1. $f(k+1) = (2k+3)7^{k+1} - 1$

$f(k+1) - f(k) = (2k+3)7^{k+1} - 1 - [(2k+1)7^k - 1]$

$= (12k+20)7^k$

$a = 12 \quad b = 20$

Basic case $n = 1$: $f(1) = 3 \times 7 - 1 = 20 = 4(5)$;

$f(1)$ is divisible by 4.

$f(k+1) - f(k) = 4(3k+5)7^k$

$\therefore$ true for $n = k+1$ if true for $n = k$.

By induction $f(n) = (2n+1)7^n - 1$ is divisible by 4 for all natural numbers n.

2. $\dfrac{r^2+3r+1}{(r+2)!}$ $\qquad \dfrac{1}{a!}-\dfrac{1}{b!}$

$$= \frac{r^2+3r+2-1}{(r+2)!}$$

$$= \frac{(r+1)(r+2)-1}{(r+2)!}$$

$$= \frac{(r+2)(r+1)r! - r!}{(r+2)!\,r!}$$

$$= \frac{(r+2)! - r!}{(r+2)!\,r!}$$

$$= \frac{(r+2)!}{r!(r+2)!} - \frac{r!}{r!(r+2)!}$$

$$= \frac{1}{r!} - \frac{1}{(r+2)!}$$

$$\therefore \sum_{r=1}^{n} \frac{r^2+3r+1}{(r+2)!} = \sum_{r=1}^{n} \frac{1}{r!} - \frac{1}{(r+2)!}$$

$$= \frac{1}{1!} - \frac{1}{3!}$$

$$+ \frac{1}{2!} - \frac{1}{4!}$$

$$+ \frac{1}{3!} - \frac{1}{5!}$$

$$\vdots$$

$$+ \frac{1}{(n-2)!} - \frac{1}{n!}$$

$$+ \frac{1}{(n-1)!} - \frac{1}{(n+1)!}$$

$$+ \frac{1}{n!} - \frac{1}{(n+2)!}$$

$$= \frac{3}{2} - \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$

$$\frac{5}{6} + \frac{11}{24} + \frac{19}{120} + \cdots = \sum_{r=1}^{\infty} \frac{r^2+3r+1}{(r+2)!}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2+3r+1}{(r+2)!}$$

$$= \lim_{n\to\infty} \frac{3}{2} - \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$

$$= \frac{3}{2}$$

3. $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} 1 \\ \sin 2\alpha \\ \cos 2\alpha \end{pmatrix} \times \begin{pmatrix} 1 \\ \sin 2\beta \\ \cos 2\beta \end{pmatrix}$

$$= \begin{pmatrix} \sin 2\alpha \cos 2\beta - \cos 2\alpha \sin 2\beta \\ \cos 2\alpha - \cos 2\beta \\ \sin 2\beta - \sin 2\alpha \end{pmatrix}$$

$$|\underset{\sim}{a} \times \underset{\sim}{b}|^2 = \sin^2(2\alpha - 2\beta)$$

$$+ \cos^2 2\alpha - 2\cos 2\alpha \cos 2\beta + \cos^2 2\beta$$

$$+ \sin^2 2\beta - 2\sin 2\alpha \sin 2\beta + \sin^2 2\alpha$$

$$= \sin^2(2\alpha - 2\beta) + 2 - 2\cos(2\alpha - 2\beta)$$

$$= 4\sin^2(\alpha - \beta) + 4\sin^2(\alpha - \beta)$$

$$= 4\sin^2(\alpha - \beta)(\cos^2(\alpha - \beta) + 1)$$

$$= 4\sin^2\theta(\cos^2\theta + 1)$$

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = 2\sin\theta\sqrt{\cos^2\theta + 1}$$

$$|\underset{\sim}{a}|^2 = 1 + \sin^2 2\alpha + \cos^2 2\alpha = 2$$

$$|\underset{\sim}{b}|^2 = 1 + \sin^2 2\beta + \cos^2 2\beta = 2$$

Given ϕ = angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

From $\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\phi . \underset{\sim}{n}$

