Topic : **P104 Curve Sketching**

Subtopic:

- *Curve of Rational Functions*

Curve Sketching in General

Terminology in curve sketching :

(a) asymptote

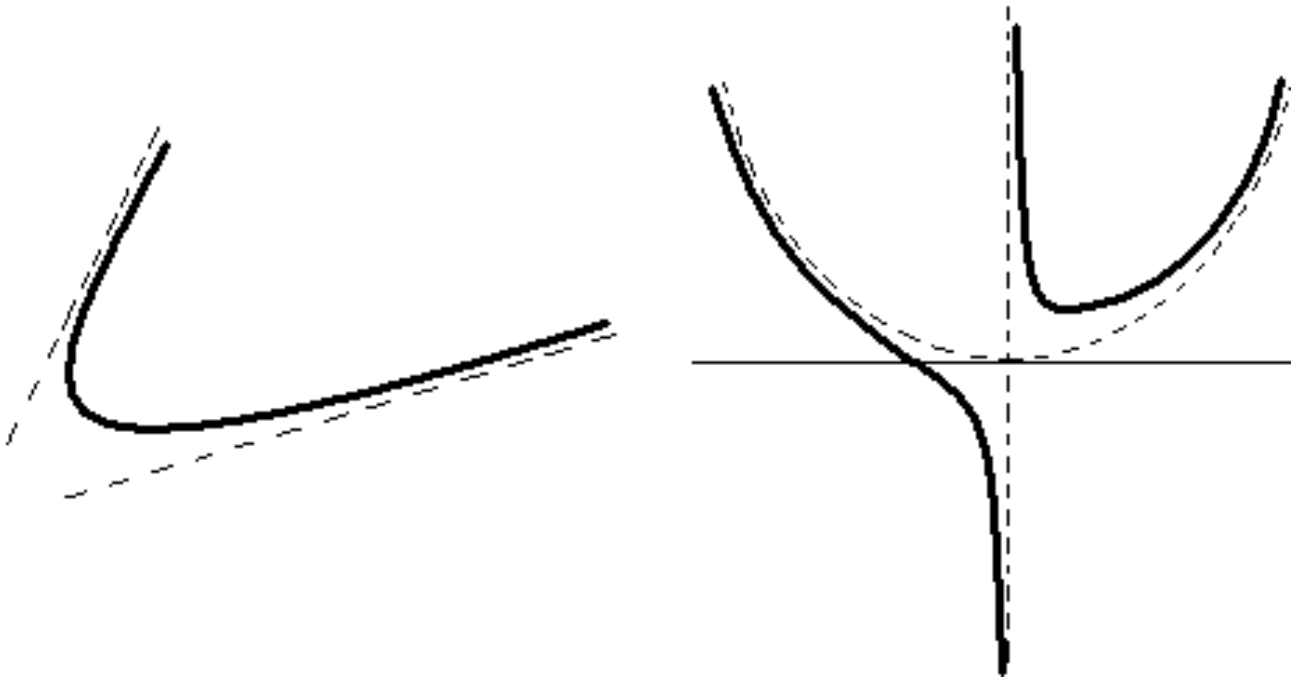
(b) stationary point / turning point

(c) increasing and decreasing function

(d) intercepts.

Asymptote

An asymptote of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity.



Asymptotes can be obtained by setting

(a) $x \to a^+$ and $x \to a^-$ (vertical asymptotes)

(b) $x \to +\infty$ and $x \to -\infty$ (horizontal asymptotes & oblique asymptotes)

