7B. Linear Spaces

7B.1 Euclidean *n*-Spaces

Background:
In the mid-seventeenth century, pairs of numbers were used to locate points in the plane and triples of numbers to locate points in 3-space. By the latter part of the 18th century, it was recognized that quadruples of numbers (a_1, a_2, a_3, a_4) could be regarded as points in "four-dimensional" space, quintuples (a_1, a_2, a_3, a_4, a_5) as points in "five-dimensional" space, and so on, an *n*-tuple of numbers being a point in "*n*-dimensional" space.

Although our geometric visualization does not extend beyond 3-space, it is nevertheless possible to extend the properties of points or vectors in 2D and 3D to *n*-dimensional analytically or numerically.

Vectors in *n*-Space

Definition:
A point in *n*-dimensional space, denoted $\mathbf{R}^n$, has coordinates $(x_1, x_2, \ldots, x_n)$.

$\mathbf{R}^n = \{(x_1, x_2, \ldots, x_n): x_1, x_2, \ldots, x_n \in \mathbf{R}\}$.

7B.1.1 Linear Transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ in Matrix Form

If $\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{R}^n$,

then T: $\mathbf{R}^n \xrightarrow{M} \mathbf{R}^m$ is a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \in \mathbf{R}^m.$$

Theorem: (Properties of Linear Transformations)

A transformation T: $\mathbf{R}^n \xrightarrow{M} \mathbf{R}^m$ is linear if and only if the following relationships hold for all vectors **u** and **v** in $\mathbf{R}^n$ and for every scalar *c*.

a) T(**u** + **v**) = T(**u**) + T(**v**)
b) T(*c***u**) = *c*T(**u**)

Example 7B.1.1:

Determine the linear transformation (in the form of T: $\mathbf{R}^n \xrightarrow{M} \mathbf{R}^m$ defined by the following:

a) $\mathbf{M} = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2$

b) $\mathbf{M} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 7 & 3 & 8 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3$

c) $\mathbf{M} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 7 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3$

d) $\mathbf{M} = \begin{pmatrix} 7 & 1 \\ -9 & -9 \\ 2 & 8 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2$

e) $\mathbf{M} = \begin{pmatrix} 1 & 4 & 3 & 4 & 5 \\ 2 & 1 & 5 & 6 & 0 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbf{R}^5$

f) $\mathbf{M} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 8 & 5 \\ 1 & 3 & 9 \\ 6 & 1 & 4 \\ 2 & 7 & 9 \\ -1 & -3 & 8 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3$

Solution:

a)
$$\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ -x_1 + 4x_2 \end{pmatrix}$$
$$M : R^2 \to R^2$$

b)
$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 7 & 3 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 4x_3 \\ x_2 + 2x_3 \\ 7x_1 + 3x_2 + 8x_3 \end{pmatrix}$$
$$M : R^3 \to R^3$$

c)
$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 + x_3 \\ x_1 + 4x_2 + 7x_3 \end{pmatrix}$$
$$M : R^3 \to R^2$$

d) $$\begin{pmatrix} 7 & 1 \\ -9 & -9 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7x_1 + x_2 \\ -9x_1 - 9x_2 \\ 2x_1 + 8x_2 \end{pmatrix}$$

$$M : R^2 \to R^3$$

e) $$\begin{pmatrix} 1 & 4 & 3 & 4 & 5 \\ 2 & 1 & 5 & 6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + 4x_2 + 3x_3 + 4x_4 + 5x_5 \\ 2x_1 + x_2 + 5x_3 + 6x_4 \end{pmatrix}$$

$$M : R^5 \to R^2$$

f) $$\begin{pmatrix} 2 & 1 & 4 \\ 0 & 8 & 5 \\ 1 & 3 & 9 \\ 6 & 1 & 4 \\ 2 & 7 & 9 \\ -1 & -3 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 + 4x_3 \\ 8x_2 + 5x_3 \\ x_1 + 3x_2 + 9x_3 \\ 6x_1 + x_2 + 4x_3 \\ 2x_1 + 7x_2 + 9x_3 \\ -x_1 - 3x_2 + 8x_3 \end{pmatrix}$$

$$M : R^3 \to R^6$$

7B.2 Linear (Vector) Spaces

Definition:
If any set of objects *V* satisfies the following conditions then *V* is called a **vector space** defined under the operations of addition and scalar multiplication.

1. If *a* and *b* are objects in *V* then *a* + *b* is an object in *V*.
2. *a* + *b* = *b* + *a*.
3. If *a*, *b* and *c* are objects in *V* then *a* + (*b* + *c*) = (*a* + *b*) + *c*
4. There exists an object 0 in *V* such that *a* + 0 = 0 + *a* = *a* for all *a* in *V*. 0 is called the zero vector.
5. For each object *a* in *V* there is an object –*a* in *V* called the negative of *a* such that *a* + (-*a*) = (-*a*) + *a* = 0.
6. If *a* is any object in *V* and *k* is any scalar then *ka* is an object in *V*.
7. *k*(*a* + *b*) = *ka* + *kb*.
8. (*k* + *l*) *a* = *ka* + *la* for any scalars *k* and *l*.
9. *k*(*la*) = (*kl*)*a*.
10. 1*u* = *u*.

❖ The set of matrices $\{[a_{ij}]_{m\times n}\}$ satisfies these conditions and forms a vector space.

❖ The set of vectors in $\mathbf{R}^n$ also satisfies these conditions and forms a vector space.

❖ The set of all real valued functions defined on $\mathbf{R}^n$ also satisfies these conditions and forms a vector space.

7B.2.1 Subspaces

Definition:
If V is a vector space and W is a subset of V then W is a subspace of V if W itself is a vector space. The set W is a subspace of the vector space V if and only if

1. If a and b are in W, $a + b$ is in W.
2. If k is any scalar and a is a vector in W, ka is in W.

Since every vector can be considered a matrix having 1 column, therefore the set of vectors in $\mathbf{R}^m$: $\{\mathbf{x} : \mathbf{x} \in \mathbf{R}^m\}$ is a subset of the set of matrices having n columns : $\{[a_{ij}]_{m\times n}\}$.

Every nonzero vector space V has at least two subspaces: V itself and the zero subspace $\{\mathbf{0}\}$ consisting only zero vector in V.

The following is the list of subspaces of $\mathbf{R}^2$ and $\mathbf{R}^3$:

Subspaces of $\mathbf{R}^2$	Subspaces of $\mathbf{R}^3$
• $\{\mathbf{0}\} = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$ • Lines through the origin. • $\mathbf{R}^2$	• $\{\mathbf{0}\} = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$ • Lines through the origin. • Planes through the origin. • $\mathbf{R}^3$

To show that a ***line through the origin*** of $\mathbf{R}^3$ is a ***subspace*** of $\mathbf{R}^3$, let W be a line thorugh the origin of $\mathbf{R}^3$. It is evident geometrically that the sum of two vectors on this line also lies on the line and that a scalar multiple of a vector on the line is on the line as well (as shown in Figure 1). Thus W is closed under addition and scalar multiplication, so it is a subspace of $\mathbf{R}^3$.

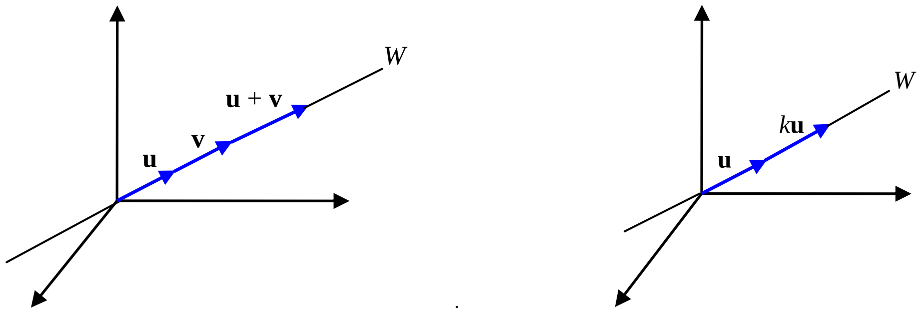


Figure 1

Let W be any plane through the origin, and let **u** and **v** be any vectors in W. then **u** + **v** must lie in W because it is diagonal of the parallelogram determined by **u** and **v**, and k**u** must lie in W for any scalar k because k**u** lies on a line through **u** (refer to Figure 2). Thus W is closed under addition and scalar multiplication, so it is a ***subspace*** of $\mathbf{R}^3$.

