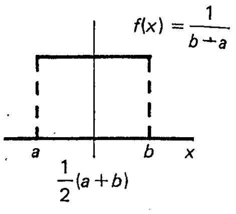
IN A-LEVEL STATISTICS

of a rectangular



$^2 + ab + a^2)$

$\frac{a^2 + 2ab + b^2)}{4}$

$^2) - 3(a^2 + 2ab + b^2)\}$

$)$

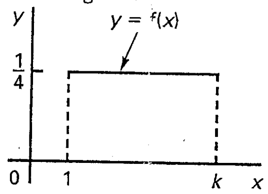
$b - a)^2.$

Exercise 6f

1. If the continuous r.v. X is such that $X \sim R(3, 6)$ find (*a*) the p.d.f. of X, (*b*) $E(X)$, (*c*) $\text{Var}(X)$, (*d*) $P(X > 5)$.

2. If the continuous r.v. X has p.d.f. $f(x)$ where $f(x) = k$ and $X \sim R(-5, -2)$ find (*a*) the value of the constant k, (*b*) $P(-4.3 < X < -2.8)$, (*c*) $E(X)$, (*d*) $\text{Var}(X)$.

3. The continuous r.v. X has p.d.f. $f(x)$ as shown in the diagram:



Find (*a*) the value of k,
(*b*) $P(2.1 < X < 3.4)$,
(*c*) $E(X)$,
(*d*) $\text{Var}(X)$.

4. The random variable X has probability density function given by

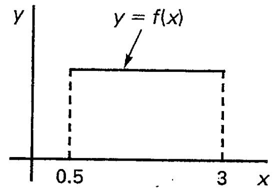
$$f(x) = \begin{cases} \dfrac{1}{(b-a)} & a \leqslant x \leqslant b \quad \text{where } b > a \\ 0 & \text{otherwise} \end{cases}$$

Show that the mean is $(b + a)/2$, and the variance is $(b - a)^2/12$ for this distribution.

Given that the mean equals 1 and the variance equals 4/3 find
(i) $P(X < 0)$,
(ii) the value of z such that

$$P(X > z + \sigma_x) = \tfrac{1}{4},$$

where σ_x is the standard deviation of X.
(AEB)

5. 

The random variable X has p.d.f. $f(x)$ as shown in the diagram.
If two independent observations of X are made, find the probability that one is less than 1.5 and the other is greater than the mean.

6. The random variable Y has probability density function given by

$$f(y) = \begin{cases} \dfrac{1}{5} & 32 \leqslant y \leqslant 37 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that Y lies within one standard deviation of the mean.

THE EXPONENTIAL DISTRIBUTION

A continuous r.v. X having p.d.f. $f(x)$ where

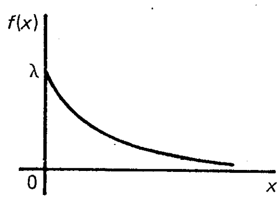
$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \geqslant 0,$$

where λ is a positive constant, is said to follow an **exponential distribution**.

NOTE: this distribution is also known as the negative exponential distribution.

The parameter of the distribution is λ.

The graph of $y = f(x)$ is as shown:



To show that X is the random variable, we need to show that the total area under the curve is 1.

Now
$$\int_{\text{all } x} f(x)\,\mathrm{d}x = \int_0^\infty \lambda \mathrm{e}^{-\lambda x}\,\mathrm{d}x$$
$$= -[\mathrm{e}^{-\lambda x}]_0^\infty$$
$$= -\mathrm{e}^{-\infty} + \mathrm{e}^0$$

But $\mathrm{e}^0 = 1$

and $\mathrm{e}^{-\infty} = \dfrac{1}{\mathrm{e}^\infty} = 0$ (try e^{-100}, e^{-270}, e^{-648}, etc. on your calculator)

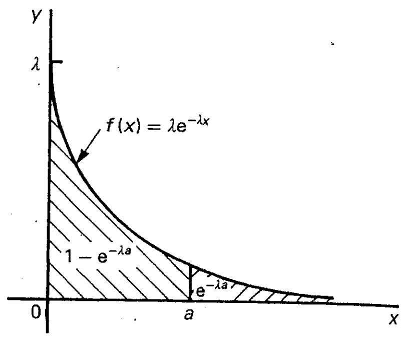
So
$$\int_0^\infty f(x)\,\mathrm{d}x = 0 + 1 = 1$$

Therefore X is a random variable.

