MAY / JUNE 2010

1.

$$A = \begin{pmatrix} 5 & -3 & 0 \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{pmatrix}$$

5 is an eigenvalue.

If $\underset{\sim}{x}$ is the corresponding eigenvector for the eigenvalue 5,

$$A\underset{\sim}{x} = 5\underset{\sim}{x}$$

$$\begin{pmatrix} 5 & -3 & 0 \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 5x - 3y \\ x + 2y + z \\ -x + 3y + 4z \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \\ 5z \end{pmatrix}$$

$$5x - 3y = 5x \quad - (1)$$
$$x + 2y + z = 5y \quad - (2)$$
$$-x + 3y + 4z = 5z \quad - (3)$$

From (1): $y = 0$

$x + z = 0$

$-x - z = 0$

Let $z = s$, $s \in R$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ The corresponding eigenvector for the eigenvalue 5 is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

$$\begin{aligned}(A + A^2)\underset{\sim}{x} &= A\underset{\sim}{x} + A^2\underset{\sim}{x} \\ &= \lambda\underset{\sim}{x} + A(A\underset{\sim}{x}) \\ &= \lambda\underset{\sim}{x} + A(\lambda\underset{\sim}{x}) \\ &= \lambda\underset{\sim}{x} + \lambda(A\underset{\sim}{x}) \\ &= \lambda\underset{\sim}{x} + \lambda(\lambda\underset{\sim}{x}) \\ &= \lambda\underset{\sim}{x} + \lambda^2\underset{\sim}{x} \\ &= (\lambda + \lambda^2)\underset{\sim}{x}\end{aligned}$$

$\therefore$ If A has an eigenvalue λ with corresponding eigenvector $\underset{\sim}{x}$, $A + A^2$ has an eigenvalue $\lambda + \lambda^2$ with corresponding eigenvector $\underset{\sim}{x}$.

Since 5 is an eigenvalue of A with corresponding eigenvector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $5 + 5^2$ is an eigenvalue of A with corresponding eigenvector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

$\therefore$ A has an eigenvalue 30 with corresponding eigenvector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

2. $\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha]$

$$\begin{aligned}&= \cos 2n\alpha \cos\alpha + \sin 2n\alpha \sin\alpha \\ &\quad - (\cos 2n\alpha \cos\alpha - \sin 2n\alpha \sin\alpha) \\ &= \cos 2n\alpha \cos\alpha + \sin 2n\alpha \sin\alpha \\ &\quad - \cos 2n\alpha \cos\alpha + \sin 2n\alpha \sin\alpha \\ &= 2\sin 2n\alpha \sin\alpha.\end{aligned}$$

$$\sin 2n\alpha = \frac{\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha]}{2\sin\alpha}, \quad \alpha \neq k\pi,\ k \in \mathbb{Z}.$$

$$\sum_{n=1}^{N} \sin 2n\alpha = \sum_{n=1}^{N} \frac{\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha]}{2\sin\alpha}$$

$$\begin{aligned}&= \frac{1}{2\sin\alpha}\Big(\cos\alpha - \cos 3\alpha \\ &\quad + \cos 3\alpha - \cos 5\alpha \\ &\quad + \cos 5\alpha - \cos 7\alpha \\ &\quad \vdots \\ &\quad + \cos(2N-5)\alpha - \cos(2N-3)\alpha \\ &\quad + \cos(2N-3)\alpha - \cos(2N-1)\alpha \\ &\quad + \cos(2N-1)\alpha - \cos(2N+1)\alpha \Big)\end{aligned}$$

$$= \frac{\cos\alpha - \cos(2N+1)\alpha}{2\sin\alpha}$$

$$= \frac{\cot\alpha}{2} - \frac{\csc\alpha \cos(2N+1)\alpha}{2}$$

when $\alpha = \frac{\pi}{3}$:

$$\sum_{n=1}^{N} \sin \frac{2n\pi}{3} = \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \cos[(2N+1)\alpha]$$

$$\sum_{n=1}^{\infty} \sin \frac{2n\pi}{3} = \lim_{N\to\infty} \left(\frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \cos[(2N+1)\alpha]\right)$$

since $-1 < \cos[(2n+1)\alpha] < 1$

$$\frac{-1}{\sqrt{3}} < \frac{-1}{\sqrt{3}} \cos[(2n+1)\alpha] < \frac{1}{\sqrt{3}}$$

$$\frac{-1}{2\sqrt{3}} < \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \cos[(2n+1)\alpha] < \frac{\sqrt{3}}{2}$$

$$\frac{-1}{2\sqrt{3}} < \sum_{n=1}^{\infty} \sin \frac{2n\pi}{3} < \frac{\sqrt{3}}{2}$$

$\therefore$ The infinite series $\sum_{n=1}^{\infty} \sin \frac{2n\pi}{3}$

does not converge.

3. $x_1, x_2, x_3, \ldots, \quad x_1 = 3$

$$x_{n+1} = \frac{2x_n^2 + 4x_n - 2}{2x_n + 3}, \quad n = 1, 2, 3, \ldots$$

$x_n > 2$

when $n=1 \quad x_1 = 3 > 2$

Assume the statement is true when $n=k$.

