# MAY / JUNE 2009

1. $x^4 - x^3 - 1 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots.

$\alpha^3, \beta^3, \gamma^3, \delta^3$

Let $y = \alpha^3$

$\alpha = y^{\frac{1}{3}}$

$\alpha$ is a root

$\alpha^4 - \alpha^3 - 1 = 0$

$(y^{\frac{1}{3}})^4 - (y^{\frac{1}{3}})^3 - 1 = 0$

$y^{\frac{4}{3}} - y - 1 = 0$

$y^{\frac{1}{3}} y = y + 1$

$(y^{\frac{1}{3}} y)^3 = (y+1)^3$

$y y^3 = y^3 + 3y^2 + 3y + 1$

$y^4 = y^3 + 3y^2 + 3y + 1$

$y^4 - y^3 - 3y^2 - 3y - 1 = 0$

The equation $y^4 - y^3 - 3y^2 - 3y - 1 = 0$ has roots $\alpha^3, \beta^3, \gamma^3, \delta^3$.

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 1$$

$$\alpha^3\beta^3 + \alpha^3\gamma^3 + \alpha^3\delta^3 + \beta^3\gamma^3 + \beta^3\delta^3 + \gamma^3\delta^3 = -3$$

$$\alpha^3\beta^3\gamma^3 + \alpha^3\beta^3\delta^3 + \alpha^3\gamma^3\delta^3 + \beta^3\gamma^3\delta^3 = 3$$

$$\alpha^3\beta^3\gamma^3\delta^3 = -1$$

$$\alpha^6 + \beta^6 + \gamma^6 + \delta^6$$

$$= (\alpha^3 + \beta^3 + \gamma^3 + \delta^3)^2$$

$$-2(\alpha^3\beta^3 + \alpha^3\gamma^3 + \alpha^3\delta^3 + \beta^3\gamma^3 + \beta^3\delta^3 + \gamma^3\delta^3)$$

$$= 1^2 - 2(-3)$$

$$= 1 + 6$$

$$= 7$$

2. $$\frac{1}{(n+2)(2n+3)} - \frac{1}{(n+3)(2n+5)}$$

$$= \frac{(n+3)(2n+5) - (n+2)(2n+3)}{(n+2)(n+3)(2n+3)(2n+5)}$$

$$= \frac{2n^2 + 11n + 15 - (2n^2 + 7n + 6)}{(n+2)(n+3)(2n+3)(2n+5)}$$

$$= \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}$$

1) $$\sum_{n=1}^{N} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}$$

$$= \sum_{n=1}^{N} \frac{1}{(n+2)(2n+3)} - \frac{1}{(n+3)(2n+5)}$$

$$= \frac{1}{3\cdot 5} - \frac{1}{4\cdot 7}$$

$$+ \frac{1}{4\cdot 7} - \frac{1}{5\cdot 9}$$

$$+ \frac{1}{5\cdot 9} - \frac{1}{6\cdot 11}$$

$$\vdots$$

$$+ \frac{1}{N(2N-1)} - \frac{1}{(N+1)(2N+1)}$$

$$+ \frac{1}{(N+1)(2N+1)} - \frac{1}{(N+2)(2N+3)}$$

$$+ \frac{1}{(N+2)(2N+3)} - \frac{1}{(N+3)(2N+5)}$$

$$= \frac{1}{15} - \frac{1}{(N+3)(2N+5)}$$

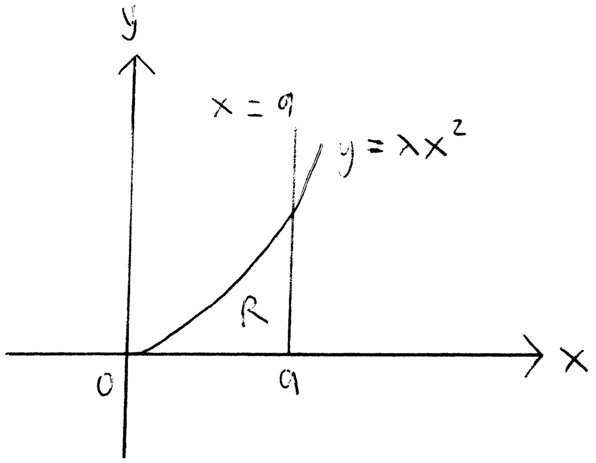
ii) $$\sum_{n=1}^{\infty} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}$$

$$= \lim_{N \to \infty} \left( \frac{1}{15} - \frac{1}{(N+3)(2N+5)} \right)$$

$$= \frac{1}{15}$$

3. $y = \lambda x^2, \; \lambda > 0$




The area of R, A, is $\displaystyle\int_0^a y \, dx$

$$= \int_0^a \lambda x^2 \, dx$$

$$= \left[ \frac{\lambda x^3}{3} \right]_0^a$$

$$= \frac{\lambda a^3}{3}$$

If the y-coordinate of the centroid of R is a,

$$a = \frac{\displaystyle\int_0^a \frac{y^2}{2} \, dx}{A}$$

$$= \frac{\displaystyle\int_0^a \frac{\lambda^2 x^4}{2} \, dx}{\frac{\lambda a^3}{3}}$$

$$\frac{\lambda a^4}{3} = \left[ \frac{\lambda^2 x^5}{10} \right]_0^a$$

$$= \frac{\lambda^2 a^5}{10}$$

$$\lambda = \frac{10}{3a}$$

4. $y = \frac{x^3}{3} + 1$

If the arc length of the curve from $x = 0$ to $x = 1$ is $s$,

$$s = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + (x^2)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + x^4}\, dx$$

If the surface area generated when the arc is rotated through one complete revolution about the x-axis is $S$,

