MAY / JUNE 2010

1.
$$A = \begin{pmatrix} 5 & -3 & 0 \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{pmatrix}$$

5 is an eigenvalue.

If $\underset{\sim}{x}$ is the corresponding eigenvector for the eigenvalue 5,

$$A\underset{\sim}{x} = 5\underset{\sim}{x}$$

$$\begin{pmatrix} 5 & -3 & 0 \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 5x - 3y \\ x + 2y + z \\ -x + 3y + 4z \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \\ 5z \end{pmatrix}$$

$$5x - 3y = 5x \quad - (1)$$
$$x + 2y + z = 5y \quad - (2)$$
$$-x + 3y + 4z = 5z \quad - (3)$$

From (1): $y = 0$

$$x + z = 0$$
$$-x - z = 0$$

Let $z = s,\ s \in R$

$$x = -s$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix}$$
$$= s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ The corresponding eigenvector for the eigenvalue 5 is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

$$\begin{aligned}(A + A^2)\underset{\sim}{x} &= A\underset{\sim}{x} + A^2\underset{\sim}{x} \\ &= \lambda\underset{\sim}{x} + A(A\underset{\sim}{x}) \\ &= \lambda\underset{\sim}{x} + A(\lambda\underset{\sim}{x}) \\ &= \lambda\underset{\sim}{x} + \lambda(A\underset{\sim}{x}) \\ &= \lambda\underset{\sim}{x} + \lambda(\lambda\underset{\sim}{x}) \\ &= \lambda\underset{\sim}{x} + \lambda^2\underset{\sim}{x} \\ &= (\lambda + \lambda^2)\underset{\sim}{x}\end{aligned}$$

$\therefore$ If A has an eigenvalue λ with corresponding eigenvector $\underset{\sim}{x}$, $A + A^2$ has an eigenvalue $\lambda + \lambda^2$ with corresponding eigenvector $\underset{\sim}{x}$.

Since 5 is an eigenvalue of A with corresponding eigenvector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $5 + 5^2$ is an eigenvalue of A with corresponding eigenvector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

$\therefore$ A has an eigenvalue 30 with corresponding eigenvector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

2. $\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha]$

$= \cos 2n\alpha \cos\alpha + \sin 2n\alpha \sin\alpha$

$\quad - (\cos 2n\alpha \cos\alpha - \sin 2n\alpha \sin\alpha)$

$= \cos 2n\alpha \cos\alpha + \sin 2n\alpha \sin\alpha$

$\quad - \cos 2n\alpha \cos\alpha + \sin 2n\alpha \sin\alpha$

$= 2\sin 2n\alpha \sin\alpha.$

$$\sin 2n\alpha = \frac{\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha]}{2\sin\alpha}, \quad \alpha \neq k\pi, \ k \in \mathbb{Z}.$$

$$\sum_{n=1}^{N} \sin 2n\alpha = \sum_{n=1}^{N} \frac{\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha]}{2\sin\alpha}$$

$$= \frac{1}{2\sin\alpha}\Big(\cos\alpha - \cos 3\alpha$$
$$+ \cos 3\alpha - \cos 5\alpha$$
$$+ \cos 5\alpha - \cos 7\alpha$$
$$\vdots$$
$$+ \cos(2N-5)\alpha - \cos(2N-3)\alpha$$
$$+ \cos(2N-3)\alpha - \cos(2N-1)\alpha$$
$$+ \cos(2N-1)\alpha - \cos(2N+1)\alpha\Big)$$

$$= \frac{\cos\alpha - \cos(2N+1)\alpha}{2\sin\alpha}$$

$$= \frac{\cot\alpha}{2} - \frac{\csc\alpha\cos(2N+1)\alpha}{2}$$

when $\alpha = \frac{\pi}{3}$:

$$\sum_{n=1}^{N} \sin \frac{2n\pi}{3} = \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \cos\left[(2N+1)\alpha\right]$$

$$\sum_{n=1}^{\infty} \sin \frac{2n\pi}{3} = \lim_{N \to \infty} \left(\frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \cos\left[(2N+1)\alpha\right] \right)$$

since $-1 < \cos\left[(2n+1)\alpha\right] < 1$

$$\frac{-1}{\sqrt{3}} < \frac{-1}{\sqrt{3}} \cos\left[(2n+1)\alpha\right] < \frac{1}{\sqrt{3}}$$

$$\frac{-1}{2\sqrt{3}} < \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \cos\left[(2n+1)\alpha\right] < \frac{\sqrt{3}}{2}$$

$$\frac{-1}{2\sqrt{3}} < \sum_{n=1}^{\infty} \sin \frac{2n\pi}{3} < \frac{\sqrt{3}}{2}$$

$\therefore$ The infinite series $\sum_{n=1}^{\infty} \sin \frac{2n\pi}{3}$

does not converge.

3. $x_1, x_2, x_3, \ldots, \quad x_1 = 3$

$$x_{n+1} = \frac{2x_n^2 + 4x_n - 2}{2x_n + 3}, \quad n = 1, 2, 3, \ldots$$

$x_n > 2$

when $n=1 \quad x_1 = 3 > 2$

Assume the statement is true when $n=k$.

$n=k$: $x_k > 2$.

when $n=k+1$: $x_{k+1} > 2$

(what needs to be proved)

If $x_k > 2$

$x_k^2 > 4$

$2x_k^2 > 8$

$2x_k^2 + 4x_k - 2 > 4x_k + 6$

$2x_k^2 + 4x_k - 2 > 2(2x_k + 3)$

$$\frac{2x_k^2 + 4x_k - 2}{2x_k + 3} > 2$$

$x_{k+1} > 2$

$\therefore x_n > 2$ for every positive integer n.

