OCTOBER / NOVEMBER 2008

1. $C: \; x = t^4 - 4\ln t, \quad y = 4t^2$

$$\frac{dx}{dt} = 4t^3 - \frac{4}{t} \qquad \frac{dy}{dt} = 8t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(4t^3 - \frac{4}{t}\right)^2 + (8t)^2$$

$$= 16t^6 - 32t^2 + \frac{16}{t^2} + 64t^2$$

$$= 16t^6 + 32t^2 + \frac{16}{t^2}$$

$$= \left(4t^3 + \frac{4}{t}\right)^2$$

The arc length of C from $t = 2$ to $t = 4$ is

$$\int_2^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_2^4 4t^3 + \frac{4}{t}\, dt$$

$$= \left[t^4 + 4\ln t\right]_2^4$$

$$= 256 + 4\ln 4 - (16 + 4\ln 2)$$

$$= 256 + 8\ln 2 - 16 - 4\ln 2$$

$$= 240 + 4\ln 2$$

2. $y = e^x$

The mean value of y over the interval $0 \le x \le 2$ is $\dfrac{1}{2-0}\displaystyle\int_0^2 y\, dx$

$$= \frac{1}{2}\int_0^2 e^x\, dx$$

$$= \left[\frac{e^x}{2}\right]_0^2$$

$$= \frac{e^2 - 1}{2}$$

$x = \ln y$

The mean value of x over the interval $1 \le y \le e^2$ is $\dfrac{1}{e^2-1}\displaystyle\int_1^{e^2} x\, dy$

$$= \frac{1}{e^2-1}\int_1^{e^2} \ln y\, dy$$

$$u = \ln y \qquad dv = dy$$

$$du = \frac{dy}{y} \qquad v = y$$

$$= \frac{1}{e^2-1}\left(\left[y\ln y\right]_1^{e^2} - \int_1^{e^2} \frac{y}{y}\, dy\right)$$

$$= \frac{1}{e^2-1}\left(e^2\ln e^2 - 1\ln 1 - \int_1^{e^2} 1\, dy\right)$$

$$= \frac{1}{e^2-1}\left(2e^2 - 0 - [y]_1^{e^2}\right)$$

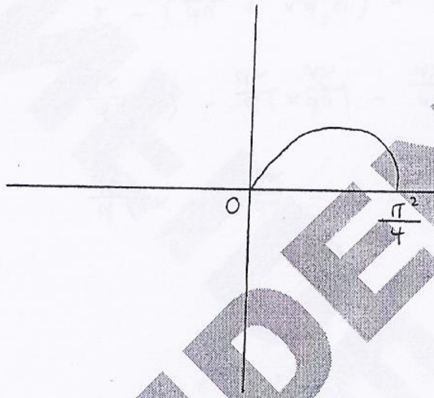
$$= \frac{1}{e^2-1}\left(2e^2 - (e^2-1)\right)$$

$$= \frac{1}{e^2-1}\left(2e^2 - e^2 + 1\right)$$

$$= \frac{e^2+1}{e^2-1}.$$

3. C : $r = \left(\frac{\pi}{2} - \theta\right)^2, \quad 0 \le \theta \le \frac{\pi}{2}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	$\frac{\pi^2}{4}$	$\frac{\pi^2}{9}$	$\frac{\pi^2}{16}$	$\frac{\pi^2}{36}$	0



The area bounded by C and the initial line is $\int_0^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)^4 d\theta$$

$$= \frac{1}{2}\left[\frac{-1}{5}\left(\frac{\pi}{2} - \theta\right)^5\right]_0^{\frac{\pi}{2}}$$

$$= \frac{-1}{10}\left(0 - \frac{\pi^5}{32}\right)$$

$$= \frac{\pi^5}{320}.$$

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4. $A\underset{\sim}{e} = \lambda \underset{\sim}{e}$

$$A^2\underset{\sim}{e} = A(A\underset{\sim}{e})$$
$$= A(\lambda\underset{\sim}{e})$$
$$= \lambda(A\underset{\sim}{e})$$
$$= \lambda(\lambda\underset{\sim}{e})$$
$$= \lambda^2\underset{\sim}{e}.$$

$\therefore$ The matrix A^2 has an eigenvalue λ^2 with corresponding eigenvector $\underset{\sim}{e}$.

Since A^2 has an eigenvalue λ^2, $A^4 = (A^2)^2$ has an eigenvalue λ^4 with corresponding eigenvector $\underset{\sim}{e}$.

$A\underset{\sim}{e} = 3\underset{\sim}{e}$

$$(A^4 + 3A^2 + 2I)\underset{\sim}{e} = A^4\underset{\sim}{e} + 3A^2\underset{\sim}{e} + 2I\underset{\sim}{e}$$
$$= A^4\underset{\sim}{e} + 3A^2\underset{\sim}{e} + 2\underset{\sim}{e}$$
$$= 3^4\underset{\sim}{e} + 3(3^2)\underset{\sim}{e} + 2\underset{\sim}{e}$$
$$= 81\underset{\sim}{e} + 27\underset{\sim}{e} + 2\underset{\sim}{e}$$
$$= 110\underset{\sim}{e}$$

$\therefore$ If one eigenvalue of A is 3, an eigenvalue of $A^4 + 3A^2 + 2I$ is 110.

