$$\underset{\sim}{x} = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + \lambda \underset{\sim}{e}_1 + \mu \underset{\sim}{e}_2 \text{ is a solution}$$

of $M\underset{\sim}{x} = \begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix}$.

iv) If $\begin{pmatrix} \alpha \\ 0 \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 14 \\ 1 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 19 \\ 2 \\ 0 \\ -3 \end{pmatrix}$

$$= \begin{pmatrix} 4 + 14\lambda + 19\mu \\ -3 + \lambda + 2\mu \\ -3\lambda \\ -3\mu \end{pmatrix}$$

$\alpha = 4 + 14\lambda + 19\mu$, $0 = -3 + \lambda + 2\mu$,

$\gamma = -3\lambda$, $\delta = -3\mu$.

$\therefore \lambda = \frac{-\gamma}{3}$, $\mu = \frac{-\delta}{3}$.

$\alpha = 4 - \frac{14\gamma}{3} - \frac{19\delta}{3}$

$\therefore$ A solution of the form $\begin{pmatrix} \alpha \\ 0 \\ \gamma \\ \delta \end{pmatrix}$ is

$$\begin{pmatrix} 4 - \frac{14\gamma}{3} - \frac{19\delta}{3} \\ 0 \\ \frac{-\gamma}{3} \\ \frac{-\delta}{3} \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} \frac{-14}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix} + \delta \begin{pmatrix} \frac{-19}{3} \\ 0 \\ 0 \\ -\frac{1}{3} \end{pmatrix}$$

$= \begin{pmatrix} 37 \\ 0 \\ -3 \\ -3 \end{pmatrix}$ when $\lambda = 1, \mu = 1,$
$\alpha = 37, \gamma = -3, \delta = -3$

1. $x^2 + y^2 + \left(\frac{dy}{dx}\right)^3 = 29$, $x = 1$ $y = -1$

$x = 1$ $y = -1$: $1^2 + (-1)^2 + \left(\frac{dy}{dx}\right)^3 = 29$

$$1 + 1 + \left(\frac{dy}{dx}\right)^3 = 29$$

$$\left(\frac{dy}{dx}\right)^3 = 27$$

$$\therefore \frac{dy}{dx} = 3$$

$$\frac{d}{dx}\left(x^2 + y^2 + \left(\frac{dy}{dx}\right)^3\right) = \frac{d}{dx}(29)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}\left(\frac{dy}{dx}\right)^3 = 0$$

$$2x + 2y\frac{dy}{dx} + 3\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} = 0$$

$x = 1$ $y = -1$, $\frac{dy}{dx} = 3$:

$$2(1) + 2(-1)3 + 3(3)^2 \frac{d^2y}{dx^2} = 0$$

$$2 - 6 + 27\frac{d^2y}{dx^2} = 0$$

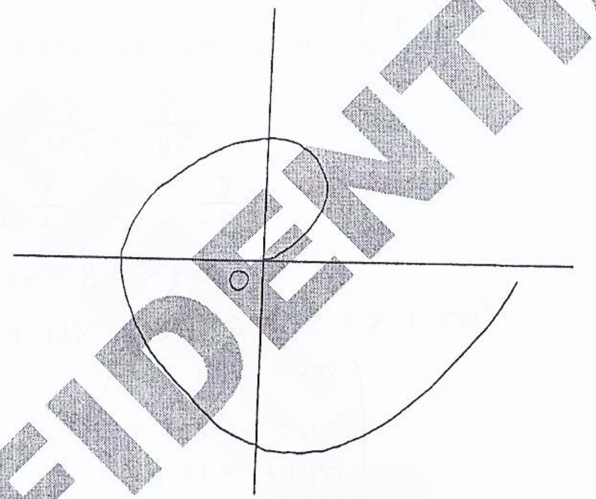
$$27\frac{d^2y}{dx^2} = 4$$

$$\therefore \frac{d^2y}{dx^2} = \frac{4}{27}$$

2. $C: r = a(1 - e^{-\theta}),\ a > 0\quad 0 \le \theta < 2\pi.$

i)

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	0	$a(1 - e^{-\frac{\pi}{2}})$	$a(1 - e^{-\pi})$	$a(1 - e^{-\frac{3\pi}{2}})$



ii) The area bounded by C and the lines $\theta = \ln 2$ and $\theta = \ln 4$ is

$$\int_{\ln 2}^{\ln 4} \frac{r^2}{2}\, d\theta$$

$$= \int_{\ln 2}^{\ln 4} \frac{a^2(1 - e^{-\theta})^2}{2}\, d\theta$$

$$= \frac{a^2}{2} \int_{\ln 2}^{\ln 4} 1 - 2e^{-\theta} + e^{-2\theta}\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + 2e^{-\theta} - \frac{e^{-2\theta}}{2}\right]_{\ln 2}^{\ln 4}$$

