MAY / JUNE 2003

1. $r = a\sin 2\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad a > 0$

$$\text{Area} = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{a^2 \sin^2 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2}\, d\theta$$

$$= \frac{a^2}{2} \left[\frac{\theta}{2} - \frac{\sin 4\theta}{8} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2} \left(\frac{\pi}{4} - 0 \right)$$

$$= \frac{\pi a^2}{8}$$

2. $$\sum_{n=1}^{N} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}, \quad N \geqslant 1$$

When $n = 1$: $$\sum_{n=1}^{1} \frac{n+2}{n(n+1)2^n}$$

$$= \frac{1+2}{1(1+1)2^1}$$

$$= \frac{3}{1 \cdot 2 \cdot 2}$$

$$= \frac{3}{4}$$

$$= 1 - \frac{1}{4}$$

$$= 1 - \frac{1}{2 \cdot 2}$$

$$= 1 - \frac{1}{(1+1)2^1}$$

Assume the statement is true when $N = k$.

$$\sum_{n=1}^{k} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(k+1)2^k}$$

when $N = k + 1$:

$$\sum_{n=1}^{k+1} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(k+2)2^{k+1}}.$$

(what needs to be proved)

$$\sum_{n=1}^{k+1} \frac{n+2}{n(n+1)2^n} = \frac{k+1+2}{(k+1)(k+1+1)2^{k+1}} + \sum_{n=1}^{k} \frac{n+2}{n(n+1)2^n}$$

$$= \frac{k+3}{(k+1)(k+2)2^{k+1}} + 1 - \frac{1}{(k+1)2^k}$$

$$= 1 + \frac{k+3}{(k+1)(k+2)2^{k+1}} - \frac{1}{(k+1)2^k}$$

$$= 1 + \frac{1}{(k+1)2^k}\left(\frac{k+3}{2(k+2)} - 1\right)$$

$$= 1 + \frac{1}{(k+1)2^k}\left(\frac{k+3-2k-4}{2(k+2)}\right)$$

$$= 1 + \frac{(-k-1)}{2^k 2(k+1)(k+2)}$$

$$= 1 - \frac{1}{(k+2)2^{k+1}}$$

$$\therefore \sum_{n=1}^{N} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}$$

for every positive integer N.

3. $v_1, v_2, v_3, \ldots$

$$u_n = nv_n - (n+1)v_{n+1}, \quad n = 1, 2, 3, \ldots$$

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} nv_n - (n+1)v_{n+1}$$

$$= v_1 - 2v_2$$
$$+ 2v_2 - 3v_3$$
$$+ 3v_3 - 4v_4$$

$$+ (N-2)v_{N-2} - (N-1)v_{N-1}$$
$$+ (N-1)v_{N-1} - Nv_N$$
$$+ Nv_N - (N+1)v_{N+1}$$

$$= v_1 - (N+1)v_{N+1}$$

i) $v_n = n^{-\frac{1}{2}}$

$$\sum_{n=1}^{N} u_n = v_1 - (N+1)v_{N+1}$$

$$= 1 - (N+1)(N+1)^{-\frac{1}{2}}$$

$$= 1 - (N+1)^{\frac{1}{2}}$$

Since $(N+1)^{\frac{1}{2}} \longrightarrow \infty$ as $N \to \infty$,

the series $\sum_{n=1}^{\infty} u_n$ is not convergent.

ii) $v_n = n^{-\frac{3}{2}}$

$$\sum_{n=1}^{N} u_n = v_1 - (N+1)v_{N+1}$$

$$= 1 - (N+1)(N+1)^{-\frac{3}{2}}$$

$$= 1 - (N+1)^{-\frac{1}{2}}$$

Since $(N+1)^{-\frac{1}{2}} \longrightarrow 0$ as $N \longrightarrow \infty$,

$$\sum_{n=1}^{\infty} u_n = \lim_{N \to \infty} \left(1 - (N+1)^{-\frac{1}{2}}\right)$$

$$= 1$$

4. C : $y = \dfrac{x^2 - 4}{x - 3}$

i)

$$\begin{array}{r|l} & x + 3 \\ \hline x - 3 & x^2 - 4 \\ & x^2 - 3x \\ \hline & \quad 3x - 4 \\ & \quad 3x - 9 \\ \hline & \qquad 5 \end{array}$$

