1. $(r+1)^6 - (r-1)^6 = r^6 + 6r^5 + 15r^4 + 20r^3 + 15r^2 + 6r + 1$

$$- (r^6 - 6r^5 + 15r^4 - 20r^3 + 15r^2 - 6r + 1)$$

$$= 12r^5 + 40r^3 + 12r$$

$$\sum_{r=1}^{n} 12r^5 + 40r^3 + 12r = \sum_{r=1}^{n} (r+1)^6 - (r-1)^6$$

$$12\sum_{r=1}^{n} r^5 + 40\sum_{r=1}^{n} r^3 + 12\sum_{r=1}^{n} = (n+1)^6 - (n-1)^6$$

$$+ n^6 - (n-2)^6$$

$$+ (n-1)^6 - (n-3)^6$$

$$+ (n-2)^6 - (n-4)^6$$

$$\vdots$$

$$+ 5^6 - 3^6$$

$$+ 4^6 - 2^6$$

$$+ 3^6 - 1^6$$

$$+ 2^6 - 0^6$$

$$= (n+1)^6 + n^6 - 1$$

$$12\sum_{r=1}^{n} r^5 + \frac{40n^2(n+1)^2}{4} + \frac{12n(n+1)}{2} = n^6 + 6n^5 + 15n^4 + 20n^3$$

$$+ 15n^2 + 6n + 1 + n^6 - 1$$

$$12\sum_{r=1}^{n} r^5 + 10n^2(n+1)^2 + 6n(n+1) = 2n^6 + 6n^5 + 15n^4$$

$$+ 20n^3 + 15n^2 + 6n$$

$$12\sum_{r=1}^{n} r^5 + 10n^4 + 20n^3 + 10n^2 + 6n^2 + 6n$$

$$= 2n^6 + 6n^5 + 15n^4 + 20n^3 + 15n^2 + 6n$$

$$12\sum_{r=1}^{n} r^5 = 2n^6 + 6n^5 + 5n^4 - n^2$$

$$\sum_{r=1}^{n} r^5 = \frac{n^2(2n^4 + 6n^3 + 5n^2 - 1)}{12}$$

2. $$\sum_{n=1}^{N} n \times 3^{n-1} = \frac{1}{4} + \frac{3^N}{4}(2N-1)$$

$N=1$: LHS $= 1 \times 3^0$ $\quad$ RHS $= \frac{1}{4} + \frac{3}{4}(1)$

$= 1$

$= 1$

LHS = RHS

Statement is true for $N=1$.

$N=K$: Assume that $$\sum_{n=1}^{K} n \times 3^{n-1} = \frac{1}{4} + \frac{3^K}{4}(2K-1)$$

$N=K+1$: Need to show that

$$\sum_{n=1}^{K+1} n \times 3^{n-1} = \frac{1}{4} + \frac{3^{K+1}}{4}(2K+1)$$

$$\sum_{n=1}^{K+1} n \times 3^{n-1} = \frac{1}{4} + \frac{3^K}{4}(2K-1) + (K+1)3^K$$

$$= \frac{1}{4} + \frac{3^K}{4}[2K - 1 + 4K + 4]$$

$$= \frac{1}{4} + \frac{3^K}{4}[3(2K+1)]$$

$$= \frac{1}{4} + \frac{3^{K+1}}{4}[2K+1]$$

If the statement is true for $N=K$ then it is true for $N=K+1$.

By induction, $$\sum_{n=1}^{N} n \times 3^{n-1} = \frac{1}{4} + \frac{3^N}{4}(2N-1)$$

3. $y = 2\sqrt{x-2}\ , \quad x \geqslant 2$

$= 2(x-2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{(x-2)^{\frac{1}{2}}}$$

$$\text{Area} = \int_3^5 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_3^5 4\pi \sqrt{x-2}\sqrt{1+\frac{1}{x-2}}\, dx$$

$$= \int_3^5 4\pi (x-2+1)^{\frac{1}{2}}\, dx$$

$$= \int_3^5 4\pi (x-1)^{\frac{1}{2}}\, dx$$

$$= 4\pi \left[\frac{2}{3}(x-1)^{\frac{3}{2}}\right]_3^5$$

$$= \frac{8\pi}{3}(8 - 2\sqrt{2})$$

$$= \frac{16\pi}{3}(4 - \sqrt{2})$$

4. i) $A = \begin{pmatrix} 1 & 2 & 3 & 4 & -5 \\ 2 & 7 & 4 & 15 & -9 \\ 3 & 9 & 7 & 24 & -23 \\ 4 & 17 & 6 & 42 & -26 \\ 5 & 22 & 7 & 58 & -39 \end{pmatrix}$

