OCTOBER / NOVEMBER 2003

1. $C: \quad r = \theta^{\frac{1}{2}} e^{\frac{\theta^2}{\pi}}, \quad 0 \leq \theta \leq \pi$

If the area of the finite region bounded by C and the line $\theta = \beta$ is π,

$$\pi = \int_0^{\beta} \frac{r^2}{2} \, d\theta$$

$$= \int_0^{\beta} \frac{\theta e^{\frac{2\theta^2}{\pi}}}{2} \, d\theta$$

$u = 2\theta^2$

$du = 4\theta \, d\theta$

$\theta = 0 \quad u = 0$

$\theta = \beta \quad u = 2\beta^2$

$$= \int_0^{2\beta^2} \frac{e^{\frac{u}{\pi}}}{4(2)} \, du$$

$$= \left[\frac{\pi e^{\frac{u}{\pi}}}{8} \right]_0^{2\beta^2}$$

$$= \frac{\pi}{8} \left(e^{\frac{2\beta^2}{\pi}} - 1 \right)$$

$$e^{\frac{2\beta^2}{\pi}} - 1 = 8$$

$$e^{\frac{2\beta^2}{\pi}} = 9$$

$$\frac{2\beta^2}{\pi} = \ln 9$$

$$= 2 \ln 3$$

$$\beta^2 = \pi \ln 3$$

$$\beta = (\pi \ln 3)^{\frac{1}{2}}$$

2. $$u_n = \frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1}$$

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1}$$

$$= \frac{1}{1} - \frac{1}{3}$$

$$+ \frac{1}{3} - \frac{1}{7}$$

$$+ \frac{1}{7} - \frac{1}{13}$$

$$\vdots$$

$$+ \frac{1}{N^2 - 5N + 7} - \frac{1}{N^2 - 3N + 3}$$

$$+ \frac{1}{N^2 - 3N + 3} - \frac{1}{N^2 - N + 1}$$

$$+ \frac{1}{N^2 - N + 1} - \frac{1}{N^2 + N + 1}$$

$$= 1 - \frac{1}{N^2 + N + 1}$$

$$S_N = \sum_{n=N+1}^{2N} u_n$$

$$= \sum_{n=1}^{2N} u_n - \sum_{n=1}^{N} u_n$$

$$= 1 - \frac{1}{4N^2 + 2N + 1} - \left(1 - \frac{1}{N^2 + N + 1}\right)$$

$$= \frac{1}{N^2 + N + 1} - \frac{1}{4N^2 + 2N + 1}$$

$$S_N = \frac{1}{N^2 + N + 1} - \frac{1}{4N^2 + 2N + 1}$$

$$< \frac{1}{N^2 + N + 1}$$

$$< \frac{1}{N^2}$$

If $M = 10^{10}$

and $N > M$

$$N^2 > 10^{20}$$

$$\frac{1}{N^2} < 10^{-20}$$

$\therefore$ $S_N < 10^{-20}$ for all $N > M$

3. If $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3$ are linearly dependent, there exist k_1, k_2, k_3 not all zero such that $k_1\underset{\sim}{x}_1 + k_2\underset{\sim}{x}_2 + k_3\underset{\sim}{x}_3 = \underset{\sim}{0}$.

$$M(k_1\underset{\sim}{x}_1 + k_2\underset{\sim}{x}_2 + k_3\underset{\sim}{x}_3) = M\underset{\sim}{0}$$

$$M(k_1\underset{\sim}{x}_1) + M(k_2\underset{\sim}{x}_2) + M(k_3\underset{\sim}{x}_3) = \underset{\sim}{0}$$

$$k_1(M\underset{\sim}{x}_1) + k_2(M\underset{\sim}{x}_2) + k_3(M\underset{\sim}{x}_3) = \underset{\sim}{0}$$

Since k_1, k_2, k_3 are not all zero, the vectors $M\underset{\sim}{x}_1, M\underset{\sim}{x}_2, M\underset{\sim}{x}_3$ are also linearly dependent.

$$\underset{\sim}{y}_1 = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \quad \underset{\sim}{y}_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \quad \underset{\sim}{y}_3 = \begin{pmatrix} 5 \\ 51 \\ 55 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -7 \end{pmatrix}$$

i) if $k_1\underset{\sim}{y}_1 + k_2\underset{\sim}{y}_2 + k_3\underset{\sim}{y}_3 = \underset{\sim}{0}$,

$$k_1\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} + k_2\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + k_3\begin{pmatrix} 5 \\ 51 \\ 55 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} k_1 + 2k_2 + 5k_3 \\ 5k_1 - 3k_2 + 51k_3 \\ 7k_1 + 4k_2 + 55k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 5 \\ 5 & -3 & 51 \\ 7 & 4 & 55 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 5 & -3 & 51 & 0 \\ 7 & 4 & 55 & 0 \end{array}\right)$$

$$\xrightarrow[-7r_1 + r_3]{-5r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -13 & 26 & 0 \\ 0 & -10 & 20 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-13},\ \frac{r_3}{-10}} \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $k_3 = s,\ s \in R$

