1. $A = \begin{pmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & -7 & -2 \\ -1 & -3 & 5 & -5 \\ 2 & 5 & -9 & 6 \end{pmatrix}$

$$\xrightarrow[-2r_1 + r_4]{\substack{-2r_1 + r_2 \\ r_1 + r_3}} \begin{pmatrix} 1 & 2 & -4 & 1 \\ 0 & -1 & 1 & -4 \\ 0 & -1 & 1 & -4 \\ 0 & 1 & -1 & 4 \end{pmatrix}$$

$$\xrightarrow{\substack{-r_2 + r_3 \\ r_2 + r_4}} \begin{pmatrix} 1 & 2 & -4 & 1 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-r_2} \begin{pmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank}(A) = 2$$

If $\begin{pmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & -7 & -2 \\ -1 & -3 & 5 & -5 \\ 2 & 5 & -9 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 0 \\ 2 & 3 & -7 & -2 & 0 \\ -1 & -3 & 5 & -5 & 0 \\ 2 & 5 & -9 & 6 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 0 \\ 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = s$ and $z = t$, $s, t \in \mathbb{R}$

$$y = t - 4s$$

$$x + 2y - 4z + w = 0$$

$$x + 2(t - 4s) - 4t + s = 0$$

$$x + 2t - 8s - 4t + s = 0$$

$$x = 7s + 2t$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 7s + 2t \\ t - 4s \\ t \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 7 \\ -4 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ The nullspace of A is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s\begin{pmatrix} 7 \\ -4 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

A basis for the range space of A is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -3 \\ 5 \end{pmatrix} \right\}$$

2. $x^2y^3 + (4x + 5y)^6 = 7$

$$\frac{d}{dx}\left(x^2y^3 + (4x+5y)^6\right) = \frac{d}{dx}(7)$$

$$\frac{d}{dx}(x^2y^3) + \frac{d}{dx}(4x+5y)^6 = 0$$

$$x^2\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x^2) + 6(4x+5y)^5\frac{d}{dx}(4x+5y) = 0$$

$$x^2\left(3y^2\frac{dy}{dx}\right) + 2xy^3 + 6(4x+5y)^5\left(4 + \frac{5dy}{dx}\right) = 0$$

$$3x^2y^2\frac{dy}{dx} + 2xy^3 + 6(4x+5y)^5\left(4 + \frac{5dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(3x^2y^2\frac{dy}{dx}\right) + \frac{d}{dx}(2xy^3) + \frac{d}{dx}\left(6(4x+5y)^5\right)\left(4 + \frac{5dy}{dx}\right)) = 0$$

$$3x^2\frac{d}{dx}\left(y^2\frac{dy}{dx}\right) + y^2\frac{dy}{dx}\frac{d}{dx}(3x^2) + 2x\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(2x)$$

$$+ 6(4x+5y)^5\frac{d}{dx}\left(4 + \frac{5dy}{dx}\right) + \left(4 + \frac{5dy}{dx}\right)\frac{d}{dx}\left(6(4x+5y)^5\right) = 0$$

$$3x^2\left(y^2\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(y^2)\right) + 6xy^2\frac{dy}{dx} + 6xy^2\frac{dy}{dx} + 2y^3 = 0$$

$$3x^2\left(y^2\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2\right) + 12xy^2\frac{dy}{dx} + 2y^3$$

$$+ 30(4x+5y)^5\frac{d^2y}{dx^2} + 30(4x+5y)^4\left(4 + \frac{5dy}{dx}\right)^2 = 0$$

3. $A = \begin{pmatrix} 7 & 4 & -8 \\ -2 & 1 & 16 \\ 0 & 0 & 9 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 7 & 4 & -8 \\ -2 & 1 & 16 \\ 0 & 0 & 9 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 7-\lambda & 4 & -8 \\ -2 & 1-\lambda & 16 \\ 0 & 0 & 9-\lambda \end{pmatrix}$

$|A - \lambda I| = (7-\lambda) \begin{vmatrix} 1-\lambda & 16 \\ 0 & 9-\lambda \end{vmatrix} - 4 \begin{vmatrix} -2 & 16 \\ 0 & 9-\lambda \end{vmatrix}$

