

$$\begin{aligned}
 1. \quad 1^{\frac{1}{5}} &= (\cos \theta + i \sin \theta)^{\frac{1}{5}} \\
 &= \left[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi) \right]^{\frac{1}{5}} \\
 &= \left(\cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} \right)^{\frac{1}{5}}, \quad k = 0, 1, 2, 3, 4 \\
 z^5 &= -16 + 16\sqrt{3}i \\
 &= 16(-1 + \sqrt{3}i) \\
 &= 16(2) \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \\
 &= 32 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \\
 &= 32 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \\
 &= 32 \left(\cos \left(\frac{2\pi}{3} + 2k\pi \right) + i \sin \left(\frac{2\pi}{3} + 2k\pi \right) \right) \\
 \therefore z &= \left(32 \left(\cos \left(\frac{2\pi}{3} + 2k\pi \right) + i \sin \left(\frac{2\pi}{3} + 2k\pi \right) \right) \right)^{\frac{1}{5}} \\
 &= 32^{\frac{1}{5}} \left(\cos \left(\frac{2\pi}{3} + 2k\pi \right) + i \sin \left(\frac{2\pi}{3} + 2k\pi \right) \right)^{\frac{1}{5}} \\
 &= 2 \left(\cos \frac{1}{5} \left(\frac{2\pi}{3} + 2k\pi \right) + i \sin \frac{1}{5} \left(\frac{2\pi}{3} + 2k\pi \right) \right) \\
 &= 2 \left(\cos \left(\frac{2\pi}{15} + \frac{2k\pi}{5} \right) + i \sin \left(\frac{2\pi}{15} + \frac{2k\pi}{5} \right) \right) \\
 &= 2e^{\left(\frac{2\pi}{15} + \frac{2k\pi}{5} \right)i}, \quad k = 0, 1, 2, 3, 4
 \end{aligned}$$

$$2. \quad u_{n+1} = -1 + \sqrt{u_n + 7}, \quad u_1 = 1$$

$$\text{i) } u_1 = 1 < 2.$$

Assume that $u_n < 2$ when $n = k$: $u_k < 2$

since $u_k < 2$

$$u_k + 7 < 9$$

$$\sqrt{u_k + 7} < 3$$

$$-1 + \sqrt{u_k + 7} < 2$$

$$u_{k+1} < 2$$

Since $u_1 < 2$ and $u_{k+1} < 2$ if $u_k < 2$

$\therefore u_n < 2$ for all $n \geq 1$

$$\text{ii) If } u_n = 2 - \varepsilon$$

$$u_{n+1} = -1 + \sqrt{u_n + 7}$$

$$= -1 + \sqrt{2 - \varepsilon + 7}$$

$$= -1 + \sqrt{9 - \varepsilon}$$

$$= -1 + \sqrt{9(1 - \frac{\varepsilon}{9})}$$

$$= -1 + \sqrt{9} \sqrt{1 - \frac{\varepsilon}{9}}$$

$$= -1 + 3 \left(1 - \frac{\varepsilon}{9}\right)^{\frac{1}{2}}$$

$$= -1 + 3 \left(1 + \frac{1}{2} \cdot \left(-\frac{\varepsilon}{9}\right) + \frac{1}{2} \left(\frac{1}{2} - 1\right) \left(-\frac{\varepsilon}{9}\right)^2 \frac{1}{2!} + \dots\right)$$

$$= -1 + 3 \left(1 - \frac{\varepsilon}{18} + \frac{1}{2} \left(-\frac{1}{2}\right) \frac{\varepsilon^2}{81} \left(\frac{1}{2}\right) + \dots\right)$$

$$= -1 + 3 \left(1 - \frac{\varepsilon}{18} - \frac{\varepsilon^2}{648} + \dots\right)$$

$$= -1 + 3 - \frac{\varepsilon}{6} - \frac{\varepsilon^2}{216} + \dots$$

$$= 2 - \frac{\varepsilon}{6} - \frac{\varepsilon^2}{216} + \dots$$

$$= 2 - \frac{\varepsilon}{6}, \quad \text{if } \varepsilon \text{ is small.}$$

$$3. C : y = \frac{x^2}{x+\lambda}, \lambda \neq 0.$$

$$x \rightarrow -\lambda \quad y \rightarrow \pm\infty$$

$\therefore x = -\lambda$ is an asymptote of C .

$$\begin{aligned} x+\lambda &\sqrt{\frac{x-\lambda}{x^2}} \\ &\frac{x^2 + \lambda x}{-x\lambda} \\ &\frac{-x\lambda - x^2}{x^2} \end{aligned}$$

$$\therefore y = x - \lambda + \frac{\lambda^2}{x+\lambda}$$

$$x \rightarrow \pm\infty \quad y \rightarrow x - \lambda$$

$y = x - \lambda$ is an asymptote of C

\therefore The asymptotes of C are $x = -\lambda$ and $y = x - \lambda$.

$$\text{when } x=0, y=0$$

$$\text{i) } \lambda > 0:$$

$$y = x - \lambda + \frac{\lambda^2}{x+\lambda}$$

$$\frac{dy}{dx} = 1 - \frac{\lambda^2}{(x+\lambda)^2}$$

$$\text{If } \frac{dy}{dx} = 0 : 1 - \frac{\lambda^2}{(x+\lambda)^2} = 0$$

$$\frac{\lambda^2}{(x+\lambda)^2} = 1$$

$$\begin{aligned} \lambda^2 &= (x+\lambda)^2 \\ &= x^2 + 2\lambda x + \lambda^2 \end{aligned}$$

$$x^2 + 2\lambda x = 0$$

$$x(x+2\lambda) = 0$$

$$x = 0, -2\lambda$$

$$y = 0, -4\lambda$$

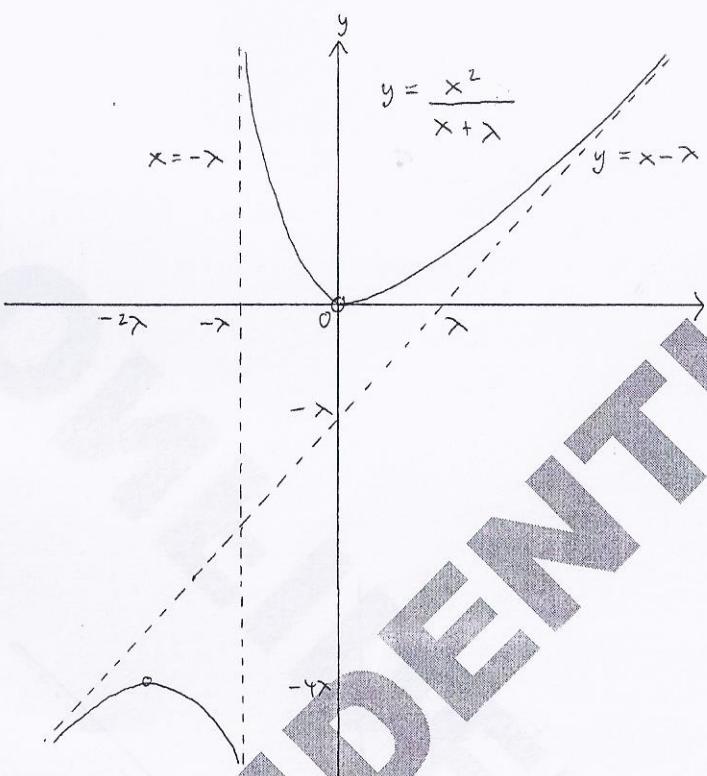
\therefore The stationary points of C are $(0,0)$ and $(-2\lambda, -4\lambda)$

$$\frac{d^2y}{dx^2} = \frac{2\lambda^2}{(x+\lambda)^3}$$

$$\text{when } x=0 \quad \frac{d^2y}{dx^2} = \frac{2\lambda^2}{\lambda^3} > 0$$

$$\text{when } x=-2\lambda \quad \frac{d^2y}{dx^2} = \frac{-2\lambda^2}{(-2\lambda+\lambda)^3} < 0$$

$\therefore (0,0)$ is a minimum point and $(-2\lambda, -4\lambda)$ is a maximum point



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$$= x^2 + 2\lambda x + \lambda^2$$

$$x^2 + 2\lambda x = 0$$

$$x(x + 2\lambda) = 0$$

$$x = 0, -2\lambda$$

$$y = 0, -4\lambda$$

∴ The stationary points of C are $(0, 0)$ and $(-2\lambda, -4\lambda)$.

$$\frac{d^2y}{dx^2} = \frac{2\lambda^2}{(x+\lambda)^3}$$

$$\text{when } x=0 \quad \frac{d^2y}{dx^2} = \frac{2}{\lambda} < 0$$

$$\text{when } x = -2\lambda \quad \frac{d^2y}{dx^2} = -\frac{2}{\lambda} > 0$$

$(0, 0)$ is a maximum point and $(-2\lambda, -4\lambda)$ is a minimum point.

