1. Let $f(n) = 80^{2n-1} + 37^{2n}$

When $n = 1$: $f(1) = 80^{2(1)-1} + 37^{2(1)}$

$$= 80^{2-1} + 37^{2}$$

$$= 80^{1} + 1369$$

$$= 1449$$

$$= 9(161)$$

$$\therefore 9 \mid f(1)$$

Assume the statement is true when $n = k$.

$n = k$: $9 \mid f(k)$

$f(k) = 9s$, s is an integer

$80^{2k-1} + 37^{2k} = 9s$

When $n = k+1$: $f(k+1) = 80^{2(k+1)-1} + 37^{2(k+1)}$

$$= 80^{2k+2-1} + 37^{2k+2}$$

$$= 80^{2k-1}80^{2} + 37^{k}37^{2}$$

$$= 80^{2k-1}6400 + 37^{2k}1369$$

$$= 80^{2k-1}(5031 + 1369) + 37^{2k}1369$$

$$= 80^{2k-1}5031 + 80^{2k-1}1369 + 37^{2k}1369$$

$$= 80^{2k-1}(9 \cdot 559) + 1369(80^{2k-1} + 37^{2k})$$

$$= 80^{2k-1}(9 \cdot 559) + 1369(9s)$$

$$= 9(80^{2k-1}559 + 1369s)$$

Since s is an integer and k is an integer,

$80^{2k-1}559 + 1369s$ is an integer.

$\therefore 9 \mid f(k+1)$

$\therefore 80^{2n-1} + 37^{2n}$ is divisible by 9 for every positive integer n.

2. $(n+1)2n + (n+2)(2n-1) + (n+3)(2n-2) + \dots + (2n-1)(n+2) + 2n(n+1)$

$$= \sum_{r=n+1}^{2n} r(3n+1-r)$$

$$= \sum_{r=n+1}^{2n} (3n+1)r - r^2$$

$$= (3n+1)\sum_{r=n+1}^{2n} r - \sum_{r=n+1}^{2n} r^2$$

$$= (3n+1)\sum_{r=1}^{2n} r - (3n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{2n} r^2 + \sum_{r=1}^{n} r^2$$

$$= (3n+1)\frac{2n(2n+1)}{2} - (3n+1)\frac{n(n+1)}{2}$$

$$\frac{-2n(2n+1)(2(2n)+1)}{6} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(3n+1)n}{2}(2(2n+1) - (n+1)) - \frac{n(2n+1)}{6}(2(4n+1) - n - 1)$$

$$= \frac{(3n+1)n}{2}(4n+2-n-1) - \frac{n(2n+1)}{6}(8n+2-n-1)$$

$$= \frac{(3n+1)n(3n+1)}{2} - \frac{n(2n+1)(7n+1)}{6}$$

$$= \frac{n(3(3n+1)^2 - (2n+1)(7n+1))}{6}$$

$$= \frac{n(3(9n^2+6n+1) - (14n^2+9n+1))}{6}$$

$$= \frac{n(27n^2+18n+3-14n^2-9n-1)}{6}$$

$$= \frac{n(13n^2+9n+2)}{6}$$

3. $3x^4 - 9x^3 + 4x^2 - 6x + 8 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$\alpha^2 - 1,\ \beta^2 - 1, \gamma^2 - 1,\ \delta^2 - 1$

