OCTOBER / NOVEMBER 2011

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1. $$\frac{1}{n^2} - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - n^2}{n^2(n+1)^2}$$

$$= \frac{n^2+2n+1-n^2}{n^2(n+1)^2}$$

$$= \frac{2n+1}{n^2(n+1)^2}$$

$$S_N = \sum_{r=1}^{N} \frac{2r+1}{r^2(r+1)^2}$$

$$= \sum_{r=1}^{N} \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

$$= \frac{1}{1^2} - \frac{1}{2^2}$$

$$+ \frac{1}{2^2} - \frac{1}{3^2}$$

$$+ \frac{1}{3^2} - \frac{1}{4^2}$$

$$\vdots$$

$$+ \frac{1}{(N-2)^2} - \frac{1}{(N-1)^2}$$

$$+ \frac{1}{(N-1)^2} - \frac{1}{N^2}$$

$$+ \frac{1}{N^2} - \frac{1}{(N+1)^2}$$

$$= 1 - \frac{1}{(N+1)^2}$$

$$S = \lim_{N\to\infty} S_N$$

$$= \lim_{N\to\infty} 1 - \frac{1}{(N+1)^2}$$

$$= 1 - 0$$

$$= 1$$

If $S - S_N < 10^{-16}$

$$1 - \left(1 - \frac{1}{(N+1)^2}\right) < 10^{-16}$$

$$\frac{1}{(N+1)^2} < 10^{-16}$$

$$(N+1)^2 > 10^{16}$$

$$N+1 > 10^8$$

$$N_{min} = 10^8$$

$\therefore$ The least value of N such that $S - S_N < 10^{-16}$ is 10^8.

2. $$\frac{d^n}{dx^n}\left(\frac{1}{2x+3}\right) = \frac{(-1)^n n! 2^n}{(2x+3)^{n+1}}$$

when $n=1$: $$\frac{d^1}{dx^1}\left(\frac{1}{2x+3}\right) = \frac{d}{dx}\left(\frac{1}{2x+3}\right)$$

$$= \frac{-2}{(2x+3)^2}$$

$$= \frac{(-1)(2)1}{(2x+3)^2}$$

$$= \frac{(-1)^1 1! 2^1}{(2x+3)^{1+1}}$$

Assume the statement is true when $n=k$.

$n=k$: $$\frac{d^k}{dx^k}\left(\frac{1}{2x+3}\right) = \frac{(-1)^k k! 2^k}{(2x+3)^{k+1}}$$

when $n=k+1$: $$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{2x+3}\right) = \frac{(-1)^{k+1}(k+1)! 2^{k+1}}{(2x+3)^{k+2}}$$

(what needs to be proved)

$$\frac{d^k}{dx^k}\left(\frac{1}{2x+3}\right) = \frac{(-1)^k k! 2^k}{(2x+3)^{k+1}}$$

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{2x+3}\right) = \frac{d}{dx}\left(\frac{d^k}{dx^k}\left(\frac{1}{2x+3}\right)\right)$$

$$= \frac{d}{dx}\left(\frac{(-1)^k k! 2^k}{(2x+3)^{k+1}}\right)$$

$$= (-1)^k k! 2^k \frac{d}{dx}\left(\frac{1}{(2x+3)^{k+1}}\right)$$

$$= (-1)^k k! 2^k \left(\frac{-(k+1)2}{(2x+3)^{k+2}}\right)$$

$$= \frac{(-1)^k(-1)(k+1)k! 2^k 2}{(2x+3)^{k+2}}$$

$$= \frac{(-1)^{k+1}(k+1)! 2^{k+1}}{(2x+3)^{k+2}}$$

$$\therefore \frac{d^n}{dx^n}\left(\frac{1}{2x+3}\right) = \frac{(-1)^n n! 2^n}{(2x+3)^{n+1}}$$

for every positive integer n.