ii) $\lambda < 0$:

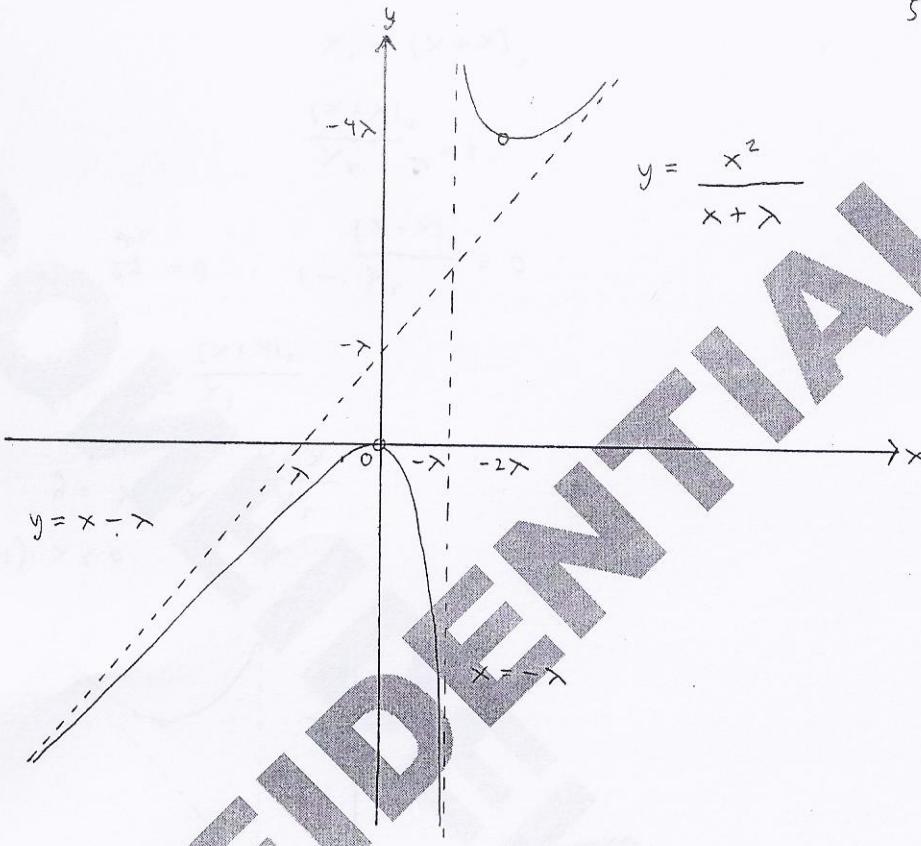
$$y = x - \lambda + \frac{\lambda^2}{x+\lambda}$$

$$\frac{dy}{dx} = 1 - \frac{\lambda^2}{(x+\lambda)^2}$$

$$\text{If } \frac{dy}{dx} = 0 \quad \therefore 1 - \frac{\lambda^2}{(x+\lambda)^2} = 0$$

$$\frac{\lambda^2}{(x+\lambda)^2} = 1$$

$$\lambda^2 = (x+\lambda)^2$$



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4. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 24e^{2x}$, $y=1$ and $\frac{dy}{dx}=9$ when $x=0$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda = -1, -2$$

$$y_c = Ae^{-x} + Be^{-2x}$$

Let $y_p = Ce^{2x}$

$$\frac{dy_p}{dx} = 2Ce^{2x}$$

$$\frac{d^2y_p}{dx^2} = 4Ce^{2x}$$

$$\begin{aligned} \therefore \frac{d^2y_p}{dx^2} + 3\frac{dy_p}{dx} + 2y_p &= 4Ce^{2x} + 6Ce^{2x} + 2Ce^{2x} \\ &= 12Ce^{2x} \\ &= 24e^{2x} \end{aligned}$$

$$\begin{aligned} 12C &= 24 \\ C &= 2 \end{aligned}$$

$$\therefore y_p = 2e^{2x}$$

$$\begin{aligned} y &= y_c + y_p \\ &= Ae^{-x} + Be^{-2x} + 2e^{2x} \end{aligned}$$

$$\frac{dy}{dx} = -Ae^{-x} - 2Be^{-2x} + 4e^{2x}$$

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when $x=0$ $y=1$: $1 = A + \beta + 2$

when $x=0$ $\frac{dy}{dx}=9$: $9 = -A - \beta + 4$

$$A + \beta + 2 = 1$$

$$-A - 2\beta + 4 = 9$$

$$A + \beta = -1$$

$$-A - 2\beta = 5$$

$$-\beta = 4$$

$$\beta = -4$$

$$A = 3$$

$$\therefore y = 3e^{-x} - 4e^{-2x} + 2e^{2x}$$

$$5. \quad x^3 + ax^2 + bx + c = 0.$$

If α, β , and γ are the roots of the equation

$$\therefore \alpha + \beta + \gamma = -a$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = b$$

$$\alpha\beta\gamma = -c$$

i) $\alpha > 1, \beta > 1$ and $\gamma > 1$

$$\therefore \alpha + \beta + \gamma > 1+1+1$$

$$-a > 3$$

$$\alpha < -3$$

since $\alpha + \beta + \gamma = -a$

ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$$= (-a)^2 - 2b$$

$$= a^2 - 2b$$

Since $\alpha > 1, \beta > 1$ and $\gamma > 1$,
 $\alpha^2 > 1, \beta^2 > 1$ and $\gamma^2 > 1$.

$$\therefore \alpha^2 + \beta^2 + \gamma^2 > 1+1+1$$

$$a^2 - 2b > 3$$

$$\alpha^2 > 2b + 3$$

iii) If $s_n = \alpha^n + \beta^n + \gamma^n$

$$\therefore s_0 = \alpha^0 + \beta^0 + \gamma^0 = 1+1+1=3,$$

$$s_1 = \alpha^1 + \beta^1 + \gamma^1 = \alpha + \beta + \gamma = -a$$

$$\text{and } S_2 = \alpha^2 + \beta^2 + \gamma^2 = \alpha^2 - 2b$$

$$x^3 + ax^2 + bx + c = 0$$

$$1S_3 + aS_2 + bS_1 + cS_0 = 0$$

$$S_3 + a(\alpha^2 - 2b) + b(-a) + 3c = 0$$

$$S_3 + \alpha^3 - 2ab - ab + 3c = 0$$

$$S_3 + \alpha^3 - 3ab + 3c = 0$$

$$\therefore S_3 = 3ab - \alpha^3 - 3c$$

Since $\alpha > 1$, $\beta > 1$ and $\gamma > 1$,

$\alpha\beta > 1$, $\alpha\gamma > 1$ and $\beta\gamma > 1$

$$\therefore \alpha\beta + \alpha\gamma + \beta\gamma > 1+1+1$$

$$b > 3$$

Since $\alpha < -3$

$$\alpha + 3 < 0$$

$$(\alpha + 3)b < 0$$

$$ab + 3b < 0$$

$$3(ab + 3b) < 0$$

$$3ab + 9b < 0$$

$$-3ab - 9b > 0$$

$$\alpha^3 - 3ab - 9b > \alpha^3$$

Since $\alpha > 1$, $\beta > 1$ and $\gamma > 1$

$\alpha^3 > 1$, $\beta^3 > 1$ and $\gamma^3 > 1$

$$\therefore \alpha^3 + \beta^3 + \gamma^3 > 1+1+1$$

$$S_3 > 3$$

$$S_3 = 3ab - \alpha^3 - 3c$$

$$\therefore 3ab - \alpha^3 - 3c > 3$$

$$-3c - 3 > \alpha^3 - 3ab$$

$$-9b - 3c - 3 > \alpha^3 - 3ab - 9b$$

Since $-9b - 3c - 3 > \alpha^3 - 3ab - 9b$ and

$$\alpha^3 - 3ab - 9b > \alpha^3$$

$$\therefore -9b - 3c - 3 > \alpha^3$$

$$6. I_n = \int_0^1 (1+x^2)^{-n} dx, n \geq 1$$

$$\begin{aligned}\frac{d}{dx} (x(1+x^2)^{-n}) &= (1+x^2)^{-n} \frac{d}{dx}(x) + x \frac{d}{dx} (1+x^2)^{-n} \\ &= (1+x^2)^{-n} (1) + x(-n)(1+x^2)^{-n-1}(2x) \\ &= (1+x^2)^{-n} - 2nx^2(1+x^2)^{-n-1}\end{aligned}$$

$$\begin{aligned}\therefore x(1+x^2)^{-n} &= \int (1+x^2)^{-n} - 2nx^2(1+x^2)^{-n-1} dx \\ &= \int (1+x^2)^{-n} dx - \int 2nx^2(1+x^2)^{-n-1} dx \\ &= \int (1+x^2)^{-n} dx - 2n \int x^2(1+x^2)^{-n-1} dx\end{aligned}$$

