15. LINEAR SPACE

1. Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 6 & -1 & -9 \\ 4 & 3 & 1 \end{pmatrix}$.

   Hence, or otherwise, solve the equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$.

   (J72/II/20)

2. Given that A is an invertible (non-singular) 3×3 matrix, show that the first column of A^{-1} is the solution of the equation
$$AX = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

   Give equations whose solutions are the second and third columns of A^{-1} respectively.

   Hence, or otherwise, find the inverse of P, where $P = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$.

   Find the matrix B, given that $BP = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & 5 \end{pmatrix}$. (J73/I/4)

3. Find the inverse of the matrix A, whereA $= \begin{pmatrix} 1 & -3 & -1 \\ 5 & -1 & 5 \\ -3 & 1 & 1 \end{pmatrix}$.

   Solve the system of equation $A\mathbf{x} = \mathbf{b}$ where

   $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and (i) $\mathbf{b} = \begin{pmatrix} 7 \\ 47 \\ 13 \end{pmatrix}$ (ii) $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$. (N73/II/20)

4. Given a real 3×1 column vector X and a real 3×3 matrix A, show that the list of vectors (A^3X, A^2X, AX, X) is linearly dependent.

   In the case $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, $X = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, find numbers a_3, a_2, a_1, a_0, not all zero, such that $a_3 A^3X + a_2 A^2X + a_1AX + a_0X = 0$, where 0 is a zero matrix.
   With these values of a_3, a_2, a_1, a_0, verify that $a_3A^3 + a_2A^2 + a_1A + a_0I = 0$.
   Hence show, without calculating the elements of the inverse explicitly, that $A^{-1} = \frac{1}{3}(A^2 - 6A + 12I)$. (N73/II/14)

5. (a) Prove that the determinant

$$\begin{vmatrix} a & a+b & a+2b \\ a+b & a+2b & a \\ a+2b & a & a+b \end{vmatrix},$$

where $b \neq 0$, is equal to zero if and only if $a = -b$.

(b) Show that

$$\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} = 2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(N73/II/5)

6. Prove that every 2×2 matrix A with real elements, satisfying the equation $AA^T = I$, where I is the 2×2 unit matrix and A^T is the transpose of A, may be written in one of the two forms:

$$M(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad N(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Describe geometrically, giving your reasons, the transformations of vectors in two dimensions given by the matrices $M(0)$, $M(\frac{1}{2}\pi)$, $N(0)$, $N(\frac{1}{2}\pi)$. (J74/I/5)

7. (a) Solve for $\begin{vmatrix} 3-\lambda & -5 & -1 \\ 2 & 2-\lambda & 0 \\ 2 & 5 & 1 \end{vmatrix} = 0.$

(b) Factorise completely $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}.$

(J74/II/5)

8. (a) Using the Gauss elimination method, solve the following equations:
$3x + 4y + z = 7$, $2x - 5y + 3z = -10$, $x + 2y + z = 1$.
Give a geometrical interpretation of your solution.

(b) Show that the following simultaneous equations are not independent and give a geometrical interpretation of the nature of the solutions:
$x + 2y + 5z = 6$, $3x - y + 6z = 2$, $9x - 10y + 9z = -10$. (J74/II/19)

9. A transformation T of the Oxy plane into itself is defined by $T: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix}$

where $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Find the image points corresponding to the points $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and hence or otherwise describe the effect of the transformation on points on the Oxy plane. What would be the geometric effect of applying this transformation twice?

Find the equation of the image of the line $y = 2x + 1$ under T. (N74/I/7)

10. If $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n$ are the 3×3 matrices corresponding to a sequence of elementary row operations which transforms a 3×3 matrix $\mathbf{A}$ to $\mathbf{I}$ (the unit matrix), find the result of operating with these same matrices (in the same order) on $\mathbf{I}$.
Find $\mathbf{A}^{-1}$, where $\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 13 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (N74/II/20)

11. The vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ form a basis for a vector space V, and $\mathbf{a} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \alpha_3\mathbf{b}_3$ is a vector in V, α_1, α_2 and α_3 being scalars. Obtain a necessary and sufficient condition that the vectors $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent.
A vector $\mathbf{x}$ in V is such that $\mathbf{x} = \lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2 + \lambda_3\mathbf{b}_3$, where the scalars λ_1, λ_2 and λ_3 are positive. If $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent and $\mathbf{x}$ is a linear combination of these vectors with non-negative scalar coefficients, show that $\alpha_1 > \mathbf{0}$.
Show, by a counter-example that if $\alpha_1 > 0$, then the coefficients expressing $\mathbf{x}$ as a linear combination of $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ need not be non-negative. (N75/II/14)

12. Show that the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ satisfies the equation $A^2 - 4A - 5I = 0$, where I and O denote the 2×2 identity and zero matrices respectively.
Prove by induction that, for each positive integer n, there are real numbers b_n and c_n such that $A^n = b_nA + c_nI$.
Hence or otherwise find the matrix B, where $B = A^4 - 3A^3 - 7A^2 - 10A - 6I$.
(J76/II/3)

13. Find a basis for the vector space spanned by the vectors $(1, 2, -1), (3, -1, 2), (2, -10, 8), (7, -7, 8)$. What is the dimension of this space? For what value (or values) of a does $(1, 4, a)$ belong to the space? (J76/II/13)

14. A 3×3 matrix of the form $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ is said to be upper triangular.
Find an upper triangular matrix R, with positive diagonal elements, such that
$R^TR = \begin{pmatrix} 4 & -6 & 2 \\ -6 & 10 & -3 \\ 2 & -3 & 10 \end{pmatrix}$, where R^T is the transpose of R.
Determine the inverse of R. (N76/II/4)

15. Find a basis for the vector space spanned by the vectors $(1, 2, -2), (2, 1, 3), (1, -4, 12), (3, -9, 29)$. What is the dimension of this space? For what value (or values) of a does $(4, 3, a)$ belong to this space. (N76/II/13)

16. Denote by Q the set of all rational numbers. Show that the set F of all numbers of the form $a + b\sqrt{2}$, where $a, b \in Q$, forms a field under the usual definitions of addition and multiplication.
F may also be considered as a vector space over the field Q. Find a basis for this vector space, and hence find its dimension. (N76/II/14)

17. A square matrix A is said to be symmetric if $A^{T} = A$ and skew-symmetric if $A^{T} = -A$, where A^{T} is the transpose of the matrix A. For each of the following statements, either prove it, or give a counter-example to show that it is false.
(a) A is a 2×2 non-singular symmetric matrix $\Rightarrow A^{-1}$ is a symmetric matrix.
(b) A is a 2×2 skew-symmetric matrix $=$ det $A \Rightarrow 0$.
(c) A is a 3×3 skew-symmetric matrix $=$ det $A \Rightarrow 0$.
(d) $A - A^{T}$ is a skew-symmetric matrix, where A is any 2×2 matrix.
(e) Any 2×2 matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix. (J77/II/5)

18. If $z = x + iy$ is any complex number, the 2×2 matrix $M(z)$ is defined to be $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

Prove that $M(zw) = M(z)\,M(w) = M(w)\,M(z)$, where $w = u + iv$ is any complex number.
Hence show that $M(z^n) = (M(z))^n$, where n is a positive integer.
Deduce, or prove otherwise, that if

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{then } T^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

Use this result to find six different 2×2 matrices A, with real elements, such that $A^6 = I$. (J77/II/14)

19. (a) A transformation T of the x-y plane into itself is defined by

$$T\colon \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix}, \text{ where } \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Show that all points that are their own image under T lie on a line, and give the equation of this line.
(b) Given that M is a 2×2 matrix such that $M^2 = I$, show that $M = M^{-1}$.

