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4 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 3 & 4 & 2 & 5 \\ 6 & 7 & 5 & 8 \\ 9 & 9 & 9 & 9 \\ 15 & 16 & 14 & 17 \end{pmatrix}.$$

Find

(i) the rank of $\mathbf{M}$ and a basis for the range space of T, [4]

(ii) a basis for the null space of T. [3]

8 The vector $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{A}$, with corresponding eigenvalue λ, and is also an eigenvector of the matrix $\mathbf{B}$, with corresponding eigenvalue μ. Show that $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{AB}$ with corresponding eigenvalue $\lambda\mu$. [2]

State the eigenvalues of the matrix $\mathbf{C}$, where

$$\mathbf{C} = \begin{pmatrix} -1 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix},$$

and find corresponding eigenvectors. [4]

Show that $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{D}$, where

$$\mathbf{D} = \begin{pmatrix} 1 & -1 & 1 \\ -6 & -3 & 4 \\ -9 & -3 & 7 \end{pmatrix},$$

and state the corresponding eigenvalue. [3]

Hence state an eigenvector of the matrix $\mathbf{CD}$ and give the corresponding eigenvalue. [2]

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11.

Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$ [7]

The linear transformation $\mathrm{T} : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\mathbf{x} \mapsto \mathbf{Ax}$. Let $\mathbf{e}$, $\mathbf{f}$ be two linearly independent eigenvectors of $\mathbf{A}$, with corresponding eigenvalues λ and μ respectively, and let Π be the plane, through the origin, containing $\mathbf{e}$ and $\mathbf{f}$. By considering the parametric equation of Π, show that all points of Π are mapped by T onto points of Π. [3]

Find cartesian equations of three planes, each with the property that all points of the plane are mapped by T onto points of the same plane. [4]

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3 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 & -2 & 4 \\ 5 & 15 & -9 & 19 \\ -2 & -6 & 3 & -7 \\ 3 & 9 & -5 & 11 \end{pmatrix}$.

(i) Find the rank of $\mathbf{M}$. [3]

(ii) Obtain a basis for the null space of T. [3]

8 Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 0 & -3 \\ 1 & -3 & 0 \end{pmatrix}$. [8]

Find a non-singular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [3]

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11 Answer only **one** of the following two alternatives.

EITHER

A 3×3 matrix $\mathbf{A}$ has eigenvalues -1, 1, 2, with corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

respectively. Find

(i) the matrix $\mathbf{A}$,

(ii) $\mathbf{A}^{2n}$, where n is a positive integer.

[14]

OR

Determine the rank of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & -1 & -4 & 3 \\ 3 & -3 & -2 & 2 \\ 5 & -4 & -6 & 5 \end{pmatrix}.$$ [3]

Show that if

$$\mathbf{Ax} = p\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} + q\begin{pmatrix} -1 \\ -1 \\ -3 \\ -4 \end{pmatrix} + r\begin{pmatrix} -1 \\ -4 \\ -2 \\ -6 \end{pmatrix},$$

where p, q and r are given real numbers, then

$$\mathbf{x} = \begin{pmatrix} p+\lambda \\ q+\lambda \\ r+\lambda \\ \lambda \end{pmatrix},$$

where λ is real. [4]

Find the values of p, q and r such that

$$p\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} + q\begin{pmatrix} -1 \\ -1 \\ -3 \\ -4 \end{pmatrix} + r\begin{pmatrix} -1 \\ -4 \\ -2 \\ -6 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 8 \\ 15 \end{pmatrix}.$$ [3]

Find the solution $\mathbf{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$ of the equation $\mathbf{Ax} = \begin{pmatrix} 3 \\ 7 \\ 8 \\ 15 \end{pmatrix}$ for which $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{11}{4}$. [4]

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6 The linear transformation $\mathrm{T}: \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & \alpha \\ 2 & 3 & -1 & 0 \\ 2 & 1 & 2 & -2 \\ 0 & 1 & -3 & -2 \end{pmatrix}.$$

Given that the dimension of the range space of T is 4, show that $\alpha \neq 1$. [3]

It is now given that $\alpha = 1$. Show that the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ -1 \\ 2 \\ -3 \end{pmatrix}$$

form a basis for the range space of T. [2]

