7. a) $$\begin{vmatrix} 3-\lambda & -5 & -1 \\ 2 & 2-\lambda & 0 \\ 2 & 5 & 1 \end{vmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 5 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 2-\lambda \\ 2 & 5 \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda-0) + 5(2-0) - (10 - 4 + 2\lambda) = 0$$

$$(3-\lambda)(2-\lambda) + 10 - (6 + 2\lambda) = 0$$

$$6 - 5\lambda + \lambda^2 + 10 - 6 - 2\lambda = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 2)(\lambda - 5) = 0$$

$$\lambda = 2, 5$$

b) $$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= x(y^2z^3 - y^3z^2) - y(x^2z^3 - x^3z^2) + z(x^2y^3 - x^3y^2)$$

$$= xy^2z^2(z-y) - x^2yz^2(z-x) + x^2y^2z(y-x)$$

$$= xyz[yz(z-y) - xz(z-x) + xy(y-x)]$$

38. $\sigma : R^4 \to R^3$

$$\begin{pmatrix} 1 & -4 & 5 & -2 \\ 0 & 9 & -3 & 3 \\ 4 & 5 & 13 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4 & 5 & -2 \\ 0 & 9 & -3 & 3 \\ 4 & 5 & 13 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -4 & 5 & -2 & 0 \\ 0 & 9 & -3 & 3 & 0 \\ 4 & 5 & 13 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{r_2}{3}]{-4r_1 + r_3,} \left(\begin{array}{cccc|c} 1 & -4 & 5 & -2 & 0 \\ 0 & 3 & -1 & 1 & 0 \\ 0 & 21 & -7 & 7 & 0 \end{array}\right)$$

$$\xrightarrow{-7r_2 + r_3} \left(\begin{array}{cccc|c} 1 & -4 & 5 & -2 & 0 \\ 0 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = 3s$, $s \in R$ and $z = 3t$, $t \in R$

$y = t - s$

$x - 4(t - s) + 5(3t) - 2(3s) = 0$

$x - 4t + 4s + 15t - 6s = 0$

$x = 2s - 11t$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2s - 11t \\ t - s \\ 3t \\ 3s \end{pmatrix}$$

$$= s\begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} -11 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4 & 5 & -2 \\ 0 & 9 & -3 & 3 \\ 4 & 5 & 13 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x - 4y + 5z - 2w \\ 9y - 3z + 3w \\ 4x + 5y + 13z - w \end{pmatrix}$$

$$= \begin{pmatrix} x \\ 0 \\ 4x \end{pmatrix} + \begin{pmatrix} -4y \\ 9y \\ 5y \end{pmatrix} + \begin{pmatrix} 5z \\ -3z \\ 13z \end{pmatrix} + \begin{pmatrix} -2w \\ 3w \\ -w \end{pmatrix}$$

$$= x\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + y\begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix} + z\begin{pmatrix} 5 \\ -3 \\ 13 \end{pmatrix} + w\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

$\therefore$ A basis for the null space of σ is

$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -11 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right\}$ and a basis for the range space

of σ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix} \right\}$

$$\begin{pmatrix} 1 & -4 & 5 & -2 \\ 0 & 9 & -3 & 3 \\ 4 & 5 & 13 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ 21 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -4 & 5 & -2 & 0 \\ 0 & 9 & -3 & 3 & 9 \\ 4 & 5 & 13 & -1 & 21 \end{array}\right)$$

$$\xrightarrow{-4r_1 + r_3;\ \frac{r_2}{3}} \left(\begin{array}{cccc|c} 1 & -4 & 5 & -2 & 0 \\ 0 & 3 & -1 & 1 & 3 \\ 0 & 21 & -7 & 7 & 21 \end{array}\right)$$

$$\xrightarrow{-7r_2 + r_3} \left(\begin{array}{cccc|c} 1 & -4 & 5 & -2 & 0 \\ 0 & 3 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = 3s, s \in R$ and $z = 3t, t \in R$

$y = t - s + 1$

$x - 4(t - s + 1) + 5(3t) - 2(3s) = 0$

$x - 4t + 4s - 4 + 15t - 6s = 0$

$x = 4 - 11t + 2s$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 - 11t + 2s \\ t - s + 1 \\ 3t \\ 3s \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -11t \\ t \\ 3t \\ 0 \end{pmatrix} + \begin{pmatrix} 2s \\ -s \\ 0 \\ 3s \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -11 \\ 1 \\ 3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}$$

The set of vectors in R^4 which are mapped to $\begin{pmatrix} 0 \\ 9 \\ 21 \end{pmatrix}$ by σ is $\left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -11 \\ 1 \\ 3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix} \right\}$.