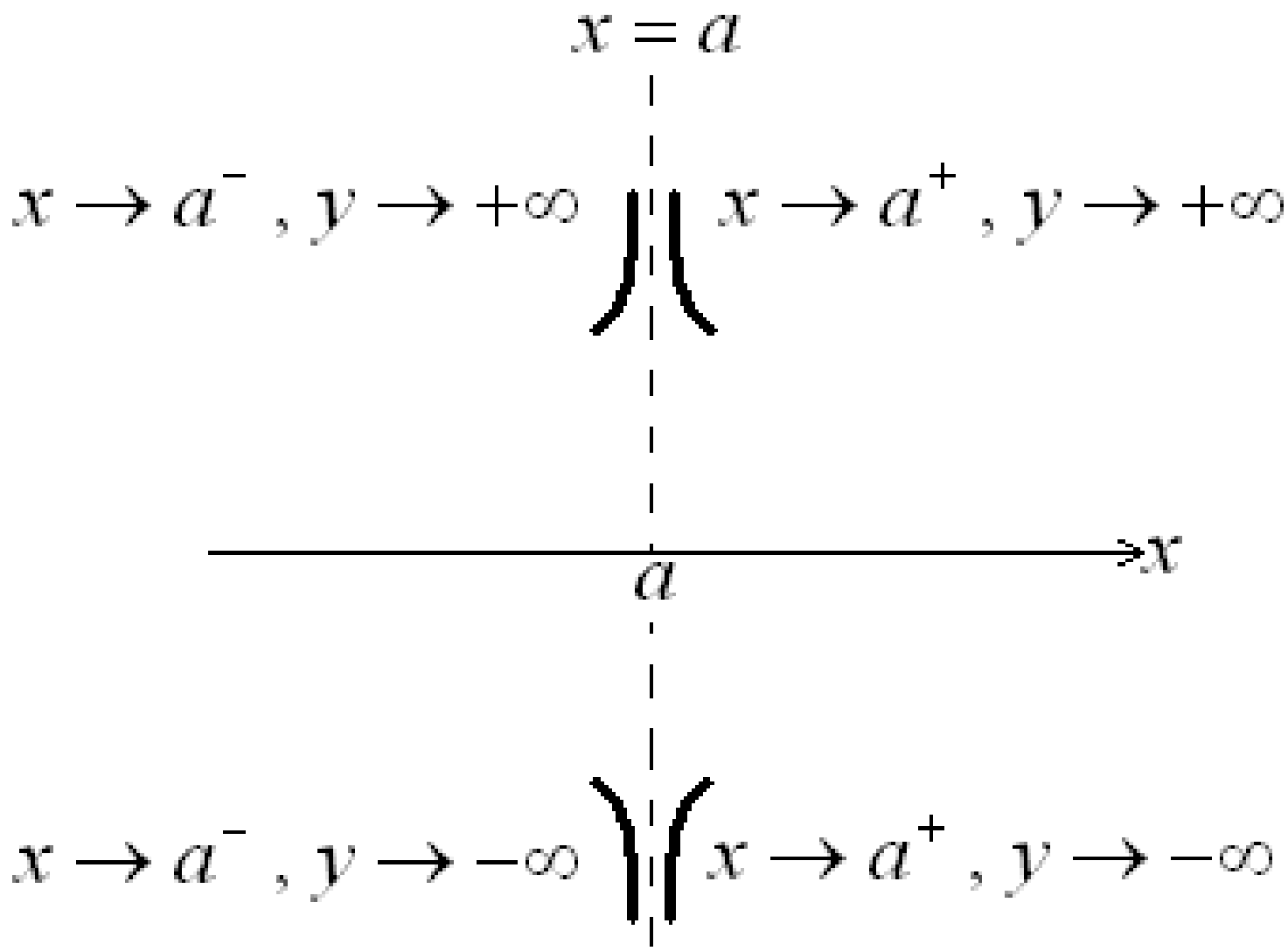
Vertical Asymptote

Definition :

$x = a$ is a *vertical asymptote* of $y = f(x)$ if

$\lim_{x \to a^+} f(x) = \pm\infty$ OR $\lim_{x \to a^-} f(x) = \pm\infty$.

In other words, $x = a$ is the value such that $f(a)$ is undefined.


$x = a$
$x \to a^-\ , y \to +\infty$
$x \to a^+\ , y \to +\infty$
x
a
$x \to a^-\ , y \to -\infty$
$x \to a^+\ , y \to -\infty$

Example :

Find and sketch the vertical asymptote(s) of

$$\text{(a)}\ y = \frac{1}{x-2} \qquad \text{(b)}\ y = \frac{2}{2x-1}$$

$$\text{(c)}\ y = 3 + \frac{4}{1-x} \qquad \text{(d)}\ y = -4 + \frac{1}{x-5}$$

Example :

Find and sketch the vertical asymptote(s) of the curves

$$(\text{a})\ y = \frac{1}{x(x+2)} \qquad (\text{b})\ y = \frac{x^2 - x - 2}{(x-1)(x+2)}$$

$$(\text{c})\ y = \frac{1}{2x-3} - \frac{1}{3x+2} \qquad (\text{d})\ y = \frac{1}{(x+1)^2} + \frac{1}{1-2x}$$

