MAY / JUNE 2007

1. $$\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} = \frac{(n+1)^2+1-(n^2+1)}{(n^2+1)((n+1)^2+1)}$$

$$= \frac{n^2+2n+1+1-n^2-1}{(n^2+1)((n+1)^2+1)}$$

$$= \frac{2n+1}{(n^2+1)((n+1)^2+1)}$$

Since

$$\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} = \frac{2n+1}{(n^2+1)((n+1)^2+1)},$$

$$\sum_{n=1}^{N} \frac{2n+1}{(n^2+1)((n+1)^2+1)}$$

$$= \sum_{n=1}^{N} \frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}$$

$$= \frac{1}{1^2+1} - \frac{1}{2^2+1}$$

$$+ \frac{1}{2^2+1} - \frac{1}{3^2+1}$$

$$+ \frac{1}{3^2+1} - \frac{1}{4^2+1}$$

$$\vdots$$

$$+ \frac{1}{(N-1)^2+1} - \frac{1}{N^2+1}$$

$$+ \frac{1}{N^2+1} - \frac{1}{(N+1)^2+1}$$

$$= \frac{1}{2} - \frac{1}{(N+1)^2+1}.$$

Since $N \geqslant 1$, $\frac{1}{(N+1)^2+1} > 0$

and $\frac{1}{2} - \frac{1}{(N+1)^2+1} < \frac{1}{2}$

$$\therefore \sum_{n=1}^{N} \frac{2n+1}{(n^2+1)((n+1)^2+1)} < \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+1)((n+1)^2+1)}$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \frac{2n+1}{(n^2+1)((n+1)^2+1)}$$

$$= \lim_{N \to \infty} \left(\frac{1}{2} - \frac{1}{(N+1)^2 + 1} \right)$$

$$= \frac{1}{2}$$

2. $C: \quad x = t - \ln t \quad y = 4t^{\frac{1}{2}}, \quad t > 0$

$$\frac{dx}{dt} = 1 - \frac{1}{t} \qquad \frac{dy}{dt} = 2t^{-\frac{1}{2}}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 - \frac{1}{t}\right)^2 + \left(2t^{-\frac{1}{2}}\right)^2$$

$$= 1 - \frac{2}{t} + \frac{1}{t^2} + \frac{4}{t}$$

$$= 1 + \frac{2}{t} + \frac{1}{t^2}$$

$$= \left(1 + \frac{1}{t}\right)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1 + \frac{1}{t}$$

The area of the surface generated by rotating the arc of C from $t = 1$ to $t = 4$ about the x-axis is

$$\int_1^4 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \int_1^4 2\pi (4t^{\frac{1}{2}})\left(1 + \frac{1}{t}\right) dt$$

$$= \int_1^4 8\pi \left(t^{\frac{1}{2}} + \frac{1}{t^{\frac{1}{2}}}\right) dt$$

$$= 8\pi \left[\frac{2t^{\frac{3}{2}}}{3} + 2t^{\frac{1}{2}}\right]_1^4$$

$$= 8\pi \left(\frac{2}{3}(8) + 2(2) - \frac{2}{3} - 2\right)$$

$$= 8\pi \left(\frac{16}{3} + 4 - \frac{2}{3} - 2\right)$$

$$= 8\pi \left(\frac{20}{3}\right)$$

$$= \frac{160\pi}{3}$$

3. $$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 29y = 58x + 37$$

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 29y = 0$$

$$m^2 + 4m + 29 = 0$$

$$(m + 2)^2 + 25 = 0$$

$$(m + 2)^2 = -25$$

$$m + 2 = \pm 5i$$

$$m = -2 \pm 5i$$

$\therefore$ The complementary function, y_c is given by

$$y_c = e^{-2x}(A\cos 5x + B\sin 5x)$$

The particular integral, y_p is

$$y_p = Ax + B$$

$$\frac{dy_p}{dx} = A$$

$$\frac{d^2y_p}{dx^2} = 0$$

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$$\therefore \frac{d^2y_p}{dx^2} + \frac{4dy_p}{dx} + 29y_p$$

$$= 0 + 4A + 29(Ax + B)$$

$$= 29Ax + 4A + 29B$$

$$= 58x + 37$$

$\therefore 29A = 58 \qquad 4A + 29B = 37$

$A = 2 \qquad 4(2) + 29B = 37$

$8 + 29B = 37$

$29B = 29$

$B = 1$

$\therefore y_p = 2x + 1$

$y = y_c + y_p$

$= e^{-2x}(A\cos 5x + B\sin 5x) + 2x + 1$

$\therefore$ The general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 29y = 58x + 37$$

is $y = e^{-2x}(A\cos 5x + B\sin 5x) + 2x + 1$

4. $y = x + e^{-xy}$

when $x = 0$ $\quad y = 0 + 1$

$= 1$

$$\frac{dy}{dx} = \frac{d}{dx}(x + e^{-xy})$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}(e^{-xy})$$

$$= 1 + e^{-xy}\frac{d}{dx}(-xy)$$

$$= 1 + e^{-xy}\left(-y - x\frac{dy}{dx}\right)$$

$$= 1 - ye^{-xy} - xe^{-xy}\frac{dy}{dx}$$

At $x = 0$ and $y = 1$:

$$\frac{dy}{dx} = 1 - 1 - 0$$

$$= 0.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(1 - ye^{-xy} - xe^{-xy}\frac{dy}{dx}\right)$$

$$= 0 - \frac{d}{dx}(ye^{-xy}) - \frac{d}{dx}\left(xe^{-xy}\frac{dy}{dx}\right)$$

