1) Triangle inequality $\|u+v\| \leq \|u\| + \|v\|$

$(u+v).(u+v) = u.u + u.v + v.u + v.v$

$\|u+v\|^2 = \|u\|^2 + 2u.v + \|v\|^2 \longrightarrow (1)$

$(\|u\| + \|v\|)^2 = \|u\|^2 + \|v\|^2 + 2(\|u\|\,\|v\|) \longrightarrow (2)$

Substitute 2 in 1

$\|u+v\|^2 = (\|u\| + \|v\|)^2 + 2[u.v - \|u\|.\|v\|]$

$\|u+v\|^2 - (\|u\| + \|v\|)^2 = 2[u.v - \|u\|\,\|v\|]$

$\downarrow$

$u.v = \|u\|\,\|v\| \cos\theta$

$= \cancel{2\|u\|\|v\|} \cos\theta \leq 1$

$\Rightarrow u.v \leq \|u\|\,\|v\|$

$u.v - \|u\|\,\|v\| \leq 0$

$\|u+v\|^2 - (\|u\| + \|v\|)^2 \leq 0$

$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$

$\Rightarrow \boxed{\|u+v\| \leq (\|u\| + \|v\|)}$ Hence proved

(2)

2) Parallelogram Law $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2(u \cdot v)$$

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2(u \cdot v)$$

$$\|u+v\|^2 + \|u-v\|^2 = \underline{\underline{2(\|u\|^2 + \|v\|^2)}}$$

Hence proved

3) Show that $\langle u+v,\ u-v \rangle = \|u\|^2 - \|v\|^2$

$$(u+v)\cdot(u-v) = u \cdot u + \underbrace{v \cdot u - u \cdot v}_{(u \cdot v = v \cdot u)} - v \cdot v$$

$$= \|u\|^2 + 0 - \|v\|^2$$

$$= \underline{\underline{\|u\|^2 - \|v\|^2}}$$

Hence proved.

$\|v\|$

4) Prove that $\langle u,v\rangle = 0$ iff $\|u\| \le \|u+av\|$

Necessity :- Assume $\langle u,v\rangle = 0$.

$$\|u+av\|^2 = \|u\|^2 + a^2\|v\|^2 + 2a\langle u,v\rangle$$

If $\langle u,v\rangle = 0$ $\Rightarrow$

$$\|u+av\|^2 = \|u\|^2 + a^2\|v\|^2$$

$$\Rightarrow \|u+av\|^2 \ge \|u\|^2$$

$$\Rightarrow \|u\|^2 \le \|u+av\|^2$$

$$\Rightarrow \|u\| \le \|u+av\|$$

Completion :-

Assume $\|u\| \le \|u+av\|$ Let $a = -\dfrac{\langle u,v\rangle}{\|v\|^2}$

$$\cancel{\|u\|^2} \le \cancel{\|u\|^2} + a^2\|v\|^2 + 2a\langle u,v\rangle$$

$$0 \le \frac{\langle u,v\rangle^2}{\cancel{\|v\|^4}}\cancel{\|v\|^2} + 2\left(\frac{-\langle u,v\rangle}{\|v\|^2}\langle u.v\rangle\right)$$

$$0 \le -\frac{\langle u,v\rangle^2}{\|v\|^2} \rightarrow (1)$$

Equation ① is only possible when
$\langle u, v \rangle = 0$

Thus we prove that $\langle u, v \rangle = 0$
if and only if $\|u\| \leq \|u + av\|$

Hence proved

④

5) $\|au+bv\| = \|bu+av\|$

$\|au+bv\|^2 = \|bu+av\|^2$

$(au+bv)\cdot(au+bv) = (bu+av)(bu+av)$

$a^2\|u\|^2 + b^2\|v\|^2 + \cancel{2ab\,u\cdot v} = b^2\|u\|^2 + a^2\|v\|^2 + \cancel{2ab\,u\cdot v}$

$\cancel{(a^2-b^2)}\|u\|^2 = \cancel{(a^2-b^2)}\|v\|^2$

$\|u\|^2 = \|v\|^2$

$\Rightarrow \underline{\underline{\|u\| = \|v\|}}$

6) Prove that $16 \leq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$

$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$

$= \left[1+\frac{a}{b}+\frac{a}{c}+\frac{a}{d}\right]+\left[1+\frac{b}{a}+\frac{b}{c}+\frac{b}{d}\right]+\left[1+\frac{c}{a}+\frac{c}{b}+\frac{c}{d}\right]$

$+\left[1+\frac{d}{a}+\frac{d}{b}+\frac{d}{c}\right]$

$= 4+\left[\frac{a}{b}+\frac{b}{a}\right]+\left[\frac{a}{c}+\frac{c}{a}\right]+\left[\frac{a}{d}+\frac{d}{a}\right]+\left[\frac{b}{c}+\frac{c}{b}\right]$

$+\left[\frac{b}{d}+\frac{d}{b}\right]+\left[\frac{c}{d}+\frac{d}{c}\right] \longrightarrow ①$

(5) minimum value of the sum of number and its reciprocal is

we know that A.M $\geq$ G.M $\rightarrow$ (2).

