1 $$k_1\begin{pmatrix}1\\2\\3\end{pmatrix} + k_2\begin{pmatrix}5\\4\\8\end{pmatrix} + k_3\begin{pmatrix}7\\-1\\6\end{pmatrix} = \underset{\sim}{0}$$

$$\begin{pmatrix}k_1 + 5k_2 + 7k_3\\2k_1 + 4k_2 - k_3\\3k_1 + 8k_2 + 6k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 5 & 7\\2 & 4 & -1\\3 & 8 & 6\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Since $\begin{vmatrix}1 & 5 & 7\\2 & 4 & -1\\3 & 8 & 6\end{vmatrix} = 32 - 75 + 28$

$= -15$

$\neq 0$

$$\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}1 & 5 & 7\\2 & 4 & -1\\3 & 8 & 6\end{pmatrix}^{-1}\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$k_1 = k_2 = k_3 = 0$$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix}$, $\begin{pmatrix} 7 \\ -1 \\ 6 \end{pmatrix}$ are linearly

independent.

2 $$y = \frac{1}{ax+b}$$

$$\frac{d^n y}{dx^n} = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$$

When $n = 1$: $\frac{dy}{dx} = \frac{(-1)^1 1!\, a^1}{(ax+b)^{1+1}} = \frac{-a}{(ax+b)^2}$

Assume the statement is true when $n = k$

$n = k$: $\frac{d^k y}{dx^k} = \frac{(-1)^k k!\, a^k}{(ax+b)^{k+1}}$

When $n = k+1$:

$$\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left(\frac{(-1)^k k!\, a^k}{(ax+b)^{k+1}}\right)$$

$$= (-1)^k k!\, a^k \frac{d}{dx}\left(\frac{1}{(ax+b)^{k+1}}\right)$$

$$= (-1)^k k!\, a^k \left(-(k+1)(ax+b)^{-k-1-1} a\right)$$

$$= \frac{(-1)^k k!\, a^k (-(k+1)a)}{(ax+b)^{k+2}}$$

$$= \frac{(-1)^{k+1}(k+1)!\,a^{k+1}}{(ax+b)^{k+2}}$$

$$\therefore \frac{d^n y}{dx^n} = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}} \text{ for every}$$

positive integer n.

3. $$\left.\begin{aligned} 3x + 2y - 8z &= 1 \\ 4x + 7y + 9z &= 0 \\ 5x - 6y + z &= 9 \end{aligned}\right\}$$

$$\left.\begin{aligned} 20 \times ①: \quad 60x + 40y - 160z &= 20 \\ 15 \times ②: \quad 60x + 105y + 135z &= 0 \\ 12 \times ③: \quad 60x - 72y + 12z &= 108 \end{aligned}\right\}$$

$$\left.\begin{aligned} -① + ②: \quad 60x + 40y - 160z &= 20 \\ -① + ③: \quad 65y + 295z &= -20 \\ -112y + 172z &= 88 \end{aligned}\right\}$$

$$\left.\begin{aligned} 112 \times ②: \quad 60x + 40y - 160z &= 20 \\ 65 \times ③: \quad 7280y + 33040z &= -2240 \\ -7280y + 11180z &= 5720 \end{aligned}\right\}$$

$$\left.\begin{aligned} ② + ③: \quad 60x + 40y - 160z &= 20 \\ 7280y + 33040z &= -2240 \\ 44220z &= 3480 \end{aligned}\right\}$$

$$44220z = 3480$$

$$z = \frac{58}{737}$$

$$y = \frac{-490}{737}$$

$$x = \frac{727}{737}$$

4. $3x^4 - x^3 + 9x^2 - 4x + 5 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots.

$\alpha + \beta + \gamma, \alpha + \beta + \delta, \alpha + \gamma + \delta, \beta + \gamma + \delta$

