1. $f(n) = 10^{3n} + 38^{n} + 35$

$n = 1$: $f(1) = 10^{3(1)} + 38^{1} + 35$

$= 10^{3} + 38 + 35$

$= 1000 + 38 + 35$

$= 1073$

$= 29 \cdot 37$

$37 \mid f(1)$

Assume that $f(n)$ is divisible by 37 when $n = k$.

$n = k$: $f(k) = 10^{3k} + 38^{k} + 35$

If $37 \mid f(k)$

$f(k) = 37s, \ s \in \mathbb{Z}$

$10^{3k} + 38^{k} + 35 = 37s$

when $n = k + 1$:

$f(k+1) = 10^{3(k+1)} + 38^{k+1} + 35$

$= 10^{3k+3} + 38^{k+1} + 35$

$= 10^{3k}10^{3} + 38^{k}38 + 35$

$= 10^{3}(37s - 38^{k} - 35) + 38^{k}38 + 35$

$= 10^{3}(37s) - 10^{3}(38^{k}) - 10^{3}(35) + 38^{k}38 + 35$

$$= 10^3(37s) + (38 - 1000)38^k + 35(1 - 1000)$$

$$= 10^3(37s) - 962 \cdot 38^k - 35 \cdot 999$$

$$= 10^3 37s - 38^k \cdot 26 \cdot 37 - 35 \cdot 27 \cdot 37$$

$$= 37(1000s - 38^k 26 - 35 \cdot 27)$$

Since $s$ is an integer and $k$ is an integer, $1000s - 38^k 26 - 35 \cdot 27$ is an integer.

$\therefore 37 \mid f(k+1)$

Since 37 divides $10^{3n} + 38^n + 35$ when $n=1$, and 37 divides $10^{3n} + 38^n + 35^n$ when $n = k+1$ if 37 divides $10^{3n} + 38^n + 35$ when $n = k$ $\therefore$ 37 divides $10^{3n} + 38^n + 35$ for all integers $n \geqslant 1$.

2. $y = \dfrac{2x^2 - x - 19}{(x+1)(x-3)}$

Expressing $y$ as partial fractions,

$$\frac{2x^2 - x - 19}{(x+1)(x-3)} = A + \frac{B}{x+1} + \frac{C}{x-3}$$

$$= \frac{A(x+1)(x-3) + B(x-3) + C(x+1)}{(x+1)(x-3)}$$

$$\therefore 2x^2 - x - 19 = A(x+1)(x-3) + B(x-3) + C(x+1)$$

$$= A(x^2 - 2x - 3) + B(x-3) + C(x+1)$$

$$= Ax^2 + (-2A + B + C)x - 3A - 3B + C$$

Equating coefficients,

$A = 2 \qquad -2A + B + C = -1 \qquad -3A - 3B + C = -19$

$B + C = 3 \qquad -3B + C = -13$

$4B = 16$

$B = 4$

$$\therefore \frac{2x^2 - x - 19}{(x+1)(x-3)} = 2 + \frac{4}{x+1} - \frac{1}{x-3}$$

$$y = 2 + \frac{4}{x+1} - \frac{1}{x-3}$$

As $x \to \pm\infty \quad y \to 2$,

$x \to -1 \quad y \to \pm\infty$

$x \to 3 \quad y \to \pm\infty$

$\therefore$ The asymptotes of $y$ are the lines $y=2$, $x=-1$ and $x=3$.