$$\xrightarrow[\substack{-2r_1+r_2 \\ -3r_1+r_3 \\ -4r_1+r_4 \\ -5r_1+r_5}]{} \begin{pmatrix} 1 & 2 & 3 & 4 & -5 \\ 0 & 3 & -2 & 7 & 1 \\ 0 & 3 & -2 & 12 & -8 \\ 0 & 9 & -6 & 26 & -6 \\ 0 & 12 & -8 & 38 & -14 \end{pmatrix}$$

$$\xrightarrow[\substack{-r_2+r_3 \\ -3r_2+r_4 \\ -4r_2+r_5}]{} \begin{pmatrix} 1 & 2 & 3 & 4 & -5 \\ 0 & 3 & -2 & 7 & 1 \\ 0 & 0 & 0 & 5 & -9 \\ 0 & 0 & 0 & 5 & -9 \\ 0 & 0 & 0 & 10 & -18 \end{pmatrix}$$

$$\xrightarrow[\substack{-r_3+r_4 \\ -2r_3+r_5}]{} \begin{pmatrix} 1 & 2 & 3 & 4 & -5 \\ 0 & 3 & -2 & 7 & 1 \\ 0 & 0 & 0 & 5 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank $(A) = 3$

ii) A basis for the range space of A is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 9 \\ 17 \\ 22 \end{pmatrix}, \begin{pmatrix} 4 \\ 15 \\ 24 \\ 42 \\ 58 \end{pmatrix} \right\}.$$

iii) The dimension of the null space of A is $5 - 3 = 2$.

5. i) $x + \frac{1}{x} = \sec\theta + \tan\theta + \frac{1}{\sec\theta + \tan\theta}$

$= \frac{\sec^2\theta + 2\sec\theta\tan\theta + \tan^2\theta + 1}{\sec\theta + \tan\theta}$

$= \frac{\sec^2\theta + 2\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}$

$= \frac{2\sec^2\theta + 2\sec\theta\tan\theta}{\sec\theta + \tan\theta}$

$= \frac{2\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}$

$= 2\sec\theta$

$y + \frac{1}{y} = \csc\theta + \cot\theta + \frac{1}{\csc\theta + \cot\theta}$

$= \frac{\csc^2\theta + 2\csc\theta\cot\theta + \cot^2\theta + 1}{\csc\theta + \cot\theta}$

$= \frac{\csc^2\theta + 2\csc\theta\cot\theta + \csc^2\theta}{\csc\theta + \cot\theta}$

$= \frac{2\csc^2\theta + 2\csc\theta\cot\theta}{\csc\theta + \cot\theta}$

$= \frac{2\csc\theta(\csc\theta + \cot\theta)}{\csc\theta + \cot\theta}$

$= 2\csc\theta$

ii) $x = \sec\theta + \tan\theta$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta + \sec^2\theta$$

$$= \sec\theta(\tan\theta + \sec\theta)$$

$$= \frac{1}{2}\left(x + \frac{1}{x}\right)x$$

$$= \frac{1+x^2}{2}$$

$y = \csc\theta + \cot\theta$

$$\frac{dy}{d\theta} = -\csc\theta\cot\theta - \csc^2\theta$$

$$= -\csc\theta(\cot\theta + \csc\theta)$$

$$= -\frac{1}{2}\left(y + \frac{1}{y}\right)y$$

$$= \frac{-(1+y^2)}{2}$$

$$\frac{dy}{dx} = \frac{-(1+y^2)}{1+x^2}$$

ALTERNATIVELY,

$$x + \frac{1}{x} = 2\sec\theta$$

$$x^2 + 1 = 2x\sec\theta$$

$$2x\frac{dx}{d\theta} = 2\sec\theta\frac{dx}{d\theta} + 2x\sec\theta\tan\theta$$

$$2(x - \sec\theta)\frac{dx}{d\theta} = 2x\sec\theta\tan\theta$$

$$2\tan\theta\frac{dx}{d\theta} = 2x\sec\theta\tan\theta$$

$$\frac{dx}{d\theta} = x\sec\theta$$

$$= \frac{x^2+1}{2}$$

$$y + \frac{1}{y} = 2\csc\theta$$

$$y^2 + 1 = 2y\csc\theta$$

$$2y\frac{dy}{d\theta} = 2\csc\theta\frac{dy}{d\theta} - 2y\csc\theta\cot\theta$$