5. $C: x^2 - xy - 2y^2 = 4$

$$\frac{d}{dx}(x^2 - xy - 2y^2) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) - \frac{d}{dx}(2y^2) = 0$$

$$2x - \left(x\frac{dy}{dx} + y\right) - 4y\frac{dy}{dx} = 0$$

$$2x - x\frac{dy}{dx} - y - 4y\frac{dy}{dx} = 0$$

At $A(2,0)$:

$$4 - 2\frac{dy}{dx} - 0 = 0$$

$$\frac{dy}{dx} = 2$$

$$\frac{d}{dx}\left(2x - x\frac{dy}{dx} - y - 4y\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}(2x) - \frac{d}{dx}\left(x\frac{dy}{dx}\right) - \frac{dy}{dx} - \frac{d}{dx}\left(4y\frac{dy}{dx}\right) = 0$$

$$2 - \left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right) - \frac{dy}{dx} - \left(4\left(\frac{dy}{dx}\right)^2 + 4y\frac{d^2y}{dx^2}\right) = 0$$

$$2 - \frac{dy}{dx} - x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4\left(\frac{dy}{dx}\right)^2 - 4y^2\frac{d^2y}{dx^2} = 0$$

At $A(2,0)$, $\frac{dy}{dx} = 2$:

$$2 - 2 - \frac{2d^2y}{dx^2} - 2 - 4(2^2) - 0 = 0$$

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$$2\frac{d^2y}{dx^2} = -18$$

$$\frac{d^2y}{dx^2} = -9$$

6. $A = \begin{pmatrix} 1 & -1 & -2 & -3 \\ -2 & 1 & 7 & 2 \\ -3 & 3 & 6 & \alpha \\ 7 & -6 & -17 & -17 \end{pmatrix}$

i) $\alpha = 9$:

$$A = \begin{pmatrix} 1 & -1 & -2 & -3 \\ -2 & 1 & 7 & 2 \\ -3 & 3 & 6 & 9 \\ 7 & -6 & -17 & -17 \end{pmatrix}$$

$$\xrightarrow[-7r_1 + r_4]{2r_1 + r_2,\ 3r_1 + r_3} \begin{pmatrix} 1 & -1 & -2 & -3 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 4 \end{pmatrix}$$

$$\xrightarrow{r_3 \Longleftrightarrow r_4} \begin{pmatrix} 1 & -1 & -2 & -3 \\ 0 & -1 & 3 & -4 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & -1 & -2 & -3 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-1 \times r_2} \begin{pmatrix} 1 & -1 & -2 & -3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank $(A) = 2$

If $\begin{pmatrix} 1 & -1 & -2 & -3 \\ -2 & 1 & 7 & 2 \\ -3 & 3 & 6 & 9 \\ 7 & -6 & -17 & -17 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ -2 & 1 & 7 & 2 & 0 \\ -3 & 3 & 6 & 9 & 0 \\ 7 & -6 & -17 & -17 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = s$ and $z = t$, $s, t \in R$

$y = 3t - 4s$

$x - (3t - 4s) - 2t - 3s = 0$

$x - 3t + 4s - 2t - 3s = 0$

$x = 5t - s$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5t - s \\ 3t - 4s \\ t \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ -4 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ A basis for the null space of A is

$$\left\{ \begin{pmatrix} -1 \\ -4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

ii) If $\alpha \neq 9$

$$A = \begin{pmatrix} 1 & -1 & -2 & -3 \\ -2 & 1 & 7 & 2 \\ -3 & 3 & 6 & \alpha \\ 7 & -6 & -17 & -17 \end{pmatrix}$$

$$\xrightarrow[-7r_1 + r_4]{2r_1 + r_2,\ 3r_1 + r_3} \begin{pmatrix} 1 & -1 & -2 & -3 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & \alpha - 9 \\ 0 & 1 & -3 & 4 \end{pmatrix}$$

$$\xrightarrow{r_2 + r_4,\ \frac{r_3}{\alpha - 9}} \begin{pmatrix} 1 & -1 & -2 & -3 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-1 \times r_2} \begin{pmatrix} 1 & -1 & -2 & -3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank $(A) = 3$

7. $I_n = \int_0^1 \frac{1}{(1+x^4)^n}\,dx$

$$\frac{d}{dx}\left(\frac{x}{(1+x^4)^n}\right) = \frac{1}{(1+x^4)^n} - \frac{nx(4x^3)}{(1+x^4)^{n+1}}$$

$$= \frac{1}{(1+x^4)^n} - \frac{4nx^4}{(1+x^4)^{n+1}}$$

$$= \frac{1}{(1+x^4)^n} - \frac{4n(1+x^4-1)}{(1+x^4)^{n+1}}$$

$$= \frac{1}{(1+x^4)^n} - \frac{4n(1+x^4)}{(1+x^4)^{n+1}} + \frac{4n}{(1+x^4)^{n+1}}$$

$$= \frac{1}{(1+x^4)^n} - \frac{4n}{(1+x^4)^n} + \frac{4n}{(1+x^4)^{n+1}}$$

$$= \frac{1-4n}{(1+x^4)^n} + \frac{4n}{(1+x^4)^{n+1}}$$

$$\frac{x}{(1+x^4)^n} = \int \frac{1-4n}{(1+x^4)^n} + \frac{4n}{(1+x^4)^{n+1}}\,dx$$

$$= \int \frac{1-4n}{(1+x^4)^n}\,dx + \int \frac{4n}{(1+x^4)^{n+1}}\,dx$$

$$\left[\frac{x}{(1+x^4)^n}\right]_0^1 = \int_0^1 \frac{1-4n}{(1+x^4)^n}\,dx + \int_0^1 \frac{4n}{(1+x^4)^{n+1}}\,dx$$

