1. $S_N = 8\sum n^3 - 6\sum n^2$

$= 8 \cdot \frac{1}{4} N^2(N+1)^2 - 6 \cdot \frac{1}{6} N(N+1)(2N+1)$

$= N(N+1)[2N(N+1) - (2N+1)]$

$= N(N+1)(2N^2-1)$

$\sum_{N+1}^{2N} (8n^3 - 6n^2)$

$= S_{2N} - S_N$

$= 2N(2N+1)(2(2N)^2-1) - N(N+1)(2N^2-1)$

$= N[2(2N+1)(8N^2-1) - (N+1)(2N^2-1)]$

$= N[30N^3 + 14N^2 - 3N - 1]$

2.

$$
\begin{array}{r|l}
 & x \\
\hline
x+3 & x^2 + ax + 1 \\
 & x^2 + 3x \\
\hline
 & (a-3)x + 1 \\
 & (a-3)x + 3a - 9 \\
\hline
 & 10 - 3a
\end{array}
$$

Asymptotes are : $x = -3$,

$y = x + a - 3$.

3. $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3$ are independent.

$\Rightarrow a\underset{\sim}{x}_1 + b\underset{\sim}{x}_2 + c\underset{\sim}{x}_3 = \underset{\sim}{0}$

where not all $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$ are zero.

$\Rightarrow aM\underset{\sim}{x}_1 + bM\underset{\sim}{x}_2 + cM\underset{\sim}{x}_3$

$= M(a\underset{\sim}{x}_1 + b\underset{\sim}{x}_2 + c\underset{\sim}{x}_3)$

$= M\underset{\sim}{0}$

$= \underset{\sim}{0}$.

$\therefore M\underset{\sim}{x}_1, M\underset{\sim}{x}_2, M\underset{\sim}{x}_3$ are linearly dependent.

a) Let $a\underset{\sim}{y}_1 + b\underset{\sim}{y}_2 + c\underset{\sim}{y}_3 = \underset{\sim}{0}$

$$\Rightarrow \begin{pmatrix} \underset{\sim}{y}_1 & \underset{\sim}{y}_2 & \underset{\sim}{y}_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 5 \\ 5 & -3 & 51 \\ 7 & 4 & 55 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve $\left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 5 & -3 & 51 & 0 \\ 7 & 4 & 55 & 0 \end{array}\right)$ by Gaussian Elimination

to obtain $a = 9$, $b = -2$, $c = -1$ i.e. $9\underset{\sim}{y}_1 - 2\underset{\sim}{y}_2 - \underset{\sim}{y}_3 = 0$

OR show that $\begin{vmatrix} 1 & 2 & 5 \\ 5 & -3 & 51 \\ 7 & 4 & 55 \end{vmatrix} = 0$ and conclude that

many solution is the nature for the system.

b) Basis of $P\underset{\sim}{y}_1, P\underset{\sim}{y}_2, P\underset{\sim}{y}_3$

$= \{ P\underset{\sim}{y}_1, P\underset{\sim}{y}_2 \}$

$$= \left\{ \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 2 \\ 45 \\ -49 \end{pmatrix}, \begin{pmatrix} 13 \\ 7 \\ -14 \end{pmatrix} \right\}$$

4 Eigenvalues are 1,3,4

$(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

when $\lambda = 1$, $\begin{pmatrix} 0 & 2 & -3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_3 = 0, \; x_2 = 0$

$\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

when $\lambda = -3$, $\begin{pmatrix} -2 & 2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_3 = 0, \; x_1 = x_2$

$\underset{\sim}{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

when $\lambda = 4$, $\begin{pmatrix} -3 & 2 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$r_2 \Rightarrow x_2 = -x_3$

$r_1 \Rightarrow x_1 = \frac{-5x_3}{3}$

let $x_3 = \lambda$,

$\underset{\sim}{e}_2 = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 4^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 243 & 0 \\ 0 & 0 & 1024 \end{pmatrix}$

