10. a)

$$2x^3 - 3x^2 - 59x + 30 = 0$$

α, β, γ are the roots.

$$\alpha + \beta + \gamma = \frac{3}{2}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{-59}{2}$$

$$\alpha\beta\gamma = -15$$

$$\gamma - \beta = \beta - \alpha$$

$$\alpha + \gamma = 2\beta$$

$$2\beta + \beta = \frac{3}{2}$$

$$3\beta = \frac{3}{2}$$

$$\beta = \frac{1}{2}$$

$$(\alpha + \gamma)\beta + \alpha\gamma = -\frac{59}{2}$$

$$(2\beta)\beta + \alpha\gamma = \frac{-59}{2}$$

$$2\beta^2 + \alpha\gamma = -\frac{59}{2}$$

$$2\left(\frac{1}{4}\right) + \alpha\gamma = \frac{-59}{2}$$

$$\frac{1}{2} + \alpha\gamma = \frac{-59}{2}$$

$$\alpha\gamma = -30$$

$$\alpha + \gamma = 1$$

$$\gamma = 1 - \alpha$$

$$\alpha(1 - \alpha) = \cancel{3}30$$

$$\alpha - \alpha^2 = -30$$

$$\alpha^2 - \alpha - 30 = 0$$

$$(\alpha - 6)(\alpha + 5) = 0$$

$$\alpha = 6, -5$$

b) $2x^3 + x^2 - 5x + 3 = 0$

α, β, γ are the roots

i) $$\alpha + \beta + \gamma = \frac{-1}{2}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{-5}{2}$$

$$\alpha\beta\gamma = \frac{-3}{2}$$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2$$

$$-2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= \left(\frac{-1}{2}\right)^2 - 2\left(\frac{-5}{2}\right)$$

$$= \frac{1}{4} + 5$$

$$= \frac{21}{4}$$

ii) Let $S_n = \alpha^n + \beta^n + \gamma^n$

$S_0 = \alpha^0 + \beta^0 + \gamma^0$

$= 1 + 1 + 1$

$= 3$

$S_1 = \alpha^1 + \beta^1 + \gamma^1$

$= \alpha + \beta + \gamma$

$= \frac{-1}{2}$

$S_2 = \alpha^2 + \beta^2 + \gamma^2$

$= \frac{21}{4}$

$aS_3 + bS_2 + cS_1 + dS_0 = 0$

$2S_3 + S_2 - 5S_1 + 3S_0 = 0$

$2S_3 + \frac{21}{4} - 5\left(\frac{-1}{2}\right) + 3(3) = 0$

$2S_3 + \frac{21}{4} + \frac{5}{2} + 9 = 0$

$2S_3 = -\frac{67}{4}$

$S_3 = -\frac{67}{8}$

$\alpha^3 + \beta^3 + \gamma^3 = -\frac{67}{8}$

iii) $\alpha + 1, \beta + 1, \gamma + 1$

Let $u = \alpha + 1$

$\alpha = u - 1$

α is a root

$\therefore \; 2\alpha^3 + \alpha^2 - 5\alpha + 3 = 0$

$2(u-1)^3 + (u-1)^2 - 5(u-1) + 3 = 0$

$2(u^3 - 3u^2 + 3u - 1)$

$+ u^2 - 2u + 1 - 5u + 5 + 3 = 0$

$2u^3 - 6u^2 + 6u - 2$

$+ u^2 - 2u + 1 - 5u + 5 + 3 = 0$

$2u^3 - 5u^2 - u + 7 = 0$

11. $x^3 + ax^2 + bx + c = 0$

x_1, x_2, x_3 are the roots

$x_1 + x_2 + x_3 = -a$

$x_1 x_2 + x_1 x_3 + x_2 x_3 = b$

$x_1 x_2 x_3 = -c$

$$x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1 x_2 + x_1 x_3 + x_2 x_3)$$
$$= (-a)^2 - 2b$$
$$= a^2 - 2b$$

a) x_1, x_2, x_3 are the roots

$x_1^3 + ax_1^2 + bx_1 + c = 0$ —— ①

$x_2^3 + ax_2^2 + bx_2 + c = 0$ —— ②

$x_3^3 + ax_3^2 + bx_3 + c = 0$ —— ③

① + ② + ③ :

$$x_1^3 + x_2^3 + x_3^3 + a(x_1^2 + x_2^2 + x_3^2) + b(x_1 + x_2 + x_3) + 3c = 0$$