$$y = x + 3 + \frac{5}{x - 3}$$

As $x \to \pm\infty$ $y \to x + 3$

As $x \to 3$ $y \to \pm\infty$

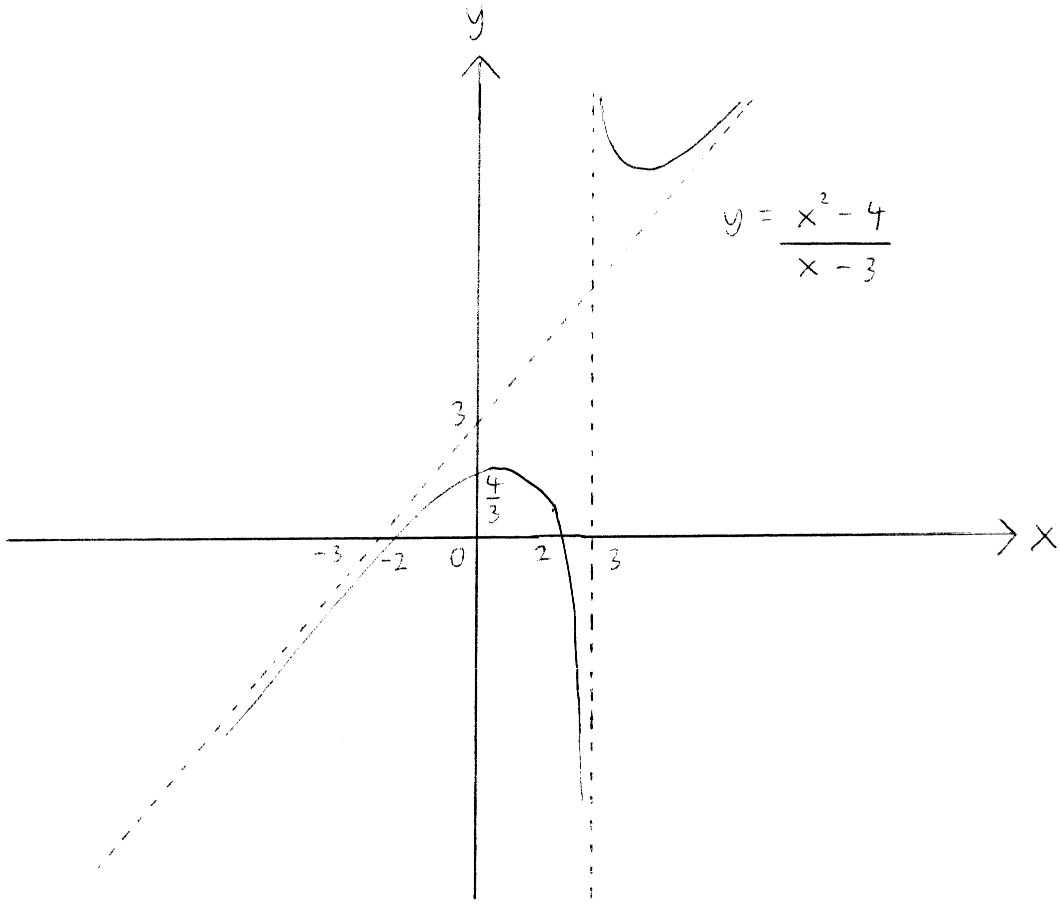
The equations of the asymptotes of C are $y = x + 3$ and $x = 3$.

ii) When $x = 0$, $y = \frac{4}{3}$

when $y = 0$: $\dfrac{x^2 - 4}{x - 3} = 0$

$$x^2 - 4 = 0$$

$$x = \pm 2$$



y
x
$y = \frac{x^2 - 4}{x - 3}$
3
$\frac{4}{3}$
-3
-2
0
2
3

5. $8x^3 + 12x^2 + 4x - 1 = 0$

α, β, γ are the roots

$2\alpha + 1,\ 2\beta + 1,\ 2\gamma + 1$

Let $y = 2\alpha + 1$

$\alpha = \dfrac{y-1}{2}$

α is a root

$8\alpha^3 + 12\alpha^2 + 4\alpha - 1 = 0$

$8\left(\dfrac{y-1}{2}\right)^3 + 12\left(\dfrac{y-1}{2}\right)^2 + 4\left(\dfrac{y-1}{2}\right) - 1 = 0$

$\dfrac{8(y^3 - 3y^2 + 3y - 1)}{8} + 12\left(\dfrac{y^2 - 2y + 1}{4}\right)$

$+ 2y - 2 - 1 = 0$

$y^3 - 3y^2 + 3y - 1 + 3y^2 - 6y + 3 + 2y - 3 = 0$

$y^3 - y - 1 = 0$

$\therefore$ The equation with roots, $2\alpha + 1, 2\beta + 1, 2\gamma + 1$

is $y^3 - y - 1 = 0$.