$$= \frac{a^2}{2}\left(\ln 4 + 2\left(\frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{16}\right) - \ln 2 - 2\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{4}\right)\right)$$

$$= \frac{a^2}{2}\left(2\ln 2 + \frac{1}{2} - \frac{1}{32} - \ln 2 - 1 + \frac{1}{8}\right)$$

$$= \frac{a^2}{2}\left(\ln 2 - \frac{13}{32}\right)$$

3. C : $\frac{dx}{dt} = t\sqrt{t^2+4}$ $\quad \frac{dy}{dt} = -t\sqrt{4-t^2}$, $0 \le t \le 2$.

The arc length of C from $t=0$ to $t=2$ is

$$\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^2 \sqrt{t^2(t^2+4) + t^2(4-t^2)}\, dt$$

$$= \int_0^2 \sqrt{t^3 + 4t^2 + 4t^2 - t^3}\, dt$$

$$= \int_0^2 \sqrt{8t^2}\, dt$$

$$= \int_0^2 2\sqrt{2}t\, dt$$

$$= \left[\sqrt{2}t^2\right]_0^2$$

$$= \sqrt{2}(4-0)$$

$$= 4\sqrt{2}$$

$$y = \int -t\sqrt{4-t^2}\, dt$$

$u = 4 - t^2$

$du = -2t\, dt$

$$= \int \frac{\sqrt{u}}{2}\, du$$

$$= \frac{u^{\frac{3}{2}}}{3} + c$$

$$= \frac{(4-t^2)^{\frac{3}{2}}}{3} + c$$

$t = 2$ $y = 0$: $0 = c$

$$\therefore y = \frac{(4-t^2)^{\frac{3}{2}}}{3}$$

The surface area of revolution from $t=0$ to $t=2$ about the x-axis is

$$\int_0^2 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^2 2\pi \frac{(4-t^2)^{\frac{3}{2}}}{3} 2\sqrt{2}t\, dt$$

$$= \frac{4\sqrt{2}\pi}{3}\int_0^2 t\,(4-t^2)^{\frac{3}{2}}\,dt$$

$u = 4 - t^2$

$du = -2t\,dt$

$t = 0 \quad u = 4$

$t = 2 \quad u = 0$

$$= \frac{4\sqrt{2}\pi}{3}\int_4^0 -\frac{u^{\frac{3}{2}}}{2}\,du$$

$$= \frac{2\sqrt{2}\pi}{3}\int_0^4 u^{\frac{3}{2}}\,du$$

$$= \frac{2\sqrt{2}\pi}{3}\left[\frac{2u^{\frac{5}{2}}}{5}\right]_0^4$$

$$= \frac{4\sqrt{2}\pi}{15}\left(4^{\frac{5}{2}} - 0\right)$$

$$= \frac{4\sqrt{2}\pi}{15}(32)$$

$$= \frac{128\sqrt{2}\pi}{15}$$

4. $S_N = \sum_{n=1}^{N} n^5$

$$\left(n+\tfrac{1}{2}\right)^6 - \left(n-\tfrac{1}{2}\right)^6 = 6n^5 + 5n^3 + \frac{3n}{8}$$

$$\sum_{n=1}^{N} \left(n+\tfrac{1}{2}\right)^6 - \left(n-\tfrac{1}{2}\right)^6 = \sum_{n=1}^{N} 6n^5 + 5n^3 + \frac{3n}{8}$$

$$\left(\tfrac{3}{2}\right)^6 - \left(\tfrac{1}{2}\right)^6 \qquad = 6\sum_{n=1}^{N} n^5 + 5\sum_{n=1}^{N} n^3 + \frac{3}{8}\sum_{n=1}^{N} n$$

$$+\left(\tfrac{5}{2}\right)^6 - \left(\tfrac{3}{2}\right)^6$$

$$+\left(\tfrac{7}{2}\right)^6 - \left(\tfrac{5}{2}\right)^6$$

$$\vdots$$

$$+\left(N-\tfrac{3}{2}\right)^6 - \left(N-\tfrac{5}{2}\right)^6$$

$$+\left(N-\tfrac{1}{2}\right)^6 - \left(N-\tfrac{3}{2}\right)^6$$

$$+\left(N+\tfrac{1}{2}\right)^6 - \left(N-\tfrac{1}{2}\right)^6$$

$$\left(N+\tfrac{1}{2}\right)^6 - \left(\tfrac{1}{2}\right)^6 = 6\sum_{n=1}^{N} n^5 + \frac{5N^2(N+1)^2}{4} + \frac{3N(N+1)}{16}$$

$$6\sum_{n=1}^{N} n^5 = \frac{(2N+1)^6}{64} - \frac{1}{64} - \frac{5N^2(N+1)^2}{4} - \frac{3N(N+1)^6}{16}$$