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\phi$$

$$2\sin\theta\sqrt{\cos^2\theta + 1} = \sqrt{2}\sqrt{2}\sin\phi$$

$$\therefore \sin\phi = \sin\theta\sqrt{\cos^2\theta + 1}$$

4. $M = \begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & -4 & -5 \\ 3 & a^2+5 & 2a-7 & 3a-9 \\ 6 & a^2+12 & 2a-14 & 3a-18 \end{pmatrix}$

$$\xrightarrow[R_4 : R_4 - 6R_1]{R_2 : R_2 - 2R_1,\ R_3 : R_3 - 3R_1} \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & a^2-1 & 2a+2 & 3a+3 \\ 0 & a^2 & 2a+4 & 3a+6 \end{pmatrix}$$

$$\xrightarrow{R_4 : R_4 - R_2} \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & a^2-1 & 2a+2 & 3a+3 \\ 0 & a^2-1 & 2a+2 & 3a+3 \end{pmatrix}$$

$$\xrightarrow[R_4 : R_4 - R_3]{R_3 : R_3 - (a^2-1)R_2} \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2(a+1)(a-2) & 3(a+1)(a-2) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The dimension of the range space of T is 3 provided $(a+1)(a-2) \neq 0$

$a \neq -1, 2$

Given that $a = 2$,

$M = \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 : R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$x_1 - 7x_3 - 10x_4 = 0 \Rightarrow x_1 = 7x_3 + 10x_4$

$x_2 + 2x_3 + 3x_4 = 0 \Rightarrow x_2 = -2x_3 - 3x_4$

Let $x_3 = s$ and $x_4 = t$

$x_2 = -2x_3 - 3x_4$
$= -2s - 3t$

$x_1 = 7x_3 + 10x_4$
$= 7s + 10t$

Basis for the null space of T is

$$\left\{ \begin{pmatrix} 7 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

5. $C: y = ax + \dfrac{b}{cx+d}$, $a, b, c > 0$

i) As $x \to \pm\infty$ $\quad y \to ax$

As $x \to -\dfrac{d}{c}$ $\quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = ax$ and $x = -\dfrac{d}{c}$.

ii) $y(cx+d) = ax(cx+d) + b$

$cxy + dy = acx^2 + adx + b$

$acx^2 + (ad - cy)x + b - dy = 0$

$A = ac \quad B = ad - cy \quad C = b - dy$

$$\begin{aligned} B^2 - 4AC &= (ad-cy)^2 - 4ac(b-dy) \\ &= a^2d^2 - 2acdy + c^2y^2 - 4abc + 4acdy \\ &= c^2y^2 + 2acdy + a^2d^2 - 4abc \\ &= (cy+ad)^2 - 4abc \end{aligned}$$

If $B^2 - 4AC \geq 0$,

$(cy+ad)^2 - 4abc \geq 0$

$(cy+ad)^2 \geq 4abc$

$cy + ad \geq 2\sqrt{abc}$ or $cy + ad \leq -2\sqrt{abc}$

$cy \geq 2\sqrt{abc} - ad$ or $cy \leq -2\sqrt{abc} - ad$

$y \geq \dfrac{2\sqrt{abc} - ad}{c}$ or $y \leq \dfrac{-2\sqrt{abc} - ad}{c}$

$\therefore$ The minimum value of y is $\dfrac{2\sqrt{abc} - ad}{c}$

and the maximum value of y is $\dfrac{-2\sqrt{abc} - ad}{c}$.

iii) $\frac{dy}{dx} = a - \frac{bc}{(cx+d)^2}$

If $\frac{dy}{dx} = 0$, $a - \frac{bc}{(cx+d)^2} = 0$

$$(cx+d)^2 = \frac{bc}{a}$$

$$cx+d = \pm\sqrt{\frac{bc}{a}}$$

$$cx = -d \pm \sqrt{\frac{bc}{a}}$$

$$x = -\frac{d}{c} \pm \sqrt{\frac{b}{ac}}$$

$\therefore$ The range of x is $x \neq -\frac{d}{c} \pm \sqrt{\frac{b}{ac}}$

6. $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$, M is non-singular

$(M^{-1})^n A M^n$

$$[(M^{-1})^n A M^n](M^{-1})^n \underset{\sim}{x} = (M^{-1})^n A M^n (M^{-1})^n \underset{\sim}{x}$$
$$= (M^{-1})^n A (MM^{-1})^n \underset{\sim}{x}$$
$$= (M^{-1})^n A I^n \underset{\sim}{x}$$
$$= (M^{-1})^n A I \underset{\sim}{x}$$
$$= (M^{-1})^n A \underset{\sim}{x}$$
$$= (M^{-1})^n \lambda \underset{\sim}{x}$$
$$= \lambda((M^{-1})^n \underset{\sim}{x})$$

$\therefore$ The matrix $(M^{-1})^n A M^n$ has an eigenvalue λ with corresponding eigenvector $(M^{-1})^n \underset{\sim}{x}$.