$S_n = (2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$

$2\alpha + 1 + 2\beta + 1 + 2\gamma + 1 = 0$

$(2\alpha + 1)(2\beta + 1) + (2\alpha + 1)(2\gamma + 1) + (2\beta + 1)(2\gamma + 1) = -1$

$(2\alpha + 1)(2\beta + 1)(2\gamma + 1) = 1$

$$S_0 = (2\alpha + 1)^0 + (2\beta + 1)^0 + (2\beta + 1)^0$$

$$= 1 + 1 + 1$$

$$= 3$$

$$S_1 = (2\alpha + 1)^1 + (2\beta + 1)^1 + (2\gamma + 1)^1$$

$$= 2\alpha + 1 + 2\beta + 1 + 2\gamma + 1$$

$$= 0$$

$$S_2 = (2\alpha + 1)^2 + (2\beta + 1)^2 + (2\gamma + 1)^2$$

$$= (2\alpha + 1 + 2\beta + 1 + 2\gamma + 1)^2$$

$$- 2\left[(2\alpha + 1)(2\beta + 1) + (2\alpha + 1)(2\gamma + 1) + (2\beta + 1)(2\gamma + 1)\right]$$

$$= 0^2 - 2(-1)$$

$$= 2$$

Since $2\alpha + 1$, $2\beta + 1$, $2\gamma + 1$ are the roots of the equation $y^3 - y - 1 = 0$,

$$S_{3+r} - S_{1+r} - S_r = 0$$

$$r = 0: \quad S_3 - S_1 - S_0 = 0$$

$$S_3 = S_1 + S_0$$

$$= 3$$

$$S_{-1} = (2\alpha + 1)^{-1} + (2\beta + 1)^{-1} + (2\gamma + 1)^{-1}$$

$$= \frac{1}{2\alpha + 1} + \frac{1}{2\beta + 1} + \frac{1}{2\gamma + 1}$$

$$= \frac{(2\alpha+1)(2\beta+1) + (2\alpha+1)(2\gamma+1) + (2\beta+1)(2\gamma+1)}{(2\alpha+1)(2\beta+1)(2\gamma+1)}$$

$$= \frac{-1}{1}$$

$$= -1$$

$r = -2 : \quad S_1 - S_{-1} - S_{-2} = 0$

$$S_{-2} = S_1 - S_{-1}$$

$$= 0 - (-1)$$

$$= 1.$$

6. $(\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6\cos^5\theta(i\sin\theta)$

$+ 15\cos^4\theta(-\sin^2\theta) + 20\cos^3\theta(-i\sin^3\theta)$

$+ 15\cos^2\theta(\sin^4\theta) + 6\cos\theta(i\sin^5\theta)$

$- \sin^6\theta$

$\cos 6\theta + i\sin 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta$

$+ 15\cos^2\theta\sin^4\theta - \sin^6\theta$

$+ i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta$

$+ 6\cos\theta\sin^5\theta)$

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

$= \cos^6\theta - 15\cos^4\theta(1 - \cos^2\theta)$

$+ 15\cos^2\theta(1 - \cos^2\theta)^2 - (1 - \cos^2\theta)^3$

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta$

$+ 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$- (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta$

$+ 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta$

$- 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

$$64x^6 - 96x^4 + 36x^2 - 1 = 0$$

$$64x^6 - 96x^4 + 36x^2 - 2 = -1$$

$$32x^6 - 48x^4 + 18x^2 - 1 = \frac{-1}{2}$$

Let $x = \cos\theta$

$$\therefore 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = \frac{-1}{2}$$

$$\cos 6\theta = \frac{-1}{2}$$

$$6\theta = \frac{2\pi}{3} + 2k\pi,\ \frac{4\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$$

$$\theta = \left(\frac{k}{3} + \frac{1}{9}\right)\pi,\ \left(\frac{k}{3} + \frac{2}{9}\right)\pi,\ k \in \mathbb{Z}$$

$$= \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$$

$$\therefore x = \cos\frac{\pi}{9}, \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}, \cos\frac{8\pi}{9}$$

$$= \pm\cos\frac{\pi}{9}, \pm\cos\frac{2\pi}{9}, \pm\cos\frac{4\pi}{9}$$

7 $x^4 + y^4 = 1, \quad 0 < x < 1, \quad 0 < y < 1.$

i) $$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(1)$$

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$x^3 + y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x^3}{y^3}$$

$$\frac{d}{dx}\left(x^3 + y^3\frac{dy}{dx}\right) = 0$$

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right)\frac{dy}{dx} + y^3\frac{d^2y}{dx^2} = 0$$

