OCTOBER / NOVEMBER 2009

1. $y = x^2 \sin x$

i) The mean value of $\frac{dy}{dx}$ over $0 \leq x \leq \frac{\pi}{2}$

is $$\frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} \frac{dy}{dx}\, dx$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d}{dx}(y)\, dx$$

$$= \frac{2}{\pi} \left[y \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi} \left[x^2 \sin x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi} \left(\frac{\pi^2}{4}(1) - 0 \right)$$

$$= \frac{\pi}{2}.$$

ii) The mean value of $\frac{d^2y}{dx^2}$ over $0 \leq x \leq \frac{\pi}{2}$

is $$\frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} \frac{d^2y}{dx^2}\, dx$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d}{dx}\left(\frac{dy}{dx}\right) dx$$

$$= \frac{2}{\pi} \left[\frac{dy}{dx} \right]_0^{\frac{\pi}{2}}$$

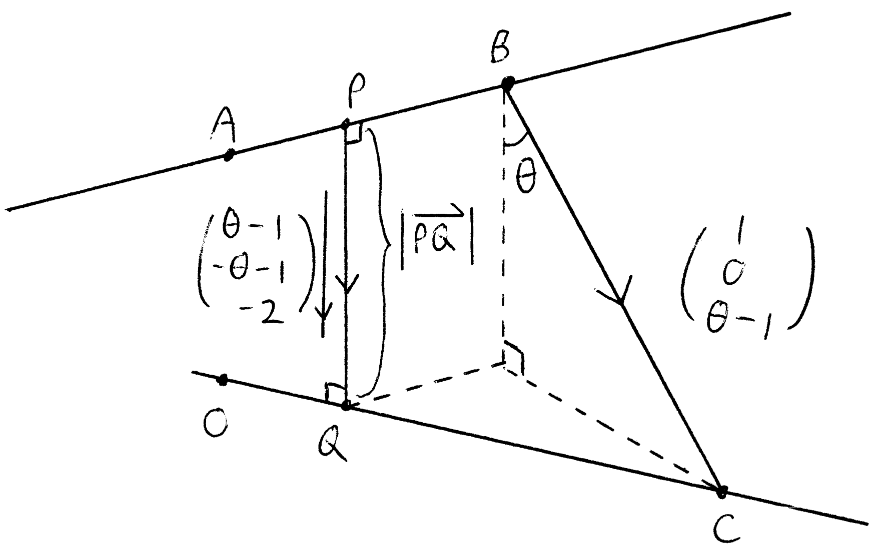
$$= \frac{2}{\pi}\left[x^2 \cos x + 2x \sin x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{4}(0) + 2\left(\frac{\pi}{2}\right)1 - 0 \right)$$

$$= \frac{2}{\pi}(\pi)$$

$$= 2$$

2.



$\overrightarrow{OA} = \underset{\sim}{i} \quad \overrightarrow{OB} = \underset{\sim}{j} + \underset{\sim}{k} \quad \overrightarrow{OC} = \underset{\sim}{i} + \underset{\sim}{j} + \theta \underset{\sim}{k}$

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

$\therefore$ The line AB has equation

$$\begin{aligned} \underset{\sim}{r} &= \overrightarrow{OA} + s\overrightarrow{AB} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

and the line OC has equation

$$\begin{aligned} \underset{\sim}{r} &= t\overrightarrow{OC} \\ &= t\begin{pmatrix} 1 \\ 1 \\ \theta \end{pmatrix} \end{aligned}$$

Since PQ is perpendicular to both AB and OC, PQ is parallel to $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ \theta \end{pmatrix}$.

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ \theta \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -1 & 1 & 1 \\ 1 & 1 & \theta \end{vmatrix} = \begin{pmatrix} \theta - 1 \\ -\theta - 1 \\ -2 \end{pmatrix}$$

$$\therefore \overrightarrow{PQ} = c \begin{pmatrix} \theta - 1 \\ -\theta - 1 \\ -2 \end{pmatrix}, \quad c \text{ is a scalar.}$$

$$\overrightarrow{PQ} \cdot \overrightarrow{BC} = |\overrightarrow{PQ}| |\overrightarrow{BC}| \cos\theta$$

$$c \begin{pmatrix} \theta - 1 \\ -\theta - 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \theta - 1 \end{pmatrix} = \left| c \begin{pmatrix} \theta - 1 \\ -\theta - 1 \\ -2 \end{pmatrix} \right| |\overrightarrow{BC}| \cos\theta$$

$$c(\theta - 1 + 0 - 2\theta + 2) = c \left| \begin{pmatrix} \theta - 1 \\ -\theta - 1 \\ -2 \end{pmatrix} \right| |\overrightarrow{BC}| \cos\theta$$

$$1 - \theta = \left| \begin{pmatrix} \theta - 1 \\ -\theta - 1 \\ -2 \end{pmatrix} \right| |\overrightarrow{BC}| \cos\theta, \quad \text{since } c \neq 0.$$