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$$= -e^{-xy}\frac{dy}{dx} - y\frac{d}{dx}(e^{-xy}) - xe^{-xy}\frac{d^2y}{dx^2}$$

$$-\frac{dy}{dx}\frac{d}{dx}(e^{-xy})$$

$$= -e^{-xy}\frac{dy}{dx} - ye^{-xy}\frac{d}{dx}(-xy) - xe^{-xy}\frac{d^2y}{dx^2}$$

$$-\frac{dy}{dx}(e^{-xy})\frac{d}{dx}(-xy)$$

$$= -e^{-xy}\frac{dy}{dx} - ye^{-xy}\left(-y - x\frac{dy}{dx}\right) - xe^{-xy}\frac{d^2y}{dx^2}$$

$$-e^{-xy}\frac{dy}{dx}\left(-y - x\frac{dy}{dx}\right)$$

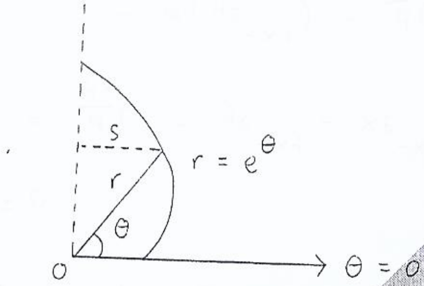
$$= -e^{-xy}\frac{dy}{dx} + y^2e^{-xy} + xye^{-xy}\frac{dy}{dx}$$

$$-xe^{-xy}\frac{d^2y}{dx^2} + ye^{-xy}\frac{dy}{dx} + xe^{-xy}\left(\frac{dy}{dx}\right)^2$$

When $x = 0$, $y = 1$ and $\frac{dy}{dx} = 0$:

$$\frac{d^2y}{dx^2} = 1$$

5.



The perpendicular distance, s, of a point of C from the line $\theta = \frac{\pi}{2}$ is given by

$$s = r\sin\left(\frac{\pi}{2} - \theta\right)$$

$$= r\cos\theta$$

$$= e^{\theta}\cos\theta$$

$$\frac{ds}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta$$

When $\frac{ds}{d\theta} = 0$

$$e^{\theta}\cos\theta - e^{\theta}\sin\theta = 0$$

$$e^{\theta}(\cos\theta - \sin\theta) = 0$$

$$\cos\theta - \sin\theta = 0$$

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$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

$$\frac{d^2s}{d\theta^2} = e^{\theta}\cos\theta - e^{\theta}\sin\theta - e^{\theta}\sin\theta - e^{\theta}\cos\theta$$

$$= -2e^{\theta}\sin\theta$$

When $\theta = \frac{\pi}{4}$: $\frac{d^2s}{d\theta^2} = -\sqrt{2}\,e^{\frac{\pi}{4}} < 0$

$\therefore$ The maximum distance of a point of C from the line $\theta = \frac{\pi}{2}$ is $\frac{e^{\frac{\pi}{4}}}{\sqrt{2}}$ when $\theta = \frac{\pi}{4}$.

The area of the region bounded by C and the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$ is

$$\int_0^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{e^{2\theta}}{2}\, d\theta$$

$$= \left[\frac{e^{2\theta}}{2(2)}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{e^{2\theta}}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{e^{\pi} - 1}{4}$$

6.

$$A = \begin{pmatrix} 7 & -4 & 6 \\ 2 & 2 & 2 \\ -3 & 4 & -2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 7 & -4 & 6 \\ 2 & 2 & 2 \\ -3 & 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7-\lambda & -4 & 6 \\ 2 & 2-\lambda & 2 \\ -3 & 4 & -2-\lambda \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & -4 & 6 \\ 2 & 2-\lambda & 2 \\ -3 & 4 & -2-\lambda \end{vmatrix}$$

The eigenvalues, λ, of A are 1, 2 and 4.

$\lambda = 1$:

$$\begin{pmatrix} 6 & -4 & 6 \\ 2 & 1 & 2 \\ -3 & 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 6 & -4 & 6 & 0 \\ 2 & 1 & 2 & 0 \\ -3 & 4 & -3 & 0 \end{array}\right)$$

$$\xrightarrow[2 \times r_3]{3 \times r_2} \left(\begin{array}{ccc|c} 6 & -4 & 6 & 0 \\ 6 & 3 & 6 & 0 \\ -6 & 8 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[r_1 + r_3]{-r_1 + r_2} \left(\begin{array}{ccc|c} 6 & -4 & 6 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s$, $s \in R$

$x = -s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 2$:

$$\begin{pmatrix} 5 & -4 & 6 \\ 2 & 0 & 2 \\ -3 & 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 5 & -4 & 6 & 0 \\ 2 & 0 & 2 & 0 \\ -3 & 4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow[10 \times ③]{6 \times ①,\ 15 \times ②} \left(\begin{array}{ccc|c} 30 & -24 & 36 & 0 \\ 30 & 0 & 30 & 0 \\ -30 & 40 & -40 & 0 \end{array}\right)$$

$$\xrightarrow[r_1 + r_3]{-r_1 + r_2} \left(\begin{array}{ccc|c} 30 & -24 & 36 & 0 \\ 0 & 24 & -6 & 0 \\ 0 & 16 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{6},\ \frac{r_2}{6},\ \frac{r_3}{4}} \left(\begin{array}{ccc|c} 5 & -4 & 6 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 5 & -4 & 6 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 4s,\ s \in R$