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right)^2 + y^3\frac{d^2y}{dx^2} = 0$$

$$3x^2 + 3y^2\left(\frac{-x^3}{y^3}\right)^2 + y^3\frac{d^2y}{dx^2} = 0$$

$$3x^2 + \frac{3x^6y^2}{y^6} + y^3\frac{d^2y}{dx^2} = 0$$

$$y^3\frac{d^2y}{dx^2} = -3x^2 - \frac{3x^6}{y^4}$$

$$= \frac{-3x^2y^4 - 3x^6}{y^4}$$

$$= \frac{-3x^2(x^4 + y^4)}{y^4}$$

$$= \frac{-3x^2}{y^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3x^2}{y^7}$$

ii) The mean value of $\frac{d^3y}{dx^3}$ over the interval

$a_1 \leq x \leq a_2$ is $\frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \frac{d^3y}{dx^3} \, dx$

$$= \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) dx$$

$$= \frac{1}{a_2 - a_1} \left[\frac{d^2y}{dx^2}\right]_{a_1}^{a_2}$$

$$= \frac{1}{a_2 - a_1} \left[\frac{-3x^2}{y^7}\right]_{(a_1, b_1)}^{(a_2, b_2)}$$

$$= \frac{1}{a_2 - a_1} \left(\frac{-3a_2^2}{b_2^7} - \left(\frac{-3a_1^2}{b_1^7}\right)\right)$$

$$= \frac{1}{a_2 - a_1} \left(\frac{3a_1^2}{b_1^7} - \frac{3a_2^2}{b_2^7}\right)$$

$$= \frac{1}{a_2 - a_1} \left(\frac{3a_1^2 b_2^7 - 3a_2^2 b_1^7}{b_1^7 b_2^7}\right)$$

$$= \frac{3(a_1^2 b_2^7 - a_2^2 b_1^7)}{b_1^7 b_2^7 (a_2 - a_1)}$$

8 $T: R^4 \longrightarrow R^4$

$$A = \begin{pmatrix} 1 & -1 & -2 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & -2 & -3 & 14 \\ 4 & -3 & -5 & 17 \end{pmatrix}$$

$$\xrightarrow[\substack{-3r_1 + r_3 \\ -4r_1 + r_4}]{-2r_1 + r_2} \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \end{pmatrix}$$

$$\xrightarrow[-r_2 + r_4]{-r_2 + r_3} \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $(A) = 2$

$$\begin{pmatrix} 1 & -1 & -2 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & -2 & -3 & 14 \\ 4 & -3 & -5 & 17 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & 0 \\ 2 & -1 & -1 & 11 & 0 \\ 3 & -2 & -3 & 14 & 0 \\ 4 & -3 & -5 & 17 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & 0 \\ 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_4 = s$ and $x_3 = t$, $s, t \in R$

$x_2 = -3t - 5s$

$x_1 - (-3t - 5s) - 2t + 3s = 0$

$x_1 + 3t + 5s - 2t + 3s = 0$

$x_1 = -8s - t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -8s - t \\ -3t - 5s \\ t \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

. A basis for the null space of T is

$$\left\{ \begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\underset{\sim}{e} = \begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}$$

Since $\underset{\sim}{e}$ is a solution of the equation $A\underset{\sim}{x} = A\underset{\sim}{e}$

and $\left\{ \begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for the

null space of T, the general solution of the

equation $A\underset{\sim}{x} = A\underset{\sim}{e}$ is $\begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$.

if $\begin{pmatrix} p \\ q \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 - 8s - t \\ -2 - 5s - 3t \\ -1 \qquad + t \\ -1 + s \end{pmatrix}$$

$$p = 1 - 8s - t$$

$$q = -2 - 5s - 3t$$

$$1 = -1 + t$$

$$1 = -1 + s$$

$$s = t = 2, \quad p = -17, \quad q = -18$$

9. $$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 5x\frac{dx}{dt} + 3x^2 = 3\sin 2t + 15\cos 2t,$$

