1 $(r+1)^5 - (r-1)^5$

$= r^5 + 5r^4 + 10r^3 + 10r^2 + 5r + 1$

$-(r^5 - 5r^4 + 10r^3 - 10r^2 + 5r - 1)$

$= 10r^4 + 20r^2 + 2$

$$\therefore \sum_{r=1}^{n} (r+1)^5 - (r-1)^5$$

$$= \sum_{r=1}^{n} 10r^4 + 20r^2 + 2$$

$$(n+1)^5 - (n-1)^5 = 10\sum_{r=1}^{n} r^4 + 20\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2$$

$+ n^5 - (n-2)^5$

$+ (n-1)^5 - (n-3)^5$

$\vdots$

$+ 4^5 - 2^5$

$+ 3^5 - 1^5$

$+ 2^5 - 0^5$

$$(n+1)^5 + n^5 - 1 = 10\sum_{r=1}^{n} r^4$$

$$+ \frac{20n(n+1)(2n+1)}{6} + 2n$$

$$n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 + n^5 - 1$$

$$= 10\sum_{r=1}^{n} r^4 + \frac{10n(n+1)(2n+1)}{3} + 2n$$

$$10\sum_{r=1}^{n} r^4 + \frac{10n(n+1)(2n+1)}{3} + 2n$$

$$= 2n^5 + 5n^4 + 10n^3 + 10n^2 + 5n$$

$$10\sum_{r=1}^{n} r^4 = 2n^5 + 5n^4 + 10n^3 + 10n^2 + 5n$$

$$- \frac{10n(n+1)(2n+1)}{3} - 2n$$

$$= 2n^5 + 5n^4 + 10n^3 + 10n^2 + 3n$$

$$- \frac{10n(n+1)(2n+1)}{3}$$

$$= \frac{n}{3}\left(6n^4 + 15n^3 + 30n^2 + 30n + 9\right.$$

$$\left. - 10(2n^2 + 3n + 1)\right)$$

$$= \frac{n}{3}\left(6n^4 + 15n^3 + 30n^2 + 30n + 9\right.$$

$$\left. - 20n^2 - 30n - 10\right)$$

$$= \frac{n}{3}(6n^4 + 15n^3 + 10n^2 - 1)$$

$$\therefore \sum_{r=1}^{n} r^4 = \frac{n}{30}(6n^4 + 15n^3 + 10n^2 - 1)$$

2. $4^n > 3n^2 + 5$

when $n = 3$: $4^3 = 64$

> 32

$= 27 + 5$

$= 3(9) + 5$

$= 3(3^2) + 5$

Assume the statement is true when $n = k$.

$n = k$: $4^k > 3k^2 + 5$

when $n = k + 1$:

$4^k > 3k^2 + 5$

$4^k 4 > 4(3k^2 + 5)$

$4^{k+1} > 12k^2 + 20$

Since $k > 0$, $(3k - 1)^2 + 11 > 0$

$9k^2 - 6k + 12 > 0$

$12k^2 + 20 > 3k^2 + 6k + 8$

$= 3k^2 + 6k + 3 + 5$

$= 3(k^2 + 2k + 1) + 5$

$= 3(k + 1)^2 + 5$

Since $4^{k+1} > 12k^2 + 20$ and

$12k^2 + 20 > 3(k+1)^2 + 5$,

$4^{k+1} > 3(k+1)^2 + 5$

Since $4^n > 3n^2 + 5$ is true when $n = 3$

and $4^n > 3n^2 + 5$ is true when $n = k+1$

if $4^n > 3n^2 + 5$ is true when $n = k$

$4^n > 3n^2 + 5$ for every positive

integer $n > 2$.

3. $7x^3 - 6x^2 + 5x - 9 = 0$

α, β, γ are the roots.

$$\frac{3}{\alpha\beta} - \frac{\gamma}{4}, \quad \frac{3}{\alpha\gamma} - \frac{\beta}{4}, \quad \frac{3}{\beta\gamma} - \frac{\alpha}{4}$$

