16. EIGENVALUES AND EIGENVECTORS

1. If P is a real matrix such that $P=\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, evaluate the product PP^T, where P^T is the transpose of P.
Hence or otherwise find the matrix P with positive diagonal elements such that

$PP^T = \begin{pmatrix} 9 & 12 \\ 12 & 41 \end{pmatrix}$ and verify that $P^TP = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix}$.

Find also the real matrix Q, where $Q=\begin{pmatrix} d & 0 \\ e & f \end{pmatrix}$ having positive diagonal elements and such that $QQ^T = P^TP$.
If $R = Q^T P^{-1}$ verify that RR^T is a unit matrix. (J72/II/5)

2. The column vector $\mathbf{x}$ is an eigenvector of the matrix $\mathbf{A}$ corresponding to eigenvalue λ.
   (a) Show that $\mathbf{x}$ is an eigenvector of $\mathbf{A}^2$ and find the corresponding eigenvalue.
   (b) Show that $\mathbf{x}$ is an eigenvector of $\mathbf{A}+3\mathbf{I}$, where $\mathbf{I}$ is the unit matrix, and find the corresponding eigenvalue.
   (c) Given that $\mathbf{A}$ possesses an inverse show that $\lambda \neq 0$. Show also that $\mathbf{x}$ is an eigenvector of $\mathbf{A}^{-1}$ and find the corresponding eigenvalue. (J72/II/14)

3. The symmetric matrix A has real elements and V and U are eigenvectors corresponding to eigenvalues λ and μ respectively. By considering V^TAU or otherwise, show that if $\lambda \neq \mu$ then $V^TU = 0$.
If $X = aV + bU$ where a and b are real, show that $(A^2 + I)X = (\lambda^2 + 1)aV + (\mu^2 + 1)bU$, where I is the unit matrix. (J73/II/14)

4. (a) If $A = \begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ x & y \end{pmatrix}$ where $x \neq y$, find x and y so that $B^{-1}AB$ is a diagonal matrix.
   (b) If $C = \begin{pmatrix} 3 & 2 \\ -8 & -3 \end{pmatrix}$, evaluate C^n where n is a positive integer, distinguishing between the cases where n is odd and n is even. (N73/I/5)

5. Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} 6 & 12 \\ 12 & -1 \end{pmatrix}$.

Hence or otherwise deduce that by a suitable rotation of axes, which should be specified, in which the coordinates (x, y) of P become (ξ, η), the equation $6x^2 + 24xy - y^2 = 1$ becomes $15\xi^2 - 10\eta^2 = 1$. (N74/II/14)

6. Let A be a 2×2 matrix. If $\mathbf{b}_1$ and $\mathbf{b}_2$ are 2×1 matrices and B is the 2×2 matrix having columns $\mathbf{b}_1$ and $\mathbf{b}_2$, show that AB has columns $A\mathbf{b}_1$ and $A\mathbf{b}_2$.
Suppose that A has distinct eigenvalues λ_1 and λ_2 with associated eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. Show that
   (a) the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent,
   (b) the 2×2 matrix P, with columns $\mathbf{x}_1$ and $\mathbf{x}_2$, is non-singular and

$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ (J75/II/14)

7. Show that the matrix $A = \begin{pmatrix} 0 & 4 & 4 \\ 1 & -3 & -1 \\ 3 & 3 & 1 \end{pmatrix}$ has a characteristic root (eigenvalue) equal to 4 and find the other two characteristic roots.
Find also eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ corresponding to the characteristic roots λ_1, λ_2, λ_3 respectively.
The columns of the matrix P are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in that order. Verify that the matrix product AP is expressible in the form PD, where D is a diagonal matrix, and deduce that $P^{-1}AP$ is a diagonal matrix. (N75/II/13)

8. The square $r \times r$ matrices A, B and P are related by $B = P^{-1}AP$, where P is non-singular. Show that $B^n = P^{-1}A^nP$ for any positive integer n.
Show also that, if A is non-singular and A^0 and B^0 are defined to be the unit matrix of order r, then $B^n = P^{-1}A^nP$ for any integer n. (N76/II/4)

9. Find the eigenvalues λ_1 and λ_2 and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} -10 & 12 \\ -9 & 11 \end{pmatrix}$$

Using your result, or otherwise, find a 2×2 matrix P such that $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Hence, or otherwise, find A^{10}. (J77/II/13)

10. The matrices $\mathbf{B}$ and $\mathbf{P}$ are given by $\mathbf{B} = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Find $\mathbf{P}^{-1}$ and show that $\mathbf{P}^{-1}\mathbf{BP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Given that n is a positive integer, show that $(\mathbf{P}^{-1}\mathbf{BP})^n = \mathbf{P}^{-1}\mathbf{B}^n\mathbf{P}$, and hence find $\mathbf{B}^n$. (J83/I/15)

11. (a) Given that $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and that the matrix $\mathbf{X}$ satisfies the equation $\mathbf{XAX}^{\mathrm{T}} = \mathbf{B}$, prove that $\mathbf{X}$ has one or other of the forms

$$\begin{pmatrix} a & -a \\ -\dfrac{1}{a} & -\dfrac{1}{a} \end{pmatrix}, \quad \begin{pmatrix} a & a \\ -\dfrac{1}{a} & \dfrac{1}{a} \end{pmatrix}$$

