Discrete random variables

S1/4/6: Uniform Distribution (Discrete)
Poisson Distribution
Geometric Distribution

S1/4/6

Uniform Distribution (Discrete)

Learning Outcome

Students should be able to:

- Identify the relevance of the uniform distribution to the distribution of random events and use the uniform distribution as a model.
- Calculate the probabilities for the distribution.
- Calculate the expected value and variance.

The Uniform Distribution

<u>Example of a discrete uniform distribution</u>

Thrown an ordinary die.

x = number on the die

The probability distribution of x

x	1	2	3	4	5	6
$P(X = x)$	1/6	1/6	1/6	1/6	1/6	1/6

$$P(X = x) = \frac{1}{6} \quad for\ x = 1, 2, 3, 4, 5, 6$$

The Uniform Distribution

<u>Condition for a discrete uniform model</u>

- Random variable *x* is defined over the set of n distinct values $x_1, x_2, \quad \ldots.., x_n$

- Each value is equally likely to occur and

$$P(X = x_r) = \frac{1}{n} \quad for\, r = 1, 2, \ldots., n$$

Example 1

The discrete variable X is such that

$$P(X = x) = c \quad for\ x = 20, 30, 45, 50.$$

Find

(a) The probability distribution of X,

(b) μ , the expectation of X,

(c) $P(X < \mu)$,

Poisson Distribution

(i) Probability density function, expected value and variance.

(ii) Use Poisson distribution as an approximation to binomial distribution .

(iii) Use Normal distribution as an approximation to Poisson distribution.

(iv) Linear combination of independent Poisson variables.

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Poisson Distribution

(i) Probability density function, expected value and variance.

Learning Outcome

Students should be able to:

- Identify the relevance of the Poisson distribution to the distribution of random events and use the Poisson distribution as a model.
- Calculate the probabilities for the distribution Po(μ).
- Use the fact the E(X) = Var(X) = μ.

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Poisson Distribution

(ii) Use Poisson distribution as an approximation to binomial distribution .

Learning Outcome

Students should be able to:

(a) To know the valid conditions for approximation: $n > 50$ and $np < 5$.

(b) Identify the relevance of this approximation and apply in problems involving Binomial distribution.

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Poisson Distribution

(iii) Use Normal distribution as an approximation to Poisson distribution.

Learning Outcome

Students should be able to:

(a) To know the valid condition for approximation: $\mu > 15$.

(b) To make continuity corrections

(c) To identify the relevance of this approximation and apply in problems involving Poisson distribution.

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Poisson Distribution

(iv) Linear combination of independent Poisson variables.

Learning Outcome

Students should be able to:

(a) To write down the distribution of a variable which is a linear combination of independent Poisson variables.

(b) To identify the relevance of the linear combination of independent Poisson variables and apply this in problems involving Poisson variables.

Conditions for a Poisson model

- occur randomly in space or time,
- Events occur singly, events cannot occur simultaneously,
- occur independently,
- occur at a constant rate, that is the mean number of events in a given time interval is proportional to the size of the interval.

Conditions for a Poisson model

- λ = the mean number of occurrences in the given interval.

  x = number of occurrences in the given interval.

$$x \sim Po(\lambda), \text{ where}$$

$$P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!} \quad \text{for } x = 0,1,2,3,\ldots \text{ to infinity}$$

Conditions for a Poisson model

Examples of variables

- the number of emergency calls received by an ambulance
- the number of flaws in a metre length of material.

If $x \sim Po(\lambda)$,

then $P(X = 0) = e^{-\lambda}$ and $P(X = 1) = \lambda e^{-\lambda}$

Example 2:

A student finds that the average number of amoebas in 10 ml of pond water from a particular pond is four. Assuming that the number of amoebas follows a Poisson distribution, find the probability that in a 10 ml sample

(a) There are exactly five amoebas,

(b) There are no amoebas,

(c) There are fewer that three amoebas.

Unit Interval

X = number of amoebas in 10 ml of pond water

$$X \sim Po(4)$$

Y = number of amoebas in 5 ml of pond water

$$Y \sim Po(2)$$

Example 3

On average the school photocopier breaks down eight times during the school week (Monday to Friday) . Assuming that the number of breakdowns can be modelled by a Poisson distribution, find the probability that it breaks down

(a) Five times in a given week,

(b) Once on Monday,

(c) Eight times in a forthnight.

