MAY / JUNE 2002

1.

$$A = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (4-\lambda)(-1-\lambda) + 6$$

$$= (\lambda - 4)(\lambda + 1) + 6$$

$$= \lambda^2 - 3\lambda + 2$$

$$= (\lambda - 1)(\lambda - 2)$$

when $|A - \lambda I| = 0$:

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1, 2.$$

when $\lambda = 1$: $\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{-r_1 + r_2} \left(\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $y = 3s$, $s \in R$

$x = 2s$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2s \\ 3s \end{pmatrix}$$

$$= s\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

when $\lambda = 2$: $\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2},\ \frac{r_2}{3}} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_1 + r_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $y = s$, $s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are $1, 2$ with corresponding eigenvectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

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2. $C_n = \int_0^1 (1-x)^n \cos x \, dx$

$u = (1-x)^n$ $\quad dv = \cos x \, dx$

$du = -n(1-x)^{n-1} dx$ $\quad v = \sin x$

$= \left[(1-x)^n \sin x\right]_0^1 - \int_0^1 -n(1-x)^{n-1} \sin x \, dx$

$= 0 - 0 + n\int_0^1 (1-x)^{n-1} \sin x \, dx$

$= nS_{n-1}$

$S_n = \int_0^1 (1-x)^n \sin x \, dx$

$u = (1-x)^n$ $\quad dv = \sin x \, dx$

$du = -n(1-x)^{n-1} dx$ $\quad v = -\cos x$

$= \left[-(1-x)^n \cos x\right]_0^1 - \int_0^1 n(1-x)^{n-1} \cos x \, dx$

$= 0 - (-1) - n\int_0^1 (1-x)^{n-1} \cos x \, dx$

$= 1 - n\int_0^1 (1-x)^{n-1} \cos x \, dx$

$= 1 - nC_{n-1}$

$\therefore C_n = nS_{n-1}, \quad S_n = 1 - nC_{n-1}, \quad n \geq 1.$

$n = 3: S_3 = 1 - 3C_2$

$C_2 = 2S_1$

$S_1 = 1 - C_0$

$C_0 = \int_0^1 \cos x \, dx$

$= [\sin x]_0^1$

$= \sin 1$

$S_1 = 1 - \sin 1$

$C_2 = 2S_1$

$= 2(1 - \sin 1)$

$= 2 - 2\sin 1$

$S_3 = 1 - 3(2 - 2\sin 1)$

$= 1 - 6 + 6\sin 1$

$= -5 + 6\sin 1$

$= 0.048826.$

3. $S_N = \sum_{n=1}^{N} (2n-1)^3$

$$= \sum_{n=1}^{N} 8n^3 - 12n^2 + 6n - 1$$

$$= 8\sum_{n=1}^{N} n^3 - 12\sum_{n=1}^{N} n^2 + 6\sum_{n=1}^{N} n - \sum_{n=1}^{N} 1$$

$$= \frac{8N^2(N+1)^2}{4} - \frac{12N(N+1)(2N+1)}{6} + \frac{6N(N+1)}{2} - N$$

$$= 2N^2(N+1)^2 - 2N(N+1)(2N+1) + 3N(N+1) - N$$

$$= 2N^2(N^2+2N+1) - 2N(2N^2+3N+1) + 3(N^2+N) - N$$

$$= 2N^4 + 4N^3 + 2N^2 - 4N^3 - 6N^2 - 2N + 3N^2 + 3N - N$$

$$= 2N^4 - N^2$$

$$= N^2(2N^2-1)$$

$$\sum_{n=N+1}^{2N} (2n-1)^3 = \sum_{n=1}^{2N} (2n-1)^3 - \sum_{n=1}^{N} (2n-1)^3$$

$$= (2N)^2(2(2N)^2-1) - N^2(2N^2-1)$$

$$= 4N^2(8N^2-1) - (2N^4-N^2)$$

$$= 32N^4 - 4N^2 - 2N^4 + N^2$$

$$= 30N^4 - 3N^2$$

$$= 3N^2(10N^2-1)$$

4. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 15x + 16$

$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$

$m^2 + 2m + 5 = 0$

$(m+1)^2 + 4 = 0$

$(m+1)^2 = -4$

$m + 1 = \pm 2i$

$m = -1 \pm 2i$

$\therefore$ The complementary function, y_c, is

$y_c = e^{-x}(A\cos 2x + B\sin 2x)$.