$$2(y - \csc\theta)\frac{dy}{d\theta} = -2y\csc\theta\cot\theta$$

$$2\cot\theta\frac{dy}{d\theta} = -2y\csc\theta\cot\theta$$

$$\frac{dy}{d\theta} = -y\csc\theta$$

$$= -\frac{(1+y^2)}{2}$$

$$\frac{dy}{dx} = \frac{\frac{-(1+y^2)}{2}}{\frac{1+x^2}{2}}$$

$$= \frac{-(1+y^2)}{1+x^2}$$

6. $$1 + \frac{e^{4\theta i}}{5} + \frac{e^{8\theta i}}{5^2} + \frac{e^{12\theta i}}{5^3} + \cdots + \frac{e^{4n\theta i}}{5^n}$$

$$a = 1 \quad r = \frac{e^{4\theta i}}{5}$$

$$\therefore 1 + \frac{e^{4\theta i}}{5} + \frac{e^{8\theta i}}{5^2} + \frac{e^{12\theta i}}{5^3} + \cdots + \frac{e^{4n\theta i}}{5^n} = \frac{1\left(1 - \left(\frac{e^{4\theta i}}{5}\right)^{n+1}\right)}{1 - \frac{e^{4\theta i}}{5}}$$

$$\sum_{r=0}^{n} \frac{e^{4r\theta i}}{5^r} = \frac{5^{n+1} - e^{4(n+1)\theta i}}{5^n(5 - e^{4\theta i})}$$

$$\sum_{r=0}^{n} \frac{\cos 4r\theta + i\sin 4r\theta}{5^r} = \frac{(5^{n+1} - e^{4\theta(n+1)i})(5 - e^{-4\theta i})}{5^n(5 - e^{4\theta i})(5 - e^{-4\theta i})}$$

$$\sum_{r=0}^{n} \frac{\cos 4r\theta}{5^r} + i\sum_{r=0}^{n} \frac{\sin 4r\theta}{5^r}$$

$$= \frac{5^{n+2} - 5e^{4(n+1)\theta i} - 5^{n+1}e^{-4\theta i} + e^{4n\theta i}}{5^n(25 - 5(e^{4\theta i} + e^{-4\theta i}) + 1)}$$

$$= \frac{5^{n+2} - 5e^{4(n+1)\theta i} - 5^{n+1}e^{-4\theta i} + e^{4n\theta i}}{5^n(26 - 5(2\cos 4\theta))}$$

$$= \frac{5^{n+2} - 5e^{4(n+1)\theta i} - 5^{n+1}e^{-4\theta i} + e^{4n\theta i}}{5^n(26 - 10\cos 4\theta)}$$

$$= \frac{5^{n+2} - 5(\cos 4(n+1)\theta + i\sin 4(n+1)\theta) - 5^{n+1}(\cos 4\theta - i\sin 4\theta) + \cos 4n\theta + i\sin 4n\theta}{5^n(26 - 10\cos 4\theta)}$$

$$= \frac{5^{n+2} - 5\cos 4(n+1)\theta - 5^{n+1}\cos 4\theta + \cos 4n\theta}{5^n(26 - 10\cos 4\theta)}$$

$$+ \frac{i(-5\sin 4(n+1)\theta + 5^{n+1}\sin 4\theta + \sin 4n\theta)}{5^n(26 - 10\cos 4\theta)}$$

$$= \frac{25 - 5^{-n+1}\cos 4(n+1)\theta - 5\cos 4\theta + 5^{-n}\cos 4n\theta}{26 - 10\cos 4\theta}$$

$$+ \frac{i(5\sin 4\theta - 5^{-n+1}\sin 4(n+1)\theta + 5^{-n}\sin 4n\theta)}{26 - 10\cos 4\theta}$$

$$\therefore \sum_{r=0}^{n} \frac{\sin 4r\theta}{5^r} = \frac{5\sin 4\theta - 5^{-n+1}\sin 4(n+1)\theta + 5^{-n}\sin 4n\theta}{26 - 10\cos 4\theta}$$

$$\sum_{r=0}^{n} \frac{\cos 4r\theta}{5^r} = \frac{25 - 5^{-n+1}\cos 4(n+1)\theta - 5\cos 4\theta + 5^{-n}\cos 4n\theta}{26 - 10\cos 4\theta}$$

$$\sum_{r=0}^{\infty} \frac{\cos 4r\theta}{5^r} = \lim_{n \to \infty} \sum_{r=0}^{n} \frac{\cos 4r\theta}{5^r}$$

$$= \frac{25 - 5\cos 4\theta}{26 - 10\cos 4\theta}$$

7. $C: y = 8 - \dfrac{4}{5(5x-3)} + \dfrac{9}{7(7x+6)}$

i) As $x \to \pm\infty$ $y \to 8$

As $x \to \dfrac{3}{5}$ $y \to \pm\infty$

As $x > \dfrac{-6}{7}$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = 8$, $x = \dfrac{3}{5}$ and $x = \dfrac{-6}{7}$.

