OCTOBER / NOVEMBER 2004

1. $T: R^4 \longrightarrow R^4$

$$\begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 0 & -1 & 7 \\ 3 & -1 & -2 & 10 \\ 4 & 10 & 13 & 29 \end{pmatrix}$$

If $$\begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 0 & -1 & 7 \\ 3 & -1 & -2 & 10 \\ 4 & 10 & 13 & 29 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 5 & 2 & 6 & 0 \\ 2 & 0 & -1 & 7 & 0 \\ 3 & -1 & -2 & 10 & 0 \\ 4 & 10 & 13 & 29 & 0 \end{array}\right)$$

$$\xrightarrow[-4r_1 + r_4]{\substack{-2r_1 + r_2 \\ -3r_1 + r_3}} \left(\begin{array}{cccc|c} 1 & 5 & 2 & 6 & 0 \\ 0 & -10 & -5 & -5 & 0 \\ 0 & -16 & -8 & -8 & 0 \\ 0 & -10 & 5 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-5},\ \frac{r_3}{-8},\ \frac{r_4}{5}} \left(\begin{array}{cccc|c} 1 & 5 & 2 & 6 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[-r_2 + r_4]{-r_2 + r_3} \left(\begin{array}{cccc|c} 1 & 5 & 2 & 6 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{r_3 \Longleftrightarrow r_4} \left(\begin{array}{cccc|c} 1 & 5 & 2 & 6 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = s, s \in R$

$z = -s$

$2y + z + w = 0$

$y = 0$

$x + 5y + 2z + 6w = 0$

$x + 5(0) + 2(-s) + 6s = 0$

$x - 2s + 6s = 0$

$x = -4s$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -4s \\ 0 \\ -s \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} -4 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ The dimension of the null space of T is 1.

2. $C: \quad x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \leqslant t \leqslant \frac{\pi}{2}, \quad a > 0$

$\frac{dx}{dt} = -3a\cos^2 t \sin t \qquad \frac{dy}{dt} = 3a\sin^2 t \cos t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= (-3a\cos^2 t \sin t)^2 + (3a\sin^2 t \cos t)^2$$

$$= 9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t$$

$$= 9a^2\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)$$

$$= 9a^2\cos^2 t \sin^2 t$$

The area of the surface generated when C is rotated through one complete revolution about the x-axis is

$$\int_0^{\frac{\pi}{2}} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} 2\pi a \sin^3 t\,(3a\cos t \sin t)\, dt$$

$$= 6\pi a^2 \int_0^{\frac{\pi}{2}} \sin^4 t \cos t \, dt$$

$$u = \sin t$$

$$du = \cos t \, dt$$

$$t = 0 \quad u = 0$$

$$t = \frac{\pi}{2} \quad u = 1$$

$$= 6\pi a^2 \int_0^1 u^4 \, du$$

$$= 6\pi a^2 \left[\frac{u^5}{5} \right]_0^1$$

$$= 6\pi a^2 \left(\frac{1}{5} - 0 \right)$$

$$= \frac{6\pi a^2}{5}$$

3. $\alpha + \beta + \gamma = 0$

$\alpha^2 + \beta^2 + \gamma^2 = 14$

$\alpha^3 + \beta^3 + \gamma^3 = -18$

If α, β, γ are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$,

$$x^3 + \frac{bx^2}{a} + \frac{cx}{a} + \frac{d}{a}$$

$$= (x - \alpha)(x - \beta)(x - \gamma)$$

$$= (x^2 - (\alpha + \beta)x + \alpha\beta)(x - \gamma)$$

$$= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma$$

Equating coefficients,

$$\alpha + \beta + \gamma = \frac{-b}{a}, \quad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}, \quad \alpha\beta\gamma = \frac{-d}{a}$$

Also,
$$a\alpha^3 + b\alpha^2 + c\alpha + d = 0 \text{ — } ①$$
$$a\beta^3 + b\beta^2 + c\beta + d = 0 \text{ — } ②$$
$$a\gamma^3 + b\gamma^2 + c\gamma + d = 0 \text{ — } ③$$

① + ② + ③ :

$$a(\alpha^3 + \beta^3 + \gamma^3) + b(\alpha^2 + \beta^2 + \gamma^2) + c(\alpha + \beta + \gamma) + 3d = 0$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{(\alpha + \beta + \gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2)}{2}$$

$$= \frac{0^2 - 14}{2}$$

$$= -7$$

If $a = 1$

$b = -a(\alpha + \beta + r)$

$= 0$

and $c = a(\alpha\beta + \alpha r + \beta r)$

$= -7$

$\therefore 1(-18) + 0(14) - 7(0) + 3d = 0$

$-18 + 3d = 0$

$d = 6$

$\therefore \alpha\beta r = -6$

$x^3 - (\alpha + \beta + r)x^2 + (\alpha\beta + \alpha r + \beta r)x - \alpha\beta r$

$= x^3 - 7x + 6$

$\therefore x^3 - 7x + 6 = 0$ is a cubic equation whose roots are α, β, r.