If the matrix A has eigenvalues a, b, c with corresponding eigenvectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

and $M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$(m^{-1})^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$$

$$(m^{-1})^n \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ 2^n a_2 \\ 3^n a_3 \end{pmatrix}$$

$$(m^{-1})^n \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ 2^n b_2 \\ 3^n b_3 \end{pmatrix}$$

$$(m^{-1})^n \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ 2^n c_2 \\ 3^n c_3 \end{pmatrix}$$

$\therefore$ The matrix $(m^{-1})^n A m^n$ has eigenvalues a, b, c with corresponding eigenvectors $\begin{pmatrix} a_1 \\ 2^n a_2 \\ 3^n a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ 2^n b_2 \\ 3^n b_3 \end{pmatrix}, \begin{pmatrix} c_1 \\ 2^n c_2 \\ 3^n c_3 \end{pmatrix}$.

If $P^{-1}[(m^{-1})^n A m^n]^k P = D^k$, where P is a square matrix and D is a diagonal matrix,

$$P = \begin{pmatrix} a_1 & b_1 & c_1 \\ 2^n a_2 & 2^n b_2 & 2^n c_2 \\ 3^n a_3 & 3^n b_3 & 3^n c_3 \end{pmatrix} \text{ and } D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^k = \begin{pmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{pmatrix}.$$

7. $x = \sin t + \cos t \qquad y = \sin t - \cos t$

$$\frac{dx}{dt} = \cos t - \sin t \qquad \frac{dy}{dt} = \cos t + \sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\cos t + \sin t}{\cos t - \sin t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{\cos t + \sin t}{\cos t - \sin t}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left(\frac{\cos t + \sin t}{\cos t - \sin t}\right)$$

$$= \frac{1}{\cos t - \sin t} \frac{\left(\begin{array}{l} (\cos t - \sin t)(-\sin t + \cos t) \\ -(\cos t + \sin t)(-\sin t - \cos t) \end{array} \right)}{(\cos t - \sin t)^2}$$

$$= \frac{\begin{array}{c} \cos^2 t - \sin t \cos t + \sin^2 t \\ + \cos^2 t + \sin t \cos t + \sin^2 t \end{array}}{(\cos t - \sin t)^3}$$

$$= \frac{2}{(\cos t - \sin t)^3}$$

ALTERNATIVELY,

$$x^2 + y^2 = \sin^2 t + 2\sin t\cos t + \cos^2 t$$
$$+ \sin^2 t - 2\sin t\cos t + \cos^2 t$$

$$= 2$$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(2)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$\frac{d}{dx}\left(x + y\frac{dy}{dx}\right) = 0$$

$$1 + y\frac{d^2y}{dx^2} + \frac{x^2}{y^2} = 0$$

$$y\frac{d^2y}{dx^2} = \frac{-(x^2+y^2)}{y^2}$$

$$= \frac{-2}{y^3}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-2}{y^3}$$

ii) $x = \sin t + \sin t\cos t + \cos t \quad y = \sin t - \sin t\cos t + \cos t$

$$x^2 = (\sin t + \sin t\cos t + \cos t)(\sin t + \sin t\cos t + \cos t)$$
$$= \sin^2 t + \sin^2 t\cos t + \sin t\cos t$$
$$+ \sin^2 t\cos t + \sin^2 t\cos^2 t + \sin t\cos^2 t$$
$$+ \sin t\cos t + \sin t\cos^2 t + \cos^2 t$$
$$= 1 + 2\sin^2 t\cos t + 2\sin t\cos t + \sin^2 t\cos^2 t + 2\sin t\cos^2 t$$

$$y^2 = (\sin t - \sin t\cos t + \cos t)(\sin t - \sin t\cos t + \cos t)$$
$$= \sin^2 t - \sin^2 t\cos t + \sin t\cos t$$
$$- \sin^2 t\cos t + \sin^2 t\cos^2 t - \sin t\cos^2 t$$
$$+ \sin t\cos t - \sin t\cos^2 t + \cos^2 t$$
$$= 1 - 2\sin^2 t\cos t - 2\sin t\cos t + \sin^2 t\cos^2 t + 2\sin t\cos^2 t$$

