} + 1$

$f(b) = f(\sqrt{3}) = 2(\sqrt{3})^3 + (\sqrt{3})^2 - 4(\sqrt{3}) - 2$

$= 2(3\sqrt{3}) + 3 - 4\sqrt{3} - 2$

$= 6\sqrt{3} - 4\sqrt{3} + 1$

$= 2\sqrt{3} + 1$

$f(a) \neq f(b)$

$\therefore$ Rolle's theorem is not applicable.

(5) (iv) $f(x) = (x+2)^3 (x-3)^4$ in $(-2, 3)$

Sol: Given $f(x) = (x+2)^3 (x-3)^4$

$a = -2$, $b = 3$

(i) $f(x)$ is continuous

(ii) $f(x) = (x+2)^3 (x-3)^4$

$f'(x) = 3(x+2)^2 (x-3)^4 + (x+2)^3 4(x-3)^3$

$= (x+2)^2 . (x-3)^3 . (3(x-3) + 4(x+2))$

$= (x+2)^2 . (x-3)^3 . (3x - 9 + 4x + 8)$

$f'(x) = (x+2)^2 . (x-3)^3 . (7x-1)$ exist for every value of x in $(-2, 3)$

(iii) $f(a) = f(-2) = ((-2)+2)^3 \cdot ((-2)-3)^4 = 0$

$f(b) = f(3) = (3+2)^3 \cdot (3-3)^4 = 0$

$f(a) = f(b)$

$f(x)$ satisfies all three conditions of rolles theorem,
then there exist atleast one value of c of x in $x(-2,3)$
such that $f'(c) = 0$

$$(c+2)^2 \cdot (c-3)^3 \cdot (7c-1) = 0$$

$(c+2)^2 = 0$ $\quad (c-3)^3 = 0.$ $\quad 7c-1 = 0$

$c = -2$ $\quad c = 3$ $\quad \boxed{c = \frac{1}{7}}$

$\boxed{c = \frac{1}{7}} \in (-2, 3).$

Lagranges Mean Value theorem:

Statement:
let $f(x)$ be a function such that
(i) It is continuous in closed interval $[a,b]$
(ii) It is derivable in (a,b) open interval
then there exist atleast one point 'c' in open interval (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

prblms

① Verify L.M.V.T for $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$

Sol: Given $f(x) = x^3 - x^2 - 5x + 3$

$a = 0, \; b = 4$

(i) $f(x)$ is continuous in $[0,4]$

(ii) $f(x) = x^3 - x^2 - 5x + 3$

$f'(x) = 3x^2 - 2x - 5$ exist for every value of x in $[0,4]$

$f(a) = f(0) = 0 + 3 = 3$

$f(b) = f(4) = (4)^3 - (4)^2 - 5(4) + 3 = 31$

$f(x)$ satisfies all the two conditions of Lagranges theorem
then, there exist atleast one value of c such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$3c^2 - 2c - 5 = \frac{31-3}{4-0}$$

$$3c^2 - 2c - 5 = 7$$

$$3c^2 - 2c - 12 = 0$$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$c = \frac{2 \pm \sqrt{4 - 4(3)(-12)}}{2(3)}$$

$$c = \frac{2 \pm \sqrt{4+144}}{6} = \frac{2 \pm \sqrt{148}}{6}$$

$$c = \frac{1 \pm \sqrt{37}}{3}$$

$$c = \frac{1+\sqrt{37}}{3}, \quad c = \frac{1-\sqrt{37}}{3}$$

$$\boxed{\therefore c = \frac{1+\sqrt{37}}{3}}$$

$\therefore$ Hence, the L.M.V.T is satisfies.

(2) Verify L.M.V.T for following functions in the interval indicated

(i) $\log_e x$ in $[1, e]$, (ii) $x(x-2)(x-3)$ in $[0,4]$

(iii) $2x^2 - 7x + 10$ in $[0,5]$

Sol: (i) Given $f(x) = \log_e x$ $\quad a = 1, \quad b = e$

$f(x)$ is continuous, since log functions are continuous for every value of x

$\rightarrow f(x) = \log_e x$

$f'(x) = 1/x$ exist for every value of x in $(1, e)$

$f(a) = f(1) = \log_e 1 = 0$

$f(b) = f(e) = \log_e e = 1 \qquad f(a) \neq f(b)$

$f(x)$ satisfies all the two conditions of L.M.V.T, so there exist atleast one value of c in $(1, e)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{c} = \frac{1-0}{e-1}$$

$$\frac{1}{c} = \frac{1}{e-1}$$

$$c = e-1 \in (1, e)$$

$\therefore$ Hence L.M.V.T is satisfies.

(ii) $f(x) = x(x-2)(x-3)$

$a=0, \ b=4$

$\Rightarrow f(x) = (x^2-2x)(x-3)$

(i) $f(x)$ is continuous, all polynomials are continuous.

$f'(x) = (x^2-2x)(x-3)$

$= x^2(x-3) - 2x(x-3)$

$= x^3 - 3x^2 - 2x^2 + 6x$

$f(x) = x^3 - 5x^2 + 6x$

$f'(x) = 3x^2 - 10x + 6$ exist for every value of x in $(0,4)$

$f(a) = f(0) = 0^3 - 3(0)^2 - 2(0) + 6(0) = 0$

$f(b) = f(4) = 4^3 - 3(4)^2 - 2(4) + 0$

$= 64 - 3(16) - 8$

$= 64 - 48 - 8$

$= 64 - 56 = 8$

$f(a) \neq f(b)$

$f(x)$ satisfies all the two conditions of L.M.V.T, so then there exist at least one value of c in $(0,4)$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$3c^2 - 10c + 6 = \frac{8-0}{4-0}$$