Let $u = \alpha^2 - 1$

$\alpha^2 = u + 1$

$\alpha = \pm\sqrt{u+1}$

α is a root

$\therefore 3\alpha^4 - 9\alpha^3 + 4\alpha^2 - 6\alpha + 8 = 0$

$3(\pm\sqrt{u+1})^4 - 9(\pm\sqrt{u+1})^3 + 4(\pm\sqrt{u+1})^2$

$- 6(\pm\sqrt{u+1}) + 8 = 0$

$3(u+1)^2 - 9(\pm(u+1)\sqrt{u+1}) + 4(u+1)$

$- 6(\pm\sqrt{u+1}) + 8 = 0$

$3(u+1)^2 + 4(u+1) + 8 = 9(\pm(u+1)\sqrt{u+1}) + 6(\pm\sqrt{u+1})$

$3(u^2 + 2u + 1) + 4u + 4 + 8 = \pm\sqrt{u+1}(9(u+1) + 6)$

$3u^2 + 6u + 3 + 4u + 4 + 8 = \pm\sqrt{u+1}(9u + 9 + 6)$

$3u^2 + 10u + 15 = \pm\sqrt{u+1}(9u + 15)$

$(3u^2 + 10u + 15)^2 = (\pm\sqrt{u+1}(9u + 15))^2$

$9u^4 + 30u^3 + 45u^2 + 30u^3 + 100u^2 + 150u$

$+ 45u^2 + 150u + 225 = (u+1)(81u^2 + 270u + 225)$

$9u^4 + 60u^3 + 190u^2 + 300u + 225$

$= 81u^3 + 270u^2 + 225u + 81u^2 + 270u + 225$

$9u^4 - 21u^3 - 161u^2 - 195u = 0$

$\therefore$ The equation having roots $\alpha^2 - 1,\ \beta^2 - 1,\ \gamma^2 - 1,\ \delta^2 - 1$ is $9u^4 - 21u^3 - 161u^2 - 195u = 0$

4.i)
$$\left.\begin{aligned} x + 4y - 9z &= -59 \\ 3x - 7y + 2z &= 81 \\ 5x + 8y - 6z &= -70 \end{aligned}\right\}$$

$$\begin{aligned} &-3 \times ① + ②: \\ &-5 \times ① + ③: \end{aligned} \quad \left.\begin{aligned} x + 4y - 9z &= -59 \\ -19y + 29z &= 258 \\ -12y + 39z &= 225 \end{aligned}\right\}$$

$$\begin{aligned} &12 \times ② \\ &19 \times ③: \end{aligned} \quad \left.\begin{aligned} x + 4y - 9z &= -59 \\ -228y + 348z &= 3096 \\ -228y + 741z &= 4275 \end{aligned}\right\}$$

$$-② + ③: \quad \left.\begin{aligned} x + 4y - 9z &= -59 \\ -228y + 348z &= 3096 \\ 393z &= 1179 \end{aligned}\right\}$$

$$\begin{aligned} z &= 3 \\ y &= -9 \\ x &= 4 \end{aligned}$$

$\therefore$ The intersection point of the planes $x + 4y - 9z = -59$, $3x - 7y + 2z = 81$ and $5x + 8y - 6z = -70$ is $(4, -9, 3)$.

ii) $\overrightarrow{OA} = a^2\underset{\sim}{i}$, $\overrightarrow{OB} = b^2\underset{\sim}{j}$, $\overrightarrow{OC} = c^2\underset{\sim}{k}$

Since A, B and C are on the plane, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel to the plane and $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the plane.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= -a^2\underset{\sim}{i} + b^2\underset{\sim}{j}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= -a^2\underset{\sim}{i} + c^2\underset{\sim}{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{matrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -a^2 & b^2 & 0 \\ -a^2 & 0 & c^2 \end{matrix}$$

$$= b^2c^2\underset{\sim}{i} + a^2c^2\underset{\sim}{j} + a^2b^2\underset{\sim}{k}$$

$\therefore$ The plane containing A, B and C has equation

$$\underset{\sim}{r} \cdot \begin{pmatrix} b^2c^2 \\ a^2c^2 \\ a^2b^2 \end{pmatrix} = \begin{pmatrix} a^2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} b^2c^2 \\ a^2c^2 \\ a^2b^2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} b^2c^2 \\ a^2c^2 \\ a^2b^2 \end{pmatrix} = a^2b^2c^2 + 0 + 0$$

$$b^2c^2x + a^2c^2y + a^2b^2z = a^2b^2c^2$$

The line perpendicular to the plane and passing through the origin has equation

$$\underset{\sim}{r} = s\begin{pmatrix} b^2c^2 \\ a^2c^2 \\ a^2b^2 \end{pmatrix}.$$

When the line meets the plane containing A, B and C,

$x = b^2c^2s$, $y = a^2c^2s$, $z = a^2b^2s$.

$$b^2c^2(b^2c^2s) + a^2c^2(a^2c^2s) + a^2b^2(a^2b^2s) = a^2b^2c^2$$

$$b^4c^4s + a^4c^4s + a^4b^4s = a^2b^2c^2$$

$$(b^4c^4 + a^4c^4 + a^4b^4)s = a^2b^2c^2$$

$$s = \frac{a^2b^2c^2}{b^4c^4 + a^4c^4 + a^4b^4}$$

The line meets the plane at the point

$$\left(\frac{a^2b^4c^4}{b^4c^4 + a^4c^4 + a^4b^4}, \frac{a^4b^2c^4}{b^4c^4 + a^4c^4 + a^4b^4}, \frac{a^4b^4c^2}{b^4c^4 + a^4c^4 + a^4b^4}\right).$$

