OCTOBER / NOVEMBER 2013

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1. $$\frac{1}{r(r+1)(r-1)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r-1}$$

$$= \frac{A(r^2-1) + B(r^2-r) + C(r^2+r)}{r(r+1)(r-1)}$$

$$1 = A(r^2-1) + B(r^2-r) + C(r^2+r)$$

$$= (A+B+C)r^2 + (C-B)r - A$$

$A+B+C=0 \quad C-B=0 \quad -A=1$

$A=-1$

$B+C=1$

$2C=1$

$C=\frac{1}{2} \quad B=\frac{1}{2}$

$$\therefore \frac{1}{r(r+1)(r-1)} = \frac{1}{2(r+1)} - \frac{1}{r} + \frac{1}{2(r-1)}$$

$$\sum_{r=2}^{n} \frac{1}{r(r+1)(r-1)} = \sum_{r=2}^{n} \frac{1}{2(r+1)} - \frac{1}{r} + \frac{1}{2(r-1)}$$

$$= \frac{1}{2(3)} - \frac{1}{2} + \frac{1}{2(1)}$$

$$+ \frac{1}{2(4)} - \frac{1}{3} + \frac{1}{2(2)}$$

$$+ \frac{1}{2(5)} - \frac{1}{4} + \frac{1}{2(3)}$$

$$\vdots$$

$$+ \frac{1}{2(n-1)} - \frac{1}{n-2} + \frac{1}{2(n-3)}$$

$$+ \frac{1}{2n} - \frac{1}{n-1} + \frac{1}{2(n-2)}$$

$$+ \frac{1}{2(n+1)} - \frac{1}{n} + \frac{1}{2(n-1)}$$

$$= -\frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{2n} + \frac{1}{2(n+1)} - \frac{1}{n}$$

$$= \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n}$$

$$\sum_{r=2}^{\infty} \frac{1}{r(r+1)(r-1)} = \lim_{n\to\infty} \sum_{r=2}^{n} \frac{1}{r(r+1)(r-1)}$$

$$= \lim_{n\to\infty} \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n}$$

$$= \frac{1}{4} + 0 + 0$$

$$= \frac{1}{4}$$

2. $\begin{pmatrix} 1 & 4 & 2 \\ 3 & 0 & -2 \\ 3 & -3 & 4 \end{pmatrix}$

$$\begin{vmatrix} 1 & 4 & 2 \\ 3 & 0 & -2 \\ 3 & -3 & -4 \end{vmatrix} = 1(-6) - 4(-12+6) + 2(-9)$$
$$= -6 + 24 - 18$$
$$= 0$$

$\therefore \begin{pmatrix} 1 & 4 & 2 \\ 3 & 0 & -2 \\ 3 & -3 & 4 \end{pmatrix}$ has no inverse.

$$x + 4y - 2z = 0$$
$$3x \quad - 2z = 4$$
$$3x - 3y - 4z = 5$$

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 0 & -2 \\ 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 3 & 0 & -2 & 4 \\ 3 & -3 & 4 & 5 \end{array}\right)$$

$$\xrightarrow[-3r_1+r_3]{-3r_1+r_2} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -12 & -8 & 4 \\ 0 & -15 & -10 & 5 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-4}, \frac{r_3}{-5}} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 3 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

let $z = s,\ s \in \mathbb{R}$

$$y = \frac{-2s - 1}{3}$$

$$x + 4y + 2z = 0$$

$$x + \frac{-8s}{3} - \frac{4}{3} + 2s = 0$$

$$x = \frac{2s + 4}{3}$$

3. $$\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 4y = 4x^2 + 8$$

$$m^2 + 2m + 4 = 0$$
$$(m+1)^2 + 3 = 0$$
$$m + 1 = \pm\sqrt{3}i$$
$$m = -1 \pm \sqrt{3}i$$