$3c^2 - 10c + 6 = 2$

$3c^2 - 10c + 6 - 2 = 0$

$3c^2 - 10c + 4 = 0$

$a=3, \ b=-10, \ c=4$

$$c = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{10 \pm \sqrt{100-4(3)(4)}}{2(3)}$$

$$= \frac{10 \pm \sqrt{100-48}}{6} = \frac{10 \pm \sqrt{52}}{6}$$

$$c = \frac{5 \pm \sqrt{13}}{3}$$

$$c = \frac{5+\sqrt{13}}{3}, \quad c = \frac{5-\sqrt{13}}{3}$$

$\therefore$ Hence, L.M.V.T is satisfies h/w

(iii) Given $2x^2 - 7x + 10 = f(x)$

$a = 0$, $b = 5$

$f(x)$ is continuous

$f(x) = 2x^2 - 7x + 10$

$f'(x) = 4x - 7$ exist for every value of x in $(0, 5)$

$f(a) = f(0) = 2(0)^2 - 7(0) + 10 = 10$

$f(b) = f(5) = 2(5)^2 - 7(5) + 10 = 2(25) - 35 + 10$

$= 60 - 35$

$= 25$

$f(a) \neq f(b)$

$f(x)$ satisfies all the two conditions of L.M.V.T, then there exist at least one value of c, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$4c - 7 = \frac{25 - 10}{5 - 0}$$

$$4c - 7 = \frac{\cancel{15}\,3}{\cancel{5}}$$

$$4c - 7 - 3 = 0$$

$$4c - 10 = 0$$

$$c = \frac{\cancel{10}\,5}{\cancel{4}\,2} = 5/2$$

$$\boxed{c = 5/2}$$

∴ Hence, L.M.V.T is satisfies

(3) If $a < b$, P.T $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$ using L.M.V.T deduce the following (i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

(ii) $\frac{5\pi + 4}{20} < \tan^{-1} 2 < \frac{\pi + 2}{4}$

Sol: let $f(x) = \tan^{-1} x$ in $[a, b]$ for $a < b$

$f(x) = \tan^{-1} x$ is continuous in $[a, b]$

$f'(x) = \frac{1}{1+x^2}$ exist for every value of x in (a, b)

$f(a) = \tan^{-1} a$ $\qquad f(b) = \tan^{-1} b$

$f(a) \neq f(b) \Rightarrow f(x)$ satisfies all the conditions of L.M.V.T then, there exist a point 'c' in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a} \longrightarrow (1)$$

$c \in (a,b)$ and given that $a < b$

$\therefore a < c < b$ (S.O. Bach term)

$a^2 < c^2 < b^2$ (adding 1 to each term)

$1+a^2 < 1+c^2 < 1+b^2$.

Taking reciprocal of each term

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$ from eq. (1)

$$\frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{b-a} > \frac{1}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \tan^{-1} b - \tan^{-1} a > \frac{b-a}{1+b^2}$$

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

Deduction

(1) put $b = 4/3$ & $a = 1$ we get

$$\frac{4/3 - 1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1^2}$$

$$\frac{1/3}{25/9} < \tan^{-1}\frac{4}{3} - \tan^{-1}(\tan \pi/4) < \frac{1/3}{1+1}$$

$$3/25 < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

adding $\pi/4$ to each term

$$\pi/4 + 3/25 < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \pi/4$$

$\therefore$ Hence it is proved.

(ii) put $b=2$ & $a=1$ in ② we get

$$\frac{2-1}{1+2^2} < \tan^{-1} 2 - \tan^{-1} 1 < \frac{2-1}{1+1^2}$$

$$\frac{1}{5} < \tan^{-1} 2 - \tan^{-1}(\tan \pi/4) < \frac{1}{2}$$

$$\frac{1}{5} < \tan^{-1} 2 - \pi/4 < \frac{1}{2}$$

adding $\pi/4$ to each term

$$\frac{1}{5} + \frac{\pi}{4} < \tan^{-1} 2 < \frac{1}{2} + \frac{\pi}{4}$$

$$\frac{5\pi+4}{20} < \tan^{-1} 2 < \frac{\pi+2}{4}$$

4) prove that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{5} < \frac{\pi}{6} + \frac{1}{8}$ (L.M.V.T)

Sol: let $f(x) = \sin^{-1} x$ in $[a,b]$

(i) $f(x)$ is continuous

(ii) $f'(x) = \frac{1}{\sqrt{1-x^2}}$ exist for every value of x in (a,b)

$f(a) = \sin^{-1} a$ $\qquad$ $f(b) = \sin^{-1}(b)$

$f(x)$ satisfies both the condition of LMVT. so, then there exist a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1} b - \sin^{-1} a}{b-a} \longrightarrow (1)$$

Since $c \in (a,b)$ we have $a<c<b$

$$a^2 < c^2 < b^2$$

$$-a^2 > -c^2 > -b^2$$

$$1-a^2 > 1-c^2 > 1-b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b-\sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$$

Put $b=\frac{3}{5}$, $a=\sin 30 = \frac{1}{2}$

$$\frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} < \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{1}{2}\right) < \frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\left(\frac{3}{5}\right)^2}}$$

$$\frac{\left(\frac{6-5}{10}\right)}{\sqrt{1-\frac{1}{4}}} < \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\sin\frac{\pi}{6}\right) < \frac{1/10}{\sqrt{1-\frac{9}{25}}}$$

$$\frac{1/10}{\sqrt{3/4}} < \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}(\sin \pi/6) < \frac{1/10}{\sqrt{16/25}}$$

$$\frac{1/10}{\sqrt{3}/2} < \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}(\sin \pi/6) < \frac{1/10}{4/5}$$

$$\frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) - \pi/6 < \frac{1}{8}$$

$$\frac{\pi}{6}+\frac{1}{5\sqrt{3}} < \sin^{-1}(3/5) < \frac{1}{8}+\pi/6$$

$\therefore$ Hence, proved.