The following results are useful:

$$P(X < a) = \int_0^a \lambda \mathrm{e}^{-\lambda x}\,\mathrm{d}x$$
$$= \left[-\mathrm{e}^{-\lambda x}\right]_0^a$$
$$= -\mathrm{e}^{-\lambda a} + \mathrm{e}^0$$
$$= 1 - \mathrm{e}^{-\lambda a}$$

and so $P(X > a) = 1 - (1 - \mathrm{e}^{-\lambda a})$
$$= \mathrm{e}^{-\lambda a}$$



$$P(X < a) = 1 - \mathrm{e}^{-\lambda a}$$
$$P(X > a) = \mathrm{e}^{-\lambda a}$$

It follows that the cumulative distribution function, $F(x)$ is given by

$$F(x) = P(X \leqslant x)$$
$$= 1 - \mathrm{e}^{-\lambda x} \quad \text{for } x \geqslant 0$$

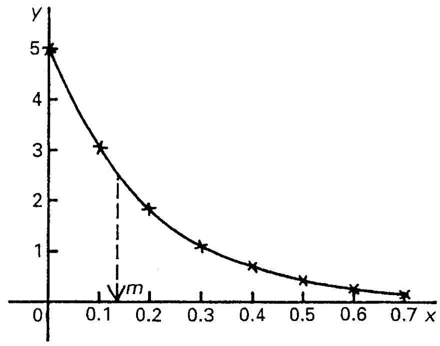
Example 6.27 If $f(x) = 5\mathrm{e}^{-5x}$ where $x \geqslant 0$, find the median value of x.

Solution 6.27 *NOTE*: it is a good idea to draw this graph accurately, particularly if you are not familiar with the exponential function.

For example, to work out the value of $f(x)$ when $x = 0.1$, key in

[0.1] [×] [5] [+/−] [e^x] [×] [5] [=]

This gives the value 3.0326...



Let the median be m, so

$$F(m) = 0.5$$

Since $F(x) = 1 - e^{-\lambda x}$,

where $\lambda = 5$,

we have $F(m) = 1 - e^{-5m}$

$$\text{so} \quad 1 - e^{-5m} = 0.5$$

$$e^{-5m} = 0.5$$

$$\frac{1}{e^{5m}} = \frac{1}{2}$$

$$\text{Therefore} \quad e^{5m} = 2$$

$$5m = \ln 2$$

$$m = \tfrac{1}{5}\ln 2$$

$$\underline{m = 0.139 \text{ (2 d.p.)}}$$

So *half* the total area under the curve lies between $x = 0$ and $x = 0.139$.

Example 6.28 The time, t seconds, between the arrivals of successive vehicles at a particular junction has p.d.f. $f(t) = 0.025\,e^{-0.025t}$, $t \geqslant 0$. A pedestrian, who takes 20 seconds to cross the road, sets off as one vehicle passes. Find the probability that she will complete the crossing before the next vehicle arrives. If she follows the same procedure on the return journey, find the probability that she completes each crossing without a vehicle arriving.

Solution 6.28 For the exponential distribution $f(t) = \lambda e^{-\lambda t}$,

$$P(T > t) = e^{-\lambda t}$$

Now we are given $f(t) = 0.025\,e^{-0.025t}$. This is an exponential distribution with $\lambda = 0.025$.

She will complete the crossing before the next vehicle arrives if $T > 20$.

$$P(T > 20) = e^{-0.025(20)}$$
$$= e^{-0.5}$$
$$= 0.6065\ldots$$

<u>So the probability that she completes the crossing before the next vehicle arrives is 0.607 (3 d.p.)</u>

Now $$P(T_1 > 20) \times P(T_2 > 20) = e^{-0.5} \times e^{-0.5}$$
$$= e^{-1}$$
$$= 0.3670\ldots$$

So the probability that she completes each crossing without a vehicle arriving whilst she is crossing is 0.367 (3 d.p.)

The following result is also useful:

If the continuous r.v. X has p.d.f. $f(x) = \lambda e^{-\lambda x}$, $x \geqslant 0$, then

$$P[(X > a + b) \mid (X > a)] = P(X > b)$$

Now $$P(X > a) = e^{-\lambda a}$$

and $$P[(X > a + b) \mid (X > a)] = \frac{P[(X > a + b) \cap (X > a)]}{P(X > a)}$$
$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$
$$= e^{-\lambda b}$$
$$= P(X > b)$$

So, for example

$$P[(X > 75) \mid (X > 30)] = P(X > 45)$$

and $$P[(X > 1200) \mid (X > 850)] = P(X > 350)$$

Also $$P[(X > q) \mid (X > p)] = P(X > (q - p)) \quad \text{where } q > p.$$

Example 6.29 The lifetime, in years, of a television tube of a certain make is a random variable T and its probability density function $f(t)$ is given by

$$f(t) = 0.25\,e^{-0.25t} \quad \text{for } 0 \leqslant t \leqslant \infty$$

If a tube lasts longer than 2 years, find the probability that it will last longer than 5 years.

