8B. DIFFERENTIAL EQUATIONS

Learning Outcomes:-

Student should be able to

- ✓ Recall the meaning of the terms 'complementary function' and 'particular integral' in the context of linear differential equations, and recall that the general solution is the sum of the complementary function and a particular integral;
- ✓ Find the complementary function for a second order linear differential equation with constant coefficients;
- ✓ Recall the form of, and find, a particular integral for a second order linear differential equation in the cases where a polynomial or e^{bx} or $a \cos px + b \sin px$ is a suitable form, and in other simple cases find the appropriate coefficient(s) given a suitable form of particular integral;
- ✓ Use a substitution to reduce a given differential equation to a second order linear equation with constant coefficients;
- ✓ Use initial conditions to find a particular solution to a differential equation, and interpret a solution in terms of a problem modeled by a differential equation.

8B.0 <u>Some Basic Concept about Differential Equations</u>

- A differential equation is any equation which contains derivatives.
- **Order**
  - The **order** of a differential equation is the highest derivative present in the differential equation.
  - Eg:

    (a) $\frac{dy}{dx} + y = x \qquad \Longrightarrow \text{ Order} =$

    (b) $x\left(\frac{dy}{dx}\right)^2 + y = 3 \qquad \Longrightarrow \text{ Order} =$

    (c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 7 \qquad \Longrightarrow \text{ Order} =$

    (d) $(\sin y)\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2e^{-5y} \qquad \Longrightarrow \text{ Order} =$
- **Linear and Non-linear**
  - A differential equation of n^{th} order is linear if it can be written in the form of

    $$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

    where the coefficients $a_0(x), a_1(x), \dots, a_n(x)$ in which $a_n(x) \neq 0$, and $f(x)$ are functions of x.

- If a differential equation cannot be written in the above form, then it is called a **non-linear** differential equation.

- The **linear differential equation of first order** can be described in the form

$$a(x)\frac{dy}{dx} + b(x)y = f(x)$$

where $\frac{dy}{dx}$ and y appear only to the first degree.

- The **linear differential equation of second order** has the general form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x)$$

where $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y appear only to the first degree.

- The important thing to note about linear differential equations is that there are no products of the function y and its derivatives and neither the function nor its derivatives occur to any power other than power of 1.
- Eg:

(a) $x\frac{dy}{dx} - 2y = x + 1$ $\Longrightarrow$

(b) $(1 - x^2)\frac{dy}{dx} = x(y + \sin^{-1}x)$ $\Longrightarrow$

(c) $\frac{dy}{dx} + y^2 = e^x$ $\Longrightarrow$

(d) $2x^2\frac{dy}{dx} + xe^y = \sin x$ $\Longrightarrow$

(e) $x\left(\frac{dy}{dx}\right)^2 + e^{-2x}y = 0$ $\Longrightarrow$

(f) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ $\Longrightarrow$

(g) $y\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = 5x^2$ $\Longrightarrow$

(h) $\frac{d^2y}{dx^2} + (1 + x)^2\frac{dy}{dx} + xe^y = \cos x$ $\Longrightarrow$

- **Homogeneous and Nonhomogeneous**
  - If the $f(x)$ in the general form of linear differential equation of n^{th} order shown above is equal to zero, i.e. $f(x) = 0$, those equations are known as linear homogeneous equations. Conversely, if $f(x) \neq 0$, they are called linear nonhomogeneous equations.
  - Eg:

(a) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 0$ $\Longrightarrow$

(b) $\frac{d^2y}{dx^2} + xy = \sin x$ $\Longrightarrow$

- **Formation of Ordinary Differential Equations**
  - Besides arise naturally in the practical situations, the formation of ordinary differential equation can also be done by eliminating arbitrary constants in a given expression.

  - In general, if y is expressed as a function of x which contains n arbitrary constants, then n differentiations are just sufficient to eliminate the constants and reduce the relation between y and x to form an ordinary differential equation of order n.