Example 4 Trial - March 07 Q5

A farmer uses a small explosive device as a bird scarer in his apple orchard. The device is programmed to go off at random intervals, between 6 am till 7 pm, so that there are four explosions in a 30 minute period.

Find the probability that there are, in a randomly chosen period of one hour,

(i) Three explosions [2]

(ii) Not more than four explosions [2]

Mean and variance of the Poisson distribution

$$\text{If} \quad X \sim Po(\lambda)$$

$$\text{mean} = \mathrm{E}(x) = \lambda$$

$$\mathrm{Var}(x) = \lambda$$

Example 5

X follows a Poisson distribution with standard deviation 1.5.

Find $P(X \geq 3)$.

S2/1/3:
Poisson as an approximation to the binomial

Learning Outcome

Students should be able to:

- Use the Poisson distribution as an approximation to the binomial distribution.

Poisson as an approximation to the binomial

If $X \sim B(n, p)$, and if $n > 50$ and $np < 5$, then X can be reasonably be approximated by the Poisson distribution $W \sim \text{Po}(np)$.

The larger n and the smaller p, the better the approximation.

Example 6

A Christmas draw aims to sell 5000 tickets, 50 of which will win a prize.

(a) A syndicate buys 200 tickets. Let X represent the number of these tickets that win a prize.

(i) Justify the use of the Poisson approximation for the distribution of X.

(ii) Calculate $P(X \leq 3)$.

(b) Calculate how many tickets should be bought in order for there to be a 90% probability of winning at least one prize.

S2/1/4:

Normal distribution as an approximation to the Poisson

Learning Outcome

Students should be able to:

- Use the Normal distribution as an approximation to the Poisson distribution.

Normal distribution as an approximation to the Poisson

If $X \sim Po(\lambda)$, and if $\lambda > 15$ then X may reasonably be approximated by the Normal distribution $Y \sim \mathrm{N}(\lambda, \lambda)$.

A continuity correction must be applied.

The larger λ the better the approximation.

Example 7:

A radioactive disintegration gives counts that follow a Poisson distribution with a mean count of 25 per second. Find the probability that in a one-second interval the count is between 23 and 27 inclusive.

Example 8

It is thought that the number of serious accidents, X, in a time interval of t weeks, on a given stretch of road, can be modelled by a Poisson distribution with mean $0.4t$. Find the probability of

(a) One or fewer accidents in a randomly chosen 2-week interval,

(b) 12 or more accidents in a randomly chosen year.

Linear combination of independent Poisson variables

For independent variables,

X and Y, if $X \sim Po(m)$ and $Y \sim Po(n)$,

$\Rightarrow X + Y \sim Po(m+n)$.

Example 9:

Two identical racing cars are being tested on a circuit. For each car, the number of mechanical breakdown can be modelled by a Poisson distribution with a mean of one breakdown in 100 laps. If a car breaks down it is attended and continues on the circuit. The first car is tested for 20 laps and the second car for 40 laps.

Find the probability that the service team is called out to attend to breakdowns

(a) once,

(b) More than twice.

S1/4/6

Geometric Distribution

Learning Outcome

Students should be able to:

- Identify the relevance of the Geometric distribution to the distribution of random events and use the Poisson distribution as a model.
- Calculate the probabilities for the distribution Geo(p).
- Calculate the expected value and variance.
- Calculate the probabilities involving Geometric Distribution.

Geometric VS Poisson

The Geometric and Poisson are two other discrete probability distributions that are related to the binomial distribution.

(i) Geometric Distribution

Useful for determining the number of trials needed to achieve success.

(ii) Poisson Distribution

Useful for describing the number of events that will occur during a specific interval of time or space (distance, area or volume).

The Geometric Distribution

Conditions for a Geometric model

- Independent trials are carried out,
- The outcome of each trial is deemed either a success or a failure,
- The number of trials is not fixed; instead the experiment continues until the first success.
- The probability, *p*, of a successful outcome is the same for each trial.

