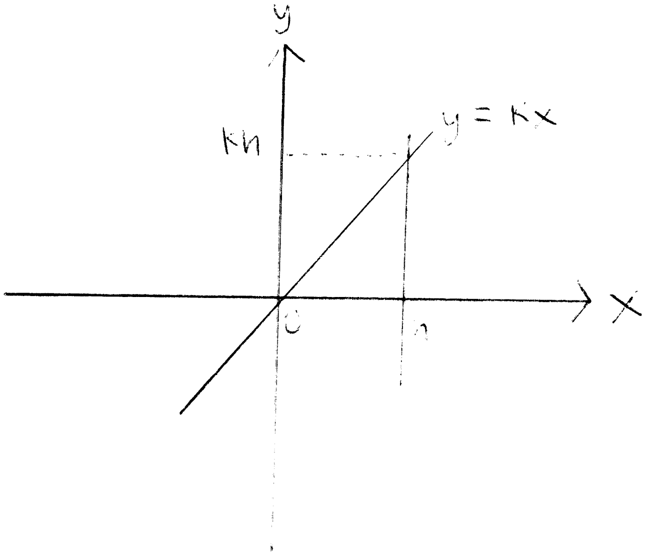
MAY / JUNE 2008

1.



The volume of the cone, V formed by rotating the finite region formed by the line $y = kx$, $k > 0$, the x-axis for $0 \leq x \leq h$ and the line $x = h$ is $\frac{\pi}{3}(kh)^2 h = \frac{\pi k^2 h^3}{3}$, since the volume of a cone of height h and base radius r is $\frac{\pi r^2 h}{3}$.

The x-coordinate of the centroid of the cone is given by

$$\bar{x} = \frac{\int_0^h \pi x y^2 \, dx}{V}$$

$$= \frac{\int_0^h \pi x (kx)^2 \, dx}{\frac{\pi k^2 h^3}{3}}$$

$$= \frac{\pi k^2 \int_0^h x^3\,dx}{\frac{\pi k^2 h^3}{3}}$$

$$= \frac{\pi k^2 \left[\frac{x^4}{4}\right]_0^h}{\frac{\pi k^2 h^3}{3}}$$

$$= \frac{\pi k^2 \left(\frac{h^4}{4} - 0\right)}{\frac{\pi k^2 h^3}{3}}$$

$$= \frac{\frac{\pi k^2 h^4}{4}}{\frac{\pi k^2 h^3}{3}}$$

$$= \frac{3h}{4}$$

2 $$u_n = \ln\left(\frac{1 + x^{n+1}}{1 + x^n}\right), \quad x > -1.$$

$$= \ln(1 + x^{n+1}) - \ln(1 + x^n)$$

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \ln(1 + x^{n+1}) - \ln(1 + x^n)$$

$$= \ln(1 + x^{N+1}) - \ln(1 + x^N)$$

$$+ \ln(1 + x^N) - \ln(1 + x^{N-1})$$

$$+ \ln(1 + x^{N-1}) - \ln(1 + x^{N-2})$$

$$\vdots$$

$$+ \ln(1 + x^4) - \ln(1 + x^3)$$

$$+ \ln(1 + x^3) - \ln(1 + x^2)$$

$$+ \ln(1 + x^2) - \ln(1 + x)$$

$$= \ln(1 + x^{N+1}) - \ln(1 + x)$$

$$= \ln\left(\frac{1 + x^{N+1}}{1 + x}\right)$$

$$u_1 + u_2 + u_3 + \cdots = \sum_{n=1}^{\infty} u_n$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} u_n$$

$$= \lim_{N \to \infty} \ln\left(\frac{1 + x^{N+1}}{1 + x}\right)$$

1) when $-1 < x < 1$,

$$\lim_{N \to \infty} x^{N+1} = 0$$

$$\lim_{N \to \infty} 1 + x^{N+1} = 1 + \lim_{N \to \infty} x^{N+1} = 1 + 0 = 1$$

$\therefore u_1 + u_2 + u_3 + \cdots$

$$= \lim_{N \to \infty} \left[\ln(1 + x^{N+1}) - \ln(1 + x)\right]$$

$$= \lim_{N \to \infty} \ln(1 + x^{N+1}) - \lim_{N \to \infty} \ln(1 + x)$$

$$= \ln \lim_{N \to \infty} (1 + x^{N+1}) - \ln(1 + x)$$

$$= \ln 1 - \ln(1 + x)$$

$$= -\ln(1 + x)$$

ii) when $x = 1$,

$$u_1 + u_2 + u_3 + \cdots = \lim_{N \to \infty} \ln\left(\frac{1 + x^{N+1}}{1 + x}\right)$$

$$= \lim_{N \to \infty} \ln 1$$

$$= \lim_{N \to \infty} 0$$

$$= 0$$

3. If the square matrix A has eigenvalue λ with corresponding eigenvector $\underset{\sim}{e}$ and the square matrix B has eigenvalue μ with corresponding eigenvector $\underset{\sim}{e}$, then

$A\underset{\sim}{e} = \lambda \underset{\sim}{e}$ and $B\underset{\sim}{e} = \mu \underset{\sim}{e}$.

$$\begin{aligned}(A+B)\underset{\sim}{e} &= A\underset{\sim}{e} + B\underset{\sim}{e} \\ &= \lambda\underset{\sim}{e} + \mu\underset{\sim}{e} \\ &= (\lambda+\mu)\underset{\sim}{e}.\end{aligned}$$

$\therefore$ The matrix $A+B$ has eigenvalue $\lambda+\mu$ with corresponding eigenvector $\underset{\sim}{e}$.

If $A = \begin{pmatrix} 3 & -1 & 0 \\ -4 & -6 & -6 \\ 5 & 11 & 10 \end{pmatrix}$ has an eigenvector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $A\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 \\ -4 & -6 & -6 \\ 5 & 11 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix}$$

$$= 4\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

the corresponding eigenvalues for the eigenvector $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ is 4.

If the other two eigenvalues of A are 1 and 2 with corresponding eigenvectors $\begin{pmatrix}1\\2\\-3\end{pmatrix}$ and $\begin{pmatrix}1\\1\\-2\end{pmatrix}$

and the matrix B has eigenvalues 2, 3 and 1 with corresponding eigenvectors $\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\-3\end{pmatrix}$ and $\begin{pmatrix}1\\1\\-2\end{pmatrix}$ respectively, the matrix $A + B$ has eigenvalues 6, 4 and 3 with corresponding eigenvectors $\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\-3\end{pmatrix}$ and $\begin{pmatrix}1\\1\\-2\end{pmatrix}$.