  - Eg:

    (a) By eliminating the constant A from equation

    $$y = x^2 + \frac{A}{x} \quad \cdots\cdots\cdots \text{(I)}$$

    form an ordinary differential equation.

    Solution:

    $$\frac{dy}{dx} = 2x - \frac{A}{x^2} \quad \cdots\cdots\cdots \text{(II)}$$

    (I) + [(II) × x]: $\quad x\frac{dy}{dx} + y = 3x^2$

    Alternatively,

    $$xy = x^3 + A$$

    $$x\frac{dy}{dx} + y = 3x^2$$

    (b) By eliminating the constants A and B from the equation

    $$y = e^{3t}(A\cos 2t + B\sin 2t)$$

    form an ordinary differential equation.

    Solution:

    $$ye^{-3t} = A\cos 2t + B\sin 2t$$

    Differentiating w. r. t. t, yields

    $$y'e^{-3t} - 3ye^{-3t} = -2A\sin 2t + 2B\cos 2t$$

    Again differentiating w. r. t. t gives

    $$y''e^{-3t} - 6y'e^{-3t} + 9ye^{-3t} = -4A\cos 2t - 4B\sin 2t$$

    $$e^{-3t}(y'' - 6y' + 9y) = -4(A\cos 2t + B\sin 2t)$$

    $$e^{-3t}(y'' - 6y' + 9y) = -4ye^{-3t}$$

    $$e^{-3t}(y'' - 6y' + 13y) = 0$$

    Since $e^{-3t} \neq 0$ for all t then

    $$y'' - 6y' + 13y = 0$$

- **Solutions of Ordinary Differential Equations**
  - The solution for a differential equation is not a number but a function. Since we have to find a function from its derivatives, the process of integration is involved.

- If the highest derivative in an equation is of order n, the solution involves n times of integrations and will contain n arbitrary constants of integration.

- The general solution of a linear differential equation is the sum of two parts:
  Particular integral – solution of the linear nonhomogeneous equation
  Complementary function – general solution of the linear homogeneous equation.
- Eg:
  (a) The general solution of

$$\frac{dy}{dx}\cos x + y\sin x = \tan x$$

is $y = \frac{1}{2}\sec x + C\cos x$, where C is an arbitrary constant.

If $\boldsymbol{y = \frac{1}{2}\sec x}$, then

$$\frac{dy}{dx}\cos x + y\sin x = \left(\frac{1}{2}\sec x\tan x\right)\cos x + \left(\frac{1}{2}\sec x\right)\sin x$$

$$= \frac{1}{2}\tan x + \frac{1}{2}\tan x = \tan x$$

$\therefore y = \frac{1}{2}\sec x$ is a solution of the linear nonhomogeneous equation $\frac{dy}{dx}\cos x + y\sin x = \tan x$ or the particular integral.

If $\boldsymbol{y = C\cos x}$, then

$$\frac{dy}{dx}\cos x + y\sin x = (-C\sin x)\cos x + (C\cos x)\sin x = 0$$

$\therefore y = C\cos x$ is a solution of the linear homogeneous equation $\frac{dy}{dx}\cos x + y\sin x = 0$ or the complimentary function.

- **Initial Value Problem**
  As mentioned earlier, the general solution of a n^{th} order differential equation has n arbitrary constants. Therefore, in order to ensure a particular solution exists, we need to have n particular conditions. These conditions are called initial conditions.

Definition:
Initial conditions are conditions which have the same value for the independent variable. ***Initial value problems*** are differential equations together with its initial conditions.

8B.1 Linear Differential Equations of First Order

We begin our discussion with the following definition.

Definition:
The differential equation
$$a(x)\frac{dy}{dx} + b(x)y = c(x), \quad \cdots\cdots\cdots (8.1)$$
Where $a(x), b(x)$and $c(x)$ are continuous functions of x or constants is called **first order linear equation**.

