General Continuous Random Variables

- Probability density function (pdf)
- Expected value, median, percentiles and variance.
- Expected value and variance of a *function* of X.
- Cumulative Distribution Function (cdf)

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Probability density function (pdf)

Learning Outcome

Students should be able to:

- To define a continuous random variable.
- Recall and use the pdf properties of a continuous random variable.
- To sketch the pdf graph.

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Expected value, median, percentiles and variance.

Learning Outcome

Students should be able to:

- Use pdf to solve problems involving probabilities.
- Calculate the mean, variance, median, percentiles and variance (by means of integration) of a continuous variable where the pdf is given.

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Expected value and variance of a *function* of X.

Learning Outcome

Students should be able to:

- Calculate the mean and variance of a *function* of X , where X is a continuous variable where the pdf is given.

Cumulative Distribution Function (cdf)

Learning Outcome

Students should be able to:

- To know the relationship between pdf and cdf.
- Recall and use the cdf properties of a continuous random variable.
- To find cdf from pdf and vice-versa with correct defined interval(s).
- To sketch the cdf graph.
- To calculate median or percentiles using cdf.

Continuous random variables

1.1 Definition

- Probability was represented by the area under a curve, known as the probability density function, $f(x)$, (p.d.f.).

- A probability density function of a random variable , X, must be non-negative for all values of X and the total area under the curve must be 1.

Continuous random variables

- Probability that an observed value of *x* lies between *a* and *b*:

$$P(a < x < b) = \int_a^b f(x)\,dx$$

- Property of a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Example 1

The continuous random variable X has pdf, $f(x)$ where

$$f(x) = \begin{cases} k(x+2)^2 & -2 \leq x < 0 \\ 4k & 0 \leq x \leq 1\frac{1}{3} \\ 0 & otherwise \end{cases}$$

(a) Find the value of the constant k.

(b) Sketch $y = f(x)$.

(c) Find $P(-1 \leq x \leq 1)$.

(d) Find $P(x > 1)$.

Continuous random variables

1.2 Finding the expectation and variance of a continuous random variable

$$E(X) = \mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

$$Var(X) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

Example 2

A charity group raises funds by collecting waste paper. A skip-full will contain an amount, X of other materials such as plastic bags and rubber bands. X may be regarded as a random variable with pdf,

$$f(x) = \begin{cases} k(x-1)(4-x) & 1 < x < 4 \\ 0 & otherwise \end{cases}$$

(All numerical values in this question are in units of 100kg)

(a) Show that $k = \frac{2}{9}$.

(b) Find the mean and standard deviation of X.

(c) Find the probability that X exceeds 3.5.

Example 2

A skip-full may normally be sold for £250 but if X exceeds 3.5 only £125 will be paid. Find the expected value of a skip-full.

Alternatively, the paper may be sorted before being placed in the skip. This will ensure a very low value of X and a skip-full may then be sold £310. However the effort put into sorting means that 25 percent fever skip-full will be sold. Advise the charity whether or not to sort the paper.

Continuous random variables

1.3 Cumulative distribution functions

- For a continuous random variable, X, the cumulative distribution function (c.d.f.), $F(x)$, is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt\,,$$

Where f is the pdf of X.

The probability that $a \leq x \leq b$ is given by

$$P(a \leq x \leq b) = F(b) - F(a).$$

Example 3

X is a continuous random variable with p.d.f., $f(x)$, where

$$f(x) = \begin{cases} \dfrac{x}{3} & 0 \le x \le 2 \\ -\dfrac{2x}{3} + 2 & 2 \le x \le 3 \\ 0 & otherwise \end{cases}$$

(a) Find the c.d.f. $F(x)$ and sketch it.

(b) Find $P(1 \le x \le 2.5)$.

Continuous random variables

Finding the median, quartiles and other percentiles

- The median is the value 50% of the way through the

$$\int_a^m f(x)\,dx = 0.5 \qquad \Rightarrow \qquad F(m) = 0.5$$

- Note : If $f(x)$ is symmetrical in the given range,

  $\Rightarrow$ mean = median

Continuous random variables

- The lower quartile, q_1 , is the value 25% of the way

$$\int_a^{q_1} f(x)\,dx = 0.25 \qquad \Rightarrow \quad F(q_1) = 0.25$$

- The upper quartile, q_3 , is the value 75% of the way

$$\int_a^{q_3} f(x)\,dx = 0.75 \qquad \Rightarrow \quad F(q_3) = 0.75$$

Continuous random variables

- The n *th* percentile, P_n , is defined by the condition

$$F(P_n) = \frac{n}{100}$$

Example 4

A continuous random variable with p.d.f. given by

$$f(x) = \begin{cases} \dfrac{x}{6} & 1 \leq x < 2 \\ \dfrac{1}{3} & 2 \leq x \leq k \\ 0 & otherwise \end{cases}$$

(a) Find the median and lower and upper quartiles.
(b) If $P(x \leq h) = 0.2$, find the value of h.
(c) Find the $60th$ percentile.