4. $x = \cos t + t\sin t \qquad y = \sin t - t\cos t$

$\frac{dx}{dt} = -\sin t + \sin t + t\cos t \qquad \frac{dy}{dt} = \cos t - \cos t + t\sin t$

$= t\cos t \qquad\qquad = t\sin t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2\cos^2 t + t^2\sin^2 t$$
$$= t^2(\cos^2 t + \sin^2 t)$$
$$= t^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t$$

The surface area of one complete revolution from $t = 0$ to $t = \frac{\pi}{2}$ about the x-axis is

$$\int_0^{\frac{\pi}{2}} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} 2\pi t(\sin t - t\cos t)\, dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} t\sin t\, dt - 2\pi \int_0^{\frac{\pi}{2}} t^2\cos t\, dt$$

$u = t \qquad dv = \sin t\, dt \qquad u = t^2 \qquad dv = \cos t\, dt$

$du = dt \qquad v = -\cos t \qquad du = 2t\, dt \qquad v = \sin t$

$$= 2\pi\left(\left[-t\cos t\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos t\, dt\right)$$
$$- 2\pi\left(\left[t^2\sin t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2t\sin t\, dt\right)$$

$$= 2\pi\left(0 + [\sin t]_0^{\frac{\pi}{2}}\right)$$

$$- 2\pi\left(\frac{\pi^2}{4} - 0 - 2\int_0^{\frac{\pi}{2}} t\sin t \, dt\right)$$

$$= 2\pi(1) - 2\pi\left(\frac{\pi^2}{4} - 2\int_0^{\frac{\pi}{2}} t\sin t \, dt\right)$$

$u = t \qquad dv = \sin t \, dt$

$du = dt \qquad v = -\cos t$

$$= 2\pi - 2\pi\left(\frac{\pi^2}{4} - 2\left([-t\cos t]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos t \, dt\right)\right)$$

$$= 2\pi - 2\pi\left(\frac{\pi^2}{4} - 2\left(0 + [\sin t]_0^{\frac{\pi}{2}}\right)\right.$$

$$= 2\pi - 2\pi\left(\frac{\pi^2}{4} - 2(1 - 0)\right)$$

$$= 2\pi - 2\pi\left(\frac{\pi^2}{4} - 2\right)$$

$$= 2\pi - \frac{\pi^3}{2} + 4\pi$$

$$= 6\pi - \frac{\pi^3}{2}.$$

5. $$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta$$
$$+ 10(-1)\cos^3\theta\sin^2\theta$$
$$+ 10(-i)\cos^2\theta\sin^3\theta$$
$$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$$

$$\cos 5\theta + i\sin 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta$$
$$+ 5\cos\theta\sin^4\theta$$
$$+ i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta$$
$$+ \sin^5\theta)$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta$$
$$+ \sin^5\theta$$
$$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta$$
$$-10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$
$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta$$
$$-10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$
$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

$$32x^5 - 40x^3 + 10x + 1 = 0$$
$$32x^5 - 40x^3 + 10x = -1$$
$$16x^5 - 20x^3 + 5x = \frac{-1}{2}$$

If $x = \sin\theta$,
$$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = \frac{-1}{2}$$

$$\sin 5\theta = -\frac{1}{2}$$

$$5\theta = \frac{7\pi}{6} + 2k\pi, \ \frac{11\pi}{6} + 2k\pi, \ k \in \mathbb{Z}$$

$$\theta = \frac{7\pi}{30} + \frac{2k\pi}{5}, \ \frac{11\pi}{30} + \frac{2k\pi}{5}$$

$k = 0$: $\theta = \frac{7\pi}{30}, \frac{11\pi}{30}$

$k = 1$: $\theta = \frac{19\pi}{30}, \frac{23\pi}{30}$

$k = 2$: $\theta = \frac{31\pi}{30}, \frac{35\pi}{30}$

$k = 3$: $\theta = \frac{43\pi}{30}, \frac{47\pi}{30}$

$k = 4$: $\theta = \frac{55\pi}{30}, \frac{59\pi}{30}$

$\therefore$ The roots of the equation

$32x^5 - 40x^3 + 10x - 1 = 0$ are

$\sin \frac{7\pi}{30}$, $\sin \frac{19\pi}{30}$, $\sin \frac{31\pi}{30}$, $\sin \frac{43\pi}{30}$ and $\sin \frac{55\pi}{30}$.

6. $C: \ y = \dfrac{x^2 - 3x - 7}{x+1}$

$$\begin{array}{r} x - 4 \\ x+1 \,\big)\, \overline{x^2 - 3x - 7} \\ \underline{x^2 + x \quad\;\;} \\ -4x - 7 \\ \underline{-4x - 4} \\ -3 \end{array}$$

$= x - 4 - \dfrac{3}{x+1}$

i) As $x \longrightarrow \pm\infty$ $\quad y \longrightarrow x - 4$

As $x \longrightarrow -1$ $\quad y \longrightarrow \pm\infty$

$\therefore$ The asymptotes of C are $y = x - 4$ and $x = -1$.