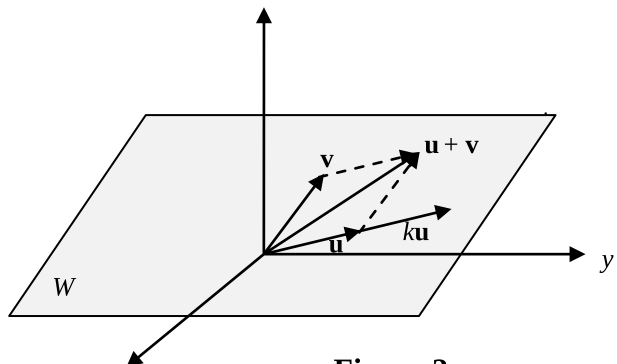


Figure 2

If $\mathbf{Ax} = \mathbf{0}$ then the set of solution vectors is a subspace of $\mathbf{R}^n$.

Example 7B.2.1 (Pg 233):
Each of the following systems has three unknowns, show that the solutions form subspaces of $\mathbf{R}^3$.

a) $\begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Solution:

Geometrically, this means that each solution space must be the origin only, a line through the origin, a plane through the origin, or all of $\mathbf{R}^3$.

a) The solutions are

$$x = 2s - 3t, y = s, z = t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s-3t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$$x - 2y + 3z = 0$$

This is the equation of a plane through origin.

b) The solution are

$$x = -5t, y = -t, z = t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5t \\ -t \\ t \end{pmatrix} = t\begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}$$

which are parametric equations for the line through the origin.

c) The solution is

$$x = 0, y = 0, z = 0.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So the solution space is the origin only, that is, $\{\mathbf{0}\}$.

d) The solutions are

$$x = r, y = s, z = t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ s \\ t \end{pmatrix} = r\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So the solution space is all of $\mathbf{R}^3$.

7B.2.2 Linear Combination

Definition:
Let V denote a vector space, and $\mathbf{w} \in$ V. The vector $\mathbf{w}$ is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

Remarks:
Because V is a vector space, we know that given $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V, the linear combination $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$ is also in V. A more difficult question is: given $\mathbf{w} \in$ V and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V, can we find $k_1, k_2, \ldots, k_r$ such that $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$?

This is simply a matter of solving the system of linear equations $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$. If the determinant of the coefficient matrix of the system is not zero, then the system has unique solution and hence there exist a linear combination. However, if the determinant is zero, a linear combination might exist since the system could have no solution or infinitely many solutions. But, for this situation, the linear combination will only exist with restrictions.

Therefore, we are starting to get answers to the questions asked in the previous paragraph. Given $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V, it is possible that not every vector of V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$.

When it is the case that every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are special and they are called the ***spanning sets*** (Refer to the next section).

Example 7B.2.2:

1. Every vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbf{R}^3$ is expressible as a linear combination of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ since

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2. Given $\mathbf{u} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$ $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, write the following in linear combination.

a) $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ b) $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ c) $\begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$ d) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

3. Express the following as linear combinations of $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$.

a) $\begin{pmatrix} -9 \\ -7 \\ -15 \end{pmatrix}$ b) $\begin{pmatrix} 6 \\ 11 \\ 6 \end{pmatrix}$ c) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ d) $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

Solution:

2a)
$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = k_1 \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} k_2 \\ -2k_1 + 3k_2 \\ 2k_1 - k_2 \end{pmatrix}$$

$$k_2 = 2$$
$$-2k_1 + 3k_2 = 2$$
$$2k_1 - k_2 = 2$$
$$\therefore k_1 = 2, k_2 = 2.$$

2b)
$$\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = k_1 \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} k_2 \\ -2k_1 + 3k_2 \\ 2k_1 - k_2 \end{pmatrix}$$

$$k_2 = 3$$
$$-2k_1 + 3k_2 = 1$$
$$2k_1 - k_2 = 5$$
$$\therefore k_2 = 3, k_1 = 4.$$

2c)
$$\begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} = k_1 \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} k_2 \\ -2k_1 + 3k_2 \\ 2k_1 - k_2 \end{pmatrix}$$

$$k_2 = 0$$
$$-2k_1 + 3k_2 = 4$$
$$2k_1 - k_2 = 5$$

2d) $\begin{pmatrix}0\\0\\0\end{pmatrix} = k_1\begin{pmatrix}0\\-2\\2\end{pmatrix} + k_2\begin{pmatrix}1\\3\\-1\end{pmatrix}$

$= \begin{pmatrix}k_2\\-2k_1+3k_2\\2k_1-k_2\end{pmatrix}$

$k_2 = 0$

$-2k_1 + 3k_2 = 0$

$2k_1 - k_2 = 0$

$\therefore k_1 = 0, k_2 = 0.$

$\begin{pmatrix}0\\0\\0\end{pmatrix} = 0\begin{pmatrix}0\\-2\\2\end{pmatrix} + 0\begin{pmatrix}1\\3\\-1\end{pmatrix}$

3. $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \in R^3 \quad ; \quad \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = k_1\begin{pmatrix}2\\1\\4\end{pmatrix} + k_2\begin{pmatrix}1\\-1\\3\end{pmatrix} + k_3\begin{pmatrix}3\\2\\5\end{pmatrix}$

$$= \begin{pmatrix}2k_1+k_2+3k_3\\k_1-k_2+2k_3\\4k_1+3k_2+5k_3\end{pmatrix}$$

$$= \begin{pmatrix}2 & 1 & 3\\1 & -1 & 2\\4 & 3 & 5\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}$$

$$= A\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}, A = \begin{pmatrix}2 & 1 & 3\\1 & -1 & 2\\4 & 3 & 5\end{pmatrix}$$

$|A| = -22 + 3 + 21 = 2$

$$adj(A) = \begin{pmatrix}-11 & 4 & 5\\3 & -2 & -1\\7 & -2 & -3\end{pmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix}-11 & 4 & 5\\3 & -2 & -1\\7 & -2 & -3\end{pmatrix}$$

$$\therefore \begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = A^{-1}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-11 & 4 & 5\\3 & -2 & -1\\7 & -2 & -3\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

7B.2.3 Spanning

Theorem:
If V is a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are vectors in V, the set of all linear combinations $S = \{k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r\}$ is a subspace of V.

Definition:
Let V denote a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ a subset of V.

(a) We say that S is a spanning set of V or that S spans V if for every vector $\mathbf{w}$ in V, $\mathbf{w}$ can be written as a linear combination of the vectors in S.

(b) The span of S, denoted span$\{S\}$, is the set of all linear combinations of vectors in S. In other words,

$$\text{span}\{S\} = \{k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r \,;\, k_1, k_2, \ldots, k_r \in \mathbf{R} \text{ and } \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \in S\}$$

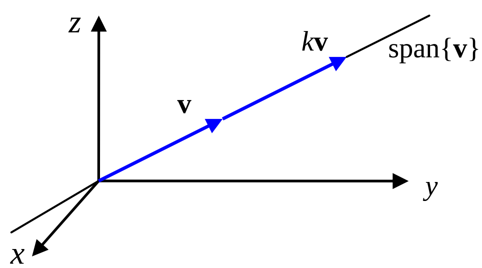
Eg: The polynomials $1, x, x^2, \ldots, x^n$ span P_n since every polynomial can be written as $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$.

To determine whether a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ span the vector space V, one must ensure an arbitrary vector $\mathbf{b} \in V$, can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$.

