1. $17^{2n+1} + 25^{n}$ is divisible by 3

Let $f(n) = 17^{2n+1} + 25^{n}$

when $n=1$: $f(1) = 17^{2(1)+1} + 25^{1}$

$= 17^{2+1} + 25^{1}$

$= 17^{3} + 25$

$= 4913 + 25$

$= 4938$

$= 1646(3)$

Assume the statement is true when $n=k$.

$n=k$: $3 \mid f(k)$

$f(k) = 3s$, s is an integer.

$17^{2k+1} + 25^{k} = 3s$

when $n=k+1$: $f(k+1) = 17^{2(k+1)+1} + 25^{k+1}$

$= 17^{2k+2+1} + 25^{k+1}$

$= 17^{2k+1}17^{2} + 25^{k}25$

$= 17^{2k+1}289 + 25^{k}25$

$= 17^{2k+1}(264+25) + 25^{k}25$

$= 17^{2k+1}264 + 17^{2k+1}25 + 25^{k}25$

$= 17^{2k+1}264 + (17^{2k+1} + 25^{k})25$

$= 17^{2k+1}264 + 3s(25)$

$= 3(17^{2k+1}88 + 25s)$

Since s is an integer and k is an integer,

$17^{2k+1}88 + 25s$ is an integer.

$\therefore 3 \mid f(k+1)$

$\therefore 17^{2n+1} + 25^n$ is divisible by 3 for every positive integer n.

2. a) $\left.\begin{array}{l} x - y + z = 2 \\ 2x + 3y - z = 4 \\ 3x + 7y - 3z = 6 \end{array}\right\}$

$\begin{array}{l} \\ -2\times ① + ②: \\ -3\times ① + ③: \end{array} \quad \left.\begin{array}{l} x - y + z = 2 \\ 5y - 3z = 0 \\ 10y - 6z = 0 \end{array}\right\}$

let $z = 5s$, $s \in \mathbb{R}$

and $y = 3s$

$x = 2 - 2s$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-2s \\ 3s \\ 5s \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$

b) $m = \begin{pmatrix} 1 & a^2 & a \\ a & 1 & a^2 \\ a^2 & a & 1 \end{pmatrix}, \ a \neq \pm 1$

$$\begin{aligned} |m| &= 1 - a^3 - a^2(a - a^4) + a(a^2 - a^2) \\ &= 1 - a^3 - a^3 + a^6 \\ &= 1 - a^3 - a^3(1 - a^3) \\ &= (1 - a^3)(1 - a^3) \\ &= (a^3 - 1)(a^3 - 1) \\ &= (a^3 - 1)^2 \end{aligned}$$

$$m^{-1} = \frac{1}{(a^3-1)(a^3-1)} \begin{pmatrix} 1-a^3 & a^4-a & 0 \\ 0 & 1-a^3 & a^4-a \\ a^4-a & 0 & 1-a^3 \end{pmatrix}^T$$

$$= \frac{1}{(a^3-1)(a^3-1)} \begin{pmatrix} 1-a^3 & 0 & a^4-a \\ a^4-a & 1-a^3 & 0 \\ 0 & a^4-a & 1-a^3 \end{pmatrix}$$

3. $\Pi_1 : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

$\Pi_2 : \underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -8 \\ 0 \\ 1 \end{pmatrix}$

$\Pi_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3s \\ 3 - 2s - 2t \\ t \end{pmatrix}$ $\quad \Pi_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - 4s - 8t \\ s \\ t \end{pmatrix}$

$y = 3 - \frac{2x}{3} - 2z$

$3y = 9 - 2x - 6z$

$2x + 3y + 6z = 9$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = 9$

$x = 2 - 4y - 8z$

$x + 4y + 8z = 2$

$x + 4y + 8z = 2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} = 2$

If the line has equation $\underset{\sim}{r} = s\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, since the angle between the line and the planes is θ

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}\right| \cos\theta$$

and $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} = \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}\right| \cos\theta$

$2a + 3b + 6c = 7\sqrt{a^2 + b^2 + c^2} \cos\theta$

$a + 4b + 8c = 9\sqrt{a^2 + b^2 + c^2} \cos\theta$

$\frac{a + 4b + 8c}{2a + 3b + 6c} = \frac{9}{7}$

$7a + 28b + 56c = 18a + 27b + 54c$

$b + 2c = 11a$

$\underset{\sim}{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$= \begin{pmatrix} a \\ 11a - 2c \\ c \end{pmatrix}$

$= a\begin{pmatrix} 1 \\ 11 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