3. $x^3 + 5x^2 - 3x - 15 = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = -5 \quad \alpha\beta + \alpha\gamma + \beta\gamma = -3 \quad \alpha\beta\gamma = 15$

$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) \\ &= (-5)^2 - 2(-3) \\ &= 25 + 6 \\ &= 31\end{aligned}$$

$$\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$$

$$\begin{aligned}\begin{vmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{vmatrix} &= 1(1-\gamma^2) - \alpha(\alpha - \beta\gamma) + \beta(\alpha\gamma - \beta) \\ &= 1 - \gamma^2 - \alpha^2 + \alpha\beta\gamma + \alpha\beta\gamma - \beta^2 \\ &= 1 - (\alpha^2 + \beta^2 + \gamma^2) + 2\alpha\beta\gamma \\ &= 1 - 31 + 30 \\ &= 0\end{aligned}$$

$\therefore$ since $\begin{vmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{vmatrix} = 0$, $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular.

4. $x = 2\sin 2t \quad y = 3\cos 2t, \quad 0 < t < \frac{\pi}{2}$.

i) $\frac{dx}{dt} = 4\cos 2t \quad \frac{dy}{dt} = -6\sin 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{6\sin 2t}{4\cos 2t} = -\frac{3}{2}\tan 2t$$

At $t = \frac{\pi}{3}$: $\frac{dy}{dx} = -\frac{3}{2}\tan\frac{2\pi}{3} = -\frac{3}{2}(-\sqrt{3}) = \frac{3\sqrt{3}}{2}$.

ii)
$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) \\ &= \frac{1}{4\cos 2t}\frac{d}{dt}\left(-\frac{3}{2}\tan 2t\right) \\ &= \frac{-3}{8\cos 2t}(2\sec^2 2t) \\ &= \frac{-3}{4\cos^3 2t}\end{aligned}$$

At $t = \frac{\pi}{3}$:
$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-3}{4\left(-\frac{1}{2}\right)^3} \\ &= \frac{-3(-8)}{4} \\ &= 6.\end{aligned}$$

5. $z = \cos\theta + i\sin\theta$

$z^{-1} = (\cos\theta + i\sin\theta)^{-1}$

$= \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

$z^{n} = (\cos\theta + i\sin\theta)^{n}$

$= \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$z + \frac{1}{z} = 2\cos\theta$

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$

$(2\cos\theta)^4 = (z + \frac{1}{z})^4$

$16\cos^4\theta = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$

$= z^4 + \frac{1}{z^4} + 4(z^2 + \frac{1}{z^2}) + 6$

$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$

$\therefore \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

$= \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}$

$$\int_0^{\frac{\pi}{4}} \cos^4\theta \, d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8} \, d\theta$$

$$= \left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3\theta}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4} + \frac{3\pi}{32}$$

6. $\frac{d^2x}{dt^2} + \frac{4dx}{dt} + 4x = \sin 2t$

The auxiliary equation is $m^2 + 4m + 4 = 0$

$(m+2)^2 = 0$

$m = -2$.

$\therefore$ The complementary function, x_c, is

$x_c = (At + B)e^{-2t}$

The particular integral, x_p, is given by

$x_p = C\cos 2t + D\sin 2t$

$\frac{dx_p}{dt} = -2C\sin 2t + 2D\cos 2t$

$\frac{d^2x_p}{dt^2} = -4C\cos 2t - 4D\sin 2t$

$\frac{d^2x_p}{dt^2} + \frac{4dx_p}{dt} + 4x_p = -4C\cos 2t - 4D\sin 2t$

$+ 4(-2C\sin 2t + 2D\cos 2t)$

$+ 4(C\cos 2t + D\sin 2t)$

$= 8D\cos 2t - 8C\sin 2t$

$= \sin 2t$

$-8C = 1 \quad 8D = 0$

$C = \frac{-1}{8} \quad D = 0$

$\therefore x_p = \frac{-\cos 2t}{8}$

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$$x = x_c + x_p$$
$$= (At+B)e^{-2t} - \frac{\cos 2t}{8}$$

$\therefore$ The general solution of the equation is

$$x = (At+B)e^{-2t} - \frac{\cos 2t}{8}.$$

As $t \to \infty$, since $e^{-2t} \to 0$, $x \to -\frac{\cos 2t}{8}$.