$$\frac{dy}{dx} = \frac{-4}{(x+1)^2} + \frac{1}{(x-3)^2}$$

If $\frac{dy}{dx}=0$ : $\frac{-4}{(x+1)^2} + \frac{1}{(x-3)^2} = 0$

$$\frac{4}{(x+1)^2} = \frac{1}{(x-3)^2}$$

$$4(x-3)^2 = (x+1)^2$$

$$\left(\frac{x+1}{x-3}\right)^2 = 4$$

$$\frac{x+1}{x-3} = \pm 2$$

$$\frac{x+1}{x-3} = 2, \quad \frac{x+1}{x-3} = -2$$

$$x+1 = 2(x-3) \qquad x+1 = -2(x-3)$$

$$= 2x-6 \qquad = -2x+6$$

$$x = 7 \qquad 3x = 5$$

$$y = \frac{9}{4} \qquad x = \frac{5}{3}$$

$$y = \frac{17}{4}$$

The critical points of $y$ are $(7, \frac{9}{4})$ and $(\frac{5}{3}, \frac{17}{4})$

$$\frac{d^2y}{dx^2} = \frac{8}{(x+1)^3} - \frac{2}{(x-3)^3}$$

$$x = 7: \quad \frac{d^2y}{dx^2} = \frac{1}{64} - \frac{1}{32} = \frac{-1}{64} < 0$$

$$x = \frac{5}{3}: \quad \frac{d^2y}{dx^2} = \frac{27}{64} + \frac{27}{32} = \frac{81}{64} > 0$$

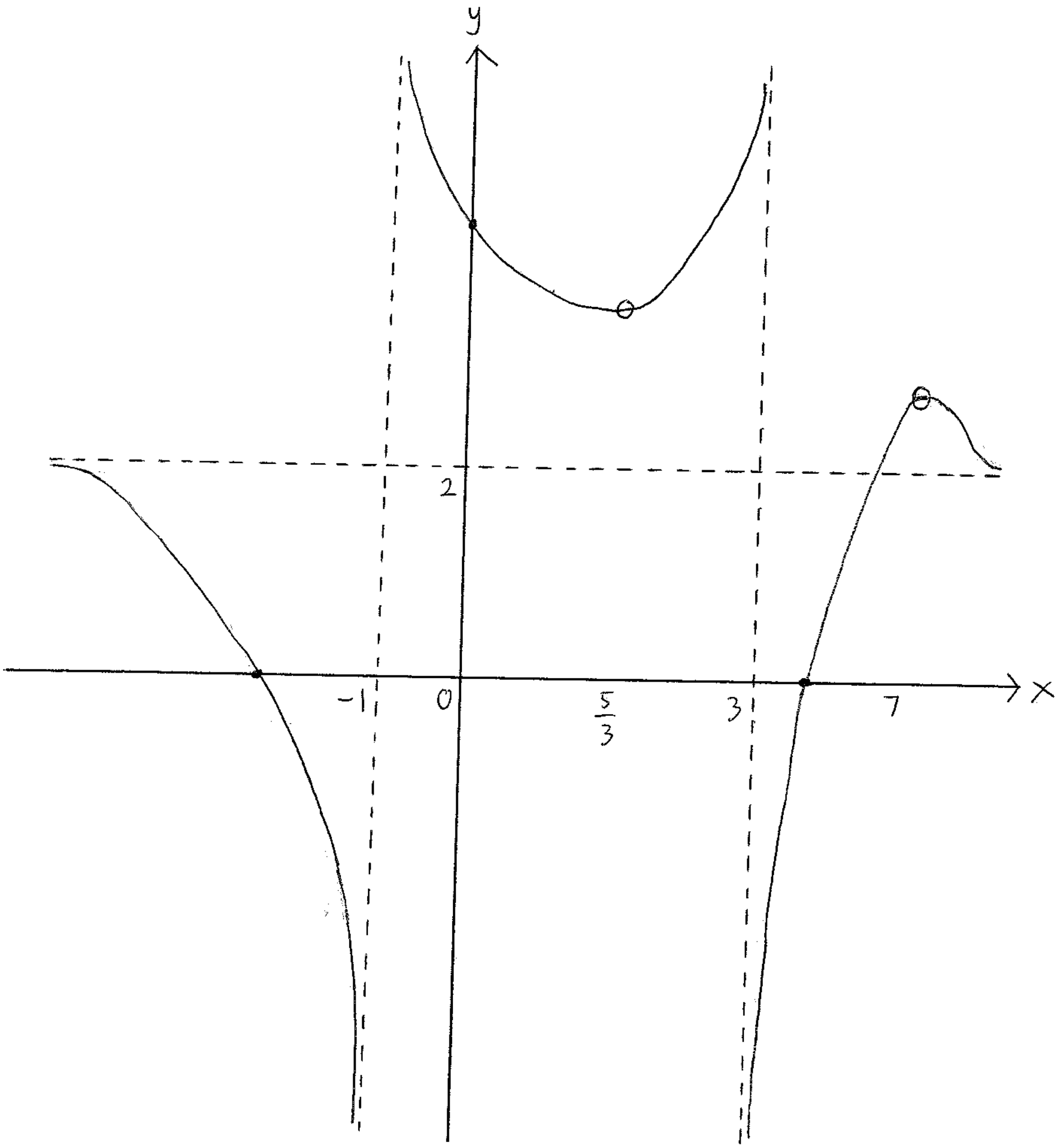
$\therefore (7, \frac{9}{4})$ is a maximum point and $(\frac{5}{3}, \frac{17}{4})$ is a minimum point

