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**CAMBRIDGE A LEVEL PROGRAMME**

# SEMESTER ONE EXAMINATION DECEMBER 2012

**(July 2012 Intake)**
**18 months & 2 year programme**

**Wednesday** **5 December 2012** **1.00 pm – 3.00 pm**

## FURTHER MATHEMATICS 9231/01

**PAPER 1** **2 hours**

Additional materials: Answer Booklet/Paper
List of formulae (MF 10)

**READ THESE INSTRUCTIONS FIRST**

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total marks for this paper is **50**.

This document consists of **3** printed pages.

**1** The roots of the equation $x^4 - ax^3 + bx^2 - abx + 5 = 0$ are $\alpha, \beta, \gamma$ and $\delta$. Show that

$$(\alpha + \beta + \gamma)(\alpha + \beta + \delta)(\alpha + \gamma + \delta)(\beta + \gamma + \delta) = 5.$$ [4]

**2** Prove, by induction, that $5n + 5 \leq n^2$ for all integers $n \geq 6$. [5]

**3** Find, in terms of $n$, the series

$$\sum_{r=1}^{n} \frac{8r^2}{(2r-1)(2r+1)}.$$ [6]

**4** Solve the system of linear equations

$$\begin{aligned} x + y + \lambda z &= 0, \\ x + \lambda y + z &= 0, \\ \lambda x + y + z &= 0, \end{aligned}$$

completely for $x, y$ and $z$ where $\lambda$ is a constant. [6]

**5** A plane $\pi_1$ contains the points $(0, 1, 1)$, $(1, 0, 0)$ and $(1, 2, 3)$.

**(i)** Find the Cartesian equation of $\pi_1$. [3]

**(ii)** Another plane $\pi_2$ has equation $x + y + 2z = 0$. Find the angle between $\pi_1$ and $\pi_2$. [3]

**(iii)** Find the perpendicular distance from the origin to $\pi_1$. [2]

**6** The curve $C$ has equation

$$y = 3x + \mu + 9 + \frac{3\mu + 12}{x - 3},$$

where $\mu$ is a constant.

**(i)** Find the set of values of $\mu$ for which $C$ has turning points. [5]

**(ii)** Draw a sketch of $C$ for the case $\mu = -6$. [4]

7 Answer only **one** of the following two alternatives.

**EITHER**

**(i)** The roots of the equation

$$2x^4 + px^2 + pq = 0,$$

where $p, q \neq 0$, are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$. Given also that

$$S_n = \alpha^{-n} + \beta^{-n} + \gamma^{-n} + \delta^{-n}.$$

**(a)** Show that $S_2 = -p$. [3]

**(b)** Find $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [3]

**(ii)** If the equation $x^3 + 3x^2 - 7x + 1 = 0$ has roots $\alpha, \beta$ and $\gamma$, find the equation with roots $\alpha^2 + \alpha$, $\beta^2 + \beta$ and $\gamma^2 + \gamma$. [6]

**OR**

**(i)** Find a vector equation of the line $l_1$ which passes through points $A$ and $B$ with position vectors **a** and **b** respectively. [1]

**(ii)** $R$ is a point on $l_1$ and a point $C$ has position vector **c**.

**(a)** Find the vectors **CR** in terms of **a**, **b** and **c**. [1]

**(b)** Prove that $\mathbf{CR} \times \mathbf{AB} = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$. [2]

**(c)** Deduce the perpendicular distance of $C$ from $l_1$. [2]

**(iii)** **(a)** Show that, the perpendicular distance of point $T(\alpha, 4\alpha, -\alpha)$ to the line $l_2$ which passes through points $(-1, 4, 7)$ and $(0, 0, 4)$ is given by

$$\sqrt{\frac{162\alpha^2 + 248\alpha + 136}{13}}.$$ [3]

**(b)** Deduce the shortest distance between skew lines $l_2$ and $l_3$, where the Cartesian equation of $l_3$ is given by $4x = y = -4z$. [3]

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Mark Scheme : **FURTHER Mathematics Paper 1**

Exam : **Semester 1 Exam** (Dec 2012)

Intake : **July 2012** (18-month & 2-Year programmes)

