problems

1) Calculate Expectation and variance of X, If the probability distribution of R.V. X is given by

X	-1	0	1	2	3
f	0.3	0.1	0.1	0.3	0.2

Sol Expectation of x, $E(x) = \sum_{i=-1}^{3} x_i f_i$

$$= (-1)(0.3) + 0(0.1) + 1(0.1) + 2(0.3) + 3(0.2)$$

$$= 1$$

variance of x, $V(x) = E(x^2) - [E(x)]^2$

$$= \sum_{i=-1}^{3} x_i^2 f_i - (1)^2$$

$$= [1(0.3) + 0(0.1) + 1(0.1) + 4(0.3) + 9(0.2)] - 1$$

$$= [0.3 + 0.1 + 1.2 + 1.8] - 1$$

$$V(x) = 3.4 - 1 \Rightarrow 2.4$$

2) The probability density function of the random variable is given by $f(x) = \begin{cases} \frac{c}{\sqrt{x}}, & \text{for } 0 < x < 4 \\ 0, & \text{elsewhere} \end{cases}$

Determine i) the value of c ii) $p(x < 1)$ and iii) $p(x \geq 1)$

Sol Given probability density function

$$f(x) = \begin{cases} \frac{c}{\sqrt{x}} & \text{for } 0 < x < 4 \\ 0, & \text{elsewhere} \end{cases}$$

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{4} f(x)dx + \int_{4}^{\infty} f(x)dx = 1$$

$$0 + \int_{0}^{4} \frac{c}{\sqrt{x}} dx + 0 = 1$$

$$c \int_{0}^{4} \frac{1}{\sqrt{x}} dx = 1$$

$$c\left[2\sqrt{x}\right]_0^4 = 1$$

$$c\left[2\sqrt{4} - 2\sqrt{0}\right] = 1$$

$$4c = 1$$

$$c = 1/4$$

$$f(x) = \begin{cases} \frac{1}{4\sqrt{x}}, & 0 < x < 4 \\ 0; & \text{elsewhere} \end{cases}$$

ii) $P(X<1) = \int_{-\infty}^{1} f(x)\,dx)$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx = 0 + \int_0^1 \frac{1}{4\sqrt{x}}\,dx$$

$$= \frac{1}{4}\left[2\sqrt{x}\right]_0^1$$

$$= \frac{1}{4} \times 2 = \frac{1}{2}$$

iii) $P(X \geq 1) = \int_{1}^{\infty} f(x)\,dx$

$$= \int_1^4 f(x)\,dx + \int_4^{\infty} f(x)\,dx$$

$$= \int_1^4 \frac{1}{4\sqrt{x}}\,dx = \frac{1}{4}\left(2\sqrt{x}\right)_1^4$$

$$= \frac{1}{4}\left[2\times 2 - 2\right] = \frac{1}{\cancel{4}_2} \times \cancel{2} = \frac{1}{2}$$

3) During the course of a day, a machine turns out 0, 1 or 2 defective pens with probabilities 1/6, 2/3 and 1/6 respectively calculate the mean value and The variance of the defective pens produced by The machine in a day.

The probability distribution for the given data

x_i:	0	1	2
P_i:	1/6	2/3	1/6

Mean, $\mu = E[x]$

$$= \sum_{i=0}^{2} x_i P_i = 0\left(\frac{1}{6}\right) + 1\left(2/3\right) + 2\left(1/6\right)$$

$$= 2/3 + 1/3 = 1$$

variance $= E[x^2] - (E(x))^2$

$= 0(1/6) + (1)^2\left(\frac{2}{3}\right) + 2^2\left(\frac{1}{6}\right) - 1^2$

$= \frac{2}{3} + \frac{2}{3} - 1$

$= 4/3 - 1$

$\sigma^2 = 1/3$

S.D $\sigma = 1/\sqrt{3}$

4. Given the following probability distribution

x_i	0	1	2	3	4	5	6	7
$p(x)$	0	2λ	2λ	λ	3λ	λ^2	$2\lambda^2$	$7\lambda^2+\lambda$

i) find λ ii) Evaluate $p(x \geq 5)$, $p(x<4)$

i) $\Sigma p(x) = 1$

$2\lambda + 2\lambda + \lambda + 3\lambda + \lambda^2 + 2\lambda^2 + 7\lambda^2 + \lambda = 1$

$10\lambda^2 + 9\lambda = 1$

$\lambda(10\lambda + 9) = 1 \Rightarrow 10\lambda^2 + 9\lambda - 1 = 0$

$10\lambda^2 + 10\lambda - 1\lambda - 1 = 0$

$10\lambda(\lambda+1) - 1(\lambda+1) = 0$

$(\lambda+1)(10\lambda-1) = 0$

$\lambda = 1/10, -1$

ii) $p(x \geq 5)$

$p(x=7) + p(x=6) + p(x=5)$

$= 7\lambda^2 + \lambda + 2\lambda^2 + \lambda^2$

$= 10\lambda^2 + \lambda$

$= \cancel{10}\lambda(10\lambda+1)$

$= \frac{1}{10}(1+1)$

$= \frac{\cancel{2}}{\cancel{10}\,5} = \frac{1}{5}$

iii) $p(x<4)$

$p(x=1) + p(x=2) + p(x=3)$

$= 2\lambda + 2\lambda + \lambda$

$= 5\lambda$

$= \frac{5}{10}$

$= 1/2$

5) The probability density function for a continuous random variable x is given by