$x^3 - 7x + 6 = 0$

$x^3 - x - 6x + 6 = 0$

$x(x^2 - 1) - 6(x - 1) = 0$

$x(x - 1)(x + 1) - 6(x - 1) = 0$

$(x - 1)(x(x + 1) - 6) = 0$

$(x - 1)(x^2 + x - 6) = 0$

$(x - 1)(x - 2)(x + 3) = 0$

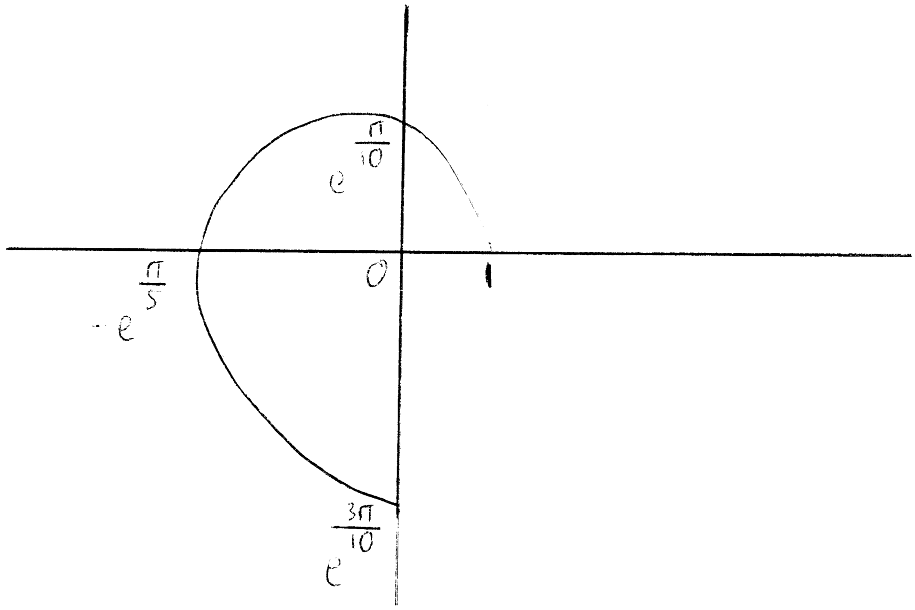
$x = 1, 2, -3$

$\therefore$ The possible values of α, β, r are $1, 2, -3$.

4. $C: \quad r = e^{\frac{\theta}{5}}, \quad 0 \leq \theta \leq \frac{3\pi}{2}$

i)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	1	$e^{\frac{\pi}{30}}$	$e^{\frac{\pi}{20}}$	$e^{\frac{\pi}{10}}$	$e^{\frac{2\pi}{15}}$	$e^{\frac{\pi}{6}}$	$e^{\frac{\pi}{5}}$	$e^{\frac{7\pi}{30}}$	$e^{\frac{\pi}{4}}$	$e^{\frac{3\pi}{10}}$



ii) The length of C from $\theta = 0$ to $\theta = \frac{3\pi}{2}$ is

$$\int_0^{\frac{3\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{3\pi}{2}} \sqrt{e^{\frac{2\theta}{5}} + \left(\frac{e^{\frac{\theta}{5}}}{5}\right)^2}\, d\theta$$

$$= \int_0^{\frac{3\pi}{2}} \sqrt{e^{\frac{2\theta}{5}} + \frac{e^{\frac{2\theta}{5}}}{25}}\, d\theta$$

$$= \int_0^{\frac{3\pi}{2}} \sqrt{\frac{26e^{\frac{2\theta}{5}}}{25}}\, d\theta$$

$$= \int_0^{\frac{3\pi}{2}} \frac{\sqrt{26}\, e^{\frac{\theta}{5}}}{5}\, d\theta$$

$$= \left[\sqrt{26} e^{\frac{\theta}{5}} \right]_0^{\frac{3\pi}{2}}$$

$$= \sqrt{26}\left(e^{\frac{3\pi}{10}} - 1\right)$$

$$\approx 7.99$$

5. $$S_N = \sum_{n=1}^{N} (-1)^{n-1} n^3$$

$$S_{2N} = \sum_{n=1}^{2N} (-1)^{n-1} n^3$$

$$= 1^3 - 2^3 + 3^3 - 4^3 + \dots + (2N-1)^3 - (2N)^3$$

$$= \sum_{n=1}^{N} (2n-1)^3 - (2n)^3$$

$$= \sum_{n=1}^{N} 8n^3 - 12n^2 + 6n - 1 - 8n^3$$

$$= \sum_{n=1}^{N} -12n^2 + 6n - 1$$

$$= -12\sum_{n=1}^{N} n^2 + 6\sum_{n=1}^{N} n - \sum_{n=1}^{N} 1$$

$$= \frac{-12N(N+1)(2N+1)}{6} + \frac{6N(N+1)}{2} - N$$

$$= -2N(N+1)(2N+1) + 3N(N+1) - N$$

$$= -2N(2N^2 + 3N + 1) + 3N^2 + 3N - N$$

$$= -4N^3 - 6N^2 - 2N + 3N^2 + 2N$$

$$= -4N^3 - 3N^2$$

$$= -N^2(4N+3)$$

$$S_{2N+1} = \sum_{n=1}^{2N+1} (-1)^{n-1} n^3$$

$$= 1^3 - 2^3 + 3^3 - 4^3 + \cdots + (2N-1)^3 - (2N)^3 + (2N+1)^3$$

$$= S_{2N} + (2N+1)^3$$

$$= -N^2(4N+3) + (2N+1)^3$$

$$= -4N^3 - 3N^2 + 8N^3 + 12N^2 + 6N + 1$$

$$= 4N^3 + 9N^2 + 6N + 1$$

$$\frac{S_{2N+1}}{N^3} = \frac{4N^3 + 9N^2 + 6N + 1}{N^3}$$

$$= 4 + \frac{9}{N} + \frac{6}{N^2} + \frac{1}{N^3}$$

As $N \longrightarrow \infty$, since $\frac{9}{N} + \frac{6}{N^2} + \frac{1}{N^3} \longrightarrow 0$,

$$\frac{S_{2N+1}}{N^3} \longrightarrow 4.$$

6. $1^{\frac{1}{8}} = (\cos 0 + i\sin 0)^{\frac{1}{8}}$

$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{8}}, \quad k \in \mathbb{Z}$

$= \cos \frac{k\pi}{4} + i\sin \frac{k\pi}{4}, \quad k = 0, 1, 2, 3, 4, 5, 6, 7$

$= e^{\frac{k\pi i}{4}}, \quad k = 0, 1, 2, 3, 4, 5, 6, 7$

$= 1, \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, i, \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -1,$

$\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, -i, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$\therefore$ The 8^{th} roots of unity are

$1, \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, i, \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -1, \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, -i, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}.$

$(z - e^{i\theta})(z - e^{-i\theta}) = z^2 - e^{i\theta}z - e^{-i\theta}z + e^{i\theta}e^{-i\theta}$