A matrix P and a diagonal matrix D such that $(A+B)^4 = PDP^{-1}$ is given by

$$P = \begin{pmatrix}1 & 1 & 1\\-1 & 2 & 1\\1 & -3 & -2\end{pmatrix}$$ and

$$D = \begin{pmatrix}6 & 0 & 0\\0 & 4 & 0\\0 & 0 & 3\end{pmatrix}^4 = \begin{pmatrix}6^4 & 0 & 0\\0 & 4^4 & 0\\0 & 0 & 3^4\end{pmatrix} = \begin{pmatrix}1296 & 0 & 0\\0 & 256 & 0\\0 & 0 & 81\end{pmatrix}$$

4. $C_1 : r = \theta + 2, \quad 0 \leq \theta \leq \pi$

$C_2 \quad r = \theta^2, \quad 0 \leq \theta \leq \pi$

i) when C_1 and C_2 intersect,

$\theta^2 = \theta + 2$

$\theta^2 - \theta - 2 = 0$

$(\theta - 2)(\theta + 1) = 0$

$\theta \neq -1 \quad \therefore \theta = 2$

$r = 4.$

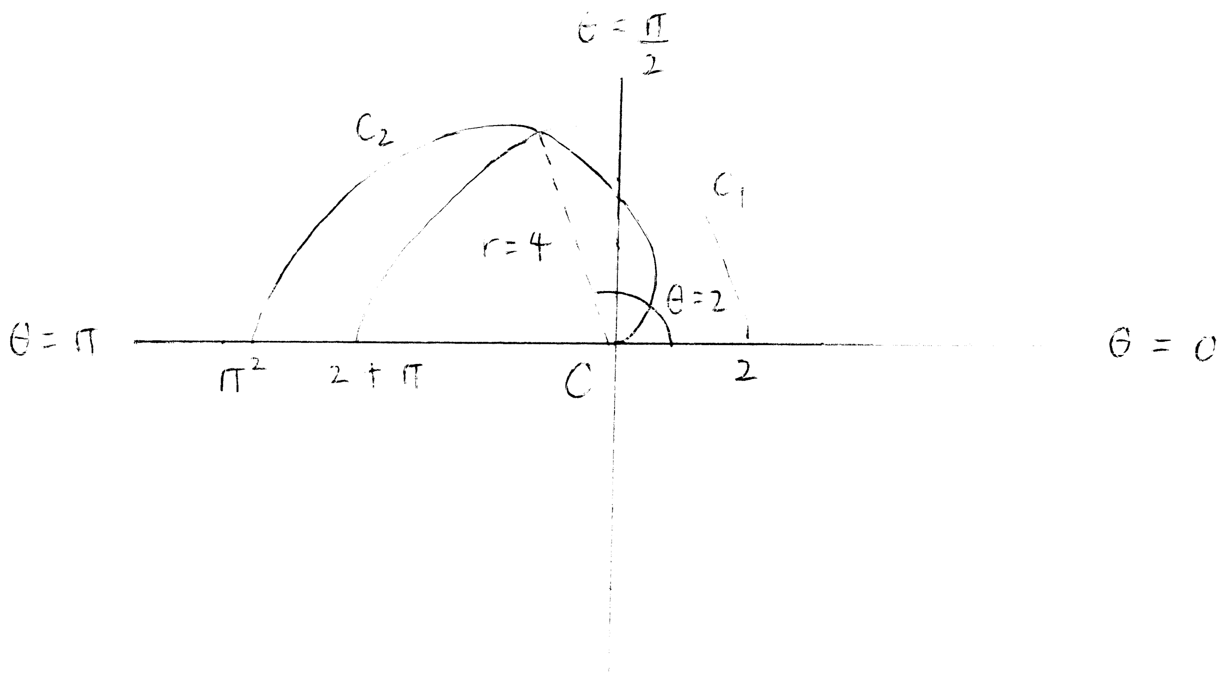
$\therefore$ The polar coordinates of the point of intersection of C_1 and C_2 is $(4, 2)$

ii)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$2 + \frac{\pi}{6}$	$2 + \frac{\pi}{4}$	$2 + \frac{\pi}{3}$	$2 + \frac{\pi}{2}$

θ	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	$2 + \frac{2\pi}{3}$	$2 + \frac{3\pi}{4}$	$2 + \frac{5\pi}{6}$	$2 + \pi$

θ	0	π	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	$\frac{\pi^2}{36}$	$\frac{\pi^2}{16}$	$\frac{\pi^2}{9}$	$\frac{\pi^2}{4}$	$\frac{4\pi^2}{9}$	$\frac{9\pi^2}{16}$	$\frac{25\pi^2}{36}$	π^2



The area bounded by C_1, C_2 and the line $\theta = 0$ is

$$\int_0^2 \frac{(\theta + 2)^2}{2} \, d\theta - \int_0^2 \frac{(\theta^2)^2}{2} \, d\theta$$

$$= \int_0^2 \frac{(\theta + 2)^2}{2} - \frac{\theta^4}{2} \, d\theta$$

$$= \left[\frac{(\theta + 2)^3}{6} - \frac{\theta^5}{10} \right]_0^2$$

$$= \frac{64}{6} - \frac{32}{10} - \left(\frac{8}{6} - 0\right)$$

$$= \frac{32}{3} - \frac{16}{5} - \frac{4}{3}$$

$$= \frac{92}{15}$$

5. $x^3 + x - 1 = 0$

α, β, γ are the roots.

$\alpha^3, \beta^3, \gamma^3$

Let $y = \alpha^3$

$\alpha = y^{\frac{1}{3}}$

α is a root

$\therefore \alpha^3 + \alpha - 1 = 0$

$(y^{\frac{1}{3}})^3 + y^{\frac{1}{3}} - 1 = 0$

$y + y^{\frac{1}{3}} - 1 = 0$

$y^{\frac{1}{3}} = 1 - y$

$y = (1 - y)^3$

$= 1 - 3y + 3y^2 - y^3$

$y^3 - 3y^2 + 4y - 1 = 0$

The equation $y^3 - 3y^2 + 4y - 1 = 0$ has roots $\alpha^3, \beta^3, \gamma^3$

$\alpha^3 + \beta^3 + \gamma^3 = 3$

$\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3 = 4$

$\alpha^3\beta^3\gamma^3 = 1$

$$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2$$
$$- 2(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3)$$
$$= 3^2 - 2(4)$$
$$= 9 - 8$$
$$= 1$$