52. $\sigma : R^3 \to R^4$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -6 & 3 \\ 6 & -4 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -6 & 3 \\ 6 & -4 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & -6 & 3 & 0 \\ 6 & -4 & 10 & 0 \end{array}\right)$$

$$\xrightarrow[-6r_1 + r_4]{\substack{-2r_1 + r_2 \\ -r_1 + r_3}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & -8 & 2 & 0 \\ 0 & -16 & 4 & 0 \end{array}\right)$$

$$\xrightarrow[-4r_2 + r_4]{-2r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 4s$, $s \in R$

$y = s$

$x + 2s + 4s = 0$

$x = -6s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6s \\ s \\ 4s \end{pmatrix}$$

$$= s \begin{pmatrix} -6 \\ 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -6 & 3 \\ 6 & -4 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + z \\ 2x + 3z \\ x - 6y + 3z \\ 6x - 4y + 10z \end{pmatrix}$$

$$= \begin{pmatrix} x \\ 2x \\ x \\ 6x \end{pmatrix} + \begin{pmatrix} 2y \\ 0 \\ -6y \\ -4y \end{pmatrix} + \begin{pmatrix} z \\ 3z \\ 3z \\ 10z \end{pmatrix}$$

$$= x \begin{pmatrix} 1 \\ 2 \\ 1 \\ 6 \end{pmatrix} + y \begin{pmatrix} 2 \\ 0 \\ -6 \\ -4 \end{pmatrix} + z \begin{pmatrix} 1 \\ 3 \\ 3 \\ 10 \end{pmatrix}$$

A basis for the null space of σ is $\left\{ \begin{pmatrix} -6 \\ 1 \\ 4 \end{pmatrix} \right\}$

and a basis for the range space of σ is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -6 \\ -4 \end{pmatrix} \right\}.$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -6 & 3 \\ 6 & -4 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -9 \\ -6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 0 & 3 & 0 \\ 1 & -6 & 3 & -9 \\ 6 & -4 & 10 & -6 \end{array}\right)$$

$$\xrightarrow[-6r_1 + r_4]{\substack{-2r_1 + r_2 \\ -r_1 + r_3}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -4 & 1 & -6 \\ 0 & -8 & 2 & -12 \\ 0 & -16 & 4 & -24 \end{array}\right)$$

$$\xrightarrow[-4r_2 + r_4]{-2r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -4 & 1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 4s, \ s \in R$

$y = \frac{3}{2} + s$

$x + 2\left(\frac{3}{2} + s\right) + 4s = 3$

$x + 3 + 2s + 4s = 3$

$x = -6s$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -6s \\ \frac{3}{2} + s \\ 4s \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{3}{2} \\ 0 \end{pmatrix} + s\begin{pmatrix} -6 \\ 1 \\ 4 \end{pmatrix}$$

The set of vectors in R^3 which are mapped to $\begin{pmatrix} 3 \\ 0 \\ -9 \\ -6 \end{pmatrix}$ by σ is $\left\{ \begin{pmatrix} 0 \\ \frac{3}{2} \\ 0 \end{pmatrix} + s\begin{pmatrix} -6 \\ 1 \\ 4 \end{pmatrix} \right\}$

58. $\sigma : R^4 \longrightarrow R^3$

$$A = \begin{pmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 4 & 5 & 5 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 4 & 5 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 2 & 1 & 3 & 3 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 4 & 5 & 5 & 8 & 0 \end{array}\right)$$

$$\xrightarrow{-2r_1 + r_3} \left(\begin{array}{cccc|c} 2 & 1 & 3 & 3 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & 3 & -1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{cccc|c} 2 & 1 & 3 & 3 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = 3s$, $s \in R$ and $z = 3t$, $t \in R$

$y = t - 2s$

$2x + t = 2s + 9t + 9s = 0$

$2x = -7s - 10t$

$x = \frac{-7s}{2} - 5t$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{-7s}{2} - 5t \\ t - 2s \\ 3t \\ 3s \end{pmatrix}$$

$$= s \begin{pmatrix} -\frac{7}{2} \\ -2 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} -5 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

A basis for the null space of σ is $\left\{ \begin{pmatrix} -\frac{7}{2} \\ -2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right\}$

. The range of σ has dimension 2.

$$\begin{pmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 4 & 5 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2x + y + 3z + 3w \\ -3y + z - 2w \\ 4x + 5y + 5z + 8w \end{pmatrix}$$

$$= \begin{pmatrix} 2x \\ 0 \\ 4x \end{pmatrix} + \begin{pmatrix} y \\ -3y \\ 5y \end{pmatrix} + \begin{pmatrix} 3z \\ z \\ 5z \end{pmatrix} + \begin{pmatrix} 3w \\ -2w \\ 8w \end{pmatrix}$$

$$= x \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + y \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + w \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix}$$