The complementary function, y_c, is

$$y_c = e^{-x}(A\cos\sqrt{3}x + B\sin\sqrt{3}x)$$

The particular integral, y_p, is given by

$$y_p = Cx^2 + Dx + E$$

$$\frac{dy_p}{dx} = 2Cx + D$$

$$\frac{d^2y_p}{dx^2} = 2C$$

$$\frac{d^2y_p}{dx^2} + \frac{2dy_p}{dx} + 4y_p = 2C + 2(2Cx + D) + 4(Cx^2 + Dx + E)$$
$$= 4Cx^2 + (4C + 4D)x + 2C + 2D + 4E$$
$$= 4x^2 + 8$$

$4C = 4$ $4C + 4D = 0$ $2C + 2D + 4E = 8$

$C = 1$ $D = -1$ $E = 2$

$$y_p = x^2 - x + 2$$

$$y = y_c + y_p$$
$$= e^{-x}(A\cos\sqrt{3}x + B\sin\sqrt{3}x) + x^2 - x + 2$$

4. $x = 2\theta - \sin 2\theta \quad y = 1 - \cos 2\theta \quad -3\pi \le \theta \le 3\pi.$

$$\frac{dx}{d\theta} = 2 - 2\cos 2\theta \quad \frac{dy}{d\theta} = 2\sin 2\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\sin 2\theta}{2 - 2\cos 2\theta}$$

$$= \frac{\sin 2\theta}{1 - \cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{1 - 1 + 2\sin^2\theta}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta, \quad \theta \neq -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d\theta}{dx}\frac{d}{d\theta}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{2 - 2\cos 2\theta}\frac{d}{d\theta}(\cot\theta)$$

$$= \frac{1}{2 - 2 + 4\sin^2\theta}(-\csc^2\theta)$$

$$= \frac{1}{4\sin^2\theta}\,\frac{-1}{\sin^2\theta}$$

$$= \frac{-1}{4\sin^4\theta}$$

At $\theta = \frac{\pi}{4}$: $\frac{d^2y}{dx^2} = -1.$

5. $8x^3 + 36x^2 + kx - 21 = 0$, k constant.

$a-d, a, a+d$ are the roots.

$$a-d+a+a+d = -\frac{9}{2}$$

$$(a-d)a + (a-d)(a+d) + a(a+d) = \frac{k}{8}$$

$$a(a-d)(a+d) = \frac{21}{8}$$

$$3a = -\frac{9}{2}$$

$$\therefore a = -\frac{3}{2}$$

since $a(a^2-d^2) = \frac{21}{8}$

$$-\frac{3}{2}\left(\frac{9}{4} - d^2\right) = \frac{21}{8}$$

$$\frac{9}{4} - d^2 = -\frac{7}{4}$$

$$d^2 = 4$$

$$d = \pm 2$$

$\therefore$ The roots are $-\frac{7}{2}, -\frac{3}{2}, \frac{1}{2}$.

$$\frac{k}{8} = a^2 - ad + a^2 - d^2 + a^2 + ad$$

$$= 3a^2 - d^2$$

$$= \frac{27}{4} - 4$$

$$= \frac{11}{4}$$

$\therefore k = 22$.

6. a) $y = \sec x$

The mean value of y over $\frac{\pi}{6} \leq x \leq \frac{\pi}{3}$ is

$$\frac{1}{\frac{\pi}{3} - \frac{\pi}{6}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} y \, dx$$

$$= \frac{6}{\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx$$

$$= \frac{6}{\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} \, dx$$

$$= \frac{6}{\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$= \frac{6}{\pi} \Big[\ln |\sec x + \tan x| \Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{6}{\pi} \left(\ln (2 + \sqrt{3}) - \ln \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \right]$$

$$= \frac{6}{\pi} \left(\ln (2 + \sqrt{3}) - \ln \sqrt{3} \right)$$

$$= \frac{6}{\pi} \ln \left(\frac{2 + \sqrt{3}}{\sqrt{3}} \right)$$

b) $C: y = -\ln \cos x \, , \quad 0 \leq x \leq \frac{\pi}{3}$

The arc length of C from $x=0$ to $x=\frac{\pi}{3}$ is

$$\int_0^{\frac{\pi}{3}} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1+\left(\frac{\sec x \ \tan x}{\sec x}\right)^2} \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x} \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x} \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec x \, dx$$

$$= \Big[\ln \left| \sec x + \tan x \right| \Big]_0^{\frac{\pi}{3}}$$

$$= \ln(2+\sqrt{3}) - \ln 1$$

$$= \ln(2+\sqrt{3})$$

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7. $C: \; y = \dfrac{2x^2+5x-1}{x+2}$

$$\begin{array}{r} 2x+1 \\ x+2\,\overline{)\,2x^2+5x-1} \\ \underline{2x^2+4x} \\ x-1 \\ \underline{x+2} \\ -3 \end{array}$$

