MAY / JUNE 2012

13

1. $\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+\dots+\frac{1}{n(n+2)}$

$= \sum_{r=1}^{n} \frac{1}{r(r+2)}$

$= \sum_{r=1}^{n} \frac{1}{2}\left(\frac{1}{r}-\frac{1}{r+2}\right)$

$= \frac{1}{2}\Big(\frac{1}{1}-\frac{1}{3}$

$+\frac{1}{2}-\frac{1}{4}$

$+\frac{1}{3}-\frac{1}{5}$

$+\frac{1}{4}-\frac{1}{6}$

$\vdots$

$+\frac{1}{n-3}-\frac{1}{n-1}$

$+\frac{1}{n-2}-\frac{1}{n}$

$+\frac{1}{n-1}-\frac{1}{n+1}$

$+\frac{1}{n}-\frac{1}{n+2}\Big)$

$= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$

$= \frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$

$\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$

$= \lim_{n\to\infty}\left(\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}\right)$

$= \frac{3}{4}$

2. $u_1, u_2, u_3, \ldots$

$u_1 = 1 \quad u_{r+1} = \dfrac{3u_r - 2}{4}$

$u_n = 4\left(\frac{3}{4}\right)^n - 2, \; n \geqslant 1$

When $n=1$: $u_1 = 1 = 3 - 2 = 4\left(\frac{3}{4}\right) - 2 = 4\left(\frac{3}{4}\right)^1 - 2$

Assume the statement is true when $n=k$.

$n=k$: $u_k = 4\left(\frac{3}{4}\right)^k - 2$

When $n=k+1$: $u_{k+1} = 4\left(\frac{3}{4}\right)^{k+1} - 2$

(what needs to be proved)

$u_k = 4\left(\frac{3}{4}\right)^k - 2$

$$\begin{aligned} u_{k+1} &= \frac{3u_k - 2}{4} \\ &= \frac{3}{4}\left(4\left(\frac{3}{4}\right)^k - 2\right) - \frac{2}{4} \\ &= 4\left(\frac{3}{4}\right)^k \frac{3}{4} - \frac{3}{2} - \frac{1}{2} \\ &= 4\left(\frac{3}{4}\right)^{k+1} - 2 \end{aligned}$$

$\therefore u_n = 4\left(\frac{3}{4}\right)^n - 2$ for every positive integer n.

3. $C: xy + (x+y)^3 = 1$

$$\frac{d}{dx}(xy + (x+y)^3) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(x+y) = 0$$

$$x\frac{dy}{dx} + y + 3(x+y)^2(1 + \frac{dy}{dx}) = 0$$

At $A(1,0)$: $\frac{dy}{dx} + 0 + 3(1 + \frac{dy}{dx}) = 0$

$$4\frac{dy}{dx} + 3 = 0$$

$$\frac{dy}{dx} = -\frac{3}{4}.$$

$$\frac{d}{dx}(x\frac{dy}{dx} + y + 3(x+y)^2(1+\frac{dy}{dx})) = 0$$

$$\frac{d}{dx}(x\frac{dy}{dx}) + \frac{dy}{dx} + \frac{d}{dx}(3(x+y)^2(1+\frac{dy}{dx})) = 0$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + 3(x+y)^2\frac{d^2y}{dx^2} + 6(x+y)(1+\frac{dy}{dx})^2 = 0$$

At $A(1,0)$, $\frac{dy}{dx} = -\frac{3}{4}$:

$$\frac{d^2y}{dx^2} + 2(-\frac{3}{4}) + 3\frac{d^2y}{dx^2} + 6(\frac{1}{4})^2 = 0$$

$$4\frac{d^2y}{dx^2} = \frac{9}{8} \quad \therefore \frac{d^2y}{dx^2} = \frac{9}{32}$$

