1. $$\text{Area} = \int_0^{2\pi} \frac{r^2}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} a^2 \sin^2\frac{\theta}{2}\, d\theta$$

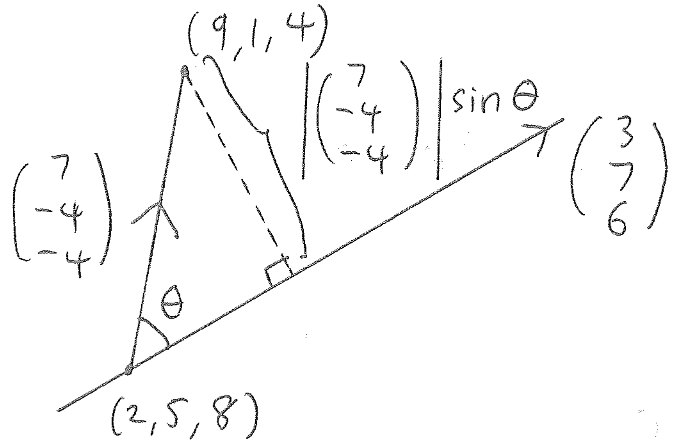
$$= \frac{a^2}{4}\int_0^{2\pi} (1-\cos\theta)\, d\theta$$

$$= \frac{a^2}{4}\Big[\theta - \sin\theta\Big]_0^{2\pi}$$

$$= \frac{a^2}{4}\Big[(2\pi - 0) - (0 - 0)\Big]$$

$$= \frac{\pi a^2}{2} \text{ unit}^2.$$

2.



$$\underset{\sim}{r} = 2\underset{\sim}{i} + 5\underset{\sim}{j} + 8\underset{\sim}{k} + s(3\underset{\sim}{i} + 7\underset{\sim}{j} + 6\underset{\sim}{k}) \quad (9,1,4)$$

$$\begin{pmatrix}7\\-4\\-4\end{pmatrix} \times \begin{pmatrix}3\\7\\6\end{pmatrix} = \left|\begin{pmatrix}7\\-4\\-4\end{pmatrix}\right| \left|\begin{pmatrix}3\\7\\6\end{pmatrix}\right| \sin\theta \, \underset{\sim}{n}$$

Since the perpendicular distance from the point to the line is $\left|\begin{pmatrix}7\\-4\\-4\end{pmatrix}\right| \sin\theta$,

$$\left|\begin{pmatrix}7\\-4\\-4\end{pmatrix}\right| \sin\theta = \frac{\left|\begin{pmatrix}7\\-4\\-4\end{pmatrix} \times \begin{pmatrix}3\\7\\6\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\7\\6\end{pmatrix}\right|}$$

$$= \frac{\left\|\begin{matrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 7 & -4 & -4 \\ 3 & 7 & 6\end{matrix}\right\|}{\sqrt{94}} = \frac{\left|\begin{pmatrix}4\\-54\\61\end{pmatrix}\right|}{\sqrt{94}} = \sqrt{\frac{6653}{94}} \approx 8.41$$

Since $\begin{pmatrix}3\\7\\6\end{pmatrix}$ and $\begin{pmatrix}7\\-4\\-4\end{pmatrix}$ are parallel to the plane, $\begin{pmatrix}3\\7\\6\end{pmatrix} \times \begin{pmatrix}7\\-4\\-4\end{pmatrix}$ is perpendicular to the plane.

$\therefore$ $\begin{pmatrix}4\\-54\\61\end{pmatrix}$ is normal to the plane.

Since $\begin{pmatrix}4\\-54\\61\end{pmatrix}$ is normal to the plane and $(9, 1, 4)$ is a point on the plane, if $\underset{\sim}{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ is any point on the plane,

$$\underset{\sim}{r} \cdot \begin{pmatrix}4\\-54\\61\end{pmatrix} = \begin{pmatrix}9\\1\\4\end{pmatrix} \cdot \begin{pmatrix}4\\-54\\61\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}4\\-54\\61\end{pmatrix} = 36 - 54 + 244$$

$$4x - 54y + 61z = 226$$

$\therefore$ The equation of the plane is $4x - 54y + 61z = 226$.