(5) P.T $\frac{\pi}{3}-\frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3}-\frac{1}{8}$ using L.M.V.T

Sol: let $f(x) = \cos^{-1}x$

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x) = \cos^{-1}x$

$f'(x) = \frac{-1}{\sqrt{1-x^2}}$ exist every value of x in (a,b)

$f(a) = \cos^{-1}a$ $\quad$ $f(b) = \cos^{-1}b$

$f(x)$ satisfies all the conditions of L.M.V.T, then there exist a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1} b - \cos^{-1} a}{b-a} \longrightarrow (1)$$

since , $c \in (a,b)$ we have $a < c < b$

$$a^2 < c^2 < b^2$$

$$-a^2 > c^2 > b^2$$

$$1-a^2 > 1-c^2 > 1-b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1} b - \cos^{-1} a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-(b-a)}{\sqrt{1-a^2}} > \cos^{-1} b - \cos^{-1} a > \frac{-(b-a)}{\sqrt{1-b^2}}$$

put $b = \frac{3}{5}$, $a = \cos\left(\frac{\pi}{3}\right) = \cos 60 = 1/2$

$a = 1/2$

$$\frac{-\left(\frac{3}{5} - \frac{1}{2}\right)}{\sqrt{1-\left(\frac{1}{2}\right)^2}} > \cos^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{1}{2}\right) > \frac{-\left(\frac{3}{5} - \frac{1}{2}\right)}{\sqrt{1-\left(\frac{3}{5}\right)^2}}$$

$$\frac{-1/10}{\sqrt{3}/2} > \cos^{-1}\frac{3}{5} - \cos^{-1}\left(\cos\frac{\pi}{3}\right) > \frac{-1/10}{4/5}$$

$$\frac{-1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{3} > \frac{-1}{8}$$

adding $\pi/3$ to each term

$$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}$$

//.

6. S.T for any $x > 0$, $1 + x < e^x < 1 + xe^x$.

Sol: let $f(x)$ is e^x in $[0, x]$

(i) $f(x)$ is continuous in $[0, x]$

(ii) $f(x) = e^x$

$f'(x) = e^x$ exist every value of x in $(0, x)$

$f(a) = e^0 = 1$ $\quad$ $f(b) = e^x = e^x$

$f(x)$ satisfies all the conditions of L.M.V.T then there exist a point $c \in (0, x)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$e^c = \frac{e^x - \cancel{0}}{x - 0} \quad \text{—— (1)}$$

Since $c \in (0, x)$ we have $a < c < b$

$$e^a < e^c < e^b$$

$$e^a < \frac{e^b - e^a}{b - a} < e^b$$

put $a = 0$, $b = x$

$$1 < \frac{e^x - 1}{x} < e^x$$

$$x < e^x - 1 < xe^x$$

$$1 + x < e^x < 1 + xe^x$$

$\therefore$ Hence proved.

7. Calculate approximately $\sqrt[5]{245}$ by using L.M.V.T

Sol: let $f(x) = \sqrt[5]{x} = x^{1/5}$ and take $b = 245$ & $a = 243 = 3^5$

(i) $f(x)$ is continuous in $[243, 245]$

(ii) $f(x) = x^{1/5}$

$f'(x) = \frac{1}{5} x^{-4/5}$ exist for every value of x in $(243, 245)$

$f(b) = 245$, $f(a) = 243 = (3^5)^{1/5} = 3$

By L.M.V.T

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{5}c^{-4/5} = \frac{f(245)-f(243)}{245-243}$$

$$\frac{1}{5}c^{-4/5} = \frac{f(245)-f(243)}{2}$$

$$f(245) = \frac{2}{5}c^{-4/5} + f(243)$$

$243 < c < 245$

put $c = 243$

$$f(245) = \frac{2}{5}(243)^{-4/5} + f(243)$$

$$= \frac{2}{5}(3^{5})^{-4/5} + 3$$

$$= \frac{2}{5}(3^{-4}) + 3$$

$$= 3 + 0.0049$$

$$f(245) = 3.0049$$

Cauchy Mean Value Theorem :

Statement : If $f: [a,b] \to R$; $g: [a,b] \to R$ are such that

(i) f,g are continuous on $[a,b]$

(ii) f,g are derivable on (a,b)

(iii) $g'(x) \neq 0$, $x \in (a,b)$

then there exist a point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

problms :

① Verify Cauchy M.V.T for $f(x) = e^{x}$, $g(x) = e^{-x}$ in $(3,7)$ & find the value of c.

Sol: Given that $f(x) = e^{x}$, $g(x) = e^{-x}$

$a = 3$, $b = 7$

(i) Since all exponentials functions are continuous, both f,g

are continuous in $[3,7]$

(ii) $f(x)=e^x$ $\quad g(x)=e^{-x}$

$f'(x)=e^x$ $\quad g'(x)=-e^{-x}$

$f(a)=f(3)=e^3$ $\quad g(a)=g(3)=e^{-3}$

$f(b)=f(7)=e^7$ $\quad g(b)=g(7)=e^{-7}$

(iii) $g'(x)=-e^{-x}\neq 0$ for any Value of x in $(3,7)$

$\therefore f,g$ Satisfies all the three conditions of C.M.V.T then there exist a point c in (a,b) such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{e^c}{-e^{-c}}=\frac{e^7-e^3}{e^{-7}-e^{-3}}$$

$$-e^{2c}=\frac{e^7-e^3}{\frac{1}{e^7}-\frac{1}{e^3}} \Rightarrow \frac{e^7-e^3}{\frac{e^3-e^7}{e^{10}}}$$

$$-e^{2c}=\frac{\cancel{(e^7-e^3)}}{1}\times\frac{e^{10}}{-\cancel{(e^3-e^7)}}$$

$$\cancel{-}\cancel{e}^{2c}=\cancel{-}\cancel{e}^{10}$$

$$2c=10$$

$$\boxed{c=5}\in(3,7)$$

Hence CMVT is proved

② find 'c' of CMVT for $f(x)=\sqrt{x}$, $g(x)=\frac{1}{\sqrt{x}}$ in $[a,b]$ $0<a<b$

Sol: Given $f(x)=\sqrt{x}$, $g(x)=\frac{1}{\sqrt{x}}$

$a=a$, $b=b$

(i) $f(x)$, $g(x)$ are continuous in $[a,b]$

(ii) $f(x)=\sqrt{x}$ $\quad g(x)=\frac{1}{\sqrt{x}}$

$f'(x)=\frac{1}{2\sqrt{x}}$ $\quad g'(x)=\frac{-1}{2x\sqrt{x}}$

$f(a)=\sqrt{a}$ $\quad g(a)=\frac{1}{\sqrt{a}}$

$f(b)=\sqrt{b}$ $\quad g(b)=\frac{1}{\sqrt{b}}$

$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)=-\frac{1}{2}x^{-1/2-1}$

$=-\frac{1}{2}x^{-3/2}$

$=\frac{-1}{2\sqrt{x^3}}$

$\frac{d}{dx}\frac{1}{x\cdot\sqrt{x}}=\frac{-1}{2x\sqrt{x}}$

(iii) $g'(x) = \frac{-1}{2x\sqrt{x}} \neq 0$ for any value of x in (a,b)

$\therefore$ f, g are satisfies all three conditions of CMVT, then there exist a point c in (a,b) Such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\frac{1}{\cancel{2\sqrt{c}}} \times \frac{\cancel{2}c\cancel{\sqrt{c}}}{-1} = \frac{\sqrt{b}-\sqrt{a}}{\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}\right)}$$

$$-c = \frac{\cancel{\sqrt{b}-\sqrt{a}}}{1} \times \frac{\sqrt{ab}}{-(\cancel{\sqrt{a}-\sqrt{b}})}$$

$$\cancel{-}c = \cancel{-}\sqrt{ab}$$

$$\boxed{c = \sqrt{ab}} \in (a,b)$$

$\therefore$ Hence Verified.

