OCTOBER / NOVEMBER 2005

1. $$1^{\frac{1}{5}} = (\cos 0 + i\sin 0)^{\frac{1}{5}}$$

$$= \left[\cos(0 + 2k\pi) + i\sin(0 + 2k\pi)\right]^{\frac{1}{5}}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{5}}$$

$$= \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}, \quad k = 0, 1, 2, 3, 4$$

$$z^5 = -16 + 16\sqrt{3}i$$

$$= 16(-1 + \sqrt{3}i)$$

$$= 16(2)\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$$

$$= 32\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$$

$$= 32\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$= 32\left(\cos\left(\frac{2\pi}{3} + 2k\pi\right) + i\sin\left(\frac{2\pi}{3} + 2k\pi\right)\right)$$

$$\therefore z = \left(32\left(\cos\left(\frac{2\pi}{3} + 2k\pi\right) + i\sin\left(\frac{2\pi}{3} + 2k\pi\right)\right)\right)^{\frac{1}{5}}$$

$$= 32^{\frac{1}{5}}\left(\cos\left(\frac{2\pi}{3} + 2k\pi\right) + i\sin\left(\frac{2\pi}{3} + 2k\pi\right)\right)^{\frac{1}{5}}$$

$$= 2\left(\cos\frac{1}{5}\left(\frac{2\pi}{3} + 2k\pi\right) + i\sin\frac{1}{5}\left(\frac{2\pi}{3} + 2k\pi\right)\right)$$

$$= 2\left(\cos\left(\frac{2\pi}{15} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{2\pi}{15} + \frac{2k\pi}{5}\right)\right)$$

$$= 2e^{\left(\frac{2\pi}{15} + \frac{2k\pi}{5}\right)i}, \quad k = 0, 1, 2, 3, 4$$

2. $u_{n+1} = -1 + \sqrt{u_n + 7}$, $u_1 = 1$

i) $u_1 = 1 < 2$.

Assume that $u_n < 2$ when $n = k$: $u_k < 2$

Since $u_k < 2$

$$u_k + 7 < 9$$

$$\sqrt{u_k + 7} < 3$$

$$-1 + \sqrt{u_k + 7} < 2$$

$$u_{k+1} < 2$$

Since $u_1 < 2$ and $u_{k+1} < 2$ if $u_k < 2$

$\therefore$ $u_n < 2$ for all $n \geq 1$

ii) If $u_n = 2 - \varepsilon$

$$\therefore u_{n+1} = -1 + \sqrt{u_n + 7}$$

$$= -1 + \sqrt{2 - \varepsilon + 7}$$

$$= -1 + \sqrt{9 - \varepsilon}$$

$$= -1 + \sqrt{9\left(1 - \frac{\varepsilon}{9}\right)}$$

$$= -1 + \sqrt{9}\sqrt{1 - \frac{\varepsilon}{9}}$$

$$= -1 + 3\left(1 - \frac{\varepsilon}{9}\right)^{\frac{1}{2}}$$

$$= -1 + 3\left(1 + \frac{1}{2}\left(-\frac{\varepsilon}{9}\right) + \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(-\frac{\varepsilon}{9}\right)^2 \frac{1}{2!} + \cdots\right)$$

$$= -1 + 3\left(1 - \frac{\varepsilon}{18} + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{\varepsilon^2}{81}\left(\frac{1}{2}\right) \cdots\right)$$

$$= -1 + 3\left(1 - \frac{\varepsilon}{18} - \frac{\varepsilon^2}{648} \cdots\right)$$

$$= -1 + 3 - \frac{\varepsilon}{6} - \frac{\varepsilon^2}{216} \cdots$$

$$= 2 - \frac{\varepsilon}{6} - \frac{\varepsilon^2}{216} - \cdots$$

$$= 2 - \frac{\varepsilon}{6}, \quad \text{if } \varepsilon \text{ is small.}$$

3. $C: \quad y = \dfrac{x^2}{x+\lambda}, \quad \lambda \neq 0.$

$x \to -\lambda \quad y \to \pm\infty$

$\therefore$ $x = -\lambda$ is an asymptote of C.

$$\begin{array}{r|l} & x - \lambda \\ \hline x+\lambda & x^2 \\ & x^2 + \lambda x \\ \hline & -\lambda x \\ & -\lambda x - \lambda^2 \\ \hline & \lambda^2 \end{array}$$

$\therefore$ $y = x - \lambda + \dfrac{\lambda^2}{x+\lambda}$

$x \to \pm\infty \quad y \to x - \lambda$

$y = x - \lambda$ is an asymptote of C

$\therefore$ The asymptotes of C are $x = -\lambda$ and $y = x - \lambda$.

when $x = 0$ $\quad y = 0$

i) $\lambda > 0$:

$$y = x - \lambda + \frac{\lambda^2}{x+\lambda}$$

$$\frac{dy}{dx} = 1 - \frac{\lambda^2}{(x+\lambda)^2}$$

If $\dfrac{dy}{dx} = 0$ $\quad 1 - \dfrac{\lambda^2}{(x+\lambda)^2} = 0$

$$\frac{\lambda^2}{(x+\lambda)^2} = 1$$

$$\lambda^2 = (x+\lambda)^2$$

$$= x^2 + 2\lambda x + \lambda^2$$

$$x^2 + 2\lambda x = 0$$

$$x(x + 2\lambda) = 0$$

$$x = 0, -2\lambda$$

$$y = 0, -4\lambda$$

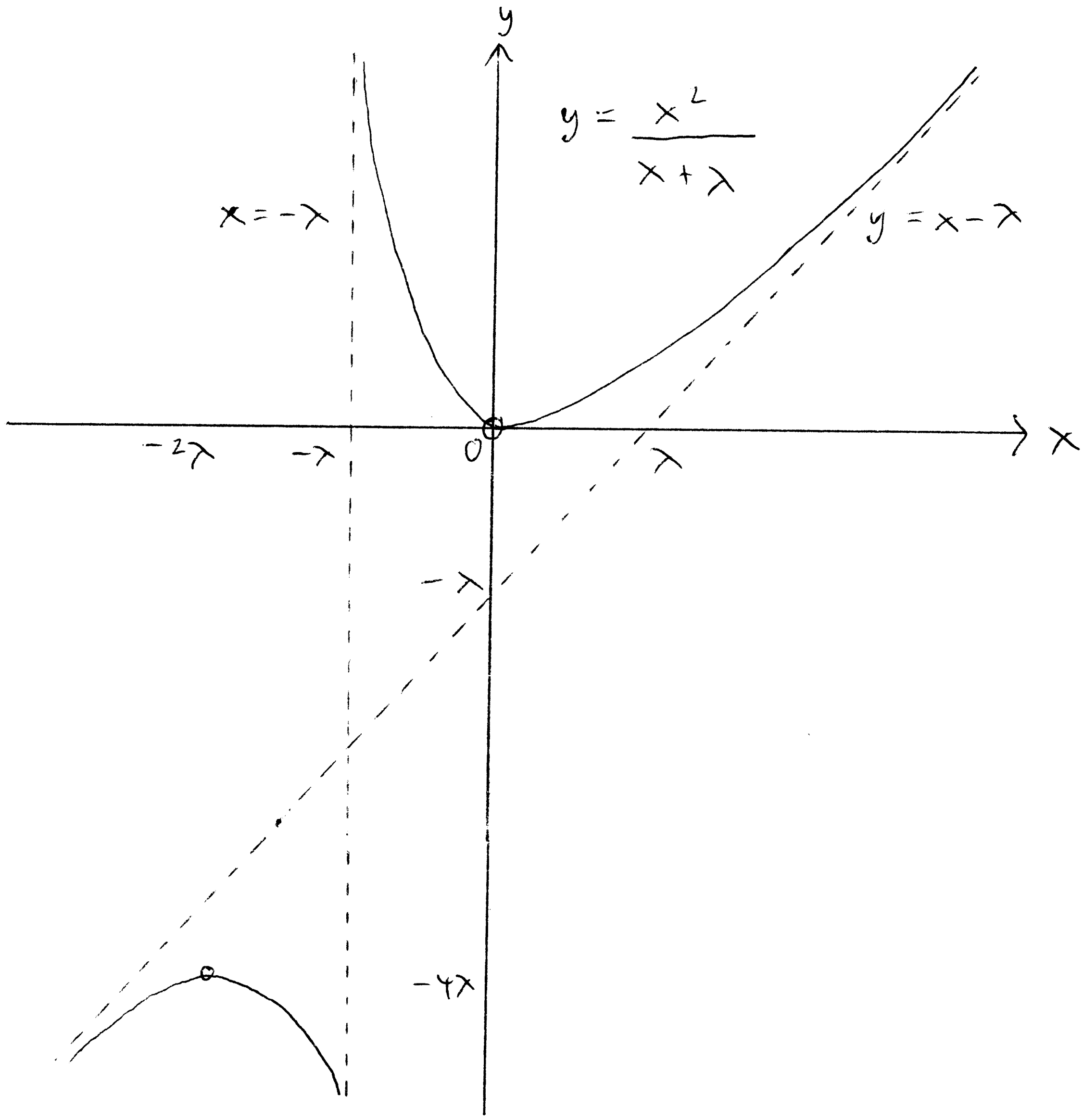
$\therefore$ The stationary points of C are $(0,0)$ and $(-2\lambda, -4\lambda)$

$$\frac{d^2y}{dx^2} = \frac{2\lambda^2}{(x+\lambda)^3}$$

when $x = 0$ $\quad \frac{d^2y}{dx^2} = \frac{2}{\lambda} > 0$

when $x = -2\lambda$ $\quad \frac{d^2y}{dx^2} = \frac{-2}{\lambda} < 0$

$\therefore$ $(0,0)$ is a minimum point and $(-2\lambda, -4\lambda)$ is a maximum point



