DIFFERENTIAL EQUATION

1. Given that y is a function of x, where $x > 0$, show that, if the substitution $x = \sqrt{t}$ is made, then

(i) $\dfrac{dy}{dx} = 2\sqrt{t}\,\dfrac{dy}{dt}$ [1]

(ii) $\dfrac{d^2y}{dx^2} = 4t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}$ [2]

Hence, or otherwise, find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{x}\left(\frac{dy}{dx}\right) + 4x^2(9y+6) = 0$$ [6]

2. i) Find the solution of the differential equation

$$\frac{d^2y}{dx^2} - 3a\frac{dy}{dx} + 2a^2y = 0, \qquad a \neq 0$$

given that $y = 4$ and $\dfrac{dy}{dx} = 5$ when $x = 0$. [4]

ii) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x.$$ [6]

3. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = 36\cos x - 12\sin x$$ [5]

4. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 32e^{x}.$$ [8]

5. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 10\cos x + 15\sin x. \qquad [9]$$

6. If $5y^4\frac{d^2y}{dx^2} + 20y^3\left(\frac{dy}{dx}\right)^2 + 25y^4\frac{dy}{dx} + 4y^5 = 3e^{7x}$, using the substitution $z = y^5$, obtain a differential equation in x and z. [5]

Hence, express y in terms of x. [9]

7. Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 3x^2 - x + 6$.

[9]

8. Show that $\cos y = x$ satisfies the differential equation

$$\left(x^2 - 1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0. \qquad [4]$$

9. The real variables x and y are related by the equation $y = e^{y(x-1)}$. Show that $\frac{dy}{dx} = 1$

and $\frac{d^2y}{dx^2} = 3$, when $x = 1$. [7]

10. Given $3x^2 + 5xy^2 + 4y^2 = 12$, find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (1, 1). [5]

11. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 90y = 4\cos 2x + 7\sin 2x. \qquad [8]$$

12. Given that $y = f(x)$, show that, if the substitution $x = \sqrt[3]{t}$ is made, then

(i) $\dfrac{dy}{dx} = 3t^{\frac{2}{3}}\dfrac{dy}{dt}$, [2]

(ii) $\dfrac{d^2y}{dx^2} = 9t^{\frac{4}{3}}\dfrac{d^2y}{dt^2} + 6t^{\frac{1}{3}}\dfrac{dy}{dt}$. [3]

Hence find the general solution of the differential equation

$$\frac{1}{x}\frac{d^2y}{dx^2} - \frac{2}{x^2}\frac{dy}{dx} + 36yx^3 = 0.$$ [5]

13. Show that $y = (a\cos 2x + b\sin 2x)x$, where a and b are constants that need to be found, is the particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4y = \sin 2x.$$ [4]

Hence solve the differential equation given that when $x = 0$, $y = \pi$ and $\dfrac{dy}{dx} = \dfrac{-1}{4}$.

[7]

14. **(i)** The curve C is defined implicitly by the equation $\sin(x + y) = \cos xy$. Find the value of $\dfrac{d^2y}{dx^2}$ at the point $(\pi, 0)$. [5]

(ii) If $y = x^3e^x$, prove that

$$\frac{d^ny}{dx^n} = x^3e^x + 3nx^2e^x + 3n(n-1)xe^x + n(n-1)(n-2)e^x$$

for every positive integer n. [4]

15.

(i) Given $x = \sin t + \cos t$ and $y = \sin t - \cos t$, find $\dfrac{d^2y}{dx^2}$. [4]

(ii) If $x = \sin t + \sin t\cos t + \cos t$ and $y = \sin t - \sin t\cos t + \cos t$, express the curve in the form $(ax + b)^2 + (cy + d)^2 = exy + f$. [5]

16. **(i)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=3e^{-x}. \qquad [5]$$

(ii) Find the particular solution given that $y=1$ and $\frac{dy}{dx}=1$ when $t=0$. [4]

17. If $5y^4\frac{d^2y}{dx^2}+20y^3(\frac{dy}{dx})^2+55y^4\frac{dy}{dx}+24y^5=181\cos x+83\sin x$ and $v=y^5$,

show that $\frac{d^2v}{dx^2}+11\frac{dv}{dx}+24v=181\cos x+83\sin x$. [3]

Find also y in terms of x. [9]

18. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+16y=144\cos 8x. \qquad [8]$$

19. If $x=\sin t\cos t+\sin^2 t$ and $y=\sin t\cos t+\cos^2 t$, show that $\frac{dy}{dx}=\frac{1-\tan 2t}{1+\tan 2t}$ and

find the exact value of $\frac{d^2y}{dx^2}$ when $t=\frac{\pi}{8}$. [7]

20. Given that $y=\frac{1}{x+2}$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$, simplifying your results as far as possible. [4]

Use induction to establish an expression for $\frac{d^ny}{dx^n}$, where n is a positive integer. [4]

21. Find y in terms of x given that

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=6\cos x\ ,$$

and that when $x=0$, $y=\frac{dy}{dx}=0$. [10]