$= 2x+1 - \dfrac{3}{x+2}$

As $x \to \pm\infty$ $\quad y \to 2x+1$

As $x \to -2$ $\quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y=2x+1$ and $x=-2$.

$$\frac{dy}{dx} = 2 + \frac{3}{(x+2)^2}$$

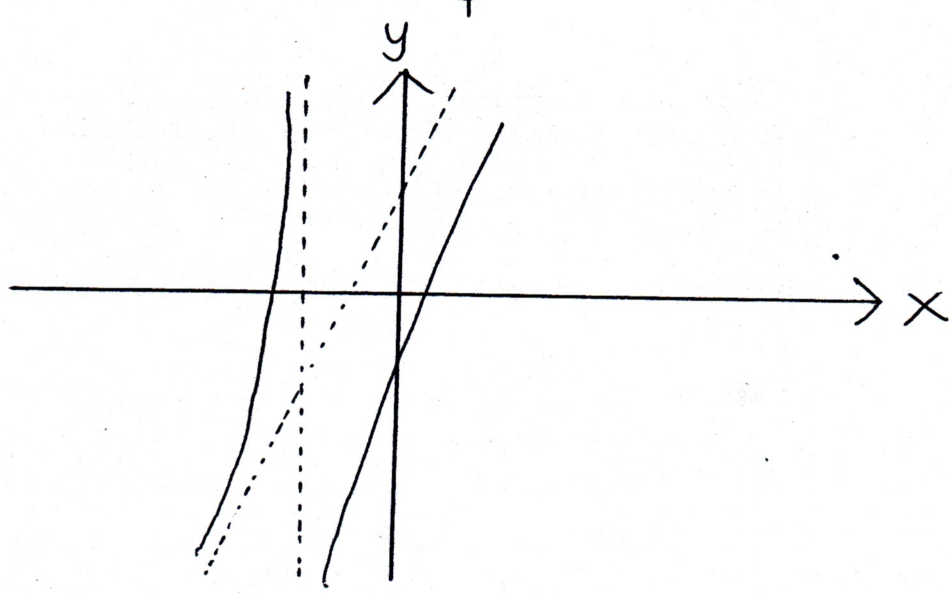
> 2 since $\dfrac{3}{(x+2)^2} > 0$.

