1. $r = \sqrt{2}\sec\left(\theta - \frac{\pi}{4}\right)$

$$= \frac{\sqrt{2}}{\cos\left(\theta - \frac{\pi}{4}\right)}$$

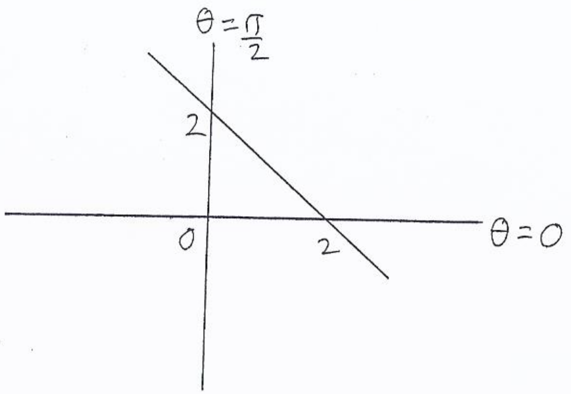
$$= \frac{\sqrt{2}}{\frac{\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{2}}}$$

$$= \frac{2}{\cos\theta + \sin\theta}$$

$r\cos\theta + r\sin\theta = 2$

$x + y = 2$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\frac{2}{\frac{1}{2}+\frac{\sqrt{3}}{2}}$	$\sqrt{2}$	$\frac{2}{\frac{1}{2}+\frac{\sqrt{3}}{2}}$	2



2. $C: y = 2x^{\frac{1}{2}}, \quad 0 \le x \le 4$

i) The mean value of y over $0 \le x \le 4$ is

$$\frac{1}{4-0}\int_0^4 y\,dx$$

$$= \frac{1}{4}\int_0^4 2x^{\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$= \frac{8}{3}$$

ii) The area of the region bounded by C, $x = 4$ and the x-axis, A, is

$$\int_0^4 y\,dx = \int_0^4 2x^{\frac{1}{2}}\,dx = \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{32}{3}.$$

$\therefore$ If $\bar{y}$ is the y coordinate of the centroid of the region bounded by C, $x=4$ and the x-axis,

$$A\bar{y} = \int_0^4 \frac{y^2}{2}\,dx = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16$$

$$\therefore \bar{y} = \frac{16}{\frac{32}{3}} = \frac{3}{2}.$$

3. $$\frac{d^2x}{dt^2} + \frac{4dx}{dt} + 13x = 26t^2 + 3t + 13$$

$$\frac{d^2x}{dt^2} + \frac{4dx}{dt} + 13x = 0$$

$$m^2 + 4m + 13 = 0$$

$$(m+2)^2 + 9 = 0$$

$$(m+2)^2 = -9$$

$$m + 2 = \pm 3i$$

$$m = -2 \pm 3i$$

$\therefore$ The complementary function, x_c, is

$$x_c = e^{-2t}(A\cos 3t + B\sin 3t).$$

The particular integral, x_p, is given by

$$x_p = Ct^2 + Dt + E$$

$$\frac{dx_p}{dt} = 2Ct + D$$

$$\frac{d^2x_p}{dt^2} = 2C$$

$$\frac{d^2x_p}{dt^2} + \frac{4dx_p}{dt} + 13x_p = 2C + 4(2Ct + D) + 13(Ct^2 + Dt + E)$$

$$= 13Ct^2 + (8C + 13D)t + 2C + 4D + 13E$$

$$= 26t^2 + 3t + 13.$$

$13C = 26$, $C = 2$

$8C + 13D = 3$, $16 + 13D = 3$, $D = -1$

$2C + 4D + 13E = 13$, $4 - 4 + 13E = 13$, $E = 1$

$$\therefore x_p = 2t^2 - t + 1$$

$$x = x_c + x_p$$

$$= e^{-2t}(A\cos 3t + B\sin 3t) + 2t^2 - t + 1.$$

4. $f(r) = r(r+1)(r+2)$

$$f(r) - f(r-1) = r(r+1)(r+2) - (r-1)r(r+1)$$
$$= r(r+1)(r+2-(r-1))$$
$$= 3r(r+1)$$

