1. $$r = \sec^n\theta + \csc^n\theta$$

$$= \frac{1}{\sin^n\theta} + \frac{1}{\cos^n\theta}$$

$$= \frac{\sin^n\theta + \cos^n\theta}{\sin^n\theta\cos^n\theta}$$

$$= \frac{r^{2n}\sin^n\theta + r^{2n}\cos^n\theta}{r^{2n}\sin^n\theta\cos^n\theta}$$

$$= \frac{r^n(r\sin\theta)^n + r^n(r\cos\theta)^n}{(r\sin\theta)^n(r\cos\theta)^n}$$

$$= \frac{r^ny^n + r^nx^n}{x^ny^n}$$

$$= \frac{r^n(x^n + y^n)}{(xy)^n}$$

$$(xy)^n = r^{n-1}(x^n + y^n)$$

$$= \left(\sqrt{x^2 + y^2}\right)^{n-1}(x^n + y^n)$$

$$= (x^2 + y^2)^{\frac{n-1}{2}}(x^n + y^n)$$

$$(xy)^{2n} = (x^2 + y^2)^{n-1}(x^n + y^n)^2$$

The Cartesian form of the polar equation $r = \sec^n\theta + \csc^n\theta$ is $(xy)^{2n} = (x^2 + y^2)^{n-1}(x^n + y^n)^2$.

2. $13^3 + 21^3 + 29^3 + \cdots + (8n + 5)^3$

$$= \sum_{r=1}^{n} (8r + 5)^3$$

$$= \sum_{r=1}^{n} 512r^3 + 960r^2 + 600r + 125$$

$$= 512\sum_{r=1}^{n} r^3 + 960\sum_{r=1}^{n} r^2 + 600\sum_{r=1}^{n} r$$

$$+ 125\sum_{r=1}^{n} 1$$

$$= \frac{512n^2(n + 1)^2}{4} + \frac{960n(n + 1)(2n + 1)}{6}$$

$$+ \frac{600n(n + 1)}{2} + 125n$$

$$= 128n^2(n + 1)^2 + 160n(n + 1)(2n + 1)$$

$$+ 300n(n + 1) + 125n$$

$$= 128n^2(n^2 + 2n + 1) + 160(2n^2 + 3n + 1)$$

$$+ 300(n^2 + n) + 125n$$

$$= 128n^4 + 256n^3 + 128n^2$$

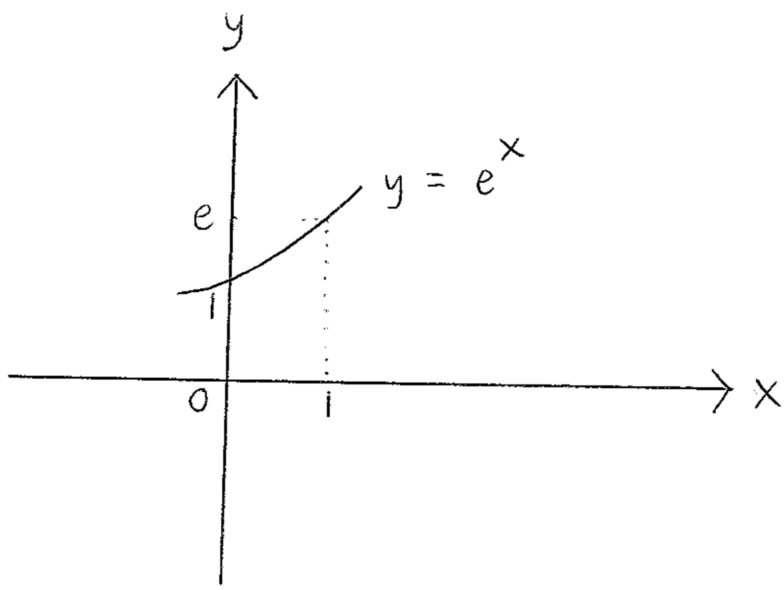
$$+ 320n^3 + 480n^2 + 160n$$

$$+ 300n^2 + 300n + 125n$$

$$= 128n^4 + 576n^3 + 908n^2 + 585n$$

$$= n(128n^3 + 576n^2 + 908n + 585)$$

3.