Show also that if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then either

(i) $M = I$ or $-I$; or (ii) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -1$ and $a + d = 0$. (N77/II/1)

20. The vector space V (over the set $\mathbf{R}$ of real numbers) consists of all vectors of the form (a, b, c) where $a, b, c \in \mathbf{R}$. Prove that in V the set of vectors $\{(1, 0, -1), (0, 2, 0), (1, 1, 1)\}$ is both a linearly independent set and a spanning set.
Determine, with reasons, whether each of the following sets of vectors forms a basis for V or not:
(a) $\{(1, 0, -1), (0, 2, 0), (1, 1, 1)\}$,
(b) $\{(1, 0, -1), (0, 2, 0), (0, 0, 0)\}$,
(c) $\{(1, 0, -1), (0, 2, 0), (1, 1, 1), (0, 1, 0)\}$,
(d) $\{(0, 2, 0), (1, 1, 1)\}$. (N77/II/13)

21. The transformation T of the x-y plane is defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

(a) Find the image of the line $y = mx + k$ under T in the case when $(a + bm)$ and $(c + dm)$ are both zero.
(b) Show that, when $(a + bm)$ and $(c + dm)$ are not both zero, the image under T of the line $y = mx + k$ is $(a + bm)y = (c + dm)x + k(ad - bc)$. (J78/I/2)

22. The linear transfromation $\sigma: \mathbf{R}^3 \to \mathbf{R}^3$ is represented by the matrix $\begin{pmatrix} 2 & -1 & 5 \\ 1 & 3 & 6 \\ 0 & 1 & 1 \end{pmatrix}$ with the respect to the standard basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Find a basis for the null space of σ and deduce that the range of σ has dimension 2. Determine a basis for the range.

Find the set of vectors in $\mathbf{R}^3$ which are mapped to $\begin{pmatrix} 10 \\ -2 \\ -2 \end{pmatrix}$ by σ. Is this set of vectors a subspace of $\mathbf{R}^3$? Give reasons for your answer. (J78/I/14)

23. (a) Let A be an $m \times n$ real matrix. Prove that the set of vectors X in $\mathbf{R}^n$ which satisfy the equation $\mathbf{AX} = \mathbf{0}$ forms a subspace of $\mathbf{R}^n$,

(b) Find the rank of the matrix $\mathbf{B}$ where $\mathbf{B} = \begin{pmatrix} 3 & 0 & -1 & 2 \\ -1 & 3 & 1 & -2 \\ 11 & -6 & -5 & 10 \end{pmatrix}$.

Find also a basis for the subspace V of $\mathbf{R}^4$ which is defined by $V = \{\mathbf{X} \in \mathbf{R}^4 : \mathbf{BX} = 0\}$ (J78/I/15)

24. (a) Show that if the equations $-x - y + 3z = 0$, $4x + ay + 6z = 0$, $-3x - y + 3z = 0$ have a solution other than $x = y = z = 0$, then $a = -2$, and find the solution in this case.
(b) Solve the equations $-x - y + 3z = b$, $4x - 2y + 6z = 1$, $-3x - y + 3z = 3$, distinguishing between the cases $b = 2$ and $b \neq 2$. (J78/II/14)

25. (a) Let V and W be subspaces of the linear (vector) space $\mathbf{R}^n$. Show that the set of vectors of the form $\mathbf{v} + \mathbf{w}$, where $\mathbf{v} \in V$ and $\mathbf{w} \in W$, is a subspace of $\mathbf{R}^n$.

(b) For each of the following subsets of $\mathbf{R}^3$ determine whether or not it is a subspace, giving reasons for your answers. Find a basis for each subset which you consider to be a subspace.
(a) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 = 1\}$,
(b) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 = x_2 = x_3\}$,
(c) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : 7x_1 = x_2\}$,
(d) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 = 0\}$, (J78/II/15)

26. (a) Let σ be a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$. Define the *range of* σ and the *null space* of σ. Prove that the range of σ is a subspace of $\mathbf{R}^m$ and that the null space of σ is a subspace of $\mathbf{R}^n$.

(b) Find the rank of the matrix A where $A = \begin{pmatrix} 2 & -1 & 10 \\ -1 & 4 & -19 \\ 1 & 0 & 3 \\ 5 & 2 & 7 \end{pmatrix}$.

Find also a basis for the subspace V of $\mathbf{R}^3$ which is defined by $V = \{\mathbf{X} \in \mathbf{R}^3 : \mathbf{AX} = \mathbf{0}\}$.

(N78/I/14)

27. The matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Prove that, if $a + d \neq 0$, then

$$\mathbf{M} = \frac{1}{a+d}\,[\mathbf{M}^2 + (ad-bc)\mathbf{I}]\,.$$

Prove also that, if $\mathbf{M}^2 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then $p + s = (a+d)^2 - 2(ad - bc)$ and $ps - qr = (ad - bc)^2$.

Hence, or otherwise, find four distinct matrices $\mathbf{M}$ such that $\mathbf{M}^2 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

(N78/II/4)

28. Show that the set of vectors $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$

forms a basis for the linear (vector) space $\mathbf{R}^3$. (N78/II/15)

29. (a) $\mathbf{A}$ is a real 3×3 matrix, $\mathbf{B}$ and X are vectors in $\mathbf{R}^3$. Prove that the equation $\mathrm{AX} = \mathbf{B}$ has a solution if, and only if, $\mathbf{B}$ is in the subspace spanned by the columns of the matrix A.

(b) Find the value of c for which the equations
$x + y + z = c$, $x - 3y + 2z = -1$, $2x + 6y + z = 4$
have a solution, and find the solution in this case.

(N78/II/14)

30. L is a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$ defined by $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y \\ x + z \\ 3x - 2y - z \end{pmatrix}$.

(a) Find the null space of L, and state its dimension.

(b) Show that $L\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, and express $L\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ and $L\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ as linear combinations of the vectors of S. (N78/II/15)

31. Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 1 & -1 & -1 & 4 \end{pmatrix}$$

V and W are subsets of $\mathbf{R}^4$ defined by $V = \{X \in \mathbf{R}^4 : AX = \mathbf{0}\}$ $W = \{X \in \mathbf{R}^4 : AX = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}\}$.

Prove that V is a subspace of $\mathbf{R}^4$ and show that W is not a subspace of $\mathbf{R}^4$.

Find a basis for V, and hence, or otherwise, solve the equation $AX = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$.

(J79/I/14)

32. (a) Show that the set of vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \right\}$ in $\mathbf{R}^3$ is both a linearly independent set and a spanning set of $\mathbf{R}^3$.

(b) For each of the following sets of vectors in $\mathbf{R}^3$, either extend the set to form a basis for $\mathbf{R}^3$, by adding a suitable vector to the set, or give a reason why it cannot be so extended to form a basis.

(i) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$,

(ii) $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$,

(iii) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} \right\}$.

(c) For each of the following sets of vectors in $\mathbf{R}^3$, either select a subset of vectors which will form a basis for $\mathbf{R}^3$, or give a reason why it is not possible to do this.

(i) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$,

(ii) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$,

(iii) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \right\}$.

(J79/II/14)

33. Let V and W be subspaces of $\mathbf{R}^n$. Show that $V \cap W$ is also a subspace of $\mathbf{R}^n$. Find a basis for $V \cap W$ in the following cases:

(a) $V \doteq \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1 \doteq x_3 \doteq 0\}$, $W \doteq \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_4 \doteq 2x_2\}$,

(b) $V \doteq \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1 + 2x_2 \doteq 0\}$, $W \doteq \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : 7x_4 \doteq 3x_2\}$,

(c) $V = \left\{ X \in \mathbf{R}^3 : AX = \mathbf{0}, \text{ where } A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \right\}$,

$W = \left\{ X \in \mathbf{R}^3 : BX = \mathbf{0}, \text{ where } B = \begin{pmatrix} 2 & 1 & -7 \\ -4 & -2 & 14 \end{pmatrix} \right\}$ (J79/II/15)

34. Find non-singular matrices $\mathbf{P}$, $\mathbf{Q}$, such that $\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$, when

(a) $\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 0 & 4 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 6 \\ 2 & 4 & 12 \end{pmatrix}$, (c) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 6 \\ 2 & 4 & 14 \end{pmatrix}$.

