Assignment 4

Tuesday 28$^{\text{th}}$ March, 2017

1. If every vertex of a connected graph G has degree 2, then G is a cycle. Give a rigorous proof.

2. Define a cut-vertex. Let v be a cut vertex of a simple graph G. Prove that $\overline{G} - v$ is connected.

3. Show that Peterson's Graph is non-planar.

4. Prove that every graph with at least two vertices contains a pair of distinct vertices of the same degree. Determine whether this also hold for multigraphs, by providing a proof or a counterexample.

5. Let $G = (V, E)$ be an undirected graph. Define a relation R on V by aRb if $a = b$ or if there is a path in G from a to b. Prove that R is an equivalence relation. Describe the partition of V induced by R.

6. Let $G = (V, E)$ be a loop-free connected undirected graph. Let $\{a, b\}$ be an edge of G. Prove that $\{a, b\}$ is part of a cycle if and only if its removal does not disconnect G.

7. Define a spanning subgraph of a graph G. Give an example of a graph G with a subgraph H which is not a spanning subgraph, but has the same number of vertices as G.

8. What are the total number of non-isomorphic induced subgraphs of a path with 10 vertices? Explain how you arrived at the answer.

9. A given graph G has the property that any induced subgraph of this graph is connected. What can one say about such a graph? Explain your answer.

10. Show that a cycle graph is never self complementary unless it is a 5-cycle.

11. If G is an undirected graph with n vertices and e edges, let $\delta = min_{v \in V}\{deg(v)\}$ and let $\Delta = max_{v \in V}\{deg(v)\}$. Prove that $\delta \le 2(e/n) \le \Delta$.

12. Let k be a fixed positive integer and let $G = (V, E)$ be a loop-free undirected graph, where $deg(v) \ge k$ for all $v \in V$. Prove that G contains a path of length k.

13. What is the maximum length of a trail in K_{2n}.

14. Carolyn and Richard attended a party with three other married couples. At this party a good deal of handshaking took place, but (1) no one shook hands with her or his spouse; (2) no one shook hands with herself or himself; and (3) no one shook hands with anyone more than once. Before leaving the party, Carolyn asked the other seven people how many hands she or he had shaken. She received a different answer from each of the seven. How many times did Carolyn shake hands at this party? How many times did Richard?

15. Show that when any edge is removed from K_5, the resulting subgraph is planar. Is this true for the graph $K_{3,3}$?

16. Show that any subgraph of a bipartite graph is bipartite.

17. Every graph on four vertices is planar. Prove this statement without enumerating all possible graphs on 4 vertices.

18. Show that every subgraph of a planar graph is planar.

19. Let $G = (V, E)$ be a loop free connected 4-regular planar graph. If $|E| = 16$, how many regions are there in a planar depiction of G?

20. Prove that every loop-free connected planar graph has a vertex v with $deg(v) < 6$.

21. Let $G = (V, E)$ be a loop-free connected planar graph. If G is isomorphic to its dual and $|V| = n$, what is $|E| =?$.

22. Let G_1, G_2 be two loop-free connected undirected graphs. If G_1, G_2 be homeomorphic, prove that (a) G_1, G_2 have the same number of vertices of odd degree; (b) G_1 has an Euler trail if and only if G_2 has an Euler trail; and (c) G_1 has an Euler circuit if and only if G_2 has an Euler circuit.

23. How many Hamilton paths are possible in C_9?

24. How many Hamilton paths are possible in K_9?

25. What are the total number of edge disjoint Hamilton cycles in a complete graph K_n where n is odd?

26. Show that the hypercube Q_n with $n > 1$ has a Hamilton cycle.

27. Let $G = (V, E)$ be a loop-free undirected n-regular graph with $|V| \geq 2n + 2$. Prove that $\overline{G}$ has a hamilton cycle.