The area, A, of the region bounded by the curve $y = e^x$, the line $x = 1$ and the x and y axes is

$$\int_0^1 y \, dx = \int_0^1 e^x \, dx$$

$$= \left[e^x \right]_0^1$$

$$= e - 1$$

The y coordinate, $\bar{y}$, of the centroid of the area bounded by the curve $y = e^x$, the line $x = 1$ and the x and y axes is

$$\frac{\int_0^1 \frac{y^2}{2}\,dx}{A} = \frac{\int_0^1 \frac{e^{2x}}{2}\,dx}{e - 1}$$

$$= \frac{\left[\frac{e^{2x}}{4}\right]_0^1}{e - 1}$$

$$= \frac{e^2 - 1}{4(e - 1)}$$

$$= \frac{e + 1}{4}$$

$$\therefore \bar{y} = \frac{e + 1}{4}$$

4. $$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 32e^x$$

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$$

The auxillary equation is

$$m^2 + 6m + 25 = 0$$

$$(m + 3)^2 + 16 = 0$$

$$(m + 3)^2 = -16$$

$$m + 3 = \pm 4i$$

$$m = -3 \pm 4i$$

The complementary function, y_c, is

$$y_c = e^{-3x}(A\cos 4x + B\sin 4x)$$

If y_p is the particular integral,

y_p is given by

$$y_p = Ce^x$$

$$\frac{dy_p}{dx} = Ce^x$$

$$\frac{d^2y_p}{dx^2} = Ce^x$$

$$\frac{d^2y_p}{dx^2} + 6\frac{dy_p}{dx} + 25y_p$$

$$= Ce^x + 6Ce^x + 25Ce^x$$

$$= 32Ce^x$$

$$= 32e^x$$

$\therefore$ $32C = 32$

$C = 1$

$y_p = e^x$

$y = y_c + y_p$

$= e^{-3x}(A\cos 4x + B\sin 4x) + e^x$

$\therefore$ The general solution of the differential equation

$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 32e^x$ is

$$y = e^{-3x}(A\cos 4x + B\sin 4x) + e^x$$

5. The square matrix A has an eigenvalue λ and eigenvector $\underset{\sim}{x}$

$\therefore A\underset{\sim}{x} = \lambda \underset{\sim}{x}$.

a) i) $A^n \underset{\sim}{x} = \lambda^n \underset{\sim}{x}$ where n is a positive integer

$$n = 1 \cdot \quad A^1 \underset{\sim}{x} = A\underset{\sim}{x}$$
$$= \lambda \underset{\sim}{x}$$
$$= \lambda^1 \underset{\sim}{x}$$

Assume the equation is true when $n = k$.

$n = k$: $A^k \underset{\sim}{x} = \lambda^k \underset{\sim}{x}$

When $n = k+1$, $A^{k+1} \underset{\sim}{x} = A^k (A\underset{\sim}{x})$

$$= A^k (\lambda \underset{\sim}{x})$$
$$= \lambda (A^k \underset{\sim}{x})$$
$$= \lambda (\lambda^k \underset{\sim}{x})$$
$$= \lambda^{k+1} \underset{\sim}{x}$$

Since $A^n \underset{\sim}{x} = \lambda^n \underset{\sim}{x}$ when $n = 1$ and $A^n \underset{\sim}{x} = \lambda^n x$ when $n = k + 1$ if $A^n \underset{\sim}{x} = \lambda^n \underset{\sim}{x}$ when $n = k$ $\therefore$ $A^n \underset{\sim}{x} = \lambda^n \underset{\sim}{x}$ for every positive integer n.

ii) $k(A\underset{\sim}{x}) = k(\lambda \underset{\sim}{x})$

$(kA)\underset{\sim}{x} = (k\lambda)\underset{\sim}{x}$

$\therefore$ kA has an eigenvalue $k\lambda$ and eigenvector $k\underset{\sim}{x}$, where k is a scalar.

b) $(a_n A^n + a_{n-1} A^{n-1} + \cdots + a_2 A^2 + a_1 A + a_0 I)\underset{\sim}{x}$

$= (a_n A^n)\underset{\sim}{x} + (a_{n-1} A^{n-1})\underset{\sim}{x} + \cdots$

$+ (a_2 A^2)\underset{\sim}{x} + (a_1 A)\underset{\sim}{x} + (a_0 I)\underset{\sim}{x}$

$= a_n(A^n \underset{\sim}{x}) + a_{n-1}(A^{n-1}\underset{\sim}{x}) + \cdots$

$+ a_2(A^2 \underset{\sim}{x}) + a_1(A\underset{\sim}{x}) + a_0(I\underset{\sim}{x})$

