TAYLOR'S COLLEGE

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CAMBRIDGE A LEVEL PROGRAMME

A2 TRIAL EXAMINATION AUGUST 2013

(July 2012 Intake)

Wednesday **21 August 2013** **12.30 pm – 3.30 pm**

FURTHER MATHEMATICS 9231/12

PAPER 1

3 hours

Additional materials: Answer Booklet/Paper
List of formulae (MF 10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total marks for this paper is **100**.

This document consists of **5** printed pages.

1 Prove by mathematical induction that, for every positive integer $n \geq 2$ and $x, y > 0$,

$$\left(\frac{x+y}{2}\right)^n \leq \frac{x^n + y^n}{2}.$$ [5]

2 The cubic equation $x^3 - 2x - 3 = 0$ has roots α, β, γ. Form a cubic equation having roots $\frac{2}{1-\alpha-\beta}, \frac{2}{1-\alpha-\gamma}, \frac{2}{1-\beta-\gamma}$. [5]

3 The curve C is defined by the equation $3y^2 = 5x^3$. Show that the length of arc of C from the point where $x = 0$ to $x = 3$ is $\frac{67}{9}$ units. [5]

4 Show that $\frac{4}{(2r+1)(2r+3)(2r+5)}$ can be written in the form of

$\frac{A}{(2r+1)(2r+3)} + \frac{B}{(2r+3)(2r+5)}$, where A and B are constants to be determined. [2]

Hence show that $\frac{1}{5 \times 7 \times 9} + \frac{1}{7 \times 9 \times 11} + \frac{1}{9 \times 11 \times 13} + \cdots = \frac{1}{140}$. [4]

5 The curve C has equation

$$x^3 - axy + 3ay^2 = 3a^3, \text{ where } a \text{ is a positive integer.}$$

(i) Show that $\frac{dy}{dx} = -\frac{2}{5}$ at the point $P(a\,, a)$ on C. [3]

(ii) Find the value of $\frac{d^2y}{dx^2}$ at point P. [4]

6 The curve C has equation

$$y = 1 + \frac{1}{x+1} + \frac{2}{(x+1)^2}.$$

(i) Find the range of y by using the algebraic method. [3]

(ii) Show that the set of values of x for which $\frac{dy}{dx} < 0$ is

$$\{x : x < -5\} \cup \{x : x > -1\}.$$ [5]

7 Let

$$I_n = \int_0^1 (1 + x^2)^{\frac{n}{2}-1}\, dx,$$

where $n \geq 0$. Use the fact that $1 + \tan^2\theta = \sec^2\theta$ to show that, for $n \geq 2$,

$$(n-1)I_n = \left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}.$$ [6]

Hence, evaluate I_3 in the exact form. [3]

8 A curve is defined parametrically by

$$x = 2t^2, \quad y = 4t.$$

The part of the curve joining the origin to the point where $t = 1$ is rotated through one complete revolution about the x-axis.

(i) Find the area of surface generated. [4]

(ii) Prove that the centroid of solid is at a distance of $\frac{4}{3}$ units from the origin O. [5]

9 The equations of two planes are

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix},$$

where s_1, s_2, t_1, t_2 are scalar multiples.

(i) Find the cosine of the acute angle between both planes. [4]

(ii) Both planes intersect at line l. Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, the equation of l. [3]

(iii) Show that the perpendicular distance from the point $(2, 1, 3)$ to line l is $\sqrt{\dfrac{5}{13}}$. [3]

10 Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}.$$ [8]

Find a non-singular matrix $\mathbf{M}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} + 3\mathbf{I})^2 = \mathbf{MDM}^{-1}$, where $\mathbf{I}$ is the 3×3 identity matrix. [3]

11 Show that the substitution $x = e^t$ reduces the differential equation

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 6x.$$

to the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 6e^t.$$ [6]

Find the general solution for y in terms of x. [6]

12 Answer only **one** of the following two alternatives.

EITHER

The linear transformation T: $\mathbb{R}^3 \to \mathbb{R}^3$ is represented by matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{pmatrix}, a \neq b, b = c \text{ and } a, b, c \in \mathbb{R}.$$

(i) The range space of T is R. In either order,

(a) show that the dimension of R is 2. [5]

(b) obtain a basis for R. [1]

(c) find a basis for the null space of T. [2]

(ii) If $a = -1$ and $b = 1$, find the general solution for $\mathbf{Mx} = \mathbf{v}$ where $\mathbf{v} = \begin{pmatrix} -6 \\ 4 \\ 4 \end{pmatrix}$. [3]

(iii) Show that there is a changes in the dimension of R if $a = b = c$. State the new dimension of R. [2]

OR

The polar curve C and line l are defined by the equations

$$C: \; r = 4\sqrt{3}\cos 2\theta$$
$$l: \; r = 3\sec\theta$$

where $0 \leq \theta \leq \dfrac{\pi}{4}$.

C and l intersect at the point with polar coordinates $P\left(2\sqrt{3}, \dfrac{\pi}{6}\right)$.

(i) Sketch both C and l on a single diagram. Show clearly the position of point P. [3]

(ii) Find the exact area enclosed by C. [4]

(iii) Show that l divides the area in **(ii)** in the ratio 1: 2. [6]