8B.1.1 Solution of First Order Linear Equations

For the purpose of finding solutions of differential equations of this type, Equation (8.1) is rearranged so that the coefficient of $\frac{dy}{dx}$ is 1. Then Equation (8.1) becomes
$$\frac{dy}{dx} + p(x)y = q(x), \quad \cdots\cdots\cdots (8.2)$$
where $p(x) = \dfrac{b(x)}{a(x)}$ and $q(x) = \dfrac{c(x)}{a(x)}$.

We can solve a first order linear differential equation as in Equation (8.2) by multiplying all its terms by an **integrating factor**, $I = e^{\int p(x)dx}$.

Observing that, by using the chain rule,
$$\frac{dI}{dx} = p(x). e^{\int p(x)dx} = p(x)I \quad \cdots\cdots\cdots (8.3)$$
Multiply both sides of Equation (8.2) by I, we have
$$I\frac{dy}{dx} + Ip(x)y = Iq(x) \quad \cdots\cdots\cdots (8.4)$$
Using Equation (8.3), and the product rule, Equation (8.4) can be written as
$$\frac{d}{dx}(Iy) = Iq(x)$$
Since I is a function of x alone and integrating this equation with respect to x gives
$$Iy = \int Iq(x)\,dx + A, \quad A \text{ constant}$$
$$y = \frac{1}{I}\left\{\int Iq(x)\,dx + A\right\}$$
which is the **general solution** of Equation (8.2).

REMARKS:

In the expression of $I = e^{\int p(x)dx}$, we do not introduce a constant of integration. Why?

If we put a constant, say k into the said expression, then

$$I = e^{\int p(x)dx+k} = e^k e^{\int p(x)dx}$$

which means that both sides of Equation (8.4) are multiplied by a constant, e^k. This constant may be omitted from the equation without having any effect on the solution. For illustration, refer to Example 8B.1.1.

Example 8B.1.1:

Find the general solution of the differential equation

$$\frac{dy}{dx} + y = x\,.$$

Solution:

$p(x) = 1$ and $q(x) = x$

$\int p(x)\,dx = \int 1\,dx = x + k$, k constant

The integrating factor is

$$I = e^{\int p(x)dx} = e^{x+k} = e^k e^x$$

Multiplying the given differential equation throughout by I yields

$$e^k e^x \frac{dy}{dx} + e^k e^x y = xe^k e^x$$

After eliminating e^k, we have

$$e^x \frac{dy}{dx} + e^x y = xe^x$$

which is same as the equation that we will obtain if the integrating factor were just $I = e^x$.

Then the equation can be written as

$$\frac{d}{dx}[e^x y] = xe^x\,.$$

Integrating both sides of the equation with respect to x gives

$$\begin{aligned}
e^x y &= \int xe^x\,dx \\
&= x\int e^x\,dx - \int\left(\int e^x\,dx\right)\left(\frac{d}{dx}(x)\right)dx \\
&= xe^x - \int e^x\,dx \\
&= xe^x - e^x + A \quad , \ A \text{ constant} \\
&= (x-1)e^x + A
\end{aligned}$$

Therefore, the required general solution is

$$y = x - 1 + Ae^{-x}$$

The **technique of finding solution for first order linear equations** can be summarized in the following table:

Step 1	Express the given equation in the form $$\frac{dy}{dx} + p(x)y = q(x)$$
Step 2	Find $p(x)$ and evaluate $\int p(x)\,dx$.
Step 3	Set the integrating factor as $$I = e^{\int p(x)dx}$$
Step 4	Multiply both sides of the equation in Step 1 by the integrating factor I and then write it in the form $$\frac{d}{dx}[Iy] = Iq(x)$$
Step 5	Integrate both sides of the equation obtained in Step 4 and then solve for y. Be sure to include a constant of integration, say A.
Step 6	If there is an initial condition, then use the values given to obtain the value for A.

