1. POLYNOMIAL EQUATIONS

1. The roots of the equation $x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ $(a_1 \neq 0)$, are $\alpha, -\alpha, \beta$ and γ. Show that
   (a) $a_1a_2a_3 = a_1^2 + a_0a_3^2$,
   (b) β, γ are the roots of the equation $a_1x^2 + a_1a_3x + a_0a_3 = 0$.
   Solve the equation $9x^4 + 6x^3 - 9x^2 - 2x + 2 = 0$ given that two of the roots are equal in magnitude and opposite in sign. (J72/I/1)

2. (a) Find all the roots, real or complex, of the equation

$$\begin{vmatrix} x^4 & x & 1 \\ 16 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

   (b) If α, β, γ are the roots of the equation $x^3 + bx^2 + ax + a = 0$, prove that

$$\begin{vmatrix} 1+\alpha & 1 & 1 \\ 1 & 1+\beta & 1 \\ 1 & 1 & 1+\gamma \end{vmatrix} = 0.$$

   (J72/II/1)

3. If $t \in R$ (the set of real numbers), find the set of values of t for which the solution set, in R, of the equation $tx^2 + x + 1 = 0$ is non-empty.
   Given that t is small and positive and the roots are denoted by α and β $(\alpha > \beta)$, expand α and β in ascending powers of t as far as the term in t and describe how α and β vary as t tends to zero. (J73/II/1)

4. (a) If a, b, and c are non-zero constants such that the roots of $x^3 + ax^2 + bx + c = 0$ are α, β and $(\alpha + \beta)$, prove that $4ab = a^3 + 8c$, and form the cubic equation whose roots are α, β and $\alpha\beta$, giving the coefficients in terms of a and c.
   (b) Make a sketch to show the regions in the x–y plane within which points (x, y) satisfy simultaneous inequalities $(x + 1)^2 \geqslant 4$ and $y^2 \geqslant 4x^2$. (N73/I/1)

5. (a) Solve for x the equation $x^2 - 2ax\cos\theta + a^2 = 0$ and show that the expression on the left-hand side is a factor of $E \equiv x^{2n} - 2a^nx^n\cos n\theta + a^{2n}$.
   Factorise E as a product of factors that are quadratic in x.
   (b) If β $(\neq 1)$ is a root of the equation $z^5 = 1$, show that $1 + \beta + \beta^2 + \beta^3 + \beta^4 = 0$.
   Hence or otherwise find a quadratic equation, with real coefficients, whose roots are $\beta + \beta^4$ and $\beta^2 + \beta^3$ (N74/II/17)

6. Let $p(x)$ be a real polynomial and a be a real number. Show that there is a real polynomial $q(x)$ such that $p(x) \equiv (x - a)^2\, q(x)$ if, and only if, $p(a) = 0 = p'(a)$.
   Hence, or otherwise, factorise the polynomial $4x^3 - 36x^2 + 81x - 54$, given that it has a repeated factor. (J75/I/1)

7. Express $(y + z)^2(y^2 - z^2) + (z + x)^2(z^2 - x^2) + (x + y)^2(x^2 - y^2)$ as a product of four first-degree factors. (J76/I/1)

8. (a) Express $\dfrac{1}{(x^2+1)(x-2)}$ in partial fractions.

(b) A cubic polynomial $P(x)$, with real coefficients, has remainder -2 when divided by $x-1$ and remainder -6 when divided by $x+1$. Find the remainder when $P(x)$ is divided by x^2-1.
If it is also known that one of the roots of the equation $P(x)=0$ is i (where $i^2 = -1$), find $P(x)$. (J77/I/1)

9. The equation $x^3 + ax^2 + bx + c = 0$, in which a, b, c are non-zero constants, has roots which form a geometric progression. Prove that $a^3c = b^3$.
Prove also that if the common ratio of the progression is 2, then $7b = 2a^2$.
Solve the equation $\sqrt{2}x^3 - 7x^2 + 7\sqrt{2}x - 4 = 0$. (N77/I/1)