$n=k: \ x_k > 2.$

when $n=k+1: \quad x_{k+1} > 2$

(what needs to be proved)

If $x_k > 2$

$x_k^2 > 4$

$2x_k^2 > 8$

$2x_k^2 + 4x_k - 2 > 4x_k + 6$

$2x_k^2 + 4x_k - 2 > 2(2x_k + 3)$

$$\frac{2x_k^2 + 4x_k - 2}{2x_k + 3} > 2$$

$x_{k+1} > 2$

$\therefore x_n > 2$ for every positive integer n.

4. $x = \cos t + t\sin t \quad y = \sin t - t\cos t$

$\frac{dx}{dt} = -\sin t + \sin t + t\cos t \quad \frac{dy}{dt} = \cos t - \cos t + t\sin t$

$= t\cos t \qquad\qquad = t\sin t$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2\cos^2 t + t^2\sin^2 t$

$= t^2(\cos^2 t + \sin^2 t)$

$= t^2$

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t$

The surface area of one complete revolution from $t=0$ to $t=\frac{\pi}{2}$ about the x-axis is

$$\int_0^{\frac{\pi}{2}} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\frac{\pi}{2}} 2\pi t(\sin t - t\cos t)\,dt$$

$$= 2\pi\int_0^{\frac{\pi}{2}} t\sin t\,dt - 2\pi\int_0^{\frac{\pi}{2}} t^2\cos t\,dt$$

$u = t \quad dv = \sin t\,dt \quad u = t^2 \quad dv = \cos t\,dt$

$du = dt \quad v = -\cos t \quad du = 2t\,dt \quad v = \sin t$

$$= 2\pi\left(\left[-t\cos t\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos t\,dt\right)$$

$$-2\pi\left(\left[t^2\sin t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2t\sin t\,dt\right)$$

$$= 2\pi\left(0 + \left[\sin t\right]_0^{\frac{\pi}{2}}\right)$$

$$-2\pi\left(\frac{\pi^2}{4} - 0 - 2\int_0^{\frac{\pi}{2}} t\sin t\,dt\right)$$

$$= 2\pi(1) - 2\pi\left(\frac{\pi^2}{4} - 2\int_0^{\frac{\pi}{2}} t\sin t\,dt\right)$$

$u = t \quad dv = \sin t\,dt$

$du = dt \quad v = -\cos t$

$$= 2\pi - 2\pi\left(\frac{\pi^2}{4} - 2\left(\left[-t\cos t\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos t\,dt\right)\right)$$

$$= 2\pi - 2\pi\left(\frac{\pi^2}{4} - 2\left(0 + \left[\sin t\right]_0^{\frac{\pi}{2}}\right)\right)$$

$$= 2\pi - 2\pi\left(\frac{\pi^2}{4} - 2(1 - 0)\right)$$

$$= 2\pi - 2\pi\left(\frac{\pi^2}{4} - 2\right)$$

$$= 2\pi - \frac{\pi^3}{2} + 4\pi$$

$$= 6\pi - \frac{\pi^3}{2}.$$

CONFIDENTIAL

5. $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta$
$+ 10(-1)\cos^3\theta\sin^2\theta$
$+ 10(-i)\cos^2\theta\sin^3\theta$
$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$

$\cos 5\theta + i\sin 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta$
$+ 5\cos\theta\sin^4\theta$
$+ i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta$
$+ \sin^5\theta)$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$
$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta$
$+ \sin^5\theta$
$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta$
$-10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$
$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta$
$-10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$
$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

$32x^5 - 40x^3 + 10x + 1 = 0$
$32x^5 - 40x^3 + 10x = -1$
$16x^5 - 20x^3 + 5x = \frac{-1}{2}$

If $x = \sin\theta$,
$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = \frac{-1}{2}$

$\sin 5\theta = \frac{-1}{2}$

$5\theta = \frac{7\pi}{6} + 2k\pi,\ \frac{11\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$

$\theta = \frac{7\pi}{30} + \frac{2k\pi}{5},\ \frac{11\pi}{30} + \frac{2k\pi}{5}$

$k=0:\ \theta = \frac{7\pi}{30}, \frac{11\pi}{30}$ $\qquad k=3:\ \theta = \frac{43\pi}{30}, \frac{47\pi}{30}$

$k=1:\ \theta = \frac{19\pi}{30}, \frac{23\pi}{30}$ $\qquad k=4:\ \theta = \frac{55\pi}{30}, \frac{59\pi}{30}$

$k=2:\ \theta = \frac{31\pi}{30}, \frac{35\pi}{30}$

$\therefore$ The roots of the equation
$32x^5 - 40x^3 + 10x - 1 = 0$ are
$\sin\frac{7\pi}{30}$, $\sin\frac{19\pi}{30}$, $\sin\frac{31\pi}{30}$, $\sin\frac{43\pi}{30}$ and $\sin\frac{55\pi}{30}$.