$k_2 = 2s$

$k_1 + 2(2s) + 5s = 0$

$k_1 + 4s + 5s = 0$

$k_1 = -9s$

$\therefore\ \underset{\sim}{y}_1, \underset{\sim}{y}_2, \underset{\sim}{y}_3$ are linearly dependent.

ii) $P\underset{\sim}{y}_1 = \begin{pmatrix} 2 \\ 45 \\ -49 \end{pmatrix}$ $\quad P\underset{\sim}{y}_2 = \begin{pmatrix} 26 \\ 14 \\ -28 \end{pmatrix}$ $\quad P\underset{\sim}{y}_3 = \begin{pmatrix} -34 \\ 377 \\ -385 \end{pmatrix}$

$\therefore$ A basis for the linear space spanned by $P\underset{\sim}{y}_1, P\underset{\sim}{y}_2, P\underset{\sim}{y}_3$ is $\left\{ \begin{pmatrix} 2 \\ 45 \\ -49 \end{pmatrix}, \begin{pmatrix} 26 \\ 14 \\ -28 \end{pmatrix} \right\}$.

4. $y = x\sin x$

$$\frac{dy}{dx} = x\cos x + \sin x$$

$$\frac{d^2y}{dx^2} = \cos x - x\sin x + \cos x$$

$$= 2\cos x - x\sin x$$

$$\frac{d^3y}{dx^3} = -2\sin x - \sin x - x\cos x$$

$$= -3\sin x - x\cos x$$

$$\frac{d^4y}{dx^4} = -3\cos x - \cos x + x\sin x$$

$$= x\sin x - 4\cos x$$

$$= 2\cos x - \frac{d^2y}{dx^2} - 4\cos x$$

$$= -2\cos x - \frac{d^2y}{dx^2}$$

$$\frac{d^5y}{dx^5} = 2\sin x - \frac{d^3y}{dx^3}$$

$$\frac{d^6y}{dx^6} = 2\cos x - \frac{d^4y}{dx^4}$$

$$= 2\cos x + 4\cos x - x\sin x$$

$$= -x\sin x + 6\cos x$$

$$\frac{d^{2n}y}{dx^{2n}} = (-1)^n x\sin x + 2(-1)^{n+1} n\cos x$$

when $n = 1$:

$$\frac{d^{2(1)}y}{dx^{2(1)}} = \frac{d^2y}{dx^2}$$

$$= -x\sin x + 2\cos x$$

$$= (-1)^1 x\sin x + 2(-1)^{1+1} 1\cos x$$

Assume the statement is true when $n = k$.

$n = k$: $$\frac{d^{2k}y}{dx^{2k}} = (-1)^k x\sin x + 2(-1)^{k+1} k\cos x$$

when $n = k+1$:

$$\frac{d^{2(k+1)}y}{dx^{2(k+1)}} = (-1)^{k+1} x\sin x + 2(-1)^{k+2}(k+1)\cos x$$

$$\frac{d^{2k}y}{dx^{2k}} = (-1)^k x\sin x + 2(-1)^{k+1} k\cos x$$

$$\frac{d}{dx}\left(\frac{d^{2k}y}{dx^{2k}}\right) = \frac{d}{dx}\left((-1)^k x\sin x + 2(-1)^{k+1} k\cos x\right)$$

$$\frac{d^{2k+1}y}{dx^{2k+1}} = (-1)^k \frac{d}{dx}(x\sin x) + 2(-1)^{k+1} k \frac{d}{dx}(\cos x)$$

$$= (-1)^k x\cos x + (-1)^k \sin x - 2(-1)^{k+1} k\sin x$$

$$\frac{d}{dx}\left(\frac{d^{2k+1}y}{dx^{2k+1}}\right) = \frac{d}{dx}\left((-1)^k x\cos x + (-1)^k \sin x - 2(-1)^{k+1} k\sin x\right)$$

$$\frac{d^{2k+2}y}{dx^{2k+2}} = (-1)^k \cos x - (-1)^k x\sin x + (-1)^k \cos x - 2(-1)^{k+1} k\cos x)$$

$$= 2(-1)^k \cos x + (-1)^{k+1} x\sin x + 2(-1)^{k+2} k\cos x$$

$$= 2(-1)^{k+2} k\cos x + 2(-1)^{k+2} \cos x + (-1)^{k+1} x\sin x$$

$$= 2(-1)^{k+2} (k+1)\cos x + (-1)^{k+1} x\sin x$$

$$\frac{d^{2n}y}{dx^{2n}} = (-1)^n x\sin x + 2(-1)^{n+1} n\cos x$$

for every positive integer n.