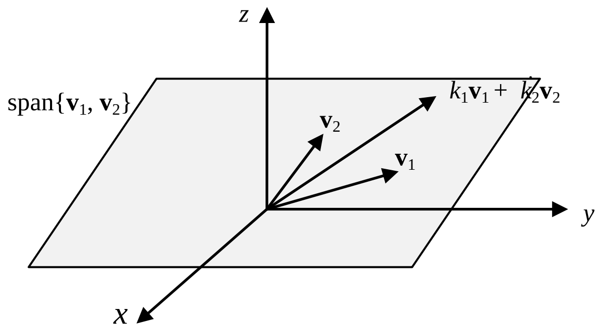
Similarly to the linear combination, it is simply a matter of solving the system of linear equation $\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$. If the determinant of the coefficient matrix of the system is not zero, then the system has unique solution and hence the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ span the vector space V. However, if the determinant is zero, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ "do not span" the ***entire*** vector space V.

Example 7B.2.3:

1. For a vector $\mathbf{v}$, span$\{\mathbf{v}\}$ is the line through the origin determined by $\mathbf{v}$ since the line is the set of all multiples $k\mathbf{v}$.



2. For two nonparallel vectors $\mathbf{v}_1, \mathbf{v}_2$, span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane passing through the origin determined by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ since the plane is the set of all linear combinations $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$.



3. Determine whether $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathrm{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ span the vector space $\mathbf{R}^3$.

Solution:

We must determine whether an arbitrary vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in $\mathbf{R}^3$ can be expressed

as a linear combination

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Expressing this equation in terms of components gives

$$(b_1, b_2, b_3) = k_1\,(1, 1, 2) + k_2\,(1, 0, 1) + k_3\,(2, 1, 3)$$

$$= (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

or

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \qquad + \ k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

The problem reduces to determining whether this system is consistent for all values of b_1, b_2 and b_3. This system is consistent if and only if the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

has a nonzero determinant. However, in this case $\det(A) = 0$, so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span $\mathbf{R}^3$.

4. Determine whether the following vectors span the vector space $\mathbf{R}^3$.

a) $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

b) $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 8 \\ -1 \\ 8 \end{pmatrix}$

c) $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$

d) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

Solution:

4a)

If $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in R^3$ and

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

$$= k_1\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + k_3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2k_1 \\ 2k_1 + k_3 \\ 2k_1 + 3k_2 + k_3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

since $\begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = -6 \neq 0$

so $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span R^3.

4b) If $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in R^3$ and

$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ = $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$

$$= k_1\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + k_2\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + k_3\begin{pmatrix} 8 \\ -1 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 2k_1 + 4k_2 + 8k_3 \\ -k_1 + k_2 - k_3 \\ 3k_1 + 2k_2 + 8k_3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

since $\begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 2(10) - 4(-5) + 8(-5) = 0$

so $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ do not span R^3.

4c) If $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in R^3$ and

$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ = $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$

$= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4(\mathbf{v}_1 - \mathbf{v}_2)$

$= (k_1 + k_4)\mathbf{v}_1 + (k_2 - k_4)\mathbf{v}_2 + k_3\mathbf{v}_3$

$= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$= c_1\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + c_3\begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 3c_1 + 2c_2 + 5c_3 \\ c_1 - 3c_2 - 2c_3 \\ 4c_1 + 5c_2 + 9c_3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 5 \\ 1 & -3 & -2 \\ 4 & 5 & 9 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

since $\begin{vmatrix} 3 & 2 & 5 \\ 1 & -3 & -2 \\ 4 & 5 & 9 \end{vmatrix} = 3(-17) - 2(17) + 5(17) = 0$

so $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ do not span R^3.

4d) Part 1:

$$\mathbf{v}_1 = k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$$

$$= k_2\begin{pmatrix}3\\4\\1\end{pmatrix} + k_3\begin{pmatrix}4\\3\\1\end{pmatrix} + k_4\begin{pmatrix}3\\3\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\6\end{pmatrix} = \begin{pmatrix}3k_2+4k_3+3k_4\\4k_2+3k_3+3k_4\\k_2+k_3+k_4\end{pmatrix}$$

$$= \begin{pmatrix}3&4&3\\4&3&3\\1&1&1\end{pmatrix}\begin{pmatrix}k_2\\k_3\\k_4\end{pmatrix}$$

$$\therefore \begin{pmatrix}k_2\\k_3\\k_4\end{pmatrix} = \begin{pmatrix}3&4&3\\4&3&3\\1&1&1\end{pmatrix}^{-1}\begin{pmatrix}1\\2\\6\end{pmatrix}$$

$$= \begin{pmatrix}0&1&-3\\1&0&-3\\-1&-1&7\end{pmatrix}\begin{pmatrix}1\\2\\6\end{pmatrix}$$

$$= \begin{pmatrix}-16\\-17\\39\end{pmatrix}$$

$$\mathbf{v}_1 = -16\mathbf{v}_2 - 17\mathbf{v}_3 + 39\mathbf{v}_4$$

4d) Part 2:

If $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \in R^3$ and

$$\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$$

$$= k_1(-16\mathbf{v}_2 - 17\mathbf{v}_3 + 39\mathbf{v}_4) + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$$

$$= (k_2 - 16k_1)\mathbf{v}_2 + (k_3 - 17k_1)\mathbf{v}_3 + (39k_1 + k_4)\mathbf{v}_4$$

$$= c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$$

$$= c_2\begin{pmatrix}3\\4\\1\end{pmatrix} + c_3\begin{pmatrix}4\\3\\1\end{pmatrix} + c_4\begin{pmatrix}3\\3\\1\end{pmatrix}$$

$$= \begin{pmatrix}3c_2+4c_3+3c_4\\4c_2+3c_3+3c_4\\c_2+c_3+c_4\end{pmatrix}$$

$$= \begin{pmatrix}3&4&3\\4&3&3\\1&1&1\end{pmatrix}\begin{pmatrix}c_2\\c_3\\c_4\end{pmatrix}$$

$$\text{Since } \begin{vmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -1 \neq 0$$

$$\begin{pmatrix} c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ span R^3.

7B.2.4 Linear Independence

Introduction:
To find an "optimal" spanning set *S* for a given vector space *V*, we begin from the Theorem below.

Theorem:
Let *V* denote a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ a spanning set of *V*. suppose further that one of the vectors of *S* is a linear combination of the others. If we remove that vector from *S*, the resulting set still spans *V*.

Remarks:
The set S in the theorem was not an "optimal" spanning set in the sense that one of its vectors was redundant. The notion of linear independence addresses this.

Definition:
If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a nonempty set of vectors, then the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$$

has at least one solution, namely

$$k_1 = k_2 = \ldots = k_r = 0$$

If this is the **only** solution, then the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_r$ are **linearly independent**. If there are **other** solutions, then $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_r$ are **linearly dependent**.

Example 7B.2.4:
Determine whether the following vectors form a linearly dependent set or a linearly independent set.

a) $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix}$

b) $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 10 \\ 2 \end{pmatrix}$

c) $\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

d) $\begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

e) $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

f) $\begin{pmatrix} -6 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

Solution:

a)

$$k_1\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + k_3\begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} + k_4\begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2k_1 + 3k_2 + 6k_3 + 7k_4 \\ 2k_2 - k_3 \\ k_1 + 5k_2 + k_3 - 2k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2k_1 + 3k_2 + 6k_3 + 7k_4 = 0$$

$$2k_2 - k_3 = 0$$

$$k_1 + 5k_2 + k_3 - 2k_4 = 0$$

$$\left(\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Leftrightarrow r_3} \left(\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -2 & 3 & 6 & 7 & 0 \end{array}\right)$$

$$\xrightarrow{2r_1 + r_3} \left(\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 13 & 8 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{2r_3} \left(\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 26 & 16 & 6 & 0 \end{array}\right)$$

$$\xrightarrow{-13r_2 + r_3} \left(\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 29 & 6 & 0 \end{array}\right)$$

$$29k_3 + 6k_4 = 0$$

$$k_4 = 29t, t \in R$$

$$k_3 = -6t, k_2 = -3t$$

$$k_1 + 5(-3t) - 6t - 2(29t) = 0$$

$$k_1 = 15t + 6t + 58t$$

$$k_1 = 79t$$

Therefore, the vectors are linearly dependent.

b)

$$k_1\begin{pmatrix}4\\-1\\2\end{pmatrix}+k_2\begin{pmatrix}-4\\10\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}4k_1-4k_2\\-k_1+10k_2\\k_1+k_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$4k_1-4k_2=0$

$-k_1+10k_2=0$

$k_1+k_2=0$

$$\left(\begin{array}{cc|c}1&-1&0\\-1&10&0\\1&1&0\end{array}\right)$$

$$\xrightarrow[-r_1+r_3]{r_1+r_2}\left(\begin{array}{cc|c}1&-1&0\\0&9&0\\0&2&0\end{array}\right)$$

Therefore, the vectors are linearly independent.