6 $C: x = 4t - t^2 \quad y = 1 - e^{-t}, \quad 0 \leq t \leq 2$

$$\frac{dx}{dt} = 4 - 2t \qquad \frac{dy}{dt} = e^{-t}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{e^{-t}}{4 - 2t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{4 - 2t}\frac{d}{dt}\left(\frac{e^{-t}}{4 - 2t}\right)$$

$$= \frac{1}{4 - 2t}\frac{((4 - 2t)(-e^{-t}) - e^{-t}(-2))}{(4 - 2t)^2}$$

$$= \frac{-4e^{-t} + 2te^{-t} + 2e^{-t}}{(4 - 2t)^3}$$

$$= \frac{2te^{-t} - 2e^{-t}}{(4 - 2t)^3}$$

$$= \frac{2(t-1)e^{-t}}{8(2-t)^3}$$

$$= \frac{(t-1)e^{-t}}{4(2-t)^3}$$

The mean value of $\frac{d^2y}{dx^2}$ with respect to x

over the interval $0 \leq x \leq \frac{7}{4}$ is

$$\frac{1}{\frac{7}{4}-0}\int_0^{\frac{7}{4}} \frac{d^2y}{dx^2}\, dx$$

when $x = 0$ $\quad 4t - t^2 = 0$

$$t(4-t) = 0$$

$$t = 0, 4.$$

when $x = \frac{7}{4}$ $\quad 4t - t^2 = \frac{7}{4}$

$$16t - 4t^2 = 7$$

$$4t^2 - 16t + 7 = 0$$

$$(2t-7)(2t-1) = 0$$

$$t = \frac{1}{2}, \frac{7}{2}$$

$$\frac{1}{\frac{7}{4}-0}\int_0^{\frac{7}{4}} \frac{d^2y}{dx^2}\,dx$$

$$= \frac{4}{7}\int_0^{\frac{7}{4}} \frac{d}{dx}\left(\frac{dy}{dx}\right)\,dx$$

$$= \frac{4}{7}\left[\frac{dy}{dx}\right]_0^{\frac{7}{4}}$$

$$= \frac{4}{7}\left[\frac{e^{-t}}{4-2t}\right]_0^{\frac{1}{2}}$$

$$= \frac{4}{7}\left(\frac{e^{-\frac{1}{2}}}{3} - \frac{1}{4}\right)$$

$$= \frac{4e^{-\frac{1}{2}}}{21} - \frac{1}{7}$$

$$= \frac{4e^{-\frac{1}{2}} - 3}{21}$$

7 $$\sum_{r=1}^{n} 3r^5 + r^3 = \frac{n^3}{2}(n+1)^3, \quad n \geq 1.$$

when $n = 1$:

$$\sum_{r=1}^{1} 3r^5 + r^3 = 3(1^5) + 1^3$$

$$= 3(1) + 1$$

$$= 3 + 1$$

$$= 4$$

$$= \frac{8}{2}$$

$$= \frac{1 \cdot 8}{2}$$

$$= \frac{1^3 2^3}{2}$$

$$= \frac{1^3(1+1)^3}{2}$$

Assume the statement is true when $n = k$

$$\sum_{r=1}^{k} 3r^5 + r^3 = \frac{k^3(k+1)^3}{2}$$

when $n = k+1$.

$$\sum_{r=1}^{k+1} 3r^5 + r^3 = 3(k+1)^5 + (k+1)^3$$

$$+ \sum_{r=1}^{k} 3r^5 + r^3$$

$$= 3(k+1)^5 + (k+1)^3 + \frac{k^3(k+1)^3}{2}$$

$$= (k+1)^3\left(3(k+1)^2 + 1 + \frac{k^3}{2}\right)$$

$$= (k+1)^3\left(3k^2 + 6k + 3 + 1 + \frac{k^3}{2}\right)$$

$$= (k+1)^3 \frac{(k^3 + 6k^2 + 12k + 6 + 2)}{2}$$

$$= \frac{(k+1)^3(k^3 + 6k^2 + 12k + 8)}{2}$$

$$= \frac{(k+1)^3(k+2)^3}{2}$$

$$= \frac{(k+1)^3(k+1+1)^3}{2}$$

$$\therefore \sum_{r=1}^{n} 3r^5 + r^3 = \frac{n^3(n+1)^3}{2}$$ for every positive integer n.

$$\sum_{r=1}^{n} 3r^5 + r^3 = \frac{n^3(n+1)^3}{2}$$

$$3\sum_{r=1}^{n} r^5 + \sum_{r=1}^{n} r^3 = \frac{n^3(n+1)^3}{2}$$

$$3\sum_{r=1}^{n} r^5 + \frac{n^2(n+1)^2}{4} = \frac{n^3(n+1)^3}{2}$$

$$3\sum_{r=1}^{n} r^5 = \frac{n^3(n+1)^3}{2} - \frac{n^2(n+1)^2}{4}$$

$$= \frac{n^2(n+1)^2}{2}\left(n(n+1) - \frac{1}{2}\right)$$

$$= \frac{n^2(n+1)^2}{2}\left(n^2 + n - \frac{1}{2}\right)$$

$$= \frac{n^2(n+1)^2}{2}\frac{(2n^2 + 2n - 1)}{2}$$

$$= \frac{n^2(n+1)^2(2n^2 + 2n - 1)}{4}$$

$$\sum_{r=1}^{n} r^5 = \frac{n^2(n+1)^2(2n^2 + 2n - 1)}{12}$$

8. i) $$I_n = \int_0^{\frac{\pi}{2}} t^n \sin t \, dt$$

$$u = t^n \qquad dv = \sin t \, dt$$

$$du = nt^{n-1} dt \qquad v = -\cos t$$

$$= \left[-t^n \cos t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -nt^{n-1} \cos t \, dt$$

$$= -\left(\frac{\pi}{2}\right)^n \cos \frac{\pi}{2} - 0 + n \int_0^{\frac{\pi}{2}} t^{n-1} \cos t \, dt$$

$$= n \int_0^{\frac{\pi}{2}} t^{n-1} \cos t \, dt$$

$$u = t^{n-1} \qquad dv = \cos t \, dt$$

$$du = (n-1) t^{n-2} dt \qquad v = \sin t$$

$$= n\left(\left[t^{n-1} \sin t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1) t^{n-2} \sin t \, dt\right)$$

$$= n\left(\left(\frac{\pi}{2}\right)^{n-1} \sin \frac{\pi}{2} - 0 - (n-1) \int_0^{\frac{\pi}{2}} t^{n-2} \sin t \, dt\right)$$