5. $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \; dx$

$$\frac{d}{dx}(\tan x \sec^n x) = \sec^2 x \sec^n x + n \tan x \sec^{n-1} x \sec x \tan x$$

$$= \sec^{n+2} x + n \tan^2 x \sec^n x$$

$$= \sec^{n+2} x + n(\sec^2 x - 1)\sec^n x$$

$$= \sec^{n+2} x + n \sec^n x \sec^2 x - n \sec^n x$$

$$= \sec^{n+2} x + n \sec^{n+2} x - n \sec^n x$$

$$= (n+1)\sec^{n+2} x - n \sec^n x$$

$$\tan x \sec^n x = \int (n+1)\sec^{n+2} x \; dx - \int n \sec^n x \; dx$$

$$\left[\tan x \sec^n x\right]_0^{\frac{\pi}{4}} = \int_0^{\frac{\pi}{4}} (n+1)\sec^{n+2} x \; dx - \int_0^{\frac{\pi}{4}} n \sec^n x \; dx$$

$$\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^n} - 0 = (n+1)\int_0^{\frac{\pi}{4}} \sec^{n+2} x \; dx - n \int_0^{\frac{\pi}{4}} \sec^n x \; dx$$

$$2^{\frac{n}{2}} = (n+1) I_{n+2} - n I_n$$

$$(n+1) I_{n+2} = 2^{\frac{n}{2}} + n I_n$$

$n=4.$ $5 I_6 = 2^2 + 4 I_4$

$$3 I_4 = 2^1 + 2 I_2$$

$$I_2 = \int_0^{\frac{\pi}{4}} \sec^2 x \; dx$$

$$= \left[\tan x\right]_0^{\frac{\pi}{4}}$$

$$= 1 - 0$$

$$= 1$$

$$3I_4 = 2 + 2$$

$$= 4$$

$$I_4 = \frac{4}{3}$$

$$5I_6 = 4 + 4\left(\frac{4}{3}\right)$$

$$= 4 + \frac{16}{3}$$

$$= \frac{28}{3}$$

$$I_6 = \frac{28}{15}$$

6. $x^3 + x + 12 = 0$

α, β, γ are the roots.

$\therefore \alpha + \beta + \gamma = 0 \quad \alpha\beta + \alpha\gamma + \beta\gamma = 1 \quad \alpha\beta\gamma = -12$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$
$$= 0^2 - 2(1)$$
$$= 0 - 2$$
$$= -2 < 0$$

Since $\alpha^2 + \beta^2 + \gamma^2 < 0$ and the equation $x^3 + x + 12 = 0$ has real coefficients, only one of the roots is real.

If $\beta = p + qi$ and $\gamma = p - qi$

$$\therefore \beta\gamma = p^2 + q^2$$
$$= |\beta|^2$$
$$= |\gamma|^2$$

Let $f(x) = x^3 + x + 12$

$$f(-3) = -27 - 3 + 12 = -18 < 0$$
$$f(-2) = -8 - 2 + 12 = 2 > 0$$

$$\therefore -3 < \alpha < -2$$

$$\frac{-1}{2} < \frac{1}{\alpha} < \frac{-1}{3}$$

Since $\alpha = \dfrac{-12}{\beta\gamma}$,

$$-\frac{1}{2} < -\frac{\beta r}{12} < -\frac{1}{3}$$

$$\frac{1}{3} < \frac{\beta r}{12} < \frac{1}{2}$$

$$4 < \beta r < 6$$

$$2 < \sqrt{\beta r} < \sqrt{6}$$

$$2 < |\beta| = |r| < \sqrt{6}$$

If $-3 < \alpha < -2$, the modulus of each of the other roots lies between 2 and $\sqrt{6}$

7. $$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-\alpha t}, \quad \alpha \neq 2$$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$$

$$m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0$$

$$m = -2$$

The complementary function, y_c, is

$$y_c = (At + B)e^{-2t}$$

The particular integral, y_p, is given by

$$y_p = Ce^{-\alpha t} \quad \text{since } \alpha \neq -2.$$

$$\frac{dy_p}{dt} = -\alpha Ce^{-\alpha t}$$

$$\frac{d^2y_p}{dt^2} = \alpha^2 Ce^{-\alpha t}$$

$$\frac{d^2y_p}{dt^2} + 4\frac{dy_p}{dt} + 4y_p = \alpha^2Ce^{-\alpha t} - 4\alpha Ce^{-\alpha t} + 4Ce^{-\alpha t}$$

$$= (\alpha^2 - 4\alpha + 4)Ce^{-\alpha t}$$

$$= e^{-\alpha t}$$

$$(\alpha^2 - 4\alpha + 4)C = 1$$

$$C = \frac{1}{\alpha^2 - 4\alpha + 4}$$

$$= \frac{1}{(\alpha - 2)^2}$$

$$y_p = \frac{e^{-\alpha t}}{(\alpha - 2)^2}$$

$$y = y_c + y_p$$

$$= (At + B)e^{-2t} + \frac{e^{-\alpha t}}{(\alpha - 2)^2}$$

The general solution of the differential equation is $y = (At + B)e^{-2t} + \dfrac{e^{-\alpha t}}{(\alpha - 2)^2}$.

