1. $S_n = 1 \times n + 2(n-1) + 3(n-2) + \cdots + n \times 1$

$$= \sum_{r=1}^{n} (n+1-r)r$$

$$= \sum_{r=1}^{n} (n+1)r - r^2$$

$$= (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2$$

$$= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left(n+1 - \frac{(2n+1)}{3}\right)$$

$$= \frac{n(n+1)}{2}\frac{(3(n+1) - 2n - 1)}{3}$$

$$= \frac{n(n+1)}{2}\frac{(3n+3-2n-1)}{3}$$

$$= \frac{n(n+1)}{2} \frac{(n+2)}{3}$$

$$= \frac{n(n+1)(n+2)}{6}$$

2. $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = 36\cos x - 12\sin x$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = 0$$

$$m^2 - 2m - 15 = 0$$

$$(m - 5)(m + 3) = 0$$

$$m = 5, -3$$

The complementary function y_c is

$$y_c = Ae^{5x} + Be^{-3x}$$

and the particular integral y_p is

$$y_p = A\cos x + B\sin x$$

$$\frac{dy_p}{dx} = -A\sin x + B\cos x$$

$$\frac{d^2y_p}{dx^2} = -A\cos x - B\sin x$$

$$\frac{d^2y_p}{dx^2} - \frac{2dy_p}{dx} - 15y_p$$

$$= -A\cos x - B\sin x - 2(-A\sin x + B\cos x)$$
$$-15(A\cos x + B\sin x)$$

$$= -A\cos x - B\sin x + 2A\sin x - 2B\cos x$$
$$-15A\cos x - 15B\sin x$$

$$= (-16A - 2B)\cos x + (2A - 16B)\sin x$$

$$= 36\cos x - 12\sin x$$

$$-16A - 2B = 36 \qquad 2A - 16B = -12$$
$$-8A - B = 18 \qquad A - 8B = -6$$
$$A = 8B - 6$$

$$-8(8B - 6) - B = 18$$
$$-64B + 48 - B = 18$$
$$65B = 30$$
$$B = \frac{6}{13}$$

$$A = 8\left(\frac{6}{13}\right) - 6$$

$$= \frac{-30}{13}$$

$$y_p = \frac{6}{13}\sin x - \frac{30}{13}\cos x$$

$$y = y_c + y_p$$

$$= Ae^{5x} + Be^{-3x} + \frac{6}{13}\sin x - \frac{30}{13}\cos x$$

$\therefore$ The general solution is

$$y = Ae^{5x} + Be^{-3x} + \frac{6}{13}\sin x - \frac{30}{13}\cos x$$

3. $3x^3 + 4x + 2 = 0$

$$\frac{1}{\alpha^2\beta^2} - 2, \quad \frac{1}{\alpha^2\gamma^2} - 2, \quad \frac{1}{\beta^2\gamma^2} - 2$$

Let $u = \dfrac{1}{\alpha^2\beta^2} - 2$

$$u + 2 = \frac{1}{\alpha^2\beta^2}$$

$$= \frac{\gamma^2}{\alpha^2\beta^2\gamma^2}$$

$$= \frac{\gamma^2}{\left(\frac{-2}{3}\right)^2} \quad \text{since } \alpha\beta\gamma = \frac{-2}{3}$$

$$= \frac{9\gamma^2}{4}$$

$$\gamma^2 = \frac{4}{9}(u + 2)$$

$$\gamma = \pm\frac{2}{3}\sqrt{u + 2}$$

r is a root

$$3r^3 + 4r + 2 = 0$$

$$3\left(\pm\frac{2}{3}\sqrt{u+2}\right)^3 + 4\left(\pm\frac{2}{3}\sqrt{u+2}\right) + 2 = 0$$

$$3\left(\pm\frac{8}{27}(u+2)\sqrt{u+2}\right) \pm \frac{8}{3}\sqrt{u+2} + 2 = 0$$

