9. DIFFERENTIAL EQUATION

1. The acceleration of a particle travelling in a straight line is given by $(2 + 4x)$, where x is the displacement of the particle from a fixed point O of its path. Given that the particle is initially at rest at O, at time $t = 0$, find expression for x and the velocity v in terms of t and show that $v^2 = 4x(1 + x)$.
Show that the particle never returns to O. (J72/I/16)

2. Water flows into a rectangular tank having a horizontal base of area A at a constant rate of n units of volume per unit time. Water flows out of the tank through a hole in the bottom at a rate which is assumed to be proportional to the square root of the depth of water in the tank. It is found that when the depth is h the level of water in the tank remains constant. Initially the tank is filled to a depth $4h$. Obtain a differential equation for the depth z at time t.
By making the substitution $z = hu^2$ show that u satisfies the differential equation

$$\frac{2Ah}{n}\frac{du}{dt} = -\frac{u-1}{u}$$

Find the time at which $z = 16h/9$ and describe how z varies with t. (J72/II/6)

3. (a) Find the general solution of $x\frac{dy}{dx} + 2y = \frac{2\sin x}{x\cos^3 x}$.

   (b) By using the substitution $y = ux$, or otherwise, find the particular solution of $x\frac{dy}{dx} - y = \frac{1}{4}x^2 - y^2$ which has $y = 0$ when $x = 1$.

   [Any correct relationship between y and x will be accepted.] (J73/I/16)

4. Find the general solution of each of the following differential equations:

   (a) $x\frac{dy}{dx} - 3y = x^5 e^{\frac{1}{2}x^2}$,

   (b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10x - 1$. (N73/I/16)

5. Find the general solution of the differential equation $\frac{dy}{dx} + 2y = \sin 3x$.
Hence or otherwise find the solution of the differential equation

$$\frac{d^2z}{dx^2} + 2\frac{dz}{dx} = 3\cos 3x$$

for which $z = 1$ and $\frac{dz}{dx} = 2$ when $x = 0$. (J74/I/15)

6. In a chemical reaction, two substances A and B react together to produce another substance C. At time t the amounts of A and B present are a-x and b-x respectively where a, b are constants and x is a function of t. The value of x is zero when $t = 0$. At any instant the rate at which the amount of A decreases is proportional to the product of the amount of A and the amount of B present at that instant. Obtain a differential equation relating x and t and solve for x as a function of t in the two cases (i) $a = b$, (ii) $a > b$. (J74/II/6)

7. (a) Sketch the family of solution curves of the differential equation $\frac{dy}{dx} = -\frac{y}{x}$.

(b) For each of the following families of curves, find a first order differential equation (relating x and y) which is satisfied by all curves of the family:

(i) $x^2 + y^2 = C\,(C > 0)$, (ii) $y = Ce^x + 2$,
(iii) the family of circles with unit radius and centre on the y-axis.

(N74/II/15)

8. Using the substitution $u = y^2$, or otherwise, find the general solution of the differential equation $2xy\frac{dy}{dx} = y^2 - 4x^2$.

Show that the solution which has $y = a\ (a > 0)$ when $x = \frac{1}{2}a$ satisfies

$$y^2 = 4x\,(a - x).$$

Give a sketch of the curve which has the above equation.
The finite region bounded by the x-axis and that part of the curve for which y is positive is rotated through four right angles about the x-axis. Find the volume of the solid of revolution so formed. (J75/I/15)

9. $N(t)$ is the number of elephants in the Tsavo National Park in Kenya at a time t. The rate of increase of N is equal to the product of k, N and $(M - N)$ where M is the maximum number of elephants that can live in the park and k is a positive constant. Treating the number of elephants as a continuous variable, formulate and solve a differential equation for $N(t)$.
Describe the behaviour of $N(t)$ as t becomes large. Given that $M = 10{,}000$, $N(0) = 5000$ and $N(10) = 6000$, find $N(100)$. (N75/I/6)

✓ 10. Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 13\sin 3t.$$

Determine the particular solution which has $x = -\frac{3}{2}$ and $\frac{dx}{dt} = -4$ when $t = 0$.