Continuous random variables

Obtaining the p.d.f. $f(x)$, from the c.d.f. $F(x)$

- Since F can be obtained by integrating f, it follows that f can be obtained by differentiating F.

$$f(x) = \frac{d}{dx} F(x) = F'(x)$$

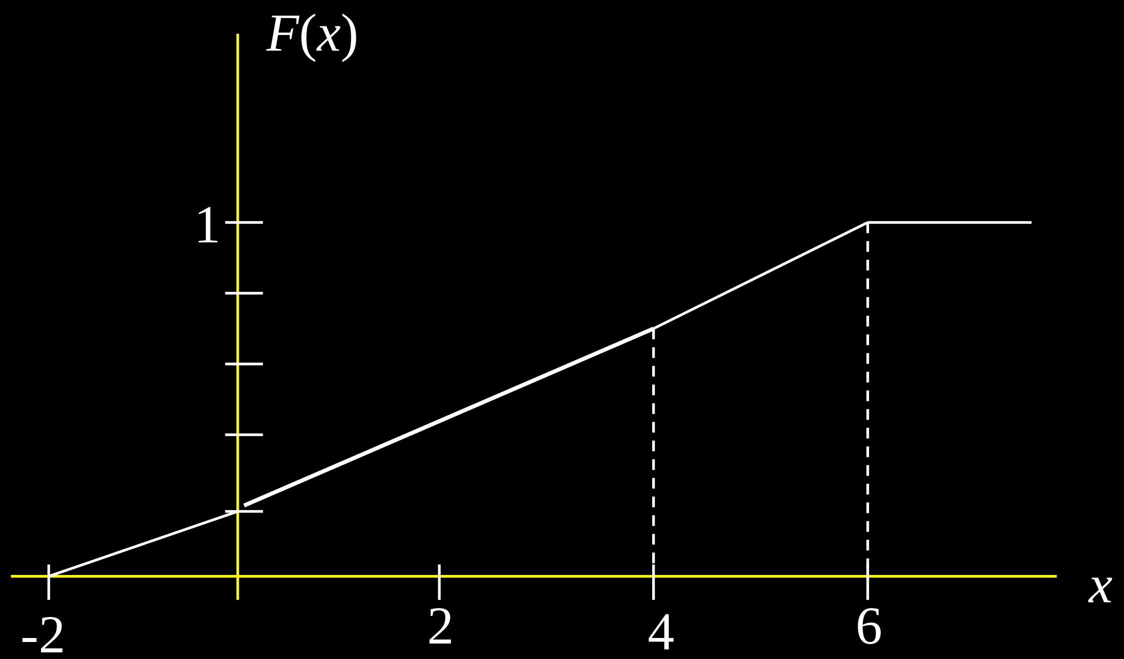
- Note : the gradient of the $F(x)$ curve gives the value of $f(x)$.

Example 5

The continuous random variable X has c.d.f. $F(x)$ as shown in the sketch.

$$\mathrm{F}(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{12}(2+x) & -2 \leq x < 0 \\ \frac{1}{6}(1+x) & 0 \leq x < 4 \\ \frac{1}{12}(6+x) & 4 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$$

Example 5



(a) Find the p.d.f of X, $f(x)$, and sketch $y = f(x)$.

(b) Find $E(x)$.

Example 6

The continuous random variable X has c.d.f. given by

$$f(x) = \begin{cases} 0 & x < 0 \\ 2x - x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

(a) Show that $P\left(x < \frac{1}{2}\right) = \frac{3}{4}$.

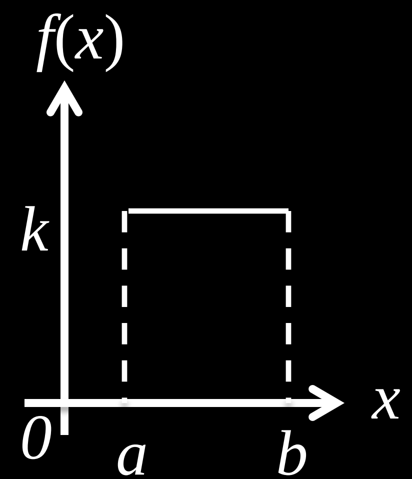
(b) Find the interquartile range of X.

1.4 Cumulative distribution functions of related variables

The continuous Uniform (or Rectangular) distribution

Consider the continuous random variable *X* with pdf

$$f(x) = k \quad for\ a \leq x \leq b$$



Since the total area under the curve is 1,

$$(b-a)k = 1 \rightarrow k = \frac{1}{b-a}$$

$$f(x) = \frac{1}{b-a}$$

$$X \sim R(a, b)$$

The steps involved in finding the pdf of a continuous

Example 7

Cubical boxes are made so that the length X (in cm), of an edge has uniform distribution over the interval $9 < x < 11$. Find the pdf of the volume, Y, of the boxes.

Example 8

The radius, X, of a circle is a random variable with pdf

$$f_X(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

Find the pdf of Y, the area of the circle.

Example 9

Find the pdf of the continuous random variable Y where $Y = X^2$ and X has pdf

1.5 The expectation of a function of a random variable

If the continuous random variable X has pdf $f(x)$ and $g(x)$ is a function of X, then

$$E(g(x)) = \int_a^b g(x).f(x)\,dx \;,$$

where a and b are the limits appropriate to the situation.

Example 10

Find the mean of the random variable Y in Example 7 and Example 8.