Topical Past Year Questions – Polynomial and Rational Functions

Oct/Nov 2011 (9231/11)

1 The equation $x^3 + px + q = 0$ has a repeated root. Prove that $4p^3 + 27q^2 = 0$. [5]

7 The curve C has equation $y = \dfrac{x^2 + px + 1}{x - 2}$, where p is a constant. Given that C has two asymptotes, find the equation of each asymptote. [3]

Find the set of values of p for which C has two distinct turning points. [5]

Sketch C in the case $p = -1$. Your sketch should indicate the coordinates of any intersections with the axes, but need not show the coordinates of any turning points. [3]

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3 The equation

$$x^3 + 5x^2 - 3x - 15 = 0$$

has roots α, β, γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

Hence show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

10 A curve C has equation

$$y = \frac{5(x^2 - x - 2)}{x^2 + 5x + 10}.$$

Find the coordinates of the points of intersection of C with the axes. [2]

Show that, for all real values of x, $-1 \leqslant y \leqslant 15$. [4]

Sketch C, stating the coordinates of any turning points and the equation of the horizontal asymptote. [7]

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2 The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are $\dfrac{\beta}{k}, \beta, k\beta$, where p, q, r, k and β are non-zero real constants. Show that $\beta = -\dfrac{q}{p}$. [4]

Deduce that $rp^3 = q^3$. [2]

11.

The curve C has equation

$$y = \frac{x^2 + \lambda x - 6\lambda^2}{x + 3},$$

where λ is a constant such that $\lambda \neq 1$ and $\lambda \neq -\frac{3}{2}$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and deduce that if C has two stationary points then $-\frac{3}{2} < \lambda < 1$. [5]

(ii) Find the equations of the asymptotes of C. [3]

(iii) Draw a sketch of C for the case $0 < \lambda < 1$. [3]

(iv) Draw a sketch of C for the case $\lambda > 3$. [3]

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3 Find a cubic equation with roots α, β and γ, given that

$$\alpha + \beta + \gamma = -6, \qquad \alpha^2 + \beta^2 + \gamma^2 = 38, \qquad \alpha\beta\gamma = 30.$$ [3]

Hence find the numerical values of the roots. [3]

9 The curve C with equation

$$y = \frac{ax^2 + bx + c}{x - 1},$$

where a, b and c are constants, has two asymptotes. It is given that $y = 2x - 5$ is one of these asymptotes.

(i) State the equation of the other asymptote. [1]

(ii) Find the value of a and show that $b = -7$. [3]

(iii) Given also that C has a turning point when $x = 2$, find the value of c. [3]

(iv) Find the set of values of k for which the line $y = k$ does not intersect C. [4]

Oct/Nov 2010 (9231/01)

7 The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ. Use the substitution $y = \frac{1}{1 + x}$ to show that the equation $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $\frac{1}{\alpha + 1}$, $\frac{1}{\beta + 1}$ and $\frac{1}{\gamma + 1}$. [2]

For the cases $n = 1$ and $n = 2$, find the value of

$$\frac{1}{(\alpha + 1)^n} + \frac{1}{(\beta + 1)^n} + \frac{1}{(\gamma + 1)^n}.$$ [2]

Deduce the value of $\frac{1}{(\alpha + 1)^3} + \frac{1}{(\beta + 1)^3} + \frac{1}{(\gamma + 1)^3}$. [2]

Hence show that $\frac{(\beta + 1)(\gamma + 1)}{(\alpha + 1)^2} + \frac{(\gamma + 1)(\alpha + 1)}{(\beta + 1)^2} + \frac{(\alpha + 1)(\beta + 1)}{(\gamma + 1)^2} = \frac{73}{36}$. [3]

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \frac{x^2 + 2\lambda x}{x^2 - 2x + \lambda},$$

where λ is a constant and $\lambda \neq -1$.

