Vieta's Formulas

Theorem:

Let $R_1, R_2, R_3, \ldots$ and R_n be the n roots of a polynomial equation of degree n, $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$.

Therefore, $\underline{\underline{\sum R_i = -\frac{a_{n-1}}{a_n}}}$, $\underline{\underline{\sum R_i R_j = \frac{a_{n-2}}{a_n}}}$, $\underline{\underline{\sum R_i R_j R_k = -\frac{a_{n-3}}{a_n}}}$,

.... OR in general, $\underline{\underline{\sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k} = (-1)^k \frac{a_{n-k}}{a_n}}}$.

Example:

Let α, β, γ and δ be the roots of the a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$. State all the Vieta's formulas.

Example:

Let α, β, γ and δ be the roots of the a quartic equation $2x^4 - 3x^2 + 5x - 8 = 0$.

(a) Find the equation with roots $\dfrac{\alpha}{2}, \dfrac{\beta}{2}, \dfrac{\gamma}{2}$ and $\dfrac{\delta}{2}$.

(b) Find $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$.

Example:

Three of the four roots of the quartic equation $ax^4 + cx^2 + dx + e = 0$ are equal. Prove that $8c^3 + 27ad^2 = 0$ and $12ae + c^2 = 0$.

Deduce the solution of the equation in terms of a and d.

Problem:

The roots of $x^2 - 2x - 3 = 0$ are α and β. Without evaluating α and β, find

(a) $\alpha + \beta$.

(b) $\alpha^2 + \beta^2$.

(c) $\alpha^3 + \beta^3$.

(d) $\alpha^4 + \beta^4$.

(e) $\alpha^{-1} + \beta^{-1}$.

Activity:

The aim of this activity is to solve the previous problem and find a general expression for $\alpha^N + \beta^N$. Follow the instructions:

(1) $\alpha^2 + \beta^2$

(a) Rewrite the equation in terms of α, and name it $Eqn\ 1$.

$Eqn\ 1$: ______________________________

(b) Do the same for β, and name the equation $Eqn\ 2$.

$Eqn\ 2$: ______________________________

(c) Add $Eqn\ 1$ and $Eqn\ 2$, and simplify your equation.

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(d) Hence, $\alpha^2 + \beta^2 =$

(2) Evaluation of $\alpha^3 + \beta^3$.

(a) Multiply Eqn 1 with α.

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(b) Multiply Eqn 2 with β.

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(c) Add both new equation and simplify your equation.

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(d) Hence, $\alpha^3 + \beta^3 =$

(3) It is your turn to evaluate

$(a)\ \alpha^4 + \beta^4$

$(b)\ \alpha^{-1} + \beta^{-1}$

(4) Does this method work for cubic equation?

You may try with the following problem:

Let α, β and γ be the roots of the cubic equation

$x^3 - 2x^2 + x - 3 = 0$. Without evaluating α, β and γ, find

(a) $\alpha^3 + \beta^3 + \gamma^3$

(b) $\alpha^4 + \beta^4 + \gamma^4$

(c) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$

Prove the following :

(a) If α and β be the roots of a quadratic equation

$ax^2 + bx + c = 0$ and $\underline{S_N = \alpha^N + \beta^N}$, then

$\underline{\underline{aS_{2+r} + bS_{1+r} + cS_r = 0}}$ where $r \in \mathrm{Z}$.

(b) If α, β and γ the the roots of a cubic equation

$ax^3 + bx^2 + cx + d = 0$ and $\underline{S_N = \alpha^N + \beta^N + \gamma^N}$,

then $\underline{\underline{aS_{3+r} + bS_{2+r} + cS_{1+r} + dS_r = 0}}$ where $r \in \mathrm{Z}$.

Newton's Sum

Theorem :

Let $R_1, R_2, ..., R_n$ be the roots of a polynomial equation of degree n, $a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0$. If $\underline{S_N = R_1{}^N + R_2{}^N + ... + R_n{}^N}$, then

$$\underline{\underline{a_n S_{n+r} + a_{n-1} S_{(n-1)+r} + ... + a_{n-i} S_{(n-i)+r} + ... + a_0 S_r = 0}}$$

Example:

The cubic equation $2x^3 + 3x^2 + 4x + 5 = 0$ has roots α, β and γ. Find the values of

(a) $\alpha^3 + \beta^3 + \gamma^3$

(b) $\alpha^4 + \beta^4 + \gamma^4$

(c) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$

Example:

The equation $x^4 - 8x + 3 = 0$ has roots α, β, γ and δ.

Find the values of

(a) $2\alpha + 2\beta + 2\gamma + 2\delta$

(b) $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$

(c) $\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}$

(d) $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example:

The product of two of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ is equal to the product of the other two roots. Prove that $ad^2 = b^2 e$.

Example:

The equation $x^4 - 8x + 3 = 0$ has roots α, β, γ and δ. Find the values of

$$(\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\alpha - \delta)^2 + (\beta - \gamma)^2 + (\beta - \delta)^2 + (\gamma - \delta)^2$$

Example:

The equation $x^4-3x^3+6x^2-9x+13=0$ has roots α, β, γ and δ. Find the values of

(a) $\alpha^2+\beta^2+\gamma^2+\delta^2$

(b) $\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1}$

(c) $\alpha^3+\beta^3+\gamma^3+\delta^3$

(d) $\alpha^{-2}+\beta^{-2}+\gamma^{-2}+\delta^{-2}$

Example:

The roots of the equation $x^4 + 9x^3 + 27x^2 + 36x + 18 = 0$ are α, β, γ and δ.

(a) Obtain an equation in u which has roots $\alpha + 1$, $\beta + 1$, $\gamma + 1$ and $\delta + 1$.

(b) By mean of substitution of $v = u + \dfrac{1}{u}$, show that the equation in (a) can be reduced to a quadratic equation in v.

(c) Hence, find α, β, γ and δ.

Example:

Solve the simultaneous equations $\begin{cases} p+q+r=2 \\ p^2+q^2+r^2=6. \\ p^3+q^3+r^3=8 \end{cases}$