$$f(x) = \begin{cases} \frac{1}{2}\sin x & 0 \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

find mean & variance

Given

$$\text{pdf} = f(x) = \begin{cases} \frac{1}{2} & 0 \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean}, \mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^{\pi} \frac{x}{4} \; \frac{1}{2} \; \frac{\sin x \, dx}{v}$$

$$= \frac{1}{2}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{\pi}$$

$$= \frac{1}{2}\left[-\pi\cos\pi + \sin\pi + 0 - \sin 0\right]$$

$$= \frac{1}{2}\left[-\pi(-1) + 0\right]$$

$$\boxed{\mu = \pi/2}$$

$$\text{Variance } \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$= \int_0^{\pi} x^2 \, \frac{1}{2}\sin x \, dx - (\pi/2)^2$$

$$= \frac{1}{2}\left[x^2(-\cos x) - 2(x)(-\sin x) + 2[\cos x]\right]_0^{\pi} - \frac{\pi^2}{4}$$

$$= \frac{1}{2}\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi} - \frac{\pi^2}{4}$$

$$= \frac{1}{2}\left[-\pi^2\cos\pi + 2\pi\sin\pi + 2\cos\pi + 0 - 0 - 2\cos 0\right] - \frac{\pi^2}{4}$$

$$= \frac{1}{2}\left[-\pi^2(-1) + 0 + 2(-1) - 2(1)\right] - \pi^2/4$$

$$= 1/2\left[\pi^2 - 4\right] - \pi^2/4$$

$$= 1/2\left[\pi^2 - 4\right] - \pi^2/4$$

$$= \frac{\pi^2}{2} - 2 - \pi^2/4$$

$$\sigma^2 = \frac{\pi^2}{4} - 2$$

6) If x is continuous random variable with probability density function $f(x) = \begin{cases} axe^{-x^2}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ find the parameter a!

Given pdf $= f(x) = \begin{cases} axe^{-x^2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

We know that $= \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$= \int_0^\infty a x e^{-x^2} dx = 1$$

$x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2} dt$

limits if $x = 0 \Rightarrow t = 0$

if $x = \infty \Rightarrow t = \infty$

$$a \int_{t=0}^{\infty} e^{-t} \tfrac{1}{2} dt = 1$$

$$a/2 \int_{t=0}^{\infty} e^{-t} dt = 1$$

$$\frac{a}{2}\left[\frac{e^{-t}}{-1}\right]_{t=0}^{\infty} = 1$$

$$-a/2\left[e^{-\infty} - e^{0}\right] = 1 \Rightarrow -\frac{a}{2}[0-1] = 1 \Rightarrow \frac{a}{2} = 1$$

$\boxed{a = 2}$

Binomial distribution

$$p(x=r) = \begin{cases} {}^{n}c_{r}\, p^{r} q^{n-r} \\ 0 \quad \text{otherwise} \end{cases}$$

1) If the probability of hitting a target is 10% & 10 shots are fired independently what is The probability that the target will be hit atleast once.

sol The probability hitting the target

$p = 10\% = \frac{10}{100} = \frac{1}{10} = 0.1$

$q = 1 - p = 0.9$

given $n = 10$

The probability density function of B.D

$P(x = r) = {}^{n}c_{r}\, p^{r} q^{n-r}$

The probability of hitting The target atleast once $P(x \geq 1)$

$= 1 - P(x < 1) \Rightarrow 1 - P(x = 0)$

$= 1 - {}^{10}c_{0} (0.1)^{0} (0.9)^{10-0}$

$= 1 - (0.9)^{10}$

$= 1 - 0.3486$

$= 0.6514$

2) The probability that a bulb produced by a factory will fuse after 100 days of use is 0.05 find the probability that out off 5 such bulbs i) None ii) Not more than one iii) more than one iv) atleast one will fuse 100 day of use.

The probability of a bulb fuse after 100 days

$p = 0.05$

$q = 1-p$

$q = 1-0.05 = 0.95$

given $n=5$

The p.d.f binomial distribution is

$P(x=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$

i) $p(x=0) = {}^{n}C_{0}\, p^{0} q^{n-0}$

$= {}^{5}C_{0} (0.05)^{0} (0.95)^{5-0} = (0.95)^{5}$

$= 0.773$

ii) $p(x \leq 1) = p(x=0) + p(x=1)$

$= 0.773 + {}^{5}C_{1} (0.05)^{1} (0.95)^{4}$

$= 5(0.05)(0.8145) + 0.773$

$= 0.9766$

iii) More than one

$p(x>1) = 1 - p(x \leq 1)$

$= 1 - \{p(x=0) + p(x=1)\}$

$= 1 - 0.9766$

$= 0.0227$

iv) atleast one

$p(x \geq 1) = 1 - p(x<1)$

$= 1 - p(x=0)$

$= 1 - 0.7737 = 0.2263$

3) five cards are drawn Successfully with replacement from a well-shuffled deck of 52 cards what is the probability that i) All the five cards are spades ii) only 3 cards are spades iii) none is spade.