$= a_n(\lambda^n \underset{\sim}{x}) + a_{n-1}(\lambda^{n-1}\underset{\sim}{x}) + \cdots$

$+ a_2(\lambda^2 \underset{\sim}{x}) + a_1(\lambda \underset{\sim}{x}) + a_0(\underset{\sim}{x})$

$$= (a_n \lambda^n)\underset{\sim}{x} + (a_{n-1}\lambda^{n-1}) + \cdots$$

$$+ (a_2 \lambda^2)\underset{\sim}{x} + (a_1 \lambda)\underset{\sim}{x} + a_0 \underset{\sim}{x}$$

$$= (a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0)\underset{\sim}{x}$$

$\therefore$ The matrix

$$a_n A^n + a_{n-1}A^{n-1} + \cdots + a_2 A^2 + a_1 A + a_0 I$$

has eigenvalue

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

and eigenvector $\underset{\sim}{x}$.

6 Π_1 : $x + y + z = 0$

Π_2 : $ax + by + cz = 0$

Π_3 $a^2x + b^2y + c^2z = 0$

$$\left.\begin{aligned} x + y + z &= 0 \\ ax + by + cz &= 0 \\ a^2x + b^2y + c^2z &= 0 \end{aligned}\right\}$$

$-a \times ① + ②$: $-a^2 \times ① + ③$:

$$\left.\begin{aligned} x + y + z &= 0 \\ (b-a)y + (c-a)z &= 0 \\ (b^2-a^2)y + (c^2-a^2)z &= 0 \end{aligned}\right\}$$

$-(b+a) \times ② + ③$

$$\left.\begin{aligned} x + y + z &= 0 \\ (b-a)y + (c-a)z &= 0 \\ [c^2 - a^2 - (c-a)(b+a)]z &= 0 \end{aligned}\right\}$$

$$[c^2 - a^2 - (c-a)(b+a)]z = 0$$

$$c^2 - a^2 - (c-a)(b+a) = 0$$

$$(c-a)[(c+a) - (b+a)] = 0$$

$$(c-a)(c-b) = 0$$

$$c = a, b$$

$c = b \neq a$: $0z = 0$

Let $z = s$, $s \in R$

$(b - a)y + (b - a)s = 0$

$y = -s$

$x = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$c = a \neq b$: $0z = 0$

Let $z = s$, $s \in R$

$y = 0$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ The possible lines of intersection of Π_1, Π_2, Π_3 are $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ and

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

7 $3x^4 - 2x^3 + x^2 - 9x + 7 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$$\frac{1}{\alpha^2 + 4}, \quad \frac{1}{\beta^2 + 4}, \quad \frac{1}{\gamma^2 + 4}, \quad \frac{1}{\delta^2 + 4}$$

Let $u = \dfrac{1}{\alpha^2 + 4}$

$$\alpha^2 + 4 = \frac{1}{u}$$

$$\alpha^2 = \frac{1}{u} - 4$$

$$\alpha = \pm\sqrt{\frac{1}{u} - 4}$$

α is a root

$\therefore 3\alpha^4 - 2\alpha^3 + \alpha^2 - 9\alpha + 7 = 0$

$$3\left(\pm\sqrt{\frac{1}{u} - 4}\right)^4 - 2\left(\pm\sqrt{\frac{1}{u} - 4}\right)^3$$

$$+ \left(\pm\sqrt{\frac{1}{u} - 4}\right)^2 - 9\left(\pm\sqrt{\frac{1}{u} - 4}\right)$$

$$+ 7 = 0.$$

$$3\left(\frac{1}{u} - 4\right)^2 \mp 2\left(\frac{1}{u} - 4\right)\sqrt{\frac{1}{u} - 4}$$

$$+ \frac{1}{u} - 4 \mp 9\sqrt{\frac{1}{u} - 4} + 7 = 0$$

$$3\left(\frac{1}{u^2} - \frac{8}{u} + 16\right) + \frac{1}{u} - 4 + 7$$

$$= \pm\sqrt{\frac{1}{u} - 4}\left(2\left(\frac{1}{u} - 4\right) + 9\right)$$

$$\frac{3}{u^2} - \frac{24}{u} + 48 + \frac{1}{u} + 3$$

$$= \pm\sqrt{\frac{1}{u} - 4}\left(\frac{2}{u} - 8 + 9\right)$$

$$\frac{3}{u^2} - \frac{23}{u} + 51 = \pm\sqrt{\frac{1}{u} - 4}\left(\frac{2}{u} + 1\right)$$