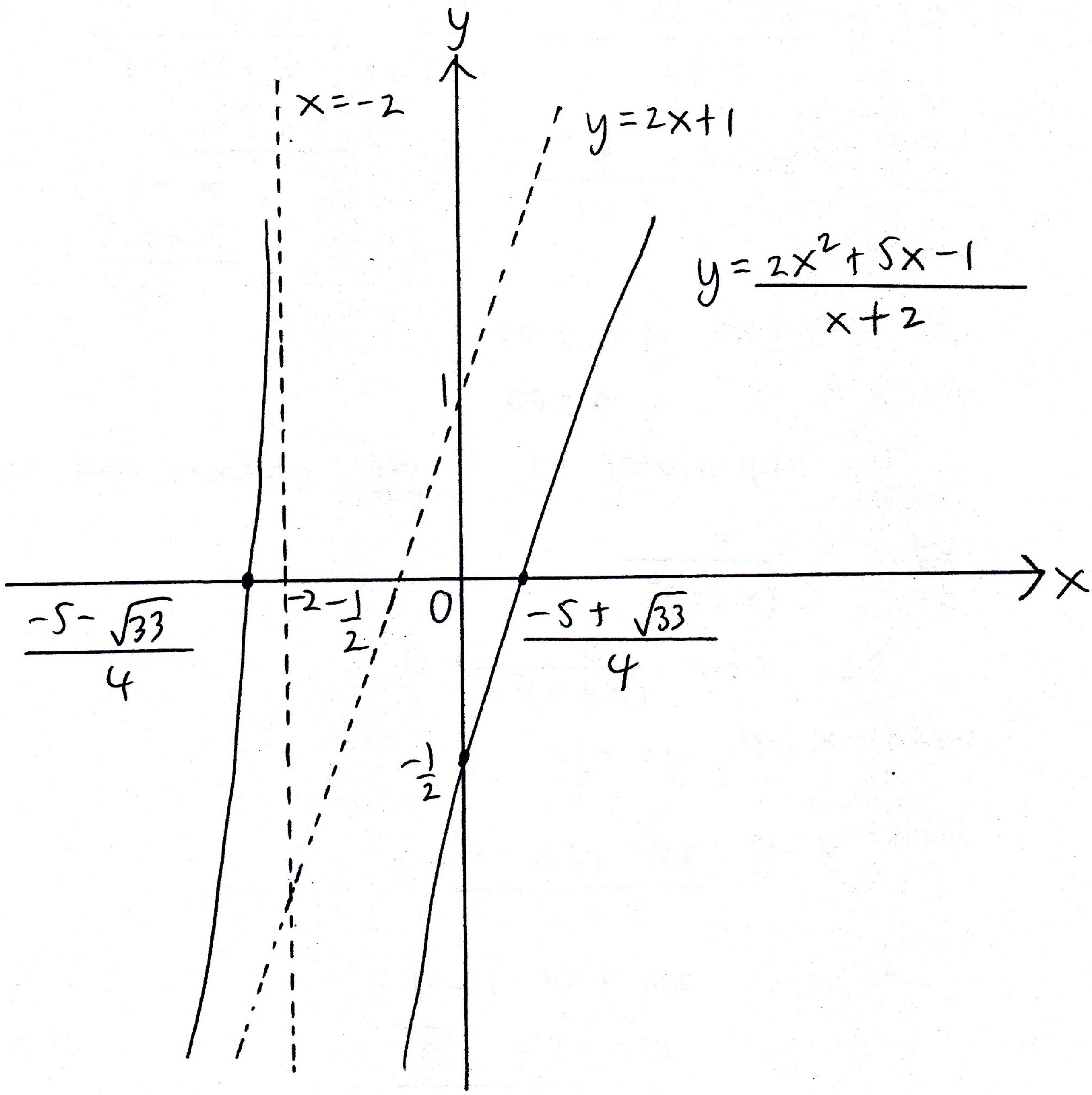
when $x=0$: $y = -\dfrac{1}{2}$

when $y=0$: $\dfrac{2x^2+5x-1}{x+2} = 0$

$$2x^2+5x-1=0$$

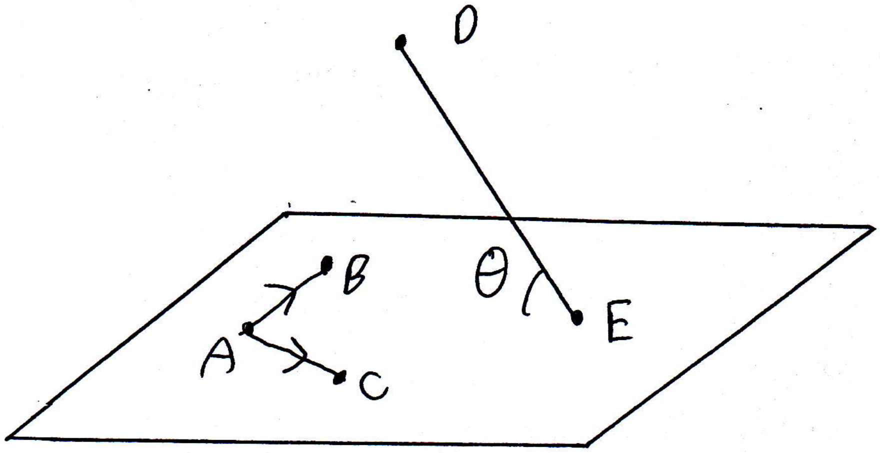
$$x = \frac{-5 \pm \sqrt{33}}{4}$$





y
x
x=-2
y=2x+1
y=\frac{2x^2+5x-1}{x+2}
1
-2
-\frac{1}{2}
0
\frac{-5-\sqrt{33}}{4}
\frac{-5+\sqrt{33}}{4}
-\frac{1}{2}