A basis for the range space of σ is

$\left\{ \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \right\}$

$$AX = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 4 & 5 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 2 & 1 & 3 & 3 & 5 \\ 0 & -3 & 1 & -2 & 3 \\ 4 & 5 & 5 & 8 & 7 \end{array}\right)$$

$$\xrightarrow{-2r_1 + r_3} \left(\begin{array}{cccc|c} 2 & 1 & 3 & 3 & 5 \\ 0 & -3 & 1 & -2 & 3 \\ 0 & 3 & -1 & 2 & -3 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{cccc|c} 2 & 1 & 3 & 3 & 5 \\ 0 & -3 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = 3s,\ s \in R$ and $z = 3t,\ t \in R$

$y = t - 2s - 1$

$2x + t - 2s - 1 + 9t + 9s = 5$

$2x = 6 - 7s - 10t$

$x = 3 - \frac{7s}{2} - 5t$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 - \frac{7s}{2} - 5t \\ t - 2s - 1 \\ 3t \\ 3s \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{7}{2} \\ -2 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} -5 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

67. $$A = \begin{pmatrix} 3 & 0 & -2 & 3 \\ -4 & 2 & 0 & -8 \\ 1 & -2 & 2 & 5 \\ 3 & -3 & 2 & 9 \end{pmatrix}$$

a) $$\xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 2 & 5 \\ -4 & 2 & 0 & -8 \\ 3 & 0 & -2 & 3 \\ 3 & -3 & 2 & 9 \end{pmatrix}$$

$$\xrightarrow[\substack{-3r_1 + r_3 \\ -3r_1 + r_4}]{4r_1 + r_2} \begin{pmatrix} 1 & -2 & 2 & 5 \\ 0 & -6 & 8 & 12 \\ 0 & 6 & -8 & -12 \\ 0 & 3 & -4 & -6 \end{pmatrix}$$

$$\xrightarrow{\frac{r_2}{2},\ \frac{r_3}{2}} \begin{pmatrix} 1 & -2 & 2 & 5 \\ 0 & -3 & 4 & 6 \\ 0 & 3 & -4 & -6 \\ 0 & 3 & -4 & -6 \end{pmatrix}$$

$$\xrightarrow[r_2 + r_4]{r_2 + r_3} \begin{pmatrix} 1 & -2 & 2 & 5 \\ 0 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 2$

b) $$\begin{pmatrix} 3 & 0 & -2 & 3 \\ -4 & 2 & 0 & -8 \\ 1 & -2 & 2 & 5 \\ 3 & -3 & 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$= \begin{pmatrix} 3x - 2z + 3w \\ -4x + 2y - 8w \\ x - 2y + 2z + 5w \\ 3x - 3y + 2z + 9w \end{pmatrix}$$

$$= \begin{pmatrix} 3x \\ -4x \\ x \\ 3x \end{pmatrix} + \begin{pmatrix} 0 \\ 2y \\ -2y \\ -3y \end{pmatrix} + \begin{pmatrix} -2z \\ 0 \\ 2z \\ 2z \end{pmatrix} + \begin{pmatrix} 3w \\ -8w \\ 5w \\ 9w \end{pmatrix}$$

$$= x\begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} 0 \\ 2 \\ -2 \\ -3 \end{pmatrix} + z\begin{pmatrix} -2 \\ 0 \\ 2 \\ 2 \end{pmatrix} + w\begin{pmatrix} 3 \\ -8 \\ 5 \\ 9 \end{pmatrix}$$

A basis for the column space of A is

$$\left\{ \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -2 \\ -3 \end{pmatrix} \right\}$$

c) $AX = 0$

$$\begin{pmatrix} 3 & 0 & -2 & 3 \\ -4 & 2 & 0 & -8 \\ 1 & -2 & 2 & 5 \\ 3 & -3 & 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 3 & 0 & -2 & 3 & 0 \\ -4 & 2 & 0 & -8 & 0 \\ 1 & -2 & 2 & 5 & 0 \\ 3 & -3 & 2 & 9 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & 5 & 0 \\ 0 & -3 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = 3s,\ s \in R$ and $z = 3t,\ t \in R$

$y = 4t + 6s$

$x - 2(4t + 6s) + 2(3t) + 5(3s) = 0$

$x - 8t - 12s + 6t + 15s = 0$

$x = 2t - 3s$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2t - 3s \\ 4t + 6s \\ 3t \\ 3s \end{pmatrix}$$

$$= t\begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 6 \\ 0 \\ 3 \end{pmatrix}$$

$$\therefore \quad V = \left\{ \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$Ax = \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & -2 & 3 \\ -4 & 2 & 0 & -8 \\ 1 & -2 & 2 & 5 \\ 3 & -3 & 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 3 & 0 & -2 & 3 & 3 \\ -4 & 2 & 0 & -8 & -4 \\ 1 & -2 & 2 & 5 & 1 \\ 3 & -3 & 2 & 9 & 3 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{cccc|c} 1 & -2 & 2 & 5 & 1 \\ -4 & 2 & 0 & -8 & -4 \\ 3 & 0 & -2 & 3 & 3 \\ 3 & -3 & 2 & 9 & 3 \end{array}\right)$$