$$\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} \frac{1}{3}\left[f(r) - f(r-1)\right]$$

$$= \frac{1}{3}\Big(f(n) - f(n-1)$$
$$+ f(n-1) - f(n-2)$$
$$+ f(n-2) - f(n-3)$$
$$\vdots$$
$$+ f(3) - f(2)$$
$$+ f(2) - f(1)$$
$$+ f(1) - f(0) \Big)$$

$$= \frac{1}{3}(f(n) - f(0))$$

$$= \frac{n(n+1)(n+2)}{3}$$

$$\sum_{r=1}^{n} r^2 + r = \frac{n(n+1)(n+2)}{3}$$

Since $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$,

$$\sum_{r=1}^{n} r^2 + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3}$$

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6}(2(n+2)-3)$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + \dots + 2(n-1)^2 + n^2$$
$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \dots + (n-1)^2 + n^2$$
$$+ 2^2 + 4^2 + 6^2 + \dots + (n-1)^2$$
$$= \sum_{r=1}^{n} r^2 + 2^2\left(1^2 + 2^2 + 3^2 + \dots + \left(\frac{n-1}{2}\right)^2\right)$$
$$= \frac{n(n+1)(2n+1)}{6} + 4\sum_{r=1}^{\frac{n-1}{2}} r^2$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{4}{6}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\left(2\left(\frac{n-1}{2}\right)+1\right)$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{2}{3}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)n$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n-1)(n+1)}{6}$$
$$= \frac{n(n+1)}{6}(2n+1+n-1)$$
$$= \frac{n^2(n+1)}{2}$$

5. $I_n = \int_0^\infty x^n e^{-2x}\,dx$

$u = x^n \qquad dv = e^{-2x}\,dx$

$du = nx^{n-1}\,dx \;:\; v = \dfrac{e^{-2x}}{-2}$

$= \left[\dfrac{x^n e^{-2x}}{-2}\right]_0^\infty - \int_0^\infty \dfrac{nx^{n-1}e^{-2x}}{-2}\,dx$

$= 0 + \dfrac{n}{2}\int_0^\infty x^{n-1}e^{-2x}\,dx$

$= \dfrac{n}{2} I_{n-1}, \; n \geqslant 1.$

$I_n = \dfrac{n!}{2^{n+1}}$

when $n=1$: $I_1 = \int_0^\infty xe^{-2x}\,dx$

$= \dfrac{I_0}{2}$

$= \dfrac{1}{2}\int_0^\infty e^{-2x}\,dx$

$= \left[\dfrac{e^{-2x}}{-4}\right]_0^\infty$

$= 0 - -\dfrac{1}{4}$

$= \dfrac{1}{4} = \dfrac{1}{2^2} = \dfrac{1!}{2^{1+1}}$

Assume the statement is true when $n=k$.

$n=k$: $I_k = \dfrac{k!}{2^{k+1}}$

when $n=k+1$: $I_{k+1} = \dfrac{(k+1)!}{2^{k+2}}$

(what needs to be proved)

$I_{k+1} = \dfrac{(k+1)I_k}{2}$

$= \dfrac{(k+1)}{2}\dfrac{k!}{2^{k+1}}$

$= \dfrac{(k+1)!}{2^{k+2}}$

$\therefore I_n = \dfrac{n!}{2^{n+1}}$ for every positive integer n.