7. $\frac{d}{dt}(t(1+t^3)^n) = (1+t^3)^n \frac{d}{dt}(t) + t\frac{d}{dt}(1+t^3)^n$

$$= (1+t^3)^n + t(n(1+t^3)^{n-1} 3t^2)$$
$$= (1+t^3)^n + 3nt^3(1+t^3)^{n-1}$$
$$= (1+t^3)^n + 3n(1+t^3-1)(1+t^3)^{n-1}$$
$$= (1+t^3)^n + 3n(1+t^3)(1+t^3)^{n-1} - 3n(1+t^3)^{n-1}$$
$$= (1+t^3)^n + 3n(1+t^3)^n - 3n(1+t^3)^{n-1}$$
$$= (3n+1)(1+t^3)^n - 3n(1+t^3)^{n-1}$$

$$I_n = \int_0^1 (1+t^3)^n \, dt$$

$$\frac{d}{dt}(t(1+t^3)^n) = (3n+1)(1+t^3)^n - 3n(1+t^3)^{n-1}$$

$$t(1+t^3)^n = \int (3n+1)(1+t^3)^n - 3n(1+t^3)^{n-1} dt$$

$$\left[t(1+t^3)^n\right]_0^1 = \int_0^1 (3n+1)(1+t^3)^n - 3n(1+t^3)^{n-1} dt$$

$$2^n - 0 = \int_0^1 (3n+1)(1+t^3)^n \, dt - \int_0^1 3n(1+t^3)^{n-1} dt$$

$$2^n = (3n+1)\int_0^1 (1+t^3)^n dt - 3n\int_0^1 (1+t^3)^{n-1} dt$$

$$2^n = (3n+1)I_n - 3n I_{n-1}$$

$$\therefore (3n+1)I_n = 2^n + 3n I_{n-1}.$$

$n=3$: $10I_3 = 8 + 9I_2$

$$7I_2 = 4 + 6I_1$$

$$4I_1 = 2 + 3I_0$$

$$I_0 = \int_0^1 (1+t^3)^0\,dt$$

$$= \int_0^1 1\,dt$$

$$= [t]_0^1$$

$$= 1 - 0$$

$$= 1$$

$$4I_1 = 5$$

$$I_1 = \frac{5}{4}$$

$$7I_2 = 4 + \frac{15}{2}$$

$$= \frac{23}{2}$$

$$I_2 = \frac{23}{14}$$

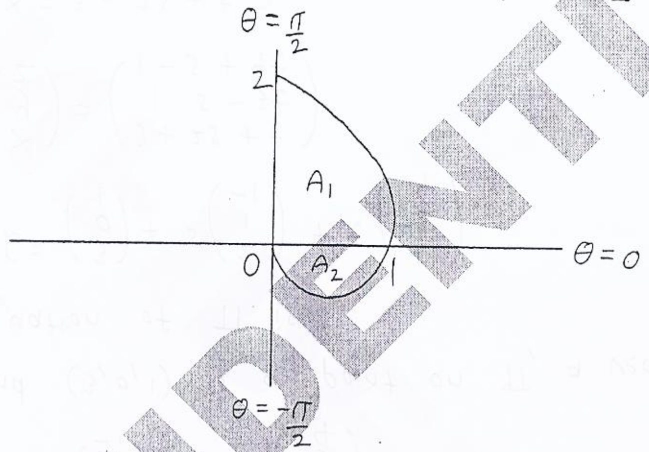
$$10I_3 = 8 + \frac{207}{14}$$

$$= \frac{319}{14}$$

$$\therefore I_3 = \frac{319}{140}$$

8. $C: r = 1 + \sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	$1-\frac{\sqrt{3}}{2}$	$1-\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$1+\frac{1}{\sqrt{2}}$	$1+\frac{\sqrt{3}}{2}$	2



$$A_1 = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1+\sin\theta)^2}{2}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} + \sin\theta + \frac{\sin^2\theta}{2}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} + \sin\theta + \frac{1}{2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{3}{4} + \sin\theta - \frac{\cos 2\theta}{4}\, d\theta$$

$$= \left[\frac{3\theta}{4} - \cos\theta - \frac{\sin 2\theta}{8}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3\pi}{8} - (-1)$$

$$= \frac{3\pi}{8} + 1$$

$$A_2 = \int_{-\frac{\pi}{2}}^{0} \frac{r^2}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0} \frac{(1+\sin\theta)^2}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0} \frac{3}{4} + \sin\theta - \frac{\cos 2\theta}{4}\, d\theta$$

