MAY/ JUNE 2011

1. $2^2 + 4^2 + \cdots + (2n)^2 = \sum_{r=1}^{n} (2r)^2$

$= \sum_{r=1}^{n} 4r^2$

$= 4\sum_{r=1}^{n} r^2$

$= \dfrac{4n(n+1)(2n+1)}{6}$

$= \dfrac{2n(n+1)(2n+1)}{3}$

$1^2 - 2^2 + 3^2 - 4^2 + \cdots - (2n)^2$

$= 1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2n-1)^2 + (2n)^2$
$\quad - 2(2^2 + 4^2 + \cdots + (2n)^2)$

$= \sum_{r=1}^{2n} r^2 - 2\sum_{r=1}^{n} (2r)^2$

$= \dfrac{2n(2n+1)(2(2n)+1)}{6} - \dfrac{2(2n)(n+1)(2n+1)}{3}$

$= \dfrac{n(2n+1)(4n+1)}{3} - \dfrac{4n(n+1)(2n+1)}{3}$

$= \dfrac{n(2n+1)}{3}\left(4n+1 - 4(n+1)\right)$

$= \dfrac{n(2n+1)}{3}\left(4n+1 - 4n - 4\right)$

$= \dfrac{n(2n+1)}{3}(-3)$

$= -n(2n+1)$

2. $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

$$A^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}, \quad n \geq 1$$

when $n = 1$: $A^1 = \begin{pmatrix} 2^1 & 3(2^1 - 1) \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 3(2-1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3(1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= A$$

Assume the statement is true when $n = k$.

$n = k$: $A^k = \begin{pmatrix} 2^k & 3(2^k - 1) \\ 0 & 1 \end{pmatrix}$

when $n = k+1$: $A^{k+1} = \begin{pmatrix} 2^{k+1} & 3(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$

(what needs to be proved)

$$A^{k+1} = A^k A$$

$$= \begin{pmatrix} 2^k & 3(2^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 3(2^k) + 3(2^k - 1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 3(2^k) + 3(2^k) - 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 6(2^k) - 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 3(2^k 2) - 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 3(2^{k+1}) - 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 3(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$$

$$\therefore A^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}$$

for every positive integer n.

CONFIDENTIAL

3. $\alpha + \beta + \gamma = -6$

$\alpha^2 + \beta^2 + \gamma^2 = 38$

$\alpha\beta\gamma = 30$

$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$(-6)^2 = 38 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$36 = 38 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$\therefore \alpha\beta + \alpha\gamma + \beta\gamma = -1$

$(x - \alpha)(x - \beta)(x - \gamma) = 0$

$(x^2 - (\alpha + \beta)x + \alpha\beta)(x - \gamma) = 0$

$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0$

$x^3 + 6x^2 - x - 30 = 0$

$\therefore$ If $\alpha + \beta + \gamma = -6$, $\alpha^2 + \beta^2 + \gamma^2 = 38$, $\alpha\beta\gamma = 30$, the equation $x^3 + 6x^2 - x - 30 = 0$ has roots α, β, γ.

$2^3 + 6(2^2) - 2 - 30 = 0$

$\therefore 2$ is a root

$$
\begin{array}{r|l}
 & x^2 + 8x + 15 \\
\hline
x - 2 & x^3 + 6x^2 - x - 30 \\
 & \underline{x^3 - 2x^2} \\
 & \quad 8x^2 - x \\
 & \quad \underline{8x^2 - 16x} \\
 & \qquad 15x - 30 \\
 & \qquad \underline{15x - 30} \\
 & \qquad\qquad 0
\end{array}
$$

$(x - 2)(x^2 + 8x + 15) = 0$

$(x - 2)(x + 3)(x + 5) = 0$

$\therefore x = 2, -3, -5.$

4. $C: 2xy^2 + 3x^2y = 1$

$$\frac{d}{dx}(2xy^2 + 3x^2y) = \frac{d}{dx}(1)$$

$$2y^2 + 4xy\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} = 0$$

At $A(-1,1)$: $2 - 4\frac{dy}{dx} - 6 + 3\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -4.$$

$$\frac{d}{dx}\left(2y^2 + 4xy\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx}\right) = 0$$

$$4y\frac{dy}{dx} + \frac{d}{dx}\left(4xy\frac{dy}{dx}\right) + \frac{d}{dx}(6xy) + \frac{d}{dx}\left(3x^2\frac{dy}{dx}\right) = 0$$

$$4y\frac{dy}{dx} + 4xy\frac{d^2y}{dx^2} + 4y\frac{dy}{dx} + 4x\left(\frac{dy}{dx}\right)^2$$

$$+ 6y + 6x\frac{dy}{dx} + 6x\frac{dy}{dx} + 3x^2\frac{d^2y}{dx^2} = 0$$

$$8y\frac{dy}{dx} + 4xy\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right)^2 + 6y + 12x\frac{dy}{dx} + 3x^2\frac{d^2y}{dx^2} = 0$$