$= z^2 - (e^{i\theta} + e^{-i\theta})z + 1$

$= z^2 - (2\cos\theta)z + 1,$

since $e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

Since $e^{\frac{7\pi i}{4}} = e^{\frac{-\pi i}{4}}$, $e^{\frac{3\pi i}{2}} = e^{\frac{-\pi i}{2}}$ and $e^{\frac{5\pi i}{4}} = e^{\frac{-3\pi i}{4}}$

$$(z - e^{\frac{0\pi i}{4}})(z - e^{\frac{\pi i}{4}})(z - e^{\frac{2\pi i}{4}})(z - e^{\frac{3\pi i}{4}})(z - e^{\frac{4\pi i}{4}})(z - e^{\frac{5\pi i}{4}})(z - e^{\frac{6\pi i}{4}})(z - e^{\frac{7\pi i}{4}})$$

$$= (z - 1)(z - e^{\frac{\pi i}{4}})(z - e^{\frac{\pi i}{2}})(z - e^{\frac{3\pi i}{4}})(z - (-1))(z - e^{\frac{-3\pi i}{4}})(z - e^{\frac{-\pi i}{2}})(z - e^{\frac{-\pi i}{4}})$$

$$= (z - 1)(z + 1)(z - e^{\frac{\pi i}{4}})(z - e^{\frac{-\pi i}{4}})(z - e^{\frac{\pi i}{2}})(z - e^{\frac{-\pi i}{2}})(z - e^{\frac{3\pi i}{4}})(z - e^{\frac{-3\pi i}{4}})$$

$$= (z - 1)(z + 1)(z^2 - (2\cos\tfrac{\pi}{4})z + 1)(z^2 - (2\cos\tfrac{\pi}{2})z + 1)(z^2 - (2\cos\tfrac{3\pi}{4})z + 1)$$

$$= (z - 1)(z + 1)(z^2 - \sqrt{2}z + 1)(z^2 + 1)(z^2 + \sqrt{2}z + 1)$$

Since $e^{\frac{k\pi i}{4}}$, $k = 0, 1, 2, 3, 4, 5, 6, 7$ are the 8^{th} roots of unity, $z - e^{\frac{k\pi i}{4}}$, $k = 0, 1, 2, 3, 4, 5, 6, 7$ are the factors of $z^8 - 1$.

$$\therefore z^8 - 1 = (z - 1)(z + 1)(z^2 - \sqrt{2}z + 1)(z^2 + 1)(z^2 + \sqrt{2}z + 1)$$

7. $C: \quad xy + (x+y)^5 = 1$

i) $\frac{d}{dx}\left(xy + (x+y)^5\right) = \frac{d}{dx}(1)$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(x+y)^5 = 0$$

$$x\frac{dy}{dx} + y + 5(x+y)^4\frac{d}{dx}(x+y) = 0$$

$$x\frac{dy}{dx} + y + 5(x+y)^4\left(1 + \frac{dy}{dx}\right) = 0$$

At $A(1,0)$:

$$\frac{dy}{dx} + 5\left(1 + \frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} + 5 + \frac{dy}{dx} = 0$$

$$6\frac{dy}{dx} = -5$$

$$\frac{dy}{dx} = -\frac{5}{6}$$

ii) $\frac{d}{dx}\left(x\frac{dy}{dx} + y + 5(x+y)^4\left(1 + \frac{dy}{dx}\right)\right) = 0$

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{dy}{dx} + \frac{d}{dx}\left(5(x+y)^4\left(1 + \frac{dy}{dx}\right)\right) = 0$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + 5(x+y)^4\frac{d}{dx}\left(1 + \frac{dy}{dx}\right)$$

$$+ \left(1 + \frac{dy}{dx}\right)\frac{d}{dx}\left(5(x+y)^4\right) = 0$$

$$x\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 5(x+y)^4\frac{d^2y}{dx^2} + \left(1 + \frac{dy}{dx}\right)\left(20(x+y)^3\right)\left(1 + \frac{dy}{dx}\right) = 0$$

$$x\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 5(x+y)^4\frac{d^2y}{dx^2} + 20(x+y)^3\left(1 + \frac{dy}{dx}\right)^2 = 0$$

At $A(1,0)$, $\frac{dy}{dx} = -\frac{5}{6}$:

$$\frac{d^2y}{dx^2} + 2\left(\frac{-5}{6}\right) + \frac{5d^2y}{dx^2} + 20\left(1 - \frac{5}{6}\right)^2 = 0$$

$$\frac{d^2y}{dx^2} - \frac{5}{3} + \frac{5d^2y}{dx^2} + \frac{5}{9} = 0$$

$$\frac{6d^2y}{dx^2} = \frac{10}{9}$$

$$\frac{d^2y}{dx^2} = \frac{5}{27}$$

8. $a_1, a_2, a_3, \ldots, \quad a_1 = 1$

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^{\lambda}, \quad \lambda > 1$$

$$a_n \geq 2^{g(n)}, \quad g(n) = \lambda^{n-1}, \quad n \geq 2$$

When $n = 1$. $a_2 = \left(a_1 + \frac{1}{a_1}\right)^{\lambda}$

$$= 2^{\lambda}$$

$$\geq 2^{\lambda^1}$$

$$= 2^{\lambda^{2-1}}$$

$$= 2^{g(2)}$$

Assume the statement is true when $n = k$.

$n = k$. $a_k \geq 2^{g(k)}, \quad g(k) = \lambda^{k-1}, \quad k \geq 2.$

when $n = k+1$

$$a_{k+1} = \left(a_k + \frac{1}{a_k}\right)^{\lambda}$$

Since $a_k \geq 2^{g(k)} > 0$,

$$a_k + \frac{1}{a_k} > 2^{g(k)}$$

$$a_k + \frac{1}{a_k} > 2^{\lambda^{k-1}}$$

$$\left(a_K + \frac{1}{a_K}\right)^{\lambda} > \left(2^{\lambda^{K-1}}\right)^{\lambda}$$

$$a_{K+1} > 2^{\lambda^{K-1}\lambda}$$

$$a_{K+1} > 2^{\lambda^{K}}$$

$$a_{K+1} > 2^{g(K+1)}, \quad g(K+1) = \lambda^{K}$$

$\therefore$ $a_n \geq 2^{g(n)}$, $g(n) = \lambda^{n-1}$ for every positive integer $n \geq 2$.