Given also that the vector $\begin{pmatrix} p \\ 1 \\ 1 \\ q \end{pmatrix}$ is in the range space of T, find a condition satisfied by p and q. [3]

9 Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$ [7]

Find a non-singular matrix $\mathbf{M}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} - 2\mathbf{I})^3 = \mathbf{MDM}^{-1}$, where $\mathbf{I}$ is the 3×3 identity matrix. [3]

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8 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix}.$$

Given that one eigenvector of $\mathbf{A}$ is $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, find the corresponding eigenvalue. [2]

Given also that another eigenvalue of $\mathbf{A}$ is 4, find a corresponding eigenvector. [2]

Given further that $\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, with corresponding eigenvalue 1, find matrices $\mathbf{P}$ and $\mathbf{Q}$, together with a diagonal matrix $\mathbf{D}$, such that $\mathbf{A}^5 = \mathbf{PDQ}$. [6]

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1 Given that 5 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -3 & 0 \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{pmatrix},$$

find a corresponding eigenvector. [2]

Hence find an eigenvalue and a corresponding eigenvector of the matrix $\mathbf{A} + \mathbf{A}^2$. [2]

12.

The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 1 & 5 & 7 \\ 3 & 9 & 17 & 25 \\ 1 & 7 & 7 & 11 \\ 3 & 6 & 16 & 23 \end{pmatrix}$.

(i) In either order,

(a) show that the dimension of R, the range space of T, is equal to 2,

(b) obtain a basis for R.

[5]

(ii) Show that the vector $\begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix}$ belongs to R. [3]

(iii) It is given that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for the null space of T, where $\mathbf{e}_1 = \begin{pmatrix} 14 \\ 1 \\ -3 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 19 \\ 2 \\ 0 \\ -3 \end{pmatrix}$.

Show that, for all λ and μ,

$$\mathbf{x} = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_2$$

is a solution of

$$\mathbf{M}\mathbf{x} = \begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix}. \qquad (*)$$

[3]

(iv) Hence find a solution of $(*)$ of the form $\begin{pmatrix} \alpha \\ 0 \\ \gamma \\ \delta \end{pmatrix}$. [2]

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One of the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & 6 \\ 2 & -4 & 2 \\ -3 & 4 & a \end{pmatrix}$$

is -2. Find the value of a. [3]

Another eigenvalue of $\mathbf{A}$ is -5. Find eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ corresponding to the eigenvalues -2 and -5 respectively. [3]

The linear space spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ is denoted by V.

(i) Prove that, for any vector $\mathbf{x}$ belonging to V, the vector $\mathbf{A}\mathbf{x}$ also belongs to V. [4]

(ii) Find a non-zero vector which is perpendicular to every vector in V, and determine whether it is an eigenvector of $\mathbf{A}$. [4]

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9 The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & -1 \\ 2 & 1 & 5 \end{pmatrix}$$

has eigenvalues 1, 5, 7. Find a set of corresponding eigenvectors. [5]

Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^n = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. [3]
[The evaluation of $\mathbf{P}^{-1}$ is not required.]

Determine the set of values of the real constant k such that $k^n\mathbf{A}^n$ tends to the zero matrix as $n \to \infty$. [3]

The linear transformations $\mathrm{T}_1 : \mathbb{R}^4 \to \mathbb{R}^4$ and $\mathrm{T}_2 : \mathbb{R}^4 \to \mathbb{R}^4$ are represented by the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively, where

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 7 & 8 \\ 1 & 7 & 11 & 13 \\ 1 & 2 & 5 & 5 \end{pmatrix}, \qquad \mathbf{M}_2 = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 5 & 1 & -3 & -3 \\ 3 & -1 & -1 & -1 \\ 13 & -1 & -6 & -6 \end{pmatrix}.$$

(i) Find a basis for R_1, the range space of T_1. [4]

(ii) Find a basis for K_2, the null space of T_2, and hence show that K_2 is a subspace of R_1. [5]

The set of vectors which belong to R_1 but do not belong to K_2 is denoted by W.