4. $I_n = \int_1^e x^2 (\ln x)^n \, dx$, $n \geq 0$

$u = (\ln x)^n$ $\qquad dv = x^2\,dx$

$du = \dfrac{n(\ln x)^{n-1}\,dx}{x}$ $\qquad v = \dfrac{x^3}{3}$

$= \left[\dfrac{x^3}{3}(\ln x)^n\right]_1^e - \int_1^e \dfrac{n(\ln x)^{n-1}}{x}\dfrac{x^3}{3}\,dx$

$= \dfrac{e^3}{3} - \dfrac{n}{3}\int_1^e x^2(\ln x)^{n-1}\,dx$

$= \dfrac{e^3}{3} - \dfrac{n}{3} I_{n-1}$

$n = 3$: $I_3 = \dfrac{e^3}{3} - \dfrac{3}{3} I_2$

$= \dfrac{e^3}{3} - I_2$

$= \dfrac{e^3}{3} - \left(\dfrac{e^3}{3} - \dfrac{2}{3} I_1\right)$

$= \dfrac{2}{3} I_1$

$= \dfrac{2}{3}\left(\dfrac{e^3}{3} - \dfrac{1}{3} I_0\right)$

$= \dfrac{2e^3}{9} - \dfrac{2}{9} I_0$

$= \dfrac{2}{9}e^3 - \dfrac{2}{9}\int_1^e x^2\,dx$

$= \dfrac{2e^3}{9} - \dfrac{2}{9}\left[\dfrac{x^3}{3}\right]_1^e$

$= \dfrac{2e^3}{9} - \dfrac{2}{9}\left(\dfrac{e^3}{3} - \dfrac{1}{3}\right)$

$= \dfrac{2e^3}{9} - \dfrac{2e^3}{27} + \dfrac{2}{27}$

$= \dfrac{4e^3}{27} + \dfrac{2}{27}$

5. $A\underset{\sim}{e} = \lambda\underset{\sim}{e}$

$(A + kI)\underset{\sim}{e} = A\underset{\sim}{e} + kI\underset{\sim}{e} = \lambda\underset{\sim}{e} + k\underset{\sim}{e} = (\lambda + k)\underset{\sim}{e}$.

$\therefore$ The matrix $A + kI$ has eigenvalue $\lambda + k$ with corresponding eigenvector $\underset{\sim}{e}$.

$$B = \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}$$

If $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B\underset{\sim}{x} = -3\underset{\sim}{x}$

$$\begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x + 2y - 3z \\ 2x + 2y + 3z \\ -3x + 3y + 3z \end{pmatrix} = \begin{pmatrix} -3x \\ -3y \\ -3z \end{pmatrix}$$

$$\left.\begin{aligned} 5x + 2y - 3z &= 0 \\ 2x + 5y + 3z &= 0 \\ -3x + 3y + 6z &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} ① + ②:\quad 5x + 2y - 3z &= 0 \\ \tfrac{1}{3} \times ③:\quad 7x + 7y \quad\; &= 0 \\ -x + y + 2z &= 0 \end{aligned}\right\}$$

Let $y = s$, $s \in R$

$x = -s$

$z = -s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ s \\ -s \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

If $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B\underset{\sim}{x} = 4\underset{\sim}{x}$,

$$\begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x + 2y - 3z \\ 2x + 2y + 3z \\ -3x + 3y + 3z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

$$\left.\begin{aligned} 2x - 2y + 3z &= 0 \\ 2x - 2y + 3z &= 0 \\ 3x - 3y + z &= 0 \end{aligned}\right\}$$

$$-3 \times ② + 2 \times ③:\quad \left.\begin{aligned} 2x - 2y + 3z &= 0 \\ 2x - 2y + 3z &= 0 \\ -7z &= 0 \end{aligned}\right\}$$

$z = 0$

Let $y = s$, $s \in R$

$x = s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ The eigenvalues -3 and 4 have corresponding eigenvectors $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

If $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is an eigenvector of B,

$$B\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ -12 \end{pmatrix} = 6\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$\therefore$ The corresponding eigenvector is 6.

$$C = \begin{pmatrix} -1 & 2 & -3 \\ 2 & -1 & 3 \\ -3 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= B - 3I$$

$\therefore$ The eigenvalues of C are $-6, 1, 3$

with corresponding eigenvectors $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

6. $C: y = \dfrac{x^2}{x-2}$.