(N75/I/16)

✓ 11. Find the general solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3\cos x$.

Find also the solution such that $y = \frac{29}{50}$ and $\frac{dy}{dx} = \frac{191}{150}$ at $x = 0$. (J76/I/16)

12. Solve the differential equation $\frac{dy}{dx} = \frac{x^2y^2(y+1)}{3y^3 + 4y^2 + 1}$ given that $y = 2$ when $x = 3$.

(N76/I/6)

✓ 13. Solve the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10$

given that $x = 2$ and $\frac{dx}{dt} = 2$ when $t = 0$.

Show that, for large t, x tends to a constant value, and find this value.
Sketch the graph of x against t for $t \geqslant 0$. (J77/I/15)

14. (a) By means of the substitution $\frac{dy}{dx} = p$ find the general solution of the differential equation $x\frac{d^2y}{dx^2} - \frac{dy}{dx} = x^3$.

(b) Find the general solution of the differential equation $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 2t$.

Give also the solution for which $x = -\frac{1}{2}$ and $\frac{dx}{dt} = \frac{1}{2}$ when $t = 0$. (N77/I/15)

15. A water tank is in the shape of a cylinder with its axis vertical. The area of the base is A m^2. Initially the tank is empty. Starting at time $t = 0$, water is poured into the tank at a constant rate of w m^{3}s^{-1}, and leaks out through a small hole in the base at a rate kx m^{3}s^{-1}, where k is a constant and x m is the depth of water in the tank. Form a differential equation and solve it to obtain an expression for x in terms of t.
Show that, however long the process continues, the depth of water never exceeds w/k m, and that if the time taken to reach half this depth is Ts, then $k = \frac{A}{T}\ln 2$.
(N77/II/2)

16. By means of the substitution $z = 1/y^2$ transform the differential equation

$$\frac{dy}{dx} + y = xy^3$$

into a differential equation containing z and x. Solve this equation for z and hence show that the solution of the given differential equation for which $y = \sqrt{2}$ when $x = 0$ is $y^2 = 2/(2x + 1)$.
Sketch this curve. (An accurately plotted graph is not required). (J78/II/5)

✓17. (a) Solve the differential equation $\frac{dy}{dx} = y\cot x$, given that $y = 1$ when $x = \frac{1}{6}\pi$.

(b) Solve the differential equation $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 3e^{-t}$,

given that $x = 1$ when $t = 0$, and that $\lim_{t\to\infty} x = 0$.
[You may assume that $\lim_{z\to\infty} ze^{-z} = 0$.] (N78/II/5)

18. The inhabitants of a country are of two types, A and B. At time t there are x million inhabitants of type A and y million inhabitants of type B. A mathematical model of the population is based on the differential equations.

$$\frac{dx}{dt} = 3x - 5y \text{ and } \frac{dy}{dt} = 2x - 4y.$$

By differentiating the first equation with respect to t, and using the given equations to eliminate $\frac{dy}{dt}$ and then y, or by any other method, show that

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 0.$$

When $t = 0$, $x = 7$ and $\frac{dx}{dt} = 1$. Show that $x = 5e^{t} + 2e^{-2t}$, and find y in terms of t. Show further that, when t is large, $2x \approx 5y$ (J79/II/5)

19. (a) Obtain the general solution of the differential equation

$$\frac{dy}{dx} = y \tan x + 2x \sec x.$$

(b) A particle is propelled from rest at the point O by a force acting along Ox. The magnitude of the force is such that the speed v m s^{-1} and the displacement x m of the particle are related by $v\frac{dv}{dx} = 50(1 - 10x)$, $(0 \leqslant x \leqslant 0.1)$.

Calculate the speed of the particle when it has travelled 0.1 m. (N79/I/7)

20. Solve the differential equation $\frac{d^2y}{dx^2} + 4y = 3\cos x$

given that $y = \frac{dy}{dx} = 0$ when $x = 0$.

Find the set of values of x for which $y = 0$.

Find also the set of values of $\cos x$ for which $\frac{dy}{dx} = 0$.