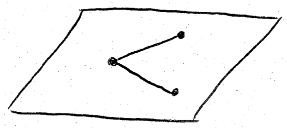
Examiner : **Chin Kok Fui and Teh Huey Ching**

| No | Solution | Mark |
|---|---|---|
| 1. | $x^4 - ax^3 + bx^2 - abx + 5 = 0$ | |
| | Roots : $\alpha, \beta, \gamma, \delta$ | |
| | $\alpha+\beta+\gamma+\delta = a$ — ① | |
| | $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = b$ | |
| | $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = ab$ | |
| | $\alpha\beta\gamma\delta = 5$ | |
| | From ①, | |
| | $(\alpha+\beta+\gamma)(\alpha+\beta+\delta)(\alpha+\gamma+\delta)(\beta+\gamma+\delta)$ | |
| | $= (a-\delta)(a-\gamma)(a-\beta)(a-\alpha)$ | 1 |
| | $= [a^2-(\gamma+\delta)a+\gamma\delta][a^2-(\alpha+\beta)a+\alpha\beta]$ | |
| | $= a^4-(\gamma+\delta)a^3+\gamma\delta a^2-(\alpha+\beta)a^3+(\alpha+\beta)(\gamma+\delta)a^2$ $-(\alpha+\beta)\gamma\delta a+\alpha\beta a^2-(\gamma+\delta)\alpha\beta a+\alpha\beta\gamma\delta$ | |
| | $= a^4-(\alpha+\beta+\gamma+\delta)a^3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)a^2$ $-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)a+\alpha\beta\gamma\delta$ | 1 |
| | $= a^4 - a(a^3) + b(a^2) - (ab)a + 5$ | 1 |
| | $= a^4 - a^4 + a^2b - a^2b + 5$ | |
| | $= 5$ (shown) | 1 |

| No | Solution | Mark |
|---|---|---|
| 2. | $5n + 5 \leq n^2 \quad , \quad n \geq 6$ | |
| | When $n = 6$ : | |
| | $5(6) + 5 = 35 \leq 36 = 6^2$ | 1 |
| | $\therefore$ True for $n = 6$. | |
| | Assume the statement is true when $n = k$ : | |
| | $5k + 5 \leq k^2 \quad , \quad k \geq 6$ | |
| | When $n = k+1$ : $5(k+1) + 5 \leq (k+1)^2$ (what needs to be proved) | |
| | $5(k+1) + 5 = 5k + 5 + 5$ | 1 |
| | $\leq k^2 + 5$ | 1 |
| | Since $k \geq 6$ | |
| | $k \geq 2$ | 1 |
| | $2k \geq 4$ | |
| | $k^2 + 5 \leq k^2 + 2k + 1$ | |
| | $k^2 + 5 \leq (k+1)^2$ | |
| | Since $5(k+1) + 5 \leq k^2 + 5$ | |
| | and $k^2 + 5 \leq (k+1)^2$ | 1 |
| | $\therefore \quad 5(k+1) + 5 \leq (k+1)^2$ | |
| | $\therefore$ True for $n = k+1$ | |
| | Therefore, $5n + 5 \leq n^2$ for all integers $n \geq 6$. | |

| No | Solution | Mark |
|---|---|---|
| 3. | $\dfrac{8r^2}{(2r-1)(2r+1)} = 2 + \dfrac{2}{(2r-1)(2r+1)}$     $4r^2-1 \overline{)\,8r^2\,}$ quotient $2$; $8r^2-2$; remainder $2$ | 1 |
| | $\dfrac{2}{(2r-1)(2r+1)} \equiv \dfrac{A}{2r-1} + \dfrac{B}{2r+1}$ | |
| | $2 \equiv A(2r+1) + B(2r-1)$ | |
| | $r = \frac{1}{2}$ : $2 = 2A \Rightarrow A = 1$ | 1 |
| | $r = -\frac{1}{2}$ : $2 = -2B \Rightarrow B = -1$ | |
| | $\therefore \dfrac{8r^2}{(2r-1)(2r+1)} = 2 + \dfrac{1}{2r-1} - \dfrac{1}{2r+1}$ | 1 |
| | Let $f(r) = \dfrac{1}{2r-1}$ | |
| | $f(r+1) = \dfrac{1}{2r+1}$ | |
| | $\displaystyle\sum_{r=1}^{n} \frac{8r^2}{(2r-1)(2r+1)} = \sum_{r=1}^{n} [2 + f(r) - f(r+1)]$ | |
| | $\displaystyle = \sum_{r=1}^{n} 2 + \sum_{r=1}^{n} [f(r) - f(r+1)]$ | 1 |
| | $= 2n + f(1) - f(n+1)$ | 1 |
| | $= 2n + 1 - \dfrac{1}{2n+1}$ | 1 |