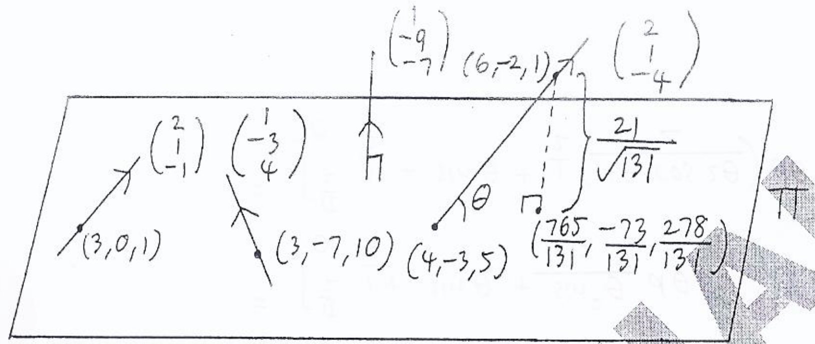
$$= \left[\frac{3\theta}{4} - \cos\theta - \frac{\sin 2\theta}{8}\right]_{-\frac{\pi}{2}}^{0}$$

$$= -1 - \left(\frac{-3\pi}{8}\right)$$

$$= \frac{3\pi}{8} - 1$$

$$\therefore \frac{A_1}{A_2} = \frac{\frac{3\pi}{8}+1}{\frac{3\pi}{8}-1} = \frac{3\pi+8}{3\pi-8} \approx 12.2$$

9.



$\underset{\sim}{r} = 3\underset{\sim}{i} + \underset{\sim}{k} + s(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$

$\underset{\sim}{r} = 3\underset{\sim}{i} - 7\underset{\sim}{j} + 10\underset{\sim}{k} + t(\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k})$

Since $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\-3\\4\end{pmatrix}$ are parallel to Π

and $(3,0,1)$ is a point on Π, a vector equation of Π is

$$\underset{\sim}{r} = \begin{pmatrix}3\\0\\1\end{pmatrix} + s\begin{pmatrix}2\\1\\-1\end{pmatrix} + t\begin{pmatrix}1\\-3\\4\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3+2s+t\\ s-3t\\ 1-s+4t\end{pmatrix}$$

$$\left.\begin{aligned} x &= 3+2s+t \\ y &= s-3t \\ z &= 1-s+4t \end{aligned}\right\}$$

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$$\left.\begin{aligned} s-3t&=y \\ 2s+t&=x-3 \\ -s+4t&=z-1 \end{aligned}\right\}$$

$$\left.\begin{aligned} -2\times ① + ②: \quad s-3t&=y \\ ① + ③: \quad 7t&=x-2y-3 \\ t&=y+z-1 \end{aligned}\right\}$$

$7(y+z-1) = x-2y-3$

$7y+7z-7 = x-2y-3$

$x-9y-7z=-4$

$\therefore$ A Cartesian equation of Π is $x-9y-7z=-4$.

$l: \underset{\sim}{r} = \begin{pmatrix}6\\-2\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-4\end{pmatrix}, \lambda \in R.$

i) when l meets Π, $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6+2\lambda\\-2+\lambda\\1-4\lambda\end{pmatrix}$,

$x=6+2\lambda,\ y=-2+\lambda,\ z=1-4\lambda$

$6+2\lambda-9(-2+\lambda)-7(1-4\lambda)=-4$

$6+2\lambda+18-9\lambda-7+28\lambda=-4$

$21\lambda=-21$

$\lambda=-1$

$\therefore l$ meets Π at $(4,-3,5)$

ii) The line perpendicular to Π through P has equation $\underset{\sim}{r} = \begin{pmatrix}6\\-2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-9\\-7\end{pmatrix}$ since $\begin{pmatrix}1\\-9\\-7\end{pmatrix}$ is perpendicular to Π.