3. $$\sum_{r=2}^{n} r(r-1)x^r = \frac{n(n-1)x^{n+3} - 2(n^2-1)x^{n+2} + (n+1)nx^{n+1} - 2x^2}{(x-1)^3}$$

when $n=2$: $$\sum_{r=2}^{2} r(r-1)x^r = 2\cdot 1x^2$$

$$= 2x^2$$

$$= \frac{2x^2(x-1)^3}{(x-1)^3}$$

$$= \frac{2x^2(x^3-3x^2+3x-1)}{(x-1)^3}$$

$$= \frac{2x^5 - 6x^4 + 6x^3 - 2x^2}{(x-1)^3}$$

$$= \frac{2\cdot 1x^{2+3} - 2\cdot 3x^{2+2} + 3\cdot 2x^{2+1} - 2x^2}{(x-1)^3}$$

$$= \frac{2\cdot 1x^{2+3} - 2(4-1)x^{2+2} + 3\cdot 2x^{2+1} - 2x^2}{(x-1)^3}$$

$$= \frac{2\cdot 1x^{2+3} - 2(2^2-1)x^{2+2} + 3\cdot 2x^{2+1} - 2x^2}{(x-1)^3}$$

Assume the statement is true when $n=k$.

$n=k$: $$\sum_{r=2}^{k} r(r-1)x^r = \frac{k(k-1)x^{k+3} - 2(k^2-1)x^{k+2} + (k+1)kx^{k+1} - 2x^2}{(x-1)^3}$$

when $n = k+1$:

$$\sum_{r=2}^{k+1} r(r-1)x^r = \frac{(k+1)kx^{k+4} - 2(k^2+2k)x^{k+3} + (k+2)(k+1)x^{k+2} - 2x^2}{(x-1)^3}$$

(what needs to be proved)

$$\sum_{r=2}^{k+1} r(r-1)x^r = (k+1)kx^{k+1} + \sum_{r=2}^{k} r(r-1)x^r$$

$$= \frac{k(k-1)x^{k+3} - 2(k^2-1)x^{k+2} + (k+1)kx^{k+1} - 2x^2}{(x-1)^3} + (k+1)kx^{k+1}$$

$$= \frac{k(k-1)x^{k+3} - 2(k^2-1)x^{k+2} + (k+1)kx^{k+1} + (k+1)kx^{k+1}(x-1)^3 - 2x^2}{(x-1)^3}$$

$$= \frac{k(k-1)x^{k+3} - 2(k^2-1)x^{k+2} + (k+1)kx^{k+1} + (k+1)kx^{k+1}(x^3-3x^2+3x-1) - 2x^2}{(x-1)^3}$$

$$= \frac{k(k-1)x^{k+3} - 2(k^2-1)x^{k+2} + (k+1)kx^{k+1} + (k+1)kx^{k+4} - 3(k+1)kx^{k+3} + 3(k+1)kx^{k+2} - (k+1)kx^{k+1} - 2x^2}{(x-1)^3}$$

$$= \frac{(k+1)kx^{k+4} + (k(k-1) - 3(k+1)k)x^{k+3} + (-2(k^2-1) + 3(k+1)k)x^{k+2} + (k+1)kx^{k+1} - (k+1)kx^{k+1} - 2x^2}{(x-1)^3}$$

$$= \frac{(k+1)kx^{k+4} + (k^2 - k - 3k^2 - 3k)x^{k+3} + (-2k^2 + 2 + 3k^2 + 3k)x^{k+2} - 2x^2}{(x-1)^3}$$

$$= \frac{(k+1)kx^{k+4} + (-2k^2 - 4k)x^{k+3} + (k^2 + 3k + 2)x^{k+2} - 2x^2}{(x-1)^3}$$

$$= \frac{(k+1)kx^{k+4} - 2(k^2 + 2k)x^{k+3} + (k+2)(k+1)x^{k+2} - 2x^2}{(x-1)^3}$$

$$\therefore \sum_{r=2}^{n} r(r-1)x^r = \frac{n(n-1)x^{n+3} - 2(n^2-1)x^{n+2} + (n+1)nx^{n+1} - 2x^2}{(x-1)^3}$$

for every positive integer $n \geq 2$.