$-8 \begin{vmatrix} -2 & 1-\lambda \\ 0 & 0 \end{vmatrix}$

$= (7-\lambda)(1-\lambda)(9-\lambda) - 4(-2(9-\lambda) - 0) - 0$

$= (7-\lambda)(\lambda^2 - 10\lambda + 9) - 4(-18 + 2\lambda)$

$= 7\lambda^2 - 70\lambda + 63 - \lambda^3 + 10\lambda^2 - 9\lambda + 72 - 8\lambda$

$= -\lambda^3 + 17\lambda^2 - 87\lambda + 135$

$= (\lambda - 3)(-\lambda^2 + 14\lambda - 45)$

$= (\lambda - 3)(\lambda - 5)(9 - \lambda)$

when $|A - \lambda I| = 0$, $(\lambda - 3)(\lambda - 5)(9 - \lambda) = 0$

$\lambda = 3, 5, 9$

$\therefore$ The eigenvalues of A are $3, 5, 9$

when $\lambda = 3$: $\begin{pmatrix} 4 & 4 & -8 \\ -2 & -4 & 16 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left(\begin{array}{ccc|c} 4 & 4 & -8 & 0 \\ -2 & -4 & 16 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right)$

$$\xrightarrow{\frac{r_1}{4},\ \frac{r_2}{-6},\ \frac{r_3}{6}} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & 2 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$z = 0$

Let $y = s, s \in R$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ s \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

When $\lambda = 5$: $\begin{pmatrix} 2 & 4 & -8 \\ -2 & -4 & 16 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 4 & -8 & 0 \\ -2 & -4 & 16 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2},\ \frac{r_2}{-2},\ \frac{r_3}{4}} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 1 & 2 & -8 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$z = 0$

Let $y = s, s \in R$

$x = -2s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s \\ s \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

When $\lambda = 9$: $\begin{pmatrix} -2 & 4 & -8 \\ -2 & -8 & 16 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -2 & 4 & -8 & 0 \\ -2 & -8 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{-2},\ \frac{r_2}{-2}} \left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 1 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 0 & 6 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s,\ s \in R$

$y = 2s$

$x = 0$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are $3, 5, 9$ with corresponding eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

$$A = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}\begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

If $A^n = PD^nP^{-1}$,

since $A^n = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}^n\begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$

$\therefore P = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 5^n & 0 \\ 0 & 0 & 9^n \end{pmatrix}$

4. $y = e^x \sin x$

$$\frac{d^{4n}y}{dx^{4n}} = (-4)^n e^x \sin x$$

$$\frac{dy}{dx} = e^x \sin x + e^x \cos x$$

$$\frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$

$$= 2e^x \cos x$$

$$\frac{d^3y}{dx^3} = 2e^x \cos x - 2e^x \sin x$$

$$\frac{d^4y}{dx^4} = 2e^x \cos x - 2e^x \sin x - 2e^x \sin x - 2e^x \cos x$$

$$= -4e^x \sin x$$

when $n = 1$: $\dfrac{d^{4(1)}y}{dx^{4(1)}} = \dfrac{d^4y}{dx^4} = -4e^x \sin x = (-4)^1 e^x \sin x$

Assume the statement is true when $n = k$

$n = k$: $\dfrac{d^{4k}y}{dx^{4k}} = (-4)^k e^x \sin x$

when $n = k+1$ $\dfrac{d^{4(k+1)}y}{dx^{4(k+1)}} = (-4)^{k+1} e^x \sin x$

(what needs to be proved)