$$ye^{\alpha t} = (At + B)e^{(\alpha - 2)t} + \frac{1}{(\alpha - 2)^2}$$

if $\alpha < 2$,

$$\lim_{t \to \infty} ye^{\alpha t} = \lim_{t \to \infty} \left((At + B)e^{(\alpha - 2)t} + \frac{1}{(\alpha - 2)^2} \right)$$

$$= \lim_{t \to \infty} (At + B)e^{(\alpha - 2)t} + \frac{1}{(\alpha - 2)^2}$$

$$= 0 + \frac{1}{(\alpha - 2)^2}$$

$$= \frac{1}{(\alpha - 2)^2}$$

If $\alpha < 2$, then $ye^{\alpha t} \longrightarrow \dfrac{1}{(2 - \alpha)^2}$ as $t \to \infty$

8. $z = e^{i\theta}$

$= \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$z^n + \frac{1}{z^n} = 2\cos n\theta, \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$

$n=1$: $z + \frac{1}{z} = 2\cos\theta, \quad z - \frac{1}{z} = 2i\sin\theta$

$(2i\sin\theta)^6 = \left(z - \frac{1}{z}\right)^6$

$-64\sin^6\theta = z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$

$= z^6 + \frac{1}{z^6} - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$

$= 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$

$\sin^6\theta = \frac{-\cos 6\theta}{32} + \frac{3\cos 4\theta}{16} - \frac{15\cos 2\theta}{32} + \frac{5}{16}$

$= p\cos 6\theta + q\sin 4\theta + r\sin 2\theta + s,$

$p = \frac{-1}{32}, \quad q = \frac{3}{16}, \quad r = \frac{-15}{32}, \quad s = \frac{5}{16}$

The mean value of $\sin^6\theta$ over the interval

$0 \leq \theta \leq \frac{\pi}{4}$ is $\dfrac{1}{\frac{\pi}{4} - 0} \displaystyle\int_0^{\frac{\pi}{4}} \sin^6\theta \, d\theta$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{-\cos 6\theta}{32} + \frac{3\cos 4\theta}{16} - \frac{15\cos 2\theta}{32} + \frac{5}{16} \, d\theta$$

$$= \frac{1}{8\pi} \int_0^{\frac{\pi}{4}} -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10 \, d\theta$$

$$= \frac{1}{8\pi} \left[\frac{-\sin 6\theta}{6} + \frac{3\sin 4\theta}{2} - \frac{15\sin 2\theta}{2} + 10\theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8\pi} \left(\frac{1}{6} - \frac{15}{2} + \frac{5\pi}{2} - 0 \right)$$

