1. $$\frac{1}{k(k+1)(k+2)} = \frac{1}{2k} + \frac{-1}{k+1} + \frac{1}{2(k+2)}$$

$n=1,\ \frac{1}{2} - \frac{1}{2} + \frac{1}{2(3)}$

$n=2,\ \frac{1}{4} - \frac{1}{3} + \frac{1}{2(4)}$

$n=3,\ \frac{1}{6} - \frac{1}{4} + \frac{1}{2(5)}$

$n=4,\ \frac{1}{8} - \frac{1}{5} + \frac{1}{2(6)}$

$\vdots$

$n=997,\ \frac{1}{2(997)} - \frac{1}{998} + \frac{1}{2(999)}$

$n=998,\ \frac{1}{2(998)} - \frac{1}{999} + \frac{1}{2(1000)}$

$n=999,\ \frac{1}{2(999)} - \frac{1}{1000} + \frac{1}{2(1001)}$

$$\sum_{k=1}^{999} \frac{1}{k(k+1)(k+2)} = \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2000} + \frac{1}{2002} - \frac{1}{1000}$$

$$= \frac{500499}{2002000}$$

2. $\underset{\sim}{n}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix}$

$\underset{\sim}{n}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

i) $\cos\theta = \dfrac{-14-2}{\sqrt{62}\sqrt{5}}$

$\cos\theta = \dfrac{16}{\sqrt{310}}$

ii) $\underset{\sim}{m} = \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Let $x = 0$, $y = 5$, $z = 2$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

3. $\alpha, \frac{1}{\alpha}, \beta$

$\Sigma\alpha = -\frac{b}{a}$ $\quad \Sigma\alpha\beta = \frac{c}{a}$ $\quad \alpha\beta\gamma = -\frac{d}{a}$

$\alpha + \frac{1}{\alpha} + \beta = -\frac{b}{a}$

$\beta = -\frac{d}{a}$

$a\beta^3 + b\beta^2 + c\beta + d = 0$

$a\left(\frac{-d}{a}\right)^3 + b\left(\frac{-d}{a}\right)^2 + c\left(\frac{-d}{a}\right) + d = 0$

$a^2 - d^2 = ac - bd$

$a = 9,\ b = 24,\ c = -11,\ d = -6$

$a^2 - d^2 = 45$ $\quad \therefore$ verified

$ac - bd = 45$

$\therefore \beta = \frac{-(-6)}{9} = \frac{2}{3},$

$\therefore (3x - 2)(3x^2 + 10x + 3) = 0$

$(3x - 2)(3x + 1)(x + 3) = 0$

$\therefore$ Roots $= \frac{2}{3}, -\frac{1}{3}, -3$

4. i) $\frac{dy}{dx} = x\left[n(1-x^2)^{n-1}(-2x)\right] + (1-x^2)^n(1)$

$= -2nx^2(1-x^2)^{n-1} + (1-x^2)^n$

$= -2nx^2(1-x^2)^{n-1} - 2n(1-x^2)^n$

$+ 2n(1-x^2)^n + (1-x^2)^n$

$= 2n[1-x^2]^{n-1}\left[-x^2 - (1-x^2)\right] + (2n+1)(1-x^2)^n$

$= -2n[1-x^2]^{n-1} + (2n+1)(1-x^2)^n$

shown.

ii) $\left[x(1-x^2)^n\right]_0^1 = (2n+1)\int_0^1 (1-x^2)^n\,dx$

$-2n\int_0^1 (1-x^2)^{n-1}\,dx$

$0 = (2n+1)I_n - 2nI_{n-1}$

$\therefore I_n = \frac{2n}{2n+1} I_{n-1}$

$\therefore I_5 = \frac{10}{11}\cdot\frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} I_0$

$= \frac{256}{693}$

5. i) $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ $\qquad x = \sqrt{t}$

$\qquad\qquad\qquad\qquad\qquad\qquad \frac{dx}{dt} = \frac{1}{2\sqrt{t}}$

$= 2\sqrt{t}\,\frac{dy}{dt}$

ii) $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$

$= \left[\frac{1}{\sqrt{t}}\frac{dy}{dt} + 2\sqrt{t}\frac{d^2y}{dt^2}\right]\cdot 2\sqrt{t}$

$= 2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}$

substitute:

$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2(9y+6) = 0$

$2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2} - \frac{1}{\sqrt{t}}(2\sqrt{t})\frac{dy}{dt} + 4t(9y+6) = 0$

