1. The formula to use is $d = \frac{|(\underset{\sim}{p} - \underset{\sim}{q})| \times \underset{\sim}{b}|}{|\underset{\sim}{b}|}$

$$\text{Distance} = \frac{\left| \left[\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right] \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right|}$$

$$= \frac{\left| \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right|}{\sqrt{1+1+1}}$$

$$= \frac{\left| \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \right|}{\sqrt{3}}$$

$$= \frac{\sqrt{38}}{\sqrt{3}}$$

$$\approx 3.56$$

2. $$\left.\begin{aligned} x + ay + a^2 z &= 0 \\ ax + y + az &= 0 \\ a^2 x + ay + z &= 0 \end{aligned}\right\}$$

$-a \times ① + ②$:
$-a^2 \times ① + ③$:

$$\left.\begin{aligned} x + ay + a^2 z &= 0 \\ (1-a^2)y + (a-a^3)z &= 0 \\ (a-a^3)y + (1-a^4)z &= 0 \end{aligned}\right\}$$

$-a \times ② + ③$:

$$\left.\begin{aligned} x + ay + a^2 z &= 0 \\ (1-a^2)y + (a-a^3)z &= 0 \\ (1-a^2)z &= 0 \end{aligned}\right\}$$

If $a = 1$: $0z = 0$

Let $z = s, s \in R$

$y = t, t \in R$

$x = -s - t$

If $a = -1$: $0z = 0$

Let $z = s, s \in R$

$y = t, t \in R$

$x = t - s$

If $a \neq 1, -1$: $z = 0$

$y = 0$

$x = 0$

3. $$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{N}{N+1}$$

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\sum_{n=1}^{N} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{N} \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{1}{2} - \frac{1}{N+2}$$

$$= \frac{N}{2(N+2)}$$

$$\frac{a}{n(n+1)} + \frac{b}{(n+1)(n+2)} = \frac{5n+6}{n(n+1)(n+2)}$$

$$a(n+2) + bn = 5n+6$$

$\therefore\ 2a = 6$ & $a + b = 5$

$a = 3$ $\qquad b = 2.$

$$\sum_{n=1}^{N} \frac{5n+6}{n(n+1)(n+2)} = \sum_{n=1}^{N} \frac{3}{n(n+1)} + \sum_{n=1}^{N} \frac{2}{(n+1)(n+2)}$$

$$= 3\left(1 - \frac{1}{N+1}\right) + 2\left(\frac{1}{2} - \frac{1}{N+2}\right)$$

$$= 4 - \frac{3}{N+1} - \frac{2}{N+2}$$

$$\sum_{n=1}^{\infty} \frac{5n+6}{n(n+1)(n+2)} = 4$$

4. a) $\sum_{r=1}^{n} (3^r + 3^{r-1})r = (2n-1)3^n + 1$

when $n=1$: $\sum_{r=1}^{1} (3^r + 3^{r-1})r = (3^1 + 3^{1-1})1$

$= 3^1 + 3^0$

$= 3 + 1$

$= 4$

$= 6 - 3 + 1$

$= 3(2-1) + 1$

$= 3(2(1)-1) + 1$

$= 3^1(2(1)-1) + 1$

Assume the statement is true when $n=k$.

$n=k$: $\sum_{r=1}^{k} (3^r + 3^{r-1})r = (2k-1)3^k + 1$.

when $n=k+1$:

$\sum_{r=1}^{k+1} (3^r + 3^{r-1})r = (2(k+1)-1)3^{k+1} + 1$

(what needs to be proved)

$\sum_{r=1}^{k+1} (3^r + 3^{r-1})r = \sum_{r=1}^{k} (3^r + 3^{r-1})r + (3^{k+1} + 3^k)(k+1)$

$= (2k-1)3^k + 1 + (3^{k+1} + 3^k)(k+1)$

$= (2k-1)3^k + 1 + (3^k 3 + 3^k)(k+1)$

$= (2k-1)3^k + 1 + 3^k 4(k+1)$

$= (2k - 1 + 4k + 4)3^k + 1$

$= (6k+3)3^k + 1$

$= 3(2k+1)3^k + 1$

$= (2k+1)3^{k+1} + 1$

$= (2(k+1) - 1)3^{k+1} + 1$

$\therefore \sum_{r=1}^{n} (3^r + 3^{r-1})r = (2n-1)3^n + 1$

for every positive integer n.

b) Let $f(n) = 7^{2n+1} + 9^n$

When $n=1$: $f(1) = 7^{2(1)+1} + 9^1$

$= 7^{2+1} + 9^1$

$= 7^3 + 9$

$= 343 + 9$

$= 352$

$= 8(44)$

$\therefore 8 \mid f(1)$

Assume the statement is true when $n=k$.

$n=k$: $8 \mid f(k)$

$f(k) = 8s$, s is an integer.

$7^{2k+1} + 9^k = 8s$.