(3) Verify C.M.V.T for $f(x)=x^2$, $g(x)=x^3$ in $[1,2]$

Sol: Given $f(x) = x^2$, $g(x)=x^3$

$a=1$, $b=2$

(i) $f(x)$, $g(x)$ are continuous in $[1,2]$

(ii) $f(x) = x^2$, $g(x) = x^3$

$f'(x) = 2x$, $g'(x) = 3x^2$

$f(a) = 1$ $\quad g(a) = 1$

$f(b) = 4$ $\quad g(b) = 8$

(iii) $g'(x) = 3x^2 \neq 0$ for any value of x in $(1,2)$

f, g satisfies all the three conditions of CMVT, then there exist a point c in (1,2), Such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{2c}{3c^2} = \frac{4-1}{8-1}$$

$$\frac{2c}{3c^2} = \frac{3}{7}$$

$14\cancel{c} = 9c^{\cancel{2}}$

$\boxed{\frac{14}{9} = c} \in (1,2)$

$\therefore$ Hence Verified.

(4). Verify C.M.V.T for $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ on $[a,b]$

Sol: given $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$

$f(x) = x^{-2}$, $g(x) = x^{-1}$

(i) $f(x), g(x)$ are continuous in $[a,b]$

(ii) $f(x) = x^{-2}$ $\quad g(x) = x^{-1}$

$f'(x) = -2x^{-3}$ $\quad g'(x) = -x^{-2}$

$f(a) = a^{-2}$ $\quad g(a) = a^{-1}$

$f(b) = b^{-2}$ $\quad g(b) = b^{-1}$

(iii) $g'(x) = -x^{-2} \neq 0$ for any value of x in (a,b)

f, g satisfies all three conditions of CMVT then, there exist a point $c \in (a,b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\cancel{-}2c^{-3}}{\cancel{-}c^{-2}} = \frac{b^{-2}-a^{-2}}{b^{-1}-a^{-1}}$$

$$2c^{-1} = \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}} \Rightarrow \frac{\frac{a^2-b^2}{a^2b^2}}{\frac{a-b}{ab}}$$

$$2c^{-1} = \frac{a^2-b^2}{a^2b^2} \times \frac{ab}{a-b}$$

$$2c^{-1} = \frac{(a+b)\cancel{(a-b)}}{ab\cancel{(a-b)}}$$

$\boxed{c = \frac{2ab}{a+b}} \in (a,b)$

$\therefore$ Hence Verified.

⑤ Verify CMVT for $f(x) = \sin x$, $g(x) = \cos x$ on $[0, \pi/2]$

Sol: given $f(x) = \sin x$, $g(x) = \cos x$

$a = 0$, $b = \pi/2$

(i) $f(x), g(x)$ are continuous in $[0, \pi/2]$

(ii) $f(x) = \sin x$ $\quad$ $g(x) = \cos x$

$f'(x) = \cos x$ $\quad$ $g'(x) = -\sin x$

$f(a) = \sin 0 = 0$ $\quad$ $g(a) = \cos 0 = 1$

$f(b) = \sin \pi/2 = 1$ $\quad$ $g(b) = \cos \pi/2 = 0$

(iii) $g'(x) = -\sin x \neq 0$ for any value of x in $(0, \pi/2)$

~~for~~ f, g satisfies all the three conditions of CMVT then,

there exist a point $c \in [0, \pi/2]$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\cos c}{-\sin c} = \frac{1-0}{0-1}$$

$$\frac{\cos c}{\cancel{-}\sin c} = \frac{\cancel{-}1}{1}$$

$$\cot c = 1$$

$$\cancel{\cot}\, c = \cancel{\cot}\, \pi/4$$

$$\boxed{c = \pi/4} \in (0, \pi/2)$$

(or)

$$\frac{\cos c}{-\sin c} = \frac{\sin b - \sin a}{\cos b - \cos a}$$

$$= \frac{\sin \pi/2 - \sin 0}{\cos \pi/2 - \cos 0}$$

$$\cancel{-}\tan c = \cancel{-}1$$

$$\cancel{\tan}\, c = \cancel{\tan}\, \pi/4$$

$$\boxed{c = \pi/4}$$

∴ Hence proved.

⑥ If $f(x) = \log x$, $g(x) = x^2$ in (a, b) with $b > a > 1$, using CMVT P.T. $\dfrac{\log b - \log a}{b - a} = \dfrac{a+b}{2c^2}$

Sol: Given $f(x) = \log x$, $g(x) = x^2$

(i) $f(x), g(x)$ are continuous on $[a, b]$

(ii) $f(x) = \log x$ $\quad$ $g(x) = x^2$

$f'(x) = 1/x$ $\quad$ $g'(x) = 2x$

$f(a) = \log a$ $\quad\quad$ $g(a) = a^2$

$f(b) = \log b$ $\quad\quad$ $g(b) = b^2$

(iii) $g'(x) = 2x \neq 0$ for any value of x in (a,b)

f, g satisfies all three conditions of CMVT. so there exist a point on $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\frac{1}{c}}{2c} = \frac{\log b - \log a}{b^2 - a^2}$$

$$\frac{1}{2c^2} = \frac{\log b - \log a}{(b-a)(b+a)}$$

$$\frac{a+b}{2c^2} = \frac{\log b - \log a}{b-a}$$

$\therefore$ Hence proved

7. state CMVT for $f(x) = e^{-x}$, $g(x) = e^{x}$ in $[2,6]$

Sol: given $f(x) = e^{-x}$, $g(x) = e^{x}$.