$$\pm\frac{8}{9}(u+2)\sqrt{u+2} \pm \frac{8}{3}\sqrt{u+2} + 2 = 0$$

$$\pm\frac{8}{9}\sqrt{u+2}\,(u+2+3) + 2 = 0$$

$$\pm\frac{8}{9}\sqrt{u+2}\,(u+5) = -2$$

$$\left[\pm\frac{8}{9}\sqrt{u+2}\,(u+5)\right]^2 = (-2)^2$$

$$\frac{64}{81}(u+2)(u+5)^2 = 4$$

$$16(u+2)(u+5)^2 = 81$$

$$16(u+2)(u^2+10u+25)=81$$

$$16(u^3+10u^2+25u+2u^2+20u+50)=81$$

$$16(u^3+12u^2+45u+50)=81$$

$$16u^3+192u^2+720u+800=81$$

$$16u^3+192u^2+720u+719=0$$

$\therefore$ The equation having roots

$\dfrac{1}{\alpha^2\beta^2}-2$, $\dfrac{1}{\alpha^2\gamma^2}-2$ and $\dfrac{1}{\beta^2\gamma^2}-2$ is

$$16u^3+192u^2+720u+719=0$$

4 C : $y = 3 - \frac{2}{x+9} + \frac{1}{2x-3}$

i) As $x \to \pm\infty$ $y \to 3$

As $x \to -9$ $y \to \pm\infty$

As $x \to \frac{3}{2}$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = 3$,

$x = -9$ and $x = \frac{3}{2}$

ii) $\frac{dy}{dx} = \frac{2}{(x+9)^2} - \frac{2}{(2x-3)^2}$

when $\frac{dy}{dx} = 0$:

$$\frac{2}{(x+9)^2} - \frac{2}{(2x-3)^2} = 0$$

$$\frac{2}{(x+9)^2} = \frac{2}{(2x-3)^2}$$

$$(x+9)^2 = (2x-3)^2$$

$$x + 9 = \pm(2x - 3)$$

$$x + 9 = 2x - 3, \quad x + 9 = -2x + 3$$

$$x = 12 \qquad\qquad 3x = -6$$

$$y = \frac{62}{21} \qquad\qquad x = -2$$

$$y = \frac{18}{7}$$

$\therefore$ The critical points of C are

$\left(12, \frac{62}{21}\right)$ and $\left(-2, \frac{18}{7}\right)$.

iii) when $x = 0$: $y = 3 - \frac{2}{9} - \frac{1}{3}$

$$= \frac{22}{9}$$

when $y = 0$: $3 - \frac{2}{x+9} + \frac{1}{2x-3} = 0$

$$\frac{2}{x+9} - \frac{1}{2x-3} = 3$$

$$\frac{2(2x-3) - (x+9)}{(x+9)(2x-3)} = 3$$

$$\frac{4x - 6 - x - 9}{(x+9)(2x-3)} = 3$$

$$\frac{3x - 15}{(x+9)(2x-3)} = 3$$

$$\frac{x - 5}{(x+9)(2x-3)} = 1$$

$$\begin{aligned} x - 5 &= (x+9)(2x-3) \\ &= 2x^2 + 15x - 27 \end{aligned}$$

$$2x^2 + 14x - 22 = 0$$

$$x^2 + 7x - 11 = 0$$

$$x = \frac{-7 \pm \sqrt{93}}{2}$$

$\therefore$ The intersection points of C are

$\left(0, \frac{22}{9}\right)$, $\left(\frac{-7 + \sqrt{93}}{2}, 0\right)$ and

$\left(\frac{-7 - \sqrt{93}}{2}, 0\right)$

5. $I_n = \int \cos^n x \, dx$

$= \int \cos^{n-2} x \cos^2 x \, dx$

$= \int \cos^{n-2} x (1 - \sin^2 x) \, dx$

$= \int \cos^{n-2} x - \cos^{n-2} x \sin^2 x \, dx$

$= \int \cos^{n-2} x \, dx - \int \cos^{n-2} x \sin^2 x \, dx$

$= I_{n-2} - \int \cos^{n-2} x \sin^2 x \, dx$

$= I_{n-2} - \int \cos^{n-2} x \sin x \sin x \, dx$

$u = \sin x$ $\qquad dv = \cos^{n-2} x \sin x \, dx$

$du = \cos x \, dx$ $\qquad v = \dfrac{-\cos^{n-1} x}{n-1}$

$= I_{n-2} - \left(-\dfrac{\sin x \cos^{n-1} x}{n-1} \right.$

$\left. - \int \dfrac{-\cos^{n-1} x \cos x}{n-1} \, dx \right)$

$$= I_{n-2} - \left(\frac{-\sin x \cos^{n-1} x}{n-1} + \int \frac{\cos^n x}{n-1}\, dx \right)$$