$y = s$

$x = -4s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4s \\ s \\ 4s \end{pmatrix}$$

$$= s \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}$$

$x = 4$: $\begin{pmatrix} 3 & -4 & 6 \\ 2 & -2 & 2 \\ -3 & 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & -4 & 6 & 0 \\ 2 & -2 & 2 & 0 \\ -3 & 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[2 \times ③]{2 \times ③,\ 3 \times ②} \left(\begin{array}{ccc|c} 6 & -8 & 12 & 0 \\ 6 & -6 & 6 & 0 \\ -6 & 8 & -12 & 0 \end{array}\right)$$

$$\xrightarrow[r_1 + r_3]{-r_1 + r_2} \left(\begin{array}{ccc|c} 6 & -8 & 12 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2},\ \frac{r_2}{2}} \left(\begin{array}{ccc|c} 3 & -4 & 6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s,\ s \in R$

$y = 3s$

$x = 2s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ 3s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$\therefore$ The eigenvectors of A are

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 10 & -4 & 6 \\ 2 & 5 & 2 \\ -3 & 4 & 1 \end{pmatrix}$$

Since $B = A + 3I$

$$B\underset{\sim}{x} = (A + 3I)\underset{\sim}{x}$$

$$= A\underset{\sim}{x} + (3I)\underset{\sim}{x}$$

$$= \lambda\underset{\sim}{x} + 3(I\underset{\sim}{x})$$

$$= \lambda\underset{\sim}{x} + 3\underset{\sim}{x}$$

$$= (\lambda + 3)\underset{\sim}{x}$$

$\therefore$ If A has an eigenvalue λ with corresponding eigenvector $\underset{\sim}{x}$, B has an eigenvalue $\lambda + 3$ with eigenvector $\underset{\sim}{x}$.

Since A has eigenvalues $1, 2, 4$ with corresponding eigenvectors

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$\therefore$ B has eigenvalues $4, 5, 7$ with corresponding eigenvectors.

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

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7. $x^3 + 3x - 1 = 0$

α, β, γ are the roots;

$\alpha^3, \beta^3, \gamma^3$

Let $y = \alpha^3$

$\alpha = y^{\frac{1}{3}}$

α is a root

$\therefore \alpha^3 + 3\alpha - 1 = 0$

$(y^{\frac{1}{3}})^3 + 3y^{\frac{1}{3}} - 1 = 0$

$y + 3y^{\frac{1}{3}} - 1 = 0$

$3y^{\frac{1}{3}} = -y + 1$

$(3y^{\frac{1}{3}})^3 = (-y+1)^3$

$27y = -y^3 + 3y^2 - 3y + 1$

$y^3 - 3y^2 + 30y - 1 = 0.$

$\therefore$ The equation having roots $\alpha^3, \beta^3, \gamma^3$ is $y^3 - 3y^2 + 30y - 1 = 0.$

$\alpha^3 + \beta^3 + \gamma^3 = 3$

$\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3 = 30$

$\alpha^3\beta^3\gamma^3 = 1$

Let $S_n = \alpha^{3n} + \beta^{3n} + \gamma^{3n}$

$S_0 = \alpha^0 + \beta^0 + \gamma^0$

$= 1 + 1 + 1$

$= 3$

$S_1 = \alpha^3 + \beta^3 + \gamma^3$

$= 3$

$S_2 = \alpha^6 + \beta^6 + \gamma^6$

$= (\alpha^3 + \beta^3 + \gamma^3)^2 - 2(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3)$

$= 3^2 - 2(30)$

$= 9 - 60$

$= -51.$

$S_3 - 3S_2 + 30S_1 - S_0 = 0$

$S_3 - 3(-51) + 30(3) - 3 = 0$

$S_3 + 153 + 90 - 3 = 0$

$S = -240$

$\alpha^9 + \beta^9 + \gamma^9 = -240$

8. $x_1, x_2, x_3, \ldots, x_1 = 1$

$$x_{n+1} = \frac{1 + 4x_n}{5 + 2x_n}$$

$x_n > \frac{1}{2}$

when $n = 1$: $x_1 = 1 > \frac{1}{2}$

Assume the statement is true when $n = k$.

$n = k$: $x_k > \frac{1}{2}$

Since $x_k > \frac{1}{2}$

$6x_k > 3$

$2 + 8x_k > 5 + 2x_k > 0$

$2(1 + 4x_k) > 5 + 2x_k$

$$\frac{1 + 4x_k}{5 + 2x_k} > \frac{1}{2}$$

$x_{k+1} > \frac{1}{2}$.

Since $x_1 > \frac{1}{2}$ and $x_{k+1} > \frac{1}{2}$ if $x_k > \frac{1}{2}$, $x_n > \frac{1}{2}$ for all $n \geqslant 1$.

$$x_n - x_{n+1} = x_n - \left(\frac{1 + 4x_n}{5 + 2x_n}\right)$$

$$= \frac{x_n(5 + 2x_n) - (1 + 4x_n)}{5 + 2x_n}$$

$$= \frac{5x_n + 2x_n^2 - 1 - 4x_n}{5 + 2x_n}$$

$$= \frac{2x_n^2 + x_n - 1}{5 + 2x_n}$$

$$= \frac{(2x_n - 1)(x_n + 1)}{5 + 2x_n}$$

Since $x_n > 1$, $2x_n - 1 > 0$

$x_n - x_{n+1} > 0$

$x_n > x_{n+1}$ for all $n \geq 1$.