$x > 0$,

$0 \le t \le \frac{\pi}{2}$

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\frac{dy}{dt}\frac{dt}{dx} = 2x$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(2x\frac{dx}{dt}\right)$$

$$= 2x\frac{d^2x}{dt^2} + 2\left(\frac{dx}{dt}\right)^2$$

$$2x\frac{d^2x}{dt^2} + 2\left(\frac{dx}{dt}\right)^2 + 10x\frac{dx}{dt} + 6x^2 = 6\sin 2t + 30\cos 2t$$

$$\frac{d^2y}{dt^2} + \frac{5dy}{dt} + 6y = 6\sin 2t + 30\cos 2t$$

$$\frac{d^2y}{dt^2} + \frac{5dy}{dt} + 6y = 0$$

$$m^2 + 5m + 6 = 0$$

$$(m + 2)(m + 3) = 0$$

$$m = -2, -3$$

The complementary function, y_c, is

$$y_c = Ae^{-2t} + Be^{-3t}$$

The particular integral, y_p, is given by

$$y_p = C\cos 2t + D\sin 2t$$

$$\frac{dy_p}{dt} = -2C\sin 2t + 2D\cos 2t$$

$$\frac{d^2y_p}{dt^2} = -4C\cos 2t - 4D\sin 2t$$

$$\frac{d^2y_p}{dt^2} + \frac{5dy_p}{dt} + 6y_p = -4C\cos 2t - 4D\sin 2t$$

$$+ 5(-2C\sin 2t + 2D\cos 2t)$$

$$+ 6(C\cos 2t + D\sin 2t)$$

$$= -4C\cos 2t - 4D\sin 2t$$

$$-10C\sin 2t + 10D\cos 2t$$

$$+ 6C\cos 2t + 6D\sin 2t$$

$$= (2C + 10D)\cos 2t$$

$$+ (-10C + 2D)\sin 2t$$

$$= 6\sin 2t + 30\cos 2t$$

$2C + 10D = 30$ $\quad$ $-10C + 2D = 6$

$C + 5D = 15$ $\quad$ $-5C + D = 3$

$$-5(15 - 5D) + D = 3$$

$$-75 + 25D + D = 3$$

$$26D = 78$$

$$D = 3$$

$$C = 0$$

$$y_p = 3\sin 2t$$

$$y = y_c + y_p$$

$$= Ae^{-2t} + Be^{-3t} + 3\sin 2t$$

$$x^2 = Ae^{-2t} + Be^{-3t} + 3\sin 2t$$

$x = 2$ and $\frac{dx}{dt} = -\frac{3}{2}$ when $t = 0$

$$2x\frac{dx}{dt} = -2Ae^{-2t} - 3Be^{-3t} + 6\cos 2t$$

$t = 0 \quad x = 2 : \quad 4 = A + B$

$t = 0 \quad x = 2 \quad \frac{dx}{dt} = -\frac{3}{2} : \quad -6 = -2A - 3B + 6$

$$A + B = 4$$

$$2A + 3B = 12$$

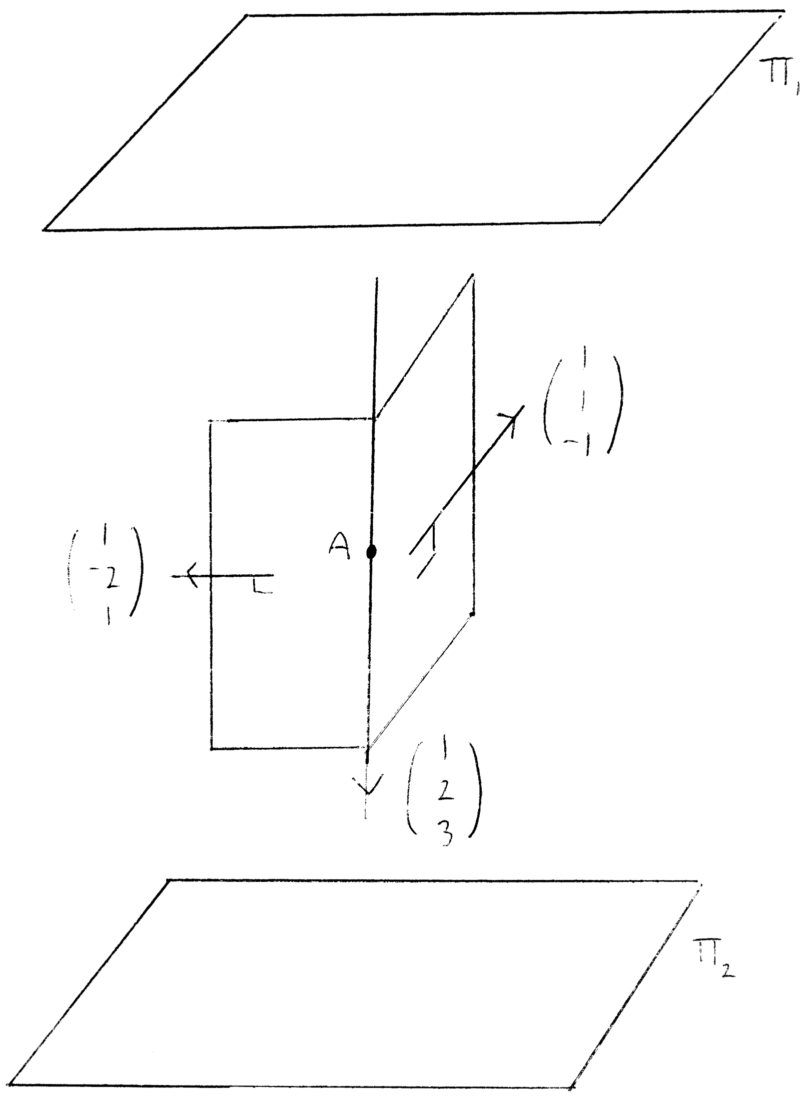
$$B = 4$$

$$A = 0$$

$$\therefore x^2 = 4e^{-3t} + 3\sin 2t$$

$$x = \sqrt{4e^{-3t} + 3\sin 2t}$$

10. $x - 2y + z - 9 = 0 \quad x + y - z + 2 = 0 \quad \overrightarrow{OA} = p\underset{\sim}{i} + q\underset{\sim}{j} + k$