At $A(-1,1)$:

$$-32 - 4\frac{d^2y}{dx^2} - 4(16) + 6 - 12(-4) + 3\frac{d^2y}{dx^2} = 0$$

$$\therefore \frac{d^2y}{dx^2} = -42$$

5. $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, \; n \geq 0$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}x \tan^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}x(\sec^2 x - 1) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}x\sec^2 x - \tan^{n-2}x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}x\sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2}x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}x\sec^2 x \, dx - I_{n-2}$$

$w = \tan x$

$dw = \sec^2 x \, dx$

$x = 0 : w = 0$

$x = \frac{\pi}{4} : w = 1$

$$= \int_0^1 w^{n-2} \, dw - I_{n-2}$$

$$= \left[\frac{w^{n-1}}{n-1}\right]_0^1 - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

$n=8$: $I_8 = \frac{1}{7} - I_6$

$I_6 = \frac{1}{5} - I_4$

$I_4 = \frac{1}{3} - I_2$

$I_2 = \frac{1}{1} - I_0$

$$I_0 = \int_0^{\frac{\pi}{4}} \tan^0 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} 1 \, dx$$

$$= [x]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

$I_2 = 1 - \frac{\pi}{4}$

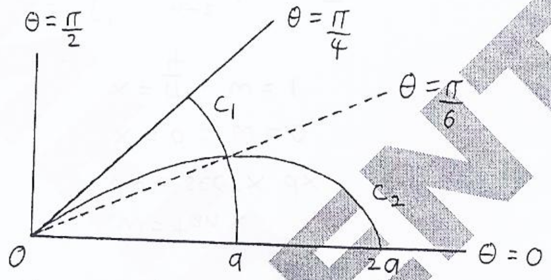
$I_4 = \frac{1}{3} - 1 + \frac{\pi}{4}$

$I_6 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$

$I_8 = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{\pi}{4}$

6. $C_1 : r = a \quad C_2 : r = 2a\cos 2\theta, \quad 0 \le \theta \le \frac{\pi}{4}, \ a > 0$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r_{C_1}	a	a	a	a
r_{C_2}	$2a$	$\sqrt{3}a$	a	0



If C_1 and C_2 intersect at (a, β),

$a = 2a\cos 2\theta$

$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{6}$

$\therefore \beta = \frac{\pi}{6}$

The area of the region bounded by $\theta = 0$, C_1 from $\theta = 0$ to $\theta = \frac{\pi}{6}$ and C_2 from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{4}$ is

$$\int_0^{\frac{\pi}{6}} \frac{r^2}{2}\, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{r^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{a^2}{2}\, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(2a\cos 2\theta)^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{a^2}{2}\, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2a^2\cos^2 2\theta\, d\theta$$

$$= \left[\frac{a^2\theta}{2}\right]_0^{\frac{\pi}{6}} + a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 1 + \cos 4\theta\, d\theta$$

$$= \frac{\pi a^2}{12} - 0 + a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{\pi a^2}{12} + a^2\left(\frac{\pi}{4} + 0 - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right)\right)$$

$$= \frac{\pi a^2}{12} + a^2\left(\frac{\pi}{4} - \frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$$

$$= \frac{\pi a^2}{12} + a^2\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)$$

$$= \frac{\pi a^2}{12} + \frac{\pi a^2}{12} - \frac{\sqrt{3}a^2}{8}$$

$$= a^2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$$

7. $C: x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \pi.$

$$\frac{dx}{dt} = e^t\cos t - e^t\sin t \qquad \frac{dy}{dt} = e^t\sin t + e^t\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2$$

$$= e^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t) + e^{2t}(\sin^2 t + 2\sin t\cos t + \cos^2 t)$$