$$\left(\frac{3}{u^2} - \frac{23}{u} + 51\right)^2 = \left(\frac{1}{u} - 4\right)\left(\frac{2}{u} + 1\right)^2$$

$$\left(\frac{3}{u^2} - \frac{23}{u}\right)^2 + 102\left(\frac{3}{u^2} - \frac{23}{u}\right) + 2601$$

$$= \left(\frac{1}{u} - 4\right)\left(\frac{4}{u^2} + \frac{4}{u} + 1\right)$$

$$\frac{9}{u^4} - \frac{138}{u^3} + \frac{529}{u^2} + \frac{306}{u^2} - \frac{2346}{u} + 2601$$

$$= \frac{4}{u^3} + \frac{4}{u^2} + \frac{1}{u} - \frac{16}{u^2} - \frac{16}{u} - 4$$

$$9 - 138u + 529u^2 + 306u^2 - 2346u^3$$

$$+ 2601u^4 = 4u + 4u^2 + u^3 - 16u^2 - 16u^3$$

$$- 4u^4$$

$$2601u^4 - 2331u^3 + 847u^2 - 142u + 9 = 0$$

$\therefore$ The equation having roots

$$\frac{1}{\alpha^2 + 4}, \frac{1}{\beta^2 + 4}, \frac{1}{\gamma^2 + 4}, \frac{1}{\delta^2 + 4} \text{ is}$$

$$2601u^4 - 2331u^3 + 847u^2 - 142u + 9 = 0.$$

8. C :

$$y = \frac{t^7}{7} + \frac{3}{t^3}, \quad x = 3t^2$$

$$\frac{dy}{dt} = t^6 - \frac{9}{t^4} \qquad \frac{dx}{dt} = 6t$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2$$

$$= \left(t^6 - \frac{9}{t^4}\right)^2 + (6t)^2$$

$$= t^{12} - 18t^2 + \frac{81}{t^8} + 36t^2$$

$$= t^{12} + 18t^2 + \frac{81}{t^8}$$

$$= \left(t^6 + \frac{9}{t^4}\right)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t^6 + \frac{9}{t^4}$$

The area of the surface generated by rotating C about the x-axis from $t = 1$ to $t = 2$ is $\int_1^2 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_1^2 2\pi \left(\frac{t^7}{7} + \frac{3}{t^3}\right)\left(t^6 + \frac{9}{t^4}\right) dt$$

$$= 2\pi \int_1^2 \frac{t^{13}}{7} + 3t^3 + \frac{9t^3}{7} + \frac{27}{t^7}\, dt$$

$$= 2\pi \int_1^2 \frac{t^{13}}{7} + \frac{30t^3}{7} + \frac{27}{t^7}\, dt$$

$$= 2\pi \left[\frac{t^{14}}{98} + \frac{15t^4}{14} - \frac{9}{2t^6}\right]_1^2$$

$$= 2\pi \left(\frac{16384}{98} + \frac{120}{7} - \frac{9}{128} \right.$$
$$\left. \frac{-1}{98} - \frac{15}{14} - \frac{9}{2}\right)$$