$$\therefore x^2 + y^2 = 2 + 2\sin^2 t\cos^2 t + 4\sin t\cos t$$

$$x - y = \sin t + \sin t\cos t + \cos t - \sin t + \sin t\cos t - \cos t$$
$$= 2\sin t\cos t$$

$$\therefore x^2 + y^2 = 2 + 2\left(\frac{x-y}{2}\right)^2 + \frac{4(x-y)}{2}$$
$$= 2 + \frac{(x-y)^2}{2} + 2(x-y)$$

$$2x^2 + 2y^2 = 4 + (x-y)^2 + 4(x-y)$$
$$= 4 + x^2 - 2xy + y^2 + 4x - 4y$$

$$x^2 + y^2 + 2xy = 4 + 4x - 4y$$
$$x^2 - 4x + 4 + y^2 + 4y + 4 = 12 - 2xy$$
$$(x-2)^2 + (y-2)^2 = 12 - 2xy$$

8. C.E/A.E. : $m^2 + 2m + 2 = 0 \rightarrow m = -1 \pm i$

C.F. is $y = e^{-t}(A\cos t + B\sin t)$

P.I. is $y = \lambda e^{-t}$ with $y' = -\lambda e^{-t}$ and $y'' = \lambda e^{-t}$

$\therefore (\lambda - 2\lambda + 2\lambda)e^{-t} = 3e^{-t} \rightarrow \lambda = 3$

$\therefore y = e^{-t}(A\cos t + B\sin t + 3)$

$y(0) = 1 : 1 = A + 3 \rightarrow A = 2$

$y' = e^{-t}(-A\sin t + B\cos t) - e^{-t}(A\cos t + B\sin t + 3)$

$y'(0) = 1 : 1 = B - A - 3 \rightarrow B = 2$

$\therefore y = e^{-t}(2\sin t - 2\cos t + 3)$

9. $x^4 - 2x - 1 = 0 \Rightarrow$ roots : $\alpha, \beta, \gamma, \delta$.

Let $y = \frac{1}{1+\alpha} \Rightarrow \alpha = \frac{1-y}{y}$

$$\left(\frac{1-y}{y}\right)^4 - 2\left(\frac{1-y}{y}\right) - 1 = 0$$

$$1 - 4y + 6y^2 - 4y^3 + y^4 - 2y^3(1-y) - y^4 = 0$$

$$2y^4 - 6y^3 + 6y^2 - 4y + 1 = 0$$

$a = 2,\ b = -6,\ c = 6,\ d = -4,\ e = 1$

Given $S_n = \frac{1}{(1+\alpha)^n} + \frac{1}{(1+\beta)^n} + \frac{1}{(1+\gamma)^n} + \frac{1}{(1+\delta)^n}$

$$S_0 = 1 + 1 + 1 + 1 = 4$$

$$S_1 = \sum \frac{1}{1+\alpha} = -\frac{(-6)}{2} = 3$$

$$S_2 = \left(\sum \frac{1}{1+\alpha}\right)^2 - 2\left(\sum \left(\frac{1}{1+\alpha}\right)\left(\frac{1}{1+\beta}\right)\right)$$

$$= 3^2 - 2 \cdot \frac{6}{2}$$

$$= 9 - 6$$

$$= 3$$

$$2S_3 - 6S_2 + 6S_1 - 4S_0 + S_{-1} = 0$$

$$S_{-1} = \sum \left(\frac{1}{1+\alpha}\right)^{-1} = \sum 1 + \alpha = 4 + \sum \alpha = 4 + 0 = 4$$

$$\therefore 2S_3 - 6(3) + 6(3) - 4(4) + 4 = 0$$

$$2S_3 = 16 - 4$$

$$= 12$$

$$S_3 = 6$$

10. $I_n = \int \sec^2 y \sec^n(\tan y) \csc^n(\tan y)\, dy$

$x = \tan y$

$dx = \sec^2 y\, dy$

$= \int \sec^n x \csc^n x\, dx$

$= 2^n \int \frac{1}{2^n \sin^n x \cos^n x}\, dx$

$= 2^n \int \frac{1}{\sin^n 2x}\, dx$

$= 2^n \int \csc^n 2x\, dx$

$= 2^n \int \csc^{n-2} 2x \csc^2 2x\, dx$

$= 2^n \int \csc^{n-2} 2x\,(1 + \cot^2 2x)\, dx$

$= 2^n \int \csc^{n-2} 2x + \csc^{n-2} 2x \cot^2 2x\, dx$

$= 2^n \int \csc^{n-2} 2x\, dx + 2^n \int \csc^{n-2} 2x \cot^2 2x\, dx$

$= 2^{n-2} 4 \int \csc^{n-2} 2x\, dx + 2^n \int \csc^{n-3} 2x \csc 2x \cot 2x \cot 2x\, dx$