$$\sum_{i=1}^{3} x_i^3 + a\sum_{i=1}^{3} x_i^2 + b\sum_{i=1}^{3} x_i + 3c = 0$$

b) $\sum_{i=1}^{3} x_i^3 + a(a^2 - 2b) + b(-a) + 3c = 0$

$$\sum_{i=1}^{3} x_i^3 + a^3 - 2ab - ab + 3c = 0$$

$$\therefore \sum_{i=1}^{3} x_i^3 = 3ab - 3c - a^3$$

$p + q + r = 4$

$p^2 + q^2 + r^2 = 12$

$p^3 + q^3 + r^3 = 64$

$$p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + pr + qr)$$

$$12 = 4^2 - 2(pq + pr + qr)$$

$$pq + pr + qr = 2$$

$$= -8$$

$$ax^3 + bx^2 + cx + d = 0$$

$$x^3 - (p + q + r)x^2 + (pq + pr + qr)x - pqr = 0$$

$a = 1 \quad b = -(p + q + r) \quad c = pq + pr + qr$

$\quad\quad\quad\;\; = -4 \quad\quad\quad\quad\quad\quad = 2$

$d = pqr$

$$S_n = p^n + q^n + r^n$$

$$aS_3 + bS_2 + cS_1 + dS_0 = 0$$

$$S_0 = p^0 + q^0 + r^0$$

$$\equiv 1 + 1 + 1$$

$$\equiv 3$$

$$S_1 \equiv p^1 + q^1 + r^1$$

$$\equiv p + q + r$$

$$= 4$$

$$S_2 = p^2 + q^2 + r^2$$

$$= 12$$

$$S_3 = p^3 + q^3 + r^3$$

$$= 64$$

$$1(64) + -4(12) + 2(4) + 3d = 0$$

$$64 - 48 + 8 + 3d = 0$$

$$3d = -24$$

$$d = -8$$

$$x^3 - 4x^2 + 2x - 8 = 0$$

$$x^2(x - 4) + 2(x - 4) = 0$$

$$(x - 4)(x^2 + 2) = 0$$

$$x = 4, \pm\sqrt{2}i$$

12 a) $(k + 3)x^2 - 2(k + 1)x + 2k - 1 = 0, \quad k \in R, \; k \neq -3$

i) $a = k + 3 \quad b = 2(k + 1) \quad c = 2k - 1$

$$b^2 - 4ac \equiv 4(k + 1)^2 - 4(k + 3)(2k - 1)$$
$$\equiv 4(k^2 + 2k + 1) - 4(k + 3)(2k - 1)$$
$$\equiv 4[(k^2 + 2k + 1) - (2k^2 + 5k - 3)]$$
$$\equiv 4(k^2 + 2k + 1 - 2k^2 - 5k + 3)$$
$$\equiv 4(-k^2 - 3k + 4)$$
$$\equiv -4(k^2 + 3k - 4)$$
$$\equiv -4(k + 4)(k - 1)$$

$$b^2 - 4ac \geqslant 0$$
$$-4(k + 4)(k - 1) \geqslant 0$$
$$(k + 4)(k - 1) \leqslant 0$$
$$-4 \leqslant k \leqslant 1.$$

ii) α, β are the roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{2(k + 1) \pm \sqrt{(k + 4)(1 - k)}}{2(k + 3)}$$

$$\alpha + \beta = \frac{2(k + 1)}{k + 3}, \quad \alpha\beta = \frac{2k - 1}{k + 3}$$

Let $\alpha = \dfrac{2(k+1) + \sqrt{(k+4)(1-k)}}{2(k+3)}$,

$\beta = \dfrac{2(k+1) - \sqrt{(k+4)(1-k)}}{2(k+3)}$

If $\alpha > 0$ and $\beta < 0$ or $\alpha < 0$ and $\beta > 0$,

$\therefore \alpha\beta < 0$

$\therefore \dfrac{2k-1}{k+3} < 0$

$-3 < k < \dfrac{1}{2}$

iii) If $\alpha, \beta > 0$, $\alpha\beta > 0$

$\therefore \dfrac{2k-1}{k+3} > 0$

$k < -3$ or $k > \dfrac{1}{2}$

b) $5x^2 - 6x + 3 = 0$

α, β are the roots

$\therefore \alpha + \beta = \frac{6}{5}$

$\alpha\beta = \frac{3}{5}$

i) $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta + \alpha\beta$

$= (\alpha + \beta)^2 - \alpha\beta$

$= \left(\frac{6}{5}\right)^2 - \frac{3}{5}$

$= \frac{36}{25} - \frac{6}{5}$

$= \frac{6}{25}$

ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{\left(\frac{6}{5}\right)^2 - 2\left(\frac{3}{5}\right)}{\frac{3}{5}}$

