1. $x^4 - ax^3 + bx^2 - abx + 5 = 0$

Roots: $\alpha, \beta, \gamma, \delta$

$\alpha + \beta + \gamma + \delta = a$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = b$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = ab$

$\alpha\beta\gamma\delta = 5$

From 1:

$(\alpha + \beta + \gamma)(\alpha + \beta + \delta)(\alpha + \gamma + \delta)(\beta + \gamma + \delta)$

$= (a - \alpha)(a - \beta)(a - \gamma)(a - \delta)$

$= (a^2 - (\alpha + \beta)a + \alpha\beta)(a^2 - (\gamma + \delta)a + \gamma\delta)$

$= a^4 - (\alpha + \beta + \gamma + \delta)a^3$

$+ (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)a^2$

$- (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)a + \alpha\beta\gamma\delta$

$= a^4 - a(a^3) + b(a^2) - ab(a) + 5$

$= a^4 - a^4 + a^2b - a^2b + 5$

$= 5$

2. $n^2 \geqslant 5n + 5, \quad n \geqslant 6$

When $n = 6$: $6^2 = 36 \geqslant 35 = 5 + 30 = 5(6) + 5$

Assume the statement is true when $n = k$.

$n = k$: $k^2 \geqslant 5k + 5, \quad k \geqslant 6$

when $n = k+1$: $(k+1)^2 \geqslant 5(k+1) + 5$

(what needs to be proved)

$$
\begin{aligned}
5(k+1) + 5 &= 5k + 5 + 5 \\
&\leqslant k^2 + 5
\end{aligned}
$$

since $k \geqslant 6$

$k \geqslant 2$

$2k \geqslant 4$

$k^2 + 5 \leqslant k^2 + 2k + 1$

$k^2 + 5 \leqslant (k+1)^2$

Since $5(k+1) + 5 \leqslant k^2 + 5$

and $k^2 + 5 \leqslant (k+1)^2$

$\therefore 5(k+1) + 5 \leqslant (k+1)^2$

$\therefore$ True for $n = k+1$

Therefore, $n^2 \geqslant 5n + 5$ for all integers $n \geqslant 6$.

3. $$\frac{8r^2}{(2r-1)(2r+1)} = 2 + \frac{2}{(2r-1)(2r+1)}$$

$$\begin{array}{r|l} & 2 \\ 4r^2-1 & 8r^2 \\ & \underline{8r^2-2} \\ & \quad 2 \end{array}$$

$$\frac{2}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

$$2 = A(2r+1) + B(2r-1)$$

$r = \frac{1}{2}$: $2 = 2A \Rightarrow A = 1$

$r = \frac{-1}{2}$: $2 = -2B \Rightarrow B = -1$

$$\therefore \frac{8r^2}{(2r-1)(2r+1)} = 2 + \frac{1}{2r-1} - \frac{1}{2r+1}$$

Let $f(r) = \frac{1}{2r-1}$

$f(r+1) = \frac{1}{2r+1}$

$$\sum_{r=1}^{n} \frac{8r^2}{(2r-1)(2r+1)} = \sum_{r=1}^{n} 2 + f(r) - f(r+1)$$

$$= \sum_{r=1}^{n} 2 + \sum_{r=1}^{n} f(r) - f(r+1)$$

$$= 2n + f(1) - f(n+1)$$

$$= 2n + 1 - \frac{1}{2n+1}$$

4. $$\left.\begin{aligned} x + y + \lambda z &= 0 \\ x + \lambda y + z &= 0 \\ \lambda x + y + z &= 0 \end{aligned}\right\}$$

$$\begin{array}{l} -r_1 + r_2 \\ -\lambda r_1 + r_3 \end{array} \quad \left.\begin{aligned} x + y + \lambda z &= 0 \\ (\lambda-1)y + (1-\lambda)z &= 0 \\ (1-\lambda)y + (1-\lambda^2)z &= 0 \end{aligned}\right\}$$

$$r_2 + r_3 \quad \begin{aligned} x + y + \lambda z &= 0 \\ (\lambda-1)y + (1-\lambda)z &= 0 \\ (\lambda+2)(\lambda-1)z &= 0 \end{aligned}$$

when $\lambda = 1$: $x + y + z = 0$

$0y + 0z = 0$

$0z = 0$

Let $y = s,\ z = t,\ s, t \in R$

$\therefore x = -s - t$

when $\lambda = -2$: $$\left.\begin{aligned} x + y - 2z &= 0 \\ -3y + 3z &= 0 \\ 0z &= 0 \end{aligned}\right\}$$