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 2\pi \left(\frac{x^3}{3} + 1\right) \sqrt{1 + (x^2)^2}\, dx$$

$$= \int_0^1 2\pi \left(\frac{x^3}{3} + 1\right) \sqrt{1 + x^4}\, dx$$

$$= \int_0^1 \frac{2\pi x^3}{3} \sqrt{1 + x^4} + 2\pi \sqrt{1 + x^4}\, dx$$

$$= \int_0^1 \frac{2\pi x^3}{3} \sqrt{1 + x^4}\, dx + \int_0^1 2\pi \sqrt{1 + x^4}\, dx$$

$$= \frac{2\pi}{3} \int_0^1 x^3 \sqrt{1 + x^4}\, dx + 2\pi \int_0^1 \sqrt{1 + x^4}\, dx$$

$$= \frac{2\pi}{3}\int_0^1 x^3\sqrt{1+x^4}\,dx + 2\pi S$$

$$u = x^4$$
$$du = 4x^3\,dx$$
$$x = 0 \quad u = 0$$
$$x = 1 \quad u = 1$$

$$= \frac{2\pi}{3}\int_0^1 \frac{\sqrt{1+u}}{4}\,du + 2\pi S$$

$$= \frac{2\pi}{3}\left[\frac{2}{3}\frac{(1+u)^{\frac{3}{2}}}{4}\right]_0^1 + 2\pi S$$

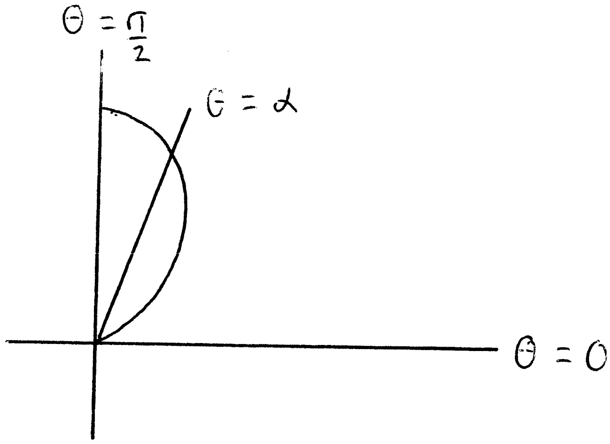
$$= \frac{2\pi}{3}\left(\frac{2\sqrt{2}-1}{6}\right) + 2\pi S$$

$$= \frac{(2\sqrt{2}-1)\pi}{9} + 2\pi S$$

$$= \frac{\pi}{9}(18S + 2\sqrt{2} - 1)$$

5. $C: r = \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $r$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |




If the line $\theta = \alpha$ divides $R$ into two regions of equal area,

$$\int_0^\alpha \frac{r^2}{2}\,d\theta = \int_\alpha^{\frac{\pi}{2}} \frac{r^2}{2}\,d\theta$$

$$\int_0^\alpha \frac{\theta^2}{2}\,d\theta = \int_\alpha^{\frac{\pi}{2}} \frac{\theta^2}{2}\,d\theta$$

$$\left[\frac{\theta^3}{6}\right]_0^\alpha = \left[\frac{\theta^3}{6}\right]_\alpha^{\frac{\pi}{2}}$$

$$\alpha^3 - 0^3 = \left(\frac{\pi}{2}\right)^3 - \alpha^3$$

$$2\alpha^3 = \frac{\pi^3}{8}$$

$$\alpha^3 = \frac{\pi^3}{16}$$

$$\alpha = \frac{\pi}{16^{\frac{1}{3}}}$$

6.

$$(x + y)(x^2 + y^2) = 1$$

$$\frac{d}{dx}\left[(x + y)(x^2 + y^2)\right] = \frac{d}{dx}(1)$$

$$(x + y)\frac{d}{dx}(x^2 + y^2) + (x^2 + y^2)\frac{d}{dx}(x + y) = 0$$

$$(x + y)(2x + 2y\frac{dy}{dx}) + (x^2 + y^2)(1 + \frac{dy}{dx}) = 0$$

At $(0,1)$: $1(2\frac{dy}{dx}) + 1(1 + \frac{dy}{dx}) = 0$

$$2\frac{dy}{dx} + 1 + \frac{dy}{dx} = 0$$

$$3\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{-1}{3}$$

$$\frac{d}{dx}\left[(x + y)(2x + 2y\frac{dy}{dx}) + (x^2 + y^2)(1 + \frac{dy}{dx})\right] = 0$$

$$\frac{d}{dx}\left[(x + y)(2x + 2y\frac{dy}{dx})\right] + \frac{d}{dx}\left[(x^2 + y^2)(1 + \frac{dy}{dx})\right] = 0$$

$$(x + y)\frac{d}{dx}(2x + 2y\frac{dy}{dx}) + (2x + 2y\frac{dy}{dx})\frac{d}{dx}(x + y)$$

$$+ (x^2 + y^2)\frac{d}{dx}(1 + \frac{dy}{dx}) + (1 + \frac{dy}{dx})\frac{d}{dx}(x^2 + y^2) = 0$$

$$(x+y)\left(2+2\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}\right)+\left(2x+2y\frac{dy}{dx}\right)\left(1+\frac{dy}{dx}\right)$$

$$+(x^2+y^2)\frac{d^2y}{dx^2}+\left(1+\frac{dy}{dx}\right)\left(2x+2y\frac{dy}{dx}\right)=0$$

At $(0,1)$, $\frac{dy}{dx}=\frac{-1}{3}$

$$1\left(2+2\left(\frac{1}{9}\right)+\frac{2d^2y}{dx^2}\right)+2\left(-\frac{1}{3}\right)\left(1-\frac{1}{3}\right)$$

$$+1\frac{d^2y}{dx^2}+\left(1-\frac{1}{3}\right)2\left(-\frac{1}{3}\right)=0$$

$$2+\frac{2}{9}+\frac{2d^2y}{dx^2}-\frac{4}{9}+\frac{d^2y}{dx^2}-\frac{4}{9}=0$$

$$\frac{3d^2y}{dx^2}=-\frac{4}{3}$$

$$\frac{d^2y}{dx^2}=\frac{-4}{9}$$

7 $I_n = \int_0^1 t^n e^{-t}\,dt, \quad n \geqslant 0$

$$u = t^n \qquad dv = e^{-t}\,dt$$
$$du = nt^{n-1} \qquad v = -e^{-t}$$

$$= \left[-t^n e^{-t}\right]_0^1 - \int_0^1 -nt^{n-1}e^{-t}\,dt$$

$$= -e^{-1} - 0 + n\int_0^1 t^{n-1}e^{-t}\,dt$$

$$= -e^{-1} + nI_{n-1}$$

$$= nI_{n-1} - e^{-1}, \quad n \geqslant 1.$$

$I_n < n!$

when $n = 1$ $\quad I_1 = 1I_0 - e^{-1}$

$$= I_0 - e^{-1}$$

$$I_0 = \int_0^1 e^{-t}\,dt$$

$$= \left[e^{-t}\right]_0^1$$

$$= e^{-1} - 1$$

$$\therefore I_1 = I_0 - e^{-1}$$

$$= -1$$

$$< 1$$

$$= 1!$$

Assume the statement is true when $n = k$.

