Topic : **P103 Mathematical Induction**

Subtopic:

- *Principle of Mathematical Induction*
- *Inductive proof involving summations*
- *Inductive proof involving recurrence relations*
- *Inductive proof involving divisibility*
- *Inductive proof involving inequalities*
- *Inductive proof involving derivatives*

Principle of Mathematical Induction

Theorem:

If a subset M of N satisies the conditions

$\underline{1 \in M \text{ and } n \in M \text{ implies } n+1 \in M}$, then

necessarily $M = N$.

Proving by Mathematical Induction

Induction can frequently be used to solve problems if the problem solution involves establishing that a certain result is true for all positive integers. First show that the result holds for the positive integer 1. Then show that the result holds for $(k+1)$ under the assumption that it holds for k. The principle of mathematical induction then guarantees the result is valid for all positive integers.

Three steps in proving a statement $P(n)$ by mathematical inductions :

(a) **Basis case**.

$\therefore$ Prove that $P(1)$ is true.

(b) **Inductive step**.

$\therefore$ Prove that the implication $P(k) \Rightarrow P(k+1)$ is true.

(c) **Completion**.

$\therefore$ Combine (a) and (b) to show that $P(2), P(3), \ldots, P(n)$ are true.

Example:

Prove by induction that $1+3+\ldots+(2n-1)=n^2$ is true for all integers $n \geq 1$.

Example:

Prove by induction, for all integers $n \geq 1$, that

$1 \cdot 2 + 2 \cdot 3 + \ldots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$ is true.

Example:

Prove by induction, for all integers $n \geq 1$, that

$$\frac{1\cdot 4}{2\cdot 3}+\frac{2\cdot 5}{3\cdot 4}+\ldots+\frac{n(n+3)}{(n+1)(n+2)}=\frac{n^2+n}{n+2}$$ is true.

Example:

Prove by induction, for all integers $n \geq 1$, that

$$\sum_{r=1}^{n} \left[\left(r^2 + 1 \right) r! \right] = n \left[\left(n + 1 \right)! \right] \text{ is true.}$$

Example:

Let $S(n)=\sum_{r=1}^{n}\frac{1}{r(r+1)}$.

(a) Evaluate $S(1)$, $S(2)$, and $S(3)$.

(b) Conjecture $S(n)$ in terms of n.

(c) Prove your conjecture by mathematical induction.

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example:

Prove by induction, for all integers $n \geq 1$, that

$$\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3} n\left(4n^2 - 1\right) \text{ is true.}$$

Example:

Prove by induction, for all integers $n \geq 1$, that

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!} \text{ is true.}$$

Example:

Prove by induction, for all integers $n \geq 1$, that

$$\sum_{r=1}^{n} (k+1)2^{k-1} = n2^n \text{ is true.}$$

Example :

Prove by induction , for all integers $n \geq 1$, that

$$\sum_{r=1}^{n} \frac{r\left(2^{r}\right)}{(r+1)(r+2)} = \frac{2^{n+1}}{n+2} - 1 \text{ is true.}$$

Example:

Prove that

$$\sin(A+B)-\sin(A-B)\equiv 2\cos A\sin B,\ \forall A,B\in \mathrm{R}.$$

Hence, or otherwise, prove by induction, for all integers

$n\geq 1$, that $\displaystyle\sum_{r=1}^{n}\sin rx=\frac{\sin\left[\frac{1}{2}(n+1)x\right]\sin\left(\frac{1}{2}nx\right)}{\sin\left(\frac{1}{2}x\right)}$ is true.