ii) $\lambda < 0$:

$$y = x - \lambda + \frac{\lambda^2}{x+\lambda}$$

$$\frac{dy}{dx} = 1 - \frac{\lambda^2}{(x+\lambda)^2}$$

If $\dfrac{dy}{dx} = 0$: $1 - \dfrac{\lambda^2}{(x+\lambda)^2} = 0$

$$\frac{\lambda^2}{(x+\lambda)^2} = 1$$

$$\lambda^2 = (x+\lambda)^2$$

$$= x^2 + 2\lambda x + \lambda^2$$

$$x^2 + 2\lambda x = 0$$

$$x(x + 2\lambda) = 0$$

$$x = 0, -2\lambda$$

$$y = 0, -4\lambda$$

$\therefore$ The stationary points of C are $(0,0)$ and $(-2\lambda, -4\lambda)$

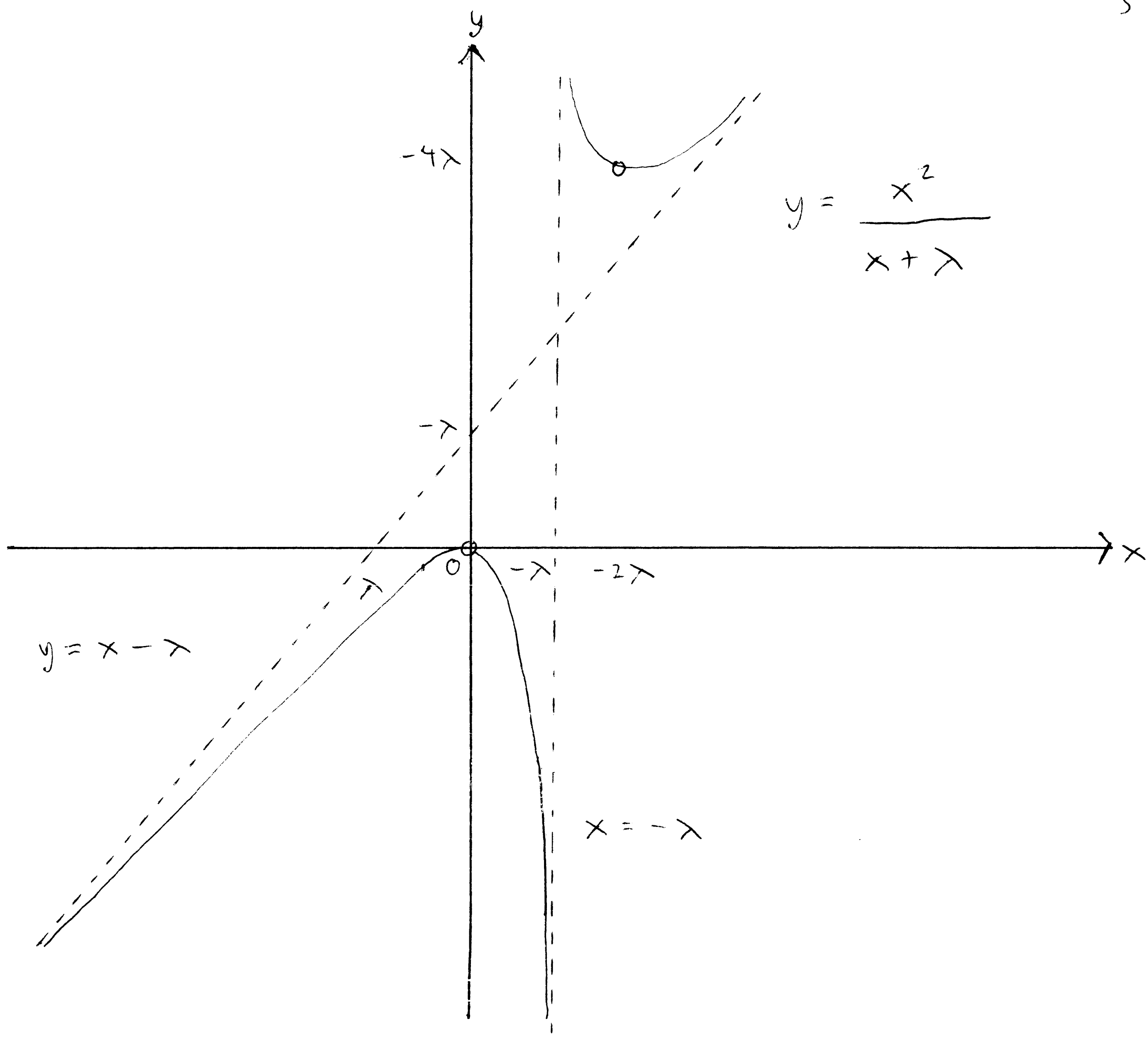
$$\frac{d^2y}{dx^2} = \frac{2\lambda^2}{(x+\lambda)^3}$$

when $x = 0$ $\quad \frac{d^2y}{dx^2} = \frac{2}{\lambda} < 0$

when $x = -2\lambda$ $\quad \frac{d^2y}{dx^2} = -\frac{2}{\lambda} > 0$

$(0,0)$ is a maximum point and $(-2\lambda, -4\lambda)$ is a

maximum point.


y
x
-4λ
$-\lambda$
$y = \frac{x^2}{x+\lambda}$
λ
O
$-\lambda$
-2λ
$y = x - \lambda$
$x = -\lambda$

4. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 24e^{2x}$, $y = 1$ and $\frac{dy}{dx} = 9$ when $x = 0$

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 1)(\lambda + 2) = 0$

$\lambda = -1, -2$

$\therefore y_c = Ae^{-x} + Be^{-2x}$

Let $y_p = Ce^{2x}$

$\frac{dy_p}{dx} = 2Ce^{2x}$

$\frac{d^2y_p}{dx^2} = 4Ce^{2x}$

$\therefore \frac{d^2y_p}{dx^2} + 3\frac{dy_p}{dx} + 2y_p = 4Ce^{2x} + 6Ce^{2x} + 2Ce^{2x}$

$= 12Ce^{2x}$

$= 24e^{2x}$

$\therefore 12C = 24$

$C = 2$

$\therefore y_p = 2e^{2x}$

$y = y_c + y_p$

$= Ae^{-x} + Be^{-2x} + 2e^{2x}$

$\frac{dy}{dx} = -Ae^{-x} - 2Be^{-2x} + 4e^{2x}$

when $x=0$ $y=1$: $1 = A + B + 2$

when $x=0$ $\frac{dy}{dx}=9$: $9 = -A - B + 4$

$A + B + 2 = 1$

$A + B = -1$

$-A - 2B + 4 = 9$

$-A - 2B = 5$

$-B = 4$

$B = -4$

$A = 3$

$\therefore y = 3e^{-x} - 4e^{-2x} + 2e^{2x}$

5. $x^3 + ax^2 + bx + c = 0.$

If α, β and γ are the roots of the equation

$\therefore \alpha + \beta + \gamma = -a$

$\alpha\beta + \alpha\gamma + \beta\gamma = b$

$\alpha\beta\gamma = -c.$

i) $\alpha > 1$, $\beta > 1$ and $\gamma > 1$

$\therefore \alpha + \beta + \gamma > 1 + 1 + 1$

$-a > 3$

$a < -3$ since $\alpha + \beta + \gamma = -a$

ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (-a)^2 - 2b$

$= a^2 - 2b$

Since $\alpha > 1$, $\beta > 1$ and $\gamma > 1$,
$\alpha^2 > 1$, $\beta^2 > 1$ and $\gamma^2 > 1$.

$\therefore \alpha^2 + \beta^2 + \gamma^2 > 1 + 1 + 1$

$a^2 - 2b > 3$

$a^2 > 2b + 3$

iii) If $S_n = \alpha^n + \beta^n + \gamma^n$

$\therefore S_0 = \alpha^0 + \beta^0 + \gamma^0 = 1 + 1 + 1 = 3,$

$S_1 = \alpha^1 + \beta^1 + \gamma^1 = \alpha + \beta + \gamma = -a$

and $S_2 = \alpha^2 + \beta^2 + \gamma^2 = a^2 - 2b$

$x^3 + ax^2 + bx + c = 0$

$\therefore 1S_3 + aS_2 + bS_1 + cS_0 = 0$

$S_3 + a(a^2 - 2b) + b(-a) + 3c = 0$

$S_3 + a^3 - 2ab - ab + 3c = 0$

$S_3 + a^3 - 3ab + 3c = 0$

$\therefore S_3 = 3ab - a^3 - 3c$

Since $\alpha > 1$, $\beta > 1$ and $\gamma > 1$,

$\alpha\beta > 1$, $\alpha\gamma > 1$ and $\beta\gamma > 1$

$\therefore \alpha\beta + \alpha\gamma + \beta\gamma > 1 + 1 + 1$

$b > 3$

Since $a < -3$

$a + 3 < 0$

$(a + 3)b < 0$

$ab + 3b < 0$

$3(ab + 3b) < 0$

$3ab + 9b < 0$

$-3ab - 9b > 0$

$a^3 - 3ab - 9b > a^3$

Since $\alpha > 1$, $\beta > 1$ and $\gamma > 1$

$\alpha^3 > 1$, $\beta^3 > 1$ and $\gamma^3 > 1$

$\therefore \alpha^3 + \beta^3 + \gamma^3 > 1 + 1 + 1$

$S_3 > 3$

$S_3 = 3ab - a^3 - 3c$

$\therefore 3ab - a^3 - 3c > 3$

$-3c - 3 > a^3 - 3ab$

$-9b - 3c - 3 > a^3 - 3ab - 9b$

Since $-9b - 3c - 3 > a^3 - 3ab - 9b$ and

$a^3 - 3ab - 9b > a^3$

$\therefore -9b - 3c - 3 > a^3$

6. $I_n = \int_0^1 (1+x^2)^{-n}\,dx, \quad n \geq 1$

$$\frac{d}{dx}\left(x(1+x^2)^{-n}\right) = (1+x^2)^{-n}\frac{d}{dx}(x) + x\frac{d}{dx}(1+x^2)^{-n}$$