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$$\therefore \sum_{n=1}^{N} n^5 = \frac{(2N+1)^6}{384} - \frac{1}{384} - \frac{5N^2(N+1)^2}{24} - \frac{3N(N+1)}{96}$$

$$S_N = \frac{(2N+1)^6}{384} - \frac{1}{384} - \frac{5N^2(N+1)^2}{24} - \frac{3N(N+1)}{96}$$

$$N^{-\lambda} S_N = \frac{(2N+1)^6}{384N^\lambda} - \frac{1}{384N^\lambda} - \frac{5N^2(N+1)^2}{24N^\lambda} - \frac{3N(N+1)}{96N^\lambda}$$

i) $\lambda = 6$:

$$N^{-6} S_N = \frac{(2N+1)^6}{384N^6} - \frac{1}{384N^6} - \frac{5N^2(N+1)^2}{24N^6} - \frac{3N(N+1)}{96N^6}$$

$$= \frac{1}{384}\left(2+\frac{1}{N}\right)^6 - \frac{1}{384N^6} - \frac{5(N+1)^2}{24N^4} - \frac{(N+1)}{32N^5}$$

$$\lim_{N\to\infty} N^{-6} S_N = \frac{2^6}{384}$$

$$= \frac{64}{384}$$

$$= \frac{1}{6}$$

ii) $\lambda > 6$:

$$\lim_{N\to\infty} N^{-\lambda} S_N = 0.$$

5. $I_n = \int_1^e x(\ln x)^n \, dx, \quad n \geq 1.$

$$I_{n+1} = \int_1^e x(\ln x)^{n+1} \, dx$$

$u = (\ln x)^{n+1}$ $\qquad dv = x\,dx$

$du = \frac{(n+1)(\ln x)^n}{x} dx$ $\qquad v = \frac{x^2}{2}$

$$= \left[\frac{x^2(\ln x)^{n+1}}{2}\right]_1^e - \int_1^e \frac{(n+1)x^2(\ln x)^n dx}{2x}$$

$$= \frac{e^2(\ln e)^{n+1}}{2} - \frac{1^2(\ln 1)^{n+1}}{2} - \frac{(n+1)}{2}\int_1^e x(\ln x)^n dx$$

$$= \frac{e^2}{2} - \frac{(n+1)I_n}{2}$$

$I_n = A_n e^2 + B_n$, A_n, B_n are rational numbers.

when $n=1$: $I_1 = \int_1^e x \ln x \, dx$

$u = \ln x$ $\qquad dv = x\,dx$

$du = \frac{1}{x} dx$ $\qquad v = \frac{x^2}{2}$

$$= \left[\frac{x^2 \ln x}{2}\right]_1^e - \int_1^e \frac{x}{2} dx$$

$$= \frac{e^2 \ln e}{2} - \frac{1^2 \ln 1}{2} - \left[\frac{x^2}{4}\right]_1^e$$

$$= \frac{e^2}{2} - 0 - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{e^2+1}{4}$$

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$= A_1 e^2 + B_1$, $A_1 = \frac{1}{4}$, $B_1 = \frac{1}{4}$

Assume the statement is true when $n = k$.

$n = k$: $I_k = A_k e^2 + B_k$, A_k, B_k are rational numbers.

when $n = k+1$: $I_{k+1} = \frac{e^2}{2} - \frac{(k+1) I_k}{2}$

$$= \frac{e^2}{2} - \frac{(k+1)(A_k e^2 + B_k)}{2}$$

$$= \frac{e^2}{2} - \frac{(k+1) A_k e^2}{2} - \frac{(k+1) B_k}{2}$$

$$= \left(\frac{1}{2} - \frac{(k+1) A_k}{2}\right) e^2 - \frac{(k+1) B_k}{2}$$

$$= A_{k+1} e^2 + B_{k+1} ,$$

$$A_{k+1} = \frac{1}{2} - \frac{(k+1) A_k}{2} ,$$

$$B_{k+1} = -\frac{(k+1) B_k}{2} .$$

$\therefore I_n = A_n e^2 + B_n$, A_n, B_n are rational numbers for every positive integer n.

6. $x^3 + x - 1 = 0$

α, β, γ are the roots.

$x = \sqrt{y}$

$(\sqrt{y})^3 + \sqrt{y} - 1 = 0$

$y\sqrt{y} + \sqrt{y} - 1 = 0$

$\sqrt{y}(y+1) = 1$

$(\sqrt{y}(y+1))^2 = 1^2$

$y(y+1)^2 = 1$

$y(y^2 + 2y + 1) = 1$

$y^3 + 2y^2 + y = 1$

$y^3 + 2y^2 + y - 1 = 0$

$\therefore$ The equation $y^3 + 2y^2 + y - 1 = 0$ has roots $\alpha^2, \beta^2, \gamma^2$.