(Arithmetic mean) (Geometric mean)

Let x & $1/x$ be two numbers

$$A.M = \frac{x+\frac{1}{x}}{2} \; ; \; G.M = \sqrt{x \cdot \frac{1}{x}} = \sqrt{1} = 1$$

$$\frac{x+\frac{1}{x}}{2} \geq 1 \quad \text{(from equation 2)}$$

$$x+\frac{1}{x} \geq 2$$

This implies that sum of number and reciprocal minimum value is 2.

In equation (1) there are 6 terms with sum of numbers and reciprocal $\therefore$ the minimum value is 12.

$$\Rightarrow 4 + \underbrace{\left[\frac{a}{b}+\frac{b}{a}\right] + \left[\frac{a}{c}+\frac{c}{a}\right] + \left[\frac{a}{d}+\frac{d}{a}\right] + \left[\frac{b}{c}+\frac{c}{b}\right] + \left[\frac{b}{d}+\frac{d}{b}\right] + \left[\frac{c}{d}+\frac{d}{c}\right]}_{(\geq 12)}$$

$$\Rightarrow \geq 16$$

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 16$$ Hence proved.

④

7) $\|u\| = 3$, $\|u+v\| = 4$, $\|u-v\| = 6$

from parallelogram law

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

$$\frac{\cancel{16}}{8} + \frac{\cancel{36}}{16} = \cancel{2}(9 + \|v\|^2)$$

$$26 - 9 = \|v\|^2 = 17 = \|v\|^2$$

$$\|v\| = \sqrt{17}$$

8) $V = \{A \in R^{n \times n}\}$; $trace(A) = \sum_{i=1}^{N} A_{ii}$

Let $B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0n} \\ b_{10} & b_{11} & \cdots & b_{1n} \\ \vdots & & \ddots & \vdots \\ b_{n0} & \cdots & & b_{nn} \end{bmatrix}$, $A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n0} & \cdots & & a_{nn} \end{bmatrix}$

The dot product is defined as

$$\langle A, B \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} b_{ij}$$

$$B^T A = \begin{bmatrix} b_{00} & \cdots & b_{n0} \\ b_{01} & \ddots & \vdots \\ \vdots & & \vdots \\ b_{0n} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n0} & \cdots & & a_{nn} \end{bmatrix}$$

$$B^T A = \begin{bmatrix} \sum_{i=1}^{N} b_{i0} a_{i0} & \sum b_{i0} a_{i1} & \cdots & \sum b_{i0} a_{in} \\ \vdots & & & \\ \sum b_{in} a_{i0} & \cdots & & \sum b_{in} a_{in} \end{bmatrix}$$

(7)

$Tr(B^T A) = \sum_{p=1}^{N} \sum_{i=1}^{N} b_{ip} a_{ip}$

$\Rightarrow Tr(B^T A) = \langle A . B \rangle = \sum_{p=1}^{N} \sum_{i=1}^{N} b_{ip} a_{ip}$

Hence Proved that $Tr(B^T A)$ is inner product.

Furthermore the function can be observed to satisfy the required properties of a inner product.

1) positivity :- $\langle v,v\rangle \geq 0 \quad \forall\, v \in V$

assume $\;$ $\text{Trace}(A) = \sum_{i=1}^{N} A_{ii}$

$$Tr(A^TA) = \sum_{i=1}^{N} A_{ii} \cdot \sum_{i=1}^{N} A_{ii}$$

$$Tr(A^TA) = \sum_{i=1}^{N} (A_{ii})^2 \geq 0$$

Thus positivity condition holds

2) Homogenity in first slot $\quad \langle \lambda u, v\rangle = \lambda \langle u,v\rangle \;\forall\, u,v \in V$

$$\langle \lambda A, B\rangle = \lambda \langle A, B\rangle \quad \text{[To prove]}$$

$$\langle A, B\rangle = Tr(B^TA) \rightarrow (1)$$

$$\Rightarrow \langle \lambda A, B\rangle = Tr(\lambda B^TA)$$

$$= \lambda\, Tr(B^TA)$$

$$\langle \lambda A, B\rangle = \lambda \langle A, B\rangle \quad \text{from (1)}$$

3) Additivity in first slot :- $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$

$$\langle A+C, B\rangle = \langle A, B\rangle + \langle C, B\rangle \quad \text{[To prove]}$$

$$Tr(B^T(A+C)) = Tr(B^TA + B^TC)$$

$$= Tr(B^TA) + Tr(B^TC)$$

$$= \langle A, B\rangle + \langle C, B\rangle$$

4) Definiteness $\langle V, V \rangle = 0$ iff $V = 0$

~~$\langle A, B \rangle = 0$ iff $AB = 0$~~

~~$Tr(B^T A)$~~ =

$$Tr(V^T V) = \sum_{i=1}^{N} V_{ii}^2$$

all $V_{ii}^2 \geq 0$, $\Rightarrow$ That $Tr(V^T V)$ can be 0 only when all elements are 0. Thus proved

Since all properties hold for $\langle A, B \rangle = trace(B^T A)$ we can conclude that $\langle A, B \rangle = trace(B^T A)$ is an inner product.