$$= 2e^{2t}(\sin^2 t + \cos^2 t)$$

$$= 2e^{2t}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2}e^t$$

The arc length of C is

$$\int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{2}e^t\, dt$$

$$= \left[\sqrt{2}e^t\right]_0^{\pi}$$

$$= \sqrt{2}(e^{\pi} - 1).$$

The surface area of revolution of C through 2π radians about the x-axis is

$$\int_0^{\pi} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} 2\pi e^t \sin t \sqrt{2} e^t \, dt$$

$$= 2\sqrt{2}\pi \int_0^{\pi} e^{2t} \sin t \, dt$$

$u = e^{2t}$ $\quad dv = \sin t \, dt$

$du = 2e^{2t} dt$ $\quad v = -\cos t$

$$= 2\sqrt{2}\pi \left(\left[-e^{2t} \cos t \right]_0^{\pi} - \int_0^{\pi} -2e^{2t} \cos t \, dt \right)$$

$$= 2\sqrt{2}\pi \left(e^{2\pi} + 1 + 2\int_0^{\pi} e^{2t} \cos t \, dt \right)$$

$u = e^{2t}$ $\quad dv = \cos t \, dt$

$du = 2e^{2t} dt$ $\quad v = \sin t$

$$= 2\sqrt{2}\pi \left(e^{2\pi} + 1 + 2\left(\left[e^{2t} \sin t \right]_0^{\pi} - \int_0^{\pi} 2e^{2t} \sin t \, dt \right) \right)$$

$$= 2\sqrt{2}\pi \left(e^{2\pi} + 1 + 2\left(0 - 2\int_0^{\pi} e^{2t} \sin t \, dt \right) \right)$$

$$= 2\sqrt{2}\pi \left(e^{2\pi} + 1 - 4\int_0^{\pi} e^{2t} \sin t \, dt \right)$$

$$= 2\sqrt{2}\pi (e^{2\pi} + 1) - 8\sqrt{2}\pi \int_0^{\pi} e^{2t} \sin t \, dt$$

$$2\sqrt{2}\pi \int_0^{\pi} e^{2t} \sin t \, dt + 8\sqrt{2}\pi \int_0^{\pi} e^{2t} \sin t \, dt = 2\sqrt{2}\pi (e^{2\pi} + 1)$$

$$10\sqrt{2}\pi \int_0^{\pi} e^{2t} \sin t \, dt = 2\sqrt{2}\pi (e^{2\pi} + 1)$$

$$\therefore \int_0^{\pi} e^{2t} \sin t \, dt = \frac{e^{2\pi} + 1}{5}$$

$\therefore$ The surface area of revolution of C through 2π radians about the x-axis is $\frac{2\sqrt{2}\pi (e^{2\pi} + 1)}{5}$.

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8. $$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10\sin t$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$

$$m^2 + 2m + 5 = 0$$

$$(m+1)^2 + 4 = 0$$

$$(m+1)^2 = -4$$

$$m+1 = \pm 2i$$

$$m = -1 \pm 2i$$

$\therefore$ The complementary function, x_c, is

$$x_c = e^{-t}(A\cos 2t + B\sin 2t).$$

The particular integral, x_p, is given by

$$x_p = C\cos t + D\sin t.$$

$$\frac{dx_p}{dt} = -C\sin t + D\cos t$$

$$\frac{d^2x_p}{dt^2} = -C\cos t - D\sin t$$

$$\frac{d^2x_p}{dt^2} + 2\frac{dx_p}{dt} + 5x_p = -C\cos t - D\sin t + 2(-C\sin t + D\cos t) + 5(C\cos t + D\sin t)$$

$$= (4C + 2D)\cos t + (-2C + 4D)\sin t$$

$$= 10\sin t$$

$$4C + 2D = 0 \qquad -2C + 4D = 10$$

$$D = -2C \qquad -C + 2D = 5$$

$$-C - 4C = 5$$

$$-5C = 5$$

$$C = -1$$

$$D = 2$$

$$x_p = 2\sin t - \cos t$$

$$x = x_c + x_p$$

$$= e^{-t}(A\cos 2t + B\sin 2t) + 2\sin t - \cos t$$

$\therefore$ The general solution of $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10\sin t$

is $x = e^{-t}(A\cos 2t + B\sin 2t) + 2\sin t - \cos t$.

$$\frac{dx}{dt} = -e^{-t}(A\cos 2t + B\sin 2t) + e^{-t}(-2A\sin 2t + 2B\cos 2t) + 2\cos t + \sin t$$