Show that, if x is amll, $y \approx \frac{1}{2}x^2(3 - \frac{5}{4}x^2)$. (N79/II/4)

21. (a) Solve the differential equation $x\frac{dy}{dx} - 2y = x^4 \cos 2x$.

(b) By means of the substitution $y = xu$ transform the differential equation

$$(x^2 + y^2)\frac{dy}{dx} = xy$$

into a differential equation containing only u, x and $\frac{du}{dx}$. Hence, or otherwise, solve the differential equation $(x^2 + y^2)\frac{dy}{dx} = xy$.

Show that the solution for which $y = 1$ when $x = 1$ is $x = y(1 + 2\ln y)^{\frac{1}{2}}$.

(J80/I/4)

22. Solve the differential equation $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 80 \sin 3t$,

given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = -6$. (J80/II/5)

23. (a) Solve the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$,

given that $y = 1$ when $x = 0$, and y has a stationary value when $x = 1$.
Sketch the graph of the solution for $x \geqslant 0$.

(b) Obtain a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2e^{-2x}.$$ (N80/I/5)

24. A colony of bacteria is being grown in a shallow dish of area 100 cm^2. After t days the area of the colony is x cm^2. The growth of the colony is modelled by the differential equation $\frac{dx}{dt} = \frac{1}{25}t(100 - x)$

Given that initially the colony occupies 1 cm^2, solve the differential equation, and calculate the number of days that will elapse before the colony has covered half the remaining area of the dish,(i.e. $x = 50.5$).
Show that the rate of growth is at its maximum when $t = 5$, and that the area of the colony is then nearly 40 cm^2.

Sketch the graph of x against t for $t \geqslant 0$. (N80/II/5)

25. A spherical tank of internal radius a contains water whose surface is at a height x above the lowest point of the tank. Show that the volume V of water in the tank is given by $V = \frac{1}{3}\pi x^2(3a - x)$.

Water flows from the tank, through an outlet at its lowest point, at a rate $\pi k \sqrt{x}$, where k is a positive constant. Show that

$(2ax - x^2)\frac{dx}{dt} = -k\sqrt{x}$, where t is the time.

Find the ratio $T_1 : T_2$, where T_1 is the time taken to empty the tank when initially it is completely full, and T_2 is the time taken to empty the tank when initially it is half full. (J81/I/5).

26. Find the general solution of the differential equation $\frac{dy}{dx} + 2xy = 2x(x^2 + 1)$.

Show that, if a particular solution curve of the differential equation passes through the point $(0, k)$, then that solution curve will have a maximum or mininum at $(0, k)$ according as k is greater than or less than 1.
On one diagram sketch the three solution curves passing respectively through the points $(0, 0)$, $(0, 1)$, $(0, 2)$. (J81/II/5)

27. (a) Indicate on a sketch the family of solution curves for the differential equation

$$\frac{dy}{dx} = x - y.$$

(b) Solve the differential equation

$$x\frac{dy}{dx} + (x+1)y = 2xe^{-x}, \text{ given that } y = 0 \text{ when } x = 1.$$ (N81/I/5)

28. (a) Given that $y = \tan(3\tan^{-1}x)$, show that $(1+x^2)\frac{dy}{dx} = 3(1+y^2)$.

(b) Solve the differential equation $x\frac{dy}{dx} = 2y + x^2 \ln x$,

given that $y = 2$ when $x = 1$. (J82/I/5)

29. Given that y is a function of x, where $x > 0$, show that, if the substitution $x = \sqrt{t}$ is made, then

(a) $\frac{dy}{dx} = 2\sqrt{t}\frac{dy}{dt}$,

(b) $\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$.

Hence, or otherwise, find the general solution of the differential equation.

$$\frac{d^2y}{dx^2} - \frac{1}{x}\left(\frac{dy}{dx}\right) + 4x^2(9y+6) = 0$$ (J82/II/5)

30. (a) Solve the differential equation $\frac{d^2x}{dx} + 5\frac{dx}{dt} + 4x = 5 + 4t$,

given that, when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$.