$Q = \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{pmatrix}$

5. $\frac{d}{dx}(\tan x \sec^n x)$

$$= \tan x \cdot n\sec^{n-1}x \cdot \sec x \tan x + \sec^2 x \cdot \sec^n x$$

$$= n\sec^n x \tan^2 x + \sec^{n+2} x$$

$$= n\sec^n x(\sec^2 x - 1) + \sec^{n+2} x$$

$$= n\sec^{n+2} x - n\sec^n x + \sec^{n+2} x$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} n\sec^{n+2} x - n\sec^n x + \sec^{n+2} x \, dx$$

$$= \tan x \sec^n x \Big|_0^{\frac{\pi}{4}}$$

$$\Rightarrow (n+1) I_{n+2} = \frac{1}{\cos^n \frac{\pi}{4}} + n I_n$$

$$\Rightarrow (n+1) I_{n+2} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^n} + n I_n$$

$$\Rightarrow (n+1) I_{n+2} = (2^{\frac{1}{2}})^n + n I_n$$

$$\Rightarrow (n+1) I_{n+2} = 2^{\frac{n}{2}} + n I_n$$

$$I_{n+2} = \frac{1}{n+1} 2^{\frac{n}{2}} + \frac{n}{n+1} I_n$$

$$I_6 = \frac{1}{5} 2^2 + \frac{4}{5}[I_4]$$

$$= \frac{4}{5} + \frac{4}{5}\left[\frac{1}{3} \cdot 2^1 + \frac{2}{3} I_2\right]$$

$$= \frac{4}{5} + \frac{8}{15} + \frac{8}{15}[1]$$

$$= \frac{28}{15}$$

6. $x^3 + x + 12 = 0$

$$\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$$
$$= 0 - 2(1)$$
$$= -2$$

- $\sum \alpha^2 < 0 \Rightarrow$ not all roots are real
- degree 3 $\Rightarrow$ 3 roots
- all coefficients are real $\Rightarrow$ complex roots occur in conjugate pairs.

$\Rightarrow$ only one real root.

Let $f(x) = x^3 + x + 12$

$$f(-3) = -27 - 3 + 12 = -18 < 0$$
$$f(-2) = -8 - 2 + 12 = 2 > 0$$

$\Rightarrow -3 < \alpha < -2$

Let $\beta = a + ib \Rightarrow \beta^* = a - ib$

$$|\beta| = \sqrt{a^2 + b^2}, \quad |\beta^*| = \sqrt{a^2 + b^2}$$

to show $|\beta| = |\beta^*| \Rightarrow (2, \sqrt{6})$

consider $\alpha\beta\beta^* = -12 \quad (\alpha\beta\gamma = -\frac{d}{a})$

$$\therefore |\alpha\beta\beta| = 12$$
$$|\alpha||\beta||\beta^*| = 12$$
$$|\alpha| = \frac{12}{|\beta||\beta^*|}$$

