3. CURVE SKETCHING

1. Find all the values of x giving the zeros and the maxima and minima of the function $f : x \to 2 \sin x + \sin 2x$ $(0 \leqslant x \leqslant 4\pi)$.
Give a sketch of the graph of the function. (J72/I/5)

2. Expand $\dfrac{9}{(3+h)^3} - \dfrac{(4+h)^{\frac{5}{2}}}{16}$ in ascending powers of h as far as the term in h^2.
Hence, or otherwise, sketch the graph of $y = \dfrac{9}{(x+2)^3} - \dfrac{(x+3)^{\frac{5}{2}}}{16} + \dfrac{5}{3}$
in the neighbourhood of $x = 1$, giving a clear indication of the concavity of the curve there. (J73/I/1)

3. Express $f(x) = \dfrac{3x-1}{x^2(x-3)}$ in partial fractions.
Show that the curve of $y = f(x)$ has only one stationary point and determine its nature. Sketch the curve. (N73/I/2)

4. Prove that the cubic curve, $y = a(x^3 - 3h^2x)$, where a and h are real, non-zero constants, has
(a) a point of inflexion, C (whose co-ordinates should be stated),
(b) two distinct turning points, A and B (whose coordinates should be stated),
(c) point symmetry about C.
Prove also that the tangent to the curve at each of the points P, $x = \dfrac{1}{2}h$, and Q, $x = -\dfrac{1}{2}h$, passes through a turning point.
Which, if any, of the above properties (a), (b), (c), are true for the general cubic curve $y = ax^3 + bx^2 + cx + d$? (N73/II/1)

5. If $y = \dfrac{9}{x+1} - \dfrac{1}{x-1}$, obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. Find the stationary values of y.
Show that there is only one point of inflexion and that this occurs for a value of x lying between 2 and 3. Sketch the graph of y against x. (J74/I/6)

6. Show that $x-3$ is a factor of $x^3 - x^2 + x - 21$. Find the point of inflexion of the curve $y = x^3 - x^2 + x - 21$.
Draw the graph of this curve paying particular attention to the nature of the curve at its point of inflexion. Sketch the graph of $y = \dfrac{1}{x^3 - x^2 + x - 21}$ (N74/I/2)

7. A curve has the parametric equations $x = b \cot\theta$, $y = a \sec\theta$ $(0 < \theta < \frac{1}{2}\pi)$.
Determine in any form the equation of the tangent at the point P where $\theta = \alpha$.
If this tangent meets the x-axis at G, and F is the foot of the perpendicular from P to Ox, find the area of triangle PFG. (N74/I/8)

8. Sketch the boundary curves and shade the solution space for the non-linear inequalities $x+1 \geqslant y^2$ and $x^2+y^2 \leqslant 7$. Find the greatest value of the expression $E=3y+2x$ subject to the given inequalities. Some attempt to justify that your solution is the greatest value should be given. (N74/II/7)

9. Show that the expression $y=\dfrac{4x^2+4x+21}{4x+3}$ (x real), cannot take values between -5 and 4. Find the coordinates of the turning points of the graph of y against x. Sketch the graph. (N74/II/8)

10. Find all the zeros and stationary points of the function
$f: x \to \cos 2x + 4\cos x - 1$, $(0 \leqslant x \leqslant 4\pi)$.
Sketch the graph of the function f. (J75/I/6)

11. Express $y \equiv \dfrac{-x^2+x+5}{x^2-x-2}$ in partial fraction form.

Show that y cannot take values in the interval $-\dfrac{7}{3}<y<-1$.

Sketch the graph of y. (N75/I/1)

12. Find the stationary points of the function

$f: x \to \dfrac{(x+1)^4}{x^4+1}$ ($x \in$ R, the set of all real numbers).
Sketch the graph of f.
Determine the dependence of the number of real roots of the equation $(x+1)^4=a(x^4+1)$ on the value of the real number a. (J76/I/6)

13. Sketch the sets

$A=\{(x, y): x \in \text{R}, y \in \text{R and } y \leqslant x+1\}$,
$B=\{(x, y): x \in \text{R}, y \in \text{R and } x^2+y^2 \leqslant 5\}$
in the plane. (R is the set of all real numbers.) The function f is defined on $A \cap B$ by $f(x, y)=3x+y$. Show that the maximum value of f is $5\sqrt{2}$. (J76/II/8)

14. A curve has parametric equations $x=t-\dfrac{1}{t}$, $y=t+\dfrac{1}{t}$, ($t \in$ R (the set of real numbers), $t \neq 0$). Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

Sketch the graph of the curve, indicating clearly the coordinates of the points at which the tangent to the curve is parallel to the x-axis. (J77/I/6)

15. (a) Sketch the graph of $y=\dfrac{4x-1}{x+5}$

Hence, or otherwise, find the set of values of x for which $-1<\dfrac{4x-1}{x+5}<1$.