when the line meets Π, $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6+\mu\\-2-9\mu\\1-7\mu\end{pmatrix}$

$x=6+\mu,\ y=-2-9\mu,\ z=1-7\mu$

$6+\mu-9(-2-9\mu)-7(1-7\mu)=-4$

$6+\mu+18+81\mu-7+49\mu=-4$

$131\mu=-21$

$\mu=\frac{-21}{131}$

$\therefore$ The line meets Π at $\left(\frac{765}{131}, \frac{-73}{131}, \frac{278}{131}\right)$

$\therefore$ The perpendicular distance from P to Π is

$$\sqrt{\left(\frac{765}{131}-6\right)^2+\left(\frac{-73}{131}+2\right)^2+\left(\frac{278}{131}-1\right)^2}$$

$$=\sqrt{\frac{21^2}{131^2}+81\left(\frac{21^2}{131^2}\right)+49\left(\frac{21^2}{131^2}\right)}$$

$$=\sqrt{\frac{131(21^2)}{131^2}}=\sqrt{\frac{21^2}{131}}=\frac{21}{\sqrt{131}}.$$

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since the normal to Π is $\begin{pmatrix}1\\-9\\-7\end{pmatrix}$ and ℓ has direction $\begin{pmatrix}2\\1\\-4\end{pmatrix}$,

$$\begin{pmatrix}1\\-9\\-7\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-4\end{pmatrix} = \left|\begin{pmatrix}1\\-9\\-7\end{pmatrix}\right|\left|\begin{pmatrix}2\\1\\-4\end{pmatrix}\right| \cos\theta$$

$$2-9+28 = \sqrt{131}\sqrt{21}\cos\theta$$

$$\cos\theta = \frac{21}{\sqrt{131}\sqrt{21}}$$

$$= \sqrt{\frac{21}{131}}$$

$$\theta = \cos^{-1}\sqrt{\frac{21}{131}}$$

$$\approx 66.4^\circ$$

$\therefore$ The acute angle between ℓ and Π is 66.4°.

10. $C:\ y = \dfrac{5(x^2-x-2)}{x^2+5x+10}$

when $x=0$: $y=-1$

when $y=0$: $\dfrac{5(x^2-x-2)}{x^2+5x+10}=0$

$$x^2-x-2=0$$

$$(x-2)(x+1)=0$$

$$x=2,-1$$

$\therefore$ The intersection points of C are $(0,-1)$, $(2,0)$ and $(-1,0)$.

$$\begin{array}{r|l} & 5 \\ \hline x^2+5x+10 & 5x^2-5x-10 \\ & 5x^2+25x+50 \\ \hline & -30x-60 \end{array}$$

$$y = 5 + \frac{-30x-60}{x^2+5x+10} = 5 - \frac{30(x+2)}{x^2+5x+10}$$

As $x \to \pm\infty$ $y \to 5$.

$\therefore$ The asymptote of C is $y=5$

$$\frac{dy}{dx} = \frac{-30}{x^2+5x+10} + \frac{30(x+2)(2x+5)}{(x^2+5x+10)^2}$$

when $\frac{dy}{dx} = 0$: $\frac{-30}{x^2+5x+10} + \frac{30(x+2)(2x+5)}{(x^2+5x+10)^2} = 0$

$$\frac{30(x+2)(2x+5)}{(x^2+5x+10)^2} = \frac{30}{x^2+5x+10}$$

$$2x^2+9x+10 = x^2+5x+10$$

$$x^2+4x=0$$

$$x(x+4)=0$$

$$x=0,-4$$

$$y=-1,15$$

$$\frac{d^2y}{dx^2} = \frac{30(2x+5)}{(x^2+5x+10)^2} + \frac{30(2x+5)}{(x^2+5x+10)^2} + \frac{60(x+2)}{(x^2+5x+10)^2} - \frac{60(x+2)(2x+5)^2}{(x^2+5x+10)^3}$$