State, with reasons, which of the given matrices have inverses, and find each inverse that exists. (N79/I/14)

35. (a) Let V and W be subspaces of the linear (vector) space $\mathbf{R}^4$. The set of vectors of the form $v + w$, where $v \in V$ and $w \in W$ is denoted by $V + W$. Show that $V + W$ is a subspace of $\mathbf{R}^4$.

(b) S is the subspace of $\mathbf{R}^4$ spanned by the vectors $\{(1, 0, 1, 1), (5, 0, 2, 2), (-2, 0, 1, 1), (3, 0, 0, 0)\}$ and T is the subspace of $\mathbf{R}^4$ spanned by the vectors $\{(1, -1, 0, 1), (0, 0, 0, 0), (0, 1, 1, 0), (0, 3, 3, 0)\}$
Give bases for the subspaces S, T, $S \cap T$, $S + T$.
Give also a basis for a subspace U of $\mathbf{R}^4$, which is such that $S \cap U = \{(0, 0, 0, 0)\}$ and $S + U = \mathbf{R}^4$. (N79/I/15)

36. The transformation T of the x-y plane is defined by $T: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Show that T leaves all lines through the origin invariant if and only if $b = c = 0$ and $a = d \neq 0$.
In this case, describe T geometrically. (N79/II/1)

37. Given that $\mathbf{A} = \begin{pmatrix} 2 & a & -3 \\ 3 & 1 & -2 \\ 1 & 4 & 1 \end{pmatrix}$ and that $\mathbf{B}$ is a given vector in $\mathbf{R}^3$, find the value of a for which the equation $AX = \mathbf{B}$ does not have exactly one solution.
Using this value of a, find the solutions, if any, of the equation $\mathbf{AX} = \mathbf{B}$ in the cases (a) $\mathbf{B} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$; (b) $\mathbf{B} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$ (N79/II/14)

38. The linear transformation $\sigma: \mathbf{R}^4 \rightarrow \mathbf{R}^3$ is represented by the matrix $\begin{pmatrix} 1 & -4 & 5 & -2 \\ 0 & 9 & -3 & 3 \\ 4 & 5 & 13 & -1 \end{pmatrix}$

with respect to the standard bases of $\mathbf{R}^4$ and $\mathbf{R}^3$.
Find bases for the null space of σ and for the range (image) space of σ.

Find the set of vectors in $\mathbf{R}^4$ which are mapped to the vector $\begin{pmatrix} 0 \\ 9 \\ 21 \end{pmatrix}$ by σ.
(N79/II/15)

39. The linear transformation $\sigma: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is represented by the matrix $\begin{pmatrix} 2 & 1 & 3 \\ 0 & 3 & -3 \\ -1 & 2 & -4 \end{pmatrix}$

with respect to the standard basis of $\mathbf{R}^3$.

(a) Show that the range of σ has dimension 2.

(b) Show that all points of the line $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

are mapped to the same point by σ, and find the position vector of this point.

(c) Find the subset of $\mathbf{R}^3$ whose image under σ is the point with position vector $\begin{pmatrix} 5 \\ -3 \\ -5 \end{pmatrix}$.
(J80/I/14)

40. The set P_2 consists of all polynomials in x, of degree less than or equal to 2, and having real coefficients, i.e. $P_2 = \{ax^2 + bx + c: a, b, c, \epsilon\, \mathbf{R}\}$.
Show that, with the usual operations of addition and of multiplication by a real number, P_2 is a linear (vector) space over $\mathbf{R}$, of dimension 3.
For each of the following subsets of P_2, determine whether or not it is a subspace, giving brief reasons for your answers. Give a basis for each subset which you consider to be a subspace.

(a) $\{f(x) \in P_2 : f(0) = 0\}$,
(b) $\{f(x) \in P_2 : f(0) = 1\}$,
(c) $\{f(x) \in P_2 : f(1) = 0\}$,
(d) $\{f(x) \in P_2 : f(-x) = f(x) \text{ for all } x \in \mathbf{R}\}$.
(J80/I/15)

41. For each of the following assertions concerning 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$, state whether it is true or false, justifying your answer.

(a) If $A^2 = A$ then A is a singular matrix.
(b) For all A, $A^2 - 5A + 6I = (A - 3I)(A - 2I)$, where I is the unit 2×2 matrix.
(c) For all A and B, $A^2 - 5AB + 6B^2 = (A - 3B)(A - 2B)$.
(d) If A and B are symmetric matrices than AB is also a symmtric matrix. (A symmetric matrix A satisfies $A^T = A$.)
(e) If $A = \begin{pmatrix} \cos\frac{1}{6}\pi & -\sin\frac{1}{6}\pi \\ \sin\frac{1}{6}\pi & \cos\frac{1}{6}\pi \end{pmatrix}$, then $A^{12} = I$.
(J80/II/4)

42. The matrix A is given by $A = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 4 & -1 & -5 & 1 \end{pmatrix}$.

Find a basis of the subspace V of $\mathbf{R}^4$ where $V = \{X : X \in \mathbf{R}^4, AX = \mathbf{0}\}$.
Hence, or otherwise find the solutions, if any, of the equation $AX = \mathbf{K}$ in each of the following cases:

(a) $K = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(b) $K = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

(c) $K = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(J80/II/14)

43. Let A be the matrix $\begin{pmatrix} 4 & 8 & 3 \\ 3 & 5 & 1 \\ 1 & 4 & 3 \end{pmatrix}$.

By performing elementary row operations on the matrix $(\mathbf{A}|\mathbf{I})$ and A^{-1}.
Hence, or otherwise, find the inverses of the following matrices:

(a) $\begin{pmatrix} 4 & 3 & 1 \\ 8 & 5 & 4 \\ 3 & 1 & 3 \end{pmatrix}$, (b) $\begin{pmatrix} 8 & 16 & 6 \\ 3 & 5 & 1 \\ 1 & 4 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} -22 & 24 & 14 \\ 16 & -18 & -10 \\ -14 & 16 & 8 \end{pmatrix}$.

(J80/II/15)

44. It is given that $\mathbf{M}$ is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $b \neq 0$, $c \neq 0$.

(a) Given that $\mathbf{A}$ is a matrix of the form $\begin{pmatrix} \lambda a + \mu & \lambda b \\ x & y \end{pmatrix}$ and is such that $\mathbf{MA} = \mathbf{AM}$, obtain expressions for x and y in terms of λ, μ and the elements of $\mathbf{M}$. Hence express $\mathbf{A}$ in terms of $\mathbf{M}$ and the unit matrix $\mathbf{I}$.

(b) Deduce that any matrix which commutes with $\mathbf{M}$ under the operation of matrix multiplication can be expressed in the form $\alpha\mathbf{M} + \beta\mathbf{I}$.

(c) Prove the converse of (b).

(d) Obtain two singular matrices each of which commutes with $\begin{pmatrix} 4 & 5 \\ 2 & 1 \end{pmatrix}$ under the operation of matrix multiplication, and such tat neither is a scalar multiple of the other.

(N80/I/4)

45. Find the dimension of the row space of the matrix A, where $A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 1 & -2 \\ 7 & -5 & 10 \end{pmatrix}$.

Find the solution, if any, of the equation $\mathbf{AX} = \mathbf{K}$ in the cases:

(a) $\mathbf{K} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (b) $\mathbf{K} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ (c) $\mathbf{K} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$ (N80/I/14)

46. A linear transformation $L: \mathbf{R}^3 \to \mathbf{R}^3$ is defined by $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+z \\ x+z \end{pmatrix}$.