10. (a) Solve the equation $2x^3 - 3x^2 - 59x + 30 = 0$, given that the roots are in arithmetic progression.
(b) The roots of the equation $2x^3 + x^2 - 5x + 3 = 0$ are α, β, γ. Find
(i) $\alpha^2 + \beta^2 + \gamma^2$,
(ii) $\alpha^3 + \beta^3 + \gamma^3$,
(iii) a cubic equation, with numerical coefficients, which has roots $\alpha+1$, $\beta+1$, $\gamma+1$. (J78/I/1)

11. The cubic equation $x^3 + ax^2 + bx + c = 0$, in which a, b, c are real constants, has roots x_1, x_2, x_3. Give expressions for $\sum_{i=1}^{3} x_i$ and $\sum_{i=1}^{3} x_i^2$ in terms of the coefficients of the cubic equation.
Obtain, in either order,

(a) the equation $\sum_{i=1}^{3} x_i^3 + a\sum_{i=1}^{3} x_i^2 + b\sum_{i=1}^{3} x_i + 3c = 0$,

(b) an expression for $\sum_{i=1}^{3} x_i^3$ in terms of a, b, c.

Hence, or otherwise, given the simultaneous equations

$$\begin{aligned} p + q + r &= 4, \\ p^2 + q^2 + r^2 &= 12, \\ p^3 + q^3 + r^3 &= 64, \end{aligned}$$

find a cubic equation which has p, q and r as its roots, and solve this equation. (N78/I/1)

12. (a) In the quadratic equation $(k+3)x^2 - 2(k+1)x + 2k - 1 = 0$, k is a real constant and $k \neq -3$.
Find the set of values of k for which the equation has
(i) real roots,
(ii) real roots, one positive and one negative,
(iii) real, positive roots.
(b) The quadratic equation $5x^2 - 6x + 3 = 0$ has roots α and β. Find the values of
(i) $\alpha^2 + \alpha\beta + \beta^2$, (ii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$. (J79/I/1)

13. If z_1 and z_2 are complex numbers prove that $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, and $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$. Deduce that if z_1 is a root of the equation $az^4 + bz^3 + cz^2 + dz + e = 0$, where a, b, c, d, e, ϵ R, then $\bar{z}_1$ is also a root.
Hence, or otherwise,
(a) find a polynomial of the fourth degree with real coefficients, which has $2 + i$ *and* $1 - 2i$ as two of its roots,
(b) solve the equation $4z^4 - 24z^3 + 39z^2 + 6z - 10 = 0$, given that $3 - i$ is one of the roots. (J79/II/1)

14. (a) Solve the equation $6x^3 + 25x^2 - 62x + 24 = 0$, given that the product of two of the roots is -8.
(b) The roots of the equation $36x^3 + 72x^2 + 23x - 6 = 0$ are α, β, γ. Find a cubic equation with numerical coefficients, which has roots $6\alpha + 1, 6\beta + 1, 6\gamma + 1$. (J81/I/1)

15. The equation $x^2 - 2x + a + bx^{-1} + 4x^{-2} = 0$ has roots $\alpha, \beta, \gamma, \delta$. Given that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 12$, and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = 1$, find the values of a and b.
Show that if $y = x + 2x^{-1}$, the given equation can be written as a quadratic equation in y.
Hence, or otherwise, solve the given equation. (N81/I/1)

16. The equation $2x^3 - 3x^2 + ax + b = 0$, where a and b are real constants, has roots α, β and γ. Given that $\alpha = 1 - i\sqrt{3}$,
(a) find β, γ, a and b,
(b) prove that, for $n \epsilon$ Z,
$$\alpha^n + \beta^n + \gamma^n = 2^{n+1}\cos(\tfrac{1}{3}n\pi) + (-\tfrac{1}{2})^n.$$
(J82/I/3)