(iii) State whether W is a vector space, justifying your answer. [1]

The linear transformation $\mathrm{T}_3 : \mathbb{R}^4 \to \mathbb{R}^4$ is the result of applying T_1 and then T_2, in that order.

(iv) Find the dimension of the null space of T_3. [3]

Oct/Nov 2008 (9231/01)

4 The matrix $\mathbf{A}$ has λ as an eigenvalue with $\mathbf{e}$ as a corresponding eigenvector. Show that $\mathbf{e}$ is an eigenvector of $\mathbf{A}^2$ and state the corresponding eigenvalue. [3]

Given that one eigenvalue of $\mathbf{A}$ is 3, find an eigenvalue of the matrix $\mathbf{A}^4 + 3\mathbf{A}^2 + 2\mathbf{I}$, justifying your answer. [3]

6 The matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 & -3 \\ -2 & 1 & 7 & 2 \\ -3 & 3 & 6 & \alpha \\ 7 & -6 & -17 & -17 \end{pmatrix}.$$

(i) Show that if $\alpha = 9$ then the rank of $\mathbf{A}$ is 2, and find a basis for the null space of $\mathbf{A}$ in this case. [5]

(ii) Find the rank of $\mathbf{A}$ when $\alpha \neq 9$. [2]

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3 Show that if λ is an eigenvalue of the square matrix $\mathbf{A}$ with $\mathbf{e}$ as a corresponding eigenvector, and μ is an eigenvalue of the square matrix $\mathbf{B}$ for which $\mathbf{e}$ is also a corresponding eigenvector, then $\lambda + \mu$ is an eigenvalue of the matrix $\mathbf{A} + \mathbf{B}$ with $\mathbf{e}$ as a corresponding eigenvector. [2]

The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -4 & -6 & -6 \\ 5 & 11 & 10 \end{pmatrix}$$

has $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ as an eigenvector. Find the corresponding eigenvalue. [1]

The other two eigenvalues of $\mathbf{A}$ are 1 and 2, with corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ respectively. The matrix $\mathbf{B}$ has eigenvalues 2, 3, 1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ respectively. Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} + \mathbf{B})^4 = \mathbf{PDP}^{-1}$. [3]

[You are not required to evaluate $\mathbf{P}^{-1}$.]

12.

The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 3 & -1 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & 3 & -4 & -1 \end{pmatrix}.$$

The range space of T is denoted by V.

(i) Determine the dimension of V. [3]

(ii) Show that the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 0 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ -1 \\ 3 \\ -4 \end{pmatrix}$ are linearly independent. [4]

(iii) Write down a basis of V. [1]

The set of elements of $\mathbb{R}^4$ which do not belong to V is denoted by W.

(iv) State, with a reason, whether W is a vector space. [1]

(v) Show that if the vector $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ belongs to W then $y - z - t \neq 0$. [5]

Oct/Nov 2007 (9231/01)

10 The vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ are defined as follows:

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The linear space spanned by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ is denoted by V_1 and the linear space spanned by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4$ is denoted by V_2.

(i) Give a reason why $V_1 \cup V_2$ is not a linear space. [1]

(ii) State the dimension of the linear space $V_1 \cap V_2$ and write down a basis. [2]

Consider now the set V_3 of all vectors of the form $q\mathbf{b}_2 + r\mathbf{b}_3 + s\mathbf{b}_4$, where q, r, s are real numbers. Show that V_3 is a linear space, and show also that it has dimension 3. [3]

Determine whether each of the vectors

$$\begin{pmatrix} 4 \\ 4 \\ 2 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 \\ 4 \\ 2 \\ 5 \end{pmatrix}$$

belongs to V_3 and justify your conclusions. [4]

11 Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 4 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

and corresponding eigenvectors. [7]

The matrix $\mathbf{B}$ is defined by

$$\mathbf{B} = \mathbf{A} - k\mathbf{I},$$

where $\mathbf{I}$ is the 3×3 identity matrix and k is a real number. Find a non-singular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{B}^3 = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$ [4]

May/June 2007 (9231/01)

6 The matrix $\mathbf{A}$, given by

$$\mathbf{A} = \begin{pmatrix} 7 & -4 & 6 \\ 2 & 2 & 2 \\ -3 & 4 & -2 \end{pmatrix},$$

has eigenvalues 1, 2, 4. Find a set of corresponding eigenvectors. [4]

Hence find the eigenvalues of $\mathbf{B}$, where

$$\mathbf{B} = \begin{pmatrix} 10 & -4 & 6 \\ 2 & 5 & 2 \\ -3 & 4 & 1 \end{pmatrix},$$

and state a set of corresponding eigenvectors. [3]

12.

