System of Linear Equations

Definition :

A system of linear equations in m equations with n unknowns is an array as follows:

$$
\begin{aligned}
&\mathrm{E}_1 : a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&\mathrm{E}_2 : a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
&\mathrm{E}_m : a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{aligned}
$$

where

$(\text{a})\ a_{ij}, b_i \in \mathrm{R}, i = 1, 2, \ldots \& m,\ j = 1, 2, \ldots \& n,$ and

$(\text{b})\ x_j$ is an variable $j = 1, 2, \ldots \& n.$

Remark:

$(\text{a})\ a_{ij}$ is called the $(i, j)-$coefficient.

$(\text{b})\ \mathrm{E}_i : a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij_i}x_{j_i} + \ldots + a_{in}x_n = b_i$ is called the i - th equation.

Examples of systems of linear equations :

$$\text{(a)}\begin{cases} 2x+3y=1 \\ 4x-y=9 \end{cases} \qquad \text{(b)}\begin{cases} p+2q+3r=6 \\ 2p+3q+r=7 \end{cases}$$

$$\text{(c)}\begin{cases} x+3y=4 \\ \quad\quad y=1 \\ x+y=3 \end{cases} \qquad \text{(d)}\begin{cases} 2x_1+x_2-x_3=2 \\ x_1 \qquad\quad +3x_3=7 \\ \qquad 5x_2-x_3=0 \end{cases}$$

Solution of a System of Linear Equations

Definition :

Let a system of linear equations in m equations with n unknowns be :

$$
\begin{aligned}
&\mathrm{E}_1 : a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&\mathrm{E}_2 : a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
&\mathrm{E}_m : a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m .
\end{aligned}
$$

$R_1, R_2, \ldots, R_n \in \mathrm{R}$ is called a <u>solution</u> of the above equation iff...

.....

$$
\begin{aligned}
a_{11}R_1 + a_{12}R_2 + \ldots + a_{1n}R_n &= b_1 \\
a_{21}R_1 + a_{22}R_2 + \ldots + a_{2n}R_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad & \;\;\vdots \\
a_{m1}R_1 + a_{m2}R_2 + \ldots + a_{mn}R_n &= b_m .
\end{aligned}
$$

Example:

Determine whether the following statements are true:

(a) $x = 2$ and $y = -1$ is the solution of $\begin{cases} 2x + 3y = 1 \\ 4x - y = 9 \end{cases}$.

(b) If $\begin{cases} 2x_1 \phantom{{}-2x_2} - x_3 = 3 \\ x_1 - 2x_2 = 2 \\ 3x_1 + 5x_2 - x_3 = 0 \end{cases}$, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Example:

Determine whether the following statements are true:

(a) $\{(x, y) \mid (t, 2-t), t \in \mathrm{R}\}$ is the solution of $\begin{cases} x+y=2 \\ 2x+2y=4 \end{cases}$.

(b) If $\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 + 3x_2 - x_3 = 7 \end{cases}$, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \lambda \in \mathrm{R}.$

Solving a System of Linear Equation

Elimination Algorithm:

(1) Starting from the first equation, express one of the variables in terms of the others.

(2) Substitute this expression into the remaining equations. This reduces the number of unknowns in the remaining equations.

(3) Repeat (1) and (2) to the remaining equations until the system is reduced to a single linear equation.

(4) Solve this equation, and then back-substitute until the entire solution is found.

Example:

Solve the system of linear equations using the Elimination Algorithm:

$$\text{(a)}\begin{cases} x - y = 3 \\ 2x + y = 6 \end{cases} \qquad \text{(b)}\begin{cases} x + 4y = -2 \\ 2x + 3y = 1 \end{cases}$$

Example:

Solve the system of linear equations using the Elimination Algorithm:

$$\text{(a)} \begin{cases} 2x - 6y = 2 \\ x - 3y = 1 \end{cases} \qquad \text{(b)} \begin{cases} 2x + 4y = 3 \\ x + 2y = 1 \end{cases}$$

Example:

Solve the system of linear equations using the Elimination Algorithm:

$$(a)\begin{cases} x+2y+3z=6 \\ 2x+3y+z=7 \end{cases} \qquad (b)\begin{cases} 3x-y-z=2 \\ x+2y+z=1 \end{cases}$$

Example:

Solve the system of linear equations $\begin{cases} x - y = 1 \\ ax + ay = 2 \end{cases}$,

where $a \in \mathrm{R}$, completely for $x, y \in \mathrm{R}$.

Example:

Solve the system of linear equations $\begin{cases} ax + y = 2a^2 \\ x + y = 2 \end{cases}$,

where $a \in \mathrm{R}$, completely for $x, y \in \mathrm{R}$.

Example:

The solution of system of linear equations $\begin{cases} x+2y=\lambda x \\ 2x-2y=\lambda y \end{cases}$ is $x=0,\ y=0$. For what value of λ are there other solutions?

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example:

Solve the system of linear equations using the Elimination Algorithm:

$$(a)\begin{cases} 3x_1 + 2x_2 + x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \end{cases} \qquad (b)\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ x_1 \qquad\quad - x_3 = 2 \end{cases}$$

Example:

Solve the system of linear equations $\begin{cases} x + ky = k + 1 \\ kx + y = k + 1 \end{cases}$,

where $k \in \mathrm{R}$, completely for $x, y \in \mathrm{R}$.

Example:

Solve the system of linear equations $\begin{cases} 2x - ky = 1 \\ (k+3)x - 9y = k \end{cases}$,

where $k \in \mathrm{R}$, completely for $x, y \in \mathrm{R}$.

Example:

Solve the system of linear equations $\begin{cases} x+y=2 \\ ax+y=a^2-3 \end{cases}$,

where $a \in \mathrm{R}$, completely for $x, y \in \mathrm{R}$.

Example:

Solve the system of linear equations $\begin{cases} ax + by = a \\ bx + ay = b \end{cases}$,

where $a, b \in \mathrm{R}$, completely for $x, y \in \mathrm{R}$.