17. (a) Determine the complex numbers v and w for which $z^2 + (4 - 2i)z = (z + v)^2 - w$, for all $z \epsilon$ C.
(b) Find the square roots of $5 - 12i$.
(c) Solve the quadratic equation $z^2 + (4 - 2i)z = 2 - 8i$.
(d) Obtain an equation, of degree four with integer coefficients, having as two of its roots $-5 + 3i$ and $1 - i$. (N82/I/1)

18. The equation $x^3 + px^2 + qx + r = 0$ has positive roots α, β, γ. The arithmetic mean of α, β, γ, is A; the geometric mean is G, where $G = (\alpha\beta\gamma)^{\frac{1}{3}}$; and the harmonic mean is H, where $H^{-1} = \frac{1}{3}(\alpha^{-1} + \beta^{-1} + \gamma^{-1})$. Express p, q, r in terms of A, G, H.
Given that three numbers have arithmetic mean 4, geometric mean 2 and harmonic mean 1, show that the numbers are the roots of the equation $x^3 - 12x^2 + 24x - 8 = 0$.
Hence, or otherwise, find the numbers. (N82/II/1)

19. Prove that if $(x - a)^2$ is a factor of the polynomial $f(x)$ then $(x - a)$ is a factor of $f'(x)$. Give a counter-example to show that the converse of this result is not true.
(a) Solve the equation $3x^3 + 29x^2 + 65x - 25 = 0$, given that two of its roots are equal.
(b) Solve the equation $54x^4 + 27x^3 - 198x^2 + 164x - 40 = 0$, given that three of its roots are equal. (J83/II/1)

20. The equation $x^3 - 2x^2 + 3x - 5 = 0$ has roots α, β, γ. Given that the equation $x^3 + px^2 + qx + r = 0$ has roots $\beta + \gamma, \gamma + \alpha, \alpha + \beta$, obtain numerical values for p, q and r. (N83/I/1)

21. (a) Given that the roots of the equation $x^3 + px^2 + qx + r = 0$ are three consecutive terms of an arithmetic progression, show that $2p^3 + 27r = 9pq$.
(b) Given that the roots of the equation $x^3 + px^2 + qx + r = 0$ are three consecutive terms of a geometric progression, find a condition that p, q and r must satisfy. (J84/I/1)

22. The equation $3x^4 + 4x^3 - x^2 - 10x - 6 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Given that $\alpha\beta = 2$, write down the value of $\gamma\delta$, and express $3x^4 + 4x^3 - x^2 - 10x - 6$ as a product of two quadratic factors.
Hence, or otherwise, solve the equation. (N84/I/1)

23. (a) Express $\cos 5\theta$ as a polynomial in $\cos\theta$.
(b) Write down the solutions of the equation $\cos 5\theta = 1$, for $0 \leqslant \theta < 2\pi$.
Deduce that $\cos\frac{2}{5}\pi$ is one root of the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$.
(c) By expressing $16x^4 + 16x^3 - 4x^2 - 4x + 1$ in the form $(ax^2 + bx + c)^2$, or otherwise, obtain the exact value of $\cos\frac{2}{5}\pi$ (N84/II/1)

24. The equation $x^3 - 9x^2 + 28x - 27 = 0$ has roots α, β and γ.
(a) Find the numerical value of $x^3 + \beta^3 + \gamma^3$.
(b) Given that the equation $y^3 + py^2 + qy + r = 0$ has roots $\alpha - 3, \beta - 3, \gamma - 3$, obtain numerical values for p, q and r.
(c) Using the result of (b), or otherwise, determine the number of real roots of the equation $x^3 - 9x^2 + 28x - 27 = 0$, and find between which pair of consecutive integers ecah real root lies. (J85/I/1)

25. The equation $x^3 + px + q = 0$ has roots α, β, γ.
(a) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of p and q.
(b) Prove that $\alpha^3 + \beta^3 + \gamma^3 = -3q$.
(c) Find the numerical value of $\alpha^4 + p\alpha^2 + q\alpha$.
(d) Prove that $2(\alpha^4 + \beta^4 + \gamma^4) - (\alpha^2 + \beta^2 + \gamma^2)^2 = 0$. (N85/I/1)