6. $C: \ y = \dfrac{x^2 - 3x - 7}{x+1}$

$$\begin{array}{r} x - 4 \\ x+1 \overline{\big) x^2 - 3x - 7} \\ \underline{x^2 + x} \\ -4x - 7 \\ \underline{-4x - 4} \\ -3 \end{array}$$

$= x - 4 \ \dfrac{-3}{x+1}$

i) As $x \longrightarrow \pm\infty$ $y \longrightarrow x - 4$

As $x \longrightarrow -1$ $y \longrightarrow \pm\infty$

$\therefore$ The asymptotes of C are $y = x - 4$ and $x = -1$.

ii) $\dfrac{dy}{dx} = 1 + \dfrac{3}{(x+1)^2}$

> 1

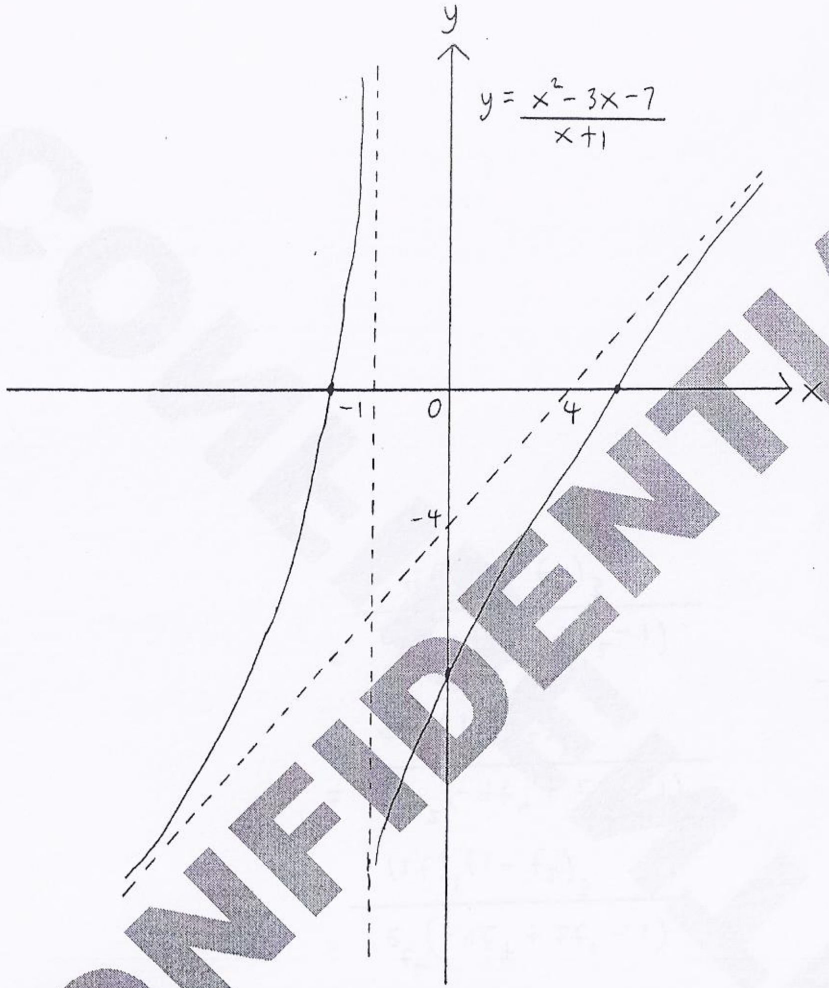
$\therefore \dfrac{dy}{dx} > 1$ at every point of C.

iii) when $x = 0$: $y = -7$

when $y = 0$: $\dfrac{x^2 - 3x - 7}{x+1} = 0$

$x^2 - 3x - 7 = 0$

$x = \dfrac{3 \pm \sqrt{37}}{2}$



• : Intersection point

7. $x = t^2 e^{-t^2}$ $\quad y = te^{-t^2}$

i) $\frac{dx}{dt} = 2te^{-t^2} - 2t^3e^{-t^2}$ $\quad \frac{dy}{dt} = e^{-t^2} - 2t^2e^{-t^2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{e^{-t^2} - 2t^2e^{-t^2}}{2te^{-t^2} - 2t^3e^{-t^2}}$$

$$= \frac{(1-2t^2)e^{-t^2}}{2t(1-t^2)e^{-t^2}}$$

$$= \frac{1-2t^2}{2t-2t^3}$$

ii) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{(2t-2t^3)e^{-t^2}}\frac{d}{dt}\left(\frac{1-2t^2}{2t-2t^3}\right)$$

$$= \frac{e^{t^2}}{2t-2t^3}\left(\frac{(2t-2t^3)(-4t)-(1-2t^2)(2-8t^2)}{(2t-2t^3)^2}\right)$$

$$= \frac{e^{t^2}}{2t-2t^3}\left(\frac{-8t^2+8t^4-2+4t^2+8t^2-16t^4}{(2t-2t^3)^2}\right)$$