$= \dfrac{x^2 - 2x + 2x}{x-2}$

$= \dfrac{x(x-2)}{x-2} + \dfrac{2x}{x-2}$

$= x + \dfrac{2(x-2+2)}{x-2}$

$= x + 2 + \dfrac{4}{x-2}$

As $x \to \pm\infty$ $y \to x+2$

As $x \to 2$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = x+2$ and $x = 2$.

$\dfrac{dy}{dx} = 1 - \dfrac{4}{(x-2)^2}$

when $\dfrac{dy}{dx} = 0: 1 - \dfrac{4}{(x-2)^2} = 0$

$(x-2)^2 = 4$

$x - 2 = \pm 2$

$x = 4, 0$

$y = 0, 0$

$\dfrac{d^2y}{dx^2} = \dfrac{8}{(x-2)^3}$

when $x = 4: \dfrac{d^2y}{dx^2} = 1 > 0$

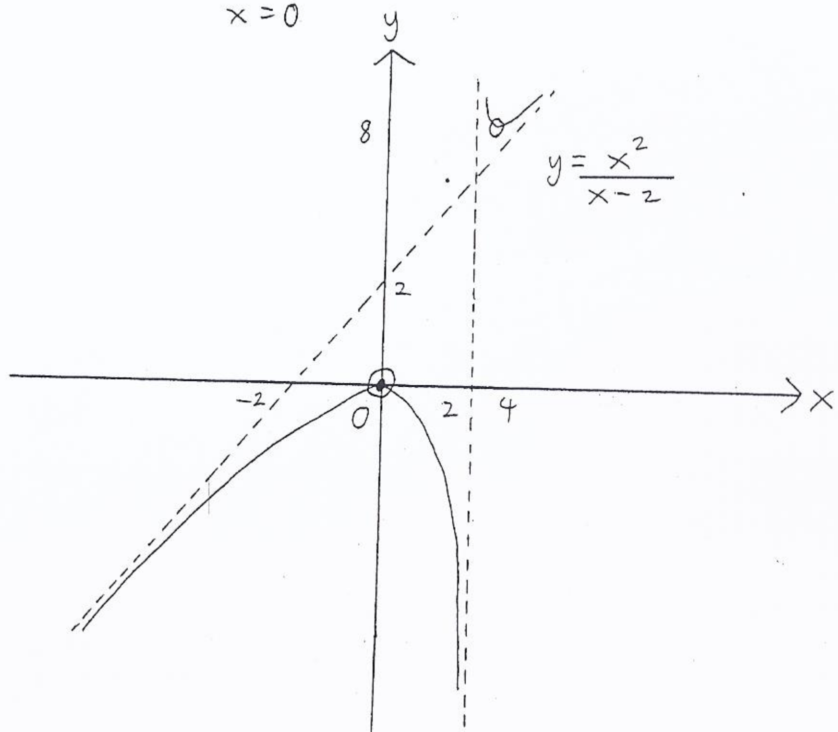
when $x = 0: \dfrac{d^2y}{dx^2} = -1 < 0$

$\therefore$ $(4, 8)$ is a minimum point and $(0,0)$ is a maximum point.

when $x = 0: y = 0$

when $y = 0: \dfrac{x^2}{x-2} = 0$.

$x = 0$



O: Critical Point.

