1. i) $u_1 = 1$

$u_2 = u_1 + 2 = 1 + 2 = 3$

$u_3 = u_2 + 2 = 3 + 2 = 5$

$u_4 = u_3 + 2 = 5 + 2 = 7$

$u_5 = u_4 + 2 = 7 + 2 = 9$

Through observation, n-th term is an odd number.
i.e. $u_n = 2n - 1$.

ii) Suppose $u_K = 2K - 1$ is true for some positive integers K.

Then $u_{K+1} = u_K + 2$
$= 2K - 1 + 2$
$= 2K + 1$
$= 2(K+1) - 1$

$\therefore u_n = 2n - 1$ for all natural numbers n.

2. Let $\alpha - h, \alpha, \alpha + h$ be the roots of $x^3 + px^2 + r = 0$, since they're in arithmetic progression.

i) $\alpha - h + \alpha + \alpha + h = -p$
$3\alpha = -p$
$\alpha = -\frac{p}{3}$

ii) $\alpha(\alpha+h) + (\alpha - h)(\alpha + h) + (\alpha - h)\alpha = 0$
$3\alpha^2 = h^2$
$h^2 = \frac{p^2}{3}$

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$\alpha(\alpha+h)(\alpha-h) = -r$
$\alpha(\alpha^2 - h^2) = -r$
$-\frac{p}{3}\left(\frac{p^2}{9} - \frac{p^2}{3}\right) = -r$
$2p^3 + 27r = 0$

2. $$\left.\begin{aligned} x + y + \lambda z &= \lambda^2 \\ x + \lambda y + z &= \lambda \\ \lambda x + y + z &= 1 \end{aligned}\right\}$$

$$\begin{aligned} &-① + ②: \\ &-\lambda \times ① + ③: \end{aligned} \quad \left.\begin{aligned} x + y + \lambda z &= \lambda^2 \\ (\lambda - 1)y + (1-\lambda)z &= \lambda - \lambda^2 \\ (1-\lambda)y + (1-\lambda^2)z &= 1 - \lambda^3 \end{aligned}\right\}$$

when $\lambda = 1$: $$\left.\begin{aligned} x &= 1 - y - z \\ 0y + 0z &= 0 \\ 0y + 0z &= 0 \end{aligned}\right\} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-s-t \\ s \\ t \end{pmatrix},$$

$s, t \in R$.

when $\lambda \neq 1$: $$\left.\begin{aligned} x &= \lambda^2 - y - \lambda z \\ y &= \frac{\lambda - \lambda^2 - (1-\lambda)z}{\lambda - 1} \\ (\lambda + 2)z &= \frac{\lambda^3 + \lambda^2 - \lambda - 1}{\lambda - 1} \end{aligned}\right\}$$

when $\lambda = -2$: $$\begin{aligned} x &= 4 - y + 2z \\ y &= \frac{-6 - 3z}{-3} \\ 0z &= 1 \end{aligned}$$

$\therefore$ no solution

$\therefore$ $\lambda = 1$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-s-t \\ s \\ t \end{pmatrix}$

$\lambda = -2$: nosolution

$\lambda \neq 1, -2$: unique solution.

3. i) $\dfrac{2}{r^2-1} = \dfrac{r+1-(r-1)}{(r+1)(r-1)} = \dfrac{1}{r-1} - \dfrac{1}{r+1}$