| No | Solution | Mark |
|---|---|---|
| 4. | $\left.\begin{aligned} x+y+\lambda z&=0\\ x+\lambda y+z&=0\\ \lambda x+y+z&=0\end{aligned}\right\} \xLeftrightarrow[-\lambda r_1+r_3\to r_3']{-r_1+r_2\to r_2'} \left.\begin{aligned} x+y+\lambda z&=0\\ (\lambda-1)y+(1-\lambda)z&=0\\ (1-\lambda)y+(1-\lambda^2)z&=0\end{aligned}\right\}$ | |
| | $\xLeftrightarrow{r_2+r_3\to r_3'} \left.\begin{aligned} x+y+\lambda z&=0\\ (\lambda-1)y+(1-\lambda)z&=0\\ (\lambda+2)(\lambda-1)z&=0\end{aligned}\right\}$ | |
| | When $\lambda=1$ : $\left.\begin{aligned} x+y+z&=0\\ 0y+0z&=0\\ 0z&=0\end{aligned}\right\}$ | 1 |
| | Let $y=s$ , $z=t$ , $s,t\in\mathbb{R}$ <br> $\therefore\ x=-s-t$ | 1 |
| | When $\lambda=-2$ : $\left.\begin{aligned} x+y-2z&=0\\ -3y+3z&=0\\ 0z&=0\end{aligned}\right\}$ | 1 |
| | Let $z=t$ , $t\in\mathbb{R}$ <br> $y=t$ | 1 |
| | $x=2t-t=t$ | |
| | When $\lambda\neq-2,1$ : $\left.\begin{aligned} x+y+\lambda z&=0\\ (\lambda-1)y+(1-\lambda)z&=0\\ (\lambda+2)(\lambda-1)z&=0\end{aligned}\right\}$ | 1 |
| | $\therefore\ z=0$ <br> $y=0$ <br> $x=0$ | 1 |
| | $\therefore$ The complete solutions are : <br> If $\lambda=1$ : $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-s-t\\s\\t\end{pmatrix}$ If $\lambda=-2$ : $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}t\\t\\t\end{pmatrix}$ If $\lambda\neq-2,1$ : $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ | |

| No | Solution | Mark |
|---|---|---|
| 5. |  $\Pi_1$ $(0,1,1)$ , $(1,0,0)$ , $(1,2,3)$ | |
| (i) | Vector equation of $\Pi_1$ : | |
| | $\underset{\sim}{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}1-0\\0-1\\0-1\end{pmatrix} + t\begin{pmatrix}1-1\\2-0\\3-0\end{pmatrix}$ | 1 |
| | $= \begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}1\\-1\\-1\end{pmatrix} + t\begin{pmatrix}0\\2\\3\end{pmatrix}$ | |
| | $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+s\\-s+2t\\-s+3t\end{pmatrix}$ | |
| | $x = 1+s$ ; $s = x-1$ —① ; $y = -s+2t$ ; $y = 1-x+2t$ ; $t = \dfrac{x+y-1}{2}$ —② ; $z = -s+3t$ ; $z = 1-x+3t$ ; $t = \dfrac{x+z-1}{3}$ —③ | 1 |
| | ② = ③ : | |
| | $\dfrac{x+y-1}{2} = \dfrac{x+z-1}{3}$ | |
| | $3x+3y-3 = 2x+2z-2$ | |
| | $x+3y-2z = 1$ | 1 |
| | ∴ The cartesian equation of $\Pi_1$ is $x+3y-2z=1$. | |