$= \frac{\frac{36}{25} - \frac{6}{5}}{\frac{3}{5}}$

$$= \frac{\frac{6}{25}}{\frac{3}{5}}$$

$$= \frac{2}{5}$$

14 a) $6x^3 + 25x^2 - 62x + 24 = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = \frac{-25}{6}$

$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{31}{3}$

$\alpha\beta\gamma = -4$

$\alpha\beta = -8$

$\therefore -8\gamma = -4$

$\gamma = \frac{1}{2}$

$\therefore \alpha + \beta = \frac{-14}{3}, \quad \alpha\beta = -8$

$\alpha = \frac{-8}{\beta}$

$\frac{-8}{\beta} + \beta = \frac{-14}{3}$

$-24 + 3\beta^2 = -14\beta$

$3\beta^2 + 14\beta - 24 = 0$

$(3\beta - 4)(\beta + 6) = 0$

$\beta = \frac{4}{3}, -6$

$\alpha = -6, \frac{4}{3}$

b) $36x^3 + 72x^2 + 23x - 6 = 0$

α, β, γ are the roots

$6\alpha + 1,\ 6\beta + 1,\ 6\gamma + 1$

Let $u = 6\alpha + 1$

$$\alpha = \frac{u-1}{6}$$

α is a root

$$\therefore 36\alpha^3 + 72\alpha^2 + 23\alpha - 6 = 0$$

$$36\left(\frac{u-1}{6}\right)^3 + 72\left(\frac{u-1}{6}\right)^2$$

$$+ 23\left(\frac{u-1}{6}\right) - 6 = 0$$

$$\frac{u^3 - 3u^2 + 3u - 1}{6} + 2(u^2 - 2u + 1)$$

$$+ \frac{23u - 23}{6} - 6 = 0$$

$$u^3 - 3u^2 + 3u - 1 + 12(u^2 - 2u + 1)$$

$$+ 23u - 23 - 36 = 0$$

$$u^3 - 3u^2 + 3u - 1 + 12u^2 - 24u + 12$$

$$+ 23u - 23 - 36 = 0$$

$$u^3 + 9u^2 + 2u - 48 = 0$$

20. $x^3 - 2x^2 + 3x - 5 = 0$

α, β, γ are the roots

$\beta + \gamma, \gamma + \alpha, \alpha + \beta$

Let $u = \beta + \gamma$

$$u + \alpha = \alpha + \beta + \gamma$$

$$= 2, \text{ since } \alpha + \beta + \gamma = 2$$

$$\alpha = 2 - u$$

α is a root

$\therefore \alpha^3 - 2\alpha^2 + 3\alpha - 5 = 0$

$$(2 - u)^3 - 2(2 - u)^2 + 3(2 - u) + 5 = 0$$

$$8 - 12u + 6u^2 - u^3 - 2(4 - 4u + u^2)$$
$$+ 6 - 3u + 5 = 0$$

$$8 - 12u + 6u^2 - u^3 - 8 + 8u - 2u^2$$
$$+ 6 - 3u + 5 = 0$$

$$u^3 - 4u^2 + 7u - 11 = 0$$

21. a)

$$x^3 + px^2 + qx + r = 0$$

α, β, γ are the roots

$$\alpha + \beta + \gamma = -p$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = q$$

$$\alpha\beta\gamma = -r$$

$$\gamma - \beta = \beta - \alpha$$

$$\alpha + \gamma = 2\beta$$

$$2\beta + \beta = -p$$

$$3\beta = -p$$

$$\beta = \frac{-p}{3}$$

$$(\alpha + \gamma)\beta + \alpha\gamma = q$$

$$(2\beta)\beta + \alpha\gamma = q$$

$$2\beta^2 + \left(\frac{-r}{\beta}\right) = q$$

$$2\beta^3 - r = q\beta$$

$$2\left(\frac{-p}{3}\right)^3 - r = \frac{-pq}{3}$$

$$\frac{-2p^3}{27} - r = \frac{-pq}{3}$$

$$-2p^3 + 27r = 9pq$$

b) $x^3 + px^2 + qx + r = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = -p$

$\alpha\beta + \alpha\gamma + \beta\gamma = q$

$\alpha\beta\gamma = -r$

$\frac{\gamma}{\beta} = \frac{\beta}{\alpha}$

$\alpha\gamma = \beta^2$

$\alpha\beta + \beta^2 + \beta\gamma = q$

$\beta(\alpha + \beta + \gamma) = q$

$\beta(-p) = q$

$\beta = \frac{-q}{p}$

$(\beta^2)\beta = -r$

$\beta^3 = -r$

$\beta = -r^{\frac{1}{3}}$

$\frac{-q}{p} = -r^{\frac{1}{3}}$

$\frac{q}{p} = r^{\frac{1}{3}}$

$\frac{q^3}{p^3} = r$

$p^3 r = q^3$

27 $x^3 + x^2 - 2x - 5 = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = -1$