$k_2=0, k_1=0$

c)

$$k_1\begin{pmatrix}-3\\0\\4\end{pmatrix}+k_2\begin{pmatrix}5\\-1\\2\end{pmatrix}+k_3\begin{pmatrix}1\\1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-3k_1+5k_2+k_3\\-k_2+k_3\\4k_1+2k_2+3k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$-3k_1+5k_2+k_3=0$

$-k_2+k_3=0$

$4k_1+2k_2+3k_3=0$

$k_2=k_3 \therefore k_1=2k_2$

$4k_1+5k_2=0$

$13k_2=0$

Therefore, the vectors are linearly independent.

$k_2=0, k_3=0, k_1=0$

d)

$$k_1\begin{pmatrix}8\\-1\\3\end{pmatrix}+k_2\begin{pmatrix}4\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}8k_1+4k_2\\-k_1\\3k_1+k_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$8k_1+4k_2=0$$
$$-k_1=0$$
$$3k_1+k_2=0$$
$$k_2=0$$
$$k_1=0$$

Therefore, the vectors are linearly independent.

e)

$$k_1\begin{pmatrix}2\\-2\\0\end{pmatrix}+k_2\begin{pmatrix}6\\1\\4\end{pmatrix}+k_3\begin{pmatrix}2\\0\\-4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}2k_1+6k_2+2k_3\\-2k_1+k_2\\4k_2-4k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}2&6&2\\-2&1&0\\0&4&-4\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{vmatrix}2&6&2\\-2&1&0\\0&4&-4\end{vmatrix}=-72\neq 0$$

Therefore, the vectors are linearly independent.

f)

$$k_1\begin{pmatrix}-6\\7\\2\end{pmatrix}+k_2\begin{pmatrix}3\\2\\4\end{pmatrix}+k_3\begin{pmatrix}4\\-1\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-6k_1+3k_2+4k_3\\7k_1+2k_2-k_3\\2k_1+4k_2+2k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{vmatrix}-6&3&4\\7&2&-1\\2&4&2\end{vmatrix}=0$$

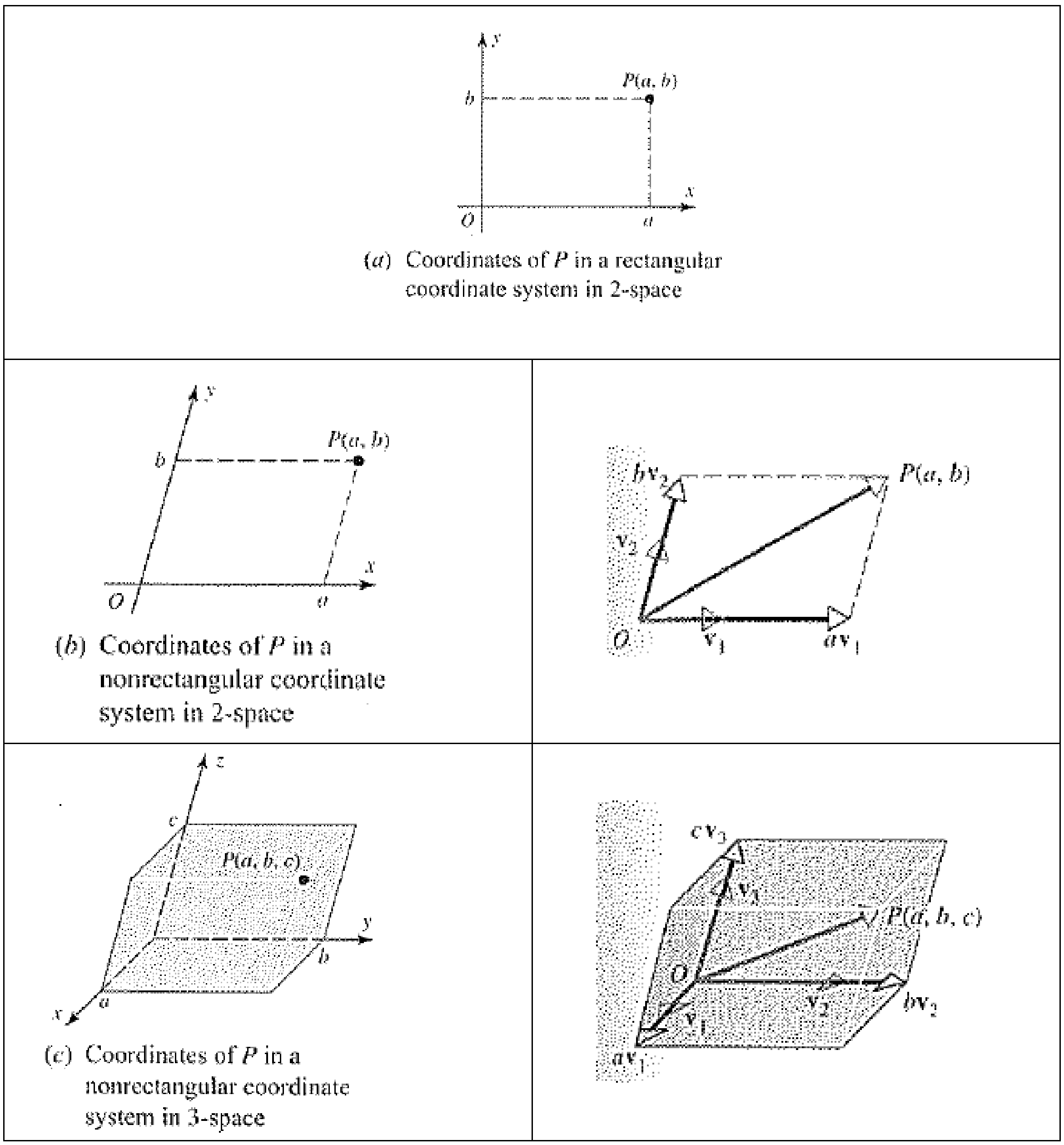
$$\begin{pmatrix}-6&3&4\\7&2&-1\\2&4&2\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Therefore, the vectors are linearly dependent.

7B.2.5 Basis and Dimension

Introduction:

In 2D, we associate a point *P* in the plane with a pair of coordinates (*a*, *b*) which by projecting *P* onto a pair of perpendicular coordinate axes on the two perpendicular lines (*x*-axis and *y*-axis) as shown in Figure 3a. Although the perpendicular axes are the most common, any two nonparallel lines can actually be used to define a coordinate system in the plane. For example in Figure 3b, we project the point *P*(*a*, *b*) parallel to the non-perpendicular coordinate axes. Similarly, in 3D any three noncoplanar coordinate axes can be used to define a coordinate system (Figure 3c).



(*a*) Coordinates of *P* in a rectangular coordinate system in 2-space

(*b*) Coordinates of *P* in a nonrectangular coordinate system in 2-space

(*c*) Coordinates of *P* in a nonrectangular coordinate system in 3-space

Figure 3

Our objective in this section is to extend the concept of a coordinate system to general vector spaces.