- Useful in business applications .

Example: how many candidates need to be interviewed before the perfect candidate for a job is found?

The discrete random variable, X, is the number of trials needed to obtain the first successful outcome.

$$X \sim \mathrm{Geo}(p)$$

The probability that the first success is obtained at the $r\ th$ attempt is

$$P(X = r) = q^{r-1}\mathrm{p} \quad \text{for } \mathrm{r} = 1, 2, 3, 4, \ldots$$

So that

$$P(X = 1) = \mathrm{p}$$
$$P(X = 2) = q\mathrm{p}$$
$$P(X = 3) = q^2\mathrm{p} \text{ and so on}$$

The Mode of the Geometric Distribution

$X \sim \mathrm{Geo}(p)$

$P(X = 1) = \mathrm{p}$

$P(X = 2) = q\mathrm{p}$

Since $0 < q < 1$ $\Rightarrow qp < \mathrm{p}$

also $P(X = 3) = q^2 p < qp < p$ *and so on*

As n increases, $P(X = n)$ decreases.

From the result,

The mode of any Geometric distribution is 1.

This means that for any value of P, one attempt is the most likely number of attempts to obtain the first success!

Example 10:

Jack is playing a board game in which he needs to throw a six with an ordinary die in order to start the game. Find the probability that

(a) Exactly four attempts are needed t obtain a six,

(b) At least two attempts are needed,

(c) He is successful in throwing a six in three or fewer attempts,

(d) He needs more than three attempts to obtain a six.

Useful results:

If $X \sim \text{Geo}(p)$, $q = 1 - p$

$$P(X \leq x) = 1 - q^x$$

$$P(X > x) = q^x$$

Example 11:

On a particular production line the probability that an item is faulty is 0.08. In a quality control test, items are selected at random from the production line. It is assumed that quality of an item is independent of that of the other items.

(a) Find the probability that the first faulty item

(i) does not occur in the first six selected,

(ii) occurs in fewer than five selections,

(b) There is to be at least 90% chance of picking a faulty item on or before the n *th* attempt. What is the smallest number n?

Expectation and Variance of the Geometric Distribution

If $X \sim \text{Geo}(p)$

$$E(X) = \frac{1}{\text{p}}$$

$$Var(X) = \frac{q}{p^2}$$

Example 12:

When I make a telephone call to an office, the probability of not getting through is 0.45. If I do not get through, then I try again later. Let X denote the number of attempts I have make in order to get through. Stating any necessary assumptions, identify the probability distribution of X. Hence, calculate

(a) $P(X \geq 4)$,

(b) $E(X)$ and $Var(X)$.

Example 13:

Identical independent trials of an experiment are carried out. The probability of a successful outcome is P. On average, five trials are required until a successful outcome occurs.

(a) Find the value of P.

(b) Find the probability that the first successful outcome occurs on the fifth trial.

Summary of Bernoulli, Binomial, Geometric & Poisson distributions

1 Bernoulli distribution

Apply when an experiment has only two possible outcomes, namely, ‘failure’ or ‘success’. The probability function:

$$f(x,p) = p^x(1-p)^{1-x} \quad \text{for } x = 0\,,1$$

where p is the probability of success.

2 Binomial distribution

When n independent Bernoulli trials are carried out, the probability distribution becomes a Binomial distribution.

$$f(x,n,p) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$

3 Geometric distribution

$$f(x,p) = p(1-p)^{x-1} \quad \text{for } x = 1, 2, \ldots$$

Distinguish the probability function of Geometric distribution from that of Bernoulli distribution.

Geometric: Prob. of getting the 1st success in the last of x trials.

Bernoulli: Prob. of getting x successes in one trial.

4 Poisson distribution

Poisson distribution is actually a binomial distribution under the limiting condition that $n \to \infty$ and $p \to 0$ with $np = \lambda$ = constant. Under this restriction, the limiting form of the Binomial distribution is

$$f(x, \lambda) = \frac{\lambda^x e^{-x}}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

Means and Variances

Distribution	Mean	Variance
Bernoulli (p)	p	$p(1 - p)$
Binomial (n , p)	np	$np(1 - p)$
Geometric (p)	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson (λ)	λ	λ