9 $I_n = \int_0^1 \frac{1}{(4 - x^2)^n} \, dx$, $n = 1, 2, 3, \ldots$

$$\frac{d}{dx}\left[\frac{x}{(4 - x^2)^n}\right] = x \frac{d}{dx}\left(\frac{1}{(4 - x^2)^n}\right) + \frac{1}{(4 - x^2)^n} \frac{d}{dx}(x)$$

$$= x\left(\frac{-n}{(4 - x^2)^{n+1}}\right)(-2x) + \frac{1}{(4 - x^2)^n}$$

$$= \frac{2nx^2}{(4 - x^2)^{n+1}} + \frac{1}{(4 - x^2)^n}$$

$$= \frac{2n(4 - 4 + x^2)}{(4 - x^2)^{n+1}} + \frac{1}{(4 - x^2)^n}$$

$$= \frac{8n + 2n(-4 + x^2)}{(4 - x^2)^{n+1}} + \frac{1}{(4 - x^2)^n}$$

$$= \frac{8n - 2n(4 - x^2)}{(4 - x^2)^{n+1}} + \frac{1}{(4 - x^2)^n}$$

$$= \frac{8n}{(4 - x^2)^{n+1}} - \frac{2n(4 - x^2)}{(4 - x^2)^{n+1}} + \frac{1}{(4 - x^2)^n}$$

$$= \frac{8n}{(4 - x^2)^{n+1}} - \frac{2n}{(4 - x^2)^n} + \frac{1}{(4 - x^2)^n}$$

$$= \frac{8n}{(4 - x^2)^{n+1}} + \frac{1 - 2n}{(4 - x^2)^n}$$

since $\frac{d}{dx}\left[\frac{x}{(4 - x^2)^n}\right] = \frac{8n}{(4 - x^2)^{n+1}} + \frac{1 - 2n}{(4 - x^2)^n}$,

$$\left[\frac{x}{(4 - x^2)^n}\right]_0^1 = \int_0^1 \frac{8n}{(4 - x^2)^{n+1}}\,dx + \int_0^1 \frac{1 - 2n}{(4 - x^2)^n}\,dx$$

$$\frac{1}{3^n} - 0 = 8n \int_0^1 \frac{1}{(4 - x^2)^{n+1}}\,dx + (1 - 2n) \int_0^1 \frac{1}{(4 - x^2)^n}\,dx$$

$$\frac{1}{3^n} = 8n\, I_{n+1} + (1 - 2n) I_n$$

$$8n\, I_{n+1} = (2n - 1) I_n + \frac{1}{3^n}$$

The y-coordinate of the centroid of the region bounded by the axes, the line $x = 1$ and the curve $y = \frac{1}{4 - x^2}$, $\bar{y}$, is

$$\frac{\int_0^1 \frac{y^2}{2}\,dx}{\int_0^1 y\,dx}$$

$$= \frac{\int_0^1 \frac{1}{(4-x^2)^2}\,dx}{2\int_0^1 \frac{1}{4-x^2}\,dx}$$

$$= \frac{I_2}{2I_1}$$

$$n=1: \quad 8(1)\,I_2 = (2(1)-1)\,I_1 + \frac{1}{3^1}$$

$$8I_2 = I_1 + \frac{1}{3}$$

$$I_1 = \int_0^1 \frac{1}{4-x^2}\,dx$$

$$= \int_0^1 \frac{1}{4}\left(\frac{1}{2-x} + \frac{1}{2+x}\right) dx$$

$$= \int_0^1 \frac{1}{4(2+x)} + \frac{1}{4(2-x)}\,dx$$

$$= \left[\frac{1}{4}\ln|2+x| - \frac{1}{4}\ln|2-x|\right]_0^1$$

$$= \left[\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right|\right]_0^1$$

$$= \frac{1}{4}(\ln 3 - \ln 1)$$

$$= \frac{1}{4}\ln 3$$

$$\therefore \quad 8I_2 = \frac{1}{4}\ln 3 + \frac{1}{3}$$

$$I_2 = \frac{1}{32}\ln 3 + \frac{1}{24}$$

$$\bar{y} = \frac{\frac{1}{32}\ln 3 + \frac{1}{24}}{2\left(\frac{1}{4}\ln 3\right)}$$

$$= \frac{1}{16} + \frac{1}{12\ln 3}$$

$$\approx 0.138$$

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10. $\ell_1 : \underset{\sim}{r} = \underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k} + s(-3\underset{\sim}{i} + 6\underset{\sim}{j} + 15\underset{\sim}{k})$

$\ell_2 : \underset{\sim}{r} = \underset{\sim}{i} - 2\underset{\sim}{j} + 8\underset{\sim}{k} + t(\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k})$

i) Since ℓ_3 passes through the point $(-1, 3, 2)$ and is perpendicular to ℓ_1 and ℓ_2, the direction of ℓ_3 is parallel to $\begin{pmatrix} -3 \\ 6 \\ 15 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$

$$\begin{pmatrix} -3 \\ 6 \\ 15 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -3 & 6 & 15 \\ 1 & -1 & -3 \end{vmatrix}$$

