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1) The constrains of the given problem are

$2x_1 + x_2 \geq 1$

$x_1 + 3x_2 \geq 1$

$x_1 \geq 0$ , $x_2 \geq 0$

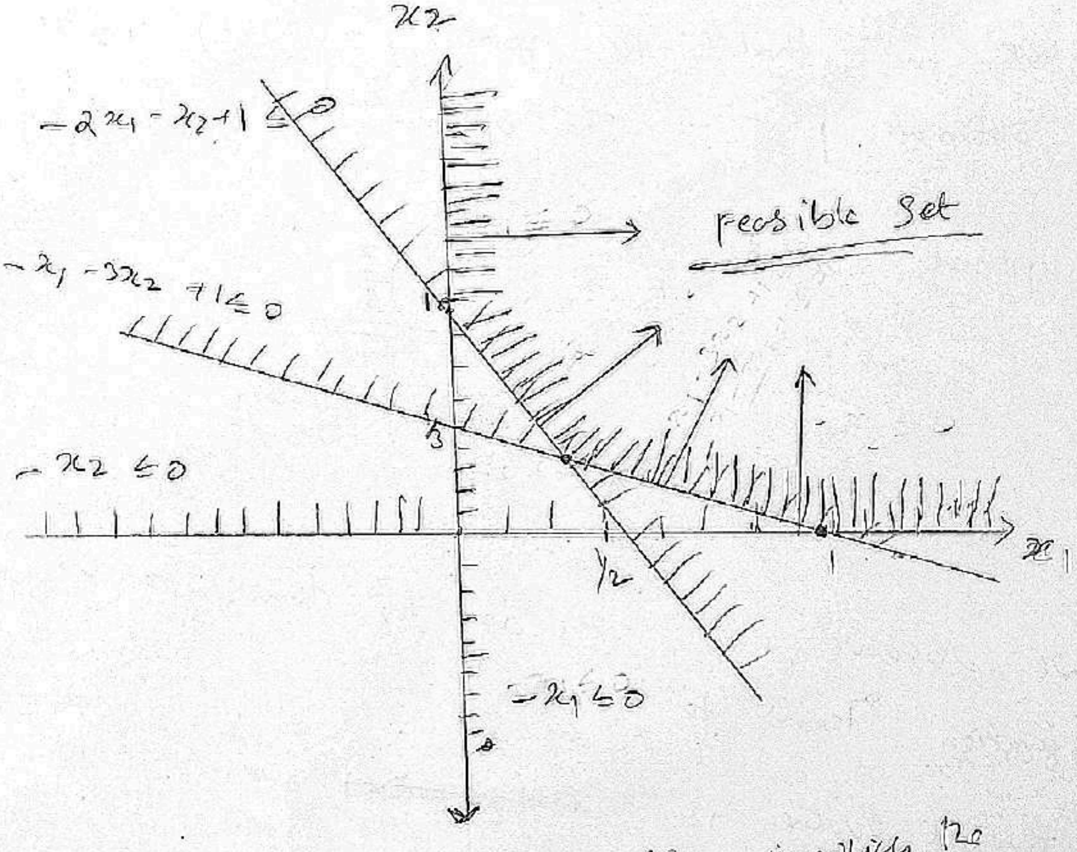
writing them in Standard form $\rightarrow$

$-2x_1 - x_2 + 1 \leq 0$

$-x_1 - 3x_2 + 1 \leq 0$

$-x_1 \leq 0$

$-x_2 \leq 0$




The arrow mark indicates the direction in which the feasible set opens according to each ~~constra~~ of the constrains.

The feasible set has 3 ~~ar~~ corners at $(0,1)$, $(2/5, 1/5)$ and $(1,0)$

1.1) $f_0(x_1, x_2) = x_1 + x_2$

→ As $x_1$ or $x_2$ tends to infinity ($\infty$). The sum would tend to $\infty$.

→ Therefore the optimal point has to be one of the corners.

→ We find that the point $(2/5, 1/5)$ is the optimal point. The optimal set is $x^* = \{(2/5, 1/5)\}$

→ optimal value $p^* = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$

1.2) $f_0(x_1, x_2) = -x_1 - x_2$

→ We observe as $x_1$ or $x_2$ tends to $\infty$, the function tends to $-\infty$.

→ Optimal value is ~~[illegible]~~ $-\infty$, the function is unbounded below

→ optimal set $x^* = \{(\infty, \infty)\}$

1.3) $f_0(x_1, x_2) = x_1$

→ We observe that the function is independent of $x_2$.

→ ~~Therefore the feasible set is~~

→ The optimal value of the function is 0,

$$f^*(x_1, x_2) = p^* = x_1 = \underline{\underline{0}}$$

→ The set $x^* = \{ (0, x_2) \mid x_2 \geq 1 \}$ would lead to this optimal value.