(i) Show that C has at most two stationary points. [3]

(ii) Show that if C has **exactly** two stationary points then $\lambda > -\frac{5}{4}$. [2]

(iii) Find the set of values of λ such that C has two vertical asymptotes. [2]

(iv) Find the x-coordinates of the points of intersection of C with

(a) the x-axis,

(b) the horizontal asymptote.

[3]

(v) Sketch C in each of the cases

(a) $\lambda < -2$,

(b) $\lambda > 2$.

[4]

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6 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Use the relation $x = \sqrt{y}$ to show that the equation

$$y^3 + 2y^2 + y - 1 = 0$$

has roots $\alpha^2, \beta^2, \gamma^2$. [2]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(i) Write down the value of S_2 and show that $S_4 = 2$. [3]

(ii) Find the values of S_6 and S_8. [4]

11.

The curve C has equation

$$y = \frac{x(x+1)}{(x-1)^2}.$$

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Show that there is exactly one point of intersection of C with the asymptotes and find its coordinates. [2]

(iii) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and hence

(a) find the coordinates of any stationary points of C,

(b) state the set of values of x for which the gradient of C is negative.

[6]

(iv) Draw a sketch of C. [3]

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6 The curve C has equation

$$y = \frac{x^2 - 3x - 7}{x + 1}.$$

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Show that $\frac{dy}{dx} > 1$ at all points of C. [2]

(iii) Draw a sketch of C. [3]

10 The equation

$$x^4 + x^3 + cx^2 + 4x - 2 = 0,$$

where c is a constant, has roots $\alpha, \beta, \gamma, \delta$.

(i) Use the substitution $y = \frac{1}{x}$ to find an equation which has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$. [2]

(ii) Find, in terms of c, the values of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [3]

(iii) Hence find

$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2$$

in terms of c. [2]

(iv) Deduce that when $c = -3$ the roots of the given equation are not all real. [3]

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3 The curve C has equation

$$y = \frac{x^2 - 5x + 4}{x + 1}.$$

(i) Obtain the coordinates of the points of intersection of C with the axes. [2]

(ii) Obtain the equation of each of the asymptotes of C. [3]

(iii) Draw a sketch of C. [3]

5 The equation

$$x^3 + 5x + 3 = 0$$

has roots α, β, γ. Use the substitution $x = -\frac{3}{y}$ to find a cubic equation in y and show that the roots of this equation are $\beta\gamma, \gamma\alpha, \alpha\beta$. [4]

Find the exact values of $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$ and $\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3$. [5]

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1 The equation

$$x^4 - x^3 - 1 = 0$$

has roots $\alpha, \beta, \gamma, \delta$. By using the substitution $y = x^3$, or by any other method, find the exact value of $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$. [5]

10 The curve C has equation

$$y = \frac{x^2}{x + \lambda},$$

where λ is a non-zero constant. Obtain the equation of each of the asymptotes of C. [3]

In separate diagrams, sketch C for the cases $\lambda > 0$ and $\lambda < 0$. In both cases the coordinates of the turning points must be indicated. [8]

Oct/Nov 2008 (9231/01)

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \frac{(x-2)(x-a)}{(x-1)(x-3)},$$

where a is a constant not equal to 1, 2 or 3.

(i) Write down the equations of the asymptotes of C. [2]

(ii) Show that C meets the asymptote parallel to the x-axis at the point where $x = \dfrac{2a-3}{a-2}$. [2]

(iii) Show that the x-coordinates of any stationary points on C satisfy

$$(a-2)x^2 + (6-4a)x + (5a-6) = 0,$$

and hence find the set of values of a for which C has stationary points. [6]

(iv) Sketch the graph of C for

(a) $a > 3$,

(b) $2 < a < 3$.