Let $z = t,\ t \in R$

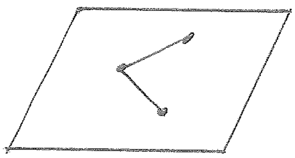
$y = t$

$x = t$

when $\lambda \neq -2, 1$: $$\left.\begin{aligned} x + y + \lambda z &= 0 \\ (\lambda-1)y + (1-\lambda)z &= 0 \\ (\lambda+2)(\lambda-1)z &= 0 \end{aligned}\right\}$$

$\therefore z = 0\quad y = 0\quad x = 0$

5.



$(0,1,1) \quad (1,0,0) \quad (1,2,3)$

i) Vector equation of Π_1 :

$$\underset{\sim}{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}1-0\\0-1\\0-1\end{pmatrix} + t\begin{pmatrix}1-1\\2-0\\3-0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}1\\-1\\-1\end{pmatrix} + t\begin{pmatrix}0\\2\\3\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+s\\-s+2t\\-s+3t\end{pmatrix}$$

$x = 1+s \qquad y = -s+2t \qquad z = -s+3t$

$s = x-1 \qquad y = 1-x+2t \qquad z = 1-x+3t$

$t = \dfrac{x+y-1}{2} \qquad t = \dfrac{x+z-1}{3}$

$$\therefore \frac{x+y-1}{2} = \frac{x+z-1}{3}$$

$3x+3y-3 = 2x+2z-2$

$x+3y-2z = 1$

$\therefore$ The Cartesian equation of Π is

$x+3y-2z=1$

ii) $\Pi_1: x + 3y - 2z = 1 \Rightarrow \underset{\sim}{n}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

$\Pi_2: x + y + 2z = 0 \Rightarrow \underset{\sim}{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 = |\underset{\sim}{n}_1||\underset{\sim}{n}_2| \cos\theta$

Let θ denote the angle between Π_1 and Π_2,

$\therefore \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \left|\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right| \cos\theta$

$1 + 3 - 4 = \sqrt{1+9+4}\sqrt{1+1+4} \cos\theta$

$\cos\theta = 0$

$\theta = 90°.$

iii) The equation of plane Π_1 is

$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 1$, $\underset{\sim}{n}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

Changing this to the form $\underset{\sim}{r} \cdot \hat{\underset{\sim}{n}}_1 = d$, where $\hat{\underset{\sim}{n}}_1$ is a unit vector,

$\hat{\underset{\sim}{n}}_1 = \dfrac{\underset{\sim}{n}_1}{|\underset{\sim}{n}_1|} = \dfrac{1}{\sqrt{1+9+4}} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \dfrac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

$\underset{\sim}{r} \cdot \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{-2}{\sqrt{14}} \end{pmatrix} = 1\left(\dfrac{1}{\sqrt{14}}\right)$

$\therefore$ The perpendicular distance from O to Π_1 is $\dfrac{1}{\sqrt{14}}$

6. C: $y = 3x + 9 + m + \frac{3m+12}{x-3}$, m is a constant.

i) $\frac{dy}{dx} = 3 - \frac{3m+12}{(x-3)^2}$

when $\frac{dy}{dx} = 0$:

$\frac{3m+12}{(x-3)^2} = 3$

$3(x^2 - 6x + 9) = 3(m+4)$

$x^2 - 6x + 9 - m - 4 = 0$

$x^2 - 6x + 5 - m = 0$

If C has turning points,

$B^2 - 4AC \geqslant 0$

$(-6)^2 - 4(1)(5-m) \geqslant 0$

$36 - 20 + 4m \geqslant 0$

$4m \geqslant -16$

$m \geqslant -4$

ii) when $m = -6$

$y = 3x + 3 - \frac{6}{x-3}$

Asymptotes: As $x \to 3$ $y \to \pm\infty$

As $x \to \pm\infty$, $y \to 3x + 3$

$\therefore$ The asymptotes are $x = 3$ and $y = 3x + 3$

No turning points since $m = -6 < -4$.