9) Prove or disprove

(a) $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$

Let $a_0, a_1, a_2 \ldots a_m$ be the basis ~~...~~ that span rowspace A and let $b_0, b_1, b_2 \ldots b_n$ be the basis vectors of rowspace B.

The sum $A+B$ would contain rows that are a linear combination of $a_0, a_1, a_2 \ldots a_n$ and $b_0, b_1, b_2 \ldots b_n$.

$\Rightarrow \text{rank}(A) = \dim(\text{Span}(a_0, a_1, \ldots a_m)) = m$

$\text{rank}(B) = \dim(\text{Span}(b_0, b_1, \ldots b_n)) = n$

~~$\text{rank}(A+B) = \dim(\text{Span}\{a_0+b_0, a_1+b_1, \ldots, a_m+\ldots$~~

~~$\text{rank}(A)+\text{rank}(B)$~~ ~~$\geq \dim(\text{Span}(\{a_0, a_1, \ldots a_m\}) + \dim(\text{Span}(b_0, b_1, \ldots b_n))]$~~

$$\text{rank}(A+B) \leq \dim(\text{Span}(a_0, a_1, \ldots a_m)) + \dim(\text{Span}(b_0, b_1, \ldots b_n))$$

$$\text{rank}(A+B) \leq \underline{\underline{m+n}}$$ Hence proved.

This implies that the matrix $A+B$ can have a maximum of $m+n$ ~~linear~~ linear independent basis.

~~a~~ b) Row rank and coloumn rank are the same (8)

Proof using Linear Combination :-

→ Consider a Matrix A of dimension $m \times n$. Let the coloumn rank be r and $c_1, c_2, \cdots c_r$ be the basis for the coloumn space.

→ If a matrix "C" contains these basis vectors of the coloumn space of A, C will be of dimension $m \times r$.

→ The remaining $r \times n$ matrix (denoted as R), act as the coefficients of the linear ~~transformation~~ combination of the coloumns of C ~~of the~~

i.e $A = CR$ $(m \times n = (m \times r)(r \times n))$

→ The matrix R contain multiples for the basis of the coloumn space of A (which is C).

→ This implies that each row of A is formed by some linear combination of r rows of R.

∴ The r rows form the spanning set of space A, and the row rank of A cannot exceed r.

→ This proves:-

row rank$(A) \leq$ coloumn rank(A). → (1)

→ Apply same analysis to the transform of A

$A^T = R^T C^T$.

A^T is now $n \times m$, R^T is $n \times r$, C^T is $r \times m$

→ Columns of A^T are rows of A, each column of A^T is a linear combination of each column of R^T

∴ R^T is basis vector for column space of A^T.

→ The row rank of A^T is column rank of A and column rank of A^T is row rank of A

→ This proves :-

row rank(A) = column rank(A^T)

column rank(A) = row rank (A^T)

column rank(A^T) ≤ row rank(A^T) → (2)

Effectively (1) and (2) establish a reverse inequality and we obtain the equality that

row rank (A) = column rank (A)

c) Eigen values of Symmetric matrix are always real

Let $A = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} (2+i-\lambda) & 0 \\ 0 & (2-i-\lambda) \end{vmatrix}$

$= (2+i-\lambda)(2-i-\lambda) = 0$

$(2-\lambda+i)(2-\lambda-i) = 0$

$\lambda = 2-i \quad , \quad \lambda = i-2$

$\lambda = \pm (i-2)$

The Statement is false, the Eigen values need not be real

d) If determinat of matrix in 0 then matrix is rank deficient. 16

(1) → The determinat of a matrix can be 0 if and only if there exist a linear relationship (dependence) between the colomns of a matrix.

(2) → The definition of rank is the maximum ~~nu~~ number of linearly independent columns in a matrix.

→ From (1) and (2) we observe that if ~~a~~ the determinat is 0 then there ~~is~~ exist linear relationship hence the ~~rank is~~ matrix has rank ~~less than (n)~~. is less than n.