when $x=0$: $\frac{d^2y}{dx^2} > 0$

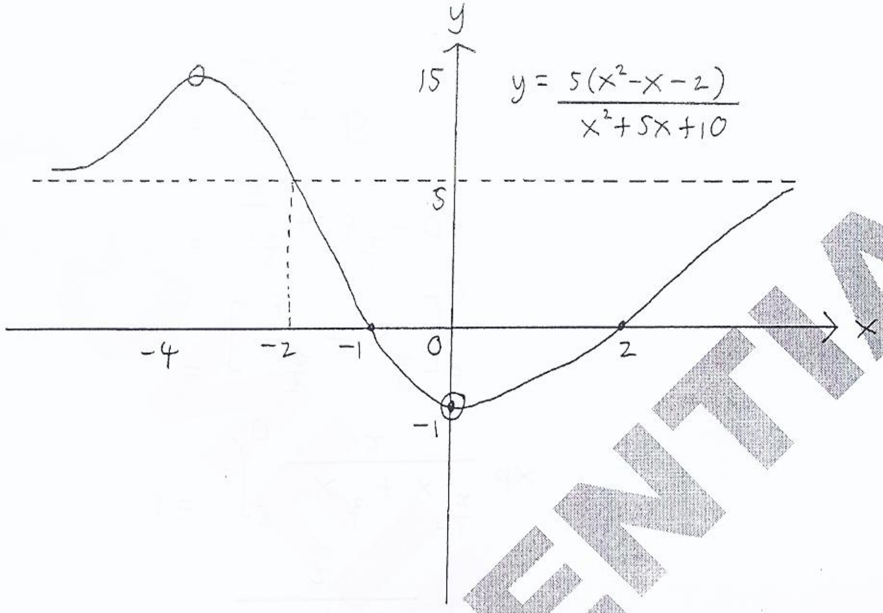
when $x=-4$: $\frac{d^2y}{dx^2} < 0$

$\therefore (0,-1)$ is a minimum point and $(-4,15)$ is a maximum point.

when $\frac{5(x^2-x-2)}{x^2+5x+10} = 5$

$$x^2-x-2 = x^2+5x+10$$

$$x=-2$$



o : Critical Point

• : Intersection Point

11 EITHER

$$C: y = \frac{x^{\frac{1}{2}}}{3}(3-x), \quad 0 \le x \le 3$$

$$= x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3}$$

The mean value of y over $0 \le x \le 3$ is

$$\frac{1}{3-0}\int_0^3 x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3}\, dx$$

$$= \frac{1}{3}\int_0^3 x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3}\, dx$$

$$= \frac{1}{3}\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}}\right]_0^3$$

$$= \frac{1}{3}\left(\frac{2}{3}(3^{\frac{3}{2}}) - \frac{2}{15}(3^{\frac{5}{2}}) - 0\right)$$

$$= \frac{1}{3}\left(2\sqrt{3} - \frac{2}{5}(3^{\frac{3}{2}})\right)$$

$$= \frac{2}{3}\sqrt{3} - \frac{2}{5}\sqrt{3}$$

$$= \frac{4\sqrt{3}}{15}$$

If s is the arc length of C,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1 + \left(\frac{x^{-\frac{1}{2}}}{2} - \frac{x^{\frac{1}{2}}}{2}\right)^2}$$

$$= \sqrt{1 + \frac{1}{4x} - \frac{1}{2} + \frac{x}{4}}$$

$$= \sqrt{\frac{x}{4} + \frac{1}{2} + \frac{1}{4x}}$$

$$= \sqrt{\left(\frac{x^{\frac{1}{2}}}{2} + \frac{x^{-\frac{1}{2}}}{2}\right)^2}$$

$$= \frac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{2}$$

$$s = \int_0^3 \frac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{2}\, dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{3} + x^{\frac{1}{2}}\right]_0^3$$

$$= \frac{3^{\frac{3}{2}}}{3} + 3^{\frac{1}{2}} - 0$$

$$= \sqrt{3} + \sqrt{3}$$

$$= 2\sqrt{3}.$$

The surface area when C is rotated one complete revolution about the x-axis is

$$\int_0^3 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^3 2\pi\left(x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3}\right)\left(\frac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{2}\right)dx$$