| No | Solution | Mark |
|---|---|---|
| 5(ii) | $\Pi_1 : x + 3y - 2z = 1 \Rightarrow \underset{\sim}{n_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ <br> $\Pi_2 : x + y + 2z = 0 \Rightarrow \underset{\sim}{n_2} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ | $\}$ 1 |
| | $\underset{\sim}{n_1} \cdot \underset{\sim}{n_2} = \lvert \underset{\sim}{n_1} \rvert \lvert \underset{\sim}{n_2} \rvert \cos\theta$ <br> Let $\theta$ denotes the angle between $\Pi_1$ and $\Pi_2$, <br> $\therefore \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} 1 \\ 3 \\ -2 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} \cos\theta$ | 1 |
| | $1 + 3 - 4 = \sqrt{1^2 + 3^2 + (-2)^2}\,\sqrt{1^2 + 1^2 + 2^2}\,\cos\theta$ <br> $\cos\theta = \dfrac{0}{\sqrt{14}\,\sqrt{6}}$ <br> $\cos\theta = 0$ <br> $\theta = \dfrac{\pi}{2}$ or $90^\circ$ | 1 |
| (iii) | The equation of plane $\Pi_1$ is <br> $\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 1$ , $\underset{\sim}{n_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ <br> Changing this to the form $\underset{\sim}{r} \cdot \underset{\sim}{\hat{n}_1} = d$ , where $\underset{\sim}{\hat{n}_1}$ is a unit vector, <br> $\underset{\sim}{\hat{n}_1} = \dfrac{\underset{\sim}{n_1}}{\lvert \underset{\sim}{n_1} \rvert} = \dfrac{1}{\sqrt{1^2 + 3^2 + (-2)^2}} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \dfrac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ <br> $\underset{\sim}{r} \cdot \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ -\frac{2}{\sqrt{14}} \end{pmatrix} = 1\left(\dfrac{1}{\sqrt{14}}\right)$ | 1 |
| | $\therefore$ The perpendicular distance from the origin to $\Pi_1$ is $\dfrac{1}{\sqrt{14}}$ . | 1 |

| No | Solution | Mark |
|---|---|---|
| 6. | $C: \quad y = 3x + \mu + 9 + \dfrac{3\mu + 12}{x-3}$ , $\mu$ is constant | |
| (i) | $\dfrac{dy}{dx} = 3 - \dfrac{3\mu+12}{(x-3)^2}$ | 1 |
| | When $\dfrac{dy}{dx} = 0$ | 1 |
| | $\dfrac{3\mu+12}{(x-3)^2} = 3$ | |
| | $3(x^2-6x+9) = 3(\mu+4)$ | |
| | $x^2-6x+9-\mu-4=0$ | |
| | $x^2-6x+5-\mu=0$ | 1 |
| | If C has turning points, | |
| | $B^2-4AC \geqslant 0$ | 1 |
| | $(-6)^2-4(1)(5-\mu) \geqslant 0$ | |
| | $36-20+4\mu \geqslant 0$ | |
| | $4\mu \geqslant -16$ | |
| | $\mu \geqslant -4$ | 1 |
| (ii) | When $\mu = -6$ | |
| | $y = 3x+3-\dfrac{6}{x-3}$ | |
| | Asymptotes : | |
| | As $x \to 3$ , $y \to \pm\infty$ | |
| | As $x \to \pm\infty$ , $y \to 3x+3$ | |
| | $\therefore$ The asymptotes are $x=3$ and $y=3x+3$ | |
| | No turning point since $\mu = -6 < -4$ . | |