The probability of selecting spades (13) from a deck of cards $52 = 13/52$

$$p = 13/52 = 1/4$$
$$q = 1 - 1/4$$
$$q = 3/4$$

given $n=5$

The PDF of binomial distribution.

$$p(x=r) = {}^nC_r\, p^r q^{n-r}$$

i) all are spade's

$$p(x=5) = {}^5C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^{5-5}$$
$$= 1(0.000976)$$
$$= 0.000976$$

ii) only 3 cards are spades

$$p(x=3) = {}^5C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2$$
$$= 10(0.015)(3/4)^2 = 0.08775$$

iii) none is spade

$$p(x=0) = {}^5C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5$$
$$= (1)(1)\left(\frac{3}{4}\right)^5 = 0.2373$$

4) A die is thrown 3 times. if get in a '6' is considered a success find the probability of i) 3 successes ii) at least 2 successes.

The probability of thrown a die

$$p = 1/6$$
$$q = 1-p$$
$$q = 1 - 1/6$$
$$q = \frac{6-1}{6}$$
$$q = 5/6$$ given $n=3$

i) 3 successes

$$p(x=3) = {}^3C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{3-3}$$
$$= {}^3C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^0$$

$= 1(0.004629)$

$= 0.004629$

ii) $p(x \geq 2) = p(x=2) + p(x=3)$

$= {}^3C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{3-2} + 0.004629$

$= 3(0.0277)\left(\frac{5}{6}\right) + 0.004629$

$= 0.06925 + 0.004629$

$= 0.073879.$

1) A box contains 200 tickets each bearing one of the numbers from 1 to 200, 20 tickets are drawn successively with replacement from the box find the probability that atmost 4 tickets bear number divisible by 20.

Here $n = 20$

$p = 10/200$ $p = p(\text{divisible by } 20 \text{ from } 1 \text{ to } 200)$

$p = 1/20 = 0.05$

Mean $\mu = np$

$= n(1/20)$

$= 20(1/20)$

$= 1$

$\lambda = 1$

The probability that almost 4 tickets bear numbers divisible by 20 $p(x \leq 4)$.

$= \frac{e^{-1}(1)^0}{0!} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!} + \frac{e^{-1}(1)^3}{3!} + \frac{e^{-1}(1)^4}{4!}$

$= e^{-1}(1 + 1 + 1/2 + 1/6 + 1/24)$

$= 0.3678(2.7083)$

$= 0.99611$

2) In a given city 6% of all drivers get atleast one parking ticket for year use the poisson's

abstraction to binomial distribution determine the probabilities that among 80 drivers

i) four will get atleast 1 parking ticket at any given year.

ii) Atleast 3 will get atleast 1 parking ticket in any given year.

iii) Between 3 & 6, both values included will get atleast one parking ticket in any given year.

Sol Given that $p = 6/100 = 0.06$

$n = 80$

Mean $\mu = np = \lambda$

$= 80(0.06) = 4.8$

$\lambda = 4.8$

p.d.f poisson distribution $p(x=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

i) probability 4 will get atleast 1 parking ticket at any given year

$$p(x=4) = \frac{e^{-4.8}(4.8)^4}{4!} = 0.182$$

$$= \frac{0.0082(530.84)}{4!}$$

$$= 0.182$$

ii) Atleast 3 will get atleast 1 parking ticket in any given year is

$$p(x \geq 3) = 1 - p(x<3)$$

$$= 1 - \{p(x=0) + p(x=1) + p(x=2)\}$$

$$= 1 - \left\{\frac{e^{-4.8}(4.8)^0}{0!} + \frac{e^{-4.8}(4.8)^1}{1!} + \frac{e^{-4.8}(4.8)^2}{2!}\right\}$$

$$= 1 - e^{-4.8}\left\{\frac{(4.8)^0}{0!} + \frac{(4.8)^1}{1!} + \frac{(4.8)^2}{2!}\right\}$$

$$= 1 - e^{-4.8}\left\{1 + \frac{4.8}{1} + \frac{23.04}{2}\right\}$$

$$= 1 - e^{-4.8}(17.32)$$

$$= 1 - 0.0082(17.32)$$

$$= 1 - 0.1420 = 0.86$$

c) $p(3 \le x \le 6) = p(x=3) + p(x=4) + p(x=5) + p(x=6)$

$$= e^{-4.8}\left\{\frac{(4.8)^3}{3!} + \frac{(4.8)^4}{4!} + \frac{(4.8)^5}{5!} + \frac{(4.8)^6}{6!}\right\}$$

$$= e^{-4.8}\left\{\frac{110.5}{6} + \frac{530.84}{24} + \frac{2548.03}{120} + \frac{12230.59}{720}\right\}$$

$$= 0.0082\{78.755\}$$

$$= 0.6457$$

3) If 0.8% of the fuses delivered to an arsenia are diffective use the poin approximation to determi the probability that four fuses will be defective in a random Sample of 400.