$$= \pi\int_0^3 x - \frac{x^2}{3} + 1 - \frac{x}{3}\,dx$$

$$= \pi\int_0^3 \frac{2x}{3} - \frac{x^2}{3} + 1\,dx$$

$$= \pi\left[\frac{x^2}{3} - \frac{x^3}{9} + x\right]_0^3$$

$$= \pi(3 - 3 + 3 - 0)$$

$$= 3\pi$$

OR

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(2-\lambda)(3-\lambda) - 2] - 1(0-(-2)) + 2(0+2-\lambda)$$

$$= (1-\lambda)(2-\lambda)(3-\lambda) - 2(1-\lambda) - 2 + 4 - 2\lambda$$

$$= (1-\lambda)(\lambda^2 - 5\lambda + 6) - 2 + 2\lambda - 2 + 4 - 2\lambda$$

$$= -(\lambda-1)(\lambda-2)(\lambda-3)$$

When $|A - \lambda I| = 0$:

$$-(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\lambda = 1, 2, 3$$

$\therefore$ The eigenvalues of A are 1, 2 and 3.

$\lambda = 1$: $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Let $z = s$, $s \in R$

$y = -2s$

$-x + y + 2z = 0$

$-x - 2s + 2s = 0$

$x = 0$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2s \\ s \end{pmatrix} = s\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$\lambda = 2$: $\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$z = 0$

Let $y = s$, $s \in R$

$x = s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 3$: $\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Leftrightarrow r_3} \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -2 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-2r_1 + r_3} \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right)$$

Let $z = s$, $s \in R$

$y = 2s$

$x = 2s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ 2s \\ s \end{pmatrix} = s\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are $1, 2, 3$ with corresponding eigenvectors $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

$T: R^3 \to R^3$

$\underset{\sim}{x} \mapsto A\underset{\sim}{x}$

If $\underset{\sim}{e}$ and $\underset{\sim}{f}$ are two linearly independent eigenvectors of A and Π is the plane containing $\underset{\sim}{e}$ and $\underset{\sim}{f}$, if $\underset{\sim}{r}$ is any point on Π,

$\underset{\sim}{r} = s\underset{\sim}{e} + t\underset{\sim}{f}$, $s, t \in R$.

$A\underset{\sim}{r} = A(s\underset{\sim}{e} + t\underset{\sim}{f}) = A(s\underset{\sim}{e}) + A(t\underset{\sim}{f})$

$= s(A\underset{\sim}{e}) + t(A\underset{\sim}{f}) = s(\lambda\underset{\sim}{e}) + t(\mu\underset{\sim}{f})$

$= (s\lambda)\underset{\sim}{e} + (t\mu)\underset{\sim}{f}$

If $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ are the eigenvectors of A, the three planes whose points are mapped onto the same planes are

$$A(s\underset{\sim}{e}_1 + t\underset{\sim}{e}_2) = (s\lambda)\underset{\sim}{e}_1 + (t\mu)\underset{\sim}{e}_2,$$
$$A(s\underset{\sim}{e}_1 + t\underset{\sim}{e}_3) = (s\lambda)\underset{\sim}{e}_1 + (t\mu)\underset{\sim}{e}_3 \text{ and}$$
$$A(s\underset{\sim}{e}_2 + t\underset{\sim}{e}_3) = (s\lambda)\underset{\sim}{e}_2 + (t\mu)\underset{\sim}{e}_3.$$

$\therefore$ The vectors $\underset{\sim}{e}_1 \times \underset{\sim}{e}_2$, $\underset{\sim}{e}_1 \times \underset{\sim}{e}_3$ and $\underset{\sim}{e}_2 \times \underset{\sim}{e}_3$ are perpendicular to the three planes

Since the eigenvectors of A are $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, the vectors

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = -2\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

are perpendicular to the three planes.
Since the planes contain the origin, the Cartesian equations of the three planes are
$x - y - 2z = 0$, $2x - y - 2z = 0$ and $x - y = 0$.