When $n = k+1$:

$f(k+1) = 7^{2(k+1)+1} + 9^{k+1}$

$= 7^{2k+2+1} + 9^{k+1}$

$= 7^{2k+1}7^2 + 9^k 9$

$= 7^{2k+1}49 + 9^k 9$

$= 7^{2k+1}(40+9) + 9^k 9$

$= 7^{2k+1}40 + 7^{2k+1}9 + 9^k 9$

$= 7^{2k+1}40 + 9(7^{2k+1} + 9^k)$

$= 7^{2k+1}40 + 9(8s)$

$= 8(7^{2k+1}5 + 9s)$

Since s is an integer and k is an integer, $7^{2k+1}s + 9s$ is an integer.

$\therefore 8 \mid f(k+1)$

$\therefore 7^{2n+1} + 9^n$ is divisible by 8 for every positive integer n.

5. i) $y = \dfrac{x^2 - 6x + 5}{x^2 - 2x - 3} = 1 + \dfrac{8 - 4x}{x^2 - 2x - 3}$

$y = 1$; $x = -1$; $x = 3$

ii) $\dfrac{dy}{dx} = \dfrac{-4(x^2 - 2x - 3) - (8 - 4x)(2x - 2)}{(x^2 - 2x - 3)^2}$

$-4(x^2 - 2x - 3) + 8(x - 2)(x - 1)$

$= 4[2(x^2 - 3x + 2) - (x^2 - 2x - 3)]$

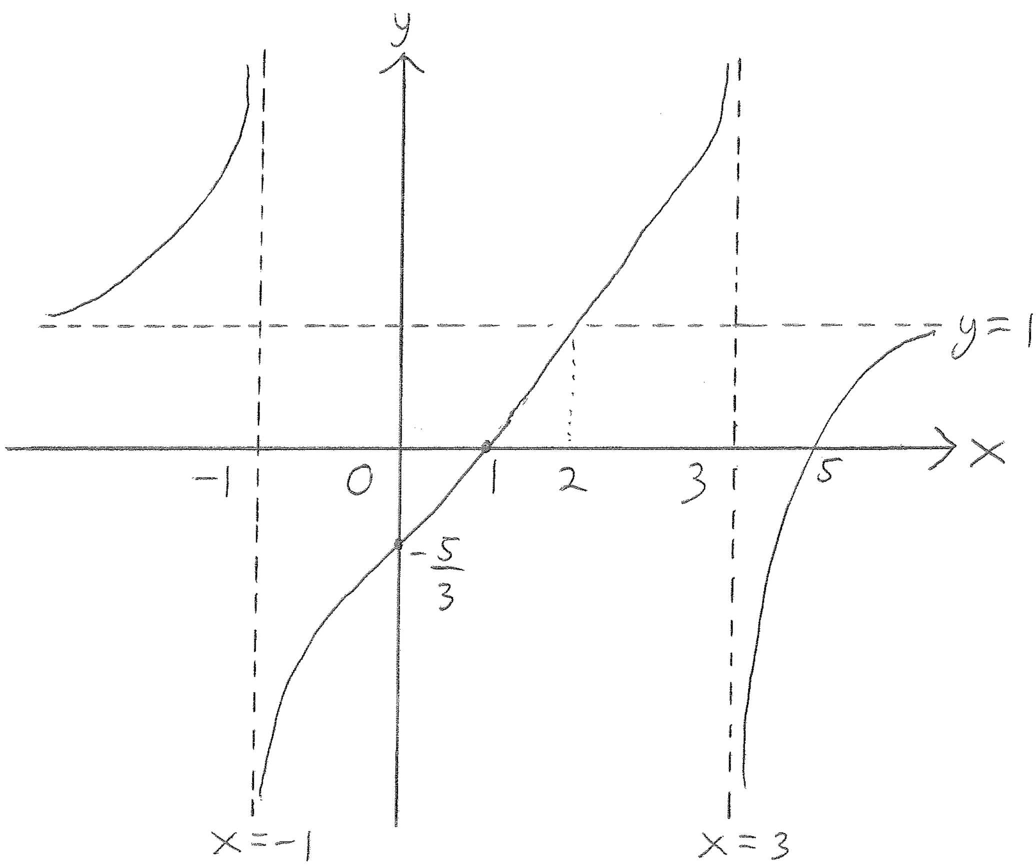
$= 4[x^2 - 4x + 7]$

Discriminant $= 16 - 28 < 0$

$\Rightarrow$ No turning points

since $\dfrac{dy}{dx} \neq 0 \quad \forall x \in R$.

iii)



6. a) $5x^4 - 6x^3 + 2 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$\alpha + \beta + \gamma + \delta = \frac{6}{5}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 0$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 0$

$\alpha\beta\gamma\delta = \frac{2}{5}$

$\alpha^2, \beta^2, \gamma^2, \delta^2$

Let $u = \alpha^2$

$\alpha = \pm\sqrt{u}$

α is a root

$\therefore 5\alpha^4 - 6\alpha^3 + 2 = 0$

$5(\pm\sqrt{u})^4 - 6(\pm\sqrt{u})^3 + 2 = 0$

$5u^2 - 6(\pm u\sqrt{u}) + 2 = 0$

$5u^2 + 2 = \pm 6u\sqrt{u}$

$(5u^2+2)^2 = (\pm 6u\sqrt{u})^2$

$25u^4 + 20u^2 + 4 = 36u^3$

$25u^4 - 36u^3 + 20u^2 + 4 = 0$

$\therefore$ The equation $25u^4 - 36u^3 + 20u^2 + 4 = 0$ has roots $\alpha^2, \beta^2, \gamma^2, \delta^2$.