Example 8B.1.2:

Find the particular solution of the differential equation

$$\frac{dy}{dx} + y\tan x = \cos x,$$

which satisfies the condition $y = 1$ when $x = 0$.

Example 8B.1.3:

Solve the differential equation

$$(1 - x^2)\frac{dy}{dx} - xy = \frac{1}{1 - x^2}.$$

8B.2 Linear Differential Equations of Second Order

Definition:

A differential equation of second order is said to be linear if it can be expressed in the form

$$a(x)\frac{d^2y}{dx^2}+b(x)\frac{dy}{dx}+c(x)y=f(x) \quad \cdots\cdots\cdots (8.5)$$

where the coefficients $a(x), b(x)$ and $c(x)$ in which $a(x)\neq 0$, and $f(x)$ are functions of x.

If there is at least one of the coefficients is not a constant, then the equation is called ***linear differential equation of second order with variable coefficients***. Conversely, if all the coefficients $a(x), b(x)$ and $c(x)$ are constants, then Equation (8.5) can be written as

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \quad \cdots\cdots\cdots (8.6)$$

and is called ***linear differential equation of second order with constant coefficients***.

If $f(x)=0$, then it is called a ***homogeneous equation***. Conversely, if $f(x)\neq 0$, it is known as ***nonhomogeneous equation***.

REMARKS:

In our syllabus, we are only dealing with the ***linear differential equation with constant coefficients***. If the given differential equation is not in this form, we shall use a **substitution** to reduce it to a second order linear equation with constant coefficients.

Recall from Section 8B.0, the **general solution** for a linear differential equation of n^{th} order is the **sum** of the particular integral and the complimentary function. In another words, for linear differential equations, we can split the problem of solution into two parts:

(i) Finding the **complementary function**, y_h – solution of the **homogeneous** equation
(ii) Finding **particular integral**, y_p – solution of the **nonhomogeneous** equation

Definition: (Complimentary function and Particular Integral)

The general solution of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

denoted by y_h , is called ***homogeneous solution*** or ***complimentary function***, while the solution of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

denoted by y_p , is called ***particular integral***.

Theorem: (General Solution of Nonhomogeneous Equation)

If y_h is a homogeneous solution of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

and y_p is a particular integral of nonhomogeneous equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

then the ***general solution*** of nonhomogeneous equation is given by

$$y=y_h+y_p\ .$$

8B.2.1 <u>Finding Complimentary Functions: Constant Coefficients</u>

We encounter linear differential equation of second order in many engineering problems. In this section, we focus on obtaining a method for finding a solution of a homogeneous linear differential equation of second order with constant coefficients. For that purpose, the equation to be considered is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0,\quad \cdots\cdots\cdots (8.7)$$

where a, b and c are constants

<u>Technique of Obtaining Solutions</u>

Let us begin with the following linear homogeneous differential equation of first order

$$b\frac{dy}{dx}+cy=0$$

$$\frac{dy}{dx}+py=0\quad ,\text{where } p=\frac{c}{b}$$

$$\frac{dy}{dx}=-py$$

$$\int\frac{dy}{y}=-p\int 1\,dx$$

$$\ln y=-px+k\ \ ,k\text{ constant}$$

$$y=Ae^{-px}\quad ,\text{where } A=e^k$$

If we let $m=-p$, the solution of $\dfrac{dy}{dx}=my$ is $y=Ae^{mx}$.

In this case , $m=-\dfrac{c}{b}$ or $bm+c=0,$ which is known as the auxiliary equation.

So the result can be summarized as:

The complimentary function for the differential equation

$$b\frac{dy}{dx} + cy = q(x)$$

is $y = Ae^{mx}$, where m is the root of the ***auxiliary/characteristic equation*** $bm + c = 0$.

Using the method above may seem an unnecessary complication since it is a separable differential equation. However, it provides the key which opens the door to the solution of the second order linear differential equation with constant coefficients.