Given that

$$p = 0.8\% = 0.008$$

$$n = 400$$

Mean $\mu = np = \lambda$

$$= 400 \times 0.008 = 3.2$$

$$\lambda = 3.2$$

p.d.f of poisson distribution is $p(x=x)$

$$= \frac{e^{-\lambda}\lambda^x}{x!}$$

probability for 4 fuses will be defective is

$$p(x=4) = \frac{e^{-3.2}(3.2)^4}{4!} = \frac{0.0407(104.857)}{24}$$

$$= 0.178$$

4) A new car, on average has 3 defective parts. If you buys new car, what is the probability that the car will have i) no defective part ii) atmost one defective part and iii) atleast 2 defective parts.

Hint i - use poison distribution with $\mu = 3$

Sol Given that $\mu = 3$

$$\lambda = \mu = 3$$

Pdf poisson distribution $p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

i) No defective

$p(x=0) = \frac{(3)}{0!} = \frac{20.0095(1)}{1}$

$= 0.0498$

ii) Almost one defective part

$p(x \le 1) = p(x=0) + p(x=1)$

$= \frac{e^{-3}(3)^0}{0!} + \frac{e^{3}(3)^1}{1!}$

$= e^{-3}\left(\frac{(3)^0}{0!} + \frac{(3)^1}{1!}\right)$

$= e^{-3}\left(\frac{1}{1} + \frac{3}{1}\right)$

$= e^{-3}(1+3)$

$= 0.0498(4) = 0.1991$

iii) atleast 2 defective parts

$p(x \ge 2) = p(x=2) + p(x=3)$

$= \frac{e^{-3}(3)^2}{2!} + \frac{e^{3}(3)^3}{3!}$

$= e^{-3}\left(\frac{(3)^2}{2!} + \frac{(3)^3}{3!}\right)$

$= 0.0498\left(\frac{9}{2} + \frac{27}{6}\right)$

$= 0.0498(9)$

$= 0.4482$

5) A medical company finds that 0.1 percent of the bottles produced in the factory are defective. If a distributor buys 100 boxes, each box containing 500 bottles, find out the how many boxes will contain.

i) no defective bottle and

ii) atleast two defective bottles

Hint : use poission distribution with $\mu = np$ $= 500(0.001) = 0.5$

Sol Given that

$\mu = \lambda = np = 0.5$

Pdf poisson's distribution $p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

i) No defective bottles :-

$$p(x=0) = \frac{e^{-0.5}(0.5)^0}{0!}$$

$$= \frac{0.6065(1)}{1!}$$

$$= 0.6065$$

∴ The no. of boxes with no defective bottles $= 0.6065 \times 100$
$= 60.65 \simeq 61$

ii) atleast two defective bottles

$$P(x \geq 2) = 1 - p(x<2)$$

$$= 1 - (p(x=0) + p(x=1))$$

$$= 1 - \left(\frac{e^{-0.5}(0.5)^0}{0!} + \frac{e^{-0.5}(0.5)^1}{1!}\right)$$

$$= 1 - e^{-0.5}\left(\frac{1}{1} + \frac{0.5}{1}\right)$$

$$= 1 - e^{-0.5}(1+0.5)$$

$$= 1 - 0.6065(1.5)$$

$$= 1 - 0.9098$$

$$= 0.0902$$

∴ No. of boxes with atleast two defective bottles
$= 0.0902 \times 100$
$= 9.02 \simeq 9$

Normal distribution:-

i) Suppose that during period of transcendental Meditation The reduction of a person's oxygen consumption with mean = 37.6 cc/min and SD = 4.6 cc/min. Determine the probabilities that during a period of transcendental meditation a person's oxygen consumption will be reduced by:

i) Atleast 44.5 cc/min ii) Atmost 35.0 cc/min
iii) Anywhere from 30.0 to 40.0 cc/min.

Sol Given that $\mu = 37.6$
$\sigma = 4.6$

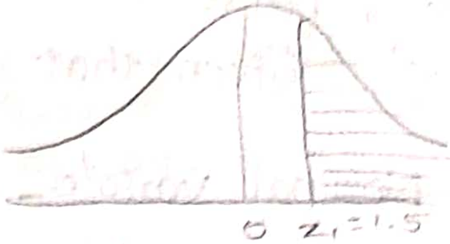
Normal variate $z = \frac{x-\mu}{\sigma} = \frac{x - 37.6}{4.6}$

i) probability that a person's oxygen consumption will be reduced by atleast 44.5 cc/min is

$$P(x \geq 44.5)$$

let $x_1 = 44.5$

$$z_1 = \frac{44.5-37.6}{4.6} = 1.5$$

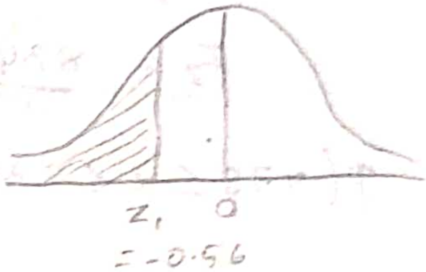


$p(x \geq 44.5) = p(z \geq z_1)$
$= 0.5 - A(1.5)$
$= 0.5 - 0.4332$
$= 0.0668$

) probability that a person's oxygen consumption will be reduced by atmost 35.0 cc/min.