$$\xrightarrow[-3r_1 + r_4]{\substack{4r_1 + r_2 \\ -3r_1 + r_3}} \left(\begin{array}{cccc|c} 1 & -2 & 2 & 5 & 1 \\ 0 & -6 & 8 & 12 & 0 \\ 0 & 6 & -8 & -12 & 0 \\ 0 & 3 & -4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{2},\ \frac{r_3}{2}} \left(\begin{array}{cccc|c} 1 & -2 & 2 & 5 & 1 \\ 0 & -3 & 4 & 6 & 0 \\ 0 & 3 & -4 & -6 & 0 \\ 0 & 3 & -4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[r_2 + r_4]{r_2 + r_3} \left(\begin{array}{cccc|c} 1 & -2 & 2 & 5 & 1 \\ 0 & -3 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = 3s, s \in R$ and $z = 3t, t \in R$

$y = 4t + 6s$

$x - 2(4t + 6s) + 2(3t) + 5(3s) = 1$

$x - 8t - 12s + 6t + 15s = 1$

$x = 2t - 3s + 1$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2t - 3s + 1 \\ 4t + 6s \\ 3t \\ 3s \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ 6 \\ 0 \\ 3 \end{pmatrix}$$

83.

$$S = \left\{ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$$

$$k_1 \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + k_2 \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + k_3 \begin{pmatrix}0\\1\\0\\1\end{pmatrix} + k_4 \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix} k_1 + k_2 \\ k_2 + k_3 \\ k_1 \\ k_3 + k_4 \end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$k_1 = 0,\ k_2 = 0,\ k_3 = 0,\ k_4 = 0$

If $\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix} \in R^4$

and $$\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix} = k_1 \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + k_2 \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + k_3 \begin{pmatrix}0\\1\\0\\1\end{pmatrix} + k_4 \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix} k_1 + k_2 \\ k_2 + k_3 \\ k_1 \\ k_3 + k_4 \end{pmatrix}$$

$k_1 = b_3,\ k_2 = b_1 - b_3,\ k_3 = b_2 - b_1 + b_3,$

$k_4 = b_4 - b_2 + b_1 - b_3$

Since S spans R^4 and is linearly independent, it forms a basis for R^4.

$$L : R^3 \to R^4$$

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - z \\ x + z \\ x + y \\ 2x + y + z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_4]{-r_1 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[-r_2 + r_4]{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s$, $s \in R$

$y = s$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore$ The null space of L is $\left\{ s\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

and has dimension 1.

$$L\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 3 \end{pmatrix} = k_1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_3\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + k_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} k_1 + k_2 \\ k_2 + k_3 \\ k_1 \\ k_3 + k_4 \end{pmatrix}$$

$k_1 = 1,\ k_2 = -2,\ k_3 = 4,\ k_4 = -1$

$$L\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = k_1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_3\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + k_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} k_1 + k_2 \\ k_2 + k_3 \\ k_1 \\ k_3 + k_4 \end{pmatrix}$$

$k_1 = 1,\ k_2 = -1,\ k_3 = 2,\ k_4 = 0$

92. $T: R^4 \to R^4$

$$\begin{pmatrix} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 1 \\ 1 & -3 & -1 & 7 \\ 2 & 1 & 5 & 4 \end{pmatrix}$$

i) $$\xrightarrow[-2r_1 + r_4]{-r_1 + r_2,\ -r_1 + r_3} \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -4 & -4 & 5 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow[r_2 + r_4]{4r_2 + r_3} \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{r_3 + r_4} \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The dimension of the range space of T is 3

ii) $$\begin{pmatrix} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 1 \\ 1 & -3 & -1 & 7 \\ 2 & 1 & 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$= \begin{pmatrix} x + y + 3z + 2w \\ x + 2y + 4z + w \\ x - 3y - z + 7w \\ 2x + y + 5z + 4w \end{pmatrix}$$

$$= \begin{pmatrix} x \\ x \\ x \\ 2x \end{pmatrix} + \begin{pmatrix} y \\ 2y \\ -3y \\ y \end{pmatrix} + \begin{pmatrix} 3z \\ 4z \\ -z \\ 5z \end{pmatrix} + \begin{pmatrix} 2w \\ w \\ 7w \\ 4w \end{pmatrix}$$

$$= x\begin{pmatrix}1\\1\\1\\2\end{pmatrix} + y\begin{pmatrix}1\\2\\-3\\1\end{pmatrix} + z\begin{pmatrix}3\\4\\-1\\5\end{pmatrix} + w\begin{pmatrix}2\\1\\7\\4\end{pmatrix}$$

