Exponential Distribution

(i) Probability density function, expected value and variance.

(ii) Relationship between an Exponential and a Poisson variable.

(i) Probability density function, expected value and variance.

Learning Outcome

Students should be able to apply:

- Identify the relevance of the Exponential distribution to the distribution of random events and use the Exponential distribution as a model.
- Calculate the probabilities for the distribution Exp(λ).
- Calculate the expected value and variance.

(ii) Relationship between an Exponential and a Poisson variable.

Learning Outcome

Students should be able to apply:

- Identify the relationship between one Exponential and one Poisson variables.
- To prove the relationship.

The Exponential Distribution (Negative Exponential Distribution)

A continuous random variable X having pdf $f(x)$ where

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad x \geq 0,$$

Where λ is a positive constant, is said to follow an exponential distribution.

Note :

Way to show that X is the random variable

$\Rightarrow$ Show that $\int_{all\ x} f(x)\ dx = 1.$

Some useful results:

$$P(X < a) = 1 - e^{-\lambda a}$$

$$P(X > a) = e^{-\lambda a}$$

$$The\ cdf, F(x)$$

$$F(x) = \mathrm{P}(\mathrm{X} \leq x) = 1 - e^{-\lambda x} \quad for\ \ x \geq 0$$

Example 1:

If $f(x) = 5e^{-5x}$ $where\ x \geq 0$,

find the median value of x.

Example 2:

The time, *t* seconds, between the arrivals of successive vehicles at a particular junction has pdf

$$f(t) = 0.025e^{-0.025t} \quad , \qquad t \geq 0.$$

A pedestrian, who takes 20 seconds to cross the road, sets off as one vehicle passes. Find the probability that she will complete the crossing before the next vehicle arrives. If she follows the same procedure on the return journey, find the probability that she completes each crossing without a vehicle arriving.

If the continuous random variable X has pdf

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad x \geq 0, \text{ then}$$

$$P[(X > a + b) | (X > a)] = P(X > b)$$

$$P(X > a) = e^{-\lambda a}$$

$$P[(X > a + b)|(X > a)] = \frac{P[(X > a + b) \cap (X > a)]}{P(X > a)}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$

$$= e^{-\lambda b}$$

$$= P(X > b)$$

$$P[(X > q)|(X > p)] = P\big(X > (q - p)\big) \; where \; q > p.$$

Example 3:

The lifetime, in years, of a television tube of a certain make is a random variable T and its pdf $f(t)$ is given by

$$f(t) = 0.25e^{-0.25t} \;, \qquad 0 \le t \le \infty$$

If a tube lasts longer than 2 years, find the probability that it will last longer than 5 years.

Expectation and variance

If X has pdf given by

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad x \geq 0$$

then

$$E(X) = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Example 4:

The lifetime of a particular type of light bulb has a negative exponential distribution with mean lifetime 1000 hours.

(a) Find the probability that a bulb is still working after 1300 hours.

(b) Given that it is still working after 1300 hours, find the probability that it is still working after 1500 hours.

(c) Find the standard deviation of the lifetime of this type of light bulb.

Example 5:

The continuous random variable X has the negative exponential distribution whose probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda a} \ , & x \geq 0 \\ 0 \ , & otherwise \end{cases}$$

where λ is a positive constant. Obtain expressions, in terms of λ, for

(a) The mean, $E(x)$, of the distribution,

Example 5:

Television sets are hired out by a rental company. The time in months, X, between major repairs has the above negative exponential distribution with $\lambda=0.05$. find, to 3 significant figures, the probability that a television set hired out by the company will not require a major repair for at least a 2-year period. Find also the median value of X.

The company agree to replace any set for which the time between major repairs is less than M months. Given that the company does not want to have to replace more than one set in 5, find M.

If X has exponential distribution with mean α, then

$E(X) = \alpha$ so that X has pdf

$$f(x) = \frac{1}{\alpha} e^{-\frac{x}{\alpha}}, \qquad \text{for} \quad x \geq 0$$

and $F(x) = 1 - e^{-\frac{x}{\alpha}}$

Example 6:

A random variable T has exponential distribution with mean α. Show that

$$P(t_1 \leq T \leq t_2) = e^{-\frac{t_1}{\alpha}} - e^{-\frac{t_2}{\alpha}}$$

The link between the exponential distribution and the Poisson distribution

X = the number of cars arriving at petrol station in one minute.
λ = average number of cars arriving per minute.

$$\Rightarrow \mathrm{X} \sim P_o(\lambda)$$

Y = the number of cars arriving in t minutes.

$$\Rightarrow \mathrm{Y} \sim P_o(\lambda t)$$

$$P(Y = 0) = e^{-\lambda t}$$

P(at least one car arrives in t minutes)

$$= P(Y \geq 1) = 1 - P(Y = 0) = 1 - e^{-\lambda t}$$

The link between the exponential distribution and the Poisson distribution

T = time in minutes between the arrival of one car and the next.
OR the length of time, in minutes, between successive cars.
OR "waiting times" between successive events.

$$P(T > t) = P(no\ car\ in\ t) = P(Y = 0) = e^{-\lambda t}$$

$$\boxed{P(T < t) = 1 - P(T > t) = 1 - e^{-\lambda t}}$$

$$\Rightarrow F(t) = 1 - e^{-\lambda t}$$

$$\Rightarrow f(t) = \lambda e^{-\lambda t}$$

T follows the exponential distribution with parameter λ.

Conclusion :

If X = number of cars arriving in one minute

$$\Rightarrow X \sim P_o(\lambda)$$

If T = the time, in minutes, between successive cars

$$pdf, \qquad f(t) = \lambda e^{-\lambda t}$$

$$E(T) = \frac{1}{\lambda}, \qquad Var(T) = \frac{1}{\lambda^2}$$

Example 7:

Cars arrive at a garage at an average rate of 0.8 per minute. Find the mean and variance of the waiting times between successive cars.

Example 8:

On a busy road, accidents occur at random at the rate of 3 per day. Find the probability that, after a particular accident has occurred, at least one day goes by without another.