$$\frac{1}{2^n} - 0 = (1-4n)\int_0^1 \frac{1}{(1+x^4)^n}\,dx + 4n\int_0^1 \frac{1}{(1+x^4)^{n+1}}\,dx$$

$$\frac{1}{2^n} = (1-4n)I_n + 4nI_{n+1}$$

$$4nI_{n+1} = \frac{1}{2^n} + (4n-1)I_n$$

$I_1 = 0.86697$

$n=2$: $8I_3 = \frac{1}{2^2} + 7I_2$

$4I_2 = \frac{1}{2} + 3I_1$

since $I_1 = 0.86697$,

$$4I_2 = \frac{1}{2} + 3(0.86697)$$

$$= 3.10091$$

$$I_2 = 0.7752275$$

$$8I_3 = \frac{1}{4} + 7I_2$$

$$= \frac{1}{4} + 5.4265925$$

$$= 5.6765925$$

$$I_3 = 0.7095740625$$

$$\approx 0.71$$

8. $$\frac{5d^2y}{dt^2} + \frac{6dy}{dt} + 5y = 15 + 12t + 5t^2,$$

$$y = \frac{dy}{dt} = 0 \quad \text{when } t = 0.$$

$$\frac{5d^2y}{dt^2} + \frac{6dy}{dt} + 5y = 0$$

$$5m^2 + 6m + 5 = 0$$

$$m^2 + \frac{6m}{5} + 1 = 0$$

$$m^2 + \frac{6m}{5} + \frac{9}{25} = \frac{9}{25} - 1$$

$$\left(m + \frac{3}{5}\right)^2 = \frac{-16}{25}$$

$$m + \frac{3}{5} = \frac{\pm 4i}{5}$$

$$m = -\frac{3}{5} \pm \frac{4i}{5}$$

$\therefore$ The complementary function, y_c, is

$$y_c = e^{\frac{-3t}{5}}\left(A\cos\frac{4t}{5} + B\sin\frac{4t}{5}\right)$$

The particular integral, y_p, is given by

$$y_p = Ct^2 + Dt + E$$

$$\frac{dy_p}{dt} = 2Ct + D$$

$$\frac{d^2y_p}{dt^2} = 2C$$

$$\frac{5d^2y_p}{dt^2} + \frac{6dy_p}{dt} + 5y_p = 5(2C) + 6(2Ct + D) + 5(Ct^2 + Dt + E)$$

$$= 5Ct^2 + (12C + 5D)t + 10C + 6D + 5E$$

$$= 5t^2 + 12t + 15$$

$5C = 5 \qquad 12C + 5D = 12 \qquad 10C + 6D + 5E = 15$

$C = 1$

$D = 0$

$E = 1$

$$\therefore \quad y_p = t^2 + 1$$

$$y = y_c + y_p$$

$$= e^{\frac{-3t}{5}}\left(A\cos\frac{4t}{5} + B\sin\frac{4t}{5}\right) + t^2 + 1$$

$$\frac{dy}{dt} = \frac{-3e^{\frac{-3t}{5}}}{5}\left(A\cos\frac{4t}{5} + B\sin\frac{4t}{5}\right) + e^{\frac{-3t}{5}}\left(-\frac{4A}{5}\sin\frac{4t}{5} + \frac{4B}{5}\cos\frac{4t}{5}\right) + 2t$$

$t = 0 \quad y = 0 : \quad 0 = A + 1$

$t = 0 \quad \frac{dy}{dt} = 0 : \quad 0 = -\frac{3A}{5} + \frac{4B}{5}$

$$A = -1$$

$$\frac{3}{5} + \frac{4B}{5} = 0$$

$$B = \frac{-3}{4}$$

$$y = e^{\frac{-3t}{5}}\left(-\cos\frac{4t}{5} - \frac{3}{4}\sin\frac{4t}{5}\right) + t^2 + 1$$

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9. $$\sum_{n=1}^{N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(N+1)(2N+1)}$$

when $N = 1$:

$$\sum_{n=1}^{1} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = \frac{4(1)+1}{1(1+1)(2(1)-1)(2(1)+1)}$$

$$= \frac{4+1}{1(2)(2-1)(2+1)}$$

$$= \frac{5}{1\cdot 2\cdot 1\cdot 3}$$

$$= \frac{5}{6}$$

$$= 1 - \frac{1}{6}$$

$$= 1 - \frac{1}{2\cdot 3}$$

$$= 1 - \frac{1}{(1+1)(2+1)}$$

$$= 1 - \frac{1}{(1+1)(2(1)+1)}$$

Assume the statement is true, when $N = K$.

$N = K$: $$\sum_{n=1}^{K} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(K+1)(2K+1)}$$

$N = K+1$: $$\sum_{n=1}^{K+1} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(K+2)(2K+3)}$$

(what needs to be proved).