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{36}{25}$$

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2 = \frac{4}{5}$$

$$\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2 = 0$$

$$\alpha^2\beta^2\gamma^2\delta^2 = \frac{4}{25}$$

$$S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

$$\begin{aligned} S_0 &= \alpha^0 + \beta^0 + \gamma^0 + \delta^0 \\ &= 1 + 1 + 1 + 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} S_1 &= \alpha^1 + \beta^1 + \gamma^1 + \delta^1 \\ &= \alpha + \beta + \gamma + \delta \\ &= \frac{6}{5} \end{aligned}$$

$$\begin{aligned} S_2 &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \\ &= (\alpha + \beta + \gamma + \delta)^2 \\ &\quad -2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) \\ &= \left(\frac{6}{5}\right)^2 - 2(0) \\ &= \frac{36}{25} \end{aligned}$$

$$S_4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4$$

$$= (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2$$

$$-2(\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2)$$

$$= \left(\frac{36}{25}\right)^2 - 2\left(\frac{4}{5}\right)$$

$$= \frac{1296}{625} - \frac{8}{5}$$

$$= \frac{296}{625}$$

$$S_{-2} = \alpha^{-2} + \beta^{-2} + \gamma^{-2} + \delta^{-2}$$

$$= \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$$

$$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right)^2$$

$$-2\left(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\alpha\delta} + \frac{1}{\beta\gamma} + \frac{1}{\beta\delta} + \frac{1}{\gamma\delta}\right)$$

$$= \left(\frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha\beta\gamma\delta}\right)^2$$

$$-2\left(\frac{\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta}{\alpha\beta\gamma\delta}\right)$$

$$= 0 - 2(0)$$

$$= 0$$

$$S_{-4} = \alpha^{-4} + \beta^{-4} + \gamma^{-4} + \delta^{-4}$$

$$= \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}$$

$$= \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}\right)^2$$

$$-2\left(\frac{1}{\alpha^2\beta^2} + \frac{1}{\alpha^2\gamma^2} + \frac{1}{\alpha^2\delta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\beta^2\delta^2} + \frac{1}{\gamma^2\delta^2}\right)$$

$$= \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}\right)^2$$

$$-2\left(\frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2}{\alpha^2\beta^2\gamma^2\delta^2}\right)$$

$$= 0 - \frac{2\left(\frac{4}{5}\right)}{\frac{4}{25}}$$

$$= -10.$$

b) $ax^3 + bx^2 + cx + d = 0$

α, β, r are the roots

$\alpha + \beta + r = -\frac{b}{a} \quad \alpha\beta + \alpha r + \beta r = \frac{c}{a} \quad \alpha\beta r = -\frac{d}{a}$

If $\alpha^2 = \frac{\alpha}{\beta r} + \beta r$, $\beta^2 = \frac{\beta}{\alpha r} + \alpha r$, $r^2 = \frac{r}{\alpha\beta} + \alpha\beta$,

$$\alpha^2 + \beta^2 + r^2 = \frac{\alpha}{\beta r} + \beta r + \frac{\beta}{\alpha r} + \alpha r + \frac{r}{\alpha\beta} + \alpha\beta$$

$$= \frac{\alpha^2 + \beta^2 + r^2}{\alpha\beta r} + \alpha\beta + \alpha r + \beta r$$

$$(\alpha^2 + \beta^2 + r^2)\alpha\beta r = \alpha^2 + \beta^2 + r^2 + (\alpha\beta + \alpha r + \beta r)\alpha\beta r$$

$$(\alpha^2 + \beta^2 + r^2)(\alpha\beta r - 1) = (\alpha\beta + \alpha r + \beta r)\alpha\beta r$$

$$[(\alpha + \beta + r)^2 - 2(\alpha\beta + \alpha r + \beta r)](\alpha\beta r - 1)$$

$$= (\alpha\beta + \alpha r + \beta r)\alpha\beta r$$

$$\left[\left(-\frac{b}{a}\right)^2 - \frac{2c}{a}\right]\left(-\frac{d}{a} - 1\right) = \frac{c}{a}\left(-\frac{d}{a}\right)$$

$$\left(\frac{b^2}{a^2} - \frac{2c}{a}\right)\left(\frac{d}{a} + 1\right) = \frac{cd}{a^2}$$

$$(b^2 - 2ac)(a + d) = acd$$

$$b^2 - 2ac = \frac{acd}{a + d}$$

$$b^2 = 2ac + \frac{acd}{a + d}$$

$$b = \pm\sqrt{\frac{2a^2c + 3acd}{a + d}}$$