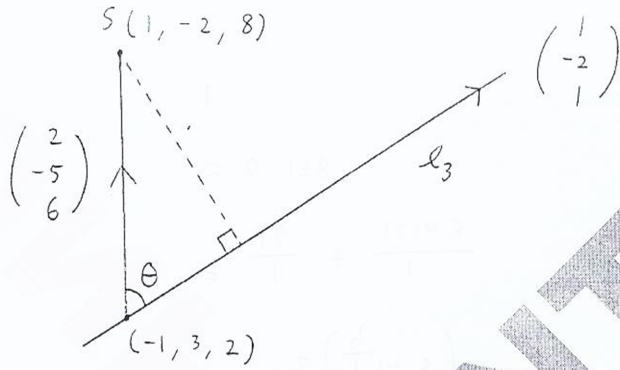
$$= -3\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}$$

$$= -3(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$$

$\therefore$ ℓ_3 has equation

$$\underset{\sim}{r} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$$

ii)



The perpendicular distance from S to ℓ_3 is $\left|\begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix}\right| \sin\theta$.

$$\begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -5 & 6 \\ 1 & -2 & 1 \end{vmatrix}$$

$$\left|\begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right| \sin\theta \, \underset{\sim}{n} = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix},$$

where $\underset{\sim}{n}$ is a unit vector.

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$$\left| \left| \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right| \sin\theta \, n \right| = \left| \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} \right|$$

$$\left| \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right| \sin\theta = \left| \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} \right|$$

$$\left| \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix} \right| \sin\theta = \frac{\left| \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right|}$$

$$= \frac{\sqrt{49 + 16 + 1}}{\sqrt{1 + 4 + 1}}$$

$$= \frac{\sqrt{66}}{\sqrt{6}}$$

$$= \sqrt{11}$$

$\therefore$ The perpendicular distance from S to ℓ_3 is $\sqrt{11}$.

iii) The pláne which contains ℓ_3 and passes through P has normal perpendicular to the vectors $\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $2\underset{\sim}{i} - 4\underset{\sim}{j} - 4\underset{\sim}{k}$ since the vector are in the direction of the plane. $\therefore$ The normal of the plane is parallel to $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -2 & 1 \\ 2 & -4 & -4 \end{vmatrix}$$

$$= 12\underset{\sim}{i} + 6\underset{\sim}{j}$$

$$= 6(2\underset{\sim}{i} + \underset{\sim}{j})$$

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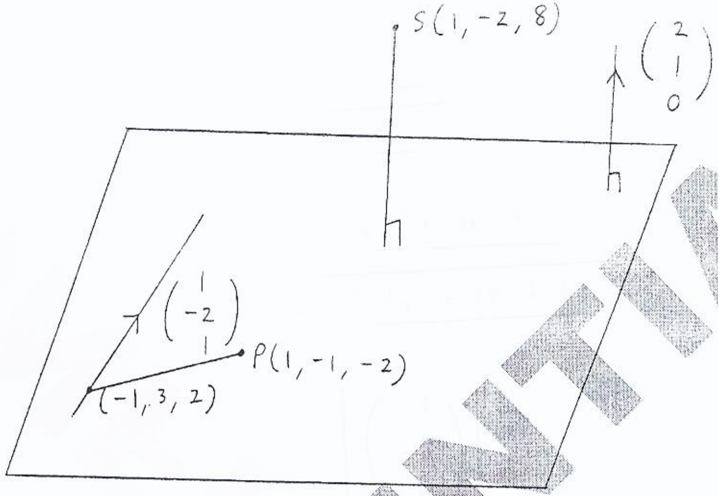

The line perpendicular to the plane and passing through S has equation

$$\underset{\sim}{r} = \underset{\sim}{i} - 2\underset{\sim}{j} + 8\underset{\sim}{k} + \mu(2\underset{\sim}{i} + \underset{\sim}{j}),$$

since it is parallel to the normal of the plane.

Since $2\underset{\sim}{i} + \underset{\sim}{j}$ is the direction of the normal of the plane and P(1, -1, -2) is a point on the plane, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is any point on the plane

$$\underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 2 - 1 + 0$$

$$2x + y = 1$$

The perpendicular to the plane passing through S intersects the plane at

$$2(2\mu + 1) + \mu - 2 = 1$$
$$4\mu + 2 + \mu - 2 = 1$$
$$5\mu = 1$$
$$\mu = \frac{1}{5}$$

$$\left(\frac{7}{5}, -\frac{9}{5}, 8\right).$$

$\therefore$ The perpendicular distance from S to the plane which contains ℓ_3 and passes through P is

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$$\sqrt{\left(\frac{7}{5}-1\right)^2+\left(\frac{-9}{5}-2\right)^2+(8-8)^2}$$

$$=\sqrt{\left(\frac{2}{5}\right)^2+\left(\frac{1}{5}\right)^2}$$

$$=\sqrt{\frac{4}{25}+\frac{1}{25}}$$

$$=\sqrt{\frac{1}{5}}$$

11\. a) $(\cos\theta + i\sin\theta)^8 = \cos^8\theta + 8\cos^7\theta(i\sin\theta)$