1.4) $f_0(x_1, x_2) = \max \{ x_1, x_2 \}$

$$f_0(x_1, x_2) = \begin{cases} x_1 & x_1 > x_2 \\ x_2 & x_2 > x_1 \end{cases}$$

→ This implies $x_1 = x_2$ is a point of cross over where the function $f_0$ will ~~return~~ begin to return $x_2$ ( Assuming we are moving increasing in $x_1$ ).

→ Substitute $x_1 = x_2$ in the constraints


$$2x + x \geq 1 = x \geq 1/3$$

$$x + 3x \geq 1 = x \geq 1/4$$

$$[1/3, \infty) \cap [1/4, \infty)$$
$$= \underline{\underline{[1/3, \infty)}}$$

→ The optimal point in the set as to be $(1/3, 1/3)$

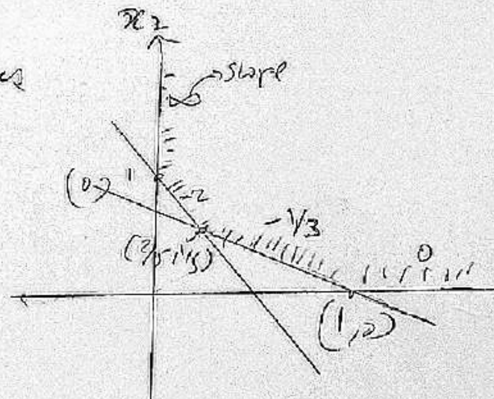
$\therefore$ $x^* = \{(1/3, 1/3)\}$

→ optimal value = $f^*(x_1, x_2) = p^* = \underline{1/3}$

1.5) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$

→ From the figure we observe that our feasible set has piecewise linear boundaries with slopes $\left(\frac{dx_2}{dx_1}\right) = [\infty, -2, -1/3, 0]$

→ We know that $x_1^2 + 9x_2^2$ increases for all $x_1$ and $x_2$ and it passes through the line $x_1 + 3x_2 = 1$




→ $\therefore$ The point at which slope is perpendicular to the line of the function $f_0(x_1, x_2)$ is $x_1 + 3x_2 = 1$ should be the optimal point.

→ $\nabla f = \cancel{\begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix}} \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 18x_2 \end{bmatrix}$

Slope $= \dfrac{\partial f/\partial x_2}{\partial f/\partial x_1} = \dfrac{18x_2}{2x_1} = \dfrac{9x_2}{x_1} \rightarrow (1)$

If two Slopes [lines] ($m_1$ and $m_2$) are prendicular it $4^1$

implies $m_1 m_2 = -1$ $\left[ m_1 = \frac{9x_2}{x_1}, m_2 = \frac{-1}{3} \right]$

$$\Rightarrow \frac{9x_2}{x_1} \cdot \left(\frac{-1}{3}\right) = -1$$

$$\frac{x_2}{x_1} = \frac{1}{3} \rightarrow (2)$$

The point at which the $(x_1, x_2)$ pair has to further

Satisfy $x_1 + 3x_2 = 1$

$$1 + 3\left(\frac{x_2}{x_1}\right) = \frac{1}{x_1} \qquad = 1 + 3\left(\frac{1}{3}\right) = \frac{1}{x_1}$$

$$\underline{\underline{x_1 = 1/2}}$$

$$\Rightarrow x_2 = \underline{\underline{1/6}}$$

→ The point $\underline{(1/2, 1/6)}$ is optimal.

→ This point further Satisfies $2x_1 + x_2 = 7/6 \underline{\geq 1}$

5 (5)

2) Show that $x^* = (1, 1/2, -1)$ is optimal

$$\min \left(\frac{1}{2}\right) x^T P x + Q^T x + r$$

s.t. $-1 \le x_i \le 1$, $i = 1, 2, 3$

→ optimality condition for constrained optimization is $\nabla f(x^*)^T (y - x^*) \geq 0$

↳ (1)

$$f(x) = \frac{1}{2} x^T P x + Q^T x + r$$

$$\nabla f(x) = P x + Q$$

$$\nabla f(x^*) = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13.0 \end{bmatrix}$$

$$= \begin{bmatrix} 21 \\ 14.5 \\ -11 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13.0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

substituting in equation (1).

$$\begin{bmatrix} -1, & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1^* - 1 \\ y_2^* - 1/2 \\ y_3^* + 1 \end{bmatrix} = -1(y_1^* - 1) + 2(y_3^* + 1)$$

↳ (2)

~~∀ $-1 \le y_i \le 1$ equation (2) is $\geq 0$~~

~~∴ $(1, 1/2, -1)$ is an optimal point~~

∀ $-1 \le y_i \le 1$ equation (2) is $\geq 0$

∴ $(1, 1/2, -1)$ is an optimal point.