A basis for V is $\left\{\begin{pmatrix}1\\1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\-3\\1\end{pmatrix}, \begin{pmatrix}2\\1\\7\\4\end{pmatrix}\right\}$

iii) W is not a ~~vector~~ space since it does not contain the zero vector.

iv) If $\begin{pmatrix}x\\y\\z\\t\end{pmatrix} \in V$, $\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = k_1\begin{pmatrix}1\\1\\1\\2\end{pmatrix} + k_2\begin{pmatrix}1\\2\\3\\-1\end{pmatrix} + k_3\begin{pmatrix}2\\1\\7\\4\end{pmatrix}$

$$= \begin{pmatrix}k_1 + k_2 + 2k_3\\k_1 + 2k_2 + k_3\\k_1 + 3k_2 + 7k_3\\2k_1 - k_2 + 4k_3\end{pmatrix}$$

$x = k_1 + k_2 + 2k_3$, $y = k_1 + 2k_2 + k_3$,
$z = k_1 + 3k_2 + 7k_3$, $t = 2k_1 - k_2 + 4k_3$

$$\begin{aligned}8x - 5y - z - t &= 8(k_1 + k_2 + 2k_3) - 5(k_1 + 2k_2 + k_3)\\&\quad - (k_1 + 3k_2 + 7k_3) - (2k_1 - k_2 + 4k_3)\\&= 8k_1 + 8k_2 + 16k_3 - 5k_1 - 10k_2 - 5k_3\\&\quad - k_1 - 3k_2 - 7k_3 - 2k_1 + k_2 - 4k_3\\&= 0\end{aligned}$$

$\therefore$ If $\begin{pmatrix}x\\y\\z\\t\end{pmatrix} \in W$, $8x - 5y - z - t \neq 0$.

95. $T: R^4 \to R^4$ $\quad T: \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 4 & -5 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 6 & \theta \end{pmatrix}$$

i) $$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 4 & -5 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 6 & \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 2 & -1 & 4 & -5 & 0 \\ 3 & 1 & 3 & -2 & 0 \\ 6 & 2 & 6 & \theta & 0 \end{array}\right)$$

$$\xrightarrow[\substack{-3r_1 + r_3 \\ -6r_1 + r_4}]{-2r_1 + r_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & -10 & 12 & \theta - 18 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_2 + r_4]{-r_2 + r_3} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta + 4 & 0 \end{array}\right)$$

If $\theta \neq -4$: $$\xrightarrow{\frac{r_4}{\theta + 4}} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

rank $(A) = 3$

$\therefore$ The dimension of the null space, K, of T is 1

If $\theta = -4$: $$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

rank $(A) = 2$

$\therefore$ The dimension of the null space, K, of T is 2

ii) When $\theta \neq -4$:
$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$t = 0$

Let $z = 5s$, $s \in R$

$y = 6s$

$x + 12s - 5s = 0$

$x = -7s$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -7s \\ 6s \\ 5s \\ 0 \end{pmatrix}$$

$$= s \begin{pmatrix} -7 \\ 6 \\ 5 \\ 0 \end{pmatrix}$$

When $\theta = -4$:
$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $t = 5\lambda$, $\lambda \in R$ and $z = 5s$, $s \in R$

$y = 6s - 11\lambda$

$x + 2(6s - 11\lambda) - 5s + 15\lambda = 0$

$x + 12s - 22\lambda - 5s + 15\lambda = 0$

$x = 7\lambda - 7s$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 7\lambda - 7s \\ 6s - 11\lambda \\ 5s \\ 5\lambda \end{pmatrix}$$

$$= s \begin{pmatrix} -7 \\ 6 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -11 \\ 0 \end{pmatrix}$$

iii) $\underset{\sim}{b} = \begin{pmatrix} 5 \\ 0 \\ 5 \\ 10 \end{pmatrix}$, $\underset{\sim}{e}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

when $\theta = -4$:

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 4 & -5 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 6 & -4 \end{pmatrix}$$

If $\underset{\sim}{x}_1 = \underset{\sim}{e}_0 + \lambda \underset{\sim}{e}_1 + \mu \underset{\sim}{e}_2$,

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 6 \\ 5 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -11 \\ 0 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 7\lambda + 7\mu \\ 1 + 6\lambda - 11\mu \\ 1 + 5\lambda \\ 1 + 5\mu \end{pmatrix}, \quad \lambda, \mu \in R$$

$$A\underset{\sim}{x}_1 = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 4 & -5 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 6 & -4 \end{pmatrix} \begin{pmatrix} 1 - 7\lambda + 7\mu \\ 1 + 6\lambda - 11\mu \\ 1 + 5\lambda \\ 1 + 5\mu \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 7\lambda + 7\mu + 2 + 12\lambda - 22\mu - 1 - 5\lambda + 3 + 15\mu \\ 2 - 14\lambda + 14\mu - 1 - 6\lambda + 11\mu + 4 + 20\lambda - 5 - 25\mu \\ 3 - 21\lambda + 21\mu + 1 + 6\lambda - 11\mu + 3 + 15\lambda - 2 - 10\mu \\ 6 - 42\lambda + 42\mu + 2 + 12\lambda - 22\mu + 6 + 30\lambda - 4 - 20\mu \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 0 \\ 5 \\ 10 \end{pmatrix}$$