5

| No | Solution | Mark |
|---|---|---|

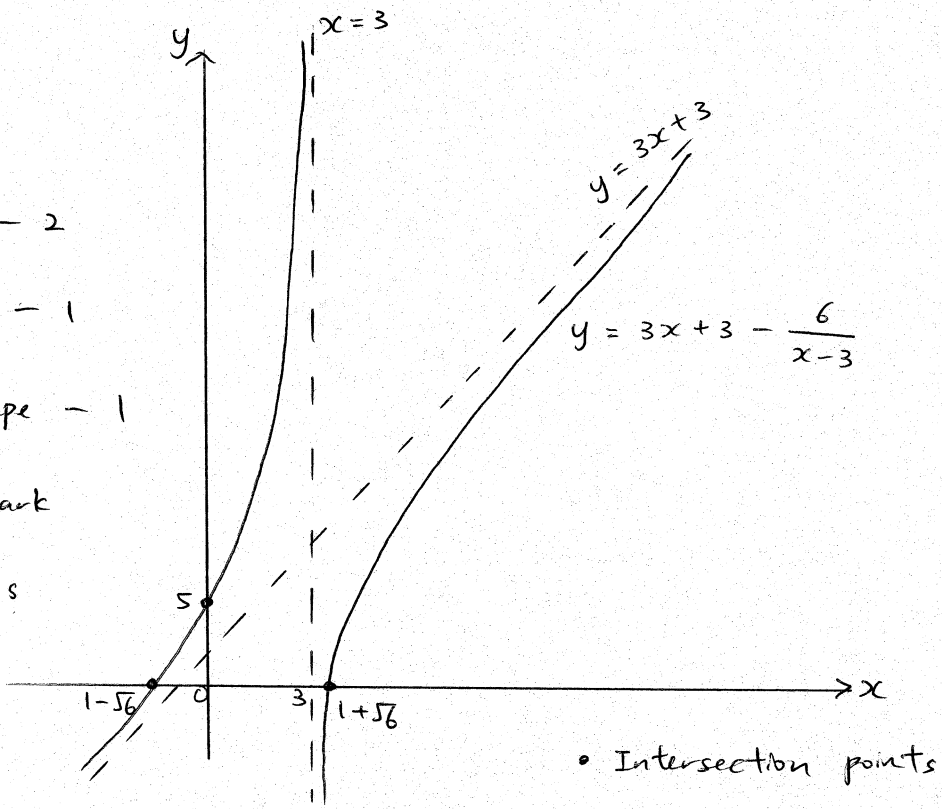
**6(ii)** Intersection points :

When $x = 0$, $y = 3 - \frac{6}{(-3)} = 5$

When $y = 0$, $3x + 3 = \frac{6}{x-3}$

$$(3x+3)(x-3) = 6$$
$$3x^2 - 6x - 9 - 6 = 0$$
$$x^2 - 2x - 5 = 0$$
$$(x-1)^2 - 1 - 5 = 0$$
$$(x-1)^2 = 6$$
$$x - 1 = \pm\sqrt{6}$$
$$x = 1 \pm \sqrt{6}$$

$\therefore$ The intersection points are $(1-\sqrt{6}, 0)$, $(1+\sqrt{6}, 0)$ and $(0, 5)$




Asymptotes — 2

Intersection points — 1

Correct shape — 1

Deduct 1 mark overall for wrong forms at infinity.

| No | Solution | Mark |
|---|---|---|
| 7 | EITHER | |
| (i) | $2x^4 + px^2 + pq = 0$ | |
| | Roots : $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ | |
| | $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} = 0$ | |
| | $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\alpha\delta}+\frac{1}{\beta\gamma}+\frac{1}{\beta\delta}+\frac{1}{\gamma\delta} = \frac{p}{2}$ | |
| | $\frac{1}{\alpha\beta\gamma}+\frac{1}{\alpha\beta\delta}+\frac{1}{\alpha\gamma\delta}+\frac{1}{\beta\gamma\delta} = 0$ | |
| | $\frac{1}{\alpha\beta\gamma\delta} = \frac{pq}{2}$ | |
| | Given $S_n = \frac{1}{\alpha^n}+\frac{1}{\beta^n}+\frac{1}{\gamma^n}+\frac{1}{\delta^n}$ | |
| | $S_0 = \frac{1}{\alpha^0}+\frac{1}{\beta^0}+\frac{1}{\gamma^0}+\frac{1}{\delta^0} = 4$ | |
| | $S_1 = 0$ | |
| a) | $S_2 = \left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)^2 - 2\left(\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\alpha\delta}+\frac{1}{\beta\gamma}+\frac{1}{\beta\delta}+\frac{1}{\gamma\delta}\right)$ | 1 |
| | $= 0^2 - 2\left(\frac{p}{2}\right)$ | 1 |
| | $= -p$ (shown) | 1 |
| b) | To find $S_{-2} = \alpha^2+\beta^2+\gamma^2+\delta^2$, | |
| | $2S_{4+r} + pS_{2+r} + pqS_r = 0$ | |
| | Let $r=-2$ : $2S_2 + pS_0 + pqS_{-2} = 0$ | 1 |
| | $2(-p) + p(4) + pqS_{-2} = 0$ | 1 |
| | $S_{-2} = -\frac{2p}{pq} = -\frac{2}{q}$ | |
| | $\therefore \alpha^2+\beta^2+\gamma^2+\delta^2 = -\frac{2}{q}$. | 1 |