$$= n\left(\left(\frac{\pi}{2}\right)^{n-1} - (n-1) \int_0^{\frac{\pi}{2}} t^{n-2} \sin t \, dt\right)$$

$$= n\left(\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) \int_0^{\frac{\pi}{2}} t^{n-2} \sin t \, dt\right.$$

$$= n\left(\frac{\pi}{2}\right) - n(n-1) I_{n-2}$$

ii) $\frac{dx}{dt} = t^4(1 - \cos 2t) \quad \frac{dy}{dt} = t^4 \sin 2t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^8(1-\cos 2t)^2 + t^8\sin^2 2t$$

$$= t^8((1-\cos 2t)^2 + \sin^2 2t)$$

$$= t^8(1 - 2\cos 2t + \cos^2 2t + \sin^2 2t)$$

$$= t^8(1 - 2\cos 2t + 1)$$

$$= t^8(2 - 2\cos 2t)$$

$$= 2t^8(1 - \cos 2t)$$

$$= 2t^8(1 - (1 - 2\sin^2 t))$$

$$= 2t^8(1 - 1 + 2\sin^2 t)$$

$$= 2t^8(2\sin^2 t)$$

$$= 4t^8 \sin^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2t^4 \sin t$$

The arc length from $t = 0$ to $t = 2$ is

$$\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} 2t^4 \sin t \; dt$$

$$= 2\int_0^{\frac{\pi}{2}} t^4 \sin t \; dt$$

$$= 2I_4$$

$$= 2\left(4\left(\frac{\pi}{2}\right)^3 - 4(3)I_2\right)$$

$$= 2\left(4\left(\frac{\pi^3}{8}\right) - 12I_2\right)$$

$$= 2\left(\frac{\pi^3}{2} - 12\left(2\left(\frac{\pi}{2}\right)^1 - 2(1)I_0\right)\right)$$

$$= 2\left(\frac{\pi^3}{2} - 12(\pi - 2I_0)\right)$$

$$= \pi^3 - 24(\pi - 2I_0)$$

$$= \pi^3 - 24\pi + 48I_0$$

$$= \pi^3 - 24\pi + 48\int_0^{\frac{\pi}{2}} \sin t \; dt$$

$$= \pi^3 - 24\pi + 48\left[-\cos t\right]_0^{\frac{\pi}{2}}$$

$$= \pi^3 - 24\pi + 48\left(-\cos\frac{\pi}{2} - (-\cos 0)\right)$$

$$= \pi^3 - 24\pi + 48(0 + 1)$$

$$= \pi^3 - 24\pi + 48$$

9 C : $y = \dfrac{x^2 - 2x + \lambda}{x + 1}$

$$
\begin{array}{r|l}
 & x - 3 \\
\hline
x + 1 & x^2 - 2x + \lambda \\
 & \underline{x^2 + x} \\
 & \quad -3x + \lambda \\
 & \quad \underline{-3x - 3} \\
 & \qquad \lambda + 3
\end{array}
$$

$$y = x - 3 + \frac{\lambda + 3}{x + 1}$$

As $x \to \pm\infty$ $y \to x - 3$

As $x \to -1$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = x - 3$ and $x = -1$.

When C meets the x-axis,

$$0 = \frac{x^2 - 2x + \lambda}{x + 1}$$

$$x^2 - 2x + \lambda = 0$$

$$(x - 1)^2 = 1 - \lambda$$

If the x-axis is a tangent to C , $\lambda = 1$

$x = 1$: $\quad y = x - 3 + \dfrac{4}{x+1}$

Intersection points.

when $y = 0$ $\quad x - 3 + \dfrac{4}{x+1} = 0$

$$\frac{(x-3)(x+1) + 4}{x+1} = 0$$

$$(x-3)(x+1) + 4 = 0$$

$$x^2 - 2x - 3 + 4 = 0$$

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

$$x = 1$$

when $x = 0$ $\quad y = 1$

Critical points: $\quad \dfrac{dy}{dx} = 1 - \dfrac{4}{(x+1)^2}$

when $\dfrac{dy}{dx} = 0$. $\quad 1 - \dfrac{4}{(x+1)^2} = 0$

$$1 = \frac{4}{(x+1)^2}$$

$$(x+1)^2 = 4$$

$$x + 1 = \pm 2$$

$$x = -3, 1$$

$$y = -8, 0$$

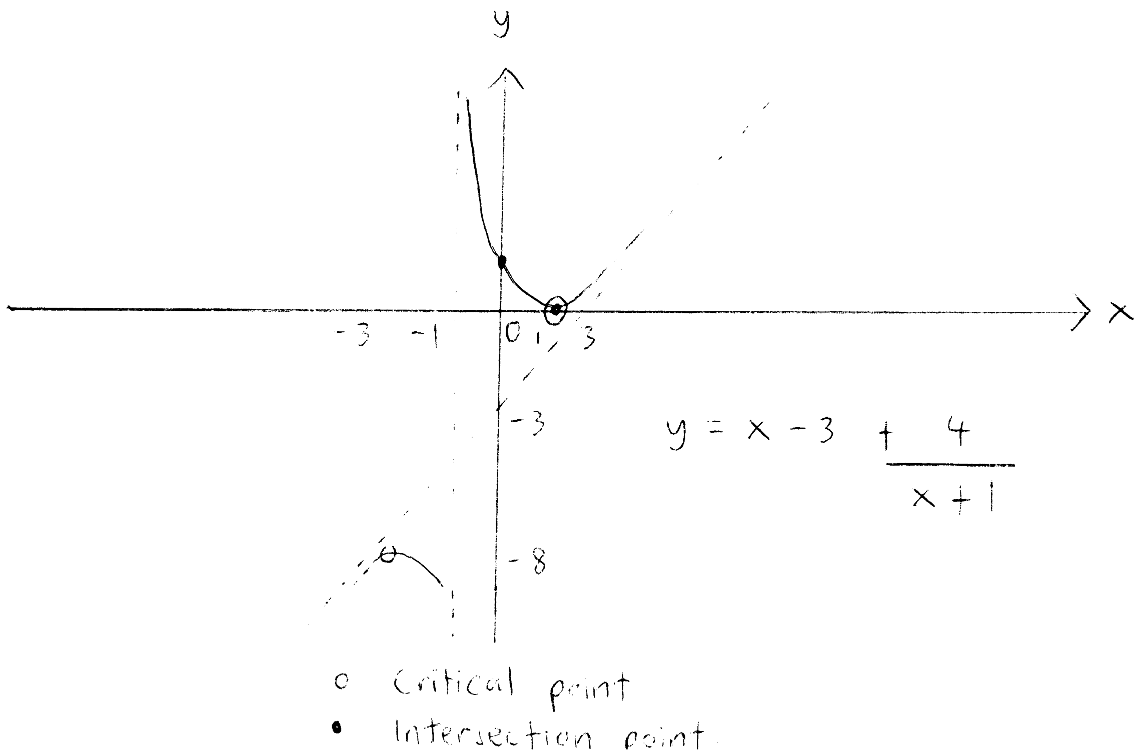
$$\frac{d^2y}{dx^2} = \frac{8}{(x+1)^3}$$

when $x = -3$: $\frac{d^2y}{dx^2} = -1 < 0$

when $x = 1$: $\frac{d^2y}{dx^2} = 1 > 0$

$(-3, -8)$ is a maximum point and

$(1, 0)$ is a minimum point.