$t = 0$, $x = 5$: $5 = A - 1$

$t = 0$, $\frac{dx}{dt} = 2$: $2 = -A + 2B + 2$

$$A = 6$$

$$B = 3$$

$$x = e^{-t}(6\cos 2t + 3\sin 2t) + 2\sin t - \cos t$$

$\therefore$ The particular solution is

$$x = e^{-t}(6\cos 2t + 3\sin 2t) + 2\sin t - \cos t.$$

As $t \to \infty$, since $-1 \le \sin t \le 1$ and $-1 \le \cos t \le 1$,
$e^{-t}(6\cos 2t + 3\sin 2t) \to 0$. $\therefore x \to 2\sin t - \cos t$.

9. C: $y = \dfrac{ax^2 + bx + c}{x - 1}$

$y = 2x - 5$ is an asymptote.

i)

$$\begin{array}{r|l} & ax + a + b \\ \hline x - 1 & ax^2 + bx + c \\ & ax^2 - ax \\ \hline & (a+b)x + c \\ & (a+b)x - a - b \\ \hline & a + b + c \end{array}$$

$y = ax + a + b + \dfrac{a+b+c}{x-1}$

As $x \to 1$, $y \to \pm\infty$.

$\therefore$ The equation of the other asymptote is $x = 1$.

ii) Since $y = 2x - 5$ is an asymptote,

$ax + a + b = 2x - 5$

$a = 2 \qquad a + b = -5$

$b = -7$

iii) $y = 2x - 5 + \dfrac{c-5}{x-1}$

$\dfrac{dy}{dx} = 2 - \dfrac{(c-5)}{(x-1)^2}$

when $\dfrac{dy}{dx} = 0$: $2 - \dfrac{(c-5)}{(x-1)^2} = 0$

$\dfrac{c-5}{(x-1)^2} = 2$

$c - 5 = 2(x-1)^2$

If C has a turning point when $x = 2$,

$c - 5 = 2$

$c = 7$.

iv) $y = \dfrac{2x^2 - 7x + 7}{x - 1}$

If $y = k$ intersects C,

$\dfrac{2x^2 - 7x + 7}{x - 1} = k$

$2x^2 - 7x + 7 = kx - k$

$2x^2 - (k+7)x + k + 7 = 0$

$a = 2 \quad b = -(k+7) \quad c = k + 7$

$b^2 - 4ac \geq 0$

$(k+7)^2 - 4(2)(k+7) \geq 0$

$(k+7)^2 - 8(k+7) \geq 0$

$(k+7)(k+7-8) \geq 0$

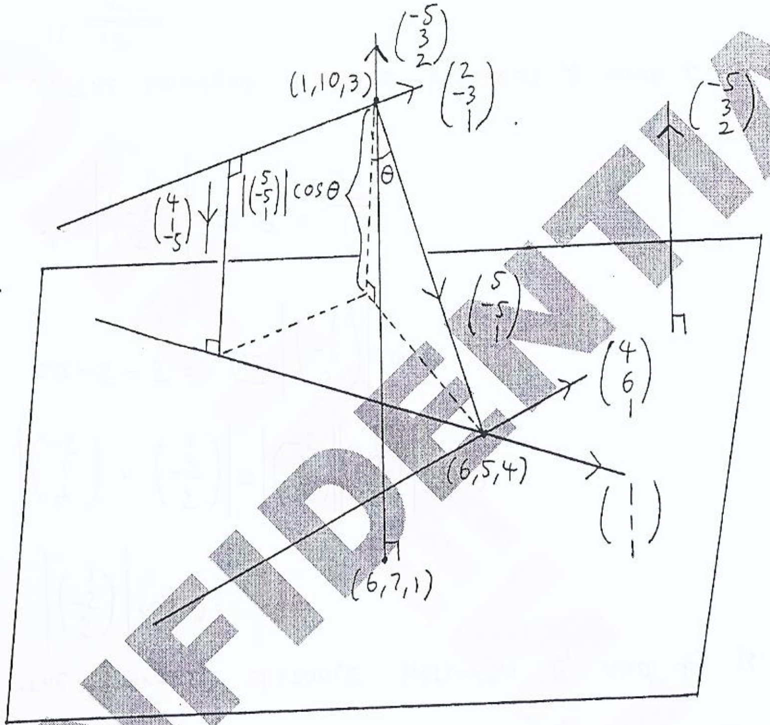
$(k+7)(k-1) \geq 0$

$k \leq -7$ or $k \geq 1$

$\therefore$ If $y = k$ does not intersect C, $-7 < k < 1$.