$S_n = \alpha^n + \beta^n + \gamma^n$

$\alpha^2 + \beta^2 + \gamma^2 = -2$

$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = 1$

$\alpha^2\beta^2\gamma^2 = 1$

i) $S_2 = \alpha^2 + \beta^2 + \gamma^2$

$= -2$

$S_4 = \alpha^4 + \beta^4 + \gamma^4$

$= (\alpha^2 + \beta^2 + \gamma^2)^2 - 2(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)$

$= (-2)^2 - 2(1)$

$= 4 - 2$

$= 2$

Since $y^3 + 2y^2 + y - 1 = 0$ has roots $\alpha^2, \beta^2, \gamma^2$, if $T_n = \alpha^{2n} + \beta^{2n} + \gamma^{2n}$,

$T_n = S_{2n}$

$T_{3+r} + 2T_{2+r} + T_{1+r} - T_r = 0$

$r = 0$: $T_3 + 2T_2 + T_1 - T_0 = 0$

$T_0 = (\alpha^2)^0 + (\beta^2)^0 + (\gamma^2)^0$

$= 1 + 1 + 1$

$= 3$

$T_1 = (\alpha^2)^1 + (\beta^2)^1 + (\gamma^2)^1$

$= \alpha^2 + \beta^2 + \gamma^2$

$= S_2$

$= -2$

$T_2 = (\alpha^2)^2 + (\beta^2)^2 + (\gamma^2)^2$

$= \alpha^4 + \beta^4 + \gamma^4$

$= S_4$

$= 2$

$T_3 + 2(2) + (-2) - 3 = 0$

$T_3 + 4 - 2 - 3 = 0$

$T_3 = 1$

$\therefore S_6 = 1$

$r = 1$: $T_4 + 2T_3 + T_2 - T_1 = 0$

$T_4 + 2(1) + 2 - (-2) = 0$

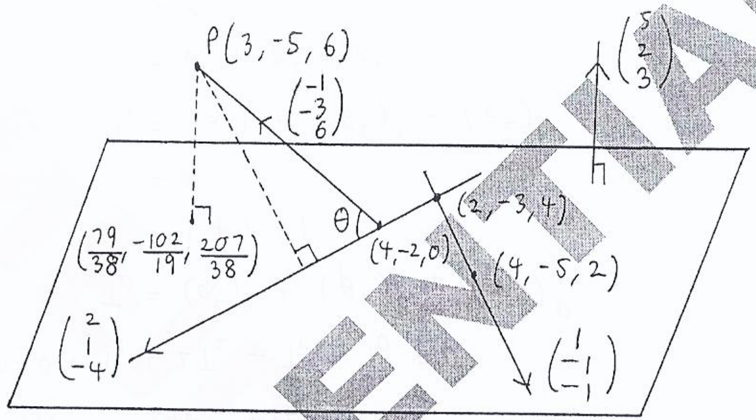
$T_4 + 2 + 2 + 2 = 0$

$T_4 = -6$

$\therefore S_8 = -6$

7. $l_1: \underset{\sim}{r} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + \lambda(2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k})$

$l_2: \underset{\sim}{r} = 4\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k} + \mu(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k})$



i) $\begin{pmatrix} 4+2\lambda \\ -2+\lambda \\ -4\lambda \end{pmatrix} = \begin{pmatrix} 4+\mu \\ -5-\mu \\ 2-\mu \end{pmatrix}$

$$\left.\begin{aligned} 4+2\lambda &= 4+\mu \\ -2+\lambda &= -5-\mu \\ -4\lambda &= 2-\mu \end{aligned}\right\}$$

$$\left.\begin{aligned} \mu - 2\lambda &= 0 \\ \mu + \lambda &= -3 \\ \mu - 4\lambda &= 2 \end{aligned}\right\}$$

$$\left.\begin{aligned} -①+②: \quad \mu - 2\lambda &= 0 \\ -①+③: \quad 3\lambda &= -3 \\ -2\lambda &= 2 \end{aligned}\right\}$$

$\lambda = -1$

$\mu = -2$

$\therefore l_1$ and l_2 intersect at $(2, -3, 4)$.

ii) Since the plane contains l_1 and l_2, the normal of the plane is perpendicular to both $\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$\therefore$ The normal of the plane is parallel to $\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 1 & -4 \\ 1 & -1 & -1 \end{vmatrix} = \begin{pmatrix} -5 \\ -2 \\ -3 \end{pmatrix}.$$

Since $\begin{pmatrix} -5 \\ -2 \\ -3 \end{pmatrix}$ is normal to the plane and $(2, -3, 4)$ is a point on the plane, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a point on the plane,

$$\underset{\sim}{r} \cdot \begin{pmatrix} -5 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -2 \\ -3 \end{pmatrix} = -10 + 6 - 12$$

$$-5x - 2y - 3z = -16$$

$$5x + 2y + 3z = 16$$

$\therefore$ The plane containing l_1 and l_2 has equation $5x + 2y + 3z = 16$.