3) 3.1) minimize $c^T x$

s.t $Ax = b$

The Lagrange function is given as

$L(x, \lambda) = c^T x + \lambda (Ax - b)$

The Dual is given by

$g(\lambda) = \inf_x ( c^T x + \lambda . (Ax - b))$

$= -\lambda b + \inf_x ( c^T x + \lambda A x)$

$\frac{d}{dt} (c^T + \lambda A) x = 0$ $c^T = -\lambda A + N(A)$

[where $N(A)$ is the null space of $A$]

[replacing $-\lambda$ by $\lambda$]

$c = A^T \lambda + N(A)$ [$A^T \lambda \perp N(A)$]

$c^T x = \lambda^T A x + N(A) x$ [If $N(A) = 0$]

$c^T x = \lambda^T b \rightarrow$ [Result 1]

Case II :- When $b \notin R(A)$ ~~then~~ when $b$ is not in range of $A$ then no solution exists. or optimal value is $\infty \rightarrow$ [Result 2].

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3)

3.1) minimize $c^T x$

s.t $Ax = b$

The Lagrange function is given as

$$L(x, \lambda) = c^T x + \lambda (Ax - b)$$

The Dual is given by

$$g(\lambda) = \inf_x \left( c^T x + \lambda (Ax - b) \right)$$

$$= -\lambda b + \inf_x \left( c^T x + \lambda A x \right)$$

$\frac{d}{dt} (c^T + \lambda A) x = 0$ $\quad c^T = -\lambda A + N(A)$

[replacing $-\lambda$ by $\lambda$] [where $N(A)$ is the null space of $A$]

$c = A^T \lambda + N(A)$ [$A^T \lambda \perp N(A)$]

$c^T x = \lambda^T A x + N(A) x$ [If $N(A) = 0$]

$c^T x = \lambda^T b \rightarrow$ [Result 1]

Case II:- When $b \notin R(A)$ ~~then~~ when $b$ is not in range of $A$ then no solution exists. or optimal value is $\infty \rightarrow$ [Result 2].

Case III :-

In case I we assumed $\mathcal{N}(A) = 0$, here we break that assumption. $\mathcal{N}(A) \neq 0$.

$$c^{*} = A^T \lambda + \mathcal{N}(A).$$

This problem is unbounded below,

$$c^T x = \lambda^T A x + \mathcal{N}^T(A) x$$

Set $x = x_0 - t\mathcal{N}(A)$ $\quad [\mathcal{N}(A) \neq 0]$

$$= \lambda^T A (x_0 - t\mathcal{N}(A)) + \mathcal{N}^T(A) [x_0 - t\mathcal{N}(A)]$$

$$= (\lambda^T A + \mathcal{N}^T(A)) x_0 - (\lambda^T t A + t\mathcal{N}^T(A)\mathcal{N}(A))$$

$$= (\lambda^T A + \mathcal{N}^T(A)) x_0 - (\lambda^T A + \mathcal{N}^T(A)\mathcal{N}(A)) t \quad \hookrightarrow (a)$$

We observe from equation (a) that as $t \to \infty$, the function tends to $-\infty$.

$\therefore$ its unbounded below and $p^* = -\infty$ $\hookrightarrow$ [Result 3]

Summarizing results

$$= \begin{cases} \lambda^T b & c \in A^T \lambda, \\ \infty & b \notin R(A) \\ -\infty & \text{otherwise} \end{cases}$$

3.2) minimize $c^T x$

s.t $a^T x \leq b$

The Lagrange is given by

$L(x, \lambda) = c^T x + \lambda (a^T x - b)$

The Dual is given by

$g(\lambda) = -\lambda b + \inf_x (c^T x + \lambda a^T x)$

$\frac{d}{dx} (c^T + \lambda a^T) x = 0$ ; $c^T + \lambda a^T = 0$

$c^T = -\lambda a^T$

or [replacing $-\lambda$ by $\lambda$]

$c = \lambda a + N(a)$

if $N(a) = 0$ then

$p^* = \lambda b$ → [Result 1]. [where $\lambda \leq 0$]

Case II :-

$c^T = -\lambda a^T + N(a)$ if $N(a) \neq 0$, then the function is unbounded below and the optimal value is $-\infty$. → Result [2].