Let $u = \dfrac{3}{\alpha\beta} - \dfrac{\gamma}{4}$

$$= \frac{3\gamma}{\alpha\beta\gamma} - \frac{\gamma}{4}$$

$$= \frac{3\gamma}{-\frac{9}{7}} - \frac{\gamma}{4}, \quad \text{since } \alpha\beta\gamma = -\frac{9}{7}$$

$$= -\frac{7\gamma}{3} - \frac{\gamma}{4}$$

$$= \frac{-31\gamma}{12}$$

$$\therefore \gamma = \frac{-12u}{31}$$

γ is a root

$$\therefore 7\gamma^3 - 6\gamma^2 + 5\gamma + 9 = 0$$

$$7\left(\frac{-12u}{31}\right)^3 - 6\left(\frac{-12u}{31}\right)^2 + 5\left(\frac{-12u}{31}\right) + 9 = 0$$

$$-\frac{12096}{29791}u^3 - \frac{864}{961}u^2 - \frac{60u}{31} + 9 = 0$$

$$-12096u^3 - 26784u^2 - 57660u + 268119 = 0$$

$$12096u^3 + 26784u^2 + 57660u - 268119 = 0$$

$\therefore$ The equation having roots

$\frac{3}{\alpha\beta} - \frac{\gamma}{4}$, $\frac{3}{\alpha\gamma} - \frac{\beta}{4}$ and $\frac{3}{\beta\gamma} - \frac{\alpha}{4}$ is

$$12096u^3 + 26784u^2 + 57660u - 268119 = 0$$

$$\therefore \frac{3}{\alpha\beta} - \frac{\gamma}{4} + \frac{3}{\alpha\gamma} - \frac{\beta}{4} + \frac{3}{\beta\gamma} - \frac{\alpha}{4}$$

$$= \frac{-26784}{12096} = \frac{-31}{14}$$

$$\left(\frac{3}{\alpha\beta} - \frac{\gamma}{4}\right)\left(\frac{3}{\alpha\gamma} - \frac{\beta}{4}\right) + \left(\frac{3}{\alpha\beta} - \frac{\gamma}{4}\right)\left(\frac{3}{\beta\gamma} - \frac{\alpha}{4}\right)$$

$$+ \left(\frac{3}{\alpha\gamma} - \frac{\beta}{4}\right)\left(\frac{3}{\beta\gamma} - \frac{\alpha}{4}\right) = \frac{57660}{12096}$$

$$= \frac{4805}{1008}$$

$$\left(\frac{3}{\alpha\beta} - \frac{\gamma}{4}\right)\left(\frac{3}{\alpha\gamma} - \frac{\beta}{4}\right)\left(\frac{3}{\beta\gamma} - \frac{\alpha}{4}\right) = \frac{268119}{12096}$$

$$\left(\frac{3}{\alpha\beta} - \frac{\gamma}{4}\right)^2 + \left(\frac{3}{\alpha\gamma} - \frac{\beta}{4}\right)^2 + \left(\frac{3}{\beta\gamma} - \frac{\alpha}{4}\right)^2$$

$$= \left(\frac{3}{\alpha\beta} - \frac{\gamma}{4} \quad \frac{3}{\alpha\gamma} - \frac{\beta}{4} + \frac{3}{\beta\gamma} - \frac{\alpha}{4}\right)^2$$

$$- 2\left[\left(\frac{3}{\alpha\beta} - \frac{\gamma}{4}\right)\left(\frac{3}{\alpha\gamma} - \frac{\beta}{4}\right) + \left(\frac{3}{\alpha\beta} - \frac{\gamma}{4}\right)\left(\frac{3}{\beta\gamma} - \frac{\alpha}{4}\right)\right.$$

$$\left. + \left(\frac{3}{\alpha\gamma} - \frac{\beta}{4}\right)\left(\frac{3}{\beta\gamma} - \frac{\alpha}{4}\right)\right]$$

$$= \left(\frac{-31}{14}\right)^2 - 2\left(\frac{4805}{1008}\right)$$

$$= \frac{961}{196} - \frac{4805}{504}$$

$$= -\frac{457436}{97894}$$

$$= -\frac{114359}{24696}$$

4. $l_1: \underset{\sim}{r} = \begin{pmatrix} 4 \\ -1 \\ -9 \end{pmatrix} + s\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 9 \\ 0 \\ -7 \end{pmatrix} + s\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

$l_2 \quad \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

Direction vector of $l_3 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \\ 14 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$l_3: \underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

i) Normal vector of the plane $= \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 4 \end{pmatrix}$