•: Intersection Point

7. $\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2$

$= \left(z+\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)^2\left(z-\frac{1}{z}\right)^2$

$= \left(z+\frac{1}{z}\right)^2\left(z^2-\frac{1}{z^2}\right)^2$

$= \left(z^2+2+\frac{1}{z^2}\right)\left(z^4-2+\frac{1}{z^4}\right)$

$= z^6-2z^2+\frac{1}{z^2}+2z^4-4+\frac{2}{z^4}+z^2-\frac{2}{z^2}+\frac{1}{z^6}$

$= z^6+\frac{1}{z^6}+2\left(z^4+\frac{1}{z^4}\right)-\left(z^2+\frac{1}{z^2}\right)-4$

$z = \cos\theta + i\sin\theta$

$z^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

$\therefore\ z+\frac{1}{z} = 2\cos\theta\ ,\ z^n+\frac{1}{z^n} = 2\cos n\theta\ ,\ z-\frac{1}{z} = 2i\sin\theta$

$\therefore\ \left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2 = z^6+\frac{1}{z^6}+2\left(z^4+\frac{1}{z^4}\right)-\left(z^2+\frac{1}{z^2}\right)-4$

$(2\cos\theta)^4(2i\sin\theta)^2 = 2\cos6\theta + 4\cos4\theta - 2\cos2\theta - 4$

$-64\cos^4\theta\sin^2\theta = 2\cos6\theta + 4\cos4\theta - 2\cos2\theta - 4$

$\therefore\ 64\cos^4\theta\sin^2\theta = 4 + 2\cos2\theta - 4\cos4\theta - 2\cos6\theta$

$\int_1^2 x^4\sqrt{4-x^2}\,dx$

$x = 2\cos\theta$

$dx = -2\sin\theta\,d\theta$

$x=1\ \ \theta = \frac{\pi}{3}$

$x=2\ \ \theta = 0$

$\therefore\ \int_1^2 x^4\sqrt{4-x^2}\,dx = \int_{\frac{\pi}{3}}^{0} 16\cos^4\theta\,(2\sin\theta)(-2\sin\theta)\,d\theta$

$= \int_0^{\frac{\pi}{3}} 64\cos^4\theta\sin^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{3}} 4 + 2\cos2\theta - 4\cos4\theta - 2\cos6\theta\,d\theta$

$= \left[4\theta + \sin2\theta - \sin4\theta - \frac{\sin6\theta}{3}\right]_0^{\frac{\pi}{3}}$

$= \frac{4\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - 0$

$= \frac{4\pi}{3} + \sqrt{3}$

8. $x^3 - x^2 - 3x - 10 = 0$

α, β, r are the roots

$\alpha + \beta + r = 1 \quad \alpha\beta + \alpha r + \beta r = -3 \quad \alpha\beta r = 10$

i) $u = -\alpha + \beta + r$

$u + 2\alpha = \alpha + \beta + r$

$= 1$

$\alpha = \dfrac{1-u}{2}$

α is a root

$\therefore \alpha^3 - \alpha^2 - 3\alpha - 10 = 0$

$\left(\dfrac{1-u}{2}\right)^3 - \left(\dfrac{1-u}{2}\right)^2 - 3\left(\dfrac{1-u}{2}\right) - 10 = 0$

$\dfrac{1 - 3u + 3u^2 - u^3}{8} - \dfrac{(1 - 2u + u^2)}{4} - \dfrac{3 + 3u}{2} - 10 = 0$

$1 - 3u + 3u^2 - u^3 - 2 + 4u - 2u^2 - 12 + 12u - 80 = 0$

$u^3 - u^2 - 13u + 93 = 0$

$\therefore$ The equation $u^3 - u^2 - 13u + 93$ has roots $-\alpha + \beta + r$, $\alpha - \beta + r$, $\alpha + \beta - r$.

