Unit - V

Beta & Gamma functions

Introduction:

Many integrals can be evaluated easily by expressing them in terms of Beta & Gama functions. To find the solution of Multidimensional heat conduction problems, we require a prior knowledge of Fourier series, Bessels functions, Legendre polynomials, Laplace transforms, and complex Variable Theory

Improper integrals:

Consider the integral $\int_a^b f(x)dx$ such as an integral for which (i) either the integral of integration is not finite i.e; $a=-\infty$ (or) $b=\infty$ (or) both

(ii) Or the function $f(x)$ is unbounded at one or more points in $[a,b]$ is called an improper integral

Note: Integrals corresponding to (i) & (ii) are called improper integrals of the first & second kinds ~~rep~~ respectively. Integrals which satisfy both the conditions (i) & (ii) are called, improper integrals of the 3rd kind

ex:- (i) $\int_0^\infty \frac{1}{1+x^2}dx$ and $\int_{-\infty}^{+\infty} \frac{1}{1+x^4}dx$ are improper integral of first kind

(ii) $\int_0^1 \frac{1}{1-x^2}dx$ is an improper integral of second kind

(iii) The Gamma function defined by the

$$\int_0^\infty e^{-x}x^{n-1}dx$$ where $n>0$ is an improper integral of third kind

* Beta function :

properties of Beta function :

1) symmetry of Beta function

i.e., $\beta(m,n) = \beta(n,m)$

proof : By the definition of Beta function we have

$$\beta(m,n) = \int_{x=0}^{1} x^{m-1} (1-x)^{n-1} \, dx$$

put $1-x = y$

$x = 1-y$

$dx = -dy$

limits if $x=0 \Rightarrow y=1$

if $x=1 \Rightarrow y=0$

$$\beta(m,n) = \int_{y=1}^{0} (1-y)^{m-1} \cdot y^{n-1} (-dy)$$

$$= \int_{y=0}^{1} (1-y)^{m-1} y^{n-1} \, dy$$

Replace y by x

$$\beta(m,n) = \int_{x=0}^{1} (1-x)^{m-1} x^{n-1} \, dx$$

$= \beta(n,m)$

∴ Hence it is proved

(2) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

By def of Beta fn.

Proof: $\beta(m,n) = \int_{x=0}^{1} x^{m-1} (1-x)^{n-1} \, dx$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta$

limits if $x=0 \Rightarrow \theta=0$; if $x=1 \Rightarrow \theta=\pi/2$

$$\beta(m,n) = \int_{\theta=0}^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \, 2\cos\theta\sin\theta \, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \sin\theta\cos\theta \, d\theta.$$

$$= 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$\therefore$ Hence, it is proved.

③ $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

(or)

$\beta(p,q) = \beta(p+1,q) + \beta(p,q+1)$

Sol:- By the def. of Beta function we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx \longrightarrow ①$$

$$\beta(m+1,n) = \int_0^1 x^m (1-x)^{n-1} \, dx$$

$$\beta(m,n+1) \int_0^1 x^{m-1} (1-x)^n \, dx$$

RHS :

$$\beta(m+1,n) + \beta(m,n+1) = \int_{x=0}^1 x^m (1-x)^{(n-1)} \, dx + \int_0^1 x^{m-1}(1-x)^n \, dx$$

$$= \int_{x=0}^1 \left(x^m(1-x)^{n-1} + x^{m-1}(1-x)^n\right) dx$$

$$= \int_{x=0}^1 \left[\frac{x^m(1-x)^n}{1-x} + \frac{x^m(1-x)^n}{x}\right] dx$$

$$= \int_{x=0}^1 x^m (1-x)^n \left(\frac{1}{1-x} + \frac{1}{x}\right) dx$$

$$= \int_{x=0}^1 x^m (1-x)^n \left(\frac{\cancel{x}+1-\cancel{x}}{x(1-x)}\right) dx$$

$$= \int_{x=0}^{1} x^{m-1}(1-x)^{n-1}\,dx$$

$$= \beta(m,n) \quad (\text{from } ①).$$

④ if m ~~and~~ and n are positive integers, then $\beta(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$

Note: (1) putting $m = 1$, we get $\beta(1,n) = \frac{(n-1)!}{n!} = \frac{1}{n}$

(2) Illy. $\beta(m,1) = \frac{1}{m}$.

Other forms of β functions:

form 1: Beta function as an infinite integral

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$(\text{or}) = \int_0^\infty \frac{y^{n-1}}{(1+x)^{m+n}}\,dy \quad (\text{or}) \quad \beta(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$$

Proof: By the defi. of Beta function, we have

$$\beta(m,n) = \int_{x=0}^{1} x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \frac{1}{1+y} \Rightarrow y = \frac{1}{x} - 1$

$$dx = \frac{-1}{(1+y)^2}\,dy$$

limits

if $x=0 \Rightarrow y=\infty$

if $x=1 \Rightarrow y=0$

$$\beta(m,n) = \int_{y=\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right)dy$$

$$= \int_{y=0}^{\infty} \frac{1}{(1+y)^{m-1}} \cdot \left(\frac{\cancel{1}+y-\cancel{1}}{1+y}\right)^{n-1}\left(\frac{1}{(1+y)^2}\right)dy$$

$$= \int_{y=0}^{\infty} \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2}\,dy$$

$$= \int_{y=0}^{\infty} \frac{y^{n-1}}{(1+y)^{m-1+n-1+2}}\,dy$$

$$\beta(m,n) = \int_{y=0}^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\, dy \rightarrow ②$$

Replace 'y' by 'x'

$$\beta(m,n) = \int_{x=0}^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx \rightarrow ③$$

By the Symmetricity of Beta function

$$\beta(m,n) = \int_{x=0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \rightarrow ④$$

$\therefore$ Hence proved.