6. $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta$
$- 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

$\cos 4\theta + i\sin 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
$+ i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$
$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta$
$+ 1 - 2\cos^2\theta + \cos^4\theta$
$= 8\cos^4\theta - 8\cos^2\theta + 1$

$\cos\frac{4\pi}{7} = \cos\left(\pi - \frac{3\pi}{7}\right) = -\cos\frac{3\pi}{7}$

$\cos\frac{12\pi}{7} = \cos\left(3\pi - \frac{9\pi}{7}\right) = \cos 3\pi\cos\frac{9\pi}{7} = \cos\pi\cos\frac{9\pi}{7} = -\cos\frac{9\pi}{7}$

$\cos\frac{20\pi}{7} = \cos\frac{6\pi}{7} = \cos\left(\pi - \frac{\pi}{7}\right) = \cos\pi\cos\frac{\pi}{7} = -\cos\frac{\pi}{7} = -\cos\frac{15\pi}{7}$

$\cos 4\pi = \cos 0 = 1 = -\cos\pi = -\cos 3\pi$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$8c^4 + 4c^3 - 8c^2 - 3c + 1 = 0$

Let $c = \cos\theta$

$\therefore 8\cos^4\theta + 4\cos^3\theta - 8\cos^2\theta - 3\cos\theta + 1 = 0$
$8\cos^4\theta - 8\cos^2\theta + 1 + 4\cos^3\theta - 3\cos\theta = 0$
$\cos 4\theta + \cos 3\theta = 0$
$\cos 4\theta = -\cos 3\theta$

Since $\theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi$ satisfy the equation $\cos 4\theta = -\cos 3\theta$,

the roots of $8c^4 + 4c^3 - 8c^2 - 3c + 1 = 0$ are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$ and -1.

Since $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} - 1 = -\frac{4}{8}$
$= -\frac{1}{2}$,

$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$.

/

7. $C: \ y = \lambda x + \dfrac{x}{x-2}, \quad \lambda \neq 0$

$= \lambda x + 1 + \dfrac{2}{x-2}$

As $x \to \pm\infty \quad y \to \lambda x + 1$

As $x \to 2 \quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = \lambda x + 1$ and $x = 2$.

$\dfrac{dy}{dx} = \lambda - \dfrac{2}{(x-2)^2}$

If $\lambda < 0$, since $\dfrac{2}{(x-2)^2} > 0$,

$\dfrac{-2}{(x-2)^2} < 0$

$\lambda - \dfrac{2}{(x-2)^2} < \lambda < 0$

$\therefore \dfrac{dy}{dx} < 0$

$\therefore$ C has no turning points.

$\lambda = -1$: $y = -x + 1 + \dfrac{2}{x-2}$

when $x = 0$: $y = 0$

when $y = 0$: $-x + 1 + \dfrac{2}{x-2} = 0$

$x - 1 = \dfrac{2}{x-2}$

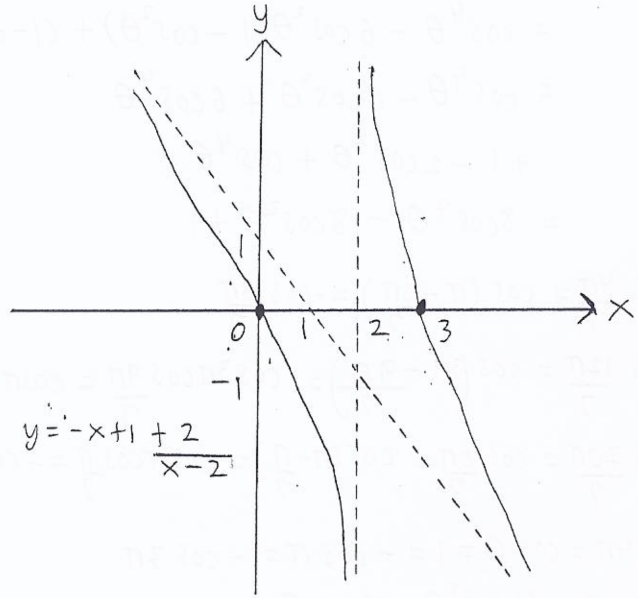
$(x-1)(x-2) = 2$

$x^2 - 3x + 2 = 2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, 3$



8. $C: x = \frac{t^3}{3} - \ln t, \quad y = \frac{4t^{\frac{3}{2}}}{3}, \quad 1 \leq t \leq 3.$