Case III :-

Let $[\lambda > 0]$, choose $x = -ta$ and $t \to \infty$

$$c^T x = -t c^T a = -\lambda t a^T a$$

$$t \to \infty \quad \Big[\underbrace{-\lambda}_{(\geq 0)} \underbrace{t}_{(\geq 0)} \underbrace{a^T a}_{(\geq 0)} \to -\infty\Big]$$

and the constrain

$$a^T x - b$$

$$-t a^T a - b \leq 0$$

for large values of $t$, the function ~~that~~ would reach $-\infty$. $\Rightarrow p^* = -\infty$

Summarizing results.

$$p^* = \begin{cases} \lambda b & c = a\lambda, \text{ where } \lambda \leq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

3.3) $\min \; c^T x$

s.t. $\; l_i \le x_i \le u_i$

-> We observe that for $(c^T) > 0$

~~The optimal~~ optimal value is when

$x^* = l_i$

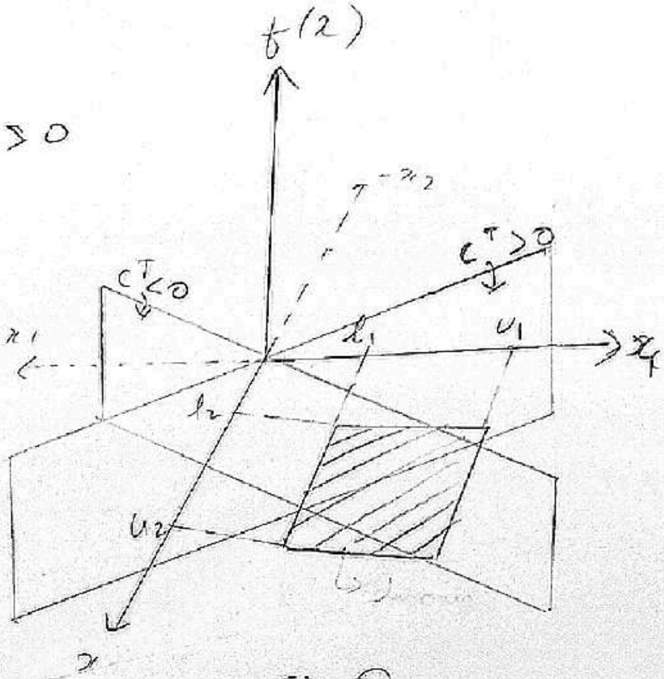



Fig ①

-> When $c^T < 0$

The optimal value is when $x^* = u_i$

-> When $c = 0$ then all values in domain are optimal

-> ∴ The optimal value is given by

$$p^* = \begin{cases} c\, l_i & c > 0 \\ c\, u_i & c < 0 \end{cases}$$

-> This can be written in a compact form as

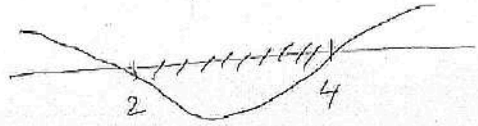
$$p^* = l^T c^+ + U^T c^-$$

where $c^+ = \max(c_i, 0)$

$c^- = \max(-c_i, 0)$

4) min $x^2+1$

s.t $(x-2)(x-4) \leq 0$




4.1)

→ The feasible set is

$$\underline{[2,4]}$$

$$= \{x \mid 2 \leq x \leq 4\}$$

→ Lagrangian

$$L(x,\lambda) = x^2+1+\lambda((x-2)(x-4))$$

$$= x^2+1+\lambda(x^2-6x+8)$$

$$L(x,\lambda) = (1+\lambda)x^2 + (1+8\lambda) - 6\lambda x$$

$$\nabla L = 0 = \begin{bmatrix} 2x(1+\lambda) - 6\lambda \\ x^2+8-6x \end{bmatrix} = 0$$

$2x + 2x\lambda - 6\lambda = 0$ | $x^2+8-6x = 0$

$x^2-4x-2x+8$

$x(x-4)-2(x-4)$

$(x-4)(x-2)=0$

$\underline{x=4,2}$

with $x=2$

$4+4\lambda-6\lambda=0$

$\underline{\lambda = 2}$

with $x=4$

$8+8\lambda-6\lambda=0$

$\underline{\lambda=-4}$

Since $\lambda$ is an inequality constraint it has to be greater than 0. ∴ $x=2$ will be the optimal point. Furthermore as $\lambda$ increases from 0 to 2 ~~g~~ $g(\lambda)$ reaches its maximum at $\lambda=2$ and decreases with further increase in $\lambda$. Hence $\underline{\lambda=2}$