$p(x \leq 35.0)$

let $x_1 = 35.0 \Rightarrow z_1 = \frac{35.0-37.6}{4.6} = -0.56$



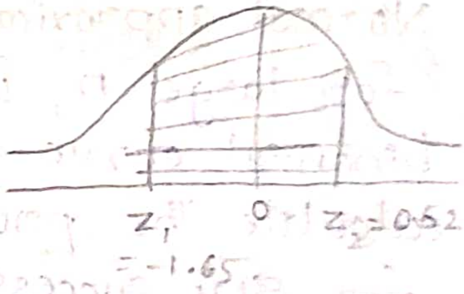
$p(x \leq 35.0) = p(z \leq z_1)$
$= p(z \leq -0.56)$
$= 0.5 - A(-0.56)$
$= 0.5 - 0.2123$
$= 0.2877$

i) probability that a persons oxygen consumption will be reduced by Anywhere from 30.0 to 40.0 cc/min.

$p(30 \leq x_1 \leq 40)$

let $x_1 = 30$, $x_2 = 40$

$$z_1 = \frac{30-37.6}{4.6} = \frac{-7.6}{4.6} = -1.65$$



$$z_2 = \frac{40-37.6}{4.6} = 0.52$$

$p(30 \leq x \leq 40) = p(z_1 \leq z \leq z_2)$
$= p(-1.65 \leq z \leq 0.52)$
$= A(z_1) + A(z_2)$
$= A(-1.65) + A(0.52)$
$= 0.4505 + 0.1985$
$= 0.649$

)A Electrical firm manufactures light bulbs that have a length of life which is normal distributed with mean $\mu = 800$ hours and A standard deviation $\sigma = 40$ hours. Find the probability that a bulb ~~an~~ burns b/w 778 and

834 hours.

Sol Given that $\mu = 800$ hours

$\sigma = 40$ hours

Normal variate $z = \frac{x-\mu}{\sigma}$

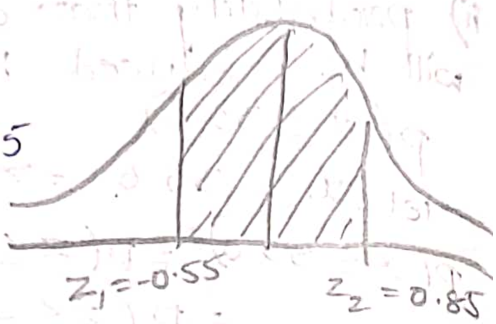
probability that a bulb burns b/w 778 and 834 hours :-

$P(778 \le x \le 834)$

let $x_1 = 778$ & $x_2 = 834$

$z_1 = \frac{778 - 800}{40} = -0.55$

$z_2 = \frac{834 - 800}{40} = 0.85$



$P(778 \le x \le 834) = P(z_1 \le z \le z_2)$

$= P(-0.55 \le z \le 0.85)$

$= A(0.55) + A(0.85)$

$= 0.2088 + 0.3023$

$= 0.5111$

Normal approximation to The Binomial distribution.

for large n, we can approximate the binomial curve by The normal curve and calculate The probability for any success real class interval is $(x - \frac{1}{2}, x + \frac{1}{2})$

Hence z_1 must correspond to the lower limit of The x_1 class and z_2 to The upper limit of the x_2 class.

Hence $z_1 = \frac{(x_1 - \frac{1}{2}) - \mu}{\sigma} = \frac{(x_1 - \frac{1}{2}) - np}{\sqrt{npq}}$

$z_2 = \frac{(x_2 + \frac{1}{2}) - \mu}{\sigma} = \frac{(x_2 + \frac{1}{2}) - np}{\sqrt{npq}}$

$\therefore$ The required probability $= \int_{z_1}^{z_2} \phi(z)\, dz$

and this can be evaluated normal tables

1) Find the probability that out of 100 patients b/w 84 and 95 inclusive will survive a heart operation given that the chances of survival is 0.9

Sol Given probability of survival after operation

$p = 0.9$

$q = 1 - p = 1 - 0.9 = 0.1$

$n = 100$

Here $\mu = np$

$= 100 \times 0.9 = 90$

$\sigma = \sqrt{npq} = \sqrt{100 \times 0.9 \times 0.1} = 3$

probability for b/w 84 & 95 inclusive will survive

$p(84 \leq x \leq 95)$

let $x_1 = 84$ $\quad x_2 = 95$

$$z_1 = \frac{(x_1 - \frac{1}{2}) - \mu}{\sigma} = \frac{(84 - \frac{1}{2}) - 90}{3} = -2.17$$

$$z_2 = \frac{(x_2 + \frac{1}{2}) - \mu}{\sigma} = \frac{(95 + \frac{1}{2}) - 90}{3} = 1.83$$