Find bases for the null space of L and for the range of L.
Find also the images under L of

(a) the line with equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(b) the line with equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(c) the plane with equation $\mathbf{r}.\begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} = 0.$ (N80/I/15)

47. The set of vectors $\{e_1, e_2, e_3\}$ is a linearly independent set in $\mathbf{R}^3$, and $\{f_1, f_2\}$ is a linearly independent set in $\mathbf{R}^2$. The linear transformation $L: \mathbf{R}^3 \to \mathbf{R}^2$ is defined by $L(e_1) = f_1 + f_2$, $L(e_2) = f_1 - f_2$, $L(e_3) = 2f_2$.
Find (a) a vector in $\mathbf{R}^3$ which is mapped to f_1 by L,
(b) a basis for the null space of L,
(c) the set of vectors in $\mathbf{R}^3$ which are mapped to f_2 by L. (N80/II/14)

48. Find non-singular matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{PA}$ and $\mathbf{QB}$ are echelon matrices,

where $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} -2 & 0 & 3 \\ 1 & 3 & 1 \\ -1 & 9 & 9 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -2 & 1 \\ 3 & -1 & 5 \\ -1 & 4 & 0 \end{pmatrix}$.

For each of $\mathbf{A}$ and $\mathbf{B}$ find its inverse if it exists.
Find numbers α, β, γ, not all zero, such that
$\alpha(-2, 0, 3) + \beta(1, 3, 1) + \gamma(-1, 9, 9) = (0, 0, 0)$. (N80/II/15)

49. (a) Show that the matrix X, where $X=\begin{pmatrix}-2 & 3\\ 5 & 4\end{pmatrix}$, belongs to the linear space (over **R**) spanned by the matrices A, B, C, where

$$A=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, \qquad B=\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}, \qquad C=\begin{pmatrix}-1 & 0\\ 1 & 2\end{pmatrix}.$$

Find the dimension of this space.

(b) Find a relationship between x, y, z equivalent to the statement that (x, y, z) belongs to the subspace of $\mathbf{R}^3$ spanned by the vectors $\mathbf{a}=(1, 2, -1)$, $\mathbf{b}=(3, -1, 3)$, $\mathbf{c}=(1, -5, 5)$.
Find the dimension of this subspace.

(c) Determine the value of a given that the vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}-1\\3\\0\end{pmatrix}, \begin{pmatrix}a\\1\\2\end{pmatrix}$) do not form a spanning set for $\mathbf{R}^3$. (J81/I/14)

50. Using elementary row operations, reduce the matrix $\begin{pmatrix}1 & 2 & -1 & 1\\ 2 & 5 & a & 3\\ 3 & a & 6 & 4\end{pmatrix}$ to echelon form.
Hence, or otherwise, determine the values of a so that the equation

$$\begin{pmatrix}1 & 2 & -1\\ 2 & 5 & a\\ 3 & a & 6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ 3\\ 4\end{pmatrix}$$

has (a) no solution, (b) more than one solution, (c) a unique solution, and solve the equation completely in case (b). (J81/I/15)

51. Prove that if **A**, **P**, **Q**, are non-singular matrices such that $\mathbf{PAQ}=\mathbf{I}$, then $A^{-1}=$ **QP**. Given that $\mathbf{P}=\begin{pmatrix}1 & 3 & 6\\ 0 & -1 & -2\\ 1 & 4 & 7\end{pmatrix}$ find a non-singular matrix **Q** such that $\mathbf{PAQ}=\begin{pmatrix}\mathbf{I}_r & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}$ in each of the following cases:

(a) $\mathbf{A}=\begin{pmatrix}1 & 3\\ -2 & 1\\ 1 & -1\end{pmatrix}$ (b) $\mathbf{A}=\begin{pmatrix}1 & 3 & 5\\ -2 & 1 & -3\\ 1 & -1 & 1\end{pmatrix}$ (c) $\mathbf{A}=\begin{pmatrix}1 & 3 & 6\\ -2 & 1 & -3\\ 1 & -1 & 1\end{pmatrix}$

State which of the matrices given in (a), (b), (c) have inverses, and find each inverse that exists. (J81/II/14)

52. The linear transformation σ: $\mathbf{R}^3 \to \mathbf{R}^4$ is represented by the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -6 & 3 \\ 6 & -4 & 10 \end{pmatrix}$

with respect to the standard bases of $\mathbf{R}^3$ and $\mathbf{R}^4$. Find bases for the null space of σ and for the range of space of σ.

Obtain the set of vectors in $\mathbf{R}^3$ which are mapped by σ to the vector $\begin{pmatrix} 3 \\ 0 \\ -9 \\ -6 \end{pmatrix}$

(J81/II/15)

53. (a) The matrices $\mathbf{A}$ and $\mathbf{C}$ are given by $\mathbf{A} = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \\ p_4 & q_4 & r_4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.

Prove that $\mathbf{AC}$ is a linear combination of the columns of $\mathbf{A}$.
Given also that $\mathbf{B}$ is a 3×3 matrix, prove that the column space of $\mathbf{AB}$ is a subspace of the column space of $\mathbf{A}$.

(b) Determine which of the following matrices have the same column space, and jusfity your answers:

$$\mathbf{P} = \begin{pmatrix} 1 & 2 & -1 \\ -3 & -5 & 1 \\ 13 & 23 & -7 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & -1 & 6 \end{pmatrix},$$

$$\mathbf{R} = \begin{pmatrix} 1 & 3 \\ -2 & -5 \\ 10 & 27 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ -2 & -7 & -15 \end{pmatrix}.$$

(N81/I/14)

54. Show that the vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}$, where $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, form a basis for the linear space $\mathbf{R}^3$.

Given that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha\mathbf{p} + \beta\mathbf{q} + \gamma\mathbf{r}$, find a matrix $\mathbf{Q}$ such that $\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \mathbf{Q}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Let L be the linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$ defined by L: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \mathrm{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

where A is the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & -4 & 1 \\ 3 & 1 & 0 \end{pmatrix}$. Show that the image of $\mathbf{p}$ under this transformation is $-8\mathbf{p} - 6\mathbf{q} + 10\mathbf{r}$, and find (also in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$) the images of $\mathbf{q}$ and $\mathbf{r}$ under the transformation.
Given that the vector $\mathbf{u}$ is such that the image of $\mathbf{u}$ under the transformation L is $\mathbf{p}$, express $\mathbf{u}$ as a linear combination of $\mathbf{p}, \mathbf{q}, \mathbf{r}$. (N81/I/15)

55. The real matrix A, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, satisfies the equation $A^2 - pA + qI = O$, where p, q are real constants, I is the 2×2 units matrix and O is the 2×2 zero matrix.

(a) Given that $bc \neq 0$, obtain the values of p and q in terms of the elements of A, and show that if the matrix $A - xI$ is singular then x satisfies $x^2 - px + q = 0$.

(b) Obtain two distinct matrices A with $a, b, c, d, \in \mathbf{N}$, such that $A^2 - 3A + I = O$.

(N81/II/3)

56. For each of the following assertions state whether it is true or false, and justify your answer.

(a) The vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ are linearly independent.

(b) The set of vectors S spans a given subspace $\Leftrightarrow S$ is a basis for the subspace.

(c) The vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are linearly independent $\Leftrightarrow x_1y_2 \neq x_2y_1$.

(d) The vectors X_1, X_2, X_3 span $\mathbf{R}^3 \Leftrightarrow X_1, + X_2, X_2 - X_3, X_1 + X_2 + X_3$ span $\mathbf{R}^3$.

