A Mathematical Investigation of Populations and Predator- Prey Dynamics

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1 Introduction

In the late 1980s, research scientists working for the National Forest Service, in the Sierra Nevada conducted a study to understand why the native population of mule deer had dramatically decreased. The investigators believed that initial reason for the decrease in deer population was due to over-population, but the deer population never recovered even after the population had been reduced to a size that the environment can support.The researches found that while the deer population had decreased the native mountain lion population had been increasing. This was contrary to popular belief as mountain lions are natural predators of the deer, so a predator species such as a mountain lion benefits the prey population by keeping the prey population in balance with the environment. Interactions between these species influence the evolution of each population over time.

Investigators wished to mathematically model the interactions between the populations of deer and mountain lion and settled on two different models:

The Logistic Equation and The Lotka-Volterra Predator-Prey Model

2 The Logistic Equation

One approach to study the mountain lions and deer is to individually model each population and see how they interact over time. Since we are studying each population, deer and mountain lion individually, we need a model that takes into account both the intrinsic growth rate of the populations and the environment's carrying capacity.Assumptions used in this model include.

1. Mountain lions are protected from hunting and have no natural predators
2. Mountain lions depend solely on deer for food,
3. The deer population is finite.
4. The population will grow exponentially in the absence of external constraints

Therefore a reasonable way to represent the change in the mountain lion population is with the *logistic equation*.

$$\frac{dx}{dt} = r(1 - \frac{x}{L})x \tag{1}$$

$x(t)$ is the size of the population in *dozens* at any given time t (in years). The underlying assumption that the population will grow exponentially in the absence of external constraints is seen by the first term of (1) $rx(t)$. A population is also naturally limited in size by environmental restrictions such as the availability of resources like food and shelter. The logistic equation takes this into account with second term in the equation $\frac{-rx(t)^2}{L}$, a correction factor modeling the effects of environmental constraints.

2.1 Classification and General Solution

2.1.1

The parameter $r > 0$ is the intrinsic growth rate of the population of mountain lions in *dozens*. In other words it is the per capita growth rate. r shows how quickly a population grows *per individual* already in the population. Its units are $\frac{1}{time}$

The parameter L is the carrying capacity of the population with units *number of mountain lions in dozens*

2.1.2

The Analytical Solution to the Logistic Differential Equation (1) is:

$$x(t) = \frac{L}{1 + (\frac{L}{x_0} - 1)e^{-rt}} \tag{2}$$

where x_0 is the initial population in dozens at time $t = 0$ with initial condition $x(0) = x_0$. The equilibrium solutions to (1) the Logistic Differential Equation are L its carrying capacity and 0.

The derivation of (2) *and its equilibrium solutions are found in the appendix.*

2.2 Euler's Method

Explicit Solutions to differential equations such as (1) are often difficult to solve, however, by utilizing Euler's Method, we were able create an approximations that accurately characterize the equation. To validate this model, solution curves generated from Euler's Method were plotted against the exact solution to the Logistic Differential Equation (1)

2.2.1

Assuming an initial condition for the mountain lion population of 6, Euler's method was used with step sizes of $h_1 = 0.5$, $h_2 = 0.1$, and $h_3 = 0.01$ with $t \in [0, 30]$. Graphs of the three approximation curves and the exact solution curve were plotted, using the `matplotlib.pyplot` package in python. See figure below:

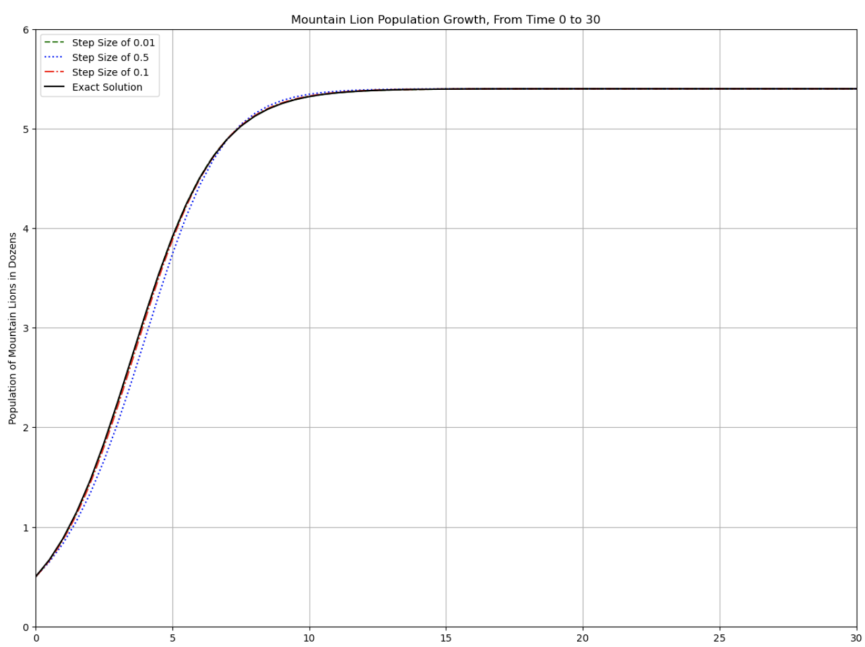

Figure 1

Euler's Method Extrapolation of Mountain Lion Population with Varying Step Sizes, including Exact Solution Curve