$$\frac{d^{4k}y}{dx^{4k}} = (-4)^k e^x \sin x$$

$$\frac{d^{4k+1}y}{dx^{4k+1}} = (-4)^k e^x \sin x + (-4)^k e^x \cos x$$

$$\frac{d^{4k+2}y}{dx^{4k+2}} = (-4)^k e^x \sin x + (-4)^k e^x \cos x$$

$$+ (-4)^k e^x \cos x - (-4)^k e^x \sin x$$

$$= 2(-4)^k e^x \cos x$$

$$\frac{d^{4k+3}y}{dx^{4k+3}} = 2(-4)^k e^x \cos x - 2(-4)^k e^x \sin x$$

$$\frac{d^{4k+4}y}{dx^{4k+4}} = 2(-4)^k e^x \cos x - 2(-4)^k e^x \sin x$$

$$- 2(-4)^k e^x \sin x - 2(-4)^k e^x \cos x$$

$$= -4(-4)^k e^x \sin x$$

$$= (-4)^{k+1} e^x \sin x$$

$\therefore \dfrac{d^{4n}y}{dx^{4n}} = (-4)^n e^x \sin x$ for every positive integer n.

5. $9y^2\frac{d^2y}{dx^2} + 18y\left(\frac{dy}{dx}\right)^2 + 45y^2\frac{dy}{dx} + 12y^3 = 4\cos x + 7\sin x$

$v = y^3$

$\frac{dv}{dy} = 3y^2$

$\frac{dx}{dy}\frac{dv}{dx} = 3y^2$

$\frac{dv}{dx} = 3y^2\frac{dy}{dx}$

$\frac{d^2v}{dx^2} = \frac{d}{dx}\left(3y^2\frac{dy}{dx}\right)$

$= 3y^2\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(3y^2)$

$= 3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2$

$9y^2\frac{d^2y}{dx^2} + 18y\left(\frac{dy}{dx}\right)^2 + 45y^2\frac{dy}{dx} + 12y^3 = 4\cos x + 7\sin x$

$3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 15y^2\frac{dy}{dx} + 4y^3 = \frac{4}{3}\cos x + \frac{7}{3}\sin x$

$3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 5\left(3y^2\frac{dy}{dx}\right) + 4y^3 = \frac{4}{3}\cos x + \frac{7}{3}\sin x$

$\frac{d^2v}{dx^2} + 5\frac{dv}{dx} + 4v = \frac{4}{3}\cos x + \frac{7}{3}\sin x$

$\frac{d^2v}{dx^2} + 5\frac{dv}{dx} + 4v = 0$

$m^2 + 5m + 4 = 0$

$(m+1)(m+4) = 0$

$m = -1, -4$

∴ The complementary function, v_c, is

$v_c = Ae^{-x} + Be^{-4x}$

The particular integral, v_p, is given by

$v_p = C\cos x + D\sin x$

$\frac{dv_p}{dx} = -C\sin x + D\cos x$

$\frac{d^2v_p}{dx^2} = -C\cos x - D\sin x$

$$\frac{d^2v_p}{dx^2} + \frac{5dv_p}{dx} + 4v_p = -C\cos x - D\sin x$$

$$+ 5(-C\sin x + D\cos x)$$

$$+ 4(C\cos x + D\sin x)$$

$$= -C\cos x - D\sin x$$

$$-5C\sin x + 5D\cos x$$

$$+ 4C\cos x + 4D\sin x$$

$$= (3C + 5D)\cos x + (-5C + 3D)\sin x$$

$$= \frac{4}{3}\cos x + \frac{7}{3}\sin x$$

$3C + 5D = \frac{4}{3}$ $\qquad -5C + 3D = \frac{7}{3}$

$9C + 15D = 4$ $\qquad -25C + 15D = \frac{35}{3}$

$34C = \frac{-23}{3}$

$C = \frac{-23}{102}$

$D = \frac{41}{102}$

$v_p = \frac{-23}{102}\cos x + \frac{41}{102}\sin x$

$$v = v_c + v_p$$

$$= Ae^{-x} + Be^{-4x} - \frac{23}{102}\cos x + \frac{41}{102}\sin x$$

since $v = y^3$,

$$y^3 = Ae^{-x} + Be^{-4x} - \frac{23}{102}\cos x + \frac{41}{102}\sin x$$

$$y = \left(Ae^{-x} + Be^{-4x} - \frac{23}{102}\cos x + \frac{41}{102}\sin x \right)^{\frac{1}{3}}$$

As $x \longrightarrow \infty$, $y \longrightarrow \left(\frac{-23}{102}\cos x + \frac{41}{102}\sin x \right)^{\frac{1}{3}}$