4. $\tan y = x^{\frac{1}{2}}$

$$\sec^2 y \frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$(\tan^2 y + 1) \frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$(x+1) \frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$(x+1) \frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{4} x^{-\frac{3}{2}}$$

$$= \frac{-1}{2x} \times \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{-1}{2x} (x+1) \frac{dy}{dx}$$

$$2x(x+1) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + (x+1) \frac{dy}{dx} = 0$$

$$2x(x+1) \frac{d^2y}{dx^2} + (3x+1) \frac{dy}{dx} = 0$$

5. $z^{3n} - (\sqrt{3}+1)z^{2n} + (\sqrt{3}+1)z^n - 1 = 0$

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, 2, \ldots, n-1.$$

$$z^n = \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)^n$$

$$= \cos 2k\pi + i\sin 2k\pi$$

$$= 1$$

Since $z^n = 1$ is a root of the equation,

$$\begin{array}{r|l} & z^{2n} - \sqrt{3}z^n + 1 \\ \hline z^n - 1 & z^{3n} - (\sqrt{3}+1)z^{2n} + (\sqrt{3}+1)z^n - 1 \\ & \underline{z^{3n} - z^{2n}} \\ & \quad -\sqrt{3}z^{2n} + (\sqrt{3}+1)z^n \\ & \quad \underline{-\sqrt{3}z^{2n} + \sqrt{3}z^n} \\ & \qquad\qquad z^n - 1 \\ & \qquad\qquad \underline{z^n - 1} \\ & \qquad\qquad\quad 0 \end{array}$$

$$(z^n - 1)(z^{2n} - \sqrt{3}z^n + 1) = 0$$

$$z^n = 1, \quad z^{2n} - \sqrt{3}z^n + 1 = 0$$

$$z^n = \frac{\sqrt{3}}{2} \pm \frac{i}{2}$$

$$= e^{\frac{\pi i}{6}}, e^{\frac{5\pi i}{6}}.$$

$$z^n = e^{2k\pi i},\ z^n = e^{(2k+\frac{1}{6})\pi i},\ z^n = e^{(2k+\frac{5}{6})\pi i},\ k=0,1,\dots,n-1$$

$$z = e^{\frac{2k\pi i}{n}},\ e^{\frac{(12k+1)\pi i}{6n}},\ e^{\frac{(12k+5)\pi i}{6n}},\ k=0,1,2,\dots,n-1$$

$\therefore$ The roots of the equation are

$$z^n = e^{\frac{2k\pi i}{n}},\ e^{\frac{(12k+1)\pi i}{6n}},\ e^{\frac{(12k+5)\pi i}{6n}},\ k=0,1,\dots,n-1$$

6. A.E. $\lambda^2+16=0 \Rightarrow \lambda=\pm 4i$

$$y_c = A\cos 4x + B\sin 4x$$

$$y_p = a\cos 8x + b\sin 8x$$

$$y_p' = -8a\sin 8x + 8b\cos 8x$$

$$y_p'' = -64a\cos 8x - 64b\sin 8x$$

$$= -64y_p$$

$$y_p'' + 16y_p = -48y_p$$

$$\therefore -48\left[a\cos 8x + b\sin 8x\right] = 144\cos 8x$$

$$a=-3,\ b=0$$

$$\therefore y_p = -3\cos 8x$$

G.S. : $y = A\cos 4x + B\sin 4x - 3\cos 8x$

7. $$\frac{7r-3}{r^3-r} = \frac{2}{r-1} + \frac{3}{r} - \frac{5}{r+1}$$

$$\frac{2[7(2)-3]}{1\times 2\times 3} + \frac{2[7(3)-3]}{2\times 3\times 4} + \cdots + \frac{2(7n-3)}{(n-1)n(n+1)}$$

$$= 2\sum_{r=2}^{n} \frac{7r-3}{r^3-r}$$

$$= 2\sum_{r=2}^{n} \left[\frac{2}{r-1} - \frac{2}{r} + \frac{5}{r} - \frac{5}{r+1}\right]$$