The particular integral, y_p, is given by

$y_p = Cx + D$

$\dfrac{dy_p}{dx} = C$

$\dfrac{d^2y_p}{dx^2} = 0$.

$\dfrac{d^2y_p}{dx^2} + 2\dfrac{dy_p}{dx} + 5y_p = 0 + 2C + 5(Cx + D)$

$= 5Cx + 2C + 5D$

$= 15x + 16$

$5C = 15$ $\quad$ $2C + 5D = 16$

$C = 3$ $\quad$ $5D = 10$

$D = 2$

$y_p = 3x + 2$

$y = y_c + y_p$

$= e^{-x}(A\cos 2x + B\sin 2x) + 3x + 2$

$\therefore$ The general solution is

$y = e^{-x}(A\cos 2x + B\sin 2x) + 3x + 2$.

As $x \to \infty$, since $-1 < \cos 2x < 1$ and $-1 < \sin 2x < 1$, $e^{-x}(A\cos 2x + B\sin 2x) \to 0$

$\therefore$ $y \to 3x + 2$.

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5. $x^3 - 3x^2 + 1 = 0$

α, β, γ are the roots.

$\alpha + \beta + \gamma = 3 \quad \alpha\beta + \alpha\gamma + \beta\gamma = 0 \quad \alpha\beta\gamma = -1$

$\frac{\alpha}{\alpha - 2}, \frac{\beta}{\beta - 2}, \frac{\gamma}{\gamma - 2}$

Let $y = \frac{\alpha}{\alpha - 2}$

$$\alpha y - 2y = \alpha$$
$$\alpha y - \alpha = 2y$$
$$\alpha(y - 1) = 2y$$
$$\alpha = \frac{2y}{y - 1}$$

α is a root

$\therefore \alpha^3 - 3\alpha^2 + 1 = 0$

$$\left(\frac{2y}{y-1}\right)^3 - 3\left(\frac{2y}{y-1}\right)^2 + 1 = 0$$

$$\frac{8y^3}{(y-1)^3} - \frac{3(4y^2)}{(y-1)^2} + 1 = 0$$

$$8y^3 - 12y^2(y-1) + (y-1)^3 = 0$$
$$8y^3 - 12y^3 + 12y^2 + y^3 - 3y^2 + 3y - 1 = 0$$
$$-3y^3 + 9y^2 + 3y - 1 = 0$$
$$3y^3 - 9y^2 - 3y + 1 = 0$$

$\therefore$ The equation $3y^3 - 9y^2 - 3y + 1 = 0$ has roots $\frac{\alpha}{\alpha - 2}, \frac{\beta}{\beta - 2}, \frac{\gamma}{\gamma - 2}$.

$$\frac{\alpha}{\alpha - 2} + \frac{\beta}{\beta - 2} + \frac{\gamma}{\gamma - 2} = 3$$

$$\left(\frac{\alpha}{\alpha - 2}\right)\left(\frac{\beta}{\beta - 2}\right) + \left(\frac{\alpha}{\alpha - 2}\right)\left(\frac{\beta}{\beta - 2}\right) + \left(\frac{\beta}{\beta - 2}\right)\left(\frac{\gamma}{\gamma - 2}\right) = -1$$

$$\frac{\alpha}{\alpha - 2}\left(\frac{\beta}{\beta - 2}\right)\frac{\gamma}{\gamma - 2} = -\frac{1}{3}$$

i) $$\frac{\alpha\beta\gamma}{(\alpha-2)(\beta-2)(\gamma-2)} = -\frac{1}{3}$$

$$\frac{-1}{(\alpha-2)(\beta-2)(\gamma-2)} = -\frac{1}{3}$$

$\therefore (\alpha-2)(\beta-2)(\gamma-2) = 3$

ii) $$\frac{\alpha(\beta-2)(\gamma-2) + \beta(\alpha-2)(\gamma-2) + \gamma(\alpha-2)(\beta-2)}{(\alpha-2)(\beta-2)(\gamma-2)} = 3$$

$$\frac{\alpha(\beta-2)(\gamma-2) + \beta(\alpha-2)(\gamma-2) + \gamma(\alpha-2)(\beta-2)}{3} = 3$$