Let $u = \alpha + \beta + \gamma$

$$u + \delta = \alpha + \beta + \gamma + \delta$$

$$= \frac{1}{3}$$

$$\therefore \delta = \frac{1}{3} - u$$

δ is a root

$$\therefore 3\delta^4 - \delta^3 + 9\delta^2 - 4\delta + 5 = 0$$

$$3\left(\frac{1}{3} - u\right)^4 - \left(\frac{1}{3} - u\right)^3 + 9\left(\frac{1}{3} - u\right)^2 - 4\left(\frac{1}{3} - u\right) + 5 = 0$$

$$3\left(\frac{1}{81} - \frac{4u}{27} + \frac{6u^2}{9} - \frac{4u^3}{3} + u^4\right)$$

$$-\left(\frac{1}{27} - \frac{3u}{9} + \frac{3u^2}{3} - u^3\right) + 9\left(\frac{1}{9} - \frac{2u}{3} + u^2\right)$$

$$-\frac{4}{3} + 4u + 5 = 0$$

$$\frac{1}{27} - \frac{4u}{9} + 2u^2 - 4u^3 + 3u^4$$

$$\frac{-1}{27} + \frac{u}{3} - u^2 + u^3 + 1 - 6 + 9u^2$$

$$-\frac{4}{3} + 4u + 5 = 0$$

$$3u^4 - 3u^3 + 10u^2 - \frac{19u}{9} + \frac{14}{3} = 0$$

$$27u^4 - 27u^3 + 90u^2 - 19u + 42 = 0$$

$\therefore$ The equation having roots $\alpha + \beta + \gamma$, $\alpha + \beta + \delta$, $\alpha + \gamma + \delta$, $\beta + \gamma + \delta$ is $27u^4 - 27u^3 + 90u^2 - 19u + 42 = 0$.

5. The area, A, bounded by the x-axis, the line $x = 1$ and the curve $y = x^3$ is

$$\int_0^1 y \, dx$$

$$= \int_0^1 x^3 \, dx$$

$$= \left[\frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{4} - 0$$

$$= \frac{1}{4}$$

$$\bar{x} = \frac{\int_0^1 xy \, dx}{A}$$

$$= \frac{\int_0^1 x^4 \, dx}{\frac{1}{4}}$$

$$= \frac{\left[\frac{x^5}{5} \right]_0^1}{\frac{1}{4}}$$

$$= \frac{\frac{1}{5} - 0}{\frac{1}{4}}$$

$$= \frac{\frac{1}{5}}{\frac{1}{4}}$$

$$= \frac{4}{5}$$

$$\bar{y} = \frac{\int_0^1 \frac{y^2}{2} \, dx}{A}$$

$$= \frac{\int_0^1 \frac{x^6}{2} \, dx}{\frac{1}{4}}$$

$$= \frac{\left[\frac{x^7}{14} \right]_0^1}{\frac{1}{4}}$$

$$= \frac{\frac{1}{14} - 0}{\frac{1}{4}}$$

$$= \frac{\frac{1}{14}}{\frac{1}{4}}$$

$$= \frac{2}{7}$$

$\therefore$ The centroid of the region bounded by the x - axis, the line $x = 1$ and the curve $y = x^3$ is $\left(\frac{4}{5}, \frac{2}{7}\right)$

6. $C: \quad y = 4 + \dfrac{1}{2x+3} - \dfrac{8}{x-6}$

i) As $x \longrightarrow \pm\infty \quad y \longrightarrow 4$

As $x \longrightarrow \frac{-3}{2} \quad y \longrightarrow \pm\infty$

As $x \longrightarrow 6 \quad y \longrightarrow \pm\infty$

$\therefore$ The asymptotes of C are $y = 4,\ x = -\frac{3}{2}$ and $x = 6$.

ii) $\dfrac{dy}{dx} = \dfrac{-2}{(2x+3)^2} + \dfrac{8}{(x-6)^2}$

When $\dfrac{dy}{dx} = 0$: $\dfrac{-2}{(2x+3)^2} + \dfrac{8}{(x-6)^2} = 0$

$$\frac{2}{(2x+3)^2} = \frac{8}{(x-6)^2}$$

$$(x-6)^2 = 4(2x+3)^2$$

$$x - 6 = \pm 2(2x+3)$$

$$= 4x + 6,\ -4x - 6$$

$$3x = -12,\quad 5x = 0$$

$$x = -4 \qquad x = 0$$

$$y = \frac{23}{5} \qquad y = \frac{7}{3}$$

$\therefore$ The critical points of C are $\left(-4, \frac{23}{5}\right)$ and $\left(0, \frac{7}{3}\right)$.