Since $-3 < \alpha < -2$

$\Rightarrow \quad 2 < |\alpha| < 3$

$\Rightarrow \quad 2 < \dfrac{12}{|\beta||\beta^*|} < 3$

$\Rightarrow \quad 2 < \dfrac{12}{a^2 + b^2} < 3$

$\Rightarrow \quad \dfrac{1}{3} < \dfrac{a^2 + b^2}{12} < \dfrac{1}{2}$

$\Rightarrow \quad 4 < a^2 + b^2 < 6$

$\Rightarrow \quad 2 < \sqrt{a^2 + b^2} < \sqrt{3}$

7. a) $S_\infty = m$

$\Rightarrow \quad m = \dfrac{9m+3}{m+7}$

$\Rightarrow \quad m^2 + 7m = 9m + 3$

$\Rightarrow \quad m^2 - 2m - 3 = 0$

$\Rightarrow \quad (m-3)(m+1) = 0$

$\Rightarrow \quad m = 3, -1 \quad$ but $\quad m > 0$

$\therefore \; m = 3$

b) $u_{n+1} = \dfrac{9u_n + 3}{u_n + 7}$

$u_n(u_{n+1} - 9) = 3 - 7u_{n+1}$

$u_n = \dfrac{3 - 7u_{n+1}}{u_{n+1} - 9}$

$\therefore \; 3 - u_n = 3 - \dfrac{(3 - 7u_{n+1})}{u_{n+1} - 9}$

$= \dfrac{10u_{n+1} - 30}{u_{n+1} - 9}$

Assumed $u_n < 3$,

$3 - u_n > 0$

$\frac{10(u_{n+1} - 3)}{u_{n+1} - 9} > 0$

$\frac{10(3 - u_{n+1})}{9 - u_{n+1}} > 0$ $\quad \because u_{n+1} - 3 < 0$

$3 - u_{n+1} > 0$

$3 > u_{n+1}$

$\Rightarrow u_{n+1} < 3$

It is true for u_{n+1}

$\therefore$ with $u_1 = 1$,

By induction, $u_n < 3$

8. a) $(x^2 + y^2)^2 = 4xy$

$\Rightarrow (r^2\cos^2\theta + r^2\sin^2\theta)^2 = 4r\cos\theta \, r\sin\theta$

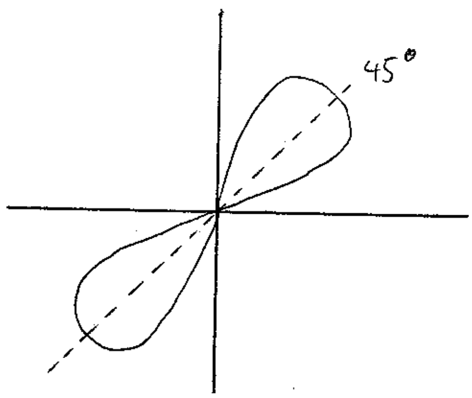
$\Rightarrow (r^2)^2 = 4r^2 \cos\theta \sin\theta$

$\Rightarrow r^2 = 2\sin 2\theta$

b) $\sin 2\theta \Rightarrow$ symmetrical about vertical line

$\Rightarrow 2\theta = 90^\circ$

$\theta = 45^\circ$



θ	0	30°	45°	60°	90°
r	0	± 1.32	± 1.41	± 1.32	0

c) $d_{max} = \sqrt{2}$

9. $x = \theta \sin\theta + \cos\theta$, $y = (\theta^2 + 4)\sin\theta$

a) $\frac{dx}{d\theta} = \theta\cos\theta + \sin\theta - \sin\theta = \theta\cos\theta$

$\frac{dy}{d\theta} = (\theta^2+4)\cos\theta + 2\theta\sin\theta$

$\Rightarrow \frac{dy}{dx} = \frac{(\theta^2+4)\cos\theta + 2\theta\sin\theta}{\theta\cos\theta}$

$= \theta + \frac{4}{\theta} + 2\tan\theta$

b) $\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$

$= \left[1 - \frac{4}{\theta^2} + 2\sec^2\theta\right] \cdot \frac{1}{\theta\cos\theta}$

OR $= \frac{1}{\theta}\sec\theta - \frac{4\sec\theta}{\theta^3} + \frac{2\sec^3\theta}{\theta}$

c) $\frac{d^2y}{dx^2} = 0$

$\Rightarrow 1 - \frac{4}{\theta^2} + 2\sec^2\theta = 0$

$\Rightarrow 1 - \frac{4}{\theta^2} + 2(1 + \tan^2\theta) = 0$

$\Rightarrow 2\tan^2\theta - \frac{4}{\theta^2} + 3 = 0$

$\Rightarrow \theta^2(2\tan^2\theta + 3) = 4$

10. $z = e^{i\theta}$

$z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos n\theta - i\sin n\theta$