$$= (1+x^2)^{-n}(1) + x(-n)(1+x^2)^{-n-1}(2x)$$

$$= (1+x^2)^{-n} - 2nx^2(1+x^2)^{-n-1}$$

$$\therefore x(1+x^2)^{-n} = \int (1+x^2)^{-n} - 2nx^2(1+x^2)^{-n-1}\,dx$$

$$= \int (1+x^2)^{-n}\,dx - \int 2nx^2(1+x^2)^{-n-1}\,dx$$

$$= \int (1+x^2)^{-n}\,dx - 2n\int x^2(1+x^2)^{-n-1}\,dx$$

$$\therefore \int (1+x^2)^{-n}\,dx = x(1+x^2)^{-n} + 2n\int x^2(1+x^2)^{-n-1}\,dx$$

$$\int_0^1 (1+x^2)^{-n}\,dx = \left[x(1+x^2)^{-n}\right]_0^1 + 2n\int_0^1 x^2(1+x^2)^{-n-1}\,dx$$

$$= 1(1+1)^{-n} - 0(1+0)^{-n} + 2n\int_0^1 x^2(1+x^2)^{-n-1}\,dx$$

$$= 2^{-n} + 2n\int_0^1 x^2(1+x^2)^{-n-1}$$

$$= 2^{-n} + 2n\int_0^1 (1+x^2-1)(1+x^2)^{-n-1}\,dx$$

$$= 2^{-n} + 2n\int_0^1 (1+x^2)(1+x^2)^{-n-1} - (1+x^2)^{-n-1}\,dx$$

$$= 2^{-n} + 2n\int_0^1 (1+x^2)^{-n} - (1+x^2)^{-n-1}\,dx$$

$$= 2^{-n} + 2n\int_0^1 (1+x^2)^{-n} - (1+x^2)^{-(n+1)}\,dx$$

$$= 2^{-n} + 2n\int_0^1 (1+x^2)^{-n}\,dx - 2n\int_0^1 (1+x^2)^{-(n+1)}\,dx$$

$$\therefore I_n = 2^{-n} + 2n I_n - 2n I_{n+1}$$

$$2n I_n - I_n + 2^{-n} = 2n I_{n+1}$$

$$(2n-1) I_n + 2^{-n} = 2n I_{n+1}$$

$n=2$: $$2(2) I_3 = (2(2)-1) I_2 + 2^{-2}$$

$$4 I_3 = 3 I_2 + \frac{1}{4}$$

$n=1$: $$2(1) I_2 = (2(1)-1) I_1 + 2^{-1}$$

$$2 I_2 = I_1 + \frac{1}{2}$$

$$I_1 = \int_0^1 (1+x^2)^{-1}\,dx$$

$x = \tan u$

$dx = \sec^2 u\,du$

$x=1 \quad u=\frac{\pi}{4}$

$x=0 \quad u=0$

$$= \int_0^{\frac{\pi}{4}} (1+\tan^2 u)^{-1} \sec^2 u\,du$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 u)^{-1} \sec^2 u\,du$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 u \sec^2 u\,du$$

$$= \int_0^{\frac{\pi}{4}} 1 \, du$$

$$= \Big[u \Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

$$2I_2 = I_1 + \frac{1}{2}$$

$$= \frac{\pi}{4} + \frac{1}{2}$$

$$\therefore I_2 = \frac{\pi}{8} + \frac{1}{4}$$

$$4I_3 = 3I_2 + \frac{1}{4}$$

$$= 3\left(\frac{\pi}{8} + \frac{1}{4}\right) + \frac{1}{4}$$

$$= \frac{3\pi}{8} + \frac{3}{4} + \frac{1}{4}$$

$$= \frac{3\pi}{8} + 1$$

$$\therefore I_3 = \frac{3\pi}{32} + \frac{1}{4}$$

7. $$\sum_{n=1}^{N} 2^{-n} z^{n} = \sum_{n=1}^{N} \frac{z^n}{2^n}$$

$$= \sum_{n=1}^{N} \left(\frac{z}{2}\right)^n$$

$$= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots + \left(\frac{z}{2}\right)^N$$

$$= \frac{z}{2}\left(\frac{1-\left(\frac{z}{2}\right)^n}{1-\frac{z}{2}}\right)$$

$$= \frac{z}{2}\left(1-\left(\frac{z}{2}\right)^n\right)\left(\frac{2}{2-z}\right)$$

$$= \frac{z}{2-z}\left(1-\left(\frac{z}{2}\right)^n\right)$$

since $\frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{3}\right)^3 + \dots + \left(\frac{z}{2}\right)^N$ is a geometric series with $a = \frac{z}{2}$ and $r = \frac{z}{2}$

If $z = \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}$ and $N = 10$

$$\sum_{n=1}^{N} 2^{-n} z^{n} = \sum_{n=1}^{10} 2^{-n}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)^n$$

$$= \sum_{n=1}^{10} 2^{-n}\left(\cos\frac{n\pi}{10} + i\sin\frac{n\pi}{10}\right)$$

$$= \sum_{n=1}^{10} 2^{-n}\cos\frac{n\pi}{10} + i2^{-n}\sin\frac{n\pi}{10}$$

$$= \sum_{n=1}^{10} 2^{-n} \cos\frac{n\pi}{10} + i\sum_{n=1}^{10} 2^{-n} \sin\frac{n\pi}{10}$$

$$= \frac{\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}}{2 - \left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)} \left(1 - \left(\frac{1}{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right)^{10}\right)$$

$$= \frac{\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}}{2 - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}} \left(1 - \left(\frac{1}{2^{10}}\left(\cos\frac{10\pi}{10} + i\sin\frac{10\pi}{10}\right)\right)\right)$$

$$= \frac{\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}}{2 - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}} \left(1 - \left(\frac{1}{1024}\left(\cos\pi + i\sin\pi\right)\right)\right)$$

$$= \frac{\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}}{2 - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}} \left(1 - \left(\frac{-1}{1024}\right)\right)$$

$$= \frac{\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}}{2 - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}} \left(\frac{1024 - -1}{1024}\right)$$

$$= \frac{1025\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)}{1024\left(2 - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)}$$

$$= \frac{1025\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\left(2 - \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)}{1024\left(2 - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)\left(2 - \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)}$$

$$= \frac{1025\left[\cos\frac{\pi}{10}\left(2 - \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right) + i\sin\frac{\pi}{10}\left(2 - \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right]}{1024\left[\left(2 - \cos\frac{\pi}{10}\right)^2 - \left(2 - \cos\frac{\pi}{10}\right)i\sin\frac{\pi}{10} + \left(2 - \cos\frac{\pi}{10}\right)i\sin\frac{\pi}{10} - i^2\sin^2\frac{\pi}{10}\right]}$$

$$= \frac{1025\left(2\cos\frac{\pi}{10} - \cos^2\frac{\pi}{10} + i\sin\frac{\pi}{10}\cos\frac{\pi}{10} + 2i\sin\frac{\pi}{10} - i\sin\frac{\pi}{10}\cos\frac{\pi}{10} + i^2\sin^2\frac{\pi}{10}\right)}{1024\left(\left(2 - \cos\frac{\pi}{10}\right)^2 + \sin^2\frac{\pi}{10}\right)}$$

$$= \frac{1025\left(2\cos\frac{\pi}{10} - \cos^2\frac{\pi}{10} - \sin^2\frac{\pi}{10} + 2i\sin\frac{\pi}{10}\right)}{1024\left(4 - 4\cos\frac{\pi}{10} + \cos^2\frac{\pi}{10} + \sin^2\frac{\pi}{10}\right)}$$

$$= \frac{1025\left(2\cos\frac{\pi}{10} - \left(\cos^2\frac{\pi}{10} + \sin^2\frac{\pi}{10}\right) + 2i\sin\frac{\pi}{10}\right)}{1024\left(4 - 4\cos\frac{\pi}{10} + 1\right)}$$

$$= \frac{1025\left(2\cos\frac{\pi}{10} - 1 + 2i\sin\frac{\pi}{10}\right)}{1024\left(5 - 4\cos\frac{\pi}{10}\right)}$$

$$= \frac{1025\left(2\cos\frac{\pi}{10} - 1\right) + 1025\left(2i\sin\frac{\pi}{10}\right)}{1024\left(5 - 4\cos\frac{\pi}{10}\right)}$$

$$= \frac{1025\left(2\cos\frac{\pi}{10} - 1\right)}{1024\left(5-4\cos\frac{\pi}{10}\right)} + \frac{1025\left(2i\sin\frac{\pi}{10}\right)}{1024\left(5 - 4\cos\frac{\pi}{10}\right)}$$

$$= \frac{1025\left(2\cos\frac{\pi}{10} - 1\right)}{1024\left(5 - 4\cos\frac{\pi}{10}\right)} + \frac{1025\, i\sin\frac{\pi}{10}}{512\left(5 - 4\cos\frac{\pi}{10}\right)}$$

$$= \frac{1025\left(2\cos\frac{\pi}{10} - 1\right)}{1024\left(5 - 4\cos\frac{\pi}{10}\right)} + i\frac{\left(1025\sin\frac{\pi}{10}\right)}{2560 - 2048\cos\frac{\pi}{10}}$$

Equating imaginary parts,

$$\therefore \sum_{n=1}^{10} 2^{-n}\sin\frac{n\pi}{10} = \frac{1025\sin\frac{\pi}{10}}{2560 - 2048\cos\frac{\pi}{10}}$$

8. $y = x^2(1-x)$

$= x^2 - x^3$

$\frac{dy}{dx} = 2x - 3x^2$

If $\frac{dy}{dx} = 0$ $\quad 2x - 3x^2 = 0$

$x(2-3x) = 0$

$x = 0, \frac{2}{3}$

$y = 0, \frac{4}{27}$

The stationary points of $y = x^2(1-x)$ are $(0,0)$ and $\left(\frac{2}{3}, \frac{4}{27}\right)$.