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^{\lambda}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{a_n}\left(a_n + \frac{1}{a_n}\right)^{\lambda}$$

$$= \frac{1}{a_n}\left[a_n\left(1 + \frac{1}{a_n^2}\right)\right]^{\lambda}$$

$$= \frac{1}{a_n} a_n^{\lambda}\left(1 + \frac{1}{a_n^2}\right)^{\lambda}$$

$$= a_n^{\lambda - 1}\left(1 + \frac{1}{a_n^2}\right)^{\lambda}$$

Since $1 + \frac{1}{a_n^2} > 1$ and $\lambda > 1$,

$$\left(1 + \frac{1}{a_n^2}\right)^\lambda > 1$$

$$a_n^{\lambda-1}\left(1 + \frac{1}{a_n^2}\right)^\lambda \geqslant a_n^{\lambda-1}$$

$$\geqslant 2^{(\lambda-1)g(n)}, \text{ since } a_n \geqslant 2^{g(n)}$$

$$\therefore \frac{a_{n+1}}{a_n} \geqslant 2^{(\lambda-1)g(n)}, \quad n \geqslant 2.$$

9. $I_n = \int_0^1 (1 + x^3)^{-n}\, dx, \quad n > 0$

i) $\frac{d}{dx}\left[x(1 + x^3)^{-n}\right]$

$= (1 + x^3)^{-n} - nx(1 + x^3)^{-n-1} 3x^2$

$= (1 + x^3)^{-n} - 3nx^3(1 + x^3)^{-n-1}$

$= (1 + x^3)^{-n} - 3n(1 + x^3 - 1)(1 + x^3)^{-n-1}$

$= (1 + x^3)^{-n} - 3n(1 + x^3)(1 + x^3)^{-n-1}$
$+ 3n(1 + x^3)^{-n-1}$

$= (1 + x^3)^{-n} - 3n(1 + x^3)^{-n} + 3n(1 + x^3)^{-n-1}$

$= (1 - 3n)(1 + x^3)^{-n} + 3n(1 + x^3)^{-n-1}$

$= -(3n - 1)(1 + x^3)^{-n} + 3n(1 + x^3)^{-n-1}$

$x(1 + x^3)^{-n}$

$= \int -(3n - 1)(1 + x^3)^{-n} + 3n(1 + x^3)^{-n-1}\, dx$

$= \int -(3n - 1)(1 + x^3)^{-n}\, dx + \int 3n(1 + x^3)^{-n-1}\, dx$

$\left[x(1 + x^3)^{-n}\right]_0^1$

$= \int_0^1 -(3n - 1)(1 + x^3)^{-n}\, dx + \int_0^1 3n(1 + x^3)^{-n-1}\, dx$

$$2^{-n} - 0 = (1 - 3n)\int_0^1 (1 + x^3)^{-n}\,dx + 3n\int_0^1 (1 + x^3)^{-(n+1)}\,dx$$

$$2^{-n} = (1 - 3n) I_n + 3n I_{n+1}$$

$$3n I_{n+1} = 2^{-n} - (1 - 3n) I_n$$

$$= 2^{-n} + (3n - 1) I_n$$

$$I_{n+1} = \frac{2^{-n}}{3n} + \left(\frac{3n - 1}{3n}\right) I_n$$

$$= \frac{2^{-n}}{3n} + \left(1 - \frac{1}{3n}\right) I_n$$

ii) $y = \dfrac{1}{1 + x^3}$

As $x \to \pm\infty$, $y \to 0$

As $x \to -1$, $y \to \pm\infty$

$y = 0$ $x = -1$

when $x = 0$ $y = 1$

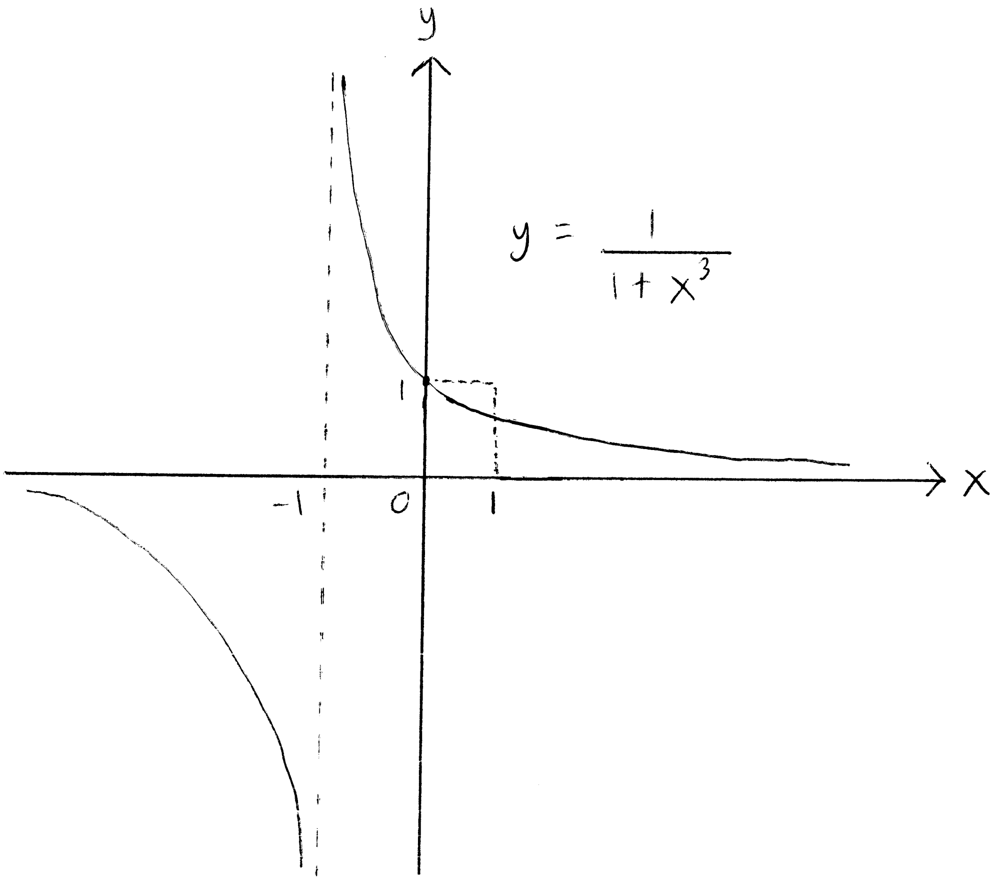
$$\frac{dy}{dx} = \frac{-3x^2}{(1 + x^3)^2}$$

when $\dfrac{dy}{dx} = 0$

$$\frac{-3x^2}{(1 + x^3)^2} = 0$$

$x = 0$

$y = 1$



$$I_1 = \int_0^1 \frac{1}{1+x^3}\,dx$$

Since the area of the region bounded by the curve $y = \frac{1}{1+x^3}$ and x and y axes from $x=0$ to $x=1$ is I_1,