Since $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ is normal to the plane,

the line passing through $P(3, -5, 6)$ and perpendicular to the plane has equation

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -5 \\ 6 \end{pmatrix} + s\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}.$$

When the line $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -5 \\ 6 \end{pmatrix} + s\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ meets the plane $5x + 2y + 3z = 16$,

$$x = 3 + 5s, \quad y = -5 + 2s, \quad z = 6 + 3s$$

$$5(3 + 5s) + 2(-5 + 2s) + 3(6 + 3s) = 16$$

$$15 + 25s - 10 + 4s + 18 + 9s = 16$$

$$38s = -7$$

$$s = \frac{-7}{38}.$$

$\therefore$ The line meets the plane at $\left(\frac{79}{38}, \frac{-102}{19}, \frac{207}{38}\right)$

$\therefore$ The distance from P to the line containing l_1 and l_2 is

$$\sqrt{\left(\frac{79}{38} - 3\right)^2 + \left(\frac{-102}{19} - (-5)\right)^2 + \left(\frac{207}{38} - 6\right)^2}$$

$$= \sqrt{\left(-\frac{35}{38}\right)^2 + \left(\frac{-7}{19}\right)^2 + \left(\frac{-21}{38}\right)^2}$$

$$= \sqrt{\frac{1225}{1444} + \frac{196}{1444} + \frac{441}{1444}}$$

$$= \sqrt{\frac{49}{38}}$$

$$= \frac{7}{\sqrt{38}}$$

iii) The perpendicular distance from P to l_1 is

$$\left|\begin{pmatrix} -1 \\ -3 \\ 6 \end{pmatrix}\right| \sin\theta$$

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$$= \frac{\left| \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \times \begin{pmatrix} -1 \\ -3 \\ 6 \end{pmatrix} \right|}{\left| \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \right|}$$

$$= \frac{\left| \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 1 & -4 \\ -1 & -3 & 6 \end{vmatrix} \right|}{\left| \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \right|}$$

$$= \frac{\left| \begin{pmatrix} -6 \\ -8 \\ -5 \end{pmatrix} \right|}{\left| \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \right|}$$

$$= \frac{\sqrt{(-6)^2 + (-8)^2 + (-5)^2}}{\sqrt{2^2 + 1^2 + (-4)^2}}$$

$$= \frac{\sqrt{36 + 64 + 25}}{\sqrt{4 + 1 + 16}}$$

$$= \frac{\sqrt{125}}{\sqrt{21}}$$

$$\approx 2.44$$

8. $A = \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix}$

$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, where λ is an eigenvalue with corresponding eigenvector $\underset{\sim}{x}$.

If $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ is an eigenvector of A,

$$A\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$\therefore$ The corresponding eigenvalue is 3.

If 4 is an eigenvalue of A with corresponding eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x + y - z \\ -4x - y + 4z \\ -y + 5z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

$$\left.\begin{aligned} 4x + y - z &= 4x \\ -4x - y + 4z &= 4y \\ -y + 5z &= 4z \end{aligned}\right\}$$

$$\left.\begin{aligned} y - z &= 0 \\ 4x + 5y - 4z &= 0 \\ y - z &= 0 \end{aligned}\right\}$$

Let $z = 4s$, $s \in R$

$y = 4s$

$x = -s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 4s \\ 4s \end{pmatrix} = s\begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}.$$

If $\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$ is an eigenvector of A with corresponding eigenvalue 1 and $A^5 = PD^5Q$, where P is a square matrix and D is a diagonal matrix, $Q = P^{-1}$.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}^5 = \begin{pmatrix} 1^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 4^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 243 & 0 \\ 0 & 0 & 1024 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 & 1 & -1 \\ -4 & -2 & 4 \\ -1 & -1 & 4 \end{pmatrix}$$

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$$|P| = 1(-8+4) - 1(-16+4) - 1(4-2)$$
$$= -4 + 12 - 2$$
$$= 6$$

$$P^T = \begin{pmatrix} 1 & -4 & -1 \\ 1 & -2 & -1 \\ -1 & 4 & 4 \end{pmatrix}$$

$$\text{adj } P = \begin{pmatrix} -4 & -3 & 2 \\ 12 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|}\text{adj } P = \frac{1}{6}\begin{pmatrix} -4 & -3 & 2 \\ 12 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} & -\frac{1}{2} & \frac{1}{3} \\ 2 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

$$\therefore Q = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{2} & \frac{1}{3} \\ 2 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

