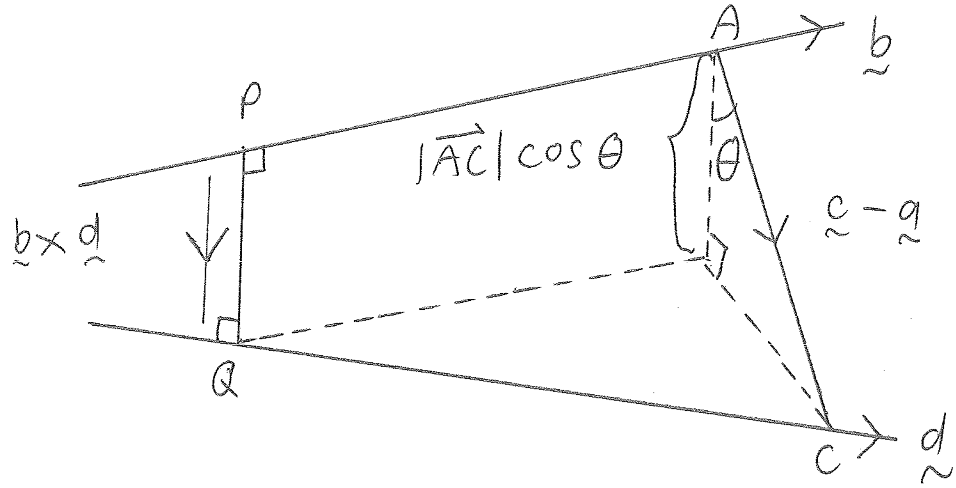
1. $l_1 : \underset{\sim}{r} = \underset{\sim}{a} + s\underset{\sim}{b} \quad l_2 : \underset{\sim}{r} = \underset{\sim}{c} + t\underset{\sim}{d}$



$$\overrightarrow{PQ} \cdot \overrightarrow{AC} = |\overrightarrow{PQ}||\overrightarrow{AC}| \cos\theta$$

since $\overrightarrow{PQ} = k(\underset{\sim}{b} \times \underset{\sim}{d}),\ k \neq 0$

and $|\overrightarrow{PQ}| = |\overrightarrow{AC}| \cos\theta$,

$$k(\underset{\sim}{b} \times \underset{\sim}{d}) \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = |\overrightarrow{PQ}||\overrightarrow{PQ}|$$

$$|\overrightarrow{PQ}|^2 = k(\underset{\sim}{b} - \underset{\sim}{d}) \cdot (\underset{\sim}{c} - \underset{\sim}{a})$$

$$|\overrightarrow{PQ}| = \sqrt{k(\underset{\sim}{b} - \underset{\sim}{d}) \cdot (\underset{\sim}{c} - \underset{\sim}{a})}$$

2. $1\times n^2 + 2\times(n-1)^2 + 3\times(n-2)^2 + \cdots + (n-1)\times 2^2 + n\times 1^2$

$$= \sum_{r=1}^{n} r^2(n+1-r)$$

$$= \sum_{r=1}^{n} (n+1)r^2 - r^3$$

$$= (n+1)\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r^3$$

$$= (n+1)\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4}$$

$$= \frac{n(n+1)^2(2n+1)}{6} - \frac{n^2(n+1)^2}{4}$$

$$= \frac{n(n+1)^2}{2}\left(\frac{2n+1}{3} - \frac{n}{2}\right)$$

$$= \frac{n(n+1)^2}{2}\left(\frac{4n+2-3n}{6}\right)$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

3. $\sum_{r=1}^{n} r2^{n-r} = 2^{n+1} - 2 - n$

when $n=1$: $\sum_{r=1}^{1} r2^{1-r} = 1 \cdot 2^{1-1} = 1 \cdot 2^0 = 1 \cdot 1 = 1$

$= 4 - 2 - 1 = 2^2 - 2 - 1 = 2^{1+1} - 2 - 1$

Assume the statement is true when $n=k$.

$n=k$: $\sum_{r=1}^{k} r2^{k-r} = 2^{k+1} - 2 - k$

when $n=k+1$: $\sum_{r=1}^{k+1} r2^{k+1-r} = 2^{k+2} - 2 - (k+1)$

(what needs to be proved)

$$\sum_{r=1}^{k+1} r2^{k+1-r} = (k+1)2^0 + \sum_{r=1}^{k} r2^{k+1-r}$$

$$= k+1 + 2\sum_{r=1}^{k} r2^{k-r}$$

$$= k+1+2(2^{k+1}-2-k)$$

$$= k+1+2^{k+2}-4-2k$$

$$= 2^{k+2}-3-k$$

$$= 2^{k+2}-2-(k+1)$$

$\therefore \sum_{r=1}^{n} r2^{n-r} = 2^{n+1} - 2 - n$ for every positive integer n.