ii) a) $S_n = \displaystyle\sum_{r=2}^{n} \frac{1}{r-1} - \frac{1}{r+1}$

$$= \frac{1}{1} - \frac{1}{3}$$

$$+ \frac{1}{2} - \frac{1}{4}$$

$$+ \frac{1}{3} - \frac{1}{5}$$

$$+ \frac{1}{4} - \frac{1}{6}$$

$$\vdots$$

$$+ \frac{1}{n-2} - \frac{1}{n}$$

$$+ \frac{1}{n-1} - \frac{1}{n+1}$$

$$= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

ii) b) $\displaystyle\lim_{n \to \infty} S_n = \frac{3}{2}.$

4. i) $y = 1 - x + \dfrac{2}{x-2}$

$\Rightarrow y(x-2) = (1-x)(x-2) + 2$

$\Rightarrow x^2 + (y-3)x - 2y = 0$

$a = 1 \quad b = y - 3 \quad c = -2y$

$b^2 - 4ac = (y-3)^2 - 4 \cdot 1 \cdot (-2y)$

$= y^2 + 2y + 9$

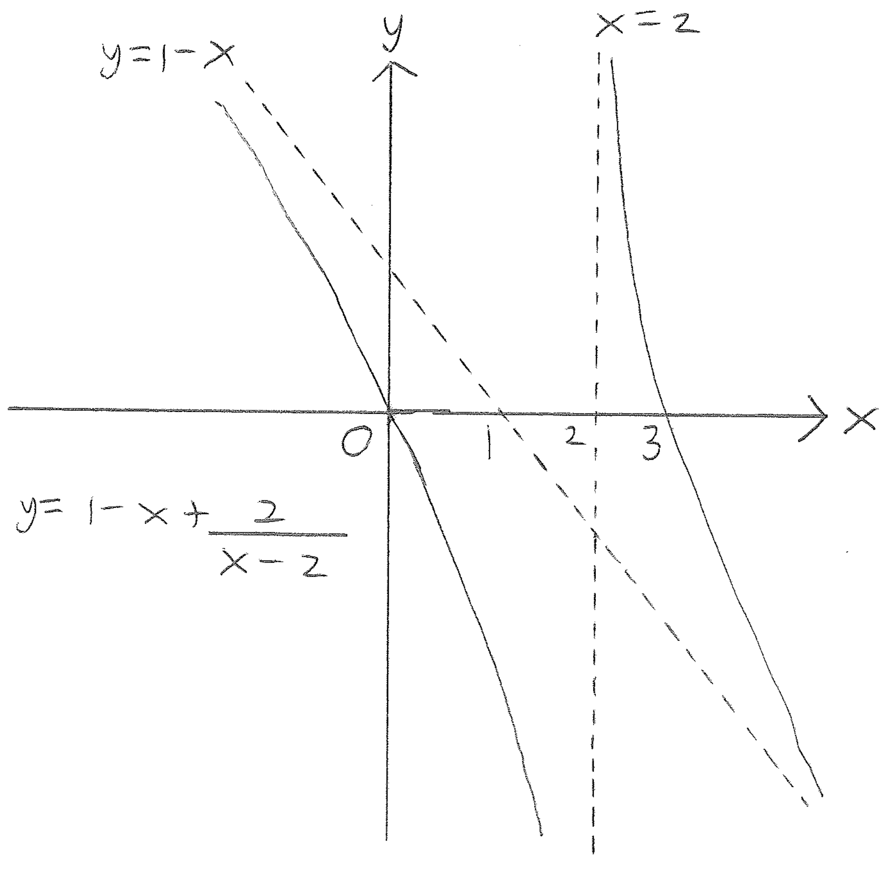
$= (y+1)^2 + 8$

Since $b^2 - 4ac > 0$, $\forall y \in \mathbb{R}$, the range of the curve is $y \in \mathbb{R}$.

ii) $\dfrac{dy}{dx} = -1 - \dfrac{2}{(x-2)^2} = -\left[1 + \dfrac{2}{(x-2)^2}\right] < 0 \quad \forall x \in \mathbb{R}$.

$\Rightarrow C$ is decreasing.

iii)



5. i) l is perpendicular to the directions of the normals of both planes.

$\therefore$ l has direction $\begin{pmatrix}1\\-1\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\5\\3\end{pmatrix}$.