ii) $\dfrac{dy}{dx} = \dfrac{4}{(5x-3)^2} - \dfrac{9}{(7x+6)^2}$

when $\dfrac{dy}{dx} = 0$: $\dfrac{4}{(5x-3)^2} = \dfrac{9}{(7x+6)^2}$

$4(7x+6)^2 = 9(5x-3)^2$

$2(7x+6) = \pm 3(5x-3)$

$= 3(5x-3), -3(5x-3)$

$14x + 12 = 15x - 9, -15x + 9$

$x = 21, \quad 29x = -3$

$x = \dfrac{-3}{29}$

$y = \dfrac{14281}{1785} \quad y = \dfrac{15121}{1785}$

$\therefore$ The critical points of C are $\left(21, \dfrac{14281}{1785}\right)$ and $\left(\dfrac{-3}{29}, \dfrac{15121}{1785}\right)$.

$$
\begin{aligned}
8.\quad I_n &= \int \sin^n x \cos^n x \, dx \\
&= 2^{-n} \int 2^n \sin^n x \cos^n x \, dx \\
&= 2^{-n} \int (2\sin x \cos x)^n \, dx \\
&= 2^{-n} \int \sin^n 2x \, dx \\
&= 2^{-n} \int \sin^{n-2} 2x \sin^2 2x \, dx \\
&= 2^{-n} \int \sin^{n-2} 2x \, (1 - \cos^2 2x) \, dx \\
&= 2^{-n} \int \sin^{n-2} 2x - \sin^{n-2} 2x \cos^2 2x \, dx \\
&= 2^{-n} \int \sin^{n-2} 2x \, dx - 2^{-n} \int \sin^{n-2} 2x \cos^2 2x \, dx \\
&= 2^{-n} \int (2\sin x \cos x)^{n-2} \, dx \\
&\quad - 2^{-n} \int \sin^{n-2} 2x \cos 2x \cos 2x \, dx \\
&= 2^{-n} \int 2^{n-2} \sin^{n-2} x \cos^{n-2} x \, dx \\
&\quad - 2^{-n} \int \sin^{n-2} 2x \cos 2x \cos 2x \, dx \\
&= \frac{1}{4} \int \sin^{n-2} x \cos^{n-2} x \, dx \\
&\quad - 2^{-n} \int \sin^{n-2} 2x \cos 2x \cos 2x \, dx
\end{aligned}
$$

$$u = \cos 2x \qquad dv = \sin^{n-2} 2x \cos 2x \, dx$$

$$du = -2 \sin 2x \, dx \qquad v = \int \sin^{n-2} 2x \cos 2x \, dx$$

$$= \frac{\sin^{n-1} 2x}{2(n-1)}$$

$$= \frac{1}{4} I_{n-2} - 2^{-n} \left(\frac{\sin^{n-1} 2x \cos 2x}{2(n-1)} - \int \frac{-2 \sin^{n-1} 2x \sin 2x \, dx}{2(n-1)} \right)$$

$$= \frac{1}{4} I_{n-2} - \frac{\sin^{n-1} 2x \cos 2x}{2^{n+1}(n-1)} - 2^{-n} \int \frac{\sin^n 2x}{n-1} \, dx$$

$$= \frac{1}{4} I_{n-2} - \frac{\sin^{n-1} 2x \cos 2x}{2^{n+1}(n-1)} - \frac{1}{n-1} I_n$$

$$\left(1 + \frac{1}{n-1}\right) I_n = \frac{1}{4} I_{n-2} - \frac{\sin^{n-1} 2x \cos 2x}{2^{n+1}(n-1)}$$