$\therefore$ The optimal point $x^* = 2$

optimal solution $f^*(x^*) = 5$

4.2) $L(x, \lambda) = (1+\lambda)x^2 - 6\lambda x + (8\lambda + 1)$

$\rightarrow$ when $\lambda = 2$

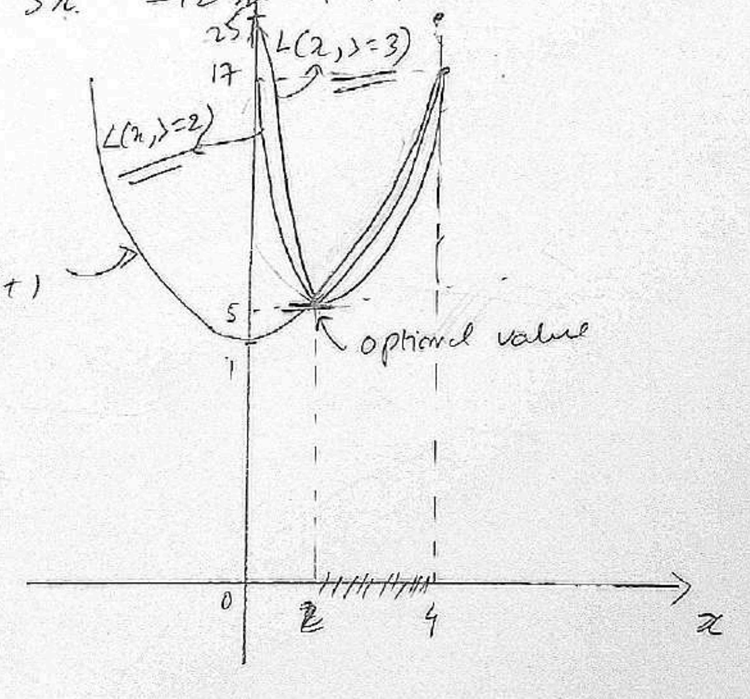
$L(x,\lambda) = \cancel{3x^2 - 18x + 17}$

$L(x, \lambda) = 3x^2 - 12x + 17$

when $\lambda = 3$

$L(x,\lambda) = 4x^2 - 18x + 25$

$\rightarrow$




$\rightarrow$ Dual function.

$g(\lambda) = \inf_x \left( (1+\lambda)x^2 - 6\lambda x + (8\lambda + 1) \right)$

$(8\lambda + 1) + \inf_x \left( (1+\lambda) x^2 - 6\lambda x \right) \rightarrow 1$

$\frac{d}{dx} (1+\lambda)x^2 - 6\lambda x = 2(1+\lambda)x - 6\lambda = 0.$

$x = \frac{3\lambda}{(1+\lambda)}$

Substitute $\lambda$ in equation ① for $\lambda \geq -1$

$$= (8\lambda+1) + \inf_{x} \left( (1+\lambda)x^2 - 6\lambda x \right)$$

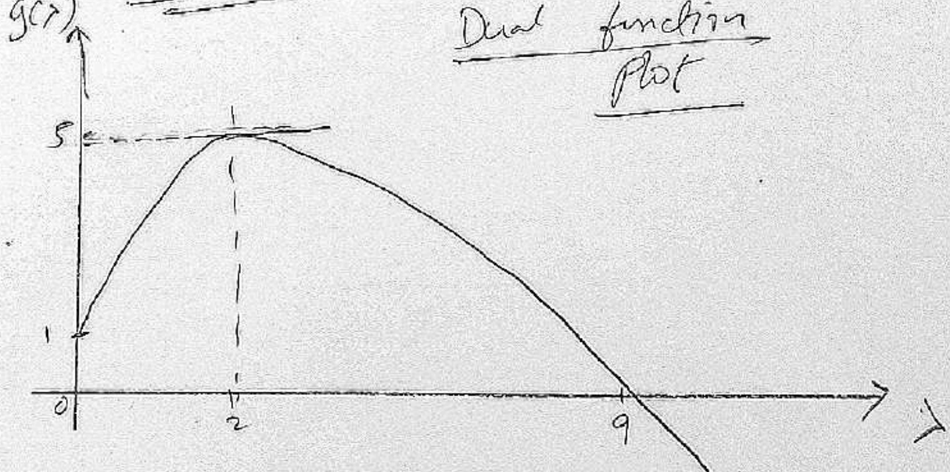
$$(8\lambda+1) + \cancel{(1+\lambda)} \frac{9\lambda^2}{(1+\lambda)^{\cancel{2}}} - 6\lambda \cdot \frac{3\lambda}{(1+\lambda)}$$

$$= (8\lambda+1) + \frac{9\lambda^2}{(1+\lambda)} - \frac{18\lambda^2}{(1+\lambda)}$$

$$(8\lambda+1) - \frac{9\lambda^2}{(1+\lambda)} = g(\lambda)$$

$$g(\lambda) = -\frac{9\lambda^2}{(1+\lambda)} + (8\lambda+1)$$

Dual function plot




If $\lambda < -1$ the function is unbounded below

$$\boxed{g(\lambda) = \begin{cases} \dfrac{-9\lambda^2}{(1+\lambda)} + (8\lambda+1) & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}}$$

4.3) Dual problem

$$\boxed{\begin{array}{ll} \max & g(\lambda) = \dfrac{-9\lambda^2}{(\lambda+1)} + (8\lambda+1) \\ \text{s.t} & \lambda \geq 0 \end{array}}$$