$$= \frac{e^{t^2}(-8t^4+4t^2-2)}{(2t-2t^3)^3}$$

$$= \frac{2e^{t^2}(-4t^4+2t^2-1)}{(2t)^3(1-t^2)^3}$$

$$= \frac{2e^{t^2}(-4t^4+2t^2-1)}{8t^3(1-t^2)^3}$$

$$= \frac{e^{t^2}(-4t^4+2t^2-1)}{4t^3(1-t^2)^3}.$$

CONFIDENTIAL

8. $\dfrac{d^2y}{dx^2} + \dfrac{5dy}{dx} + 4y = 10\sin 3x - 20\cos 3x$

$\dfrac{d^2y}{dx^2} + \dfrac{5dy}{dx} + 4y = 0$

$m^2 + 5m + 4 = 0$

$(m+1)(m+4) = 0$

$m = -1, -4$

The complementary function, y_c, is given by $y_c = Ae^{-x} + Be^{-4x}$.

The particular integral, y_p, is given by

$y_p = C\cos 3x + D\sin 3x$

$\dfrac{dy_p}{dx} = -3C\sin 3x + 3D\cos 3x$

$\dfrac{d^2y_p}{dx^2} = -9C\cos 3x - 9D\sin 3x$

$\dfrac{d^2y_p}{dx^2} + \dfrac{5dy_p}{dx} + 4y_p$

$= -9C\cos 3x - 9D\sin 3x$
$\quad + 5(-3C\sin 3x + 3D\cos 3x)$
$\quad + 4(C\cos 3x + D\sin 3x)$

$= -9C\cos 3x - 9D\sin 3x$
$\quad -15C\sin 3x + 15D\cos 3x$
$\quad +4C\cos 3x + 4D\sin 3x$

$= (-5C + 15D)\cos 3x + (-15C - 5D)\sin 3x$

$= 10\sin 3x - 20\cos 3x$

$-5C + 15D = -20 \qquad -15C - 5D = 10$

$C - 3D = 4 \qquad -3C - D = 2$

$C = 3D + 4$

$-3(3D + 4) - D = 2$

$-9D - 12 - D = 2$

$-10D = 14$

$D = -\dfrac{7}{5}$

$C = -\dfrac{1}{5}$

$\therefore y_p = -\dfrac{\cos 3x}{5} - \dfrac{7\sin 3x}{5}$

$y = y_c + y_p$

$= Ae^{-x} + Be^{-4x} - \dfrac{\cos 3x}{5} - \dfrac{7\sin 3x}{5}.$

If $-\dfrac{\cos 3x}{5} - \dfrac{7\sin 3x}{5} = R\sin(3x + \phi)$

$= R\sin 3x\cos\phi + R\cos 3x\sin\phi$

$R\cos\phi = \dfrac{-7}{5} \qquad R\sin\phi = \dfrac{-1}{5}$

$$\frac{R\sin\phi}{R\cos\phi} = \frac{1}{7}$$

$$\tan\phi = \frac{1}{7}$$

$$\phi = \pi + \tan^{-1}\frac{1}{7}$$
$$\approx 3.28$$

$$R^2\cos^2\phi + R\sin^2\phi = \frac{49}{25} + \frac{1}{25}$$

$$R^2(\cos^2\phi + \sin^2\phi) = 2$$

$$R^2 = 2$$
$$R = \sqrt{2}$$
$$\approx 1.41$$

$$\therefore \frac{-\cos 3x}{5} - \frac{7\sin 3x}{5} = \sqrt{2}\sin\left(3x + \pi + \tan^{-1}\frac{1}{7}\right)$$

$$y = Ae^{-x} + Be^{-4x} + \sqrt{2}\sin\left(3x + \pi + \tan^{-1}\frac{1}{7}\right)$$

As $x \to \infty$, $y \to \sqrt{2}\sin\left(3x + \pi + \tan^{-1}\frac{1}{7}\right)$

9. $I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta, \; n \geqslant 0$

$$I_{n+2} = \int_0^{\frac{\pi}{2}} \sin^{n+2}\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^n\theta\sin^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^n\theta(1-\cos^2\theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^n\theta - \sin^n\theta\cos^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta - \int_0^{\frac{\pi}{2}} \sin^n\theta\cos^2\theta \, d\theta$$

$$= I_n - \int_0^{\frac{\pi}{2}} \sin^n\theta\cos\theta\cos\theta \, d\theta$$

$u = \cos\theta$ $\quad$ $dv = \sin^n\theta\cos\theta \, d\theta$

$du = -\sin\theta \, d\theta$ $\quad$ $v = \int \sin^n\theta\cos\theta \, d\theta$

