12.

$$A = \begin{pmatrix} -3 & 5 & 5 \\ -4 & 6 & 5 \\ 4 & -4 & -3 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -3 & 5 & 5 \\ -4 & 6 & 5 \\ 4 & -4 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3-\lambda & 5 & 5 \\ -4 & 6-\lambda & 5 \\ 4 & -4 & -3-\lambda \end{pmatrix}$$

$$A - \lambda I = (-3-\lambda)\left[(6-\lambda)(-3-\lambda) + 20\right]$$

$$-5\left[-4(-3-\lambda) - 20\right] + 5(16 - 4(6-\lambda))$$

$$= (\lambda+3)^2(6-\lambda) + 20(-3-\lambda) + 20(-3-\lambda)$$

$$+ 100 + 5(16 - 24 + 4\lambda)$$

$$= (\lambda^2 + 6\lambda + 9)(6-\lambda) + 40(-3-\lambda)$$

$$+ 100 + 5(4\lambda - 8)$$

$$= 6\lambda^2 + 36\lambda + 54 - \lambda^3 - 6\lambda^2 - 9\lambda$$

$$- 120 - 40\lambda + 100 + 20\lambda - 40$$

$$= -\lambda^3 + 7\lambda - 6$$

$$= (1-\lambda)(\lambda^2 + \lambda - 6)$$

$$= -(\lambda-1)(\lambda-2)(\lambda+3)$$

$$|A - \lambda I| = 0$$

$$-(\lambda-1)(\lambda-2)(\lambda+3) = 0$$

$$\lambda = 1, 2, -3.$$

$\lambda = 1$: $\begin{pmatrix} -4 & 5 & 5 \\ 4 & 5 & 5 \\ 4 & -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -4 & 5 & 5 & 0 \\ 4 & 5 & 5 & 0 \\ 4 & -4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow[r_1 + r_3]{r_1 + r_2} \left(\begin{array}{ccc|c} -4 & 5 & 5 & 0 \\ 0 & 10 & 10 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{10}} \left(\begin{array}{ccc|c} -4 & 5 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} -4 & 5 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s$, $s \in R$

$y = -s$

$-4x - 5s + 5s = 0$

$-4x = 0$

$x = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -s \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda = 2: \quad \begin{pmatrix} -5 & 5 & 5 \\ -4 & 4 & 5 \\ 4 & -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -5 & 5 & 5 & 0 \\ -4 & 4 & 5 & 0 \\ 4 & -4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{5},\ \frac{r_3}{4}} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ -4 & 4 & 5 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[r_1 + r_3]{-4r_1 + r_2} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = 0$

Let $y = s$, $s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix}$$

$$= s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = -3$: $\begin{pmatrix} 0 & 5 & 5 \\ -4 & 9 & 5 \\ 4 & -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 0 & 5 & 5 & 0 \\ -4 & 9 & 5 & 0 \\ 4 & -4 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 4 & -4 & 0 & 0 \\ -4 & 9 & 5 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{4}, \frac{r_3}{5}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -4 & 9 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{4r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -4 & 9 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

Let $z = s$, $s \in R$

$y = -s$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ -s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are $1, 2, -3$

with corresponding eigenvectors $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\begin{aligned} E &= D^3 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}^3 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -27 \end{pmatrix} \end{aligned}$$

15. $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, $B\underset{\sim}{x} = \mu \underset{\sim}{x}$

$$(AB)\underset{\sim}{x} = A(B\underset{\sim}{x})$$
$$= A(\mu \underset{\sim}{x})$$
$$= \mu (A\underset{\sim}{x})$$
$$= \mu (\lambda \underset{\sim}{x})$$
$$= (\lambda \mu) \underset{\sim}{x}$$

$$C = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$C - \lambda I = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 1 & 4 \\ 1 & 3-\lambda & -1 \\ 2 & 1 & 3-\lambda \end{pmatrix}$$

$$|C - \lambda I| = (1-\lambda)\left[(3-\lambda)^2 + 1\right]$$
$$-(3 - \lambda + 2) + 4(1 - 2(3-\lambda))$$
$$= (1-\lambda)(3-\lambda)^2 + 1 - \lambda$$
$$-(5-\lambda) + 4(1 - 6 + 2\lambda)$$
$$= (1-\lambda)(9 - 6\lambda + \lambda^2) + 1 - \lambda$$
$$-5 + \lambda + 4(2\lambda - 5)$$