Informally stated, vectors that specify a coordinate system are called "basis vectors" for that system. Any nonzero vectors of any length will suffice (not necessarily must be basis vectors of length 1).

In the section on spanning sets and linear independence, we were trying to understand what the elements of a vector space looked like by studying how they could be generated. We learned that some subsets of a vector space could generate the entire vector space. Such subsets were called spanning sets. Other subsets did not generate the entire space, but their span was still a subspace of the underlying vector space. In some cases, the number of vectors in such a set was redundant in the sense that one or more of the vectors could be removed, without changing the span of the set. In other cases, there was not a unique way to generate some vectors in the space. In this section, we want to make this process of generating all the elements of a vector space more reliable, more efficient.

Definition (Basis):
If V is a vector space and S is a set of vectors in V, $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, S is called a **basis** of V if

(i) the vectors in S are linearly independent
(ii) S spans V

This definition tells us that a basis has to contain enough vectors to generate the entire vector space. But it does not contain too many. In other words, if we removed one of the vectors, it would no longer generate the space.

Example 7B.2.5:

If $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a linearly independent set in $\mathbf{R}^3$. S also spans $\mathbf{R}^3$ since every vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbf{R}^3$ can be expressed as $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

S is a basis for $\mathbf{R}^3$ and is called the standard basis for $\mathbf{R}^3$.

Definition (Dimension):
The **dimension** of a vector space V, $\dim(V)$ is the number of vectors in a basis for V.
The zero vector space is defined to have dimension 0.

Example 7B.2.6:

1. Determine whether the following set of vectors form a basis for $\mathbf{R}^3$.

a) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}, \begin{pmatrix}3\\3\\3\end{pmatrix}$

b) $\begin{pmatrix}3\\1\\-4\end{pmatrix}, \begin{pmatrix}2\\5\\6\end{pmatrix}, \begin{pmatrix}1\\4\\8\end{pmatrix}$

c) $\begin{pmatrix}2\\-3\\1\end{pmatrix}, \begin{pmatrix}4\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-7\\1\end{pmatrix}$

d) $\begin{pmatrix}1\\6\\4\end{pmatrix}, \begin{pmatrix}2\\4\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\5\end{pmatrix}$

2. Find the basis and dimension of the solution space of the following homogeneous systems:

a) $$\begin{aligned} x-3y+z&=0\\ 2x-6y+2z&=0\\ 3x-9y+3z&=0 \end{aligned}$$

b) $$\begin{aligned} x+y+z&=0\\ 3x+2y-2z&=0\\ 4x+3y-z&=0\\ 6x+5y+z&=0 \end{aligned}$$

c) $$\begin{aligned} 2x+y+3z&=0\\ x+5z&=0\\ y+z&=0 \end{aligned}$$

d) $$\begin{aligned} x+y-z&=0\\ -2x-y+2z&=0\\ -x+z&=0 \end{aligned}$$

e) $$\begin{aligned} 3x+y+z+w&=0\\ 5x-y+z-w&=0 \end{aligned}$$

f) $$\begin{aligned} x-4y+3z-w&=0\\ 2x-8y+6z-2w&=0 \end{aligned}$$

Solution:

1a)

$$k_1\begin{pmatrix}1\\0\\0\end{pmatrix}+k_2\begin{pmatrix}2\\2\\0\end{pmatrix}+k_3\begin{pmatrix}3\\3\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}k_1+2k_2+3k_3\\2k_2+3k_3\\3k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1&2&3\\0&2&3\\0&0&3\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{vmatrix}1&2&3\\0&2&3\\0&0&3\end{vmatrix}=6$$

$$\therefore\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}1&2&3\\0&2&3\\0&0&3\end{pmatrix}^{-1}\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

If $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in R^3$

$$k_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + k_3\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} k_1 + 2k_2 + 3k_3 \\ 2k_2 + 3k_3 \\ 3k_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}^{-1}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

b)

$$k_1\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} + k_2\begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} + k_3\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3k_1 + 2k_2 + k_3 \\ k_1 + 5k_2 + 4k_3 \\ -4k_1 + 6k_2 + 8k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 26$$

$$\therefore \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in R^3$

$$k_1\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} + k_2\begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} + k_3\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} 3k_1 + 2k_2 + k_3 \\ k_1 + 5k_2 + 4k_3 \\ -4k_1 + 6k_2 + 8k_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{pmatrix}^{-1}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

c)

$$k_1\begin{pmatrix}2\\-3\\1\end{pmatrix}+k_2\begin{pmatrix}4\\1\\1\end{pmatrix}+k_3\begin{pmatrix}0\\-7\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}2k_1+4k_2\\-3k_1+k_2-7k_3\\k_1+k_2+k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}2&4&0\\-3&1&-7\\1&1&1\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{vmatrix}2&4&0\\-3&1&-7\\1&1&1\end{vmatrix}=0$$

The vectors are linearly dependent.

If $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\in R^3$

$$k_1\begin{pmatrix}2\\-3\\1\end{pmatrix}+k_2\begin{pmatrix}4\\1\\1\end{pmatrix}+k_3\begin{pmatrix}0\\-7\\1\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\begin{pmatrix}2k_1+4k_2\\-3k_1+k_2-7k_3\\k_1+k_2+k_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\begin{pmatrix}2&4&0\\-3&1&-7\\1&1&1\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\begin{vmatrix}2&4&0\\-3&1&-7\\1&1&1\end{vmatrix}=0$$

The vectors do not span R^3.

d)

$$k_1\begin{pmatrix}1\\6\\4\end{pmatrix}+k_2\begin{pmatrix}2\\4\\-1\end{pmatrix}+k_3\begin{pmatrix}-1\\2\\5\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}k_1+2k_2-k_3\\6k_1+4k_2+2k_3\\4k_1-k_2+5k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1&2&-1\\6&4&2\\4&-1&5\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{vmatrix}1&2&-1\\6&4&2\\4&-1&5\end{vmatrix}=0$$

The vectors are linearly dependent.

If $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\in R^3$

$$k_1\begin{pmatrix}1\\6\\4\end{pmatrix}+k_2\begin{pmatrix}2\\4\\-1\end{pmatrix}+k_3\begin{pmatrix}-1\\2\\5\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\begin{pmatrix}k_1+2k_2-k_3\\6k_1+4k_2+2k_3\\4k_1-k_2+5k_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\begin{pmatrix}1&2&-1\\6&4&2\\4&-1&5\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\begin{vmatrix}1&2&-1\\6&4&2\\4&-1&5\end{vmatrix}=0$$

The vectors do not span R^3.

2a) $$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right) \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = s, y = t, x = 3t - s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3t - s \\ t \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

Basis: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ Dimension: 2

b) $$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right) \xrightarrow{\substack{-3r_1+r_2 \\ -4r_1+r_3 \\ -6r_1+r_4}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right) \xrightarrow{-1\times r_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right)$$

$$\xrightarrow[r_2+r_4]{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = t, y = -5t, x = 4t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4t \\ -5t \\ t \end{pmatrix} = t\begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$$

$Basis: \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$

$Dimension: 1$

c) $$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{r_1 \Leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{-2r_1+r_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 8 & 0 \end{array}\right)$$

$x = 0, y = 0, z = 0$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

No basis.
Dimension: 0

d) $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[r_1+r_3]{2r_1+r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$

$y = 0, x - z = 0$

$z = t, x = t$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ Basis: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ Dimension: 1

e) $\left(\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{\frac{r_1}{3}} \left(\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{-5r_1+r_2} \left(\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{array}\right)$

$w = s, z = t$

$y = -\frac{t}{4} - s$

$x = -\frac{t}{4}$

$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{t}{4} \\ -\frac{t}{4} - s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ Basis: $\begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

Dimension: 2

f) $\left(\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{array}\right) \xrightarrow{-2r_1+r_2} \left(\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$w = r, z = s, y = t$

$x = 4t - 3s + r$

$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4t - 3s + r \\ t \\ s \\ r \end{pmatrix} = r\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ Basis: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Dimension: 3

7B.2.6 Row Space, Column Space, and Nullspace

Introduction:

In application of linear algebra, subspaces of $\mathbf{R}^n$ typically arise in one of two situations:

1) As the set of solutions of a linear homogeneous system ($\mathbf{Ax} = \mathbf{0}$)
2) As the set of solutions of all linear combinations of a given set of vectors ($\mathbf{Ax} = \mathbf{b}$)

In this section, we will study, compare and contrast these two situations. We will try to answer the following questions:

1. How does one find the row space of a matrix **A**?
2. How does one find the column space of a matrix **A**?
3. How does one find the null space of a matrix **A**?
4. Is there a relationship between them?
5. Since matrices are closely related to solving systems of linear equations, is there a relationship between the row space, column space, null space of a matrix **A** and the linear system $\mathbf{Ax} = \mathbf{b}$?