10: $l_1: \underset{\sim}{r} = 6\underset{\sim}{i} + 5\underset{\sim}{j} + 4\underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

$l_2: \underset{\sim}{r} = 6\underset{\sim}{i} + 5\underset{\sim}{j} + 4\underset{\sim}{k} + \mu(4\underset{\sim}{i} + 6\underset{\sim}{j} + \underset{\sim}{k})$



Since Π contains l_1 and l_2, the normal of Π is parallel to $\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}4\\6\\1\end{pmatrix}$.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}4\\6\\1\end{pmatrix} = \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 1 & 1 & 1\\ 4 & 6 & 1\end{vmatrix} = \begin{pmatrix}-5\\3\\2\end{pmatrix}$$

Since $\begin{pmatrix}-5\\3\\2\end{pmatrix}$ is perpendicular to the Π and (6,5,4) is a point on Π, the cartesian equation of Π is

$$\underset{\sim}{r} \cdot \begin{pmatrix}-5\\3\\2\end{pmatrix} = \begin{pmatrix}6\\5\\4\end{pmatrix} \cdot \begin{pmatrix}-5\\3\\2\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-5\\3\\2\end{pmatrix} = -30 + 15 + 8$$

$$-5x + 3y + 2z = -7$$

$$5x - 3y - 2z = 7.$$

Since $\begin{pmatrix}-5\\3\\2\end{pmatrix}$ is normal to Π, the line perpendicular to Π passing through (1,10,3) is $\underset{\sim}{r} = \begin{pmatrix}1\\10\\3\end{pmatrix} + s\begin{pmatrix}-5\\3\\2\end{pmatrix}$.

When $\underset{\sim}{r} = \begin{pmatrix}1\\10\\3\end{pmatrix} + s\begin{pmatrix}-5\\3\\2\end{pmatrix}$ meets Π,

$$x = 1 - 5s, \quad y = 10 + 3s, \quad z = 2s + 3$$

$$5(1 - 5s) - 3(10 + 3s) - 2(2s + 3) = 7$$

$$5 - 25s - 30 - 9s - 4s - 6 = 7$$

$$-38s = 38$$

$$s = -1$$

$\therefore$ The foot of the perpendicular from (1,10,3) to Π has position vector $\begin{pmatrix}6\\7\\1\end{pmatrix}$.

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$l_3: \underset{\sim}{r} = \underset{\sim}{i} + 10\underset{\sim}{j} + 3\underset{\sim}{k} + v(2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k})$

The line perpendicular to l_1 and l_3 has direction

$\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}2\\-3\\1\end{pmatrix}$.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 1 & 1 & 1\\ 2 & -3 & 1\end{vmatrix} = \begin{pmatrix}4\\1\\-5\end{pmatrix}.$$

The shortest distance between l_1 and l_3 is

$\left|\begin{pmatrix}5\\-5\\1\end{pmatrix}\right| \cos\theta.$

$$\left|\begin{pmatrix}4\\1\\-5\end{pmatrix} \cdot \begin{pmatrix}5\\-5\\1\end{pmatrix}\right| = \left|\begin{pmatrix}4\\1\\-5\end{pmatrix}\right|\left|\begin{pmatrix}5\\-5\\1\end{pmatrix}\right| \cos\theta$$

$$20 - 5 - 5 = \sqrt{42}\left|\begin{pmatrix}5\\-5\\1\end{pmatrix}\right| \cos\theta$$

$$\therefore \left|\begin{pmatrix}5\\-5\\1\end{pmatrix}\right| \cos\theta = \frac{10}{\sqrt{42}}.$$

$\therefore$ The shortest distance between l_1 and l_3 is $\dfrac{10}{\sqrt{42}}$.