7. $x = t^2 \cos t \quad y = t^2 \sin t, \quad t = 0, \ t = 2$

$$\frac{dx}{dt} = 2t \cos t - t^2 \sin t$$

$$\frac{dy}{dt} = 2t \sin t + t^2 \cos t$$

$$\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (2t \cos t - t^2 \sin t)^2 \\
&\quad + (2t \sin t - t^2 \cos t)^2 \\
&= 4t^2 \cos^2 t + t^4 \sin^2 t \\
&\quad - 4t^3 \sin t \cos t \\
&\quad + 4t^2 \sin^2 t + t^4 \cos^2 t \\
&\quad + 4t^3 \sin t \cos t \\
&= 4t^2 (\cos^2 t + \sin^2 t) \\
&\quad + t^4 (\cos^2 t + \sin^2 t) \\
&= 4t^2 + t^4 \\
&= t^2 (4 + t^2)
\end{aligned}$$

$$\begin{aligned}
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{t^2 (4 + t^2)} \\
&= t\sqrt{4 + t^2}
\end{aligned}$$

The arc length from $t = 0$ to $t = 2$ is given by

$$\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \int_0^2 t\sqrt{4 + t^2} \, dt$$

$$= \left[\frac{(4 + t^2)^{\frac{3}{2}}}{3} \right]_0^2$$

$$= \frac{16\sqrt{2} - 8}{3}$$

8. $I_n = \int \csc^n x \; dx$

$= \int \csc^{n-2} x \csc^2 x \; dx$

$= \int \csc^{n-2} x (1 + \cot^2 x) \; dx$

$= \int \csc^{n-2} x + \csc^{n-2} x \cot^2 x \; dx$

$= I_{n-2} + \int \csc^{n-2} x \cot^2 x \; dx$

$= I_{n-2} + \int \csc^{n-3} x \csc x \cot x \cot x \; dx$

$u = \cot x \qquad dv = \csc^{n-3} x \csc x \cot x \; dx$

$du = -\csc^2 x \; dx \qquad v = \int \csc^{n-3} x \csc x \cot x \; dx$

$= -\int \csc^{n-3} x (-\csc x \cot x) \; dx$

$= \dfrac{-\csc^{n-2} x}{n-2}$

$$= I_{n-2} + \left(\frac{-\csc^{n-2} x \cot x}{n-2} - \int \frac{(-\csc^{n-2} x)(-\csc^2 x) \, dx}{n-2} \right)$$

$$= I_{n-2} - \frac{\csc^{n-2} x \cot x}{n-2} - \int \frac{\csc^n x}{n-2} \, dx$$

$$= I_{n-2} - \frac{\csc^{n-2} x \cot x}{n-2} - \frac{1}{n-2} I_n$$

$$\left(1 + \frac{1}{n-2}\right) I_n = I_{n-2} - \frac{\csc^{n-2} x \cot x}{n-2}$$

$$\left(\frac{n-1}{n-2}\right) I_n = I_{n-2} - \frac{\csc^{n-2} x \cot x}{n-2}$$

$$\therefore (n-1) I_n = (n-2) I_{n-2} - \csc^{n-2} x \cot x$$

$$n = 3: \quad 2I_3 = I_1 - \csc x \cot x$$

$$I_1 = \int \csc x \, dx$$

$$= \int \frac{\csc x (\csc x + \cot x)}{\csc x + \cot x} dx$$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$$

$$= -\int \frac{-\csc^2 x - \csc x \cot x}{\csc x + \cot x} dx$$

$$= -\ln|\csc x + \cot x|$$

$$2I_3 = -\ln|\csc x + \cot x| - \csc x \cot x$$

$$\therefore I_3 = -\frac{1}{2} \ln|\csc x + \cot x| - \frac{\csc x \cot x}{2}$$

$$n = 4: \quad 3I_4 = 2I_2 - \csc^2 x \cot x$$

$$I_2 = \int \csc^2 x \, dx$$

$$= -\int -\csc^2 x \, dx$$

$$= -\cot x$$

$$3I_4 = -2\cot x - \csc^2 x \cot x$$

$$I_4 = \frac{-2\cot x}{3} - \frac{\csc^2 x \cot x}{3} + C$$

9. $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin n\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$z^n + \frac{1}{z^n} = 2\cos n\theta, \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$

$z - \frac{1}{z} = 2i\sin\theta$

$\left(z - \frac{1}{z}\right)^6 = (2i\sin\theta)^6$

$z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$

$= -64\sin^6\theta$

$z^6 + \frac{1}{z^6} - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$

$= -64\sin^6\theta$

$-64\sin^6\theta = 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$

$\therefore \sin^6\theta = \frac{-1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$