$g(\lambda)$ is concave if $\underline{\nabla^2 g(\lambda) \leq 0}$

$$\nabla g(\lambda) = \cancel{-18\lambda(\lambda+1)}$$

$$= -\left[\frac{18\lambda(\lambda+1) - 9\lambda^2}{(\lambda+1)^2}\right] + 8$$

$$\nabla_\lambda g(\lambda) = -\left[\frac{9\lambda^2 + 18\lambda}{(\lambda+1)^2}\right] + 8$$

Second derivative

$$\nabla^2_{\lambda\lambda} g(\lambda) = -\left[\frac{(18\lambda+18)(\lambda+1)^2 - 2(\lambda+1)(9\lambda^2+18\lambda)}{(1+\lambda)^4}\right]$$

$$= -\left[\frac{18(\lambda+1)^3 - 2(9\lambda^3 + 18\lambda^2 + 9\lambda^2 + 18\lambda)}{(1+\lambda)^4}\right] \hookrightarrow (1)$$

Solving only numerator

$18(\lambda+1)^3 - 2(9\lambda^3 + 27\lambda^2 + 18\lambda)$

$18(\lambda+1)^3 - 2(9\lambda)(\lambda^2 + 3\lambda + 2)$

$18(\lambda+1)^3 - 18\lambda(\lambda+2)(\lambda+1)$

$= 18(\lambda+1)^3 - 18\lambda(\lambda+2)(\lambda+1)$

$= 18(\lambda+1)\left[(\lambda+1)^2 - \lambda(\lambda+2)\right]$

$= 18(\lambda+1)\left[\cancel{\lambda^2}+\cancel{2\lambda}+1-\cancel{\lambda^2}-\cancel{2\lambda}\right]$

$= \underline{18(\lambda+1)}$

Subsitute in equation ①.

$$= \frac{-18(\lambda+1)}{(\lambda+1)^4} = \frac{-18}{(1+\lambda)^3} \leq 0$$

$\nabla^2 g(\lambda) \leq 0$ the function is concave

$\rightarrow \quad g(\lambda) = \dfrac{-9\lambda^2}{(\lambda+1)} + (8\lambda+1)$

$\nabla g(\lambda) = 0 \quad = -\left[\dfrac{9\lambda^2+18\lambda}{(\lambda+1)^2}\right] + 8 = 0$

$-(9\lambda^2+18\lambda) + 8(\lambda+1)^2 = 0$

$-9\lambda(\lambda+2) + 8(\lambda+1)^2 = 0$

$-9\lambda^2 - 18\lambda + 8(\lambda^2+2\lambda+1) = 0$

$-9\lambda^2 - 18\lambda + 8\lambda^2 + 16\lambda + 8 = 0$

$-\lambda^2 - 2\lambda + 8 = 0$

$\lambda^2 + 2\lambda - 8 = 0$

(1d)

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$\lambda^2 + 2\lambda - 8 = 0$

$\lambda^2 + 4\lambda - 2\lambda - 8 = 0$

$\lambda(\lambda + 4) - 2(\lambda + 4) = 0 \; ; \; \lambda = \underline{2, -4}$

The optimal $\underline{\lambda^* = 2}$

$\underline{\underline{g(\lambda^*) = 5 = d^*}}$

Since $p^* = 5 = d^*$ strong duality holds

5) minimize $c^T x$

s.t $\quad Gx \leq x$

$\quad\quad Ax = b$

→ The Lagrange function is given by

~~$L(x, \lambda)$ =~~

$$L(x, \lambda, \nu) = c^T x + \lambda(Gx - x) + \nu(Ax - b)$$

→ The Lagrange Dual function is

$$g(\lambda, \nu) = \inf_x \left( c^T x + \lambda(Gx - x) + \nu(Ax - b) \right)$$

$$= -\nu b + \inf_x \left( [c^T + \lambda(G - I) + \nu A] x \right)$$

$$\frac{d}{dx} [c^T + \lambda(G - I) + \nu A] x = c^T + \lambda(G - I) + \nu A = 0 \quad \hookrightarrow (1)$$

The Dual function is

$$g(\lambda, \nu) = \begin{cases} -\nu b & c^T + \lambda(G - I) + \nu A = 0 \\ -\infty & \text{otherwise} \end{cases}$$

→ The Dual problem is given by



$$\boxed{\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{Subject to} & \lambda \geq 0 \end{array}}$$

By making the implicit constrains explicit we obtain

$$\boxed{\begin{array}{ll} \text{maximize} & -\nu b \\ \text{Subject to} & c^T + \lambda(G - I) + \nu A = 0 \\ & \lambda \geq 0 \end{array}}$$

b) $\min \max_{i=1,\dots m} (a_i^T x + b)$

$\Downarrow$

$\min \max_{i=1,\dots m} y_i$

s.t $a_i^T x + b = y_i \quad i=1,\dots m$

$\Rightarrow L(x, \lambda) = \max_{i=1,2\dots m} y_i + \sum_{i=1}^{m} \lambda_i (a^T x_i + b - y_i)$