Example:

State the vertical asymptotes of the following curves:

$$\text{(a)}\ y = \frac{1}{x^2+1} \qquad \text{(b)}\ y = \frac{x-1}{x^2+x+1}$$

Horizontal Asymptote

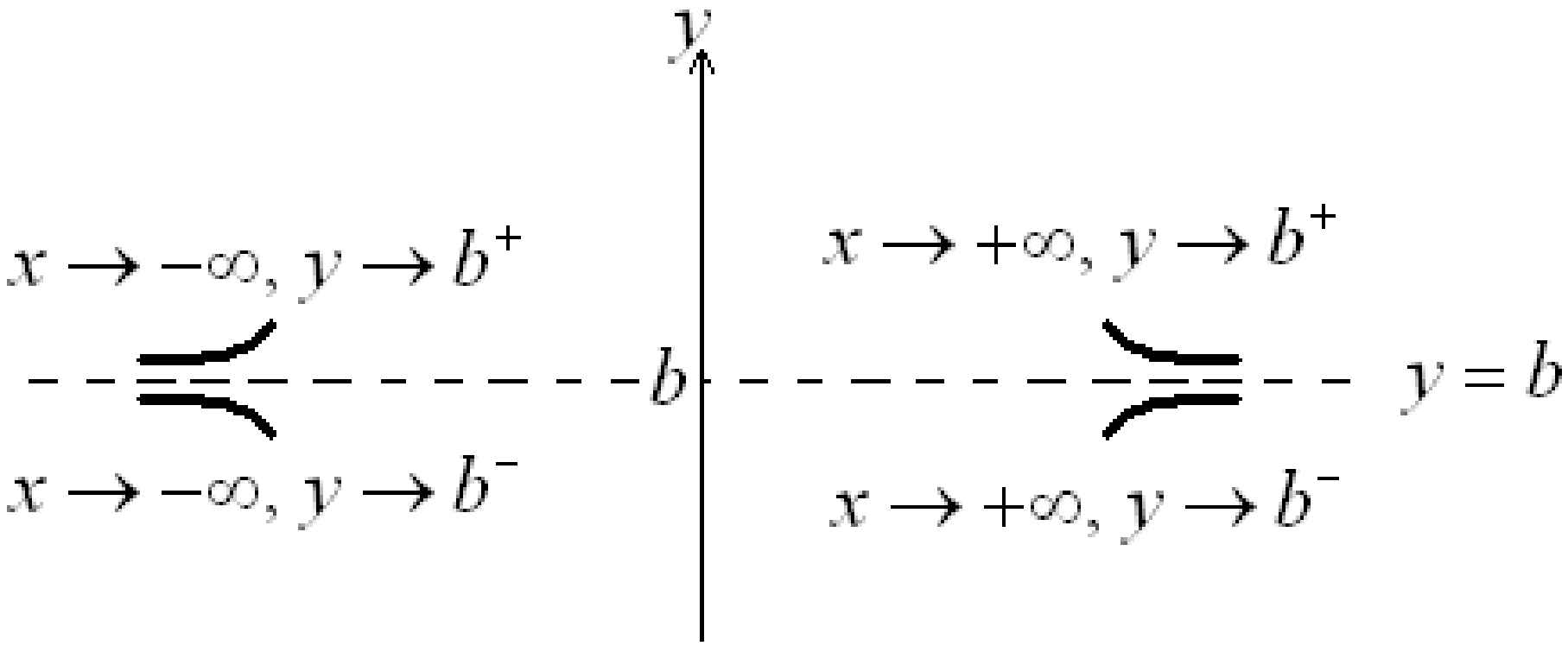
Definition :

$y = b$ is a *horizontal asymptote* of $y = f(x)$

if $\lim_{x \to -\infty} f(x) = b$ OR $\lim_{x \to +\infty} f(x) = b.$

In other words, $y = b$ is the value when

$x \to -\infty$ and $x \to \infty.$


y
$x \to -\infty,\ y \to b^+$
$x \to +\infty,\ y \to b^+$
b
$y = b$
$x \to -\infty,\ y \to b^-$
$x \to +\infty,\ y \to b^-$

Example:

Find and sketch the horizontal asymptote(s) of

$$(\text{a})\ y = \frac{1}{x-2} \qquad (\text{b})\ y = \frac{2}{1-2x}$$

$$(\text{c})\ y = 3 + \frac{4}{x-1} \qquad (\text{d})\ y = -4 + \frac{1}{2-x}$$

Example:

Find and sketch the horizontal asymptote(s) of

$$\text{(a)}\ y = \frac{1+x}{x} \qquad \text{(b)}\ y = \frac{2+x}{1-x}$$

$$\text{(c)}\ y = \frac{2+3x}{2+x} \qquad \text{(d)}\ y = \frac{1-x}{3-x}$$

Example:

Find and sketch the horizontal asymptote(s) of

$$(\text{a})\ y = \frac{1}{x^2 + 2x + 2} \qquad (\text{b})\ y = \frac{1}{(2 - x)(x + 1)}$$

$$(\text{c})\ y = \frac{x}{x^2 + 1} \qquad (\text{d})\ y = \frac{x}{(x - 1)^2}$$

Example:

Find and sketch the horizontal asymptote(s) of

$$\text{(a)}\ y = \frac{x^2 + 2}{x^2 + 2x - 3} \qquad \text{(b)}\ y = \frac{2x^2 - 1}{x^2 - 2x + 1}$$

$$\text{(c)}\ y = \frac{x^2 + 1}{1 - 2x^2} \qquad \text{(d)}\ y = \frac{-x^2 + 7x - 6}{-x^2 - 1}$$