(b) Find the general solution of the differential equation

$$(x^2+1)\frac{dy}{dx} + y = \tan^{-1}x.$$ (N82/II/5)

31. Find the general solution of the differential equation $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + x = t$,

for the cases (i) $k > 1$, (ii) $0 < k < 1$.

In the case when $k = 1$, find the solution for which $x = 0$ and $\frac{dx}{dt} = 1$ when $t = 0$. (J83/I/5)

32. (a) Given that $y = Axe^{Bx}$, where A and B are constants, show that

$$x^2y\frac{d^2y}{dx^2} - x^2\left(\frac{dy}{dx}\right)^2 + y^2 = 0.$$

(b) Find the general solution of the differential equation $\frac{dy}{dx} + 2y\tan x = \sin x$.

Given that $y = 0$ when $x = 0$, find the maximum value of y. (J83/II/4)

33. Find the general solution of the differential equation $\frac{dy}{dx} + 3y = \cos 2x$.

Hence or otherwise solve the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = -2\sin 2x$,

given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. (N83/II/5)

34. (a) Find the derivative of $\tan^{-1}\left(\frac{2}{x}\right) + \tan^{-1}\left(\frac{x}{2}\right)$, $(x > 0)$.

(b) Find the general solution of the differential equation

$$(x^2+4)\frac{dy}{dx} = 2y + x^2 e^{\tan^{-1}(\frac{1}{2}x)},$$

expressing your answer in the form $y = f(x)$. Find also the particular solution for which $y = e^{\frac{1}{4}\pi}$ when $x = 2$. (J84/I/4)

35. Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 4e^{-x}$.

Find also the particular solution for which $y = 1$ and $\frac{dy}{dx} = 3$ when $x = 0$. Show that, for this particular solution, $y = 0$ when $x = \frac{7}{12}\pi$, and find the other values of x in $[0, 2\pi]$ for which $y = 0$. (J84/II/5)

36. (a) Find $\int e^{2x}\cos x\, dx$.

(b) Solve the differential equation $\frac{dy}{dx} + 2y = \cos x$, given that $y = 1$ when $x = 0$.

(c) Obtain the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = \sin t.$$ (N84/II/6)

37. (a) Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 9x$.

(b) Find the general solution of the differential equation $(a^2 - x^2)\frac{dy}{dx} - xy + a^2 = 0$,

where a is a positive constant and $|x| < a$. (J85/II/5)

38. (a) Obtain the general solution of the differential equation

$$\frac{dy}{dx} + \frac{4y}{1-4x^2} = 1 - 2x.$$

(b) Show that $x = -\frac{1}{4}t\cos 2t$ is a particular integral of the differential equation

$$\frac{d^2x}{dt^2} + 4x = \sin 2t.$$

Hence solve the differential equation given that when $t = 0$, $x = \pi$ and $\frac{dx}{dt} = -\frac{1}{4}$

(J85/I/5)

39. (a) Given that $(\sin x)\dfrac{dy}{dx} - (\cos x)y = \sin^3 x$, and that $y = 1$ when $x = \frac{1}{2}\pi$, show that $y = \frac{1}{2}(1 + \sqrt{2})$ when $x = \frac{3}{4}\pi$.

(b) Show that, if y is a function of x and $x = e^u$, then $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2} - \dfrac{dy}{du}$.

Given that y satisfies the differential equation $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + 5y = x^3$, $(x > 0)$,

use the substitution $x = e^u$ to show that $\dfrac{d^2y}{du^2} + 2\dfrac{dy}{du} + 5y = e^{3u}$.

Hence find the general solution for y in terms of x. (J86/I/5)

40. (a) Solve the differential equation

$$\frac{dy}{dx} + \left(\frac{e^x}{4 + e^x}\right)y = 1, \text{ given that } y = 1 \text{ when } x = 0.$$

(b) Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin 2x + 3\cos 2x.$$

Hence show that, when x is large and positive, $y \approx R\sin(2x - \Phi)$, where the values of the constants R and Φ, such that $R > 0$ and $0 < \Phi < \frac{1}{2}\pi$, are to be found.