When dealing with Equation (8.7), we might well guess a solution $y = e^{mx}$. This solution satisfies Equation (8.7) if and only if

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

Since $e^{mx} \neq 0$ for all real values of x, then

$$am^2 + bm + c = 0.$$

Hence, $y = e^{mx}$ will be a solution of Equation (8.7) if and only if the ***auxiliary/characteristic equation*** $am^2 + bm + c = 0$.

From the above discussion, we should also notice that the characteristic equation can be obtained from the differential equation by replacing

$$\frac{d^2y}{dx^2} \to m^2$$

$$\frac{dy}{dx} \to m^1$$

$$y \to m^0$$

By this substitution, the problem of solving homogeneous linear differential equation has apparently been reduced to the purely algebraic problem of finding the roots of $am^2 + bm + c = 0$. There are three types of roots for this quadratic equation:

(i)	Distinct real roots	if	$b^2 - 4ac > 0$
(ii)	Equal real roots	if	$b^2 - 4ac = 0$
(iii)	Complex roots	if	$b^2 - 4ac < 0$

Case I: **Distinct Real Roots** $(b^2 - 4ac > 0)$

If the auxiliary equation has two distinct real roots m_1 and m_2,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad , \qquad m_1 + m_2 = -\frac{b}{a} \ \text{ and } \ m_1 m_2 = \frac{c}{a}$$

The complimentary function is the most general function satisfying

$$\frac{d^2y}{dx^2} - (m_1 + m_2)\frac{dy}{dx} + m_1 m_2 y = 0\,.$$

This can be rearranged as

$\frac{d^2y}{dx^2} - m_2\frac{dy}{dx} = m_1\left(\frac{dy}{dx} - m_2 y\right)$ $\frac{d}{dx}\left(\frac{dy}{dx} - m_2 y\right) = m_1\left(\frac{dy}{dx} - m_2 y\right)$ Let $u = \frac{dy}{dx} - m_2 y$ $\frac{du}{dx} = m_1 u$ So $u = Ce^{m_1 x}$, that is $\frac{dy}{dx} - m_2 y = Ce^{m_1 x}$.	or	$\frac{d^2y}{dx^2} - m_1\frac{dy}{dx} = m_2\left(\frac{dy}{dx} - m_1 y\right)$ $\frac{d}{dx}\left(\frac{dy}{dx} - m_1 y\right) = m_2\left(\frac{dy}{dx} - m_1 y\right)$ Let $u = \frac{dy}{dx} - m_1 y$ $\frac{du}{dx} = m_2 u$ So $u = De^{m_2 x}$, that is $\frac{dy}{dx} - m_1 y = De^{m_2 x}$.

Subtract the two final equations, we then get the **complimentary function** as

$$(m_1 - m_2)y = Ce^{m_1 x} - De^{m_2 x}$$

$$y = \left(\frac{C}{m_1 - m_2}\right)e^{m_1 x} + \left(\frac{-D}{m_1 - m_2}\right)e^{m_2 x}$$

$$y = Ae^{m_1 x} + Be^{m_2 x} \quad \text{where} \quad A = \frac{C}{m_1 - m_2} \ \text{ and } \ B = \frac{-D}{m_1 - m_2}\,.$$

Case II: **Equal Real Roots** $(b^2 - 4ac = 0)$

If the auxiliary equation has equal real roots m,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad , \qquad 2m = -\frac{b}{a} \ \text{ and } \ m^2 = \frac{c}{a}$$