Solution 6.29 Now $$P[(T > 5) \mid (T > 2)] = P(T > 3)$$
$$= e^{-0.25(3)}$$
$$= e^{-0.75}$$
$$= 0.4723\ldots$$

Given that the tube lasts longer than 2 years, the probability that it will last longer than 5 years is 0.472 (3 d.p.).

EXPECTATION AND VARIANCE

If X has p.d.f. given by $f(x) = \lambda e^{-\lambda x}$, $x \geqslant 0$, then

$$E(X) = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

The proof of this is on page 735 in Appendix 2.

Example 6.30 The lifetime of a particular type of light bulb has a negative exponential distribution with mean lifetime 1000 hours.

(a) Find the probability that a bulb is still working after 1300 hours.

(b) Given that it is still working after 1300 hours, find the probability that it is still working after 1500 hours.

(c) Find the standard deviation of the lifetime of this type of light bulb.

Solution 6.30 Let X be the r.v. 'the lifetime of a light bulb in hours',

Then $\quad f(x) = \lambda e^{-\lambda x} \; x \geqslant 0.$

Now $\quad E(X) = \frac{1}{\lambda}$

But $\quad E(X) = 1000$, therefore $\frac{1}{\lambda} = 1000$

$$\lambda = 0.001$$

So $\quad f(x) = 0.001e^{-0.001x}$

(a)
$$\begin{aligned} P(X > x) &= e^{-\lambda x} \quad \text{(see p. 354)} \\ P(X > 1300) &= e^{-0.001(1300)} \\ &= e^{-1.3} \\ &= 0.273 \quad (3 \text{ S.F.}) \end{aligned}$$

The probability that a bulb is still working after 1300 hours is 0.273 (3 S.F.).

(b)
$$\begin{aligned} P[(X > 1500) \mid (X > 1300)] &= P(X > 200) \quad \text{(see p. 356)} \\ &= e^{-0.001(200)} \\ &= e^{-0.2} \\ &= 0.819 \quad (3 \text{ S.F.}) \end{aligned}$$

The probability that the bulb is still working after 1500 hours, given that it is still working after 1300 hours, is 0.819 (3 S.F.).

(c) $$\text{Var}(X) = \frac{1}{\lambda^2}$$

so $$\text{standard deviation} = \sqrt{1/\lambda^2}$$
$$= \frac{1}{\lambda}$$

Standard deviation $= 1000$ hours

NOTE: For the exponential distribution,
standard deviation of X is equal to the mean of X.

Example 6.31 The continuous random variable X has the negative exponential distribution whose probability density function is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geqslant 0$$
$$f(x) = 0, \quad \text{otherwise}$$

where λ is a positive constant. Obtain expressions, in terms of λ, for

(**a**) the mean, $E(X)$, of the distribution,

(**b**) $F(x)$, the (cumulative) distribution function.

Television sets are hired out by a rental company. The time in months, X, between major repairs has the above negative exponential distribution with $\lambda = 0.05$. Find, to 3 significant figures, the probability that a television set hired out by the company will not require a major repair for at least a 2-year period. Find also the median value of X.

The company agrees to replace any set for which the time between major repairs is less than M months. Given that the company does not want to have to replace more than one set in 5, find M. (L)

Solution 6.31 $f(x) = \lambda e^{-\lambda x}, \; x \geqslant 0$

(**a**) $E(X) = \dfrac{1}{\lambda}$ (see p. 357)

(**b**)
$$F(t) = \int_0^t \lambda e^{-\lambda x}\, dx \qquad t \geqslant 0$$
$$= \left[-e^{-\lambda x}\right]_0^t$$
$$= -(e^{-\lambda t} - 1)$$
$$= 1 - e^{-\lambda t}$$

Therefore $F(x) = 1 - e^{-\lambda x}, \; x \geqslant 0.$

Let X be the r.v. 'the time, in months, between major repairs'.

$$f(x) = 0.05e^{-0.05x}$$

$$\begin{aligned} P(X > 24) &= 1 - F(24) \\ &= e^{-0.05(24)} \\ &= e^{-1.2} \\ &= 0.301 \quad (3 \text{ S.F.}) \end{aligned}$$

The probability that a television set will not need major repair in a 2-year period is 0.301 (3 S.F.).