$p(84 \leq x \leq 95)$

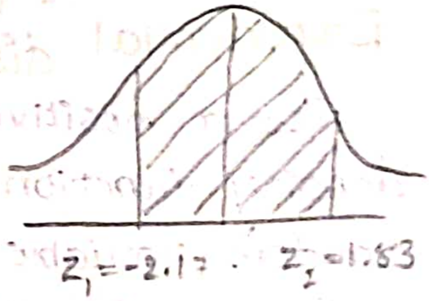
$= p(z_1 \leq z \leq z_2)$

$= p(-2.17 \leq z \leq 1.83)$

$= A(2.17) + A(1.83)$

$= 0.4850 + 0.4664$

$= 0.9514$



2) Find the probability of getting an even number on face 3 to 5 times in throwing 10 dice together

Sol let p = probability of getting an even number on face

$= \frac{3}{6} = \frac{1}{2}$

$q = 1 - \frac{1}{2} = \frac{1}{2}$

Here $n = 10$

Mean $\mu = np = 10 \times \frac{1}{2} = 5$

variance $\sigma^2 = npq = 5 \times \frac{1}{2} = 2.5$

$\sigma = \sqrt{2.5} = 1.58$

probability of getting even number on face 3 to 5 times :-

$P(3 \le x \le 5)$

let $x_1 = 3$ & $x_2 = 5$

Now $z_1 = \frac{(x_1 - 1/2) - \mu}{\sigma} = \frac{(3 - \frac{1}{2}) - 5}{1.58}$

$= -1.58$

$z_2 = \frac{(x_2 + 1/2) - \mu}{\sigma} = \frac{5 + 1/2 - 5}{1.58} = 0.32$

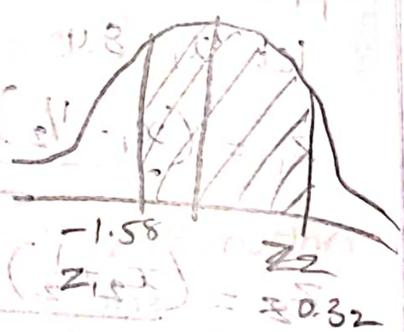
$P(3 \le x \le 5) = P(z_1 \le z \le z_2)$

$= P(-1.58 \le z \le 0.32)$

$= A(-1.58) + A(0.32)$

$= 0.4430 + 0.1256$

$= 0.5686$



Exponential distribution :- (arrivals)

For a positive real number 'λ'. The probability density function of a exponential distributed random variable is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \in [0, \infty] = R_X \\ 0 & \text{otherwise} \end{cases}$$

Here λ is a parameter

Note :-

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{0} f_X(x)\,dx + \int_{0}^{\infty} f_X(x)\,dx$$

$$= 0 + \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty} = -\left[e^{-\lambda \infty} - e^{-\lambda 0} \right]$$

$$= -[0 - 1] = 1$$

Constants of exponential distribution:-

The probability density function of exponential function, $p(X=x) = \lambda e^{-\lambda x}; x \in [0, \infty]$

Mean:-

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} x e^{-\lambda x} dx$$

$$= \lambda \left[x \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 1 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) \right]_0^{\infty}$$

$$= \lambda \left[0 - 0 - 0 + \frac{e^0}{\lambda^2} \right]$$

$$= \cancel{\lambda} \left(\frac{1}{\lambda^{\cancel{2}}} \right)$$

$$\mu = 1/\lambda$$

② variance:-

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx - \left(\frac{1}{\lambda} \right)^2$$

$$= \lambda \left[x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) \right]_0^{\infty} - \frac{1}{\lambda^2}$$

$$\sigma^2 = \lambda \left[x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) - 2 \left(\frac{e^{-\lambda x}}{\lambda^3} \right) \right]_0^{\infty} - \frac{1}{\lambda^2}$$

$$= \lambda \left[0 - 0 - 0 - 0 - 0 + 2 \left(\frac{e^0}{\lambda^3} \right) \right] - \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\sigma^2 = \frac{1}{\lambda^2}$$

$$\boxed{\sigma = \frac{1}{\lambda}} \quad \boxed{S.D = \frac{1}{\lambda}}$$

1) Customers arrive at a certain shop according to an approximate poission process at a mean frequency of 20 per hour. What is the probability that the shopkeeper will have to wait for more than 5 minutes for the arrival of the first customer.

Sol let x be the waiting time in minutes for the shopkeeper until the first customer arrives

$$\therefore \mu = \frac{60}{20} = 3$$

$$\therefore \mu = \frac{1}{\lambda}$$

$$\lambda = \frac{1}{\mu} = \frac{1}{3}$$

P.d.f of exponential distribution is $f_x(x) = \lambda e^{-\lambda x}$

Where $x \in [0, \infty]$

$$p(x > 5) = \int_5^\infty \frac{1}{3} e^{-\frac{1}{3}x} dx$$

$$= \frac{1}{3} \int_5^\infty e^{-\frac{1}{3}x} dx$$

$$= \frac{1}{3} \left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}} \right]_5^\infty$$

$$= \frac{1}{\cancel{3}} \left[-\cancel{3} e^{-\frac{1}{3}x} \right]_5^\infty$$

$$= -\left[0 - e^{-5/3} \right]$$

$$= e^{-5/3} = 0.1888.$$

2) Telephone calls arrive at a college switch board according to a poisson process on an average of two every three minutes. What is the probability that the waiting time is more than 2 minutes till the first call arrives after 10.A.M?