$$= \frac{1}{8\pi} \left(\frac{5\pi}{2} - \frac{22}{3} \right)$$

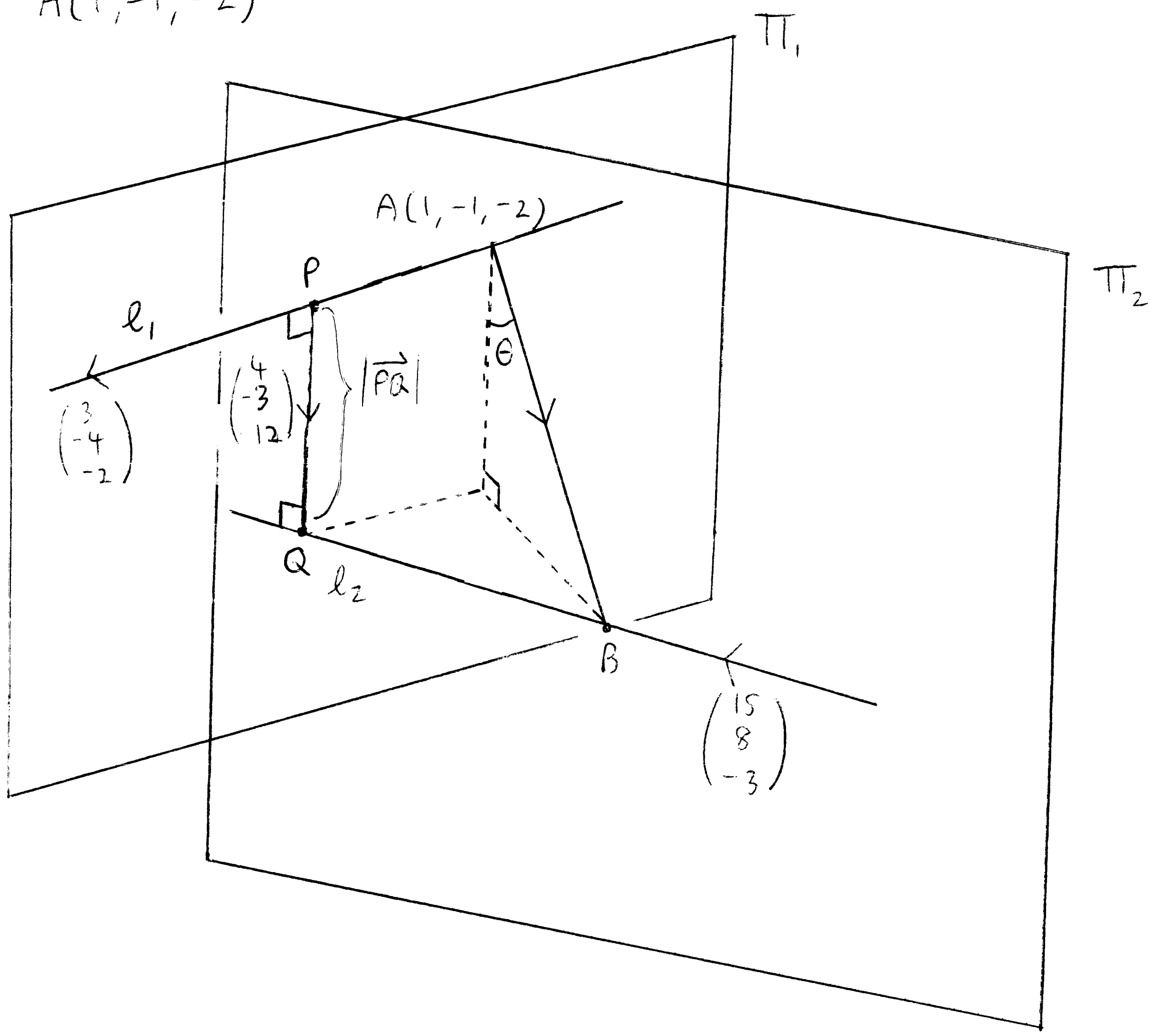
$$= \frac{5}{16} - \frac{11}{12\pi}$$

9. $l_1 \quad \underset{\sim}{r} = \underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k} + s(3\underset{\sim}{i} - 4\underset{\sim}{j} - 2\underset{\sim}{k})$

$l_2 \quad \underset{\sim}{r} = (1 + 5\cos t)\underset{\sim}{i} - (1 + 5\sin t)\underset{\sim}{j} - 14\underset{\sim}{k} + t(15\underset{\sim}{i} + 8\underset{\sim}{j} - 3\underset{\sim}{k}),$

$0 \leq t \leq 2\pi$

A(1, -1, -2)



i) Let B denote the point $(1 + 5\cos t, -(1 + 5\sin t), -14)$

Since PQ is perpendicular to both l_1 and l_2,

the direction of $\overrightarrow{PQ}$ is parallel to $\begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 15 \\ 8 \\ -3 \end{pmatrix}$.

$$\begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 15 \\ 8 \\ -3 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -4 & -2 \\ 15 & 8 & -3 \end{vmatrix} = 28\underset{\sim}{i} - 21\underset{\sim}{j} + 84\underset{\sim}{k}$$

$$= 7(4\underset{\sim}{i} - 3\underset{\sim}{j} + 12\underset{\sim}{k})$$

$$\overrightarrow{AB} \cdot \overrightarrow{PQ} = |\overrightarrow{AB}|\,|\overrightarrow{PQ}| \cos\theta$$

$$\left| \left[\begin{pmatrix} 1 + 5\cos t \\ -(1 + 5\sin t) \\ -14 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right] \cdot \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \right| = |\overrightarrow{AB}| \left| \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \right| \cos\theta$$

$$\left| \begin{pmatrix} 5\cos t \\ -5\sin t \\ -12 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \right| = |\overrightarrow{AB}| \sqrt{169} \cos\theta$$

$$|20\cos t + 15\sin t - 144| = 13|\overrightarrow{AB}|\cos\theta$$

$$|\overrightarrow{AB}|\cos\theta = \frac{|20\cos t + 15\sin t - 144|}{13}$$

since $|\overrightarrow{PQ}| = |\overrightarrow{AB}|\cos\theta$,

$$|\overrightarrow{PQ}| = \frac{|20\cos t + 15\sin t - 144|}{13}$$

ii) since $20\cos t + 15\sin t = 25\sin(t + \tan^{-1}\frac{4}{3})$

$$|\overrightarrow{PQ}| = \frac{\left|25\sin(t + \tan^{-1}\frac{4}{3}) - 144\right|}{13}$$

$$-25 \leqslant 25\sin\left(t + \tan^{-1}\frac{4}{3}\right) \leqslant 25$$

$$-169 \leqslant 25\sin\left(t + \tan^{-1}\frac{4}{3}\right) - 144 \leqslant -119$$

$$119 \leqslant \left|25\sin\left(t + \tan^{-1}\frac{4}{3}\right) - 144\right| \leqslant 169$$

$$119 \leqslant 13\,|\overrightarrow{PQ}| \leqslant 169$$

$$\frac{119}{13} \leqslant |\overrightarrow{PQ}| \leqslant 13$$

Since the minimum distanie between the lines l_1 and l_2 is $\frac{119}{13}$, the lines do not intersect.