$\frac{d^2y}{dx^2} = 2 - 6x$

$x = 0: \quad \frac{d^2y}{dx^2} = 2 > 0$

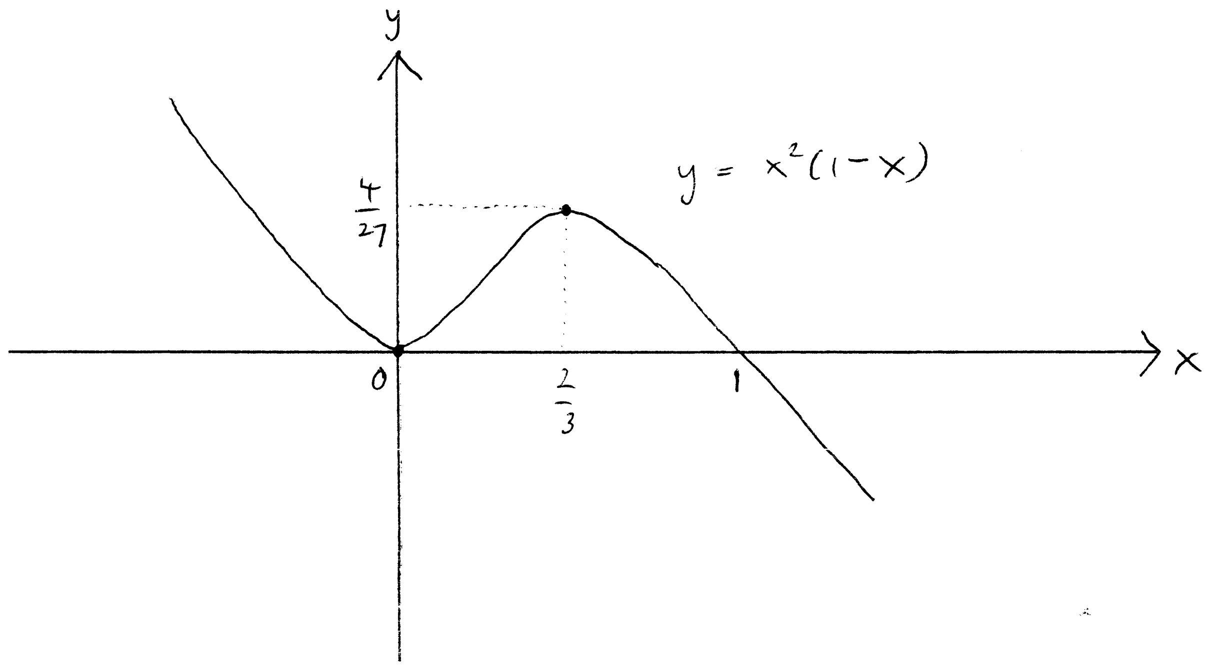
$x = \frac{2}{3}: \quad \frac{d^2y}{dx^2} = -2 < 0$

$(0,0)$ is a minimum point and $\left(\frac{2}{3}, \frac{4}{27}\right)$ is a maximum point.

when $x = 0$ $y = 0$

when $y = 0$: $x^2(1-x) = 0$

$x = 0, 1$



The area A of the finite region bounded by the x-axis and the curve is $\int_0^1 x^2(1-x)\,dx$

$$= \int_0^1 x^2 - x^3 \, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - 0$$

$$= \frac{1}{12}$$

If the centroid has coordinates $(\bar{x}, \bar{y})$,

$$\bar{x} = \frac{\int_0^1 xy \, dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 x^3(1-x) \, dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 x^3 - x^4 \, dx}{\frac{1}{12}}$$

$$= \frac{\left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1}{\frac{1}{12}}$$

$$= \frac{\frac{1}{4} - \frac{1}{5} - 0}{\frac{1}{12}}$$

$$= \frac{\frac{1}{20}}{\frac{1}{12}}$$

$$= \frac{12}{20}$$

$$= \frac{3}{5}$$

and $$\bar{y} = \frac{\int_0^1 \frac{y^2}{2}\,dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{x^4}{2}(1-x)^2\,dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{x^4}{2}(x^2-2x+1)\,dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{x^6}{2} - x^5 + \frac{x^4}{2}\,dx}{\frac{1}{12}}$$

$$= \frac{\left[\frac{x^7}{14} - \frac{x^6}{6} + \frac{x^5}{10}\right]_0^1}{\frac{1}{12}}$$

$$= \frac{\frac{1}{14} - \frac{1}{6} + \frac{1}{10} - 0}{\frac{1}{12}}$$

$$= \frac{\frac{1}{210}}{\frac{1}{12}}$$

$$= \frac{12}{210}$$

$= \frac{2}{35}$

$\therefore$ The coordinates of the centroid are $\left(\frac{3}{5}, \frac{2}{35}\right)$.

$y = x(1-x)^2$

Using the substitution $x = 1-u$

$y = (1-u)\left[1-(1-u)\right]^2$

$= (1-u)(1-1+u)^2$

$= u^2(1-u)$

$\therefore$ If the centroid of the finite region bounded by the x-axis and the curve whose equation is $y = x(1-x)^2$ has coordinates $(\bar{x}_1, \bar{y}_1)$

$\bar{x}_1 = 1 - \bar{x}$

$= 1 - \frac{3}{5}$

$= \frac{2}{5}$

and $\bar{y}_1 = \bar{y}$

$= \frac{2}{35}$.

$\therefore$ The coordinates of the centroid are $\left(\frac{2}{5}, \frac{2}{35}\right)$.

$$y = x(1-x)^2$$

$$= x(1-2x+x^2)$$

$$= x - 2x^2 + x^3$$

$$\frac{dy}{dx} = 1 - 4x + 3x^2$$

If $\frac{dy}{dx} = 0$ $\quad 3x^2 - 4x + 1 = 0$

$$(3x-1)(x-1) = 0$$

$$x = 1, \frac{1}{3}$$

$$y = 0, \frac{4}{27}$$

The stationary points of $y = x(1-x)^2$ are $(1, 0)$ and $\left(\frac{1}{3}, \frac{4}{27}\right)$.

$$\frac{d^2y}{dx^2} = 6x - 4$$

$x = 1$: $\frac{d^2y}{dx^2} = 2 > 0$

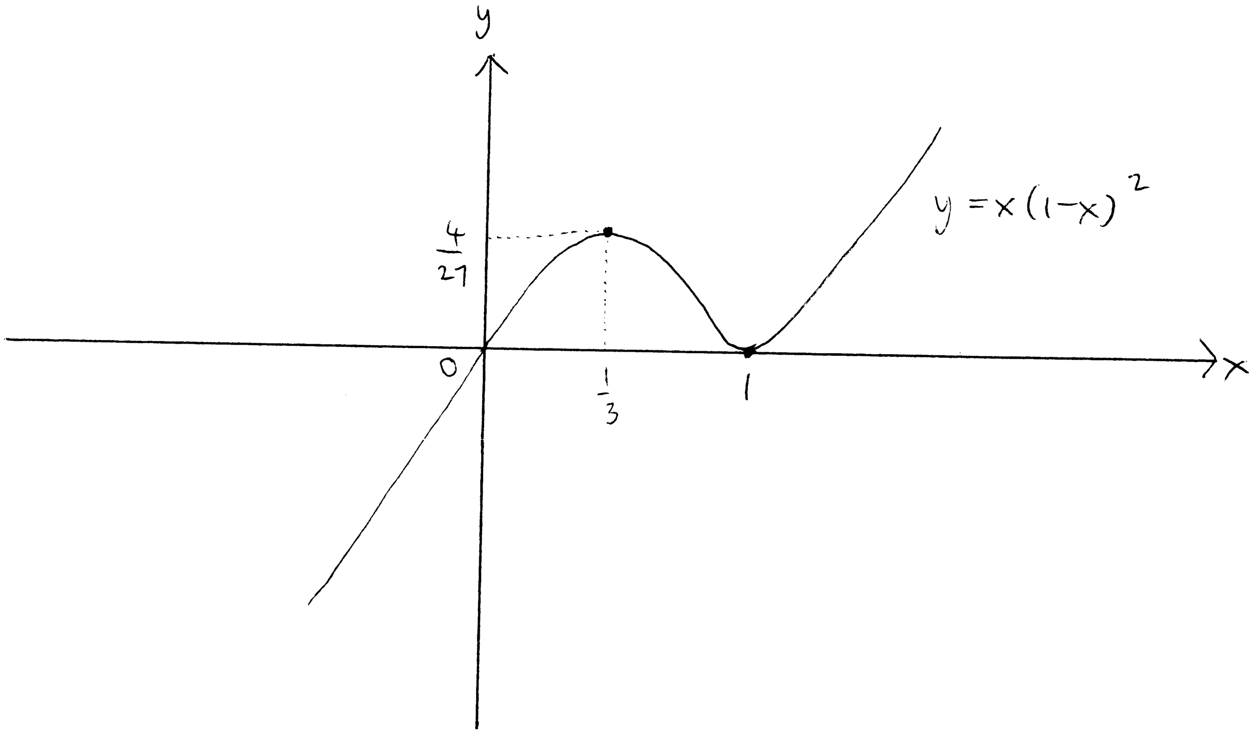
$x = \frac{1}{3}$: $\frac{d^2y}{dx^2} = -2 < 0$