when $x = 0$: $y = \frac{19}{3}$

when $y = 0$: $\dfrac{2x^2 - x - 19}{(x+1)(x-3)} = 0$

$$2x^2 - x - 19 = 0$$

$$x = \frac{1 \pm \sqrt{153}}{4}$$




• Intersection point

o Critical point

3. $\overrightarrow{OA} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ $\quad \overrightarrow{OB} = -4\underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k}$ $\quad \overrightarrow{OC} = 4\underset{\sim}{i} - \underset{\sim}{k}$

i) Let $\overrightarrow{OD} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

Since $2AD = DB$

$$2\left[\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right] = \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$2\begin{pmatrix} d_1 - 1 \\ d_2 - 2 \\ d_3 - 3 \end{pmatrix} = \begin{pmatrix} -4 - d_1 \\ 5 - d_2 \\ -1 - d_3 \end{pmatrix}$$

$$\begin{pmatrix} 2d_1 - 2 \\ 2d_2 - 4 \\ 2d_3 - 6 \end{pmatrix} = \begin{pmatrix} -4 - d_1 \\ 5 - d_2 \\ -1 - d_3 \end{pmatrix}$$

$2d_1 - 2 = -4 - d_1$

$2d_2 - 4 = 5 - d_2$

$2d_3 - 6 = -1 - d_3$

$3d_1 = -2$

$3d_2 = 9$

$3d_3 = 5$

$d_1 = \frac{-2}{3}, \quad d_2 = 3, \quad d_3 = \frac{5}{3}$

$\therefore D\left(-\frac{2}{3}, 3, \frac{5}{3}\right)$

$$\overrightarrow{OD} = \begin{pmatrix} -\frac{2}{3} \\ 3 \\ \frac{5}{3} \end{pmatrix}$$

ii) If $\underset{\sim}{n}$ is a normal to the plane, since AB is perpendicular to the plane $\therefore \underset{\sim}{n} \parallel \overrightarrow{AB}$

$$\overrightarrow{AB} = \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix}$$

$$\therefore \underset{\sim}{n} = c\begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix}, \quad c \in R$$

Since $\begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix}$ is a vector perpendicular to the plane and O is a point on the plane, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the position vector of a point on the plane

$$\underset{\sim}{r} \cdot \begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ 3 \\ \frac{5}{3} \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix} = \frac{10}{3} + 9 - \frac{20}{3}$$

$$-5x + 3y - 4z = \frac{17}{3}$$

$$15x - 9y + 12z = -17$$

The equation of the plane which passes through D and is perpendicular to AB is

$$15x - 9y + 12z = -17$$

iii) The equation of the line through C parallel to the vector $\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$ is

$$\underset{\sim}{r} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \quad s \in R$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 + s \\ -2s \\ -1 - s \end{pmatrix}$$