$x - 2y + z = 9 \qquad x + y - z = -2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 9 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = -2$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \left|\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right| \cos\theta$$

$$1 - 2 - 1 = \sqrt{6}\sqrt{3}\cos\theta$$

$$-2 = 3\sqrt{2}\cos\theta$$

$$\cos\theta = \frac{-\sqrt{2}}{3}$$

$$\theta = 118.1$$

$\therefore$ The acute angle between the planes

$x - 2y + z = 9$ and $x + y - z = -2$ is $61.9°$.

i) Since l lies in both the planes and A is a point on l,

$p - 2q + 1 = 9$

$p + q - 1 = -2$

$p - 2q = 8$

$p + q = -1$

$3q = -9$

$q = -3$

$p = 2$

ii) Since l is perpendicular to both the normals $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, it is parallel to $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Since l has direction $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and A is a point on l, a vector equation for l is

$$\underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k} + s(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$$

Since ℓ is perpendicular to both the planes Π_1 and Π_2, Π_1 and Π_2 have the form

$\Pi_1 : \ x + 2y + 3z = d_1$

$\Pi_2 : \ x + 2y + 3z = d_2$

when ℓ meets the plane Π_1,

$$2 + s + 2(-3 + 2s) + 3(1 + 3s) = d_1$$
$$2 + s - 6 + 4s + 3 + 9s = d_1$$
$$14s = d_1 + 1$$
$$s = \frac{d_1 + 1}{14}$$

$\therefore$ ℓ meets Π_1 at the point

$$\left(\frac{d_1 + 29}{14}, \frac{2d_1 - 40}{14}, \frac{3d_1 + 17}{14}\right).$$

Also, when ℓ meets the plane Π_2,

$$2 + s + 2(-3 + 2s) + 3(1 + 3s) = d_2$$
$$2 + s - 6 + 4s + 3 + 9s = d_2$$
$$14s = d_2 + 1$$
$$s = \frac{d_2 + 1}{14}$$

$\therefore$ ℓ meets Π_2 at the point

$$\left(\frac{d_2 + 29}{14}, \frac{2d_2 - 40}{14}, \frac{3d_2 + 17}{14}\right).$$