$w = \sin\theta$

$dw = \cos\theta \, d\theta$

$$= \int w^n \, dw$$

$$= \frac{w^{n+1}}{n+1}$$

$$= \frac{\sin^{n+1}\theta}{n+1}$$

$$= I_n - \left(\left[\frac{\sin^{n+1}\theta \cos\theta}{n+1} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1}\theta(-\sin\theta)\, d\theta}{n+1} \right)$$

$$= I_n - \left(0 + \frac{1}{n+1} \int_0^{\frac{\pi}{2}} \sin^{n+2}\theta \, d\theta \right)$$

$$= I_n - \frac{1}{n+1} \int_0^{\frac{\pi}{2}} \sin^{n+2}\theta \, d\theta$$

$$= I_n - \frac{1}{n+1} I_{n+2}$$

$$\left(1 + \frac{1}{n+1}\right) I_{n+2} = I_n$$

$$\left(\frac{n+2}{n+1}\right) I_{n+2} = I_n$$

$$I_{n+2} = \left(\frac{n+1}{n+2}\right) I_n$$

The y-coordinate of the centroid of the region bounded by the x-axis, the line $x = \frac{\pi}{2m}$ and the curve $y = \sin^4 mx$, $m > 0$, R is

$$\frac{\int_0^{\frac{\pi}{2m}} \frac{y^2}{2} \, dx}{\int_0^{\frac{\pi}{2m}} y \, dx}$$

$$= \frac{\int_0^{\frac{\pi}{2m}} \frac{\sin^8 mx}{2} \, dx}{\int_0^{\frac{\pi}{2m}} \sin^4 mx \, dx}$$

$u = mx$

$du = m\,dx$

$x = 0 \quad u = 0$

$x = \frac{\pi}{2m} \quad u = \frac{\pi}{2}$

$$= \frac{\int_0^{\frac{\pi}{2}} \frac{\sin^8 u}{2m} \, du}{\int_0^{\frac{\pi}{2}} \frac{\sin^4 u}{m} \, du}$$

$$= \frac{\frac{1}{2m} \int_0^{\frac{\pi}{2}} \sin^8 u \, du}{\frac{1}{m} \int_0^{\frac{\pi}{2}} \sin^4 u \, du}$$

$$= \frac{I_8}{2 I_4}$$

$$= \frac{\frac{7}{8} I_6}{2 \cdot \frac{3}{4} I_2}$$

$$= \frac{\frac{7}{8}\cdot\frac{5}{6} I_4}{2\cdot\frac{3}{4}\cdot\frac{1}{2} I_0}$$

$$= \frac{\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4} I_2}{2\cdot\frac{3}{4}\cdot\frac{1}{2} I_0}$$

$$= \frac{\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2} I_0}{2\cdot\frac{3}{4}\cdot\frac{1}{2} I_0}$$

$$= \frac{1}{2}\left(\frac{7}{8}\cdot\frac{5}{6}\right)$$

$$= \frac{1}{2}\left(\frac{35}{48}\right)$$

$$= \frac{35}{96}$$

10. $x^4 + x^3 + cx^2 + 4x - 2 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots.

i) Let $y = \frac{1}{x}$

$x = \frac{1}{y}$

$$\frac{1}{y^4} + \frac{1}{y^3} + \frac{c}{y^2} + \frac{4}{y} - 2 = 0$$

$$1 + y + cy^2 + 4y^3 - 2y^4 = 0$$

$$2y^4 - 4y^3 - cy^2 - y - 1 = 0$$

$\therefore$ The equation $2y^4 - 4y^3 - cy^2 - y - 1 = 0$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$.

ii) $\alpha + \beta + \gamma + \delta = -1$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -4$

$\alpha\beta\gamma\delta = -2$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$= (-1)^2 - 2c$$

$$= 1 - 2c$$

Also, $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=2$

$$\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\alpha\delta}+\frac{1}{\beta\gamma}+\frac{1}{\beta\delta}+\frac{1}{\gamma\delta}=-\frac{c}{2}$$

$$\frac{1}{\alpha\beta\gamma}+\frac{1}{\alpha\beta\delta}+\frac{1}{\alpha\gamma\delta}+\frac{1}{\beta\gamma\delta}=\frac{1}{2}$$

$$\frac{1}{\alpha\beta\gamma\delta}=\frac{-1}{2}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}=\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)^2-2\left(\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\alpha\delta}+\frac{1}{\beta\gamma}+\frac{1}{\beta\delta}+\frac{1}{\gamma\delta}\right)$$

$$=2^2-2\left(\frac{-c}{2}\right)$$

$$=4+c$$

iii) $\left(\alpha-\frac{1}{\alpha}\right)^2+\left(\beta-\frac{1}{\beta}\right)^2+\left(\gamma-\frac{1}{\gamma}\right)^2+\left(\delta-\frac{1}{\delta}\right)^2$

$$=\alpha^2-2+\frac{1}{\alpha^2}+\beta^2-2+\frac{1}{\beta^2}+\gamma^2-2+\frac{1}{\gamma^2}+\delta^2-2+\frac{1}{\delta^2}$$

$$=\alpha^2+\beta^2+\gamma^2+\delta^2+\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}-8$$

$$=1-2c+4+c-8$$

$$=-c-3$$

iv) when $c=-3$:

$$\left(\alpha-\frac{1}{\alpha}\right)^2+\left(\beta-\frac{1}{\beta}\right)^2+\left(\gamma-\frac{1}{\gamma}\right)^2+\left(\delta-\frac{1}{\delta}\right)^2=0$$

