5. SUMMATION OF FINITE SERIES AND MATHEMATICAL INDUCTION

1. Evaluate the integral $\int_0^a x\, e^x dx$.

   Prove, by induction or otherwise, that

   $$\int_0^a x^n e^x dx = (-1)^n e^a (n!) \left[1 - a + \frac{a^2}{2!} + \ldots + \frac{(-1)^n a^n}{n!} - e^{-a}\right].$$ (J72/I/6)

2. Show, by induction or otherwise, that, for each positive integer n, both $\cos nx$ and $\frac{\sin nx}{\sin x}$ can be expressed as polynomials in $\cos x$ with integer coefficients. (J75/II/2)

3. Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

   Express $\frac{5n+2}{n(n+1)(n+2)}$ in the form $\frac{a}{n(n+1)} + \frac{b}{(n+1)(n+2)}$,

   where a and b are constants to be determined.

   Hence, or otherwise, evaluate S_n, where $S_n = \sum_{k=1}^{n} \frac{5k+2}{k(k+1)(k+2)}$ (N79/I/1)

4. Prove that, when n is a positive integer, $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ na & 1 \end{pmatrix}$

   Prove that the result also holds when n is a negative integer. (N79/II/1)

5. Prove that $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$, for $n \in Z_+$.

   Deduce that $nx(1+x)^{n-1} = \binom{n}{1}x + 2\binom{n}{2}x^2 + \ldots + n\binom{n}{n}x^n$, for $n \in Z_+$.

   Hence, or otherwise, find

   (a) $\sum_{r=1}^{n} (-1)^r \binom{n}{r}$, (b) $\sum_{r=1}^{n} r\binom{n}{r}$, (c) $\sum_{r=1}^{n} r^2\binom{n}{r}$. (N81/II/1)

6. Prove by induction, or otherwise, that

   $$\begin{pmatrix} \cos\theta - \sin\theta & -\sqrt{2}\sin\theta \\ \sqrt{2}\sin\theta & \cos\theta + \sin\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta - \sin n\theta & -\sqrt{2}\sin n\theta \\ \sqrt{2}\sin n\theta & \cos n\theta + \sin n\theta \end{pmatrix},$$

   where $n \in Z^+$.

   Find the smallest positive integer n such that

   $$\begin{pmatrix} \frac{1}{2}(\sqrt{3}-1) & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}(\sqrt{3}+1) \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ (N83/II/3)

7. (a) Prove the binomial theorem for a positive integral index.
Given that

$$y = x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \ldots + \frac{1}{r}\binom{n}{r-1}x^r + \ldots + \frac{1}{n+1}\binom{n}{n}x^{n+1},$$

show that $\dfrac{dy}{dx} = (1 + x)^n$.

Hence obtain another expression for y in terms of n and x, and deduce that

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{r}\binom{n}{r-1} + \ldots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1} - 1}{n+1}$$

(b) The term containing x^r in the binomial expansion of $(1 + x)^n$, where $n < 0$ and $|x| < 1$, is denoted by T_r. Obain, in terms of n, r and x, the value of T_r / T_{r-1}. Hence, or otherwise, show that if $-1 < x < 0$ and $n < 0$, the binomial expansion of $(1 + x)^n$ contains only positive terms.

In the case where $x = -\dfrac{1}{3}$ and $n = -20$ find the value of r such that T_r is the greatest term in the expansion, and show also that, for $r > 38$, $T_r < \dfrac{1}{2} T_{r-1}$.

(N85/II/2)

8. Show that, for all positive integers n and k, with $k \geqslant 2$,
$n(n + 1)(n + 2) \ldots (n + k - 1) - (n - 1)n(n + 1) \ldots (n + k - 2) = kn(n + 1) \ldots (n + k - 2)$.

Deduce that $\sum_{n=1}^{N} n(n + 1) = \frac{1}{3}N(N + 1)(N + 2)$, and write down similar expressions for $\sum_{n=1}^{N} n(n + 1)(n + 2)$ and $\sum_{n=1}^{N} n(n + 1)(n + 2)(n + 3)$.

Find numerical values for the constants a, b, c, d, given that
$an(n + 1)(n + 2)(n + 3) + bn(n + 1)(n + 2) + cn(n + 1) + dn \equiv n^4$.

Hence find (but **do not simplify**) an expression in terms of N for $\sum_{n=1}^{N} n^4$ (J86/I/2)

9. Prove by induction that, for all $n \in \mathbb{Z}$,

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + x^n$$

[The result $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, if used in your proof, must be justified.]

Deduce that

$$1 - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^r\binom{n}{r} + \ldots + (-1)^n = 0.$$

By considering $\int_0^1 (1 + x)^n \, dx$, show that

$$1 + \frac{1}{2}\binom{n}{1}x + \frac{1}{3}\binom{n}{2} \ldots + \frac{1}{r+1}\binom{n}{r} + \ldots + \frac{1}{n+1} = \frac{2^{n+1} - 1}{n+1}.$$

(N86/I/2)

7. (a) Prove the binomial theorem for a positive integral index.
Given that

$$y = x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \ldots + \frac{1}{r}\binom{n}{r-1}x^r + \ldots + \frac{1}{n+1}\binom{n}{n}x^{n+1},$$

show that $\frac{dy}{dx} = (1+x)^n$.

Hence obtain another expression for y in terms of n and x, and deduce that

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{r}\binom{n}{r-1} + \ldots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$$

(b) The term containing x^r in the binomial expansion of $(1+x)^n$, where $n < 0$ and $|x| < 1$, is denoted by T_r. Obain, in terms of n, r and x, the value of T_r/T_{r-1}.
Hence, or otherwise, show that if $-1 < x < 0$ and $n < 0$, the binomial expansion of $(1+x)^n$ contains only positive terms.

