Situation :

We know that if $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$,

then $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 & a_3 b_1 - a_1 b_3 & a_1 b_2 - a_2 b_1 \end{pmatrix}$, and

hence the magnitude of $\mathbf{a} \times \mathbf{b}$ i.e.

$$|\,\mathbf{a} \times \mathbf{b}\,| = \sqrt{(a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2}.$$

Problem : IS THERE ANY RELATIONSHIP BETWEEN $|\,\mathbf{a} \times \mathbf{b}\,|$, $|\,\mathbf{a}\,|$ and $|\,\mathbf{b}\,|$?

This aim of this activity is to find the relationship between
$|\mathbf{a} \times \mathbf{b}|$, $|\mathbf{a}|$ and $|\mathbf{b}|$. *Try the following procedure:*

Let $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$, and the angle between the vectors be θ.

(i) By considering $\mathbf{a} \cdot \mathbf{b}$, express $\cos\theta$ in terms of a_1, a_2, a_3, b_1, b_2 and b_3.

(ii) By using the identity $\cos^2\theta + \sin^2\theta \equiv 1$, show that

$$\sin\theta = \frac{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{(a_1{}^2 + a_2{}^2 + a_3{}^2)}\sqrt{(b_1{}^2 + b_2{}^2 + b_3{}^3)}}$$

(iii) Relate (ii) with $|\,\mathbf{a}\times\mathbf{b}\,|$, $|\,\mathbf{a}\,|$ and $|\,\mathbf{b}\,|$, and express $|\,\mathbf{a}\times\mathbf{b}\,|$ as the subject.

Magnitude of Cross Product

Theorem:

Let $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix},$

and the angle between the vectors be θ.

Then, $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

Example:

Given that $\mathbf{v} = \begin{pmatrix} 4 & -6 & 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -3 & 2 & 7 \end{pmatrix}$.

Find $\mathbf{v} \times \mathbf{w}$. Hence, find the angle between $\mathbf{v}$ and $\mathbf{w}$.

Example:

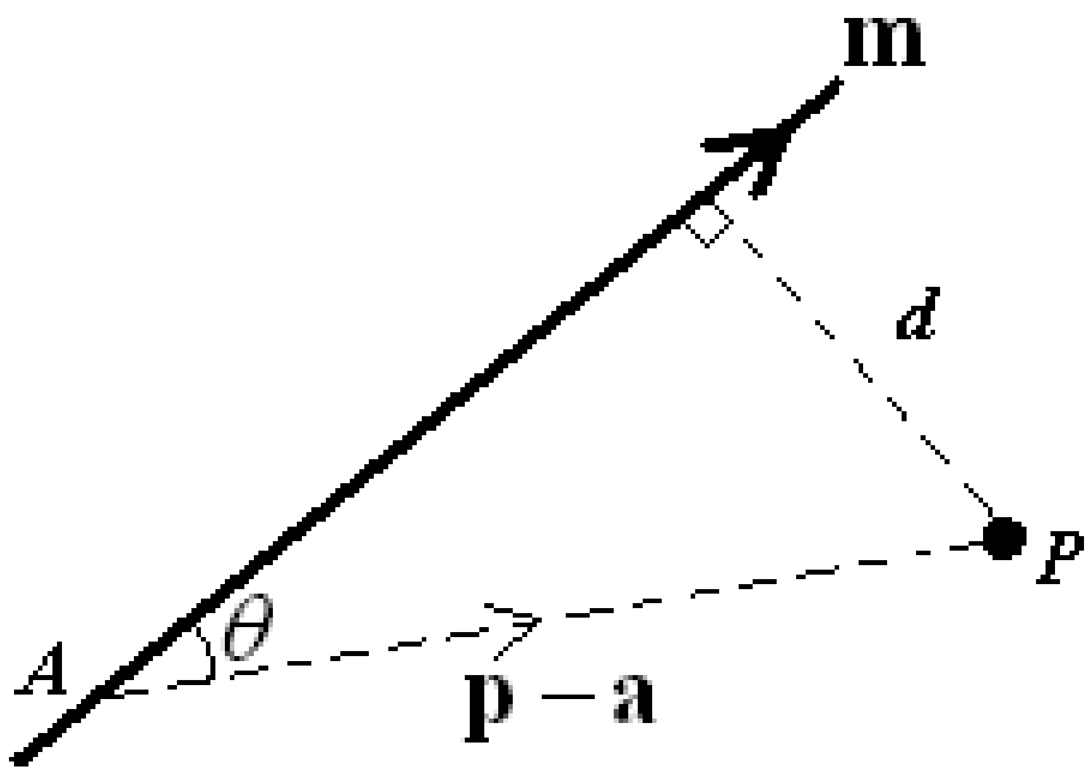
Given that $\mathbf{u} = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 & -6 & 2 \end{pmatrix}$