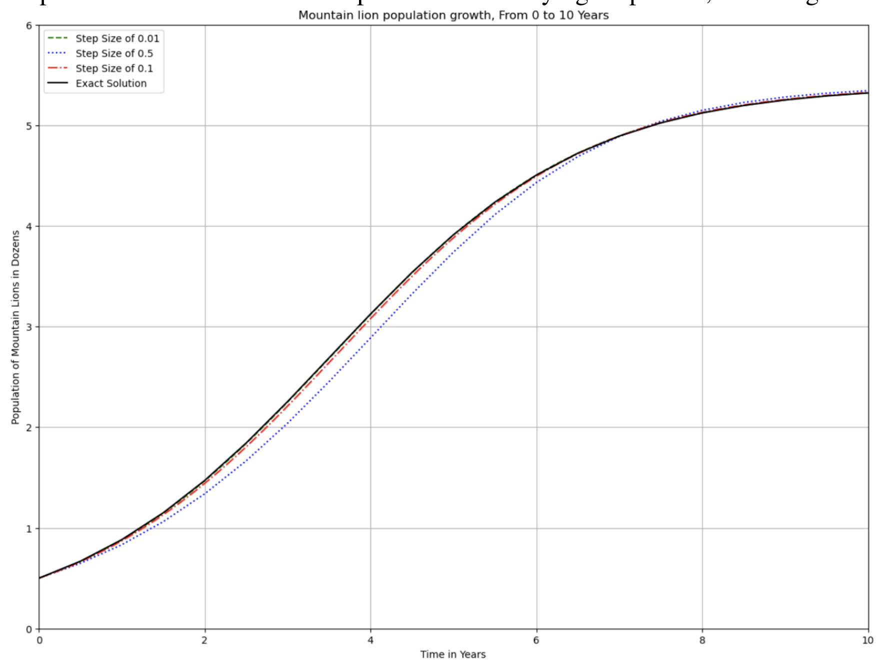

Figure 2
Euler's Method Extrapolation of Mountain Lion Population with Varying Step Sizes, including Exact Solution Curve, *Zoomed In* to Provide Better Visualization

As seen from both graphs, Euler's method provided an extremely accurate approximation to the solution of (1), the rate of change of the mountain lion population. Analysis of the graphs show that the solutions converge to the carrying capacity of $L = 5.4$ supporting our analysis from the previous section.

2.2.2

In order to determine how accurate the solution curve's from Euler's method are the absolute error was computed and plotted using a *semilog* where one axis the *absolute error* is plotted on a logarithmic scale, while the other t *time in years* is plotted on a linear scale. It is useful for data with exponential relationships where one variable covers a large range of values. Semilog was applied to the error graphs because they differ from each other on exponential scale.

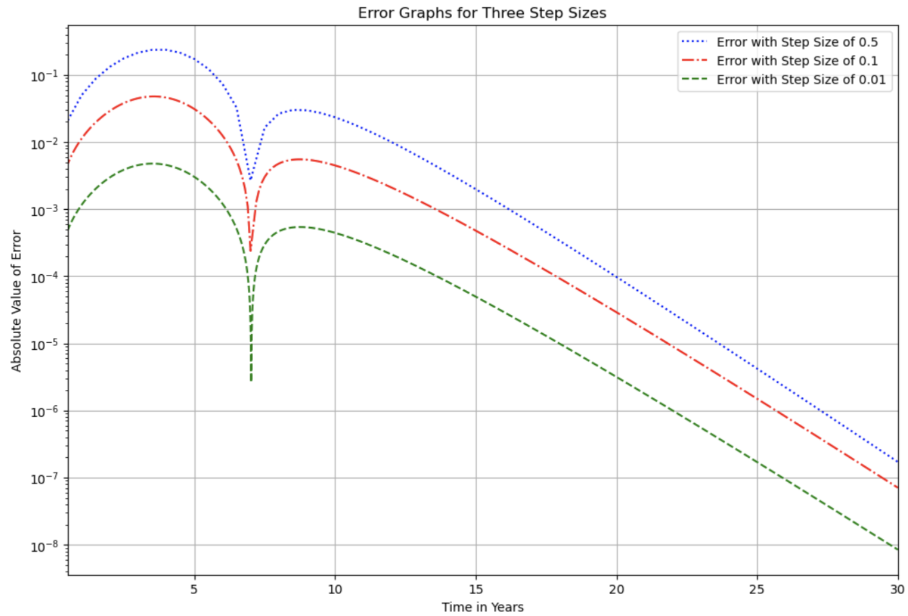

Figure 3
Absolute Error of Euler's Method Curves

The formula to compute absolute error is:

$$\text{Absolute Error} = |x_a - x_e| \text{ where } x_a \text{ is the approximate solution and } x_e \text{ is the exact solution}$$

From figure 3 it is clear that the solution curve that used a step size of 0.01 was the most accurate. Euler's method is an iterative numerical solution and relies on the fact that close to a point, a function and its tangent have nearly the same value. Because of this a smaller step size will result in a more accurate approximation.

As seen from the graph there is a spike in the error curves around $t = 7$. This can be attributed to the absolute value function on the absolute error function. The Euler's method graphs provide an overestimate of the exact solution around $t = 7$ which causes a negative error value. The spike occours when the Euler's method plots shift from an underestimate of the exact solution to an overestimate.

2.2.3

Often time there is a trade off between accuracy and computational cost when computing numerical solutions to differential equations such as the logistic equation (1) Although the solution curve with the step size of 0.01 provided the most accurate approximation, it required 3000 calculations to provide a marginal improvement in accuracy. With this in consideration the solution that provides the best "balance" between computation cost and accuracy would be the step size of 0.1. It required 300 calculations to provide a solution with max absolute error of 0.047.

2.3 Harvest Function

A logistic-type model is a reasonable starting point to model the deer population due to the limited resources in their environment. However, the presence of mountain lions, which are natural predators of deer, is a secondary factor that affects the deer population. Therefore, it is sensible to modify the logistic equation to include a "harvesting" $H(x)$ term that represents the number of deer killed by mountain lions.

$$H(x) = \frac{px^2}{q + x^2} \tag{3}$$

The parameters p and q represent how skilled mountain lions are at catching their prey and r and L represent the same parameters as in the equation modeling the mountain lion population (1). This modification allows for the inclusion of the effect of predation on the deer population. The modified logistic equation provides a relatively simple way to model the deer population in the presence of predators. Therefore the differential equation to model the deer population is:

$$\frac{dx}{dt} = r(1-\frac{x}{L})x - \frac{px^2}{q+x^2} \tag{4}$$

Equation (4) is a non-linear first-order autonomous differential equation. It is non-linear because it includes x^2 terms and first order because the equation only contains a first-order derivative and is autonomous because the equation does not contain any terms with its dependent variable t time. In the context of a population model this means that the rate of change of the population deer over time does not depend on time itself, but on factors such as predation and environmental factors such as food and shelter.
Analysis of the Harvesting Equation (3) shows that as x, the population of deer becomes large, the equation approaches the parameter value of p. Near $x = 0$ the harvesting equation is near 0. This makes sense physically because the harvesting equation is a model of the predation of deer. If there were unlimited deer the i.e as x approached ∞ the only factor affecting predation would be the skill of the mountain lions to predate the deer. The converse is true as well, if the population of deer approached 0 the ability of the mountain lions to predate the deer would decrease as well because there are simply less deer.