$= 4 I_{n-2} + 2^n \int \csc^{n-3} 2x \csc 2x \cot 2x \cot 2x\, dx$

$u = \cot 2x$ $\qquad dv = \csc^{n-3} 2x \csc 2x \cot 2x \cot 2x\, dx$

$du = -2\csc^2 2x\, dx$ $\qquad v = \int \csc^{n-3} 2x \csc 2x \cot 2x \cot 2x\, dx$

$w = \csc 2x$

$dw = -2 \csc 2x \cot 2x\, dx$

$= \int \frac{w^{n-3}}{-2}\, dw$

$$= \frac{w^{n-2}}{-2(n-2)}$$

$$= \frac{\csc^{n-2} 2x}{-2(n-2)}$$

$$= 4I_{n-2} + 2^n \left(\frac{\csc^{n-2} 2x \cot 2x}{-2(n-2)} - \int \frac{\csc^{n-2} 2x \csc^2 2x \, dx}{n-2} \right)$$

$$= 4I_{n-2} - \frac{2^{n-1} \csc^{n-2} 2x \cot 2x}{n-2} - 2^{-n} \int \frac{\csc^n 2x}{n-2} \, dx$$

$$= 4I_{n-2} - \frac{2^{n-1} \csc^{n-2} 2x \cot 2x}{n-2} - \frac{1}{n-2} I_n$$

$$\left(1 + \frac{1}{n-2}\right) I_n = 4I_{n-2} - \frac{2^{n-1} \csc^{n-2} 2x \cot 2x}{n-2}$$

$$\left(\frac{n-1}{n-2}\right) I_n = 4I_{n-2} - \frac{2^{n-1} \csc^{n-2} 2x \cot 2x}{n-2}$$

$$(n-1) I_n = 4(n-2) I_{n-2} - 2^{n-1} \csc^{n-2} 2x \cot 2x$$

$$I_n = \frac{4(n-2) I_{n-2}}{n-1} - \frac{2^{n-1} \csc^{n-2} 2x \cot 2x}{n-1}$$

$$n=3: \; 2I_3 = 4I_1 - 4\csc 2x \cot 2x$$

$$I_1 = 2\int \csc 2x \, dx$$

$$= -2 \int \frac{-\csc^2 2x - \csc 2x \cot 2x}{\csc 2x + \cot 2x} \, dx$$

$$= -\ln |\csc 2x + \cot 2x| + C$$

$$I_3 = 2I_1 - 2\csc 2x \cot 2x$$

$$= -2\csc 2x \cot 2x - 2\ln|\csc 2x + \cot 2x| + c$$

$$n=4:\ 3I_4 = 8I_2 - 8\csc^2 2x \cot 2x$$

$$I_2 = 4\int \csc^2 2x \; dx$$

$$= -2\cot 2x + c$$

$$3I_4 = -8\csc^2 2x \cot 2x - 16\cot 2x + c$$

$$I_4 = \frac{-8\csc^2 2x \cot 2x}{3} - \frac{16\cot 2x}{3} + c$$

11. EITHER

i) $\int e^{ax} \cos bx$

$u = e^{ax}$ $\quad dv = \cos bx \, dx$

$du = ae^{ax} dx$ $\quad v = \dfrac{\sin bx}{b}$

$$= \frac{e^{ax} \sin bx}{b} - \int \frac{ae^{ax} \sin bx \, dx}{b}$$

$$= \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \int e^{ax} \sin bx \, dx$$