Find $\mathbf{u} \times \mathbf{v}$. Hence, find the angle between $\mathbf{u}$ and $\mathbf{v}$.

Perpendicular Distance from a Point to a Line

Theorem:

Let a straight line l have equation $\mathbf{r} = \mathbf{a} + \lambda \mathbf{m}$ and the point P have position vector $\mathbf{p}$. Then, the perpendicular distance (or shortest distance) from P to l is given by $d = \dfrac{\left|\mathbf{m} \times (\mathbf{p} - \mathbf{a})\right|}{|\,\mathbf{m}\,|}$.


m
d
A
θ
p
p − a

Example:

Find the perpendicular distance from point P to line l.

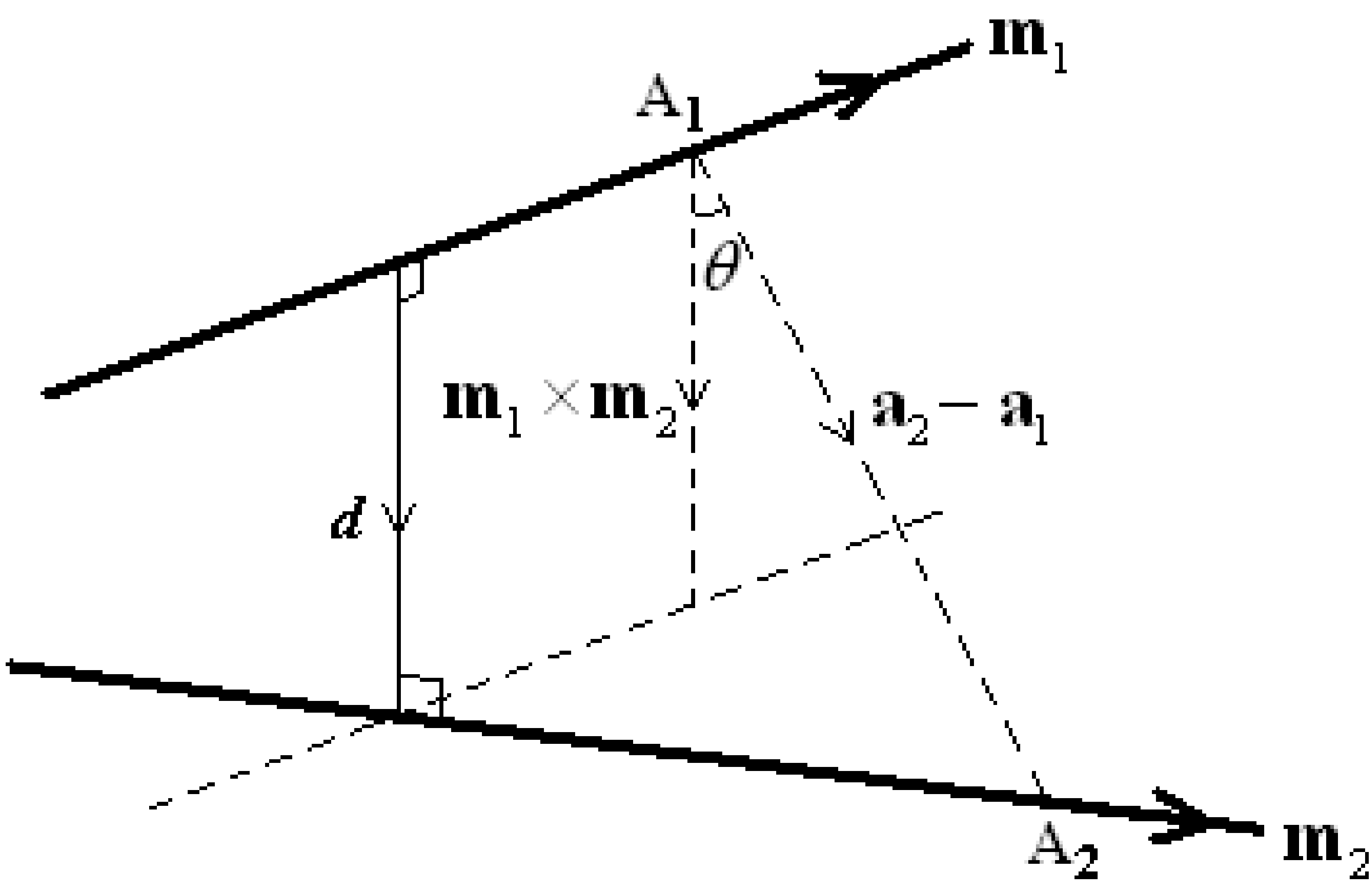
(a) $l : \mathbf{r} = 5\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ & $P : (1, 5, 1)$.