Since $(0,1,1)$ is a point on both planes, the equation of l is $\underset{\sim}{r} = \begin{pmatrix}0\\1\\1\end{pmatrix} + t\begin{pmatrix}-1\\5\\3\end{pmatrix}$.

$$\Rightarrow \frac{x}{-1} = \frac{y-1}{5} = \frac{z-1}{3}$$

ii) The direction of the normal of the plane is $\underset{\sim}{n} = \overrightarrow{AB} \times \underset{\sim}{m}_l$

$$= \begin{pmatrix}-1\\1\\0\end{pmatrix} \times \begin{pmatrix}-1\\5\\3\end{pmatrix} = \begin{pmatrix}3\\3\\-4\end{pmatrix}$$

$\therefore$ The equation of the plane passing through A and containing l is

$$\underset{\sim}{r} \cdot \begin{pmatrix}3\\3\\-4\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} \cdot \begin{pmatrix}3\\3\\-4\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}3\\3\\-4\end{pmatrix} = -1$$

$$3x + 3y - 4z = -1.$$

iii) The perpendicular distance from A to l is $\dfrac{|\overrightarrow{AB} \times m_l|}{|m_l|} = \dfrac{\sqrt{3^2+3^2+(-4)^2}}{\sqrt{(-1)^2+5^2+3^2}} = \sqrt{\dfrac{34}{35}}$

6. i) Let $u = -\alpha + \beta + \gamma$

$$u + 2\alpha = \alpha + \beta + \gamma$$
$$= 1$$
$$\alpha = \frac{1-u}{2}$$

$$\therefore \left(\frac{1-u}{2}\right)^3 + \left(\frac{1-u}{2}\right)^2 - 3\left(\frac{1-u}{2}\right) - 10 = 0$$

$$\Rightarrow (1-u)^3 - 2(1-u)^2 - 12(1-u) - 10 = 0$$

$$1 - 3u + 3u^2 - u^3 - 2 + 4u - 2u^2 - 12 + 12u - 10 = 0$$

$$u^3 - u^2 - 13u + 23 = 0$$

ii) Let the cubic equation whose roots are p, q, r be $x^3 + bx^2 + cx + d = 0$.

$$p + q + r = -b = 0$$
$$b = 0$$

$$p^2 + q^2 + r^2 = (p+q+r)^2 - 2(pq + pr + qr)$$
$$2 = 0^2 - 2c$$
$$c = -1$$

$$S_n = \alpha^n + \beta^n + \gamma^n$$

Then $S_3 - S_1 + dS_0 = 0$

$$0 - 0 + 3d = 0$$
$$\therefore d = 0$$

$\therefore$ The equation is $x^3 - x = 0$.

with roots $x = 0, 1, -1$.

$\therefore$ p, q, r is any permutation of $0, 1, -1$.

7. EITHER

i) a) $(0,0,0,0)$ is a solution of P_0.

b) i) Since
$$\left.\begin{aligned} a_{11}s_1 + a_{12}s_2 + a_{13}s_3 + a_{14}s_4 = 0 \\ a_{21}s_1 + a_{22}s_2 + a_{23}s_3 + a_{24}s_4 = 0 \end{aligned}\right\} - ①$$

$$\left.\begin{aligned} \lambda(a_{11}s_1 + a_{12}s_2 + a_{13}s_3 + a_{14}s_4) = \lambda(0) \\ \lambda(a_{21}s_1 + a_{22}s_2 + a_{23}s_3 + a_{24}s_4) = \lambda(0) \end{aligned}\right\}$$

$$\left.\begin{aligned} a_{11}(\lambda s_1) + a_{12}(\lambda s_2) + a_{13}(\lambda s_3) + a_{14}(\lambda s_4) = 0 \\ a_{21}(\lambda s_1) + a_{22}(\lambda s_2) + a_{23}(\lambda s_3) + a_{14}(\lambda s_4) = 0 \end{aligned}\right\}$$

$\therefore (\lambda s_1, \lambda s_2, \lambda s_3, \lambda s_4)$ is a solution of P_1.

(b) (ii) Since
$$\left.\begin{aligned} a_{11}t_1 + a_{12}t_2 + a_{13}t_3 + a_{14}t_4 = 0 \\ a_{21}t_1 + a_{22}t_2 + a_{23}t_3 + a_{24}t_4 = 0 \end{aligned}\right\} - ②$$

① + ② :

$$\left.\begin{aligned} a_{11}(s_1 + t_1) + a_{12}(s_2 + t_2) + a_{13}(s_3 + t_3) + a_{14}(s_4 + t_4) = 0 \\ a_{21}(s_1 + t_1) + a_{12}(s_2 + t_2) + a_{13}(s_3 + t_3) + a_{24}(s_4 + t_4) = 0 \end{aligned}\right\}$$

$\Rightarrow (s_1 + t_1, s_2 + t_2, s_3 + t_3, s_4 + t_4)$ is a solution of P_0.