$\Rightarrow$ Form : 2 $\because$ symmetric integration form

To show that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx$ (or)

$$\beta(p,q) = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\, dx$$

Proof : We know that

$$\beta(m,n) = \int_{x=0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$\beta(m,n) = \int_{x=0}^{1} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_{x=1}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \rightarrow ①$$

consider

$$\int_{x=1}^{+\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \qquad \text{put } x = \frac{1}{y} \Rightarrow y = \frac{1}{x}$$

$$dx = \frac{-1}{y^2}\, dy$$

limits

if $x=1 \Rightarrow y=1$

if $x=\infty \Rightarrow y = \frac{1}{\infty} = 0$

$$= \int_{y=1}^{0} \frac{(1/y)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \left(\frac{-1}{y^2}\right) dy = \int_{y=0}^{1} \frac{\frac{1}{y^{m-1}}}{\left(\frac{y+1}{y}\right)} \left(\frac{+1}{y^2}\right) dy$$

$$= \int_{y=0}^{1} \frac{1}{y^{m-1}} \cdot \frac{y^{m+n}}{(y+1)^{m+n}} \cdot \frac{1}{y^2}\, dy$$

$$= \int_{y=0}^{1} \frac{y^{m+n} \cdot y^{-m+1} \cdot y^{-2}}{(1+y)^{m+n}}\, dy = \int_{y=0}^{1} \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$\therefore$ Replace 'y' by 'x'.

$$\int_{x=1}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_{x=0}^{1} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$(1) \Rightarrow \beta(m,n) = \int_{x=0}^{1} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_{x=0}^{1} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \int_{x=0}^{1} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$$

$\therefore$ Hence it is proved.

Form :- 3 : Improper integral form

To show $\beta(m,n) = a^m b^n \int_0^{\infty} \frac{x^{m-1}}{(ax+b)^{m+n}}\, dx.$

proof :

RHS

$$a^m b^n \int_0^{\infty} \frac{x^{m-1}}{(ax+b)^{m+n}}\, dx = a^m b^n \int_{x=0}^{\infty} \frac{x^{m-1}}{b^{m+n}\left(\frac{ax}{b}+1\right)^{m+n}}\, dx$$

put $\frac{ax}{b} = t \Rightarrow x = \frac{b}{a} t$

$dx = \frac{b}{a} dt$

limits

if $x = 0 \Rightarrow t = 0$

if $x = \infty \Rightarrow t = \infty$

$$= \frac{a^m b^n}{b^{m+n}} \int_{t=0}^{\infty} \frac{\left(\frac{b}{a} t\right)^{m-1}}{(1+t)^{m+n}} \left(\frac{b}{a}\right) dt$$

$$= \frac{a^m \cancel{b^n}}{b^m \cdot \cancel{b^n}} \int_{t=0}^{\infty} \frac{(b/a)^{m-1}\, t^{m-1}}{(1+t)^{m+n}} \left(\frac{b}{a}\right) dt$$

$$= \frac{a^m}{b^m} \cdot \frac{b^m}{a^m} \int_{t=0}^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}}\, dt$$

Replace 't' by x

$$\Rightarrow \int_{x=0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \beta(m,n) \qquad \text{(from eq (1))}$$

$\Rightarrow$ **form :- 4 :-** Integral form, 0 to 1 form

To show $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\, dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

proof: By the defi. of Beta function, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \longrightarrow (1)$$

put $x = \dfrac{(1+a)y}{(y+a)}$

$$dx = (1+a)\left(\frac{(y+a)1 - y(1+0)}{(y+a)^2}\right) dy$$

$$= (1+a)\left[\frac{\cancel{y}+a-\cancel{y}}{(y+a)^2}\right] dy$$

$$dx = \frac{a(1+a)}{(y+a)^2}\, dy$$

limits:- if $x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$$\beta(m,n) = \int_{y=0}^{1} \left(\frac{(1+a)y}{y+a}\right)^{m-1}\left(1 - \frac{(1+a)y}{y+a}\right)^{n-1} \frac{a(1+a)}{(y+a)^2}\, dy$$

$$= \int_{y=0}^{1} \frac{(1+a)^{m-1} y^{m-1}}{(y+a)^{m-1}} \left(\frac{\cancel{y}+a-\cancel{y}-ay}{y+a}\right)^{n-1} \frac{a(1+a)}{(y+a)^2}\, dy$$

$$= \int_{y=0}^{1} \frac{a(1+a)^m . y^{m-1}(a-ay)^{n-1}}{(y+a)^{m-1}(y+a)^{n-1}(y+a)^2}\, dy$$

$$= (1+a)^m \int_{y=0}^{1} \frac{a\, y^{m-1}\, a^{n-1} (1-y)^{n-1}}{(y+a)^{m+n}}\, dy$$

$$\beta(m,n) = a^n (1+a)^m \int_{y=0}^{1} \frac{y^{m-1}(1-y)^{n-1}}{(y+a)^{m+n}}\, dy$$