→ Further proof:

consider $|A - \lambda I| = 0$ if $|A| = 0 \Rightarrow \lambda = 0$.

one of eigen value is 0

$AV = \lambda V$ [for eigen vector V]

$AV = 0$ [when $\lambda = 0$]

A vetor "v" has been projected into a null space

⇒ there exist linear dependence in A and

colomn rank(A) $< n$

e) Sum of two SPD is SPD

if matrix A is SPD $\Rightarrow$ $x^T A x \geq 0$
$\{\forall x \in R^n, x \neq 0\}$.

If matrix B is SPD $\Rightarrow$ $x^T B x \geq 0$
$\{\forall x, \in R^n, x \neq 0\}$.

$A + B = C$

$$\underbrace{x^T A x}_{\geq 0} + \underbrace{x^T B x}_{\geq 0} = x^T C x \Rightarrow \geq 0.$$

Thus if A & B are SPD $\underline{C \text{ is SPD}}$

Hence Sum of two SPD is SPD

f) For a Given eigen vector "x" ~~the eigen~~ of matrix A

$$A x = \lambda x$$

$$x^T A x = x^T x \lambda$$

$$\lambda = \frac{x^T A x}{x^T x}$$

If A is SPD

$x^T A x \geq 0$

$x^T x = \|x\|^2 \geq 0$

$$\lambda = \frac{x^T A x \ (\geq 0)}{x^T x \ (\geq 0)}$$

$$\Rightarrow \underline{\lambda \geq 0}$$

the eigen values are ~~positive~~ ≥ 0

2) Give example of function $f: X \longrightarrow Y$ 12

a) $X = R^{n \times 1}$, $Y = R$

$f(x) = AX = Y$

where $A \in R^{1 \times n}$

[Example value of A

$A = [1, 1, 1 \cdots \cdots 1]$]

~~$X = E$~~ $Y \in R$

a) $Y = \|x\|_2$ L_2 norm is also an example for the same function

b) $X = R$, $Y = R^n$

$f(x) = Ax = y$

$A \in R^{n \times 1}$

$x \in R \Rightarrow Y = R^{n \times 1}$.

$$\left[A = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right]$$

c) $X = R^m$, $Y = R^n$

$f(x) = Ax = y$

$x \in R^m$

$A \in R^{n \times m} \Rightarrow y \in R^n$

d) $X = R^{m \times n}$, $Y = R^n$

$f(x) = x^T A = y$

$x^T \in R^{n \times m}$

$A \in R^m \Rightarrow y \in R^n$

e) $X = R^{m \times n}$, $Y = R$

$f(x) = AXB$

$X \in R^{m \times n}$

$A \in R^{1 \times m}$

$B \in R^{n \times 1} \Rightarrow Y \in R$

f) $X = R^{m \times n}$, $Y = R^{m \times n}$

$f(X) = aX = Y$

$a \in R$ (Scalar),

$X \in R^{m \times n} \Rightarrow Y \in R^{m \times n}$ Scaling operation.

~~Y~~

2) a) $f(x) = Ax$

$A \in R^{n \times n}$

~~x~~ x has to of dimension $x \in R^{n \times k}$

$\Rightarrow Y \in R^{n \times k}$ $\left[Ax : R^{n \times n} \cdot R^{n \times k} = \underline{\underline{R^{n \times k}}} \right]$

b) $f(x) = x^T A x$

$A \in R^{n \times n}$

x is of dimension $x \in R^{n \times p}$

$R^{p \times n} \cdot R^{n \times n} \cdot R^{n \times p} \Rightarrow \underline{\underline{Y \in R^{p \times p}}}$

c) $f(x) = Ax - w$

$A \in R^{n \times n}$, $w \in R^n$

$\Rightarrow x \in R^{n \times 1}$

hence $\underline{\underline{Y \in R^n}}$

3) find derivatives of the following

a) $f(x) = \sin x + \cos x$

$= \underline{\cos x - \sin x}$

b) $f(x) = e^x \sin x$

$= \underline{e^x \sin x + \cos x \, e^x}$

c) $f(x) = \dfrac{e^x}{x}$

$= \dfrac{e^x x - e^x}{x^2}$

d) $f(x) = \log(\sin(x))$

$= \dfrac{1}{\sin x} \times \cos x = \cot(x)$

e) $f(x) = \dfrac{e^{\sin x^2}}{x^2} + \dfrac{x}{\sin x}$

$= \dfrac{e^{\sin x^2} \cdot \cos x^2 \cdot 2x(x^2) - 2x\, e^{\sin x^2}}{x^4} + \dfrac{\sin x - x\cos x}{\sin^2 x}$

$f'(x) = \dfrac{2 e^{\sin^2 x}(x^2 \cos x^2 - 1)}{x^3} + \dfrac{\sin x - x\cos x}{\sin^2 x}$