4. a) $\displaystyle\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)}$

Let $\dfrac{2}{r(r+1)(r+2)} = \dfrac{A}{r} + \dfrac{B}{r+1} + \dfrac{C}{r+2}$

$$= \frac{A(r+1)(r+2) + Br(r+2) + Cr(r+1)}{r(r+1)(r+2)}$$

$$2 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$$
$$= A(r^2+3r+2) + B(r^2+2r) + C(r^2+r)$$
$$= (A+B+C)r^2 + (3A+2B+C)r + 2A$$

$A+B+C=0$ $\quad$ $3A+2B+C=0$ $\quad$ $2A=2$

$B+C=-1$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $A=1$

$2B+C=-1$

$B=-2$

$C=1$

$$\therefore \frac{2}{r(r+1)(r+2)} = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$$

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$$

$$= \frac{1}{1} - \frac{2}{2} + \frac{1}{3}$$
$$+ \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$
$$+ \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$$
$$+ \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$$
$$\vdots$$

$$+ \frac{1}{n-3} - \frac{2}{n-2} + \frac{1}{n-1}$$

$$+ \frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n}$$

$$+ \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$$

$$+ \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$= \frac{1}{1} - \frac{2}{2} + \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$\therefore \sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

b) $a + a+d + a + 4d + a + 9d + \cdots + a + (n-1)^2 d$

$$= \sum_{r=1}^{n} a + (r-1)^2 d$$

$$= \sum_{r=1}^{n} a + \sum_{r=1}^{n} (r-1)^2 d$$

$$= an + d \sum_{r=1}^{n} r^2 - 2r + 1$$

$$= an + d \sum_{r=1}^{n} r^2 - 2d \sum_{r=1}^{n} r + d \sum_{r=1}^{n} 1$$

$$= an + \frac{dn(n+1)(2n+1)}{6} - \frac{2dn(n+1)}{2} + dn$$

$$= an + \frac{dn(n+1)(2n+1)}{6} - dn(n+1) + dn$$

$$= an + \frac{dn(2n^2+3n+1)}{6} - dn^2 - dn + dn$$

$$= an + \frac{dn(2n^2+3n+1)}{6} - dn^2$$

$$= an + d\left[\frac{n(2n^2+3n+1) - 6n^2}{6}\right]$$

$$= an + \frac{d}{6}\left(n(2n^2+3n+1) - 6n^2\right)$$

$$= an + \frac{d}{6}(2n^3+3n^2+n-6n^2)$$

$$= an + \frac{d}{6}(2n^3-3n^2+n)$$

5. $3x^4 + 5x^3 + 8x^2 + 4x + 9 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$\alpha + \beta + \gamma + \delta = -\frac{5}{3}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{8}{3}$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{4}{3}$

$\alpha\beta\gamma\delta = 3$

$\alpha + 1, \beta + 1, \gamma + 1, \delta + 1$

Let $u = \alpha + 1$

$\alpha = u - 1$

α is a root

$\therefore 3\alpha^4 + 5\alpha^3 + 8\alpha^2 + 4\alpha + 9 = 0$

$3(u-1)^4 + 5(u-1)^3 + 8(u-1)^2 + 4(u-1) + 9 = 0$

$3(u^4 - 4u^3 + 6u^2 - 4u + 1)$

$+ 5(u^3 - 3u^2 + 3u - 1)$

$+ 8(u^2 - 2u + 1)$

$+ 4u - 4 + 9 = 0$

$3u^4 - 12u^3 + 18u^2 - 12u + 3$

$5u^3 - 15u^2 + 15u - 5$

$+ 8u^2 - 16u + 8$

$+ 4u + 5 = 0$

$3u^4 - 7u^3 + 11u^2 - 9u + 11 = 0.$

$\therefore$ The equation $3u^4 - 7u^3 + 11u^2 - 9u + 11 = 0$

has roots $\alpha+1, \beta+1, \gamma+1, \delta+1$.