$$\frac{dx}{dt} = t^2 - \frac{1}{t} \qquad \frac{dy}{dt} = 2t^{\frac{1}{2}}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(t^2 - \frac{1}{t}\right)^2 + \left(2t^{\frac{1}{2}}\right)^2$$

$$= t^4 - 2t + \frac{1}{t^2} + 4t$$

$$= t^4 + 2t + \frac{1}{t^2}$$

$$= \left(t^2 + \frac{1}{t}\right)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t^2 + \frac{1}{t}$$

The arc length of C is

$$\int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^3 t^2 + \frac{1}{t}\, dt$$

$$= \left[\frac{t^3}{3} + \ln t\right]_1^3$$

$$= 9 + \ln 3 - \frac{1}{3} - \ln 1$$

$$= \frac{26}{3} + \ln 3$$

The surface area of revolution about the x-axis is

$$\int_1^3 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^3 2\pi\left(\frac{4}{3}t^{\frac{3}{2}}\right)\left(t^2 + \frac{1}{t}\right) dt$$

$$= \frac{8\pi}{3}\int_1^3 t^{\frac{7}{2}} + t^{\frac{1}{2}}\, dt$$

$$= \frac{8\pi}{3}\left[\frac{2}{9}t^{\frac{9}{2}} + \frac{2}{3}t^{\frac{3}{2}}\right]_1^3$$

$$= \frac{8\pi}{3}\left(\frac{2}{9}(3^{\frac{9}{2}}) + \frac{2}{3}(3^{\frac{3}{2}}) - \frac{2}{9} - \frac{2}{3}\right)$$

$$= \frac{8\pi}{3}\left(\frac{2}{9}(81\sqrt{3}) + \frac{2}{3}(3\sqrt{3}) - \frac{8}{9}\right)$$

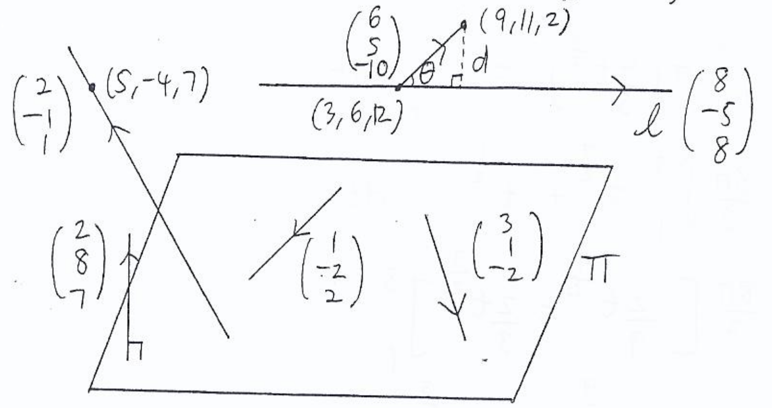
$$= \frac{8\pi}{3}\left(18\sqrt{3} + 2\sqrt{3} - \frac{8}{9}\right)$$

$$= \frac{8\pi}{3}\left(20\sqrt{3} - \frac{8}{9}\right)$$

$$= \pi\left(\frac{160\sqrt{3}}{3} - \frac{64}{27}\right).$$

9. $\Pi : \underset{\sim}{r} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k} + \lambda(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}) + \mu(3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$

$l : \underset{\sim}{r} = 3\underset{\sim}{i} + 6\underset{\sim}{j} + 12\underset{\sim}{k} + t(8\underset{\sim}{i} + 5\underset{\sim}{j} - 8\underset{\sim}{k})$



since $\begin{pmatrix}1\\-2\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\\-2\end{pmatrix}$ are parallel to the plane $\begin{pmatrix}1\\-2\\2\end{pmatrix} \times \begin{pmatrix}3\\1\\-2\end{pmatrix}$ is perpendicular to the plane.