$+ 28\cos^6\theta(i\sin\theta)^2$

$+ 56\cos^5\theta(i\sin\theta)^3$

$+ 70\cos^4\theta(i\sin\theta)^4$

$+ 56\cos^3\theta(i\sin\theta)^5$

$+ 28\cos^2\theta(i\sin\theta)^6$

$+ 8\cos\theta(i\sin\theta)^7$

$+ (i\sin\theta)^8$

$\cos 8\theta + i\sin 8\theta = \cos^8\theta + 8i\cos^7\theta\sin\theta$

$- 28\cos^6\theta\sin^2\theta$

$- 56i\cos^5\theta\sin^3\theta$

$+ 70\cos^4\theta\sin^4\theta$

$+ 56i\cos^3\theta\sin^5\theta$

$- 28\cos^2\theta\sin^6\theta$

$- 8i\cos\theta\sin^7\theta$

$+ \sin^8\theta$

$$= \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta$$
$$- 28\cos^2\theta\sin^6\theta + \sin^8\theta$$
$$+ i(8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta$$
$$+ 56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta)$$

$$\therefore \sin 8\theta = 8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta$$
$$+ 56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta$$
$$= \sin\theta\cos\theta(8\cos^6\theta - 56\cos^4\theta\sin^2\theta$$
$$+ 56\cos^2\theta\sin^4\theta - 8\sin^6\theta)$$

$$= \sin\theta\cos\theta\,(8(1-\sin^2\theta)^3$$
$$-56(1-\sin^2\theta)^2\sin^2\theta$$
$$+56(1-\sin^2\theta)\sin^4\theta$$
$$-8\sin^6\theta)$$

$$= \sin\theta\cos\theta(8(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)$$
$$-56(1 - 2\sin^2\theta + \sin^4\theta)\sin^2\theta$$
$$+56\sin^4\theta - 56\sin^6\theta$$
$$-8\sin^6\theta)$$

$$= \sin\theta\cos\theta(8 - 24\sin^2\theta + 24\sin^4\theta - 8\sin^6\theta$$
$$-56\sin^2\theta + 112\sin^4\theta - 56\sin^6\theta$$
$$+56\sin^4\theta - 64\sin^6\theta)$$

$$= \sin\theta\cos\theta(-128\sin^6\theta + 192\sin^4\theta - 80\sin^2\theta + 8)$$

$$= \sin\theta\cos\theta(a\sin^6\theta + b\sin^4\theta + c\sin^2\theta + d)$$

$a = -128$, $b = 192$, $c = -80$, $d = 8$

b) $$\frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots + \left(\frac{z}{2}\right)^n = \frac{\frac{z}{2}\left(1 - \left(\frac{z}{2}\right)^n\right)}{1 - \frac{z}{2}}$$

$$\sum_{n=1}^{N}\left(\frac{z}{2}\right)^n = \frac{\frac{z}{2}\left(1 - \frac{z^N}{2^N}\right)}{1 - \frac{z}{2}}$$

$$= \frac{z}{2-z}\left(1 - \frac{z^N}{2^N}\right)$$

$$z = e^{i\theta}$$
$$= \cos\theta + i\sin\theta$$

$$\sum_{n=1}^{N} \left(\frac{e^{i\theta}}{2}\right)^n = \frac{e^{i\theta}}{2 - e^{i\theta}} \left(1 - \frac{e^{iN\theta}}{2^N}\right)$$

$$\sum_{n=1}^{N} \frac{e^{in\theta}}{2^n} = \frac{e^{i\theta}(2^N - e^{iN\theta})}{2^N(2 - e^{i\theta})}$$

$$= \frac{e^{i\theta}(2^N - e^{iN\theta})(2 - e^{-i\theta})}{2^N(2 - e^{i\theta})(2 - e^{-i\theta})}$$

$$= \frac{e^{i\theta}(2^{N+1} - 2e^{iN\theta} - 2^N e^{-i\theta} + e^{i(N-1)\theta})}{2^N(4 - 2(e^{i\theta} + e^{-i\theta}) + 1)}$$

$$= \frac{e^{i\theta}(2^{N+1} - 2e^{iN\theta} - 2^N e^{-i\theta} + e^{i(N-1)\theta})}{2^N(5 - 2(2\cos\theta))}$$

$$= \frac{e^{i\theta}(2^{N+1} - 2e^{iN\theta} - 2^N e^{-i\theta} + e^{i(N-1)\theta})}{2^N(5 - 4\cos\theta)}$$

$$\sum_{n=1}^{N} \frac{\cos n\theta + i\sin n\theta}{2^n} = \frac{e^{i\theta}(2^{N+1} - 2e^{iN\theta} - 2^N e^{-i\theta} + e^{i(N-1)\theta})}{2^N(5 - 4\cos\theta)}$$

$$= \frac{(\cos\theta + i\sin\theta)(2^{N+1} - 2(\cos N\theta + i\sin N\theta) - 2^N(\cos\theta - i\sin\theta) + \cos(N-1)\theta + i\sin(N-1)\theta)}{2^N(5 - 4\cos\theta)}$$

$$= \frac{2^{N+1}\cos\theta - 2\cos\theta\cos N\theta + 2\sin\theta\sin N\theta - 2^N\cos^2\theta - 2^N\sin^2\theta + \cos\theta\cos(N-1)\theta - \sin\theta\sin(N-1)\theta}{2^N(5 - 4\cos\theta)}$$

$$+ \frac{i(2^{N+1}\sin\theta - 2\cos N\theta\sin\theta - 2\cos\theta\sin N\theta - 2^N\cos\theta\sin\theta + 2^N\cos\theta\sin\theta + \sin\theta\cos(N-1)\theta + \cos\theta\sin(N-1)\theta)}{2^N(5 - 4\cos\theta)}$$