$$\therefore \int (1+x^2)^{-n} dx = x(1+x^2)^{-n} + 2n \int x^2(1+x^2)^{-n-1} dx$$

$$\begin{aligned}\int_0^1 (1+x^2)^{-n} dx &= \left[x(1+x^2)^{-n} \right]_0^1 + 2n \int_0^1 x^2(1+x^2)^{-n-1} dx \\ &= 1(1+1)^{-n} - 0(1+0)^{-n} + 2n \int_0^1 x^2(1+x^2)^{-n-1} dx \\ &= 2^{-n} + 2n \int_0^1 x^2(1+x^2)^{-n-1} dx \\ &= 2^{-n} + 2n \int_0^1 (1+x^2-1)(1+x^2)^{-n-1} dx \\ &= 2^{-n} + 2n \int_0^1 (1+x^2)(1+x^2)^{-n-1} - (1+x^2)^{-n-1} dx \\ &= 2^{-n} + 2n \int_0^1 (1+x^2)^{-n} - (1+x^2)^{-n-1} dx \\ &= 2^{-n} + 2n \int_0^1 (1+x^2)^{-n} - (1+x^2)^{-(n+1)} dx\end{aligned}$$

$$= 2^{-n} + 2n \int_0^1 (1+x^2)^{-n} dx - 2n \int_0^1 (1+x^2)^{-(n+1)} dx$$

$$\therefore I_n = 2^{-n} + 2n I_n - 2n I_{n+1}$$

$$2n I_n - I_n + 2^{-n} = 2n I_{n+1}$$

$$(2n-1) I_n + 2^{-n} = 2n I_{n+1}$$

$$n=2 : 2(2) I_3 = (2(2)-1) I_2 + 2^{-2}$$

$$4 I_3 = 3 I_2 + \frac{1}{4}$$

$$n=1 : 2(1) I_2 = (2(1)-1) I_1 + 2^{-1}$$

$$2 I_2 = I_1 + \frac{1}{2}$$

$$I_1 = \int_0^1 (1+x^2)^{-1} dx$$

$$\begin{aligned}x &= \tan u \\ dx &= \sec^2 u du\end{aligned}$$

$$x=1 \quad u=\frac{\pi}{4}$$

$$x=0 \quad u=0$$

$$= \int_0^{\frac{\pi}{4}} (1+\tan^2 u)^{-1} \sec^2 u du$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 u)^{-1} \sec^2 u du$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 u \sec^2 u du$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 dy$$

$$= [y]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

$$2I_2 = I_1 + \frac{1}{2}$$

$$= \frac{\pi}{4} + \frac{1}{2}$$

$$\therefore I_2 = \left(\frac{\pi}{8} + \frac{1}{4} \right)$$

$$\begin{aligned} 4I_3 &= 3I_2 + \frac{1}{4} \\ &= 3\left(\frac{\pi}{8} + \frac{1}{4}\right) + \frac{1}{4} \\ &= \frac{3\pi}{8} + \frac{3\pi}{4} + \frac{1}{4} \\ &= \frac{3\pi}{8} + 1 \end{aligned}$$

$$I_3 = \frac{3\pi}{32} + \frac{1}{4}$$

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$$7. \sum_{n=1}^N z^{-n} z^n = \sum_{n=1}^N \frac{z^n}{z^n}$$

$$= \sum_{n=1}^N \left(\frac{z}{z}\right)^n$$

$$= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots + \left(\frac{z}{2}\right)^N$$

$$= \frac{z}{2} \left(\frac{1 - \left(\frac{z}{2}\right)^N}{1 - \frac{z}{2}} \right)$$

$$= \frac{z}{2} \left(1 - \left(\frac{z}{2}\right)^N \right) \left(\frac{z}{2 - z} \right)$$

$$= \frac{z}{z - z} \left(1 - \left(\frac{z}{2}\right)^N \right)$$

since $\frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots + \left(\frac{z}{2}\right)^N$ is a geometric series with $a = \frac{z}{2}$ and $r = \frac{z}{2}$

If $z = \cos \frac{n\pi}{10} + i \sin \frac{n\pi}{10}$ and $N = 10$

$$\sum_{n=1}^N z^{-n} z^n = \sum_{n=1}^{10} z^{-n} \left(\cos \frac{n\pi}{10} + i \sin \frac{n\pi}{10} \right)^n$$

$$= \sum_{n=1}^{10} z^{-n} \left(\cos \frac{n\pi}{10} + i \sin \frac{n\pi}{10} \right)$$

$$= \sum_{n=1}^{10} z^{-n} \cos \frac{n\pi}{10} + i z^{-n} \sin \frac{n\pi}{10}$$

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$$= \sum_{n=1}^{10} z^{-n} \cos \frac{n\pi}{10} + i \sum_{n=1}^{10} z^{-n} \sin \frac{n\pi}{10}$$

$$= \frac{\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}}{2 - (\cos \frac{\pi}{10} + i \sin \frac{\pi}{10})} \left(1 - \left(\frac{1}{2} \left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right) \right)^{10} \right)$$

$$= \frac{\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}}{2 - \cos \frac{\pi}{10} - i \sin \frac{\pi}{10}} \left(1 - \left(\frac{1}{2^{10}} \left(\cos \frac{10\pi}{10} + i \sin \frac{10\pi}{10} \right) \right) \right)$$

$$= \frac{\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}}{2 - \cos \frac{\pi}{10} - i \sin \frac{\pi}{10}} \left(1 - \left(\frac{1}{1024} (\cos \pi + i \sin \pi) \right) \right)$$

$$= \frac{\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}}{2 - \cos \frac{\pi}{10} - i \sin \frac{\pi}{10}} \left(1 - \left(-\frac{1}{1024} \right) \right)$$

$$= \frac{\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}}{2 - \cos \frac{\pi}{10} - i \sin \frac{\pi}{10}} \left(\frac{1024 - 1}{1024} \right)$$

$$= \frac{1025 \left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right)}{1024 \left(2 - \cos \frac{\pi}{10} - i \sin \frac{\pi}{10} \right)}$$

$$= \frac{1025 \left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right) \left(2 - \cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right)}{1024 \left(2 - \cos \frac{\pi}{10} - i \sin \frac{\pi}{10} \right) \left(2 - \cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right)}$$

$$= \frac{1025 \left[\cos \frac{\pi}{10} \left(2 - \cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right) + i \sin \frac{\pi}{10} \left(2 - \cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right) \right]}{1024 \left[\left(2 - \cos \frac{\pi}{10} \right)^2 - \left(2 - \cos \frac{\pi}{10} \right) i \sin \frac{\pi}{10} + \left(2 - \cos \frac{\pi}{10} \right) i \sin \frac{\pi}{10} - i^2 \sin^2 \frac{\pi}{10} \right]}$$

$$= \frac{1025 \left(2 \cos \frac{\pi}{10} - \cos^2 \frac{\pi}{10} + i \sin \frac{\pi}{10} \cos \frac{\pi}{10} + 2 i \sin \frac{\pi}{10} - i \sin \frac{\pi}{10} \cos \frac{\pi}{10} + i^2 \sin^2 \frac{\pi}{10} \right)}{1024 \left(\left(2 - \cos \frac{\pi}{10} \right)^2 + \sin^2 \frac{\pi}{10} \right)}$$

$$= \frac{1025 \left(2 \cos \frac{\pi}{10} - \cos^2 \frac{\pi}{10} - \sin^2 \frac{\pi}{10} + 2 i \sin \frac{\pi}{10} \right)}{1024 \left(4 - 4 \cos \frac{\pi}{10} + \cos^2 \frac{\pi}{10} + \sin^2 \frac{\pi}{10} \right)}$$

$$= \frac{1025 \left(2 \cos \frac{\pi}{10} - \left(\cos^2 \frac{\pi}{10} + \sin^2 \frac{\pi}{10} \right) + 2 i \sin \frac{\pi}{10} \right)}{1024 \left(4 - 4 \cos \frac{\pi}{10} + 1 \right)}$$

$$= \frac{1025 \left(2 \cos \frac{\pi}{10} - 1 + 2 i \sin \frac{\pi}{10} \right)}{1024 \left(5 - 4 \cos \frac{\pi}{10} \right)}$$

$$= \frac{1025 \left(2 \cos \frac{\pi}{10} - 1 \right) + 1025 \left(2i \sin \frac{\pi}{10} \right)}{1024 \left(5 - 4 \cos \frac{\pi}{10} \right)}$$

$$= \frac{1025 \left(2 \cos \frac{\pi}{10} - 1 \right)}{1024 \left(5 - 4 \cos \frac{\pi}{10} \right)} + \frac{1025 \left(2i \sin \frac{\pi}{10} \right)}{1024 \left(5 - 4 \cos \frac{\pi}{10} \right)}$$

$$= \frac{1025 \left(2 \cos \frac{\pi}{10} - 1 \right)}{1024 \left(5 - 4 \cos \frac{\pi}{10} \right)} + \frac{1025 i \sin \frac{\pi}{10}}{512 \left(5 - 4 \cos \frac{\pi}{10} \right)}$$

$$= \frac{1025 \left(2 \cos \frac{\pi}{10} - 1 \right)}{1024 \left(5 - 4 \cos \frac{\pi}{10} \right)} + i \left(\frac{1025 \sin \frac{\pi}{10}}{2560 - 2048 \cos \frac{\pi}{10}} \right)$$