4) a

a) A function is linear if

$$f(\alpha X + \beta Y) = \alpha f(x) + \beta f(Y) \rightarrow \text{(0)}$$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \sum_{i=1}^{n} x_i$$

$$f(X) = \sum_{i=1}^{n} x_i \quad \text{where } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\cancel{f(\alpha X + \beta Y)} = \alpha X + \beta Y = \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \vdots \\ \alpha x_n + \beta y_n \end{bmatrix}$$

$$f(\alpha X + \beta Y) = \sum_{i=1}^{n} \alpha x_i + \beta y_i \rightarrow \text{(1)}$$

$$f(x) = \sum_{i=1}^{n} x_i \quad ; \quad f(y) = \sum_{i=1}^{n} y_i \rightarrow \text{(2)}$$

~~from (2)~~ substitute (2) in (0) we get

$$f(\alpha X + \beta Y) = \alpha \sum_{i=1}^{n} x_i + \beta \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \alpha x_i + \beta y_i \rightarrow \text{(3)}$$

Since (1) = (3) the system is linear

→ $A = [1, 1, 1 \ldots \ldots, 1_n]$ A is $1 \times n$ dimensional matrix of ones. ~~A~~ multiplying with A will sum all the elements of X.

b) $f\left(\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ \vdots \\ x_n - x_1 \end{bmatrix}$ (16)

$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_1 \end{bmatrix} = X - x'$

~~Tht~~ where x' is obtained by rotating ~~x~~ by one position.

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$\Rightarrow \cancel{f(\alpha(x-x') + \beta(y-y') = \alpha f(x-x')}$

$f(x) = x - x'$

$f(\alpha x + \beta y) = (\alpha x + \beta y) - (\alpha x' + \beta y')$

$= \alpha(x - x') + \beta(y - y') \rightarrow ①$

$\rightarrow \alpha f(x) = \alpha (x - x')$

$\beta f(y) = \beta (y - y')$

$\alpha f(x) + \beta f(y) = \alpha(x - x') + \beta(y - y') \rightarrow ②$

equation ① = ② hence the system is linear.

The matrix A can be

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ \vdots & & & & & & 1 \\ -1 & 0 & 0 & \cdots & & & \end{bmatrix}$$

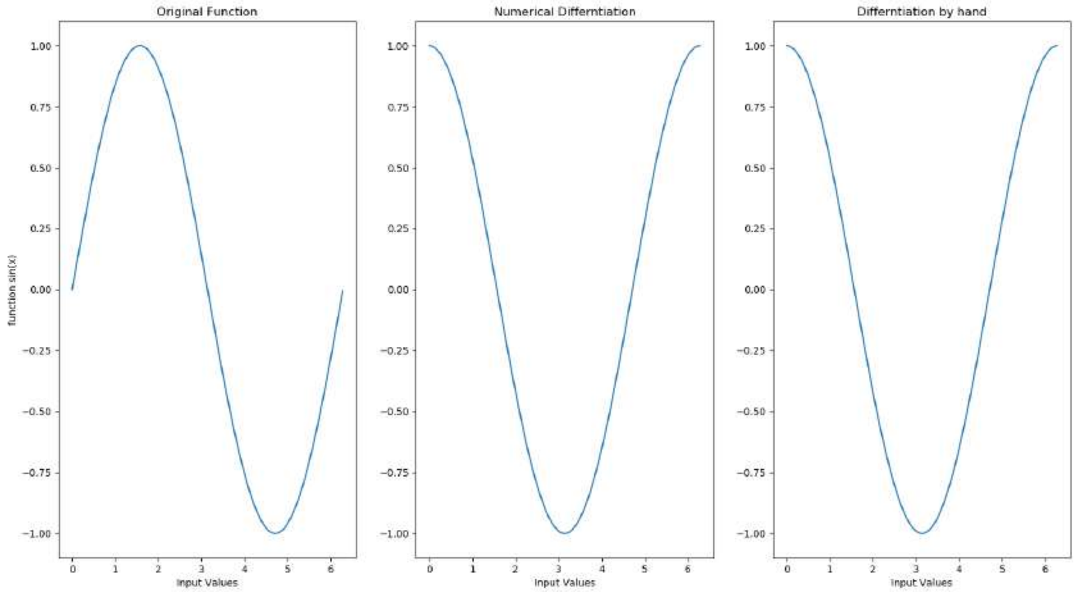
16 1/2

c) If $f(x)$ is bijective, then every element of x is paired with exactly one element of set Y. and each element of set Y is paired with one element of X. i.e there is a one-to-one mapping between X and Y.