ii) $\dfrac{dy}{dx} = 1 + \dfrac{3}{(x+1)^2}$

> 1

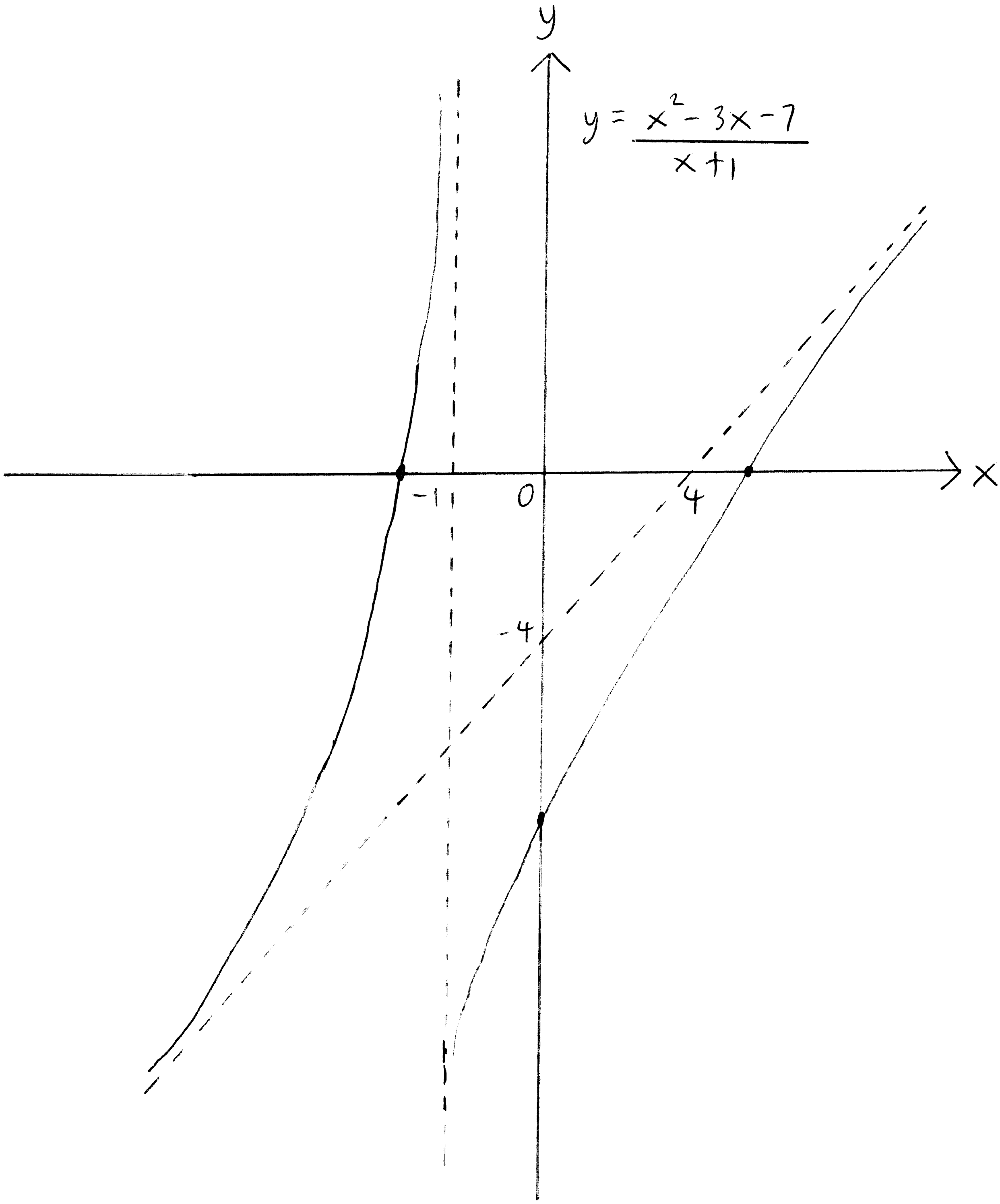
$\therefore \dfrac{dy}{dx} > 1$ at every point of C.

iii) when $x = 0$: $y = -7$

when $y = 0$: $\dfrac{x^2 - 3x - 7}{x+1} = 0$

$x^2 - 3x - 7 = 0$

$x = \dfrac{3 \pm \sqrt{37}}{2}$



• : Intersection point

7. $x = t^2 e^{-t^2}$ $\qquad y = te^{-t^2}$

i) $\frac{dx}{dt} = 2te^{-t^2} - 2t^3 e^{-t^2}$ $\qquad \frac{dy}{dt} = e^{-t^2} - 2t^2 e^{-t^2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{e^{-t^2} - 2t^2 e^{-t^2}}{2te^{-t^2} - 2t^3 e^{-t^2}}$$

$$= \frac{(1-2t^2)e^{-t^2}}{2t(1-t^2)e^{-t^2}}$$

$$= \frac{1-2t^2}{2t-2t^3}$$

ii) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{(2t-2t^3)e^{-t^2}} \frac{d}{dt}\left(\frac{1-2t^2}{2t-2t^3}\right)$$

$$= \frac{e^{t^2}}{2t-2t^3}\left(\frac{(2t-2t^3)(-4t)-(1-2t^2)(2-8t^2)}{(2t-2t^3)^2}\right)$$

$$= \frac{e^{t^2}}{2t - 2t^3}\left(\frac{-8t^2 + 8t^4 - 2 + 4t^2 + 8t^2 - 16t^4}{(2t - 2t^3)^2}\right)$$