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$$\therefore \sum_{n=1}^{N} \frac{\cos n\theta}{2^n} + i\sum_{n=1}^{N} \frac{\sin n\theta}{2^n}$$

$$= \frac{2^{N+1}\cos\theta - 2\cos(N+1)\theta - 2^N + \cos N\theta}{2^N(5 - 4\cos\theta)}$$

$$+ \frac{i(2^{N+1}\sin\theta - 2\sin(N+1)\theta + \sin N\theta)}{2^N(5 - 4\cos\theta)}$$

$$\therefore \sum_{n=1}^{N} \frac{\sin n\theta}{2^n}$$

$$= \frac{2^{N+1}\sin\theta - 2\sin(N+1)\theta + \sin N\theta}{2^N(5 - 4\cos\theta)}$$

12. EITHER

$C: \ y = \lambda x + \dfrac{x}{x+2}, \quad \lambda \neq 0.$

i) $y = \lambda x + \dfrac{x+2-2}{x+2}$

$= \lambda x + 1 - \dfrac{2}{x+2}$

As $x \to \pm\infty \quad y \to \lambda x + 1$

As $x \to -2 \quad y \to \pm\infty$

The asymptotes of C are

$y = \lambda x + 1$ and $x = -2$.

ii) $y = \lambda x + 1 - \dfrac{2}{x+2}$

$\dfrac{dy}{dx} = \lambda + \dfrac{2}{(x+2)^2}$

Since $\dfrac{2}{(x+2)^2} > 0$, if $\lambda > 0$

then $\dfrac{dy}{dx} > 0$ at all points of C.

iii) When $\frac{dy}{dx} = 0$:

$$\lambda + \frac{2}{(x+2)^2} = 0$$

$$\frac{2}{(x+2)^2} = -\lambda$$

$$(x+2)^2 = \frac{-2}{\lambda}$$

If $\lambda < \frac{-1}{2}$,

$$\frac{-2}{\lambda} < 4$$

$$(x+2)^2 < 4$$

$$-2 < x + 2 < 2$$

$$-4 < x < 0$$

Also, since $\frac{-2}{\lambda} > 0$

$$x = -2 \pm \sqrt{\frac{-2}{\lambda}}$$

$\therefore$ If $\lambda < \frac{-1}{2}$, C has two distinct stationary points, both to the left of the y-axis.

If $\lambda < \frac{-1}{2}$, when $\frac{dy}{dx} = 0$:

$$x = -2 \pm \sqrt{\frac{-2}{\lambda}}$$

$$\frac{d^2y}{dx^2} = \frac{-4}{(x+2)^3}$$

when $x = -2 + \sqrt{\frac{-2}{\lambda}}$: $\frac{d^2y}{dx^2} = \frac{-4}{\frac{-2}{\lambda}\sqrt{\frac{-2}{\lambda}}}$

$$= 2\lambda\sqrt{\frac{-\lambda}{2}} < 0$$

When $x = -2 - \sqrt{\frac{-2}{\lambda}}$: $\frac{d^2y}{dx^2} = \frac{-4}{\frac{-2}{\lambda}\left(-\sqrt{\frac{-2}{\lambda}}\right)}$

$$= -2\lambda\sqrt{\frac{-\lambda}{2}} > 0$$

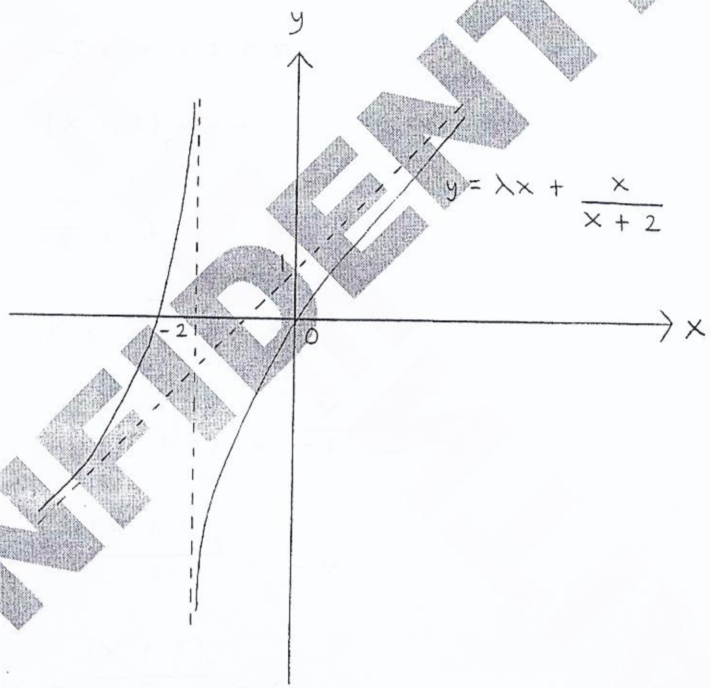
$\therefore$ C has a maximum point when $x = -2 + \sqrt{\frac{-2}{\lambda}}$ and a minimum point when $x = -2 - \sqrt{\frac{-2}{\lambda}}$.