(e) Given that $X_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $X_2 = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$, the only vectors of the form $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$ that are linear combinations of X_1 and X_2 are $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(N81/II/14)

57. The matrix A is given by $A = \begin{pmatrix} 8 & 5 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

By performing elementary row operations on the matrix $(A|I)$ find A^{-1}.

(a) The linear transformation $L: \mathbf{R}^3 \to \mathbf{R}^3$ is defined by $L: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Find the element in $\mathbf{R}^3$ whose image under L is $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) Solve the equation $XA = K$, where $K = (1, -1, 2)$.

(N81/II/15)

58. The linear transformation $\sigma: \mathbf{R}^4 \to \mathbf{R}^3$ is represented by the matrix $A = \begin{pmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 4 & 5 & 5 & 8 \end{pmatrix}$ with respect to the standard bases in $\mathbf{R}^4$ and $\mathbf{R}^3$. Find a basis for the null space of σ, and deduce that the range of σ has dimension 2.
Find a basis for the range.

Solve the equation $AX = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix}$, and state, with a reason, whether the set of solutions forms a subspace of $\mathbf{R}^4$.

(J82/I/14)

59. Show that, under suitable operations, which should be stated, the set of all polynomials of the form $ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbf{R}$, forms a linear space. State the dimension of this space.
Show that the set of polynomials $\{(x+1)^3, (x+1)^2, (x+1), 1\}$ forms a basis for this space, and express the polynomial x^3 in terms of this basis.
Determine the value of k, given that the polynomial $(x-1)^3 + k$ belongs to the subspace spanned by $\{(x+1)^3, (x+1)^2, (x+1)\}$.
The mapping L is defined by $L(ax^3 + bx^2 + cx + d) = ax^2 + cx + c$.
Give a basis for the null space of L. (J82/I/15)

60. (a) The matrices A and B are such that AB = BA. Show that $(A+B)^3 = A^3 + 3A^2B + 3AB^3 + B^3$.
(b) Let C be a given matrix, where $C \neq O$. Find two different matrices P satisfying both the equations $CP = PC$ and $P^2 - PC - 6C^2 = 0$.
(c) Givent hat $D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ show that all the solutions of the equation $QD = DQ$ are of the form $Q = \alpha I + \beta D$ where α and β are scalars. (J82/II/3)

61. Show that $\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix}\right\}$ is a basis for $\mathbf{R}^2$, and express $\begin{pmatrix} x \\ y \end{pmatrix}$ in the form $\alpha\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, where α, β are to be found in terms of x, y.
Let $L: \mathbf{R}^3 \to \mathbf{R}^2$ be the linear mapping defined by

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y - z \\ 2x - y + z \end{pmatrix}$$

Find the images under L of the vectors $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence, or otherwise, find the matrix of L with respect to the bases

$$\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ and } \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix}\right\}$$

of $\mathbf{R}^3$ and $\mathbf{R}^2$ respectively. (J82/II/14)

62. The matrix A is given by $A = \begin{pmatrix} 8 & 3 & -12 \\ -5 & -2 & 8 \\ 10 & 4 & -15 \end{pmatrix}$.

By performing elementary row operations on the matrix $(\mathbf{A}|\mathbf{I})$, find $\mathbf{A}^{-1}$.

(a) Solve the equation $\mathbf{AX} = \mathbf{K}$, where $\mathbf{K} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

(b) By multiplying both sides of the equation by $\mathbf{A}^{-1}$, or otherwise, solve the equation

$$\begin{pmatrix} 8 & 3 & -12 & 2 \\ -5 & -2 & 8 & 3 \\ 10 & 4 & -15 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

(J82/II/15)

63. (a) Let $\mathbf{A}$ be the matrix $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}$.

Write down matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ such that

$$\mathbf{P}_1\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 + a_1 & b_2 + a_2 & b_3 + a_3 & b_4 + a_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix};$$

$$\mathbf{P}_2\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 + a_1 & b_2 + a_2 & b_3 + a_3 & b_4 + a_4 \\ c_1 + ka_1 & c_2 + ka_2 & c_3 + ka_3 & c_4 + ka_4 \end{pmatrix}.$$

(b) Given that $\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & -3 & -6 \\ 4 & -9 & -13 & -25 \end{pmatrix}$, find a matrix $\mathbf{P}$ such that

$\mathbf{PA} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & x & y \\ 0 & 0 & s & t \end{pmatrix}$, where x, y, s, t are to be determined.

Solve the equation $\mathbf{AX} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$.

(N82/I/14)

64. In each of the following cases, state whether, with the usual operations, the given set forms a linear space over the field $\mathbf{R}$. For each of those which you consider to be a linear space, give a basis for the space, and for each of those which you consider not to be a linear space, justify your answer.

(a) $S_1 = \left\{\text{solutions of } \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0\right\}$. (It may be assumed that the general solution of the differential equation is $y = Ae^{2x} + Be^{x}$.)

(b) $S_2 = \left\{\text{solutions of } \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x - 3\right\}$. (It may be assumed that the general solution of the differential equation is $y = Ae^{2x} + Be^{x} + x$.)

(c) $S_3 = \{(x, y): y = 2x + 1\}$.

(d) $S_4 = \{(x, y, z): x + y + z = 0\}$.

(e) $S_5 = \{z: z \in \mathbf{C}\}$.

(N82/I/15)

65. (a) The matrix M is given by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a + d = -1$ and $ad - bc = 1$.

Show that $M^2 = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and deduce that $\mathbf{M}^2 + \mathbf{M} + \mathbf{I} = \mathbf{O}$.

(b) A transformation is given by $X \to AX$, where A is a 2×2 matrix and $X = \begin{pmatrix} x \\ y \end{pmatrix}$
The points (1, 0) and (0, 1) are mapped to (1, 0) and (p, 1) respectively. Write down the matrix A.
A second transformation is given by $X \to BX$ and is an anticlockwise rotation about the origin through an obtuse angle θ, where $\cos\theta = -\frac{4}{5}$. Write down the matrix B.
Given that the matrix M in (a) above is such that $M = BA$, find the value of p.
(N82/II/3)

66. The matrix Q is such that $\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 2 \\ 2 & 3 & -2 \\ 2 & 4 & -2 \end{pmatrix} Q = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \\ 2 & 0 & 0 \end{pmatrix}$.

Find, in each of the following cases, a non-singular matrix $\mathbf{P}$ such that $\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$, where A is given by

(a) $A = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$,

(b) $A = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 2 \end{pmatrix}$,

(c) $A = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 2 \\ 2 & 3 & -2 \end{pmatrix}$,

(d) $A = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 2 \\ 2 & 4 & -2 \end{pmatrix}$.

In each case state the rank of $\mathbf{A}$, and find the inverse of $\mathbf{A}$ when it exists.
(N82/II/14)

67. Given the matrix A, where $A = \begin{pmatrix} 3 & 0 & -2 & 3 \\ -4 & 2 & 0 & -8 \\ 1 & -2 & 2 & 5 \\ 3 & -3 & 2 & 9 \end{pmatrix}$,
find, in any order,
(a) the rank of A;
(b) a basis for the column-space of A;
(c) a basis for the subspace V defined by $V = \{X \in \mathbf{R}^4 : AX = \mathbf{0}\}$.

Find the general solution of the equation $AX = \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}$,

and state, justifying your answer, whether the set of all solutions of this equation forms a subspace of $\mathbf{R}^4$.
(J83/I/14)

71. A linear transformation $L: \mathbf{R}^3 \to \mathbf{R}^3$ is defined by $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Find a basis for the range of L and a basis for the null space of L.

(a) Find the image under L of the line $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$

(b) Find, in the form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$, an equation for the line whose image under L is point $\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$

(c) Show that the image under L of the plane with equation $x - y + z = 2$ is the plane $x - y + z = 0$. (N83/II/14)

72. The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 6 & 3 & 2 \\ 3 & 2 & 1 \\ 8 & 4 & 3 \end{pmatrix}$.

By performing elementary row operations on the matrix $(\mathbf{A}|\mathbf{I})$ find $\mathbf{A}^{-1}$.

