1. $f(n) = 2^{2n} - 1$,

Basis case $n=1$: $f(1) = 2^{2(1)} - 1 = 4 - 1 = 3$

$f(1)$ is divisible by 3.

$n=k$: Assume $f(k) = 2^{2k} - 1$ is divisible by 3.

$n=k+1$:

$$\begin{aligned} f(k+1) &= 2^{2(k+1)} - 1 \\ &= 2^{2k+2} - 1 \\ &= 4(2^{2k} - 1 + 1) - 1 \\ &= 4(2^{2k} - 1) + 4 - 1 \\ &= 4f(k) + 3 \end{aligned}$$

If $f(k)$ is divisible by 3, then $f(k+1)$ is also divisible by 3.

By induction $f(n) = 2^{2n} - 1$ is divisible by 3 for all positive integers n.

2. i) $16^2 + 25^2 + 34^2 + \cdots + (9n+7)^2$

$$= \sum_{r=1}^{n} (9r+7)^2$$

$$= \sum_{r=1}^{n} (81r^2 + 126r + 49)$$

$$= 81\sum_{r=1}^{n} r^2 + 126\sum_{r=1}^{n} r + 49\sum_{r=1}^{n} 1$$

$$= \frac{81n(n+1)(2n+1)}{6} + \frac{126n(n+1)}{2} + 49n$$

$$= \frac{27n(2n^2+3n+1)}{2} + \frac{126(n^2+n)}{2} + 49n$$

$$= \frac{54n^3 + 81n^2 + 27n + 126n^2 + 126n + 98n}{2}$$

$$= \frac{54n^3 + 207n^2 + 251n}{2}$$

$$= \frac{n}{2}(54n^2 + 207n + 251)$$

If $16^2 + 23^2 + 30^2 + \cdots + (9n+7)^2 > \frac{54n^3 + 201n^2 + 307n}{2}$,

$$\frac{54n^3 + 207n^2 + 251n}{2} > \frac{54n^3 + 201n^2 + 307n}{2}$$

$$3n^2 - 28n > 0$$

$$n(3n - 28) > 0$$

$$n > \frac{28}{3}$$

$\therefore$ The minimum value of n if

$16^2 + 23^2 + 30^2 + \cdots + (9n+7)^2 > \frac{54n^3 + 201n^2 + 307n}{2}$

is 10.

ii) $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\cdots+\frac{n}{(n+1)!}$

$$=\sum_{r=1}^{n}\frac{r}{(r+1)!}$$

$$=\sum_{r=1}^{n}\frac{r+1-1}{(r+1)!}$$

$$=\sum_{r=1}^{n}\left(\frac{r+1}{(r+1)!}-\frac{1}{(r+1)!}\right)$$

$$=\sum_{r=1}^{n}\left(\frac{r+1}{(r+1)r!}-\frac{1}{(r+1)!}\right)$$

$$=\sum_{r=1}^{n}\left(\frac{1}{r!}-\frac{1}{(r+1)!}\right)$$

$$=\frac{1}{1!}-\frac{1}{2!}$$

$$+\frac{1}{2!}-\frac{1}{3!}$$

$$+\frac{1}{3!}-\frac{1}{4!}$$

$$\vdots$$

$$+\frac{1}{(n-2)!}-\frac{1}{(n-1)!}$$

$$+\frac{1}{(n-1)!}-\frac{1}{n!}$$

$$+\frac{1}{n!}-\frac{1}{(n+1)!}=1-\frac{1}{(n+1)!}$$

3\. i) $l_1 : \underset{\sim}{r} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

ii) $l_2 : \underset{\sim}{r} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

iii) $\underset{\sim}{i} : 3 + \lambda = 2 + \mu \quad - (1)$

$\underset{\sim}{j} : 1 + 2\lambda = 5 - \mu \quad - (2)$

$\underset{\sim}{k} : -1 + 3\lambda = \mu \quad - (3)$

$(1) + (2) : 4 + 3\lambda = 7 \Rightarrow \lambda = 1$

$(1) : \mu = 1 + \lambda = 2$

$(3) : \text{LHS} = -1 + 3\lambda = -1 + 3 = 2 = \mu = \text{RHS}$

Consistent therefore lines l_1 and l_2 intersect.