$\therefore \alpha(\beta-2)(\gamma-2) + \beta(\alpha-2)(\gamma-2) + \gamma(\alpha-2)(\beta-2) = 9$

6. $u_1, u_2, u_3, \dots$, $u_1 < 4$, $u_n > 0$; $n = 1, 2, 3, \dots$

$$u_{n+1} = \frac{5u_n + 4}{u_n + 2}$$

$u_n < 4$

when $n = 1$: $u_1 < 4$

Assume the statement is true when $n = k$.

$n = k$: $u_k < 4$

when $n = k+1$: $u_{k+1} < 4$

(what needs to be proved)

$$4 - u_{k+1} = 4 - \left(\frac{5u_k + 4}{u_k + 2}\right)$$

$$= \frac{4(u_k + 2) - (5u_k + 4)}{u_k + 2}$$

$$= \frac{4u_k + 8 - 5u_k - 4}{u_k + 2}$$

$$= \frac{4 - u_k}{u_k + 2}$$

> 0, since $u_k < 4$

$\therefore u_{k+1} < 4$.

$\therefore u_n < 4$ for all $n \geq 1$.

$$u_{n+1} - u_n = \frac{5u_n + 4}{u_n + 2} - u_n$$

$$= \frac{5u_n + 4 - u_n(u_n + 2)}{u_n + 2}$$

$$= \frac{5u_n + 4 - u_n^2 - 2u_n}{u_n + 2}$$

$$= \frac{-u_n^2 + 3u_n + 4}{u_n + 2}$$

$$= \frac{(4 - u_n)(1 + u_n)}{u_n + 2}$$

> 0, since $u_n < 4$.

$\therefore u_{n+1} > u_n$ for all $n \geq 1$.

7. $x = 1 + \frac{1}{t} \qquad y = t^3 e^{-t}, \quad t \neq 0$

$$\frac{dx}{dt} = -\frac{1}{t^2} \qquad \frac{dy}{dt} = 3t^2e^{-t} - t^3e^{-t}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{3t^2e^{-t} - t^3e^{-t}}{-\frac{1}{t^2}}$$

$$= (t-3)t^4e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left((t-3)t^4e^{-t}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left((t-3)t^4e^{-t}\right)$$

$$= -t^2\left(t^4e^{-t} + 4(t-3)t^3e^{-t} - (t-3)t^4e^{-t}\right)$$

$$= -t^2e^{-t}\left(t^4 + 4t^4 - 12t^3 - t^5 + 3t^4\right)$$

$$= -t^2e^{-t}\left(-t^5 + 8t^4 - 12t^3\right)$$

$$= (t^7 - 8t^6 + 12t^5)e^{-t}$$

when $\frac{d^2y}{dx^2} = 0$: $(t^7 - 8t^6 + 12t^5)e^{-t} = 0$

$$t^7 - 8t^6 + 12t^5 = 0$$

$$t^5(t^2 - 8t + 12) = 0$$

$$t^5(t-2)(t-6) = 0$$

$$t \neq 0 \quad \therefore \quad t = 2, 6$$

8. C: $y = \frac{3x^{\frac{4}{3}}}{8} - \frac{3x^{\frac{2}{3}}}{4}$, $x = 1$ $x = 8$

i) $\frac{dy}{dx} = \frac{x^{\frac{1}{3}}}{2} - \frac{x^{-\frac{1}{3}}}{2}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x^{\frac{1}{3}}}{2} - \frac{x^{-\frac{1}{3}}}{2}\right)^2$$

$$= 1 + \frac{x^{\frac{2}{3}}}{4} - \frac{1}{2} + \frac{x^{-\frac{2}{3}}}{4}$$

$$= \frac{x^{\frac{2}{3}}}{4} + \frac{1}{2} + \frac{x^{-\frac{2}{3}}}{4}$$

$$= \left(\frac{x^{\frac{1}{3}}}{2} + \frac{x^{-\frac{1}{3}}}{2}\right)^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{x^{\frac{1}{3}}}{2} + \frac{x^{-\frac{1}{3}}}{2}$$