If the line intersects the plane at the point E,

$$15(4 + s) - 9(-2s) + 12(-1 - s) = -17$$

$$60 + 15s + 18s - 12 - 12s = -17$$

$$21s = -65$$

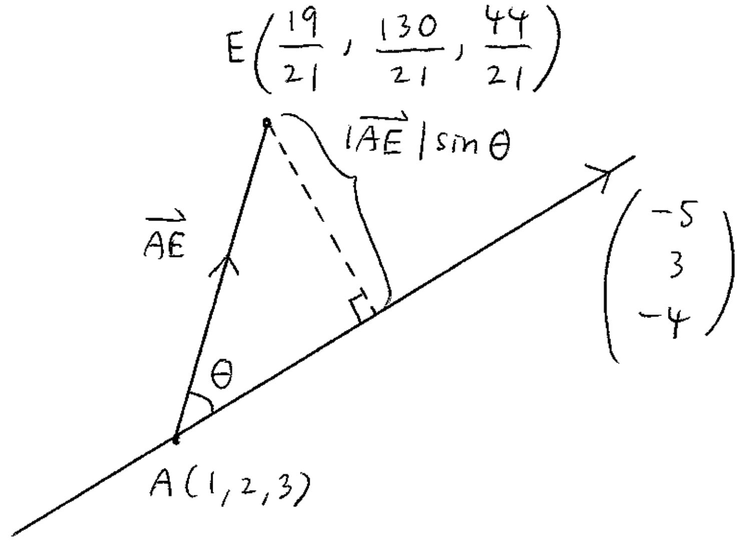
$$s = -\frac{65}{21}$$

$$E\left(\frac{19}{21}, \frac{130}{21}, \frac{44}{21}\right)$$

$\therefore$ The position vector of the point E is

$$\begin{pmatrix} \frac{19}{21} \\ \frac{130}{21} \\ \frac{44}{21} \end{pmatrix}$$

iv)




The perpendicular distance from E to the line AB is $|\overrightarrow{AE}|\sin\theta$

$$\overrightarrow{AE} = \begin{pmatrix}\frac{19}{21}\\[4pt]\frac{130}{21}\\[4pt]\frac{44}{21}\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}\frac{-2}{21}\\[4pt]\frac{88}{21}\\[4pt]\frac{-19}{21}\end{pmatrix}$$

$$\overrightarrow{AE} \times \begin{pmatrix}-5\\3\\-4\end{pmatrix} = |\overrightarrow{AE}|\left|\begin{pmatrix}-5\\3\\-4\end{pmatrix}\right|\sin\theta\,\underset{\sim}{n}$$, where $\underset{\sim}{n}$ is a unit vector

$$\begin{pmatrix}\frac{-2}{21}\\[4pt]\frac{88}{21}\\[4pt]\frac{-19}{21}\end{pmatrix} \times \begin{pmatrix}-5\\3\\-4\end{pmatrix} = |\overrightarrow{AE}|\left|\begin{pmatrix}-5\\3\\-4\end{pmatrix}\right|\sin\theta\,\underset{\sim}{n}$$

$$\begin{pmatrix} \frac{-2}{21} \\ \frac{88}{21} \\ \frac{-19}{21} \end{pmatrix} \times \begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{-2}{21} & \frac{88}{21} & \frac{-19}{21} \\ -5 & 3 & -4 \end{vmatrix}$$

$$= \frac{-295}{21}\underset{\sim}{i} + \frac{87}{21}\underset{\sim}{j} + \frac{434}{21}\underset{\sim}{k}$$

$$\frac{-295}{21}\underset{\sim}{i} + \frac{87}{21}\underset{\sim}{j} + \frac{434}{21}\underset{\sim}{k} = |\overrightarrow{AE}| \left| \begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix} \right| \sin\theta\, \underset{\sim}{n}$$

$$= |\overrightarrow{AE}| \sqrt{(-5)^2 + 3^2 + (-4)^2} \sin\theta\, \underset{\sim}{n}$$