$$n I_n = \frac{(n-1) I_{n-2}}{4} - \frac{\sin^{n-1} 2x \cos 2x}{2^{n+1}}$$

$$n = 3: \; 48 I_3 = 8 I_1 - \sin^2 2x \cos 2x$$

$$I_1 = \int \sin x \cos x \, dx$$

$$= \int \frac{\sin 2x}{2} \, dx$$

$$= \frac{-\cos 2x}{4} + C$$

$$48 I_3 = -2\cos 2x - \sin^2 2x \cos 2x$$

$$I_3 = \frac{-\cos 2x}{24} - \frac{\sin^2 2x \cos 2x}{48} + C$$

$$n = 4:\ 128 I_4 = 24 I_2 - \sin^3 2x \cos 2x$$

$$I_2 = \int \sin^2 x \cos^2 x \, dx$$

$$= \int \frac{\sin^2 2x}{4} \, dx$$

$$= \int \frac{1 - \cos 4x}{8} \, dx$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

$$128 I_4 = 3x - \frac{3\sin 4x}{4} - \sin^3 2x \cos 2x$$

$$I_4 = \frac{3x}{128} - \frac{3\sin 4x}{512} - \frac{\sin^3 2x \cos 2x}{128} + C$$

9. $A = \begin{pmatrix} 4 & -4 & 6 \\ 2 & -1 & 2 \\ -3 & 4 & -5 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 4 & -4 & 6 \\ 2 & -1 & 2 \\ -3 & 4 & -5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & -4 & 6 \\ 2 & -1-\lambda & 2 \\ -3 & 4 & -5-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (4-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 4 & -5-\lambda \end{vmatrix} - (-4)\begin{vmatrix} 2 & 2 \\ -3 & -5-\lambda \end{vmatrix}$$

$$+ 6 \begin{vmatrix} 2 & -1-\lambda \\ -3 & 4 \end{vmatrix}$$

$$= (4-\lambda)\left[(1+\lambda)(5+\lambda) - 8\right] + 4(-10-2\lambda+6)$$

$$+ 6(8-3-3\lambda)$$

$$= (4-\lambda)(\lambda^2+6\lambda-3) + 4(-2\lambda-4) + 6(-3\lambda+5)$$

$$= 4\lambda^2 + 24\lambda - 12 - \lambda^3 - 6\lambda^2 + 3\lambda$$

$$-8\lambda - 16 - 18\lambda + 30$$

$$= -\lambda^3 - 2\lambda^2 + \lambda + 2$$

$$= -\lambda^2(\lambda+2) + \lambda + 2$$

$$= (\lambda+2)(1-\lambda^2)$$

$$= (\lambda+2)(1-\lambda)(1+\lambda)$$

$$|A - \lambda I| = 0$$

$$(\lambda + 2)(1 - \lambda)(1 + \lambda) = 0$$

$$\lambda = 1, -1, -2$$

$\lambda = 1$: $$\begin{pmatrix} 3 & -4 & 6 \\ 2 & -2 & 2 \\ -3 & 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & -4 & 6 & 0 \\ 2 & -2 & 2 & 0 \\ -3 & 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 3 & -4 & 6 & 0 \\ -3 & 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & -4 & 6 & 0 \\ -3 & 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[3r_1 + r_3]{-3r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s,\ s \in \mathbb{R}$

$y = 3s$

$x - 3s + s = 0$

$x = 2s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ 3s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$\lambda = -1$: $$\begin{pmatrix} 5 & -4 & 6 \\ 2 & 0 & 2 \\ -3 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 5 & -4 & 6 & 0 \\ 2 & 0 & 2 & 0 \\ -3 & 4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 5 & -4 & 6 & 0 \\ -3 & 4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 5 & -4 & 6 & 0 \\ -3 & 4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow[3r_1 + r_3]{-5r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 4s, s \in \mathbb{R}$

$y = s$

$x = -4s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4s \\ s \\ 4s \end{pmatrix}$$

$$= s \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}$$

$\lambda = -2$:
$$\begin{pmatrix} 6 & -4 & 6 \\ 2 & 1 & 2 \\ -3 & 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 6 & -4 & 6 & 0 \\ 2 & 1 & 2 & 0 \\ -3 & 4 & -3 & 0 \end{array}\right)$$

$$\xrightarrow[2 \times r_3]{3 \times r_2} \left(\begin{array}{ccc|c} 6 & -4 & 6 & 0 \\ 6 & 3 & 6 & 0 \\ -6 & 8 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[r_1 + r_3]{-r_1 + r_2} \left(\begin{array}{ccc|c} 6 & -4 & 6 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2}, \frac{r_2}{7}, \frac{r_3}{4}} \left(\begin{array}{ccc|c} 3 & -2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 3 & -2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s$, $s \in R$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are $1, -1, -2$ with corresponding eigenvectors $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$A^{-1}(A\underset{\sim}{x}) = A^{-1}(\lambda \underset{\sim}{x})$$