Since the perpendicular distance from A to both Π_1 and Π_2 is $\sqrt{14}$,

$$\sqrt{\left(\frac{d_1 + 29}{14} - 2\right)^2 + \left(\frac{2d_1 - 40}{14} + 3\right)^2 + \left(\frac{3d_1 + 17}{14} - 1\right)^2} = \sqrt{14}$$

and

$$\sqrt{\left(\frac{d_1 + 29}{14} - 2\right)^2 + \left(\frac{2d_1 - 40}{14} + 3\right)^2 + \left(\frac{3d_1 + 17}{14} - 1\right)^2} = \sqrt{14}$$

$$\left(\frac{d_1+1}{14}\right)^2 + \left(\frac{2d_2+2}{14}\right)^2 + \left(\frac{3d_1+3}{14}\right)^2 = 14$$

$$\left(\frac{d_1+1}{14}\right)^2 + 4\left(\frac{d_2+1}{14}\right)^2 + 9\left(\frac{d_1+1}{14}\right)^2 = 14$$

$$14\left(\frac{d_1+1}{14}\right)^2 = 14$$

$$\left(\frac{d_1+1}{14}\right)^2 = 1$$

$$\frac{d_1+1}{14} = \pm 1$$

$$d_1 + 1 = \pm 14$$

$$d_1 = 13, -15$$

$$d_2 = -15, 13$$

. The equations of the planes Π_1 and Π_2 are $x + 2y + 3z = 13$ and $x + 2y + 3z = -15$

11. EITHER

$$I_n = \int_0^1 x^n e^{-\alpha x}\, dx, \quad \alpha > 0, \quad n \geq 0$$

$u = x^n \qquad dv = e^{-\alpha x}\, dx$

$du = nx^{n-1} \qquad v = \dfrac{e^{-\alpha x}}{-\alpha}$

$$= \left[\frac{x^n e^{-\alpha x}}{-\alpha}\right]_0^1 - \int_0^1 \frac{nx^{n-1}e^{-\alpha x}}{-\alpha}\, dx$$

$$= \frac{e^{-\alpha}}{-\alpha} - 0 + \frac{n}{\alpha}\int_0^1 x^{n-1}e^{-\alpha x}\, dx$$

$$= \frac{e^{-\alpha}}{-\alpha} + \frac{n}{\alpha} I_{n-1}$$

$$\therefore \alpha I_n = n I_{n-1} - e^{-\alpha}, \quad n \geq 1.$$

The area, A, of the finite region bounded by the x-axis, the line $x = 1$ and the curve $y = xe^{-x}$ is

$$\int_0^1 xe^{-x}\, dx, \quad \alpha = 1$$

$$= I_1$$

$$= I_0 - e^{-1}$$

$$= \int_0^1 e^{-x}\, dx - e^{-1}$$

$$= \left[-e^{-x}\right]_0^1 - e^{-1}$$

$$= -e^{-1} - (-1) - e^{-1}$$

$$= 1 - 2e^{-1}$$

If the centroid has coordinates $(\bar{x}, \bar{y})$,

$$\bar{x} = \frac{\int_0^1 xy \, dx}{A}$$

$$= \frac{\int_0^1 x^2 e^{-x} \, dx}{1 - \frac{2}{e}}, \quad \alpha = 1$$

$$= \frac{I_2}{1 - \frac{2}{e}}$$

$$= \frac{2I_1 - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2(I_0 - e^{-1}) - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2\left(\int_0^1 e^{-x} \, dx - e^{-1}\right) - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2\left(\left[-e^{-x}\right]_0^1 - e^{-1}\right) - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2(-e^{-1} - (-1) - e^{-1}) - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2\left(1 - \frac{2}{e}\right) - \frac{1}{e}}{1 - \frac{2}{e}}$$

$$= \frac{2 - \frac{4}{e} - \frac{1}{e}}{1 - \frac{2}{e}}$$

$$= \frac{2 - \frac{5}{e}}{1 - \frac{2}{e}}$$

$$= \frac{2e - 5}{e - 2}$$

$$\bar{y} = \frac{\int_0^1 \frac{y^2}{2}\, dx}{A},$$

$$= \frac{\int_0^1 \frac{x^2 e^{-2x}}{2}\, dx}{1 - \frac{2}{e}}, \quad \alpha = 2$$

$$= \frac{\frac{I_2}{2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{1}{2}\left(\frac{2I_1 - e^{-2}}{2}\right)}{1 - \frac{2}{e}}$$