4. $\int_0^1 y^n(1-y)^r\,dy = \dfrac{n!}{(r+1)(r+2)\dots(r+n+1)}, n \geq 1, r > -1.$

when $n=1$: $\int_0^1 y(1-y)^r\,dy$

$$u = y \qquad dv = (1-y)^r dy$$
$$du = dy \qquad v = \frac{-(1-y)^{r+1}}{r+1}$$

$$= \left[\frac{-y(1-y)^{r+1}}{r+1}\right]_0^1 - \int_0^1 \frac{-(1-y)^{r+1}}{r+1}\,dy$$

$$= 0 - 0 + \int_0^1 \frac{(1-y)^{r+1}}{r+1}\,dy$$

$$= \left[\frac{-(1-y)^{r+2}}{(r+2)(r+1)}\right]_0^1$$

$$= 0 - -\frac{1}{(r+1)(r+2)}$$

$$= \frac{1}{(r+1)(r+2)}$$

$$= \frac{1!}{(r+1)(r+1+1)}$$

Assume the statement is true when $n=k$.

$n=k$: $\int_0^1 y^k(1-y)^r\,dy = \dfrac{k!}{(r+1)(r+2)\dots(r+k+1)}$

when $n=k+1$: $\int_0^1 y^{k+1}(1-y)^r dy = \dfrac{(k+1)!}{(r+1)(r+2)\cdots(r+k+2)}$

(what needs to be proved)

$$\int_0^1 y^{k+1}(1-y)^r dy$$

$$= \int_0^1 y^k y (1-y)^r dy$$

$$= \int_0^1 y^k (1-(1-y))(1-y)^r dy$$

$$= \int_0^1 (y^k - y^k(1-y))(1-y)^r dy$$

$$= \int_0^1 y^k(1-y)^r - y^k(1-y)^{r+1} dy$$

$$= \int_0^1 y^k(1-y)^r dy - \int_0^1 y^k(1-y)^{r+1} dy$$

$$= \frac{k!}{(r+1)(r+2)\cdots(r+k+1)} - \frac{k!}{(r+2)(r+3)\cdots(r+k+1)(r+k+2)}$$

$$= \frac{k!}{(r+2)(r+3)\cdots(r+k+1)}\left(\frac{1}{r+1} - \frac{1}{r+k+2}\right)$$

$$= \frac{k!}{(r+2)(r+3)\cdots(r+k+1)} \frac{(r+k+2-(r+1))}{(r+1)(r+k+2)}$$

$$= \frac{(k+1)k!}{(r+1)(r+2)\cdots(r+k+1)(r+k+2)}$$

$$= \frac{(k+1)!}{(r+1)(r+2)\cdots(r+k+2)}$$

$$\therefore \int_0^1 y^n(1-y)^r\,dy = \frac{n!}{(r+1)(r+2)\cdots(r+n+1)}$$

for every positive integer n.

5. $x = \dfrac{t^2+1}{t}$ $\quad y = 2\ln t,\quad t = 1, 2$

$$= t + \frac{1}{t}$$

$$\frac{dx}{dt} = 1 - \frac{1}{t^2} \quad \frac{dy}{dt} = \frac{2}{t}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 - \frac{1}{t^2}\right)^2 + \left(\frac{2}{t}\right)^2$$

$$= 1 - \frac{2}{t^2} + \frac{1}{t^4} + \frac{4}{t^2}$$

$$= 1 + \frac{2}{t^2} + \frac{1}{t^4}$$

$$= \left(1 + \frac{1}{t^2}\right)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1 + \frac{1}{t^2}$$

Surface area of revolution about the x-axis

$$= \int_1^2 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^2 2\pi (2\ln t)\left(1 + \frac{1}{t^2}\right) dt$$

$$= 4\pi \int_1^2 \ln t \left(1 + \frac{1}{t^2}\right) dt$$

$$u = \ln t \qquad dv = \left(1 + \frac{1}{t^2}\right) dt$$

$$du = \frac{dt}{t} \qquad v = t - \frac{1}{t}$$

$$= 4\pi \left(\left[\left(t - \frac{1}{t}\right) \ln t \right]_1^2 - \int_1^2 1 - \frac{1}{t^2} dt \right)$$

$$= 4\pi \left(\left(2 - \frac{1}{2}\right) \ln 2 - \left[t + \frac{1}{t} \right]_1^2 \right)$$

$$= 4\pi \left(\frac{3}{2} \ln 2 - \left(\frac{5}{2} - 2 \right) \right)$$

$$= 4\pi \left(\frac{3 \ln 2}{2} - \frac{1}{2} \right)$$

$$= 2\pi (3 \ln 2 - 1)$$

6. $1+\frac{\cos 3\theta}{7}+\frac{\cos 6\theta}{7^2}+\frac{\cos 9\theta}{7^3}+\cdots+\frac{\cos 3n\theta}{7^n}$

$$1+\frac{e^{3\theta i}}{7}+\frac{e^{6\theta i}}{7^2}+\frac{e^{9\theta i}}{7^3}+\cdots+\frac{e^{3n\theta i}}{7^n}$$

$$a=1 \quad r=\frac{e^{3\theta i}}{7}$$

$$\sum_{K=0}^{n}\frac{e^{3\theta K i}}{7^K}=\frac{1\left(1-\frac{e^{3\theta(n+1)i}}{7^{K+1}}\right)}{1-\frac{e^{3\theta n i}}{7}}$$