Let m be the median value, then

$$F(m) = 0.5$$

So
$$\begin{aligned} 1 - e^{-\lambda m} &= 0.5 \\ e^{-\lambda m} &= 0.5 \\ -\lambda m &= \ln 0.5 \\ m &= -\frac{1}{0.05} \ln 0.5 \\ &= 13.9 \text{ months} \quad (3 \text{ S.F.}) \end{aligned}$$

The median is 13.9 months (3 S.F.).

We require $\quad P(X < M) \leqslant 0.2$

Therefore
$$\begin{aligned} 1 - e^{-0.05M} &\leqslant 0.2 \\ e^{-0.05M} &\geqslant 0.8 \\ -0.05M &\geqslant \ln 0.8 \\ M &\leqslant -\frac{\ln 0.8}{0.05} \\ M &\leqslant 4.46 \end{aligned}$$

Since M is an integer, $M = 4$

The company agrees to replace any set for which the time between major repairs is less than 4 months.

If X has an exponential distribution with mean α, then

$$E(X) = \alpha, \quad \text{so that } X \text{ has p.d.f. } f(x) = \frac{1}{\alpha}e^{-x/\alpha}, \quad x \geqslant 0$$

and $F(x) = 1 - e^{-x/\alpha}$

For example, if $E(X) = 0.6$

then
$$\begin{aligned} f(x) &= \frac{1}{0.6}e^{-x/0.6} \\ &= \tfrac{5}{3}e^{-5x/3} \qquad x \geqslant 0 \end{aligned}$$

Example 6.32 A random variable T has exponential distribution with mean α.

Show that $P(t_1 \leqslant T \leqslant t_2) = e^{-t_1/\alpha} - e^{-t_2/\alpha}$.

Solution 6.32 If $E(T) = \alpha$

then $$f(t) = \frac{1}{\alpha}e^{-t/\alpha}$$

and $$F(t) = 1 - e^{-t/\alpha}$$

So $$\begin{aligned} P(t_1 \leqslant T \leqslant t_2) &= F(t_2) - F(t_1) \\ &= 1 - e^{-t_2/\alpha} - \left(1 - e^{-t_1/\alpha}\right) \\ &= e^{-t_1/\alpha} - e^{-t_2/\alpha} \end{aligned}$$

THE LINK BETWEEN THE EXPONENTIAL DISTRIBUTION AND THE POISSON DISTRIBUTION

The 'waiting times' between successive events in a Poisson distribution can be shown to follow an exponential distribution.

For example, suppose that X is the discrete random variable 'the number of cars arriving at a petrol station in one minute', and suppose that the average number of cars arriving per minute is λ.

Then $$X \sim \text{Po}(\lambda).$$

We now need to find a probability model for the time in minutes between the arrival of one car and the next. Time is a continuous variable, so we need to find a continuous function for our model.

In one minute we 'expect' λ cars to arrive, so in t minutes we 'expect' λt cars to arrive.

So if Y is 'the number of cars arriving in t minutes' then $Y \sim \text{Po}(\lambda t)$.

So $$P(Y = 0) = e^{-\lambda t}$$

Therefore $P(\textit{at least one}$ car arrives in t minutes)

$$\begin{aligned} &= P(Y \geqslant 1) \\ &= 1 - P(Y = 0) \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Now let T be the random variable 'the length of time, in minutes, between successive cars'.

Then $$\begin{aligned} P(\text{waiting time} < t) &= P(\text{at least one car arrives in } t \text{ minutes}) \\ &= 1 - e^{-\lambda t} \end{aligned}$$

But $P(\text{waiting time} < t)$ is the same as the distribution function for T.

Therefore $$F(t) = 1 - e^{-\lambda t}$$

Differentiating, we get $$f(t) = \lambda e^{-\lambda t}$$

Therefore T follows the exponential distribution with parameter λ.

Remember that if T has p.d.f. $f(t) = \lambda e^{-\lambda t}, \ t \geqslant 0$

then $$E(T) = \frac{1}{\lambda} \quad \text{and} \quad \text{Var}(T) = \frac{1}{\lambda^2}.$$ (see p. 357)

To sum up:

If X is 'the number of cars arriving in one minute'

then $$X \sim \text{Po}(\lambda)$$

and if T is 'the time, in minutes, between successive cars'

then p.d.f. is $f(t) = \lambda e^{-\lambda t}$ where $E(T) = \frac{1}{\lambda}$.