$u = e^{ax}$ $\quad dv = \sin bx \, dx$

$du = ae^{ax} dx$ $\quad v = \dfrac{-\cos bx}{b}$

$$= \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \left(\frac{-e^{ax} \cos bx}{b} - \int \frac{-ae^{ax}}{b} \cos bx \, dx \right)$$

$$= \frac{e^{ax} \sin bx}{b} + \frac{ae^{ax} \cos bx}{b^2} - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx$$

$$\left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{b^2}$$

$$\therefore \int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$$

ii) $C: x = e^t \cos t \quad y = e^t \sin t, \ t = 0, 1.$

$$\frac{dx}{dt} = e^t \cos t - e^t \sin t \qquad \frac{dy}{dt} = e^t \sin t + e^t \cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t)$$
$$+ e^{2t}(\cos^2 t + 2\cos t \sin t + \sin^2 t)$$
$$= 2e^{2t}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2}e^t$$

The arc length of C from $t=0$ to $t=1$ is

$$\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{2}e^t \, dt$$

$$= \left[\sqrt{2}e^t\right]_0^1$$

$$= \sqrt{2}e - \sqrt{2}$$

$$= \sqrt{2}(e-1)$$

iii) The surface area of revolution of C from $t=0$ to $t=1$ about the y-axis is

$$\int_0^1 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 2\pi e^t \cos t \left(\sqrt{2} e^t\right) dt$$

$$= 2\sqrt{2}\pi \int_0^1 e^{2t} \cos t \, dt$$

$$= 2\sqrt{2}\pi \left[\frac{e^{2t}(2\cos t + \sin t)}{5} \right]_0^1$$

$$= \frac{2\sqrt{2}\pi}{5}\left(e^2(2\cos 1 + \sin 1) - 2e^0 \right)$$

$$= \frac{2\sqrt{2}\pi}{5}\left(2e^2 \cos 1 + e^2 \sin 1 - 2 \right)$$

iv) $y = e^{2x}\sin 3x$, $x = 4, 5$

The y-coordinate of the centroid of the region bounded by the x-axis, the curve $y = e^{2x}\sin 3x$ and the lines $x = 4$ and $x = 5$, c, is $\int_4^5 \frac{y^2}{2}\,dx$.

If A is the area of the region,

$$\therefore Ac = \int_4^5 \frac{e^{4x}\sin^2 3x}{2}\,dx$$

$$= \int_4^5 \frac{e^{4x}}{2}\left(\frac{1-\cos 6x}{2}\right)dx$$

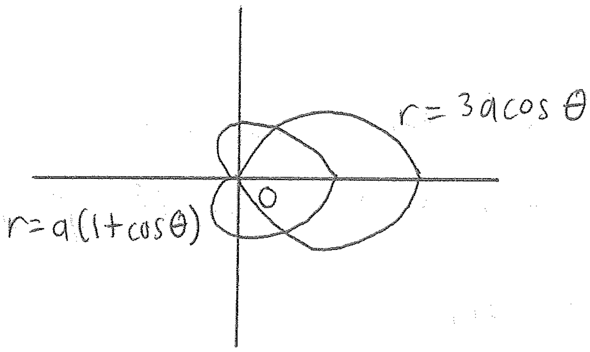
$$= \int_4^5 \frac{e^{4x} - e^{4x}\cos 6x}{4}\,dx$$

$$= \left[\frac{e^{4x}}{16} - \frac{e^{4x}(4\cos 6x + 6\sin 6x)}{208}\right]_4^5$$

$$= \frac{e^{20}}{16} - \frac{e^{20}(4\cos 30 + 6\sin 30)}{208}$$

$$- \frac{e^{16}}{16} + \frac{e^{16}(4\cos 24 + 6\sin 24)}{208}$$

OR



$$3a\cos\theta = a(1+\cos\theta)$$
$$2\cos\theta = 1$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \pm\frac{\pi}{3}$$

$$\therefore\ r = 3a\cos\theta = 3a\left(\frac{1}{2}\right) = \frac{3a}{2}$$

$$P\left(\frac{3a}{2}, \frac{\pi}{3}\right) \quad Q\left(\frac{3a}{2}, \frac{-\pi}{3}\right)$$

$$\text{Area} = \int_0^{\frac{\pi}{3}} \frac{a^2}{2}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{a^2}{2}\left[1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{3}} \left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\left[\frac{3\theta}{2}+2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\left(\frac{3}{2}\times\frac{\pi}{3}\right)+2\sin\frac{\pi}{3}+\frac{1}{4}\sin\frac{2\pi}{3}\right) - (0)\Big]$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+2\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\pi a^2}{4} + \frac{9\sqrt{3}a^2}{16}$$