6

| No | Solution | Mark |
|---|---|---|

7(ii)

$$x^3+3x^2-7x+1=0$$

Roots : $\alpha, \beta, \gamma$

New roots : $\alpha^2+\alpha,\ \beta^2+\beta,\ \gamma^2+\gamma$

Let $u=\alpha^2+\alpha$

$$\left(\alpha+\tfrac{1}{2}\right)^2-\tfrac{1}{4}=u$$

$$\left(\alpha+\tfrac{1}{2}\right)^2=u+\tfrac{1}{4}$$

$$\alpha+\tfrac{1}{2}=\pm\sqrt{u+\tfrac{1}{4}}$$

$$\alpha=-\tfrac{1}{2}\pm\sqrt{u+\tfrac{1}{4}}$$ 1

Since $\alpha$ is one of the roots,

$$\alpha^3+3\alpha^2-7\alpha+1=0$$

$$\left(-\tfrac{1}{2}\pm\sqrt{u+\tfrac{1}{4}}\right)^3+3\left(-\tfrac{1}{2}\pm\sqrt{u+\tfrac{1}{4}}\right)^2-7\left(-\tfrac{1}{2}\pm\sqrt{u+\tfrac{1}{4}}\right)+1=0$$ 1

$$-\tfrac{1}{8}+\tfrac{3}{4}\left(\pm\sqrt{u+\tfrac{1}{4}}\right)-\tfrac{3}{2}\left(u+\tfrac{1}{4}\right)+\left(u+\tfrac{1}{4}\right)\left(\pm\sqrt{u+\tfrac{1}{4}}\right)$$
$$+3\left[\tfrac{1}{4}-\left(\pm\sqrt{u+\tfrac{1}{4}}\right)+u+\tfrac{1}{4}\right]+\tfrac{7}{2}-7\left(\pm\sqrt{u+\tfrac{1}{4}}\right)+1=0$$ 1, 1, 1

$$-\tfrac{1}{8}+\tfrac{3}{4}\left(\pm\sqrt{u+\tfrac{1}{4}}\right)-\tfrac{3}{2}u-\tfrac{3}{8}+\left(u+\tfrac{1}{4}\right)\left(\pm\sqrt{u+\tfrac{1}{4}}\right)$$
$$+\tfrac{3}{4}-3\left(\pm\sqrt{u+\tfrac{1}{4}}\right)+3u+\tfrac{3}{4}+\tfrac{7}{2}-7\left(\pm\sqrt{u+\tfrac{1}{4}}\right)+1=0$$

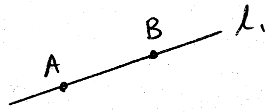
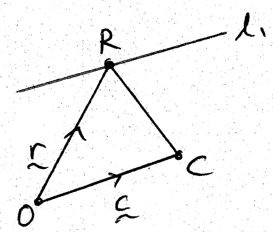
$$\tfrac{11}{2}+\tfrac{3}{2}u+\left(\pm\sqrt{u+\tfrac{1}{4}}\right)\left(\tfrac{3}{4}+u+\tfrac{1}{4}-3-7\right)=0$$