The distance from the point

$$\left(\frac{a^2b^4c^4}{b^4c^4 + a^4c^4 + a^4b^4}, \frac{a^4b^2c^4}{b^4c^4 + a^4c^4 + a^4b^4}, \frac{a^4b^4c^2}{b^4c^4 + a^4c^4 + a^4b^4}\right)$$

to the origin is

$$\sqrt{\left(\frac{a^2b^4c^4}{b^4c^4+a^4c^4+a^4b^4} - 0\right)^2 + \left(\frac{a^4b^2c^4}{b^4c^4+a^4c^4+a^4b^4} - 0\right)^2 + \left(\frac{a^4b^4c^2}{b^4c^4+a^4c^4+a^4b^4} - 0\right)^2}$$

$$= \sqrt{\frac{a^4b^8c^8 + a^8b^4c^8 + a^8b^8c^4}{(b^4c^4 + a^4c^4 + a^4b^4)^2}}$$

$$= \sqrt{\frac{a^4b^4c^4(b^4c^4 + a^4c^4 + a^4b^4)}{(b^4c^4 + a^4c^4 + a^4b^4)^2}}$$

$$= \sqrt{\frac{a^4b^4c^4}{b^4c^4 + a^4c^4 + a^4b^4}}$$

5. C: $y = \dfrac{7x - 51}{6(2x-9)(3x-7)}$

i) Expressing y in terms of partial fractions,

$$\frac{7x-51}{(2x-9)(3x-7)} = \frac{A}{2x-9} + \frac{B}{3x-7}$$

$$= \frac{A(3x-7) + B(2x-9)}{(2x-9)(3x-7)}$$

$$7x - 51 = A(3x-7) + B(2x-9)$$

$$= 3Ax - 7A + 2Bx - 9B$$

$$= (3A + 2B)x - 7A - 9B$$

$$3A + 2B = 7 \qquad -7A - 9B = -51$$

$$A = \frac{7-2B}{3} \qquad 7A + 9B = 51$$

$$7\left(\frac{7-2B}{3}\right) + 9B = 51$$

$$49 - 14B + 27B = 153$$

$$13B = 104$$

$$B = 8$$

$$A = -3$$

$$\frac{7x-51}{(2x-9)(3x-7)} = \frac{-3}{2x-9} + \frac{8}{3x-7}$$

$$y = \frac{-1}{2(2x-9)} + \frac{4}{3(3x-7)}$$

As $x \longrightarrow \frac{9}{2}$, $y \longrightarrow \pm\infty$

As $x \longrightarrow \frac{7}{3}$, $y \longrightarrow \pm\infty$

As $x \longrightarrow \pm\infty$, $y \longrightarrow 0$

$\therefore$ The asymptotes of C are $x = \frac{9}{2}$, $x = \frac{7}{3}$ and $y = 0$

ii) When $x = 0$: $y = \frac{-17}{126}$

when $y = 0$: $\frac{7x - 51}{6(2x-9)(3x-7)} = 0$

$$7x - 51 = 0$$

$$x = \frac{51}{7}$$

$\therefore$ The intersection points are $\left(0, \frac{-17}{126}\right)$ and $\left(\frac{51}{7}, 0\right)$.