$$\sum_{n=1}^{K+1} \frac{4n+1}{n(n+1)(2n-1)(2n+1)}$$

$$= \frac{4(K+1)+1}{(K+1)(K+1+1)(2(K+1)-1)(2(K+1)+1)} + \sum_{n=1}^{K} \frac{4n+1}{n(n+1)(2n-1)(2n+1)}$$

$$= \frac{4K+5}{(K+1)(K+2)(2K+1)(2K+3)} + 1 - \frac{1}{(K+1)(2K+1)}$$

$$= 1 + \frac{4K+5}{(K+1)(K+2)(2K+1)(2K+3)} - \frac{1}{(K+1)(2K+1)}$$

$$= 1 + \frac{1}{(K+1)(2K+1)}\left(\frac{4K+5}{(K+2)(2K+3)} - 1\right)$$

$$= 1 + \frac{1}{(K+1)(2K+1)}\left(\frac{4K+5-(K+2)(2K+3)}{(K+2)(2K+3)}\right)$$

$$= 1 + \frac{1}{(K+1)(2K+1)} \frac{(4K+5-(2K^2+7K+6))}{(K+2)(2K+3)}$$

$$= 1 + \frac{-2K^2-3K-1}{(K+1)(K+2)(2K+1)(2K+3)}$$

$$= 1 - \frac{(2K+1)(K+1)}{(K+1)(K+2)(2K+1)(2K+3)}$$

$$= 1 - \frac{1}{(k+2)(2k+3)}$$

$$\sum_{n=1}^{N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(N+1)(2N+1)}$$

for every positive integer N.

$$\sum_{n=N+1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)}$$

$$= \sum_{n=1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} - \sum_{n=1}^{N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)}$$

$$= 1 - \frac{1}{(2N+1)(2(2N)+1)} - \left(1 - \frac{1}{(N+1)(2N+1)}\right)$$

$$= 1 - \frac{1}{(2N+1)(4N+1)} - 1 + \frac{1}{(N+1)(2N+1)}$$

$$= \frac{1}{(N+1)(2N+1)} - \frac{1}{(2N+1)(4N+1)}$$

$$= \frac{1}{2N+1}\left(\frac{1}{N+1} - \frac{1}{4N+1}\right)$$

$$= \frac{1}{2N+1}\left(\frac{4N+1-N-1}{(N+1)(4N+1)}\right)$$

$$= \frac{3N}{(N+1)(2N+1)(4N+1)}$$

$$= \frac{3N}{(2N^2+3N+1)(4N+1)}$$

$$= \frac{3N}{8N^3+14N^2+7N+1}$$

$$= \frac{3}{\frac{8N^3+14N^2+7N+1}{N}}$$

$$= \frac{3}{8N^2+14N+7+\frac{1}{N}}$$

$$< \frac{3}{8N^2}, \quad \text{since } 14N+7+\frac{1}{N} > 0$$

10. $(\cos\theta + i\sin\theta)^8$

$$\begin{aligned}
&= \cos^8\theta + 8i\cos^7\theta\sin\theta - 28\cos^6\theta\sin^2\theta \\
&\quad - 56i\cos^5\theta\sin^3\theta + 70\cos^4\theta\sin^4\theta + 56i\cos^3\theta\sin^5\theta \\
&\quad - 28\cos^2\theta\sin^6\theta - 8i\cos\theta\sin^7\theta + \sin^8\theta
\end{aligned}$$

$\cos 8\theta + i\sin 8\theta$

$$\begin{aligned}
&= \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta \\
&\quad + i(8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta + 56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta)
\end{aligned}$$

Equating real parts,

$$\begin{aligned}
\cos 8\theta &= \cos^8\theta - 28\cos^6\theta\sin^2\theta \\
&\quad + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta \\
&= \cos^8\theta - 28\cos^6\theta(1-\cos^2\theta) \\
&\quad + 70\cos^4\theta(1-\cos^2\theta)^2 \\
&\quad - 28\cos^2\theta\cdot(1-\cos^2\theta)^3 + (1-\cos^2\theta)^4 \\
&= \cos^8\theta - 28\cos^6\theta + 28\cos^8\theta \\
&\quad + 70\cos^4\theta(1 - 2\cos^2\theta + \cos^4\theta) \\
&\quad - 28\cos^2\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta) \\
&\quad + 1 - 4\cos^2\theta + 6\cos^4\theta - 4\cos^6\theta + \cos^8\theta \\
&= \cos^8\theta - 28\cos^6\theta + 28\cos^8\theta \\
&\quad + 70\cos^4\theta - 140\cos^6\theta + 70\cos^8\theta \\
&\quad - 28\cos^2\theta + 84\cos^4\theta - 84\cos^6\theta + 28\cos^8\theta \\
&\quad + 1 - 4\cos^2\theta + 6\cos^4\theta - 4\cos^6\theta + \cos^8\theta \\
&= 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1
\end{aligned}$$

i)

$$\begin{aligned}
\cos 8\theta &= \cos^8\theta - 28\cos^6\theta\sin^2\theta \\
&\quad + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta \\
&= (1-\sin^2\theta)^4 - 28(1-\sin^2\theta)^3\sin^2\theta \\
&\quad + 70(1-\sin^2\theta)^2\sin^4\theta - 28(1-\sin^2\theta)\sin^6\theta \\
&\quad + \sin^8\theta \\
&= 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta \\
&\quad - 28(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)\sin^2\theta \\
&\quad + 70(1 - 2\sin^2\theta + \sin^4\theta)\sin^4\theta \\
&\quad - 28\sin^6\theta + 28\sin^8\theta + \sin^8\theta \\
&= 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta \\
&\quad - 28\sin^2\theta + 84\sin^4\theta - 84\sin^6\theta + 28\sin^8\theta \\
&\quad + 70\sin^4\theta - 140\sin^6\theta + 70\sin^8\theta \\
&\quad - 28\sin^6\theta + 28\sin^8\theta + \sin^8\theta \\
&= 128\sin^8\theta - 256\sin^6\theta + 160\sin^4\theta - 32\sin^2\theta + 1
\end{aligned}$$