9. i) $1^{\frac{1}{5}} = (\cos 0 + i\sin 0)^{\frac{1}{5}}$

$$= (\cos(0+2k\pi) + i\sin(0+2k\pi))^{\frac{1}{5}}, \quad k \in \mathbb{Z}$$
$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{5}}$$
$$= \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}, \quad k = 0,1,2,3,4$$
$$= e^{\frac{2k\pi i}{5}}$$
$$= 1, e^{\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}}$$

ii) $z^5 + 16 + 16\sqrt{3}i = 0$

$$z^5 = -16 - 16\sqrt{3}i$$
$$= 32\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$$
$$= 32\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$
$$= 32\left(\cos\left(\frac{4\pi}{3} + 2k\pi\right) + i\sin\left(\frac{4\pi}{3} + 2k\pi\right)\right),$$
$$k \in \mathbb{Z}$$

$$z = \left(32\left(\cos\left(\frac{4\pi}{3} + 2k\pi\right) + i\sin\left(\frac{4\pi}{3} + 2k\pi\right)\right)\right)^{\frac{1}{5}}$$

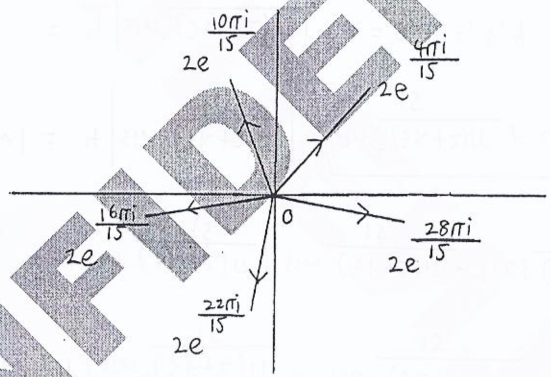
$$= 2\left(\cos\left(\frac{4\pi}{15} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{4\pi}{15} + \frac{2k\pi}{5}\right)\right),$$

$$k = 0, 1, 2, 3, 4.$$

$$= 2e^{\left(\frac{4\pi}{15} + \frac{2k\pi}{5}\right)i}, \quad k = 0, 1, 2, 3, 4$$

$$= 2e^{\frac{2(3k+2)\pi i}{15}}, \quad k = 0, 1, 2, 3, 4$$

$$= 2e^{\frac{4\pi i}{15}},\ 2e^{\frac{10\pi i}{15}},\ 2e^{\frac{16\pi i}{15}},\ 2e^{\frac{22\pi i}{15}},\ 2e^{\frac{28\pi i}{15}}$$



If w is a root of the equation,

$w^5 = -16 - 16\sqrt{3}i$

iii) $$\sum_{k=0}^{4}\left(\frac{w}{2}\right)^k = 1 + \frac{w}{2} + \frac{w^2}{2^2} + \frac{w^3}{2^3} + \frac{w^4}{2^4}$$

$$= \frac{1\left(1 - \left(\frac{w}{2}\right)^5\right)}{1 - \frac{w}{2}}$$

$$= \frac{2(2^5 - w^5)}{32(2 - w)}$$

$$= \frac{32 - w^5}{16(2 - w)}$$

$$= \frac{32 - (-16 - 16\sqrt{3}i)}{16(2 - w)}$$

$$= \frac{48 + 16\sqrt{3}i}{16(2 - w)}$$

$$= \frac{3 + \sqrt{3}i}{2 - w}$$

iv) $$w = 2e^{\frac{2(3k+2)\pi i}{15}}$$

$$= 2\left(\cos\frac{2(3k+2)\pi}{15} + i\sin\frac{2(3k+2)\pi}{15}\right)$$

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$$z - w = z - 2\cos\frac{2(3k+2)\pi}{15} - 2i\sin\frac{2(3k+2)\pi}{15}$$

$$= 2\left(1 - \cos\frac{2(3k+2)\pi}{15} - i\sin\frac{2(3k+2)\pi}{15}\right)$$

$$= 2\left(1 - 1 + 2\sin^2\frac{(3k+2)\pi}{15} - 2i\sin\frac{(3k+2)\pi}{15}\cos\frac{(3k+2)\pi}{15}\right)$$

$$= 4\left(\sin^2\frac{(3k+2)\pi}{15} - i\sin\frac{(3k+2)\pi}{15}\cos\frac{(3k+2)\pi}{15}\right)$$

$$= 4\sin\frac{(3k+2)\pi}{15}\left(\sin\frac{(3k+2)\pi}{15} - i\cos\frac{(3k+2)\pi}{15}\right)$$

$$|z - w| = 4\left|\sin\frac{(3k+2)\pi}{15}\right|\sqrt{\sin^2\frac{(3k+2)\pi}{15} + \cos^2\frac{(3k+2)\pi}{15}}$$

$$= 4\left|\sin\frac{(3k+2)\pi}{15}\right|, \quad k = 0, 1, 2, 3, 4$$

$$= 4\left|\sin\frac{2\pi}{15}\right|,\ 4\left|\sin\frac{5\pi}{15}\right|,\ 4\left|\sin\frac{8\pi}{15}\right|,\ 4\left|\sin\frac{11\pi}{15}\right|,\ 4\left|\sin\frac{14\pi}{15}\right|.$$

$$= 4\sin\frac{2\pi}{15},\ 4\sin\frac{5\pi}{15},\ 4\sin\frac{7\pi}{15},\ 4\sin\frac{4\pi}{15},\ 4\sin\frac{\pi}{15}$$