The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & -2 & 2 & 4 \\ 2 & -4 & 5 & 9 \\ 3 & -6 & 8 & 14 \\ 5 & -10 & 12 & 22 \end{pmatrix}.$$

(i) Find the rank of $\mathbf{M}$. [3]

(ii) Obtain a basis for the null space, K, of T. [3]

(iii) Evaluate

$$\mathbf{M}\begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \end{pmatrix},$$

and hence show that any solution of

$$\mathbf{Mx} = \begin{pmatrix} 5 \\ 11 \\ 17 \\ 27 \end{pmatrix} \qquad (*)$$

has the form

$$\begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \end{pmatrix} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2,$$

where λ and μ are constants and $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for K. [3]

(iv) Find the solution $\mathbf{x}_1$ of $(*)$ such that the first component of $\mathbf{x}_1$ is A, and the sum of all the components of $\mathbf{x}_1$ is B. [5]

Oct/Nov 2006 (9231/01)

1 It is given that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{pmatrix}.$$

Write down the eigenvalues of $\mathbf{A}$ and find corresponding eigenvectors. [5]

May/June 2006 (9231/01)

4 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -3 & 4 & 5 \\ 5 & -6 & 10 & 14 \\ 4 & -5 & 8 & 11 \end{pmatrix}.$$

Show that the dimension of the range space of T is 2. [3]

Let $\mathbf{M}$ be a given 4×4 matrix and let S be the vector space consisting of vectors of the form $\mathbf{MAx}$, where $\mathbf{x} \in \mathbb{R}^4$. Show that if $\mathbf{M}$ is non-singular then the dimension of S is 2. [4]

11.

The square matrix $\mathbf{A}$ has λ as an eigenvalue with corresponding eigenvector $\mathbf{x}$. The non-singular matrix $\mathbf{M}$ is of the same order as $\mathbf{A}$. Show that $\mathbf{Mx}$ is an eigenvector of the matrix $\mathbf{B}$, where $\mathbf{B} = \mathbf{MAM}^{-1}$, and that λ is the corresponding eigenvalue. [3]

It is now given that

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ a & -3 & 0 \\ b & c & -5 \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Write down the eigenvalues of $\mathbf{A}$ and obtain corresponding eigenvectors in terms of a, b, c. [4]

(ii) Find the eigenvalues and corresponding eigenvectors of $\mathbf{B}$. [4]

(iii) Hence find a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{B}^n = \mathbf{QDQ}^{-1}$. [3]

[You are not required to find $\mathbf{Q}^{-1}$.]

Oct/Nov 2005 (9231/01)

10 It is given that the eigenvalues of the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix},$$

are 1, 3, 4. Find a set of corresponding eigenvectors. [4]

Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{M}^n = \mathbf{PDP}^{-1},$$

where n is a positive integer. [2]

Find $\mathbf{P}^{-1}$ and deduce that

$$\lim_{n\to\infty} 4^{-n}\mathbf{M}^n = \begin{pmatrix} -\frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \end{pmatrix}.$$ [5]

11 Find the rank of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 4 & 3 & 5 & 16 \\ 6 & 6 & 13 & 13 \\ 14 & 12 & 23 & 45 \end{pmatrix}.$$ [3]

Find vectors $\mathbf{x}_0$ and $\mathbf{e}$ such that any solution of the equation

$$\mathbf{Ax} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix} \qquad (*)$$

can be expressed in the form $\mathbf{x}_0 + \lambda\mathbf{e}$, where $\lambda \in \mathbb{R}$. [5]

Hence show that there is no vector which satisfies (*) and has all its elements positive. [3]

May/June 2005

9 Find the eigenvalues and a corresponding set of eigenvectors of the matrix **M** given by

$$\mathbf{M} = \begin{pmatrix} a & 2 & 1 \\ 0 & b & -1 \\ 0 & 0 & c \end{pmatrix},$$

where a, b, c are all different. [6]

Find a matrix **P** and a diagonal matrix **D** such that

$$(\mathbf{M} - k\mathbf{I})^n = \mathbf{PDP}^{-1},$$

where **I** is the identity matrix, k is a constant scalar and n is a positive integer. [4]

[You are not required to evaluate $\mathbf{P}^{-1}$.]