$\therefore (1, 0)$ is a minimum point and $\left(\frac{1}{3}, \frac{4}{27}\right)$ is a maximum point.

when $x = 0 \quad y = 0$

when $y = 0 \quad x(1-x)^2 = 0$

$x = 0, 1$



The area of the finite region bounded by the x-axis and the curve is $\int_0^1 x(1-x)^2 \, dx$

$$= \int_0^1 x - 2x^2 + x^3 \, dx$$

$$= \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} - 0$$

$$= \frac{1}{12}$$

If the centroid has coordinates $(\bar{x}_1, \bar{y}_1)$,

$$\bar{x}_1 = \frac{\int_0^1 xy \, dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 x^2(1-x)^2 \, dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 x^2 - 2x^3 + x^4 \, dx}{\frac{1}{12}}$$

$$= \frac{\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1}{\frac{1}{12}}$$

$$= \frac{\frac{1}{3} - \frac{1}{2} + \frac{1}{5} - 0}{\frac{1}{12}}$$

$$= \frac{\frac{1}{30}}{\frac{1}{12}}$$

$$= \frac{12}{30}$$

$$= \frac{2}{5}$$

and $\bar{y}_1 = \dfrac{\int_0^1 \frac{y^2}{2}\, dx}{\frac{1}{12}}$

$$= \frac{\int_0^1 \frac{x^2(1-x)^4}{2}\, dx}{\frac{1}{12}}$$

$u = 1 - x$

$du = -dx$

$x = 1 \quad u = 0$

$x = 0 \quad u = 1$

$$= \frac{\int_1^0 \frac{(1-u)^2 u^4 (-du)}{2}}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{(1-u)^2 u^4\, du}{2}}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{1}{2} u^4 (1 - 2u + u^2)\, du}{\frac{1}{12}}$$

$$= \frac{\frac{1}{2}\left[\frac{u^5}{5} - \frac{u^6}{3} + \frac{u^7}{7}\right]_0^1}{\frac{1}{12}}$$

$$= \frac{\frac{1}{2}\left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7} - 0\right)}{\frac{1}{12}}$$

$$= \frac{\frac{1}{2}\left(\frac{1}{105}\right)}{\frac{1}{12}}$$

$$= \frac{\frac{1}{210}}{\frac{1}{12}}$$

$$= \frac{12}{210}$$

$$= \frac{2}{35}$$

$\therefore$ The coordinates of the centroid are $\left(\frac{2}{5}, \frac{2}{35}\right)$

9. $\Pi_1 : \underset{\sim}{r} = \lambda_1(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}) + \mu_1(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$

$\Pi_2 : \underset{\sim}{r} = \lambda_2(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) + \mu_2(3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$

Expressing Π_1 and Π_2 in Cartesian form:

$\Pi_1 : \underset{\sim}{r} = \lambda_1(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}) + \mu_1(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mu_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 + 2\mu_1 \\ \lambda_1 - \mu_1 \\ -\lambda_1 + \mu_1 \end{pmatrix}$$

$x = \lambda_1 + 2\mu_1, \quad y = \lambda_1 - \mu_1, \quad z = -\lambda_1 + \mu_1$

$$\left.\begin{aligned} \lambda_1 + 2\mu_1 &= x \\ \lambda_1 - \mu_1 &= y \\ -\lambda_1 + \mu_1 &= z \end{aligned}\right\}$$

(2) + (3). $y + z = 0$.

$\therefore$ The equation of Π_1 in Cartesian form is

$y + z = 0$.

$\Pi_2 : \underset{\sim}{r} = \lambda_2(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) + \mu_2(3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_2 + 3\mu_2 \\ 2\lambda_2 + \mu_2 \\ \lambda_2 - \mu_2 \end{pmatrix}$$

$x = \lambda_2 + 3\mu_2, \quad y = 2\lambda_2 + \mu_2, \quad z = \lambda_2 - \mu_2$

$$\left.\begin{aligned} \lambda_2 + 3\mu_2 &= x \\ 2\lambda_2 + \mu_2 &= y \\ \lambda_2 - \mu_2 &= z \end{aligned}\right\}$$

$$\begin{aligned} -2\times ① + ②: \\ -① + ③: \end{aligned} \quad \left.\begin{aligned} \lambda_2 + 3\mu_2 &= x \\ -5\mu_2 &= y - 2x \\ -4\mu_2 &= z - x \end{aligned}\right\}$$

$$-5\mu_2 = y - 2x \qquad -4\mu_2 = z - x$$

$$\mu_2 = \frac{y - 2x}{-5} \qquad \mu_2 = \frac{z - x}{-4}$$

$$\therefore \frac{y - 2x}{-5} = \frac{z - x}{-4}$$

$$-4(y - 2x) = -5(z - x)$$

$$-4y + 8x = -5z + 5x$$

$$3x - 4y + 5z = 0.$$

The equation of Π_2 in Cartesian form is

$3x - 4y + 5z = 0$.

Since ℓ is parallel to both Π_1 and Π_2, ℓ must be perpendicular to the normals of both Π_1 and Π_2. $\therefore$ If $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ are the normal vectors of Π_1 and Π_2, ℓ must be parallel to $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$.

$$\underset{\sim}{n}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \underset{\sim}{n}_2 = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

$$\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 1 & 1 \\ 3 & -4 & 5 \end{vmatrix}$$

$$= \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix}$$

$$= 3\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

Since l is parallel to $3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and passes through the point with position vector $4\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}$, a vector equation for l is

$$\underset{\sim}{r} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k} + \lambda(3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}).$$

If m is the line of intersection of Π_1 and Π_2, since the planes intersect in m, m must be perpendicular to $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$. $\therefore$ The direction vector of m is parallel to $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$ since m lies in both planes.

$$\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix}$$

$$= 3\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

Since m lies in both planes, a point on the line m satisfies the equations $y + z = 0$ and $3x - 4y + 5z = 0$.

$x = 0$.

$y + z = 0$

$y = -z$ — ①

$3x - 4y + 5z = 0$

$-4y + 5z = 0$ — ②

$4 \times ① + ②$: $4y - 4y + 5z = -4z$

$5z = -4z$

$9z = 0$

$z = 0$

$y = 0$

$\therefore (0, 0, 0)$ is a point on m.

Since m is parallel to $3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $(0,0,0)$ is a point on m, a vector equation for m is

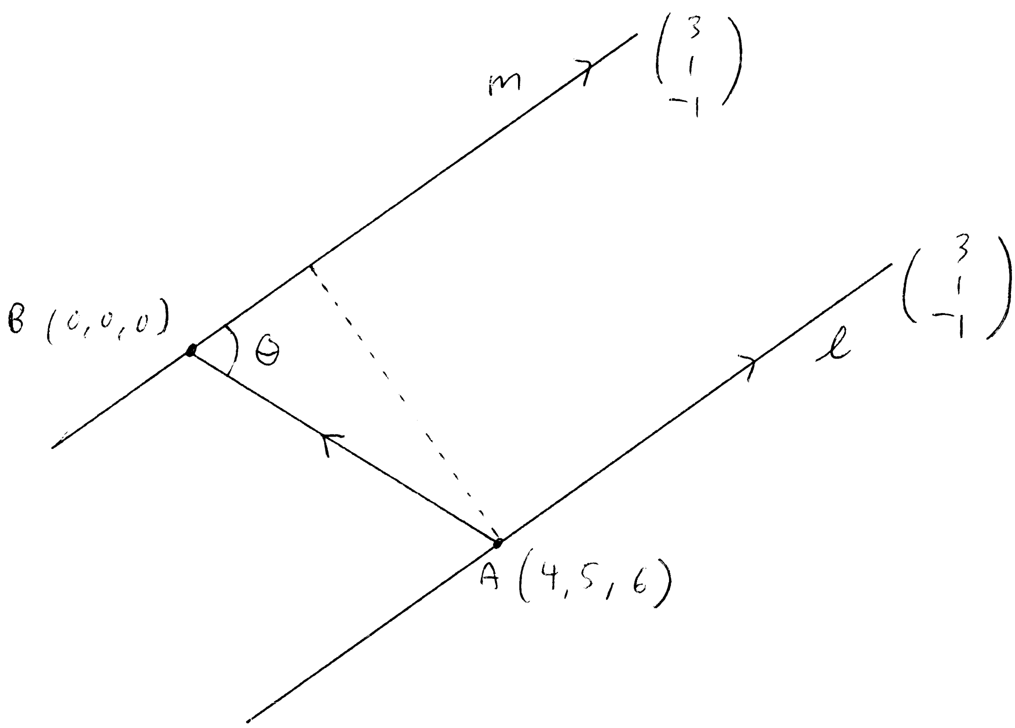
$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$= s(3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$$

If A and B are the points with position vectors $4\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}$ on l and $0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k}$ on m

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -5 \\ -6 \end{pmatrix}$$



The shortest distance between ℓ and m is

$|AB| \sin\theta$.

$$\left|\overrightarrow{AB} \times \begin{pmatrix}3\\1\\-1\end{pmatrix}\right| = |\overrightarrow{AB}| \left|\begin{pmatrix}3\\1\\-1\end{pmatrix}\right| \sin\theta$$

$$\left|\begin{pmatrix}-4\\-5\\-6\end{pmatrix} \times \begin{pmatrix}3\\1\\-1\end{pmatrix}\right| = |\overrightarrow{AB}| \sqrt{9+1+1} \sin\theta$$

$$= |\overrightarrow{AB}| \sqrt{11} \sin\theta$$

$$\therefore |\overrightarrow{AB}| \sqrt{11} \sin\theta = \left\| \begin{matrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -4 & -5 & -6 \\ 3 & 1 & -1 \end{matrix} \right\|$$

$$= |11\underset{\sim}{i} - 22\underset{\sim}{j} + 11\underset{\sim}{k}|$$

$$= 11\,|\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}|$$

$$= 11\sqrt{1+4+1}$$

$$= 11\sqrt{6}$$

$$\therefore |\overrightarrow{AB}| \sin\theta = \frac{11\sqrt{6}}{\sqrt{11}}$$

$$= \sqrt{66}.$$

$\therefore$ The shortest distance between ℓ and m is $\sqrt{66}$.