In the case where $x = -\frac{1}{3}$ and $n = -20$ find the value of r such that T_r is the greatest term in the expansion, and show also that, for $r > 38$, $T_r < \frac{1}{2}T_{r-1}$.

(N85/II/2)

8. Show that, for all positive integers n and k, with $k \geqslant 2$,
$n(n+1)(n+2)\ldots(n+k-1) - (n-1)n(n+1)\ldots(n+k-2) = kn(n+1)\ldots(n+k-2)$.

Deduce that $\sum_{n=1}^{N} n(n+1) = \frac{1}{3}N(N+1)(N+2)$, and write down similar expressions for $\sum_{n=1}^{N} n(n+1)(n+2)$ and $\sum_{n=1}^{N} n(n+1)(n+2)(n+3)$.

Find numerical values for the constants a, b, c, d, given that
$an(n+1)(n+2)(n+3) + bn(n+1)(n+2) + cn(n+1) + dn \equiv n^4$.

Hence find (but **do not simplify**) an expression in terms of N for $\sum_{n=1}^{N} n^4$. (J86/I/2)

9. Prove by induction that, for all $n \in \mathbb{Z}$,

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + x^n.$$

[The result $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, if used in your proof, must be justified.]

Deduce that

$$1 - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^r\binom{n}{r} + \ldots + (-1)^n = 0.$$

By considering $\int_0^1 (1+x)^n\,dx$, show that

$$1 + \frac{1}{2}\binom{n}{1}x + \frac{1}{3}\binom{n}{2} \ldots + \frac{1}{r+1}\binom{n}{r} + \ldots + \frac{1}{n+1} = \frac{2^{n+1}-1}{n+1}.$$

(N86/I/2)

10. (a) Express $\frac{2}{x(x+1)(x+2)}$ in partial fractions.

By using your result, or otherwise, show that

$$\sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(N+1)(N+2)}.$$

(b) Prove by induction that $3^{4n-2} + 17^n + 22$ is divisible by 16 for every positive integer n. (J87/I/2)

11. The sequence of real numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = \frac{5u_n + 4}{u_n + 2}$ for all $n \geqslant 1$. Prove by induction that $u_n n < 4$ for all $n \geqslant 1$. (N87/I/2)

12. (a) Given that $y = \frac{1}{1+x}$, prove by induction that $\frac{d^n y}{dx^n} = \frac{(-1)^n(n!)}{(1+x)^{n+1}}$ for every positive integer n.

(b) (i) Verify that $\frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1} = \frac{2n}{n^4 + n^2 + 1}$.

(ii) Express $n^2 + n + 1$ in the form $(n+a)^2+b$, and express $n^2 - n + 1$ in a similar form.

(iii) Hence, or otherwise, find an expression in terms of N for the sum S_N, where

$$S_N = \sum_{n=1}^{N} \frac{n}{n^4 + n^2 + 1}$$

Deduce that $S_N < \frac{1}{2}$.

(J88/I/2)

13. (a) Let $a_n = e^{1-n} - e^{-n} + \frac{1}{n+1} - \frac{1}{n}$.

(i) Find, in terms of N, an expression for S_N, where $S_N = a_1 + a_2 + \ldots + a_N$, simplifying your result as far as possible.

(ii) Show that $S_N > 0$ for all $N \geq 1$.

(b) Given that $y = e^x \sin x$, prove by induction that $\frac{d^n y}{dx^n} = 2^{\frac{1}{2}n} e^x \sin(x + \frac{1}{4}n\pi)$ for every positive integer n. (N88/I/4)

14. (a) Prove that $\sum_{n=1}^{N} \frac{1}{\sqrt{n} + \sqrt{n-1}} = \sqrt{N}$

Deduce, or prove otherwise, that $\sum_{n=1}^{N} \frac{1}{\sqrt{n}} < 2\sqrt{N}$.

(b) The sequence of real numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5$ and $u_{n+1} = (u_n + \frac{1}{u_n})^2$ for all $n \geq 1$. Prove by induction that, for every positive $u_n > 2^m$, where $m = 2^n$. (J89/I/2)

15. (a) By considering $\frac{1}{1+a^{n-1}} - \frac{1}{1+a^n}$ or otherwise, show that

$$\sum_{n=1}^{N} \frac{a^{n-1}}{(1+a^{n-1})(1+a^n)} = \frac{a^N - 1}{2(a-1)(a^N+1)},$$ where a is positive and $a \neq 1$.

Deduce that $\sum_{n=1}^{N} \frac{2^n}{(1+2^{n-1})(1+2^n)} < 1$.

(b) Prove by induction, or otherwise, that $10^{3n} + 38^n + 35$ is divisible by 37 for every non-negative integer n. (N89/I/2)

16. By considering $\sum_{n=1}^{N} [\cos(n - \frac{1}{2})x - \cos(n + \frac{1}{2})x]$, or otherwise, show that $\sum_{n=1}^{N} \sin nx = \operatorname{cosec}(\frac{1}{2}x)\sin[\frac{1}{2}(N + 1)x]\sin(\frac{1}{2} N x)$, provided that $\sin(\frac{1}{2}x) \neq 0$.

Deduce that $\sum_{n=1}^{N-1} \sin\frac{n\pi}{N} < \operatorname{cosec}\frac{\pi}{2N}$ for all $N \geq 2$. (J90/I/2)