$$= \frac{\frac{I_1}{2} - \frac{e^{-2}}{4}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{1}{4}(I_0 - e^{-2}) - \frac{e^{-2}}{4}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{I_0}{4} - \frac{e^{-2}}{4} - \frac{e^{-2}}{4}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{\int_0^1 e^{-2x}\,dx}{4} - \frac{e^{-2}}{2}}{1 - \frac{2}{e}}$$

$$= \frac{\left[\frac{e^{-2x}}{-8}\right]_0^1 - \frac{1}{2e^2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{e^{-2}}{-8} - \left(-\frac{1}{8}\right) - \frac{1}{2e^2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{1}{8} - \frac{1}{8e^2} - \frac{1}{2e^2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{1}{8} - \frac{5}{8e^2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{e^2 - 5}{8e^2}}{\frac{e - 2}{e}}$$

$$= \frac{e^2 - 5}{8e(e - 2)}$$

$\therefore$ The centroid of the region bounded by the x-axis, the line $x = 1$ and the curve $y = xe^{-x}$ is $\left(\frac{2e - 5}{e(e - 2)}, \frac{e^2 - 5}{8e(e - 2)} \right)$.

OR

$A\underset{\sim}{e} = \lambda \underset{\sim}{e}, \quad B\underset{\sim}{e} = \mu \underset{\sim}{e}$

$$
\begin{aligned}
AB\underset{\sim}{e} &= A(B\underset{\sim}{e}) \\
&= A(\mu \underset{\sim}{e}) \\
&= \mu (A\underset{\sim}{e}) \\
&= \mu (\lambda \underset{\sim}{e}) \\
&= (\lambda \mu) \underset{\sim}{e}
\end{aligned}
$$

$\therefore$ If $\underset{\sim}{e}$ is an eigenvector of the $n \times n$ matrices A and B with corresponding eigenvalues λ and μ respectively, $\underset{\sim}{e}$ is an eigenvector of the matrix AB.

$$
C = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{pmatrix}
$$

$$
\begin{aligned}
C - \lambda I &= \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -\lambda & 1 & 4 \\ 1 & 2-\lambda & -1 \\ 2 & 1 & 2-\lambda \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
|C - \lambda I| &= -\lambda\left[(2-\lambda)^2 + 1\right] - (2 - \lambda + 2) + 4(1 - 2(2-\lambda)) \\
&= -\lambda(\lambda^2 - 4\lambda + 4 + 1) + \lambda - 4 + 4(2\lambda - 3) \\
&= -\lambda^3 + 4\lambda^2 - 5\lambda + \lambda - 4 + 8\lambda - 12 \\
&= -\lambda^3 + 4\lambda^2 + 4\lambda - 16
\end{aligned}
$$

$$= -\lambda^2(\lambda - 4) + 4(\lambda - 4)$$

$$= (\lambda - 4)(4 - \lambda^2)$$

$$= (\lambda - 4)(\lambda + 2)(2 - \lambda)$$

when $|C - \lambda I| = 0$,

$$(\lambda - 4)(\lambda + 2)(2 - \lambda) = 0$$

$$\lambda = 2, 4, -2$$

when $\lambda = 2$: $\begin{pmatrix} -2 & 1 & 4 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -2 & 1 & 4 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s, s \in R$

$y = -2s$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -2s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

when $\lambda = 4$. $\begin{pmatrix} -4 & 1 & 4 \\ 1 & -2 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -4 & 1 & 4 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -4 & 1 & 4 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{4r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-7},\ \frac{r_3}{5}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s$, $s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

when $\lambda = -2$: $\begin{pmatrix} 2 & 1 & 4 \\ 1 & 4 & -1 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{-2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -7 & 6 & 0 \\ 0 & -7 & 6 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -7 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 7s$, $s \in \mathbb{R}$

$y = 6s$

$x + 24s - 7s = 0$

$x = -17s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -17s \\ 6s \\ 7s \end{pmatrix}$$

$$= s\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$$

$\therefore$ The eigenvalues of C are $2, 4, -2$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$

$$D = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}$$

$$D\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -4 \end{pmatrix} = -4\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$D\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$$

$$D\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 64 \\ 16 \\ -25 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of D with eigenvalue -4.

Since $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of C and D with eigenvalues 2 and -4 respectively, the matrix CD has an eigenvector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ with eigenvalue -8.