Hence, or otherwise, obtain $\mathbf{B}^{-1}$, where $\mathbf{B} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{8} \end{pmatrix}$.

Given that the real numbers x_1, x_2, x_3 satisfy the equation $\mathbf{B}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$,

show that the solution of the equation: $\mathbf{BX} = \begin{pmatrix} c_1 + \delta \\ c_2 - \delta \\ c_3 - \delta \end{pmatrix}$ is $\mathbf{X} = \begin{pmatrix} x_1 + 42\delta \\ x_2 - 18\delta \\ x_3 - 96\delta \end{pmatrix}$.

(N83/II/15)

73. Using elementary row operations, reduce the matrix $\begin{pmatrix} 1 & -2 & 2 & 2 \\ 4 & -7 & a & 5 \\ 3 & a & -7 & 3 \end{pmatrix}$ to echelon form.

Hence, or otherwise, determine the values of a so that the equation

$$\begin{pmatrix} 1 & -2 & 2 \\ 4 & -7 & a \\ 3 & a & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

has (a) no solution, (b) more than one solution, (c) a unique solution.
Solve the equation in case (b). (J84/I/14)

68. Given the matrix $\mathbf{A}$, where $\mathbf{A}=\begin{pmatrix}1 & 1 & -2 & 3 & -2\\ 1 & 4 & -2 & -3 & -5\\ 2 & -1 & -4 & 12 & -1\end{pmatrix}$, find a matrix $\mathbf{P}$ such that $\mathbf{PA}$ is in echelon form with the first non-zero element in each row equal to 1.

Find the solution space S of the equation $\mathbf{A}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$.

State the dimension of S, and give a basis for S.

Given that the equation $\mathbf{A}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix}=\begin{pmatrix}a\\ 8\\ -2\end{pmatrix}$ has solutions, find the value of a and, with this value of a, solve the equation. (J83/II/14)

69. The subspaces V and W of $\mathbf{R}^3$ are spanned by the sets

$$\left\{\begin{pmatrix}1\\ 3\\ -2\end{pmatrix},\begin{pmatrix}2\\ 0\\ 5\end{pmatrix},\begin{pmatrix}3\\ 5\\ 0\end{pmatrix}\right\} \text{ and } \left\{\begin{pmatrix}1\\ 3\\ -3\end{pmatrix},\begin{pmatrix}2\\ 4\\ -3\end{pmatrix},\begin{pmatrix}2\\ 0\\ 3\end{pmatrix}\right\} \text{ respectively.}$$

(a) Find the dimension of V and of W.

(b) Give that $\begin{pmatrix}x\\ y\\ z\end{pmatrix}\in V$, obtain a linear relationship between x, y and z.

(c) Find a 1×3 matrix A such that $\{X: AX=0\}=W$.

(d) Find a basis for the subspace $V\cap W$. (J83/II/15)

70. Find the inverse of the matrix $\begin{pmatrix}0 & 1 & 2\\ 2 & 0 & 1\\ 1 & 2 & 0\end{pmatrix}$.

Hence, or otherwise, show that the set S, where $S=\left\{\begin{pmatrix}0\\ 2\\ 1\end{pmatrix},\begin{pmatrix}1\\ 0\\ 2\end{pmatrix},\begin{pmatrix}2\\ 1\\ 0\end{pmatrix}\right\}$, is a basis for the linear space $\mathbf{R}^3$.

Given that L is a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$ defined by

$$L\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}2x+z\\ x-2y\\ 3x+2y+2z\end{pmatrix},$$

(a) find the null space of L, and state its dimension;

(b) show that $L\begin{pmatrix}0\\ 2\\ 1\end{pmatrix}=-\frac{4}{3}\begin{pmatrix}0\\ 2\\ 1\end{pmatrix}+\frac{11}{3}\begin{pmatrix}1\\ 0\\ 2\end{pmatrix}-\frac{4}{3}\begin{pmatrix}2\\ 1\\ 0\end{pmatrix}$,

and express $L\begin{pmatrix}1\\ 0\\ 2\end{pmatrix}$ and $L\begin{pmatrix}2\\ 1\\ 0\end{pmatrix}$ as linear combinations of the vectors of S. (N83/I/14)

74. (a) Given that $(a\ b\ c)$ belongs to the row space of the matrix $\begin{pmatrix} 3 & 2 & 1 \\ -2 & -2 & 1 \\ 1 & -2 & 7 \end{pmatrix}$,

find a linear relation that must be satisfied by a, b, c.

(b) Given that **P** and **Q** are 3×3 matrices,

(i) prove that the column space of **PQ** is a subspace of the column space of **P**,

(ii) state a similar result concerning the row space of **PQ**,

(iii) deduce that rank(**PQ**) cannot exceed the smaller of rank(**P**) and rank(**Q**).

(J84/I/15)

75. (a) Given that $\left\{ \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}, \begin{pmatrix} b \\ 1 \\ a \end{pmatrix}, \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \right\}$

is not a basis for $\mathbf{R}^3$, prove that $a^3 - 3ab + b^3 + 1 = 0$,

(b) Express $\begin{pmatrix} x \\ y \end{pmatrix}$ in the form $\alpha\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \beta\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, where α, β are to be given in terms of x, y.

Let L: $\mathbf{R}^2 \to \mathbf{R}^2$ be the linear mapping defined by $\mathrm{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 3y \\ 2x - y \end{pmatrix}$.

Find the image under L of the vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Hence, or otherwise, find the matrix of L with respect to the basis

$$\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$$

(J84/II/14)

76. Find the dimension of the row space of the matrix A, where $\mathrm{A} = \begin{pmatrix} 4 & -2 & 1 \\ 3 & 1 & -2 \\ 6 & -8 & 7 \end{pmatrix}$.

Find the solutions, if any, of the equation $\mathbf{AX} = \mathbf{K}$ in the cases

(a) $\mathbf{K} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, (b) $\mathbf{K} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, (c) $\mathbf{K} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$.

The linear transformation L: $\mathbf{R}^3 \to \mathbf{R}^3$ is given by $\mathrm{L(X)} = \mathrm{AX}$.
Give a basis for the range space of L.

(J84/II/15)

77. (a) Let S be the set of all real 2×2 matrices, and let **M** be $\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$. The transformation L: $S \to S$ is given by $\mathrm{L(A)} = \mathrm{AM} - \mathrm{MA}$ where $\mathrm{A} \in \mathrm{S}$. Find bases for the null space and the range space of L.
Obtain the set of matrices in S which are mapped by L to the matrix $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

(b) The linear transformation σ: $\mathbf{R}^2 \to \mathbf{R}^2$ is such that $\sigma\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and

$\sigma\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -11 \end{pmatrix}$ Find $\sigma\begin{pmatrix} x \\ y \end{pmatrix}$.

(N84/I/14)

78. (a) Find the dimension of the column space of the matrix $\mathbf{L}$, where

$$\mathbf{L} = \begin{pmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & -7 & -2 \\ -1 & -3 & 5 & -5 \\ 2 & 5 & -9 & 6 \end{pmatrix},$$

and give a basis for the column space. Find also a basis for the null space of the transformation $\mathbf{R}^4 \to \mathbf{R}^4$ given by $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \mathbf{L}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$.

(b) For each of the following subsets of $\mathbf{R}^3$ determine whether it is a subspace, giving reasons for your answers. Find a basis for each subset that you consider to be a subspace.

(i) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon x_1 + x_2 + x_3 = 0\}$,
(ii) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon x_1 \geqslant 0\}$,
(iii) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon x_1^2 = x_2^2\}$. (N84/I/15)

79. The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} -5 & 4 & 3 \\ 10 & -7 & -6 \\ -8 & 6 & 5 \end{pmatrix}$

By performing elementary row operations on the matrix $(\mathbf{A}|\mathbf{I})$, find $\mathbf{A}^{-1}$.