$$I_1 < 1$$

iii) $I_{n+1} = \frac{2^{-n}}{3n} + \left(1 - \frac{1}{3n}\right) I_n$

$n=2$: $I_3 = \frac{2^{-2}}{3(2)} + \left(1 - \frac{1}{3(2)}\right) I_2$

$= \frac{1}{24} + \frac{5}{6} I_2$

$$I_2 = \frac{2^{-1}}{3(1)} + \left(1 - \frac{1}{3(1)}\right) I_1$$

$$= \frac{1}{6} + \frac{2}{3} I_1$$

Since $I_1 < 1$

$$\frac{2}{3} I_1 < \frac{2}{3}$$

$$\frac{1}{6} + \frac{2}{3} I_1 < \frac{5}{6}$$

$$I_2 < \frac{5}{6}$$

$$\frac{5}{6} I_2 < \frac{25}{36}$$

$$\frac{1}{24} + \frac{5}{6} I_2 < \frac{53}{72}$$

$$\therefore I_3 < \frac{53}{72}.$$

10. $C: \quad y = \dfrac{x^2 + 2x - 3}{(\lambda x + 1)(x + 4)}$

i) $\lambda = 0$:

$$y = \frac{x^2 + 2x - 3}{x + 4}$$

$$\begin{array}{r|l} & x - 2 \\ \hline x + 4 & x^2 + 2x - 3 \\ & x^2 + 4x \\ \hline & -2x - 3 \\ & -2x - 8 \\ \hline & 5 \end{array}$$

$$= x - 2 + \frac{5}{x + 4}$$

As $x \longrightarrow \pm\infty$, $y \longrightarrow x - 2$

As $x \longrightarrow -4$, $y \longrightarrow \pm\infty$

The asymptotes of C when $\lambda = 0$ are $y = x - 2$ and $x = -4$.

ii) $\lambda \neq -1, 0, \frac{1}{4}, \frac{1}{3}$

$$y = \frac{x^2 + 2x - 3}{(\lambda x + 1)(x + 4)}$$

$$\begin{array}{r|l} & \frac{1}{\lambda} \\ \hline \lambda x^2 + (4\lambda + 1)x + 4 & x^2 + 2x - 3 \\ & x^2 + (4 + \frac{1}{\lambda})x + \frac{4}{\lambda} \\ \hline & (-2 - \frac{1}{\lambda})x - 3 - \frac{4}{\lambda} \end{array}$$

$$= \frac{1}{\lambda} + \frac{(-2 - \frac{1}{\lambda})x - 3 - \frac{4}{\lambda}}{(\lambda x + 1)(x + 4)}$$

As $x \longrightarrow \frac{-1}{\lambda}$, $y \longrightarrow \pm\infty$

As $x \longrightarrow -4$, $y \longrightarrow \pm\infty$

As $x \longrightarrow \pm\infty$, $y \longrightarrow \frac{1}{\lambda}$

$\therefore$ The asymptotes of C when $\lambda \neq -1, 0, \frac{1}{4}, \frac{1}{3}$

are $x = \frac{-1}{\lambda}$, $x = -4$ and $y = \frac{1}{\lambda}$.

iii) $\lambda = -1$:

$$y = \frac{x^2 + 2x - 3}{(-x+1)(x+4)}$$

$$= \frac{(x+3)(x-1)}{(-x+1)(x+4)}$$

$$= -\frac{x+3}{x+4} \quad \text{when } x \neq 1.$$

$$= -1 + \frac{1}{x+4}$$

$$\begin{array}{r|l} & -1 \\ \hline -x-3 & x+4 \\ & x+3 \\ \hline & 1 \end{array}$$

As $x \longrightarrow \pm\infty$ $y \longrightarrow -1$

As $x \longrightarrow -4$ $y \longrightarrow \pm\infty$

$\therefore$ The asymptotes of C when $\lambda = -1$

are $y = -1$ and $x = -4$.