$x = -4$. $\quad y = x - 3 - \dfrac{1}{x+1}$

Intersection points:

when $x = 0$ $\quad y = -4$

when $y = 0$ $\quad x - 3 - \dfrac{1}{x+1} = 0$

$$\frac{(x-3)(x+1) - 1}{x+1} = 0$$

$$(x-3)(x+1) - 1 = 0$$

$$x^2 - 2x - 3 - 1 = 0$$

$$x^2 - 2x - 4 = 0$$

$$(x-1)^2 - 5 = 0$$

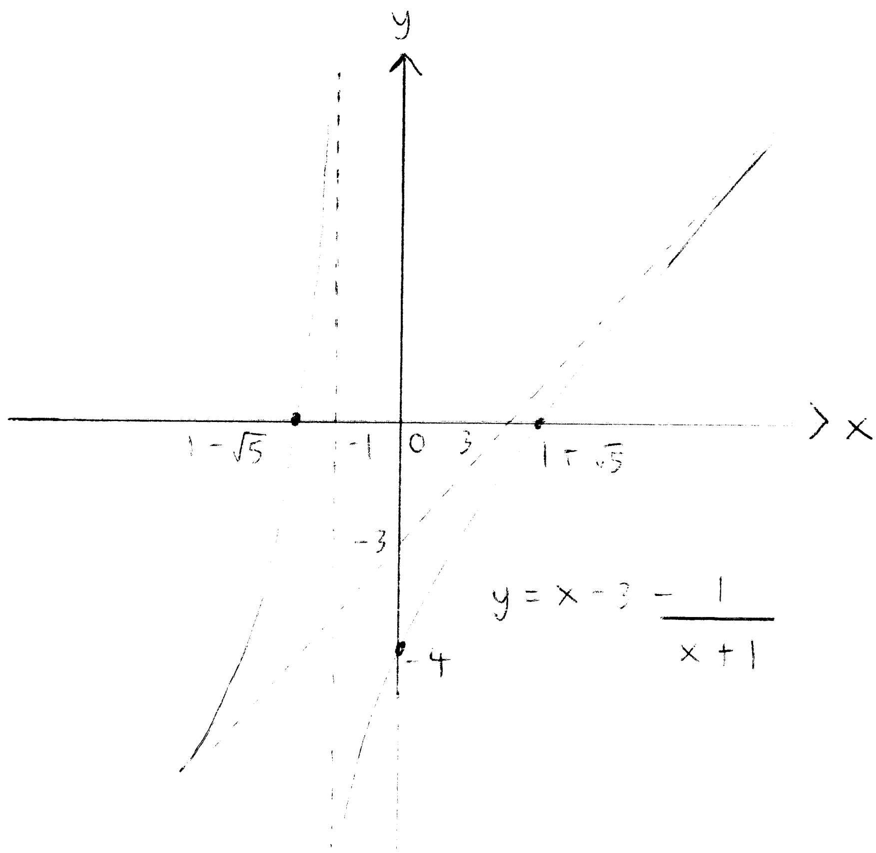
$$x = 1 \pm \sqrt{5}$$

Critical points:

$$\frac{dy}{dx} = 1 + \frac{1}{(x+1)^2} > 1$$

$$\frac{dy}{dx} > 0$$

no critical points.



• Intersection points

10. $$\sum_{n=1}^{N} z^{2n-1} = z + z^3 + z^5 + \cdots + z^{2N-1}$$

$$= \frac{z(1 - z^{2N})}{1 - z^2}, \quad z = e^{i\theta}$$

$$\sum_{n=1}^{N} (e^{i\theta})^{2n-1} = \frac{e^{i\theta}(1 - e^{2iN\theta})}{1 - e^{2i\theta}}$$

$$\sum_{n=1}^{N} (\cos\theta + i\sin\theta)^{2n-1} = \frac{e^{i\theta}(1 - e^{2iN\theta})}{1 - e^{2i\theta}}$$

$$\sum_{n=1}^{N} \cos(2n-1)\theta + i\sin(2n-1)\theta$$

$$= \frac{e^{i\theta}(1 - e^{2iN\theta})(1 - e^{-2i\theta})}{(1 - e^{2i\theta})(1 - e^{-2i\theta})}$$

$$\sum_{n=1}^{N} \cos(2n-1)\theta + i\sum_{n=1}^{N} \sin(2n-1)\theta$$

$$= \frac{e^{i\theta}(1 - e^{2iN\theta} - e^{-2i\theta} + e^{2i(N-1)\theta})}{1 - (e^{2\theta i} + e^{-2\theta i}) + 1}$$

$$= \frac{e^{i\theta}(1 - e^{2iN\theta} - e^{-2i\theta} + e^{2i(N-1)\theta})}{2 - 2\cos 2\theta}$$

$$= \frac{(\cos\theta + i\sin\theta)}{2 - 2\cos 2\theta}\Big(1 - (\cos 2N\theta + i\sin 2N\theta) - (\cos(-2\theta) + i\sin(-2\theta)) + \cos 2(N-1)\theta + i\sin 2(N-1)\theta)\Big)$$

$$= \frac{(\cos\theta + i\sin\theta)}{2 - 2(1 - 2\sin^2\theta)}(1 - \cos 2N\theta - i\sin 2N\theta - \cos 2\theta + i\sin 2\theta + \cos 2(N-1)\theta + i\sin 2(N-1)\theta)$$

$$= \frac{(\cos\theta + i\sin\theta)}{4\sin^2\theta}(1 - \cos 2N\theta - i\sin 2N\theta - \cos 2\theta + i\sin 2\theta + \cos 2(N-1)\theta + i\sin 2(N-1)\theta)$$

$$= \frac{\cos\theta - \cos\theta\cos 2N\theta - \cos\theta\cos 2\theta + \cos\theta\cos 2(N-1)\theta + \sin\theta\sin 2N\theta - \sin\theta\sin 2\theta - \sin\theta\sin 2(N-1)\theta}{4\sin^2\theta}$$

$$+ \frac{i(-\cos\theta\sin 2N\theta + \cos\theta\sin 2\theta + \cos\theta\sin 2(N-1)\theta + \sin\theta - \sin\theta\cos 2N\theta - \sin\theta\cos 2\theta + \sin\theta\cos 2(N-1)\theta)}{4\sin^2\theta}$$