$a = 2$, $b = 6$

(i) $f(x), g(x)$ are continuous in $[2,6]$

(ii) $f(x) = e^{-x}$ $\quad\quad$ $g(x) = e^{x}$

$f'(x) = -e^{-x}$ $\quad\quad$ $g'(x) = e^{x}$

$f(a) = e^{-2}$ $\quad\quad$ $g(a) = e^{2}$

$f(b) = e^{-6}$ $\quad\quad$ $g(b) = e^{+6}$

f, g satisfies all three conditions of CMVT then there exist a point on $c \in (2,6)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{-e^{-c}}{e^{c}} = \frac{e^{-6} - e^{-2}}{e^{6} - e^{2}}$$

$$\cancel{+}\frac{1}{e^{2c}} = \frac{e^2 \cancel{-} e^6}{e^6 \cdot e^2} \times \frac{1}{\cancel{+}(e^2 \cancel{-} e^6)}$$

$$\frac{1}{e^{2c}} = \frac{1}{e^8}$$

$2c = 8$

$c = 4$ $\quad$ $4 \in (2,6)$

Hence Verified.

Generalized Mean Value Theorem :

Mean Value theorems, especially Taylors theorem play an important role in differentiation. the Values of a function and it Successive derivatives at a point help us in finding the Value of the function in the neighbourhood of that point Using Taylors theorem.

i.e., Taylors theorem provides expansion of $f(a+h)$ in ascending powers of h and the derivatives of f and a

Taylor's Theorem :

Statement : If $f:[a,b]\to R$ is such that

ⓐ f^{n-1} is continuous on $[a,b]$

ⓑ f^{n-1} is derivable on (a,b) &

ⓒ $p\in z^+$ then three exist a point $c\in(a,b)$

Such that

$$f(b) = f(a) + \frac{b-a}{1!}f'(a) + \frac{(b-a)^2}{2!}f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + R_n$$

$$\text{Where, } R_n = \frac{(b-a)^p(b-c)^{n-p}f^n(c)}{(n-1)!\,p}$$

→ schlomilch - Roches form of Remainder

* Lagranges form of remainder

Puting $p=n$, we get

$$R_n = \frac{(b-a)^n}{n!}f^n(c)$$

$(n(n-1))! = n!$

* Cauchy's form of remainder :

putting $p=1$, we get

$$R_n = \frac{(b-a)(b-c)^{n-1}f^n(c)}{(n-1)!}$$

Another form of Taylors theorem:

If $f:[a, a+h] \to R$ is such that

(i) f^{n-1} is continuous on $[a, a+h]$

(ii) f^{n-1} is derivable on $(a, a+h)$ &

(iii) p is a ~~gi~~ given +ve integer, then there exist atleast one number θ in $(0,1)$ such that

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{(n-1)}}{(n-1)!} f^{n-1}(a) + R_n$$

Where,

$$R_n = \frac{h^n (1-\theta)^{n-p} f^n(a+\theta h)}{p(n-1)!}$$ → schlomilch - Roche's form of Remainder

imp. * **Lagranges form of Remainder:** $(p=n)$

$$R_n = \frac{h^n f^n(a+\theta h)}{n!} \quad (\text{or}) \quad \frac{h^n}{n!} f^n(a+\theta h)$$

* **Cauchy's ~~Mean~~ form of Remainder:** $(p=1)$

~~If f: [a, a+h] → R is such that~~

~~(i) f~~

$$R_n = \frac{h^n (1-\theta)^{n-1} f^n(a+\theta h)}{(n-1)!}$$

Maclaurin's Theorem:

If $f:[0,x] \to R$ is such that

(i) f^{n-1} is continuous on $[0,x]$

(ii) f^{n-1} is derivable on $(0,x)$ & $p \in Z^+$

then there exist a real number $\theta \in (0,1)$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{0!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n.$$

where

$$R_n = \frac{x^n (1-\theta)^{n-p} f^n(\theta x)}{p(n-1)!}$$ → schlomilch - Roches form of Remainder

* Lagrange's form of Remainder : (p=n)

$$R_n = \frac{x^n}{n!} f^n(\theta x)$$

* Cauchy's form of Remainder : (p=1)

$$R_n = \frac{x^n (1-\theta)^{n-1} f^n(\theta x)}{(n-1)!}$$

Note : If we put $a=0$, $b=x$ and $c=\theta x$ where $0<\theta<1$ in Taylors theorem, we get Maclaurins theorem, so there is no basic difference b/w Taylors and Maclaurins theorem.

(1) Obtain the Maclaurins series expansion of the following functions.

(i) e^x (ii) $\cos x$ (iii) $\sin x$ (iv) $\cosh x$ (v) $\sinh x$

sol: (ii) $\cos x = f(x)$

The Maclaurins series expansion of $f(x)$ is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots \rightarrow (1)$$

$f(x) = \cos x \Rightarrow f(0) = \cos 0 = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -\cos 0 = -1$

$f'''(x) = -(-\sin x) \Rightarrow f'''(0) = \sin 0 = 0$

$f^{iv}(x) = -(-\cos x) \Rightarrow f^{iv}(0) = \cos 0 = 1$

Substituting above values in $\rightarrow (1)$, we get

$$\cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

⇒ Verify rolles theorem for $f(x) = x^{2/3} - 2x^{1/3}$ in $(0,8)$

Sol: Given $f(x) = x^{2/3} - 2x^{1/3}$

$f(x)$ is continuous in $[0,8]$

$$f'(x) = \frac{2}{3} x^{2/3 - 1} - 2 \cdot \frac{1}{3} x^{1/3 - 1}$$

$$= \frac{2}{3} x^{-1/3} - \frac{2}{3} x^{-2/3}$$

$$= \frac{2}{3}\left(x^{-1/3} - x^{-2/3}\right)$$ exist for all values of x in $(0,8)$