when $x = 0$: $y = -\frac{3}{4}$

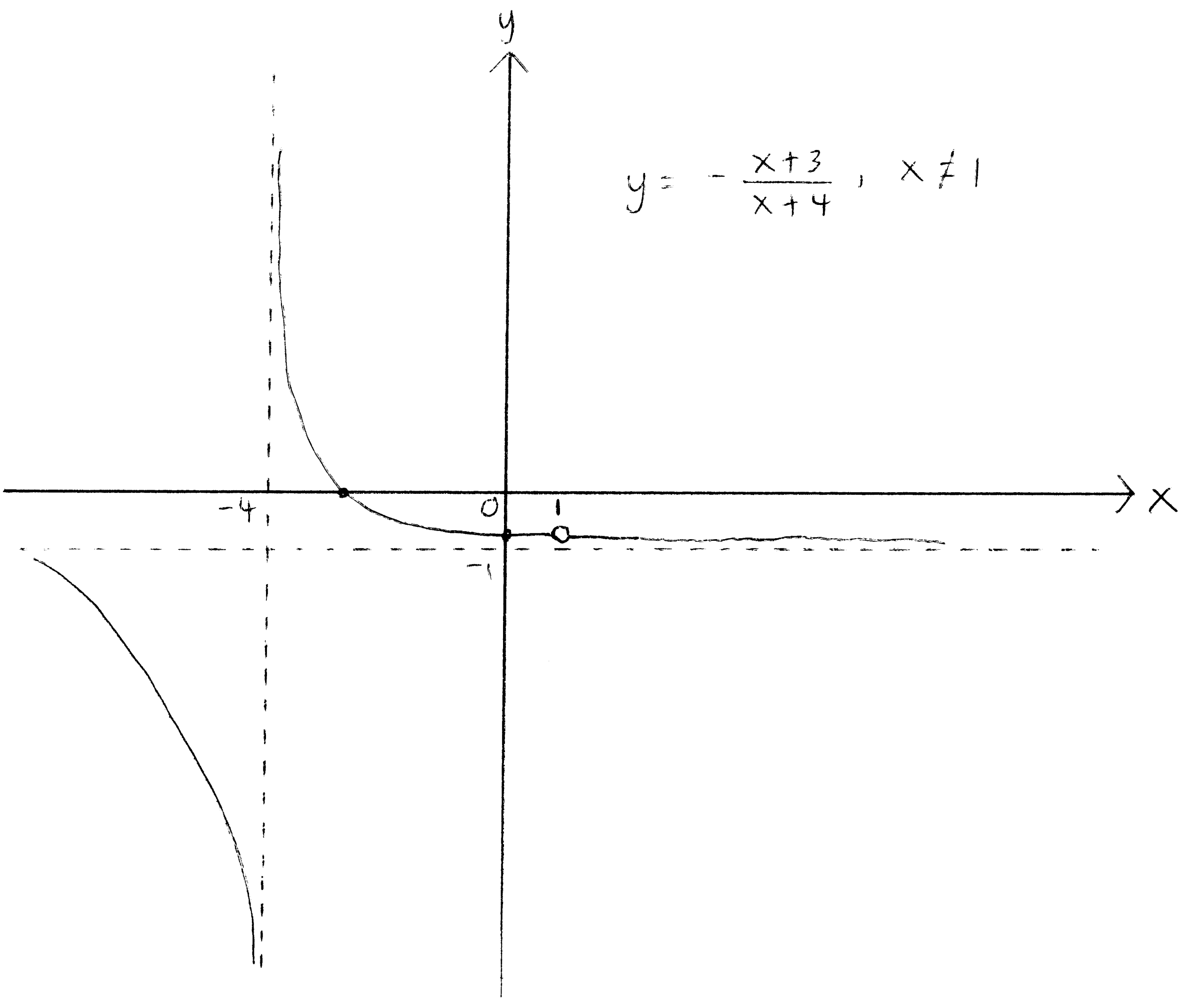
when $y = 0$: $-\frac{x+3}{x+4} = 0$

$$x + 3 = 0$$

$$x = -3$$

$$\frac{dy}{dx} = \frac{-1}{(x+4)^2} < 0$$

$\therefore$ no critical points.

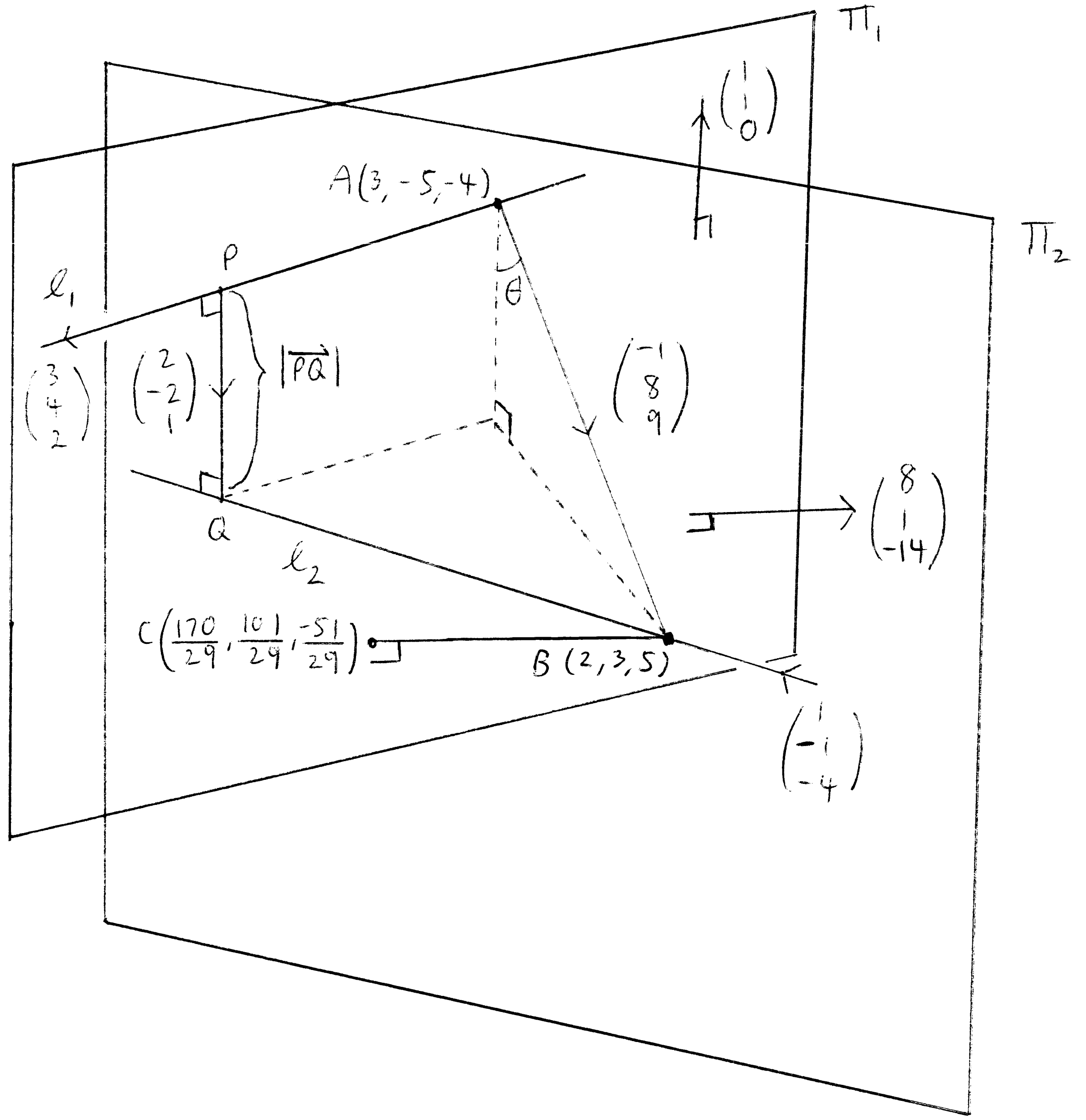


• Intersection point

11. $l_1: \underset{\sim}{r} = 3\underset{\sim}{i} - 5\underset{\sim}{j} - 4\underset{\sim}{k} + s(3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k})$

$l_2: \underset{\sim}{r} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k} + t(\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k})$.