NOTE: the units of time are the same in both distributions.

Example 6.33 Cars arrive at a garage at an average rate of 0.8 per minute. Find the mean and the variance of the waiting times between successive cars.

Solution 6.33 If X is 'the number of cars arriving in one minute', then $X \sim \text{Po}(0.8)$.

Therefore, if T is 'the time, in minutes, between successive cars'

then $$f(t) = 0.8e^{-0.8t}$$

and $$E(T) = \frac{1}{0.8} = 1.25$$

$$\text{Var}(T) = \frac{1}{(0.8)^2} = 1.5625$$

So the mean waiting time is 1.25 minutes and the variance is 1.5625 min^2.

Example 6.34 On a busy road, accidents occur at random at the rate of 3 per day. Find the probability that, after a particular accident has occurred, at least one day goes by without another.

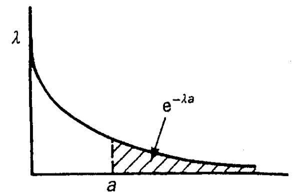
Solution 6.34 Let X be 'the number of accidents in *1 day*'

Then $$X \sim \text{Po}(3)$$

If T is 'the time *in days* between successive accidents', then $f(t) = 3e^{-3t}$.

P(at least one day goes by without an accident)

$$= P(T > 1)$$
$$= e^{-3}$$
$$= 0.050 \quad (3 \text{ d.p.})$$



(see p. 354)

So there is only a 5% chance that one whole day goes by between successive accidents.

Exercise 6g

1. A continuous r.v. X has p.d.f. $f(x)$ where $f(x) = 5e^{-5x}$, $x \geqslant 0$. Find (*a*) $P(X > 0.5)$, (*b*) $E(X)$, (*c*) $P(X < E(X))$, (*d*) the standard deviation of X, (*e*) the median, (*f*) the mode.

2. The lifetime, in thousands of hours, of Extralight light bulbs follows an exponential distribution with p.d.f. $f(x) = 0.5e^{-0.5x}$
(*a*) Find the mean lifetime.
(*b*) A bulb is selected at random. Find the probability that it lasts (i) more than 2500 hours, (ii) less than 1800 hours.
(*c*) Two light bulbs are selected at random. Find the probability that one lasts more than the mean number of hours and the other lasts for less than the mean number of hours.
(*d*) A random sample of 6 light bulbs is chosen. Find the probability that exactly 4 will each last more than 2500 hours.

3. A batch of high-power light bulbs is such that the probability that any bulb fails before x hours, when kept on continuously, is $F(x) = 1 - e^{-x/10}$, $x \geqslant 0$. Find
(*a*) the median time to failure,
(*b*) the density function of the distribution of the time to failure,
(*c*) the mean and the variance of the distribution,
(*d*) the probability that a bulb will fail between five and ten hours. (O)

4. The lifetime T, in years, of articles produced by a manufacturer can be modelled by the probability density function given by
$$f(t) = ae^{-at}, \quad t \geqslant 0$$
$$f(t) = 0, \quad t < 0$$
The articles are produced at a unit cost of £10 and sold for £25. Research shows that 50% of those produced fail within the first five years of life. Find the value of a.
After some time in business the manufacturer decides to guarantee free replacement of items which fail during their first year, but at the same time he raises the price so that the increase covers the expected cost of providing the guarantee. What should the new price be?
If two items are purchased what is the probability that just one will be replaced under guarantee?

5. A continuous random variable T has a negative exponential distribution given by
$$f(t) = \lambda e^{-\lambda t} \quad t > 0$$
The life in hours of a type of electric battery can be modelled by the above distribution and when a sample of 800 is tested the mean life is found to be 92.2 h. What is the value of λ based on this figure?
(*a*) What is the probability that a battery will last for at least 200 h?
(*b*) If a battery has lasted 200 h what is the probability that it will last for at least a further 100 h?
(*c*) If two batteries are bought what is the probability that one fails before 200 h and the other after 200 h?