$f(a) = f(0) = 0$

$$f(b) = f(8) = 8^{2/3} - 2(8)^{1/3}$$

$$= (2^{3})^{2/3} - 2(2^{3})^{1/3}$$

$$= 4 - 4 = 0$$

$f(x)$ satisfies all conditions of Rolles Theorem

then there exist a value $c \in (0,8)$ such that

$$f'(c) = 0$$

$$\frac{2}{3}\left(c^{-1/3} - c^{-2/3}\right) = 0$$

$$c^{-1/3} - c^{-1/3} \cdot c^{-1/3} = 0$$

$$c^{-1/3}\left(1 - c^{-1/3}\right) = 0$$

$$1 - c^{-1/3} = 0$$

$$c^{-1/3} = 1$$

$$\sqrt[3]{c^{-1}} = 1$$

$$\sqrt[3]{\frac{1}{c}} = 1$$

$$\frac{1}{c} = 1$$

$$\boxed{c = 1} \in (0,8)$$

L.M.V.T

$\sqrt[6]{65}$

$f(x) = x^{1/6}$

$a = 64, \ b = 65$

$f'(x) = \frac{1}{6} x^{1/6 - 1}$

$= \frac{1}{6} x^{-5/6}$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

put $c = 64$

$f(a) = f(64)$

$= (2^6)^{1/6}$

$= 2$

$\frac{1}{6} c^{-5/6} = \frac{f(65) - 2}{65 - 64}$

$\frac{2}{6} c^{-5/6} = f(65) - 2$

$f(65) = \frac{2}{6} c^{-5/6} + 2$

$c = 2.0104$

---

1. e^x

The maclaurins series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots \rightarrow (1)$$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^x = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^x = e^0 = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = e^x = e^0 = 1$

$f^{iv}(x) = e^x \Rightarrow f^{iv}(0) = e^x = e^0 = 1$

Substituting above values in (1).

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

(iii) $\sin x$

The maclaurins series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) \rightarrow (1)$$

$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{iv}(x) = -(-\sin x) \Rightarrow f^{iv}(0) = \sin 0 = 0$

$f^{v}(x) = \cos x \Rightarrow f^{v}(0) = \cos 0 = 1$

Sub in (1).

$$\sin x = 0 + \frac{x}{1!} + 0 - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

(iv) $\cosh x$

The maclaurins series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) \rightarrow (1)$$

$f(x) = \cosh x \Rightarrow f(0) = \cosh 0 = 1$

$f'(x) = \sinh x \Rightarrow f'(0) = \sinh 0 = 0$

$f''(x) = \cosh x \Rightarrow f''(0) = \cosh 0 = 1$

$f'''(x) = \sinh x \Rightarrow f'''(0) = \sinh 0 = 0$

$f^{iv}(x) = \cosh x \Rightarrow f^{iv}(0) = \cosh 0 = 1$

Sub in eq (1)

$\sin x = 1 +$

② Obtain the Taylor's Series expansion of $\sin x$ in powers of $x - \frac{\pi}{4}$

Sol: let $f(x) = \sin x$

The taylor series expansion for $f(x)$ in powers of $(x-a)$ is

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(f-a)^3}{3!} f'''(a) + \ldots \text{①}$$

here $a = \pi/4$

$f(x) = \sin x \Longrightarrow f(a) = \sin \pi/4 = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Rightarrow f'(a) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f''(a) = -\sin \pi/4 = \frac{-1}{\sqrt{2}}$

$f'''(x) = -\cos x \Rightarrow f'''(a) = -\cos \pi/4 = \frac{-1}{\sqrt{2}}$

$f^{IV}(x) = -(-\sin x) = \sin x \Rightarrow f^{IV}(a) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

⋮

Sub above values in ①.

~~f(x)=~~

$$\sin x = \frac{1}{\sqrt{2}} + \frac{(x-\pi/4)}{1!}\left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^3}{3!}\left(\frac{-1}{\sqrt{2}}\right)$$

$$+ \frac{(x-\frac{\pi}{4})^4}{4!}\left(\frac{1}{\sqrt{2}}\right) + \ldots$$

$$\sin x = \frac{1}{\sqrt{2}}\left(1 + \frac{x-\pi/4}{1!} - \frac{(x-\pi/4)^2}{2!} - \frac{(x-\pi/4)^3}{3!} + \frac{(x-\pi/4)^4}{4!} + \ldots\right)$$

is req. sol.

③ Obtain the taylor's Series expansion of $\sin 2x$ in powers of $x-\frac{\pi}{4}$.

sol: given $f(x) = \sin 2x$

let $x-\frac{\pi}{4} = t$

$x = t+\frac{\pi}{4}$

$\sin 2x = \sin 2\left(t+\frac{\pi}{4}\right)$

$= \sin\left(2t+\frac{\pi}{2}\right)$

$= \sin\left(\frac{\pi}{2}+2t\right)$

$= \cos 2t$

$$\boxed{\text{w.k.T} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

$\sin 2x = 1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} + \cdots$

$= 1 - \frac{4t^2}{2!} + \frac{16}{4!}t^4 \cdots$

put $t = x-\pi/4$

$\sin 2x = 1 - \frac{4}{2!}(x-\pi/4)^2 + \frac{16}{4!}(x-\pi/4)^2 + \cdots$

④ Obtain the taylor's ~~theo~~ series expansion of e^x about $x=-1$

sol: let $f(x) = e^x$

let $t = x+1$

$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

put $t = x+1$

$e^{x+1} = 1 + \frac{(x+1)}{1!} + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \cdots$

$e^x . e = $ "

$e^x = \frac{1}{e}\left(\text{"} \ldots \right)$

5) Verify Taylor's Theorem for $f(x) = (1-x)^{5/2}$ with Lagranges form of Remainder upto 2 terms in interval $[0,1]$

Sol: let $f(x) = (1-x)^{5/2}$

(i) f, f' are continuous in $[0,1]$

(ii) $f'(x)$ is differentiable in $(0,1)$

Then The Taylor's theory expansion with Lagranges form of remainder upto 2 terms is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(\theta x) \longrightarrow ①$$