$$= \frac{e^{t^2}(-8t^4 + 4t^2 - 2)}{(2t - 2t^3)^3}$$

$$= \frac{2e^{t^2}(-4t^4 + 2t^2 - 1)}{(2t)^3(1 - t^2)^3}$$

$$= \frac{2e^{t^2}(-4t^4 + 2t^2 - 1)}{8t^3(1 - t^2)^3}$$

$$= \frac{e^{t^2}(-4t^4 + 2t^2 - 1)}{4t^3(1 - t^2)^3}.$$

8. $$\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 4y = 10\sin 3x - 20\cos 3x$$

$$\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 4y = 0$$

$$m^2 + 5m + 4 = 0$$

$$(m+1)(m+4) = 0$$

$$m = -1, -4$$

The complementary function, y_c, is given by $y_c = Ae^{-x} + Be^{-4x}$

The particular integral, y_p, is given by

$$y_p = C\cos 3x + D\sin 3x$$

$$\frac{dy_p}{dx} = -3C\sin 3x + 3D\cos 3x$$

$$\frac{d^2y_p}{dx^2} = -9C\cos 3x - 9D\sin 3x$$

$$\frac{d^2y_p}{dx^2} + \frac{5dy_p}{dx} + 4y_p$$

$$= -9C\cos 3x - 9D\sin 3x$$
$$+ 5(-3C\sin 3x + 3D\cos 3x)$$
$$+ 4(C\cos 3x + D\sin 3x)$$

$$= -9C\cos 3x - 9D\sin 3x$$
$$-15C\sin 3x + 15D\cos 3x$$
$$+4C\cos 3x + 4D\sin 3x$$

$$= (-5C + 15D)\cos 3x + (-15C - 5D)\sin 3x$$

$$= 10\sin 3x - 20\cos 3x$$

$$-5C + 15D = -20 \qquad -15C - 5D = 10$$
$$C - 3D = 4 \qquad -3C - D = 2$$
$$C = 3D + 4$$

$$-3(3D+4) - D = 2$$
$$-9D - 12 - D = 2$$
$$-10D = 14$$
$$D = \frac{-7}{5}$$

$$C = \frac{-1}{5}$$

$$\therefore y_p = -\frac{\cos 3x}{5} - \frac{7\sin 3x}{5}$$

$$y = y_c + y_p$$
$$= Ae^{-x} + Be^{-4x} - \frac{\cos 3x}{5} - \frac{7\sin 3x}{5}.$$

If $-\frac{\cos 3x}{5} - \frac{7\sin 3x}{5} = R\sin(3x + \phi)$

$$= R\sin 3x\cos\phi + R\cos 3x\sin\phi$$

$$R\cos\phi = \frac{-7}{5} \qquad R\sin\phi = \frac{-1}{5}$$

$$\frac{R\sin\phi}{R\cos\phi} = \frac{1}{7}$$

$$\tan\phi = \frac{1}{7}$$

$$\phi = \pi + \tan^{-1}\frac{1}{7}$$
$$\approx 3.28$$

$$R^2\cos^2\phi + R\sin^2\phi = \frac{49}{25} + \frac{1}{25}$$

$$R^2(\cos^2\phi + \sin^2\phi) = 2$$

$$R^2 = 2$$

$$R = \sqrt{2}$$
$$\approx 1.41$$

$$-\frac{\cos 3x}{5} - \frac{7\sin 3x}{5} = \sqrt{2}\sin\left(3x + \pi + \tan^{-1}\frac{1}{7}\right)$$

$$y = Ae^{-x} + Be^{-4x} + \sqrt{2}\sin\left(3x + \pi + \tan^{-1}\frac{1}{7}\right)$$

As $x \to \infty$, $y \to \sqrt{2}\sin\left(3x + \pi + \tan^{-1}\frac{1}{7}\right)$

9. $I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta \,, \quad n \geq 0$

$$I_{n+2} = \int_0^{\frac{\pi}{2}} \sin^{n+2} \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^n \theta \sin^2 \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^n \theta (1 - \cos^2 \theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^n \theta - \sin^n \theta \cos^2 \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta - \int_0^{\frac{\pi}{2}} \sin^n \theta \cos^2 \theta \, d\theta$$

$$= I_n - \int_0^{\frac{\pi}{2}} \sin^n \theta \cos \theta \cos \theta \, d\theta$$

$u = \cos \theta \qquad dv = \sin^n \theta \cos \theta \, d\theta$

$du = -\sin \theta \, d\theta \qquad v = \int \sin^n \theta \cos \theta \, d\theta$

$w = \sin \theta$

$dw = \cos \theta \, d\theta$

$$= \int w^n \, dw$$

$$= \frac{w^{n+1}}{n+1}$$

$$= \frac{\sin^{n+1} \theta}{n+1}$$

$$= I_n - \left(\left[\frac{\sin^{n+1}\theta \cos\theta}{n+1} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1}\theta(-\sin\theta)\,d\theta}{n+1} \right)$$

$$= I_n - \left(0 + \frac{1}{n+1} \int_0^{\frac{\pi}{2}} \sin^{n+2}\theta \, d\theta \right)$$

$$= I_n - \frac{1}{n+1} \int_0^{\frac{\pi}{2}} \sin^{n+2}\theta \, d\theta$$

$$= I_n - \frac{1}{n+1} I_{n+2}$$

$$\left(1 + \frac{1}{n+1}\right) I_{n+2} = I_n$$

$$\left(\frac{n+2}{n+1}\right) I_{n+2} = I_n$$

$$I_{n+2} = \left(\frac{n+1}{n+2}\right) I_n$$

The y-coordinate of the centroid of the region bounded by the x-axis, the line $x = \frac{\pi}{2m}$ and the curve $y = \sin^4 mx$, $m > 0$, R is