6. The random variable X can take all values between 0 and a inclusive, where $a > 0$. Its probability density function $f(x)$ is zero for $x < 0$ and $x > a$, and, for $0 \leqslant x \leqslant a$, satisfies
$$f(x) = (A/a)\exp(-x/a),$$

$e^{-\lambda a}$

p. 354)

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$e^{-\lambda t} \quad t > 0$

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y function $f(x)$ is zero
, and, for $0 \leqslant x \leqslant a$,

$a)\exp(-x/a)$,

where A is a positive constant. Show by integration that $A = 1.582$ to 3 decimal places.
Also use integration to find to 2 decimal places
(i) the probability that X is less than $\frac{1}{2}a$;
(ii) the number λ for which there is a probability $\frac{1}{2}$ that X is less than λa. (MEI)

7. Explain briefly, from your projects if possible, a real-life situation that can be modelled by an exponential distribution.
An archer shoots arrows at a target. The distance X cm from the centre of the target at which an arrow strikes the target has probability density function, f, defined by

$$f(x) = \tfrac{1}{10}e^{-x/10} \quad x > 0$$
$$f(x) = 0 \quad \text{otherwise}$$

An arrow scores 8 points if $X \leqslant 2$, 5 points if $2 < X \leqslant 5$, one point if $5 < X \leqslant 15$ and no points otherwise. Find, to 3 decimal places, the expected score when one arrow is shot at the target. (L)

8. Find the mean of the random variable X which has an exponential distribution with probability density function

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \geqslant 0 \text{ where } \lambda > 0$$
$$f(x) = 0 \quad \text{for } x < 0$$

For people suffering from a mental illness, the time in days from the end of a treatment to the occurrence of renewed symptoms is an exponential random variable with parameter $\lambda > 0$. Find, in terms of λ and t, the probability that neither of two randomly chosen sufferers from the illness will show renewed symptoms for a time t days after a treatment.
Given that two patients have no renewed symptoms for a time t days after a treatment, find, in terms of λ and t, the probability that both will remain free of symptoms for a further t days.
During a routine check at time t days after his treatment, another patient is found to be showing renewed symptoms. Find, in terms of λ, k and t, the probability that the renewed symptoms first showed in this patient less than kt days before the day of the routine check, where $0 \leqslant k \leqslant 1$. (L)

9. A random variable X has the probability density function f given by

$$f(x) = ce^{-2x} \quad x > 0$$
$$f(x) = 0 \quad \text{otherwise.}$$

Find the value of c. Find also the mean and the variance of X.
[You may assume that $\int_0^\infty x^2 e^{-2x}\,dx = \frac{1}{4}$.]
Find the distribution function of X. Hence, or otherwise, show that, for positive t and k,

$$P[(X > t + k) \mid (X > k)] = P(X > t)$$

Given that X is the lifetime in years of a particular type of indicator lamp that is alight continuously, explain in words the meaning of the above result.
Given that 2 such lamps, A and B, have already been alight for 3 months and 4 months respectively, find the probability that both will still be alight in 3 months' time. (L)

10. On a stretch of road, breakdowns occur at an average rate of 2 per day, and the number of breakdowns follows a Poisson distribution. Find (*a*) the mean time between breakdowns, (*b*) the median time between breakdowns.

11. In any match, the number of goals scored by Rovers follows a Poisson distribution with mean 3 per 90 minutes. Find
(*a*) the mean and variance of the waiting times between successive goals scored by the team,
(*b*) the probability that, following a goal by Rovers in the tenth minute, they will not score another goal in the next 35 minutes.

12. Describe the conditions under which it is appropriate to use the exponential distribution, supporting your answer with reference to an experiment you may have carried out.
A major road construction project is underway. In the site supervisor's office, there is an average of two telephone calls every 5 minutes. Stating any assumptions you make, write down the probability that in a period of t minutes there is
(*a*) no telephone call,
(*b*) at least one telephone call.
Presenting a carefully reasoned argument, give the cumulative distribution function, $F(t)$, for the length of time between telephone calls. Hence establish that the probability density function, $f(t)$, is

$$f(t) = 0.4e^{-0.4t} \quad t > 0$$

Calculate
(*c*) the mean time between calls,
(*d*) the median time between calls.
Given that the supervisor has had no call in the last 3 minutes, what is the probability that she could leave the office for 5 minutes without missing a call? (O)

SUMMARY — CONTINUOUS RANDOM VARIABLES

For a continuous random variable X,
with p.d.f. $f(x)$ for $a \leqslant x \leqslant b$

$$\int_{\text{all } x} f(x)\,\mathrm{d}x = 1$$

$$P(c \leqslant X \leqslant d) = \int_c^d f(x)\,\mathrm{d}x \qquad a \leqslant c < d \leqslant b$$

$$E(X) = \int_{\text{all } x} x f(x)\,\mathrm{d}x$$

$$\operatorname{Var}(X) = \int_{\text{all } x} x^2 f(x)\,\mathrm{d}x - E^2(X)$$

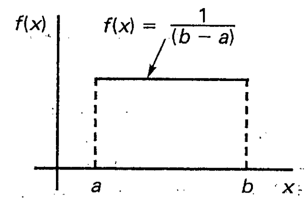
$$F(t) = \int_a^t f(x)\,\mathrm{d}x \qquad a \leqslant t \leqslant b$$

where $F(t)$ is the cumulative distribution function

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x)$$

The rectangular distribution

If $f(x) = \dfrac{1}{b-a}$ $\quad a \leqslant x \leqslant b$, then $X \sim \mathrm{R}(a, b)$