The mean value of $\sin^6\theta$ over the interval

$(-\pi, \pi)$ is $\dfrac{1}{\pi - (-\pi)} \displaystyle\int_{-\pi}^{\pi} \sin\theta \; d\theta$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^6\theta \; d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{-1}{32}\left(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right) d\theta$$

$$= \frac{1}{2\pi} \left[\frac{-1}{32}\left(\frac{\sin 6\theta}{6} - \frac{3\sin 4\theta}{2} + \frac{15\sin 2\theta}{2} - 10\theta \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{-1}{32}\left(-10\pi - 10\pi\right)\right)$$

$$= \frac{5}{16}$$

10 $(r+1)^5 - (r-1)^5$

$= r^5 + 5r^4 + 10r^3 + 10r^2 + 5r + 1$

$\quad - (r^5 - 5r^4 + 10r^3 - 10r^2 + 5r - 1)$

$= 10r^4 + 20r^2 + 2$

$$\sum_{r=1}^{n} 10r^4 + 20r^2 + 2 = \sum_{r=1}^{n} (r+1)^5 - (r-1)^5$$

$$10\sum_{r=1}^{n} r^4 + 20\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} 1 = (n+1)^5 - (n-1)^5$$

$+ n^5 - (n-2)^5$

$+ (n-1)^5 - (n-3)^5$

$\vdots$

$+ 4^5 - 2^5$

$+ 3^5 - 1^5$

$+ 2^5 - 0^5$

$$10\sum_{r=1}^{n} r^4 + \frac{20n(n+1)(2n+1)}{6} = (n+1)^5 + n^5 - 1$$

$+ 2n$

$= n^5 + 5n^4 + 10n^3$

$+ 10n^2 + 5n + 1$

$+ n^5 - 1$

$= 2n^5 + 5n^4 + 10n^3 + 10n^2 + 5$

$$10\sum_{r=1}^{n} r^4 + \frac{10n(n+1)(2n+1)}{3} + 2n$$

$$= 2n^5 + 5n^4 + 10n^3 + 10n^2 + 5n$$

$$30\sum_{r=1}^{n} r^4 + 10n(n+1)(2n+1) + 6n$$

$$= 6n^5 + 15n^4 + 30n^3 + 30n^2 + 15n$$

$$30\sum_{r=1}^{n} r^4 + 20n^3 + 30n^2 + 10n + 6n$$

$$= 6n^5 + 15n^4 + 30n^3 + 30n^2 + 15n$$

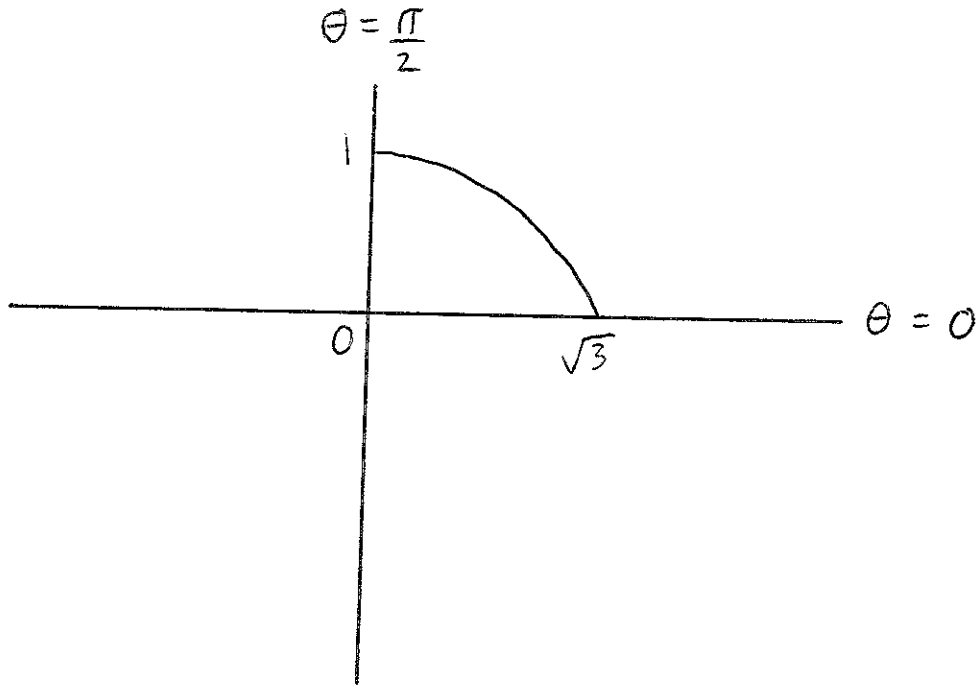
$$30\sum_{r=1}^{n} r^4 = 6n^5 + 15n^4 + 10n^3 - n$$

$$\therefore \sum_{r=1}^{n} r^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