$$\sum_{K=0}^{n}\frac{\cos 3K\theta+i\sin 3K\theta}{7^K}=\frac{7^{n+1}-e^{3\theta(n+1)i}}{7^n\left(7-e^{3\theta n i}\right)}$$

$$\sum_{K=0}^{n}\frac{\cos 3K\theta}{7^K}+i\sum_{K=0}^{n}\frac{\sin 3K\theta}{7^K}$$

$$=\frac{\left(7^{n+1}-e^{3\theta(n+1)i}\right)\left(7-e^{-3\theta n i}\right)}{7^n\left(7-e^{3\theta n i}\right)\left(7-e^{-3\theta n i}\right)}$$

$$=\frac{7^{n+1}-7e^{3\theta(n+1)i}-7^{n+1}e^{3\theta n i}+e^{3\theta i}}{7^n\left(49-7e^{3\theta n i}-7e^{-3n\theta i}+1\right)}$$

$$= \frac{7^{n+1} - 7(\cos 3(n+1)\theta + i\sin 3(n+1)\theta) - 7^{n+1}(\cos 3n\theta + i\sin 3n\theta) + \cos 3\theta + i\sin 3\theta}{7^n(50 - 7(e^{3n\theta i} + e^{-3n\theta i}))}$$

$$= \frac{7^{n+1} - 7\cos 3(n+1)\theta - 7^{n+1}\cos 3n\theta + \cos 3\theta + i(-7\sin 3(n+1)\theta - 7^{n+1}\sin 3n\theta + \sin 3\theta)}{7^n(50 - 14\cos 3n\theta)}$$

$$= \frac{7^{n+1} - 7\cos 3(n+1)\theta - 7^{n+1}\cos 3n\theta + \cos 3\theta}{7^n(50 - 14\cos 3n\theta)}$$

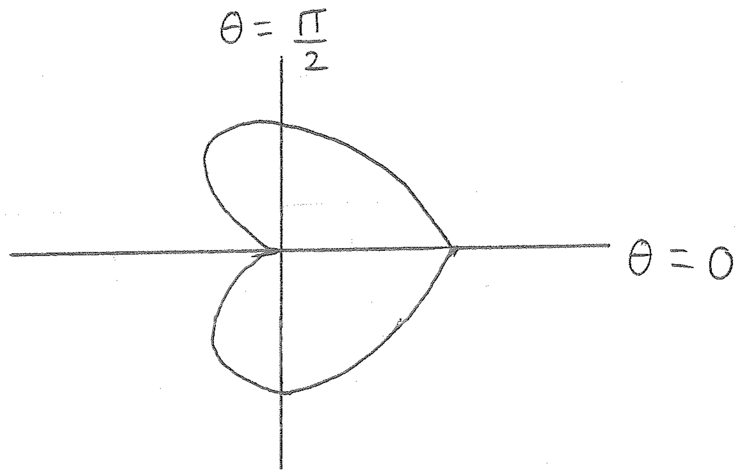
$$+ \frac{i(-7\sin 3(n+1)\theta - 7^{n+1}\sin 3n\theta + \sin 3\theta)}{7^n(50 - 14\cos 3n\theta)}$$

$$\therefore \sum_{k=0}^{n} \frac{\cos 3k\theta}{7^k} = \frac{7^{n+1} - 7\cos 3(n+1)\theta - 7^{n+1}\cos 3n\theta + \cos 3\theta}{7^n(50 - 14\cos 3n\theta)}$$

7i) $r = a(1+\cos\theta)$, $0 \leq \theta \leq 2\pi$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	2π
r	a	$a(1+\frac{\sqrt{3}}{2})$	$a(1+\frac{1}{\sqrt{2}})$	$\frac{3a}{2}$	a	$\frac{a}{2}$	$a(1-\frac{1}{\sqrt{2}})$	$a(1-\frac{\sqrt{3}}{2})$	0

$$a(1+\cos(-\theta)) = a(1+\cos\theta)$$



ii) $$\text{Area} = 2\int_0^{\pi} \frac{r^2}{2}\, d\theta$$

$$= 2\int_0^{\pi} \frac{a(1+\cos\theta)^2}{2}\, d\theta$$

$$= a^2 \int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta \, d\theta$$

$$= a^2 \int_0^{\pi} 1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= a^2 \int_0^{\pi} \frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\, d\theta$$

$$= a^2 \left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4} \right]_0^{\pi} = \frac{3\pi a^2}{2}$$

iii) Arc Length $= 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= 2\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{a^2(1+2\cos\theta+\cos^2\theta) + a^2\sin^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{a^2(2+2\cos\theta)}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{a^2(2+4\cos^2\tfrac{\theta}{2}-2)}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{4a^2\cos^2\tfrac{\theta}{2}}\, d\theta$$