(a) Solve the equation $\mathbf{XA} = \mathbf{K}$, where $\mathbf{K} = (-1 \;\; 2 \;\; 3)$.

(b) Solve, by multiplying both sides of the equation by $\mathbf{A}^{-1}$, or otherwise, the equation $(x \;\; y \;\; z \;\; t) \begin{pmatrix} -1 & 2 & 3 \\ -5 & 4 & 3 \\ 10 & -7 & -6 \\ -8 & 6 & 5 \end{pmatrix} = (2 \;\; -2 \;\; 1)$.

(N84/II/14)

80. (a) Obtain a basis for the solution space of the equations
$2x + 7y + z = 0,\ x - 4y - 2z = 0,\ 3x - 3y - 3z = 0.$

(b) Let V be the linear space spanned by the polynomials
$x^3 + 3x^2 - 3,\ 2x^3 + 4x^2 - x - 4,\ 3x^3 - 5x^2 - 7x + 5,\ x^3 - x^2 - 2x + 1.$
Obtain the dimension of V and a basis for V.

(c) Prove that the vectors $(3 + 2\sqrt{2}, \sqrt{2} + 1)$ and $(1, \sqrt{2} - 1)$ in $\mathbf{R}^2$ are linearly dependent over $\mathbf{R}$. (N84/II/15)

81. The matrix $\mathrm{M}(z)$, where $z = x + \mathrm{i}y$, is defined by $\mathrm{M}(z) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$.

Prove that, for any complex numbers z_1 and z_2,

(a) $\mathrm{M}(z_1 z_2) = \mathrm{M}(z_1)\,\mathrm{M}(z_2)$,
(b) $\mathrm{M}(z_1^3) = (\mathrm{M}(z_1))^3$.

Hence, or otherwise, find the distinct 2×2 matrices A such that $\mathrm{A}^3 = \mathbf{I}$, where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find also $\mathbf{B}^7$, where $\mathbf{B} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. (J85/I/5)

82. Given that the matrix A is singular, where $A = \begin{pmatrix} 1 & 2 & 1 \\ a & -1 & -11 \\ -2 & a & 12 \end{pmatrix}$

find the possible values of a.

For each of these values, determine whether the equation $\mathbf{AX} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

has (a) no solution,
(b) a unique solution,
(c) more than one solution.

Obtain the general solution of the given equation for each value of a for which (c) is true. (J85/I/14)

83. Show that the set S of vectors given by $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

forms a basis for the linear space $\mathbf{R}^4$.

The linear transformation L: $\mathbf{R}^3 \to \mathbf{R}^4$ is defined by $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - z \\ x + z \\ x + y \\ 2x + y + z \end{pmatrix}$

Find the null space of L, and state its dimension.

Show that $L\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, and express $L\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $L\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ each as

a linear combinations of the vectors of S. (J85/I/15)

84. The linear space V consists of all real 2×2 matrices under the operations of matrix addition and multiplication by a scalar. The mapping L: $V \to V$ is defined

by L: $A \to AM - MA$, where $M = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$

Find the dimension of the null space of L, and give a basis for this null space.

Find also a basis for the range of L. Given that L: $X \to \begin{pmatrix} 6 & -6 \\ 6 & -6 \end{pmatrix}$

show that X is of the form $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} + \lambda P + \mu Q$, ($\lambda \in \mathbf{R}$, $\mu \in \mathbf{R}$), where P, Q are 2×2 matrices which are to be found. (J85/II/14)

85. The matrix A is given by $A = \begin{pmatrix} -15 & 2 & 7 \\ 5 & 0 & -2 \\ -24 & 3 & 11 \end{pmatrix}$

By performing elementary row operations on the matrix $(\mathbf{A}|\mathbf{I})$, find $\mathbf{A}^{-1}$.

(a) Solve the equation $\mathbf{XA} = \mathbf{K}$, where $\mathbf{K} = (-2\ \ 1\ \ 3)$.

(b) By multiplying both sides of the equation by A^{-1}, or otherwise, solve the equation $(x\ \ y\ \ z\ \ w)\begin{pmatrix} -15 & 2 & 7 \\ 5 & 0 & -2 \\ -24 & 3 & 11 \\ -2 & 1 & 3 \end{pmatrix} = (1\ \ 2\ \ 1)$.

(J85/II/15)

86. The matrix $\mathbf{P}$ is such that $\mathbf{P}\begin{pmatrix} 1 & 1 & 2 & 2 \\ -1 & 0 & -1 & -2 \\ 2 & 3 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

In each of the following cases find a non-singular matrix $\mathbf{Q}$ such that $\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$, where A is given by

(a) $A = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, (b) $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 3 \end{pmatrix}$,

(c) $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ 2 & 3 & 4 \end{pmatrix}$ (d) $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -2 \\ 2 & 3 & 4 \end{pmatrix}$.

In cases (c) and (d) find A^{-1} if it exists. (N85/I/14)

87. The linear transformation $\sigma: \mathbf{R}^3 \to \mathbf{R}^3$ is represented by the matrix $\begin{pmatrix} 2 & 1 & 4 \\ -1 & 3 & -9 \\ 3 & 1 & 7 \end{pmatrix}$ with respect to the standard basis of $\mathbf{R}^3$.

(a) Show that the range of σ has dimension 2, and state the nullity of σ.

(b) Given that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in the range of σ, obtain integers a, b, c, not all zero, such that

$ax + by + cz = 0$.

(c) Find the subset P of $\mathbf{R}^3$ whose image under σ is the line $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 5 \end{pmatrix}$

Show that P is a plane and give its equation in the form $kx + ly + mz = n$, where k, l, m, n are integers. (N85/I/15)

88. Given matrices **A** and **B**, where $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \\ 3 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 12 \\ 3 & 1 & 0 \end{pmatrix}$,

find non-singular matrices **P** and **Q** such that **PA** and **QB** are echelon matrices in which the first non-zero entry in any row is unity.
For each of **A** and **B** find its inverse, if it exists.

For each of (i) and (ii) below, solve, if possible, the equation:

(a) $\mathbf{AX} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$, (b) $\mathbf{BX} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$ (N85/II/14)

89. Show that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ may be expressed in the form $\alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, giving α, β, γ in terms of x, y, z. Deduce, or show otherwise, that $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right\}$ is a basis for $\mathbf{R}^3$.

Given that L: $\mathbf{R}^2 \to \mathbf{R}^3$ is the linear mapping defined by $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-2y \\ 3x-y \end{pmatrix}$

find the images under L of the vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Find α, β, γ such that $L\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and express $L\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ in a similar form.
Hence, or otherwise, express each of $L\begin{pmatrix} 4 \\ 6 \end{pmatrix}$, $L\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ and $L\begin{pmatrix} 9 \\ 13 \end{pmatrix}$ as linear combinations of the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. (N85/II/15)

90. The linear transformation T:$R^4 \to R^3$ is defined by

T: $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to A\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$, where $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 3 & 0 & 1 & 2 \end{pmatrix}$.

(i) Find, in either order, the rank of A and the dimension of the null space, K, of T.

(ii) Find the basis vectors e_1 and e_2 of K such the e_1 is of the form $\begin{pmatrix} 1 \\ 0 \\ a \\ b \end{pmatrix}$ and e_2 is of the form $\begin{pmatrix} 0 \\ 1 \\ c \\ d \end{pmatrix}$.

