1. $9^3 + 16^3 + 23^3 + \cdots + (7n+2)^3$

$$= \sum_{r=1}^{n} (7r+2)^3$$

$$= \sum_{r=1}^{n} 343r^3 + 294r^2 + 84r + 8$$

$$= 343\sum_{r=1}^{n} r^3 + 294\sum_{r=1}^{n} r^2 + 84\sum_{r=1}^{n} r + 8\sum_{r=1}^{n} 1$$

$$= \frac{343n^2(n+1)^2}{4} + \frac{294n(n+1)(2n+1)}{6} + \frac{84n(n+1)}{2} + 8n$$

$$= \frac{343n^2}{4}(n^2+2n+1) + 49n(2n^2+3n+1) + 42n(n+1) + 8n$$

$$= \frac{343}{4}(n^4+2n^3+n^2) + 49(2n^3+3n^2+n) + 42n^2 + 42n + 8n$$

$$= \frac{343(n^4+2n^3+n^2) + 196(2n^3+3n^2+n) + 168n^2 + 200n}{4}$$

$$= \frac{343n^4 + 686n^3 + 343n^2 + 392n^3 + 588n^2 + 196n + 168n^2 + 200n}{4}$$

$$= \frac{343n^4 + 1078n^3 + 1099n^2 + 396n}{4}$$

2. Let $f(n) = 6^{2n} + 34^{2n-1}$

when $n=1$: $f(1) = 6^{2(1)} + 34^{2(1)-1}$

$= 6^2 + 34^{2-1}$

$= 36 + 34^1$

$= 36 + 34$

$= 70$

$= 35(2)$

$\therefore 35 \mid f(1)$

Assume the statement is true when $n=k$.

$n=k$: $35 \mid f(k)$

$f(k) = 35s$, s is an integer.

$6^{2k} + 34^{2k-1} = 35s$

When $n=k+1$:

$f(k+1) = 6^{2(k+1)} + 34^{2(k+1)-1}$

$= 6^{2k+2} + 34^{2k+2-1}$

$= 6^{2k}6^2 + 34^{2k-1}34^2$

$= 6^{2k}36 + 34^{2k-1}1156$

$= 6^{2k}36 + 34^{2k-1}(36 + 1120)$

$= 6^{2k}36 + 34^{2k-1}36 + 34^{2k-1}1120$

$= 36(6^{2k} + 34^{2k-1}) + 34^{2k-1}(35)32$

$= 36(35s) + 34^{2k-1}(35)32$

$= 35(36s + 34^{2k-1}32)$

Since s is an integer and k is an integer,

$36s + 34^{2k-1}32$ is an integer.

$35 \mid f(k+1)$

$\therefore$ $6^{2n} + 34^{2n-1}$ is divisible by 35 for every positive integer n.

3. $9x^4 - 4x^3 + 8x^2 + 3x - 2 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$\alpha^2 + 3, \beta^2 + 3, \gamma^2 + 3, \delta^2 + 3$

Let $u = \alpha^2 + 3$

$\alpha^2 = u - 3$

$\alpha = \pm\sqrt{u-3}$

α is a root

$\therefore 9\alpha^4 - 4\alpha^3 + 8\alpha^2 + 3\alpha - 2 = 0$

$9(\pm\sqrt{u-3})^4 - 4(\pm\sqrt{u-3})^3$

$+ 8(\pm\sqrt{u-3})^2 + 3(\pm\sqrt{u-3}) - 2 = 0$

$9(u-3)^2 \mp 4\sqrt{u-3}(u-3)$

$+ 8(u-3) \pm 3\sqrt{u-3} - 2 = 0$

$9(u-3)^2 + 8u - 24 - 2$

$= \pm 4\sqrt{u-3}(u-3) \mp 3\sqrt{u-3}$

$9(u^2 - 6u + 9) + 8u - 26$

$= \pm\sqrt{u-3}(4(u-3) - 3)$

$9u^2 - 54u + 81 + 8u - 26$

$= \pm\sqrt{u-3}(4u - 12 - 3)$

$$9u^2 - 46u + 55 = \pm\sqrt{u-3}\,(4u - 15)$$

$$(9u^2 - 46u + 55)^2 = (u-3)(4u-15)^2$$

$$81u^4 - 414u^3 + 495u^2 - 414u^3 + 2116u^2 - 2530u$$

$$+ 495u^2 - 2530u + 3025$$

$$= (u-3)(16u^2 - 120u + 225)$$

$$81u^4 - 828u^3 + 3106u^2 - 5060u + 3025$$

$$= 16u^3 - 120u^2 + 225u - 48u^2 + 360u - 675$$

$$81u^4 - 844u^3 + 3274u^2 - 5645u + 3700 = 0$$

$\therefore$ The equation having roots

$\alpha^2 + 3,\ \beta^2 + 3,\ \gamma^2 + 3,\ \delta^2 + 3$ is

$$81u^4 - 844u^3 + 3274u^2 - 5645u + 3700 = 0$$

4. $y = \dfrac{3x + 8}{4x + 9}$

$$\begin{array}{r|l} & \frac{3}{4} \\ \hline 4x + 9 & 3x + 8 \\ & 3x + \frac{27}{4} \\ \hline & \quad \frac{5}{4} \end{array}$$