$$= 2\int_0^{\pi} 2a\cos\tfrac{\theta}{2}\, d\theta$$

$$= 2\left[4a\sin\tfrac{\theta}{2}\right]_0^{\pi}$$

$$= 2(4a-0)$$

$$= 8a.$$

8. $I_n = \int \cos^n x \, dx$

$= \int \cos^{n-2} x \cos^2 x \, dx$

$= \int \cos^{n-2} x \, (1 - \sin^2 x \, dx)$

$= \int \cos^{n-2} x - \cos^{n-2} x \sin^2 x \, dx$

$= \int \cos^{n-2} x \, dx - \int \cos^{n-2} x \sin^2 x \, dx$

$= I_{n-2} - \int \cos^{n-2} x \sin x \sin x \, dx$

$u = \sin x \qquad dv = \cos^{n-2} x \sin x \, dx$

$du = \cos x \, dx \qquad v = \int \cos^{n-2} x \sin x \, dx$

$w = \cos x$

$dw = -\sin x \, dx$

$= \int -w^{n-2} \, dw$

$= \dfrac{-w^{n-1}}{n-1}$

$= -\dfrac{\cos^{n-1} x}{n-1}$

$= I_{n-2} - \left(\dfrac{-\sin x \cos^{n-1} x}{n-1} + \int \dfrac{\cos^{n-1} x \cos x \, dx}{n-1} \right)$

$= I_{n-2} + \dfrac{\sin x \cos^{n-1} x}{n-1} - \int \dfrac{\cos^n x \, dx}{n-1}$

$= I_{n-2} + \dfrac{\sin x \cos^{n-1} x}{n-1} - \dfrac{1}{n-1} I_n$

$$\left(1+\frac{1}{n-1}\right) I_n = I_{n-2} + \frac{\sin x \cos^{n-1} x}{n-1}$$

$$\left(\frac{n}{n-1}\right) I_n = I_{n-2} + \frac{\sin x \cos^{n-1} x}{n-1}$$

$$n I_n = (n-1) I_{n-2} + \sin x \cos^{n-1} x$$

$n=3$: $3I_3 = 2I_1 + \sin x \cos^2 x$

$$I_1 = \int \cos x \, dx$$

$$= \sin x + c$$

$$3I_3 = 2\sin x + \sin x \cos^2 x$$

$$I_3 = \frac{2\sin x}{3} + \frac{\sin x \cos^2 x}{3} + c$$

$n=2$: $2I_2 = \int \cos^2 x \, dx$

$$= \int \frac{\cos 2x + 1}{2} \, dx$$

$$= \frac{\sin 2x}{4} + \frac{x}{2} + c$$

$$= \frac{\sin x \cos x}{2} + \frac{x}{2} + c$$

9. $S: R^3 \to R^3$ $T: R^3 \to R^3$

$$M = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{pmatrix} \qquad N = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{pmatrix}$$

i) $$M = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{pmatrix}$$

$$\xrightarrow[-bcr_1+r_3]{-ar_1+r_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & ac-bc & ab-bc \end{pmatrix}$$

$$\xrightarrow{cr_2+r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & ab-bc+c^2-ac \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$$

Since $a > b > c$, $\text{rank}(M) = 3$

A basis for the row space of M is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b-a \\ c-a \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ (c-a)(c-b) \end{pmatrix} \right\}.$$

ii) $N = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{pmatrix}$

$$\xrightarrow[-(b+c)r_1 + r_2]{-ar_1 + r_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & a-b & a-c \end{pmatrix}$$

$$\xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank$(A) = 2$

A basis for the range space of N is

$$\left\{ \begin{pmatrix} 1 \\ a \\ b+c \end{pmatrix}, \begin{pmatrix} 1 \\ b \\ a+c \end{pmatrix} \right\}$$

If $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ a & b & c & 0 \\ b+c & a+c & a+b & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & b-a & c-a & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \qquad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (c-b)s \\ (a-c)s \\ (b-a)s \end{pmatrix} = s \begin{pmatrix} c-b \\ a-c \\ b-a \end{pmatrix}$$

Let $z = (b-a)s,\ s \in \mathbb{R}$

$y = -(c-a)s$

$x = (c-b)s$

$\therefore$ A basis for the nullspace is $\left\{ \begin{pmatrix} c-b \\ a-c \\ b-a \end{pmatrix} \right\}$.

10. $5y^4\frac{d^2y}{dx^2} + 20y^3\left(\frac{dy}{dx}\right)^2 + 55y^4\frac{dy}{dx} + 24y^5 = 181\cos x + 83\sin x$

$v = y^5$

$\frac{dv}{dy} = 5y^4$

$\frac{dv}{dx} = 5y^4\frac{dy}{dx}$

$\frac{d^2v}{dx^2} = 5y^4\frac{d^2y}{dx^2} + 20y^3\left(\frac{dy}{dx}\right)^2$

$5y^4\frac{d^2y}{dx^2} + 20y^3\left(\frac{dy}{dx}\right)^2 + 55y^4\frac{dy}{dx} + 24y^5 = 181\cos x + 83\sin x$

$5y^4\frac{d^2y}{dx^2} + 20y^3\left(\frac{dy}{dx}\right)^2 + 11\left(5y^4\frac{dy}{dx}\right) + 24y^5 = 181\cos x + 83\sin x$