The length of C is

$$\int_1^8 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^8 \frac{x^{\frac{1}{3}}}{2} + \frac{x^{-\frac{1}{3}}}{2}\, dx$$

$$= \left[\frac{3x^{\frac{4}{3}}}{8} + \frac{3x^{\frac{2}{3}}}{4}\right]_1^8$$

$$= \frac{3(16)}{8} + \frac{3(4)}{4} - \frac{3}{8} - \frac{3}{4}$$

$$= \frac{45}{8} + \frac{9}{4}$$

$$= \frac{63}{8}$$

ii) The surface area of revolution when C is rotated about the y-axis is

$$\int_1^8 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^8 2\pi x \left(\frac{x^{\frac{1}{3}}}{2} + \frac{x^{-\frac{1}{3}}}{2}\right) dx$$

$$= \pi \int_1^8 x^{\frac{4}{3}} + x^{\frac{2}{3}}\, dx$$

$$= \pi\left[\frac{3x^{\frac{7}{3}}}{7} + \frac{3x^{\frac{5}{3}}}{5}\right]_1^8$$

$$= \pi\left(\frac{3(128)}{7} + \frac{3(32)}{5} - \frac{3}{7} - \frac{3}{5}\right)$$

$$= \pi\left(\frac{381}{7} + \frac{93}{5}\right)$$

$$= \frac{2556\pi}{35}$$

9. $w_n = 3^{-n}\cos 2n\theta,\quad n = 1, 2, 3, \ldots$

$$\sum_{n=0}^{N-1} \frac{e^{2n\theta i}}{3^n} = 1 + \frac{e^{2\theta i}}{3} + \frac{e^{4\theta i}}{3^2} + \frac{e^{6\theta i}}{3^3} + \cdots + \frac{e^{2(N-1)\theta i}}{3^{N-1}}$$

$$= \frac{1 - \left(\frac{e^{2\theta i}}{3}\right)^N}{1 - \frac{e^{2\theta i}}{3}}$$

$$\sum_{n=0}^{N-1} \frac{\cos 2n\theta + i\sin 2n\theta}{3^n} = \frac{1 - \frac{e^{2N\theta i}}{3^N}}{1 - \frac{e^{2\theta i}}{3}}$$

$$\sum_{n=0}^{N-1} \frac{\cos 2n\theta}{3^n} + i\sum_{n=0}^{N-1} \frac{\sin 2n\theta}{3^n}$$

$$= \frac{3(3^N - e^{2N\theta i})}{3^N(3 - e^{2\theta i})}$$

$$= \frac{3^N - e^{2N\theta i}}{3^{N-1}(3 - e^{2\theta i})}$$

$$= \frac{(3^N - e^{2N\theta i})(3 - e^{-2\theta i})}{3^{N-1}(3 - e^{2\theta i})(3 - e^{-2\theta i})}$$

$$= \frac{3^{N+1} - 3e^{2N\theta i} - 3^N e^{-2\theta i} + e^{2N\theta i - 2\theta i}}{3^{N-1}(9 - 3(e^{2\theta i} + e^{-2\theta i}) + 1)}$$

$$= \frac{3^{N+1} - 3e^{2N\theta i} - 3^N e^{-2\theta i} + e^{2\theta(N-1)i}}{3^{N-1}(10 - 3(2\cos 2\theta))}$$

$$= \frac{3^{N+1} - 3e^{2N\theta i} - 3^N e^{-2\theta i} + e^{2\theta(N-1)i}}{3^{N-1}(10 - 6\cos 2\theta)}$$

$$= \frac{9 - 3^{-N+2} e^{2N\theta i} - 3e^{-2\theta i} + 3^{-N+1} e^{2\theta(N-1)i}}{10 - 6\cos 2\theta}$$

$$= \frac{9 - 3^{-N+2}(\cos 2N\theta + i\sin 2N\theta) - 3(\cos(-2\theta) + i\sin(-2\theta)) - 3^{-N+1}(\cos 2(N-1)\theta + i\sin 2(N-1)\theta)}{10 - 6\cos 2\theta}$$