Equating imaginary parts,

$$\sum_{n=1}^{10} 2^{-n} \sin \frac{n\pi}{10} = \frac{1025 \sin \frac{\pi}{10}}{2560 - 2048 \cos \frac{\pi}{10}}$$

8. $y = x^2(1-x)$

$$= x^2 - x^3$$

$$\frac{dy}{dx} = 2x - 3x^2$$

$$\text{If } \frac{dy}{dx} = 0 \quad 2x - 3x^2 = 0 \\ x(2 - 3x) = 0$$

$$x = 0, \frac{2}{3}$$

$$y = 0, \frac{4}{27}$$

The stationary points of $y = x^2(1-x)$ are $(0,0)$ and $\left(\frac{2}{3}, \frac{4}{27}\right)$.

$$\frac{d^2y}{dx^2} = 2 - 6x$$

$$x = 0 : \frac{d^2y}{dx^2} = 2 > 0$$

$$x = \frac{2}{3} : \frac{d^2y}{dx^2} = -2 < 0$$

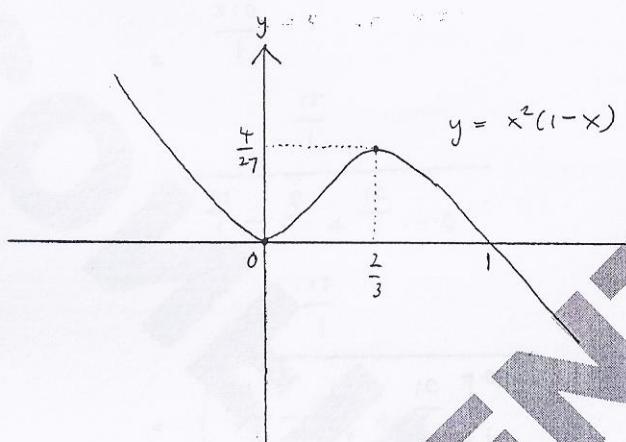
$(0,0)$ is a minimum point and $\left(\frac{2}{3}, \frac{4}{27}\right)$ is a maximum point.

when $x = 0 : y = 0$

when $y = 0 : x^2(1-x) = 0$

$$x = 0, 1 \text{ or } 1-x = 0$$

$$y = 0, x = 0 \text{ or } x = 1$$



The area A of the finite region bounded by the x -axis and the curve is

$$\int_0^1 x^2(1-x) dx$$

$$= \int_0^1 x^2 - x^3 dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4} - 0 \right)$$

$$= \frac{1}{12}$$

If the centroid has coordinates (\bar{x}, \bar{y}) ,

$$\bar{x} = \frac{\int_0^1 xy dx}{\int_0^1 y dx}$$

$$\frac{1}{12}$$

$$= \frac{\int_0^1 x^3(1-x) dx}{\int_0^1 y dx}$$

$$\frac{1}{12}$$

$$= \frac{\int_0^1 x^3 - x^4 dx}{\int_0^1 y dx}$$

$$\frac{1}{12}$$

$$= \frac{\left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1}{\int_0^1 y dx}$$

$$\frac{1}{12}$$

$$= \frac{\frac{1}{4} - \frac{1}{5}}{\int_0^1 y dx}$$

$$\frac{1}{12}$$

$$= \frac{\frac{1}{20}}{\int_0^1 y dx}$$

$$\frac{1}{12}$$

$$= \frac{\frac{1}{20}}{\frac{1}{12}}$$

$$\frac{3}{5}$$

$$\text{and } \bar{y} = \frac{\int_0^1 \frac{y^2}{2} dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{x^4}{2}(1-x)^2 dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{x^4}{2}(x^2 - 2x + 1) dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{x^6}{2} - x^5 + \frac{x^4}{2} dx}{\frac{1}{12}}$$

$$= \left[\frac{x^7}{14} - \frac{x^6}{6} + \frac{x^5}{10} \right]_0^1$$

$$= \frac{\frac{1}{14} - \frac{1}{6} + \frac{1}{10} - 0}{\frac{1}{12}}$$

$$= \frac{\frac{1}{210}}{\frac{1}{12}}$$

$$= \frac{12}{210}$$

$$= \frac{2}{35}$$

\therefore The coordinates of the centroid are $\left(\frac{3}{5}, \frac{2}{35}\right)$.

$$y = x(1-x)^2$$

Using the substitution $x = 1-u$

$$\begin{aligned} y &= (1-u)[1-(1-u)]^2 \\ &= (1-u)(1-1+u)^2 \\ &= u^2(1-u) \end{aligned}$$

\therefore If the centroid of the finite region bounded by the x -axis and the curve whose equation is

$$y = x(1-x)^2 \text{ has coordinates } (\bar{x}_1, \bar{y}_1)$$

$$\bar{x}_1 = 1-\bar{x}$$

$$= 1 - \frac{3}{5}$$

$$= \frac{2}{5}$$

$$\begin{aligned} \text{and } \bar{y}_1 &= \bar{y} \\ &= \frac{2}{35}. \end{aligned}$$

\therefore The coordinates of the centroid are $\left(\frac{2}{5}, \frac{2}{35}\right)$.

$$y = x(1-x)^2$$

$$= x(1-2x+x^2)$$

$$= x - 2x^2 + x^3$$

$$\frac{dy}{dx} = 1 - 4x + 3x^2$$

If $\frac{dy}{dx} = 0$ $3x^2 - 4x + 1 = 0$
 $(3x-1)(x-1) = 0$

$$x = 1, \frac{1}{3}$$

$$y = 0, \frac{4}{27}$$

The stationary points of $y = x(1-x)^2$ are $(1, 0)$

and $(\frac{1}{3}, \frac{4}{27})$.

$$\frac{d^2y}{dx^2} = 6x - 4$$

$$x = 1 : \frac{d^2y}{dx^2} = 2 > 0$$

$$x = \frac{1}{3} : \frac{d^2y}{dx^2} = -2 < 0$$

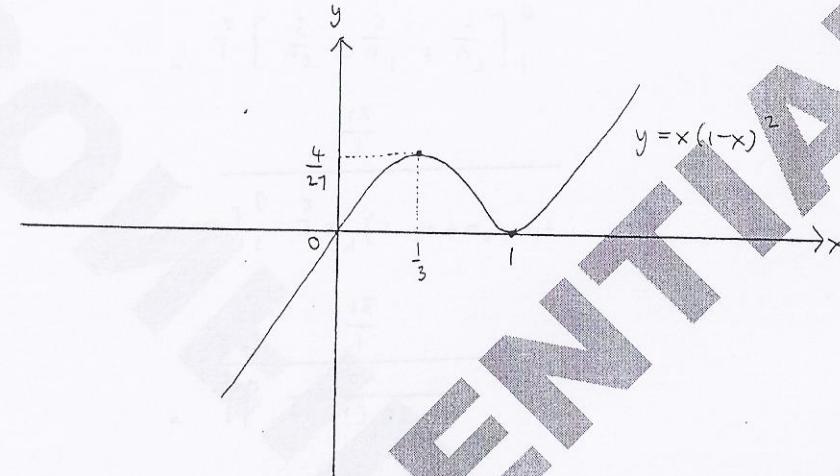
$(1, 0)$ is a minimum point and $(\frac{1}{3}, \frac{4}{27})$

is a maximum point.