10. $M = \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix}$

$$M - \lambda I = \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & 1 & -1 \\ -4 & -1-\lambda & 4 \\ 0 & -1 & 5-\lambda \end{pmatrix}$$

$$|M - \lambda I| = \begin{vmatrix} 4-\lambda & 1 & -1 \\ -4 & -1-\lambda & 4 \\ 0 & -1 & 5-\lambda \end{vmatrix}$$

$$= (4-\lambda) \begin{vmatrix} -1-\lambda & 4 \\ -1 & 5-\lambda \end{vmatrix} - 1 \begin{vmatrix} -4 & 4 \\ 0 & 5-\lambda \end{vmatrix}$$

$$-1 \begin{vmatrix} -4 & -1-\lambda \\ 0 & -1 \end{vmatrix}$$

$$= (4-\lambda)\left[(-1-\lambda)(5-\lambda) + 4\right] - (-4)(5-\lambda) - (-4)(-1)$$

$$= (4-\lambda)(-5+\lambda-5\lambda+\lambda^2+4) + 4(5-\lambda) - 4$$

$$= (4-\lambda)(\lambda^2-4\lambda-1) + 4(5-\lambda) - 4$$

$$= 4\lambda^2 - 16\lambda - 4 - \lambda^3 + 4\lambda^2 + \lambda + 20 - 4\lambda - 4$$

$$= -\lambda^3 + 8\lambda^2 - 19\lambda + 12$$

$$= -(\lambda^3 - 8\lambda^2 + 19\lambda - 12)$$

$\lambda = 1, 3, 4$

$\therefore$ when $\lambda = 1$: $\begin{pmatrix} 3 & 1 & -1 \\ -4 & -2 & 4 \\ 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ -4 & -2 & 4 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow[3 \times r_2]{4 \times r_1} \left(\begin{array}{ccc|c} 12 & 4 & -4 & 0 \\ -12 & -6 & 12 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 + r_2} \left(\begin{array}{ccc|c} 12 & 4 & -4 & 0 \\ 0 & -2 & 8 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{4};\ \frac{r_2}{-2}} \left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$0z = 0$

Let $z = s$, $s \in R$

$$y - 4z = 0$$

$$y = 4s$$

$$3x + y - z = 0$$

$$3x + 4s - s = 0$$

$$x = -s$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 4s \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

when $\lambda = 3$: $\begin{pmatrix} 1 & 1 & -1 \\ -4 & -4 & 4 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -4 & -4 & 4 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{4r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right)$$

$$-y + 2z = 0$$

Let $z = s$, $s \in \mathbb{R}$

$$-y + 2s = 0$$

$$y = 2s$$

$$x + y - z = 0$$

$$x + 2s - s = 0$$

$$x = -s$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 2s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

when $\lambda = 4$: $\begin{pmatrix} 0 & 1 & -1 \\ -4 & -5 & 4 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -4 & -5 & 4 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Leftrightarrow r_2} \left(\begin{array}{ccc|c} -4 & -5 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} -4 & -5 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y - z = 0$

Let $z = s,\ s \in \mathbb{R}$

$y - s = 0$

$y = s$

$-4x - 5y + 4z = 0$

$-4x - 5s + 4s = 0$

$-4x = s$

$x = -\frac{s}{4}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{s}{4} \\ s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -\frac{1}{4} \\ 1 \\ 1 \end{pmatrix}$$

$\therefore$ A set of eigenvectors for the eigenvalues $1, 3, 4$

are $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$

If $M^n = PDP^{-1}$ where D is a diagonal matrix

and n is a positive integer

$$\therefore P = \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}$$

and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}^n$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}$$

$$|P| = -1 \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} - (-1) \begin{vmatrix} 4 & 4 \\ 1 & 4 \end{vmatrix} - 1 \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= -(8-4) + 16 - 4 - (4-2)$$

$$= -4 + 12 - 2$$

$$= 6$$

$$\text{adj } P = \begin{pmatrix} 4 & -12 & 2 \\ 3 & -3 & 0 \\ -2 & 0 & 2 \end{pmatrix}^T$$

$$= \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\therefore P^{-1} = \frac{1}{|P|} \text{adj } P$$

$$= \frac{1}{6} \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$4^{-n} M^n = 4^{-n} P D P^{-1}$$

$$= \frac{1}{4^n} P D P^{-1}$$

$$= \frac{1}{4^n}\begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}\frac{1}{6}\begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{4^n 6}\begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}\begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}\frac{1}{4^n}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}\begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} \left(\frac{1}{4}\right)^n & 0 & 0 \\ 0 & \left(\frac{3}{4}\right)^n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\therefore \lim_{n\to\infty} 4^{-n} M^n = \lim_{n\to\infty} 4^{-n} P D P^{-1}$$

$$= \lim_{n\to\infty} \frac{1}{6}\begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} \left(\frac{1}{4}\right)^n & 0 & 0 \\ 0 & \left(\frac{3}{4}\right)^n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}\lim_{n\to\infty}\begin{pmatrix} \left(\frac{1}{4}\right)^n & 0 & 0 \\ 0 & \left(\frac{3}{4}\right)^n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -2 & 0 & -2 \\ 8 & 0 & 8 \\ 8 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-2}{6} & 0 & \frac{-2}{6} \\ \frac{8}{6} & 0 & \frac{8}{6} \\ \frac{8}{6} & 0 & \frac{8}{6} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{3} & 0 & \frac{-1}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \end{pmatrix}$$

11.

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 4 & 3 & 5 & 16 \\ 6 & 6 & 13 & 13 \\ 14 & 12 & 23 & 45 \end{pmatrix}$$

$$\xrightarrow[-14r_1+r_4]{\substack{-4r_1+r_2 \\ -6r_1+r_3}} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & -2 & -5 & 3 \end{pmatrix}$$

$$\xrightarrow{-2r_2+r_4} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & -5 \end{pmatrix}$$

$$\xrightarrow{-r_2,\ -r_3+r_4} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank $(A) = 3$

If $A\underset{\sim}{x} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix}$, where $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 4 & 3 & 5 & 16 \\ 6 & 6 & 13 & 13 \\ 14 & 12 & 23 & 45 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 4 & 3 & 5 & 16 & 2 \\ 6 & 6 & 13 & 13 & -1 \\ 14 & 12 & 23 & 45 & 3 \end{array}\right)$$

$$\xrightarrow[-14r_1 + r_4]{-4r_1 + r_2,\ -6r_1 + r_3} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & -1 & -3 & 4 & 2 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & -2 & -5 & 3 & 3 \end{array}\right)$$

$$\xrightarrow{-2r_2 + r_4} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & -1 & -3 & 4 & 2 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & 0 & 1 & -5 & -1 \end{array}\right)$$

$$\xrightarrow{-r_2,\ -r_3 + r_4} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & -4 & -2 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$z - 5w = -1$$

Let $w = s,\ s \in \mathbb{R}$

$$z - 5s = -1$$

$$z = 5s - 1$$

$$y + 3z - 4w = -2$$

$$y + 15s - 3 - 4s = -2$$

$$y = 1 - 11s$$

$$x + y + 2z + 3w = 0$$

$$x + 1 - 11s + 10s - 2 + 3s = 0$$

$$x + 2s = 1$$

$$x = -2s + 1$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2s + 1 \\ -11s + 1 \\ 5s - 1 \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} -2 \\ -11 \\ 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$\therefore$ Any solution of the equation $A\underset{\sim}{x} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix}$

can be expressed in the form $\underset{\sim}{x}_0 + \lambda \underset{\sim}{e}$, $\lambda \in R$

where $\underset{\sim}{x}_0 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ and $\underset{\sim}{e} = \begin{pmatrix} -2 \\ -11 \\ 5 \\ 1 \end{pmatrix}$

$$\underset{\sim}{x} = \underset{\sim}{x}_0 + \lambda \underset{\sim}{e}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -11 \\ 5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2\lambda \\ 1-11\lambda \\ -1+5\lambda \\ \lambda \end{pmatrix}$$

If $\underset{\sim}{x}$ has all it's elements positive,

$1-2\lambda > 0$, $1-11\lambda > 0$, $-1+5\lambda > 0$, $\lambda > 0$

$\lambda < \frac{1}{2}$, $\lambda < \frac{1}{11}$, $\lambda > \frac{1}{5}$

Since $\lambda < \frac{1}{11}$ and $\lambda > \frac{1}{5}$ is impossible

$\therefore$ there is no vector which satisfies the equation $A\underset{\sim}{x} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix}$ and has all it's elements positive.