The maximum length of PQ as t varies is 13.

iii) Since Π_1 contains l_1 and PQ, the normal of Π_1 is perpendicular to both l_1 and PQ.

$$\begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -4 & -2 \\ 4 & -3 & 12 \end{vmatrix} = \begin{pmatrix} -54 \\ -44 \\ 7 \end{pmatrix}$$

Also, since Π_2 contains l_2 and PQ, the normal of Π_2 is perpendicular to both l_2 and PQ.

$$\begin{pmatrix} 15 \\ 8 \\ -3 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 15 & 8 & -3 \\ 4 & -3 & 12 \end{vmatrix} = \begin{pmatrix} 87 \\ -192 \\ -77 \end{pmatrix}$$

Since $\begin{pmatrix} -54 \\ -44 \\ 7 \end{pmatrix}$ is a vector perpendicular to Π_1

and $\begin{pmatrix} 87 \\ -192 \\ -77 \end{pmatrix}$ is a vector perpendicular to Π_2,

if θ is the angle between $\begin{pmatrix} -54 \\ -44 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 87 \\ -192 \\ -77 \end{pmatrix}$

$$\begin{pmatrix}-54\\-44\\7\end{pmatrix}\cdot\begin{pmatrix}87\\-192\\-77\end{pmatrix} = \left|\begin{pmatrix}-54\\-44\\7\end{pmatrix}\right|\left|\begin{pmatrix}87\\-192\\-77\end{pmatrix}\right|\cos\theta$$

$$-4698 + 8448 - 539 = \sqrt{4901}\sqrt{50362}\cos\theta$$

$$\cos\theta = \frac{3211}{\sqrt{4901}\sqrt{50362}}$$

$$= 0.204$$

$$\theta = 78.2^\circ$$

10. $A = \begin{pmatrix} 6 & 4 & 1 \\ -6 & -1 & 3 \\ 8 & 8 & 4 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 6 & 4 & 1 \\ -6 & -1 & 3 \\ 8 & 8 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 - \lambda & 4 & 1 \\ -6 & -1 - \lambda & 3 \\ 8 & 8 & 4 - \lambda \end{pmatrix}$$

$$|A - \lambda I| = (6 - \lambda)\left[(-1-\lambda)(4-\lambda) - 24\right]$$

$$-4(-6(4-\lambda) - 24) + 1(-48 - 8(-1 - \lambda))$$

$$= (6 - \lambda)(-4 + \lambda - 4\lambda + \lambda^2 - 24)$$

$$-4(-24 + 6\lambda - 24) \quad -48 + 8 + 8\lambda$$

$$= (6 - \lambda)(\lambda^2 - 3\lambda - 28) - 4(6\lambda - 48) + 8\lambda - 40$$

$$= 6\lambda^2 - 18\lambda - 168 - \lambda^3 + 3\lambda^2 + 28\lambda - 24\lambda + 192$$

$$+ 8\lambda - 40$$

$$= -\lambda^3 + 9\lambda^2 - 6\lambda - 16$$

$$\begin{array}{r|l} & -\lambda^2 + 10\lambda - 16 \\ \hline \lambda + 1 & -\lambda^3 + 9\lambda^2 - 6\lambda - 16 \\ & \underline{-\lambda^3 - \lambda^2} \\ & 10\lambda^2 - 6\lambda \\ & \underline{10\lambda^2 + 10\lambda} \\ & -16\lambda - 16 \\ & \underline{-16\lambda - 16} \\ & 0 \end{array}$$

$$= (\lambda + 1)(-\lambda^2 + 10\lambda - 16)$$

$$= -(\lambda+1)(\lambda-2)(\lambda-8)$$

when $|A - \lambda I| = 0$,

$$-(\lambda+1)(\lambda-2)(\lambda-8) = 0$$

$$\lambda = 2, 8, -1$$

when $\lambda = 2$: $\begin{pmatrix} 4 & 4 & 1 \\ -6 & -3 & 3 \\ 8 & 8 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 4 & 4 & 1 & 0 \\ -6 & -3 & 3 & 0 \\ 8 & 8 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-3},\ \frac{r_3}{2}} \left(\begin{array}{ccc|c} 4 & 4 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 4 & 4 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \longleftrightarrow r_2} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 4 & 4 & 1 & 0 \\ 4 & 4 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{-2r_1 + r_2} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 4s$, $s \in R$