$$= 357\frac{1095}{3136}\pi$$

9 $$I_n = \int \sec^n x \; dx$$

$$= \int \sec^{n-2} x \sec^2 x \; dx$$

$$= \int \sec^{n-2} x (1 + \tan^2 x) \; dx$$

$$= \int \sec^{n-2} x + \sec^{n-2} x \tan^2 x \; dx$$

$$= \int \sec^{n-2} x \; dx + \int \sec^{n-2} x \tan^2 x \; dx$$

$$= I_{n-2} + \int \sec^{n-2} x \tan^2 x \; dx$$

$$= I_{n-2} + \int \sec^{n-3} x \sec x \tan x \tan x \; dx$$

$u = \tan x$

$du = \sec^2 x \; dx$

$dv = \sec^{n-3} x \sec x \tan x \; dx$

$v = \int \sec^{n-3} x \sec x \tan x \; dx$

$w = \sec x$

$dw = \sec x \tan x \; dx$

$= \int w^{n-3} \; dw$

$$= \frac{w^{n-2}}{n-2}$$

$$= \frac{\sec^{n-2} x}{n-2}$$

$$= I_{n-2} + \frac{\tan x \sec^{n-2} x}{n-2}$$

$$- \int \frac{\sec^{n-2} x \sec^2 x}{n-2} \, dx$$

$$= I_{n-2} + \frac{\tan x \sec^{n-2} x}{n-2}$$

$$- \int \frac{\sec^n x}{n-2} \, dx$$

$$= I_{n-2} + \frac{\tan x \sec^{n-2} x}{n-2} - \frac{1}{n-2} I_n$$

$$\left(1 + \frac{1}{n-2}\right) I_n = I_{n-2} + \frac{\tan x \sec^{n-2} x}{n-2}$$

$$\left(\frac{n-1}{n-2}\right) I_n = I_{n-2} + \frac{\tan x \sec^{n-2} x}{n-2}$$

$$(n-1)I_n = (n-2)I_{n-2} + \tan x \sec^{n-2} x$$

$$n=3: \quad 2I_3 = I_1 + \tan x \sec x$$

$$I_1 = \int \sec x \, dx$$

$$= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \ln|\sec x + \tan x|$$

$$2I_3 = \tan x \sec x$$

$$+ \ln|\sec x + \tan x|$$

$$I_3 = \frac{\tan x \sec x}{2}$$

$$+ \frac{\ln|\sec x + \tan x|}{2} + C$$

$$n=4: \quad 3I_4 = 2I_2 + \tan x \sec^2 x$$

$$I_2 = \int \sec^2 x \, dx$$

$$= \tan x$$

$$\therefore\ 3I_4 = 2\tan x + \tan x \sec^2 x$$

$$I_4 = \frac{2\tan x}{3} + \frac{\tan x \sec^2 x}{3} + C$$

10 $$y = 5 - \frac{3}{x-1} + \frac{48}{x+9}$$

As $x \longrightarrow \pm\infty$ $y \longrightarrow 5$

As $x \longrightarrow 1$ $y \longrightarrow \pm\infty$

As $x \longrightarrow -9$ $y \longrightarrow \pm\infty$

$\therefore$ The asymptotes of C are $y = 5$, $x = 1$ and $x = -9$.

when $x = 0$: $y = \frac{40}{3}$

when $y = 0$ $$5 - \frac{3}{x-1} + \frac{48}{x+9} = 0$$

$$5(x-1)(x+9) - 3(x+9) + 48(x-1) = 0$$

$$5(x^2 + 8x - 9) - 3x - 27 + 48x - 48 = 0$$

$$5x^2 + 40x - 45 + 45x - 75 = 0$$

$$5x^2 + 85x - 120 = 0$$

$$x = \frac{-85 \pm \sqrt{9625}}{10}$$

$$= \frac{-17 \pm \sqrt{385}}{2}$$

The intersection points of C are

$\left(0, \frac{40}{3}\right)$, $\left(\frac{-17 + \sqrt{385}}{2}, 0\right)$ and

$\left(\frac{-17 - \sqrt{385}}{2}, 0\right)$.

$$\frac{dy}{dx} = \frac{3}{(x-1)^2} - \frac{48}{(x+9)^2}$$

when $\frac{dy}{dx} = 0$

$$\frac{3}{(x-1)^2} - \frac{48}{(x+9)^2} = 0$$

$$\frac{3}{(x-1)^2} = \frac{48}{(x+9)^2}$$

$$(x+9)^2 = 16(x-1)^2$$

$$x + 9 = \pm 4(x-1)$$

$$x + 9 = 4(x-1), \quad x + 9 = -4(x-1)$$

$$= 4x - 4 \qquad\qquad = -4x + 4$$

$$5x = 13$$

$$x = \frac{13}{3}$$

$$y = \frac{77}{10}$$

$$5x = -5$$

$$x = -1$$

$$y = \frac{25}{2}$$

$\therefore$ The critical points of C are

$\left(\frac{13}{3}, \frac{7}{10}\right)$ and $\left(-1, \frac{25}{2}\right)$.