Since -1 and 1 are not roots of the equation $x^4+x^3-3x^2+4x-2=0$,

$\alpha^2\neq 1,\ \beta^2\neq 1,\ \gamma^2\neq 1,\ \delta^2\neq 1.$

$\alpha\neq\frac{1}{\alpha},\ \beta\neq\frac{1}{\beta},\ \gamma\neq\frac{1}{\gamma},\ \delta\neq\frac{1}{\delta}$

$\left(\alpha-\frac{1}{\alpha}\right)^2>0,\ \left(\beta-\frac{1}{\beta}\right)^2>0,\ \left(\gamma-\frac{1}{\gamma}\right)^2>0,\ \left(\delta-\frac{1}{\delta}\right)^2>0.$

$\therefore \left(\alpha-\frac{1}{\alpha}\right)^2+\left(\beta-\frac{1}{\beta}\right)^2+\left(\gamma-\frac{1}{\gamma}\right)^2+\left(\delta-\frac{1}{\delta}\right)^2>0.$

$\therefore$ when $c=-3$, the roots of the equation $x^4+x^3-3x^2+4x-2=0$ are not all real.

11. $C: r = \dfrac{a}{1+\theta}, \quad 0 \le \theta \le \dfrac{\pi}{2}, \quad a > 0.$

i) $\dfrac{dr}{d\theta} = \dfrac{-a}{(1+\theta)^2}$

Since $\dfrac{dr}{d\theta} < 0$, r decreases as θ increases.

ii) $y = r\sin\theta$

$= \dfrac{a\sin\theta}{1+\theta}$

$\dfrac{dy}{d\theta} = \dfrac{a\cos\theta}{1+\theta} - \dfrac{a\sin\theta}{(1+\theta)^2}$

when $\dfrac{dy}{d\theta} = 0$: $\dfrac{a\cos\theta}{1+\theta} - \dfrac{a\sin\theta}{(1+\theta)^2} = 0$

$\dfrac{\cos\theta}{1+\theta} = \dfrac{\sin\theta}{(1+\theta)^2}$

$\tan\theta = 1+\theta$

$\therefore$ If P is the point on C which is furthest from the initial line, at P $\tan\theta = 1+\theta$.

If $f(\theta) = \tan\theta - 1 - \theta$

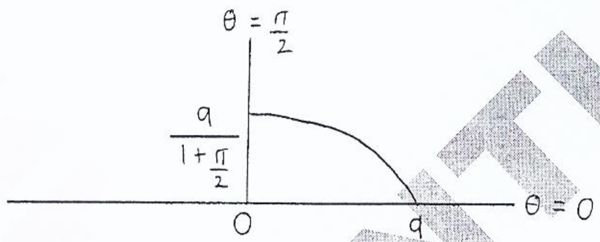
$f(1.1) = -0.1352 < 0$

$f(1.2) = 0.3722 > 0$

$\therefore$ The equation $\tan\theta = 1+\theta$ has a root between 1.1 and 1.2.

iii)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	a	$\frac{a}{1+\frac{\pi}{6}}$	$\frac{a}{1+\frac{\pi}{4}}$	$\frac{a}{1+\frac{\pi}{3}}$	$\frac{a}{1+\frac{\pi}{2}}$



iv) The area of the region bounded by the initial line, the line $\theta = \frac{\pi}{2}$ and C is