$$= |\overrightarrow{AE}| \sqrt{25 + 9 + 16} \sin\theta\, \underset{\sim}{n}$$

$$= \sqrt{50}\, |\overrightarrow{AE}| \sin\theta\, \underset{\sim}{n}$$

$$\left| \frac{-295}{21}\underset{\sim}{i} + \frac{87}{21}\underset{\sim}{j} + \frac{434}{21}\underset{\sim}{k} \right| = \left| \sqrt{50}\, |\overrightarrow{AE}| \sin\theta\, \underset{\sim}{n} \right|$$

$$\frac{1}{21}\sqrt{(-295)^2 + 87^2 + 434^2} = \sqrt{50}\, |\overrightarrow{AE}|\, |\sin\theta|\, |\underset{\sim}{n}|$$

$$\frac{1}{21}\sqrt{87325 + 7569 + 188356} = \sqrt{50}\, |\overrightarrow{AE}| \sin\theta\,(1)$$

$$\frac{1}{21}\sqrt{283250} = 5\sqrt{2}\, |\overrightarrow{AE}| \sin\theta$$

$$\therefore |\overrightarrow{AE}| \sin\theta = \frac{\sqrt{283250}}{105\sqrt{2}}$$

The perpendicular distance of E from the line AB is $\dfrac{\sqrt{283250}}{105\sqrt{2}}$

4. $$u_j = \frac{7^j(j-2)}{5(j+3)}$$

$$u_{j+1} - u_j = \frac{7^{j+1}(j+1-2)}{5(j+1+3)} - \frac{7^j(j-2)}{5(j+3)}$$

$$= \frac{7^{j+1}(j-1)}{5(j+4)} - \frac{7^j(j-2)}{5(j+3)}$$

$$= \frac{7^j 7(j-1)}{5(j+4)} - \frac{7^j(j-2)}{5(j+3)}$$

$$= \frac{7^j}{5}\left(\frac{7(j-1)}{j+4} - \frac{(j-2)}{j+3}\right)$$

$$= \frac{7^j}{5}\left(\frac{7(j-1)(j+3) - (j-2)(j+4)}{(j+3)(j+4)}\right)$$

$$= \frac{7^j}{5}\left(\frac{7(j^2+2j-3) - (j^2+2j-8)}{(j+3)(j+4)}\right)$$

$$= \frac{7^j}{5}\left(\frac{7j^2+14j-21-j^2-2j+8}{(j+3)(j+4)}\right)$$

$$= \frac{7^j(6j^2+12j-13)}{5(j+3)(j+4)}$$

since $$u_{j+1} - u_j = \frac{7^j(6j^2 + 12j - 13)}{5(j+3)(j+4)}$$

$$S_n = \sum_{j=0}^{n} \frac{7^j(6j^2 + 12j - 13)}{5(j+3)(j+4)}$$

$$= \sum_{j=0}^{n} u_{j+1} - u_j$$

$$\begin{aligned} = \; & u_1 - u_0 \\ & + u_2 - u_1 \\ & + u_3 - u_2 \\ & \vdots \\ & + u_{n-1} - u_{n-2} \\ & + u_n - u_{n-1} \\ & + u_{n+1} - u_n \end{aligned}$$

$$= u_{n+1} - u_0$$

$$= \frac{7^{n+1}(n-1)}{5(n+4)} - \frac{1(-2)}{5(3)}$$

$$= \frac{7^{n+1}(n-1)}{5(n+4)} + \frac{2}{15}$$

$$7^{-n} S_n = 7^{-n}\left(\frac{7^{n+1}(n-1)}{5(n+4)} + \frac{2}{15}\right)$$

$$= \frac{7(n-1)}{5(n+4)} + \frac{7^{-n}2}{15}$$

$$= \frac{7}{5}\left(\frac{n-1}{n+4}\right) + \frac{2}{7^n 15}$$

$$= \frac{7}{5}\left(\frac{n+4-5}{n+4}\right) + \frac{2}{7^n 15}$$

$$= \frac{7}{5}\left(1 - \frac{5}{n+4}\right) + \frac{2}{7^n 15}$$

$$\therefore \lim_{n\to\infty} 7^{-n} s_n = \lim_{n\to\infty}\left(\frac{7}{5}\left(1-\frac{5}{n+4}\right) + \frac{2}{7^n 15}\right)$$