11 EITHER

i) $r = \sin\theta + \sqrt{3}\cos\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	$\sqrt{3}$	2	$\frac{1+\sqrt{3}}{\sqrt{2}}$	$\sqrt{3}$	1



ii) $\frac{dr}{d\theta} = \cos\theta - \sqrt{3}\sin\theta$

When $\frac{dr}{d\theta} = 0$:

$$\cos\theta - \sqrt{3}\sin\theta = 0$$

$$\sqrt{3}\sin\theta = \cos\theta$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

$$r = 2$$

$$\frac{d^2r}{d\theta^2} = -\sin\theta - \sqrt{3}\cos\theta$$

when $\theta = \frac{\pi}{6}$: $\frac{d^2r}{d\theta^2} = \frac{-1}{2} - \frac{3}{2} = -2 < 0$

$\therefore$ The maximum value of r is 2

when $\theta = \frac{\pi}{6}$.

iii) The area bounded by the curve from $\theta = 0$ to $\theta = \frac{\pi}{2}$ is

$$\int_0^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\sin\theta + \sqrt{3}\cos\theta)^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 3\cos^2\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1 + 2\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1 + 1 + \cos 2\theta + 2\sqrt{3}\sin\theta\cos\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{2 + \cos 2\theta + \sqrt{3}\sin 2\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 1 + \frac{\cos 2\theta}{2} + \frac{\sqrt{3}\sin 2\theta}{2}\, d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{4} - \frac{\sqrt{3}\cos 2\theta}{4} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + 0 - \frac{\sqrt{3}(-1)}{4} - \left(0 + 0 - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{\pi}{2} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}$$

$$= \frac{\pi}{2} + \frac{\sqrt{3}}{2}$$

OR

$$5y^4\frac{d^2y}{dx^2} + 20y^3\frac{dy}{dx} + 25y^4\frac{dy}{dx} + 4y^5 = 3e^{7x}$$

$$z = y^5$$

$$\frac{dz}{dy} = 5y^4$$

$$\frac{dz}{dx} = 5y^4\frac{dy}{dx}$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}(5y^4\frac{dy}{dx})$$

$$= 5y^4\frac{d}{dx}(\frac{dy}{dx}) + \frac{dy}{dx}\frac{d}{dx}(5y^4)$$

$$= 5y^4\frac{d^2y}{dx^2} + \frac{dy}{dx}(20y^3\frac{dy}{dx})$$

$$= 5y^4\frac{d^2y}{dx^2} + 20y^3(\frac{dy}{dx})^2$$

$$\frac{d^2z}{dx^2} + \frac{5dz}{dx} + 4z = 3e^{7x}$$

$$\frac{d^2z}{dx^2} + \frac{5dz}{dx} + 4z = 0$$

$$m^2 + 5m + 4 = 0$$

$$(m + 1)(m + 4) = 0$$

$$m = -1, -4$$

The complementary function, z_c, is

$$z_c = Ae^{-x} + Be^{-4x}$$

The particular integral, z_p, is given by

$$z_p = Ce^{7x}$$

$$\frac{dz_p}{dx} = 7Ce^{7x}$$

$$\frac{d^2z_p}{dx^2} = 49Ce^{7x}$$

$$\frac{d^2z_p}{dx^2} + \frac{5dz_p}{dx} + 4z_p$$

$$= 49Ce^{7x} + 35Ce^{7x} + 4Ce^{7x}$$

$$= 88Ce^{7x}$$

$$= 3e^{7x}$$

$$\therefore 88C = 3$$

$$C = \frac{3}{88}$$

$$z_p = \frac{3e^{7x}}{88}$$

$$z = z_c + z_p$$

$$= Ae^{-x} + Be^{-4x} + \frac{3e^{7x}}{88}$$

Since $z = y^5$,

$$y^5 = Ae^{-x} + Be^{-4x} + \frac{3e^{7x}}{88}$$

$$y = \left(Ae^{-x} + Be^{-4x} + \frac{3e^{7x}}{88}\right)^{\frac{1}{5}}$$