$$= I_{n-2} + \frac{\sin x \cos^{n-1} x}{n-1} - \int \frac{\cos^n x}{n-1}\, dx$$

$$= I_{n-2} + \frac{\sin x \cos^{n-1} x}{n-1} - \frac{1}{n-1} I_n$$

$$\left(1 + \frac{1}{n-1}\right) I_n = I_{n-2} + \frac{\sin x \cos^{n-1} x}{n-1}$$

$$\left(\frac{n}{n-1}\right) I_n = I_{n-2} + \frac{\sin x \cos^{n-1} x}{n-1}$$

$$\therefore n I_n = (n-1) I_{n-2} + \sin x \cos^{n-1} x$$

When $n = 4$:

$$I_4 = \int \cos^4 x \, dx$$

$$4 I_4 = 3 I_2 + \sin x \cos^3 x$$

$$I_2 = \int \cos^2 x \, dx$$

$$= \int \frac{\cos 2x + 1}{2}\, dx$$

$$= \frac{\sin 2x}{4} + \frac{x}{2}$$

$$= \frac{\sin x \cos x}{2} + \frac{x}{2}$$

$$4 I_4 = \frac{3 \sin x \cos x}{2} + \frac{3x}{2} + \sin x \cos^3 x$$

$$\therefore I_4 = \frac{\sin x \cos^3 x}{4} + \frac{3 \sin x \cos x}{8} + \frac{3x}{8} + c$$

6. $3^n > 2n^2 - 1$

when $n = 1$:

$3^1 = 3 > 1 = 2 - 1 = 2(1) - 1 = 2(1^2) - 1$

Assume the statement is true when $n = k$.

$n = k$: $3^k > 2k^2 - 1$

when $n = k + 1$:

$3^{k+1} = 3^k 3 = 3^k + 2 \cdot 3^k$

$3^k \geqslant 2k + 1$

when $k = 1$: $3^1 = 3 = 2 + 1 = 2(1) + 1.$

Assume the statement is true when $k = r$.

$k = r$: $3^r \geqslant 2r + 1$

when $k = r + 1$:

$3^{r+1} = 3^r 3 = 3^r + 2 \cdot 3^r > 2r + 1 + 2$

$= 2r + 3$

$= 2(r + 1) + 1.$

$\therefore 3^k \geq 2k+1$ for every positive integer k.

Since $3^k > 2k^2 - 1$ and $3^k \geq 2k+1$

$$3^k + 2\cdot 3^k > 2k^2 - 1 + 2(2k+1)$$

$$3\cdot 3^k > 2k^2 + 4k + 2 - 1$$

$$3^{k+1} > 2(k^2 + 2k + 1) - 1$$

$$= 2(k+1)^2 - 1$$

$\therefore$ Since $3^n > 2n^2 - 1$ when $n = 1$ and $3^n > 2n^2 - 1$ when $n = k+1$ if $3^n > 2n^2 - 1$ when $n = k$, $3^n > 2n^2 - 1$ for every positive integer n.

7. $z^3 = 4 - 4\sqrt{3}i$ $\qquad |z^3| = \sqrt{4^2 + (4\sqrt{3})^2} = 8$

$\arg z^3 = \frac{5\pi}{3} \left(\frac{-\pi}{3}\right)$

Let a root be $r(\cos\theta + i\sin\theta)$: $r^3 = 8 \Rightarrow r = 2$

$3\theta = \frac{5\pi}{3} + 2\pi k \Rightarrow \theta = \frac{5\pi}{9} + \frac{2\pi k}{3}$

$\therefore z_1 = 2e^{\frac{i5\pi}{9}}, \quad z_2 = 2e^{\frac{i11\pi}{9}}, \quad z_3 = 2e^{\frac{i17\pi}{9}}$

$z_1^3 + z_2^3 + z_3^3 = 8\left(e^{\frac{i5\pi}{3}} + e^{\frac{i11\pi}{3}} + e^{\frac{i17\pi}{3}}\right)$