$\alpha\beta + \alpha\gamma + \beta\gamma = -2$

$\alpha\beta\gamma = 5$

i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (-1)^2 - 2(-2)$

$= 1 + 4$

$= 5$

ii) Let $S_n = \alpha^n + \beta^n + \gamma^n$

$S_0 = \alpha^0 + \beta^0 + \gamma^0$

$= 1 + 1 + 1$

$= 3$

$S_1 = \alpha^1 + \beta^1 + \gamma^1$

$= \alpha + \beta + \gamma$

$= -1$

$S_2 = \alpha^2 + \beta^2 + \gamma^2$

$= 5$

$$aS_3 + bS_2 + cS_1 + dS_0 = 0$$

$$S_3 + S_2 - 2S_1 - 5S_0 = 0$$

$$S_3 + 5 - 2(-1) - 5(3) = 0$$

$$S_3 + 5 + 2 - 15 = 0$$

$$S_3 = 8$$

28. $x^3 - 4x - 1 = 0$

α, β, γ are the roots.

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = -4$

$\alpha\beta\gamma = 1$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= 0^2 - 2(-4)$$

$$= 8$$

Let $S_n = \alpha^n + \beta^n + \gamma^n$

$$S_0 = \alpha^0 + \beta^0 + \gamma^0$$

$$= 1 + 1 + 1$$

$$= 3$$

$$S_1 = \alpha^1 + \beta^1 + \gamma^1$$

$$= \alpha + \beta + \gamma$$

$$= 0$$

$$S_2 = \alpha^2 + \beta^2 + \gamma^2$$

$$= 8$$

$$aS_{3+r} + bS_{2+r} + cS_{1+r} + dS_r = 0$$

$$S_{3+r} - 4S_{1+r} - S_r = 0$$

$$r = 0: \quad S_{3+r} - 4S_1 \; 4S_{10} = S0 = 0$$

$$S_3 - 4(0) - 3 = 0$$

$$S_3 = 3$$

$$r = 1: \quad S_4 - 4S_2 - S_1 = 0$$

$$S_4 - 4(8) - 0 = 0$$

$$S_4 = 32$$

$$\therefore \alpha^4 + \beta^4 + \gamma^4 = 32$$

30 $x^3 + 3x - 1 = 0$

α, β, γ are the roots.

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = 3$

$\alpha\beta\gamma = 1$

Let $S_n = \alpha^n + \beta^n + \gamma^n$

$S_0 = \alpha^0 + \beta^0 + \gamma^0$

$= 1 + 1 + 1$

$= 3$

$S_1 = \alpha^1 + \beta^1 + \gamma^1$

$= \alpha + \beta + \gamma$

$= 0$

$S_2 = \alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 0^2 - 2(3)$

$= -6$

$aS_3 + bS_2 + cS_1 + dS_0 = 0$

$S_3 + 3S_1 - S_0 = 0$

$S_3 + 3(0) - 3 = 0$

$S_3 = 3$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = 3$

32 $x^3 + x^2 + 2x - 2 = 0$

$$y = \frac{x}{x + 2}$$

$$(x + 2)y = x$$

$$xy + 2y = x$$

$$xy - x = -2y$$

$$x(y - 1) = -2y$$

$$x = \frac{2y}{1 - y}$$

$$\left(\frac{2y}{1 - y}\right)^3 + \left(\frac{2y}{1 - y}\right)^2 + 2\left(\frac{2y}{1 - y}\right) - 2 = 0$$

$$\frac{8y^3}{(1 - y)^3} + \frac{4y^2}{(1 - y)^2} + \frac{4y}{1 - y} - 2 = 0$$

$$8y^3 + 4y^2(1 - y) + 4y(1 - y)^2 - 2(1 - y)^3 = 0$$

$$4y^3 + 2y^2(1 - y) + 2(1 - y)^2 - (1 - y)^3 = 0$$

$$4y^3 + 2y^2 - 2y^3 + 2(1 - 2y + y^2)$$

$$-(1 - 3y + 3y^2 - y^3) = 0$$

$$4y^3 + 2y^2 - 2y^3 + 2 - 4y + 2y^2$$

$$-1 + 3y - 3y^2 + y^3 = 0$$

$$3y^3 + y^2 - y + 1 = 0$$

Since α, β, γ are the roots of the equation $x^3 + x^2 + 2x - 2 = 0$, $\frac{1\alpha}{\alpha + 2}, \frac{\beta}{\beta + 2}, \frac{\gamma}{\gamma + 2}$