(b) Given that $\mathbf{Y} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and that $\mathbf{YY}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, show that, if p, q, r, s are real, they all lie in the interval $[-\sqrt{2}, \sqrt{2}]$. (N84/II/4)

12. Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} -3 & 5 & 5 \\ -4 & 6 & 5 \\ 4 & -4 & -3 \end{pmatrix}$$

Hence find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{DP}$.
Find also a diagonal matrix $\mathbf{E}$ such that $\mathbf{A}^3 = \mathbf{P}^{-1}\mathbf{EP}$. (J86/I/10)

13. Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 4 & -4 & 6 \\ 2 & -1 & 2 \\ -3 & 4 & -5 \end{pmatrix}.$$

Hence find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P}$. Find also a diagonal matric $\mathbf{E}$ such that $\mathbf{A}^4 = \mathbf{P}^{-1}\mathbf{E}\mathbf{P}$. (N86/I/10)

14. Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & 0 \\ 1 & -1 & 1 \\ -1 & 3 & 1 \end{pmatrix}.$$

Hence, or otherwise,

(i) find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A} + 10\mathbf{I}$, where $\mathbf{I}$ is the unit matrix of order three,

(ii) find a matrix $\mathbf{P}$ such that $\mathbf{PAP}^{-1}$ is a diagonal matrix. (J87/I/10)

15. The vector $\mathbf{x}$ is an eigenvector of each of the matrices $\mathbf{A}$ and $\mathbf{B}$, with corresponding eigenvalues λ and μ, respectively. Show that $\mathbf{x}$ is an eigenvector of $\mathbf{AB}$ with eigenvalue $\lambda\mu$.
Find the eigenvalues and corresponding eigenvectors of each of the matrics $\mathbf{C}$ and $\mathbf{D}$, where

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & -3 \end{pmatrix}.$$

Hence, or otherwise, find *one* eigenvector of the matrix $\mathbf{CD}$ and its corresponding eigenvalue. (N87/I/10)

16. Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & 4 \\ 0 & -1 & 3 \end{pmatrix}$.

Hence write down a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{QDQ}^{-1}$.

Obtain the exact value of the element in the first row and the first column of $\mathbf{M}^7$. (J88/I/10)

17. The set of all 3×3 non-singular matrices with elements in $\mathbf{R}$ is denoted by S. The relation ρ on S is defined as follows.

Given that $\mathbf{A}$ and $\mathbf{B}$ are any two elements of S, then $\mathbf{A}$ is said to be ρ related to $\mathbf{B}$ (written $\mathbf{A}\ \rho\ \mathbf{B}$) if there exists an element $\mathbf{P}$ of S such that $\mathbf{A} = \mathbf{PBP}^{-1}$.

(i) Show that ρ is an equivalence relation on S.

(ii) Given that $\mathbf{A} = \begin{pmatrix} 4 & -1 & 1 \\ 10 & -5 & -1 \\ -2 & 2 & 1 \end{pmatrix}$, find a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}\ \rho\ \mathbf{D}$.

[Note that in (ii) explicit expressions for $\mathbf{P}$ or $\mathbf{P}^{-1}$ are not required.]

(N88/I/10)

18. Show that if e is an eigenvector of the square matrix $\mathbf{A}$, with corresponding eigenvalue λ, then e is an eigenvector of $\mathbf{A}^2$ with corresponding eigenvalue λ^2.

Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{B} = \begin{pmatrix} 2 & -5 & 6 \\ 2 & 3 & 2 \\ -1 & 5 & -5 \end{pmatrix}$.

Find a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{B}^2 = \mathbf{QDQ}^{-1}$.

(J89/I/10)

19. Given two square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order, show that if there exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PBP}^{-1}$, then $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues.

Hence, by considering the product $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 1 & 5 & 0 \\ 2 & 3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, or otherwise, find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 2 & -3 & -5 \\ 1 & 5 & 1 \\ 2 & 3 & 9 \end{pmatrix}$.

Find a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{QDQ}^{-1}$, and deduce that if n is a positive integer, then $\mathbf{M}n = \mathbf{Q}\begin{pmatrix} 4^n & 0 & 0 \\ 0 & 5^n & 0 \\ 0 & 0 & 7^n \end{pmatrix}\mathbf{Q}^{-1}$.

(N89/I/10)

20. The vector $\mathbf{x}$ is an eigenvector of the 3×3 matrix $\mathbf{A}$, with corresponding eigenvalue λ. Show that if $\mathbf{A}$ is non-singular then

(i) $\lambda \neq 0$,

(ii) the vector $\mathbf{x}$ is an eigenvector of the matrix $\mathbf{A}^{-1}$, with corresponding eigenvalue λ^{-1}.

~~Find the eigenvalues and corresponding eigenvectors~~ of the matrices $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 8 \\ 0 & 0 & -3 \end{pmatrix}$ and $\mathbf{B} = (\mathbf{A} + 5\mathbf{I})^{-1}$.

(J90/I/10)

21. Show that the eigenvalues of the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} -\frac{3}{2} & 4 & \frac{11}{3} \\ 1 & -\frac{1}{2} & -2 \\ -1 & 2 & \frac{7}{3} \end{pmatrix}$, are the roots of the equation $12\lambda^3 - 4\lambda^2 - 3\lambda + 1 = 0$.

Hence find a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{QDQ}^{-1}$.

Deduce that, as the positive integer n tends to infinity, all the elements of $\mathbf{M}^n$ tend to zero. (N90/I/10)