$\therefore 7x - y + 4z = d$

$d = 7(2) - (1) + 4(-5) = -7$

$7x - y + 4z = -7$

ii) To show l_1 & l_3 intersect

$x: \; 4 + 5s = 2 + \lambda$

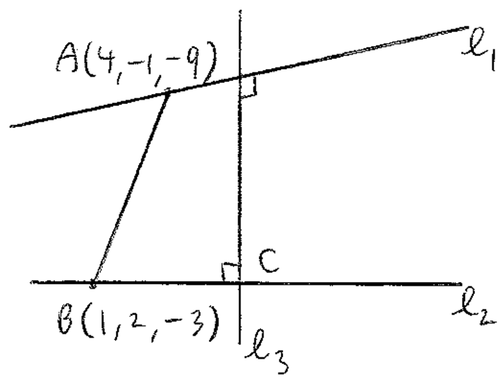
$y: \; -1 + s = 1 - \lambda$

$3 + 6s = 3 \Rightarrow s = 0$

$\lambda = 2$

$$\left.\begin{aligned} z &= -9 + 2(0) = 9 \\ z &= -5 + 2(-2) = -9 \end{aligned}\right\} \text{consistent}$$

iii) l_1 meets l_3 at $(4,-1,-9)$



Shortest dist between l_1 & l_2

$$= \frac{\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}}{\sqrt{1+1+4}} = \frac{\begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}}{\sqrt{6}}$$

$$= 3\sqrt{6}$$

5. $x + y - z + 2 = 0$

$x - 2y + z - 9 = 0$

$$\cos\theta = \frac{\begin{pmatrix}1\\1\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\1\end{pmatrix}}{\sqrt{3}\ \sqrt{6}}$$

$$= \frac{-2}{\sqrt{18}}$$

$\theta = 118.1°$

Acute ∡ between planes $\approx 61.9°$

i) $\vec{OP} = \begin{pmatrix}\alpha\\\beta\\10\end{pmatrix}$

$\alpha + \beta - 10 + 2 = 0$

$\alpha - 2\beta + 10 - 9 = 0$

$3\beta - 9 = 0 \Rightarrow \beta = 3$

$\alpha = 5$

ii) l : $\begin{pmatrix}1\\-2\\1\end{pmatrix} \times \begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

$$\underset{\sim}{r} = \begin{pmatrix}5\\3\\10\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix}$$

Since ℓ is $\perp$ to both Π_1 & Π_2

Let $\left.\begin{array}{l} \Pi_1 \text{ be } x + 2y + 3z = d_1 \\ \Pi_2 \text{ be } x + 2y + 3z = d_2 \end{array}\right\}$

But dist fr $P(5,3,10)$ to the planes can be

$$\pm\sqrt{14} = \frac{(5) + 2(3) + 3(10) - d}{\sqrt{1^2 + 2^2 + 3^2}}$$

$$\pm 14 = 41 - d$$

$$\left.\begin{array}{l} \Rightarrow d_1 = 41 + 14 = 55 \\ \Rightarrow d_2 = 41 - 14 = 27 \end{array}\right\}$$

and

$\therefore$ $\Pi_1 : x + 2y + 3z = 55$

$\Pi_2 : x + 2y + 3z = 27$

6 $$y = \frac{2x^2 - 6x + 8}{x^2 - 5x + 4} = 2 + \frac{4x}{(x-1)(x-4)}$$

i) Asymptotes : $y = 2$, $x = 1$, $x = 4$

ii) $$\frac{dy}{dx} = \frac{(x^2 - 5x + 4)(4) - 4x(2x - 5)}{(x^2 - 5x + 4)^2}$$

$$= \frac{16 - 4x^2}{(x^2 - 5x + 4)^2}$$

$16 - 4x^2 = 0 \Rightarrow x = \pm 2$

when $x = 2$, $y = 2 + \frac{8}{-2} = -2$

When $x = -2$, $y = 2 + \frac{(-8)}{(-3)(-6)} = \frac{14}{9}$

$$\frac{d^2y}{dx^2} = \frac{(x^2 - 5x + 4)^2(-8x) - (16 - 4x^2)(2x - 5)}{(x^2 - 5x + 4)^4}$$

when $x = 2$, $\dfrac{d^2y}{dx^2} < 0 \Rightarrow (2,-2)$ is a max pt

$x = -2$, $\dfrac{d^2y}{dx^2} > 0 \Rightarrow \left(-2, \dfrac{14}{9}\right)$ is a min pt

$x = 0 \quad y = 2$

$y = 0 \quad x = \dfrac{3 \pm \sqrt{-7}}{2}$

C intercepts y - axis at $(0, 2)$

C does not intercept x - axis.

$x \to +\infty, \; y \to 2^+ \qquad x \to 1^+, \; y \to -\infty$

$x \to -\infty, \; y \to 2^- \qquad x \to 1^-, \; y \to +\infty$