when $x = 0 : y = 0$

when $y = 0 : x(1-x)^2 = 0$

$$x = 0, 1$$



The area of the finite region bounded by the x -axis and the curve is

$$\int_0^1 x(1-x)^2 dx$$

$$= \int_0^1 x - 2x^2 + x^3 dx$$

$$= \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} - 0$$

$$= \frac{1}{12}$$

If the centroid has coordinates (\bar{x}_1, \bar{y}_1) ,

$$\bar{x}_1 = \frac{\int_0^1 xy \, dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 x^2(1-x)^2 \, dx}{\frac{1}{12}}$$

$$= \frac{\int_0^1 x^2 - 2x^3 + x^4 \, dx}{\frac{1}{12}}$$

$$= \frac{\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1}{\frac{1}{12}}$$

$$= \frac{\frac{1}{3} - \frac{1}{2} + \frac{1}{5}}{\frac{1}{12}} = \frac{1}{30}$$

$$= \frac{1}{30} \cdot \frac{1}{12} = \frac{1}{360}$$

$$= \frac{2}{5}$$

and $\bar{y}_1 = \frac{\int_0^1 \frac{y^2}{2} \, dx}{\frac{1}{12}}$

$$= \frac{\int_0^1 \frac{x^2 (1-x)^4}{2} \, dx}{\frac{1}{12}}$$

$u = 1-x$
 $du = -dx$
 $x = 1 \quad u = 0$
 $x = 0 \quad u = 1$

$$= \frac{\int_1^0 \frac{(1-u)^2 u^4 (-du)}{2}}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{(1-u)^2 u^4}{2} du}{\frac{1}{12}}$$

$$= \frac{\int_0^1 \frac{1}{2} u^4 (1-2u+u^2) du}{\frac{1}{12}}$$

$$= \frac{\frac{1}{2} \left[\frac{u^5}{5} - \frac{u^6}{3} + \frac{u^7}{7} \right]_0^1}{\frac{1}{12}}$$

$$= \frac{1}{2} \left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7} - 0 \right)$$

$$\frac{1}{12}$$

$$= \frac{1}{2} \left(\frac{1}{105} \right)$$

$$\frac{1}{12}$$

$$= \frac{1}{210}$$

$$\frac{1}{12}$$

$$= \frac{12}{210}$$

$$= \frac{2}{35}$$

\therefore The coordinates of the centroid are $\left(\frac{2}{5}, \frac{2}{35}\right)$

$$9. \quad \Pi_1 : r = \lambda_1(i + j - k) + m_1(2i - j + k)$$

$$\Pi_2 : r = \lambda_2(i + 2j + k) + m_2(3i + j - k)$$

Expressing Π_1 and Π_2 in Cartesian form:

$$\Pi_1 : r = \lambda_1(i + j - k) + m_1(2i - j + k)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + m_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 + 2m_1 \\ \lambda_1 - m_1 \\ -\lambda_1 + m_1 \end{pmatrix}$$

$$x = \lambda_1 + 2m_1, y = \lambda_1 - m_1, z = -\lambda_1 + m_1$$

$$\lambda_1 + 2m_1 = x$$

$$\lambda_1 - m_1 = y$$

$$-\lambda_1 + m_1 = z$$

$$\textcircled{2} + \textcircled{3} \quad y + z = 0.$$

The equation of Π_1 in Cartesian form is

$$y + z = 0.$$

$$\Pi_2 : r = \lambda_2(i + 2j + k) + m_2(3i + j - k)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + m_2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_2 + 3m_2 \\ 2\lambda_2 + m_2 \\ \lambda_2 - m_2 \end{pmatrix}$$

$$x = \lambda_2 + 3M_2, \quad y = 2\lambda_2 + M_2, \quad z = \lambda_2 - M_2$$

$$\begin{aligned} \lambda_2 + 3M_2 &= x \\ 2\lambda_2 + M_2 &= y \\ \lambda_2 - M_2 &= z \end{aligned}$$

$$-2 \times ① + ②: \quad \lambda_2 + 3M_2 = x$$

$$\begin{aligned} -① + ③: \quad -5M_2 &= y - 2x \\ &\quad \left. \begin{aligned} -4M_2 &= z - x \end{aligned} \right\} \end{aligned}$$

$$-5M_2 = y - 2x$$

$$M_2 = \frac{y - 2x}{-5}$$

$$-4M_2 = z - x$$

$$M_2 = \frac{z - x}{-4}$$

$$\therefore \frac{y - 2x}{-5} = \frac{z - x}{-4}$$

$$-4(y - 2x) = -5(z - x)$$

$$-4y + 8x = -5z + 5x$$

$$3x - 4y + 5z = 0.$$

The equation of Π_2 in Cartesian form is

$$3x - 4y + 5z = 0.$$

Since l is parallel to both Π_1 and Π_2 , l must be perpendicular to the normals of both Π_1 and Π_2 . \therefore If \tilde{n}_1 and \tilde{n}_2 are the normal vectors of Π_1 and Π_2 , l must be parallel to $\tilde{n}_1 \times \tilde{n}_2$.

$$\tilde{n}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \tilde{n}_2 = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

$$\tilde{n}_1 \times \tilde{n}_2 = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 3 & -4 & 5 \end{vmatrix}$$

$$= \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix}$$

$$= 3 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

Since l is parallel to $3\hat{i} + \hat{j} - \hat{k}$ and passes through the point with position vector $4\hat{i} + 5\hat{j} + 6\hat{k}$, a vector equation for l is

$$\tilde{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \lambda(3\hat{i} + \hat{j} - \hat{k}).$$

If m is the line of intersection of Π_1 and Π_2 , since the planes intersect in m , m must be perpendicular to \tilde{n}_1 and \tilde{n}_2 . \therefore The direction vector of m is parallel to $\tilde{n}_1 \times \tilde{n}_2$ since m lies in both planes.

$$\tilde{n}_1 \times \tilde{n}_2 = \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix}$$

$$= 3 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

Since m lies in both planes, a point on the line m satisfies the equations $y+z=0$ and

$$3x - 4y + 5z = 0.$$

$$x=0 : \quad y+z=0 \\ y = -z \quad \text{---(1)}$$

$$3x - 4y + 5z = 0 \\ -4y + 5z = 0 \quad \text{---(2)}$$

$$4 \times \text{(1)} + \text{(2)} : \quad 4y - 4y + 5z = -4z \\ 5z = -4z \\ 9z = 0 \\ z = 0 \\ y = 0$$

$\therefore (0, 0, 0)$ is a point on m .

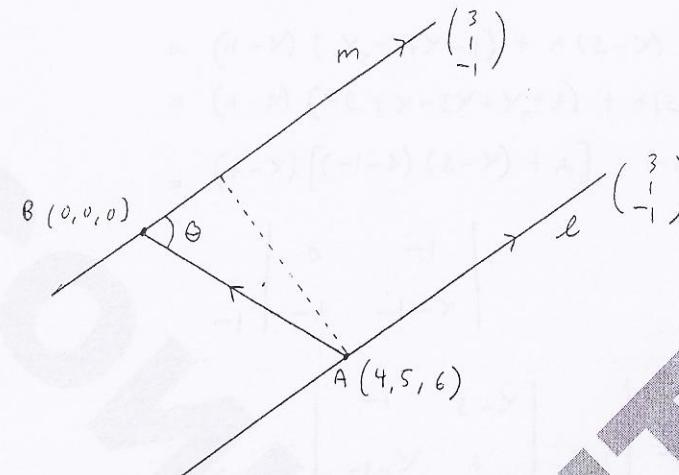
Since m is parallel to $3\hat{i} + \hat{j} - \hat{k}$ and $(0, 0, 0)$ is a point on m , a vector equation for m

$$\begin{aligned} \vec{r} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \\ &= s(3\hat{i} + \hat{j} - \hat{k}) \end{aligned}$$

If A and B are the points with position vectors $4\hat{i} + 5\hat{j} + 6\hat{k}$ on l and $0\hat{i} + 0\hat{j} + 0\hat{k}$ on m

$$\vec{AB} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -5 \\ -6 \end{pmatrix}$$



The shortest distance between l and m is

$$|\vec{AB}| \sin \theta.$$

$$|\vec{AB} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}| = |\vec{AB}| \left| \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \right| \sin \theta$$

$$\begin{vmatrix} \begin{pmatrix} -4 \\ -5 \\ -6 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \end{vmatrix} = |\vec{AB}| \sqrt{9+1+1} \sin \theta \\ = |\vec{AB}| \sqrt{11} \sin \theta$$

$$|\vec{AB}| \sqrt{11} \sin \theta = \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -5 & -6 \\ 3 & 1 & -1 \end{vmatrix} \right|$$

$$= \left| 11\hat{i} - 22\hat{j} + 11\hat{k} \right|$$

$$= 11 \left| \hat{i} - 2\hat{j} + \hat{k} \right|$$

$$= 11 \sqrt{1+4+1}$$

$$= 11\sqrt{6}$$

$$\therefore |\overrightarrow{AB}| \sin \theta = \frac{11\sqrt{6}}{\sqrt{11}} \\ = \sqrt{66}.$$

\therefore The shortest distance between l and m is $\sqrt{66}$.

