Oct/Nov 2011 (9231/11)

3 Prove by mathematical induction that, for all positive integers n,

/ done

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \sin x) = 2^{\frac{1}{2}n}\mathrm{e}^x \sin\left(x + \tfrac{1}{4}n\pi\right).$$ [7]

Oct/Nov 2011 (9231/13)

2 Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{2x+3}\right) = (-1)^n \frac{n!2^n}{(2x+3)^{n+1}}.$$ [6]

May/June 2011 (9231/11)

4 It is given that $\mathrm{f}(n) = 3^{3n} + 6^{n-1}$.

(i) Show that $\mathrm{f}(n+1) + \mathrm{f}(n) = 28(3^{3n}) + 7(6^{n-1})$. [2]

(ii) Hence, or otherwise, prove by mathematical induction that $\mathrm{f}(n)$ is divisible by 7 for every positive integer n. [4]

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2 Let $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Prove by mathematical induction that, for every positive integer n,

Skip

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}.$$ [5]

Oct/Nov 2010 (9231/01)

4 Prove by mathematical induction that, for all non-negative integers n, $7^{2n+1} + 5^{n+3}$ is divisible by 44. [5]

May/June 2010 (9231/11)

5 Let

$$I_n = \int_1^{\mathrm{e}} x(\ln x)^n \,\mathrm{d}x,$$

where $n \geqslant 1$. Show that

$$I_{n+1} = \tfrac{1}{2}\mathrm{e}^2 - \tfrac{1}{2}(n+1)I_n.$$ [3]

Hence prove by induction that, for all positive integers n, I_n is of the form $A_n\mathrm{e}^2 + B_n$, where A_n and B_n are rational numbers. [6]

May/June 2010 (9231/13)

3 The sequence $x_1, x_2, x_3, \ldots$ is such that $x_1 = 3$ and

$$x_{n+1} = \frac{2x_n^2 + 4x_n - 2}{2x_n + 3}$$

for $n = 1, 2, 3, \ldots$. Prove by induction that $x_n > 2$ for all n. [6]

Oct/Nov 2009 (9231/01)

11 Answer only **one** of the following two alternatives.

EITHER

Prove by induction that

$$\sum_{n=1}^{N} n^3 = \tfrac{1}{4}N^2(N+1)^2. \qquad [5]$$

Use this result, together with the formula for $\sum_{n=1}^{N} n^2$, to show that

$$\sum_{n=1}^{N}(20n^3 + 36n^2) = N(N+1)(N+3)(5N+2). \qquad [3]$$

Let

$$S_N = \sum_{n=1}^{N}(20n^3 + 36n^2 + \mu n).$$

Find the value of the constant μ such that S_N is of the form $N^2(N+1)(aN+b)$, where the constants a and b are to be determined. [3]

Show that, for this value of μ,

$$5 + \frac{22}{N} < N^{-4}S_N < 5 + \frac{23}{N},$$

for all $N \geqslant 18$. [3]

May/June 2009 (9231/01)

7 Let

$$I_n = \int_0^1 t^n e^{-t}\,dt,$$

where $n \geqslant 0$. Show that, for all $n \geqslant 1$,

$$I_n = nI_{n-1} - e^{-1}. \qquad [3]$$

Hence prove by induction that, for all positive integers n,

$$I_n < n!. \qquad [5]$$

Oct/Nov 2008 (9231/01)

9 Use induction to prove that

$$\sum_{n=1}^{N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(N+1)(2N+1)}. \qquad [6]$$

Show that

$$\sum_{n=N+1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} < \frac{3}{8N^2}. \qquad [4]$$

May/June 2008 (9231/01)

7 Prove by induction that

$$\sum_{r=1}^{n} \left(3r^5 + r^3\right) = \tfrac{1}{2}n^3(n+1)^3,$$

for all $n \geqslant 1$. [5]

Use this result together with the List of Formulae (MF10) to prove that

$$\sum_{r=1}^{n} r^5 = \tfrac{1}{12}n^2(n+1)^2\mathrm{Q}(n),$$

where $\mathrm{Q}(n)$ is a quadratic function of n which is to be determined. [3]