$$(A^{-1}A)\underset{\sim}{x} = \lambda(A^{-1}\underset{\sim}{x})$$

$$I\underset{\sim}{x} = \lambda(A^{-1}\underset{\sim}{x})$$

$$\underset{\sim}{x} = \lambda(A^{-1}\underset{\sim}{x})$$

$$\therefore A^{-1}\underset{\sim}{x} = \frac{1}{\lambda}\underset{\sim}{x}$$

$\therefore$ If A has an eigenvalue λ with corresponding eigenvector $\underset{\sim}{x}$ then A^{-1} has an eigenvalue $\frac{1}{\lambda}$ with corresponding eigenvector $\underset{\sim}{x}$.

If $P^{-1}A^{-1}P = D$, where P is a square matrix and D is a diagonal matrix,

$$\therefore P = \begin{pmatrix} 2 & -4 & -1 \\ 3 & 1 & 0 \\ 1 & 4 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

If $P^{-1}(A^{-1})^n P = D^n$, where P is a square matrix and D is a diagonal matrix,

$$\therefore P = \begin{pmatrix} 2 & -4 & -1 \\ 3 & 1 & 0 \\ 1 & 4 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & \left(-\frac{1}{2}\right)^n \end{pmatrix}.$$

10. $x = t^2$; $\frac{dx}{dt} = 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{2t}\frac{d^2y}{dt^2} - \frac{1}{2t^2}\frac{dy}{dt}}{2t} = \frac{1}{4t^2}\left(\frac{d^2y}{dt^2} - \frac{1}{t}\frac{dy}{dt}\right)$$

From

$$4x\frac{d^2y}{dx^2} + 2(1-2\sqrt{x})\frac{dy}{dx} + y = 3\sqrt{x} \quad \text{— ①}$$

$$4t^2 \times \frac{1}{4t^2}\left(\frac{d^2y}{dt^2} - \frac{1}{t}\frac{dy}{dt}\right) + 2(1-2t)\times\frac{1}{2t}\frac{dy}{dt} + y = 3t$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 3t \quad \text{— ②}$$

CE/AE: $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$

$\therefore y_c = (A + Bt)e^t$

PI: $y_p = at + b$; $y_p' = a$; $y_p'' = 0$

②: $0 - 2a + at + b = 3t$

$\Rightarrow a = 3$ & $b - 2a = 0 \Rightarrow b = 6$

$\therefore y_p = 3t + 6$

G.S. to ②: $y = (A + Bt)e^t + 3t + 6$

G.S. to ①: $y = (A + B\sqrt{x})e^{\sqrt{x}} + 3\sqrt{x} + 6$

11. $x^5 + y^5 = 5x^2y^2$

$$5x^4 + 5y^4 \frac{dy}{dx} = 10xy^2 + 10x^2y\frac{dy}{dx}$$

$$(y^4 - 2x^2y)\frac{dy}{dx} = 2xy^2 - x^4$$

$$\frac{dy}{dx} = \frac{2xy^2 - x^4}{y^4 - 2x^2y}$$

①: $r^5(\sin^5\theta + \cos^5\theta) = 5r^4\sin^2\theta\cos^2\theta$

$$r = \frac{5\cos^2\theta\sin^2\theta}{\cos^5\theta + \sin^5\theta}$$

$$\text{Area} = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \times \frac{25\cos^4\theta\sin^4\theta}{(\cos^5\theta + \sin^5\theta)^2}\,d\theta$$

$$= \frac{25}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^4\theta\sin^4\theta}{(\cos^5\theta + \sin^5\theta)^2}\,d\theta$$

$$= \frac{25}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\tan^4\theta}{(1 + \tan^5\theta)^2}\,d\theta$$

$$= \frac{25}{2}\int_0^{\frac{\pi}{2}} \sec^2\theta\tan^4\theta(1 + \tan^5\theta)^{-2}\,d\theta$$

$$= \frac{5}{2}\left[\frac{1}{-1}(1 + \tan^5\theta)^{-1}\right]_0^{\frac{\pi}{2}} = \frac{5}{2}\left[\frac{-1}{1 + \tan^5\theta}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{5}{2}\left[0 - \left(\frac{-1}{1}\right)\right]$$

$$= \frac{5}{2}$$

12. EITHER

i) $l_{AB}: \underset{\sim}{r} = \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