The point of intersection is $(4, 3, 2)$.

iv) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$

$5x + 2y - 3z = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 20$

$\frac{x}{4} + \frac{y}{10} - \frac{3z}{20} = 1$

4.i) $C : y = \dfrac{2x^2 + 3x - 19}{x - 3}$

$$
\begin{array}{r|l}
 & 2x + 9 \\
\hline
x - 3 & 2x^2 + 3x - 19 \\
 & \underline{2x^2 - 6x} \\
 & \qquad 9x - 19 \\
 & \qquad \underline{9x - 27} \\
 & \qquad\qquad 8
\end{array}
$$

$$= 2x + 9 + \frac{8}{x-3}$$

As $x \longrightarrow \pm\infty$, $y \longrightarrow 2x + 9$

As $x \longrightarrow 3$, $y \longrightarrow \pm\infty$

$\therefore$ The asymptotes of C are $y = 2x + 9$ and $x = 3$.

ii) $\dfrac{dy}{dx} = 2 - \dfrac{8}{(x-3)^2}$

when $\dfrac{dy}{dx} = 0$:

$$2 - \frac{8}{(x-3)^2} \quad 0$$

$$\frac{8}{(x-3)^2} = 2$$

$$(x-3)^2 = 4$$

$$x - 3 = \pm 2$$

$x = 1, 5$

$y = 7, 23$

$\therefore$ The critical points of C are $(1,7)$ and $(5,23)$.

$$\frac{d^2y}{dx^2} = \frac{16}{(x-3)^3}$$

when $x = 1$: $\frac{d^2y}{dx^2} = -2 < 0$

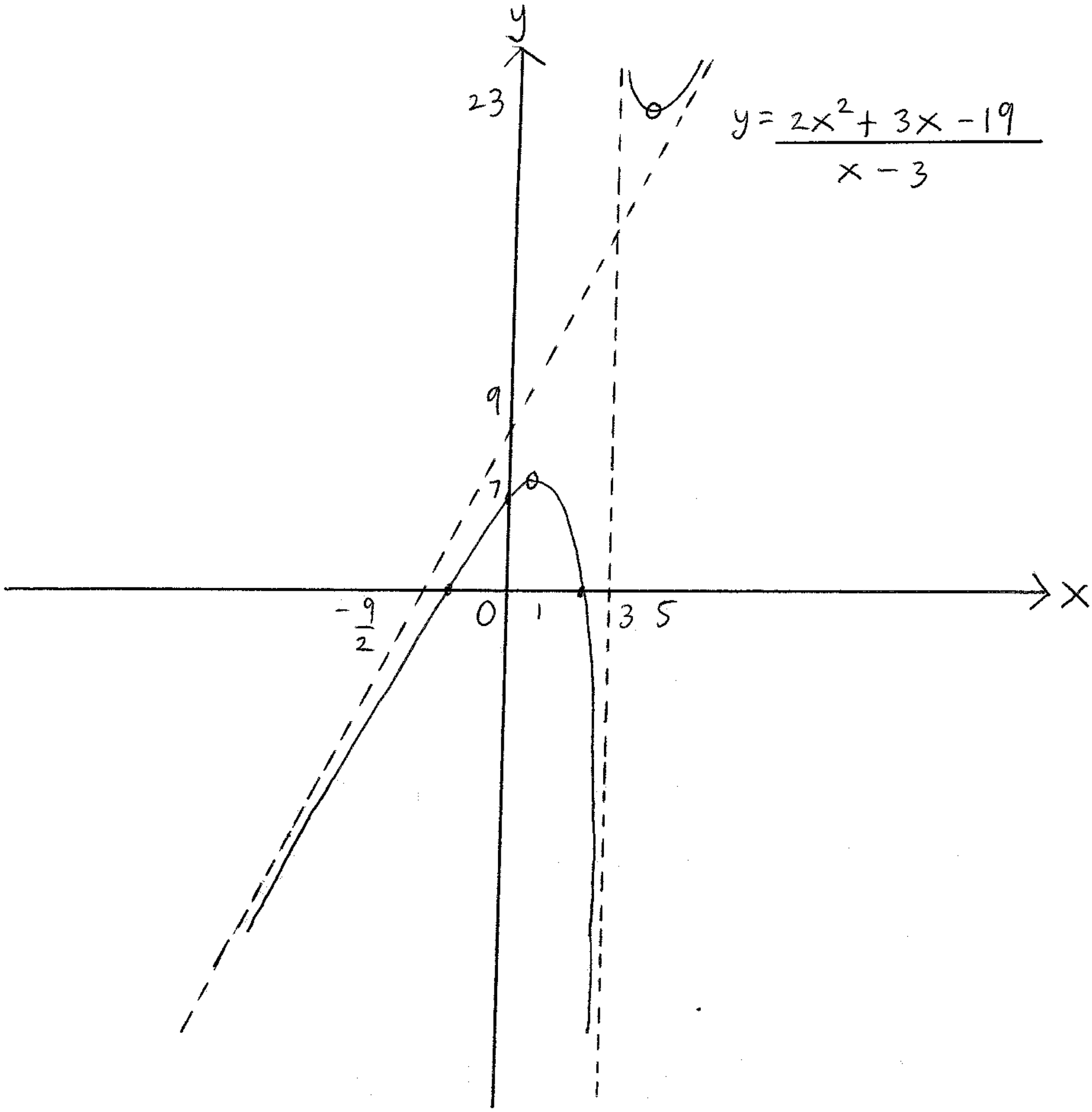
when $x = 5$: $\frac{d^2y}{dx^2} = 2 > 0$