[4]

OR

The roots of the equation

$$x^4 - 5x^2 + 2x - 1 = 0$$

are $\alpha, \beta, \gamma, \delta$. Let $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

(i) Show that

$$S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0. \quad [2]$$

(ii) Find the values of S_2 and S_4. [3]

(iii) Find the value of S_3 and hence find the value of S_6. [6]

(iv) Hence find the value of

$$\alpha^2(\beta^4 + \gamma^4 + \delta^4) + \beta^2(\gamma^4 + \delta^4 + \alpha^4) + \gamma^2(\delta^4 + \alpha^4 + \beta^4) + \delta^2(\alpha^4 + \beta^4 + \gamma^4). \quad [3]$$

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5 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $\alpha^3, \beta^3, \gamma^3$ is

$$y^3 - 3y^2 + 4y - 1 = 0. \qquad [4]$$

Hence find the value of $\alpha^6 + \beta^6 + \gamma^6$. [3]

9 The curve C has equation

$$y = \frac{x^2 - 2x + \lambda}{x + 1},$$

where λ is a constant. Show that the equations of the asymptotes of C are independent of λ. [3]

Find the value of λ for which the x-axis is a tangent to C, and sketch C in this case. [4]

Sketch C in the case $\lambda = -4$, giving the exact coordinates of the points of intersection of C with the x-axis. [3]

Oct/Nov 2007 (9231/01)

4 The roots of the equation

$$x^3 - 8x^2 + 5 = 0$$

are α, β, γ. Show that

$$\alpha^2 = \frac{5}{\beta + \gamma}. \qquad [4]$$

It is given that the roots are all real. Without reference to a graph, show that one of the roots is negative and the other two roots are positive. [3]

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \frac{ax^2 + bx + c}{x + 4},$$

where a, b and c are constants. It is given that $y = 2x - 5$ is an asymptote of C.

(i) Find the values of a and b. [3]

(ii) Given also that C has a turning point at $x = -1$, find the value of c. [3]

(iii) Find the set of values of y for which there are no points on C. [4]

(iv) Draw a sketch of the curve with equation

$$y = \frac{2(x-7)^2 + 3(x-7) - 2}{x - 3}. \qquad [3]$$

[You should state the equations of the asymptotes and the coordinates of the turning points.]

May/June 2007 (9231/01)

7 The equation

$$x^3 + 3x - 1 = 0$$

has roots α, β, γ. Use the substitution $y = x^3$ to show that the equation whose roots are $\alpha^3, \beta^3, \gamma^3$ is

$$y^3 - 3y^2 + 30y - 1 = 0. \qquad [2]$$

Find the value of $\alpha^9 + \beta^9 + \gamma^9$. [5]

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \lambda x + \frac{x}{x+2},$$

where λ is a non-zero constant.

(i) Find the asymptotes of C. [3]

(ii) Show that if $\lambda > 0$ then $\frac{dy}{dx} > 0$ at all points of C. [2]

(iii) Show that, for $\lambda < -\frac{1}{2}$, C has two distinct stationary points, both to the left of the y-axis. [3]

(iv) In separate diagrams draw sketches of C for each of the cases $\lambda > 0$ and $\lambda < -\frac{1}{2}$. [6]

Oct/Nov 2006 (9231/01)

6 The roots of the equation

$$x^3 + x + 1 = 0$$

are α, β, γ. Show that the equation whose roots are

$$\frac{4\alpha+1}{\alpha+1}, \quad \frac{4\beta+1}{\beta+1}, \quad \frac{4\gamma+1}{\gamma+1}$$

is of the form

$$y^3 + py + q = 0,$$

where the numbers p and q are to be determined. [5]

Hence find the value of

$$\left(\frac{4\alpha+1}{\alpha+1}\right)^n + \left(\frac{4\beta+1}{\beta+1}\right)^n + \left(\frac{4\gamma+1}{\gamma+1}\right)^n,$$

for $n = 2$ and for $n = 3$. [4]

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \frac{x^2 + qx + 1}{2x + 3},$$

where q is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the value of q for which the x-axis is a tangent to C, and sketch C in this case. [4]

(iii) Sketch C for the case $q = 3$, giving the exact coordinates of the points of intersection of C with the x-axis. [3]

