1. Let $f(n) = 3^{4n-2} + 17^n + 22$

when $n=1$: $f(1) = 3^{4(1)-2} + 17^1 + 22$

$= 3^{4-2} + 17 + 22$

$= 3^2 + 17 + 22$

$= 9 + 17 + 22$

$= 48$

$= 16(3)$

$\therefore f(1)$ is divisible by 16.

Assume that $f(n)$ is divisible by 16 when $n=k$

$n=k$: $f(k) = 3^{4k-2} + 17^k + 22$

$16 \mid f(k) \quad \therefore f(k) = 3^{4k-2} + 17^k + 22 = 16s, \; s \in N$

when $n=k+1$: $f(k+1) = 3^{4(k+1)-2} + 17^{k+1} + 22$

$= 3^{4k-2+4} + 17^k 17 + 22$

$= 3^{4k-2} 3^4 + 17^k 17 + 22$

$= 3^{4k-2} 81 + 17^k 17 + 22$

$= 3^{4k-2}(80+1) + 17^k(16+1) + 22$

$= 3^{4k-2} 80 + 3^{4k-2} + 17^k 16 + 17^k + 22$

$= 3^{4k-2} 80 + 17^k 16 + 3^{4k-2} + 17^k + 22$

$$= 3^{4k-2} 16(5) + 17^{k} 16 + 16s$$

$$= 16(3^{4k-2} 5 + 17^{k} + s)$$

since s is a positive integer and k is a positive integer $\therefore$ $3^{4k-2}5 + 17^{k} + s$ is an integer

$$16 \mid f(k+1)$$

since $f(k+1)$ is divisible by 16 if $f(k)$ is divisible by 16 and since $f(1)$ is divisible by 16,

$\therefore$ $f(n)$ is divisible by 16 for all positive integers n.

2. a) Expressing $\dfrac{2}{x(x+1)(x+2)}$ as partial fractions

$$\frac{2}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

$$= \frac{A(x+1)(x+2) + Bx(x+2) + Cx(x+1)}{x(x+1)(x+2)}$$

$$\therefore 2 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$$

$$= A(x^2+3x+2) + B(x^2+2x) + C(x^2+x)$$

$$= (A+B+C)x^2 + (3A+2B+C)x + 2A$$

Equating coefficients of powers of x,

$A+B+C=0$ $\quad$ $3A+2B+C=0$ $\quad$ $2A=2$

$A=1$

$B+C=-1$ $\quad$ $2B+C=-3$

$B=-2$

$C=1$

$$\therefore \frac{2}{x(x+1)(x+2)} = \frac{1}{x} - \frac{2}{x+1} + \frac{1}{x+2}$$

b) Since $\dfrac{2}{x(x+1)(x+2)} = \dfrac{1}{x} - \dfrac{2}{x+1} + \dfrac{1}{x+2}$

$$\therefore \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}\right)$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3}\right.$$

$$+ \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$

$$+ \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$$

$$+ \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$$

$$\vdots$$

$$+ \frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n}$$

$$+ \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$$

$$\left.+ \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{n+1-n-2}{(n+1)(n+2)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right)$$

$$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

3. $x^3 + 4x + 1 = 0$

$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \alpha\gamma + \beta\gamma = 4 \quad \alpha\beta\gamma = -1$

i) $S_n = \alpha^n + \beta^n + \gamma^n$

$S_2 = \alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 0^2 - 2(4)$

$= -8$

ii) Since α, β, γ are roots of the equation

$x^3 + 4x + 1 = 0$,

$\alpha^3 + 4\alpha + 1 = 0$ —— (1)

$\beta^3 + 4\beta + 1 = 0$ —— (2)

$\gamma^3 + 4\gamma + 1 = 0$ —— (3)

(1) $\times \alpha^n$: $\alpha^{3+n} + 4\alpha^{1+n} + \alpha^n = 0$ — (4)