$$= 9 - 6\lambda + \lambda^2 - 9\lambda + 6\lambda^2 - \lambda^3 + 1 - \lambda$$

$$-5 + \lambda + 8\lambda - 20$$

$$= -\lambda^3 + 7\lambda^2 - 7\lambda - 15$$

$$= (\lambda + 1)(-\lambda^2 + 8\lambda + 15)$$

$$= -(\lambda + 1)(\lambda - 3)(\lambda - 5)$$

$$|A - \lambda I| = 0$$

$$-(\lambda + 1)(\lambda - 3)(\lambda - 5) = 0$$

$$\lambda = -1, 3, 5$$

$\lambda = -1$: $\begin{pmatrix} 2 & 1 & 4 \\ 1 & 4 & -1 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -7 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 7s, s \in R$

$y = 6s$

$x + 24s - 7s = 0$

$x = -17s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -17s \\ 6s \\ 7s \end{pmatrix}$$

$$= s \begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$$

$\lambda = 3$: $$\begin{pmatrix} -2 & 1 & 4 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -2 & 1 & 4 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s, s \in R$

$y = -2s$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -2s \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\lambda = 5$: $\begin{pmatrix} -4 & 1 & 4 \\ 1 & -2 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -4 & 1 & 4 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -4 & 1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow[4r_1 + r_3]{-2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & -7 & 0 & 0 \end{array}\right)$$

$-7y = 0$

$y = 0$

Let $z = s, s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & -3 \end{pmatrix}$$

$$D - \lambda I = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 - \lambda & 1 & 1 \\ 0 & -1 - \lambda & 4 \\ 0 & 0 & -3 - \lambda \end{pmatrix}$$

$$|D - \lambda I| = (-2 - \lambda)\left[(-1 - \lambda)(-3 - \lambda) - 0\right]$$

$$-1 \cdot 0 + 1 \cdot 0$$

$$= (-2 - \lambda)(-1 - \lambda)(-3 - \lambda)$$

$$= -(\lambda + 1)(\lambda + 2)(\lambda + 3)$$

$$|D - \lambda I| = 0$$

$$-(\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

$$\lambda = -1, -2, -3$$

$$\lambda = -1: \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

$z = 0$

Let $y = s,\ s \in \mathbb{R}$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = -2: \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$z = 0$

$y = 0$

Let $x = s,\ s \in \mathbb{R}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$$

$$\equiv s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = -3$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s$, $s \in R$

$y = -2s$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -2s \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

An eigenvalue of the matrix CD is -9 with corresponding eigenvector $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

18. $A\underset{\sim}{e} = \lambda \underset{\sim}{e}$

$$A^2\underset{\sim}{e} = A(A\underset{\sim}{e})$$
$$= A(\lambda \underset{\sim}{e})$$
$$= \lambda(A\underset{\sim}{e})$$
$$= \lambda(\lambda\underset{\sim}{e})$$
$$= \lambda^2 \underset{\sim}{e}.$$

$$B = \begin{pmatrix} 2 & -5 & 6 \\ 2 & 3 & 2 \\ -1 & 5 & -5 \end{pmatrix}$$

$$B - \lambda I = \begin{pmatrix} 2 & -5 & 6 \\ 2 & 3 & 2 \\ -1 & 5 & -5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2-\lambda & -5 & 6 \\ 2 & 3-\lambda & 2 \\ -1 & 5 & -5-\lambda \end{pmatrix}$$

$$|B - \lambda I| = (2-\lambda)\left[(3-\lambda)(-5-\lambda) - 10\right]$$
$$+ 5\left[2(-5-\lambda) + 2\right] + 6(10 + 3 - \lambda)$$
$$= (2-\lambda)(3-\lambda)(-5-\lambda) - 10(2-\lambda)$$
$$+ 5(-10 - 2\lambda + 2) + 6(13 - \lambda)$$
$$= (\lambda^2 - 5\lambda + 6)(-5-\lambda) - 20 + 10\lambda$$
$$+ 5(-2\lambda - 8) + 78 - 6\lambda$$
$$= -5\lambda^2 + 25\lambda - 30 - \lambda^3 + 5\lambda^2 - 6\lambda$$
$$-20 + 10\lambda - 10\lambda - 40 + 78 - 6\lambda$$

