1. $S_n = \sum_{k=1}^{n} 8k^3 - 6k^2$

$$= \sum_{k=1}^{n} 8k^3 - \sum_{k=1}^{n} 6k^2$$

$$= 8\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2$$

$$= \frac{8n^2(n+1)^2}{4} - \frac{6n(n+1)(2n+1)}{6}$$

$$= 2n^2(n+1)^2 - n(n+1)(2n+1)$$

$$= n(n+1)[2n(n+1) - (2n+1)]$$

$$= n(n+1)(2n^2 + 2n - 2n - 1)$$

$$= n(n+1)(2n^2 - 1)$$

2. i) $y = x^2$

$x = \pm\sqrt{y}$

$x^3 - x - 1 = 0$

$(\pm\sqrt{y})^3 - (\pm\sqrt{y}) - 1 = 0$

$\pm y\sqrt{y} - (\pm\sqrt{y}) - 1 = 0$

$\pm\sqrt{y}(y - 1) = 1$

$y(y-1)^2 = 1$

$y(y^2 - 2y + 1) = 1$

$y^3 - 2y^2 + y = 1$

$y^3 - 2y^2 + y - 1 = 0$

ii) $a = 1 \quad b = 0 \quad c = -1 \quad d = -1$

$aS_{3+r} + bS_{2+r} + cS_{1+r} + dS_r = 0$

$S_{3+r} = S_{1+r} + S_r$

$S_0 = 3$

$r = 1: \; S_4 = S_2 + S_1$

$r=0: \; s_3 = s_1 + s_0 = 0+3 = 3$

$r=2: \; s_5 = s_3 + s_2 = 3+2 = 5$

$r=3: \; s_6 = s_4 + s_3 = 2+3 = 5$

$r=4: \; s_7 = s_5 + s_4 = 5+2 = 7$

$r=5: \; s_8 = s_6 + s_5 = 5+5 = 10$

$r=6: \; s_9 = s_7 + s_6 = 7+5 = 12$

$r=7: \; s_{10} = s_8 + s_7 = 10+7 = 17$

$r=8: \; s_{11} = s_9 + s_8 = 12+10 = 22$

$r=9: \; s_{12} = s_{10} + s_9 = 17+12 = 29$

$r=10: \; s_{13} = s_{11} + s_{10} = 22+17 = 39$

$r=11: \; s_{14} = s_{12} + s_{11} = 29+22 = 51$

$r=12: \; s_{15} = s_{13} + s_{12} = 39+29 = 68$

$r=13: \; s_{16} = s_{14} + s_{13} = 51+39 = 90$

$s_8 + s_{12} = 10 + 29 = 39$

$s_{16} - s_4 = 90 - 2 = 88$

$$\alpha^{-4} + \beta^{-4} + \gamma^{-4} = \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4}$$

$$= \frac{\alpha^4\beta^4 + \alpha^4\gamma^4 + \beta^4\gamma^4}{\alpha^4\beta^4\gamma^4}$$

$$= \frac{-3}{1}$$

$$= -3$$

$$aT_3 + bT_2 + cT_1 + dT_0 = 0$$

$$T_3 - 2T_2 - 3T_1 - T_0 = 0$$

$$T_3 - 2(10) - 3(2) - 3 = 0$$

$$T_3 - 20 - 6 - 3 = 0$$

$$T_3 = 29$$

$$\therefore S_{12} = \alpha^{12} + \beta^{12} + \gamma^{12} = 29$$

$$aT_4 + bT_3 + cT_2 + dT_1 = 0$$

$$T_4 - 2T_3 - 3T_2 - T_1 = 0$$

$$T_4 - 2(29) - 3(10) - 2 = 0$$

$$T_4 = 58 + 30 + 2$$

$$= 90$$

$$\therefore S_{16} = \alpha^{16} + \beta^{16} + \gamma^{16} = 90$$

$$S_8 + S_{12} = 10 + 29 = 39$$

$$S_{16} - S_4 = 90 - 2 = 88$$

$$\therefore y = x + c - a - b + \frac{ab + c^2 - ac - bc}{x - c}$$

$$= x + c - a - b + \frac{(c-a)(c-b)}{x-c}$$

ii) $x \longrightarrow \pm\infty \qquad y \longrightarrow x + c - a - b$

$x \longrightarrow c \qquad y \longrightarrow \pm\infty$

Asymptotes: $y = x + c - a - b$

$x = c$

iii) $$y = x + c - a - b + \frac{(c-a)(c-b)}{x-c}$$

$$\frac{dy}{dx} = 1 - \frac{(c-a)(c-b)}{(x-c)^2}$$

If $\frac{dy}{dx} = 0$: $\quad 1 - \frac{(c-a)(c-b)}{(x-c)^2} = 0$

$$\frac{(c-a)(c-b)}{(x-c)^2} = 1$$

$$(x-c)^2 = (c-a)(c-b)$$

$$x - c = \pm\sqrt{(c-a)(c-b)}$$

$$\therefore \left(c + \sqrt{(c-a)(c-b)}, \frac{(c-a+\sqrt{(c-a)(c-b)})(c-b+\sqrt{(c-a)(c-b)})}{\sqrt{(c-a)(c-b)}} \right)$$

is a minimum point.

$$\left(c - \sqrt{(c-a)(c-b)}, \frac{(c-a+\sqrt{(c-a)(c-b)})(c-b+\sqrt{(c-a)(c-b)})}{-\sqrt{(c-a)(c-b)}} \right)$$

is a maximum point.

when $x = 0$: $y = -\dfrac{ab}{c}$

$y = 0$: $x = a$ or $x = b$

iv) $a + b > c$

Since $0 < a < b < c$

$a + b < 2b < b + c$

$a + b < a + c$

$a + b < a + c < b + c$

$0 < a + b - c < a < b < c$