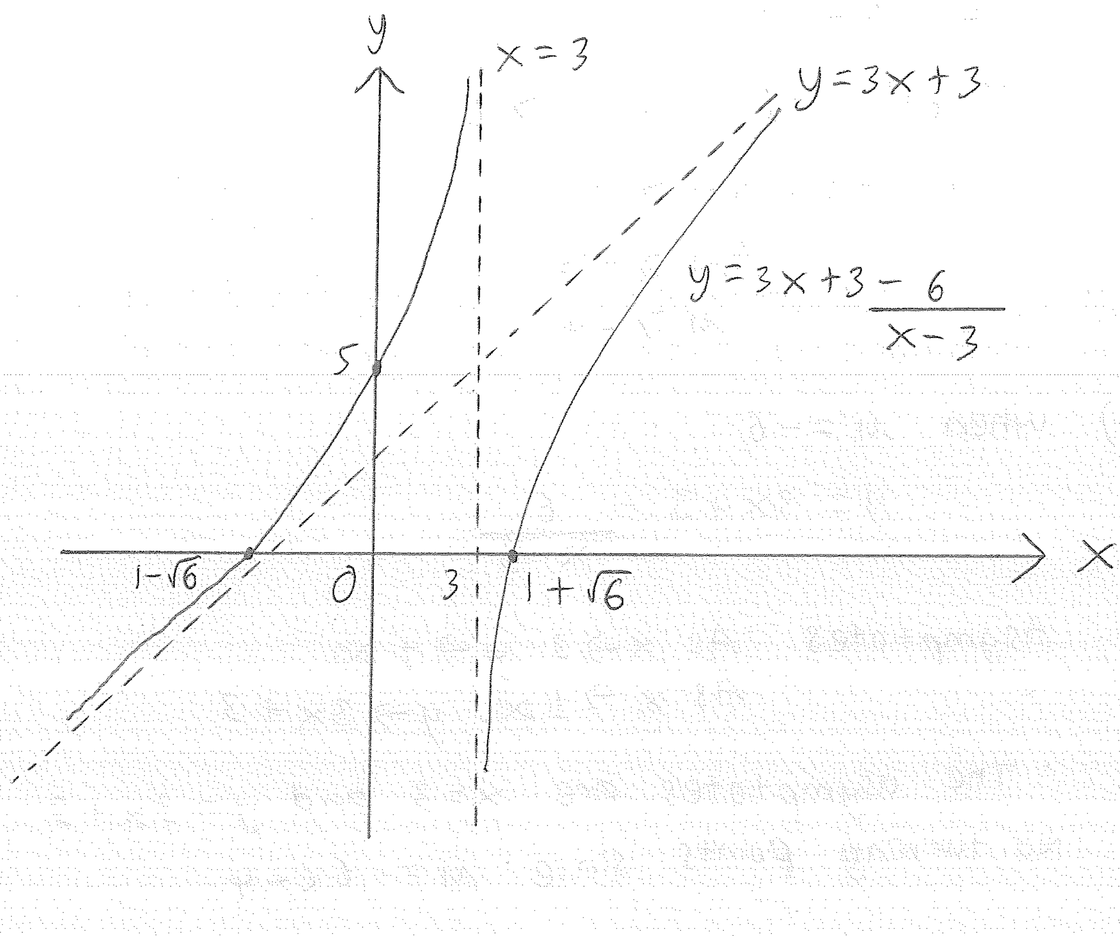
ii) Intersection Points:

when $x=0$: $y = 3 - \frac{6}{-3} = 5$

when $y=0$: $3x+3 = \frac{6}{x-3}$

$$(3x+3)(x-3) = 6$$
$$(x+1)(x-3) = 2$$
$$x^2-2x-3 = 2$$
$$x^2-2x+1 = 6$$
$$(x-1)^2 = 6$$
$$x = 1 \pm \sqrt{6}$$

$\therefore$ The intersection points are $(1-\sqrt{6}, 0)$, $(1+\sqrt{6}, 0)$ and $(0, 5)$.