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ii) $4x^4 - 8x^3 + 5x^2 - x, \quad x = \cos^2\frac{\pi}{8}$

$$= 4x^4 - 8x^3 + 5x^2 - x + \frac{1}{32} - \frac{1}{32}$$

$$= \frac{1}{32}\cdot 32\left(4x^4 - 8x^3 + 5x^2 - x + \frac{1}{32} - \frac{1}{32}\right)$$

$$= \frac{1}{32}\left(128x^4 - 256x^3 + 160x^2 - 32x + 1 - 1\right)$$

$$= \frac{1}{32}\left(128\cos^8\frac{\pi}{8} - 256\cos^6\frac{\pi}{8} + 160\cos^4\frac{\pi}{8} - 32\cos^2\frac{\pi}{8} + 1 - 1\right)$$

$$= \frac{1}{32}\left(\cos 8\left(\frac{\pi}{8}\right) - 1\right)$$

$$= \frac{1}{32}\left(\cos\pi - 1\right)$$

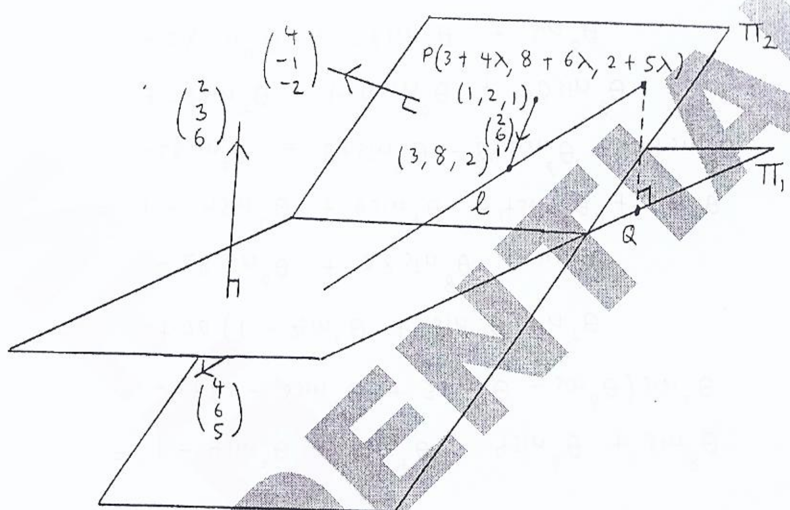
$$= \frac{1}{32}(-1 - 1)$$

$$= \frac{1}{32}(-2)$$

$$= \frac{-1}{16}$$

$\therefore$ The exact value of $4x^4 - 8x^3 + 5x^2 - x$, $x = \cos^2\frac{\pi}{8}$ is $\frac{-1}{16}$.

11 $\Pi_1 : \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \theta(2\mathbf{j} - \mathbf{k}) + \phi(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$



Expressing Π_1 in Cartesian form,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \theta\begin{pmatrix}0\\2\\-1\end{pmatrix} + \phi\begin{pmatrix}3\\2\\-2\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1 + 3\phi\\ 2 + 2\theta + 2\phi\\ 1 - \theta - 2\phi\end{pmatrix}$$

$$\left.\begin{aligned} x &= 1 + 3\phi \\ y &= 2 + 2\theta + 2\phi \\ z &= 1 - \theta - 2\phi \end{aligned}\right\}$$

$$\phi = \frac{x - 1}{3}$$

$y = 2 + 2\theta + 2\left(\frac{x-1}{3}\right)$

$z = 1 - \theta - 2\left(\frac{x-1}{3}\right)$

$\theta = \frac{y-2}{2} - \frac{(x-1)}{3}$

$= 1 - 2 - \frac{2(x-1)}{3}$

$\frac{y-2}{2} - \frac{(x-1)}{3} = 1 - 2 - \frac{2(x-1)}{3}$

$\frac{y-2}{2} + \frac{x-1}{3} = 1 - z$

$3y - 6 + 2x - 2 = 6 - 6z$

$2x + 3y + 6z = 14.$

The line through P perpendicular to Π_1 has equation $\underset{\sim}{r} = \overrightarrow{OP} + s\begin{pmatrix}2\\3\\6\end{pmatrix}$ since $\begin{pmatrix}2\\3\\6\end{pmatrix}$ is a normal to Π_1. Since P lies on ℓ,

$\overrightarrow{OP} = \begin{pmatrix}3+4\lambda\\8+6\lambda\\2+5\lambda\end{pmatrix}.$

$\underset{\sim}{r} = \begin{pmatrix}3+4\lambda\\8+6\lambda\\2+5\lambda\end{pmatrix} + s\begin{pmatrix}2\\3\\6\end{pmatrix}$

If the line through P perpendicular to Π_1 meets Π_1 at Q,

$2(3+4\lambda+2s) + 3(8+6\lambda+3s)$
$+ 6(2+5\lambda+6s) = 14$

$6 + 8\lambda + 4s + 24 + 18\lambda + 9s + 12 + 30\lambda + 36s = 14$

$56\lambda + 49s = -28$

$49s = -56\lambda - 28$

$s = \frac{-8\lambda - 4}{7}$

$\therefore Q\left(\frac{12\lambda+13}{7}, \frac{18\lambda+44}{7}, \frac{-13\lambda-10}{7}\right)$

$PQ = \sqrt{\left(\frac{12\lambda+13}{7} - 3 - 4\lambda\right)^2 + \left(\frac{18\lambda+44}{7} - 8 - 6\lambda\right)^2 + \left(\frac{-13\lambda-10}{7} - 2 - 5\lambda\right)^2}$