$\therefore$ $(1,7)$ is a maximum point and $(5,23)$ is a minimum.

iii) when $x = 0$: $y = \frac{19}{3}$

when $y = 0$: $\frac{2x^2 + 3x - 19}{x-3} = 0$

$$x = \frac{-3 \pm \sqrt{161}}{4}$$



o : Critical points

• : Intersection points

5. $5x^4 - 9x^3 + 6 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots.

$\alpha^4, \beta^4, \gamma^4, \delta^4$

Let $u = \alpha^4$

$\alpha = \pm u^{\frac{1}{4}}$

α is a root

$\therefore 5\alpha^4 - 9\alpha^3 + 6 = 0$

$5(\pm u^{\frac{1}{4}})^4 - 9(\pm u^{\frac{1}{4}})^3 + 6 = 0$

$5u - 9(\pm u^{\frac{3}{4}}) + 6 = 0$

$9(\pm u^{\frac{3}{4}}) = 5u + 6$

$(9(\pm u^{\frac{3}{4}}))^4 = (5u + 6)^4$

$6561u^3 = 625u^4 + 3000u^3 + 5400u^2$
$+ 4320u + 1296$

$625u^4 - 3561u^3 + 5400u^2 + 4320u + 1296 = 0.$

$\therefore$ The equation
$625u^4 - 3561u^3 + 5400u^2 + 4320u + 1296 = 0$
has roots $\alpha^4, \beta^4, \gamma^4, \delta^4$.

$$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = \frac{3561}{625}$$

$$\alpha^4\beta^4 + \alpha^4\gamma^4 + \alpha^4\delta^4 + \beta^4\gamma^4 + \beta^4\delta^4 + \gamma^4\delta^4 = \frac{216}{25}$$

$$\alpha^4\beta^4\gamma^4 + \alpha^4\beta^4\delta^4 + \alpha^4\gamma^4\delta^4 + \beta^4\gamma^4\delta^4 = -\frac{864}{125}$$

$$\alpha^4\beta^4\gamma^4\delta^4 = \frac{1296}{625}$$

$$S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

$$S_4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 = \frac{3561}{625}$$

$$\begin{aligned} S_8 &= \alpha^8 + \beta^8 + \gamma^8 + \delta^8 \\ &= (\alpha^4 + \beta^4 + \gamma^4 + \delta^4)^2 \\ &\quad -2(\alpha^4\beta^4 + \alpha^4\gamma^4 + \alpha^4\delta^4 + \beta^4\gamma^4 + \beta^4\delta^4 + \gamma^4\delta^4) \\ &= \left(\frac{3561}{625}\right)^2 - 2\left(\frac{216}{25}\right) \\ &= \frac{5930721}{390625} \end{aligned}$$

$$S_{-4} = \alpha^{-4} + \beta^{-4} + \gamma^{-4} + \delta^{-4}$$

$$= \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}$$

$$= \frac{\alpha^4\beta^4\gamma^4 + \alpha^4\beta^4\delta^4 + \alpha^4\gamma^4\delta^4 + \beta^4\gamma^4\delta^4}{\alpha^4\beta^4\gamma^4\delta^4}$$