ii) $\alpha\beta r = 10$

$\dfrac{1}{\beta r}, \dfrac{1}{r\alpha}, \dfrac{1}{\alpha\beta}$

Let $u = \dfrac{1}{\beta r}$

$= \dfrac{\alpha}{\alpha\beta r}$

$= \dfrac{\alpha}{10}$

$\alpha = 10u$

α is a root

$\therefore \alpha^3 - \alpha^2 - 3\alpha - 10 = 0$

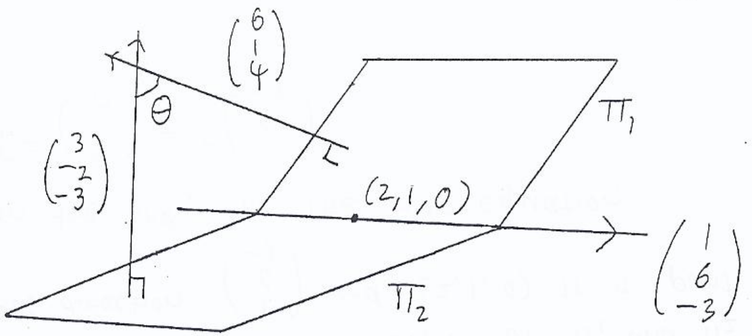
$(10u)^3 - (10u)^2 - 3(10u) - 10 = 0$

$1000u^3 - 100u^2 - 30u - 10 = 0$

$100u^3 - 10u^2 - 3u - 1 = 0$

$\therefore$ The equation $100u^3 - 10u^2 - 3u - 1$ has roots $\dfrac{1}{\alpha\beta}, \dfrac{1}{\alpha r}, \dfrac{1}{\beta r}$

9. $\Pi_1: \underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k} + \lambda(\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}) + \mu(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 + \lambda + \mu \\ -3 - 2\lambda + 2\mu \\ 1 - \lambda - 2\mu \end{pmatrix}.$$

$$\left.\begin{aligned} x &= 2 + \lambda + \mu \\ y &= -3 - 2\lambda + 2\mu \\ z &= 1 - \lambda - 2\mu \end{aligned}\right\}$$

$$\left.\begin{aligned} \lambda + \mu &= x - 2 \\ -2\lambda + 2\mu &= y + 3 \\ -\lambda - 2\mu &= z - 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} 2 \times ① + ②: \quad \lambda + \mu &= x - 2 \\ ① + ③: \quad 4\mu &= 2x + y - 1 \\ -\mu &= x + z - 3 \end{aligned}\right\}$$

$$\left.\begin{aligned} ② + 4 \times ③: \quad \lambda + \mu &= x - 2 \\ 4\mu &= 2x + y - 1 \\ 0 &= 6x + y + 4z - 13 \end{aligned}\right\}.$$

$$6x + y + 4z = 13$$

$\therefore$ The Cartesian equation of Π_1 is $6x + y + 4z = 13$.

$\Pi_2: 3x - 2y - 3z = 4$

Since $\begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix}$ is the normal of Π_1 and $\begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$ is the normal of Π_2

$$\begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix} = \left|\begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}\right| \left|\begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix}\right| \cos\theta$$

$$18 - 2 - 12 = \sqrt{22}\sqrt{53}\cos\theta$$

$$\therefore \cos\theta = \frac{4}{\sqrt{22}\sqrt{53}}$$

$$\theta \cong 83.3°.$$

$\therefore$ The acute angle between Π_1 and Π_2 is 83.3^0.

Since the line of intersection of Π_1 and Π_2 is perpendicular to both Π_1 and Π_2, the line has direction $\begin{pmatrix}6\\1\\4\end{pmatrix} \times \begin{pmatrix}3\\-2\\-3\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 6 & 1 & 4 \\ 3 & -2 & -3 \end{vmatrix}$

$$= \begin{pmatrix}5\\30\\-15\end{pmatrix}$$

$$= 5\begin{pmatrix}1\\6\\-3\end{pmatrix}.$$

$6x + y + 4z = 13 \quad 3x - 2y - 3z = 4$

$z=0: \quad 6x + y = 13 \quad 3x - 2y = 4$

$12x + 2y = 26 \quad 3x - 2y = 4$

$15x = 30$

$x = 2$

$y = 1$

Since the line of intersection of Π_1 and Π_2 has direction $\begin{pmatrix}1\\6\\-3\end{pmatrix}$ and $(2,1,0)$ is a point on the line, the line has equation

$$\underset{\sim}{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} + s\begin{pmatrix}1\\6\\-3\end{pmatrix}.$$