Here $n=2, \ a=0$

$$f(x) = (1-x)^{5/2} \Rightarrow f(0) = (1-0)^{5/2} = 1$$

$$f'(x) = \frac{5}{2}(1-x)^{5/2-1}(-1) = \frac{5}{2}(1-x)^{3/2}(-1) \Rightarrow f'(0) = -\frac{5}{2}(1-0)^{3/2} = -5/2$$

$$f''(x) = \left(-\frac{5}{2}\right)\left(\frac{3}{2}\right)(1-x)^{3/2-1}(-1)$$

$$f''(x) = \frac{15}{4}(1-x)^{1/2}$$

$$f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}$$

here $x=1$

$$f''(\theta) = \frac{15}{4}(1-\theta)^{1/2}$$

$$f(1) = (1-1)^{5/2} = 0$$

$$① \rightarrow f(1) = 1 + \frac{1}{1!}\left(\frac{-5}{2}\right) + \frac{1^2}{2!}\,\frac{15}{4}(1-\theta)^{1/2}$$

$$0 = 1 - \frac{5}{2} + \frac{15}{8}(1-\theta)^{1/2}$$

$$\frac{15}{8}(1-\theta)^{1/2} = \frac{3}{2}$$

$$(1-\theta)^{1/2} = \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{8}\,4}{\cancel{15}\,5}$$

$$(1-\theta)^{1/2} = \frac{4}{5}$$

$$1-\theta = \frac{16}{25}$$

$$\theta = 1 - \frac{16}{25}$$

$$\theta = \frac{9}{25}$$

$\theta = 0.36 \in (0,1)$

$\therefore$ Taylor's verified.

6) Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$ & hence deduce that $\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots$

Sol: let $f(x) = \log(1+e^x)$

Maclaurin's series expansion for $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots \rightarrow (1)$$

$$f(x) = \log(1+e^x) \Rightarrow f(0) = \log(1+e^0) = \log(1+1) = \log 2$$

$$f'(x) = \frac{1}{1+e^x} \cdot e^x = \frac{e^x}{1+e^x} \Rightarrow f'(0) = \frac{e^0}{1+e^0} = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \frac{e^x(1+e^x) - e^{2x}}{(1+e^x)^2}$$

$$= \frac{e^x + \cancel{e^{2x}} - \cancel{e^{2x}}}{(1+e^x)^2}$$

$$f''(x) = \frac{e^x}{(1+e^x)^2} \Rightarrow f''(0) = \frac{e^0}{(1+e^0)^2}$$

$$f''(0) = \frac{1}{4}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - 2(1+e^x) \cdot e^x e^x}{((1+e^x)^2)^2}$$

$$= \frac{(1+e^x)^2 e^x - (2e^{2x} + 2e^{3x})}{(1+e^x)^4}$$

$$= \frac{(1+e^x)^2 e^x - 2e^{2x}(1+e^x)}{(1+e^x)^4}$$

$$= \cancel{(1+e^x)^2 e^x - 2e^x}$$

$$= \frac{\cancel{(1+e^x)}\left[(1+e^x)e^x - 2e^{2x}\right]}{(1+e^x)^{\cancel{4}\,3}}$$

$$= \frac{e^x + e^{2x} - 2e^{2x}}{(1+e^x)^3}$$

$$f'''(x) = \frac{e^x - e^{2x}}{(1+e^x)^3} \Rightarrow f'''(0) = \frac{e^0 - e^0}{(1+e^0)^3} = 0.$$

$$f^{iv}(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x})\,3(1+e^x)^2 \cdot e^x}{((1+e^x)^3)^2}$$

$$= \frac{(1+e^x)^3(e^x - 2e^{2x}) - (3e^{2x} - 3e^{3x})(1+e^x)^2}{(1+e^x)^6}$$

$$= \frac{\cancel{(1+e^x)^2}\left[(1+e^x)(e^x - 2e^{2x}) - (3e^{2x} - 3e^{3x})\right]}{(1+e^x)^{\cancel{6}\,4}}$$

$$= \frac{e^x - 2e^{2x} + e^{2x} - 2e^{3x} - 3e^{2x} + 3e^{3x}}{(1+e^x)^4}$$

$$f^{iv}(x) = \frac{e^{3x} - 4e^{2x} + e^x}{(1+e^x)^4} \Rightarrow f^{iv}(0) = \frac{e^0 - 4e^0 + e^0}{(1+e^0)^4}$$

$$= \frac{1-4+1}{24} = \frac{-2}{16}$$

$$f^{iv}(0) = \frac{-1}{8}$$

Sub. above Values in (1).

$$\log(1+e^x) = \log 2 + \frac{x}{1!}\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(\frac{-1}{8}\right) + \ldots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots \longrightarrow (2)$$

deduction:

Differentiate (2) w.r.t to 'x' O.B.S

$$\frac{1}{1+e^x}\cdot e^x = 0 + \frac{1}{2} + \frac{\cancel{2}x}{\cancel{8}\,4} - \frac{\cancel{4}x^3}{\cancel{192}\,48} + \cdots$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

(7). imp * S.T $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + \dots$ (or) Expand $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$ in power of x

Sol: let $f(x) = \frac{\sin^{-1}x}{\sqrt{1-x^2}} \Rightarrow f(0) = \frac{\sin^{-1}(0)}{\sqrt{1-0}} = 0$

$\sqrt{1-x^2}\, f(x) = \sin^{-1}x \longrightarrow (1)$

~~(Diff w.r.t x (1))~~

Diff eq (1) w.r.t x

$\sqrt{1-x^2}\, f'(x) + f(x)\frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{(1-x^2) f'(x) - x f(x)}{\cancel{\sqrt{1-x^2}}} = \frac{1}{\cancel{\sqrt{1-x^2}}}$

$(1-x^2) f'(x) - x f(x) = 1 \longrightarrow (2)$

put $x = 0$

$(1-0) f'(0) - 0 = 1$

$f'(0) = 1$

Diff eq (2) w.r.t x

$(1-x^2) f''(x) + f'(x)(-2x) - (x f'(x) + f(x)) = 0$

$(1-x^2) f''(x) - 2x f'(x) - x f'(x) - f(x) = 0$

$(1-x^2) f''(x) - 3x f'(x) - f(x) = 0 \longrightarrow (3)$

put $x = 0$

$(1-0) f''(0) - 0 - f(0) = 0 \qquad (\because f(0) = 0)$

$f''(0) = 0$

Diff eq (3) w.r.t x

$(1-x^2) f'''(x) - 2x f''(x) - 3(x f''(x) + f'(x)) - f'(x) = 0$

$(1-x^2) f'''(x) - 2x f''(x) - 3x f''(x) - 3 f'(x) - f'(x) = 0$

$(1-x^2) f'''(x) - 5x f''(x) - 4 f'(x) = 0$

put $x = 0$

$(1-0) f'''(0) - 5(0) f''(0) - 4 f'(0) = 0 \qquad \because f'(0) = 1$

$$f''(0) - 4(1) = 0$$
$$f'''(0) = 4$$

Maclaurins series of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \rightarrow (4)$$