② + ③ :

$$\left.\begin{aligned} a_{11}(d_1 + t_1) + a_{12}(d_2 + t_2) + a_{13}(d_3 + t_3) + a_{14}(d_4 + t_4) = b_1 \\ a_{21}(d_1 + t_1) + a_{12}(d_2 + t_2) + a_{23}(d_3 + t_3) + a_{24}(d_4 + t_4) = b_2 \end{aligned}\right\}$$

$(d_1 + t_1, d_2 + t_2, d_3 + t_3, d_4 + t_4)$ is a solution of P_1.

ii) h) $(1,0,0,0)$ is a solution of

$$\left.\begin{aligned} x_1+x_2-2x_3+x_4&=1 \\ 2x_1+x_2-x_3+x_4&=2 \end{aligned}\right\}.$$

$$\left.\begin{aligned} x_1+x_2-2x_3+x_4&=1 \\ 2x_1+x_2-x_3+x_4&=2 \end{aligned}\right\}$$

$-2\times(1)+(2)$: $$\left.\begin{aligned} x_1+x_2-2x_3+x_4&=1 \\ -x_2+3x_3-x_4&=0 \end{aligned}\right\}$$

Let $x_3=s$ & $x_4=t$, $s,t\in\mathbb{R}$.

$$x_2=3s-t$$
$$x_1=1-s$$

$\therefore$ The solution is

$$\begin{pmatrix} 1-s \\ 3s-t \\ s \\ t \end{pmatrix}+\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 2-s \\ 3s-t \\ s \\ t \end{pmatrix}.$$

OR

i) Given that $\sum_{r=1}^{2} \frac{1}{r^2} = \frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4} < 2 - \frac{1}{2} = \frac{3}{2}$

Suppose $\sum_{r=1}^{k} \frac{1}{r^2} < 2 - \frac{1}{k}$ for $n = k \geqslant 2$.

$$\therefore \sum_{r=1}^{k+1} \frac{1}{r^2} = \sum_{r=1}^{k} \frac{1}{r^2} + \frac{1}{(k+1)^2}$$

$$< 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$= 2 + \frac{-(k+1)^2 + k}{k(k+1)^2}$$

$$= 2 - \frac{k^2+k+1}{k(k+1)^2}$$

$$< 2 - \frac{k^2+k}{k(k+1)^2}$$

$$= 2 - \frac{k(k+1)}{k(k+1)^2}$$

$$= 2 - \frac{1}{k+1}$$

$\therefore \sum_{r=1}^{n} \frac{1}{r^2} < 2 - \frac{1}{n}$ for every positive integer $n \geqslant 2$.

ii) a) $u_n = n^2(2n-1)$

b)
$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} r^2(2r-1)$$

$$= 2\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2$$

$$= 2\left(\frac{1}{4}n^2(n+1)^2\right) - \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{n(n+1)}{6}\left(3n(n+1) - (2n+1)\right)$$

$$= \frac{n(n+1)}{6}(3n^2+n-1)$$

$$S_n = \ln 2 + 4(3)\ln 2 + 9(5)\ln 2 + 16(7)\ln 2 + \dots$$

$$= [1(1) + 4(3) + 9(5) + 16(7) + \dots]\ln 2$$

$$= \ln 2 \sum_{r=1}^{n} u_r$$

$$= \frac{n}{6}(n+1)(3n^2+n-1)\ln 2$$

when $n \to \infty$, $$\frac{\frac{n}{6}(n+1)(3n^2+n-1)\ln 2}{n^4}$$

$$= \frac{1}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{3n^2+n-1}{n^2}\right)\ln 2$$

$$= \frac{1}{6}(1)\left(1+\frac{1}{n}\right)\left(3+\frac{1}{n}-\frac{1}{n^2}\right)\ln 2$$

$$= \frac{1}{6}(1)(1)\,3\ln 2$$

$$= \frac{1}{2}\ln 2.$$