$$\frac{\int_0^{\frac{\pi}{2m}} \frac{y^2}{2} \, dx}{\int_0^{\frac{\pi}{2m}} y \, dx}$$

$$= \frac{\displaystyle\int_0^{\frac{\pi}{2m}} \frac{\sin^8 mx}{2}\,dx}{\displaystyle\int_0^{\frac{\pi}{2m}} \sin^4 mx\,dx}$$

$$u = mx$$
$$du = m\,dx$$
$$x = 0 \quad u = 0$$
$$x = \frac{\pi}{2m} \quad u = \frac{\pi}{2}$$

$$= \frac{\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^8 u}{2m}\,du}{\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^4 u}{m}\,du}$$

$$= \frac{\displaystyle\frac{1}{2m}\int_0^{\frac{\pi}{2}} \sin^8 u\,du}{\displaystyle\frac{1}{m}\int_0^{\frac{\pi}{2}} \sin^4 u\,du}$$

$$= \frac{I_8}{2I_4}$$

$$= \frac{\frac{7}{8}I_6}{2\cdot\frac{3}{4}I_2}$$

$$= \frac{\frac{7}{8} \cdot \frac{5}{6} I_4}{2 \cdot \frac{3}{4} \cdot \frac{1}{2} I_0}$$

$$= \frac{\frac{7}{8} \cdot \frac{5}{6} \frac{3}{4} I_2}{2 \cdot \frac{3}{4} \cdot \frac{1}{2} I_0}$$

$$= \frac{\frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0}{2 \cdot \frac{3}{4} \frac{1}{2} I_0}$$

$$= \frac{1}{2}\left(\frac{7}{8} \cdot \frac{5}{6}\right)$$

$$= \frac{1}{2}\left(\frac{35}{48}\right)$$

$$= \frac{35}{96}$$

10. $x^4 + x^3 + cx^2 + 4x - 2 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

i) Let $y = \frac{1}{x}$

$x = \frac{1}{y}$

$\frac{1}{y^4} + \frac{1}{y^3} + \frac{c}{y^2} + \frac{4}{y} - 2 = 0$

$1 + y + cy^2 + 4y^3 - 2y^4 = 0$

$2y^4 - 4y^3 - cy^2 - y - 1 = 0$

$\therefore$ The equation $2y^4 - 4y^3 - cy^2 - y - 1 = 0$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$

ii) $\alpha + \beta + \gamma + \delta = -1$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -4$

$\alpha\beta\gamma\delta = -2$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$= (-1)^2 - 2c$$

$$= 1 - 2c$$

Also, $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=2$

$$\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\alpha\delta}+\frac{1}{\beta\gamma}+\frac{1}{\beta\delta}+\frac{1}{\gamma\delta}=\frac{-c}{2}$$

$$\frac{1}{\alpha\beta\gamma}+\frac{1}{\alpha\beta\delta}+\frac{1}{\alpha\gamma\delta}+\frac{1}{\beta\gamma\delta}=\frac{1}{2}$$

$$\frac{1}{\alpha\beta\gamma\delta}=\frac{-1}{2}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}=\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)^2$$

$$-2\left(\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\alpha\delta}+\frac{1}{\beta\gamma}+\frac{1}{\beta\delta}+\frac{1}{\gamma\delta}\right)$$

$$=2^2-2\left(\frac{-c}{2}\right)$$

$$=4+c$$

iii) $\left(\alpha-\frac{1}{\alpha}\right)^2+\left(\beta-\frac{1}{\beta}\right)^2+\left(\gamma-\frac{1}{\gamma}\right)^2+\left(\delta-\frac{1}{\delta}\right)^2$

$$=\alpha^2-2+\frac{1}{\alpha^2}+\beta^2-2+\frac{1}{\beta^2}+\gamma^2-2+\frac{1}{\gamma^2}+\delta^2-2+\frac{1}{\delta^2}$$

$$=\alpha^2+\beta^2+\gamma^2+\delta^2+\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}-8$$

$$=1-2c+4+c-8$$

$$=-c-3$$

iv) when $c = -3$:

$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2 = 0$$

Since -1 and 1 are not roots of the equation $x^4 + x^3 - 3x^2 + 4x - 2 = 0$,

$\alpha^2 \neq 1$, $\beta^2 \neq 1$, $\gamma^2 \neq 1$, $\delta^2 \neq 1$.

$\alpha \neq \frac{1}{\alpha}$, $\beta \neq \frac{1}{\beta}$, $\gamma \neq \frac{1}{\gamma}$, $\delta \neq \frac{1}{\delta}$

$\left(\alpha - \frac{1}{\alpha}\right)^2 > 0$, $\left(\beta - \frac{1}{\beta}\right)^2 > 0$, $\left(\gamma - \frac{1}{\gamma}\right)^2 > 0$, $\left(\delta - \frac{1}{\delta}\right)^2 > 0$.

$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2 > 0.$$

when $c = -3$, the roots of the equation $x^4 + x^3 - 3x^2 + 4x - 2 = 0$ are not all real.