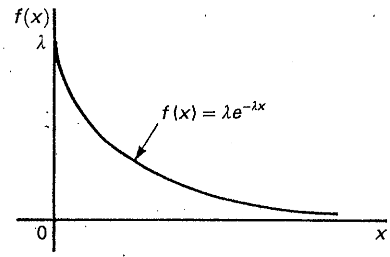


$$E(X) = \tfrac{1}{2}(a+b)$$

$$\operatorname{Var}(X) = \tfrac{1}{12}(b-a)^2$$

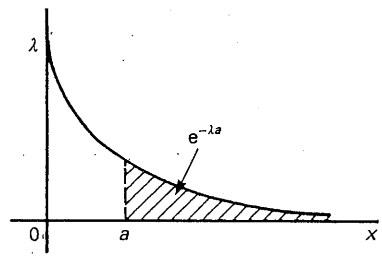
The exponential distribution

If $f(x) = \lambda \mathrm{e}^{-\lambda x} \quad x \geqslant 0$



$$E(X) = \frac{1}{\lambda}$$

$$\operatorname{Var}(X) = \frac{1}{\lambda^2}$$



$$F(x) = 1 - \mathrm{e}^{-\lambda x}$$

$$P(X > a) = \mathrm{e}^{-\lambda a}$$

11. (a) 0.455, 3 (b) 3.64, 4.95

(c) $F(x) = \begin{cases} \frac{1}{\ln 9} \ln x & 1 \leqslant x \leqslant 9 \\ 1 & x \geqslant 9 \end{cases}$

12. $\frac{3}{4}, \frac{19}{80}$

$F(x) = \begin{cases} \frac{3}{4}x - \frac{1}{16}x^3 & 0 \leqslant x \leqslant 2 \\ 1 & x \geqslant 2 \end{cases}$

0.007

13. $F(x) = \begin{cases} \frac{1}{4}x & 0 \leqslant x \leqslant 1 \\ \frac{1}{5} + \frac{x^4}{20} & 1 \leqslant x \leqslant 2 \\ 1 & x \geqslant 2 \end{cases}$

1.565, 0.821

14. (a) 5 (b) $\frac{1}{6}$ (c) $\frac{5}{252}$; 543 tonnes

15. (iii) $F(x) = \begin{cases} \frac{1}{4}x^2 & 0 \leqslant x \leqslant 1 \\ \frac{2}{3}x - \frac{x^2}{12} - \frac{1}{3} & 1 \leqslant x \leqslant 4 \\ 1 & x \geqslant 4 \end{cases}$

(iv) £283.33 (v) $\frac{1}{3}$

16. (ii) 0.9342

(iii) $F(x) = \begin{cases} \frac{1}{\pi}(x + \sin x) & 0 \leqslant x \leqslant \pi \\ 1 & x \geqslant \pi \end{cases}$

17. (b) $2\frac{1}{9}$

(c) $F(x) = \begin{cases} \frac{1}{6}x + \frac{1}{12}x^2 - \frac{1}{4} & 1 \leqslant x \leqslant 3 \\ 1 & x \geqslant 3 \end{cases}$

(d) 2.16

18. (ii) 1, 2 (iii) 0 (iv) $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

19. (a) 1.5, 0.75 (b) 0.2

(c) $F(x) = \begin{cases} 0.75x^2 - 0.25x^3 & 0 \leqslant x \leqslant 2 \\ 1 & x \geqslant 2 \end{cases}$

(d) 0.288

20. (ii) $1\frac{2}{9}, \frac{31}{18}$ (iii) $\frac{17}{48}$

21. (i) $F(x) = 1 - \dfrac{1}{(x+1)^3}$ $x \geqslant 0$; $x = 1$

(ii) 1.5 (iii) 0.75

22. (a) $F(w) = \dfrac{w^4}{5^5}(25 - 4w)$, $0 \leqslant w \leqslant 5$

(b) 0.650 (c) 0.794 (d) 3.75

(f) Negatively skewed

23. (a) (i) $f(x) \geqslant 0$ for all values of x;