$$= 2\left[2 - \frac{2}{n} + \frac{5}{2} - \frac{5}{n+1}\right]$$

$$= 9 - \frac{4}{n} - \frac{10}{n+1}$$

8. $m = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ $T: R^3 \xrightarrow{m} R^3$, $a > b > c$

i) $|m| = 1(bc^2 - b^2c) - 1(ac^2 - a^2c) + 1(ab^2 - a^2b)$

$= bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b$

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}^T = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

$$\text{adj } m = \begin{pmatrix} bc^2 - b^2c & b^2 - c^2 & c - b \\ a^2c - ac^2 & c^2 - a^2 & a - c \\ ab^2 - a^2b & a^2 - b^2 & b - a \end{pmatrix}$$

$$\therefore m^{-1} = \frac{1}{|m|} \begin{pmatrix} bc^2 - b^2c & b^2 - c^2 & c - b \\ a^2c - ac^2 & c^2 - a^2 & a - c \\ ab^2 - a^2b & a^2 - b^2 & b - a \end{pmatrix}$$

ii) If $\left.\begin{array}{l} x + y + z = |m| \\ ax + by + cz = |m| \\ a^2x + b^2y + c^2z = |m| \end{array}\right\}$,

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} |m| \\ |m| \\ |m| \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}^{-1} \begin{pmatrix} |m| \\ |m| \\ |m| \end{pmatrix}$$

$$= \frac{1}{|M|}\begin{pmatrix} bc^2-b^2c & b^2-c^2 & c-b \\ a^2c-ac^2 & c^2-a^2 & a-c \\ ab^2-a^2b & a^2-b^2 & b-a \end{pmatrix} |M| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} bc(c-b) & (b-c)(b+c) & c-b \\ ac(a-c) & (c-a)(c+a) & a-c \\ ab(b-a) & (a-b)(a+b) & b-a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} bc(c-b)+(b-c)(b+c)+c-b \\ ac(a-c)+(c-a)(c+a)+a-c \\ ab(b-a)+(a-b)(a+b)+b-a \end{pmatrix}$$

$$= \begin{pmatrix} (c-b)(bc-b-c+1) \\ (a-c)(ac-a-c+1) \\ (b-a)(ab-a-b+1) \end{pmatrix}$$

$$= \begin{pmatrix} (c-b)(b-1)(c-1) \\ (a-c)(a-1)(c-1) \\ (b-a)(a-1)(b-1) \end{pmatrix}$$

$\therefore\ x = (c-b)(b-1)(c-1)$

$y = (a-c)(a-1)(c-1)$

$z = (b-a)(a-1)(b-1)$

iii) $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$

$$\xrightarrow[-a^2r_1+r_3]{-ar_1+r_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix}$$

$$\xrightarrow{-(b+a)r_2+r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$$

Since $a > b > c > 0$, a basis for the range space of T is

$$\left\{ \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix}, \begin{pmatrix} 1 \\ b \\ b^2 \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ c^2 \end{pmatrix} \right\}.$$

