Topic : **<u>P105 Roots of Polynomial Equations</u>**

Subtopic:

- *Roots of a Quadratic Equation*
- *Roots of a Cubic Equation*
- *Roots of a Polynomial Equation*

<u>Roots and Factors</u>

Definition :

α is said to be a <u>ROOT</u> of the equation $f(x)=0$ if $f(\alpha)=0$.

The <u>FACTOR</u> of the function $f(x)$ associated to the root α is $(x-\alpha)$.

Example: Determine whether

(a) 3 is a root of $x^3 - 2x^2 - x - 6 = 0$.

(b) $(x-1)$ is a factor of $f(x) = 3x - 3$.

(c) -3 is a root of $x^2 - 2x + 6 = 0$.

(d) $(x+2)$ is a factor of $f(x) = x^3 + 2x^2 + x + 5$.

Roots of a Quadratic Equation

Theorem:

A quadratic equation $ax^2 + bx + c = 0$ always has two roots.

If α and β are the roots of the equation, then

$$\underline{\underline{\alpha + \beta = -\frac{b}{a}}} \quad \text{and} \quad \underline{\underline{\alpha\beta = \frac{c}{a}}}.$$

Example:

Let α and β be the roots of the following equations.

Determine the values of $\alpha + \beta$ and $\alpha\beta$.

(a) $x^2 - 2x - 1 = 0$

(b) $2x^2 + 4x - 3 = 0$

(c) $3x^2 - x + 1 = 0$

Example:

If the roots of $ax^2 + bx + c = 0$ are α and α^2, prove that $b^3 = ac(3b - a - c)$.

Example:

Given that α and β are the roots of $x^2+18x+36=0$, find the value of $\left(\sqrt{\alpha}+\sqrt{\beta}\right)^2$.

Example:

Given that $\alpha, \beta \in \mathrm{R}$ are roots of $x^2 + 2kx + k^2 + 2k - 5 = 0$

Prove that $k \le \dfrac{5}{2}$. If find the lowest value of $\alpha^2 + \beta^2$.

Example:

Given that α and β are real roots of $2x^2 - bx + c = 0$ where $\alpha > \beta$. If the equation $x^2 - qx + r = 0$ has roots $(\alpha + 2)$ and $(\beta + 1)$, find q and r in terms of b and c.

In the case $\alpha = \beta$, show that $q^2 - 4r - 1 = 0$.

Formation of a Quadratic Equation

Theorem :

If the roots of a quadratic equation are R_1 and R_2, then its equation is given by

$$x^2 - (R_1 + R_2)x + R_1R_2 = 0.$$

Example:

α and β are the roots of the quadratic equation $ax^2+bx+c=0$, where $a:b:c$ is in the simplest ratio.

Find a, b and c if

(a) $\alpha+\beta=2$ and $\alpha\beta=3$.

(b) $\alpha+\beta=-3$ and $\alpha\beta=\dfrac{2}{3}$

(c) $\alpha+\beta=-\dfrac{1}{3}$ and $\alpha\beta=-\dfrac{1}{2}$.

Example:

Each pair of the following are the roots of the quadratic equation $ax^2 + bx + c = 0$, where $a : b : c$ is in the simplest ratio. Use the sum and product of roots to find a, b and c.

(a) $2\,,3$

(b) $-3\,,1$

(c) $2\,,-2$

(d) $1\,,-\dfrac{1}{2}$

Example:

Let α and β are the roots of $x^2+4x+5=0$. Without evaluating α and β, find $f(x)$ which has roots

(a) $1+\alpha$ and $1+\beta$ (b) 3α and 3β

Example:

Let α and β are the roots of $x^2 - 3x - 5 = 0$. Without evaluating α and β, find $f(x)$ which has roots

(a) $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ (b) α^2 and β^2

Example:

If $3x^2 + x + 2 = 0$ has the roots α and β, find the quadratic equation having the roots $\alpha + 2\beta$ and $2\alpha + \beta$.

Example:

The roots of $ax^2 + bx + c = 0$ are α and β. Find the quadratic equation having roots $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example:

Determine the set of values of k such that equation $x^2 - (k-3)x + k^2 + 2k + 5 = 0$ has real roots. If α and β are two real roots of the equation, find the maximum value of $\alpha^2 + \beta^2$.

Example:

If α and β are the roots of $6x^2+2x-3=0$, without evaluating α and β, find $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$, $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ and $\dfrac{1}{\alpha\beta}-\dfrac{1}{\beta}-\dfrac{1}{\alpha}$.

Example:

Given that α and β are two roots of $x^2+5x+3=0$.

Without explicit calculation of α and β, find

(a) $\dfrac{\beta^2}{2\alpha-3}+\dfrac{\alpha^2}{2\beta-3}$ (b) $\left(\sqrt{\alpha}+\sqrt{\beta}\right)^2$

(c) a quadratic equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.

Example:

Solve the simultaneous equation $\begin{cases} p+q=-1 \\ pq=-2 \end{cases}$.

Example: (HSC 1966 Jan P1 Q3)

If α and β are the roots of the equation $x^2 + px + q = 0$, express $\alpha^3 + \beta^3$ in terms of p and q.

Express $\left(\alpha - \beta^2\right)\left(\beta - \alpha^2\right)$ in terms of p and q, and deduce that the condition for one root of the equation to be the square of the other is $p^3 - 3pq + q^2 + q = 0$.

Example: (HSC 1965 Jan P1 Q5)

If α and β are the roots of $x^2 + px + q = 0$, show that p and q are the roots of $x^2 + (\alpha + \beta - \alpha\beta)x - \alpha\beta(\alpha + \beta) = 0.$

Find the non-zero values of p and q if the roots of the second equations are (i) α and β, (ii) α^2 and β^2.

Find also the non-zero values of p and q in terms of k if the roots of the second equation $k\alpha$ and $k\beta$.