8. A(4,5,6) B(5,7,8) C(2,6,4)



$\vec{AB} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$

$\vec{AB} \times \vec{AC} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ 5 \end{pmatrix}$

Since $\vec{AB}$ and $\vec{AC}$ are parellel to the plane ABC, $\vec{AB} \times \vec{AC}$ is perpendicular to ABC.

Since A is a point on ABC and $\begin{pmatrix} -6 \\ -2 \\ 5 \end{pmatrix}$ is normal to ABC, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a point on ABC,

$\underset{\sim}{r} \cdot \begin{pmatrix} -6 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ -2 \\ 5 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -6 \\ -2 \\ 5 \end{pmatrix} = -24 - 10 + 30$

$-6x - 2y + 5z = -4$

$6x + 2y - 5z = 4$

$\therefore$ ABC has Cartesian equation $6x + 2y - 5z = 4$

$D(6,3,6)$

The equation of the line OD is $\underset{\sim}{r} = s\begin{pmatrix}6\\3\\6\end{pmatrix}$

when the line meets the plane ABC,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6s\\3s\\6s\end{pmatrix}$$

$x = 6s \quad y = 3s \quad z = 6s$

$6x + 2y - 5z = 4$

$6(6s) + 2(3s) - 5(6s) = 4$

$36s + 6s - 30s = 4$

$12s = 4$

$s = \frac{1}{3}$

The line meets the plane at $(2,1,2)$.

$\therefore E(2,1,2)$.

The equation of the line ED is

$$\underset{\sim}{r} = \begin{pmatrix}2\\1\\2\end{pmatrix} + s\left[\begin{pmatrix}6\\3\\6\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix}\right] = \begin{pmatrix}2\\1\\2\end{pmatrix} + s\begin{pmatrix}4\\2\\4\end{pmatrix}$$

$$\begin{pmatrix}4\\2\\4\end{pmatrix} \cdot \begin{pmatrix}6\\2\\-5\end{pmatrix} = \left|\begin{pmatrix}4\\2\\4\end{pmatrix}\right| \left|\begin{pmatrix}6\\2\\-5\end{pmatrix}\right| \cos\theta$$

$24 + 4 - 20 = \sqrt{36}\sqrt{65}\cos\theta$

$8 = 6\sqrt{65}\cos\theta$

$\therefore \cos\theta = \frac{4}{3\sqrt{65}}$

$\theta \cong 9.5°$

9. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

when $n=1$: $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

$= \cos 1\theta + i\sin 1\theta$

Assume the statement is true when $n=k$.

$n=k$: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

when $n=k+1$: $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

(what needs to be proved).

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$$
$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$
$$= \cos k\theta\cos\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta - \sin k\theta\sin\theta$$
$$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$
$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

for every integer $n \geqslant 1$.

If $z = e^{i\theta} = \cos\theta + i\sin\theta$

$z^{-1} = (\cos\theta + i\sin\theta)^{-1}$

$= \cos\theta - i\sin\theta$

$z + \frac{1}{z} = 2\cos\theta \qquad z - \frac{1}{z} = 2i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

$$(2i\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$$

$$32i\sin^5\theta = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$= 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

$$\sin^5\theta = \frac{\sin 5\theta}{16} - \frac{5\sin 3\theta}{16} + \frac{5\sin\theta}{8}$$