11. EITHER

If A has eigenvalues $-1, 1, 2$ with corresponding eigenvectors $\begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$.

i) $A = PDP^{-1}$, where $P = \begin{pmatrix}0 & -1 & 1\\ 1 & 0 & 1\\ -1 & 1 & 0\end{pmatrix}$

and $D = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$.

$$P = \begin{pmatrix}0 & -1 & 1\\ 1 & 0 & 1\\ -1 & 1 & 0\end{pmatrix}$$

$$|P| = 0(0-1) - (-1)(0+1) + 1(1-0)$$
$$= 0 + 1 + 1$$
$$= 2$$

$$P^T = \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$$

$$\operatorname{adj} P = \begin{pmatrix}-1 & 1 & -1\\ -1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$$

$$\therefore P^{-1} = \frac{1}{|P|}\operatorname{adj} P$$

$$= \frac{1}{2}\begin{pmatrix}-1 & 1 & -1\\ -1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

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$$A = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 3 \\ -2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ -1 & 1 & 0 \end{pmatrix}$$

ii) $A^{2n} = PD^{2n}P^{-1}$

$$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{2n}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} (-1)^{2n} & 0 & 0 \\ 0 & 1^{2n} & 0 \\ 0 & 0 & 2^{2n} \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{2n} \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 2^{2n} \\ 1 & 0 & 2^{2n} \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2^{2n}+1 & 2^{2n}-1 & 2^{2n}-1 \\ 2^{2n}-1 & 2^{2n}+1 & 2^{2n}-1 \\ 0 & 0 & 2 \end{pmatrix}$$

OR

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & -1 & -4 & 3 \\ 3 & -3 & -2 & 2 \\ 5 & -4 & -6 & 5 \end{pmatrix}$$

$$\xrightarrow[\substack{-2r_1 + r_2 \\ -3r_1 + r_3 \\ -5r_1 + r_4}]{} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{r_3 \Longleftrightarrow r_4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{-r_2 + r_3} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{-r_3 + r_4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank$(A) = 3$

If $A\underset{\sim}{x} = p\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} + q\begin{pmatrix} -1 \\ -1 \\ -3 \\ -4 \end{pmatrix} + r\begin{pmatrix} -1 \\ -4 \\ -2 \\ -6 \end{pmatrix}$, $p, q, r \in R$

$\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

$$A\underset{\sim}{x} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & -1 & -4 & 3 \\ 3 & -3 & -2 & 2 \\ 5 & -4 & -6 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$= \begin{pmatrix} x - y - z + w \\ 2x - y - 4z + 3w \\ 3x - 3y - 2z + 2w \\ 5x - 4y - 6z + 5w \end{pmatrix}$$

$$= \begin{pmatrix} x \\ 2x \\ 3x \\ 5x \end{pmatrix} + \begin{pmatrix} -y \\ -y \\ -3y \\ -4y \end{pmatrix} + \begin{pmatrix} -z \\ -4z \\ -2z \\ -6z \end{pmatrix} + \begin{pmatrix} w \\ 3w \\ 2w \\ 5w \end{pmatrix}$$

$$= x\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} + y\begin{pmatrix} -1 \\ -1 \\ -3 \\ -4 \end{pmatrix} + z\begin{pmatrix} -1 \\ -4 \\ -2 \\ -6 \end{pmatrix} + w\begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} p - q - r \\ 2p - q - 4r \\ 3p - 3q - 2r \\ 5p - 4q - 6r \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & p - q - r \\ 2 & -1 & -4 & 3 & 2p - q - 4r \\ 3 & -3 & -2 & 2 & 3p - 3q - 2r \\ 5 & -4 & -6 & 5 & 5p - 4q - 6r \end{array}\right)$$

$$\xrightarrow[\substack{-2r_1 + r_2 \\ -3r_1 + r_3 \\ -5r_1 + r_4}]{} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & p - q - r \\ 0 & 1 & -2 & 1 & q - 2r \\ 0 & 0 & 1 & -1 & r \\ 0 & 1 & -1 & 0 & q - r \end{array}\right)$$

$$\xrightarrow{r_3 \Leftrightarrow r_4} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & p - q - r \\ 0 & 1 & -2 & 1 & q - 2r \\ 0 & 1 & -1 & 0 & q - r \\ 0 & 0 & 1 & -1 & r \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & p - q - r \\ 0 & 1 & -2 & 1 & q - 2r \\ 0 & 0 & 1 & -1 & r \\ 0 & 0 & 1 & -1 & r \end{array}\right)$$

$$\xrightarrow{-r_3 + r_4} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & p - q - r \\ 0 & 1 & -2 & 1 & q - 2r \\ 0 & 0 & 1 & -1 & r \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = s, s \in R$