$\therefore \frac{d^2y}{dt^2} + 9y + 6 = 0$

AE: $m^2 + 9 = 0$, $m = \pm 3i$

CF: $y = A\sin 3t + B\cos 3t$

PI: $y = k$ $\quad\therefore\quad k = -\frac{2}{3}$

GS: $y = A\sin 3x^2 + B\cos 3x^2 - \frac{2}{3}$

6 i) $u_{n+1} = m = u_n$

$$m = \frac{9m+3}{m+7}$$

$$\therefore m^2 + 7m - 9m - 3 = 0$$

$$m = 3, -1, m > 0$$

$$\therefore m = 3$$

ii) $$u_{n+1} = \frac{9u_n + 3}{u_n + 7}$$

$$u_n(u_{n+1} - 9) = 3 - 7u_{n+1}$$

$$u_n = \frac{3 - 7u_{n+1}}{u_{n+1} - 9}$$

$$\therefore 3 - u_n = 3 - \left(\frac{3 - 7u_{n+1}}{u_{n+1} - 9}\right)$$

$$= \frac{10u_{n+1} - 30}{u_{n+1} - 9}$$

Proposition: $u_n < 3$ for $n \in \mathbb{Z}^+$

Basis case: $n = 1$, $u_1 = 1 < 3$

Induction step: Assume $u_K < 3$

when $n = k+1$,

$3 - u_K > 0$

$$\therefore \frac{10u_{k+1} - 30}{u_{k+1} - 9} > 0$$

$\therefore u_{k+1} < \frac{30}{10}$ since $u_{k+1} - 9 < 0$ because u_{k+1} is always converging to 3.

$\therefore u_{k+1} < 3$

Completion: $u_n < 3$ for $n \in \mathbb{Z}^+$.

7. $\frac{dx}{dt} = 4 - t^2 \qquad \frac{dy}{dt} = 4t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (4 + t^2)^2$$

i) Arc length $= \int_0^{2\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^{2\sqrt{3}} 4 + t^2 \, dt$$

$$= \left[4t + \frac{t^3}{3} \right]_0^{2\sqrt{3}}$$

$$= 16\sqrt{3}$$

ii) Area $= 2\pi \int_0^{2\sqrt{3}} \times \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= 2\pi \int_0^{2\sqrt{3}} (4t - \tfrac{1}{3}t^3)(4 + t^2)\, dt$$

$$= 2\pi \int_0^{2\sqrt{3}} 16t + \frac{8}{3}t^3 - \frac{1}{3}t^5 \, dt$$

$$= 2\pi \left[8t + \frac{2}{3}t^4 - \frac{1}{18}t^6 \right]_0^{2\sqrt{3}}$$

$$= 192\pi.$$

iii) Mean value $= \frac{1}{1} \int_0^1 y \, dt$

$$= \int_0^1 2t^2 - 8 \, dt$$

$$= \left[\frac{2t^3}{3} - 8t \right]_0^1$$

$$= -7\frac{1}{3}$$

8. i) det $A \neq 0$

$$\begin{vmatrix} 1 & 2 & a \\ 2 & -1 & 5 \\ 3 & 1 & 8 \end{vmatrix} \neq 0$$

$\therefore a \neq 3$

ii) $$\left(\begin{array}{ccc|c} 1 & 2 & a & 1 \\ 2 & -1 & 5 & 11 \\ 3 & 1 & 8 & b \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & a & 1 \\ 0 & -5 & 5-2a & a \\ 0 & 0 & 3-a & b-12 \end{array}\right)$$

if $a = 3$, Rank $A = 2$

& $b \neq 12$ Rank $A|B = 3$

$\therefore$ no common point / no solution

iii) when $a = 3$, $b = 12$

Rank A = Rank $(A|B)$ < no. of rows

$\therefore$ infinitely many solutions.

Let $y = \lambda$,

$\therefore z = -5\lambda - 9$

$x = -2\lambda - 3(-5\lambda - 9) + 1$

$= 13\lambda + 28$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 28 \\ 0 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 13 \\ 1 \\ 5 \end{pmatrix}$$

9. $r = a\sin\theta\cos^2\theta$

$$\frac{dr}{d\theta} = a\cos\theta\left[\cos^2\theta - 2\sin^2\theta\right]$$

$$\frac{d^2r}{d\theta^2} = a\sin\theta\left[2\sin^2\theta - 7\cos^2\theta\right]$$

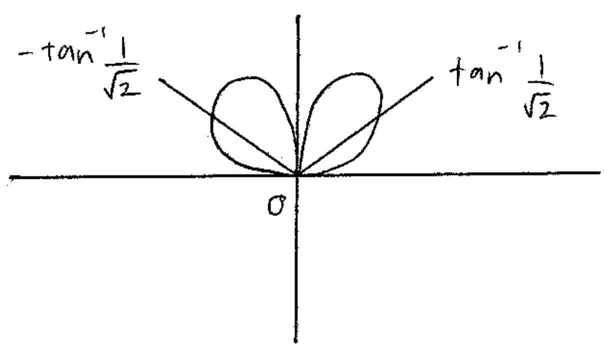
$$\frac{dr}{d\theta} = 0$$

$\cos\theta = 0$, $\tan\theta = \pm\frac{1}{\sqrt{2}}$

$$\left.\frac{d^2r}{d\theta^2}\right|_{\cos\theta = 0} > 0 \Rightarrow \text{min}$$

$$\left.\frac{d^2r}{d\theta^2}\right|_{\tan\theta = \pm\frac{1}{\sqrt{2}}} < 0 \Rightarrow \text{max}$$

max of $r = \frac{2a\sqrt{3}}{9}$



10. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Basis case: RHS $= (\cos\theta + i\sin\theta)^1$

LHS $= \cos\theta + i\sin\theta$

Inductive:

Assume: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

when $n = k+1$,

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$$

$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= \cos k\theta\cos\theta - \sin k\theta\sin\theta$$
$$+ i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

True for $n = k+1$.