$\therefore$ $\underset{\sim}{x}_1 = \underset{\sim}{e}_0 + \lambda \underset{\sim}{e}_1 + \mu \underset{\sim}{e}_2$ is a solution of $A\underset{\sim}{x} = \underset{\sim}{b}$ for all $\lambda, \mu \in R$.

96. $T: R^4 \to R^4$ $\quad T: \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$

$$A = \begin{pmatrix} a & 1 & 1 & 2 \\ a & 1 & 2 & 3 \\ 2a & b+2 & 1 & 3 \\ 3a & 2b+3 & 0 & 3 \end{pmatrix}, \quad a, b \in R, \ a \neq 0.$$

i) $$\begin{pmatrix} a & 1 & 1 & 2 \\ a & 1 & 2 & 3 \\ 2a & b+2 & 1 & 3 \\ 3a & 2b+3 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} a & 1 & 1 & 2 & 0 \\ a & 1 & 2 & 3 & 0 \\ 2a & b+2 & 1 & 3 & 0 \\ 3a & 2b+3 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow[-3r_1 + r_4]{\substack{-r_1 + r_2 \\ -2r_1 + r_3}} \left(\begin{array}{cccc|c} a & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & b & -1 & -1 & 0 \\ 0 & 2b & -3 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{-2r_3 + r_4} \left(\begin{array}{cccc|c} a & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & b & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

If $b \neq 0$; $$\left(\begin{array}{cccc|c} a & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & b & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\text{rank}(A) = 3$

$\therefore$ The dimension of k is 1.

If $b = 0$: $\left(\begin{array}{cccc|c} a & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{cccc|c} a & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

rank $(A) = 2$

. The dimension of K is 2

ii) When $b \neq 0$: $\left(\begin{array}{cccc|c} a & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & b & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Let $t = bs$, $s \in R$

$z = -bs$

$y = 0$

$ax + 0 - bs + 2bs = 0$

$ax = -bs$

$x = -\frac{bs}{a}$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -\frac{bs}{a} \\ 0 \\ -bs \\ bs \end{pmatrix}$$

$$= -bs \begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

∴ A basis vector $\underset{\sim}{e}_1$ for K is $\begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix}$.

iii) when $b = 0$: $\left(\begin{array}{cccc|c} a & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Let $t = s, s \in R$ and $y = \lambda, \lambda \in R$

$z = -s$

$ax + \lambda - s + 2s = 0$

$ax = -\lambda - s$

$x = -\frac{\lambda}{a} - \frac{s}{a}$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -\frac{\lambda}{a} - \frac{s}{a} \\ \lambda \\ -s \\ s \end{pmatrix}$$

$$= \lambda \begin{pmatrix} -\frac{1}{a} \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{1}{a} \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

If $\underset{\sim}{e}_2 = \begin{pmatrix} -\frac{1}{a} \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ is a basis for K.

iv) If $b = 0$ and $\begin{pmatrix} a + \theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix} = \lambda \begin{pmatrix} -\frac{1}{a} \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{1}{a} \\ 0 \\ -1 \\ 1 \end{pmatrix}$,

$\theta \in R$ $\qquad \begin{pmatrix} a + \theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix} = \begin{pmatrix} -\frac{\lambda}{a} - \frac{s}{a} \\ \lambda \\ -s \\ s \end{pmatrix}$

$$a+\theta = -\frac{\lambda}{a} - \frac{s}{a}, \quad -a\theta = \lambda, \quad a^2 = -s$$

$$\therefore \; s = -a^2, \quad \theta = -\frac{\lambda}{a}$$

If $b \neq 0$ and $\begin{pmatrix} a+\theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix} = -bs \begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix}$, $\theta \in R$

$$\begin{pmatrix} a+\theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix} = \begin{pmatrix} -\frac{bs}{a} \\ 0 \\ -bs \\ bs \end{pmatrix}$$

$$a+\theta = -\frac{bs}{a}, \quad -\theta a = 0, \quad a^2 = -bs$$

$$\therefore \; s = \frac{-a^2}{b}, \quad \theta = 0$$

$\therefore$ If $b = 0$, then the vector $\begin{pmatrix} a+\theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix}$, $\theta \in R$

belongs to K for all $\theta \in R$, but if $b \neq 0$,

then the vector $\begin{pmatrix} a+\theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix}$ belongs to K

for only one value of θ.