(iv) It is given that, for all values of the constant λ, the line

$$y = \lambda x + \tfrac{3}{2}\lambda + \tfrac{1}{2}(q - 3)$$

passes through the point of intersection of the asymptotes of C. Use this result, with the diagrams you have drawn, to show that if $\lambda < \frac{1}{2}$ then the equation

$$\frac{x^2 + qx + 1}{2x + 3} = \lambda x + \tfrac{3}{2}\lambda + \tfrac{1}{2}(q - 3)$$

has no real solution if q has the value found in part **(ii)**, but has 2 real distinct solutions if $q = 3$. [4]

May/June 2006 (9231/01)

5 The curve C has equation

$$y = 2x + \frac{3(x - 1)}{x + 1}.$$

(i) Write down the equations of the asymptotes of C. [2]

(ii) Find the set of values of x for which C is above its oblique asymptote and the set of values of x for which C is below its oblique asymptote. [3]

(iii) Draw a sketch of C, stating the coordinates of the points of intersection of C with the coordinate axes. [4]

11 Answer only **one** of the following two alternatives.

EITHER

Obtain the sum of the squares of the roots of the equation

$$x^4 + 3x^3 + 5x^2 + 12x + 4 = 0.$$ [2]

Deduce that this equation does not have more than 2 real roots. [3]

Show that, in fact, the equation has exactly 2 real roots in the interval $-3 < x < 0$. [5]

Denoting these roots by α and β, and the other 2 roots by γ and δ, show that $|\gamma| = |\delta| = \dfrac{2}{\sqrt{(\alpha\beta)}}$. [4]

Oct/Nov 2005 (9231/01)

3 The curve C has equation

$$y = \frac{x^2}{x + \lambda},$$

where λ is a non-zero constant. Obtain the equations of the asymptotes of C. [3]

In separate diagrams, sketch C for the cases where

(i) $\lambda > 0$,

(ii) $\lambda < 0$.

[4]

5 In the equation

$$x^3 + ax^2 + bx + c = 0,$$

the coefficients a, b and c are real. It is given that all the roots are real and greater than 1.

(i) Prove that $a < -3$. [1]

(ii) By considering the sum of the squares of the roots, prove that $a^2 > 2b + 3$. [2]

(iii) By considering the sum of the cubes of the roots, prove that $a^3 < -9b - 3c - 3$. [4]

May/June 2005 (9231/01)

4 Show that the sum of the cubes of the roots of the equation

$$x^3 + \lambda x + 1 = 0$$

is -3. [3]

Show also that there is no real value of λ for which the sum of the fourth powers of the roots is negative. [3]

12 Answer only **one** of the following two alternatives.

EITHER

The curve Γ, which has equation

$$y = \frac{ax^2 + bx + c}{x^2 + px + q},$$

has asymptotes $x = 1$, $x = 4$ and $y = 2$. Find the values of a, p and q. [4]

It is given that Γ has a stationary point at $x = 2$.

(i) Find the value of c. [3]

(ii) Show that if $b \neq -10$ then Γ has exactly 2 stationary points. [2]

(iii) Draw a sketch of Γ for the case where $b = -6$. [4]

Oct/Nov 2004 (9231/01)

3 Given that

$$\alpha + \beta + \gamma = 0, \quad \alpha^2 + \beta^2 + \gamma^2 = 14, \quad \alpha^3 + \beta^3 + \gamma^3 = -18,$$

find a cubic equation whose roots are α, β, γ. [4]

Hence find possible values for α, β, γ. [2]

10 The curve C has equation

$$y = \frac{x^2 + 2x - 3}{(\lambda x + 1)(x + 4)},$$

where λ is a constant.