(2) $\times \beta^n$: $\beta^{3+n} + 4\beta^{1+n} + \beta^n = 0$ — (5)

(3) $\times \gamma^n$: $\gamma^{3+n} + 4\gamma^{1+n} + \gamma^n = 0$ — (6)

(4) + (5) + (6):

$\alpha^{3+n} + \beta^{3+n} + \gamma^{3+n} + 4\alpha^{1+n} + 4\beta^{1+n} + 4\gamma^{1+n}$

$+ \alpha^n + \beta^n + \gamma^n = 0$

$\alpha^{3+n} + \beta^{3+n} + \gamma^{3+n} + 4(\alpha^{1+n} + \beta^{1+n} + \gamma^{1+n}) + \alpha^n + \beta^n + \gamma^n = 0$

$\therefore S_{n+3} + 4S_{n+1} + S_n = 0$

iii) when $n=1$ $\therefore S_4 + 4S_2 + S_1 = 0$

$S_1 = \alpha^1 + \beta^1 + \gamma^1 = \alpha + \beta + \gamma = 0$

$\therefore S_4 + 4(-8) + 0 = 0$

$S_4 = 32$

4. i) $C: \quad y = \dfrac{x^2+5}{x+1}$

$$
\begin{array}{r|l}
 & \quad x - 1 \\
\cline{2-2}
x+1 & x^2 + 5 \\
 & \underline{x^2 + x} \\
 & \quad -x + 5 \\
 & \quad \underline{-x - 1} \\
 & \qquad\ 6
\end{array}
$$

$$y = x - 1 + \frac{6}{x+1}$$

$x \to \pm\infty \quad y \to x - 1$

$x \to -1 \quad y \to \pm\infty$

$y = x - 1$ and $x = -1$ are the asymptotes of the curve C.

ii) $\dfrac{dy}{dx} = 1 - \dfrac{6}{(x+1)^2}$

If $\dfrac{dy}{dx} = 0$, $1 - \dfrac{6}{(x+1)^2} = 0$

$$\frac{6}{(x+1)^2} = 1$$

$$(x+1)^2 = 6$$

$$x + 1 = \pm\sqrt{6}$$

$$x = -1 \pm \sqrt{6}$$

$x = -1 + \sqrt{6}$: $y = \dfrac{(-1 + \sqrt{6})^2 + 5}{-1 + \sqrt{6} + 1}$

$= \dfrac{1 + 6 - 2\sqrt{6} + 5}{\sqrt{6}}$

$= \dfrac{12 - 2\sqrt{6}}{\sqrt{6}}$

$= \dfrac{12}{\sqrt{6}} - 2$

$= 2\sqrt{6} - 2$

$x = -1 - \sqrt{6}$: $y = \dfrac{(-1 - \sqrt{6})^2 + 5}{-1 - \sqrt{6} + 1}$

$= \dfrac{1 + 6 + 2\sqrt{6} + 5}{-\sqrt{6}}$

$= \dfrac{12 + 2\sqrt{6}}{-\sqrt{6}}$

$= -\dfrac{12}{\sqrt{6}} - 2$

$= -2\sqrt{6} - 2$

$(-1 + \sqrt{6}, 2\sqrt{6} - 2)$ and $(-1 - \sqrt{6}, -2\sqrt{6} - 2)$ are the stationary points of y.

$$\frac{d^2y}{dx^2} = \frac{12}{(x+1)^3}$$

$x = -1 + \sqrt{6}$: $\frac{d^2y}{dx^2} = \frac{12}{(\sqrt{6})^3} = \frac{2}{\sqrt{6}} > 0$