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta$

$= \displaystyle\int_0^{\frac{\pi}{2}} \frac{a^2}{2(1+\theta)^2}\, d\theta$

$= \left[\dfrac{-a^2}{2(1+\theta)}\right]_0^{\frac{\pi}{2}}$

$= \dfrac{-a^2}{2\left(1+\frac{\pi}{2}\right)} - \dfrac{(-a^2)}{2}$

$= \dfrac{-2a^2}{2(2+\pi)} + \dfrac{a^2}{2}$

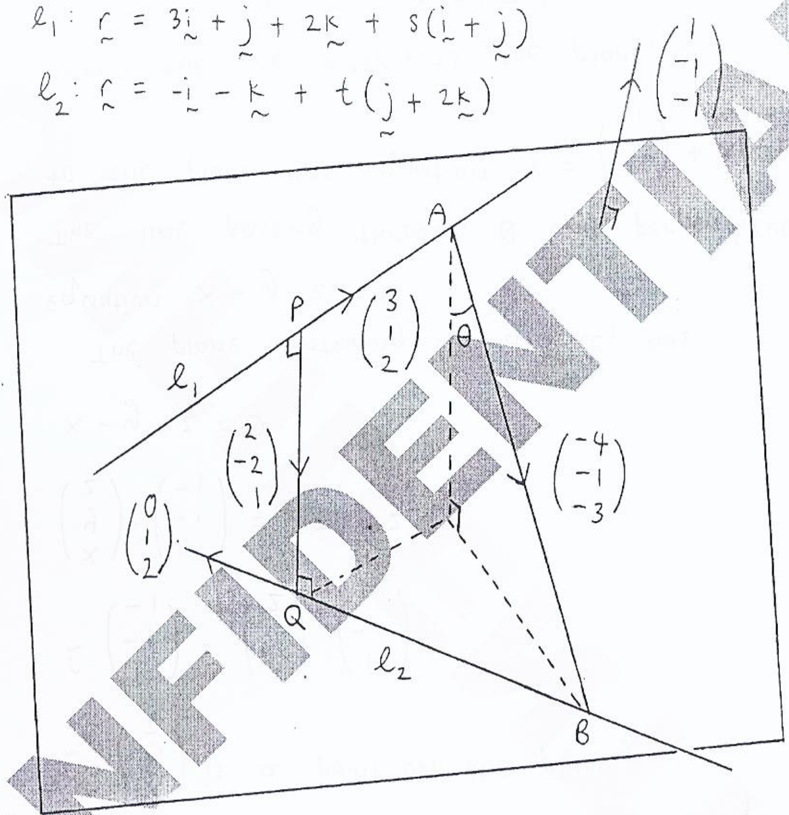
$= \dfrac{\pi a^2}{2(\pi+2)}$

12. EITHER

$A(3,1,2) \quad B(-1,0,-1)$

$\ell_1: \underset{\sim}{r} = 3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} + s(\underset{\sim}{i} + \underset{\sim}{j})$

$\ell_2: \underset{\sim}{r} = -\underset{\sim}{i} - \underset{\sim}{k} + t(\underset{\sim}{j} + 2\underset{\sim}{k})$



i) since P is a point on ℓ_1 and Q is a point on ℓ_2, let $P(3+s, 1+s, 2)$ and $Q(-1, t, -1+2t)$.

Since PQ is perpendicular to both ℓ_1 and ℓ_2, PQ is parallel to $\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\2\end{pmatrix}$

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 1 & 1 & 0\\ 0 & 1 & 2\end{vmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$\therefore \overrightarrow{PQ} = c\begin{pmatrix}2\\-2\\1\end{pmatrix}, \ c \in R.$

$$|\overrightarrow{PQ} \cdot \overrightarrow{AB}| = \left| |\overrightarrow{PQ}||\overrightarrow{AB}| \cos\theta \right|$$

$$\left|c\begin{pmatrix}2\\-2\\1\end{pmatrix} \cdot \begin{pmatrix}-4\\-1\\-3\end{pmatrix}\right| = \left|c\begin{pmatrix}2\\-2\\1\end{pmatrix}\begin{pmatrix}-4\\-3\\-1\end{pmatrix}\cos\theta\right|$$

$$|c(-8+2-3)| = 3c\,|\overrightarrow{AB}|\cos\theta$$

$$|\overrightarrow{AB}|\cos\theta = 3$$

$\therefore |\overrightarrow{PQ}| = 3$, since $|\overrightarrow{PQ}| = |\overrightarrow{AB}|\cos\theta$.

ii) Since $\overrightarrow{PQ} = \begin{pmatrix}-4-s\\ t-1-s\\ -3+2t\end{pmatrix}$,

$$\begin{pmatrix}-4-s\\ t-1-s\\ -3+2t\end{pmatrix} = c\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$\begin{pmatrix}-4\\-3\\-1\end{pmatrix} \times \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -4 & -3 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}.$$

Since $\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ is a normal to the plane

and A is a point on the plane, if

$\underset{\sim}{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ is a point on the plane,

$$\underset{\sim}{r} \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix} = 3-1-2$$

$$x - y - z = 0$$

$\therefore$ The plane containing AB and ℓ_1 has equation $x - y - z = 0$.

The line passing through Q and perpendicular to the plane has equation $\underset{\sim}{r} = \begin{pmatrix}-1\\1\\1\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\-1\end{pmatrix}$

When the line intersects the plane,

$x = -1 + \alpha,\ y = 1 - \alpha,\ z = 1 - \alpha.$

$\therefore -1 + \alpha - (1 - \alpha) - (1 - \alpha) = 0$

$$\left.\begin{aligned} -4 - s &= 2c \\ t - 1 - s &= -2c \\ -3 + 2t &= c \end{aligned}\right\}$$

$$\left.\begin{aligned} s \quad + 2c &= -4 \\ s - t - 2c &= -1 \\ 2t - c &= 3 \end{aligned}\right\}$$

$-①+②:$
$$\left.\begin{aligned} s \quad + 2c &= -4 \\ -t - 4c &= 3 \\ 2t - c &= 3 \end{aligned}\right\}$$

$2 \times ② + ③:$
$$\left.\begin{aligned} s \quad + 2c &= -4 \\ -t - 4c &= 3 \\ -9c &= 9 \end{aligned}\right\}$$

$$c = -1$$

$$t = 1$$

$$s = -2$$

$\therefore Q(-1, 1, 1)$

iii) The plane containing AB and ℓ_1 is parallel to $\begin{pmatrix}-4\\-3\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

$\therefore$ The normal of the plane is parallel to $\begin{pmatrix}-4\\-3\\-1\end{pmatrix} \times \begin{pmatrix}1\\1\\0\end{pmatrix}$ since it is perpendicular to both $\begin{pmatrix}-4\\-3\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

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$-1+\alpha-1+\alpha-1+\alpha=0$

$3\alpha=3$

$\alpha=1$

$\therefore$ The line meets the plane at $(0,0,0)$.