$$= -\lambda^3 + 13\lambda - 12$$

$$= (\lambda - 1)(\lambda^2 + \lambda - 12)$$

$$= (\lambda - 1)(\lambda - 3)(\lambda + 4)$$

$$|B - \lambda I| = 0$$

$$(\lambda - 1)(\lambda - 3)(\lambda + 4) = 0$$

$$\lambda = 1, 3, -4$$

$$\lambda = 1: \begin{pmatrix} 1 & -5 & 6 \\ 2 & 2 & 2 \\ -1 & 5 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -5 & 6 & 0 \\ 2 & 2 & 2 & 0 \\ -1 & 5 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[r_1 + r_3]{-2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -5 & 6 & 0 \\ 0 & 12 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 6s$, $s \in R$

$y = 5s$

$x - 25s + 36s = 0$

$x = -11s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11s \\ 5s \\ 6s \end{pmatrix}$$

$$= s\begin{pmatrix} -11 \\ 5 \\ 6 \end{pmatrix}$$

$$\lambda = 3: \quad \begin{pmatrix} -1 & -5 & 6 \\ 2 & 0 & 2 \\ -1 & 5 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & -5 & 6 & 0 \\ 2 & 0 & 2 & 0 \\ -1 & 5 & -8 & 0 \end{array}\right)$$

$$\xrightarrow[-r_1 + r_3]{2r_1 + r_2} \left(\begin{array}{ccc|c} -1 & -5 & 6 & 0 \\ 0 & -10 & 14 & 0 \\ 0 & 10 & -14 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} -1 & -5 & 6 & 0 \\ 0 & -10 & 14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 5s,\ s \in R$

$y = 7s$

$-x - 35s + 30s = 0$

$x = -5s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5s \\ 7s \\ 5s \end{pmatrix}$$

$$= s \begin{pmatrix} -5 \\ 7 \\ 5 \end{pmatrix}$$

$$\lambda = -4: \quad \begin{pmatrix} 6 & -5 & 6 \\ 2 & 7 & 2 \\ -1 & 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 6 & -5 & 6 & 0 \\ 2 & 7 & 2 & 0 \\ -1 & 5 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 5 & -1 & 0 \\ 2 & 7 & 2 & 0 \\ 6 & -5 & 6 & 0 \end{array}\right)$$

$$\xrightarrow[6r_1 + r_3]{2r_1 + r_2} \left(\begin{array}{ccc|c} -1 & 5 & -1 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 25 & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s,\ s \in R$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

If $B^2 = QDQ^{-1}$,

$$Q = \begin{pmatrix} -11 & -5 & -1 \\ 5 & 7 & 0 \\ 6 & 5 & 1 \end{pmatrix}$$

and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix}^2$

$$= \begin{pmatrix} 1^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & (-4)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

20. $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$. If A^{-1} exists,

i) $A^{-1}(A\underset{\sim}{x}) = A^{-1}(\lambda \underset{\sim}{x})$

$$(A^{-1}A)\underset{\sim}{x} = \lambda (A^{-1}\underset{\sim}{x})$$

$$I\underset{\sim}{x} = \lambda A^{-1}\underset{\sim}{x}$$

$$\underset{\sim}{x} = \lambda A^{-1}\underset{\sim}{x}$$

If $\lambda = 0$, $\underset{\sim}{x} = \underset{\sim}{0}$

$\therefore \lambda \neq 0$

ii) $A^{-1}\underset{\sim}{x} = \frac{1}{\lambda}\underset{\sim}{x}$

$\therefore \underset{\sim}{x}$ is an eigenvector of A^{-1} with corresponding eigenvalue $\frac{1}{\lambda}$.