(N86/I/5)

41. (a) By considering $\dfrac{d}{dx}(x^2y\ln x)$, or otherwise, find y in terms of x given that

$(x\ln x)\dfrac{dy}{dx} + (1 + 2\ln x)y = 1$ $(x > 1)$, and that $y = 1$ when $x = e$.

(b) Given that

$x^2\dfrac{d^2y}{dx^2} + (2x^2 + x)\dfrac{dy}{dx} + (2x^2 + x - \frac{1}{4})y = x^{\frac{5}{2}}$ $(x > 0)$, and that $y = x^{-\frac{1}{2}}z$, show that

$$\frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 2z = x.$$

Hence, or otherwise, find the general solution for y in terms of x. (J87/I/5)

42. (a) By using the substitution $z = y^2$, or otherwise, obtain the solution of the differential equation $(1 + x^2)y\dfrac{dy}{dx} + 2xy^2 = 3$ $(y \geqslant 0)$ such that $y = 1$ when $x = 0$.

Hence find the value of y when $x = 1$.

(b) Obtain the solution of the differential equation $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 20y = 60x + 152$ such that $y = 8$ and $\dfrac{dy}{dx} = 5$ when $x = 0$. (N87/I/5)

43. (i) Obtain the solution of the differential equation $\dfrac{dy}{dx} + (\tanh x)y = 1$ such that $y = 1$ when $x = 0$.

(ii) Show, by means of the substitution $z = y^3$, that the differential equation $3y^2\dfrac{d^2y}{dx^2} + 9y^2\dfrac{dy}{dx} + 6y\left(\dfrac{dy}{dx}\right)^2 + 2y^3 = 0$ can be reduced to the form $\dfrac{d^2z}{dx^2} + a\dfrac{dz}{dx} + bz = 0$, where a and b are numbers which are to be determined.

Hence, or otherwise, find the general solution of the given differential equation, expressing y in terms of x. (J88/I/5)

44. (a) Find the solution of the differential equation $x\frac{dy}{dx} - y = x^3$ such that $y = 1$ when $x = 1$.

(b) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin x + 3\cos x$.

(N88/I/5)

45. (a) Find the solution of the differential equation $(1 + x^2)\frac{dy}{dx} + xy = 1$, such that $y = 1$ when $x = 0$.

(b) Find the general solution of $\frac{d^2y}{dx^2} + (1 + a)\frac{dy}{dx} + ay = (2 - a)e^{-2x}$, where the real constant a is such that $a \neq 1$ and $a \neq 2$.

Find the solution for which $y = 1$ and $\frac{dy}{dx} = -1$ when $x = 0$. (J89/I/5)

46. (a) Given that $(x \sin x)\frac{dy}{dx} + (3 \sin x - x \cos x)y = \sin^2 x$, and that $y = z \sin x$, show that $x\frac{dz}{dx} + 3z = 1$.

Hence, or otherwise, find the general solution for y in terms of x.

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 21 \sin 2x + 12 \cos 2x.$$

(N89/I/5)

47. (a) Show, by means of the substitution $y = \frac{1}{z}$, that the differential equation $(1 + x)\frac{dy}{dx} - 2y + (1 + x)y^2 = 0$ reduces to $\frac{dz}{dx} + (\frac{2}{1 + x})z = 1$.

Hence find y in terms of x, given that $y = 1$ when $x = 0$.

(b) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 65 \sin 2x$.

Hence show that for the large positive x, $y \approx -8 \cos 2x - \sin 2x$, whatever the initial conditions. (J90/I/5)

48. (a) Solve the differential equation $\frac{dy}{dx} - (\cot x)y = \sin x$, $(0 < x < \pi)$, given that $y = \pi$ when $x = \frac{1}{2}\pi$.

(b) Given that $x = e^u$ and that y is a function of x, show that $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

Deduce that the differential equation $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 10y = 10\ln x + 42$, $(x > 0)$ can be reduced to the form $\frac{d^2y}{du^2} + a\frac{dy}{du} + by = cu + d$, where the constants a, b, c and d are to be determined.

Hence find the general solution for y in terms of x. (N90/I/5)