**Monsoon Semester (Aug-Nov)**
**Topics in Applied Optimization (TAO)**

**Tutorial III**

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**INSTRUCTIONS:**

**Solution to this assignment problems will be discussed in Tutorial**

1. Consider the following optimization problem.

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{array}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

  1. $f_0(x_1, x_2) = x_1 + x_2$
  2. $f_0(x_1, x_2) = -x_1 - x_2$
  3. $f_0(x_1, x_2) = x_1$
  4. $f_0(x_1, x_2) = \max\{x_1, x_2\}$
  5. $f_0(x_0, x_2) = x_1^2 + 9x_2^2$

2. Show that $x^* = (1, 1/2, -1)$ is optimal for the optimization problem

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & -1 \leq x_i \leq 1, i = 1, 2, 3, \end{array} \quad \text{where,}$$

$$P = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}, \quad q = \begin{pmatrix} -22.0 \\ -14.5 \\ 13.0 \end{pmatrix}, \quad r = 1.$$

3. Give an explicit solution to each of the LP.

  1. Minimize a linear function over an affine set.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \end{array}$$

  2. Minimize a linear function over a half space.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a^T x \leq b, \end{array}$$

  where $a \neq 0$.

3. Minimize a linear function over a rectangle.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & l \le x \le u, \end{array}$$

where $l$ and $u$ satisfy $k \le u$.

4. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \le 0, \end{array} \quad \text{with variable } x \in \mathbb{R}.$$

   1. Give the feasible set, the optimal value, and the optimal solution.
   2. Plot the objective $x^2 + 1$ versus $x$. On the same plot show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus $x$ for a few positive values of $\lambda$. Derive and sketch the Lagrange dual function $g$.
   3. State the dual problem and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution $\lambda^*$. Does strong duality hold?

5. Find the dual function of the LP.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \le x \\ & Ax = b. \end{array}$$

Give the dual problem and make the implicit equality constraint explicit.

6. Consider the convex piecewise-linear minimization problem

$$\text{minimize} \quad \max_{i=1,\ldots,m} (a_i^T x + b_i)$$

with variable $x \in \mathbb{R}^n$. Derive a dual problem based on the Lagrange dual of the equivalent problem

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\ldots,m} y_i \\ \text{subject to} & a_i^T x + b_i = y_i, i = 1, \ldots, m, \end{array} \quad \text{with variables } x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

7. Derive a dual problem for

$$\text{minimize} \quad \sum_{i=1}^{N} \|A_i x + b_i\|_2 + (1/2)\|x - x_0\|_2^2.$$

The problem data are $A_i \in \mathbb{R}^{m_i \times n}, b_i \in \mathbb{R}^{m_i}$, and $x_0 \in \mathbb{R}^n$. First introduce new vartiables $y_i \in \mathbb{R}^{m_i}$ and equality constraints $y_i = A_i x + b_i$.

8. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \le 0 \end{array}$$

with variables $x$ and $y$, and domain $\mathcal{D} = \{(x, y) \mid y > 0\}$.

1. Verify that this is a convex optimization problem. Find the optimal value.
2. Give the Lagrange dual problem and find the optimal solution $\lambda^*$ and optimal value $d^*$ of the dual problem. What is the optimal duality gap?
3. Does Slater's condition hold for this problem?
4. What is the optimal value $p^*(u)$ of the perturbed problem

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \leq u \end{array}$$

as a function of $u$.

9. Derive the KKT conditions for the following optimization problem.

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1 \end{array}$$

with vartiable $x \in \mathbb{R}^2$.

1. Sketch the feasible set and level sets of the objective. Find the optimal $x^*$ and optimal value $p^*$.
2. Give the KKT conditions. Do there exists Lagrange's multipliers $\lambda_1^*$ and $\lambda_2^*$ that prove that $x^*$ is optimal.
3. Derive and solve the Lagrange dual problem. Does strong duality hold?

10. Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & Gx = h \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ with rank$(A)$= $n$, and $G \in \mathbb{R}^{p \times n}$ with rank$(G) = p$. Give the KKT conditions, and derive expressions for the primal solution $x^*$ and the dual solution $\nu^*$.

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