$① + ② \Rightarrow z^n + \frac{1}{z^n} = 2\cos n\theta$

$① - ② \Rightarrow z^n - \frac{1}{z^n} = 2i\sin n\theta$

By $z - \frac{1}{z} = 2i\sin\theta$

$$\left(z - \frac{1}{z}\right)^6 = (2i\sin\theta)^6$$

$$\Rightarrow 2^6 i^6 \sin^6\theta = z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$$

$$\Rightarrow -2^6\sin^6\theta = \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

$$\sin^6\theta = \frac{-1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta - \frac{15}{32}\cos 2\theta + \frac{5}{16}$$

$$y_m = \frac{1}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \sin^6\theta \, d\theta$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \frac{-1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta - \frac{15}{32}\cos 2\theta + \frac{5}{16} \, d\theta$$

$$= \frac{4}{\pi}\left[\frac{-1}{32}\frac{\sin 6\theta}{6} + \frac{3}{16}\frac{\sin 4\theta}{4} - \frac{15}{32}\frac{\sin 2\theta}{2} + \frac{5\theta}{16}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{4}{\pi}\left[\frac{1}{192} - \frac{15}{64} + \frac{5}{16}\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{5}{16} - \frac{11}{12\pi}$$

11 a) $\underset{\sim}{n} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 1 & 0 \\ 0 & 1 & -2 \end{vmatrix} = \begin{pmatrix} -2 \\ 6 \\ 3 \end{pmatrix}$

$\underset{\sim}{n} \cdot \underset{\sim}{b}_1 = \begin{pmatrix} -2 \\ 6 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = 0$

$\Rightarrow \underset{\sim}{n} \perp \underset{\sim}{b}_1 \Rightarrow \underset{\sim}{n} \perp l_1 \Rightarrow \Pi \,//\, l_1$

b) let $l_1 = \Pi$

$\Rightarrow \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} + \theta \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \phi \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$\underset{\sim}{i} \Rightarrow 2 + \mu = 6 + 3\theta \quad - (1)$

$\underset{\sim}{j} \Rightarrow 1 + 3\mu = 2 + \theta + \phi \quad - (2)$

$\underset{\sim}{k} \Rightarrow -3 - 2\mu = 1 - 2\phi \quad - (3)$

substitute (3) & (1) into (2) $\Rightarrow \mu = 1$

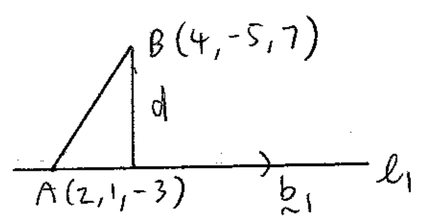
$\therefore$ from (1) $\Rightarrow \theta = -1$

from (3) $\Rightarrow \phi = 3$

$\therefore$ position vector $= \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$

c)



Let $B = (4, -5, 7)$, $A = (2, 1, -3)$

$d = |\overrightarrow{AB}| \sin\theta$

$= \dfrac{|\overrightarrow{AB} \times \underset{\sim}{b}_1|}{|\underset{\sim}{b}_1|} = \dfrac{\left\| \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -6 & 10 \\ 3 & -1 & 4 \end{vmatrix} \right\|}{\sqrt{9+1+16}} = \dfrac{1}{\sqrt{26}} \left| \begin{pmatrix} -14 \\ 22 \\ 16 \end{pmatrix} \right| = \dfrac{\sqrt{936}}{\sqrt{26}} = 6$

12. EITHER

a) $$S_R = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \frac{2}{3}x^{\frac{3}{2}} \Rightarrow \frac{dy}{dx} = x^{\frac{1}{2}}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + x$$

$$\Rightarrow S_R = \int_0^3 (1+x)^{\frac{1}{2}}\, dx$$

$$= \left.\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^3$$

$$= \frac{2}{3}\left[4^{\frac{3}{2}} - 1\right]$$

$$= \frac{14}{3}$$

b) $$\bar{y} = \frac{\frac{1}{2}\int_0^3 y^2\, dx}{\int_0^3 y\, dx}$$

$$= \frac{\frac{1}{2}\int_0^3 \frac{4}{9}x^3\, dx}{\int_0^3 \frac{2}{3}x^{\frac{3}{2}}\, dx}$$

$$= \frac{1}{3}\,\frac{\left.\frac{x^4}{4}\right|_0^3}{\left.\frac{2}{5}x^{\frac{5}{2}}\right|_0^3}$$