$$= \lim_{n\to\infty} \frac{7}{5}\left(1 - \frac{5}{n+4}\right) + \lim_{n\to\infty} \frac{2}{7^n 15}$$

$$= \frac{7}{5}(1-0) + 0$$

$$= \frac{7}{5}$$

5. i) $2x^3 - x^2 + x + 4 = 0$

$\alpha, \beta, \gamma$ are the roots

$\alpha^2, \beta^2, \gamma^2$

Let $u = \alpha^2$

$\therefore \quad \alpha = \pm\sqrt{u}$

$\alpha$ is a root

$\therefore 2\alpha^3 - \alpha^2 + \alpha + 4 = 0$

$2(\pm\sqrt{u})^3 - (\pm\sqrt{u})^2 \pm \sqrt{u} + 4 = 0$

$\pm 2u\sqrt{u} - u \pm \sqrt{u} + 4 = 0$

$\pm 2u\sqrt{u} \pm \sqrt{u} = u - 4$

$\pm\sqrt{u}(2u + 1) = u - 4$

$(\pm\sqrt{u})^2(2u + 1)^2 = (u - 4)^2$

$u(4u^2 + 4u + 1) = u^2 - 8u + 16$

$4u^3 + 4u^2 + u = u^2 - 8u + 16$

$4u^3 + 3u^2 + 9u - 16 = 0$

$\therefore$ The equation with roots $\alpha^2, \beta^2$ and $\gamma^2$ is

$4u^3 + 3u^2 + 9u - 16 = 0$

$\alpha^2 + \beta^2 + \gamma^2 = -\frac{3}{4}$

$\alpha^2\beta^2 + \beta^2\gamma^2 + \beta^2\gamma^2 = \frac{9}{4}$

$\alpha^2\beta^2\gamma^2 = \frac{16}{4} = 4$

If $S_n = \alpha^n + \beta^n + \gamma^n$ and $T_n = \alpha^{2n} + \beta^{2n} + \gamma^{2n}$

$\therefore S_0 = \alpha^0 + \beta^0 + \gamma^0$

$= 1 + 1 + 1$

$= 3$

$T_1 = \alpha^2 + \beta^2 + \gamma^2$

$= \frac{-3}{4}$

$T_2 = (\alpha^2)^2 + (\beta^2)^2 + (\gamma^2)^2$

$= \alpha^4 + \beta^4 + \gamma^4$

$= (\alpha^2 + \beta^2 + \gamma^2)^2 - 2(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)$