7. EITHER

i) $2x^4 + px^2 + pq = 0$

Roots : $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = 0$$

$$\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\alpha\delta} + \frac{1}{\beta\gamma} + \frac{1}{\beta\delta} + \frac{1}{\gamma\delta} = \frac{p}{2}$$

$$\frac{1}{\alpha\beta\gamma} + \frac{1}{\alpha\beta\delta} + \frac{1}{\alpha\gamma\delta} + \frac{1}{\beta\gamma\delta} = 0$$

$$\frac{1}{\alpha\beta\gamma\delta} = \frac{pq}{2}$$

Given $S_n = \frac{1}{\alpha^n} + \frac{1}{\beta^n} + \frac{1}{\gamma^n} + \frac{1}{\delta^n}$

$S_0 = \frac{1}{\alpha^0} + \frac{1}{\beta^0} + \frac{1}{\gamma^0} + \frac{1}{\delta^0} = 4$

$S_1 = 0$

a) $S_2 = \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right)^2 - 2\left(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\alpha\delta} + \frac{1}{\beta\gamma} + \frac{1}{\beta\delta} + \frac{1}{\gamma\delta}\right)$

$= 0^2 - 2\left(\frac{p}{2}\right)$

$= -p$

b) To find $S_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$2S_{4+r} + pS_{2+r} + pqS_r = 0$

Let $r = -2$: $2S_2 + pS_0 + pqS_{-2} = 0$

$$2(-p) + p(4) + pq S_{-2} = 0$$

$$S_{-2} = \frac{-2p}{pq} = \frac{-2}{q}$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{-2}{q}.$$

7-ii) $x^3 + 3x^2 - 7x + 1 = 0$

Roots : α, β, γ

New Roots: $\alpha^2 + \alpha,\ \beta^2 + \beta,\ \gamma^2 + \gamma$

Let $u = \alpha^2 + \alpha$

$$\left(\alpha + \frac{1}{2}\right)^2 - \frac{1}{4} = u$$

$$\left(\alpha + \frac{1}{2}\right)^2 = u + \frac{1}{4}$$

$$\alpha + \frac{1}{2} = \pm\sqrt{u + \frac{1}{4}}$$

$$\alpha = -\frac{1}{2} \pm \sqrt{u + \frac{1}{4}}$$

Since α is one of the roots,

$$\alpha^3 + 3\alpha^2 - 7\alpha + 1 = 0$$

$$\left(-\frac{1}{2} \pm \sqrt{u - \frac{1}{4}}\right)^3 + 3\left(-\frac{1}{2} \pm \sqrt{u - \frac{1}{4}}\right)^2 - 7\left(-\frac{1}{2} \pm \sqrt{u - \frac{1}{4}}\right) + 1 = 0$$

$$-\frac{1}{8} + \frac{3}{4}\left(\pm\sqrt{u + \frac{1}{4}}\right) - \frac{3}{2}\left(u + \frac{1}{4}\right) + \left(u + \frac{1}{4}\right)\left(\pm\sqrt{u + \frac{1}{4}}\right)$$

$$+ 3\left(\frac{1}{4} - \left(\pm\sqrt{u + \frac{1}{4}}\right) + u + \frac{1}{4}\right) + \frac{7}{2} - 7\left(\pm\sqrt{u + \frac{1}{4}}\right) + 1 = 0$$