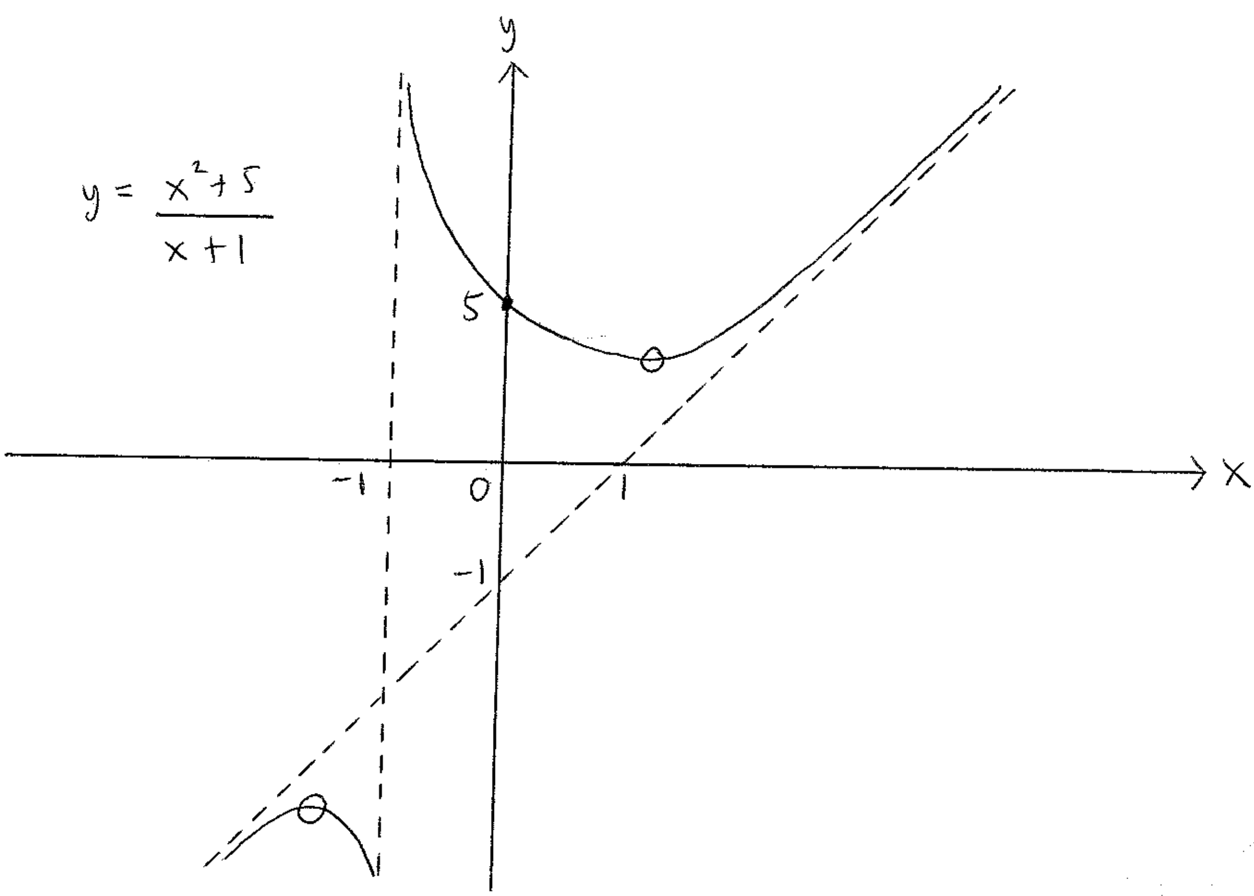
$x = -1 - \sqrt{6}$: $\frac{d^2y}{dx^2} = \frac{12}{(-\sqrt{6})^3} = \frac{-2}{\sqrt{6}} < 0$

$\therefore$ $(-1+\sqrt{6}, 2+2\sqrt{6})$ is a minimum point and

$(-1-\sqrt{6}, 2-2\sqrt{6})$ is a maximum point.

iii) when $x = 0$ $y = 5$

$$y = \frac{x^2+5}{x+1} \neq 0$$



• : intersection point

O : critical point