$$(u-9)\left(\pm\sqrt{u+\tfrac{1}{4}}\right)=-\tfrac{11}{2}-\tfrac{3}{2}u$$

$$(u-9)^2\left(\pm\sqrt{u+\tfrac{1}{4}}\right)^2=\tfrac{1}{4}(3u+11)^2$$

$$(u^2-18u+81)\left(u+\tfrac{1}{4}\right)=\tfrac{1}{4}(9u^2+66u+121)$$

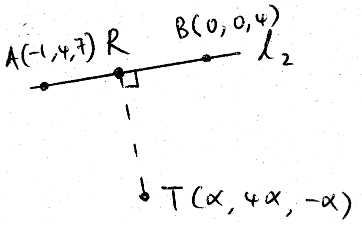
| No | Solution | Mark |
|---|---|---|
| 7(ii) | $\frac{1}{4}(u^2-18u+81)(4u+1) = \frac{1}{4}(9u^2+66u+121)$<br>$4u^3-72u^2+324u+u^2-18u+81 = 9u^2+66u+121$<br>$4u^3-80u^2+240u-40=0$<br>$u^3-20u^2+60u-10=0$<br>$\therefore$ The equation which has roots $\alpha^2+\alpha$, $\beta^2+\beta$, $\gamma^2+\gamma$ is $u^3-20u^2+60u-10=0$. | 1 |

| No | Solution | Mark |
|---|---|---|
| 7 | OR | |
| (i) | $\overrightarrow{OA} = \underset{\sim}{a} \qquad \overrightarrow{OB} = \underset{\sim}{b}$<br>$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$<br>$= \underset{\sim}{b} - \underset{\sim}{a}$<br><br>$l_1 : \underset{\sim}{r} = \overrightarrow{OA} + s\,\overrightarrow{AB}$<br>$= \underset{\sim}{a} + s(\underset{\sim}{b} - \underset{\sim}{a})$<br>$= (1-s)\underset{\sim}{a} + s\underset{\sim}{b}$ | $\left.\right\}$ 1 |
| (ii)<br>(a) | $R$ is on $l_1$ and $\overrightarrow{OC} = \underset{\sim}{c}$<br><br>$\overrightarrow{CR} = \overrightarrow{OR} - \overrightarrow{OC}$<br>$= \underset{\sim}{r} - \underset{\sim}{c}$<br>$= (1-s)\underset{\sim}{a} + s\underset{\sim}{b} - \underset{\sim}{c}$ | 1 |
| (b) | $\overrightarrow{CR} \times \overrightarrow{AB} = [(1-s)\underset{\sim}{a} + s\underset{\sim}{b} - \underset{\sim}{c}] \times (\underset{\sim}{b} - \underset{\sim}{a})$<br>$= (1-s)(\underset{\sim}{a} \times \underset{\sim}{b}) + s(\underset{\sim}{b} \times \underset{\sim}{b}) - (\underset{\sim}{c} \times \underset{\sim}{b})$<br>$- (1-s)(\underset{\sim}{a} \times \underset{\sim}{a}) - s(\underset{\sim}{b} \times \underset{\sim}{a}) + (\underset{\sim}{c} \times \underset{\sim}{a})$<br>$= (1-s)(\underset{\sim}{a} \times \underset{\sim}{b}) + s(\underset{\sim}{a} \times \underset{\sim}{b}) + (\underset{\sim}{b} \times \underset{\sim}{c}) + (\underset{\sim}{c} \times \underset{\sim}{a})$<br>$= (\underset{\sim}{a} \times \underset{\sim}{b}) - s(\underset{\sim}{a} \times \underset{\sim}{b}) + s(\underset{\sim}{a} \times \underset{\sim}{b}) + (\underset{\sim}{b} \times \underset{\sim}{c}) + (\underset{\sim}{c} \times \underset{\sim}{a})$<br>$= \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}$ (Proven) | 1: use law of distribut<br>1: use anti commut |
| (c) | The shortest distance of the point C from line $l_1$ can be denoted as $\lvert\overrightarrow{CR}\rvert$ where $\overrightarrow{CR} \perp \overrightarrow{AB}$, i.e.<br>$\lvert\overrightarrow{CR} \times \overrightarrow{AB}\rvert = \lvert\overrightarrow{CR}\rvert\lvert\overrightarrow{AB}\rvert \sin 90^\circ$<br>$\therefore \lvert\overrightarrow{CR}\rvert = \dfrac{\lvert\overrightarrow{CR} \times \overrightarrow{AB}\rvert}{\lvert\overrightarrow{AB}\rvert} = \dfrac{\lvert\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}\rvert}{\lvert\underset{\sim}{b} - \underset{\sim}{a}\rvert}$ | 1<br>1 |

