CHALLENGING INDUCTION AND SERIES QUESTIONS SET 2

1. **(i)** Using the method of differences, show that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = A + \frac{B}{n+2} + \frac{C}{n+3},$$

where A, B, and C are constants to be determined. [4]

(ii) Explain why the series $\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$ converges, and write down its value. [2]

(iii) Hence show that $\sum_{r=1}^{\infty} \frac{1}{(r+2)^2} < \frac{5}{12}$. [2]

2. **(i)** Show that $\frac{r^2+3r+1}{(r+2)!} = \frac{1}{r!} - \frac{1}{(r+2)!}$. [2]

(ii) Hence find $\sum_{r=1}^{n} \frac{r^2+3r+1}{(r+2)!}$. (There is no need to express your answer as a single algebraic fraction.) [3]

(iii) Using the result in part **(ii)** and the standard series expansion for e^x, find the exact value of $\sum_{r=1}^{\infty} \frac{r^2+3r+3}{(r+2)!}$. [4]

3. The rth term of a sequence is given by $u_r = \frac{8r}{(2r-1)^2(2r+1)^2}$, for $r = 1, 2, 3, \ldots$..

The sum of the first n terms is denoted by S_n.

(i) Given that $S_1 = \frac{8}{9}$ and $S_2 = \frac{24}{25}$, find the exact values of S_3 and S_4. [1]

(ii) By considering the values of $1 - S_n$ for n = 1, 2, 3 and 4, or otherwise, make a conjecture for a formula for S_n in terms of n. [1]

(iii) Prove your conjecture by the method of mathematical induction. [4]

(iv) Find the smallest integer n for which the sum S_n differs from 1 by less than 10^{-5}. [2]

4. Let $f(r)=(r-1)(r)(r+1)$. Show that $f(r)-f(r-1)=3r(r-1)$. [1]

(i) Find $\sum_{r=1}^{n} r(r-1)$. [2]

(ii) Using the result obtained in part **(i)**, deduce that $\sum_{r=1}^{n} r^2=\frac{1}{6}n(n+1)(2n+1)$. [2]

(iii) Find the sum of the series

$$1^2+3\times2^2+3^2+3\times4^2+5^2+3\times6^2+\ldots+3(n-1)^2+n^2,\ \text{where } n \text{ is odd.}$$

Give your answer in a fully factorised form. [4]

5. **(a)** A sequence $u_1,u_2,u_3,\ldots$ is given by

$$u_1=0 \quad \text{and} \quad u_{n+1}=u_n+2n-1 \quad \forall n\in\mathbb{Z}^+.$$

Use the method of mathematical induction to prove that

$$u_n=(n-1)^2.$$ [4]

(b) **(i)** By using the identity $4\sin^3\theta\equiv3\sin\theta-\sin3\theta$, show that

$$\sum_{r=0}^{n}\frac{1}{3^r}\sin^3\left(3^r\theta\right)=\frac{1}{4}\left[3\sin\theta-\frac{1}{3^n}\sin\left(3^{n+1}\theta\right)\right]$$ [3]

(ii) Hence, find the infinite sum

$$\sin^3\left(\frac{\pi}{2}\right)+\frac{1}{3}\sin^3\left(\frac{3\pi}{2}\right)+\frac{1}{3^2}\sin^3\left(\frac{3^2\pi}{2}\right)+\frac{1}{3^3}\sin^3\left(\frac{3^3\pi}{2}\right)+\ldots$$ [2]

6. **(a)** **(i)** Using the method of differences, find $\sum_{r=1}^{n}\left[(r+1)^3 - r^3\right]$ in terms of n. **[2]**

(ii) Express $(r+1)^3 - r^3$ in the form of $ar^2 + br + c$ where a, b and c are constants to be determined. **[1]**

(iii) Hence show that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. **[4]**

(b) Prove by mathematical induction that

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A sequence u_1, u_2, u_3, ... is such that $u_1 = \frac{1}{9}$ and $u_{n+1} = u_n - \frac{n^2+n-4}{(n+3)^2(n+2)^2}$, for $n \geq 1$.

7. **(i)** Use the method of mathematical induction to prove that $u_n = \frac{n}{(n+2)^2}$. [4]

(ii) Hence find $\sum_{n=1}^{N} \frac{n^2+n-4}{(n+3)^2(n+2)^2}$ in terms of N and state its value as $N \to \infty$. [3]

8. Prove by mathematical induction that

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+\cdots+n} = \frac{2n}{n+1}.$$

Hence find an expression for

$$\frac{1}{2+4+6} + \frac{1}{2+4+6+8} + \frac{1}{2+4+6+8+10} + \cdots + \frac{1}{2+4+6+\cdots+2n}.$$ **[6]**