$\therefore$ True for $n \in \mathbb{Z}^+$

$$(\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta$$
$$- 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} \div \frac{\cos^3\theta}{\cos^3\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

Let $\theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$

$$\tan 3\left(\frac{\pi}{12}\right) = \frac{3t - t^3}{1 - 3t^2}$$

$$t^3 - 3t^2 - 3t + 1 = 0$$

$$(t+1)(t^2 - 4t + 1) = 0$$

$$t = -1 / t = 2 \pm \sqrt{3}$$

$$\therefore \tan\frac{\pi}{12} = 2 - \sqrt{3} \qquad \tan\frac{5\pi}{12} = 2 + \sqrt{3}$$

$$\frac{\tan\frac{5\pi}{12}}{\tan\frac{\pi}{12}} = \left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right)$$

$$= 7 + 4\sqrt{3}$$

11. EITHER

$$y = x + 4 + \frac{k}{x+a}$$

$$= \frac{x^2 + (4+a)x + 4a + k}{x+a}$$

$4 + a = 3$ $\qquad b = k - 4$

$a = -1$ $\qquad \therefore k = 1,\ b = -3$

$$\frac{dy}{dx} = 1 - \frac{k}{(x-1)^2} = 0 \quad \text{when } x = 2$$

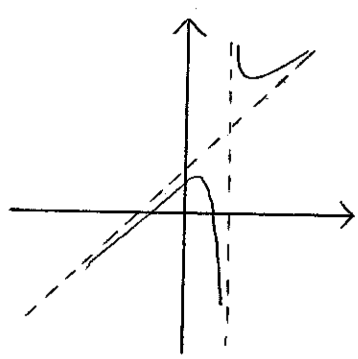
$k = 1,\ b = -3$

i) $\frac{dy}{dx} = 0 \Rightarrow (x-1)^2 = 1$

$\therefore x = 2,\ y = 7$

or $x = 0,\ y = 3$

ii)



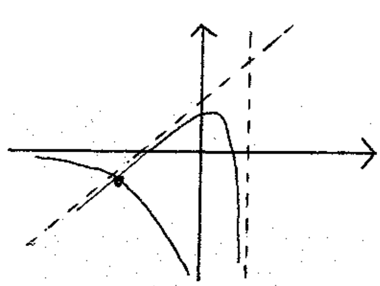
$y = 0$

$$\therefore x = \frac{-3 \pm \sqrt{21}}{2}$$

$$x^3 + 3x^2 + (b-k)x - ka = 0$$

$$x^3 + 3x^2 + bx = k(x+a)$$

$$\frac{x^2 + 3x + b}{x+a} = \frac{k}{x}$$



1 intersection point

$\therefore$ only 1 real root

11. OR

$$A = \begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & 1 & 3 \\ 3 & 7 & -2 & -1 \\ 7 & 16 & -7 & q \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 7 & 11 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 14 & q+28 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q+28}{2} - 11 \end{pmatrix}$$

i) when $q = -6$, $\frac{q+28}{2} - 11 = 0$, $\dim = 2$

$q \neq -6$, $\frac{q+28}{2} - 11 \neq 0$, $\dim = 3$

ii) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 7 \\ 16 \end{pmatrix} \right\}$

Let $\begin{pmatrix} 2 \\ 7 \\ 9 \\ 20 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \\ 7 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 5 \\ 7 \\ 10 \end{pmatrix}$

$\lambda + 2\beta = 2$

$2\lambda + 5\beta = 7$

$\therefore \beta = 3, \lambda = -4$

$\therefore \underset{\sim}{x} = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

iii) If $q \neq -6$, $x_4 = 0$ $x_3 = \lambda$

Basis for $K = \left\{ \begin{pmatrix} 17 \\ -7 \\ 1 \\ 0 \end{pmatrix} \right\}$

iv) If $q = -6$, $x_3 = \mu$, $x_4 = \beta$

Basis for $K' = \left\{ \begin{pmatrix} 17 \\ -7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 26 \\ 11 \\ 0 \\ 1 \end{pmatrix} \right\}$

Since $\underset{\sim}{x} = \mu \begin{pmatrix} 17 \\ -7 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 26 \\ 11 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ $K \subseteq K'$