$n = k$ $I_k < k!$

when $n = k + 1$: $I_{k+1} < (k+1)!$

(what needs to be proved)

$I_{k+1} = (k+1) I_k - e^{-1}$

$I_k < k!$

Since $k > 0$ $I_k < k! + \dfrac{1}{e(k+1)}$

$(k+1) I_k < (k+1) k! + e^{-1}$

$(k+1) I_k - e^{-1} < (k+1)!$

$I_{k+1} < (k+1)!$

$\therefore$ $I_n < n!$ for every positive integer $n$.

8. $$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 65y = 65x^2 + 8x + 73$$

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 65y = 0$$

$$4m^2 + 4m + 65 = 0$$

$$(2m+1)^2 + 64 = 0$$

$$(2m+1)^2 = -64$$

$$2m+1 = \pm 8i$$

$$m = -\frac{1}{2} \pm 4i$$

The complementary function, $y_c$, is

$$y_c = e^{-\frac{x}{2}}(A\cos 4x + B\sin 4x)$$

The particular integral, $y_p$, is given by

$$y_p = Cx^2 + Dx + E$$

$$\frac{dy_p}{dx} = 2Cx + D$$

$$\frac{d^2y_p}{dx^2} = 2C$$

$$4\frac{d^2y_p}{dx^2} + 4\frac{dy_p}{dx} + 65y_p = 4(2C) + 4(2Cx + D) + 65(Cx^2 + Dx + E)$$

$= 65Cx^2 + (8C + 65D)x + (8C + 4D + 65E)$

$= 65x^2 + 8x + 73$

$65C = 65 \quad 8C + 65D = 8 \quad 8C + 4D + 65E = 73$

$C = 1$

$D = 0$

$E = 1$

$\therefore y_p = x^2 + 1$

$y = y_c + y_p$

$= e^{-\frac{x}{2}}(A\cos 4x + B\sin 4x) + x^2 + 1$

The general solution is

$y = e^{-\frac{x}{2}}(A\cos 4x + B\sin 4x) + x^2 + 1$

$$\frac{y}{x^2} = \frac{e^{-\frac{x}{2}}(A\cos 4x + B\sin 4x)}{x^2} + 1 + \frac{1}{x^2}$$

As $x \longrightarrow \infty$,

since $-1 < \cos 4x < 1$ and $-1 < \sin 4x < 1$,

$\frac{\cos 4x}{x^2 e^{\frac{x}{2}}} \to 0$ and $\frac{\sin x}{x^2 e^{\frac{x}{2}}} \to 1$.

$\frac{y}{x^2} \longrightarrow 1$

9. $A = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & -1 \\ 2 & 1 & 5 \end{pmatrix}$

If $1, 5, 7$ are the eigenvalues of $A$,

since $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

$$\begin{pmatrix} 2 & 1 & 4 \\ 1 & 4 & -1 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{-2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -7 & 6 & 0 \\ 0 & -7 & 6 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -7 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 7s,\ s \in R$

$y = 6s$

$x + 4(6s) - 7s = 0$

$x + 24s - 7s = 0$

$x = -17s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -17s \\ 6s \\ 7s \end{pmatrix}$$

$$= s\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 4 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -2 & 1 & 4 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s,\ s \in R$

$y = -2s$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -2s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 1 & 4 \\ 1 & -2 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -4 & 1 & 4 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -4 & 1 & 4 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{4r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-7},\ \frac{r_3}{5}} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s,\ s \in R$

$x = s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ If the eigenvalues of $A$ are $1, 5, 7$, the corresponding eigenvectors are

$$\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

If $P$ is a square matrix and $D$ is a diagonal matrix such that $A^n = PD^nP^{-1}$,

$$P = \begin{pmatrix} -17 & 1 & 1 \\ 6 & -2 & 0 \\ 7 & 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}.$$

$$A^n = \begin{pmatrix} -17 & 1 & 1 \\ 6 & -2 & 0 \\ 7 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}^n\begin{pmatrix} -17 & 1 & 1 \\ 6 & -2 & 0 \\ 7 & 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -17 & 1 & 1 \\ 6 & -2 & 0 \\ 7 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5^n & 0 \\ 0 & 0 & 7^n \end{pmatrix}\frac{1}{24}\begin{pmatrix} -1 & 0 & 1 \\ -3 & -12 & 3 \\ 10 & 12 & 14 \end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix} -17 & 1 & 1 \\ 6 & -2 & 0 \\ 7 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5^n & 0 \\ 0 & 0 & 7^n \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ -3 & -12 & 3 \\ 10 & 12 & 14 \end{pmatrix}$$

$$k^nA^n = \frac{1}{24}\begin{pmatrix} -17 & 1 & 1 \\ 6 & -2 & 0 \\ 7 & 1 & 1 \end{pmatrix}\begin{pmatrix} k^n & 0 & 0 \\ 0 & (5k)^n & 0 \\ 0 & 0 & (7k)^n \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ -3 & -12 & 3 \\ 10 & 12 & 14 \end{pmatrix}$$

As $n \to \infty$, if $\begin{pmatrix} k^n & 0 & 0 \\ 0 & (5k)^n & 0 \\ 0 & 0 & (7k)^n \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$|k| < 1, \; |5k| < 1, \; |7k| < 1$

