Situation :

We know that if $\mathbf{a} = (a_1 \quad a_2 \quad a_3)$ and $\mathbf{b} = (b_1 \quad b_2 \quad b_3)$, then $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2 \quad a_3b_1 - a_1b_3 \quad a_1b_2 - a_2b_1)$, and hence the magnitude of $\mathbf{a} \times \mathbf{b}$ i.e.

$$|\,\mathbf{a} \times \mathbf{b}\,| = \sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}.$$

Problem : IS THERE ANY RELATIONSHIP BETWEEN $|\,\mathbf{a} \times \mathbf{b}\,|$, $|\,\mathbf{a}\,|$ and $|\,\mathbf{b}\,|$?





*This aim of this activity is to find the relationship between* $|\,\mathbf{a} \times \mathbf{b}\,|$, $|\,\mathbf{a}\,|$ and $|\,\mathbf{b}\,|$. *Try the following procedure:*

Let $\mathbf{a} = (a_1 \quad a_2 \quad a_3)$, $\mathbf{b} = (b_1 \quad b_2 \quad b_3)$, and the angle between the vectors be $\theta$.

(i) By considering $\mathbf{a} \cdot \mathbf{b}$, express $\cos\theta$ in terms of $a_1$, $a_2$, $a_3$, $b_1$, $b_2$ and $b_3$.





(ii) By using the identity $\cos^2\theta + \sin^2\theta \equiv 1$, show that

$$\sin\theta = \frac{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{(a_1^{\,2} + a_2^{\,2} + a_3^{\,2})}\sqrt{(b_1^{\,2} + b_2^{\,2} + b_3^{\,3})}}$$





(iii) Relate (ii) with $|\,\mathbf{a} \times \mathbf{b}\,|$, $|\,\mathbf{a}\,|$ and $|\,\mathbf{b}\,|$, and express $|\,\mathbf{a} \times \mathbf{b}\,|$ as the subject.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$$

$$\Rightarrow a_1 b_2 + a_2 b_2 + a_3 b_3 = \sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}\,\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{a_1 b_2 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

From $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow \quad \sin^2\theta = 1 - \cos^2\theta$$

$$\Rightarrow \quad \sin^2\theta = 1 - \frac{(a_1 b_1 + a_2 b_2 + a_3 b_3)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

$$\Rightarrow \quad \sin^2\theta = \frac{\left(\cancel{a_1^2 b_1^2} + \overset{3}{a_1^2 b_2^2} + \overset{2}{a_1^2 b_3^2} + \overset{3}{a_2^2 b_1^2} + \cancel{a_2^2 b_2^2} + \overset{1}{a_2^2 b_3^2} + \overset{2}{a_3^2 b_1^2} + \overset{1}{a_3^2 b_2^2} + \cancel{a_3^2 b_3^2}\right) - \left(\cancel{a_1^2 b_1^2} + \cancel{a_2^2 b_2^2} + \cancel{a_3^2 b_3^2} + \overset{3}{2 a_1 b_1 a_2 b_2} + \overset{2}{2 a_1 b_1 a_3 b_3} + \overset{1}{2 a_2 b_2 a_3 b_3}\right)}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

$$\Rightarrow \quad \sin^2\theta = \frac{\overset{1}{(a_2 b_3 - a_3 b_2)^2} + \overset{2}{(a_3 b_1 - a_1 b_3)^2} + \overset{3}{(a_1 b_2 - a_2 b_1)^2}}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

$$\Rightarrow \quad \sin\theta = \frac{|\underset{\sim}{a} \times \underset{\sim}{b}|}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$\Rightarrow \quad |\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin\theta .$$