10. $$\left.\begin{aligned} x - 2y - 2z &= -7 \\ 2x + (a-9)y - 10z &= -11 \\ 3x - 6y + 2az &= -29 \end{aligned}\right\}$$

$$\left.\begin{aligned} -2\times ① + ②: \quad x - 2y - 2z &= -7 \\ -3\times ① + ③: \quad (a-5)y - 6z &= 3 \\ (2a+6)z &= -8 \end{aligned}\right\}$$

If the system has a unique solution,

$2a + 6 \neq 0$ and $a - 5 \neq 0$

$a \neq -3.$ $\qquad a \neq 5$

$\therefore$ The system has a unique solution for all real values of a except -3 or 5.

If $a = -3$: $$\left.\begin{aligned} x - 2y - 2z &= -7 \\ -8y - 6z &= 3 \\ 0z &= -8 \end{aligned}\right\}$$

$\therefore$ The system has no solution if $a = -3$:

$a = 5$:

i) $$\left.\begin{aligned} x - 2y - 2z &= -7 \\ 0y - 6z &= 3 \\ 16z &= -8 \end{aligned}\right\}$$

$z = -\frac{1}{2}$

$0y = 0$

Let $y = s,\ s \in R$

$\therefore x = 2s - 8$

$\therefore$ If $a = 5$ the number of solutions is infinite.

ii) If $x + y + z = 2$

$2s - 8 + s - \frac{1}{2} = 2$

$3s = \frac{21}{2}$

$s = \frac{7}{2}$

$\therefore x = -1,\ y = \frac{7}{2},\ z = -\frac{1}{2}$

11. EITHER

$C: (x^2+y^2)^2 = a^2(x^2-y^2),\ a>0$

If $x = r\cos\theta$, $y = r\sin\theta$

$(r^2\cos^2\theta + r^2\sin^2\theta)^2 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$

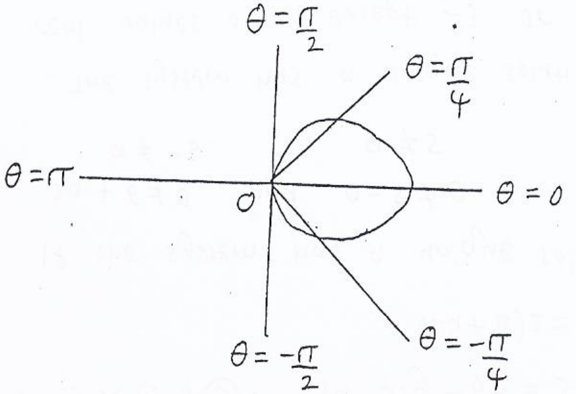
$(r^2)^2 = a^2r^2(\cos^2\theta - \sin^2\theta)$

$r^4 = a^2r^2\cos 2\theta$

$r^2 = a^2\cos 2\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	a	$\frac{a}{\sqrt{2}}$	0

$a^2\cos 2(-\theta) = a^2\cos 2\theta$



The area between $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$ is

$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{r^2}{2}\, d\theta$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a^2\cos 2\theta}{2}\, d\theta$

$= \left[\frac{a^2\sin 2\theta}{4}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \frac{a^2}{4}(1-(-1))$

$= \frac{a^2}{2}.$

$\frac{d}{dx}((x^2+y^2)^2) = \frac{d}{dx}(a^2(x^2-y^2))$

$2(x^2+y^2)(2x+2y\frac{dy}{dx}) = a^2(2x - 2y\frac{dy}{dx})$

$2(x^2+y^2)(x+y\frac{dy}{dx}) = a^2(x - y\frac{dy}{dx})$

If $\frac{dy}{dx} = 0$: $2(x^2+y^2)x = a^2x$

$x(a^2 - 2(x^2+y^2)) = 0$

$x = 0$ or $x^2+y^2 = \frac{a^2}{2}.$

$r^2 = \frac{a^2}{2}$

$r = \frac{a}{\sqrt{2}}$

$\frac{a^2}{2} = a^2\cos 2\theta$

$\cos 2\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$

$\therefore \theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$

$\therefore$ The points where the tangent is parallel to the initial line are $(\frac{a}{\sqrt{2}}, \pm\frac{\pi}{6}), (\frac{a}{\sqrt{2}}, \pm\frac{5\pi}{6})$.