11. $C: r = \dfrac{a}{1+\theta}, \quad 0 \le \theta \le \dfrac{\pi}{2}, \quad a > 0.$

i) $$\frac{dr}{d\theta} = \frac{-a}{(1+\theta)^2}$$

Since $\dfrac{dr}{d\theta} < 0$, r decreases as θ increases.

ii) $$y = r\sin\theta$$
$$= \frac{a\sin\theta}{1+\theta}$$

$$\frac{dy}{d\theta} = \frac{a\cos\theta}{1+\theta} - \frac{a\sin\theta}{(1+\theta)^2}$$

when $\dfrac{dy}{d\theta} = 0$. $$\frac{a\cos\theta}{1+\theta} - \frac{a\sin\theta}{(1+\theta)^2} = 0$$

$$\frac{\cos\theta}{1+\theta} = \frac{\sin\theta}{(1+\theta)^2}$$

$$\tan\theta = 1+\theta$$

$\therefore$ If P is the point on C which is furthest from the initial line, at P $\tan\theta = 1+\theta$.

If $f(\theta) = \tan\theta - 1 - \theta$

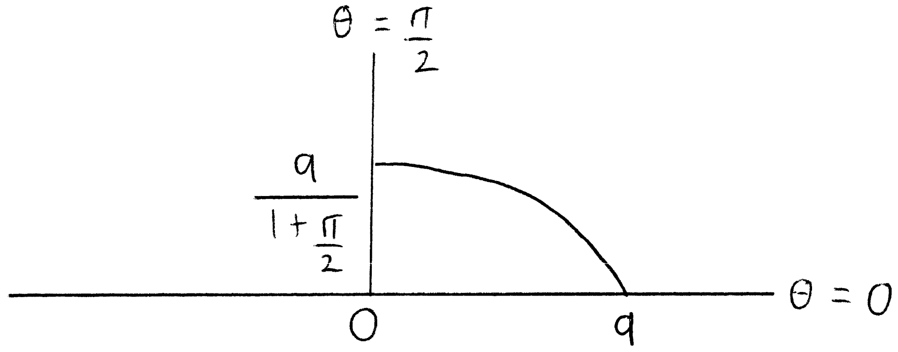
$f(1.1) = -0.1352 < 0$

$f(1.2) = 0.3722 > 0$

The equation $\tan\theta = 1+\theta$ has a root between 1.1 and 1.2.

iii)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	a	$\frac{a}{1+\frac{\pi}{6}}$	$\frac{a}{1+\frac{\pi}{4}}$	$\frac{a}{1+\frac{\pi}{3}}$	$\frac{a}{1+\frac{\pi}{2}}$



iv) The area of the region bounded by the initial line, the line $\theta = \frac{\pi}{2}$ and C is

$$\int_0^{\frac{\pi}{2}} \frac{r^2}{2} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{a^2}{2(1+\theta)^2} \, d\theta$$

$$= \left[\frac{-a^2}{2(1+\theta)} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{-a^2}{2\left(1+\frac{\pi}{2}\right)} - \frac{(-a^2)}{2}$$