9. i) $y = \dfrac{x^2+2x-3}{x+2} = x - \dfrac{3}{x+2}$

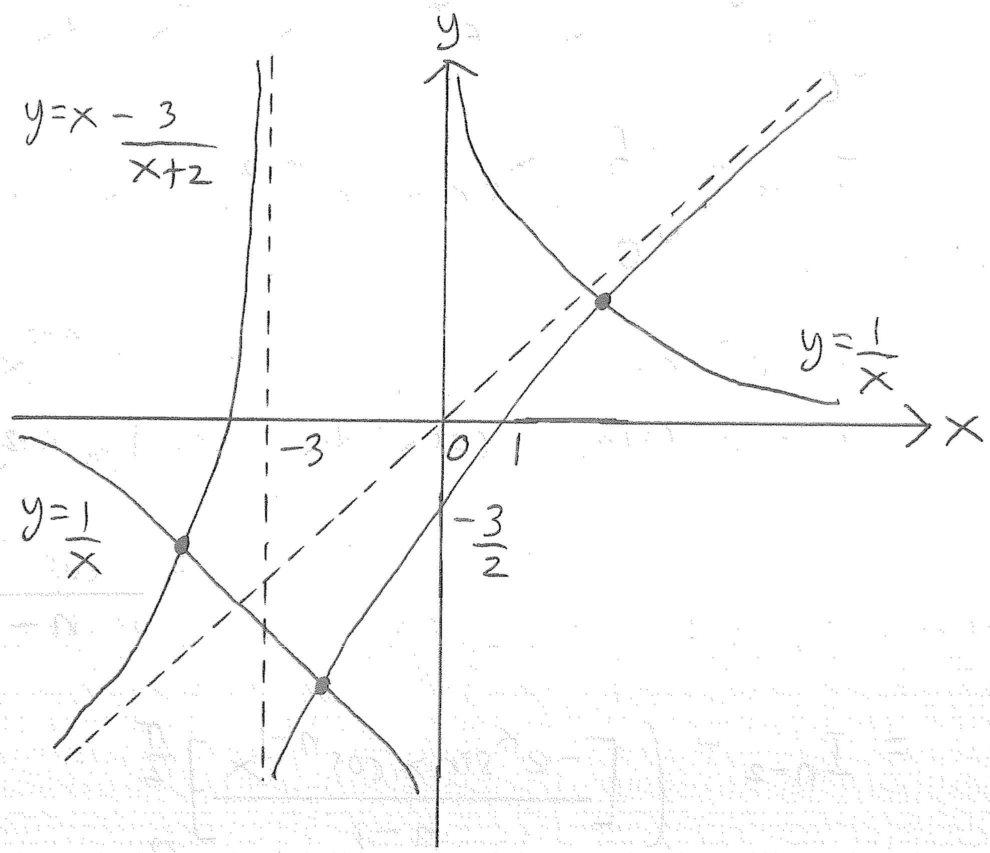
Asymptotes : $y = x$, $x = -2$

ii) $\dfrac{dy}{dx} = 1 + \dfrac{3}{(x+2)^2}$

Since $(x+2)^2 > 0 \quad \forall x \in R \Rightarrow \dfrac{dy}{dx} > 0$

$\therefore$ No turning points.

iii)



iv) $\dfrac{x^2+2x-3}{x+2} = \dfrac{1}{x}$

$x^3+2x^2-3x = x+2$

$x^3+2x^2-4x-2=0$

$\therefore$ 3 real roots.

10. $I_n = \int_0^{\frac{\pi}{2}} e^x \cos^n x \, dx$

$= \int_0^{\frac{\pi}{2}} e^x \cos^{n-2} x \cos^2 x \, dx$

$= \int_0^{\frac{\pi}{2}} e^x \cos^{n-2} x (1-\sin^2 x) \, dx$

$= \int_0^{\frac{\pi}{2}} e^x \cos^{n-2} x - e^x \cos^{n-2} x \sin^2 x \, dx$

$= I_{n-2} - \int_0^{\frac{\pi}{2}} e^x \sin x \cos^{n-2} x \sin x \, dx$

$u = e^x \sin x$ $\qquad dv = \cos^{n-2} x \sin x \, dx$

$du = e^x (\sin x + \cos x) \, dx$ $\qquad v = \int \cos^{n-2} x \sin x \, dx$

$= \frac{-\cos^{n-1} x}{n-1}$

$= I_{n-2} - \left(\left[\frac{-e^x \sin x \cos^{n-1} x}{n-1} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-\cos^{n-1} x (e^x \sin x + e^x \cos x) \, dx}{n-1} \right)$

$= I_{n-2} - \left[\frac{-e^x \sin x \cos^{n-1} x}{n-1} \right]_0^{\frac{\pi}{2}}$

$- \int_0^{\frac{\pi}{2}} \frac{e^x \sin x \cos^{n-1} x + e^x \cos^n x}{n-1} \, dx$

$$= I_{n-2} - 0^{n-1} - 0 - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} e^x \sin x \cos^{n-1} x + e^x \cos^n x \, dx$$

$$= I_{n-2} - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} e^x \sin x \cos^{n-1} x \, dx - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} e^x \cos^n x \, dx$$

$$= I_{n-2} - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} e^x \sin x \cos^{n-1} x \, dx - \frac{1}{n-1} I_n$$