$$= \frac{9 - 3^{-N+2}\cos 2N\theta - 3\cos 2\theta - 3^{-N+1}\cos 2(N-1)\theta}{10 - 6\cos 2\theta}$$

$$+ \frac{i(-3^{-N+2}\sin 2N\theta + 3\sin 2\theta - 3^{-N+1}\sin 2(N-1)\theta)}{10 - 6\cos 2\theta}$$

$$\therefore \sum_{n=0}^{N-1} \frac{\cos 2n\theta}{3^n} = \frac{9 - 3^{-N+2}\cos 2N\theta - 3\cos 2\theta - 3^{-N+1}\cos 2(N-1)\theta}{10 - 6\cos 2\theta}$$

$$1 + w_1 + w_2 + \dots + w_{N-1} = \frac{9 - 3^{-N+2}\cos 2N\theta - 3\cos 2\theta - 3^{-N+1}\cos 2(N-1)\theta}{10 - 6\cos 2\theta}$$

$$1 + w_1 + w_2 + \dots = \lim_{N\to\infty} \sum_{n=0}^{N-1} \frac{\cos 2n\theta}{3^n}$$

$$= \lim_{N\to\infty} \left(\frac{9 - 3^{-N+2}\cos 2N\theta - 3\cos 2\theta - 3^{-N+1}\cos 2(N-1)\theta}{10 - 6\cos 2\theta} \right)$$

$$= \frac{9 - 3\cos 2\theta}{10 - 6\cos 2\theta}.$$

10. $\underset{\sim}{a}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ $\underset{\sim}{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\underset{\sim}{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\underset{\sim}{b}_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \end{pmatrix}$, $\underset{\sim}{b}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ $\underset{\sim}{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$V_1 = \{ c_1 \underset{\sim}{a}_1 + c_2 \underset{\sim}{a}_2 + c_3 \underset{\sim}{a}_3 : c_1, c_2, c_3 \in R \}$

$V_2 = \{ c_1 \underset{\sim}{b}_1 + c_2 \underset{\sim}{b}_2 + c_3 \underset{\sim}{b}_3 : c_1, c_2, c_3 \in R \}$

If $k_1 \underset{\sim}{a}_1 + k_2 \underset{\sim}{a}_2 + k_3 \underset{\sim}{a}_3 = \underset{\sim}{0}$,

$$k_1 \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3k_1 + k_2 \\ 2k_1 + k_2 \\ k_1 + k_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{3\times r_2,\ 3\times r_3}\left(\begin{array}{ccc|c} 3 & 1 & 0 & 0\\ 6 & 3 & 0 & 0\\ 3 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-r_1+r_3]{-2r_1+r_2,}\left(\begin{array}{ccc|c} 3 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & 3 & 0\\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_2+r_3}\left(\begin{array}{ccc|c} 3 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 \end{array}\right)$$

$k_3 = 0$

$k_2 = 0$

$k_1 = 0$

$\therefore \{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3\}$ are linearly independent.

If $k_1\underset{\sim}{b}_1 + k_2\underset{\sim}{b}_2 + k_3\underset{\sim}{b}_3 = \underset{\sim}{0}$

$$k_1\begin{pmatrix}3\\2\\0\\2\end{pmatrix} + k_2\begin{pmatrix}2\\2\\0\\1\end{pmatrix} + k_3\begin{pmatrix}1\\1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}3k_1 + 2k_2 + k_3\\ 2k_1 + 2k_2 + k_3\\ 0\\ 2k_1 + k_2 + k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}3 & 2 & 1\\ 2 & 2 & 1\\ 0 & 0 & 0\\ 2 & 1 & 1\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0\\ 2 & 2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 2 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{3\times r_2,\ 3\times r_3}\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0\\ 6 & 6 & 3 & 0\\ 0 & 0 & 0 & 0\\ 6 & 3 & 3 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1+r_4]{-2r_1+r_2}\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0\\ 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{2\times r_4}\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0\\ 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & -2 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{r_2+r_4}\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0\\ 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 3 & 0 \end{array}\right)$$

$k_3 = 0$

$k_2 = 0$

$k_1 = 0$

$\therefore \{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ are linearly independent.