$$= \frac{-2a^2}{2(2+\pi)} + \frac{a^2}{2}$$

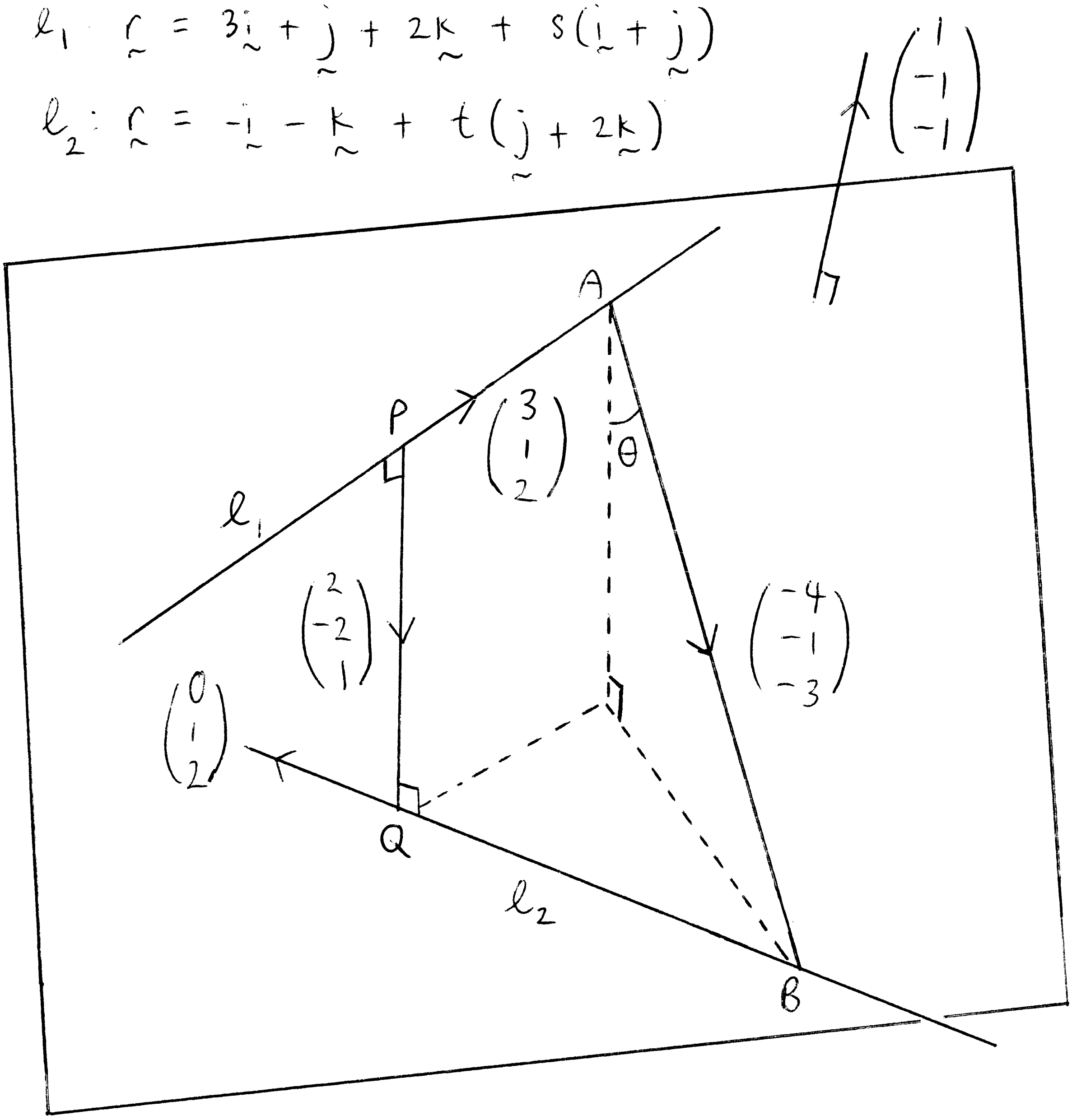
$$= \frac{\pi a^2}{2(\pi+2)}$$

12. EITHER

$A(3,1,2)$ $B(-1,0,-1)$

$l_1: \underset{\sim}{r} = 3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} + s(\underset{\sim}{i} + \underset{\sim}{j})$

$l_2: \underset{\sim}{r} = -\underset{\sim}{i} - \underset{\sim}{k} + t(\underset{\sim}{j} + 2\underset{\sim}{k})$



i) since P is a point on l_1 and Q is a point on l_2, let $P(3+s, 1+s, 2)$ and $Q(-1, t, -1+2t)$.

Since PQ is perpendicular to both l_1 and l_2,

PQ is parallel to $\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\2\end{pmatrix}$.

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$\therefore \overrightarrow{PQ} = c\begin{pmatrix}2\\-2\\1\end{pmatrix}, \quad c \in R.$$

$$|\overrightarrow{PQ} \cdot \overrightarrow{AB}| = ||\overrightarrow{PQ}||\overrightarrow{AB}|\cos\theta|$$

$$\left|c\begin{pmatrix}2\\-2\\1\end{pmatrix} \cdot \begin{pmatrix}-4\\-1\\-3\end{pmatrix}\right| = \left|c\begin{pmatrix}2\\-2\\1\end{pmatrix} \begin{pmatrix}-4\\-3\\-1\end{pmatrix}\cos\theta\right|$$

$$|c(-8+2-3)| = 3c\,|\overrightarrow{AB}|\cos\theta$$

$$|\overrightarrow{AB}|\cos\theta = 3$$

$\therefore |\overrightarrow{PQ}| = 3$, since $|\overrightarrow{PQ}| = |\overrightarrow{AB}|\cos\theta$.

ii) Since $\overrightarrow{PQ} = \begin{pmatrix}-4-s\\t-1-s\\-3+2t\end{pmatrix}$,

$$\begin{pmatrix}-4-s\\t-1-s\\-3+2t\end{pmatrix} = c\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$\begin{pmatrix}-4\\-3\\-1\end{pmatrix} \times \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -4 & -3 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}.$$

Since $\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ is a normal to the plane

and A is a point on the plane, if

$\underset{\sim}{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ is a point on the plane,

$$\underset{\sim}{r} \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix} = 3 - 1 - 2$$

$$x - y - z = 0$$

The plane containing AB and l_1 has equation $x - y - z = 0$.

The line passing through Q and perpendicular to the plane has equation $\underset{\sim}{r} = \begin{pmatrix}-1\\1\\1\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\-1\end{pmatrix}$

When the line intersects the plane,

$x = -1 + \alpha$, $y = 1 - \alpha$, $z = 1 - \alpha$.

$$-1 + \alpha - (1 - \alpha) - (1 - \alpha) = 0$$

$$\left.\begin{aligned} -4 - s &= 2c \\ t - 1 - s &= -2c \\ -3 + 2t &= c \end{aligned}\right\}$$

$$\left.\begin{aligned} s \quad + 2c &= -4 \\ s - t - 2c &= -1 \\ 2t - c &= 3 \end{aligned}\right\}$$

$-① + ②$: $$\left.\begin{aligned} s \quad + 2c &= -4 \\ -t - 4c &= 3 \\ 2t - c &= 3 \end{aligned}\right\}$$

$2 \times ② + ③$ $$\left.\begin{aligned} s \quad + 2c &= -4 \\ -t - 4c &= 3 \\ -9c &= 9 \end{aligned}\right\}$$

$c = -1$

$t = 1$

$s = -2$

$\therefore Q(-1, 1, 1)$

iii) The plane containing AB and ℓ_1 is parallel to $\begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

. The normal of the plane is parallel to $\begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ since it is perpendicular to both $\begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$$-1+\alpha-1+\alpha-1+\alpha=0$$

$$3\alpha=3$$

$$\alpha=1$$

$\therefore$ The line meets the plane at $(0,0,0)$.