$$\begin{pmatrix}1\\-2\\2\end{pmatrix} \times \begin{pmatrix}3\\1\\-2\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix} = \begin{pmatrix}2\\8\\7\end{pmatrix}$$

since the line's direction is $\begin{pmatrix}8\\5\\-8\end{pmatrix}$ and

$$\begin{pmatrix}8\\5\\-8\end{pmatrix} \cdot \begin{pmatrix}2\\8\\7\end{pmatrix} = 16 + 40 - 56 = 0, \ l \text{ is parallel to } \Pi.$$

$$\underset{\sim}{r} = 5\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k} + s(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$$

$$\begin{pmatrix}2\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\2\end{pmatrix} + \mu\begin{pmatrix}3\\1\\-2\end{pmatrix} = \begin{pmatrix}5\\-4\\7\end{pmatrix} + t\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}2+\lambda+3\mu\\3-2\lambda+\mu\\-1+2\lambda-2\mu\end{pmatrix} = \begin{pmatrix}5+2t\\-4-t\\7+t\end{pmatrix}$$

$$\left.\begin{aligned} 2+\lambda+3\mu &= 5+2t \\ 3-2\lambda+\mu &= -4-t \\ -1+2\lambda-2\mu &= 7+t \end{aligned}\right\}$$

$$\left.\begin{aligned} \lambda+3\mu-2t &= 3 \\ -2\lambda+\mu+t &= -7 \\ 2\lambda-2\mu-t &= 8 \end{aligned}\right\}$$

$$\begin{aligned} 2\times① + ②: \quad \lambda+3\mu-2t &= 3 \\ -2\times① + ③: \quad 7\mu-3t &= -1 \\ -8\mu+3t &= 2 \end{aligned}$$

$$\begin{aligned} ② + ③: \quad \lambda+3\mu-2t &= 3 \\ 7\mu-3t &= -1 \\ -\mu &= 1 \end{aligned}$$

$\mu = -1 \quad t = -2 \quad \lambda = 2$

$\therefore$ The line meets Π at $(1, -2, 5)$.

$$\left|\begin{pmatrix}6\\5\\-10\end{pmatrix}\times\begin{pmatrix}8\\5\\-8\end{pmatrix}\right| = \left|\begin{pmatrix}6\\5\\-10\end{pmatrix}\right|\left|\begin{pmatrix}8\\5\\-8\end{pmatrix}\right|\sin\theta$$

since $d = \left|\begin{pmatrix}6\\5\\-10\end{pmatrix}\right|\sin\theta$

$$d = \frac{\left|\begin{pmatrix}6\\5\\-10\end{pmatrix}\times\begin{pmatrix}8\\5\\-8\end{pmatrix}\right|}{\left|\begin{pmatrix}8\\5\\-8\end{pmatrix}\right|}$$

$$= \frac{\left|\begin{pmatrix}10\\-32\\-10\end{pmatrix}\right|}{\left|\begin{pmatrix}8\\5\\-8\end{pmatrix}\right|}$$

$$= \frac{\sqrt{1224}}{\sqrt{153}}$$

$$= \sqrt{8}$$

$\therefore$ The perpendicular distance from $(9,11,2)$ to l is $\sqrt{8}$.