iii) $\frac{dy}{dx} = \frac{1}{(2x-9)^2} - \frac{4}{(3x-7)^2}$

When $\frac{dy}{dx} = 0$: $\frac{1}{(2x-9)^2} - \frac{4}{(3x-7)^2} = 0$

$$\frac{1}{(2x-9)^2} = \frac{4}{(3x-7)^2}$$

$$(3x-7)^2 = 4(2x-9)^2$$

$$3x - 7 = \pm 2(2x-9)$$

$$= 4x - 18, \ -4x + 18$$

$$x = 11, \ 7x = 25$$

$$x = \frac{25}{7}$$

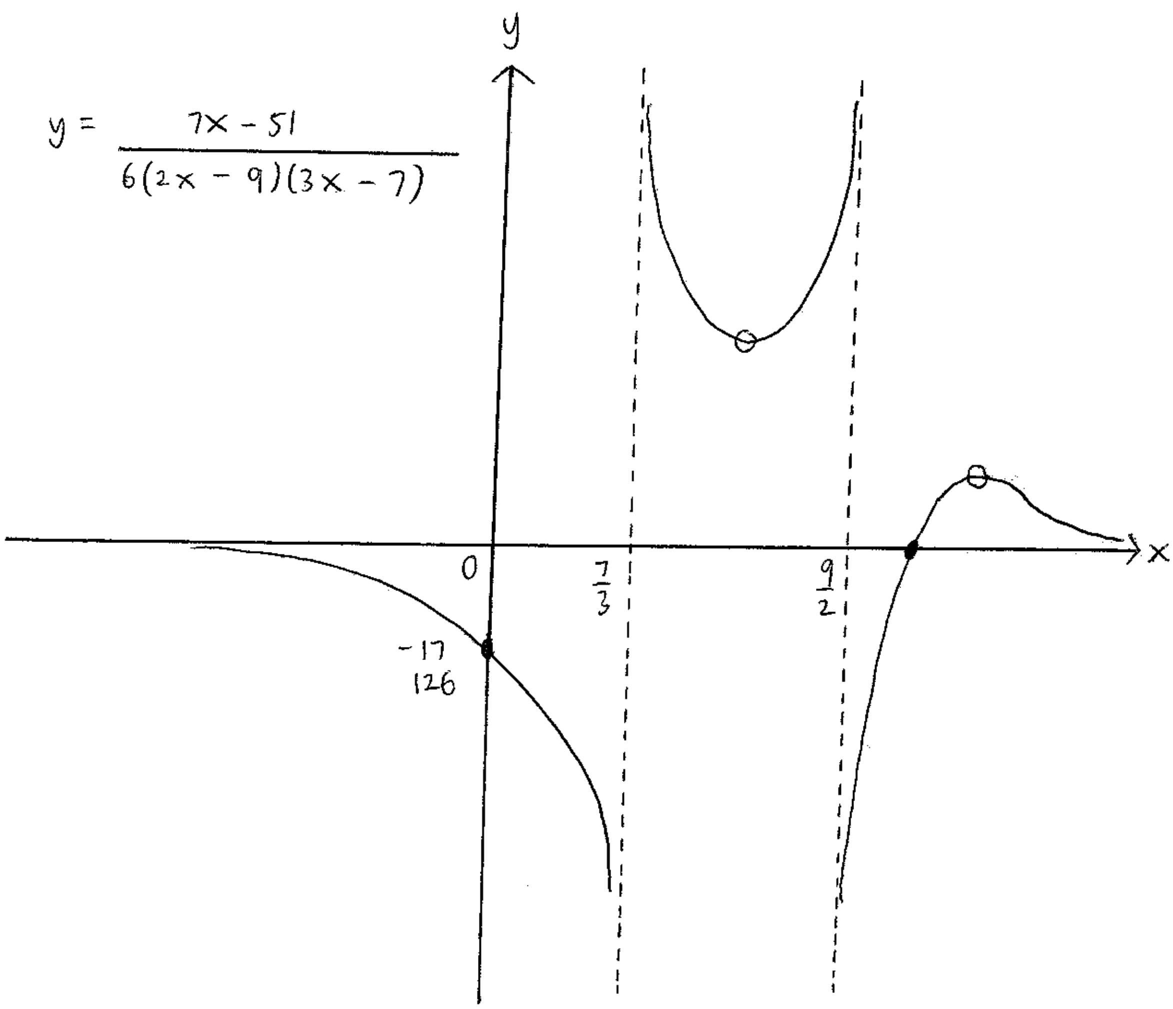
$$y = \frac{1}{78}, \frac{49}{78}$$

$$\frac{d^2y}{dx^2} = \frac{-4}{(2x-9)^3} + \frac{24}{(3x-7)^3}$$

when $x = 11$: $\frac{d^2y}{dx^2} = \frac{-1}{2197} < 0$

when $x = \frac{25}{7}$. $\frac{d^2y}{dx^2} = \frac{2401}{2197} > 0$

$\therefore$ $\left(11, \frac{1}{78}\right)$ is a maximum point and $\left(\frac{25}{7}, \frac{49}{78}\right)$ is a minimum point.



o : critical points

• : intersection points.