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Since V_1 is spanned by $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$ and V_2 is spanned by $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$, $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3\}$ forms a basis of V_1 and $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ forms a basis of V_2.

$\therefore$ V_1 and V_2 each have dimension 3.

Since a vector in V_1 has the form $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{pmatrix}$ and a vector in V_2 has the form $\begin{pmatrix} a_1 \\ a_2 \\ 0 \\ a_4 \end{pmatrix}$, a vector in both V_1 and V_2 has the form

$$\begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

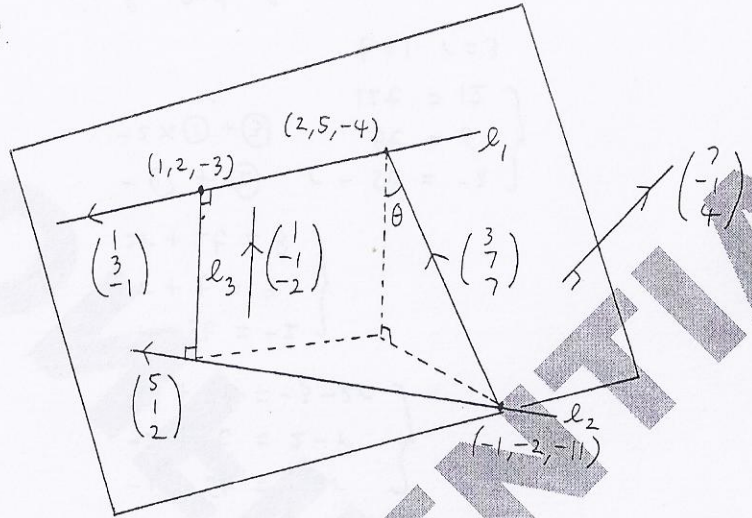
$\therefore$ A basis for V_3 is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

i) Since $W = V_1 \cup V_2 \setminus (V_1 \cap V_2)$, two linearly independent vectors in W are $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

ii) $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \notin W$ since $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \notin V_1$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \notin V_2$

$\therefore$ W is not a linear space.

11.



(2,5,-4) (1,2,-3)

$\ell_2 : \underset{\sim}{r} = -\underset{\sim}{i} - 2\underset{\sim}{j} - 11\underset{\sim}{k} + t(5\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k})$

i) Since ℓ_1 passes through $(2,5,-4)$ and $(1,2,-3)$, the equation of ℓ_1 is

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + s\left[\begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

Since ℓ_3 is perpendicular to both ℓ_1 and ℓ_2, it is parallel to $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$.

$$\begin{pmatrix}1\\3\\-1\end{pmatrix} \times \begin{pmatrix}5\\1\\2\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 3 & -1 \\ 5 & 1 & 2 \end{vmatrix} = \begin{pmatrix}7\\-7\\-14\end{pmatrix} = 7\begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

since ℓ_3 has direction $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ and passes through the point $(1,2,-3)$, ℓ_3 has equation

$$\underset{\sim}{r} = \begin{pmatrix}1\\2\\-3\end{pmatrix} + r\begin{pmatrix}1\\-1\\-2\end{pmatrix}.$$

Since $\begin{pmatrix}1\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ are the directions of ℓ_1 and ℓ_3 respectively, the normal of the plane containing ℓ_1 and ℓ_3 has direction

$$\begin{pmatrix}1\\3\\-1\end{pmatrix} \times \begin{pmatrix}1\\-1\\-2\end{pmatrix}.$$

$$\begin{pmatrix}1\\3\\-1\end{pmatrix} \times \begin{pmatrix}1\\-1\\-2\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix} = \begin{pmatrix}-7\\1\\-4\end{pmatrix}.$$