11 The matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & -1 \\ 2 & 2 & \theta \end{pmatrix}.$$

Find the rank of **A**, distinguishing between the cases $\theta \neq 1$ and $\theta = 1$. [4]

Consider the system S of equations:

$$\begin{aligned} x + 3y + 2z &= 1, \\ x - \ \ y - \ \ z &= 0, \\ 2x + 2y + \theta z &= 3\theta + \phi - 2. \end{aligned}$$

(i) Show that if $\theta \neq 1$ then S has a unique solution. Find this solution in the case $\phi = 0$. [3]

(ii) Show that if $\theta = 1$ and $\phi = 0$ then S has an infinite number of solutions. [3]

(iii) Show that if $\theta = 1$ and $\phi \neq 0$ then S has no solution. [2]

Oct/Nov 2004 (9231/01)

1 The linear transformation $\mathbf{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 0 & -1 & 7 \\ 3 & -1 & -2 & 10 \\ 4 & 10 & 13 & 29 \end{pmatrix}.$$

Find the dimension of the null space of **T**. [4]

12.

The matrix **A** has λ as an eigenvalue with **e** as a corresponding eigenvector. Show that if **A** is non-singular then

(i) $\lambda \neq 0$, [2]

(ii) the matrix $\mathbf{A}^{-1}$ has λ^{-1} as an eigenvalue with **e** as a corresponding eigenvector. [2]

The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & -3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = (\mathbf{A} + 4\mathbf{I})^{-1}.$$

Find a non-singular matrix **P**, and a diagonal matrix **D**, such that $\mathbf{B} = \mathbf{PDP}^{-1}$. [10]

May/June 2004 (9231/01)

5 Write down the eigenvalues of the matrix $\mathbf{A}$, given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{pmatrix},$$

and obtain a set of corresponding eigenvectors. [6]

Find a non-singular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [2]

10 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$\mathrm{T} : \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \mapsto \mathbf{A} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 3 & -2 \\ 5 & 0 & 7 & -7 \\ 6 & 2 & 6 & \theta + 2 \\ 9 & 3 & 9 & \theta \end{pmatrix}.$$

(i) Show that when $\theta \neq -6$, the dimension of the null space K of T is 1, and that when $\theta = -6$, the dimension of K is 2. [4]

(ii) For the case $\theta \neq -6$, determine a basis vector $\mathbf{e}_1$ for K of the form $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 0 \end{pmatrix}$, where x_1, y_1, z_1 are integers. [2]

(iii) For the case $\theta = -6$, determine a vector $\mathbf{e}_2$ of the form $\begin{pmatrix} x_2 \\ y_2 \\ 0 \\ t_2 \end{pmatrix}$, where x_2, y_2, t_2 are integers, such that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis of K. [3]

(iv) Given that $\theta = -6$, $\mathbf{b} = \begin{pmatrix} 5 \\ 5 \\ 10 \\ 15 \end{pmatrix}$, $\mathbf{e}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, show that $\mathbf{x} = \mathbf{e}_0 + k_1\mathbf{e}_1 + k_2\mathbf{e}_2$ is a solution of the equation $\mathbf{Ax} = \mathbf{b}$ for all real values of k_1 and k_2. [3]

Oct/Nov 2003 (9231/01)

3 Three $n \times 1$ column vectors are denoted by $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{M}$ is an $n \times n$ matrix. Show that if $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly dependent then the vectors $\mathbf{Mx}_1$, $\mathbf{Mx}_2$, $\mathbf{Mx}_3$ are also linearly dependent. [2]

The vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ and the matrix $\mathbf{P}$ are defined as follows:

$$\mathbf{y}_1 = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \quad \mathbf{y}_3 = \begin{pmatrix} 5 \\ 51 \\ 55 \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -7 \end{pmatrix}.$$