Definition (Row and column vector):

If $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

the vectors in $\mathbf{R}^n$,

$$\mathbf{r}_1 = [a_{11} \quad a_{12} \ldots a_{1n}],\ \mathbf{r}_2 = [a_{21} \quad a_{22} \ldots a_{2n}],\ \ldots,\ \mathbf{r}_m = [a_{m1} \quad a_{m2} \ldots a_{mn}]$$

are called the ***row vectors*** of **A** and the vectors in $\mathbf{R}^m$,

$\mathbf{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$, $\mathbf{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}$, …, $\mathbf{c}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$ are called the ***column vectors*** of **A**.

Definition (Row Space, Column/Range Space and Null Space):

Let **A** be an *m* x *n* matrix.

a) The subspace of $\mathbf{R}^n$ spanned by the row vectors of **A** is called the ***row space*** of **A.**
b) The subspace of $\mathbf{R}^m$ spanned by the column vectors of **A** is called the ***column space/range space*** of **A**.
c) The solution space of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ (a subspace of $\mathbf{R}^n$), is called the ***nullspace*** of **A**.

Remarks:

- The elements of nullspace of a *m* x *n* matrix **A** are the vectors in $\mathbf{R}^n$ (where it depends only on the number of columns of **A**). Therefore it is a subspace of $\mathbf{R}^n$. Eg: The elements of null space of **A** if **A** is 3 x 5 are vectors of $\mathbf{R}^5$.

- When asked to "find the null space" of a matrix, one is asked to find a basis for it.

  Eg: Find the null space of $\mathbf{A} = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$.

Theorem:
If a matrix **R** is in ***row-echelon form***, then

a) The row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of **R**.
b) The column vectors with the leading 1's of the row vectors form a basis for the column space of **R**.

Theorem:
Elementary row operations **do not** change the ***nullspace*** and the ***row space*** of a matrix.

Theorem:
If **A** and **B** are row equivalent matrices (we got from one to the other by row operations), then

a) A given set of columns vectors of **A** is linearly independent iff. the corresponding column vectors of **B** are linearly independent.
b) A given set of column vectors of **A** forms a basis for the column space of **A** iff. the corresponding column vectors of **B** form a basis for the column space of **B**.

Remarks:
Solving $\mathbf{Ax} = \mathbf{b}$ amounts to solving $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \ldots + x_n\mathbf{c}_n = \mathbf{b}$. so, we see that for the system to have a solution, **b** must be in the span of $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$. Thus, a system of linear equations $\mathbf{Ax} = \mathbf{b}$ is consistent iff. **b** is in the column space of **A**.

Theorem: (Relationship between Nonhomogeneous Linear Systems and Homogeneous Systems)
If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ form the basis of the nullspace of **A**, that is form the solution space of the system $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x}_0$ is a solution of the inhomogeneous system $\mathbf{Ax} = \mathbf{b}$, that is $\mathbf{Ax}_0 = \mathbf{b}$ then every solution **x** of $\mathbf{Ax} = \mathbf{b}$ has the form $\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r$.

7B.2.7 Rank and Nullity

Definition:
The rank of **A**, denoted *rank*(**A**), is the dimension of the row and column space of **A**.
The nullity of **A**, denoted *nullity*(**A**), is the dimension of the nullspace of **A**.

Theorem: (Dimension Theorem for matrices)
If **A** is a matrix with *n* columns, then

$$rank(\mathbf{A}) + nullity(\mathbf{A}) = n$$

Example 7B.2.8:

1. The following nonhomogeneous systems can be written in the form of **Ax** = **b**. Show that every solution **x** to these nonhomogeneous systems can be written as $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution of **Ax** = **b** and $\mathbf{x}_h$ is a solution associated to homogeneous system **Ax** = **0**.

a)
$$\begin{aligned} x + y + 2z &= 5 \\ x + z &= -2 \\ 2x + y + 3z &= 3 \end{aligned} \qquad \begin{aligned} x + y + 2z &= 0 \\ x + z &= 0 \\ 2x + y + 3z &= 0 \end{aligned}$$

b)
$$\begin{aligned} x - 2y + z + 2w &= -1 \\ 2x - 4y + 2z + 4w &= -2 \\ -x + 2y - z - 2w &= 1 \\ 3x - 6y + 3z + 6w &= -3 \end{aligned} \qquad \begin{aligned} x - 2y + z + 2w &= 0 \\ 2x - 4y + 2z + 4w &= 0 \\ -x + 2y - z - 2w &= 0 \\ 3x - 6y + 3z + 6w &= 0 \end{aligned}$$

c)
$$\begin{aligned} x + 2y - 3z + w &= 4 \\ -2x + y + 2z + w &= -1 \\ -x + 3y - z + 2w &= 3 \\ 4x - 7y - 5w &= -5 \end{aligned} \qquad \begin{aligned} x + 2y - 3z + w &= 0 \\ -2x + y + 2z + w &= 0 \\ -x + 3y - z + 2w &= 0 \\ 4x - 7y - 5w &= 0 \end{aligned}$$

2. For the following matrices, do the following:
   (i) Find bases for the row and column space.
   (ii) Find a basis for the nullspace.
   (iii) Determine its rank and nullity; then verify that the values obtained satisfy the Dimension Theorem.

a) $\begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{pmatrix}$

e) $\begin{pmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{pmatrix}$

3. Discuss how the rank of **A** varies with t.

a) $\mathbf{A} = \begin{pmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{pmatrix}$ b) $\mathbf{A} = \begin{pmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{pmatrix}$

Solution:

1a) $$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & 3 \end{array}\right) \xrightarrow[-2r_1+r_3]{-r_1+r_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & -1 & -1 & -7 \\ 0 & -1 & -1 & -7 \end{array}\right) \xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & -1 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = t, y = 7 - t$

$x + 7 - t + 2t = 5$

$x = -2 - t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-t \\ 7-t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

b) $$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 2 & -4 & 2 & 4 & -2 \\ -1 & 2 & -1 & -2 & 1 \\ 3 & -6 & 3 & 6 & -3 \end{array}\right) \xrightarrow[\substack{r_1+r_3 \\ -3r_1+r_4}]{-2r_1+r_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$w = t, z = s, y = r$

$x = 2r - s - 2t - 1$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2r-s-2t-1 \\ r \\ s \\ t \end{pmatrix} = r\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = r\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

c) $$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ -2 & 1 & 2 & 1 & -1 \\ -1 & 3 & -1 & 2 & 3 \\ 4 & -7 & 0 & -5 & -5 \end{array}\right) \xrightarrow[\substack{r_1+r_3 \\ -4r_1+r_4}]{2r_1+r_2} \left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & -15 & 12 & -9 & -21 \end{array}\right) \xrightarrow[3r_2+r_4]{-r_2+r_3} \left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$w = s, z = t$

$y = \frac{7}{5} + \frac{4t}{5} - \frac{3s}{5}$

$x + 2(\frac{7}{5} + \frac{4t}{5} - \frac{3s}{5}) - 3t + s = 4$

$x + \frac{14}{5} + \frac{8t}{5} - \frac{6s}{5} - 3t + s = 4$

$x = \frac{6}{5} + \frac{7t}{5} + \frac{s}{5}$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{6}{5} + \frac{7t}{5} + \frac{s}{5} \\ \frac{7}{5} + \frac{4t}{5} - \frac{3s}{5} \\ t \\ s \end{pmatrix} = \begin{pmatrix} \frac{6}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t\begin{pmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{pmatrix}$$