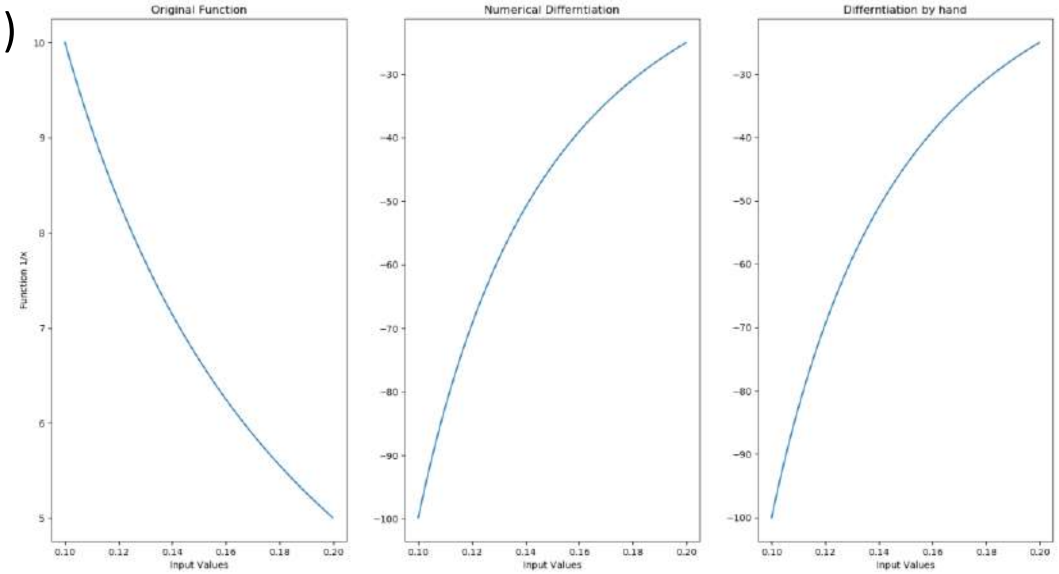
Given $f(x)$ is bijective it implies that $f(x)$ is invertible if ~~$f(x) = Ax$~~ if $f(x) = Ax$ this implies that A matrix is invertiable.

If A is invertiable then A is a <u>full Rank</u> matrix.

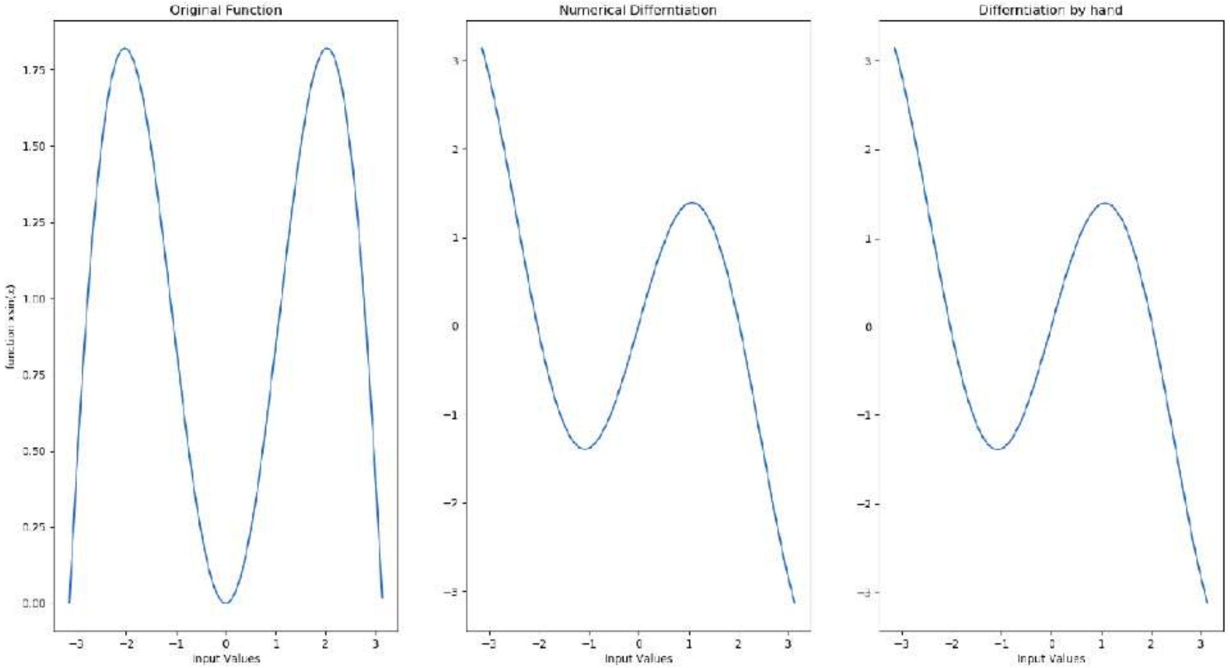
5a


Original Function
Numerical Differntiation
Differntiation by hand
function sin(x)
Input Values
Input Values
Input Values

5b (I)


Original Function
Numerical Differntiation
Differntiation by hand
Function 1/x
Input Values
Input Values
Input Values