10. $A = \begin{pmatrix}1 & 4 & -16\\0 & 2 & 3\\0 & 0 & 3\end{pmatrix}$

$\therefore$ The eigenvalues of A are $1, 2, 3$.

$$\begin{pmatrix}1 & 4 & -16\\0 & 2 & 3\\0 & 0 & 3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x+4y-16z\\2y+3z\\3z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\left.\begin{aligned}4y-16z&=0\\y+3z&=0\\2z&=0\end{aligned}\right\}$$

$z=0 \quad y=0 \quad x=s$

$$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}s\\0\\0\end{pmatrix} = s\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4 & -16\\0 & 2 & 3\\0 & 0 & 3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 2\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x+4y-16z\\2y+3z\\3z\end{pmatrix} = \begin{pmatrix}2x\\2y\\2z\end{pmatrix}$$

$$\left.\begin{aligned}-x+4y-16z&=0\\3z&=0\\2z&=0\end{aligned}\right\}$$

$z=0 \quad y=s \quad x=4s$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4s\\s\\0\end{pmatrix} = s\begin{pmatrix}4\\1\\0\end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & -16 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+4y-16z \\ 2y+3z \\ 3z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$\left.\begin{aligned} -2x+4y-16z &= 0 \\ -y+3z &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} -x+2y-8z &= 0 \\ -2y+6z &= 0 \end{aligned}\right\}$$

$$-x-2z=0$$

$z=s$ $y=3s$ $x=-2s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s \\ 3s \\ s \end{pmatrix} = s\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

If $A^n = PDP^{-1}$, $n \in \mathbb{N}$

$$P = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$$

$$|P| = 1 \quad P^T = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix} \quad \text{adj } P = \begin{pmatrix} 1 & -4 & 14 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore P^{-1} = \frac{1}{|P|}\text{adj } P = \begin{pmatrix} 1 & -4 & 14 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^n = PDP^{-1}$$

$$= \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}\begin{pmatrix} 1 & -4 & 14 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4(2^n) & -2(3^n) \\ 0 & 2^n & 3(3^n) \\ 0 & 0 & 3^n \end{pmatrix}\begin{pmatrix} 1 & -4 & 14 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4(2^n)-4 & 14-12(2^n)-2(3^n) \\ 0 & 2^n & 3(3^n)-3(2^n) \\ 0 & 0 & 3^n \end{pmatrix}$$

$$3^{-n}A^n = 3^{-n}\begin{pmatrix} 1 & 4(2^n)-4 & 14-12(2^n)-2(3^n) \\ 0 & 2^n & 3(3^n)-3(2^n) \\ 0 & 0 & 3^n \end{pmatrix}$$

$$= \begin{pmatrix} 3^{-n} & 4\left(\frac{2}{3}\right)^n-4(3^{-n}) & 14(3^{-n})-12\left(\frac{2}{3}\right)^n-2 \\ 0 & \left(\frac{2}{3}\right)^n & 3-3\left(\frac{2}{3}\right)^n \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lim_{n\to\infty} 3^{-n}A^n = \lim_{n\to\infty}\begin{pmatrix} 3^{-n} & 4\left(\frac{2}{3}\right)^n-4(3^{-n}) & 14(3^{-n})-12\left(\frac{2}{3}\right)^n-2 \\ 0 & \left(\frac{2}{3}\right)^n & 3-3\left(\frac{2}{3}\right)^n \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

11. EITHER

$x^4 - 3x^2 + 5x - 2 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$

$\alpha, \beta, \gamma, \delta$ are the roots

$\therefore \alpha^4 - 3\alpha^2 + 5\alpha - 2 = 0 \quad - (1)$

$\beta^4 - 3\beta^2 + 5\beta - 2 = 0 \quad - (2)$

$\gamma^4 - 3\gamma^2 + 5\gamma - 2 = 0 \quad - (3)$

$\delta^4 - 3\delta^2 + 5\delta - 2 = 0 \quad - (4)$

$\alpha^n \times (1): \alpha^{4+n} - 3\alpha^{2+n} + 5\alpha^{1+n} - 2\alpha^n = 0 \quad - (5)$

$\beta^n \times (2): \beta^{4+n} - 3\beta^{2+n} + 5\beta^{1+n} - 2\beta^n = 0 \quad - (6)$

$\gamma^n \times (3): \gamma^{4+n} - 3\gamma^{2+n} + 5\gamma^{1+n} - 2\gamma^n = 0 \quad - (7)$