$\therefore |z - w|_{min} = 4\sin\frac{\pi}{15}$ when $w = 2e^{\frac{28\pi i}{15}}$

10. $x + 4y + 12z = 5$

$2x + ay + 12z = a - 1$

$3x + 12y + 2az = 10$

$$\left(\begin{array}{ccc|c} 1 & 4 & 12 & 5 \\ 2 & a & 12 & a-1 \\ 3 & 12 & 2a & 10 \end{array}\right)$$

$$\xrightarrow[-3r_1 + r_3]{-2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 4 & 12 & 5 \\ 0 & a-8 & -12 & a-11 \\ 0 & 0 & 2a-36 & -5 \end{array}\right)$$

$x + 4y + 12z = 5$

$(a-8)y - 12z = a - 11$

$(2a-36)z = -5$

$\therefore$ If $a \neq 8$ and $a \neq 18$, the system has a unique solution.

If $a = 18$: $0z = -5$

no solution.

If $a = 8$: $x + 4y + 12z = 5$

$0y - 12z = -3$

$-20z = -5$

$z = \frac{1}{4}$

Let $y = s, s \in R$

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$$x + 4s + 3 = 5$$
$$x = 2 - 4s$$

If $x + y + z = 1$,

$$2 - 4s + s + \frac{1}{4} = 1$$
$$3s = \frac{5}{4}$$
$$s = \frac{5}{12}$$

$$\therefore x = \frac{1}{3},\ y = \frac{5}{12},\ z = \frac{1}{4}$$

11. EITHER

$$3z^2\frac{d^2z}{dx^2} + 6z^2\frac{dz}{dx} + 6z\left(\frac{dz}{dx}\right)^2 + 5z^3 = 5x + 2$$

$$y = z^3$$

$$\frac{dy}{dz} = 3z^2$$

$$\frac{dy}{dx}\frac{dx}{dz} = 3z^2$$

$$\frac{dy}{dx} = 3z^2\frac{dz}{dx}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(3z^2\frac{dz}{dx}\right)$$

$$\frac{d^2y}{dx^2} = 3z^2\frac{d}{dx}\left(\frac{dz}{dx}\right) + \frac{dz}{dx}\frac{d}{dx}(3z^2)$$

$$= 3z^2\frac{d^2z}{dx^2} + \frac{dz}{dx}\left(6z\frac{dz}{dx}\right)$$

$$= 3z^2\frac{d^2z}{dx^2} + 6z\left(\frac{dz}{dx}\right)^2$$

$$3z^2\frac{d^2z}{dx^2} + 6z\left(\frac{dz}{dx}\right)^2 + 2\left(3z^2\frac{dz}{dx}\right) + 5z^3 = 5x + 2$$

$$\therefore \frac{d^2y}{dx^2} + \frac{2dy}{dx} + 5y = 5x + 2.$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

$$m^2 + 2m + 5 = 0$$

$$(m+1)^2 + 4 = 0$$

$$(m+1)^2 = -4$$

$$m+1 = \pm 2i$$

$$m = -1 \pm 2i$$

$\therefore$ The complementary function, y_c, is

$y_c = e^{-x}(A\cos 2x + B\sin 2x)$.

The particular integral, y_p, is given by

$$y_p = Cx + D$$

$$\frac{dy_p}{dx} = C$$

$$\frac{d^2y_p}{dx^2} = 0$$

$$\frac{d^2y_p}{dx^2} + 2\frac{dy_p}{dx} + 5y_p = 0 + 2C + 5(Cx + D)$$

$$= 5Cx + 2C + 5D$$

$$= 5x + 2$$

$5C = 5 \quad 2C + 5D = 2$

$C = 1 \quad D = 0$

$$y_p = x$$

$$y = y_c + y_p$$

$$= e^{-x}(A\cos 2x + B\sin 2x) + x$$

$$z^3 = e^{-x}(A\cos 2x + B\sin 2x) + x$$

$$\frac{d}{dx}(z^3) = \frac{d}{dx}\left(e^{-x}(A\cos 2x + B\sin 2x) + x\right)$$

$$3z^2\frac{dz}{dx} = e^{-x}(-2A\sin 2x + 2B\cos 2x)$$

$$-e^{-x}(A\cos 2x + B\sin 2x) + 1$$

$x=0 \quad z=1 : 1 = A$

$x=0 \quad \frac{dz}{dx} = -\frac{2}{3} : 3(1^2)\left(-\frac{2}{3}\right) = 2B - A + 1$

$$B = -1$$

$$z^3 = e^{-x}(\cos 2x - \sin 2x) + x$$

$$z = \left(e^{-x}(\cos 2x - \sin 2x) + x\right)^{\frac{1}{3}}$$

As $x \longrightarrow \infty$, since $-1 < \cos 2x < 1$ and $-1 < \sin 2x < 1$, $e^{-x}(\cos 2x - \sin 2x) \longrightarrow 0$