5b (Ii)

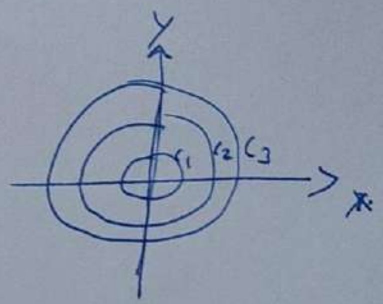

Original Function
Numerical Differntiation
Differntiation by hand
Input Values
Input Values
Input Values

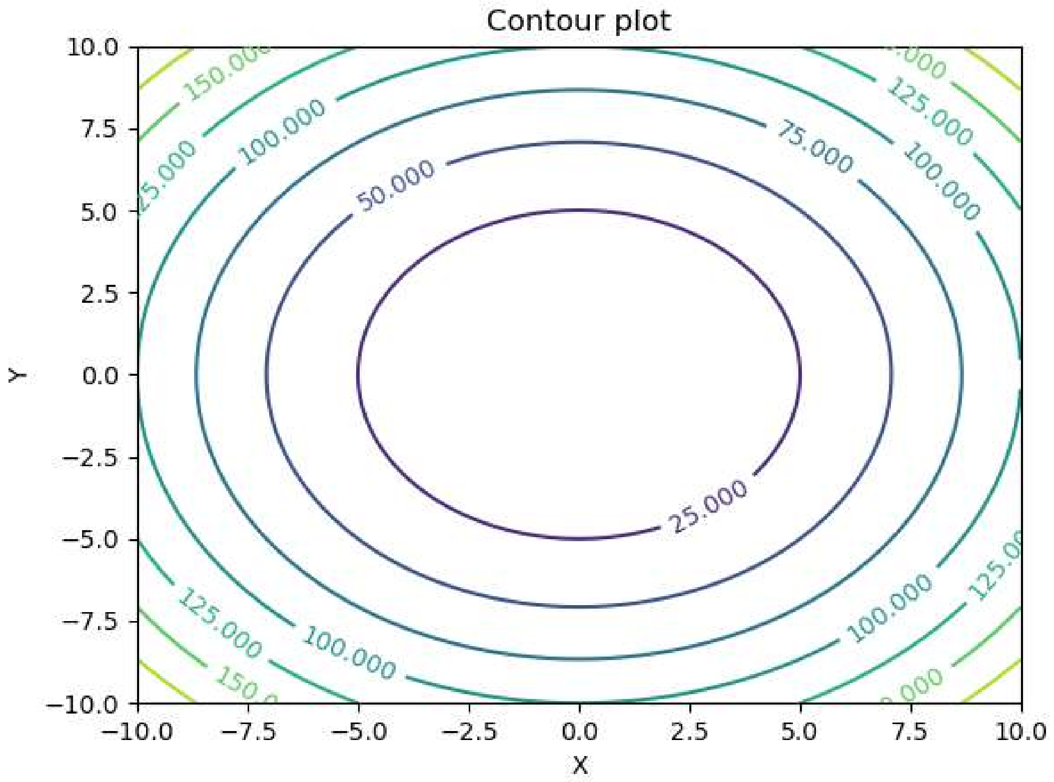
6) Gradients.