$\alpha+1+\beta+1+\gamma+1+\delta+1 = \frac{7}{3}$

$(\alpha+1)(\beta+1) + (\alpha+1)(\gamma+1) + (\alpha+1)(\delta+1)$
$+ (\beta+1)(\gamma+1) + (\beta+1)(\delta+1) + (\gamma+1)(\delta+1) = \frac{11}{3}$

$(\alpha+1)(\beta+1)(\gamma+1) + (\alpha+1)(\beta+1)(\delta+1)$
$+ (\alpha+1)(\gamma+1)(\delta+1) + (\beta+1)(\gamma+1)(\delta+1) = 3$

$(\alpha+1)(\beta+1)(\gamma+1)(\delta+1) = 3$

$S_n = (\alpha+1)^n + (\beta+1)^n + (\gamma+1)^n + (\delta+1)^n$

$S_0 = (\alpha+1)^0 + (\beta+1)^0 + (\gamma+1)^0 + (\delta+1)^0$

$= 1+1+1+1$

$= 4$

$S_1 = (\alpha+1)^1 + (\beta+1)^1 + (\gamma+1)^1 + (\delta+1)^1$

$= \alpha+1+\beta+1+\gamma+1+\delta+1$

$= \frac{7}{3}$

$S_2 = (\alpha+1)^2 + (\beta+1)^2 + (\gamma+1)^2 + (\delta+1)^2$

$= (\alpha+1+\beta+1+\gamma+1+\delta+1)^2$

$-2[(\alpha+1)(\beta+1) + (\alpha+1)(\gamma+1) + (\alpha+1)(\delta+1)$
$+ (\beta+1)(\gamma+1) + (\beta+1)(\delta+1) + (\gamma+1)(\delta+1)]$

$$= \left(\frac{7}{3}\right)^2 - 2\left(\frac{11}{3}\right)$$

$$= \frac{49}{9} - \frac{22}{3}$$

$$= \frac{-17}{9}$$

$$S_{-1} = (\alpha+1)^{-1} + (\beta+1)^{-1} + (\gamma+1)^{-1} + (\delta+1)^{-1}$$

$$= \frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} + \frac{1}{\delta+1}$$

$$= \frac{(\alpha+1)(\beta+1)(\gamma+1) + (\alpha+1)(\beta+1)(\delta+1) + (\alpha+1)(\gamma+1)(\delta+1) + (\beta+1)(\gamma+1)(\delta+1)}{(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)}$$

$$= \frac{3}{\frac{11}{3}}$$

$$= \frac{9}{11}$$

$$aS_{4+r} + bS_{3+r} + cS_{2+r} + dS_{1+r} + eS_r = 0$$

$$3S_{4+r} - 7S_{3+r} + 11S_{2+r} - 9S_{1+r} + 11S_r = 0$$

$$r=-1:\ 3S_3 - 7S_2 + 11S_1 - 9S_0 + 11S_{-1} = 0$$

$$3S_3 + \frac{119}{9} + \frac{77}{3} - 36 + 9 = 0$$

$$3S_3 = -\frac{107}{9}$$

$$S_3 = -\frac{107}{27}$$

6. $C: \quad y = \dfrac{1}{2(2x-9)} - \dfrac{1}{3(3x-8)}$

As $x \to \pm\infty \quad y \to 0$

As $x \to \dfrac{9}{2} \quad y \to \pm\infty$

As $x \to \dfrac{8}{3} \quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = 0$, $x = \dfrac{9}{2}$ and $x = \dfrac{8}{3}$.

when $x = 0$: $y = \dfrac{-1}{72}$

when $y = 0$: $\dfrac{1}{2(2x-9)} - \dfrac{1}{3(3x-8)} = 0$

$$\frac{1}{2(2x-9)} = \frac{1}{3(3x-8)}$$

$$9x - 24 = 4x - 18$$

$$5x = 6$$

$$x = \frac{6}{5}$$

$\therefore$ The intersection points are $\left(0, \dfrac{1}{72}\right)$ and $\left(\dfrac{6}{5}, 0\right)$.

$$\frac{dy}{dx} = \frac{-1}{(2x+9)^2} + \frac{1}{(3x-8)^2}$$

when $\dfrac{dy}{dx} = 0$: $\dfrac{-1}{(2x-9)^2} + \dfrac{1}{(3x-8)^2} = 0$

$$(2x-9)^2 = (3x-8)^2$$

$$2x-9 = 3x-8,\ -3x+8$$

$$x = -1,\ \frac{17}{5}$$

$$y = \frac{-1}{66},\ \frac{-25}{66}$$

$$\frac{d^2y}{dx^2} = \frac{4}{(2x-9)^3} - \frac{6}{(3x-8)^3}$$

$$x=-1:\ \frac{d^2y}{dx^2} = \frac{10}{1331} > 0$$

$$x=\frac{17}{5}:\ \frac{d^2y}{dx^2} = \frac{-1250}{1331} < 0$$

$\therefore \left(-1, \frac{-1}{66}\right)$ is a minimum point and

$\left(\frac{17}{5}, \frac{-25}{66}\right)$ is a maximum point.