The complimentary function is the most general function satisfying

$$\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2 y = 0\,.$$

This can be rearranged as

$$\frac{d^2y}{dx^2} - m\frac{dy}{dx} = m\left(\frac{dy}{dx} - my\right)$$

$$\frac{d}{dx}\left(\frac{dy}{dx} - my\right) = m\left(\frac{dy}{dx} - my\right)$$

$$\text{Let } u = \frac{dy}{dx} - my$$

$$\frac{du}{dx} = mu$$

$$\text{So } u = Ce^{mx}, \text{that is } \frac{dy}{dx} - my = Ce^{mx}.$$

Multiplying the integrating factor, $I = e^{-mx}$ with the final equation gives

$$e^{-mx}\frac{dy}{dx} - mye^{-mx} = Ce^{mx}e^{-mx}$$

$$\frac{d}{dx}(ye^{-mx}) = C$$

Integrating this gives the **complimentary function** is

$$\begin{aligned} ye^{-mx} &= \int C\,dx \\ &= Cx + D \\ y &= (Cx + D)e^{mx} \end{aligned}$$

Case II: <u>Complex Roots</u> $(b^2 - 4ac < 0)$

If the auxiliary equation has complex roots m_1 and m_2, the complementary function can be derived exactly the same as in Case I, but now

$$m_1 = \alpha + \beta i \quad \text{and} \quad m_2 = \alpha - \beta i$$

By using the Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

So, the **complimentary function** is

$$\begin{aligned} y &= Ae^{m_1 x} + Be^{m_2 x} \\ &= Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x} \\ &= e^{\alpha x}\left(Ae^{i\beta x} + Be^{-\beta ix}\right) \\ &= e^{\alpha x}[A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x)] \\ &= e^{\alpha x}[(A + B)\cos\beta x + (A - B)i\sin\beta x] \\ &= e^{\alpha x}(C\cos\beta x + D\sin\beta x) \end{aligned}$$

where $C = A + B$ and $D = (A - B)i$

Therefore, the solution of $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$ depends on the types of the roots of the characteristic equation. The general solution for each case can be summarized as in the table below:

Table 8.1: Roots of Characteristic Equation and General Solutions for the Differential Equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.

Auxiliary/Characteristic Equation	**:**	$\boldsymbol{am^2 + bm + c = 0}$
Case	**Types of Roots**	**General Solutions**
I	m_1 and m_2 : real and distinct	$y = Ae^{m_1 x} + Be^{m_2 x}$
II	m_1 and m_2 : real and equal $m = m_1 = m_2$	$y = (A + Bx)e^{mx}$
III	m_1 and m_2 : complex $m_1 = \alpha + \beta i$ $m_2 = \alpha - \beta i$	$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

From the above discussion, the procedure of finding solution for a homogeneous differential equation with constant coefficients can be summarized as follows:

Step 1	Express the given equation in the form $$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
Step 2	Express the corresponding auxiliary/characteristic equation as $am^2 + bm + c = 0$.
Step 3	Find the roots of the characteristic equation.
Step 4	Find a general solution depending on the types of roots of the characteristic equations by referring to Table 8.1.
Step 5	If there are initial conditions, then substitute the conditions into the general solution to obtain the values for constants A and B.

We illustrate our discussion with the following examples.

Example 8B.2.1:

Find the general solution of each of the following differential equations.

(a)	$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$	(d)	$y'' - 4y = 0$
(b)	$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$	(e)	$4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0$
(c)	$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$	(f)	$y'' + 4y = 0$

Example 8B.2.2:

Find the particular solution of each of the following differential equations that satisfies the given initial conditions.

(a)	$2y'' + y' = 0$; $y(0) = 3, \quad y'(0) = 2$	(d)	$y'' - 2y' - 2y = 0$; $y(0) = 0, \quad y'(0) = \sqrt{3}$
(b)	$y'' + 4y' + 4y = 0$; $y(0) = 2, \quad y'(0) = 1$	(e)	$9y'' - 6y' + y = 0$; $y(0) = 6, \quad y'(0) = 9$
(c)	$y'' - 4y' + 13y = 0$; $y(0) = -1, \quad y'(0) = 2$	(f)	$2y'' - 2y' + y = 0$; $y(0) = 1, \quad y'(0) = 1$