$y = -6s$

$2x - 6s - 4s = 0$

$x = 5s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5s \\ -6s \\ 4s \end{pmatrix}$$

$$= s\begin{pmatrix} 5 \\ -6 \\ 4 \end{pmatrix}$$

when $\lambda = 8$: $\begin{pmatrix} -2 & 4 & 1 \\ -6 & -9 & 3 \\ 8 & 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -2 & 4 & 1 & 0 \\ -6 & -9 & 3 & 0 \\ 8 & 8 & -4 & 0 \end{array}\right)$$

$$\xrightarrow[4r_1 + r_3]{-3r_1 + r_2} \left(\begin{array}{ccc|c} -2 & 4 & 1 & 0 \\ 0 & -21 & 0 & 0 \\ 0 & 24 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-21},\ \frac{r_3}{24}} \left(\begin{array}{ccc|c} -2 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} -2 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 0$

let $z = 2s$, $s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 2s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

when $\lambda = -1$: $\begin{pmatrix} 7 & 4 & 1 \\ -6 & 0 & 3 \\ 8 & 8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 7 & 4 & 1 & 0 \\ -6 & 0 & 3 & 0 \\ 8 & 8 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-3}} \left(\begin{array}{ccc|c} 7 & 4 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 8 & 8 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 7 & 4 & 1 & 0 \\ 8 & 8 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 \times 2} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 14 & 8 & 2 & 0 \\ 8 & 8 & 5 & 0 \end{array}\right)$$

$$\xrightarrow[-4r_1 + r_3]{-7r_1 + r_2} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 8 & 9 & 0 \\ 0 & 8 & 9 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 8s,\ s \in \mathbb{R}$

$y = -9s$

$x = 4s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4s \\ -9s \\ 8s \end{pmatrix}$$

$$= s \begin{pmatrix} 4 \\ -9 \\ 8 \end{pmatrix}$$

The eigenvalues of A are $2, 8, -1$ with corresponding eigenvectors $\begin{pmatrix} 5 \\ -6 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -9 \\ 8 \end{pmatrix}$.

If $\underset{\sim}{e}$ is an eigenvector of A with corresponding eigenvalue λ,

$$
\begin{aligned}
A^2\underset{\sim}{e} &= A(A\underset{\sim}{e}) \\
&= A(\lambda\underset{\sim}{e}) \\
&= \lambda(A\underset{\sim}{e}) \\
&= \lambda(\lambda\underset{\sim}{e}) \\
&= \lambda^2\underset{\sim}{e}
\end{aligned}
$$

and
$$
\begin{aligned}
A^3\underset{\sim}{e} &= A(A^2\underset{\sim}{e}) \\
&= A(\lambda^2\underset{\sim}{e}) \\
&= \lambda^2(A\underset{\sim}{e}) \\
&= \lambda^2(\lambda\underset{\sim}{e}) \\
&= \lambda^3\underset{\sim}{e}.
\end{aligned}
$$

$$
\begin{aligned}
(A + A^2 + A^3)\underset{\sim}{e} &= A\underset{\sim}{e} + A^2\underset{\sim}{e} + A^3\underset{\sim}{e} \\
&= \lambda\underset{\sim}{e} + \lambda^2\underset{\sim}{e} + \lambda^3\underset{\sim}{e} \\
&= (\lambda + \lambda^2 + \lambda^3)\underset{\sim}{e}.
\end{aligned}
$$

$\therefore$ The matrix $A + A^2 + A^3$ has eigenvalue $\lambda + \lambda^2 + \lambda^3$ with corresponding eigenvector $\underset{\sim}{e}$.

If P is a non-singular matrix and D is a diagonal matrix such that $A + A^2 + A^3 = PDP^{-1}$,

let $P = \begin{pmatrix} 5 & 1 & 4 \\ -6 & 0 & -9 \\ 4 & 2 & 8 \end{pmatrix}$

and $D = \begin{pmatrix} 2 + 2^2 + 2^3 & 0 & 0 \\ 0 & 8 + 8^2 + 8^3 & 0 \\ 0 & 0 & -1 + (-1)^2 + (-1)^3 \end{pmatrix}$

$= \begin{pmatrix} 14 & 0 & 0 \\ 0 & 584 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

11. EITHER

$$C: \quad y = \frac{5(x-1)(x+2)}{(x-2)(x+3)}$$

i)

$$\begin{array}{r|l} & \quad 5 \\ x^2+x-6 & 5x^2+5x-10 \\ & \underline{5x^2+5x-30} \\ & \qquad\qquad 20 \end{array}$$

$$= 5 + \frac{20}{(x-2)(x+3)}$$

$$\frac{20}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$$= \frac{A(x+3) + B(x-2)}{(x-2)(x+3)}$$