$\therefore \frac{d^2v}{dx^2} + 11\frac{dv}{dx} + 24v = 181\cos x + 83\sin x$

$\frac{d^2v}{dx^2} + 11\frac{dv}{dx} + 24v = 0$

$m^2 + 11m + 24 = 0$

$(m+3)(m+8) = 0$

$m = -3, -8$

$\therefore$ The complementary function, v_c, is

$$v_c = Ae^{-3x} + Be^{-8x}.$$

The particular integral, v_p, is given by

$$v_p = C\cos x + D\sin x$$

$$\frac{dv_p}{dx} = -C\sin x + D\cos x$$

$$\frac{d^2v_p}{dx^2} = -C\cos x - D\sin x$$

$$\frac{d^2v_p}{dx^2} + 11\frac{dv_p}{dx} + 24v_p = -C\cos x - D\sin x$$
$$-11C\sin x + 11D\cos x$$
$$+24C\cos x + 24D\sin x$$
$$= (23C + 11D)\cos x$$
$$+ (23D - 11C)\sin x$$
$$= 181\cos x + 83\sin x$$

$$23C + 11D = 181$$
$$23D - 11C = 83$$
$$C = \frac{23D - 83}{11}$$

$$23\left(\frac{23D - 83}{11}\right) + 11D = 181$$

$$5290 - 1909 + 1210 = 1991$$

$$6500 = 3900$$

$$0 = 6$$

$$c = 5$$

$$\therefore\ v_p = 5\cos x + 6\sin x$$

$$v = v_c + v_p$$

$$= Ae^{-3x} + Be^{-8x} + 5\cos x + 6\sin x$$

$$y^5 = Ae^{-3x} + Be^{-8x} + 5\cos x + 6\sin x$$

11. $x^3 + ax^2 + bx + c = 0$

i) p, q, r are the roots

$\therefore\ p+q+r = -a \quad pq + pr + qr = b \quad pqr = -c$

$\therefore\ a = -p-q-r \quad b = pq + pr + qr \quad c = -pqr.$

ii) $p^2q^2 + p^2r^2 + q^2r^2$

$= (pq + pr + qr)^2 - 2[(pq)pr + (pq)qr + (pr)qr]$

$= (pq + pr + qr)^2 - 2(p^2qr + pq^2r + pqr^2)$

$= (pq + pr + qr)^2 - 2pqr(p+q+r)$

$= b^2 - 2(-a)(-c)$

$= b^2 - 2ac$

iii) If $p+q+r = \frac{7}{2}$,

$pq + pr + qr = -\frac{5}{2}$,

and $pqr = -2$

$\therefore\ a = -\frac{7}{2} \quad b = -\frac{5}{2} \quad c = 2$

$x^3 - \frac{7x^2}{2} - \frac{5x}{2} + 2 = 0$

$2x^3 - 7x^2 - 5x + 4 = 0$

$2(-1)^3 - 7(-1)^2 - 5(-1) + 4 = -2 - 7 + 5 + 4 = 0$

$\therefore x = -1$ is a root

$$
\begin{array}{r|l}
 & 2x^2 - 9x + 4 \\
\hline
x+1 & 2x^3 - 7x^2 - 5x + 4 \\
 & \underline{2x^3 + 2x^2} \\
 & \quad -9x^2 - 5x \\
 & \quad \underline{-9x^2 - 9x} \\
 & \qquad 4x + 4 \\
 & \qquad \underline{4x + 4} \\
 & \qquad\quad 0
\end{array}
$$

$\therefore 2x^3 - 7x^2 - 5x + 4 = (x+1)(2x^2 - 9x + 4)$

$2x^3 - 7x^2 - 5x + 4 = 0$

$(x+1)(2x^2 - 9x + 4) = 0$

$(x+1)(2x-1)(x-4) = 0$

$\therefore x = -1, 4, \frac{1}{2}.$

12. EITHER

i) $C: \ y = \dfrac{a^2x^2 - b^2}{c^2x^2 - d^2}, \quad a, b, c, d > 0$

$$\begin{array}{r|l} & \dfrac{a^2}{c^2} \\ \hline c^2x^2 - d^2 & a^2x^2 - b^2 \\ & a^2x^2 - \dfrac{a^2d^2}{c^2} \\ \hline & \dfrac{a^2d^2}{c^2} - b^2 \end{array}$$

$$= \frac{a^2}{c^2} + \frac{a^2d^2 - b^2c^2}{c^2(c^2x^2 - d^2)}$$

$$= \frac{a^2}{c^2} + \frac{a^2d^2 - b^2c^2}{c^2(cx - d)(cx + d)}$$

$$= \frac{d^2}{c^2} + \frac{a^2d^2 - b^2c^2}{2c^2d}\left(\frac{1}{cx - d} - \frac{1}{cx + d}\right)$$

$$= \frac{a^2}{c^2} + \frac{a^2d^2 - b^2c^2}{2c^2d(cx - d)} - \frac{(a^2d^2 - b^2c^2)}{2c^2d(cx + d)}$$

As $x \to \pm\infty$, $y \to \dfrac{a^2}{c^2}$

As $x \to \dfrac{d}{c}$, $y \to \pm\infty$

As $x \to \dfrac{-d}{c}$, $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = \frac{a^2}{c^2}$, $x = \frac{d}{c}$

and $x = -\frac{d}{c}$.