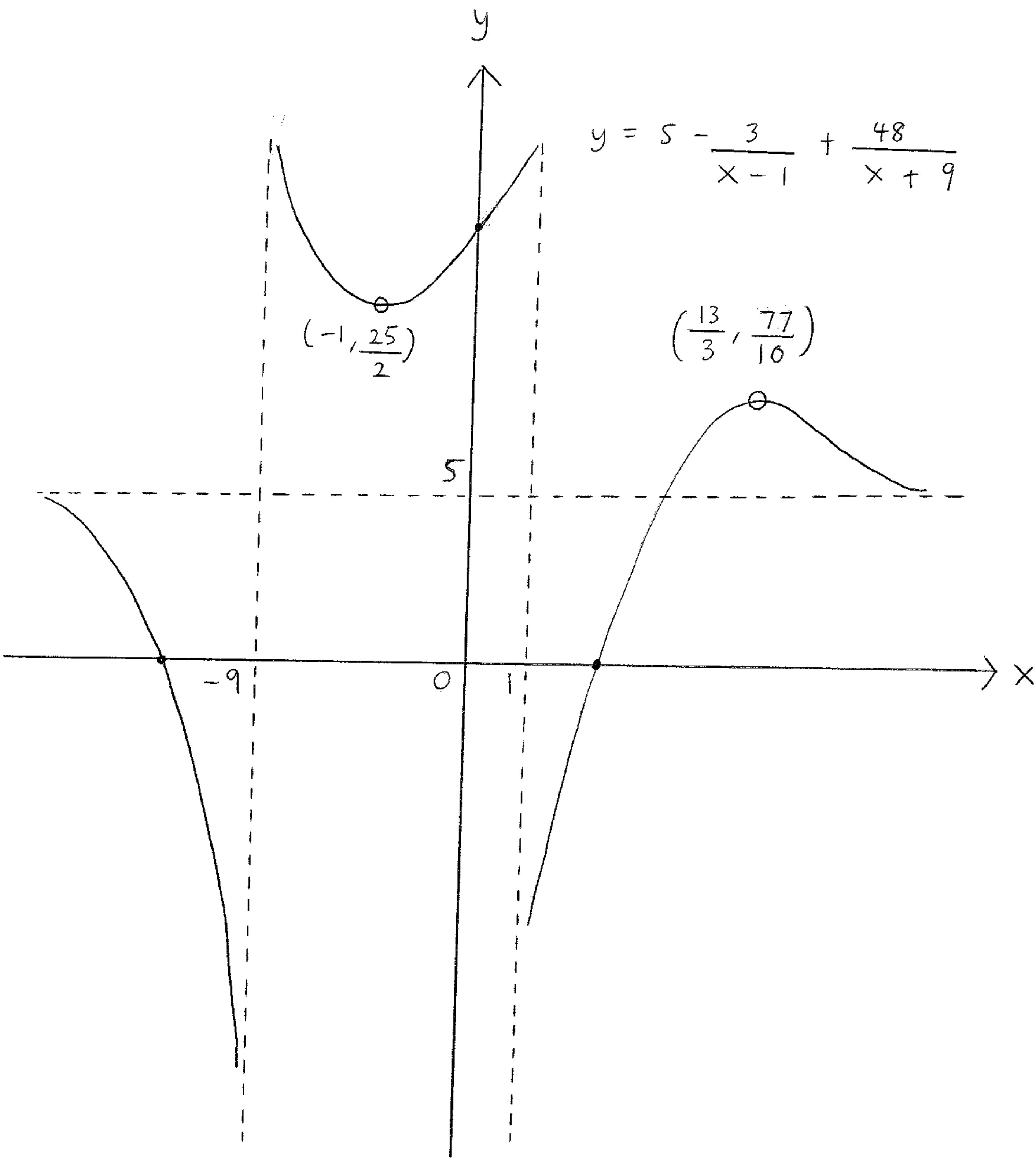
$$\frac{d^2y}{dx^2} = \frac{-6}{(x-1)^3} + \frac{96}{(x+9)^3}$$

$x = \frac{13}{3}$: $\frac{d^2y}{dx^2} = \frac{-243}{2000} < 0$

$x = -1$: $\frac{d^2y}{dx^2} = \frac{15}{16} > 0$

$\therefore$ $\left(\frac{13}{3}, \frac{77}{10}\right)$ is a maximum point

and $\left(-1, \frac{25}{2}\right)$ is a minimum point.



o : Critical point

• Intersection point.