Sub. obtained values in (4)

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = 0 + \frac{x}{1!} + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(4) + \dots$$

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + \dots$$

(8) Obtain Maclaurins Series (~~theory~~) expansion for $\log_e(1+x)$

Sol: $f(x) = \log_e(1+x) \Rightarrow f(0) = \log(1) = 0$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-1}{1+x^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{1+x^3} \Rightarrow f'''(0) = 2$$

Maclaurins series expansion for $\left(\log_e(1+x)\right)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\log_e(1+x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \dots$$

$$\therefore \log_e(1+x) = \frac{x}{1!} - \frac{x^2}{2!} + 2\frac{x^3}{3!} - \dots$$

(9) Obtain Taylor's Series expansion of $\cos x$ in powers of $(x+\pi)$

Sol: given $f(x) = \cos x$

let $(x+\pi) = t$ $\quad \left(\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$x = t - \pi$

$\cos x = \cos(t-\pi)$

$= \cos(-(\pi - t))$

$= -\cos t$

$= -\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)$

put $t = (x+\pi)$

$= -\left[1 - \frac{(x+\pi)^2}{2!} + \frac{(x+\pi)^4}{4!} - \cdots\right]$

$\cos x = -1 + \frac{(x+\pi)^2}{2!} - \frac{(x+\pi)^4}{4!} + \cdots$

(10) Expand $e^{x \sin x}$ in powers of x

Sol: let $f(x) = e^{x\sin x} \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^{x\sin x}(x\cos x + \sin x) \Rightarrow f'(0) = e^0(0+0) = 0$

$f'(x) = e^{x\sin x}(x\cos x) + e^{x\sin x}(\sin x)$

$f''(x) = e^{x\sin x}(-x\sin x + \cos x) + (x\cos x)\, e^{x\sin x}$

$(x\cos x + \sin x) + e^{x\sin x}(\cos x) + \sin x \cdot e^{x\sin x}$

$(x\cos x + \sin x)$

$f''(x) = -x\sin e^{x\sin x} + e^{x\sin x}\cos x + x^2\cos^2 x\, e^{x\sin x} +$

$-x\sin x\cos x\, e^{x\sin x} + e^{x\sin x}\cos x + x\cos x\sin x$

$e^{x\sin x} + \sin^2 x\, e^{x\sin x}$

$f''(0) = 0 + 1 + 0 + 1 + 0 + 0 = 2$

Maclaurin's series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

$$e^{x\sin x} = 1 + 0 + \frac{x^2}{2!}(2) + \dots$$

$$e^{x\sin x} = 1 + 2\frac{x^2}{2!} + \dots$$

is req. sol.

(OR)

$$e^{x\sin x} = 1 + x\sin x + \frac{(x\sin x)^2}{2!} + \frac{(x\sin x)^3}{3!} + \dots$$

$$= 1 + x\sin x + \frac{x^2}{2!}(\sin x)^2 + \dots$$

$$= 1 + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) + \frac{x^2}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) + \dots$$

$$= 1 + \left(x^4 - \frac{x^4}{6} + \dots\right) + \frac{x^2}{2}\left(x^2 - \frac{x^4}{3!} - \frac{x^4}{3!} + \frac{x^6}{36} + \dots\right) + \dots$$

$$= 1 + \left(x^2 - \frac{x^4}{6} + \dots\right) + \left(\frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{72}\right) + \dots$$

$$= 1 + x^2 - \frac{x^4}{6} + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{72} + \dots$$

$$= 1 + x^2 + \frac{2x^4}{6} - \frac{x^6}{6} + \frac{x^8}{72} + \dots$$

Theorem :-

Find Maclaurins theorem with Lagranges form of remainders for $f(x) = \cos x$

Sol: Maclaurins theorem with Lagranges form of remainder is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

(i)

we have $f(x) = \cos x$

$f(x) = \cos x$; $f(0) = 1$

$f'(x) = -\sin x$; $f'(0) = 0$

$f''(x) = -\cos x$; $f''(0) = -1$

$f'''(x) = \sin x$; $f'''(0) = 0$

$f^{iv}(x) = \cos x$; $f^{iv}(0) = 1$

$$f^n(x) = \frac{d^n}{dx^n} \cos x$$

$$= \cos\left(\frac{n\pi}{2} + x\right)$$

at

$x = 0$;

$$f^n(0) = \cos\left(\frac{n\pi}{2} + 0\right) = \cos\left(\frac{n\pi}{2}\right)$$

$$f(0) = \cos 0 = 1$$

$$f(2n) = \cos\left(\cancel{2}n \frac{\pi}{\cancel{2}}\right) = \cos n\pi = (-1)^n \rightarrow \text{even terms}$$

$$f(2n+1) = \cos\left(\frac{(2n+1)\pi}{2}\right) = \cos\left(\frac{\cancel{2}n\pi}{\cancel{2}} + \frac{\pi}{2}\right)$$

$$= \cos\left(\frac{\pi}{2} + n\pi\right)$$

$$= -\sin n\pi = 0 \rightarrow \text{odd terms.}$$

$\left(\because \cos n\pi = (-1)^n \quad \because \sin n\pi = 0 \right)$

i.e., if n is even, coeff. of x will be $(-1)^n$ and if n is odd, coeffs of x are '0'.

Substuting these value in eq ①. we get

$$① \Rightarrow \cos x = 1 + 0 + \frac{x^2}{2!}(-1) + 0 + \frac{x^4}{4!}(1) + 0 + \frac{x^6}{6!}(-1) + \cdots$$

$$+ \frac{x^{2n}}{2n!}(-1)^n f^{2n}(0) + \frac{x^{2n+1}}{(2n+1)!}(-1)^{n+1} f^{n+1}(\theta x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{2n!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}$$

$$(-1)^{n+1} \cos \theta x \,/.$$

* 2nd Unit Completed *