$= \sqrt{\left(\frac{-16\lambda-8}{7}\right)^2 + \left(\frac{-24\lambda-12}{7}\right)^2 + \left(\frac{-48\lambda-24}{7}\right)^2}$

$= \sqrt{\frac{64}{49}(2\lambda+1)^2 + \frac{144}{49}(2\lambda+1)^2 + \frac{576}{49}(2\lambda+1)^2}$

$= \sqrt{\frac{784}{49}(2\lambda+1)^2}$

$= \sqrt{16(2\lambda+1)^2}$

$$= 4|2\lambda + 1|$$

If the perpendicular distance of P from Π_1 is not greater than 4,

$$PQ \leq 4$$
$$4|2\lambda + 1| \leq 4$$
$$|2\lambda + 1| \leq 1$$
$$-1 \leq 2\lambda + 1 \leq 1$$
$$-2 \leq 2\lambda \leq 0$$
$$-1 \leq \lambda \leq 0$$

$\therefore$ If the perpendicular distance of P from Π_1 is not greater than 4, $-1 \leq \lambda \leq 0$.

Since Π_2 contains ℓ and the point $(1,2,1)$, $\begin{pmatrix}3\\8\\2\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}2\\6\\1\end{pmatrix}$ and $\begin{pmatrix}4\\6\\5\end{pmatrix}$ are both perpendicular to the normal of Π_2.

$\therefore \begin{pmatrix}2\\6\\1\end{pmatrix} \times \begin{pmatrix}4\\6\\5\end{pmatrix}$ is parallel to the normal of Π_2.

$$\begin{pmatrix}2\\6\\1\end{pmatrix} \times \begin{pmatrix}4\\6\\5\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 6 & 1 \\ 4 & 6 & 5 \end{vmatrix} = 24\underset{\sim}{i} - 6\underset{\sim}{j} - 12\underset{\sim}{k}$$
$$= 6(4\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k})$$

Since $\begin{pmatrix}2\\3\\6\end{pmatrix}$ is normal to Π_1 and $\begin{pmatrix}4\\-1\\-2\end{pmatrix}$ is normal to Π_2, if θ is the angle between $\begin{pmatrix}2\\3\\6\end{pmatrix}$ and $\begin{pmatrix}4\\-1\\-2\end{pmatrix}$,

$$\begin{pmatrix}2\\3\\6\end{pmatrix} \cdot \begin{pmatrix}4\\-1\\-2\end{pmatrix} = \left|\begin{pmatrix}2\\3\\6\end{pmatrix}\right| \left|\begin{pmatrix}4\\-1\\-2\end{pmatrix}\right| \cos\theta$$
$$8 - 3 - 12 = 7\sqrt{21}\cos\theta$$
$$7\sqrt{21}\cos\theta = -7$$
$$\cos\theta = \frac{-1}{\sqrt{21}}$$
$$\theta \approx 102.6$$

$\therefore$ The angle between Π_1 and Π_2 is $180^\circ - 102.6^\circ = 77.4^\circ$.

12. EITHER

$$C: \quad y = \frac{(x-2)(x-a)}{(x-1)(x-3)}, \quad a \neq 1, 2, 3.$$

i)

$$\begin{array}{r} 1 \\ x^2 - 4x + 3 \overline{\smash{\big)}\, x^2 - (a+2)x + 2a} \\ x^2 - 4x + 3 \\ \hline (2-a)x + 2a - 3 \end{array}$$

$$y = 1 + \frac{(2-a)x + 2a - 3}{(x-1)(x-3)}$$

$$\frac{(2-a)x + 2a - 3}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

$$= \frac{A(x-3) + B(x-1)}{(x-1)(x-3)}$$

$$(2-a)x + 2a - 3 = A(x-3) + B(x-1)$$

$$= (A+B)x - 3A - B$$

$$A + B = 2 - a$$

$$-3A - B = 2a - 3$$

$$-2A = a - 1$$

$$A = \frac{1-a}{2}$$

$$B = \frac{3-a}{2}$$

$$\therefore y = 1 + \frac{1-a}{2(x-1)} + \frac{3-a}{2(x-3)}$$

As $x \to 1$, $y \to \pm\infty$

As $x \to 3$, $y \to \pm\infty$

As $x \to \pm\infty$, $y \to 1$

$\therefore$ The asymptotes of C are $y = 1$, $x = 1$ and $x = 3$.

ii) When C meets the line $y = 1$,

$$\frac{(x-2)(x-a)}{(x-1)(x-3)} = 1$$

$$x^2 - (a+2)x + 2a = x^2 - 4x + 3$$

$$(a-2)x = 2a - 3$$

$$x = \frac{2a-3}{a-2}.$$

iii)

$$\frac{dy}{dx} = \frac{-(1-a)}{2(x-1)^2} - \frac{(3-a)}{2(x-3)^2}$$

$$= \frac{a-1}{2(x-1)^2} + \frac{a-3}{2(x-3)^2}$$

When $\frac{dy}{dx} = 0$:

$$\frac{a-1}{2(x-1)^2} + \frac{a-3}{2(x-3)^2} = 0$$

$$\frac{a-1}{2(x-1)^2} = \frac{-(a-3)}{2(x-3)^2}$$

$$(a-1)(x-3)^2 = -(a-3)(x-1)^2$$

$$(a-1)(x^2-6x+9) = -(a-3)(x^2-2x+1)$$

$$(a-1)x^2 - 6(a-1)x + 9(a-1) = -(a-3)x^2 + 2(a-3)x - (a-3)$$

$$(a-1+a-3)x^2 + (-6a+6-2a+6)x + 9a-9+a-3 = 0$$

$$(2a-4)x^2 + (-8a+12)x + 10a-12 = 0$$

$$(a-2)x^2 + (6-4a)x + 5a-6 = 0$$

$\therefore$ The x-coordinates of any stationary points on C satisfy $(a-2)x^2 + (6-4a)x + 5a - 6 = 0$.