2a) $$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array}\right) \xrightarrow[-7r_1+r_3]{-5r_1+r_2} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 1 & -19 & 0 \end{array}\right) \xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = t, y = 19t, x = 16t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16t \\ 19t \\ t \end{pmatrix} = t\begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{pmatrix}$$

$r_1 = (1, -1, 3)$
$r_2 = (0, 1, -19)$

$c_1 = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \quad c_2 = \begin{pmatrix} -1 \\ -4 \\ -6 \end{pmatrix}$

rank (A) = 2
nullity (A) = 1

b) $$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-2r_1+r_2} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{r_1}{2}} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = 2t, y = s, x = t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ s \\ 2t \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad r_1 = \begin{pmatrix} 1, & 0, & -\frac{1}{2} \end{pmatrix} \qquad c_1 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

rank (A) = 1
nullity (A) = 2

c)

$$\left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right) \xrightarrow[r_1+r_3]{-2r_1+r_2} \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{array}\right) \xrightarrow{r_2+r_3} \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-1\times r_2} \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 7 & 7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$w = 7s, z = t, y = -t - 4s$

$7y + 7t + 28s = 0$

$x - 4t - 16s + 5t + 14s = 0$

$x = 2s - t$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2s - t \\ -t - 4s \\ t \\ 7s \end{pmatrix} = t\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ -4 \\ 0 \\ 7 \end{pmatrix}$$

$$\xrightarrow{\frac{r_1}{7}} \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} r_1 &= \begin{pmatrix} 1, & 4, & 5, & 2 \end{pmatrix} \\ r_2 &= \begin{pmatrix} 0, & 1, & 1, & \frac{4}{7} \end{pmatrix} \end{aligned}$$

$$c_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

rank (A) = 2
nullity (A) = 2

d)
$$\left(\begin{array}{ccccc|c} 1 & 4 & 5 & 6 & 9 & 0 \\ 3 & -2 & 1 & 4 & -1 & 0 \\ -1 & 0 & -1 & -2 & -1 & 0 \\ 2 & 3 & 5 & 7 & 8 & 0 \end{array}\right) \xrightarrow[\substack{r_1+r_3 \\ -2r_1+r_4}]{-3r_1+r_2} \left(\begin{array}{ccccc|c} 1 & 4 & 5 & 6 & 9 & 0 \\ 0 & -14 & -14 & -14 & -28 & 0 \\ 0 & 4 & 4 & 4 & 8 & 0 \\ 0 & -5 & -5 & -5 & -10 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-14}, \frac{r_3}{4}, \frac{r_4}{-5}} \left(\begin{array}{ccccc|c} 1 & 4 & 5 & 6 & 9 & 0 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow[-r_2+r_4]{-r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 4 & 5 & 6 & 9 & 0 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$u = r, v = s, z = t$

$y + t + r + 2s = 0$

$y = -t - r - 2s$

$x - 4t - 4r - 8s + 5t + 6r + 9s = 0$

$x + t + s + 2r = 0$

$x = -t - s - 2r$

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} -t-2r-s \\ -t-r-2s \\ t \\ r \\ s \end{pmatrix} = t\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{r}_1 = (1, 4, 5, 6, 9)$
$\mathbf{r}_2 = (0, 1, 1, 1, 2)$

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \end{pmatrix}$$

rank (A) = 2
nullity (A) = 3

e)
$$\left(\begin{array}{ccccc|c} 1 & -3 & 2 & 2 & 1 & 0 \\ 0 & 3 & 6 & 0 & -3 & 0 \\ 2 & -3 & -2 & 4 & 4 & 0 \\ 3 & -6 & 0 & 6 & 5 & 0 \\ -2 & 9 & 2 & -4 & -5 & 0 \end{array}\right) \xrightarrow[\substack{-3r_1+r_4 \\ 2r_1+r_5}]{-2r_1+r_3} \left(\begin{array}{ccccc|c} 1 & -3 & 2 & 2 & 1 & 0 \\ 0 & 3 & 6 & 0 & -3 & 0 \\ 0 & 3 & -6 & 0 & 2 & 0 \\ 0 & 3 & -6 & 0 & 2 & 0 \\ 0 & 3 & 6 & 0 & -3 & 0 \end{array}\right)$$

$$\xrightarrow[-r_2+r_5]{\substack{-r_2+r_3\\-r_2+r_4}} \left(\begin{array}{ccccc|c} 1 & -3 & 2 & 2 & 1 & 0\\ 0 & 3 & 6 & 0 & -3 & 0\\ 0 & 0 & -12 & 0 & 5 & 0\\ 0 & 0 & -12 & 0 & 5 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[-r_3+r_4]{\frac{r_2}{3},} \left(\begin{array}{ccccc|c} 1 & -3 & 2 & 2 & 1 & 0\\ 0 & 1 & 2 & 0 & -1 & 0\\ 0 & 0 & -12 & 0 & 5 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$u = s, v = 12t, z = 5t$$

$$y = 2t, x = 6t - 10t - 2s - 12t = -16t - 2s$$

$$\begin{pmatrix} x\\ y\\ z\\ u\\ v \end{pmatrix} = \begin{pmatrix} -16t-2s\\ 2t\\ 5t\\ s\\ 12t \end{pmatrix} = t\begin{pmatrix} -16\\ 2\\ 5\\ 0\\ 12 \end{pmatrix} + s\begin{pmatrix} -2\\ 0\\ 0\\ 1\\ 0 \end{pmatrix}$$

$$\xrightarrow{\frac{r_3}{-12}} \left(\begin{array}{ccccc|c} 1 & -3 & 2 & 2 & 1 & 0\\ 0 & 1 & 2 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & -\frac{5}{12} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & -3 & 2 & 2 & 1\\ 0 & 1 & 2 & 0 & -1\\ 0 & 0 & 1 & 0 & -\frac{5}{12}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \mathbf{r}_1 = (1, \ -3, \ 2, \ 2, \ 1)\\ \mathbf{r}_2 = (0, \ 1, \ 2, \ 0, \ -1)\\ \mathbf{r}_3 = \left(0, \ 0, \ 1, \ 0, \ -\frac{5}{12}\right) \end{array}$$

$$\mathbf{c}_1 = \begin{pmatrix} 1\\ 0\\ 2\\ 3\\ -2 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} -3\\ 3\\ -3\\ -6\\ 9 \end{pmatrix}, \quad \mathbf{c}_3 = \begin{pmatrix} 2\\ 6\\ -2\\ 0\\ 2 \end{pmatrix}$$

rank (A) = 3
nullity (A) = 2

3a)

$$\xrightarrow[-tr_1+r_3]{-r_1+r_2}\begin{pmatrix}1 & 1 & t\\ 0 & t-1 & 1-t\\ 0 & 1-t & 1-t^2\end{pmatrix}\xrightarrow{r_2+r_3}\begin{pmatrix}1 & 1 & t\\ 0 & t-1 & 1-t\\ 0 & 0 & 2-t-t^2\end{pmatrix}$$

$$2-t-t^2=0$$
$$t^2+t-2=0$$
$$(t+2)(t-1)=0$$
$$t=1,-2$$

$$t=1:\begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$ rank (A) = 1

$$t=-2:\begin{pmatrix}1 & 1 & -2\\ 0 & -3 & 3\\ 0 & 0 & 0\end{pmatrix}$$ rank (A) = 2

$$\xrightarrow{\frac{r_2}{-3}}\begin{pmatrix}1 & 1 & -2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}$$

$$t\neq 1,-2:\begin{pmatrix}1 & 1 & t\\ 0 & t-1 & 1-t\\ 0 & 0 & 2-t-t^2\end{pmatrix}$$ rank (A) = 3