$$20 = A(x+3) + B(x-2)$$

$$= (A+B)x + 3A - 2B$$

$A + B = 0 \qquad 3A - 2B = 20$

$2A + 2B = 0$

$$5A = 20$$

$$A = 4$$

$$B = -4$$

$$y = 5 + \frac{4}{x-2} - \frac{4}{x+3}$$

ii) $$\frac{dy}{dx} = \frac{-4}{(x-2)^2} + \frac{4}{(x+3)^2}$$

when $\frac{dy}{dx} = 0$: $$\frac{-4}{(x-2)^2} + \frac{4}{(x+3)^2} = 0$$

$$\frac{4}{(x-2)^2} = \frac{4}{(x+3)^2}$$

$$(x-2)^2 = (x+3)^2$$

$$x^2 - 4x + 4 = x^2 + 6x + 9$$

$$10x = -5$$

$$x = -\frac{1}{2}$$

$$y = \frac{9}{5}$$

iii) As $x \longrightarrow \pm\infty$ $y \longrightarrow 5$

As $x \longrightarrow 2$ $y \longrightarrow \pm\infty$

As $x \longrightarrow -3$ $y \longrightarrow \pm\infty$

$\therefore$ The asymptotes of C are $y = 5$, $x = 2$ and $x = -3$

iv) If C intersects the line $y = k$,

$$\frac{5(x-1)(x+2)}{(x-2)(x+3)} = k$$

$$5(x^2 + x - 2) = k(x^2 + x - 6)$$

$$5x^2 + 5x - 10 = kx^2 + kx - 6k$$

$(5-k)x^2 + (5-k)x + 6k - 10 = 0$

$a = 5-k \quad b = 5-k \quad c = 6k-10$

$b^2 - 4ac = (5-k)^2 - 4(5-k)(6k-10)$

$= (5-k)(5-k-4(6k-10))$

$= (5-k)(5-k-24k+40)$

$= (5-k)(-25k+45)$

$= 5(k-5)(5k-9)$

when $b^2 - 4ac \geqslant 0$

$5(k-5)(5k-9) \geqslant 0$

$\frac{9}{5} \leq k \leq 5$

$\therefore$ The line $y=k$ does not intersect C

if $k < \frac{9}{5}$ or $k > 5$.

OR

$$y = \frac{2x^{\frac{3}{2}}}{3}, \quad x \geqslant 0$$

i) Since R is the arc of the curve joining the origin to the point where $x = 3$, the length of R is,

$$\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_0^3 \sqrt{1 + \left(x^{\frac{1}{2}}\right)^2} \, dx$$

$$= \int_0^3 \sqrt{1 + x} \, dx$$

$$= \left[\frac{2(1 + x)^{\frac{3}{2}}}{3} \right]_0^3$$

$$= \frac{2}{3}\left(4^{\frac{3}{2}} - 1\right)$$

$$= \frac{2}{3}(8 - 1)$$

$$= \frac{2(7)}{3}$$

$$= \frac{14}{3}$$

ii) The area of the region bounded by the x - axis, the line $x = 3$ and R, A, is

$$\int_0^3 y \, dx$$

$$= \int_0^3 \frac{2x^{\frac{3}{2}}}{3} \, dx$$

$$= \left[\frac{2}{3} \left(\frac{2x^{\frac{5}{2}}}{5} \right) \right]_0^3$$

$$= \frac{4}{15} \left(3^{\frac{5}{2}} - 0 \right)$$

$$= \frac{4 (9\sqrt{3})}{15}$$

$$= \frac{12\sqrt{3}}{5}$$

$\therefore$ The y - coordinate of the centroid of the region bounded by the x - axis, the line $x = 3$ and R is

$$\frac{\int_0^3 \frac{y^2}{2} \, dx}{A}$$

$$= \frac{\int_0^3 \frac{4x^3}{9(2)} \, dx}{\frac{12\sqrt{3}}{5}}$$

$$= \frac{4}{18}\left(\frac{5}{12\sqrt{3}}\right)\left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{4}{18}\left(\frac{5}{12\sqrt{3}}\right)\left(\frac{81-0}{4}\right)$$

$$= \frac{5\sqrt{3}}{8}$$

iii) The area of the surface generated when R is rotated through one revolution about the y-axis is

$$\int_0^3 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^3 2\pi x \sqrt{1 + \left(x^{\frac{1}{2}}\right)^2}\, dx$$

$$= 2\pi \int_0^3 x\sqrt{1+x}\, dx$$

$u = 1 + x$

$du = dx$

$x = 0 \quad u = 1$

$x = 3 \quad u = 4$

$$= 2\pi \int_1^4 (u-1)u^{\frac{1}{2}}\, du$$

$$= 2\pi \int_1^4 u^{\frac{3}{2}} - u^{\frac{1}{2}}\, du$$

$$= 2\pi \left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} \right]_1^4$$

$$= 2\pi \left(\frac{64}{5} - \frac{16}{3} - \left(\frac{2}{5} - \frac{2}{3} \right) \right)$$

$$= 2\pi \left(\frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3} \right)$$

$$= 2\pi \left(\frac{116}{15} \right)$$

$$= \frac{232\pi}{15}$$