26. (a) Solve the equation $x^3 + x^2 - 19x + 5 = 0$, given that the product of two of its roots is equal to 1.
(b) The roots of the equation $x^4 - 6x^3 + 5x^2 - 6x + 4 = 0$ are $\alpha, \beta, \gamma, \delta$. By using the substitution $x = \sqrt{y}$, or otherwise, show that the equation having roots $\alpha^2, \beta^2, \gamma^2, \delta^2$ is $y^4 - 26y^3 - 39y^2 + 4y + 16 = 0$
Hence, or otherwise, find the exact value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. (J86/I/1)

27. Given that α, β and γ are the roots of the cubic equation $x^3 + x^2 - 2x - 5 = 0$, *find the exact values of*
(i) $\alpha^2 + \beta^2 + \gamma^2$,
(ii) $\alpha^3 + \beta^3 + \gamma^3$, (N86/I/1)

28. The roots of the equation $x^3 - 4x - 1 = 0$ are α, β and γ. Find the exact value of $\alpha^2 + \beta^2 + \gamma^2$ and of $\alpha^4 + \beta^4 + \gamma^4$. (J87/I/1)

29. Show, by considering the relation $y = (1 + x)^{-1}$ between the variables x and y, or otherwise, that if α, β, λ and δ are the roots of the equation $x^4 - 2x - 1 = 0$, then $(1 + \alpha)^{-1}$, $(1 + \beta)^{-1}$, $(1 + \lambda)^{-1}$ are the roots of the equation
$2y^4 - 6y^3 + 6y^2 - 4y + 1 = 0$.
Hence find the exact value of $(1 + \alpha)^{-2} + (1 + \beta)^{-2} + (1 + \lambda)^{-2} + (1 + \delta)^{-2}$.
(N87/I/1)

30. Given that α, β and γ are the roots of the equation $x^3 + 3x - 1 = 0$, find the exact value of $\alpha^3 + \beta^3 + \gamma^3$. (J88/I/1)

31. For the equation $x^4 + 2x^3 + 3x^2 + 5x + 1 = 0$,

(i) obtain the sum of the squares of the roots of the equation;

(ii) show that the equation has two negative roots, α and β, such that $-2 < \alpha < \beta < 0$;

(iii) deduce from (i) and (ii), or prove otherwise, that the equation has no other real roots;

(iv) show that γ and δ, the complex roots of the equation, satisfy $|\gamma| = |\delta| = 1/\surd(\alpha\beta)$. (N88/I/1)

32. Given that $x^3 + x^2 + 2x - 2 = 0$ and that $y = \frac{x}{x + 2}$, show that

$$4y^3 + 2y^2(1 - y) + 2y(1 - y)^2 - (1 - y)^3 = 0.$$

Hence find the exact value of $(\frac{\alpha}{\alpha + 2})^2 + (\frac{\beta}{\beta + 2})^2 + (\frac{\gamma}{\gamma + 2})^2$, where α, β and γ are the roots of the given cubic equation in x. (J89/I/1)

33. The cubic equation $x^3 + px + q = 0$, where $q \neq 0$, has roots α, β and γ. By putting $y = -\frac{q}{x}$, or otherwise, show that the cubic equation $y^3 - py^2 - q^2 = 0$ has roots $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$.

Show that $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 = p^2$. (N89/I/1)

34. Given that α, β and γ are the roots of the equation $x^3 - 2x - 5 = 0$, find a cubic equation which has roots α^3, β^3 and γ^3.

Hence, or otherwise, find the values of $\alpha^3 + \beta^3 + \gamma^3$ and $\alpha^6 + \beta^6 + \gamma^6$. (J90/I/2)

35. The roots of the equation $x^3 - 3x^2 - 4x - 1 = 0$ are α, β and γ. Find the exact values of $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^3 + \beta^3 + \gamma^3$. (N90/I/1)