Since the shortest distance between AB and OC is $\frac{1}{\sqrt{2}}$, $|\overrightarrow{BC}| \cos\theta = \frac{1}{\sqrt{2}}$

$$1 - \theta = \sqrt{(\theta - 1)^2 + (-\theta - 1)^2 + (-2)^2} \; \frac{1}{\sqrt{2}}$$

$$(1-\theta)^2 = \frac{\theta^2 - 2\theta + 1 + \theta^2 + 2\theta + 1 + 4}{2}$$

$$1 - 2\theta + \theta^2 = \frac{2\theta^2 + 6}{2}$$

$$= \theta^2 + 3$$

$$2\theta = -2$$

$$\theta = -1$$

3. $C: \; y = \dfrac{x^2 - 5x + 4}{x+1}$

i) When $x = 0$: $y = 4$

when $y = 0$: $\dfrac{x^2 - 5x + 4}{x+1} = 0$

$$x^2 - 5x + 4 = 0$$

$$(x-1)(x-4) = 0$$

$$x = 1, 4$$

$\therefore$ The intersection points of C are $(0,4)$, $(1,0)$ and $(4,0)$.

ii)

$$\begin{array}{r|l} & x - 6 \\ \hline x+1 & x^2 - 5x + 4 \\ & x^2 + x \\ \hline & \quad -6x + 4 \\ & \quad -6x - 6 \\ \hline & \qquad\quad 10 \end{array}$$

$$y = x - 6 + \frac{10}{x+1}$$

As $x \longrightarrow \pm\infty$, $y \longrightarrow x - 6$

As $x \longrightarrow -1$, $y \longrightarrow \pm\infty$

$\therefore$ The asymptotes of C are $y = x - 6$ and $x = -1$

iii) $\dfrac{dy}{dx} = 1 - \dfrac{10}{(x+1)^2}$

when $\dfrac{dy}{dx} = 0$:

$$1 - \frac{10}{(x+1)^2} = 0$$

$$\frac{10}{(x+1)^2} = 1$$

$$(x+1)^2 = 10$$

$$x + 1 = \pm\sqrt{10}$$

$$x = -1 \pm \sqrt{10}$$

$$y = -7 \pm \sqrt{10} + \frac{10}{\pm\sqrt{10}}$$

$$= -7 \pm \sqrt{10} \pm \sqrt{10}$$

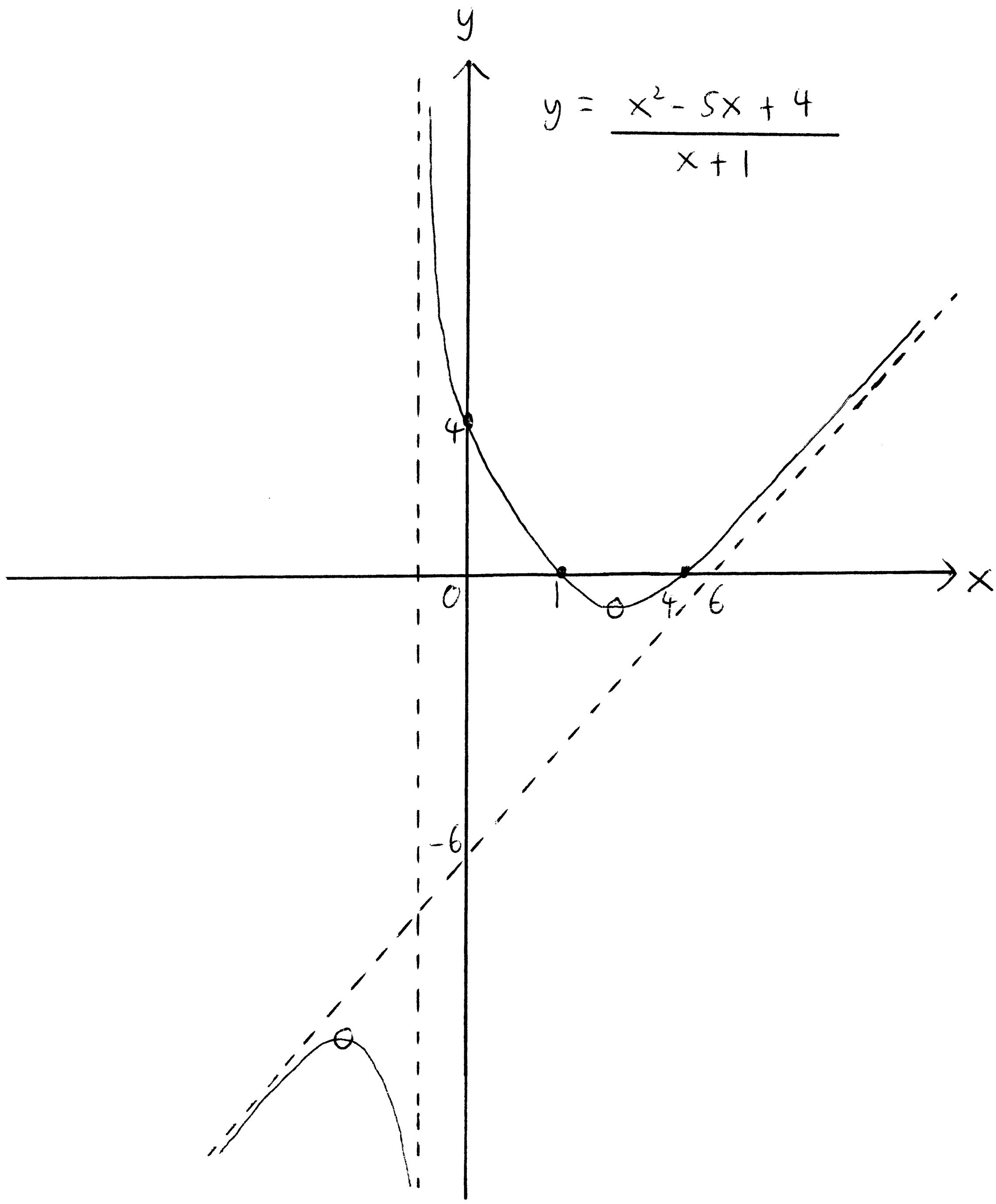
$$= -7 \pm 2\sqrt{10}$$

$$\frac{d^2y}{dx^2} = \frac{20}{(x+1)^3}$$

when $x = -1 + \sqrt{10}$: $\dfrac{d^2y}{dx^2} = \dfrac{2}{\sqrt{10}} > 0$

when $x = -1 - \sqrt{10}$: $\dfrac{d^2y}{dx^2} = \dfrac{-2}{\sqrt{10}} < 0$

$\therefore$ $(-1 + \sqrt{10}, -7 + 2\sqrt{10})$ is a minimum point and $(-1 - \sqrt{10}, -7 - 2\sqrt{10})$ is a maximum point.



o : Critical point

• Intersection point.

4. $x = t + \sin t, \quad y = t^2 + 2\cos t, \quad -\pi < t < \pi$

$$\frac{dx}{dt} = 1 + \cos t \qquad \frac{dy}{dt} = 2t - 2\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{2t - 2\sin t}{1 + \cos t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{2t - 2\sin t}{1 + \cos t}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left(\frac{2t - 2\sin t}{1 + \cos t}\right)$$

$$= \frac{1}{1 + \cos t}\frac{d}{dt}\left(\frac{2t - 2\sin t}{1 + \cos t}\right)$$

$$= \frac{1}{1 + \cos t}\left(\frac{(1 + \cos t)(2 - 2\cos t) - (2t - 2\sin t)(-\sin t)}{(1 + \cos t)^2}\right)$$

$$= \frac{2 + 2\cos t - 2\cos t - 2\cos^2 t + 2t\sin t - 2\sin^2 t}{(1 + \cos t)^3}$$

$$= \frac{2 - 2(\cos^2 t + \sin^2 t) + 2t\sin t}{(1 + \cos t)^3}$$

$$= \frac{2t\sin t}{(1+\cos t)^3}$$

when $-\pi < t < 0$, since $\sin t < 0$, $t\sin t > 0$

when $0 < t < \pi$, since $\sin t > 0$, $t\sin t > 0$.

Also since $(1+\cos t)^3 > 0$,

$$\frac{2t\sin t}{(1+\cos t)^3} > 0$$

$$\frac{d^2y}{dx^2} > 0$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$$

$\therefore$ $\frac{dy}{dx}$ increases with x when $-\pi < t < \pi$.