A graph of the harvesting equation with parameter values $p = 1.2$ and $q = 1$ are provided for reference

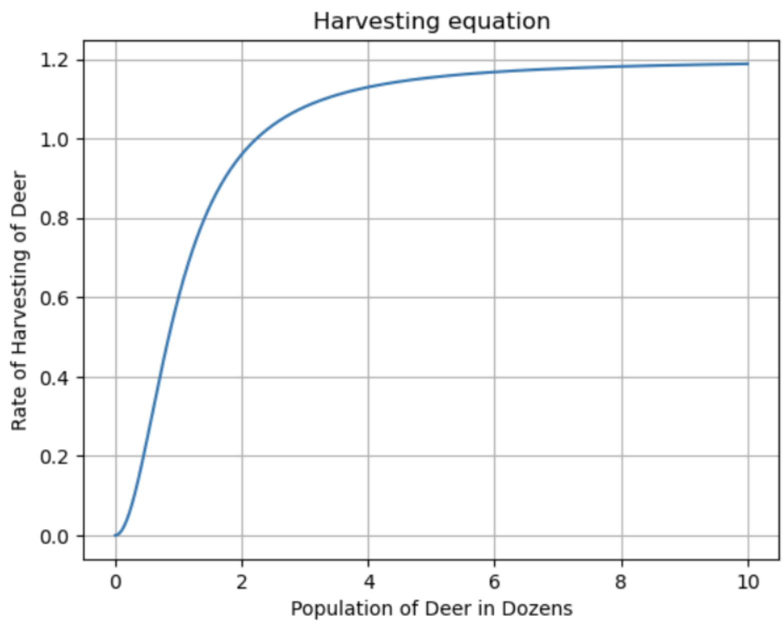

Figure 4
Harvesting Equation

2.4 Modification of the Logistic Equation

2.4.1

Equilibrium solutions to the modified logistic equation (4) modeling the deer population were found by graphing the equation using the python package `matplotlib.pyplot` and using `scipy.optimize`. The equilibrium solutions are

Equilibrium Solution	Stability
0	Unstable
8.018×10^{-1}	Stable
1.856	Unstable
5.442	Stable

The physical meaning behind the equilibrium solutions shown above, are the tendency for a population of deer to approach a constant value based on a certain initial condition. For example, since a 5.442 is a stable equilibrium solution, initial deer populations above 1.856 and 5.442 dozen will tend to toward a population number of 5.442 dozen or around *65* deer.

2.4.2

The Direction Field describing the modified logistic equation (4) is:

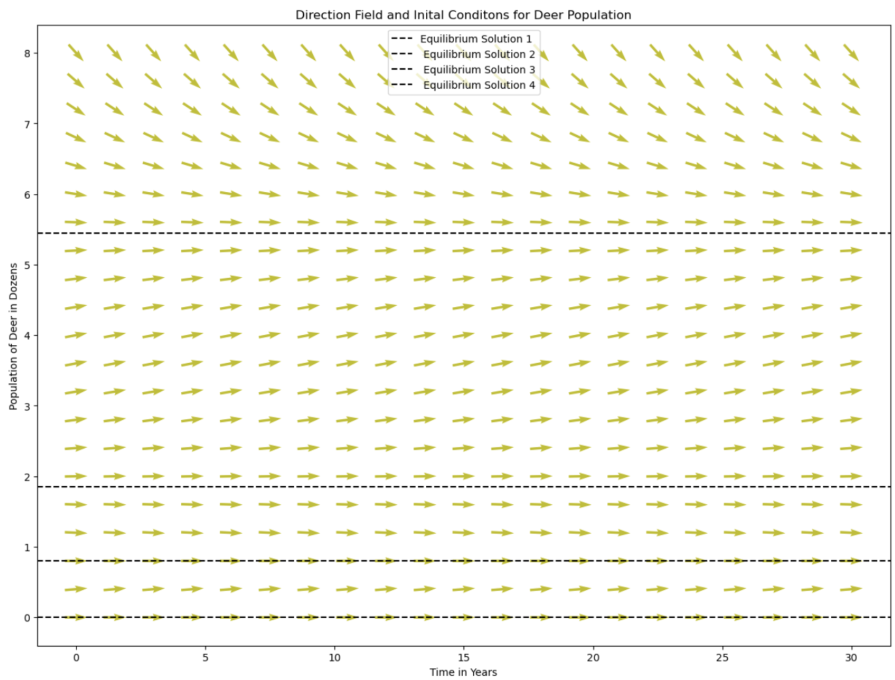

Figure 5
Direction Field of the Modified Logistic Equation *without* Euler's Method Curves

2.4.3

To emphasize the analysis from the prior section 2.4.2 Euler's method curves over the interval $t \in [0,30]$ with a step size $h = 0.1$ and varying initial conditions of the deer population of 84, 24, 18, and 6, were plotted using the `matplotlib.pylplot` package in python. The graph describing this figure is below:

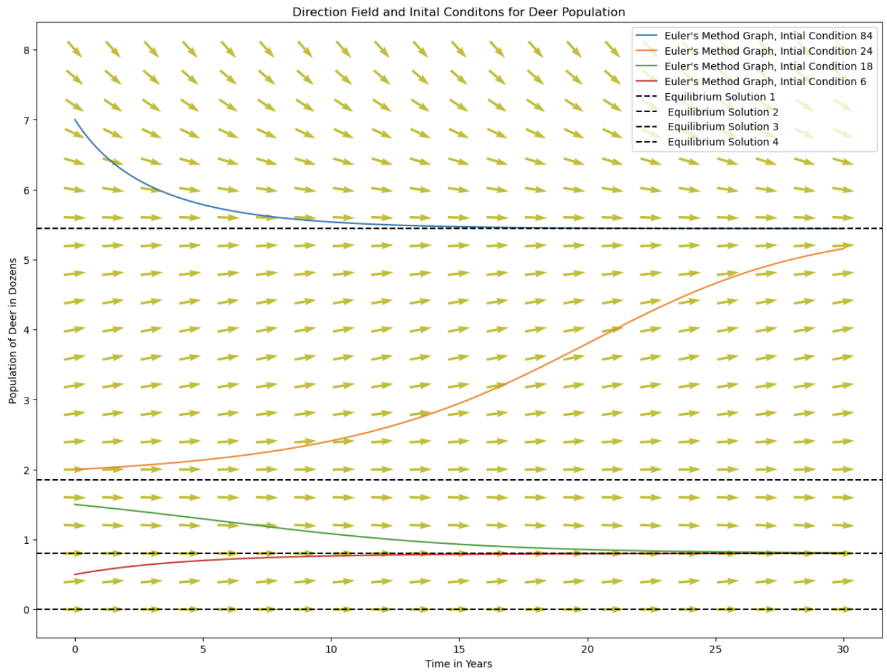

Figure 6
Direction Field of the Modified Logistic Equation *with* Euler's Method Curves

As seen from the asymptotic behavior of the Euler's method plots, the stability of the equilibrium solutions matches our analysis from the previous section.

3 The Lotka-Volterra System

The logistic equations (1) and (2) do not fully consider the impact of both internal competition and external predator-prey interactions on the populations. Although the harvesting term is used to describe the effect of predation on the deer population in equation, there is a need to modify the models to formally investigate how the populations interact with each other. To do this, the models should be adjusted to include the interactions between predator and prey and then solved simultaneously as a system. There are several methods to incorporate the interdependence of the two species. One of the simplest models describing these interactions is the *the Lotka-Volterra System*

$$\frac{dx_1}{dt} = -\alpha x_1 + \beta x_1 x_2 \tag{5}$$

$$\frac{dx_2}{dt} = \gamma x_2 - \delta x_1 x_2$$

The Lotka-Volterra system is based on the Balance Law, which states that the net rate of change of a population is determined by the difference between the rate of incoming members (birth/immigration) and the rate of outgoing members (death/emigration). In the Lotka-Volterra system, the *predator* population is represented by the variable x_1 and the prey population by x_2. Both of these variables are located in the first quadrant of the $x_1 - x_2$ plane, which is known as the population quadrant, as negative populations are physically meaningless.

The positive parameters α,β,δ, and γ can be interpreted as follows:

Parameter	Meaning
α	Predator Mortality Rate
β	Predator Attack Rate
γ	Prey Growth Rate
δ	Prey Mortality Rate

The interactions between the predator and prey populations in the Lotka-Volterra System (3) are modeled through the cross terms $\beta x_1 x_2$ and -$\delta x_1 x_2$. Specifically, the predator population is positively affected by interactions while the prey population is negatively affected by these interactions. In other words, the presence of food (prey) promotes the growth rate of predators, but the presences of predators reduces the growth rate of prey.
Key assumptions of this model are:

1. Only two species exist: mountain lions and deer
2. Deer (prey) are born and then die through only predation or natural causes
3. Mountain lions (predators) are born and their birth rate is positively affected by the rate of predation.
4. Mountain lions only die of natural causes.
5. Each population will obeys exponential growth, where the predators decay to zero in the absence of prey, and the prey grow exponentially in the absence of predators.
6. When both population's exist the number of interactions is proportional to the product of the population sizes.

3.1 Classification, Nullclines and Equilibrium Solutions

The Lotka-Volterra system (3) is a non-linear, autonomous, first-order differential equation. Meaning that the rate of change of the population is independent of time and implicitly related to the population sizes.
Nullclines are isoclines of a system of differential equations where either the x or y components are equal to 0. v-nullclines are defined as isoclines of vertical slope or where $\frac{dx}{dt} = 0$ and h-nullclines are analogously defined as isoclines of horizontal slope where $\frac{dy}{dt} = 0$. In the context of the Lotka-Volterra System (3), v-nullclines occur where $\frac{dx_1}{dt} = 0$ and h-nullclines occur where $\frac{dx_2}{dt} = 0$. Physically nullclines describe where the rate of change of either the predator or prey population do not change.

Derivation of the h and v nullclines as well as the equilibrium solutions are found in the appendix

The v-nullclines of the Lotka-Volterra System occur at $x_1 = 0$ and $x_2 = \frac{\alpha}{\beta}$. The h-Nullclines of the system occur at $x_1 = \frac{\gamma}{\delta}$ and $x_2 = 0$.

Equilibrium solutions or solutions where the system is equal to 0 occur when the h-nullclines equal the v-nullclines. Physically this represents solutions where neither the predator or prey population changes over time. Equilibrium solutions to the Lotka-Volterra System are the points $(0,0)$ and $(\frac{\alpha}{\beta}, \frac{\gamma}{\delta})$

3.2

Oftentimes it is convenient to represent the behavior of a system of differential equations with a direction field. Similar to a vector field a direction field consists of small line segments in the solution plane at various points where the slope of each line segment corresponds to the value of the derivative of the solution function at the point. Physically each line segment is the rate of change of the predator-prey populations. The direction field of the Lotka-Volterra System is below:

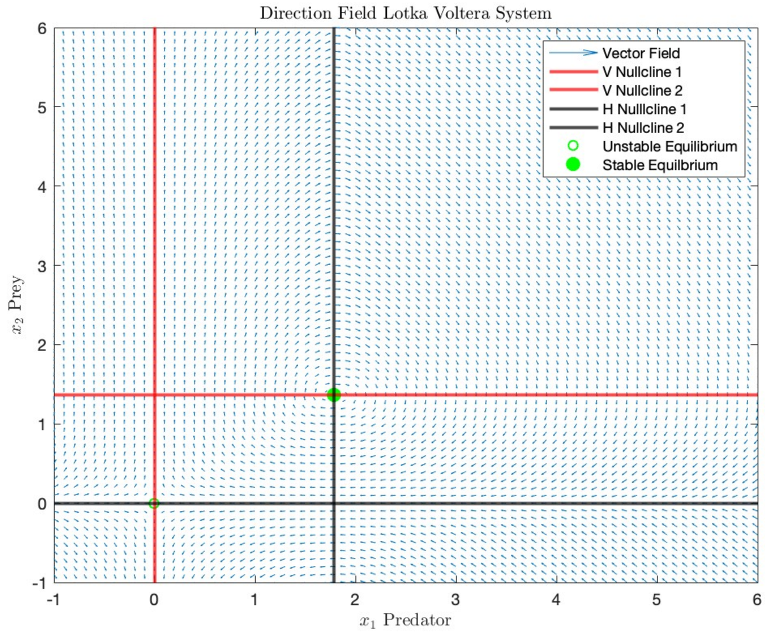

Figure 7
Direction Field of the Lotka-Volterra System

3.3 Direction Field and Component Curves

3.3.1

Using the Matlab function `ODE45` solutions to the Lotka-Volterra system were simulated with the initial conditions $x_1(0) = 0.5$ and $x_2(0) = 1.0$ over the time interval $t \in [0, 30]$ a 30 year span with domain restrictions $-1 < x_1 < 6$, $-1 < x_2 < 6$.

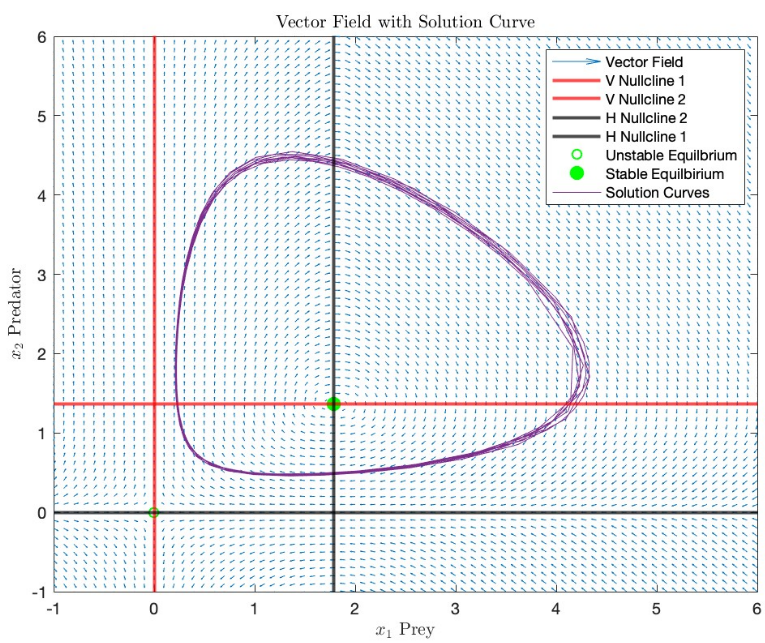

Figure 8

Direction Field of the Lotka-Volterra System including Initial Conditions

The solution curve behaves as expected, every trajectory in the first quadrant is a closed curve and thus the predator and prey populations oscillate. Furthermore, they are never zero which supports the physical meaning of the model. The phase portrait indicates that the system is out of phase.

3.3.2

Similar to the above section, the Matlab function `ODE45` was used to simulate solutions to the Lotka-Volterra system with the initial conditions $x_1(0) = 0.5$ and $x_2(0) = 1.0$ over the time interval $t \in [0, 30]$ a 30 year span with domain restrictions $-1 < x_1 < 6$, $-1 < x_2 < 6$. However, this time the component curves of the Lotka Volterra system (3) $x_1(t)$ and $x_2(t)$ were plotted together against time t

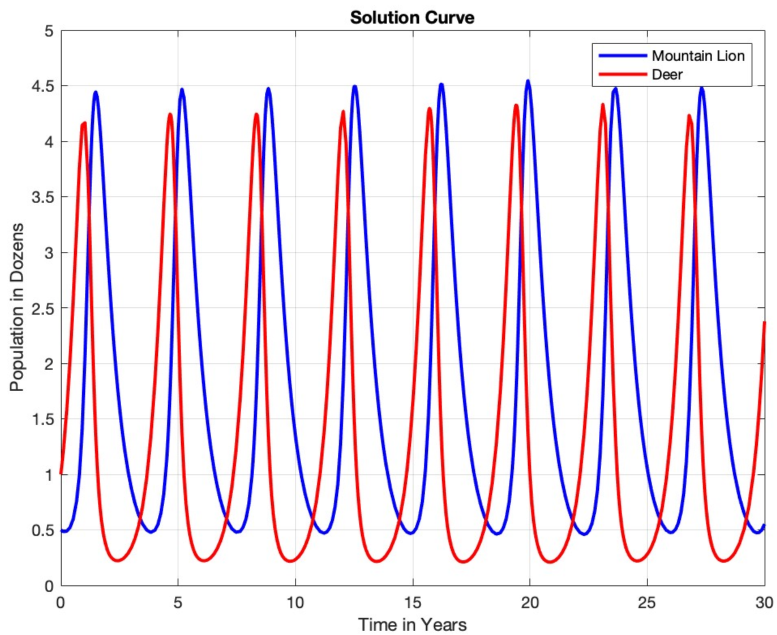

Figure 9
Direction Field of the Lotka-Volterra System including Initial Conditions

The component curves, $x_1(t)$ and $x_2(t)$, represent the population sizes of the predator and prey over time, respectively. If the two curves are in phase, it means that the populations of predator and prey rise and fall together over time. Meaning the graphs of the component curves would be indistinguishable. In other words, when the predator population increases, so does the prey population, which in turn supports the growth of the predator population. When the predator population declines, the prey population also decreases, leading to a decrease in predator population as well. However, this interpretation violates the assumptions made when constructing the Lotka-Volterra System.

On the other hand, if the curves are out of phase, as evidenced by figure 7, it means that the populations of predator and prey are not synchronized in their growth and decline patterns. When the predator population increases, the prey population will decrease due to predation pressure. However, if the predator population grows too much, it will eventually lead to a decline in the prey population, which can in turn lead to a decline in the predator population. This type of population dynamic leads to oscillations in the populations of predator and prey, which were seen when plotting the phase portrait.