$-1 < k < 1, \; -1 < 5k < 1, \; -1 < 7k < 1$

$-\frac{1}{5} < k < \frac{1}{5}, \; -\frac{1}{7} < k < \frac{1}{7}$

$\therefore$ If $-\frac{1}{7} < k < \frac{1}{7}$, $k^nA^n \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as $n \to \infty$

10. $C: \quad y = \dfrac{x^2}{x+\lambda}, \quad \lambda \neq 0.$

$$\begin{array}{r|l} & x - \lambda \\ \hline x+\lambda & x^2 \\ & x^2 + \lambda x \\ \hline & -\lambda x \\ & -\lambda x - \lambda^2 \\ \hline & \lambda^2 \end{array}$$

$$= x - \lambda + \frac{\lambda^2}{x+\lambda}$$

As $x \longrightarrow \pm\infty \quad y \longrightarrow x - \lambda$

As $x \longrightarrow -\lambda \quad y \longrightarrow \pm\infty$

$y = x - \lambda$

$x = -\lambda$

The asymptotes of $C$ are $y = x - \lambda$ and $x = -\lambda$.

when $x = 0 \quad y = 0$

when $y = 0 \quad \dfrac{x^2}{x+\lambda} = 0$

$$x = 0$$

$$\frac{dy}{dx} = 1 - \frac{\lambda^2}{(x+\lambda)^2}$$

When $\dfrac{dy}{dx} = 0 \quad 1 - \dfrac{\lambda^2}{(x+\lambda)^2} = 0$

$$\frac{\lambda^2}{(x+\lambda)^2} = 1$$

$$(x+\lambda)^2 = \lambda^2$$

$$x + \lambda = -\lambda, \lambda$$

$$x = 0, -2\lambda$$

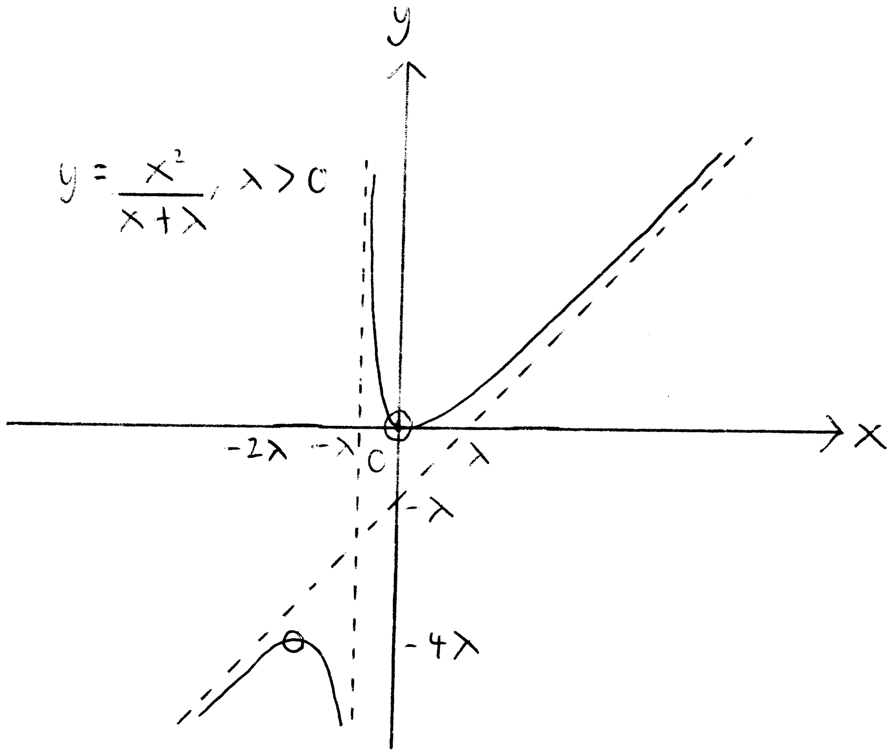
$$y = 0, -4\lambda$$

$$\frac{d^2y}{dx^2} = \frac{2\lambda^2}{(x+\lambda)^3}$$

when $x = 0$: $\frac{d^2y}{dx^2} = \frac{2}{\lambda}$

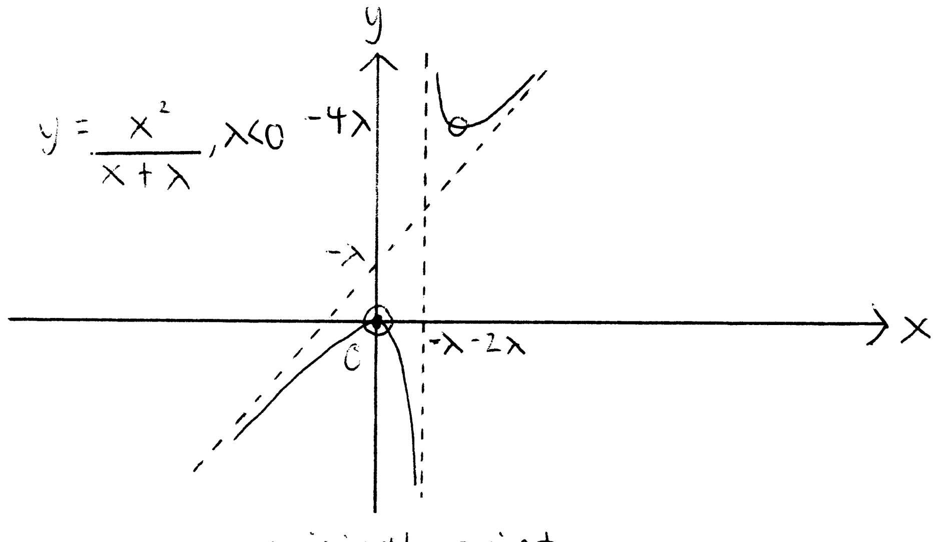
when $x = -2\lambda$: $\frac{d^2y}{dx^2} = \frac{-2}{\lambda}$

if $\lambda > 0$, $(0,0)$ is a minimum point
and $(-2\lambda, -4\lambda)$ is a maximum point