$$= p\sin 5\theta + q\sin 3\theta + r\sin\theta$$

$$p = \frac{1}{16} \quad q = -\frac{5}{16} \quad r = \frac{5}{8}$$

10. $r = 2\sin\theta(1-\cos\theta) \quad 0 \leq \theta \leq \pi.$

$$\frac{dr}{d\theta} = 2\cos\theta(1-\cos\theta) + 2\sin\theta\sin\theta$$
$$= 2\cos\theta - 2\cos^2\theta + 2\sin^2\theta$$
$$= 2\cos\theta - 2\cos^2\theta + 2(1-\cos^2\theta)$$
$$= 2 + 2\cos\theta - 4\cos^2\theta$$

when $\frac{dr}{d\theta} = 0$: $2 + 2\cos\theta - 4\cos^2\theta = 0$

$$1 + \cos\theta - 2\cos^2\theta = 0$$
$$2\cos^2\theta - \cos\theta - 1 = 0$$
$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$
$$\cos\theta = -\frac{1}{2}, 1$$
$$\theta = 0, \frac{2\pi}{3}$$

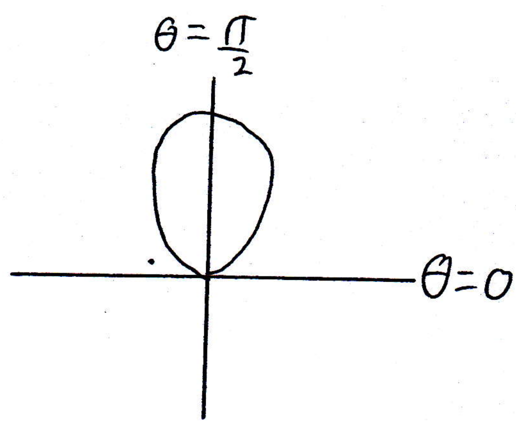
$$\frac{d^2r}{d\theta^2} = -2\sin\theta + 8\cos\theta\sin\theta$$

$\theta = 0$: $\frac{d^2r}{d\theta^2} = 0$

$\theta = \frac{2\pi}{3}$: $\frac{d^2r}{d\theta^2} = -3\sqrt{3} < 0$

$\therefore$ The furthest point from the pole is $\left(\frac{3\sqrt{3}}{2}, \frac{2\pi}{3}\right)$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	0	$\sqrt{2}-1$	2	$\sqrt{2}-1$	0



The ared from $\theta = 0$ to $\theta = \frac{\pi}{4}$ is

$$\int_0^{\frac{\pi}{4}} \frac{r^2}{2}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{4\sin^2\theta(1-\cos\theta)^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\sin^2\theta\,(1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\sin^2\theta - 4\sin^2\theta\cos\theta + 2\sin^2\theta\cos^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 1 - \cos 2\theta - 4\sin^2\theta\cos\theta + \frac{\sin^2 2\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 1 - \cos 2\theta - 4\sin^2\theta\cos\theta + \frac{1-\cos 4\theta}{4}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{5}{4} - \cos 2\theta - 4\sin^2\theta\cos\theta - \frac{\cos 4\theta}{4}\, d\theta$$

$$= \left[\frac{5\theta}{4} - \frac{\sin 2\theta}{2} - \frac{4\sin^3\theta}{3} - \frac{\sin 4\theta}{16}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{5\pi}{16} - \frac{1}{2} - \frac{4}{3}\frac{1}{2\sqrt{2}} - 0$$