Equating real parts,

$$\sum_{n=1}^{N} \cos(2n-1)\theta$$

$$= \frac{\cos\theta - \cos\theta\cos 2N\theta - \cos\theta\cos 2\theta + \cos\theta\cos 2(N-1)\theta + \sin\theta\sin 2N\theta - \sin\theta\sin 2\theta - \sin\theta\sin 2(N-1)\theta}{4\sin^2\theta}$$

$$= \frac{\cos\theta - (\cos 2N\theta\cos\theta - \sin\theta\sin 2N\theta) - (\cos 2\theta\cos\theta + \sin 2\theta\sin\theta) + \cos(2N-1)\theta\cos\theta - \sin(2N-1)\theta\sin\theta}{4\sin^2\theta}$$

$$= \frac{\cos\theta - \cos(2N+1)\theta - \cos\theta + \cos(2N-1)\theta}{4\sin^2\theta}$$

$$= \frac{\cos(2N-1)\theta - \cos(2N+1)\theta}{4\sin^2\theta}$$

$$= \frac{2\sin 2N\theta\sin\theta}{4\sin^2\theta}$$

$$= \frac{\sin 2N\theta}{2\sin\theta},$$

$$\frac{d}{d\theta}\sum_{n=1}^{N}\cos(2n-1)\theta = \frac{d}{d\theta}\left(\frac{\sin 2N\theta}{2\sin\theta}\right)$$

$$\sum_{n=1}^{N} -(2n-1)\sin(2r-1)\theta$$

$$= \frac{2\sin\theta\,(2N\cos 2N\theta) - \sin 2N\theta\,(2\cos\theta)}{4\sin^2\theta}$$

$$= \frac{4N\sin\theta\cos 2N\theta - 4\cos\theta\sin 2N\theta}{4\sin^2\theta}$$

$$\therefore \sum_{n=1}^{N} (2n-1)\sin(2n-1)\theta$$

$$= \frac{4\cos\theta\sin 2N\theta - 4N\sin\theta\cos 2N\theta}{4\sin^2\theta}$$

when $\theta = \frac{\pi}{N}$,

$$\sum_{n=1}^{N} (2n-1)\sin(2n-1)\theta$$

$$= \frac{4\cos\frac{\pi}{N}\sin 2\pi - 4N\sin\frac{\pi}{N}\cos 2\pi}{4\sin^2\frac{\pi}{N}}$$

$$= \frac{0 - 4N\sin\frac{\pi}{N}}{4\sin^2\frac{\pi}{N}}$$

$$= \frac{-N}{\sin\frac{\pi}{N}}$$

$$= -N\csc\frac{\pi}{N}$$

11 $$2x^2\frac{d^2y}{dx^2} + (3x^2 + 8x)\frac{dy}{dx} + (x^2 + 6x + 4)y = f(x)$$

$$y = x^\alpha w$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^\alpha w)$$

$$= x^\alpha\frac{dw}{dx} + w\frac{d}{dx}(x^\alpha)$$

$$= x^\alpha\frac{dw}{dx} + w\alpha x^{\alpha-1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(x^\alpha\frac{dw}{dx} + w\alpha x^{\alpha-1}\right)$$

$$= \frac{d}{dx}\left(x^\alpha\frac{dw}{dx}\right) + \frac{d}{dx}(w\alpha x^{\alpha-1})$$

$$= x^\alpha\frac{d}{dx}\left(\frac{dw}{dx}\right) + \frac{dw}{dx}\frac{d}{dx}(x^\alpha)$$

$$+ w\alpha\frac{d}{dx}(x^{\alpha-1}) + \alpha x^{\alpha-1}\frac{dw}{dx}$$

$$= x^\alpha\frac{d^2w}{dx^2} + \alpha x^{\alpha-1}\frac{dw}{dx}$$

$$+ w\alpha(\alpha-1)x^{\alpha-2} + \alpha x^{\alpha-1}\frac{dw}{dx}$$

$$2x^2 \frac{d^2y}{dx^2} + (3x^2 + 8x)\frac{dy}{dx} + (x^2 + 6x + 4)y$$

$$= 2x^2\left(x^\alpha \frac{d^2w}{dx^2} + \alpha x^{\alpha-1}\frac{dw}{dx} + w\alpha(\alpha-1)x^{\alpha-2} + \alpha x^{\alpha-1}\frac{dw}{dx}\right)$$

$$+ (3x^2 + 8x)\left(x^\alpha \frac{dw}{dx} + w\alpha x^{\alpha-1}\right)$$

$$+ (x^2 + 6x + 4)x^\alpha w$$

$$= 2x^{\alpha+2}\frac{d^2w}{dx^2} + 2\alpha x^{\alpha+1}\frac{dw}{dx} + 2\alpha(\alpha-1)wx^\alpha + 2\alpha x^{\alpha+1}\frac{dw}{dx}$$

$$+ 3x^{\alpha+2}\frac{dw}{dx} + 3\alpha w x^{\alpha+1} + 8x^{\alpha+1}\frac{dw}{dx} + 8\alpha w x^\alpha$$

$$+ (x^2 + 6x + 4)x^\alpha w$$

$$= 2x^{\alpha+2}\frac{d^2w}{dx^2} + (4\alpha x^{\alpha+1} + 3x^{\alpha+2} + 8x^{\alpha+1})\frac{dw}{dx}$$

$$+ wx^{\alpha+2} + 6wx^{\alpha+1} + 3\alpha w x^{\alpha+1}$$

$$+ 4wx^\alpha + 2\alpha(\alpha-1)wx^\alpha + 8\alpha w x^\alpha$$

$$= 2x^{\alpha+2}\frac{d^2w}{dx^2} + (4\alpha x^{\alpha+1} + 3x^{\alpha+2} + 8x^{\alpha+1})\frac{dw}{dx}$$