$\delta^n \times (4): \delta^{4+n} - 3\delta^{2+n} + 5\delta^{1+n} - 2\delta^n = 0 \quad - (8)$

$(5) + (6) + (7) + (8)$:

$\alpha^{4+n} + \beta^{4+n} + \gamma^{4+n} + \delta^{4+n}$

$-3(\alpha^{2+n} + \beta^{2+n} + \gamma^{2+n} + \delta^{2+n})$

$+5(\alpha^{1+n} + \beta^{1+n} + \gamma^{1+n} + \delta^{1+n})$

$-2(\alpha^n + \beta^n + \gamma^n + \delta^n) = 0$

$\therefore S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0$

$\alpha + \beta + \gamma + \delta = 0$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -3$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -5$

$\alpha\beta\gamma\delta = -2$

i) $S_0 = \alpha^0 + \beta^0 + \gamma^0 + \delta^0 = 1 + 1 + 1 + 1 = 4$

$S_1 = \alpha^1 + \beta^1 + \gamma^1 + \delta^1 = \alpha + \beta + \gamma + \delta = 0$

$S_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= 0^2 - 2(-3)$

$= 6$

$n = 0: S_4 - 3S_2 + 5S_1 - 2S_0 = 0$

$S_4 - 3(6) + 5(0) - 2(4) = 0$

$\therefore S_4 = 26$

ii) $S_{-1} = \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}$

$= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

$= \frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha\beta\gamma\delta}$

$= \frac{5}{2}$

$n=-1$: $S_3 - 3S_1 + 5S_0 - 2S_{-1} = 0$

$S_3 - 3(0) + 5(4) - 2\left(\frac{5}{2}\right) = 0$

$S_3 - 0 + 20 - 5 = 0$

$S_3 = -15$

$n=1$: $S_5 - 3S_3 + 5S_2 - 2S_1 = 0$

$S_5 - 3(-15) + 5(6) - 2(0) = 0$

$S_5 + 45 + 30 = 0$

$S_5 = -75$

$\alpha^2(\beta^3+\gamma^3+\delta^3) + \beta^2(\gamma^3+\delta^3+\alpha^3) + \gamma^2(\delta^3+\alpha^3+\beta^3) + \delta^2(\alpha^3+\beta^3+\gamma^3)$

$= \alpha^2(\alpha^3+\beta^3+\gamma^3+\delta^3-\alpha^3) + \beta^2(\alpha^3+\beta^3+\gamma^3+\delta^3-\beta^3)$

$+ \gamma^2(\alpha^3+\beta^3+\gamma^3+\delta^3-\gamma^3) + \delta^2(\alpha^3+\beta^3+\gamma^3+\delta^3-\delta^3)$

$= \alpha^2(S_3-\alpha^3) + \beta^2(S_3-\beta^3) + \gamma^2(S_3-\gamma^3) + \delta^2(S_3-\delta^3)$

$= \alpha^2 S_3 - \alpha^5 + \beta^2 S_3 - \beta^5 + \gamma^2 S_3 - \gamma^5 + \delta^2 S_3 - \delta^5$

$= (\alpha^2+\beta^2+\gamma^2+\delta^2)S_3 - (\alpha^5+\beta^5+\gamma^5+\delta^5)$

$= S_2 S_3 - S_5$

$= 6(-15) - (-75)$

$= -90 + 75$

$= -15$

OR

$M = \begin{pmatrix} 2 & 1 & -1 & 4 \\ 3 & 4 & 6 & 1 \\ -1 & 2 & 8 & -7 \end{pmatrix}$

i) $\xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} -1 & 2 & 8 & -7 \\ 3 & 4 & 6 & 1 \\ 2 & 1 & -1 & 4 \end{pmatrix}$ or $\xrightarrow{2r_2,\ 2r_3} \begin{pmatrix} 2 & 1 & -1 & 4 \\ 6 & 8 & 12 & 2 \\ -2 & 4 & 16 & -14 \end{pmatrix}$