a) $f\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2$

$$\nabla f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

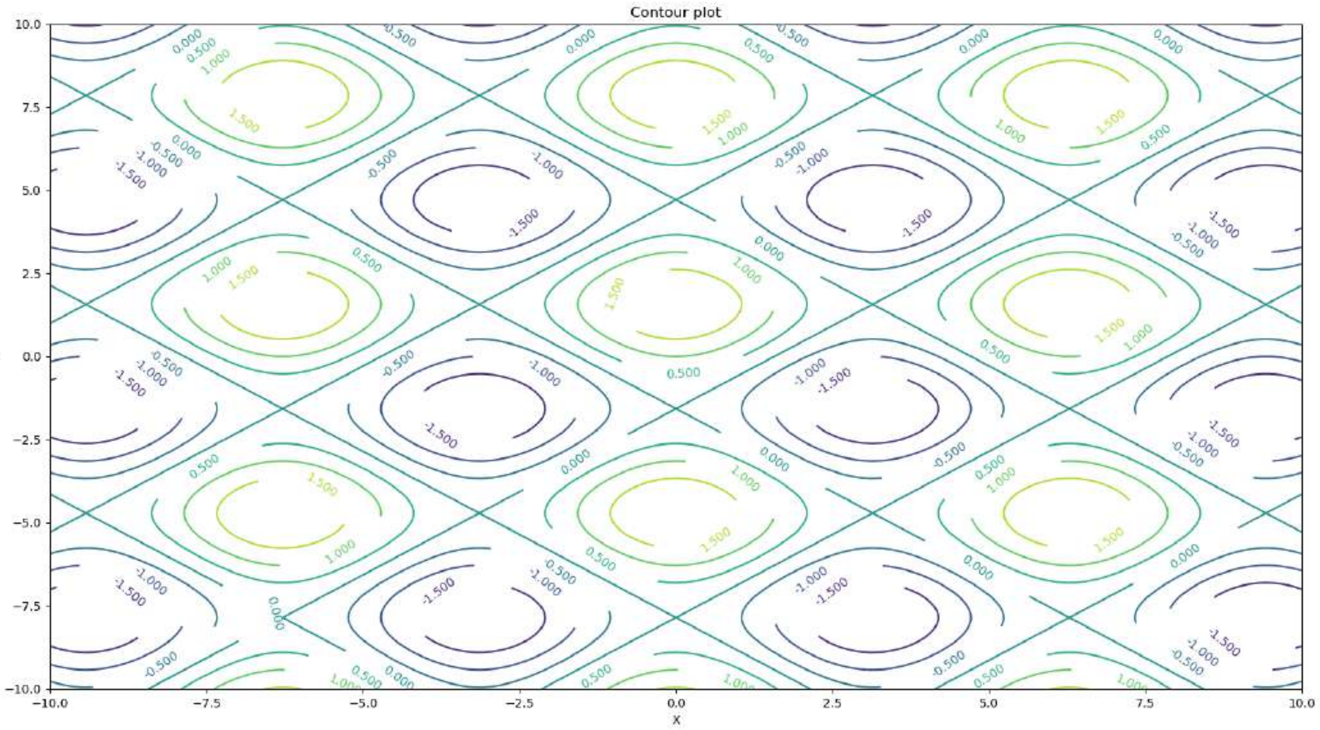
$x^2 + y^2 = c$





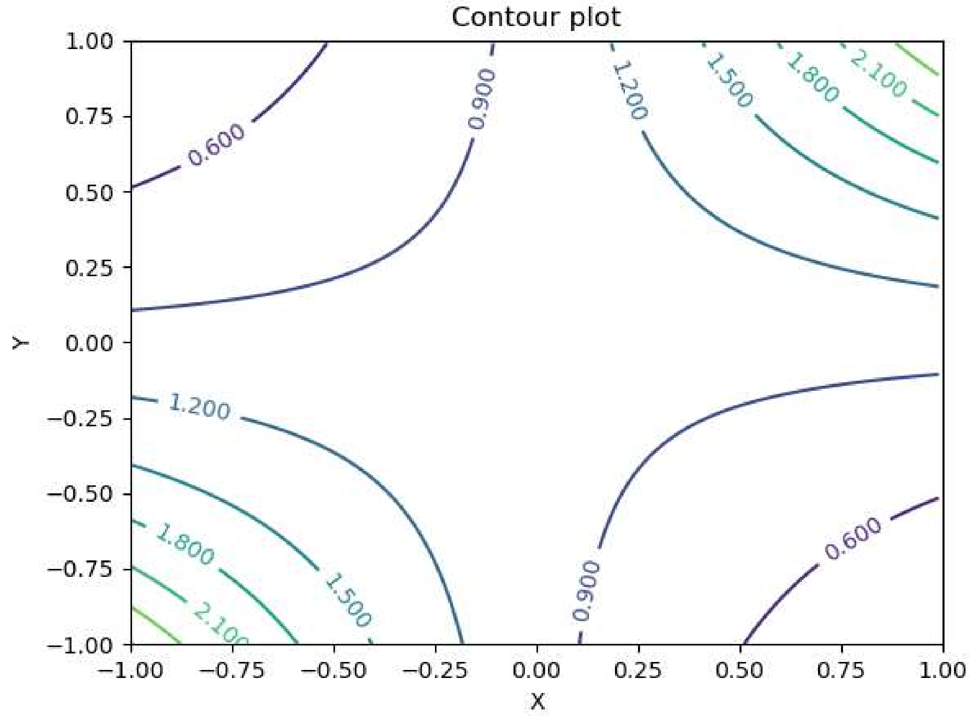
b) $f\begin{bmatrix} x \\ y \end{bmatrix} = \sin x + \cos y$

$$Df = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = \begin{bmatrix} \cos x \\ -\sin y \end{bmatrix}$$



c) $f\begin{bmatrix} x \\ y \end{bmatrix} = e^{xy}$

$$Df = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = \begin{bmatrix} e^{xy} y \\ e^{xy} x \end{bmatrix}$$



Numerical derivative of multivariable functions.

$$\frac{\partial f}{\partial x} = \frac{f(a+h_1, b) - f(a, b)}{h_1}$$

$$\frac{\partial f}{\partial y} = \frac{f(a, b+h_2) - f(a, b)}{h_2}$$