(iii) Write down two vectors e_3 and e_4 such that the set $\{_1, e_2, e_3, e_4\}$ forms a basis for R^4, and show clearly that your four vectors are linearly independent. (J86/I/9)

91. (i) Show that the vectors **a**, **b**, **c** given by

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix}, \text{ are linearly independent.}$$

(ii) The linear transformation $T: \mathrm{R}^3 \to \mathrm{R}^3$ is represented by the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 4 & 2 & -1 \end{pmatrix}.$$

Find the dimension of the range of T.

(iii) Determine whether or not the vectors **Ma**, **Mb**, **Mc** are linearly independent, justifying your conclusion.

(iv) Determine whether or not the vector $\begin{pmatrix} -2 \\ 23 \\ 17 \end{pmatrix}$ belongs to the range of T.

(N86/I/9)

92. The linear transformation $T: \mathrm{R}^4 \to \mathrm{R}^4$ is represented by the matrix

$$\begin{pmatrix} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 1 \\ 1 & -3 & -1 & 7 \\ 2 & 1 & 5 & 4 \end{pmatrix}.$$ The range space of T is denoted by V, and the set of elements of R^4 which do not belong to V is denoted by W.

(i) Determine the dimension of V.

(ii) Find a basis for V and justify your choice.

(iii) Determine whether W is a vector space and give a reason for your conclusion.

(iv) Show that if the vector $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ belongs to W then $8x - 5y - z - t \neq 0$.

(J87/I/9)

93. The linear transformation $T: \mathrm{R}^4 \to \mathrm{R}^4$ is represented by the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 & 4 \\ 1 & -2 & 3 & 5 \\ 2 & -4 & 5 & 9 \\ 3 & -6 & 7 & 13 \end{pmatrix}.$$

(i) Determine the rank of **A** and show that the dimension of the null space, K, of T is 2.

(ii) Find a basis for K.

The vectors $\begin{pmatrix} -1 \\ 1 \\ -2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 6 \\ 11 \\ 16 \end{pmatrix}$ are denoted by $\mathbf{e}_0$ and $\mathbf{b}$ respectively. Show that, if $\{\mathbf{e}_1, \mathbf{e}_2\}$ is *any* basis for K, then $\mathbf{e}_0 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$, where $\lambda, \mu \in R$, is a solution of the equation $\mathbf{Ax} = \mathbf{b}$.
Hence show that there is a solution of the equation $\mathbf{Ax} = \mathbf{b}$ of the form $\begin{pmatrix} 100 \\ 200 \\ r \\ s \end{pmatrix}$ where r and s are to be found. (N87/I/9)

94. The linear transformation $T: \mathbf{R}^4 \to \mathbf{R}^4$ is represented by the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 3 & 9 & 8 & 17 \\ 1 & 7 & 4 & 6 \\ 3 & 6 & 7 & 16 \end{pmatrix}$.

(i) Show that the dimension R_T, the range space of T, is 2.

(ii) Write down two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, with integer components, such that the set $\{\mathbf{e}_1, \mathbf{e}_2\}$ forms a basis for R_T.

(iii) Show that the vector $\begin{pmatrix} -1 \\ 15 \\ 17 \\ 6 \end{pmatrix}$ belongs to R_T.

(iv) Find a vector x such that $\mathbf{Mx} = \begin{pmatrix} -1 \\ 15 \\ 17 \\ 6 \end{pmatrix}$. (J88/I/9)

95. The linear transformation $T: \mathbf{R}^4 \to \mathbf{R}^4$ is defined by $T: \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to A\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 4 & -5 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 6 & \theta \end{pmatrix}$.

(i) Show that when $\theta \neq -4$, the dimension of the null space, K, of T is 1, and that when $\theta = -4$, the dimension of this space is 2.

(ii) When $\theta \neq -4$, determine a basis vector $\mathbf{e}_1$ for K of the form $\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$, where x, y and z are integers.

When $\theta = -4$, determine a vector $\mathbf{e}_2$ of the form $\begin{pmatrix} x' \\ y' \\ 0 \\ t' \end{pmatrix}$, where x', y' and t' are integers, such that $\{\mathbf{e}_1, \mathbf{e}_2\}$ forms a set of basis vectors for K.

(iii) Given that $\mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ 5 \\ 10 \end{pmatrix}$, $\mathbf{e}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, show that when $\theta = -4$, the vector $\mathbf{x}_1$, where $\mathbf{x}_1 = \mathbf{e}_0 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$, is a solution of the equation $\mathbf{Ax} = \mathbf{b}$ for all real values of λ and μ. (N88/I/9)

96. The linear transformation $T: \mathbf{R}^4 \to \mathbf{R}^4$ is defined by $T: \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \mathbf{A}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$, where $\mathbf{A} = \begin{pmatrix} a & 1 & 1 & 2 \\ a & 1 & 2 & 3 \\ 2a & b+2 & 1 & 3 \\ 3a & 2b+3 & 0 & 3 \end{pmatrix}$, $(a, b \in \mathbf{R}, a \neq 0)$.

The null space of T is denoted by K.

(i) Show that when $b \neq 0$ the dimension of K is 1, and that when $b = 0$ the dimension of K is 2.

(ii) For the case $b \neq 0$, find, in terms of a only, a basis vector $\mathbf{e}_1$ for K.

(iii) For the case $b = 0$, find a vector $\mathbf{e}_2$ such that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for K.

(iv) Hence show that if $b = 0$, then the vector $\begin{pmatrix} a+\theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix}$, where $\theta \in \mathbf{R}$, belongs to K for all values of θ, but if $b \neq 0$, then this vector belongs to K for only one value of θ. (J89/I/9)

97. The linear transformation $T: \mathbf{R}^4 \to \mathbf{R}^4$ is represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 0 & -1 & 7 \\ 3 & -1 & -2 & 10 \\ -1 & 5 & 3 & -1 \end{pmatrix}$.

(i) Find the rank of $\mathbf{M}$, and hence show that the dimension of the null space, K, of T is 2.

(ii) Show that there is a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for K, such that $\mathbf{e}_1 = \begin{pmatrix} p \\ q \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} r \\ s \\ 1 \\ -1 \end{pmatrix}$, where p, q, r and s are integers which are to be found.

(iii) Show that every vector of the form $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$ has the same image under T.

(iv) Hence, or otherwise, find the vector $\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix}$, such that $\mathbf{Mx} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}$, $\alpha + \beta + \gamma = 11$ and α, β, γ are integers. (N89/I/9)

98. The linear transformation T: $\mathbf{R}^4 \to \mathbf{R}^4$ is represented by the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 & 6 \\ -1 & -1 & -2 & -4 \\ 2 & 3 & 6 & 11 \\ 3 & 8 & 10 & 21 \end{pmatrix}$.

(i) Show that the dimension of R, the range space of T is 3.

(ii) Find a set of basis vectors, with integer components, for R and justify your answer.

(iii) Show that the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}$ belongs to R.

(iv) Determine whether there are any values of θ, other than zero, for which the vector $\begin{pmatrix} 1+\theta \\ 2 \\ 3 \\ 5 \end{pmatrix}$ belongs to R. (J90/I/9)

99. The linear transformation T_1: $\mathbf{R}^4 \to \mathbf{R}^4$, T_2: $\mathbf{R}^4 \to \mathbf{R}^4$ and T_3: $\mathbf{R}^4 \to \mathbf{R}^4$ are represented by the matrices $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_2\mathbf{M}_1$ respectively, where $\mathbf{M}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -3 & 3 & 5 \\ 1 & 4 & 2 & 2 \end{pmatrix}$, $\mathbf{M}_2 = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & -4 & 3 \\ 3 & -3 & -2 & 2 \\ 5 & 4 & -6 & 5 \end{pmatrix}$.

(i) Find a basis for R_1, the range space of T_1.

(ii) Find a basis for K_2, the null space of T_2.

(iii) Show that K_2 is a subspace of R_1 and hence, or otherwise, find a non-zero vector in the null space of T_3. (N90/I/9)