TAYLOR'S COLLEGE

Wisdom • Integrity • Excellence

CAMBRIDGE 'A' LEVEL PROGRAMME

A2 TRIAL EXAMINATION MARCH/APRIL 2006

(Jan/March 2005 Intake)

Thursday **6 April 2006** **8.30 am – 11.30 am**

FURTHER MATHEMATICS 9231/01

PAPER 1 **3 hours**

Additional materials: Answer Booklet/Paper
List of formulae (MF 10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The use of an electronic calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

This document consists of **6** printed pages.

[Turn over

1 Show that $\sum_{k=1}^{999} \frac{1}{k(k+1)(k+2)} = \frac{500499}{2002000}$ [3]

2 The equations of two planes are

$$\mathbf{r} = \mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \mu(3\mathbf{j} + 2\mathbf{k}) \qquad \text{and} \qquad 2x + y = 5$$

(i) Find the cosine of the acute angle between the two planes. [3]

(ii) The line of intersection of the two planes is L. Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, the equation of L. [3]

3 The cubic equation $ax^3 + bx^2 + cx + d = 0$ has the property that two of its roots are the reciprocals of each other. Prove that $a^2 - d^2 = ac - bd$. Verify that this condition holds for the equation $9x^3 + 24x^2 - 11x - 6 = 0$, hence solve the equation. [6]

4 (i) Show that

$$\frac{d}{dx}\left[x(1-x^2)^n\right] = (2n+1)(1-x^2)^n - 2n(1-x^2)^{n-1}$$ [3]

(ii) It is given that

$$I_n = \int_0^1 (1-x^2)^n \, dx \qquad (n \geq 0)$$

Use the result in part (i) to show that

$$I_n = \frac{2n}{2n+1} I_{n-1} \qquad (n \geq 1)$$ [2]

Hence evaluate I_5, leaving your answer as a fraction in its lowest terms. [2]

5 Given that y is a function of x, where $x > 0$, show that, if the substitution $x = \sqrt{t}$ is made, then

(i) $$\frac{dy}{dx} = 2\sqrt{t}\,\frac{dy}{dt}$$ [1]

(ii) $$\frac{d^2y}{dx^2} = 4t\,\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$$ [2]

Hence, or otherwise, find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{x}\left(\frac{dy}{dx}\right) + 4x^2(9y+6) = 0$$ [6]

6 The sequence of positive numbers $u_1, u_2, u_3, \ldots$, is known to converge to the limit m is such that $u_{n+1} = \dfrac{9u_n + 3}{u_n + 7}$.

(i) Find the value of m. [2]

(ii) It is given that $u_1 = 1$. By first writing u_n in terms of u_{n+1}, and considering $3 - u_n$, prove by induction that $u_n < 3$ for all positive integers n. [7]

7 A curve is defined by

$$x = 4t - \frac{1}{3}t^3, \qquad y = 2t^2 - 8$$

The arc of the curve given by $0 \le t \le 2\sqrt{3}$ is denoted by C.

(i) Find the length of the arc C. [4]

(ii) Find the area of the curved surface generated when the arc C is rotated about the y-axis. [4]

(iii) Find the mean value of y with respect to t for $0 \le t \le 1$. [3]

[Turn over

8 The linear transformation $T : R^4 \to R^4$ is represented by the matrix $\boldsymbol{M}$, where

$$M = \begin{pmatrix} 1 & 2 & 3 & 6 \\ -1 & -1 & -2 & -4 \\ 2 & 3 & 6 & 11 \\ 3 & 8 & 10 & 21 \end{pmatrix}.$$

(i) Show that the dimension of G, the range space of T, is 3. [4]

(ii) Find a set of basis vectors, with integer components, for G. [1]

(iii) Show that $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}$ belongs to G. [6]

9 The equation, in polar coordinates, of a curve is

$$r = a \sin\theta \cos^2\theta$$

where a is a positive constant and $0 \le \theta \le \pi$.

(i) Show that the greatest value of r is $\dfrac{2a\sqrt{3}}{9}$. [7]

(ii) Sketch the curve. [4]

10 Prove by induction $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where $n \in Z^+$ and $\theta \in \Re$. [3]

Hence, or otherwise, express $\tan 3\theta$ in terms of $\tan\theta$. [4]

By using a suitable value for θ in your result, find the exact values of $\tan\dfrac{1}{12}\pi$ and $\tan\dfrac{5}{12}\pi$, and show that

$$\frac{\tan\frac{5}{12}\pi}{\tan\frac{1}{12}\pi} = 7 + 4\sqrt{3}$$ [6]

Answer only **one** of the following two alternatives.

EITHER

11 The curve C has equation

$$y = \frac{x^2 + 3x + b}{x + a} \qquad (x \neq -a),$$

where a and b are constants. It is given that the line $y = x + 4$ is an asymptote of C and that C has a stationary point at $x = 2$.

Show that $a = -1$ and find the value of b. [6]

Using these values of a and b,

(i) Show that C has exactly two stationary points, and find the coordinates of each point. [3]

(ii) Draw a sketch of C. [2]

Show that the equation, $x^3 + 3x^2 + (b - k)x - ka = 0$ can be written as

$$\frac{x^2 + 3x + b}{x + a} = \frac{k}{x}.$$

Deduce that, for all positive k, the equation, $x^3 + 3x^2 + (b - k)x - ka = 0$ has exactly one negative root. [3]

[Turn over

OR

11 The linear transformation $T : \mathrm{R}^4 \to \mathrm{R}^4$ is defined by

$$T : \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \quad \text{where} \quad \mathrm{A} = \begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & 1 & 3 \\ 3 & 7 & -2 & -1 \\ 7 & 16 & -7 & q \end{pmatrix}$$

The range space of T is denoted by R.

(i) Show that if q = -6 the dimension of R is 2, but if q ≠ - 6 the dimension of R is 3. [4]

(ii) For the case where q = - 6, write down a basis for R, and hence find a vector x such that

$$Ax = \begin{pmatrix} 2 \\ 7 \\ 9 \\ 20 \end{pmatrix}$$ [5]

(iii) The null space of T, for the case where q ≠ - 6, is denoted by K. Find a basis for K. [2]

(iv) The null space of T, for the case where q = - 6, is denoted by K'. Show that K is a subspace of K'. [3]