$$10. \quad M = \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix}$$

$$M - \lambda I = \begin{pmatrix} 4-\lambda & 1 & -1 \\ -4 & -1-\lambda & 4 \\ 0 & -1 & 5-\lambda \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1-\lambda & 4 \\ 0 & -1 & 5-\lambda \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & 1 & -1 \\ -4 & -1-\lambda & 4 \\ 0 & -1 & 5-\lambda \end{pmatrix}$$

$$|M - \lambda I| = \begin{vmatrix} 4-\lambda & 1 & -1 \\ -4 & -1-\lambda & 4 \\ 0 & -1 & 5-\lambda \end{vmatrix}$$

$$= (4-\lambda) \begin{vmatrix} -1-\lambda & 4 & -1 \\ -1 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

$$-1 \begin{vmatrix} -4 & -1-\lambda \\ 0 & -1 \end{vmatrix}$$

$$= (4-\lambda) [(-1-\lambda)(5-\lambda) + 4] - (-4)(5-\lambda) - (-4)(-1)$$

$$= (4-\lambda)(-5+\lambda-5\lambda+\lambda^2+4) + 4(5-\lambda) - 4$$

$$= (4-\lambda)(\lambda^2-4\lambda-1) + 4(5-\lambda) - 4$$

$$= 4x^2 - 16x - 4 - x^3 + 4x^2 + x + 20 - 4x - 4$$

$$= -x^3 + 8x^2 - 19x + 12$$

$$= -(x^3 - 8x^2 + 19x - 12)$$

$$\lambda = 1, 3, 4$$

∴ when $\lambda = 1$: $\begin{pmatrix} 3 & 1 & -1 \\ -4 & -2 & 4 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ -4 & -2 & 4 & 0 \\ 0 & -1 & 4 & 0 \end{array} \right)$$

$$\xrightarrow{+4 \times r_1} \left(\begin{array}{ccc|c} 12 & 4 & -4 & 0 \\ -12 & -6 & 12 & 0 \\ 0 & -1 & 4 & 0 \end{array} \right)$$

$$\xrightarrow{-3 \times r_2} \left(\begin{array}{ccc|c} 12 & 4 & -4 & 0 \\ 0 & -2 & 8 & 0 \\ 0 & -1 & 4 & 0 \end{array} \right)$$

$$\xrightarrow{\frac{r_1}{12}, \frac{r_2}{-2}} \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & 4 & 0 \end{array} \right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{Let } z = s, s \in \mathbb{R} \quad 0z = 0$$

$$\text{Let } z = s, s \in \mathbb{R} \quad \text{Let } z = s, s \in \mathbb{R}$$

$$y - 4z = 0$$

$$y = 4s$$

$$3x + y - 2z = 0$$

$$3x + 4s - 2z = 0$$

$$x = -s$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 4s \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

$$\text{when } \lambda = 3:$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -4 & -4 & 4 & 0 \\ 0 & -1 & 2 & 0 \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{+4r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array} \right)$$

$$\xrightarrow{4r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array} \right)$$

$$-y + 2z = 0$$

$$\text{Let } z = s, s \in \mathbb{R}$$

$$-y + 2s = 0$$

$$y = 2s$$

$$x + y - 2z = 0$$

$$x + 2s - 2s = 0$$

$$x = -s$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 2s \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

when $\lambda = 4$:

$$\begin{pmatrix} 0 & 1 & -1 \\ -4 & -5 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -4 & -5 & 4 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} -4 & -5 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} -4 & -5 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$y - z = 0$$

Let $z = s, s \in \mathbb{R}$

$$y - s = 0$$

$$y = s$$

$$-4x - 5y + 4z = 0$$

$$-4x - 5s + 4s = 0$$

$$-4x = s$$

$$x = -\frac{s}{4}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} \\ s \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} -\frac{1}{4} \\ 1 \\ 1 \end{pmatrix}$$

\therefore A set of eigenvectors for the eigenvalues 1, 3, 4

$$\text{are } \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$$

If $M^n = PDP^{-1}$ where D is a diagonal matrix and n is a positive integer

$$\therefore P = \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}$$

$$\text{and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}^n$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}.$$

$$|P| = -1 \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} - (-1) \begin{vmatrix} 4 & 4 \\ 1 & 4 \end{vmatrix} - 1 \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= - (8 - 4) + 16 - 4 - (4 - 2)$$

$$= -4 + 12 - 2$$

$$= 6$$

$$\text{adj } P = \begin{pmatrix} 4 & -12 & 2 \\ 3 & -3 & 0 \\ -2 & 0 & 2 \end{pmatrix}^T$$

$$= \begin{pmatrix} 4 & -3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\therefore P^{-1} = \frac{1}{|P|} \text{adj } P$$

$$= \frac{1}{6} \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\lim_{n \rightarrow \infty} 4^{-n} M^n = \lim_{n \rightarrow \infty} 4^{-n} P P^{-1} P P^{-1} P P^{-1} \dots$$

$$= \frac{1}{4^n} \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix}$$

$$= \frac{1}{4^n} \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix} \frac{1}{6} \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{4^n 6} \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix} \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix} \frac{1}{4^n} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix} \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix} \left(\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \left(\frac{3}{4}\right)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\therefore \lim_{n \rightarrow \infty} 4^{-n} M^n = \lim_{n \rightarrow \infty} 4^{-n} P P^{-1}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{6} \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} \left(\frac{1}{4}\right)^n & 0 & 0 \\ 0 & \left(\frac{3}{4}\right)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix} \lim_{n \rightarrow \infty} \begin{pmatrix} \left(\frac{1}{4}\right)^n & 0 & 0 \\ 0 & \left(\frac{3}{4}\right)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} -1 & -1 & -1 \\ 4 & 2 & 4 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 4 & 3 & -2 \\ -12 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} -2 & 0 & -2 \\ 8 & 0 & 8 \\ 8 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \end{pmatrix}$$

11.

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 4 & 3 & 5 & 16 \\ 6 & 6 & 13 & 13 \\ 14 & 12 & 23 & 45 \end{pmatrix}$$

$$\xrightarrow{-r_1 + 4r_2} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & -2 & -5 & 3 \end{pmatrix}$$

$$\xrightarrow{-2r_2 + r_4} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & -5 \end{pmatrix}$$

$$\xrightarrow{-r_2, -5r_3 + r_4} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank (A) = 3

If $A \underline{x} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix}$, where $\underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 4 & 3 & 5 & 16 & 2 \\ 6 & 6 & 13 & 13 & -1 \\ 14 & 12 & 23 & 45 & 3 \end{array} \right) \left(\begin{array}{c} x \\ y \\ z \\ w \end{array} \right) = \left(\begin{array}{c} 0 \\ 2 \\ -1 \\ 3 \end{array} \right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 4 & 3 & 5 & 16 & 2 \\ 6 & 6 & 13 & 13 & -1 \\ 14 & 12 & 23 & 45 & 3 \end{array} \right)$$

$$-4r_1 + r_2$$

$$-6r_1 + r_3$$

$$-14r_1 + r_4$$

$$\xrightarrow{-4r_1 + r_2} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & -1 & -3 & 4 & 2 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & -2 & -5 & 3 & 3 \end{array} \right)$$

$$-2r_2 + r_4$$

$$\xrightarrow{-2r_2 + r_4} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 4 & 2 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & 0 & 1 & -5 & -1 \end{array} \right)$$

$$-r_2, -r_3 + r_4$$

$$\xrightarrow{-r_2, -r_3 + r_4} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & -4 & -2 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$z - 5w = -1$$

$$\text{Let } w = s, s \in \mathbb{R}$$

$$z - 5s = -1$$

$$z = 5s - 1$$

$$y + 3z - 4w = -2$$

$$y + 15s - 3 - 4s = -2$$

$$y = 1 - 11s$$

$$x + y + 2z + 3w = 0$$

$$x + 1 - 11s + 10s - 2 + 3s = 0$$

$$x + 2s = -1$$

$$x = -2s - 1$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2s + 1 \\ -11s + 1 \\ 5s - 1 \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} -2 \\ -11 \\ 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

Aug = solution of the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix}$

can be expressed in the form $x_0 + \lambda e_n, \lambda \in \mathbb{R}$

$$\text{where } x_0 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \text{ and } e_n = \begin{pmatrix} -2 \\ -11 \\ 5 \\ 1 \end{pmatrix}$$

$$x = x_0 + \lambda e_n$$

$$= \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -11 \\ 5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2\lambda \\ 1-11\lambda \\ -1+5\lambda \\ \lambda \end{pmatrix}$$

If \tilde{x} has all it's elements positive,

$$1-2\lambda > 0, 1-11\lambda > 0, -1+5\lambda > 0, \lambda > 0$$

$$\lambda < \frac{1}{2}, \quad \lambda < \frac{1}{11}, \quad \lambda > \frac{1}{5}$$

Since if $\lambda < \frac{1}{11}$, and $\lambda > \frac{1}{5}$ is impossible

\therefore there is no vector which satisfies the equation $A\tilde{x} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix}$ and has all it's elements positive.

