EIGENVALUES AND EIGENVECTORS DIAGONALIZATION

1. (a) The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 3 & -4 & 2 \\ -4 & -1 & 6 \\ 2 & 6 & -2 \end{pmatrix}$$

Find the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. [6]

Hence find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^3 = \mathbf{PDP}^{-1}$. [2]

2. Given that $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{2} & \frac{-1}{6} \\ \frac{1}{6} & 0 & \frac{-1}{6} \\ \frac{-1}{6} & \frac{1}{4} & \frac{5}{12} \end{pmatrix}$.

(i) Find the eigenvalue that corresponds to this eigenvector. [2]

(ii) Find a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{QDQ}^{-1}$. [10]

(iii) Deduce that, as the positive integer n tends to infinity, all the elements of matrix $\mathbf{A}^n$ tend to zero. [2]

3. If $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix}$, using the eigenvalues and eigenvectors of $\mathbf{A}$, find $\mathbf{A}^{10}$. [9]

4. If $\mathbf{A}$ is a square matrix, show that $\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x}$ for $k = 2, 3$, where λ is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{x}$. Given $\mathbf{A} = \begin{pmatrix} 5 & 2 & -4 \\ 7 & -8 & -7 \\ -4 & 12 & 5 \end{pmatrix}$, find a square matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \mathbf{PDP}^{-1}$. [10]

5. The matrix **A** has eigenvalue λ with corresponding eigenvector **x**. If **M** is a non-singular matrix and $(\mathbf{M}^{-1})^n\mathbf{A}\mathbf{M}^n$ has an eigenvector $(\mathbf{M}^{-1})^n\mathbf{x}$, find the corresponding eigenvalue. [3]

If the matrix **A** has eigenvalues a, b, c with corresponding eigenvectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ and $\mathbf{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, find the eigenvalues and corresponding eigenvectors of $(\mathbf{M}^{-1})^n\mathbf{A}\mathbf{M}^n$. [3]

If $\mathbf{P}^{-1}[(\mathbf{M}^{-1})^n\mathbf{A}\mathbf{M}^n]^k\mathbf{P} = \mathbf{D}^k$, where **P** is a square matrix and **D** is a diagonal matrix, find **P** and **D**. [2]

6.

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $bc - ad > 0$, find the eigenvalues of **A**. [4]

The eigenvalues of **A** are λ_1, λ_2 with corresponding eigenvectors $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Find u_1, u_2, v_1 and v_2 in terms of λ_1, λ_2, a and b. [6]

If $\mathbf{P} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, show that $\mathbf{AP} = \mathbf{PD}$. [4]

7. If $\mathbf{A} = \begin{pmatrix} 1 & 5 & 7 \\ 0 & 2 & 9 \\ 0 & 0 & 3 \end{pmatrix}$, find the eigenvalues and corresponding eigenvectors of **A**. [6]

Find also a square matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [2]

8. Find the eigenvalues of the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}.$$ [3]

Find the corresponding eigenvectors. [3]

Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$. [2]

Find $\mathbf{M}^{10}$ as a single matrix. [4]

9. The vector $\mathbf{x}$ is an eigenvector of the square matrix $\mathbf{A}$ with corresponding eigenvalue λ. Show that if $\mathbf{A}^{-1}$ exists, then $\lambda \neq 0$. Show also that the vector $\mathbf{x}$ is an eigenvector of $\mathbf{A}^{-1}$ and find the corresponding eigenvalue. [4]