OR.

$$\frac{1}{x}\frac{d^2y}{dx^2} + \left(\frac{6}{x} - \frac{2}{x^2}\right)\frac{dy}{dx} + \left(\frac{9}{x} - \frac{6}{x^2} + \frac{2}{x^3}\right) y = 169\sin 2x$$

$$y = xz$$

$$\frac{dy}{dx} = x\frac{dz}{dx} + z$$

$$\frac{d^2y}{dx^2} = x\frac{d^2z}{dx^2} + \frac{dz}{dx} + \frac{dz}{dx} = x\frac{d^2z}{dx^2} + 2\frac{dz}{dx}$$

$$\frac{1}{x}\left(x\frac{d^2z}{dx^2} + 2\frac{dz}{dx}\right) + \left(\frac{6}{x} - \frac{2}{x^2}\right)\left(x\frac{dz}{dx} + z\right)$$

$$+ \left(\frac{9}{x} - \frac{6}{x^2} + \frac{2}{x^3}\right) xz = 169\sin 2x$$

$$\frac{d^2z}{dx^2} + \frac{2}{x}\frac{dz}{dx} + 6\frac{dz}{dx} + \frac{6z}{x} - \frac{2}{x}\frac{dz}{dx} - \frac{2z}{x^2}$$

$$+ 9z - \frac{6z}{x} + \frac{2z}{x^2} = 169\sin 2x$$

$$\frac{d^2z}{dx^2} + 6\frac{dz}{dx} + 9z = 169\sin 2x$$

$$\frac{d^2z}{dx^2} + 6\frac{dz}{dx} + 9z = 0$$

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$m = -3$$

$\therefore$ The complementary function, z_c, is

$$z_c = (Ax+B)e^{-3x}.$$

The particular integral, z_p, is given by

$$z_p = C\cos 2x + D\sin 2x$$

$$\frac{dz_p}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\frac{d^2z_p}{dx^2} = -4C\cos 2x - 4D\sin 2x$$

$$\frac{d^2z_p}{dx^2} + 6\frac{dz_p}{dx} + 9z_p = -4C\cos 2x - 4D\sin 2x$$

$$+ 6(-2C\sin 2x + 2D\cos 2x)$$

$$+ 9(C\cos 2x + D\sin 2x)$$

$$= (5C + 12D)\cos 2x$$

$$+ (-12C + 5D)\sin 2x$$

$$= 169\sin 2x$$

$5C + 12D = 0 \qquad 5D - 12C = 169$

$25C + 60D = 0 \qquad 60D - 144C = 2028$

$$-169C = 2028$$

$$C = -12$$

$$D = 5$$

$$z_p = 5\sin 2x - 12\cos 2x$$

$$z = z_c + z_p$$

$$= (Ax+B)e^{-3x} + 5\sin 2x - 12\cos 2x$$

$$\frac{dz}{dx} = Ae^{-3x} - 3(Ax+B)e^{-3x} + 10\cos 2x + 24\sin 2x$$

$x=0 \quad z = -10: \quad -10 = B - 12 \quad \therefore B = 2$

$x=0 \quad \frac{dz}{dx} = 5: \quad 5 = A - 3B + 10 \quad \therefore A = 1$

$$z = (x+2)e^{-3x} + 5\sin 2x - 12\cos 2x$$

$$y = (x^2+2x)e^{-3x} + 5x\sin 2x - 12x\cos 2x$$