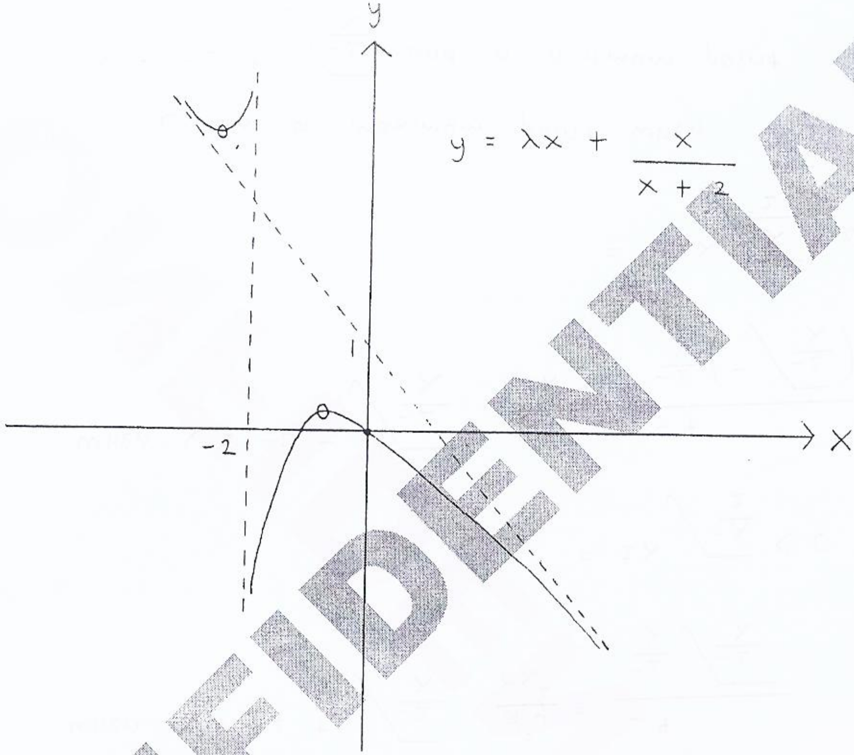
iv) The asymptotes of C are

$y = \lambda x + 1$ and $x = -2$.

When $x = 0$, y 0.

If $\lambda > 0$, $\frac{dy}{dx} = 0$ at all points of C.





o : Critical point

• : Intersection point.

OR

$$M = \begin{pmatrix} 1 & -2 & 2 & 4 \\ 2 & -4 & 5 & 9 \\ 3 & -6 & 8 & 14 \\ 5 & -10 & 12 & 22 \end{pmatrix}$$

i) $$\xrightarrow[-5r_1 + r_4]{-2r_1 + r_2,\ -3r_1 + r_3} \begin{pmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow[-r_2 + r_4]{-r_2 + r_3} \begin{pmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The rank of M is 2.

ii) $M\underset{\sim}{x} = \underset{\sim}{0}$

$$\begin{pmatrix} 1 & -2 & 2 & 4 \\ 2 & -4 & 5 & 9 \\ 3 & -6 & 8 & 14 \\ 5 & -10 & 12 & 22 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & 4 & 0 \\ 2 & -4 & 5 & 9 & 0 \\ 3 & -6 & 8 & 14 & 0 \\ 5 & -10 & 12 & 22 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_4 = t$, $t \in R$

$\therefore x_3 = -t$

Let $x_2 = s$, $s \in R$

$\therefore x_1 = 2s + 2t - 4t$

$= 2s - 2t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2s - 2t \\ s \\ t \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ A basis for the null space, K, of T is $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

iii)

$$M\begin{pmatrix}-1\\2\\-3\\4\end{pmatrix} = \begin{pmatrix}1 & -2 & 2 & 4\\2 & -4 & 5 & 9\\3 & -6 & 8 & 14\\5 & -10 & 12 & 22\end{pmatrix}\begin{pmatrix}-1\\2\\-3\\4\end{pmatrix}$$

$$= \begin{pmatrix}5\\11\\17\\27\end{pmatrix}.$$

$\therefore$ Any solution of

$$M\underset{\sim}{x} = \begin{pmatrix}5\\11\\17\\27\end{pmatrix}$$

has the form $\begin{pmatrix}-1\\2\\-3\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + \mu\begin{pmatrix}-2\\0\\-1\\1\end{pmatrix}$,

where λ and μ are constants and since $\left\{\begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\-1\\1\end{pmatrix}\right\}$ is a basis for K.

iv) If $\underset{\sim}{x}_1 = \begin{pmatrix}-1\\2\\-3\\4\end{pmatrix} + \lambda_1\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + \mu_1\begin{pmatrix}-2\\0\\-1\\1\end{pmatrix}$

$$= \begin{pmatrix}-1 + 2\lambda_1 - 2\mu_1\\2 + \lambda_1\\-3 - \mu_1\\4 + \mu_1\end{pmatrix}$$

If $-1 + 2\lambda_1 - 2\mu_1 = A$

and $-1 + 2\lambda_1 - 2\mu_1 + 2 + \lambda_1 - 3 - \mu_1 + 4 + \mu_1 = B$

$$\therefore A + 2 + \lambda_1 + 1 = B$$

$$\lambda_1 = B - A - 3$$

$$-1 + 2(B - A - 3) - 2\mu_1 = A$$

$$-1 + 2B - 2A - 6 - 2\mu_1 = A$$

$$2\mu_1 = 2B - 3A - 7$$

$$\mu_1 = \frac{2B - 3A - 7}{2}$$