4 The Logistic Predator-Prey Equations

An underlying assumption of the Lotka-Volterra System (3) is that both species will exhibit exponential behavior if there are no inter-species interactions, however, this assumption ignore environmental limits imposed on a prey-population such as finite food. A more realistic way to model predator-prey interactions is to adjust the prey component of the Lotka-Volterra model to include these environmental constraints. This gives rise to the *Logistic Predator Prey Model*.

$$\frac{dx_1}{dt} = -\alpha x_1 + \beta x_1 x_2 \tag{6}$$
$$\frac{dx_2}{dt} = \gamma(1-\kappa x_2)x_2 - \delta x_1 x_2$$

An underlying assumption of this model is that the predation the prey population will obey a logistic growth model similar to that of (1) instead of an exponential growth model. The parameters α,β,δ, and γ are the same as in the Lotka-Volterra system. The parameter κ is now included to represent the carrying capacity of the prey population. The parameter values are summarized in the table below:

Parameter	Meaning
α	Predator Mortality Rate
β	Predator Attack Rate
γ	Prey Growth Rate
δ	Prey Mortality Rate
κ	Carrying Capacity of the Prey Population

The assumptions used in the Logistic Predator-Prey Model are enumerated below:

1. Only two species exist: mountain lions and deer
2. The deer (prey) population grows logistically in the absence of predation, meaning that its growth rate is proportional to its population size until it reaches a carrying capacity determined by the environment.
3. Mountain lions (predators) are born and their birth rate is positively affected by the rate of predation.
4. Mountain lions only die of natural causes.
5. The carrying capacity of the environment for the prey population is constant and independent of the predator population.
6. When both population's exist the number of interactions is proportional to the product of the population sizes.
7. External constraints such as disease do not affect the growth or survival of the predator or prey populations.

4.1 Classification, Nullclines and Equilibrium Solutions

4.1.1

The v-nullclines of the Logistic Predator Prey model (4) are $x_1 = 0$ and $x_2 = \frac{\alpha}{\beta}$ and the h-nullclines are $x_2 = 0$ and $x_1 = \frac{\gamma(1-\kappa x_2)}{\delta}$.

4.1.2

The equilibrium solutions of the Logistic Predator Prey model (4) are:

1. $(0,0)$
2. $\left(0, \frac{1}{\kappa}\right)$
3. $\left(\frac{-\gamma(\alpha\kappa-\beta)}{\delta\beta}, \frac{\alpha}{\beta}\right)$

4.2 Direction Field and Component Curves

4.2.1

Using the Matlab function ODE45 and parameter conditions $\alpha = 1.5$ $\beta = 1.1$, $\gamma = 2.5$, $\delta = 1.4$ and $\kappa = 0.5$ over a time interval of 30 years ($t \in [0,30]$) with initial conditions $(x_1(0), x_2(0)) = (5,1)$ and $(x_1(0), x_2(0)) = (1,5)$ and domain restrictions $-1 < x_1 < 6$ and $-1 < x_2 < 6$ the direction field, nullclines and trajectories of the system were graphed.

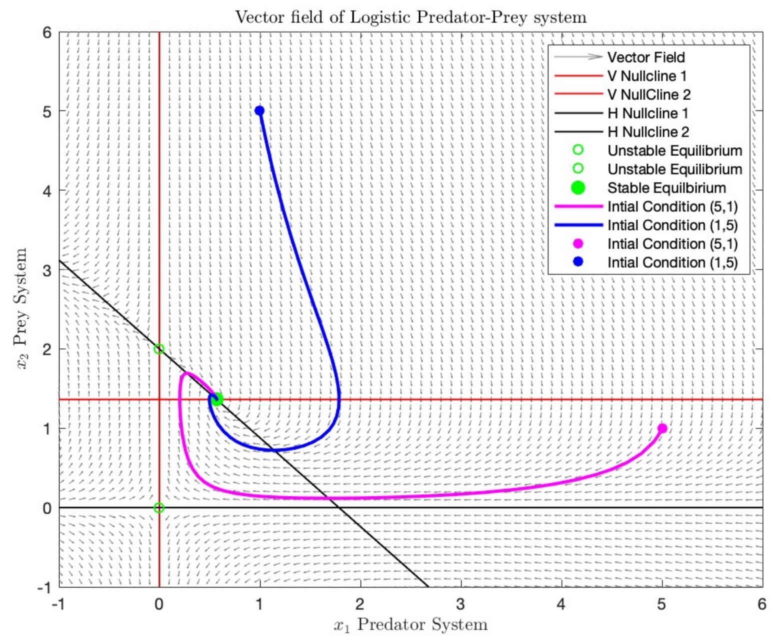

Figure 10
Direction Field of the Logistic Predator-Prey Model including Initial Conditions

As seen from the curve the only stable solution to the Logistic Predator-Prey model is $\left(\frac{-\gamma(\alpha\kappa-\beta)}{\delta\beta}, \frac{\alpha}{\beta}\right)$ meaning that any set of initial conditions on the domain will tend towards that population size of predator and prey. In contrast if an equilibrium solution is unstable, initial conditions near the equilibrium solution will move away from this value over time.