$$= \frac{\frac{-864}{125}}{\frac{1296}{625}}$$

$$= -\frac{10}{3}.$$

$$S_{-8} = \alpha^{-8} + \beta^{-8} + \gamma^{-8} + \delta^{-8}$$

$$= \frac{1}{\alpha^8} + \frac{1}{\beta^8} + \frac{1}{\gamma^8} + \frac{1}{\delta^8}$$

$$= \left(\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}\right)^2$$

$$-2\left(\frac{1}{\alpha^4\beta^4} + \frac{1}{\alpha^4\gamma^4} + \frac{1}{\alpha^4\delta^4} + \frac{1}{\beta^4\gamma^4} + \frac{1}{\beta^4\delta^4} + \frac{1}{\gamma^4\delta^4}\right)$$

$$= \left(\frac{\alpha^4\beta^4\gamma^4 + \alpha^4\beta^4\delta^4 + \alpha^4\gamma^4\delta^4 + \beta^4\gamma^4\delta^4}{\alpha^4\beta^4\gamma^4\delta^4}\right)^2$$

$$-2\left(\frac{\alpha^4\beta^4 + \alpha^4\gamma^4 + \alpha^4\delta^4 + \beta^4\gamma^4 + \beta^4\delta^4 + \gamma^4\delta^4}{\alpha^4\beta^4\gamma^4\delta^4}\right)$$

$$= \left(\frac{\frac{-864}{125}}{\frac{1296}{625}}\right)^2 - 2\left(\frac{\frac{216}{25}}{\frac{1296}{625}}\right)$$

$$= \left(\frac{-10}{3}\right)^2 - 2\left(\frac{25}{6}\right)$$

$$= \frac{100}{9} - \frac{25}{3}$$

$$= \frac{25}{9}$$

Alternatively,

$\frac{1}{\alpha^4}, \frac{1}{\beta^4}, \frac{1}{\gamma^4}, \frac{1}{\delta^4}$

Let $y = \frac{1}{u}$

$u = \frac{1}{y}$

$$625\left(\frac{1}{y}\right)^4 - 3561\left(\frac{1}{y}\right)^3 + 5400\left(\frac{1}{y}\right)^2 + 4320\left(\frac{1}{y}\right) + 1296 = 0$$

$$\frac{625}{y^4} - \frac{3561}{y^3} + \frac{5400}{y^2} + \frac{4320}{y} + 1296 = 0$$

$$625 - 3561y + 5400y^2 + 4320y^3 + 1296y^4 = 0$$

$\therefore$ The equation

$$1296y^4 + 4320y^3 + 5400y^2 - 3561y + 625 = 0$$

has roots $\frac{1}{\alpha^4}, \frac{1}{\beta^4}, \frac{1}{\gamma^4}, \frac{1}{\delta^4}$.

$$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} = \frac{-10}{3}$$

$$\frac{1}{\alpha^4\beta^4} + \frac{1}{\alpha^4\gamma^4} + \frac{1}{\alpha^4\delta^4} + \frac{1}{\beta^4\gamma^4} + \frac{1}{\beta^4\delta^4} + \frac{1}{\gamma^4\delta^4} = \frac{25}{6}$$

$$\frac{1}{\alpha^4\beta^4\gamma^4} + \frac{1}{\alpha^4\beta^4\delta^4} + \frac{1}{\alpha^4\gamma^4\delta^4} + \frac{1}{\beta^4\gamma^4\delta^4} = \frac{1187}{432}$$

$$\frac{1}{\alpha^4\beta^4\gamma^4\delta^4} = \frac{625}{1296}.$$

$$S_{-4} = \alpha^{-4} + \beta^{-4} + \gamma^{-4} + \delta^{-4}$$

$$= \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}$$

$$= -\frac{10}{3}$$

$$S_{-8} = \alpha^{-8} + \beta^{-8} + \gamma^{-8} + \delta^{-8}$$

$$= \frac{1}{\alpha^8} + \frac{1}{\beta^8} + \frac{1}{\gamma^8} + \frac{1}{\delta^8}$$

$$= \left(\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}\right)^2$$

$$-2\left(\frac{1}{\alpha^4\beta^4} + \frac{1}{\alpha^4\gamma^4} + \frac{1}{\alpha^4\delta^4} + \frac{1}{\beta^4\gamma^4} + \frac{1}{\beta^4\delta^4} + \frac{1}{\gamma^4\delta^4}\right)$$

$$= \left(\frac{-10}{3}\right)^2 - 2\left(\frac{25}{6}\right)$$

$$= \frac{25}{9}$$