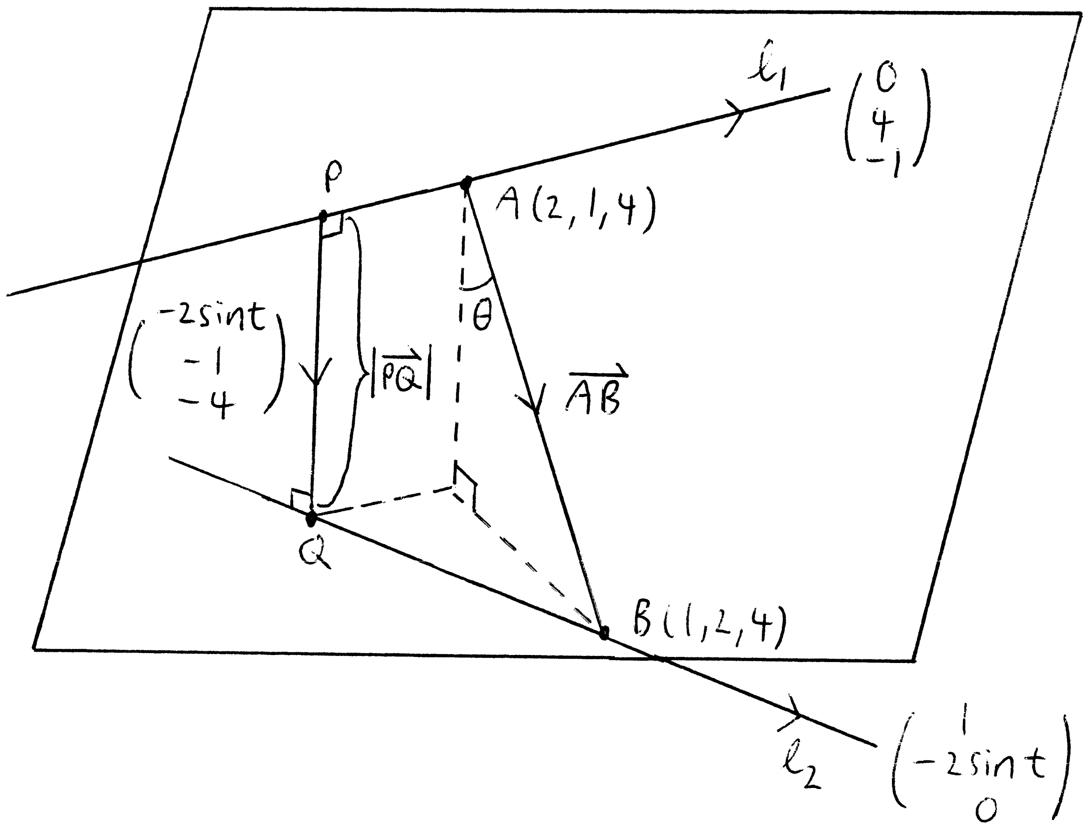

o : critical point
• : intersection point.

If $\lambda < 0$, $(0,0)$ is a maximum point
and $(-2\lambda, -4\lambda)$ is a minimum point.




o : critical point
• intersection point.

11.




$l_1: \underset{\sim}{r} = 2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k} + \lambda(4\underset{\sim}{j} - \underset{\sim}{k})$

$l_2: \underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k} + \mu(\underset{\sim}{i} - (2\sin t)\underset{\sim}{j}), \ 0 \leq t < 2\pi$

i) Since $P$ is on $l_1$ and $Q$ is on $l_2$,

let $P$ be $(2, 1+4\lambda, 4-\lambda)$ and $Q$ be

$(1+\mu, 2-2\mu\sin t, 4)$.

Since $PQ$ is perpendicular to both $l_1$ and $l_2$,

$\vec{PQ} = \begin{pmatrix}-1+\mu\\1-4\lambda-2\mu\sin t\\\lambda\end{pmatrix}$ is parallel to

$\begin{pmatrix}0\\4\\-1\end{pmatrix} \times \begin{pmatrix}1\\-2\sin t\\0\end{pmatrix}$.

$$\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2\sin t \\ 0 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 4 & -1 \\ 1 & -2\sin t & 0 \end{vmatrix} = \begin{pmatrix} -2\sin t \\ -1 \\ -4 \end{pmatrix}$$

$$\overrightarrow{PQ} \parallel \begin{pmatrix} -2\sin t \\ -1 \\ -4 \end{pmatrix}$$

$$\overrightarrow{PQ} = c\begin{pmatrix} -2\sin t \\ -1 \\ -4 \end{pmatrix}, \quad c \in \mathbb{R}$$

$$\begin{pmatrix} -1 + \mu \\ 1 - 4\lambda - 2\mu\sin t \\ \lambda \end{pmatrix} = \begin{pmatrix} -2c\sin t \\ -c \\ -4c \end{pmatrix}$$

$$\left.\begin{aligned} -1 + \mu &= -2c\sin t \\ 1 - 4\lambda - 2\mu\sin t &= -c \\ \lambda &= -4c \end{aligned}\right\}$$

$$\left.\begin{aligned} \mu + 2c\sin t &= 1 \\ 2\mu\sin t + c + 4\lambda &= 1 \\ 4c + \lambda &= 0 \end{aligned}\right\}$$

$-2\sin t \times$ ① $+$ ②:

$$\left.\begin{aligned} \mu + 2c\sin t &= 1 \\ (-4\sin^2 t + 1)c + 4\lambda &= 1 - 2\sin t \\ 4c + \lambda &= 0 \end{aligned}\right\}$$

$(-4\sin^2 t + 1) \times ③$:

$$\left.\begin{aligned} \mu \quad + 2c\sin t \qquad &= 1 \\ (-4\sin^2 t + 1)c + 4\lambda &= 1 - 2\sin t \\ 4(-4\sin^2 t + 1)c + (-4\sin^2 t + 1)\lambda &= 0 \end{aligned}\right\}$$