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 8 \\ 0 & 0 & -3 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 8 \\ 0 & 0 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 8 \\ 0 & 0 & -3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda)[(2-\lambda)(-3-\lambda) - 0]$$

$$-3 \cdot 0 + 4 \cdot 0$$

$$= (1-\lambda)(2-\lambda)(-3-\lambda) - 0 + 0$$

$$= -(\lambda - 1)(\lambda - 2)(\lambda + 3)$$

$$|A - \lambda I| = 0$$

$$-(\lambda - 1)(\lambda - 2)(\lambda + 3) = 0$$

$$\lambda = 1, 2, -3$$

$\lambda = 1$: $$\begin{pmatrix} 0 & 3 & 4 \\ 0 & 1 & 8 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 3 & 4 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$z = 0$$
$$y = 0$$

Let $x = s$, $s \in \mathbb{R}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 2: \quad \begin{pmatrix} -1 & 3 & 4 \\ 0 & 0 & 8 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 3 & 4 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right)$$

$z = 0$

Let $y = s, s \in R$

$x = 3s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3s \\ s \\ 0 \end{pmatrix}$$

$$= s \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = -3: \quad \begin{pmatrix} 4 & 3 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 4 & 3 & 4 & 0 \\ 0 & 5 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 5s, s \in R$

$y = -8s$

$4x - 24s + 20s = 0$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -8s \\ 5s \end{pmatrix}$$

$$= s \begin{pmatrix} 1 \\ -8 \\ 5 \end{pmatrix}$$

If $B = A + 5I$,

$$(A + 5I)\underset{\sim}{x} = A\underset{\sim}{x} + 5I\underset{\sim}{x}$$
$$= \lambda\underset{\sim}{x} + 5\underset{\sim}{x}$$
$$= (\lambda + 5)\underset{\sim}{x}$$

$$(A + 5I)^{-1}(A + 5I)\underset{\sim}{x} = (A + 5I)^{-1}(\lambda + 5)\underset{\sim}{x}$$
$$I\underset{\sim}{x} = (\lambda + 5)(A + 5I)^{-1}\underset{\sim}{x}$$
$$\underset{\sim}{x} = (\lambda + 5)(A + 5I)^{-1}\underset{\sim}{x}$$

$$\therefore (A + 5I)^{-1}\underset{\sim}{x} = \frac{1}{\lambda + 5}\underset{\sim}{x}$$

$\therefore$ The eigenvalues of B are $\frac{1}{6}, \frac{1}{7}, \frac{1}{2}$

with corresponding eigenvectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-8\\5\end{pmatrix}$.

21. $$M = \begin{pmatrix} -\frac{3}{2} & 4 & \frac{11}{3} \\ 1 & -\frac{1}{2} & -2 \\ -1 & 2 & \frac{7}{3} \end{pmatrix}$$

$$M - \lambda I = \begin{pmatrix} -\frac{3}{2} & 4 & \frac{11}{3} \\ 1 & -\frac{1}{2} & -2 \\ -1 & 2 & \frac{7}{3} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{2} - \lambda & 4 & \frac{11}{3} \\ 1 & -\frac{1}{2} - \lambda & -2 \\ -1 & 2 & \frac{7}{3} - \lambda \end{pmatrix}$$

$$|M - \lambda I| = \left(\frac{-3}{2} - \lambda\right)\left[\left(\frac{-1}{2} - \lambda\right)\left(\frac{7}{3} - \lambda\right) + 4\right]$$

$$-4\left(\frac{7}{3} - \lambda - 2\right) + \frac{11}{3}\left(2 - \frac{1}{2} - \lambda\right)$$

$$= \left(\frac{-3}{2} - \lambda\right)\left(\frac{-7}{6} + \frac{\lambda}{2} - \frac{7\lambda}{3} + \lambda^2 + 4\right)$$

$$-4\left(\frac{1}{3} - \lambda\right) + \frac{11}{3}\left(\frac{3}{2} - \lambda\right)$$

$$= \left(\frac{-3}{2} - \lambda\right)\left(\lambda^2 - \frac{11\lambda}{6} + \frac{17}{6}\right)$$

$$-\frac{4}{3} + 4\lambda + \frac{11}{2} - \frac{11\lambda}{3}$$

$$= \frac{-3\lambda^2}{2} + \frac{11\lambda}{4} - \frac{17}{4} - \lambda^3 + \frac{11\lambda^2}{6} - \frac{17\lambda}{6}$$

$$-\frac{4}{3} + 4\lambda + \frac{11}{2} - \frac{11\lambda}{3}$$

$$= -\lambda^3 + \frac{\lambda^2}{3} + \frac{\lambda}{4} - \frac{1}{12}$$

$|M - \lambda I| = 0$

$$-\lambda^3 + \frac{\lambda^2}{3} + \frac{\lambda}{4} - \frac{1}{12} = 0$$

$$-12\lambda^3 + 4\lambda^2 + 3\lambda - 1 = 0$$

$$12\lambda^3 - 4\lambda^2 - 3\lambda + 1 = 0$$

$\therefore$ The eigenvalues of M are the roots of the equation $12\lambda^3 - 4\lambda^2 - 3\lambda + 1 = 0$.