$$+ wx^{\alpha+2} + 6wx^{\alpha+1} + 3\alpha w x^{\alpha+1}$$

$$+ 2(\alpha+1)(\alpha+2)wx^\alpha$$

when $\alpha = -2$

$$2x^2\frac{d^2y}{dx^2} + (3x^2 + 8x)\frac{dy}{dx} + (x^2 + 6x + 4)y$$

$$= \frac{2d^2w}{dx^2} + \frac{3dw}{dx} + w$$

$\therefore$ The substitution $y = x^{-2}w$ reduces the differential equation

$$2x^2\frac{d^2y}{dx^2} + (3x^2 + 8x)\frac{dy}{dx} + (x^2 + 6x + 4)y = f(x)$$

to $$\frac{2d^2w}{dx^2} + \frac{3dw}{dx} + w = f(x)$$

If $f(x) = 6\sin 2x + 7\cos 2x$,

$$\frac{2d^2w}{dx^2} + \frac{3dw}{dx} + w = 6\sin 2x + 7\cos 2x$$

$$\frac{2d^2w}{dx^2} + \frac{3dw}{dx} + w = 0$$

$$2m^2 + 3m + 1 = 0$$

$$(2m + 1)(m + 1) = 0$$

$$m = -\frac{1}{2}, -1$$

∴ The complementary function, w_c, is

$$w_c = Ae^{-\frac{x}{2}} + Be^{-x}$$

The particular integral, w_p, is given by

$$w_p = C\cos 2x + D\sin 2x$$

$$\frac{dw_p}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\frac{d^2w_p}{dx^2} = -4C\cos 2x - 4D\sin 2x$$

$$\frac{2d^2w}{dx^2} + \frac{3dw_p}{dx} + w_p$$

$$= 2(-4C\cos 2x - 4D\sin 2x)$$

$$+ 3(-2C\sin 2x + 2D\cos 2x)$$

$$+ C\cos 2x + D\sin 2x$$

$$= -8C\cos 2x - 8D\sin 2x$$

$$-6C\sin 2x + 6D\cos 2x$$

$$+ C\cos 2x + D\sin 2x$$

$$= (6D - 7C)\cos 2x + (-6C - 7D)\sin 2x$$

$$= 6\sin 2x + 7\cos 2x$$

$6D - 7C = 7 \qquad -6C - 7D = 6$

$42D - 49C = 49 \qquad -36C - 42D = 36$

$-85C = 85$

$C = -1$

$D = 0$

$\therefore \; w_p = -\cos 2x$

$w = w_c + w_p$

$= Ae^{-\frac{x}{2}} + Be^{-x} - \cos 2x$

since $w = x^2 y$,

$x^2 y = Ae^{-\frac{x}{2}} + Be^{-x} - \cos 2x$

$y = (Ae^{-\frac{x}{2}} + Be^{-x} - \cos 2x)x^{2}$

12. EITHER

$\overrightarrow{OA} = 7\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$, $\overrightarrow{OB} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$,

$\overrightarrow{OC} = 2\underset{\sim}{i} + 6\underset{\sim}{j} + 3\underset{\sim}{k}$, $\overrightarrow{OD} = 2\underset{\sim}{i} + 7\underset{\sim}{j} + \lambda\underset{\sim}{k}$

i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k} - (7\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$

$= -4\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$\underset{\sim}{r} = \overrightarrow{OA} + s\overrightarrow{AB}$

$= 7\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k} + s(-4\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= 2\underset{\sim}{i} + 7\underset{\sim}{j} + \lambda\underset{\sim}{k} - (2\underset{\sim}{i} + 6\underset{\sim}{j} + 3\underset{\sim}{k})$

$= \underset{\sim}{j} + (\lambda - 3)\underset{\sim}{k}$

$\underset{\sim}{r} = \overrightarrow{OC} + t\overrightarrow{CD}$

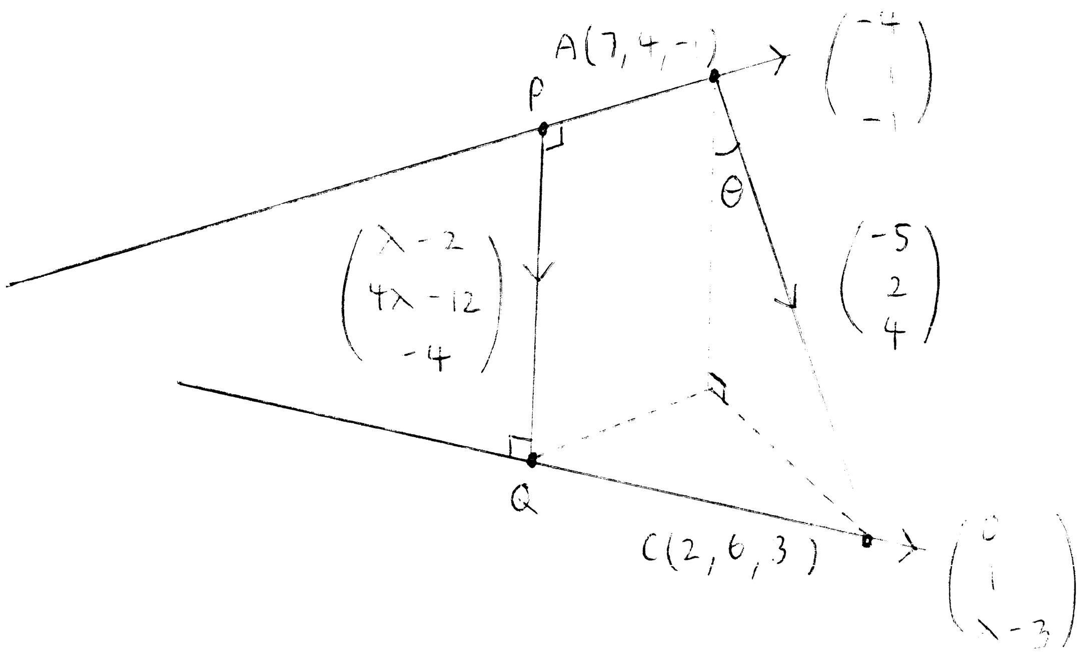
$= 2\underset{\sim}{i} + 6\underset{\sim}{j} + 3\underset{\sim}{k} + t(\underset{\sim}{j} + (\lambda - 3)\underset{\sim}{k})$

The line AB has equation

$\underset{\sim}{r} = 7\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k} + s(-4\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$

and the line CD has equation

$\underset{\sim}{r} = 2\underset{\sim}{i} + 6\underset{\sim}{j} + 3\underset{\sim}{k} + t(\underset{\sim}{j} + (\lambda - 3)\underset{\sim}{k})$.