The Dual function is given by

$\Rightarrow g(\lambda) = \inf_{x,y} \left( \max_{i=1\dots m} y_i + \sum_{i=1}^{m} \lambda_i (a^T x_i + b - y_i) \right)$

→ Taking infimum w.r.t $x$ we get

$\sum_{i=1}^{m} \lambda_i a_i = 0 \rightarrow (1)$

→ minizing over $y$ we get

$\inf_y \left( \max_i (y_i - \lambda^T y) \right)$

Case I:-

$\lambda \geq 0 \quad , 1^T\lambda = 1$

$$\lambda^T y = \sum_j \lambda_j y_j \leq \sum_j \lambda_j \max_i y_i = \max_i y_i$$

With equality if $y = 0$, in that case

$$\inf_y \left( \max_i y_i - \lambda^T y \right) = 0 \rightarrow \text{[Result 1]}.$$

Case II :-

$\lambda \not\geq 0 \quad , \lambda_j < 0$ then choosing $y_i = 0$

$i \neq j$ and $y_j = -t$ with $t \geq 0$ and $t \rightarrow \infty$

We get

$$\max_i \left( y_i - \lambda^T y \right) = 0 + t\lambda_k \quad \left[ \begin{array}{l} t \rightarrow \infty \\ \therefore t\lambda_k \rightarrow -\infty \end{array} \right]$$

$\hookrightarrow$ [Result 2]

Case III :-

If $1^T\lambda \neq 1$, $y = t1$ gives

$$\max_i \left( y_i - \lambda^T y \right) = t(1 - 1^T\lambda) \longrightarrow -\infty \quad \text{[Result 3]}$$

if $t \rightarrow \infty$ and $1 < 1^T\lambda$ or if $t \rightarrow -\infty$, $1 > 1^T\lambda$

Summarizing results.

$$\boxed{\inf_y \left(\max_i y_i - \lambda^T y\right) = \begin{cases} 0 & \lambda \geq 0, \ 1^T\lambda = 1 \\ -\infty & \text{otherwise} \end{cases}}$$

The dual function

$$g(\lambda) = b^T\lambda + \inf_{x,y}\left(\max_{i=1..m} y_i + \sum_{i=1}^{m} \lambda_i a_i^T x_i - y_i\right)$$

$$\Rightarrow \boxed{g(\lambda) = \begin{cases} b^T\lambda & \sum_i \lambda_i a_i = 0, \ 1^T\lambda = 1, \ \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}}$$

The resulting Dual problem is

$$\boxed{\begin{aligned} &\text{maximize} && b^T\lambda \\ &\text{s.t} && A^T\lambda = 0 \\ &&& 1^T\lambda = 1 \\ &&& \lambda \geq 0 \end{aligned}}$$

7) minize $\sum_{i=1}^{N} \|A x_i + b_i\|_2 + \frac{1}{2} \|x - x_0\|^2$ 

s.t $\quad y_i = A_i x + b_i$

$L(x_1, \lambda_1, \ldots \lambda_n) = \sum_{i=1}^{N} \|y_i\|_2 + \frac{1}{2} \|x - x_0\|^2 - \sum_{i=1}^{N} \lambda_i^T (y_i - A_i x - b_i)$

① we first minize over y

$\cancel{\inf_{y_i} (\|y_i\|_2 + \lambda_i^T y_i)}$ $\quad \inf_{y_i} (\|y_i\|_2 + \lambda_i^T y_i)$

Case I :-

if $\|\lambda_i\|_2 > 1$, choose $y_i = -t \lambda_i$

and let $t \longrightarrow \infty$ $\Rightarrow$ the function is unbounded below.

Case II :-

if $\|\lambda_i\|_2 \leq 1$, from Cauchy-Schwarz inequality $[(a^T a)(b^T b) \geq (a^T b)^2]$ that

$\|y_i\|_2 + z_i^T y_i \geq 0$, $\therefore$ the minimum is reached at $y_i = 0$

$$\boxed{\inf_{y_i} \left( \|y_i\|_2 + \lambda_i^T y_i \right) = \begin{cases} 0 & \|\lambda_i\| \le 1 \\ -\infty & \text{Otherwise} \end{cases}}$$

(2) Minimizing over $x$ and set gradient $= 0$.