(b) Find the set of values of x for which $3x^3+2x^2+6 \geqslant 19x$. (J80/I/1)

16. Sketch the curve with equation $y=x^3(1-x^2)$.
Hence, or otherwise, sketch the curve with equation $y^2=x^3(1-x^2)$.
Show that, if the line $y=ax$ is to touch the curve $y^2=x^3(1-x^2)$ at a point other than the origin, then $a^2=(2\sqrt{3})/9$. (J80/I/5)

17. By considering $y = \dfrac{x-3}{(x-2)(x+1)}$ as a quadratic equation in x, or otherwise, prove that for real values of x the value of y cannot lie between $\frac{1}{9}$ and 1.

Find the values of x for which $y = \frac{1}{9}$ and $y = 1$, and sketch the graph given by the above equation. (N83/I/1)

18. Write down the equations of asymptotes of the curve $y = x + \dfrac{4}{x^2}$ and give a sketch of the curve. (J86/I/4)

19. Obtain the equations of the asymptotes of the curve $y = \dfrac{x^2}{x+1}$, and give a sketch of the curve. (N86/I/1)

20. The curve C has equation $y = \dfrac{2x^2 + 3x + 1}{x - 1}$.

(i) Verify that $y = 2x + 5 + \dfrac{6}{x-1}$, and hence write down the equations of the asymptotes of C.

(ii) Show that $\dfrac{dy}{dx} = 0$ for just two values of x.

(iii) Draw a sketch of C. (J87/I/1)

21. Write down the equations of the asymptotes of the curve $y = \dfrac{2x+1}{x-1}$, and sketch the graph of y.

In the same diagram, sketch the graph of $y = \dfrac{1}{x^2}$ and deduce, or prove otherwise, that the equation $2x^3 + x^2 - x + 1 = 0$ has exactly one real root. (N87/I/1)

22. The curve C has equation $y = 2x + 1 - \dfrac{5}{2x+1}$.

(i) Write down the equations of the asymptotes of C.

(ii) Show that $\dfrac{dy}{dx}$ is positive at all points of C.

(iii) Draw a sketch of C. (J88/I/1)

23. The curve C has equation $y = \dfrac{(x-1)^2}{x+1}$.

(i) Obtain the equations of the asymptotes of C.

(ii) Show that C has two stationary points and find their coordinates.

(iii) Draw a sketch of C.

(iv) In the same diagram draw a sketch of the curve $y = -\dfrac{1}{x^2}$ and deduce that the equation $x^2(x-1)^2 + x + 1 = 0$ has no real roots. (N88/I/2)

24. Given that the curve $y = \frac{4 - ax^2}{b + x}$ has asymptotes $x = -1$ and $y = 1 - x$, find the values of a and b.

Show that, at all points of the curve, $\frac{dy}{dx}$ is negative.

Sketch the curve. (J89/I/1)

25. The curve C has equation $y = \frac{(x - 1)^2}{x + 1}$.

(i) Verify that $y = x - 3 + \frac{4}{x + 1}$, and hence write down the equations of the asymptotes of C.

(ii) Find the values of x for which $\frac{dy}{dx} = 0$.

(iii) Sketch C. (N89/I/1)

26. The curve C has equation $y = \frac{x^2 + 3x}{x - 1}$.

(i) Find the equations of the asymptotes of C.

(ii) Find the coordinates of the points C at which $\frac{dy}{dx} = 0$.

(iii) Draw a sketch of C.

(iv) In the same diagram, draw a sketch of the curve $y = (x - 1)^2 + 3$ and hence find the number of real roots of the equation $x^3 - 4x^2 + 3x - 4 = 0$.

(J90/I/1)

27. The curve C has equation $y = \frac{x^2 + 3}{x + 1}$.

(i) Find the equations of the asymptotes of C.

(ii) Find the values of x for which $\frac{dy}{dx} = 0$.

(iii) Draw a sketch of C, marking the coordinates of the turning points.

(N90/I/1)