(i) Find the equations of the asymptotes of C for the case where $\lambda = 0$. [4]

(ii) Find the equations of the asymptotes of C for the case where λ is not equal to any of -1, 0, $\frac{1}{4}$, $\frac{1}{3}$. [3]

(iii) Sketch C for the case where $\lambda = -1$. Show, on your diagram, the equations of the asymptotes and the coordinates of the points of intersection of C with the coordinate axes. [4]

May/June 2004 (9231/01)

2 The curve C has equation

$$y = \frac{x - ax^2}{x - 1},$$

where a is a constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that the x-coordinates of both the turning points of C are positive. [4]

11.

The roots of the equation

$$x^3 - x - 1 = 0$$

are α, β, γ, and

$$S_n = \alpha^n + \beta^n + \gamma^n.$$

(i) Use the relation $y = x^2$ to show that α^2, β^2, γ^2 are the roots of the equation

$$y^3 - 2y^2 + y - 1 = 0. \quad [3]$$

(ii) Hence, or otherwise, find the value of S_4. [2]

(iii) Find the values of S_8, S_{12} and S_{16}. [9]

Oct/Nov 2003 (9231/01)

6 Find the sum of the squares of the roots of the equation

$$x^3 + x + 12 = 0,$$

and deduce that only one of the roots is real. [4]

The real root of the equation is denoted by α. Prove that $-3 < \alpha < -2$, and hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$. [5]

11 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation $y = \dfrac{5(x-1)(x+2)}{(x-2)(x+3)}$.

(i) Express y in the form $P + \dfrac{Q}{x-2} + \dfrac{R}{x+3}$. [3]

(ii) Show that $\dfrac{dy}{dx} = 0$ for exactly one value of x and find the corresponding value of y. [4]

(iii) Write down the equations of all the asymptotes of C. [3]

(iv) Find the set of values of k for which the line $y = k$ does not intersect C. [4]

May/June 2003 (9231/01)

4 The curve C has equation $y = \dfrac{x^2 - 4}{x - 3}$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Draw a sketch of C and its asymptotes. Give the coordinates of the points of intersection of C with the coordinate axes. [4]

[You are not required to find the coordinates of any turning points.]

5 The equation

$$8x^3 + 12x^2 + 4x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $2\alpha + 1$, $2\beta + 1$, $2\gamma + 1$ is

$$y^3 - y - 1 = 0. \quad [3]$$

The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$ is denoted by S_n. Find the values of S_3 and S_{-2}. [5]

Oct/Nov 2002 (9231/01)

2 The equation

$$x^4 + x^3 + Ax^2 + 4x - 2 = 0,$$

where A is a constant, has roots $\alpha, \beta, \gamma, \delta$. Find a polynomial equation whose roots are

$$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}.$$ [2]

Given that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2},$$

find the value of A. [3]

11.

The curve C has equation

$$y = \frac{(x-a)(x-b)}{x-c},$$

where a, b, c are constants, and it is given that $0 < a < b < c$.

(i) Express y in the form

$$x + P + \frac{Q}{x-c},$$

giving the constants P and Q in terms of a, b and c. [3]

(ii) Find the equations of the asymptotes of C. [2]

(iii) Show that C has two stationary points. [5]

(iv) Given also that $a + b > c$, sketch C, showing the asymptotes and the coordinates of the points of intersection of C with the axes. [4]

May/June 2002 (9231/01)

5 The roots of the equation $x^3 - 3x^2 + 1 = 0$ are denoted by α, β, γ. Show that the equation whose roots are

$$\frac{\alpha}{\alpha-2}, \frac{\beta}{\beta-2}, \frac{\gamma}{\gamma-2}$$

is $3y^3 - 9y^2 - 3y + 1 = 0$. [3]

Find the value of

(i) $(\alpha-2)(\beta-2)(\gamma-2)$, [3]

(ii) $\alpha(\beta-2)(\gamma-2) + \beta(\gamma-2)(\alpha-2) + \gamma(\alpha-2)(\beta-2)$. [2]

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \frac{a(x-a)^2}{x^2 - 4a^2},$$

where a is a positive constant.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C has one maximum point and one minimum point and find their coordinates. [6]

(iii) Sketch C, and give the coordinates of any points where C meets the axes. [4]