12.

$$(n + \frac{1}{2})^3 - (n - \frac{1}{2})^3$$

$$= n^3 + 3n^2(\frac{1}{2}) + 3n(\frac{1}{2})^2 + (\frac{1}{2})^3 - \left(n^3 + 3n^2(-\frac{1}{2}) + 3n(-\frac{1}{2})^2 + (-\frac{1}{2})^3 \right)$$

$$= n^3 + \frac{3n^2}{2} + \frac{3n}{4} + \frac{1}{8} - \left(n^3 - \frac{3n^2}{2} + \frac{3n^2}{4} - \frac{1}{8} \right)$$

$$= n^3 + \frac{3n^2}{2} + \frac{3n}{4} + \frac{1}{8} - n^3 + \frac{3n^2}{2} - \frac{3n^2}{4} + \frac{1}{8}$$

$$= 3n^2 + \frac{1}{4}$$

$$\text{Since } n^2 = \frac{1}{3} \left[(n + \frac{1}{2})^3 - (n - \frac{1}{2})^3 - \frac{1}{4} \right]$$

$$\therefore \sum_{n=1}^N n^2 = \sum_{n=1}^N \frac{1}{3} \left[(n + \frac{1}{2})^3 - (n - \frac{1}{2})^3 - \frac{1}{4} \right]$$

$$= \frac{1}{3} \sum_{n=1}^N (n + \frac{1}{2})^3 - (n - \frac{1}{2})^3 - \frac{1}{4}$$

$$= \frac{1}{3} \sum_{n=1}^N (n + \frac{1}{2})^3 - (n - \frac{1}{2})^3 + \frac{1}{3} \sum_{n=1}^N -\frac{1}{4}$$

$$= \frac{1}{3} \left[(1 + \frac{1}{2})^3 - (1 - \frac{1}{2})^3 \right]$$

$$+ (2 + \frac{1}{2})^3 - (2 - \frac{1}{2})^3$$

$$+ (3 + \frac{1}{2})^3 - (3 - \frac{1}{2})^3$$

$$+ \left(N - 1 + \frac{1}{2} \right)^3 - \left(N - 1 - \frac{1}{2} \right)^3$$

$$+ \left(N + \frac{1}{2} \right)^3 - \left(N - \frac{1}{2} \right)^3 \Big]$$

$$+ \frac{1}{3} \sum_{n=1}^N -\frac{1}{4}$$

$$= \frac{1}{3} \left[\left(\frac{3}{2} \right)^3 - \left(\frac{1}{2} \right)^3 \right]$$

$$+ \left(\frac{5}{2} \right)^3 - \left(\frac{3}{2} \right)^3$$

$$+ \left(\frac{7}{2} \right)^3 - \left(\frac{5}{2} \right)^3$$

$$+ \left(N - \frac{1}{2} \right)^3 - \left(N - \frac{3}{2} \right)^3 \\ + \left(N + \frac{1}{2} \right)^3 - \left(N - \frac{1}{2} \right)^3 \Big]$$

$$+ \frac{1}{3} \left(-\frac{N}{4} \right)$$

$$= \frac{1}{3} \left(-\frac{1}{8} + \left(N + \frac{1}{2} \right)^3 \right) + \frac{1}{3} \left(-\frac{N}{4} \right)$$

$$= \frac{1}{3} \left(-\frac{1}{8} + N^3 + 3N^2 \left(\frac{1}{2} \right) + 3N \left(\frac{1}{2} \right)^2 + \left(\frac{1}{2} \right)^3 \right) - \frac{N}{12}$$

$$= \frac{1}{3} \left(-\frac{1}{8} + N^3 + \frac{3N^2}{2} + \frac{3N}{4} + \frac{1}{8} \right) - \frac{N}{12}$$

$$= \frac{1}{3} \left(N^3 + \frac{3N^2}{2} + \frac{3N}{4} \right) - \frac{1}{3} \left(\frac{N}{4} \right)$$

$$= \frac{1}{3} \left(N^3 + \frac{3N^2}{2} + \frac{3N}{4} - \frac{N}{4} \right)$$

$$= \frac{N}{3} \left(N^2 + \frac{3N}{2} + \frac{3}{4} - \frac{1}{4} \right)$$

$$= \frac{N}{3} \left(N^2 + \frac{3N}{2} + \frac{1}{2} \right)$$

$$= \frac{N}{3} \left(\frac{2N^2 + 3N + 1}{2} \right)$$

$$= \frac{N}{6} (2N^2 + 3N + 1)$$

$$= \frac{N(N+1)(2N+1)}{6}$$

$$S = 1^2 + 2^2 + 3^2 + 4^2 + \dots + (2N)^2 + (2N+1)^2$$

$$T = 1^2 + 3^2 + 5^2 + 7^2 + \dots + (2N-1)^2 + (2N+1)^2$$

$$U = 1^2 - 2^2 + 3^2 - 4^2 + \dots - (2N)^2 + (2N+1)^2$$

$$S = 1^2 + 2^2 + 3^2 + 4^2 + \dots + (2N)^2 + (2N+1)^2$$

$$= \sum_{n=1}^{2N+1} n^2$$

$$= \frac{(2N+1)(2N+1+1)(2(2N+1)+1)}{6}$$

$$= \frac{(2N+1)(2N+2)(4N+2+1)}{6}$$

$$= \frac{2(2N+1)(N+1)(4N+3)}{6}$$

$$= \frac{(N+1)(2N+1)(4N+3)}{3}$$

$$T = 1^2 + 3^2 + 5^2 + 7^2 + \dots + (2N-1)^2 + (2N+1)^2$$

$$= \sum_{n=1}^{N+1} (2n-1)^2$$

$$= \sum_{n=1}^{N+1} (4n^2 - 4n + 1)$$

$$= 4 \sum_{n=1}^{N+1} n^2 - 4 \sum_{n=1}^{N+1} n + \sum_{n=1}^{N+1} 1$$

$$= \frac{4}{6} (N+1)(N+1+1)((2(N+1)+1)) - \frac{4}{2} (N+1)(N+1+1) + N+1$$

$$= \frac{2}{3} (N+1)(N+2)(2N+2+1) - 2(N+1)(N+2) + N+1$$

$$= \frac{2}{3} (N+1)(N+2)(2N+3) - 2(N+1)(N+2) + N+1$$

$$= \frac{2}{3} (N+1) \left[(N+2)(2N+3) - 3(N+2) + \frac{3}{2} \right]$$

$$= \frac{2(N+1)}{3} \left(2N^2 + 7N + 6 - 3N - 6 + \frac{3}{2} \right)$$

$$= \frac{2(N+1)}{3} \left(2N^2 + 4N + \frac{3}{2} \right)$$

$$= \frac{2(N+1)}{3} \frac{(4N^2 + 8N + 3)}{2}$$

$$= \frac{(N+1)(4N^2 + 8N + 3)}{3}$$

$$= \frac{(N+1)(2N+3)(2N+1)}{3}$$

$$U = 1^2 - 2^2 + 3^2 - 4^2 + \dots - (2N)^2 + (2N+1)^2$$

$$= 1^2 - 2^2 + 3^2 - 4^2 + \dots + (2N-1)^2 - (2N)^2 + (2N+1)^2$$

$$= \sum_{n=1}^N [(2n-1)^2 - (2n)^2] + (2N+1)^2$$

$$= \sum_{n=1}^N (4n^2 - 4n + 1 - 4n^2) + (2N+1)^2$$

$$= \sum_{n=1}^N (-4n+1) + (2N+1)^2$$

$$= -4 \sum_{n=1}^N n + \sum_{n=1}^N 1 + (2N+1)^2$$

$$= \frac{-4N(N+1)}{2} + N + (2N+1)^2$$

$$= -2N(N+1) + N + (2N+1)^2$$

$$= -2N^2 - 2N + N + 4N^2 + 4N + 1$$

$$= 2N^2 + 3N + 1$$

$$= (2N+1)(N+1)$$

$$\text{i) } \frac{S}{T} = \frac{(N+1)(2N+1)(4N+3)}{3}$$

$$= \frac{(N+1)(2N+3)(2N+1)}{3}$$

$$= \frac{(N+1)(2N+1)(4N+3)}{(N+1)(2N+3)(2N+1)}$$

$$= \frac{4N+3}{2N+3}$$

$$\lim_{n \rightarrow N \rightarrow \infty} \left(\frac{S}{T} \right) = \lim_{N \rightarrow \infty} \left(\frac{4N+3}{2N+3} \right)$$

$$= \lim_{N \rightarrow \infty} \left(\frac{\frac{4N+3}{N}}{\frac{2N+3}{N}} \right)$$

$$= \lim_{N \rightarrow \infty} \left(\frac{4 + \frac{3}{N}}{2 + \frac{3}{N}} \right)$$

$$= \frac{4 + 0}{2 + 0}$$

$$= \frac{4}{2}$$

$$= 2$$

$$\therefore \text{As } N \rightarrow \infty \quad \frac{S}{T} \rightarrow 2$$

$$\text{ii) } \frac{S}{U} = \frac{(N+1)(2N+1)(4N+3)}{3}$$

$$= \frac{(N+1)(2N+1)}{3(N+1)(2N+1)}$$

$$= \frac{4N+3}{3}$$

If $\frac{S}{U}$ is an integer, $\frac{4N+3}{3}$ is an integer.