$= 3 \times 8e^{\frac{i5\pi}{3}}$

$= 24e^{\frac{i5\pi}{3}}$ $\qquad \alpha = 24, \ \beta = \frac{5}{3}$

8. (i) $x = 3t^2 + 1 \Rightarrow \dfrac{dx}{dt} = 6t$

$y = t^3 - 3t \qquad \dfrac{dy}{dt} = 3t^2 - 3$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{6t} = \frac{t^2 - 1}{2t}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{2t(2t) - (t^2 - 1)(2)}{4t^2} = \frac{t^2 + 1}{2t^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{t^2+1}{2t^2} \times \frac{1}{6t} = \frac{t^2+1}{12t^3}$$

Since $t > 0 \Rightarrow t^3 > 0$ and $t^2 + 1 > 0$

$\therefore \dfrac{d^2y}{dx^2} > 0$ for all $t > 0$

(ii) Surface area $= 2\pi \int x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= 2\pi \int (3t^2 + 1)\sqrt{3^2(t^2-1)^2 + (6t)^2}\, dt$$

$$= 2\pi \int (3t^2+1)(3)(t^2+1)\, dt$$

$$= 6\pi \left[\frac{3t^5}{5} + \frac{4t^3}{3} + t\right]_1^2$$

$$= 6\pi \left[\left(\frac{96}{5} + \frac{32}{3} + 2\right) - \left(\frac{3}{5} + \frac{4}{3} + 1\right)\right]$$

$$= 173\tfrac{3}{5}\pi$$

9. (i) Direction vector $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$

Vector $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$

$\therefore$ Length of $PQ = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}}{\sqrt{1+49+4}}$

$= \dfrac{35}{18}\sqrt{6}$ or 4.76

(ii) Vector $\perp$ to $\Pi_1 = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -11 \end{pmatrix}$ or $8\underset{\sim}{i} + 2\underset{\sim}{j} - 11\underset{\sim}{k}$

(iii) Distance from B to $\Pi_1 = \dfrac{|\overrightarrow{AB} \cdot \underset{\sim}{n}|}{|\underset{\sim}{n}|} = \dfrac{\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 2 \\ -11 \end{pmatrix}}{\sqrt{64+4+121}}$

$= \dfrac{17}{\sqrt{189}}$ or 1.24

(iv) Normal vector to $\Pi_2 = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 23 \\ -1 \\ -8 \end{pmatrix}$

$\cos\theta = \dfrac{\begin{pmatrix} 8 \\ 2 \\ -11 \end{pmatrix} \cdot \begin{pmatrix} 23 \\ -1 \\ -8 \end{pmatrix}}{\sqrt{189}\,\sqrt{594}} = \dfrac{270}{3\sqrt{21} \cdot 3\sqrt{66}} = \dfrac{10}{\sqrt{154}}$

$\theta = 36.3^\circ$.

10 To prove $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

when $n=1$ $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

It is true when $n=1$.

Assume true when $n=k$.

$\Rightarrow (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Consider $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)\cdot(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta$
$+ i\sin k\theta\cos\theta - \sin k\theta\sin\theta$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta)$
$+ i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

$\therefore$ It is also true when $n=k+1$ $\therefore$ true for all positive integers n.

$z^5 = (\cos\theta + i\sin\theta)^5$

$= c^5 + 5c^4 is + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4$
$+ (is)^5$

$\mathrm{Re}(z^5) = c^5 - 10c^3s^2 + 5cs^4$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta$
$+ 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos^5\theta$
$-10\cos^3\theta + 5\cos\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$$2\cos 5\theta = 32\cos^5\theta - 40\cos^3\theta + 10\cos\theta$$

Let $x = \cos\theta$

$$\therefore 32x^5 - 40x^3 + 10x - 2\cos 5\theta = 0$$

$$\Rightarrow 2\cos 5\theta = -1 \Rightarrow \cos 5\theta = -\frac{1}{2}$$

$$\therefore 5\theta = \frac{2\pi}{3} + 2\pi r \Rightarrow \theta = \frac{2\pi}{15} + \frac{2\pi r}{5}$$

$$\therefore x = \cos\frac{2\pi}{15}, \cos\frac{8\pi}{15}, \cos\frac{14\pi}{15}, \cos\frac{20\pi}{15}, \cos\frac{26\pi}{15}$$