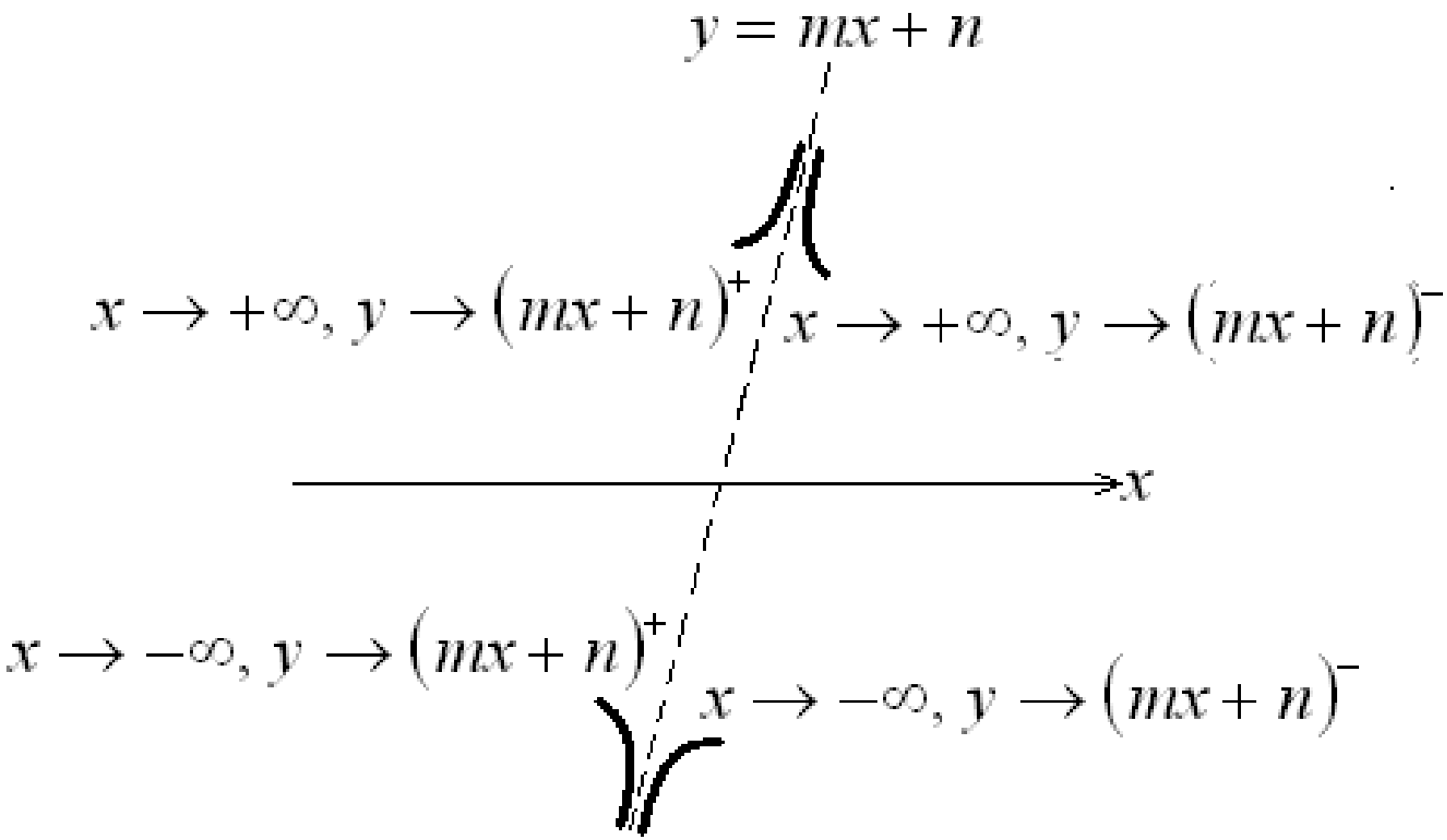
Oblique Asymptote

Definition :

$y = mx + n$ is an oblique asymptote of $y = f(x)$ if

$$\lim_{x \to -\infty} \left[f(x) - (mx + n) \right] = 0 \text{ OR } \lim_{x \to +\infty} \left[f(x) - (mx + n) \right] = 0$$

In other words, $y \to mx + n$ when $x \to \pm\infty$.


$y = mx + n$
$x \to +\infty,\ y \to (mx+n)^+$
$x \to +\infty,\ y \to (mx+n)^-$
x
$x \to -\infty,\ y \to (mx+n)^+$
$x \to -\infty,\ y \to (mx+n)^-$

Example:

Find and sketch all the asymptote(s) of the curves:

$(\text{a})\ y = x + \dfrac{1}{x}$

$(\text{b})\ y = 1 - x + \dfrac{1}{1-x}$

$(\text{c})\ y = x^2 + \dfrac{1}{x-1}$

$(\text{d})\ y = x^2 + \dfrac{1}{2-x} + \dfrac{1}{3+x}$

Example :

Find and sketch all the asymptote(s) of the curves :

$$(\text{a})\ y = \frac{4x^2 + 6x + 1}{2x} \qquad (\text{b})\ y = \frac{2x^2 - 3x}{1 - 2x}$$

$$(\text{c})\ y = \frac{x^2 - 3x + 3}{x - 2} \qquad (\text{d})\ y = \frac{x^3 + 2x^2 + x + 1}{(x-1)^2}$$

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example :

Find and sketch all the asymptotes of the curves

$$(\text{a})\ y = \frac{3}{x+2} \qquad (\text{b})\ y = 4 + \frac{1}{2x-1}$$

$$(\text{c})\ y = \frac{2-x}{1-x} \qquad (\text{c})\ y = \frac{2+3x}{2+x}$$

Example :

Find and sketch all the asymptotes of the curves

$$(\mathrm{a})\ y = 3 + 2x + \frac{1}{2x} \qquad (\mathrm{b})\ y = (x-1)^2 + \frac{1}{3-x}$$

$$(\mathrm{c})\ y = \frac{2x^2}{2x-1} \qquad (\mathrm{d})\ y = \frac{x^3 + 4x^2 + x + 5}{x+4}$$

Example :

Find and sketch all the asymptotes of the curves :

$$(\text{a})\ y = \frac{1-x}{(x+1)^2} \qquad (\text{b})\ y = \frac{x+1}{x^2+2x+2}$$

$$(\text{c})\ y = 2 - \frac{1}{(2-x)^2} \qquad (\text{d})\ y = x + \frac{1}{x+3} + \frac{2}{x-2}$$

Example:

Find and sketch all the asymptotes of the curves:

$$(\text{a})\ y = \frac{x^2 - x - 2}{x^2 - 4x + 3} \qquad (\text{b})\ y = \frac{x^2 - 7x + 6}{x^2 + 1}$$

$$(\text{c})\ y = \frac{4x^3 - 8x^2 + 3x + 2}{(3 - 2x)(1 - 2x)}$$

Example:

Find and sketch the vertical asymptote(s) of

$y = \tan\theta, -\pi \leq \theta \leq \pi.$