ii) $$\frac{dy}{dx} = \frac{a^2d^2 - b^2c^2}{2cd}\left(\frac{-1}{(cx-d)^2} + \frac{1}{(cx+d)^2}\right)$$

when $\frac{dy}{dx} = 0$: $$\frac{a^2d^2 - b^2c^2}{2cd}\left(\frac{-1}{(cx-d)^2} + \frac{1}{(cx+d)^2}\right) = 0$$

$$\frac{-1}{(cx-d)^2} + \frac{1}{(cx+d)^2} = 0$$

$$\frac{1}{(cx-d)^2} = \frac{1}{(cx+d)^2}$$

$$(cx-d)^2 = (cx+d)^2$$

$$c^2x^2 + 2cdx + d^2 = c^2x^2 - 2cdx + d^2$$

$$4cdx = 0$$

$$cdx = 0$$

$\therefore x = 0$ since $c, d > 0$

$$y = \frac{b^2}{d^2}$$

$\therefore$ The critical point of C has coordinates $\left(0, \frac{b^2}{d^2}\right)$.

$$\frac{d^2y}{dx^2} = \frac{a^2d^2 - b^2c^2}{d}\left(\frac{1}{(cx-d)^3} - \frac{1}{(cx+d)^3}\right)$$

At $\left(0, \frac{b^2}{d^2}\right)$: $\frac{d^2y}{dx^2} = \frac{2(a^2d^2 - b^2c^2)}{-d^4}$

If $a^2d^2 - b^2c^2 > 0$, $\frac{d^2y}{dx^2} < 0$

$\therefore \left(0, \frac{b^2}{d^2}\right)$ is a maximum point.

If $a^2d^2 - b^2c^2 < 0$, $\frac{d^2y}{dx^2} > 0$

$\therefore \left(0, \frac{b^2}{d^2}\right)$ is a minimum point.