$A(3, -5, -4)$ $B(2, 3, 5)$



Since PQ is perpendicular to both l_1 and l_2, the direction of $\overrightarrow{PQ}$ is parallel to $\begin{pmatrix}3\\4\\2\end{pmatrix} \times \begin{pmatrix}1\\-1\\-4\end{pmatrix}$

$$\begin{pmatrix}3\\4\\2\end{pmatrix} \times \begin{pmatrix}1\\-1\\-4\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 4 & 2 \\ 1 & -1 & -4 \end{vmatrix} = -14\underset{\sim}{i} + 14\underset{\sim}{j} - 7\underset{\sim}{k}$$

$$= -7(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$$

$$\overrightarrow{AB} \cdot \overrightarrow{PQ} = |\overrightarrow{AB}||\overrightarrow{PQ}| \cos\theta$$

$$\left| \left[\begin{pmatrix}2\\3\\5\end{pmatrix} - \begin{pmatrix}3\\-5\\-4\end{pmatrix} \right] \cdot \begin{pmatrix}2\\-2\\1\end{pmatrix} \right| = |\overrightarrow{AB}| \left| \begin{pmatrix}2\\-2\\1\end{pmatrix} \right| \cos\theta$$

$$\left| \begin{pmatrix}-1\\8\\9\end{pmatrix} \cdot \begin{pmatrix}2\\-2\\1\end{pmatrix} \right| = |\overrightarrow{AB}| \sqrt{9} \cos\theta$$

$$|-2 - 16 + 9| = 3|\overrightarrow{AB}| \cos\theta$$

$$|\overrightarrow{AB}| \cos\theta = \frac{|-9|}{3}$$

$$= 3$$

since $|\overrightarrow{PQ}| = |\overrightarrow{AB}| \cos\theta$,

$$|\overrightarrow{PQ}| = 3$$

ii) Since Π_1 contains PQ and ℓ_1, a vector perpendicular to Π_1 is parallel to $\begin{pmatrix}3\\4\\2\end{pmatrix} \times \begin{pmatrix}2\\-2\\1\end{pmatrix}$.

$$\begin{pmatrix}3\\4\\2\end{pmatrix} \times \begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 4 & 2 \\ 2 & -2 & 1 \end{vmatrix} = 8\underset{\sim}{i} + \underset{\sim}{j} - 14\underset{\sim}{k}$$

$\therefore$ A vector perpendicular to Π_1 is $8\underset{\sim}{i} + \underset{\sim}{j} - 14\underset{\sim}{k}$

iii) The line through B perpendicular to Π_1 is $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix}$ since $\begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix}$ is a normal to Π_1.

Since $\begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix}$ is normal to Π_1 and $A(3,-5,-4)$ is a point on Π_1, if $\underset{\sim}{r} = (x, y, z)$ is a point on Π_1,

$$\underset{\sim}{r} \cdot \begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix} = 24 - 5 + 56$$

$$8x + y - 14z = 75$$

$\therefore$ The equation of Π_1 is $8x + y - 14z = 75$.

If the line through B perpendicular to Π_1 meets Π_1 at C,

$$8(2 + 8\lambda) + 3 + \lambda - 14(5 - 14\lambda) = 75$$

$$16 + 64\lambda + 3 + \lambda - 70 + 196\lambda = 75$$

$$261\lambda = 126$$

$$\lambda = \frac{126}{261}$$

$$\therefore C\left(\frac{170}{29}, \frac{101}{29}, \frac{-51}{29}\right)$$

$$BC = \sqrt{\left(\frac{170}{29} - 2\right)^2 + \left(\frac{101}{29} - 3\right)^2 + \left(\frac{-51}{29} - 5\right)^2}$$

$$= \sqrt{\frac{12544}{841} + \frac{196}{841} + \frac{38416}{841}}$$

$$= \sqrt{\frac{51156}{841}}$$

$$= \frac{42}{\sqrt{29}}$$

$\therefore$ The perpendicular distance from B to Π_1 is $\frac{42}{\sqrt{29}}$

iv) Since Π_2 contains PQ and l_2, a vector perpendicular to Π_2 is parallel to $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$.

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -2 & 1 \\ 1 & -1 & 4 \end{vmatrix} = -7\underset{\sim}{i} - 7\underset{\sim}{j}$$

$$= -7(\underset{\sim}{i} + \underset{\sim}{j})$$

Since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is normal to Π_2 and $B(2,3,5)$ is a point on Π_2, if $\underset{\sim}{r} = (x, y, z)$ is a point on Π_2, $\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2 + 3 + 0$$

$$x + y = 5$$

$\therefore$ The equation of Π_2 is $x + y = 5$.

Since $\begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix}$ is normal to Π_1 and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is normal to Π_2, if θ is the angle between $\begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \left|\begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right| \cos\theta$$

$$8 + 1 - 0 = \sqrt{261}\sqrt{2}\cos\theta$$

$$\cos\theta = \frac{9}{3\sqrt{29}\sqrt{2}}$$

$$= \frac{3}{\sqrt{58}}$$

$$\theta \approx 66.8^\circ$$

$\therefore$ The angle between Π_1 and Π_2 is $180^\circ - 66.8^\circ = 113.2^\circ$.

12. EITHER

$$y = f(x), \quad x = e^t$$

$$\frac{dx}{dt} = e^t$$

$$\frac{dx}{dy}\frac{dy}{dt} = x$$

$$x\frac{dy}{dx} = \frac{dy}{dt}$$

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(x\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dt^2} = x\frac{d}{dt}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{dx}{dt}$$

$$= x\frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dt}$$

$$= x^2\frac{d^2y}{dx^2} + \frac{dy}{dt}$$

$$x^2\frac{d^2y}{dt^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

i) $$4x^2\frac{d^2y}{dx^2} + 16x\frac{dy}{dx} + 25y = 50\ln x - 1$$

Since $x = e^t$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dt^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

$$4\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 16\frac{dy}{dt} + 25y = 50t - 1$$