Let $A = a-2$, $B = 6-4a$, $C = 5a-6$

If C has stationary points,

$$B^2 - 4AC \geq 0$$

$$(6-4a)^2 - 4(a-2)(5a-6) \geq 0$$

$$36 - 48a + 16a^2 - 4(5a^2 - 16a + 12) \geq 0$$

$$36 - 48a + 16a^2 - 20a^2 + 64a - 48 \geq 0$$

$$-4a^2 + 16a - 12 \geq 0$$

$$-a^2 + 4a - 3 \geq 0$$

$$a^2 - 4a + 3 \leq 0$$

$$(a-1)(a-3) \leq 0$$

$a - 1 \leq 0$, and $a - 3 \geq 0$

or

$a - 1 \geq 0$ and $a - 3 \leq 0$

$\therefore 1 \leq a \leq 3$

iv) when $x = 0$: $y = \frac{2a}{3}$

when $y = 0$: $\frac{(x-2)(x-a)}{(x-1)(x-3)} = 0$

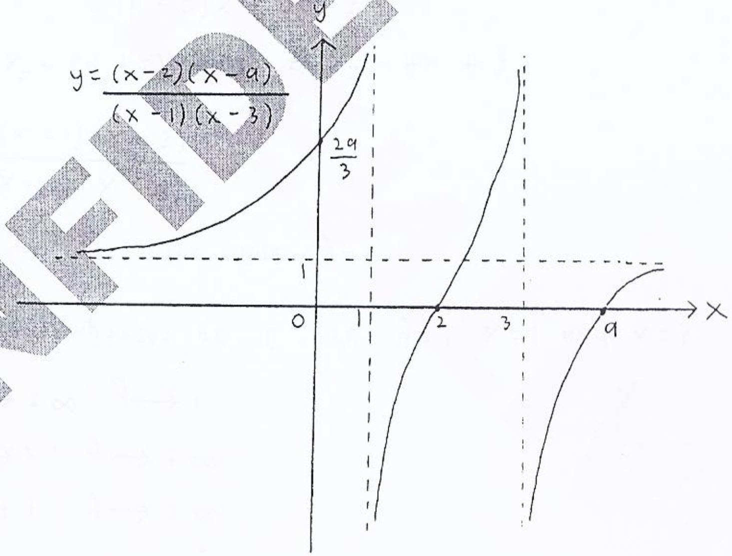
$$(x-2)(x-a) = 0$$

$$x = a, 2$$

a) If $a > 3$,

$B^2 - 4AC < 0$, $\frac{dy}{dx} > 0$

$\therefore$ no stationary points



b) If $2 < a < 3$,

$B^2 - 4AC \geq 0$

when $\frac{dy}{dx} = 0$:

$(a-2)x^2 + (6-4a)x + 5a - 6 = 0$

$$x = \frac{4a - 6 \pm \sqrt{(1-a)(a-3)}}{2(a-2)}$$

$$y = \frac{(x-2)(x-a)}{(x-1)(x-3)}$$

$(x-2)(x-a) = y(x-1)(x-3)$

$x^2 - (a+2)x + 2a = x^2y - 4xy + 3y$

$(y-1)x^2 + (a+2-4y)x + 3y - 2a = 0$

let $A = y-1$, $B = a+2-4y$, $C = 3y - 2a$

If $B^2 - 4AC \geq 0$

$(a+2-4y)^2 - 4(y-1)(3y-2a) \geq 0$

$(a+2)^2 - 8y(a+2) + 16y^2 - 4(3y^2 - (2a+3)y + 2a) \geq 0$

$(a+2)^2 - 8ay - 16y + 16y^2 - 12y^2 + 4(2a+3)y - 8a \geq 0$

$4y^2 - 4y + (a+2)^2 - 8a \geq 0$

$4y^2 - 4y + a^2 + 4a + 4 - 8a \geq 0$

$4y^2 - 4y + a^2 - 4a + 4 \geq 0$

$4y^2 - 4y + (a-2)^2 \geq 0$

$4y^2 - 4y + 1 \geq 1 - (a-2)^2$

$(2y-1)^2 \geq 1 - (a-2)^2$

$(2y-1)^2 \geq 1 - (a^2 - 4a + 4)$

$(2y-1)^2 \geq -a^2 + 4a - 3$

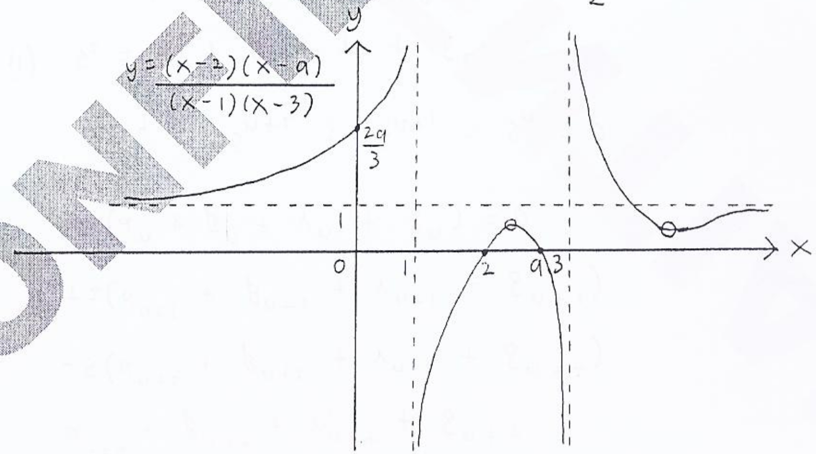
$(2y-1)^2 \geq (1-a)(a-3)$

$2y - 1 \geq \sqrt{(1-a)(a-3)}$ or $2y - 1 \leq -\sqrt{(1-a)(a-3)}$

$y \geq \frac{1 + \sqrt{(1-a)(a-3)}}{2}$ or $y \leq \frac{1 - \sqrt{(1-a)(a-3)}}{2}$