$\int_a^b f(x)\,dx = 1$

(ii) $f(x) = F'(x)$

(b) (iii) $F(x) = \begin{cases} 0 & x < 0 \\ 0.0125x^2 & 0 \leqslant x \leqslant 8 \\ 0.2x - 0.8 & 8 \leqslant x \leqslant 9 \\ 1 & x \geqslant 9 \end{cases}$

(iv) 0.55 (v) 0.166 (vi) 0.575

Exercise 6e (page 347)

1. (a) $1, -\frac{1}{27}$

(b) $F(x) = \begin{cases} \frac{x^3}{27} & 0 \leqslant x \leqslant 3 \\ 1 & x \geqslant 3 \end{cases}$

(c) $f(x) = \frac{x^2}{9}$ $0 \leqslant x \leqslant 3$

(e) 0.581

2. (a) $\frac{1}{3}$ (b) $f(x) = \begin{cases} \frac{2}{3} & 0 \leqslant x < 1 \\ \frac{1}{3} & 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$

(c) $\frac{5}{6}$ (d) $\frac{\sqrt{11}}{6}$ (e) 0.608 (3 S.F.)

3. (a) $f(x) = \begin{cases} \frac{x-1}{6} & 1 \leqslant x \leqslant 3 \\ \frac{7-x}{12} & 3 \leqslant x \leqslant 7 \\ 0 & \text{otherwise} \end{cases}$

(b) $\frac{11}{3}, \frac{14}{9}$ (c) 3.54 (3 S.F.) (d) 0.595

4. (a) $\frac{1}{3}$ (b) $f(x) = \begin{cases} 0 & x < -1 \\ \alpha & -1 \leqslant x < 0 \\ 2\alpha & 0 \leqslant x < 1 \\ 0 & x \geqslant 1 \end{cases}$

(c) $\frac{1}{6}$ (d) 0.553 (e) $\frac{11}{18}$

5. (a) 2 (b) $f(x) = \begin{cases} 2 & 0 \leqslant x \leqslant 0.5 \\ 0 & \text{otherwise} \end{cases}$

(c) 0.25 (d) 0.144

6. (a) $-\frac{1}{16}, 1$ (b) $f(x) = \begin{cases} \frac{x}{8} & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$

7. $1, \frac{5}{6}, \frac{1}{4}$

8. (i) 2.93

(ii) $F(x) = \begin{cases} 0 & x < 0 \\ 1 - \frac{1}{100}(x-10)^2 & 0 \leqslant x \leqslant 10 \\ 1 & x \geqslant 10 \end{cases}$

(iii) $f(x) = \begin{cases} \frac{1}{5} - \frac{1}{50}x & 0 \leqslant x \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$

0.25

9. (i) $\frac{1}{3}$ (iii) 4.39 (iv) 12.5

Exercise 6f (page 353)

1. (a) $f(x) = \begin{cases} \frac{1}{3} & 3 \leqslant x \leqslant 6 \\ 0 & \text{otherwise} \end{cases}$

(b) 4.5 (c) 0.75 (d) $\frac{1}{3}$

2. (a) $\frac{1}{3}$ (b) 0.5 (c) −3.5 (d) 0.75
3. (a) 5 (b) 0.325 (c) 3 (d) $\frac{4}{3}$
4. (i) 0.25 (ii) 0.845
5. 0.4
6. 0.144

Exercise 6g (page 362)

1. (a) 0.0821 (b) 0.2 (c) 0.632
   (d) 0.2 (e) 0.139 (f) 0
2. (a) 2000 h (b) (i) 0.287 (ii) 0.593
   (c) 0.465 (d) 0.0515
3. (a) 6.93 (b) $0.1e^{-0.1x}$ (c) 10, 100
   (d) 0.24 (2 S.F.)
4. 0.1386, £26.30, 0.225
5. $a = 92.2$, $A = 0.0108$
   (a) 0.114 (b) 0.338 (c) 0.202
6. (i) 0.62 (ii) 0.38
7. 2.895
8. $\frac{1}{\lambda}, e^{-2\lambda t}, e^{-2\lambda t}, \dfrac{1 - e^{-\lambda(1-k)t}}{1 - e^{-\lambda t}}$
9. $2, \frac{1}{2}, \frac{1}{4}, 1 - e^{-2x}, 0.368$
10. (a) Half a day (b) 8 hours
11. (a) 30; 900 (b) 0.311
12. (a) $e^{-0.4t}$ (b) $1 - e^{-0.4t}$ (c) 2.5
    (d) 1.73; 0.135