$$= \frac{5}{24}\left[\frac{81}{3\cdot 3\sqrt{3}}\right]$$

$$= \frac{5\sqrt{3}}{8}$$

c) $A_s = 2\pi \int_0^3 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= 2\pi \int_0^3 x \sqrt{1 + x}\, dx$

let $u^2 = 1 + x \Rightarrow x = u^2 - 1$

$2u\, du = dx$

when $x = 0$, $u = 1$

When $x = 3$, $u = 2$

$\therefore A_s = 2\pi \int_1^2 (u^2 - 1)\, u \cdot 2u\, du$

$= 4\pi \int_1^2 u^4 - u^2\, du$

$= 4\pi \left[\frac{u^5}{5} - \frac{u^3}{3} \right]_1^2$

$= 4\pi \left(\frac{32}{5} - \frac{8}{3} - \frac{1}{5} + \frac{1}{3} \right)$

$= \frac{232\pi}{15}$

OR

CF: $m^2 + 0.2m + 1.01 = 0$

$$m = \frac{-0.2 \pm \sqrt{0.2^2 - 4(1.01)}}{2}$$

$$= -0.1 \pm i$$

$$\Rightarrow \quad y = e^{-\frac{t}{10}}(A\cos t + B\sin t)$$

PI: $f(t) = 10^{10} e^{-\frac{t}{5}}$

Let $y = ae^{-\frac{t}{5}}$

$$\frac{dy}{dt} = -\frac{a}{5}e^{-\frac{t}{5}}$$

$$\frac{d^2y}{dt^2} = \frac{a^2}{25}e^{-\frac{t}{5}}$$

$$\Rightarrow \frac{1}{25}ae^{-\frac{t}{5}} + 0.2\left(-\frac{1}{5}ae^{-\frac{t}{5}}\right) + 1.01\,ae^{-\frac{t}{5}} = 10^{10}e^{-\frac{t}{5}}$$

$$\Rightarrow \frac{1}{25}a - \frac{1}{25}a + 1.01a = 10^{10}$$

$$\Rightarrow \qquad a = \frac{10^{10}}{1.01}$$

$$\Rightarrow \qquad y = \frac{10^{10}}{1.01}e^{-\frac{t}{5}}$$

$\therefore$ GS: $y = \frac{10^{10}}{1.01}e^{-\frac{t}{5}} + e^{-\frac{t}{10}}(A\cos t + B\sin t)$

$t = 0,\ y = 5.0 \times 10^9$

$$\Rightarrow 5.0 \times 10^9 = \frac{10^{10}}{1.01} + A$$

$$\Rightarrow A = 10^9\left(5 - \frac{10}{1.01}\right)$$

$$\Rightarrow A = -4.901 \times 10^9$$

$$\frac{dy}{dt} = -\frac{1}{5}\left(\frac{10^{10}}{1.01}\right)e^{-\frac{t}{5}} + e^{-\frac{t}{10}}(-A\sin t + B\cos t)$$

$$-\frac{1}{10}e^{-\frac{t}{10}}(A\cos t + B\sin t)$$

$$t=0,\ \frac{dy}{dt} = 10^8$$

$$\Rightarrow 10^8 = -\frac{1}{5}\left(\frac{10^{10}}{1.01}\right) + B - \frac{A}{10}$$

$$\Rightarrow \cdots$$

$$\Rightarrow B = 1.590 \times 10^9$$

$$t = \frac{100}{20} = 5,$$

$$y = \frac{10^{10}}{1.01}e^{-\frac{1}{5}\cdot 5} + e^{-\frac{1}{10}\cdot 5}\left[-4.901\times 10^9\cos 5 + 1.590\times 10^9\sin 5\right]$$

$$= 10^9\left[\frac{10}{1.01}e^{-\frac{1}{5}\cdot 5} + e^{-\frac{1}{10}\cdot 5}\left[-4.901\cos 5 + 1.590\sin 5\right]\right]$$

$$= 1.874 \times 10^9$$