Since $\frac{4N+3}{3} = \frac{4N}{3} + 1$ if $\frac{S}{U}$ is an integer

N is divisible by 3.

Let $N = 3k$, where k is an integer.

$$\begin{aligned} \frac{T}{U} &= \frac{(N+1)(2N+3)(2N+1)}{3} \\ &= \frac{(N+1)(2N+1)}{3(N+1)(2N+1)} \\ &= \frac{2N+3}{3} \\ &= \frac{2(3k)+3}{3} \end{aligned}$$

$$= \frac{6k+3}{3}$$

$$= 2k+1$$

Since k is an integer $\therefore 2k+1$ is an integer.

\therefore If $\frac{s}{u}$ is an integer then $\frac{T}{u}$ is an integer.

$$C_1: r = 4\cos\theta, C_2: r = 1 + \cos\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

i) If C_1 and C_2 meet,

$$4\cos\theta = 1 + \cos\theta$$

$$3\cos\theta = 1$$

$$\cos\theta = \frac{1}{3}$$

$\theta = d, -d$ where d is the acute angle

such that $\cos d = \frac{1}{3}$.

Substituting $\theta = d$ or $-d$ into $r = 4\cos\theta$ or $r = 1 + \cos\theta$

$$\therefore r = 4\cos\theta$$

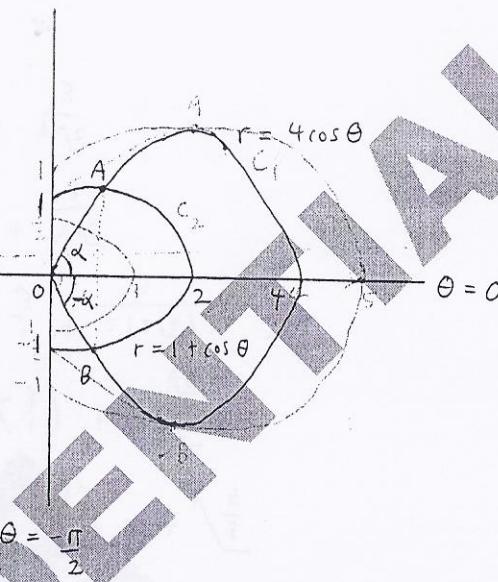
$$= \frac{4}{3}$$

$\therefore C_1$ and C_2 meet at the points $A\left(\frac{4}{3}, d\right)$ and

$$B\left(\frac{4}{3}, -d\right)$$

θ	$\frac{2\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$4\cos\theta$	0	2	$\frac{4}{\sqrt{2}}$	$2\sqrt{3}$	4	$2\sqrt{3}$	$\frac{4}{\sqrt{2}}$	2	0
$1 + \cos\theta$	1	$\frac{3}{2}$	$1 + \frac{1}{\sqrt{2}}$	$1 + \frac{\sqrt{3}}{2}$	2	$1 + \frac{\sqrt{3}}{2}$	$1 + \frac{1}{\sqrt{2}}$	$\frac{3}{2}$	1

$$\theta = \frac{\pi}{2}$$



iii) The area of the region bounded by the arcs OA and OB of C_1 and the arc AB of C_2 is

$$2 \left(\int_0^{\alpha} \frac{(1 + \cos \theta)^2}{2} d\theta + \int_{\alpha}^{\frac{\pi}{2}} \frac{(4 \cos \theta)^2}{2} d\theta \right)$$

$$= 2 \int_0^{\alpha} \frac{(1 + \cos \theta)^2}{2} d\theta + 2 \int_{\alpha}^{\frac{\pi}{2}} \frac{(4 \cos \theta)^2}{2} d\theta$$

$$= \int_0^{\alpha} (1 + \cos \theta)^2 d\theta + \int_{\alpha}^{\frac{\pi}{2}} (4 \cos \theta)^2 d\theta$$

$$= \int_0^{\alpha} 1 + 2\cos \theta + \cos^2 \theta d\theta + \int_{\alpha}^{\frac{\pi}{2}} 16 \cos^2 \theta d\theta$$

$$= \int_0^{\alpha} 1 + 2\cos \theta + \frac{\cos 2\theta + 1}{2} d\theta + \int_{\alpha}^{\frac{\pi}{2}} 16 \left(\frac{\cos 2\theta + 1}{2} \right) d\theta$$

$$= \int_0^{\alpha} 1 + 2\cos \theta + \frac{\cos 2\theta}{2} + \frac{1}{2} d\theta + \int_{\alpha}^{\frac{\pi}{2}} 8(\cos 2\theta + 1) d\theta$$

$$= \int_0^{\alpha} \frac{3}{2} + 2\cos \theta + \frac{\cos 2\theta}{2} d\theta + \int_{\alpha}^{\frac{\pi}{2}} 8\cos 2\theta + 8 d\theta$$

$$= \left[\frac{3\theta}{2} + 2\sin \theta + \frac{\sin 2\theta}{2(2)} \right]_0^{\alpha} + \left[\frac{8\sin 2\theta}{2} + 8\theta \right]_{\alpha}^{\frac{\pi}{2}}$$

$$= \left[\frac{3\theta}{2} + 2\sin \theta + \frac{2\sin \theta \cos \theta}{4} \right]_0^{\alpha} + \left[4\sin 2\theta + 8\theta \right]_{\alpha}^{\frac{\pi}{2}}$$

$$= \left[\frac{3\theta}{2} + 2\sin \theta + \frac{\sin \theta \cos \theta}{2} \right]_0^{\alpha} + \left[4(2\sin \theta \cos \theta) + 8\theta \right]_{\alpha}^{\frac{\pi}{2}}$$

$$= \left[\frac{3\theta}{2} + 2\sin \theta + \frac{\sin \theta \cos \theta}{2} \right]_0^{\alpha} + \left[8\sin \theta \cos \theta + 8\theta \right]_{\alpha}^{\frac{\pi}{2}}$$

$$= \left(\frac{3\alpha}{2} + 2\sin \alpha + \frac{\sin \alpha \cos \alpha}{2} \right) - 0$$

$$+ 8\sin \frac{\pi}{2} \cos \frac{\pi}{2} + 8\left(\frac{\pi}{2}\right) - (8\sin \alpha \cos \alpha + 8\alpha)$$

$$= \frac{3d}{2} + 2\sin d + \frac{\sin d \cos d}{2} + 4\pi - 8\sin d \cos d - 8d$$

$$= 4\pi + \frac{3d}{2} - 8d - \frac{15}{2} \sin d \cos d + 2\sin d$$

$$= 4\pi - \frac{13d}{2} - \frac{15}{2} \sin d \cos d + 2\sin d$$

$$= 4\pi - \frac{13d}{2} - \frac{15}{2} \sqrt{1 - \cos^2 d} \cos d + 2\sqrt{1 - \cos^2 d}$$

$$= 4\pi - \frac{13d}{2} - \frac{15}{2} \sqrt{1 - \left(\frac{1}{3}\right)^2} \left(\frac{1}{3}\right) + 2\sqrt{1 - \left(\frac{1}{3}\right)^2}$$

$$= 4\pi - \frac{13d}{2} - \frac{15}{2} \sqrt{1 - \frac{1}{9}} \left(\frac{1}{3}\right) + 2\sqrt{1 - \frac{1}{9}}$$

$$= 4\pi - \frac{13d}{2} - \frac{15}{2} \sqrt{\frac{8}{9}} \left(\frac{1}{3}\right) + 2\sqrt{\frac{8}{9}}$$

$$= 4\pi - \frac{13d}{2} - \frac{15}{2} \frac{\sqrt{8}}{\sqrt{9}} \left(\frac{1}{3}\right) + 2\frac{\sqrt{8}}{\sqrt{9}}$$

$$= 4\pi - \frac{13d}{2} - \frac{15}{2} \left(\frac{2\sqrt{2}}{3}\right) \frac{1}{3} + \frac{2(2\sqrt{2})}{3}$$

$$= 4\pi - \frac{13d}{2} - \frac{15\sqrt{2}}{3(3)} + \frac{4\sqrt{2}}{3}$$

$$= 4\pi - \frac{13d}{2} - \frac{5\sqrt{2}}{3} + \frac{4\sqrt{2}}{3}$$

$$= 4\pi - \frac{13d}{2} - \frac{\sqrt{2}}{3}$$