$$(2\lambda - 1)(6\lambda^2 + \lambda - 1) = 0$$

$$(2\lambda - 1)(2\lambda + 1)(3\lambda - 1) = 0$$

$$\lambda = -\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$$

$\lambda = -\frac{1}{2}$: $\begin{pmatrix} -1 & 4 & \frac{11}{3} \\ 1 & 0 & -2 \\ -1 & 2 & \frac{17}{6} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -1 & 4 & \frac{11}{3} & 0 \\ 1 & 0 & -2 & 0 \\ -1 & 2 & \frac{17}{6} & 0 \end{array}\right)$$

$$\xrightarrow[-r_1 + r_3]{r_1 + r_2} \left(\begin{array}{ccc|c} -1 & 4 & \frac{11}{6} & 0 \\ 0 & 4 & \frac{5}{3} & 0 \\ 0 & -2 & -\frac{5}{6} & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{2}} \left(\begin{array}{ccc|c} -1 & 4 & \frac{11}{6} & 0 \\ 0 & 2 & \frac{5}{6} & 0 \\ 0 & -2 & -\frac{5}{6} & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} -1 & 4 & \frac{11}{6} & 0 \\ 0 & 2 & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 12s, \ s \in R$

$y = -5s$

$-x - 20s + 22s = 0$

$x = 2s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ -5s \\ 12s \end{pmatrix}$$

$$= s \begin{pmatrix} 2 \\ -5 \\ 12 \end{pmatrix}$$

$\lambda = \frac{1}{2}$:

$$\begin{pmatrix} -2 & 4 & \frac{11}{3} \\ 1 & -1 & -2 \\ -1 & 2 & \frac{11}{6} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & 4 & \frac{11}{3} & 0 \\ 1 & -1 & -2 & 0 \\ -1 & 2 & \frac{11}{6} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & 2 & \frac{11}{6} & 0 \\ -2 & 4 & \frac{11}{3} & 0 \end{array}\right)$$

$$\xrightarrow[2r_1 + r_3]{r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & \frac{-1}{6} & 0 \\ 0 & 2 & \frac{-1}{3} & 0 \end{array}\right)$$

$$\xrightarrow{-2r_2 + r_3} \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & \frac{-1}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 6s, s \in R$

$y = s$

$x - s - 12s = 0$

$x = 13s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13s \\ s \\ 6s \end{pmatrix}$$

$$= s \begin{pmatrix} 13 \\ 1 \\ 6 \end{pmatrix}$$

$$\lambda = \frac{1}{3} \cdot \begin{pmatrix} -\frac{11}{6} & 4 & \frac{11}{3} \\ 1 & -\frac{5}{6} & -2 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -\frac{11}{6} & 4 & \frac{11}{3} & 0 \\ 1 & -\frac{5}{6} & -2 & 0 \\ -1 & 2 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 2 & 2 & 0 \\ 1 & -\frac{5}{6} & -2 & 0 \\ -\frac{11}{6} & 4 & \frac{11}{3} & 0 \end{array}\right)$$

$$\xrightarrow[-\frac{11}{6}r_1 + r_3]{r_1 + r_2} \left(\begin{array}{ccc|c} -1 & 2 & 2 & 0 \\ 0 & \frac{7}{6} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s, s \in R$

$x = 2s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ 0 \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

If $M = QDQ^{-1}$,

$$Q = \begin{pmatrix} 13 & 2 & 2 \\ 1 & -5 & 0 \\ 6 & 12 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$M^n = QD^nQ^{-1}$$

$$D^n = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}^n$$

$$= \begin{pmatrix} \left(-\frac{1}{2}\right)^n & 0 & 0 \\ 0 & \left(\frac{1}{2}\right)^n & 0 \\ 0 & 0 & \left(\frac{1}{3}\right)^n \end{pmatrix}$$

Since $D^n \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as $n \to \infty$,

$$\lim_{n\to\infty} M^n = \lim_{n\to\infty} QD^nQ^{-1}$$

$$= QOQ^{-1}$$

$$= 0.$$