$$= \frac{5\pi}{16} - \frac{1}{2} - \frac{\sqrt{2}}{3}.$$

11. EITHER

$I_n = \int_0^1 (1+x^2)^n dx$

$u = (1+x^2)^n \qquad dv = dx$

$du = n(1+x^2)^{n-1} 2x \, dx \qquad v = x$

$= [x(1+x^2)^n]_0^1 - \int_0^1 2nx^2(1+x^2)^{n-1} dx$

$= 2^n - 0 - 2n\int_0^1 (1+x^2-1)(1+x^2)^{n-1} dx$

$= 2^n - 2n\int_0^1 (1+x^2)^n - (1+x^2)^{n-1} dx$

$= 2^n - 2n\int_0^1 (1+x^2)^n dx + 2n\int_0^1 (1+x^2)^{n-1} dx$

$= 2^n - 2n I_n + 2n I_{n-1}$

$\therefore (2n+1) I_n = 2n I_{n-1} + 2^n$

$I_0 = \int_0^1 1 \, dx = [x]_0^1 = 1$

$n=3: 7I_3 = 6I_2 + 8$

$n=2: 5I_2 = 4I_1 + 4$

$n=1: 3I_1 = 2I_0 + 2$

Since $I_0 = 1$, $I_1 = \frac{4}{3}$

$5I_2 = \frac{28}{3}$

$I_2 = \frac{28}{15}$

$7I_3 = \frac{96}{5}$

$I_3 = \frac{96}{35}$

$I_{-1} = \frac{\pi}{4}$ $\quad n=-2: -3I_{-2} = -4I_{-3} + \frac{1}{4}$

$n=-1: \quad -I_{-1} = -2I_{-2} + \frac{1}{2}$

$$2I_{-2} = I_{-1} + \frac{1}{2}$$

$$= \frac{\pi}{4} + \frac{1}{2}$$

$$I_{-2} = \frac{\pi}{8} + \frac{1}{4}$$

$$4I_{-3} = 3I_{-2} + \frac{1}{4}$$

$$= 3\left(\frac{\pi}{8} + \frac{1}{4}\right) + \frac{1}{4}$$

$$= \frac{3\pi}{8} + 1$$

$$I_{-3} = \frac{3\pi}{32} + \frac{1}{4}$$

11. OR

$A\underset{\sim}{e} = \lambda \underset{\sim}{e} \qquad B\underset{\sim}{e} = \mu \underset{\sim}{e}$

$(A+B)\underset{\sim}{e} = A\underset{\sim}{e} + B\underset{\sim}{e} = \lambda\underset{\sim}{e} + \mu\underset{\sim}{e} = (\lambda+\mu)\underset{\sim}{e}$

$\therefore$ $A+B$ has an eigenvalue $\lambda+\mu$.

$$A = \begin{pmatrix} 6 & -1 & -6 \\ 1 & 0 & -2 \\ 3 & -1 & -3 \end{pmatrix}$$

If $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}$ are the eigenvectors,

$$\begin{pmatrix} 6 & -1 & -6 \\ 1 & 0 & -2 \\ 3 & -1 & -3 \end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix} = -1\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix} 6 & -1 & -6 \\ 1 & 0 & -2 \\ 3 & -1 & -3 \end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix} = 1\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix} 6 & -1 & -6 \\ 1 & 0 & -2 \\ 3 & -1 & -3 \end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}6\\0\\3\end{pmatrix} = 3\begin{pmatrix}2\\0\\1\end{pmatrix}$$

$\therefore$ The corresponding eigenvalues are $-1, 1, 3$.

$B = \begin{pmatrix} 8 & -2 & -8 \\ 2 & 0 & -4 \\ 4 & -2 & -4 \end{pmatrix}$ has eigenvectors $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}$

with corresponding eigenvalues $-2, 2, 4$.

$M = A + B - 5I$

$M\underset{\sim}{e} = (A+B-5I)\underset{\sim}{e} = A\underset{\sim}{e} + B\underset{\sim}{e} - 5I\underset{\sim}{e} = \lambda\underset{\sim}{e} + \mu\underset{\sim}{e} - 5\underset{\sim}{e}$

$= (\lambda + \mu - 5)\underset{\sim}{e}$.

$\therefore$ M has eigenvalues $-8, -2, 2$.

If $M^5 = RDS$

$$\therefore D = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^5 = \begin{pmatrix} (-8)^5 & 0 & 0 \\ 0 & (-2)^5 & 0 \\ 0 & 0 & 2^5 \end{pmatrix} = \begin{pmatrix} -32768 & 0 & 0 \\ 0 & -32 & 0 \\ 0 & 0 & 32 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and $S = R^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$

$$= \frac{1}{2} \begin{pmatrix} -1 & 1 & 2 \\ -1 & -1 & 2 \\ 2 & 0 & -2 \end{pmatrix}$$