$= \left(\frac{-3}{4}\right)^2 - 2\left(\frac{9}{4}\right)$

$= \frac{9}{16} - \frac{9}{2}$

$= \frac{-63}{16}$

Since $T_2 = (\alpha^2)^2 + (\beta^2)^2 + (\gamma^2)^2$

$= \alpha^4 + \beta^4 + \gamma^4$

$= S_4$

$\therefore S_4 = \frac{-63}{16}$

If $A = 4, B = 3, C = 9, D = -16$

Since $T_0 = \alpha^0 + \beta^0 + \gamma^0$

$= 1 + 1 + 1$

$= 3$

$\therefore AT_{3+r} + BT_{2+r} + CT_{1+r} + DT_r = 0$

$4T_{3+r} + 3T_{2+r} + 9T_{1+r} - 16T_r = 0$

$r=0$ : $4T_3 + 3T_2 + 9T_1 - 16T_0 = 0$

$4T_3 + 3\left(-\frac{63}{16}\right) + 9\left(\frac{-3}{4}\right) - 16(3) = 0$

$4T_3 - \frac{189}{16} - \frac{27}{4} - 48 = 0$

$4T_3 = \frac{1065}{16}$

$T_3 = \frac{1065}{64}$

Since $T_3 = (\alpha^2)^3 + (\beta^2)^3 + (\gamma^2)^3$

$= \alpha^6 + \beta^6 + \gamma^6$

$= S_6$

$S_6 = \frac{1065}{64}$

Let $v = \alpha^4$

$\therefore \alpha^2 = \pm\sqrt{v}$

Since $\alpha^2$ is a root of the equation

$4u^3 + 3u^2 + 9u - 16 = 0$

$\therefore 4(\alpha^2)^3 + 3(\alpha^2)^2 + 9\alpha^2 - 16 = 0$

$$4(\pm\sqrt{v})^3 + 3(\pm\sqrt{v})^2 + 9(\pm\sqrt{v}) - 16 = 0$$

$$\pm 4v\sqrt{v} + 3v \pm 9\sqrt{v} - 16 = 0$$

$$\pm 4v\sqrt{v} \pm 9\sqrt{v} = -3v + 16$$

$$\pm\sqrt{v}(4v + 9) = -3v + 16$$

$$(\pm\sqrt{v})^2(4v + 9)^2 = (-3v + 16)^2$$

$$v(16v^2 + 72v + 81) = 9v^2 - 96v + 256$$

$$16v^3 + 72v^2 + 81v = 9v^2 - 96v + 256$$

$$16v^3 + 63v^2 + 177v - 256 = 0$$

$\therefore$ The equation with roots $\alpha^4$, $\beta^4$ and $\gamma^4$ is

$$16v^3 + 63v^2 + 177v - 256 = 0$$

$$\alpha^4 + \beta^4 + \gamma^4 = -\frac{63}{16}$$

$$\alpha^4\beta^4 + \alpha^4\beta^4 + \beta^4\gamma^4 = \frac{177}{16}$$

$$\alpha^4\beta^4\gamma^4 = \frac{256}{16} = 16$$

$$S_8 = \alpha^8 + \beta^8 + \gamma^8$$

$$= (\alpha^4)^2 + (\beta^4)^2 + (\gamma^4)^2$$

$$= (\alpha^4 + \beta^4 + \gamma^4)^2 - 2(\alpha^4\beta^4 + \alpha^4\gamma^4 + \beta^4\gamma^4)$$

$$= \left(-\frac{63}{16}\right)^2 - 2\left(\frac{177}{16}\right)$$

$$= \frac{3969}{256} - \frac{177}{8}$$

$$= \frac{-1695}{256}$$

ii) $ax^3 + bx^2 + cx + d = 0$

$\alpha, \beta, \gamma$ are the roots

$\therefore \alpha + \beta + \gamma = \frac{-b}{a}$ $\quad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ $\quad \alpha\beta\gamma = \frac{-d}{a}$

If $\alpha\beta = \gamma^3$, $\alpha\gamma = \beta^3$, $\beta\gamma = \alpha^3$

$\therefore \alpha\beta(\beta\gamma)\beta\gamma = \gamma^3\beta^3\alpha^3$

$\alpha^2\beta^2\gamma^2 = \alpha^3\beta^3\gamma^3$

$\alpha^3\beta^3\gamma^3 - \alpha^2\beta^2\gamma^2 = 0$

$\alpha^2\beta^2\gamma^2(\alpha\beta\gamma - 1) = 0$

$\alpha^2\beta^2\gamma^2 = 0$ or $\alpha\beta\gamma - 1 = 0$

$\alpha\beta\gamma = 0$ or $\alpha\beta\gamma = 1$

Since $\alpha\beta\gamma = \frac{-d}{a}$,

$\frac{-d}{a} = 0$ or $\frac{-d}{a} = 1$

$d = 0$ or $d = -a$