Sol let x be the waiting time to the first call arrive

Here $\mu = \frac{2}{3} \Rightarrow \lambda = \frac{1}{\mu} = \frac{3}{2}$

p.d.f of exponential distribution is $f_x(x) = \lambda e^{-\lambda x}$

Where $x \in [0, \infty)$

The probability that the waiting time is more than 2 minutes

$$P(x > 2) = \int_2^{\infty} \frac{3}{2} e^{-3/2 x} dx$$

$$= \frac{3}{2} \left[\frac{e^{-3/2 x}}{-3/2} \right]_2^{\infty}$$

$$= \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \left[-e^{-3/2 x} \right]_2^{\infty}$$

$$= -[0 - e^{-3}]$$

$$= e^{-3}$$

$$= 0.049$$

Uniform distribution (continuous)

A continuous random variable x is said to have uniform distribution iff that corresponding probability density function is given by.

$$f(x; a, b) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

A random variable having uniform distribution is called uniform random variable

Constants of uniform distribution:-

If 'x' is a uniform random variable in the interval (a, b) then p.d.f of uniform distribution is $f(x; a, b) = \frac{1}{b-a}; a < x < b$

① Mean:-

$$\mu = E(x)$$

$$= \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_a^b x f(x) dx$$

$$= \int_a^b x \frac{1}{b-a} dx \Rightarrow \frac{1}{b-a} \left[x^2/2 \right]_a^b$$

$$\Rightarrow \frac{1}{b-a} \left[\frac{b^2}{2} - \frac{a^2}{2} \right]$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{1}{\cancel{b-a}}\left[\frac{(b+a)\cancel{(b-a)}}{2}\right]$$

$$\mu = \frac{b+a}{2}$$

② variance $\sigma^2 = E(x^2) - [E(x)]^2$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$= \int_a^b x^2 \frac{1}{b-a}\,dx - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \left[\frac{b+a}{2}\right]^2$$

$$= \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right] - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{a^2+b^2+ab}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4a^2+4b^2+4ab-(3a^2+3b^2+6ab)}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$\sigma^2 = \frac{(a-b)^2}{12}$$

③ Standard deviation $\sigma = \sqrt{\frac{(a-b)^2}{12}}$

$$= \frac{(a-b)}{\sqrt{12}} = \frac{(a-b)}{2\sqrt{3}}$$

Cumulative distribution function:-

on the Support $a < x < b$

$$F(x) = \int_a^x \frac{1}{b-a}\,dw = \left[\frac{w}{b-a}\right]_a^x = \frac{x-a}{b-a}$$

so

$$F(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$$

1) If 'x' is a uniform random variable in $(-2,4)$ and 'y' $= x^2 - 4$, find the probability density function $g(y)$ of y

Sol Given that x is a uniform random variable in $(-2,4)$.

Here $a=-2$, $b=4$

cumulative distribution function $F(x)$ of x

$$F(x) = \begin{cases} 0 & ; x \le a \\ \frac{x-a}{b-a} & ; a < x < b \\ 1 & x \ge b \end{cases}$$

$$F(x) = \begin{cases} 0 & ; x \le -2 \\ \frac{x+2}{6} & ; -2 < x < 4 \\ 1 & ; x \ge 4 \end{cases}$$

$x = -2, -1, 0, 1, 2, 3, 4$

$y = x^2 - 4$

$= 0, -3, -4, -3, 0, 5, 12 = (-4, 12)$

clearly y is a random variable in $(-4, 12)$

According to the definition of cumulative distribution function, the cumulative distribution function $G(y)$ of the random variable y is given by

for $-4 < y \le 0$

$$G(y) = P(Y \le y)$$
$$= P(x^2 - 4 \le y)$$
$$= P(x^2 \le y + 4)$$
$$= P(x \le \pm\sqrt{y+4})$$
$$= P(-\sqrt{y+4} \le x \le \sqrt{y+4})$$
$$= F(\sqrt{y+4}) - F(-\sqrt{y+4})$$
$$= \frac{\sqrt{y+4}+2}{6} - \frac{-\sqrt{y+4}+2}{6}$$

$$G(y) = \frac{\sqrt{y+4}+\cancel{2}+\sqrt{y+4}-\cancel{2}}{6}$$

$$= \frac{2\sqrt{y+4}}{6}$$

$$G(y) = \frac{\sqrt{y+4}}{3}$$

Diff w.r. to y

$$G'(y) = g(y) = \frac{1}{6\sqrt{y+4}} \quad \text{for } -4 < y \le 0$$

for $0 \le y < 12$

$$G(y) = P(Y \le y)$$

$$= P(x^2 - 4 \le y)$$

$$= P(x^2 \le y+4)$$

$$= P(x \le \sqrt{y+4})$$

$$= F(\sqrt{y+4})$$

$$G(y) = \frac{\sqrt{y+4}+2}{6}$$

$$= \frac{\sqrt{y+4}}{6} + \frac{2}{6}$$

Diff w.r.to y on both sides

$$G'(y) = g(y) = \frac{1}{12\sqrt{y+4}} \quad \text{for } 0 \le y < 12$$

$\therefore$ The probability density function of y is

$$g(y) = \begin{cases} \frac{1}{6\sqrt{y+4}} & \text{if } -4 < y \le 0 \\ \frac{1}{12\sqrt{y+4}} & \text{if } 0 \le y < 12 \end{cases}$$