12.

$$\left(n+\frac{1}{2}\right)^3-\left(n-\frac{1}{2}\right)^3$$

$$= n^3+3n^2\left(\frac{1}{2}\right)+3n\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3-\left(n^3+3n^2\left(-\frac{1}{2}\right)+3n\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^3\right)$$

$$= n^3+\frac{3n^2}{2}+\frac{3n}{4}+\frac{1}{8}-\left(n^3-\frac{3n^2}{2}+\frac{3n^2}{4}-\frac{1}{8}\right)$$

$$= n^3+\frac{3n^2}{2}+\frac{3n}{4}+\frac{1}{8}-n^3+\frac{3n^2}{2}-\frac{3n^2}{4}+\frac{1}{8}$$

$$= 3n^2+\frac{1}{4}$$

Since $n^2 = \frac{1}{3}\left[\left(n+\frac{1}{2}\right)^3-\left(n-\frac{1}{2}\right)^3-\frac{1}{4}\right]$

$$\therefore \sum_{n=1}^{N} n^2 = \sum_{n=1}^{N}\frac{1}{3}\left[\left(n+\frac{1}{2}\right)^3-\left(n-\frac{1}{2}\right)^3-\frac{1}{4}\right]$$

$$= \frac{1}{3}\sum_{n=1}^{N}\left(n+\frac{1}{2}\right)^{\frac{1}{3}}-\left(n-\frac{1}{2}\right)^3-\frac{1}{4}$$

$$= \frac{1}{3}\sum_{n=1}^{N}\left(n+\frac{1}{2}\right)^3-\left(n-\frac{1}{2}\right)^3+\frac{1}{3}\sum_{n=1}^{N}-\frac{1}{4}$$

$$= \frac{1}{3}\Big[\left(1+\frac{1}{2}\right)^3-\left(1-\frac{1}{2}\right)^3$$

$$+\left(2+\frac{1}{2}\right)^3-\left(2-\frac{1}{2}\right)^3$$

$$+\left(3+\frac{1}{2}\right)^3-\left(3-\frac{1}{2}\right)^3$$

$$+ \left(N-1+\frac{1}{2}\right)^3 - \left(N-1-\frac{1}{2}\right)^3$$

$$+ \left(N+\frac{1}{2}\right)^3 - \left(N-\frac{1}{2}\right)^3 \Bigg]$$

$$+ \frac{1}{3} \sum_{n=1}^{N} -\frac{1}{4}$$

$$= \frac{1}{3} \Bigg[\left(\frac{3}{2}\right)^3 - \left(\frac{1}{2}\right)^3$$

$$+ \left(\frac{5}{2}\right)^3 - \left(\frac{3}{2}\right)^3$$

$$+ \left(\frac{7}{2}\right)^3 - \left(\frac{5}{2}\right)^3$$

$$\vdots$$

$$+ \left(N-\frac{1}{2}\right)^3 - \left(N-\frac{3}{2}\right)^3$$

$$+ \left(N+\frac{1}{2}\right)^3 - \left(N-\frac{1}{2}\right)^3 \Bigg]$$

$$+ \frac{1}{3}\left(-\frac{N}{4}\right)$$

$$= \frac{1}{3}\left(-\frac{1}{8} + \left(N+\frac{1}{2}\right)^3\right) + \frac{1}{3}\left(-\frac{N}{4}\right)$$

$$= \frac{1}{3}\left(-\frac{1}{8} + N^3 + 3N^2\left(\frac{1}{2}\right) + 3N\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right) - \frac{N}{12}$$

$$= \frac{1}{3}\left(-\frac{1}{8} + N^3 + \frac{3N^2}{2} + \frac{3N}{4} + \frac{1}{8}\right) - \frac{N}{12}$$

$$= \frac{1}{3}\left(N^3 + \frac{3N^2}{2} + \frac{3N}{4}\right) - \frac{1}{3}\left(\frac{N}{4}\right)$$

$$= \frac{1}{3}\left(N^3 + \frac{3N^2}{2} + \frac{3N}{4} - \frac{N}{4}\right)$$

$$= \frac{N}{3}\left(N^2 + \frac{3N}{2} + \frac{3}{4} - \frac{1}{4}\right)$$

$$= \frac{N}{3}\left(N^2 + \frac{3N}{2} + \frac{1}{2}\right)$$

$$= \frac{N}{3}\left(\frac{2N^2 + 3N + 1}{2}\right)$$

$$= \frac{N}{6}\left(2N^2 + 3N + 1\right)$$

$$= \frac{N}{6}(N+1)(2N+1)$$

$$S = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2N)^2 + (2N+1)^2$$

$$T = 1^2 + 3^2 + 5^2 + 7^2 + \cdots + (2N-1)^2 + (2N+1)^2$$

$$U = 1^2 - 2^2 + 3^2 - 4^2 + \cdots - (2N)^2 + (2N+1)^2$$

$$S = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2N)^2 + (2N+1)^2$$

$$= \sum_{n=1}^{2N+1} n^2$$

$$= \frac{(2N+1)(2N+1+1)(2(2N+1)+1)}{6}$$

$$= \frac{(2N+1)(2N+2)(4N+2+1)}{6}$$

$$= \frac{2(2N+1)(N+1)(4N+3)}{6}$$

$$= \frac{(N+1)(2N+1)(4N+3)}{3}$$

$$T = 1^2 + 3^2 + 5^2 + 7^2 + \dots + (2N-1)^2 + (2N+1)^2$$

$$= \sum_{n=1}^{N+1} (2n-1)^2$$

$$= \sum_{n=1}^{N+1} (4n^2 - 4n + 1)$$

$$= 4\sum_{n=1}^{N+1} n^2 - 4\sum_{n=1}^{N+1} n + \sum_{n=1}^{N+1} 1$$

$$= \frac{4}{6}(N+1)(N+1+1)(2(N+1)+1) - \frac{4}{2}(N+1)(N+1+1) + N+1$$

$$= \frac{2}{3}(N+1)(N+2)(2N+2+1) - 2(N+1)(N+2) + N+1$$

$$= \frac{2}{3}(N+1)(N+2)(2N+3) - 2(N+1)(N+2) + N+1$$

$$= \frac{2}{3}(N+1)\left[(N+2)(2N+3) - 3(N+2) + \frac{3}{2}\right]$$

$$= \frac{2(N+1)}{3}\left(2N^2 + 7N + 6 - 3N - 6 + \frac{3}{2}\right)$$

$$= \frac{2(N+1)}{3}\left(2N^2 + 4N + \frac{3}{2}\right)$$

$$= \frac{2(N+1)}{3} \frac{(4N^2 + 8N + 3)}{2}$$

$$= \frac{(N+1)}{3}(4N^2 + 8N + 3)$$

$$= \frac{(N+1)(2N+3)(2N+1)}{3}$$

$$U = 1^2 - 2^2 + 3^2 - 4^2 + \cdots - (2N)^2 + (2N+1)^2$$

$$= 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (2N-1)^2 - (2N)^2 + (2N+1)^2$$

$$= \sum_{n=1}^{N} \left[(2n-1)^2 - (2n)^2\right] + (2N+1)^2$$

$$= \sum_{n=1}^{N} (4n^2 - 4n + 1 - 4n^2) + (2N+1)^2$$

$$= \sum_{n=1}^{N} (-4n+1) + (2N+1)^2$$

$$= -4\sum_{n=1}^{N} n + \sum_{n=1}^{N} 1 + (2N+1)^2$$

$$= \frac{-4N(N+1)}{2} + N + (2N+1)^2$$

$$= -2N(N+1) + N + (2N+1)^2$$

$$= -2N^2 - 2N + N + 4N^2 + 4N + 1$$

$$= 2N^2 + 3N + 1$$

$$= (2N+1)(N+1)$$

i) $$\frac{S}{T} = \frac{\dfrac{(N+1)(2N+1)(4N+3)}{3}}{\dfrac{(N+1)(2N+3)(2N+1)}{3}}$$

$$= \frac{(N+1)(2N+1)(4N+3)}{(N+1)(2N+3)(2N+1)}$$

$$= \frac{4N+3}{2N+3}$$

$$\therefore \lim_{N \to \infty} \left(\frac{S}{T}\right) = \lim_{N \to \infty} \left(\frac{4N+3}{2N+3}\right)$$

$$= \lim_{N \to \infty} \left(\frac{\dfrac{4N+3}{N}}{\dfrac{2N+3}{N}}\right)$$

$$= \lim_{N \to \infty} \left(\frac{4 + \dfrac{3}{N}}{2 + \dfrac{3}{N}}\right)$$

$$= \frac{4+0}{2+0}$$

$$= \frac{4}{2}$$

$$= 2$$

$\therefore$ As $N \to \infty \quad \frac{S}{T} \to 2$

ii) $$\frac{S}{U} = \frac{\frac{(N+1)(2N+1)(4N+3)}{3}}{(N+1)(2N+1)}$$

$$= \frac{(N+1)(2N+1)(4N+3)}{3(N+1)(2N+1)}$$

$$= \frac{4N+3}{3}$$

If $\frac{S}{U}$ is an integer, $\frac{4N+3}{3}$ is an integer.

Since $\frac{4N+3}{3} = \frac{4N}{3} + 1$ .. if $\frac{S}{U}$ is an integer

N is divisible by 3.

Let $N = 3k$, where k is an integer.

$$\frac{T}{U} = \frac{\frac{(N+1)(2N+3)(2N+1)}{3}}{(N+1)(2N+1)}$$

$$= \frac{(N+1)(2N+1)(2N+3)}{3(N+1)(2N+1)}$$

$$= \frac{2N+3}{3}$$

$$= \frac{2(3k)+3}{3}$$

...

$$= \frac{6k+3}{3}$$

$$= 2k+1$$

Since k is an integer $\therefore$ $2k+1$ is an integer.

$\therefore$ If $\frac{S}{U}$ is an integer then $\frac{T}{U}$ is an integer.