Replace 'y' by 'x'

$$\beta(m,n) = a^n (1+a)^m \int_{x=0}^{1} \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\, dx$$

$$= \frac{\beta(m,n)}{a^n (1+a)^m}$$ //

Form:- 5 :- integral form a-to b form

To show $\int_b^a (x-a)^{m-1} (a-x)^{n-1}\, dx = (a-b)^{m+n-1}\, \beta(m,n)$, $m>0$, $n>0$.

proof:- By the defi of Beta function, we have

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx \longrightarrow (1)$$

put $x = \frac{t-b}{a-b} \Rightarrow x = \frac{t}{a-b} - \frac{b}{a-b}$

$$dx = \frac{dt}{a-b} = 0$$

limits :

if $x=0 \Rightarrow t=b$

if $x=1 \Rightarrow t=a$

$$\beta(m,n) = \int_{t=b}^{a} \left(\frac{t-b}{a-b}\right)^{m-1} \left(1-\frac{t-b}{a-b}\right)^{n-1} \frac{dt}{a-b}$$

$$= \int_{t=b}^{a} \frac{(t-b)^{m-1}}{(a-b)^{m-1}} \left(\frac{a-\cancel{b}-t+\cancel{b}}{a-b}\right) \frac{1}{(a-b)}\, dt$$

$$= \int_{t=b}^{a} \frac{(t-b)^{m-1}}{(a-b)^{m-1}} \frac{(a-t)^{n-1}}{(a-b)^{n-1}} \frac{1}{(a-b)}\, dt$$

$$= \int_{t=b}^{a} \frac{(t-b)^{m-1}(a-t)^{n-1}}{(a-b)^{m+n-1}}\, dt$$

Replace t by x

$$\int_{x=b}^{a} (x-b)^{m-1}(a-x)^{n-1}\, dx = (a-b)^{m+n-1}\, \beta(m,n)$$

Relation b/w the Beta & Gamma function

★★★ $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ $m>0,\ n>0$

proof: By the defn of Gamma function, we have

$$\Gamma(m) = \int_{x=0}^{\infty} e^{-x} x^{m-1}\,dx \rightarrow (1)$$

put $x = yt$

$dx = y\,dt$

if $x=0 \Rightarrow t=0$

if $x=\infty \Rightarrow t=\infty$

$$\Gamma(m) = \int_{t=0}^{\infty} e^{-yt} (yt)^{m-1}\, y\,dt$$

$$= \int_{t=0}^{\infty} e^{-yt}\, t^{m-1}\, y^m\,dt$$

$$\Gamma(m) = y^m \int_{t=0}^{\infty} e^{-yt}\, t^{m-1}\,dt$$

Replace 't' by 'x'

$$\frac{\Gamma(m)}{y^m} = \int_{x=0}^{\infty} e^{-xy} x^{m-1}\,dx \rightarrow (2)$$

mutiply with $e^{-y} y^{m+n-1}$ on both sides of eq (2)

$$\frac{\Gamma(m)\, e^{-y}\, y^{m+n-1}}{y^m} = \int_{x=0}^{\infty} e^{-xy}\, x^{m-1} \cdot e^{-y}\, y^{m+n-1}\,dx$$

$$\Gamma(m)\, e^{-y} y^{n-1} = \int_{x=0}^{\infty} e^{-(x+1)y}\, x^{m-1}\, y^{m+n-1}\,dx$$

Taking integral from (0) to ∞ w.r.t 'y' on Both sides

$$\Gamma(m) \int_{y=0}^{\infty} e^{-y} y^{n-1}\,dy = \int_{y=0}^{\infty} \left\{ \int_{x=0}^{\infty} e^{-(x+1)y}\, x^{m-1}\, y^{m+n-1}\,dx \right\} dy$$

changing the Order of integration,

$$\Gamma(m)\,\Gamma(n) = \int_{x=0}^{\infty} \left\{ \int_{y=0}^{\infty} e^{-(1+x)y}\, y^{m+n-1}\,dy \right\} x^{m-1}\,dx$$

$k = 1+x$

$n = m+n$

$$\Gamma(m)\,\Gamma(n) = \int_{x=0}^{\infty} \frac{\Gamma(m+n)}{(1+x)^{m+n}}\, x^{m-1}\,dx$$

$$\Gamma(m)\,\Gamma(n) = \Gamma(m+n) \int_{x=0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) \quad \left(\text{from eq (1)}\right)$$