$\therefore$ The perpendicular distance from Q to the plane containing AB and ℓ_1 is

$\sqrt{(-1-0)^2+(1-0)^2+(1-0)^2}$

$=\sqrt{3}$.

OR

$T: R^4 \to R^4 \quad M = \begin{pmatrix} 1 & 1 & 5 & 7 \\ 3 & 9 & 17 & 25 \\ 1 & 7 & 7 & 11 \\ 3 & 6 & 16 & 23 \end{pmatrix}$

i) a) $\begin{pmatrix} 1 & 1 & 5 & 7 \\ 3 & 9 & 17 & 25 \\ 1 & 7 & 7 & 11 \\ 3 & 6 & 16 & 23 \end{pmatrix}$

$\xrightarrow[-3r_1+r_4]{-3r_1+r_2,\ -r_1+r_3} \begin{pmatrix} 1 & 1 & 5 & 7 \\ 0 & 6 & 2 & 4 \\ 0 & 6 & 2 & 4 \\ 0 & 3 & 1 & 2 \end{pmatrix}$

$\xrightarrow{\frac{r_2}{2},\ \frac{r_3}{3}} \begin{pmatrix} 1 & 1 & 5 & 7 \\ 0 & 3 & 1 & 2 \\ 0 & 3 & 1 & 2 \\ 0 & 3 & 1 & 2 \end{pmatrix}$

$\xrightarrow{-r_2+r_3,\ -r_2+r_4} \begin{pmatrix} 1 & 1 & 5 & 7 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ The dimension of R, the range space of T is 2.

b) A basis for R is $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ 7 \\ 6 \end{pmatrix} \right\}$.

ii) If $\begin{pmatrix}1\\-15\\-17\\-6\end{pmatrix} \in R$,

$$\begin{pmatrix}1\\-15\\-17\\-6\end{pmatrix} = \alpha\begin{pmatrix}1\\3\\1\\3\end{pmatrix} + \beta\begin{pmatrix}1\\9\\7\\6\end{pmatrix}, \quad \alpha,\beta \in R$$

$$= \begin{pmatrix}\alpha+\beta\\3\alpha+9\beta\\\alpha+7\beta\\3\alpha+6\beta\end{pmatrix}$$

$$\left.\begin{aligned}\alpha+\beta&=1\\3\alpha+9\beta&=-15\\\alpha+7\beta&=-17\\3\alpha+6\beta&=-6\end{aligned}\right\}$$

$$\left.\begin{aligned}\alpha+\beta&=1\\\alpha+3\beta&=-5\\\alpha+7\beta&=-17\\\alpha+2\beta&=-2\end{aligned}\right\}$$

$\alpha=4$, $\beta=-3$

$\therefore \begin{pmatrix}1\\-15\\-17\\-6\end{pmatrix}$ belongs to R.

iii) If $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ is a basis for the null space of T, $\underset{\sim}{e}_1 = \begin{pmatrix}14\\1\\-3\\0\end{pmatrix}$, $\underset{\sim}{e}_2 = \begin{pmatrix}19\\2\\0\\-3\end{pmatrix}$ and

$$\underset{\sim}{x} = \begin{pmatrix}4\\-3\\0\\0\end{pmatrix} + \lambda\underset{\sim}{e}_1 + \mu\underset{\sim}{e}_2, \quad \lambda,\mu \in R,$$

since $M\begin{pmatrix}4\\-3\\0\\0\end{pmatrix} = \begin{pmatrix}1&1&5&7\\3&9&17&25\\1&7&7&11\\3&6&16&23\end{pmatrix}\begin{pmatrix}4\\-3\\0\\0\end{pmatrix} = \begin{pmatrix}1\\-15\\-17\\-6\end{pmatrix}$

and $M(\lambda\underset{\sim}{e}_1 + \mu\underset{\sim}{e}_2)$

$$= \begin{pmatrix}1&1&5&7\\3&9&17&25\\1&7&7&11\\3&6&16&23\end{pmatrix}\left(\lambda\begin{pmatrix}14\\1\\-3\\0\end{pmatrix} + \mu\begin{pmatrix}19\\2\\0\\-3\end{pmatrix}\right)$$

$$= \begin{pmatrix}1&1&5&7\\3&9&17&25\\1&7&7&11\\3&6&16&23\end{pmatrix}\begin{pmatrix}14\lambda+19\mu\\\lambda+2\mu\\-3\lambda\\-3\mu\end{pmatrix}$$

$$= \begin{pmatrix}14\lambda+19\mu+\lambda+2\mu-15\lambda-21\mu\\42\lambda+57\mu+9\lambda+18\mu-51\lambda-75\mu\\14\lambda+19\mu+7\lambda+14\mu-21\lambda-33\mu\\42\lambda+57\mu+6\lambda+12\mu-48\lambda-69\mu\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$