$z = s + r$

$y - 2z + w = q - 2r$

$y - 2s - 2r + s = q - 2r$

$y = q + s$

$x - q - s - s - r + s = p - q - r$

$x = p + s$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} p + s \\ q + s \\ r + s \\ s \end{pmatrix}$$

$$\underset{\sim}{x} = \begin{pmatrix} p + s \\ q + s \\ r + s \\ s \end{pmatrix}$$

$$= \begin{pmatrix} p \\ q \\ r \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$p\begin{pmatrix}1\\2\\3\\5\end{pmatrix}+q\begin{pmatrix}-1\\-1\\-3\\-4\end{pmatrix}+r\begin{pmatrix}-1\\-4\\-2\\-6\end{pmatrix}=\begin{pmatrix}3\\7\\8\\15\end{pmatrix}$$

$$\begin{pmatrix}p-q-r\\2p-q-4r\\3p-3q-2r\\5p-4q-6r\end{pmatrix}=\begin{pmatrix}3\\7\\8\\15\end{pmatrix}.$$

$$\begin{pmatrix}1&-1&-1\\2&-1&-4\\3&-3&-2\\5&-4&-6\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}3\\7\\8\\15\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&-1&-1&3\\2&-1&-4&7\\3&-3&-2&8\\5&-4&-6&15\end{array}\right)$$

$$\xrightarrow[-5r_1+r_4]{\substack{-2r_1+r_2\\-3r_1+r_3}}\left(\begin{array}{ccc|c}1&-1&-1&3\\0&1&-2&1\\0&0&1&-1\\0&1&-1&0\end{array}\right)$$

$$\xrightarrow{r_3\Leftrightarrow r_4}\left(\begin{array}{ccc|c}1&-1&-1&3\\0&1&-2&1\\0&1&-1&0\\0&0&1&-1\end{array}\right)$$

$$\xrightarrow{-r_2+r_3}\left(\begin{array}{ccc|c}1&-1&-1&3\\0&1&-2&1\\0&0&1&-1\\0&0&1&-1\end{array}\right)$$

$$\xrightarrow{-r_3+r_4}\left(\begin{array}{ccc|c}1&-1&-1&3\\0&1&-2&1\\0&0&1&-1\\0&0&0&0\end{array}\right)$$

$r=-1$

$q=-1$

$p=1$

$$A\underset{\sim}{x}=\begin{pmatrix}3\\7\\8\\15\end{pmatrix},\ \underset{\sim}{x}=\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix},\ \alpha^2+\beta^2+\gamma^2+\delta^2=\frac{11}{4}$$

If $A\underset{\sim}{x}=p\begin{pmatrix}1\\2\\3\\5\end{pmatrix}+q\begin{pmatrix}-1\\-1\\-3\\-4\end{pmatrix}+r\begin{pmatrix}-1\\-4\\-2\\-6\end{pmatrix}=\begin{pmatrix}3\\7\\8\\15\end{pmatrix},\ \underset{\sim}{x}=\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}$

then $p=1,\ q=-1,\ r=-1.$

$$\therefore\ \underset{\sim}{x}=\begin{pmatrix}p+\lambda\\q+\lambda\\r+\lambda\\\lambda\end{pmatrix}=\begin{pmatrix}\lambda+1\\\lambda-1\\\lambda-1\\\lambda\end{pmatrix}$$

since $\underset{\sim}{x}=\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}$, $\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}=\begin{pmatrix}\lambda+1\\\lambda-1\\\lambda-1\\\lambda\end{pmatrix}$

$\alpha=\lambda+1$

$\beta=\lambda-1$

$\gamma=\lambda-1$

$\delta=\lambda$

$$\alpha^2+\beta^2+\gamma^2+\delta^2=\frac{11}{4}$$

$$(\lambda+1)^2+(\lambda-1)^2+(\lambda-1)^2+\lambda^2=\frac{11}{4}$$

$$\lambda^2 + 2\lambda + 1 + \lambda^2 - 2\lambda + 1 + \lambda^2 - 2\lambda + 1 + \lambda^2 = \frac{11}{4}$$

$$4\lambda^2 - 2\lambda + 3 = \frac{11}{4}$$

$$4\lambda^2 - 2\lambda + \frac{1}{4} = 0$$

$$16\lambda^2 - 8\lambda + 1 = 0$$

$$(4\lambda - 1)^2 = 0$$

$$\lambda = \frac{1}{4}$$

$$\therefore \underset{\sim}{x} = \begin{pmatrix} \lambda + 1 \\ \lambda - 1 \\ \lambda - 1 \\ \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$$