We get

$$\boxed{x = x_0 + \sum_{i=1}^{N} A_i^T \lambda_i}$$

Substituting results in Lagrangian we get the dual function..

$$\cancel{g(\lambda_1, \lambda_2 \dots \lambda_n) = \begin{cases} \sum_{i=1}^{N} (A_i x_0 + b_i)^T \lambda_i - \frac{1}{2} \left\| \sum_{i=1}^{N} A_i^T \lambda_i \right\|_2^2 & \left[ \|\lambda_i\| \le 1, \; i = 1, 2 \dots N \right] \\ -\infty & \text{otherwise} \end{cases}}$$

$$\boxed{g(\lambda_1, \lambda_2 \dots \lambda_n) = \begin{cases} \sum_{i=1}^{N} (A_i x_0^T + b_i)^T \lambda_i - \frac{1}{2} \left\| \sum_{i=1}^{N} A^T \lambda_i \right\|^2 & \left[ \|\lambda_i\|_2 \le 1, \; i = 1 \dots N \right] \\ -\infty & \text{Otherwise} \end{cases}}$$

[ Dual function ]

Dual problem :-

$$\boxed{\begin{array}{ll} \text{maximize} & \sum_{i=1}^{N} (A_i x_0 + b_i)^T \lambda_i - \frac{1}{2} \| \sum_{i=1}^{N} A^T \lambda_i \|^2 \\ \text{Subject to} & \|\lambda_i\|_2 \leq 1 \ , \ i=1 \ldots N \end{array}}$$

8) minimize $e^{-x}$ (6)

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s.t $\frac{x^2}{y} \leq 0$ $\quad D = \{(x,y) \mid y>0\}$

→ Consider $a, b \in D$.

$\theta a + (1-\theta) b = z$

$$\theta \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta) \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix} = z$$

→ $\theta y_1 + (1-\theta) y_2$

$y_1 > 0, \; y_2 > 0 \Rightarrow \theta y_1 + (1-\theta) y_2 > 0$.

$\therefore z \in D$ the set <u>Domain is convex.</u>

→ $f(x) = e^{-x}$ ; $\nabla f(x) = -e^{-x}$ $\quad \underline{\nabla^2 f(x) = e^{-x}}$

$\nabla^2 f(x) \succeq 0$

→ Feasible set:- $\frac{x^2}{y} \leq 0$ ; given $y > 0$

$x^2 \leq 0$ this only holds when $\underline{x=0}$

$\therefore$ Since $\nabla^2 f(x) \succeq 0$, domain of function and feasible set are convex, The problem is convex optimization problem.

→ optimal Value $= e^{-0} = \underline{1}$

$\underline{p^* = 1}$

8.2) Lagrange Dual

$L(x, y, \lambda) = e^{-x} + \lambda \dfrac{x^2}{y}$

Dual function

$g(\lambda) = \inf_{x, y > 0} \left( e^{-x} + \dfrac{\lambda x^2}{y} \right)$

minizing over $y$ $= \dfrac{\partial}{\partial y} \left[ e^{-x} + \dfrac{\lambda x^2}{y} \right] = \dfrac{-\lambda x^2}{y^2}$ ↳ (1)

Taking Second derivative $\dfrac{\partial^2}{\partial y^2} \left[ e^{-x} + \lambda \dfrac{x^2}{y} \right] = \dfrac{2 \lambda x^2}{y^3}$

When $\lambda \geq 0$, the function ~~#~~ attains minima and given $x = 0$, we obtain optimal value of $g(\lambda) = 0$. [By Substituting in 1].

When $\lambda < 0$ the function is unbounded below and optimal value is $g(\lambda) = \cancel{-\infty}\ \underline{-\infty}$

$$\boxed{g(\lambda) = \inf_{x,y>0} \left( e^{-x} + \lambda \frac{x^2}{y} \right) = \begin{cases} 0 & \lambda \geq 0 \\ -\infty & \lambda < 0 \end{cases}}$$ (8) 25

Therefore the dual problem is given by

$$\boxed{\begin{array}{ll} \text{maximize} & 0 \\ \text{s.t} & \lambda \geq 0 \end{array}}$$

→ The dual optimal value $d^* = 0$

→ $p^* = 1$ ∴ the ~~optimality Gap~~ optimality Gap $p^* - d^* = 1$

8.3) Slater condition is not satisfied for ~~this~~ this problem as $x = 0$ for any feasible pair $(x, y)$.

optimal value of perturbed problem.

8.4) minize $e^{-x}$

s.t $\frac{x^2}{y} \le u$





Case I:- $u \geq 0$

$\frac{x^2}{y} \le u \implies x^2 \le uy$

$-\sqrt{uy} \le x \le \sqrt{uy}$

To minize $e^{-x}$ choose $x = \sqrt{uy}$

$f(u) = e^{-\sqrt{uy}}$

as $u \to \infty$, $e^{-\sqrt{uy}} \to 0$

$p^*(u) = 0$

Case II:- $u < 0$

$\frac{x^2}{y} \le u$ ; $-\sqrt{uy} \le x \le \sqrt{uy}$

Since $u < 0$ (-ve), the solution for $x$ would be on the imaginary plane. but $x \in R$.

hence the solution is <u>infeasible</u> when $u < 0$

$p^*(u) = \infty$

Summarizing results



$$\phi^*(u) = \begin{cases} 0 & u > 0 \\ \infty & u < 0 \end{cases}$$