iii) when $x = 0$: $y = \frac{b^2}{d^2}$

when $y = 0$: $\frac{a^2x^2 - b^2}{c^2x^2 - d^2} = 0$

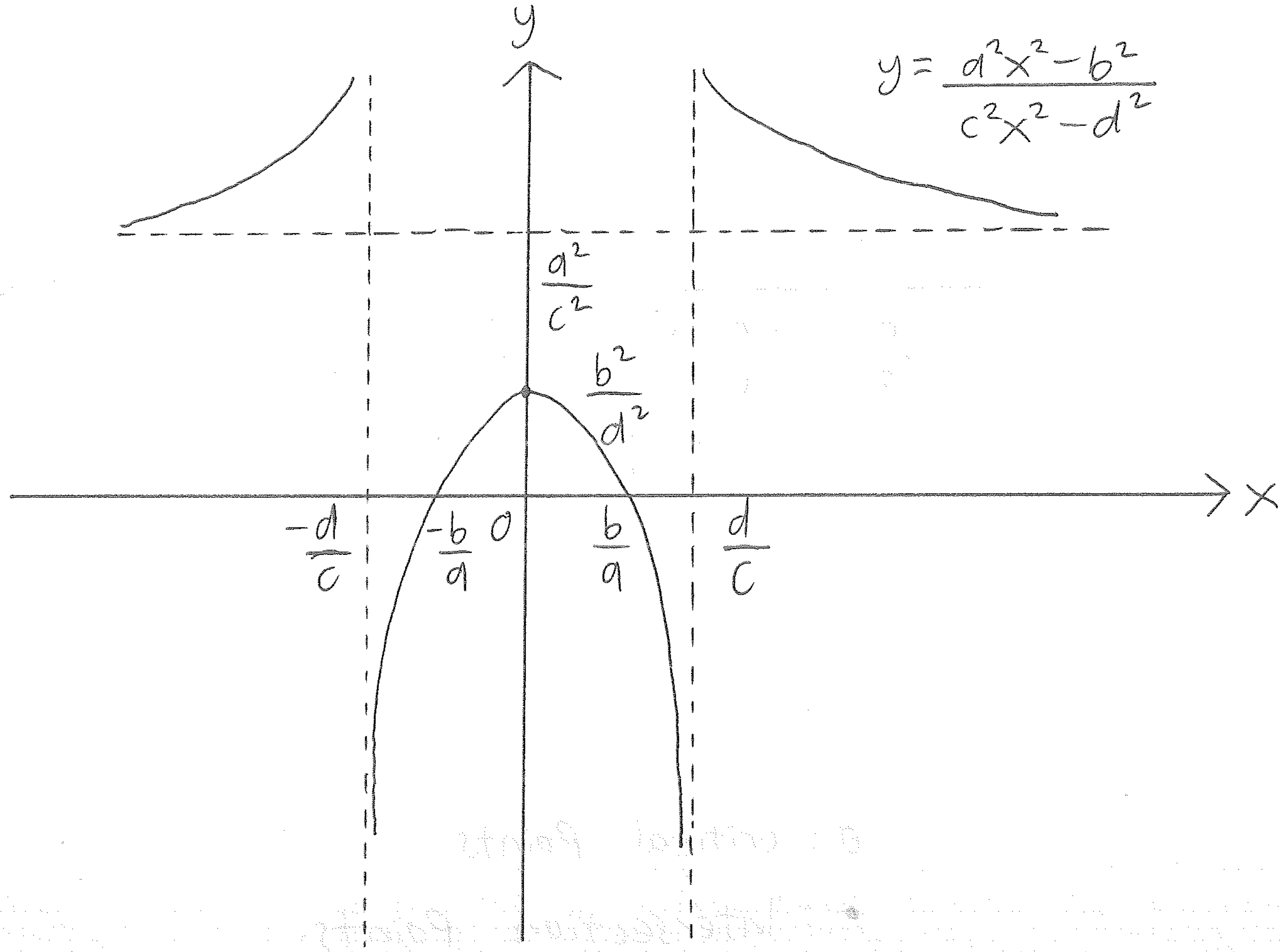
$$a^2x^2 - b^2 = 0$$

$$(ax - b)(ax + b) = 0$$

$$x = \frac{-b}{a}, \frac{b}{a}$$

$\therefore$ The intersection points of C are $\left(-\frac{b}{a}, 0\right)$ $\left(\frac{b}{a}, 0\right)$ and $\left(0, \frac{b^2}{d^2}\right)$.

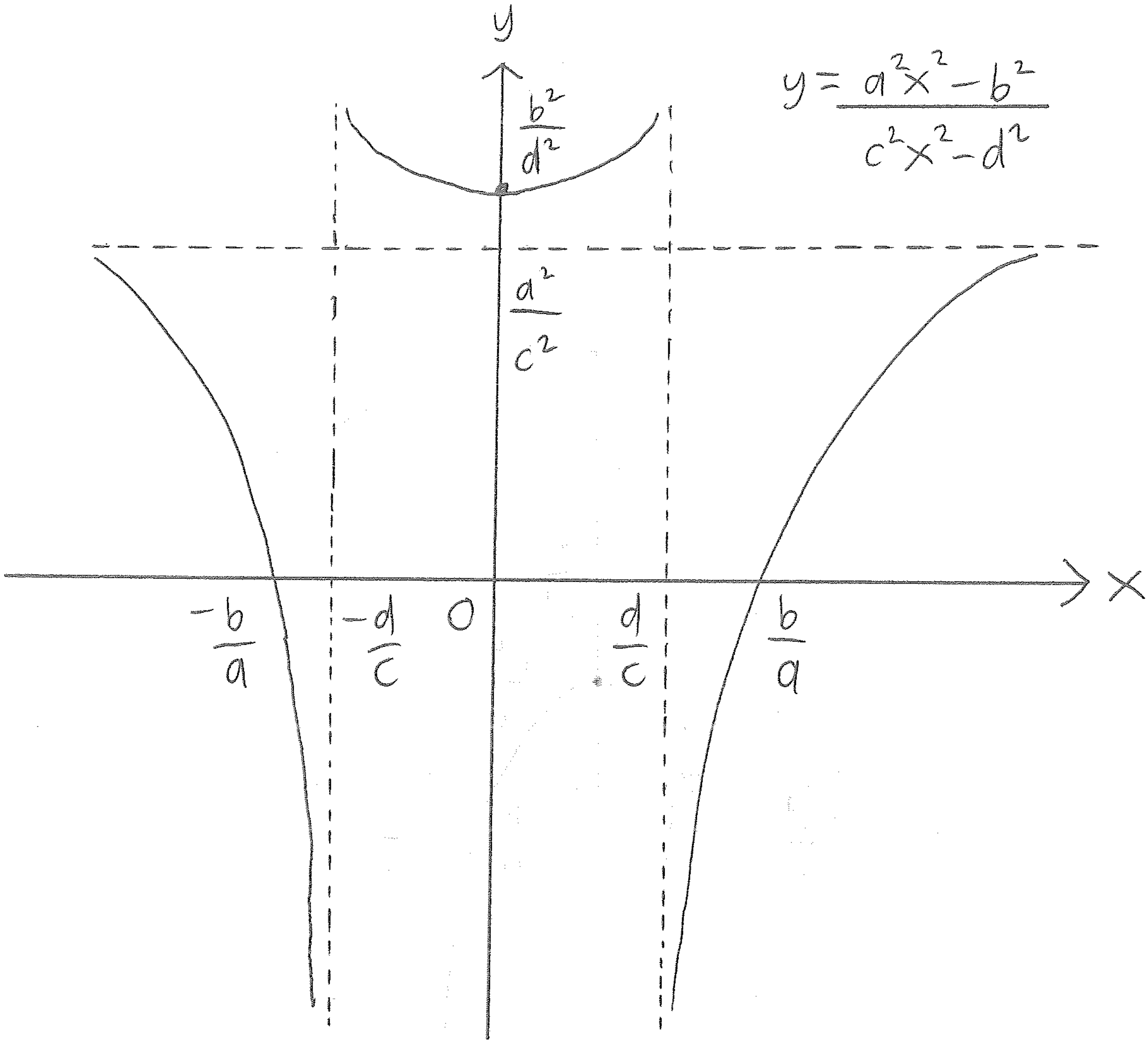
If $a^2d^2 - b^2c^2 > 0$



o : critical points

• : Intersection Points.