97. $T: R^4 \longrightarrow R^4$

$$M = \begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 0 & -1 & 7 \\ 3 & -1 & -2 & 10 \\ -1 & 5 & 3 & -1 \end{pmatrix}$$

i) $$\xrightarrow[r_1 + r_4]{-2r_1 + r_2,\ -3r_1 + r_3} \begin{pmatrix} 1 & 5 & 2 & 6 \\ 0 & -10 & -5 & -5 \\ 0 & -16 & -8 & -8 \\ 0 & 10 & 5 & 5 \end{pmatrix}$$

$$\xrightarrow{\frac{r_2}{-5},\ \frac{r_3}{-8},\ \frac{r_4}{5}} \begin{pmatrix} 1 & 5 & 2 & 6 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[-r_2 + r_4]{-r_2 + r_3} \begin{pmatrix} 1 & 5 & 2 & 6 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $(M) = 2$

$\therefore$ The dimension of the null space, K, of T is 2.

ii) $$\begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 0 & -1 & 7 \\ 3 & -1 & -2 & 10 \\ -1 & 5 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 5 & 2 & 6 & 0 \\ 2 & 0 & -1 & 7 & 0 \\ 3 & -1 & -2 & 10 & 0 \\ -1 & 5 & 3 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 5 & 2 & 6 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = 2s,\ s \in \mathbb{R}$ and $z = 2t;\ t \in \mathbb{R}$.

$$y = -s - t$$

$$x + 5(-s - t) + 2(2t) + 6(2s) = 0$$

$$x - 5s - 5t + 4t + 12s = 0$$

$$x = t - 7s$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} t - 7s \\ -s - t \\ 2t \\ 2s \end{pmatrix}$$

$$= s\begin{pmatrix} -7 \\ -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ 1 \\ 1 \end{pmatrix} = s\begin{pmatrix} -7 \\ -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -7s + t \\ -s - t \\ 2t \\ 2s \end{pmatrix}$$

$$s = \frac{1}{2},\ t = \frac{1}{2},\ p = -3,\ q = -1$$

$$\begin{pmatrix} r \\ s \\ 1 \\ -1 \end{pmatrix} = s\begin{pmatrix} -7 \\ -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -7s + t \\ -s - t \\ 2t \\ 2s \end{pmatrix}$$

$$s = \frac{-1}{2},\ t = \frac{1}{2},\ r = 4,\ s = 0$$

$$\underset{\sim}{e}_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 1 \end{pmatrix} \quad \underset{\sim}{e}_2 = \begin{pmatrix} 4 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ is a basis for K.

iii) $$\begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 0 & -1 & 7 \\ 3 & -1 & -2 & 10 \\ -1 & 5 & 3 & -1 \end{pmatrix} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 0 & -1 & 7 \\ 3 & -1 & -2 & 10 \\ -1 & 5 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 - 3\lambda + 4\mu \\ -\lambda \\ \lambda + \mu \\ \lambda - \mu \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 3\lambda + 4\mu - 5\lambda + 2\lambda + 2\mu + 6\lambda - 6\mu \\ 2 - 6\lambda + 8\mu - \lambda - \mu + 7\lambda - 7\mu \\ 3 - 9\lambda + 12\mu + \lambda - 2\lambda - 2\mu + 10\lambda - 10\mu \\ -1 + 3\lambda - 4\mu - 5\lambda + 3\lambda + 3\mu - \lambda + \mu \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}$$

every vector of the form $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \underset{\sim}{e}_1 + \mu \underset{\sim}{e}_2$

has the same image under T.

iv) $\underset{\sim}{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix}$

$M\underset{\sim}{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}$, $\alpha + \beta + \gamma = 11$

Since $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a particular solution of the equation $M\underset{\sim}{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}$ and $\left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis of the null space, K, every vector of the form

$\underset{\sim}{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ is the general solution of the equation $M\underset{\sim}{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}$.

$\underset{\sim}{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix}$

$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - 3\lambda + 4\mu \\ -\lambda \\ \lambda + \mu \\ \lambda - \mu \end{pmatrix}$

$\alpha = 1 - 3\lambda + 4\mu$, $\beta = -\lambda$, $\gamma = \lambda + \mu$, $\lambda = \mu$

$1 - 3\lambda + 4\mu - \lambda + \lambda + \mu = 11$

$5\mu - 3\lambda = 10$

$\lambda = \mu \quad \therefore \; 2\mu = 10$

$\mu = 5$

$\lambda = 5$

$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 10 \end{pmatrix}$

98. $T : R^4 \longrightarrow R^4$

$$M = \begin{pmatrix} 1 & 2 & 3 & 6 \\ -1 & -1 & -2 & -4 \\ 2 & 3 & 6 & 11 \\ 3 & 8 & 10 & 21 \end{pmatrix}$$

i) $$\xrightarrow{\substack{r_1 + r_2 \\ -2r_1 + r_3 \\ -3r_1 + r_4}} \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 0 & -1 \\ 0 & 2 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{\substack{r_2 + r_3 \\ -2r_2 + r_4}} \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\xrightarrow{r_3 + r_4} \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The dimension of R, the range space of T is 3.