$-4 \times ② + ③$:

$$\left.\begin{aligned} \mu \quad + 2c\sin t \qquad &= 1 \\ (-4\sin^2 t + 1)c + 4\lambda &= 1 - 2\sin t \\ (-4\sin^2 t - 17)\lambda &= 8\sin t - 4 \end{aligned}\right\}$$

$$(-4\sin^2 t - 17)\lambda = 8\sin t - 4$$

$$\lambda = \frac{4 - 8\sin t}{4\sin^2 t + 17}$$

$$c = \frac{-\lambda}{4}$$

$$= \frac{2\sin t - 1}{4\sin^2 t + 17}$$

$$\mu = 1 - 2c\sin t$$

$$= 1 - 2\sin t\left(\frac{2\sin t - 1}{4\sin^2 t + 17}\right)$$

$$= \frac{4\sin^2 t + 17 - 4\sin^2 t + 2\sin t}{4\sin^2 t + 17}$$

$$= \frac{2\sin t + 17}{4\sin^2 t + 17}$$

$$P\left(2,\ 1+\frac{16-32\sin t}{17+4\sin^2 t},\ 4-\left(\frac{4-8\sin t}{17+4\sin^2 t}\right)\right)$$

and $$Q\left(1+\frac{2\sin t+17}{4\sin^2 t+17},\ 2-2\sin t\left(\frac{2\sin t+17}{4\sin^2 t+17}\right),\ 4\right)$$

$$PQ=\sqrt{\left(1-\left(\frac{2\sin t+17}{4\sin^2 t+17}\right)\right)^2+\left(-1+2\sin t\left(\frac{2\sin t+17}{4\sin^2 t+17}\right)+\frac{16-32\sin t}{4\sin^2 t+17}\right)^2+\left(-\left(\frac{4-8\sin t}{4\sin^2 t+17}\right)\right)^2}$$

$$=\sqrt{\left(\frac{4\sin^2 t-2\sin t}{4\sin^2 t+17}\right)^2+\left(\frac{-4\sin^2 t-17+4\sin^2 t+34\sin t+16-32\sin t}{4\sin^2 t+17}\right)^2+\left(\frac{8\sin t-4}{4\sin^2 t+17}\right)^2}$$

$$=\frac{\sqrt{(4\sin^2 t-2\sin t)^2+(2\sin t-1)^2+(8\sin t-4)^2}}{4\sin^2 t+17}$$

$$=\frac{\sqrt{4\sin^2 t(2\sin t-1)^2+(2\sin t-1)^2+16(2\sin t-1)^2}}{4\sin^2 t+17}$$

$$= \frac{\sqrt{(4\sin^2 t + 17)(2\sin t - 1)^2}}{4\sin^2 t + 17}$$

$$= \frac{\sqrt{(2\sin t - 1)^2}}{\sqrt{4\sin^2 t + 17}}$$

$$= \frac{|2\sin t - 1|}{\sqrt{4\sin^2 t + 17}}$$

OR

$$\vec{PQ}\cdot\vec{AB} = |\vec{PQ}||\vec{AB}|\cos\theta$$

$$|\vec{PQ}\cdot\vec{AB}| = ||\vec{PQ}||\vec{AB}|\cos\theta|$$

$$\left|c\begin{pmatrix}-2\sin t\\ -1\\ -4\end{pmatrix}\cdot\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}\right| = |PQ||AB||\cos\theta|$$

$$|AB||\cos\theta| = \frac{|c(-2\sin t + 1)|}{|PQ|}$$

$$= \frac{|c||-2\sin t + 1|}{\left|c\begin{pmatrix}-2\sin t\\ -1\\ -4\end{pmatrix}\right|}$$

$$= \frac{|c||2\sin t - 1|}{|c(4\sin^2 t + 17)|}$$

$$= \frac{|c||2\sin t - 1|}{|c|(4\sin^2 t + 17)}$$

$$= \frac{|2\sin t - 1|}{4\sin^2 t + 17}$$

ii) when $PQ = 0$:

$$\frac{|2\sin t - 1|}{\sqrt{4\sin^2 t + 17}} = 0$$

$$|2\sin t - 1| = 0$$

$$2\sin t - 1 = 0$$

$$\sin t = \frac{1}{2}$$

$$t = \frac{\pi}{6}, \frac{5\pi}{6}$$

iii) when $t = \frac{\pi}{4}$,

$$l_2: \underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k} + \mu(\underset{\sim}{i} - \sqrt{2}\underset{\sim}{j})$$

Since $\overrightarrow{PQ}$ and $\overrightarrow{BQ}$ are parallel to the plane BPQ and $\overrightarrow{BQ}$ is parallel to the direction of $\ell_2$,

$\begin{pmatrix} -\sqrt{2} \\ -1 \\ -4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -\sqrt{2} \\ 0 \end{pmatrix}$ is parallel to the normal of the plane BPQ.

$$\begin{pmatrix} -\sqrt{2} \\ -1 \\ -4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -\sqrt{2} \\ 0 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -\sqrt{2} & -1 & -4 \\ 1 & -\sqrt{2} & 0 \end{vmatrix} = \begin{pmatrix} -4\sqrt{2} \\ -4 \\ 3 \end{pmatrix}$$

Since $\begin{pmatrix} -4\sqrt{2} \\ -4 \\ 3 \end{pmatrix}$ is perpendicular to the plane

BPQ and B is a point on the plane, if

$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is any point on the plane,

$$\underset{\sim}{r} \cdot \begin{pmatrix} -4\sqrt{2} \\ -4 \\ 3 \end{pmatrix} = \overrightarrow{OB} \cdot \begin{pmatrix} -4\sqrt{2} \\ -4 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -4\sqrt{2} \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -4\sqrt{2} \\ -4 \\ 3 \end{pmatrix}$$

$$-4\sqrt{2}x - 4y + 3z = -4\sqrt{2} - 8 + 12$$

$$= -4\sqrt{2} + 4$$

$$4\sqrt{2}x + 4y - 3z = 4\sqrt{2} - 4$$

$\therefore$ The plane BPQ has equation

$4\sqrt{2}x + 4y - 3z = 4\sqrt{2} - 4$

The line perpendicular to the plane and passing through A has equation

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + s\begin{pmatrix} 4\sqrt{2} \\ 4 \\ -3 \end{pmatrix}, \quad s \in R$$