$\xrightarrow{3r_1+r_2,\ 2r_1+r_3} \begin{pmatrix} -1 & 2 & 8 & -7 \\ 0 & 10 & 30 & -20 \\ 0 & 5 & 15 & -10 \end{pmatrix}$ $\xrightarrow{-3r_1+r_2,\ r_1+r_3} \begin{pmatrix} 2 & 1 & -1 & 4 \\ 0 & 5 & 15 & -10 \\ 0 & 5 & 15 & -10 \end{pmatrix}$

$\xrightarrow{\frac{r_2}{10},\ \frac{r_3}{5}} \begin{pmatrix} -1 & 2 & 8 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{pmatrix}$ $\xrightarrow{-r_2+r_3} \begin{pmatrix} 2 & 1 & -1 & 4 \\ 0 & 5 & 15 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{-r_2+r_3} \begin{pmatrix} -1 & 2 & 8 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\xrightarrow{\frac{r_2}{5}} \begin{pmatrix} 2 & 1 & -1 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ The dimension of the range space, R, is 2.

ii) A basis for R is $\left\{ \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}$

If $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, $s, t \in \mathbb{R}$

Since $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in R$,

a Cartesian equation of R is $2x - y + z = 0$.

iii) $\begin{pmatrix} 2 & 1 & -1 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left(\begin{array}{cccc|c} 2 & 1 & -1 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

let $z = s$, $w = t$, $s, t \in \mathbb{R}$

$\therefore y = -3s + 2t$

$2x + y - z + 4w = 0$

$2x - 3s + 2t - s + 4t = 0$

$2x = 4s - 6t$

$x = 2s - 3t$

$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2s - 3t \\ -3s + 2t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ A basis for the null space of T is

$\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\begin{pmatrix} 8 \\ 7 \\ k \end{pmatrix} \in R \quad \therefore 2x - y + z = 0$

$16 - 7 + k = 0$

$k = -9$

$M\underset{\sim}{x} = \begin{pmatrix} 8 \\ 7 \\ -9 \end{pmatrix}$

$M\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ -9 \end{pmatrix}$

$\left(\begin{array}{cccc|c} 2 & 1 & -1 & 4 & 8 \\ 3 & 4 & 6 & 1 & 7 \\ -1 & 2 & 8 & -7 & -9 \end{array}\right)$

$\xrightarrow{2r_2,\ 2r_3} \left(\begin{array}{cccc|c} 2 & 1 & -1 & 4 & 8 \\ 6 & 8 & 12 & 2 & 14 \\ -2 & 4 & 16 & -14 & -18 \end{array}\right)$

$\xrightarrow{-3r_1 + r_2,\ r_1 + r_3} \left(\begin{array}{cccc|c} 2 & 1 & -1 & 4 & 8 \\ 0 & 5 & 15 & -10 & -10 \\ 0 & 5 & 15 & -10 & -10 \end{array}\right)$

$\xrightarrow{-r_2 + r_3} \left(\begin{array}{cccc|c} 2 & 1 & -1 & 4 & 8 \\ 0 & 5 & 15 & -10 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\xrightarrow{\frac{r_2}{5}} \left(\begin{array}{cccc|c} 2 & 1 & -1 & 4 & 8 \\ 0 & 1 & 3 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Let $z = s$, $w = t$, $s, t \in \mathbb{R}$

$y + 3s - 2t = -2$

$y = 2t - 3s - 2$

$2x + y - z + 4w = 8$

$2x + 2t - 3s - 2 - s + 4t = 8$

$2x + 6t - 4s = 10$

$x + 3t - 2s = 5$

$x = 5 + 2s - 3t$

$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 + 2s - 3t \\ 2t - 3s - 2 \\ s \\ t \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$