C_1 : $r = 4\cos\theta$, C_2 $r = 1 + \cos\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

i) If C_1 and C_2 meet,

$4\cos\theta = 1 + \cos\theta$

$3\cos\theta = 1$

$\cos\theta = \frac{1}{3}$

$\theta = \alpha, -\alpha$ where α is the acute angle such that $\cos\alpha = \frac{1}{3}$.

Substituting $\theta = \alpha$ or $-\alpha$ into $r = 4\cos\theta$ or $r = 1 + \cos\theta$

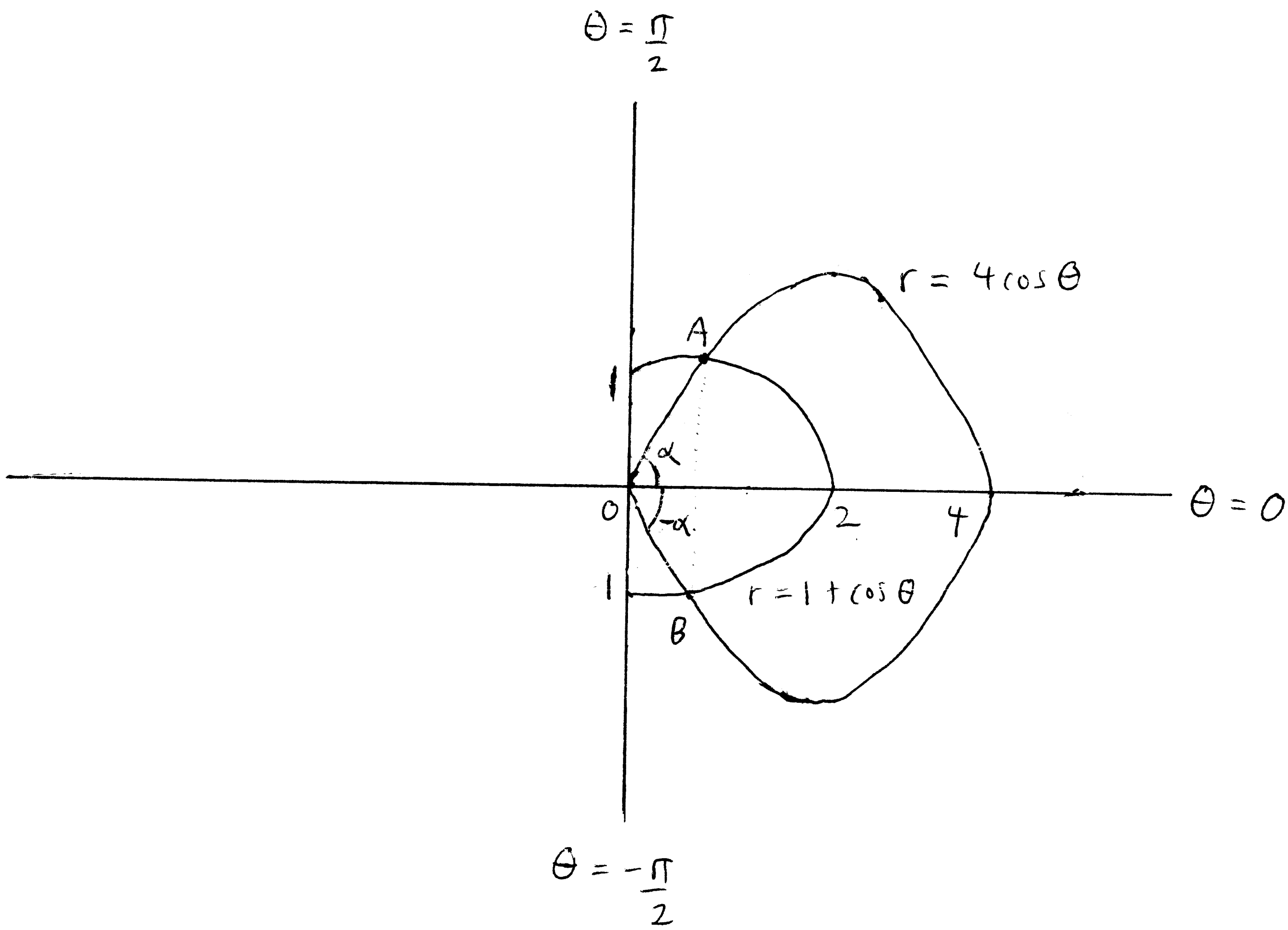
$r = 4\cos\theta$

$= \frac{4}{3}$.

C_1 and C_2 meet at the points $A\left(\frac{4}{3}, \alpha\right)$ and $B\left(\frac{4}{3}, -\alpha\right)$

ii)

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$4\cos\theta$	0	2	$\frac{4}{\sqrt{2}}$	$2\sqrt{3}$	4	$2\sqrt{3}$	$\frac{4}{\sqrt{2}}$	2	0
$1+\cos\theta$	1	$\frac{3}{2}$	$1+\frac{1}{\sqrt{2}}$	$1+\frac{\sqrt{3}}{2}$	2	$1+\frac{\sqrt{3}}{2}$	$1+\frac{1}{\sqrt{2}}$	$\frac{3}{2}$	1



iii) $\therefore$ The area of the region bounded by the arcs OA and OB of C_1 and the arc AB of C_2 is

$$2\left(\int_0^{\alpha} \frac{(1+\cos\theta)^2}{2}\, d\theta + \int_{\alpha}^{\frac{\pi}{2}} \frac{(4\cos\theta)^2}{2}\, d\theta \right)$$

$$= 2\int_0^{\alpha} \frac{(1+\cos\theta)^2}{2}\, d\theta + 2\int_{\alpha}^{\frac{\pi}{2}} \frac{(4\cos\theta)^2}{2}\, d\theta$$

$$= \int_0^{\alpha} (1+\cos\theta)^2\, d\theta + \int_{\alpha}^{\frac{\pi}{2}} (4\cos\theta)^2\, d\theta$$

$$= \int_0^{\alpha} 1 + 2\cos\theta + \cos^2\theta \; d\theta + \int_{\alpha}^{\frac{\pi}{2}} 16\cos^2\theta \; d\theta$$

$$= \int_0^{\alpha} 1 + 2\cos\theta + \frac{\cos 2\theta + 1}{2} \; d\theta + \int_{\alpha}^{\frac{\pi}{2}} 16\left(\frac{\cos 2\theta + 1}{2}\right) d\theta$$

$$= \int_0^{\alpha} 1 + 2\cos\theta + \frac{\cos 2\theta}{2} + \frac{1}{2} \; d\theta + \int_{\alpha}^{\frac{\pi}{2}} + 8(\cos 2\theta + 1) \; d\theta$$

$$= \int_0^{\alpha} \frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2} \; d\theta + \int_{\alpha}^{\frac{\pi}{2}} 8\cos 2\theta + 8 \; d\theta$$

$$= \left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{2(2)}\right]_0^{\alpha} + \left[\frac{8\sin 2\theta}{2} + 8\theta\right]_{\alpha}^{\frac{\pi}{2}}$$

$$= \left[\frac{3\theta}{2} + 2\sin\theta + \frac{2\sin\theta\cos\theta}{4}\right]_0^{\alpha} + \left[4\sin 2\theta + 8\theta\right]_{\alpha}^{\frac{\pi}{2}}$$

$$= \left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin\theta\cos\theta}{2}\right]_0^{\alpha} + \left[4(2\sin\theta\cos\theta) + 8\theta\right]_{\alpha}^{\frac{\pi}{2}}$$

$$= \left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin\theta\cos\theta}{2}\right]_0^{\alpha} + \left[8\sin\theta\cos\theta + 8\theta\right]_{\alpha}^{\frac{\pi}{2}}$$

$$= \left(\frac{3\alpha}{2} + 2\sin\alpha + \frac{\sin\alpha\cos\alpha}{2}\right) - 0$$

$$+ 8\sin\frac{\pi}{2}\cos\frac{\pi}{2} + 8\left(\frac{\pi}{2}\right) - \left(8\sin\alpha\cos\alpha + 8\alpha\right)$$

$$= \frac{3\alpha}{2} + 2\sin\alpha + \frac{\sin\alpha\cos\alpha}{2} + 4\pi - 8\sin\alpha\cos\alpha - 8\alpha$$

$$= 4\pi + \frac{3\alpha}{2} - 8\alpha - \frac{15}{2}\sin\alpha\cos\alpha + 2\sin\alpha$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{15}{2}\sin\alpha\cos\alpha + 2\sin\alpha$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{15}{2}\sqrt{(1-\cos^2\alpha)}\ \cos\alpha + 2\sqrt{1-\cos^2\alpha}$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{15}{2}\sqrt{1-\left(\frac{1}{3}\right)^2}\left(\frac{1}{3}\right) + 2\sqrt{1-\left(\frac{1}{3}\right)^2}$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{15}{2}\sqrt{1-\frac{1}{9}}\left(\frac{1}{3}\right) + 2\sqrt{1-\frac{1}{9}}$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{15}{2}\sqrt{\frac{8}{9}}\left(\frac{1}{3}\right) + 2\sqrt{\frac{8}{9}}$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{15}{2}\frac{\sqrt{8}}{\sqrt{9}}\left(\frac{1}{3}\right) + 2\frac{\sqrt{8}}{\sqrt{9}}$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{15}{2}\left(\frac{2\sqrt{2}}{3}\right)\frac{1}{3} + \frac{2(2\sqrt{2})}{3}$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{15\sqrt{2}}{3(3)} + \frac{4\sqrt{2}}{3}$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{5\sqrt{2}}{3} + \frac{4\sqrt{2}}{3}$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{\sqrt{2}}{3}$$