When the line meets the plane,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 + 4\sqrt{2}s \\ 1 + 4s \\ 4 - 3s \end{pmatrix}$$

$x = 2 + 4\sqrt{2}s, \; y = 1 + 4s, \; z = 4 - 3s$

$4\sqrt{2}(2 + 4\sqrt{2}s) + 4(1 + 4s) - 3(4 - 3s) = 4\sqrt{2} - 4$

$8\sqrt{2} + 32s + 4 + 16s - 12 + 9s = 4\sqrt{2} - 4$

$57s = 4 - 4\sqrt{2}$

$s = \dfrac{4(1 - \sqrt{2})}{57}$

$\therefore$ The line meets the plane at

$$\left(2 + 4\sqrt{2}\left(\frac{4 - 4\sqrt{2}}{57}\right), 1 + 4\left(\frac{4 - 4\sqrt{2}}{57}\right), 4 - 3\left(\frac{4 - 4\sqrt{2}}{57}\right)\right)$$

$$= \left(\frac{82 + 16\sqrt{2}}{57}, \frac{73 - 16\sqrt{2}}{57}, \frac{214 + 12\sqrt{2}}{57}\right)$$

$\therefore$ The perpendicular distance from A to the plane BPQ is

$$\sqrt{\left(\frac{82+16\sqrt{2}}{57}-2\right)^2+\left(\frac{73-16\sqrt{2}}{57}-1\right)^2+\left(\frac{214+12\sqrt{2}}{57}-4\right)^2}$$

$$=\sqrt{\left(\frac{4\sqrt{2}(4-4\sqrt{2})}{57}\right)^2+\left(\frac{4(4-4\sqrt{2})}{57}\right)^2+\left(\frac{-3(4-4\sqrt{2})}{57}\right)^2}$$

$$=\sqrt{\frac{32(4-4\sqrt{2})^2}{57^2}+\frac{16(4-4\sqrt{2})^2}{57^2}+\frac{9(4-4\sqrt{2})^2}{57^2}}$$

$$=\sqrt{\frac{57(4-4\sqrt{2})^2}{57^2}}$$

$$=\frac{4\sqrt{2}-4}{\sqrt{57}}$$

$$\approx 0.219$$

12. EITHER

$$\sum_{K=0}^{n-1} (1 + i\tan\theta)^K = \frac{1 - (1 + i\tan\theta)^n}{1 - (1 + i\tan\theta)}$$

$$\sum_{K=0}^{n-1} \left(1 + \frac{i\sin\theta}{\cos\theta}\right)^k = \frac{1 - (1 + i\tan\theta)^n}{-i\tan\theta}$$

$$\sum_{K=0}^{n-1} \frac{(\cos\theta + i\sin\theta)^K}{\cos^K\theta} = \frac{\left(1 - (1 + i\tan\theta)^n\right) i\tan\theta}{\tan^2\theta}$$

$$\sum_{K=0}^{n-1} \frac{\cos K\theta + i\sin K\theta}{\cos^k\theta} = \frac{i}{\tan\theta}\left(1 - \left(1 + \frac{i\sin\theta}{\cos\theta}\right)^n\right)$$

$$\sum_{K=0}^{n-1} \frac{\cos K\theta}{\cos^k\theta} + \frac{i\sin K\theta}{\cos^K\theta} = \frac{i}{\tan\theta}\left(1 - \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n\right)$$

$$\sum_{K=0}^{n-1} \frac{\cos K\theta}{\cos^k\theta} + i\sum_{K=0}^{n-1} \frac{\sin K\theta}{\cos^k\theta} = \frac{i}{\tan\theta}\left(1 - \frac{(\cos n\theta + i\sin n\theta)}{\cos^n\theta}\right)$$

$$\sum_{K=0}^{n-1} \cos K\theta \sec^k\theta + i\sum_{K=0}^{n-1} \sin k\theta \sec^K\theta$$

$$= \frac{i(\cos^n\theta - \cos n\theta - i\sin n\theta)}{\tan\theta\cos^n\theta}$$

$$= \frac{\sin n\theta}{\tan\theta\cos^n\theta} + \frac{i(\cos^n\theta - \cos n\theta)}{\tan\theta\cos^n\theta}$$

$$= \cot\theta\sin n\theta\sec^n\theta + i\cot\theta(\cos^n\theta - \cos n\theta)\sec^n\theta.$$

$$\therefore \sum_{k=0}^{n-1} \cos k\theta \sec^k\theta = \cot\theta \sin n\theta \sec^n\theta, \quad \theta \neq \frac{r\pi}{2}, r \in \mathbb{Z}.$$

when $\theta = \frac{\pi}{3}$:

$$\sum_{k=0}^{n-1} \cos\left(\frac{k\pi}{3}\right) \frac{1}{\left(\frac{1}{2}\right)^k} = \frac{1}{\sqrt{3}} \sin\left(\frac{n\pi}{3}\right) \frac{1}{\left(\frac{1}{2}\right)^n}$$

$$\sum_{k=0}^{n-1} 2^k \cos\left(\frac{k\pi}{3}\right) = \frac{2^n}{\sqrt{3}} \sin\left(\frac{n\pi}{3}\right).$$

If $0 < x < 1$ and $\theta = \cos^{-1}x$,

$$\sum_{k=0}^{n-1} \cos(k\cos^{-1}x) \left(\frac{1}{x}\right)^k = \frac{x}{\sqrt{1-x^2}} \sin(n\cos^{-1}x) \left(\frac{1}{x}\right)^n$$

$$\sum_{k=0}^{n-1} \frac{\cos(k\cos^{-1}x)}{x^k} = \frac{\sin(n\cos^{-1}x)}{x^{n-1}\sqrt{1-x^2}}$$