$l_{CD}: \underset{\sim}{r} = \begin{pmatrix} -3 \\ 10 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$

$\underset{\sim}{i}: -2 + 3\lambda = -3 + 3\mu$

$3\lambda = -1 + 3\mu$

$\lambda - \mu = -\frac{1}{3}$

$\underset{\sim}{j}: -3 + 2\lambda = 10 - 4\mu$

$2\lambda + 4\mu = 13$

$\lambda + 2\mu = \frac{13}{2}$

$(2) - (1): 3\mu = \frac{41}{6} \Rightarrow \mu = \frac{41}{18} = 2\frac{5}{18}$

$(1): \lambda = \frac{41}{18} - \frac{1}{3} = \frac{35}{18} = 1\frac{17}{18}$

$\underset{\sim}{k}:$ LHS $= 4 - 2\left(\frac{35}{18}\right) = \frac{1}{9}$

RHS $= 6 + 2\frac{5}{18} = 8\frac{5}{18}$

Not consistent.

$\therefore$ The lines AB and CD do not intersect.

ii) Let, $\underset{\sim}{n} \perp \overrightarrow{AB}$ and $\underset{\sim}{n} \perp \overrightarrow{CD}$

$$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -9 \\ -18 \end{pmatrix} = -6\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

$$\therefore \underset{\sim}{n} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

$$|\overrightarrow{PQ}| = \left| \frac{\left[\begin{pmatrix} -3 \\ 10 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}}{\sqrt{2^2+3^2+6^2}} \right| = \left| \frac{\begin{pmatrix} -1 \\ 13 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}}{\sqrt{4+9+36}} \right|$$

$$= \frac{-2+39+12}{\sqrt{49}} = \frac{49}{7} = 7$$

iii) Let $\overrightarrow{OP} = \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

Let $\overrightarrow{OQ} = \begin{pmatrix} -3 \\ 10 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$

$$\overrightarrow{PQ} = \begin{pmatrix} -1+3\mu-3\lambda \\ 13-4\mu-2\lambda \\ 2+\mu+2\lambda \end{pmatrix}$$

$$\begin{pmatrix} -1+3\mu-3\lambda \\ 13-4\mu-2\lambda \\ 2+\mu+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = 0$$

$$(-3+9\mu-9\lambda)+(26-8\mu-4\lambda)+(-4-2\mu-4\lambda)=0$$

$$19-\mu-17\lambda=0 \quad - \quad 3$$

$$\begin{pmatrix} -1 + 3\mu - 3\lambda \\ 13 - 4\mu - 2\lambda \\ 2 + \mu + 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = 0$$

$$(-3 + 9\mu - 9\lambda) + (-52 + 16\mu + 8\lambda) + (2 + \mu + 2\lambda) = 0$$

$$-53 + 26\mu + \lambda = 0 \quad - \quad ②$$

$② : \quad \lambda = 53 - 26\mu \quad - \quad ③$

$③$ in $①$: $19 - \mu - 901 + 442\mu = 0$

$$\mu = 2$$

$③ : \quad \lambda = 1$

$$\therefore \quad \overrightarrow{OP} = \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\overrightarrow{OQ} = \begin{pmatrix} -3 \\ 10 \\ 6 \end{pmatrix} + 2\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix}$$

OR

a) i) $ax^4 + bx^3 + cx^2 + dx + e = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$$\alpha + \beta + \gamma + \delta = \frac{-b}{a}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$$

$$\alpha\beta\gamma\delta = \frac{e}{a}$$

$$S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

$$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0 \text{ --- } ①$$

$$a\beta^4 + b\beta^3 + c\beta^2 + d\beta + e = 0 \text{ --- } ②$$

$$a\gamma^4 + b\gamma^3 + c\gamma^2 + d\gamma + e = 0 \text{ --- } ③$$

$$a\delta^4 + b\delta^3 + c\delta^2 + d\delta + e = 0 \text{ --- } ④$$

$$\alpha^r \times ① : a\alpha^{4+r} + b\alpha^{3+r} + c\alpha^{2+r} + d\alpha^{1+r} + e\alpha^r = 0 \text{ --- } ⑤$$

$$\beta^r \times ② : a\beta^{4+r} + b\beta^{3+r} + c\beta^{2+r} + d\beta^{1+r} + e\beta^r = 0 \text{ --- } ⑥$$

$$\gamma^r \times ③ : a\gamma^{4+r} + b\gamma^{3+r} + c\gamma^{2+r} + d\gamma^{1+r} + e\gamma^r = 0 \text{ --- } ⑦$$

$$\delta^r \times ④ : a\delta^{4+r} + b\delta^{3+r} + c\delta^{2+r} + d\delta^{1+r} + e\delta^r = 0 \text{ --- } ⑧$$