Oct/Nov 2007 (9231/01)

3 Prove by induction that, for all $n \geqslant 1$,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{x^2}\right) = \mathrm{P}_n(x)\mathrm{e}^{x^2},$$

where $\mathrm{P}_n(x)$ is a polynomial in x of degree n with the coefficient of x^n equal to 2^n. [6]

May/June 2007 (9231/01)

8 The sequence $x_1, x_2, x_3, \ldots$ is such that $x_1 = 1$ and

$$x_{n+1} = \frac{1 + 4x_n}{5 + 2x_n}.$$

Prove by induction that $x_n > \frac{1}{2}$ for all $n \geqslant 1$. [5]

Prove also that $x_n > x_{n+1}$ for all $n \geqslant 1$. [3]

Oct/Nov 2006 (9231/01)

4 Prove by mathematical induction that, for all positive integers n, $10^{3n} + 13^{n+1}$ is divisible by 7. [5]

May/June 2006 (9231/01)

3 Prove by induction, or otherwise, that

$$23^{2n} + 31^{2n} + 46$$

is divisible by 48, for all integers $n \geqslant 0$. [6]

Oct/Nov 2005 (9231/01)

2 The sequence $u_1, u_2, u_3, \dots$ is such that $u_1 = 1$ and

$$u_{n+1} = -1 + \sqrt{(u_n + 7)}.$$

(i) Prove by induction that $u_n < 2$ for all $n \geqslant 1$. [4]

(ii) Show that if $u_n = 2 - \varepsilon$, where ε is small, then

$$u_{n+1} \approx 2 - \tfrac{1}{6}\varepsilon.$$ [2]

May/June 2005 (9231/01)

8 The integral I_n, where n is a non-negative integer, is defined by

$$I_n = \int_0^1 e^{-x}(1-x)^n \, dx.$$

(i) Show that $I_{n+1} = 1 - (n+1)I_n$. [3]

(ii) Use induction to show that I_n is of the form $A_n + B_n e^{-1}$, where A_n and B_n are integers. [4]

(iii) Express B_n in terms of n. [2]

Oct/Nov 2004 (9231/01)

8 The sequence of real numbers $a_1, a_2, a_3, \dots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^{\lambda},$$

where λ is a constant greater than 1. Prove by mathematical induction that, for $n \geqslant 2$,

$$a_n \geqslant 2^{g(n)},$$

where $g(n) = \lambda^{n-1}$. [6]

Prove also that, for $n \geqslant 2$, $\dfrac{a_{n+1}}{a_n} > 2^{(\lambda-1)g(n)}$. [3]

May/June 2004 (9231/01)

4 It is given that

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{a_n \ln x + b_n}{x^{n+1}},$$

where a_n and b_n depend only on n.

(i) Find a_1, a_2 and a_3. [3]

(ii) Use mathematical induction to establish a formula for a_n. [5]

Oct/Nov 2003 (9231/01)

4 Given that $y = x \sin x$, find $\frac{d^2y}{dx^2}$ and $\frac{d^4y}{dx^4}$, simplifying your results as far as possible, and show that

$$\frac{d^6y}{dx^6} = -x \sin x + 6 \cos x.$$ [3]

Use induction to establish an expression for $\frac{d^{2n}y}{dx^{2n}}$, where n is a positive integer. [5]

May/June 2003 (9231/01)

2 Prove by induction that, for all $N \geqslant 1$,

$$\sum_{n=1}^{N} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}.$$ [5]

Oct/Nov 2002 (9231/01)

3 It is given that, for $n = 0, 1, 2, 3, \ldots,$

$$a_n = 17^{2n} + 3(9)^n + 20.$$

Simplify $a_{n+1} - a_n$, and hence prove by induction that a_n is divisible by 24 for all $n \geqslant 0$. [6]

May/June 2002 (9231/01)

6 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 4$ and

$$u_{n+1} = \frac{5u_n + 4}{u_n + 2}.$$

By considering $4 - u_{n+1}$, or otherwise, prove by induction that $u_n < 4$ for all $n \geqslant 1$. [5]

Prove also that $u_{n+1} > u_n$ for all $n \geqslant 1$. [3]