$= \dfrac{3}{4} + \dfrac{5}{4(4x + 9)}$

As $x \to \pm\infty$ $y \to \frac{3}{4}$

As $x \to -\frac{9}{4}$ $y \to \pm\infty$

$\therefore$ The asymptotes of the curve are $y = \frac{3}{4}$ and $x = -\frac{9}{4}$.

when $x = 0$: $y = \frac{8}{9}$

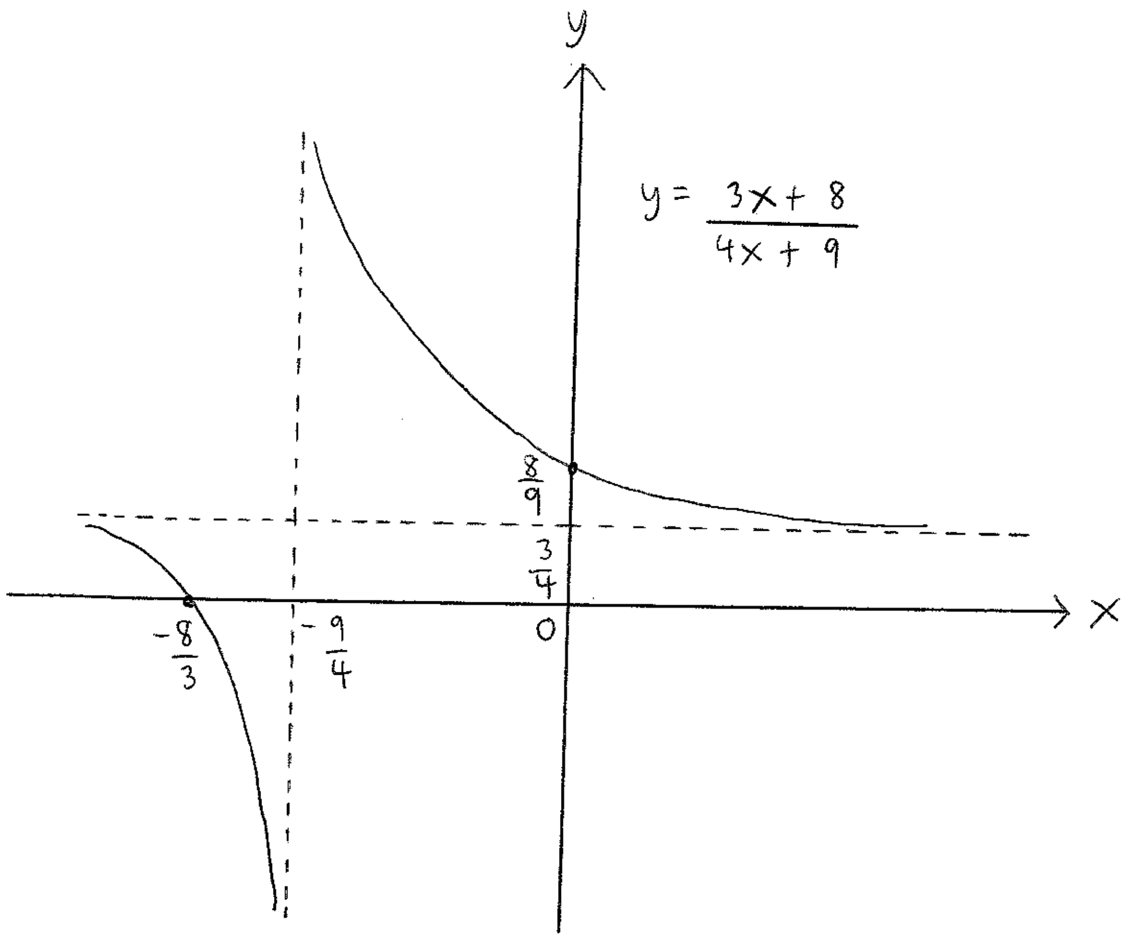
when $y = 0$: $\dfrac{3x + 8}{4x + 9} = 0$

$3x + 8 = 0$

$x = -\frac{8}{3}$

$\dfrac{dy}{dx} = \dfrac{-5}{(4x + 9)^2} < 0$

$\therefore$ no critical points.



y
$y = \frac{3x+8}{4x+9}$
$\frac{8}{9}$
$\frac{3}{4}$
x
0
$-\frac{8}{3}$
$-\frac{9}{4}$