2) let 'x' be a uniform random variable in the interval (0,4). If $Y = x^2$, find the cumulative distribution function of the random variable Y

Sol Given 'x' is a uniform random variable in (0,4) $a = 0, b = 4$

cumulative distribution function of x is

$$F(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$$

$$F(x) = \begin{cases} \frac{x}{4} & 0 < x < 4 \\ 1 & x \geq 4 \end{cases}$$

The cumulative function $G(y)$ for the random variable y is given by

$G(y) = P(Y \leq y)$

$= P(x^2 \leq y)$

$= P(x \leq \sqrt{y})$

$= F(\sqrt{y})$

$G(y) = \frac{\sqrt{y}}{4}$

$= P(x \leq \pm\sqrt{y})$

$= P(-\sqrt{y} \leq x \leq \sqrt{y})$

$= F(\sqrt{y}) - F(-\sqrt{y})$

$= \frac{\sqrt{y}}{4} - \frac{-\sqrt{y}}{4}$

$= \frac{2\sqrt{y}}{4} = \frac{\sqrt{y}}{2}$

3) let 'x' be a uniform random variable in the interval $(-3,3)$. If $Y = x^2$, find the cumulative distribution function of the random variable Y $G(y)$ $\frac{\sqrt{y}}{3}$

Sol Given 'x' is a uniform random variable in $(-3,3)$ $a = -3$, $b = 3$

cumulative distribution function of x is

$$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$$

$$F(x) = \begin{cases} 0 & x \leq -3 \\ \frac{x+3}{6} & -3 < x < 3 \\ 1 & x \geq 3. \end{cases}$$

The cumulative function $G(y)$ for the random variable y is given by

for $\cancel{-3 < x \leq a}$ $0 < y < 9$

$G(y) = P(Y \leq y)$

$= P(x^2 \leq y)$

$= P(x \leq \pm\sqrt{y})$

$= P(-\sqrt{y} \leq x \leq \sqrt{y})$

$= F(\sqrt{y}) - F(-\sqrt{y})$

$$= \frac{\sqrt{y}+3}{6} - \left(\frac{-\sqrt{y}+3}{6}\right)$$

$$= \frac{\sqrt{y}+3+\sqrt{y}-3}{6}$$

$$G(y) = \frac{\cancel{2}\sqrt{y}}{\cancel{6}\,3} = \frac{\sqrt{y}}{3}$$

Diff w.r.to y

$$G'(y) = g(y) = \frac{1}{6\sqrt{y}} \quad \text{for } \cancel{-0 < x \le 0} \quad 0 \le x < 9$$

$$\therefore g(y) = \begin{cases} \dfrac{1}{6\sqrt{y}} & 0 < y < 9 \end{cases}$$

for $0 \le x < 3$

$$G(y) = P(Y \le y)$$
$$= P(X^2 \le y)$$
$$= P(X \le \sqrt{y})$$
$$= F(\sqrt{y})$$
$$= \frac{\sqrt{y}+3}{6}$$
$$G(y) = \frac{\sqrt{y}}{6} + \frac{3}{6}$$

diff w.r.to to y

$$G'(y) = g(y) = \frac{1}{12\sqrt{y}}$$

$$\text{Mean} = \int_0^9 y \frac{1}{6\sqrt{y}}\, dy$$
$$= \frac{1}{6}\int_0^9 \frac{y\sqrt{y}}{\sqrt{y}\sqrt{y}}\, dy$$
$$= \frac{1}{6}\int_0^9 \sqrt{y}\, dy$$
$$= \frac{1}{6}\left[\frac{y^{1/2+1}}{1/2+1}\right]_0^9$$
$$= \frac{1}{\cancel{6}\,3}\left[\frac{\cancel{2}}{3} y^{3/2}\right]_0^9$$
$$= \frac{1}{9}\left[(3^2)^{3/2} - 0\right]$$
$$= \frac{1}{9}\left[\cancel{27}\,3\right] = 3$$

$$\text{variance } \sigma^2 = \int_0^9 y^2 g(y)\, dy - (\mu)^2$$
$$= \int_0^9 y^2 \frac{1}{6\sqrt{y}}\, dy - 3^2$$
$$= \frac{1}{6}\int_0^9 y\sqrt{y}\, dy - 9$$
$$= \frac{1}{6}\left[\frac{y^{3/2+1}}{3/2+1}\right]_0^9 - 9$$
$$= \frac{1}{\cancel{6}\,3}\left[\frac{\cancel{2}}{5} y^{5/2}\right]_0^9 - 9 = \frac{1}{15}\left[(3^2)^{5/2}\right] - 9$$

$$= \frac{1}{15}\left(3^5\right) - 9$$

$$= \frac{81}{5} - 9$$

$$= \frac{81-45}{5} = \frac{36}{5} = 7.2$$