$$-\frac{1}{8}+\frac{3}{4}\left(\pm\sqrt{u+\frac{1}{4}}\right)-\frac{3u}{2}-\frac{3}{8}+\left(u+\frac{1}{4}\right)\left(\pm\sqrt{u+\frac{1}{4}}\right)$$

$$+\frac{3}{4}-3\left(\pm\sqrt{u+\frac{1}{4}}\right)+3u+\frac{3}{4}+\frac{7}{2}-7\left(\pm\sqrt{u+\frac{1}{4}}\right)+1=0$$

$$\frac{11}{2}+\frac{3u}{2}+\left(\pm\sqrt{u+\frac{1}{4}}\right)\left(\frac{3}{4}+u+\frac{1}{4}-3-7\right)=0$$

$$(u-9)\left(\pm\sqrt{u+\frac{1}{4}}\right)=-\frac{11}{2}-\frac{3u}{2}$$

$$(u-9)^2\left(\pm\sqrt{u+\frac{1}{4}}\right)^2=\frac{(3u+11)^2}{4}$$

$$(u^2-18u+81)\left(u+\frac{1}{4}\right)=\frac{9u^2+66u+121}{4}$$

$$\frac{1}{4}(u^2-18u+81)(4u+1)=\frac{1}{4}(9u^2+66u+121)$$

$$4u^3-72u^2+324u+u^2-18u+81=9u^2+66u+121$$

$$4u^3-80u^2+240u-40=0$$

$$u^3-20u^2+60u-10=0$$

$\therefore$ The equation $u^3-20u^2+60u-10=0$ has roots $\alpha^2+\alpha$, $\beta^2+\beta$ and $\gamma^2+\gamma$.

7. OR

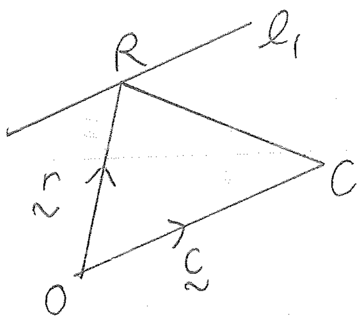
i) $\overrightarrow{OA} = \underset{\sim}{a} \qquad \overrightarrow{OB} = \underset{\sim}{b}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \underset{\sim}{b} - \underset{\sim}{a}$$

$$l_1 : \underset{\sim}{r} = \overrightarrow{OA} + s\overrightarrow{AB}$$
$$= \underset{\sim}{a} + s(\underset{\sim}{b} - \underset{\sim}{a})$$
$$= (1-s)\underset{\sim}{a} + s\underset{\sim}{b}$$

ii) R is on l_1 and $\overrightarrow{OC} = \underset{\sim}{c}$

a)



$$\overrightarrow{CR} = \overrightarrow{OR} - \overrightarrow{OC}$$
$$= \underset{\sim}{r} - \underset{\sim}{c}$$
$$= (1-s)\underset{\sim}{a} + s\underset{\sim}{b} - \underset{\sim}{c}$$

b) $\overrightarrow{CR} \times \overrightarrow{AB} = [(1-s)\underset{\sim}{a} + s\underset{\sim}{b} - \underset{\sim}{c}] \times (\underset{\sim}{b} - \underset{\sim}{a})$

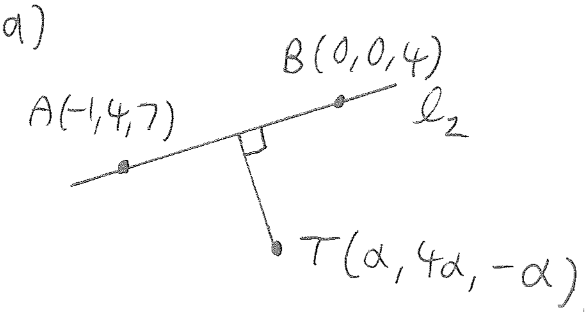
$$= (1-s)(\underset{\sim}{a} \times \underset{\sim}{b}) + s(\underset{\sim}{b} \times \underset{\sim}{b}) - (\underset{\sim}{c} \times \underset{\sim}{b})$$
$$- (1-s)(\underset{\sim}{a} \times \underset{\sim}{a}) - s(\underset{\sim}{b} \times \underset{\sim}{a}) + \underset{\sim}{c} \times \underset{\sim}{a}$$
$$= (1-s)(\underset{\sim}{a} \times \underset{\sim}{b}) + s(\underset{\sim}{a} \times \underset{\sim}{b}) + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}$$
$$= \underset{\sim}{a} \times \underset{\sim}{b} - s(\underset{\sim}{a} \times \underset{\sim}{b}) + s(\underset{\sim}{a} \times \underset{\sim}{b}) + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}$$
$$= \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}$$