ii) $$\begin{pmatrix} 1 & 2 & 3 & 6 \\ -1 & -1 & -2 & -4 \\ 2 & 3 & 6 & 11 \\ 3 & 8 & 10 & 21 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$= \begin{pmatrix} x + 2y + 3z + 6w \\ -x - y - 2z - 4w \\ 2x + 3y + 6z + 11w \\ 3x + 8y + 10z + 21w \end{pmatrix}$$

$$= \begin{pmatrix} x \\ -x \\ 2x \\ 3x \end{pmatrix} + \begin{pmatrix} 2y \\ -y \\ 3y \\ 8y \end{pmatrix} + \begin{pmatrix} 3z \\ -2z \\ 6z \\ 10z \end{pmatrix} + \begin{pmatrix} 6w \\ -4w \\ 11w \\ 21w \end{pmatrix}$$

$$= x\begin{pmatrix}1\\-1\\2\\3\end{pmatrix} + y\begin{pmatrix}2\\-1\\3\\8\end{pmatrix} + z\begin{pmatrix}3\\-2\\6\\10\end{pmatrix} + w\begin{pmatrix}6\\-4\\11\\21\end{pmatrix}$$

A basis for R is $\left\{ \begin{pmatrix}1\\-1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\3\\8\end{pmatrix}, \begin{pmatrix}3\\-2\\6\\10\end{pmatrix} \right\}$

iii) $$\begin{pmatrix}1\\2\\3\\5\end{pmatrix} = k_1\begin{pmatrix}1\\-1\\2\\3\end{pmatrix} + k_2\begin{pmatrix}2\\-1\\3\\8\end{pmatrix} + k_3\begin{pmatrix}3\\-2\\6\\10\end{pmatrix}$$

$$= \begin{pmatrix} k_1 + 2k_2 + 3k_3 \\ -k_1 - k_2 - 2k_3 \\ 2k_1 + 3k_2 + 6k_3 \\ 3k_1 + 8k_2 + 10k_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & -2 \\ 2 & 3 & 6 \\ 3 & 8 & 10 \end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & -1 & -2 & 2 \\ 2 & 3 & 6 & 3 \\ 3 & 8 & 10 & 5 \end{array}\right)$$

$$\xrightarrow[\substack{-2r_1 + r_3 \\ -3r_1 + r_4}]{r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[-2r_2 + r_4]{r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & -4 \end{array}\right)$$

$$\xrightarrow{r_3 + r_4} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$k_3 = 4$

$k_2 = -1$

$k_1 = -9$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} \in R$$

iv) If $\begin{pmatrix} 1+\theta \\ 2 \\ 3 \\ 5 \end{pmatrix} \in R$,

$$\begin{pmatrix} 1+\theta \\ 2 \\ 3 \\ 5 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ -1 \\ 3 \\ 8 \end{pmatrix} + k_3 \begin{pmatrix} 3 \\ -2 \\ 6 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} k_1 + 2k_2 + 3k_3 \\ -k_1 - k_2 - 2k_3 \\ 2k_1 + 3k_2 + 6k_3 \\ 3k_1 + 8k_2 + 10k_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & -2 \\ 2 & 3 & 6 \\ 3 & 8 & 10 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1+\theta \\ -1 & -1 & -2 & 2 \\ 2 & 3 & 6 & 3 \\ 3 & 8 & 10 & 5 \end{array}\right)$$

$$\xrightarrow[-3r_1+r_4]{\substack{r_1+r_2 \\ -2r_1+r_3}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1+\theta \\ 0 & 1 & 1 & 3+\theta \\ 0 & -1 & 0 & 1-2\theta \\ 0 & 2 & 1 & 2-3\theta \end{array}\right)$$

$$\xrightarrow[-2r_2+r_4]{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1+\theta \\ 0 & 1 & 1 & 3+\theta \\ 0 & 0 & 1 & 4-\theta \\ 0 & 0 & -1 & -4-5\theta \end{array}\right)$$

$$\xrightarrow{r_3+r_4} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1+\theta \\ 0 & 1 & 1 & 3+\theta \\ 0 & 0 & 1 & 4-\theta \\ 0 & 0 & 0 & -6\theta \end{array}\right)$$

$\therefore$ If $\theta \neq 0$, $\begin{pmatrix} 1+\theta \\ 2 \\ 3 \\ 5 \end{pmatrix} \notin R$.