If $a^2d^2 - b^2c^2 < 0$



o : critical Points

• : Intersection Points.

OR

i) $y = \dfrac{1}{\sqrt{a^2-b^2(x-c)^2}}$, $a, b, c > 0$

The mean value of y over the interval

$c + \dfrac{a}{2b} \leq x \leq c + \dfrac{\sqrt{3}a}{2b}$ is

$$\frac{1}{\frac{a(\sqrt{3}-1)}{2b}} \int_{c+\frac{a}{2b}}^{c+\frac{\sqrt{3}a}{2b}} \frac{1}{\sqrt{a^2-b^2(x-c)^2}}\, dx$$

Let $b(x-c) = a\sin\theta$

$b\,dx = a\cos\theta\, d\theta$

$dx = \dfrac{a}{b}\cos\theta\, d\theta$

$x = c + \dfrac{\sqrt{3}a}{2b}$: $\theta = \dfrac{\pi}{3}$

$x = c + \dfrac{a}{2b}$: $\theta = \dfrac{\pi}{6}$

$$= \frac{2b}{a(\sqrt{3}-1)} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sqrt{a^2-a^2\sin^2\theta}} \frac{a}{b}\cos\theta\, d\theta$$

$$= \frac{2b}{a(\sqrt{3}-1)} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{a\cos\theta}{b(a\cos\theta)}\, d\theta = \frac{2}{a(\sqrt{3}-1)} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\, d\theta$$

$$= \frac{2}{a(\sqrt{3}-1)} [\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{2}{a(\sqrt{3}-1)}\left(\frac{\pi}{6}\right) = \frac{\pi}{3a(\sqrt{3}-1)}$$

ii) If the centroid of the region R bounded by the curve, the x-axis and the lines $x = c + \frac{a}{2b}$ and $x = c + \frac{\sqrt{3}a}{2b}$ has coordinates $(\bar{x}, \bar{y})$,

$$\bar{x} = \frac{1}{A} \int_{c+\frac{a}{2b}}^{c+\frac{\sqrt{3}a}{2b}} xy \, dx, \quad A \text{ is the area of } R$$

$$= \frac{1}{A} \int_{c+\frac{a}{2b}}^{c+\frac{\sqrt{3}a}{2b}} \frac{x}{\sqrt{a^2 - b^2(x-c)^2}} \, dx$$

Let $b(x-c) = a\sin\theta$

$b\,dx = a\cos\theta \, d\theta$

$dx = \frac{a}{b}\cos\theta \, d\theta$

$x = c + \frac{a}{2b} : \theta = \frac{\pi}{6}$

$x = c + \frac{\sqrt{3}a}{2b} : \theta = \frac{\pi}{3}$

$$= \frac{1}{A} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\left(c + \frac{a\sin\theta}{b}\right)}{a\cos\theta} \frac{a}{b}\cos\theta \, d\theta$$

$$= \frac{1}{bA} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} c + \frac{a}{b}\sin\theta \, d\theta$$

$$= \frac{1}{bA} \left[c\theta - \frac{a}{b}\cos\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{c}{6} - \frac{3a}{b\pi}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$$

$$\bar{y} = \frac{1}{A}\int_{c+\frac{a}{2b}}^{c+\frac{\sqrt{3}a}{2b}} \frac{y^2}{2}\,dx, \text{ A is the area of R.}$$

$$= \frac{1}{2A}\int_{c+\frac{a}{2b}}^{c+\frac{\sqrt{3}a}{2b}} \frac{1}{a^2-b^2(x-c)^2}\,dx$$

Let $b(x-c) = a\sin\theta$

$b\,dx = a\cos\theta\,d\theta$

$dx = \frac{a\cos\theta}{b}\,d\theta$

$x = c + \frac{\sqrt{3}a}{2b} : \theta = \frac{\pi}{3}$

$x = c + \frac{a}{2b} : \theta = \frac{\pi}{6}$

$$= \frac{1}{2A}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{a^2-a^2\sin^2\theta}\,\frac{a}{b}\cos\theta\,d\theta$$

$$= \frac{1}{2A}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{ab\cos\theta}\,d\theta$$

$$= \frac{1}{2A}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec\theta}{ab}\,d\theta$$

$$= \frac{1}{2abA} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{2abA} \Big[\ln(\sec\theta + \tan\theta) \Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2abA} \left[\ln\left(\frac{1+\sin\theta}{\cos\theta} \right) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2abA} \left(\ln\left| \frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}} \right| - \ln\left| \frac{1+\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right| \right)$$

$$= \frac{1}{2abA} \left(\ln(2+\sqrt{3}) - \ln\sqrt{3} \right)$$

$$= \frac{1}{2abA} \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}} \right)$$

$$= \frac{3}{ab\pi} \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}} \right)$$

$\therefore$ The coordinates of the centroid of the region R is $\left(\frac{c}{b} - \frac{3a}{\pi b}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right), \frac{3}{ab\pi} \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) \right)$.