(b) $l : \dfrac{x-6}{7} = \dfrac{y+7}{-6} = z$ & $P(3, 5, 7)$.

Shortest Distance Between Two Skew Lines

Theorem:

Let two skew lines be $l_1 : \mathbf{r} = \mathbf{a}_1 + \lambda_1 \mathbf{m}_1$ and $l_2 : \mathbf{r} = \mathbf{a}_2 + \lambda_2 \mathbf{m}_2$. Then, the shortest distance between l_1 and l_2 is $d = \left| \dfrac{(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{m}_1 \times \mathbf{m}_2)}{|\mathbf{m}_1 \times \mathbf{m}_2|} \right|$


$\mathbf{m}_1$
A_1
θ
$\mathbf{m}_1 \times \mathbf{m}_2$
$\mathbf{a}_2 - \mathbf{a}_1$
d
A_2
$\mathbf{m}_2$

Example:

Find the shortest distance between l_1 and l_2.

(a) $l_1 : \mathbf{r} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(-9\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ &

$l_2 : \mathbf{r} = 7\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \lambda(-15\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$

(b) $l_1 : \dfrac{x-1}{-2} = y - 2 = \dfrac{z-3}{-5}$ & $l_2 : x - 3 = y + 4 = z + 1.$

<u>Homework</u>

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example:

Find the shortest distance from point P to line l.

(a) $l:\mathbf{r}=(1-\lambda)\mathbf{i}+(2-3\lambda)\mathbf{j}+2\mathbf{k}$, $P:(1,3,2)$.

(b) $l:\dfrac{x-7}{-1}=\dfrac{y+1}{3}, z=4$, $P:(0,0,0)$.

Example:

A, B, C and D are four points with position vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{k}$ and $\mathbf{d} = 3\mathbf{j}$ respectively. Find the shortest distance between line AB and CD.

Example:

Let line $l_1 : x-1 = y = z+5$ and line $l_2 : \frac{x}{3} = \frac{y+2}{2} = z+2$.

P and Q are two points on l_1 and l_2 respectively such that $PQ \perp l_1$ and l_2.

(a) Show that the distance $PQ = \sqrt{6}$ units.

(b) Find the coordinates of P and Q, and