4.2.2

Similar to the above problem the Matlab function ODE45 was used with parameter conditions $\alpha = 1.5$ $\beta = 1.1$, $\gamma = 2.5$, $\delta = 1.4$ and $\kappa = 0.5$ over a time interval of 30 years ($t \in [0,30]$) and initial conditions $(x_1(0), x_2(0)) = (5,1)$ and $(x_1(0), x_2(0)) = (1,5)$ with domain restrictions $-1 < x_1 < 6$ and $-1 < x_2 < 6$ to plot the component curves $x_1(t)$ and $x_2(t)$ together against t. The figure is included below:

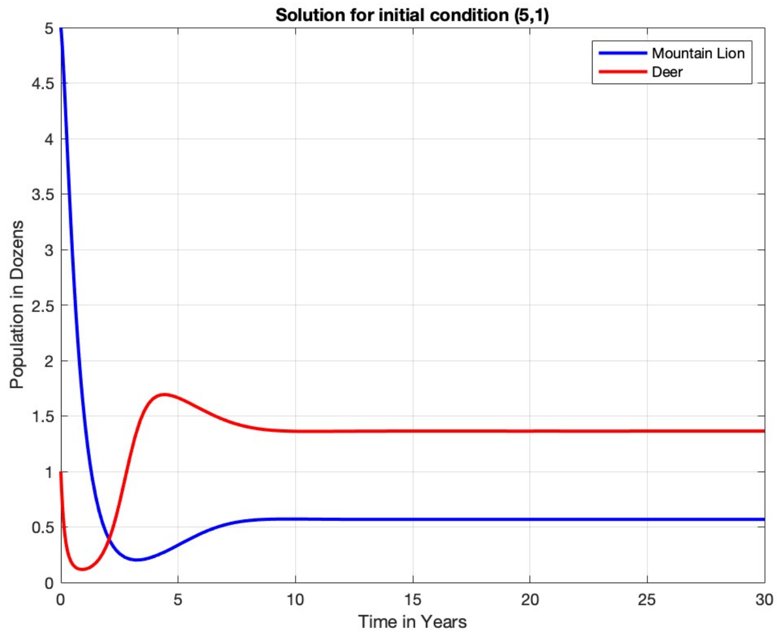

Figure 11
Component Curves with Initial Condition (5,1)

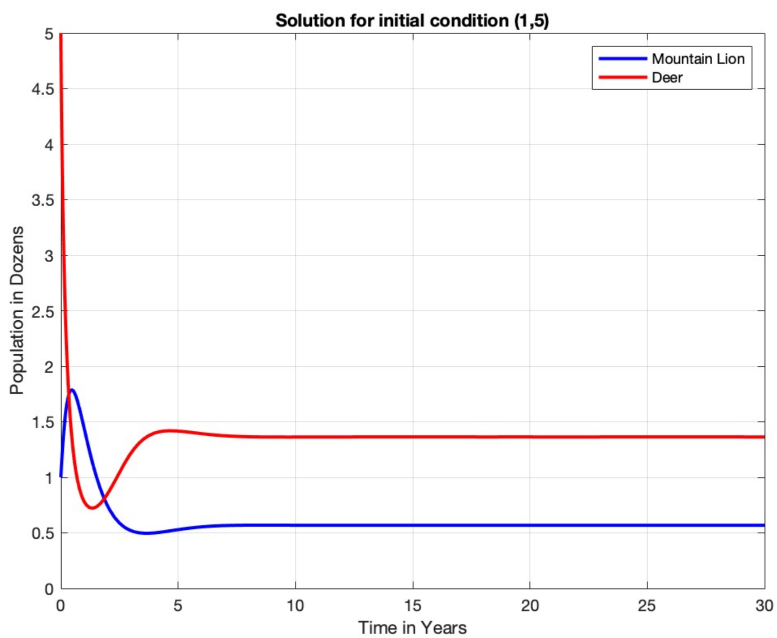

Figure 12
Component Curves with Initial Condition (1,5)

As seen from the graphs the solution curves exhibit asymptotic behavior where the predator and prey populations approach and the equilibrium value of $\left(\frac{-\gamma(\alpha\kappa-\beta)}{\delta\beta}, \frac{\alpha}{\beta}\right)$. This supports the analysis from Figure 8, where the solution curves tended toward the equilibrium solution in the phase plane.

5 Model Comparison

The Lotka-Volterra (LV) and the Logistic Predator-Prey (LPP) models are widely used to describe predator-prey interactions in ecological systems. While both models capture the essential dynamics of predator and prey populations, some critical differences between the models can impact their strengths and weaknesses.

One of the main differences between the LV and LPP models is the assumption about the carrying capacity of the prey population. The LV model assumes that the prey population has an unlimited carrying capacity, whereas the LPP model considers the environment's finite carrying capacity. This can make the LPP model more realistic in situations where resource availability limits the growth of the prey population.

Another difference between the models is the assumption about the functional response of the predator population. The LV model assumes that the predation rate is proportional to the product of the predator and prey populations. In contrast, the LPP model adopts a more complex functional response that considers the population-dependent effects of predation. This can make the LPP model more accurate when the predation rate is impacted by factors such as prey density and the availability of alternative food sources.

Strengths and weaknesses of the LV model:

- Strengths: Simple and easy to understand; captures the essential dynamics of predator and prey populations; useful for studying simple ecological systems.
- Weaknesses: Assumes an unlimited carrying capacity for the prey population, which is unrealistic in most ecological systems; needs to consider population-dependent effects of predation; can lead to unrealistic predictions in complex environmental systems.

Strengths and weaknesses of the LPP model:

- Strengths: Takes into account the finite carrying capacity of the environment; includes a more complex functional response that captures density-dependent effects of predation; can make more accurate predictions in complex ecological systems.
- Weaknesses: More complex and difficult to understand than the LV model; requires more parameter estimates than the LV model; may not accurately capture all aspects of predator-prey interactions in some ecological systems.

A modification to the LV model that might increase its accuracy in predicting predator-prey dynamics in more complex ecological systems would include a finite carrying capacity for the prey population. This would make the model more similar to the LPP model and make it more realistic in situations where resource availability limits the growth of the prey population.

A modification to the LPP model that might increase its accuracy in predicting predator-prey dynamics would be to include more complex predator behavior. For example, the model could be modified to include multiple predator species with different feeding preferences or to include predator learning and adaptation to changing prey populations. This would make the model more complex but more accurate in situations where predator behavior is an important factor in shaping predator-prey dynamics.

References

[1] Jerry Farlow, James Hall, Jean McDill, Beverly West (2007) *Differential Equations and Linear Algebra*, Pearson.