If P and Q are the points on the lines AB and CD such that the distance PQ is the shortest distance between the lines AB and CD, the vector $\overrightarrow{PQ}$ is perpendicular to $-4\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{j} + (\lambda - 3)\underset{\sim}{k}$. Since $\overrightarrow{PQ}$ is perpendicular to both $-4\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{j} + (\lambda - 3)\underset{\sim}{k}$, it is parallel to $\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \lambda - 3 \end{pmatrix}$.

$$\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \lambda - 3 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -4 & 1 & -1 \\ 0 & 1 & \lambda - 3 \end{vmatrix}$$

$$(\lambda - 2)\underset{\sim}{i} + 4(\lambda - 3)\underset{\sim}{j} - 4\underset{\sim}{k}.$$

Since $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}$ and

the shortest distance between the lines AB and CD is 3, $|\overrightarrow{PQ}| = 3$.

$$AC \cdot PQ = |AC||PQ| \cos\theta$$

$$\begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} \lambda - 2 \\ 4\lambda - 12 \\ -4 \end{pmatrix} = \left|\begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}\right| \left|\begin{pmatrix} \lambda - 2 \\ 4\lambda - 12 \\ -4 \end{pmatrix}\right| \cos\theta$$

$$-5\lambda + 10 + 8\lambda - 24 - 16 =$$

$$= \left|\begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}\right| \sqrt{(\lambda - 2)^2 + (4\lambda - 12)^2 + (-4)^2} \cos\theta$$

$$\therefore \left|\begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}\right| \cos\theta = \frac{3\lambda - 30}{\sqrt{(\lambda - 2)^2 + (4\lambda - 12)^2 + 16}}$$

Since $|\overrightarrow{AC}| \cos\theta = |\overrightarrow{PQ}|$,

$$|PQ| = \frac{3\lambda - 30}{\sqrt{(\lambda - 2)^2 + (4\lambda - 12)^2 + 16}} = 3$$

$$\sqrt{(\lambda - 2)^2 + (4\lambda - 12)^2 + 16} = \frac{3\lambda - 30}{3}$$

$$= \lambda - 10$$

$$(\lambda - 2)^2 + (4\lambda - 12)^2 + 16 = (\lambda - 10)^2$$

$$\lambda^2 - 4\lambda + 4 + 16\lambda^2 - 96\lambda + 144 + 16$$

$$= \lambda^2 - 20\lambda + 100$$

$$16\lambda^2 - 80\lambda + 64 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

ii) $(\lambda - 1)(\lambda - 4) = 0$

$$\lambda = 1, 4$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 2 \\ 1 \\ \lambda \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 3 \\ \lambda + 1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -5 \\ 3 \\ \lambda + 1 \end{pmatrix}$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -4 & 1 & -1 \\ -5 & 3 & \lambda + 1 \end{vmatrix}$$

$$= \begin{pmatrix} \lambda + 4 \\ 4\lambda + 9 \\ -7 \end{pmatrix}$$

Since the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are parallel to the plane containing A, B and D, the vector $\overrightarrow{AB} \times \overrightarrow{AD}$ is perpendicular to the plane.

The normals to the planes through A, B, D corresponding to the values $\lambda = 1$ and $\lambda = 4$ are $\begin{pmatrix} 5 \\ 13 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 25 \\ -7 \end{pmatrix}$.

$$\begin{pmatrix} 5 \\ 13 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 25 \\ -7 \end{pmatrix} = \left|\begin{pmatrix} 5 \\ 13 \\ -7 \end{pmatrix}\right| \left|\begin{pmatrix} 8 \\ 25 \\ -7 \end{pmatrix}\right| \cos\theta$$

$$40 + 325 + 49 = \sqrt{243}\sqrt{738} \cos\theta$$

$$\cos\theta = \frac{414}{\sqrt{243}\sqrt{738}}$$

$= 0.9776$

$\theta = 12.1^{\circ}$

$\therefore$ The acute angle between the planes through A, B, C corresponding to $\lambda = 1$ and $\lambda = 4$ is 12.1°.

12. OR

$T : R^4 \to R^4$

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 3 & -1 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & 3 & -4 & -1 \end{pmatrix}$$

i) $$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 3 & -1 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & 3 & -4 & -1 \end{pmatrix}$$

$$\xrightarrow[-r_1 + r_3]{-r_1 + r_2} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 4 & 2 \\ 0 & 3 & -4 & -1 \end{pmatrix}$$

$$\xrightarrow[-3r_2 + r_4]{2r_2 + r_4} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -4 & -4 \end{pmatrix}$$

$$\xrightarrow{r_3 + r_4} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{r_3}{4}} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

∴ The dimension of V, the range space of T is 3.

ii) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ -4 \end{pmatrix}$

$$k_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \\ 0 \\ 3 \end{pmatrix} + k_3 \begin{pmatrix} -1 \\ -1 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} k_1 + 2k_2 - k_3 \\ k_1 + 3k_2 - k_3 \\ k_1 + 3k_3 \\ 3k_2 - 4k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$k_1 + 2k_2 - k_3 = 0$$

$$k_1 + 3k_2 - k_3 = 0$$

$$k_1 + 3k_3 = 0$$

$$3k_2 - 4k_3 = 0$$

$k_3 = 0$, $k_2 = 0$, $k_1 = 0$.

$\therefore$ The vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ -4 \end{pmatrix}$ are linearly independent.

iii) Since the vectors $\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\\3\end{pmatrix}, \begin{pmatrix}-1\\-1\\3\\-4\end{pmatrix}$

linearly independent, a basis of V is

$$\left\{ \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\\3\end{pmatrix}, \begin{pmatrix}-1\\-1\\3\\-4\end{pmatrix} \right\}$$

iv) W is not a vector space since it does not contain the zero vector.

v) If a vector $\begin{pmatrix}x\\y\\z\\t\end{pmatrix}$ belongs to V,

$$\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = s\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\3\\0\\3\end{pmatrix} + r\begin{pmatrix}-1\\-1\\3\\-4\end{pmatrix}$$

$$= \begin{pmatrix} s + 2t - r \\ s + 3t - r \\ s + 3r \\ 3t - 4r \end{pmatrix}$$

$$\begin{aligned} y - z - t &= s + 3t - r - (s + 3r) - (3t - 4r) \\ &= s + 3t - r - s - 3r - 3t + 4r \\ &= 0 \end{aligned}$$

. . If the vector $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ belongs to W

then $y - z - t \neq 0$.