Since the normal of the plane containing ℓ_1 and ℓ_3 has direction $\begin{pmatrix}-7\\1\\-4\end{pmatrix}$ and $(1,2,-3)$ is a point on the plane,

the equation of the plane is

$$\underset{\sim}{r} \cdot \begin{pmatrix}-7\\1\\-4\end{pmatrix} = \begin{pmatrix}1\\2\\-3\end{pmatrix} \cdot \begin{pmatrix}-7\\1\\-4\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-7\\1\\-4\end{pmatrix} = -7 + 2 + 12$$

$$-7x + y - 4z = 7$$

$$7x - y + 4z = -7$$

ii) $\ell_2 : \underset{\sim}{r} = \begin{pmatrix}-1\\-2\\-11\end{pmatrix} + t\begin{pmatrix}5\\1\\2\end{pmatrix}$ $\ell_3 : \underset{\sim}{r} = \begin{pmatrix}1\\2\\-3\end{pmatrix} + r\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

If $\begin{pmatrix}-1+5t\\-2+t\\-11+2t\end{pmatrix} = \begin{pmatrix}1+r\\2-r\\-3-2r\end{pmatrix}$

$$\left.\begin{array}{l} -1+5t = 1+r \\ -2+t = 2-r \\ -11+2t = -3-2r \end{array}\right\}$$

$$\left.\begin{array}{l} r-5t = -2 \\ r+t = 4 \\ 2r+2t = 8 \end{array}\right\}$$

$$\left.\begin{array}{rl} -①+② : & r-5t = -2 \\ -2\times① + ③ : & 6t = 6 \\ & 12t = 12 \end{array}\right\}$$

$$t = 1 \quad r = 3$$

$\therefore$ ℓ_2 and ℓ_3 intersect.

iii) $\left|\begin{pmatrix}3\\7\\7\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right| = \left|\begin{pmatrix}3\\7\\7\end{pmatrix}\right|\left|\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right|\cos\theta$

$|3-7-14| = \sqrt{6}\left|\begin{pmatrix}3\\7\\7\end{pmatrix}\right|\cos\theta$

$\therefore \left|\begin{pmatrix}3\\7\\7\end{pmatrix}\right|\cos\theta = \dfrac{18}{\sqrt{6}}$

$= 3\sqrt{6}$

$\therefore$ The shortest distance between ℓ_1 and ℓ_2 is $3\sqrt{6}$.

12. EITHER.

$C: y = \dfrac{a(x-a)^2}{x^2-4a^2}, \quad a > 0.$

i)
$$\begin{array}{r|l} & \quad a \\ \hline x^2-4a^2 & ax^2 - 2a^2x + a^3 \\ & ax^2 \qquad\quad - 4a^3 \\ \hline & -2a^2x + 5a^3 \end{array}$$

$y = a + \dfrac{-2a^2x + 5a^3}{x^2-4a^2}$

$\dfrac{-2a^2x+5a^3}{x^2-4a^2} = \dfrac{A}{x-2a} + \dfrac{B}{x+2a}$

$= \dfrac{A(x+2a) + B(x-2a)}{x^2-4a^2}$

$-2a^2x + 5a^3 = A(x+2a) + B(x-2a)$

$= (A+B)x + 2aA - 2aB$

$A + B = -2a^2 \qquad 2a(A-B) = 5a^3$

$A - B = \dfrac{5a^2}{2}$

$2A = \dfrac{a^2}{2}$

$A = \dfrac{a^2}{4} \qquad B = \dfrac{-9a^2}{4}$

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$$\therefore y = a + \frac{a^2}{4(x-2a)} - \frac{9a^2}{4(x+2a)}$$

As $x \to \pm\infty \quad y \to a$

As $x \to 2a \quad y \to \pm\infty$

As $x \to -2a \quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = a$, $x = 2a$ and $x = -2a$.

ii) $\dfrac{dy}{dx} = \dfrac{-a^2}{4(x-2a)^2} + \dfrac{9a^2}{4(x+2a)^2}$

when $\dfrac{dy}{dx} = 0$: $\dfrac{-a^2}{4(x-2a)^2} + \dfrac{9a^2}{4(x+2a)^2} = 0$

$$\frac{9a^2}{4(x+2a)^2} = \frac{a^2}{4(x-2a)^2}$$

$$(x+2a)^2 = 9(x-2a)^2$$

$$x + 2a = \pm 3(x-2a)$$

$$= 3x - 6a, \; -3x + 6a$$

$$2x = 8a, \quad 4x = 4a$$

$$x = 4a, \quad x = a$$

$$y = \frac{3a}{4}, \quad y = 0$$

$$\frac{d^2y}{dx^2} = \frac{a^2}{2(x-2a)^3} - \frac{9a^2}{2(x+2a)^3}$$

when $x = 4a$: $\dfrac{d^2y}{dx^2} = \dfrac{1}{16a} - \dfrac{1}{48a} = \dfrac{1}{24a} > 0$