$u = e^x \qquad dv = \sin x \cos^{n-1} x \, dx$

$du = e^x dx \qquad v = \int \sin x \cos^{n-1} x \, dx$

$$= \frac{-\cos^n x}{n}$$

$$= I_{n-2} - \frac{1}{n-1}\left(\left[\frac{-e^x \cos^n x}{n}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-e^x \cos^n x \, dx}{n} \right)$$

$$- \frac{1}{n-1} I_n$$

$$= I_{n-2} - \frac{1}{n-1}\left[\frac{-e^x \cos x}{n}\right]_0^{\frac{\pi}{2}} - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} \frac{e^x \cos^n x \, dx}{n}$$

$$- \frac{1}{n-1} I_n$$

$$= I_{n-2} - \frac{1}{n-1}\left(0^n - \frac{-1}{n}\right) - \frac{1}{n(n-1)}\int_0^{\frac{\pi}{2}} e^x \cos^n x \, dx$$

$$- \frac{1}{n-1} I_n$$

$$= I_{n-2} - \frac{1}{n-1}\left(\frac{1}{n}\right) - \frac{1}{n(n-1)} I_n - \frac{1}{n-1} I_n$$

$$\left(1 + \frac{1}{n(n-1)} + \frac{1}{n-1}\right) I_n = I_{n-2} - \frac{1}{n(n-1)}$$

$$\frac{(n^2 - n + 1 + n) I_n}{n(n-1)} = I_{n-2} - \frac{1}{n(n-1)}$$

$$\frac{(n^2+1) I_n}{n(n-1)} = I_{n-2} - \frac{1}{n(n-1)}$$

$$(n^2+1) I_n = n(n-1) I_{n-2} - 1 \, , \; n > 1$$

$$n = 0 : \; I_0 = \int_0^{\frac{\pi}{2}} e^x \, dx$$

$$= \left[e^x \right]_0^{\frac{\pi}{2}}$$

$$= e^{\frac{\pi}{2}} - 1$$

$$n = 1 : \; I_1 = \int_0^{\frac{\pi}{2}} e^x \cos x \; dx$$

$$u = e^x \qquad dv = \cos x \, dx$$

$$du = e^x \, dx \qquad v = \sin x$$

$$= \left[e^x \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$$= e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$u = e^x \qquad dv = \sin x \, dx$

$du = e^x \, dx \qquad v = -\cos x$

$$= e^{\frac{\pi}{2}} - \left(\left[-e^x \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -e^x \cos x \, dx \right)$$

$$= e^{\frac{\pi}{2}} - \left[-e^x \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$= e^{\frac{\pi}{2}} - (0 - -1) - I_1$$

$$= e^{\frac{\pi}{2}} - 1 - I_1$$

$$2I_1 = e^{\frac{\pi}{2}} - 1$$

$$I_1 = \frac{e^{\frac{\pi}{2}} - 1}{2}$$

$n=2: \quad 5I_2 = 2I_0 - 1$

$$= 2(e^{\frac{\pi}{2}} - 1) - 1$$

$$= 2e^{\frac{\pi}{2}} - 2 - 1$$

$$= 2e^{\frac{\pi}{2}} - 3$$

$$I_2 = \frac{2e^{\frac{\pi}{2}}}{5} - \frac{3}{5}$$

$n=3$: $10 I_3 = 6 I_1 - 1$

$$= \frac{6(e^{\frac{\pi}{2}} - 1)}{2} - 1$$

$$= 3e^{\frac{\pi}{2}} - 3 - 1$$

$$= 3e^{\frac{\pi}{2}} - 4$$

$$I_3 = \frac{3e^{\frac{\pi}{2}} - 4}{10}$$

11. EITHER

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad bc - ad > 0$$

$$A - \lambda I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (a-\lambda)(d-\lambda) - bc$$

$$= (\lambda - a)(\lambda - d) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

when $|A - \lambda I| = 0$:

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\lambda^2 - (a+d)\lambda = bc - ad$$

$$\lambda^2 - (a+d)\lambda + \left(\frac{a+d}{2}\right)^2 = bc - ad + \left(\frac{a+d}{2}\right)^2$$

$$\left(\lambda - \left(\frac{a+d}{2}\right)\right)^2 = bc - ad + \left(\frac{a+d}{2}\right)^2$$

$$\lambda - \left(\frac{a+d}{2}\right) = \pm \sqrt{bc - ad + \left(\frac{a+d}{2}\right)^2}$$

$$\lambda = \frac{a+d}{2} \pm \sqrt{bc - ad + \left(\frac{a+d}{2}\right)^2}$$

$\therefore$ The eigenvalues of A are

$$\frac{a+d}{2} \pm \sqrt{bc - ad + \left(\frac{a+d}{2}\right)^2}.$$

λ_1, λ_2 are the eigenvalues

when $\lambda = \lambda_1$: $\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} a-\lambda_1 & b & 0 \\ c & d-\lambda_1 & 0 \end{array}\right)$$

since $\dfrac{c}{a-\lambda_1} = \dfrac{d-\lambda_1}{b}$,

Let $y = (\lambda_1 - a)s$, $s \in R$

$\therefore x = bs$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} bs \\ (\lambda_1 - a)s \end{pmatrix} = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix} s$$

when $\lambda = \lambda_2$: $\begin{pmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} a-\lambda_2 & b & 0 \\ c & d-\lambda_2 & 0 \end{array}\right)$$

since $\dfrac{c}{a-\lambda_2} = \dfrac{d-\lambda_2}{b}$,

Let $y = (\lambda_2 - a)s$, $s \in R$

$\therefore x = bs$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} bs \\ (\lambda_2 - a)s \end{pmatrix} = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix} s$$

$\therefore$ The eigenvalues of A are λ_1, λ_2 with corresponding eigenvectors $\begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}, \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix}$.

If $P = \begin{pmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$,

$$AP = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{pmatrix}$$

$$= \begin{pmatrix} ab + b\lambda_1 - ab & ab + b\lambda_2 - ab \\ bc + d\lambda_1 - ad & bc + d\lambda_2 - ad \end{pmatrix}$$

$$= \begin{pmatrix} b\lambda_1 & b\lambda_2 \\ bc - ad + d\lambda_1 & bc - ad + d\lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} b\lambda_1 & b\lambda_2 \\ \lambda_1^2 - a\lambda_1 & \lambda_2^2 - a\lambda_2 \end{pmatrix},$$

since $\lambda_1^2 - (a+d)\lambda_1 + ad - bc$

$= \lambda_2^2 - (a+d)\lambda_2 + ad - bc = 0$

$$= \begin{pmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$= PD.$

OR

let $y = 2\alpha - 1 \Rightarrow \alpha = \frac{y+1}{2}$

$$2x^4 + 4x^3 - 6x^2 + x - 1 = 0$$

$$2\left(\frac{y+1}{2}\right)^4 + 4\left(\frac{y+1}{2}\right)^3 - 6\left(\frac{y+1}{2}\right)^2 + \left(\frac{y+1}{2}\right) - 1 = 0$$

$$\frac{1}{8}(y^4 + 4y^3 + 6y^2 + 4y + 1) + \frac{1}{2}(y^3 + 3y^2 + 3y + 1)$$

$$-\frac{3}{2}(y^2 + 2y + 1) + \frac{1}{2}(y - 1) = 1$$

$$\therefore y^4 + 8y^3 + 6y^2 - 4y - 11 = 0$$

$$\therefore S_{n+4} + 8S_{n+3} + 6S_{n+2} - 4S_{n+1} - 11S_n = 0$$

$S_0 = 4$, $S_1 = -8$

$$S_{-1} = \frac{\sum(2\alpha-1)(2\beta-1)(2\gamma-1)}{(2\alpha-1)(2\beta-1)(2\gamma-1)} = -\frac{4}{11}$$

$$S_2 = \left[\sum(2\alpha-1)\right]^2 - 2\sum(2\alpha-1)(2\beta-1)$$

$$= 64 - 12 = 52$$

let $n = -1$,

$$\therefore S_3 + 8S_2 + 6S_1 - 4S_0 - 11S_{-1} = 0$$

$$S_3 = -8(52) - 6(-8) + 4(4) + 11\left(\frac{-4}{11}\right)$$

$$= -356$$