6

| No | Solution | Mark |
|---|---|---|

7(iii) a)




$\overrightarrow{TR}$ is similar with $\overrightarrow{CR}$ in part (ii)(c). Hence the perpendicular distance of point T to line $l_2$ is $|\overrightarrow{TR}| = |\overrightarrow{CR}|$

Let $\underset{\sim}{a} = \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$

$\underset{\sim}{c} = \begin{pmatrix} \alpha \\ 4\alpha \\ -\alpha \end{pmatrix}$, $\underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}$

$$|\overrightarrow{TR}| = |\overrightarrow{CR}| = \frac{|\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}|}{|\underset{\sim}{b} - \underset{\sim}{a}|}$$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -1 & 4 & 7 \\ 0 & 0 & 4 \end{vmatrix} = 16\underset{\sim}{i} + 4\underset{\sim}{j} = \begin{pmatrix} 16 \\ 4 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a} = (\underset{\sim}{b} - \underset{\sim}{a}) \times \underset{\sim}{c} = \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \times \begin{pmatrix} \alpha \\ 4\alpha \\ -\alpha \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -4 & -3 \\ \alpha & 4\alpha & -\alpha \end{vmatrix}$$ 1

$$= 4\alpha\underset{\sim}{i} - 3\alpha\underset{\sim}{j} + 4\alpha\underset{\sim}{k} + 4\alpha\underset{\sim}{k} + \alpha\underset{\sim}{j} + 12\alpha\underset{\sim}{i} = \begin{pmatrix} 16\alpha \\ -2\alpha \\ 8\alpha \end{pmatrix}$$

$$|\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}|$$
$$= \sqrt{(16\alpha+16)^2 + (4-2\alpha)^2 + (8\alpha)^2}$$
$$= \sqrt{256\alpha^2 + 512\alpha + 256 + 16 - 16\alpha + 4\alpha^2 + 64\alpha^2}$$ 1
$$= \sqrt{2(162\alpha^2 + 248\alpha + 136}$$

2

| No | Solution | Mark |
|---|---|---|
| | $\therefore \ \lvert\overrightarrow{TR}\rvert = \dfrac{\sqrt{2(162\alpha^2+248\alpha+136)}}{\sqrt{1^2+(-4)^2+(-3)^2}}$ | 1 |
| | $= \sqrt{\dfrac{162\alpha^2+248\alpha+136}{13}}$ | |
| (b) | $l_3: \quad x = \dfrac{y}{4} = \dfrac{z}{(-1)}$ | |
| | $\underset{\sim}{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \mu\begin{pmatrix}1\\4\\-1\end{pmatrix}$ | |
| | Since $T(\alpha, 4\alpha, -\alpha)$ is a point on $l_3$, thus the shortest distance between skew lines $l_2$ and $l_3$ is $\lvert\overrightarrow{TR}\rvert$. | |
| | $13\lvert\overrightarrow{TR}\rvert^2 = 162\alpha^2+248\alpha+136$ | |
| | $13\dfrac{d\lvert\overrightarrow{TR}\rvert^2}{d\alpha} = 324\alpha+248$ | 1 |
| | $\lvert\overrightarrow{TR}\rvert$ is minimum when $\dfrac{d\lvert\overrightarrow{TR}\rvert^2}{d\alpha} = 0$ | |
| | when $\dfrac{d\lvert\overrightarrow{TR}\rvert^2}{d\alpha} = 0$, | |
| | $324\alpha+248 = 0$ | 1 |
| | $\alpha = -\dfrac{62}{81}$ | |
| | $\therefore \ \lvert\overrightarrow{TR}\rvert = \sqrt{\dfrac{256}{81}} = \dfrac{16}{9}$ | 1 |

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