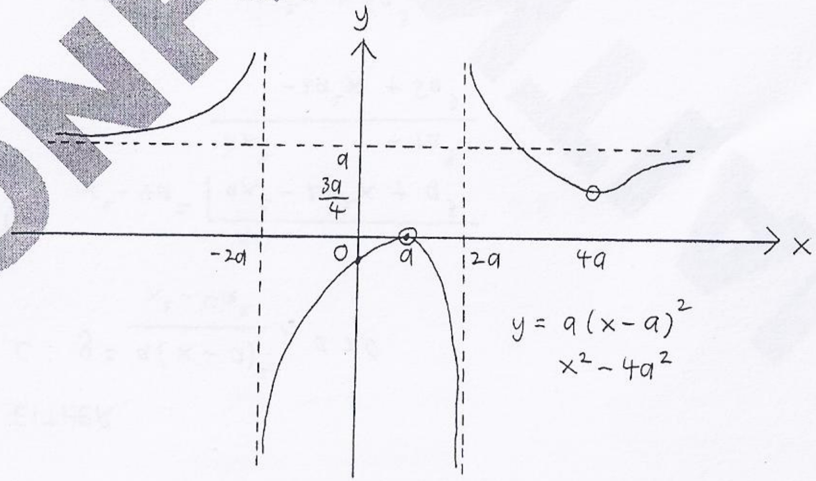
when $x = a$: $\dfrac{d^2y}{dx^2} = \dfrac{-1}{2a} - \dfrac{1}{6a} = \dfrac{-2}{3a} < 0$

$\therefore \left(4a, \dfrac{3a}{4}\right)$ is a minimum point and $(a, 0)$ is a maximum point.

iii) when $x = 0$: $y = -\dfrac{a}{4}$

when $y = 0$: $\dfrac{a(x-a)^2}{x^2 - 4a^2} = 0$

$$x = a.$$



o: Critical points

•: Intersection points.

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OR

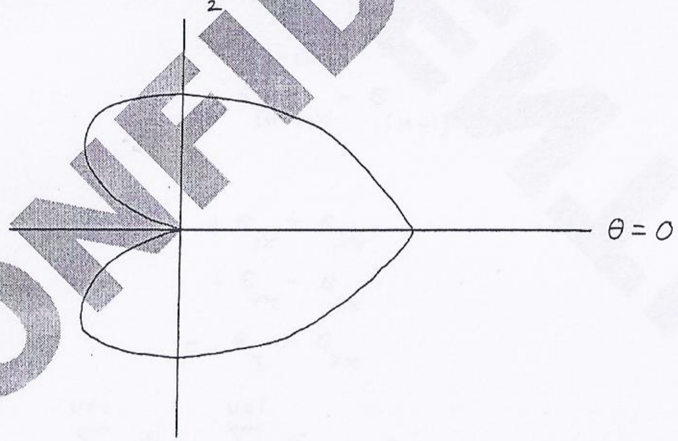
$C: r = a(1+\cos\theta)\,,\ -\pi < \theta \le \pi\,,\ a > 0$

i)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	$2a$	$(1+\frac{\sqrt{3}}{2})a$	$(1+\frac{1}{\sqrt{2}})a$	$\frac{3a}{2}$	a

θ	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	$\frac{a}{2}$	$(1-\frac{1}{\sqrt{2}})a$	$(1-\frac{\sqrt{3}}{2})a$	0

$\cos(-\theta) = \cos\theta$



when $\theta = -\frac{\pi}{3}$: $\frac{d^2y}{d\theta^2} = \frac{3a}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{3a}{2}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{3a\sqrt{3}}{2} > 0.$

when $\theta = \pi$: $\frac{d^2y}{d\theta^2} = 0$

$\therefore$ The minimum value of y is $\frac{-3\sqrt{3}\,a}{4}$

when $\theta = -\frac{\pi}{3}$.