5 + 6 + 7 + 8 :

$$a(\alpha^{4+r} + \beta^{4+r} + \gamma^{4+r} + \delta^{4+r})$$
$$+ b(\alpha^{3+r} + \beta^{3+r} + \gamma^{3+r} + \delta^{3+r})$$
$$+ c(\alpha^{2+r} + \beta^{2+r} + \gamma^{2+r} + \delta^{2+r})$$
$$+ d(\alpha^{1+r} + \beta^{1+r} + \gamma^{1+r} + \delta^{1+r}) + e(\alpha^{r} + \beta^{r} + \gamma^{r} + \delta^{r}) = 0$$

$$aS_{4+r} + bS_{3+r} + cS_{2+r} + dS_{1+r} + eS_r = 0$$

ii) $S_0 = \alpha^0 + \beta^0 + \gamma^0 + \delta^0$

$= 1 + 1 + 1 + 1$

$= 4$

$S_1 = \alpha^1 + \beta^1 + \gamma^1 + \delta^1$

$= \alpha + \beta + \gamma + \delta$

$= \frac{-b}{a}$

$S_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$= (\alpha + \beta + \gamma + \delta)^2$

$- 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= \left(\frac{-b}{a}\right)^2 - \frac{2c}{a}$

$= \frac{b^2}{a^2} - \frac{2c}{a}$

$$S_{-1} = \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}$$

$$= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$$

$$= \frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha\beta\gamma\delta}$$

$$= \frac{\frac{-d}{a}}{\frac{e}{a}}$$

$$= \frac{-d}{e}$$

iii) $aS_{4+r} + bS_{3+r} + cS_{2+r} + dS_{1+r} + eS_r = 0$

$r=-1$: $aS_3 + bS_2 + cS_1 + dS_0 + eS_{-1} = 0$

$$aS_3 + b\left(\frac{b^2}{a^2} - \frac{2c}{a}\right) - \frac{bc}{a} + 4d + e\left(\frac{-d}{e}\right) = 0$$

$$aS_3 + \frac{b^3}{a^2} - \frac{2bc}{a} - \frac{bc}{a} + 4d - d = 0$$

$$aS_3 = \frac{3bc}{a} - \frac{b^3}{a^2} - 3d$$

$$S_3 = \frac{3bc}{a^2} - \frac{b^3}{a^3} - \frac{3d}{a}$$

b) $ax^3 + bx^2 + cx + d = 0$, $a, b, c, d \neq 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = \frac{-b}{a}$ $\quad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ $\quad \alpha\beta\gamma = \frac{-d}{a}$

$\alpha^{\alpha\gamma}\beta^{\beta\gamma}\gamma^{\alpha\beta} = k^{\gamma}$ $\quad \alpha^{\alpha\beta}\beta^{\alpha\gamma}\gamma^{\beta\gamma} = k^{\beta}$ $\quad \alpha^{\beta\gamma}\beta^{\alpha\beta}\gamma^{\alpha\gamma} = k^{\alpha}$

$$\alpha^{\alpha\gamma}\beta^{\beta\gamma}\gamma^{\alpha\beta}(\alpha^{\alpha\beta}\beta^{\alpha\gamma}\gamma^{\beta\gamma})\alpha^{\beta\gamma}\beta^{\alpha\beta}\gamma^{\alpha\gamma} = k^{\gamma}k^{\beta}k^{\alpha}$$

$$\alpha^{\alpha\beta + \alpha\gamma + \beta\gamma}\beta^{\alpha\beta + \alpha\gamma + \beta\gamma}\gamma^{\alpha\beta + \alpha\gamma + \beta\gamma} = k^{\alpha + \beta + \gamma}$$

$$(\alpha\beta\gamma)^{\alpha\beta + \alpha\gamma + \beta\gamma} = k^{\alpha + \beta + \gamma}$$

$$\left(\frac{-d}{a}\right)^{\frac{c}{a}} = k^{-\frac{b}{a}}$$

$$\left(\frac{-d}{a}\right)^{c} = k^{-b}$$

$$k = \left(\frac{-d}{a}\right)^{-\frac{c}{b}}$$

$$= \left(\frac{-a}{d}\right)^{\frac{c}{b}}$$