11 a) $|A - \lambda I| = 0$ or $|\lambda I - A| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -4 & 2 \\ -4 & -1-\lambda & 6 \\ 2 & 6 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} -1-\lambda & 6 \\ 6 & -2-\lambda \end{vmatrix} - (-4)\begin{vmatrix} -4 & 6 \\ 2 & -2-\lambda \end{vmatrix}$$

$$+ 2\begin{vmatrix} -4 & -1-\lambda \\ 2 & 6 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 63\lambda + 162 = 0$$

$$(\lambda - 3)(\lambda^2 + 3\lambda - 54) = 0$$

$$(\lambda - 3)(\lambda + 9)(\lambda - 6) = 0$$

$$\lambda = 3, -9, 6$$

$$\lambda = 3: \quad \begin{pmatrix} 0 & 4 & -2 \\ 4 & 4 & -6 \\ -2 & -6 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\lambda = 6: \quad \begin{pmatrix} 3 & 4 & -2 \\ 4 & 7 & -6 \\ -2 & -6 & 8 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$\lambda = -9: \quad \begin{pmatrix} -12 & 4 & -2 \\ 4 & -8 & -6 \\ -2 & -6 & -7 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix} \qquad D = \begin{pmatrix} 27 & 0 & 0 \\ 0 & 216 & 0 \\ 0 & 0 & -729 \end{pmatrix}$$

11. (b) $$\begin{pmatrix} 1 & 2 & 4 & 9 \\ 2 & 4 & 7 & 15 \\ 1 & 3 & 5 & 13 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 4 & 9 \\ 0 & 1 & \frac{9}{5} & \frac{22}{5} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

(i) There are 3 leading 1's $\Rightarrow$ Dimension $= 3$

(ii) Basis for $R^3 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix} \right\}$

(iii) $$\left.\begin{aligned} a + 2b + 4c + 9d &= 0 \\ 5b + 9c + 22d &= 0 \\ c + 3d &= 0 \end{aligned}\right\}$$ Let $d = t$

$a = t$

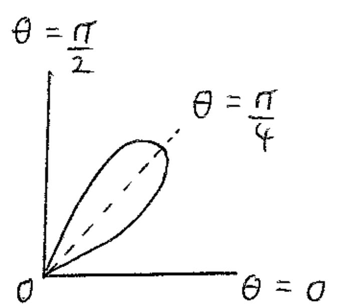
$b = t$

$c = -3t$

$\therefore \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}$ $\quad \therefore$ Basis is $\begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}$

OR

11. (a) (i) $r = a \sin 2\theta$



$$\sqrt{x^2 + y^2} = a \cdot 2 \sin\theta \cos\theta$$

$$= a \cdot 2 \left(\frac{y}{r}\right)\left(\frac{x}{r}\right)$$

$$\therefore (x^2 + y^2)^{\frac{3}{2}} = 2axy$$

(ii) $$\text{Area} = \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 \, d\theta$$

$$= \frac{1}{2} a^2 \int \sin^2 2\theta \, d\theta$$

$$= \frac{1}{4}a^2 \int (1 - \cos 4\theta)\ d\theta$$

$$= \frac{1}{4}a^2 \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{1}{8}a^2\pi$$

11 (b) Area $= \frac{1}{2}\int_0^{\beta} \left(\theta^{\frac{1}{2}} e^{\frac{\theta^2}{\pi}}\right)^2 d\theta$

$$\therefore 3\pi = \frac{1}{2}\int \theta \cdot e^{\frac{2\theta^2}{\pi}}\ d\theta$$

$$= \frac{1}{2} \times \frac{\pi}{4} \int \frac{4\theta}{\pi}\ e^{\frac{2\theta^2}{\pi}}\ d\theta$$

$$= \frac{\pi}{8}\left[e^{\frac{2\theta^2}{\pi}}\right]_0^{\beta}$$

$$\therefore 24 = e^{\frac{2\beta^2}{\pi}} - 1$$

$$e^{\frac{2\beta^2}{\pi}} = 25$$

$$\Rightarrow \frac{2}{\pi}\beta^2 = \ln 25 = 2\ln 5$$

$$\therefore \beta^2 = \pi \ln 5$$

$$\beta = \sqrt{\pi \ln 5}$$