$\therefore$ C has a minimum at $y = \frac{1 + \sqrt{(1-a)(a-3)}}{2}$

and a maximum at $y = \frac{1 - \sqrt{(1-a)(a-3)}}{2}$



o: Critical point

•: Intersection point.

CONFIDENTIAL

OR

$$x^4 - 5x^2 + 2x - 1 = 0$$

$\alpha, \beta, \gamma, \delta$ are the roots

$$S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

$$\alpha + \beta + \gamma + \delta = 0$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -5$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -2$$

$$\alpha\beta\gamma\delta = -1$$

i) Since $\alpha, \beta, \gamma, \delta$ are the roots,

$$\alpha^4 - 5\alpha^2 + 2\alpha - 1 = 0 \text{ — } ①$$

$$\beta^4 - 5\beta^2 + 2\beta - 1 = 0 \text{ — } ②$$

$$\gamma^4 - 5\gamma^2 + 2\gamma - 1 = 0 \text{ — } ③$$

$$\delta^4 - 5\delta^2 + 2\delta - 1 = 0 \text{ — } ④$$

$$\alpha^n \times ① : \alpha^{n+4} - 5\alpha^{n+2} + 2\alpha^{n+1} - \alpha^n = 0 \text{ — } ⑤$$

$$\beta^n \times ② : \beta^{n+4} - 5\beta^{n+2} + 2\beta^{n+1} - \beta^n = 0 \text{ — } ⑥$$

$$\gamma^n \times ③ : \gamma^{n+4} - 5\gamma^{n+2} + 2\gamma^{n+1} - \gamma^n = 0 \text{ — } ⑦$$

$$\delta^n \times ④ : \delta^{n+4} - 5\delta^{n+2} + 2\delta^{n+1} - \delta^n = 0 \text{ — } ⑧$$

⑤ + ⑥ + ⑦ + ⑧ :

$$\alpha^{n+4} + \beta^{n+4} + \gamma^{n+4} + \delta^{n+4}$$
$$-5(\alpha^{n+2} + \beta^{n+2} + \gamma^{n+2} + \delta^{n+2})$$
$$+2(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1} + \delta^{n+1})$$
$$-(\alpha^n + \beta^n + \gamma^n + \delta^n) = 0$$

$$S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0$$

ii) $S_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$= 0^2 - 2(-5)$$

$$= 10$$

$S_0 = \alpha^0 + \beta^0 + \gamma^0 + \delta^0$

$$= 1 + 1 + 1 + 1$$

$$= 4$$

$S_1 = \alpha^1 + \beta^1 + \gamma^1 + \delta^1$

$$= \alpha + \beta + \gamma + \delta$$

$$= 0$$

CONFIDENTIAL

$n=0: \; S_4 - 5S_2 + 2S_1 - S_0 = 0$

$S_4 - 5(10) + 2(0) - 4 = 0$

$S_4 - 50 + 0 - 4 = 0$

$S_4 = 54$

iii) $S_{-1} = \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}$

$= \dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$

$= \dfrac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha\beta\gamma\delta}$

$= \dfrac{-2}{-1}$

$= 2$

$n=-1: \; S_3 - 5S_1 + 2S_0 - S_{-1} = 0$

$S_3 - 5(0) + 2(4) - 2 = 0$

$S_3 - 0 + 8 - 2 = 0$

$S_3 = -6$

$n=2: \; S_6 - 5S_4 + 2S_3 - S_2 = 0$

$S_6 - 5(54) + 2(-6) - 10 = 0$

$S_6 - 270 - 12 - 10 = 0$

$S_6 = 292$

iv) $\alpha^2(\beta^4 + \gamma^4 + \delta^4) + \beta^2(\gamma^4 + \delta^4 + \alpha^4)$
$+ \gamma^2(\delta^4 + \alpha^4 + \beta^4) + \delta^2(\alpha^4 + \beta^4 + \gamma^4)$

$= \alpha^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4 - \alpha^4)$
$+ \beta^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4 - \beta^4)$
$+ \gamma^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4 - \gamma^4)$
$+ \delta^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4 - \delta^4)$

$= \alpha^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) - \alpha^2\alpha^4$
$+ \beta^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) - \beta^2\beta^4$
$+ \gamma^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) - \gamma^2\gamma^4$
$+ \delta^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) - \delta^2\delta^4$

$= \alpha^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) + \beta^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4)$
$+ \gamma^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) + \delta^2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4)$
$- \alpha^6 - \beta^6 - \gamma^6 - \delta^6$

$= (\alpha^4 + \beta^4 + \gamma^4 + \delta^4)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$
$- (\alpha^6 + \beta^6 + \gamma^6 + \delta^6)$

$= S_2 S_4 - S_6$

$= 10(54) - 292$

$= 540 - 292$

$= 248$