$$\frac{4d^2y}{dt^2} - \frac{4dy}{dt} + \frac{16dy}{dt} + 25y = 50t - 1$$

$$\frac{4d^2y}{dt^2} + \frac{12dy}{dt} + 25y = 50t - 1$$

ii) $$\frac{4d^2y}{dt^2} + \frac{12dy}{dt} + 25y = 0$$

$$4m^2 + 12m + 25 = 0$$

$$4m^2 + 12m + 9 + 16 = 0$$

$$(2m + 3)^2 = -16$$

$$2m + 3 = \pm 4i$$

$$m = \frac{-3}{2} \pm 2i$$

$\therefore$ The complementary function, y_c, is

$$y_c = e^{\frac{-3t}{2}}(A\cos 2t + B\sin 2t)$$

If $A\cos 2t + B\sin 2t = R\sin(2t + \phi)$,

$$A\cos 2t + B\sin 2t = R\sin 2t\cos\phi + R\cos 2t\sin\phi$$

$$R\sin\phi = A, \quad R\cos\phi = B$$

$$R^2\sin^2\phi + R^2\cos^2\phi = A^2 + B^2$$

$$R^2(\sin^2\phi + \cos^2\phi) = A^2 + B^2$$

$$R^2 = A^2 + B^2$$

$$R = \sqrt{A^2 + B^2}$$

$$\frac{R\sin\phi}{R\cos\phi} = \frac{A}{B}$$

$$\tan\phi = \frac{A}{B}$$

$$\phi = \tan^{-1}\frac{A}{B}$$

$$\therefore\ y_c = Re^{\frac{-3t}{2}}\sin(2t + \phi),\ R = \sqrt{A^2 + B^2},\ \phi = \tan^{-1}\frac{A}{B}$$

iii) The particular integral, y_p, is given by

$$y_p = Ct + D$$

$$\frac{dy_p}{dt} = C$$

$$\frac{d^2y_p}{dt^2} = 0$$

$$4\frac{d^2y_p}{dt^2} + 12\frac{dy_p}{dt} + 25y_p = 0 + 12C + 25(Ct + D)$$

$$= 25Ct + 12C + 25D$$

$$= 50t - 1$$

$$25C = 50,\quad 12C + 25D = -1$$

$$C = 2 \qquad 25D = -25$$

$$D = -1$$

$$\therefore\ y_p = 2t - 1$$

iv) $y = y_c + y_p$

$= Re^{-\frac{3t}{2}} \sin(2t + \phi) + 2t - 1$, $R = \sqrt{A^2 + B^2}$, $\phi = \tan^{-1}\frac{A}{B}$

Since $t = \ln x$,

$y = Re^{-\frac{3t}{2}} \sin(2\ln x + \phi) + 2\ln x - 1$

The general solution of the differential equation is $y = Re^{-\frac{3t}{2}} \sin(2\ln x + \phi) + 2\ln x - 1$,

$R = \sqrt{A^2 + B^2}$, $\phi = \tan^{-1}\frac{A}{B}$

OR

$A\underset{\sim}{e} = \lambda \underset{\sim}{e}$

If A is non-singular,

i) If $\lambda = 0$, since $|A - \lambda I| = 0$,

$$|A| = 0.$$

Also, since A^{-1} exists, $|A| \neq 0$

$$\therefore \lambda \neq 0.$$

ii) $A\underset{\sim}{e} = \lambda \underset{\sim}{e}$

$$A^{-1}(A\underset{\sim}{e}) = A^{-1}(\lambda \underset{\sim}{e})$$

$$(A^{-1}A)\underset{\sim}{e} = \lambda(A^{-1}\underset{\sim}{e})$$

$$I\underset{\sim}{e} = \lambda A^{-1}\underset{\sim}{e}$$

$$\underset{\sim}{e} = \lambda A^{-1}\underset{\sim}{e}$$

$$A^{-1}\underset{\sim}{e} = \frac{1}{\lambda}\underset{\sim}{e}$$

$\therefore$ The matrix A^{-1} has an eigenvalue λ^{-1} with corresponding eigenvector $\underset{\sim}{e}$.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & -3 \end{pmatrix} \qquad B = (A + 4I)^{-1}.$$

$$A - \lambda I = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -1 & 2 \\ 0 & -2-\lambda & 4 \\ 0 & 0 & -3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda)\left[(-2-\lambda)(-3-\lambda) - 0\right] - 1(0) + 2(0)$$

$$= (1-\lambda)(\lambda+2)(\lambda+3)$$

when $|A - \lambda I| = 0$.

$$(1-\lambda)(\lambda+2)(\lambda+3) = 0$$

$$\lambda = 1, -2, -3$$

when $\lambda = 1$: $\begin{pmatrix} 0 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{-3r_1 + r_2,\ \frac{r_3}{-4}} \left(\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$z = 0$

$y = 0$

Let $x = s$, $s \in \mathbb{R}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

when $\lambda = -2$: $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$z = 0$

Let $y = 3s,\ s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 3s \\ 0 \end{pmatrix}$$

$$= s \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

when $\lambda = -3$: $\begin{pmatrix} 4 & -1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 4 & -1 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 2s,\ s \in R$

$y = -8s$

$4x - (-8s) + 2(2s) = 0$

$4x = -12s$

$x = -3s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3s \\ -8s \\ 2s \end{pmatrix}$$

$$= s \begin{pmatrix} -3 \\ -8 \\ 2 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are $1, -2, -3$ with corresponding eigenvectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\3\\0\end{pmatrix}, \begin{pmatrix}-3\\-8\\2\end{pmatrix}$.

If λ is an eigenvalue of A with corresponding eigenvector $\underset{\sim}{e}$,

$$\begin{aligned}(A + 4I)\underset{\sim}{e} &= A\underset{\sim}{e} + 4I\underset{\sim}{e}\\ &= \lambda\underset{\sim}{e} + 4\underset{\sim}{e}\\ &= (\lambda + 4)\underset{\sim}{e}\end{aligned}$$

$\therefore$ The eigenvalues of $A + 4I$ are $1, 2, 5$ with corresponding eigenvectors $\begin{pmatrix}-3\\-8\\2\end{pmatrix}, \begin{pmatrix}1\\3\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}$.

Since $B = (A + 4I)^{-1}$, the eigenvalues of B are $1, \frac{1}{2}, \frac{1}{5}$ with corresponding eigenvectors $\begin{pmatrix}-3\\-8\\2\end{pmatrix}, \begin{pmatrix}1\\3\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}$.

If P is a non-singular matrix and D is a diagonal matrix such that $B = PDP^{-1}$,

let $P = \begin{pmatrix}-3 & 1 & 1\\ -8 & 3 & 0\\ 2 & 0 & 0\end{pmatrix}$

and $D = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{5}\end{pmatrix}$.