$$\xrightarrow{\frac{r_2}{t-1},\frac{r_3}{2-t-t^2}}\begin{pmatrix}1 & 1 & t\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$$

b)

$$\xrightarrow{r_1\Leftrightarrow r_3}\begin{pmatrix}-1 & -3 & t\\ 3 & 6 & -2\\ t & 3 & -1\end{pmatrix}\xrightarrow[tr_1+r_3]{3r_1+r_2}\begin{pmatrix}-1 & -3 & t\\ 0 & -3 & 3t-2\\ 0 & 3-3t & t^2-1\end{pmatrix}$$

$$\xrightarrow[(1-t)r_2+r_3]{-r_1,}\begin{pmatrix}1 & 3 & -t\\ 0 & -3 & 3t-2\\ 0 & 0 & (3t-2)(1-t)+t^2-1\end{pmatrix}\xrightarrow{\frac{r_2}{-3}}\begin{pmatrix}1 & 3 & -t\\ 0 & 1 & \frac{2}{3}-t\\ 0 & 0 & (t-1)(3-2t)\end{pmatrix}$$

$$(t-1)(3-2t)=0$$

$$t=1,\frac{3}{2}$$

$$\begin{pmatrix} 1 & 3 & -t \\ 0 & -3 & 3t-2 \\ 0 & 0 & (t-1)(3-2t) \end{pmatrix}$$

$$t=1:\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

rank (A) = 2

$$t=\frac{3}{2}:\begin{pmatrix} 1 & 3 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{pmatrix}$$

rank (A) = 2

$$t\neq 1,\frac{3}{2}:\begin{pmatrix} 1 & 3 & -t \\ 0 & 1 & \frac{2}{3}-t \\ 0 & 0 & (t-1)(3-2t) \end{pmatrix}$$

rank (A) = 3

$$\xrightarrow{\frac{r_3}{(t-1)(3-2t)}}\begin{pmatrix} 1 & 3 & -t \\ 0 & 1 & \frac{2}{3}-t \\ 0 & 0 & 1 \end{pmatrix}$$

7C. Eigenvalues, Eigenvectors and Diagonalization

Backgrounds:
If **A** is an *n* x *n* matrix and **x** is a vector in $\mathbf{R}^n$, then **Ax** is also a vector in $\mathbf{R}^n$, but usually there is no simple geometric relationship between **x** and **Ax**. However, when **Ax** is a scalar multiple of **x** (where **x** is a nonzero vector), a simple geometric relationship occurs. For example, if **A** is a 2 x 2 matrix, $\mathbf{Ax} = \lambda\mathbf{x}$, then each vector on the line through the origin determined by **x** gets mapped back onto the same line under multiplication by **A**.

Nonzero vectors that get mapped into scalar multiples of themselves under a linear operator arise naturally in the study of vibrations, genetics, population dynamics, quantum mechanics, and economics, as well as in geometry. In this section we will study such vectors.

7C.1 Eigenvalues and Eigenvectors

Definition:
If **A** is an *n* x *n* matrix, then the nonzero vector **x** in $\mathbf{R}^n$ is called an eigenvector of **A** if **Ax** is a scalar multiple of **x**; that is, if

$$\mathbf{Ax} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue/characteristic value** of **A**, and **x** is said to be an eigenvector of **A** corresponding to λ.

To find the eigenvalues of an *n* x *n* matrix **A**, we do the following:

$$\mathbf{Ax} = \lambda\mathbf{x}$$
$$\mathbf{Ax} = \lambda\mathbf{Ix}$$
$$\lambda\mathbf{Ix} - \mathbf{Ax} = \mathbf{0}$$
$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$$

For λ to be an eigenvalue, there must be a nonzero solution of this equation. That is,

$$|\lambda\mathbf{I} - \mathbf{A}| = 0$$

This is called the ***characteristic equation/characteristic polynomial*** of **A** and its solution is called the ***eigenvalue/characteristic value*** of **A**.

To find eigenvectors, we solve $\mathbf{Ax} = \lambda\mathbf{x}$ for each value of λ.

NOTE:
We can also **find** a corresponding **eigenvector** by using the ***cross product*** of any ***two linearly independent row vectors*** in the matrix $\lambda\mathbf{I} - \mathbf{A}$ after substitute an eigenvalue.

Example 7C.1.1:

Find the eigenvalues and the corresponding eigenvectors of the following matrices.

a) $\mathbf{A} = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}$

b) $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 2 & -1 \end{pmatrix}$

c) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

d) $\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}$

e) $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

f) $\mathbf{A} = \begin{pmatrix} 5 & -1 & 1 \\ -1 & 1 & -3 \\ 1 & -3 & 1 \end{pmatrix}$

g) $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution:

7C.2 Diagonalization

Background:

In this section we shall be concerned with the problem of finding a basis for $\mathbf{R}^n$ consists of eigenvectors of a given n x n matrix **A**. Such bases can be used to study geometric properties of **A** and to simplify various numerical computations involving **A**. These bases are also significance in a wide variety of applications.

Definition:

A square matrix **A** is said to be diagonalizable if there is an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix; the matrix **P** is said to diagonalize **A**.

Theorem:

If **A** is an n x n matrix, then the following are equivalent.

a) A is diagonalizable
b) A has n linearly independent eigenvectors.

Procedure for Diagonalizing a Matrix:

Step 1 Find n linearly independent eigenvectors of **A**, say $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$.

Step 2 Form the matrix **P** having $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ as its column vectors.

Step 3 The matrix $\mathbf{P}^{-1}\mathbf{AP}$ will then be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to $\mathbf{p}_i$, for $i = 1, 2, \ldots, n$.

NOTES:

- If there is a total of n basis vectors, then **A** is diagonalizable, and the n basis vectors can be used to form the diagonalizing matrix **P**.
- If there are fewer than n basis vectors, then **A** is not diagonalizable.

Computing Powers of a Matrix:

Diagonalization can be used to simplify the computation of high power of a diagonalizable square matrix.

If **A** is an n x n diagonalizable matrix, **P** is an invertible matrix, and $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ is a diagonal matrix, then

$\mathbf{P}^{-1}\mathbf{A}^k\mathbf{P} = (\mathbf{P}^{-1}\mathbf{AP})^k = \mathbf{D}^k$ for every positive integer k.

$\mathbf{PP}^{-1}\mathbf{A}^k\mathbf{P} = \mathbf{PD}^k$

$\mathbf{IA}^k\mathbf{P} = \mathbf{PD}^k$

$\mathbf{A}^k\mathbf{P} = \mathbf{PD}^k$

$\mathbf{A}^k\mathbf{PP}^{-1} = \mathbf{PD}^k\mathbf{P}^{-1}$

$\mathbf{A}^k\mathbf{I} = \mathbf{PD}^k\mathbf{P}^{-1}$

$\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$

This last equation expresses the k^{th} power of **A** in terms of the k^{th} power of the diagonal matrix **D**. But $\mathbf{D}^k$ is easy to compute, for if

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}, \text{ then } \quad \mathbf{D}^k = \begin{pmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{pmatrix}.$$

Example 7C.2.1:

For the following matrices, determine whether A is diagonalizable. If so, find a matrix **P** that diagonalizes **A**, and determine $\mathbf{P}^{-1}\mathbf{AP}$.

a) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$

e) $\mathbf{A} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

b) $\mathbf{A} = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$

f) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

c) $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

d) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

g) $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

h) $\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix}$

Example 7C.2.2:

Compute $\mathbf{D}^9$ and $\mathbf{A}^{10}$ for the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$.

Example 7C.2.3:

Compute $\mathbf{D}^8$ and $\mathbf{A}^{11}$ for the matrix $\mathbf{A} = \begin{pmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{pmatrix}$.

Example 7C.2.4:

Find $\mathbf{A}^n$ if n is a positive integer and $\mathbf{A} = \begin{pmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{pmatrix}$.

References:

1. Elementary Linear Algebra, 7^{th} Edition
   Howard Anton
   John Wiley & Sons, Inc.