[2] Mountain Lions vs. Deer. (n.d.). www.fs.usda.gov. https://www.fs.usda.gov/psw/publications/Popular/mtnlions.html

6 Appendix

6.1 Derivations

Derivation of the Analytical Solution to the Logistic Equation
Beginning with the Logistic Differential Equation

$$\frac{dx}{dt} = r(1-\frac{x}{L})x$$

Separating variables yields

$$\frac{dx}{(1-\frac{x}{L})x} = rdt$$

This equation can be solved using *partial fraction decomposition*

$$\frac{1}{x(1-\frac{x}{L})} = \frac{1}{x} + \frac{\frac{1}{x}}{(1-\frac{x}{L})}$$

Apply this to the separated form of the logistic equation yields

$$(\frac{1}{x} + \frac{\frac{1}{L}}{(1-\frac{x}{L})})dx = rdt$$

Integrating the above equation yields

$$\ln|x| - \ln|1-\frac{x}{L}| = rt + C$$

The constant of integration c is determined from the initial condition $x(0) = x_0$. Physically we know that if $0 < x_0 < L$, then $0 < x(t) < L$ for all future time which means that $0 < \frac{x}{L} < 1$ which means that both x and $1-\frac{x}{L}$ are positive and we can drop the absolute values in the above equation. Therefore:

$$\ln\frac{x}{(1-\frac{x}{L})} = rt + C$$

If we write $C = e^c$ the implicit general solution becomes

$$\frac{x}{(1-\frac{x}{L})} = Ce^{rt}$$

Using the initial condition $x(0) = x_0$, so that $x = x_0$ when $t = 0$ the above equation becomes

$$\frac{x_0}{(1-\frac{x_0}{L})} = C$$

Substituting this value for C into the implicit solution and solving for x we obtain (2)

$$x(t) = \frac{L}{1+(\frac{L}{x_0}-1)e^{-rt}}$$

Derivation of the Equilibrium Solutions to the Logistic Equation
Starting with (1)

$$\frac{dx}{dt} = r(1-\frac{x}{L})x$$

Equilibrium Solutions are found by setting $\frac{dx}{dt} = 0$ yielding

$$0 = r(1-\frac{x}{L})x$$

Dividing both sides by r yields

$$0 = (1 - \frac{x}{L})x$$

This implies there are exists two equilibrium solutions to the logistic equation, one where $(1 - \frac{x}{L}) = 0$ and $x = 0$. Adding $\frac{x}{L}$ to both sides on the former equation yields

$$\frac{x}{L} = 1$$

Which means that $x = L$.

Therefore the two equilibrium solutions are $x = L \; and \; x = 0$

Derivation of the *v* and *h* nullclines of the Lotka-Volterra System
The v-nullclines of the system are found by setting the first component of (3) equal to zero.

$$0 = -\alpha x_1 + \beta x_1 x_2$$

Factoring out an x_1 from the equation above yeilds

$$0 = x_1(-\alpha + \beta x_2)$$

This implies two v- nullclines to the system one where $x_1 = 0$ and where $x_2 = \frac{\alpha}{\beta}$

The h-nullclines of the system are found by setting the second component of (3) equal to zero.

$$0 = \gamma x_2 - \delta x_1 x_2$$

Factoring out an x_2 from the equation yields

$$0 = x_2(\gamma - \delta x_1)$$

This implies there are two h-nullclines to the system, one where x_2=0 and where $x_1 = \frac{\gamma}{\delta}$

Derivation of the Equilibrium Solutions of the Lotka-Volterra Predator-Prey System
Equilibrium Solutions occur when v-nullclines intersect with h-nullclines. So by plugging the v-nullcline x_1=0 into the second component of the system we obtain:

$$0 = \gamma x_2$$

Which implies an equilibrium solution of $(x_1,x_2)=(0,0)$

Plugging the second v-nullcline $x_2=\frac{\alpha}{\beta}$ into the second component equation yields

$$0 = \gamma\frac{\alpha}{\beta} - \delta x_1 \frac{\alpha}{\beta}$$

Factoring out $\frac{\alpha}{\beta}$ from the above equation yeilds

$$0 = \frac{\alpha}{\beta}(\gamma - \delta x_1)$$

Meaning that the second equilibrium solution is $(x_1,x_2) =(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$

Derivation of the equilibrium solutions as well as *v* and *h* nullclines of the Logistic Predator-Prey System
To find the nullclines and equilibrium solutions of the Logistic Predator-Prey system (4), we first set both equations equal to zero and solve for x_1 and x_2:

$$\frac{dx_1}{dt} = -\alpha x_1 + \beta x_1 x_2 = 0$$

$$x_1(\beta x_2 - \alpha) = 0$$

$$\frac{dx_2}{dt} = \gamma(1 - \kappa x_2)x_2 - \delta x_1 x_2 = 0$$

$$x_2(\gamma(1 - \kappa x_2) - \delta x_1) = 0$$

Now, we need to find the equilibrium solutions by solving the system of equations obtained by setting both derivatives to zero:

$$-\alpha x_1 + \beta x_1 x_2 = 0$$
$$\gamma(1 - \kappa x_2)x_2 - \delta x_1 x_2 = 0$$

From the first equation, we can solve for x_1 and x_2:

$$x_1 = 0$$

$$x_2 = \frac{\alpha}{\beta}$$

From second equation we solve for x_1 and x_2:

$$x_1 = \frac{\gamma}{\delta}(1 - \kappa(\frac{\alpha}{\beta}))$$

$$x_2 = 0$$

$$x_2 = \frac{1}{\kappa}$$

Simplifying and rearranging, we get an equation for x_1:

$$x_1 = \frac{\gamma(\beta - \kappa\alpha)}{\beta\delta}$$

From x_1 and x_2 of both equations we get equilibrium solutions:

1. Equilibrium 1: $(0,0)$
2. Equilibrium 2: $\left(0, \frac{1}{\kappa}\right)$
3. Equilibrium 3: $\left(\frac{-\gamma(\alpha\kappa-\beta)}{\delta\beta}, \frac{\alpha}{\beta}\right)$

Furthermore from these two equation, we have two v-nullclines and two h-nullclines:

1. v-nullcline $x_1 = 0$
2. v-nullcline $x_2 = \frac{\alpha}{\beta}$
3. h-nullcline $x_1 = \frac{\gamma(\beta-\kappa\alpha)}{\beta\delta}$
4. h-nullcline $x_2 = 0$

6.2 Code Files