are the roots of the equation

$3y^3 + y^2 - y + 1 = 0.$

$$\frac{\alpha}{\alpha + 2} + \frac{\beta}{\beta + 2} + \frac{\gamma}{\gamma + 2} = \frac{-1}{3}$$

$$\left(\frac{\alpha}{\alpha + 2}\right)\left(\frac{\beta}{\beta + 2}\right) + \left(\frac{\alpha}{\alpha + 2}\right)\left(\frac{\gamma}{\gamma + 2}\right)$$

$$+ \left(\frac{\beta}{\beta + 2}\right)\left(\frac{\gamma}{\gamma + 2}\right) = \frac{-1}{3}$$

$$\left(\frac{\alpha}{\alpha + 2}\right)\left(\frac{\beta}{\beta + 2}\right)\left(\frac{\gamma}{\gamma + 2}\right) = \frac{-1}{3}$$

$$\therefore \left(\frac{\alpha}{\alpha + 2}\right)^2 + \left(\frac{\beta}{\beta + 2}\right)^2 + \left(\frac{\gamma}{\gamma + 2}\right)^2$$

$$= \left(\frac{\alpha}{\alpha + 2} + \frac{\beta}{\beta + 2} + \frac{\gamma}{\gamma + 2}\right)^2$$

$$-2\left[\left(\frac{\alpha}{\alpha + 2}\right)\left(\frac{\beta}{\beta + 2}\right) + \left(\frac{\alpha}{\alpha + 2}\right)\left(\frac{\gamma}{\gamma + 2}\right)\right.$$

$$\left. + \left(\frac{\beta}{\beta + 2}\right)\left(\frac{\gamma}{\gamma + 2}\right)\right]$$

$$= \left(\frac{-1}{3}\right)^2 - 2\left(\frac{-1}{3}\right)$$

$$= \frac{1}{9} + \frac{2}{3}$$

$$= \frac{7}{9}$$

34. $x^3 - 2x - 5 = 0$

α, β, γ are the roots

$\alpha^3, \beta^3, \gamma^3$

Let $u = \alpha^3$

$$\alpha = u^{\frac{1}{3}}$$

α is a root

$$\therefore \alpha^3 - 2\alpha - 5 = 0$$

$$\left(u^{\frac{1}{3}}\right)^3 - 2u^{\frac{1}{3}} - 5 = 0$$

$$u - 2u^{\frac{1}{3}} - 5 = 0$$

$$u - 5 = 2u^{\frac{1}{3}}$$

$$(u - 5)^3 = \left(2u^{\frac{1}{3}}\right)^3$$

$$u^3 - 15u^2 + 75u - 125 = 8u$$

$$u^3 - 15u^2 + 67u - 125 = 0$$

$\therefore$ The equation having roots $\alpha^3, \beta^3, \gamma^3$ is $u^3 - 15u^2 + 67u - 125 = 0$.

$$\alpha^3 + \beta^3 + \gamma^3 = 15$$

$$\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3 = 67$$

$$\alpha^3\beta^3\gamma^3 = 125$$

$$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2$$

$$- 2(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3)$$

$$= 15^2 - 2(67)$$

$$(15$$

$$= 225 - 134$$

$$= 91$$

35. $x^3 - 3x^2 - 4x - 1 = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = 3$

$\alpha\beta + \alpha\gamma + \beta\gamma = -4$

$\alpha\beta\gamma = -1$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= 3^2 - 2(-4)$$

$$= 9 + 8$$

$$= 17.$$

Let $S_n = \alpha^n + \beta^n + \gamma^n$

$$S_0 = \alpha^0 + \beta^0 + \gamma^0$$

$$= 1 + 1 + 1$$

$$= 3$$

$$S_1 = \alpha^1 + \beta^1 + \gamma^1$$

$$= \alpha + \beta + \gamma$$

$$= 3$$

$$S_2 = \alpha^2 + \beta^2 + \gamma^2$$

$$= 17$$

$$aS_3 + bS_2 + cS_1 + dS_0 = 0$$

$$S_3 - 3S_2 - 4S_1 - S_0 = 0$$

$$S_3 - 3(17) - 4(3) - 3 = 0$$

$$S_3 - 51 - 12 - 3 = 0$$

$$S_3 = 66$$