c) The shortest distance of the point C from l can be denoted $|\vec{CR}|$ where $\vec{CR} \perp \vec{AB}$, i.e.

$$|\vec{CR} \times \vec{AB}| = |\vec{CR}||\vec{AB}| \sin 90^\circ$$

$$\therefore |\vec{CR}| = \frac{|\vec{CR} \times \vec{AB}|}{|\vec{AB}|} = \frac{|\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}|}{|\underset{\sim}{b} - \underset{\sim}{a}|}$$

iii) a)



$\vec{TR}$ is similar with $\vec{CR}$ in part ii) c).

Hence, the perpendicular distance of point T to line l_2 is $|\vec{TR}| = |\vec{CR}|$.

Let $\underset{\sim}{a} = \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$ $\underset{\sim}{b} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ $\underset{\sim}{c} = \begin{pmatrix} \alpha \\ 4\alpha \\ -\alpha \end{pmatrix}$, $\underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}$

$$|\vec{TR}| = |\vec{CR}|$$

$$= \frac{|\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}|}{|\underset{\sim}{b} - \underset{\sim}{a}|}$$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -1 & 4 & 7 \\ 0 & 0 & 4 \end{vmatrix} = \begin{pmatrix} 16 \\ 4 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a} = (\underset{\sim}{b} - \underset{\sim}{a}) \times \underset{\sim}{c}$$

$$= \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \times \begin{pmatrix} \alpha \\ 4\alpha \\ -\alpha \end{pmatrix}$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -4 & -3 \\ \alpha & 4\alpha & -\alpha \end{vmatrix}$$

$$= \begin{pmatrix} 16\alpha \\ -2\alpha \\ 8\alpha \end{pmatrix}$$

$$|\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}|$$

$$= \sqrt{(16\alpha + 16)^2 + (4 - 2\alpha)^2 + (8\alpha)^2}$$

$$= \sqrt{256\alpha^2 + 512\alpha + 256 + 16 - 16\alpha + 4\alpha^2 + 64\alpha^2}$$

$$= \sqrt{2(162\alpha^2 + 248\alpha + 136)}$$

$$\therefore |\overrightarrow{TR}| = \frac{\sqrt{2(162\alpha^2 + 248\alpha + 136)}}{\sqrt{1 + 16 + 9}}$$

$$= \frac{\sqrt{162\alpha^2 + 248\alpha + 136}}{\sqrt{13}}$$

b) $l_3: x = \frac{y}{4} = \frac{z}{-1}$

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

Since $T(\alpha, 4\alpha, -\alpha)$ is a point on l_3, thus the shortest distance between the skew lines l_2 and l_3 is $|\overrightarrow{TR}|$.

$$13|\overrightarrow{TR}|^2 = 162\alpha^2 + 248\alpha + 136$$

$$13\frac{d|\overrightarrow{TR}|^2}{d\alpha} = 324\alpha + 248$$

$|\overrightarrow{TR}|$ is minimum when $\frac{d|\overrightarrow{TR}|^2}{d\alpha} = 0$

when $\frac{d|\overrightarrow{TR}|^2}{d\alpha} = 0$,

$$324\alpha + 248 = 0$$

$$\alpha = -\frac{62}{81}$$

$$\therefore |\overrightarrow{TR}| = \sqrt{\frac{256}{81}} = \frac{16}{9}$$