$\therefore$ The perpendicular distance from Q to the plane containing AB and ℓ_1 is

$$\sqrt{(-1-0)^2+(1-0)^2+(1-0)^2}$$

$$=\sqrt{3}.$$

OR

$T: R^4 \to R^4 \quad M = \begin{pmatrix} 1 & 1 & 5 & 7 \\ 3 & 9 & 17 & 25 \\ 1 & 7 & 7 & 11 \\ 3 & 6 & 16 & 23 \end{pmatrix}$

i) a) $\begin{pmatrix} 1 & 1 & 5 & 7 \\ 3 & 9 & 17 & 25 \\ 1 & 7 & 7 & 11 \\ 3 & 6 & 16 & 23 \end{pmatrix}$

$$\xrightarrow[-3r_1 + r_4]{-3r_1 + r_2,\ -r_1 + r_3} \begin{pmatrix} 1 & 1 & 5 & 7 \\ 0 & 6 & 2 & 4 \\ 0 & 6 & 2 & 4 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{\frac{r_2}{3},\ \frac{r_3}{3}} \begin{pmatrix} 1 & 1 & 5 & 7 \\ 0 & 3 & 1 & 2 \\ 0 & 3 & 1 & 2 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow[-r_2 + r_4]{-r_2 + r_3} \begin{pmatrix} 1 & 1 & 5 & 7 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The dimension of R, the range space of T is 2.

b) A basis for R is $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ 7 \\ 6 \end{pmatrix} \right\}$.

ii) If $\begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix} \in R$,

$$\begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 9 \\ 7 \\ 6 \end{pmatrix}, \quad \alpha, \beta \in R$$

$$= \begin{pmatrix} \alpha + \beta \\ 3\alpha + 9\beta \\ \alpha + 7\beta \\ 3\alpha + 6\beta \end{pmatrix}$$

$$\left.\begin{aligned} \alpha + \beta &= 1 \\ 3\alpha + 9\beta &= -15 \\ \alpha + 7\beta &= -17 \\ 3\alpha + 6\beta &= -6 \end{aligned}\right\}$$

$$\left.\begin{aligned} \alpha + \beta &= 1 \\ \alpha + 3\beta &= -5 \\ \alpha + 7\beta &= -17 \\ \alpha + 2\beta &= -2 \end{aligned}\right\}$$

$\alpha = 4$, $\beta = -3$

$\therefore \begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix}$ belongs to R.

iii) If $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ is a basis for the null space of T, $\underset{\sim}{e}_1 = \begin{pmatrix} 14 \\ 1 \\ -3 \\ 0 \end{pmatrix}$, $\underset{\sim}{e}_2 = \begin{pmatrix} 19 \\ 2 \\ 0 \\ -3 \end{pmatrix}$ and

$$\underset{\sim}{x} = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + \lambda \underset{\sim}{e}_1 + \mu \underset{\sim}{e}_2, \quad \lambda, \mu \in R,$$

since

$$M\begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 & 7 \\ 3 & 9 & 17 & 25 \\ 1 & 7 & 7 & 11 \\ 3 & 6 & 16 & 23 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix}$$

and $M(\lambda \underset{\sim}{e}_1 + \mu \underset{\sim}{e}_2)$

$$= \begin{pmatrix} 1 & 1 & 5 & 7 \\ 3 & 9 & 17 & 25 \\ 1 & 7 & 7 & 11 \\ 3 & 6 & 16 & 23 \end{pmatrix}\left(\lambda\begin{pmatrix} 14 \\ 1 \\ -3 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 19 \\ 2 \\ 0 \\ -3 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 & 1 & 5 & 7 \\ 3 & 9 & 17 & 25 \\ 1 & 7 & 7 & 11 \\ 3 & 6 & 16 & 23 \end{pmatrix}\begin{pmatrix} 14\lambda + 19\mu \\ \lambda + 2\mu \\ -3\lambda \\ -3\mu \end{pmatrix}$$

$$= \begin{pmatrix} 14\lambda + 19\mu + \lambda + 2\mu - 15\lambda - 21\mu \\ 42\lambda + 57\mu + 9\lambda + 18\mu - 51\lambda - 75\mu \\ 14\lambda + 19\mu + 7\lambda + 14\mu - 21\lambda - 33\mu \\ 42\lambda + 57\mu + 6\lambda + 12\mu - 48\lambda - 69\mu \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{x} = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + \lambda \underset{\sim}{e}_1 + \mu \underset{\sim}{e}_2 \text{ is a solution}$$

$$\text{of } M\underset{\sim}{x} = \begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix}.$$

iv) If $\begin{pmatrix} \alpha \\ 0 \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 14 \\ 1 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 19 \\ 2 \\ 0 \\ -3 \end{pmatrix}$

$$= \begin{pmatrix} 4 + 14\lambda + 19\mu \\ -3 + \lambda + 2\mu \\ -3\lambda \\ -3\mu \end{pmatrix}$$

$\alpha = 4 + 14\lambda + 19\mu$, $0 = -3 + \lambda + 2\mu$,

$\gamma = -3\lambda$, $\delta = -3\mu$.

$\therefore \lambda = \frac{-\gamma}{3}$, $\mu = \frac{-\delta}{3}$.

$$\alpha = 4 - \frac{14\gamma}{3} - \frac{19\delta}{3}$$

$\therefore$ A solution of the form $\begin{pmatrix} \alpha \\ 0 \\ \gamma \\ \delta \end{pmatrix}$ is

$$\begin{pmatrix} 4 - \frac{14\gamma}{3} - \frac{19\delta}{3} \\ 0 \\ -\frac{\gamma}{3} \\ -\frac{\delta}{3} \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} \frac{-14}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix} + \delta \begin{pmatrix} \frac{-19}{3} \\ 0 \\ 0 \\ -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 37 \\ 0 \\ -3 \\ -3 \end{pmatrix} \text{ when } \lambda = 1, \mu = 1,$$

$\alpha = 37$, $\gamma = -3$, $\delta = -3$