11 EITHER

$$z + z^2 + z^3 + \dots + z^n = \frac{z(1 - z^n)}{1 - z}$$

$$z = e^{i\theta}$$

$$\therefore \sum_{r=1}^{n} e^{ir\theta} = \frac{e^{i\theta}(1 - e^{in\theta})}{1 - e^{i\theta}}$$

Since $e^{ir\theta} = \cos r\theta + i\sin r\theta$

$$\sum_{r=1}^{n} \cos r\theta + i\sin r\theta = \frac{e^{i\theta}(1 - e^{in\theta})}{1 - e^{i\theta}}$$

$$\sum_{r=1}^{n} \cos r\theta + i\sum_{r=1}^{n} \sin r\theta = \frac{e^{i\theta}(e^{in\theta} - 1)}{e^{i\theta} - 1}$$

$$= \frac{e^{i\theta}(e^{in\theta} - 1)\, e^{\frac{in\theta}{2}} e^{\frac{i\theta}{2}}}{e^{\frac{in\theta}{2}} e^{\frac{i\theta}{2}} (e^{i\theta} - 1)}$$

$$= \frac{e^{i\theta} e^{\frac{in\theta}{2}} (e^{\frac{in\theta}{2}} - e^{-\frac{in\theta}{2}})}{e^{\frac{i\theta}{2}} (e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}})}$$

$$= e^{\frac{i(n+1)\theta}{2}} \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}$$

$$= \left(\cos \left(\frac{n+1}{2}\right)\theta + i\sin \left(\frac{n+1}{2}\right)\theta \right) \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}$$

$$= \frac{\cos \left(\frac{n+1}{2}\right)\theta \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} + \frac{i\sin \left(\frac{n+1}{2}\right)\theta \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}$$

$$\therefore \sum_{r=1}^{n} \cos r\theta = \frac{\cos \left(\frac{n+1}{2}\right)\theta \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}$$

OR

a) $\tan x + \csc y = \cot x + \sec y$

$$\frac{d}{dx}(\tan x + \csc y) = \frac{d}{dx}(\cot x + \sec y)$$

$$\sec^2 x - \csc y \cot y \frac{dy}{dx} = -\csc^2 x + \sec y \tan y \frac{dy}{dx}$$

$$\sec y \tan y \frac{dy}{dx} + \csc y \cot y \frac{dy}{dx} = \sec^2 x + \csc^2 x$$

$$(\sec y \tan y + \csc y \cot y)\frac{dy}{dx} = \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cos^2 x}$$

$$= \frac{1}{\sin^2 x \cos^2 x}$$

At $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$: $(\sqrt{2} + \sqrt{2})\frac{dy}{dx} = \frac{1}{\frac{1}{4}}$

$$2\sqrt{2}\frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \sqrt{2}$$

$$y - \frac{\pi}{4} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$= \sqrt{2}x - \frac{\sqrt{2}\pi}{4}$$

$$y = \sqrt{2}x + \frac{\pi}{4}(1 - \sqrt{2})$$

$\therefore$ The equation of the tangent to the curve $\tan x + \csc y = \cot x + \sec y$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ is

$$y = \sqrt{2}x + \frac{\pi}{4}(1 - \sqrt{2})$$

b) $e^x \ln y = x^e$

$$\frac{d}{dx}(e^x \ln y) = \frac{d}{dx}(x^e)$$

$$e^x \ln y + \frac{e^x}{y}\frac{dy}{dx} = ex^{e-1}$$

At (e, e) : $e^e + e^{e-1}\dfrac{dy}{dx} = e^e$

$$\frac{dy}{dx} = 0$$

$$e^x\left(\ln y + \frac{1}{y}\frac{dy}{dx}\right) = ex^{e-1}$$

$$\frac{d}{dx}\left[e^x\left(\ln y + \frac{1}{y}\frac{dy}{dx}\right)\right] = \frac{d}{dx}(ex^{e-1})$$

$$e^x\left(\ln y + \frac{1}{y}\frac{dy}{dx}\right)$$

$$+ e^x\left(\frac{1}{y}\frac{dy}{dx} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2}\right)$$

$$= e(e-1)x^{e-2}$$

At (e, e) $\quad e^{e}(1) + e^{e}\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)$

$$= e(e-1)e^{e-2}$$

$$e^{e} + e^{e-1}\frac{d^2y}{dx^2} = (e-1)e^{e-1}$$

$$= e^{e} - e^{e-1}$$

$$e^{e-1}\frac{d^2y}{dx^2} = -e^{e-1}$$

$$\frac{d^2y}{dx^2} = -1$$

The value of $\frac{d^2y}{dx^2}$ at the point (e, e) is -1