$\therefore z \longrightarrow x^{\frac{1}{3}}$

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OR

$C: \; y = \dfrac{x(x+1)}{(x-1)^2}$

i) $\dfrac{x(x+1)}{(x-1)^2} = A + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

$= \dfrac{A(x-1)^2 + B(x-1) + C}{(x-1)^2}$

$x(x+1) = A(x-1)^2 + B(x-1) + C$

$x^2 + x = A(x^2 - 2x + 1) + Bx - B + C$

$= Ax^2 + (-2A + B)x + A - B + C$

$A = 1 \quad -2A + B = 1 \quad A - B + C = 0$

$B = 3 \quad C = 2$

$\dfrac{x(x+1)}{(x-1)^2} = 1 + \dfrac{3}{x-1} + \dfrac{2}{(x-1)^2}$

$\therefore y = 1 + \dfrac{3}{x-1} + \dfrac{2}{(x-1)^2}$

As $x \to \pm\infty \quad y \to 1$

As $x \to 1 \quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = 1$ and $x = 1$.

ii) If $\dfrac{x(x+1)}{(x-1)^2} = 1$

$x^2 + x = (x-1)^2$

$= x^2 - 2x + 1$

$3x = 1$

$x = \dfrac{1}{3}$

$\therefore$ C intersects $y = 1$ at $\left(\frac{1}{3}, 1\right)$.

iii) $y = 1 + \dfrac{3}{x-1} + \dfrac{2}{(x-1)^2}$

$\dfrac{dy}{dx} = \dfrac{-3}{(x-1)^2} - \dfrac{4}{(x-1)^3}$

a) when $\dfrac{dy}{dx} = 0$:

$\dfrac{-3}{(x-1)^2} - \dfrac{4}{(x-1)^3} = 0$

$-3(x-1) - 4 = 0$

$-3x + 3 - 4 = 0$

$3x = -1$

$x = -\dfrac{1}{3}$

$y = -\dfrac{1}{8}$

$$\frac{d^2y}{dx^2} = \frac{6}{(x-1)^3} + \frac{12}{(x-1)^4}$$

At $\left(-\frac{1}{3}, -\frac{1}{8}\right)$: $\frac{d^2y}{dx^2} = \frac{81}{64} > 0$.

$\therefore \left(-\frac{1}{3}, -\frac{1}{8}\right)$ is a minimum point.

b) when $\frac{dy}{dx} < 0$:

$$\frac{-3}{(x-1)^2} - \frac{4}{(x-1)^3} < 0$$

$$\frac{3(x-1) + 4}{(x-1)^3} > 0$$

$$\frac{3x+1}{(x-1)^3} > 0$$

$$\frac{3x+1}{x-1} > 0$$

$3x+1 > 0$, $x-1 > 0$

or

$3x+1 < 0$, $x-1 < 0$

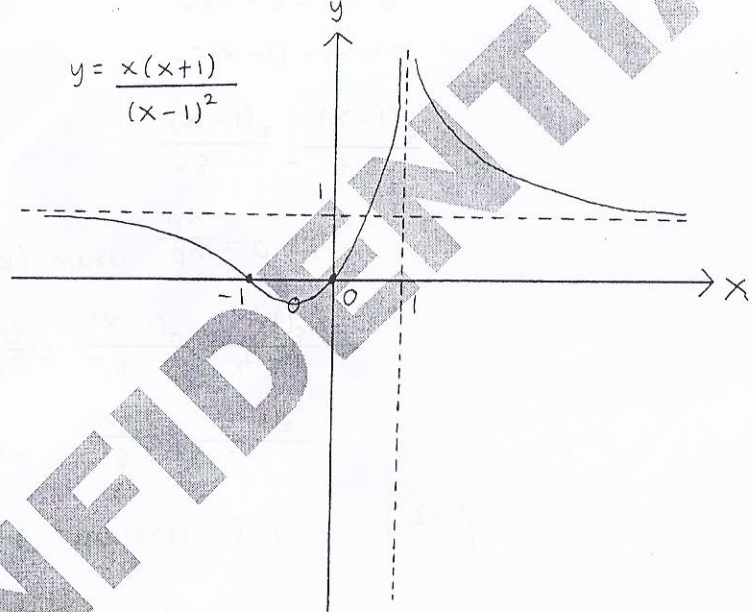
$x > 1$ or $x < -\frac{1}{3}$.

iv) when $x = 0$: $y = 0$

when $y = 0$: $\frac{x(x+1)}{(x-1)^2} = 0$

$x(x+1) = 0$

$x = 0, -1$



0 : Critical point

• : stationary point.

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