OR

$T_1 : R^4 \xrightarrow{M_1} R^4 \qquad T_2 : R^4 \xrightarrow{M_2} R^4$

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 7 & 8 \\ 1 & 7 & 11 & 13 \\ 1 & 2 & 5 & 5 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 5 & 1 & -3 & -3 \\ 3 & -1 & -1 & -1 \\ 13 & -1 & -6 & -6 \end{pmatrix}$$

i) $$M_1 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 7 & 8 \\ 1 & 7 & 11 & 13 \\ 1 & 2 & 5 & 5 \end{pmatrix}$$

$$\xrightarrow[-r_1 + r_4]{\substack{-r_1 + r_2 \\ -r_1 + r_3}} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 3 & 6 & 6 \\ 0 & 6 & 10 & 11 \\ 0 & 1 & 4 & 3 \end{pmatrix}$$

$$\xrightarrow{\frac{r_2}{3}} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 6 & 10 & 11 \\ 0 & 1 & 4 & 3 \end{pmatrix}$$

$$\xrightarrow{\substack{-6r_2 + r_3 \\ -r_2 + r_4}} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_3 + r_4} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-2 \times r_3} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ A basis for $R_1$, the range space of $T_1$ is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 11 \\ 5 \end{pmatrix} \right\}$$

ii) If

$$M_2 \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 & -1 & -1 \\ 5 & 1 & -3 & -3 \\ 3 & -1 & -1 & -1 \\ 13 & -1 & -6 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 2 & 0 & -1 & -1 & 0 \\ 5 & 1 & -3 & -3 & 0 \\ 3 & -1 & -1 & -1 & 0 \\ 13 & -1 & -6 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{2 \times r_2 \\ 2 \times r_3 \\ 2 \times r_4}]{} \left(\begin{array}{cccc|c} 2 & 0 & -1 & -1 & 0 \\ 10 & 2 & -6 & -6 & 0 \\ 6 & -2 & -2 & -2 & 0 \\ 26 & -2 & -12 & -12 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{-5r_1 + r_2 \\ -3r_1 + r_3 \\ -13r_1 + r_4}]{} \left(\begin{array}{cccc|c} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{r_2 + r_3 \\ r_2 + r_4}]{} \left(\begin{array}{cccc|c} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = 2s$ and $z = 2t$, $s, t \in R$

$$2y - 2t - 2s = 0$$

$$y = s + t$$

$$2x - 2t - 2s = 0$$

$$x = s + t$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} s + t \\ s + t \\ 2t \\ 2s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

$\therefore$ A basis for $K_2$, the null space of $T_2$ is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

If $\underset{\sim}{x} \in K_2$, $\underset{\sim}{x} = \alpha\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$, $\alpha, \beta \in R$.

Also, if $\alpha\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 4 \\ 7 \\ 2 \end{pmatrix} + c\begin{pmatrix} 1 \\ 7 \\ 11 \\ 5 \end{pmatrix}$

$$\begin{pmatrix} \alpha + \beta \\ \alpha + \beta \\ 2\beta \\ 2\alpha \end{pmatrix} = \begin{pmatrix} a + b + c \\ a + 4b + 7c \\ a + 7b + 11c \\ a + 2b + 5c \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 7 \\ 1 & 7 & 11 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha+\beta \\ \alpha+\beta \\ 2\beta \\ 2\alpha \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha+\beta \\ 1 & 4 & 7 & \alpha+\beta \\ 1 & 7 & 11 & 2\beta \\ 1 & 2 & 5 & 2\alpha \end{array}\right)$$

$$\xrightarrow[-r_1+r_4]{\substack{-r_1+r_2 \\ -r_1+r_3}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha+\beta \\ 0 & 3 & 6 & 0 \\ 0 & 6 & 10 & -\alpha+\beta \\ 0 & 1 & 4 & \alpha-\beta \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{3}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha+\beta \\ 0 & 1 & 2 & 0 \\ 0 & 6 & 10 & -\alpha+\beta \\ 0 & 1 & 4 & \alpha-\beta \end{array}\right)$$

$$\xrightarrow[-r_2+r_4]{-6r_2+r_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha+\beta \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -\alpha+\beta \\ 0 & 0 & 2 & \alpha-\beta \end{array}\right)$$

$$\xrightarrow{r_3+r_4} \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha+\beta \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -\alpha+\beta \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_3}{-2}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha+\beta \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & \frac{\alpha-\beta}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$c = \frac{\alpha - \beta}{2}$$

$$b = -2c$$

$$= -\alpha + \beta$$

$$a + b + c = \alpha + \beta$$

$$a - \alpha + \beta + \frac{\alpha - \beta}{2} = \alpha + \beta$$

$$a = \frac{3\alpha + \beta}{2}$$

$$\underset{\sim}{x} = \left(\frac{3\alpha + \beta}{2}\right)\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + (-\alpha + \beta)\begin{pmatrix}1\\4\\7\\2\end{pmatrix} + \frac{(\alpha - \beta)}{2}\begin{pmatrix}1\\7\\11\\5\end{pmatrix}$$

$\therefore \underset{\sim}{x} \in R_1$

$\therefore K_2$ is a subspace of $R_1$

iii) Since $K_2$ contains the zero vector, $W$ is not a vector space since it does not contain the zero vector.

iv) $T_3 : R^4 \xrightarrow{M_2 M_1} R^4$

$$M_2 M_1 = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 5 & 1 & -3 & -3 \\ 3 & -1 & -1 & -1 \\ 13 & -1 & -6 & -6 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 7 & 8 \\ 1 & 7 & 11 & 13 \\ 1 & 2 & 5 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -7 & -14 & -14 \\ 0 & -18 & -36 & -36 \\ 0 & -10 & -20 & -20 \\ 0 & -45 & -90 & -90 \end{pmatrix}$$

$$\xrightarrow{\frac{r_1}{-7},\ \frac{r_2}{-18},\ \frac{r_3}{-10},\ \frac{r_4}{-45}} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow[-r_1 + r_4]{\substack{-r_1 + r_2 \\ -r_1 + r_3}} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The dimension of the range space of $T_3$ is 1.

$\therefore$ The dimension of the null space of $T_3$ is $4 - 1 = 3$.