(i) Show that $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ are linearly dependent. [2]

(ii) Find a basis for the linear space spanned by the vectors $\mathbf{Py}_1$, $\mathbf{Py}_2$, $\mathbf{Py}_3$. [2]

10 Find the eigenvalues and corresponding eigenvectors of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 6 & 4 & 1 \\ -6 & -1 & 3 \\ 8 & 8 & 4 \end{pmatrix}. \qquad [8]$$

Hence find a non-singular matrix **P** and a diagonal matrix **D** such that $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \mathbf{PDP}^{-1}$. [4]

May/June 2003 (9231/01)

8 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & -2 & -3 & 14 \\ 4 & -3 & -5 & 17 \end{pmatrix}.$$

Find the rank of **A** and a basis for the null space of T. [7]

The vector $\begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}$ is denoted by **e**. Show that there is a solution of the equation $\mathbf{Ax} = \mathbf{Ae}$ of the form

$\mathbf{x} = \begin{pmatrix} p \\ q \\ 1 \\ 1 \end{pmatrix}$, where p and q are to be found. [4]

11.

The vector **e** is an eigenvector of each of the $n \times n$ matrices **A** and **B**, with corresponding eigenvalues λ and μ respectively. Prove that **e** is an eigenvector of the matrix **AB** with eigenvalue $\lambda\mu$. [3]

Find the eigenvalues and corresponding eigenvectors of the matrix **C**, where

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{pmatrix}. \qquad [8]$$

Verify that one of the eigenvectors of **C** is an eigenvector of the matrix **D**, where

$$\mathbf{D} = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}. \qquad [2]$$

Hence find an eigenvalue of the matrix **CD**. [1]

Oct/Nov 2002

11 Answer only **one** of the following two alternatives.

EITHER

The vector **e** is an eigenvector of the square matrix **G**. Show that

(i) **e** is an eigenvector of $\mathbf{G} + k\mathbf{I}$, where k is a scalar and **I** is an identity matrix,

(ii) **e** is an eigenvector of $\mathbf{G}^2$.

[5]

Find the eigenvalues, and corresponding eigenvectors, of the matrices **A** and $\mathbf{B}^2$, where

$$\mathbf{A} = \begin{pmatrix} 3 & -3 & 0 \\ 1 & 0 & 1 \\ -1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -5 & -3 & 0 \\ 1 & -8 & 1 \\ -1 & 3 & -6 \end{pmatrix}. \qquad [9]$$

10 The linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{H}$, where

$$\mathbf{H} = \begin{pmatrix} 1 & 2 & -3 & -5 \\ -1 & 4 & 5 & 1 \\ 2 & 3 & 0 & -3 \\ -3 & 5 & 7 & 2 \end{pmatrix}.$$

(i) Find the dimension of the range space of T. [3]

(ii) Find a basis for the null space of T. [3]

(iii) It is given that $\mathbf{x}$ satisfies the equation

$$\mathbf{Hx} = \begin{pmatrix} 2 \\ -10 \\ -1 \\ -15 \end{pmatrix}.$$

Using the fact that

$$\mathbf{H}\begin{pmatrix} 1 \\ -3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -1 \\ -15 \end{pmatrix},$$

find the least possible value of $|\mathbf{x}|$. [7]

[For the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $|\mathbf{x}| = \sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2)}$.]

May/June 2002 (9231/01)

1 Find the eigenvalues of the matrix

$$\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}.$$

Find also corresponding eigenvectors. [5]

10 The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are given by

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

The subspace of $\mathbb{R}^4$ spanned by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is denoted by V_1, and the subspace of $\mathbb{R}^4$ spanned by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ is denoted by V_2. Show that V_1 and V_2 each have dimension 3. [3]

The set of vectors which belong to both V_1 and V_2 is denoted by V_3. Find a basis for V_3. [2]

The set of vectors which consists of the zero vector and all vectors which belong to only one of V_1 and V_2 is denoted by W.

(i) Write down two linearly independent vectors which belong to W. [2]

(ii) Show that W is not a linear space. [3]