5. $x^3 + 5x + 3 = 0$

α, β, r are the roots

$\alpha + \beta + r = 0 \quad \alpha\beta + \alpha r + \beta r = 5 \quad \alpha\beta r = -3$

$$x = \frac{-3}{y}$$

$$\left(\frac{-3}{y}\right)^3 + 5\left(\frac{-3}{y}\right) + 3 = 0$$

$$\frac{-27}{y^3} - \frac{15}{y} + 3 = 0$$

$$\frac{-9}{y^3} - \frac{5}{y} + 1 = 0$$

$$-9 - 5y^2 + y^3 = 0$$

$$y^3 - 5y^2 - 9 = 0$$

Since $\alpha\beta = \frac{-3}{r}$, $\alpha r = \frac{-3}{\beta}$, $\beta r = \frac{-3}{\alpha}$,

the equation $y^3 - 5y^2 - 9$ has roots

$\alpha\beta, \alpha r, \beta r$.

$$\alpha\beta + \alpha r + \beta r = 5$$

$$\alpha\beta(\alpha r) + \alpha\beta(\beta r) + \alpha r(\beta r) = 0$$

$$\alpha\beta(\alpha r)\beta r = 9$$

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2$$

$$-2[\alpha\beta(\alpha\gamma) + \alpha\beta(\beta\gamma) + \alpha\gamma(\beta\gamma)]$$

$$= 5^2 - 2(0)$$

$$= 25$$

Let $S_n = (\alpha\beta)^n + (\alpha\gamma)^n + (\beta\gamma)^n$

$$S_0 = (\alpha\beta)^0 + (\alpha\gamma)^0 + (\beta\gamma)^0$$

$$= 1 + 1 + 1$$

$$= 3$$

$$S_1 = (\alpha\beta)^1 + (\alpha\gamma)^1 + (\beta\gamma)^1$$

$$= \alpha\beta + \alpha\gamma + \beta\gamma$$

$$= 5$$

$$S_2 = (\alpha\beta)^2 + (\alpha\gamma)^2 + (\beta\gamma)^2$$

$$= 25$$

$$S_3 - 5S_2 - 9S_0 = 0$$

$$S_3 - 5(25) - 9(3) = 0$$

$$S_3 - 125 - 27 = 0$$

$$S_3 = 152$$

6. $$\frac{d}{dx}\left[x^{n-1}\sqrt{4-x^2}\right] = (n-1)x^{n-2}\sqrt{4-x^2} + \frac{x^{n-1}}{2}\frac{1}{\sqrt{4-x^2}}(-2x)$$

$$= (n-1)x^{n-2}\sqrt{4-x^2} - \frac{x^n}{\sqrt{4-x^2}}$$

$$= \frac{(n-1)x^{n-2}(4-x^2) - x^n}{\sqrt{4-x^2}}$$

$$= \frac{4(n-1)x^{n-2} - (n-1)x^n - x^n}{\sqrt{4-x^2}}$$

$$= \frac{4(n-1)x^{n-2} - nx^n}{\sqrt{4-x^2}}$$

$$I_n = \int_0^1 \frac{x^n}{\sqrt{4-x^2}}\,dx, \quad n \geqslant 0$$

Since $\frac{d}{dx}\left[x^{n-1}\sqrt{4-x^2}\right] = \frac{4(n-1)x^{n-2}}{\sqrt{4-x^2}} - \frac{nx^n}{\sqrt{4-x^2}}$,

$$x^{n-1}\sqrt{4-x^2} = \int \frac{4(n-1)x^{n-2}}{\sqrt{4-x^2}} - \frac{nx^n}{\sqrt{4-x^2}}\,dx$$

$$= 4(n-1)\int \frac{x^{n-2}}{\sqrt{4-x^2}}\,dx - n\int \frac{x^n}{\sqrt{4-x^2}}\,dx$$

$$\left[x^{n-1}\sqrt{4-x^2}\right]_0^1 = 4(n-1)\int_0^1 \frac{x^{n-2}}{\sqrt{4-x^2}}\,dx$$

$$-n\int_0^1 \frac{x^n}{\sqrt{4-x^2}}\,dx$$

$$\sqrt{3} - 0 = 4(n-1)I_{n-2} - nI_n$$

$$\therefore nI_n = 4(n-1)I_{n-2} - \sqrt{3},\ n \geqslant 2.$$

$$I_0 = \frac{\pi}{6}$$

when $n = 4$. $4I_4 = 4(3)I_2 - \sqrt{3}$

$$= 12I_2 - \sqrt{3}$$

when $n = 2$: $2I_2 = 4(1)I_0 - \sqrt{3}$

$$= 4I_0 - \sqrt{3}$$

$$= 4\left(\frac{\pi}{6}\right) - \sqrt{3}$$

$$= \frac{2\pi}{3} - \sqrt{3}$$

$$I_2 = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$$4I_4 = 12\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) - \sqrt{3}$$

$$= 4\pi - 6\sqrt{3} - \sqrt{3}$$

$$= 4\pi - 7\sqrt{3}$$

$$\therefore I_4 = \pi - \frac{7\sqrt{3}}{4}$$

7. if $z = \cos\theta + i\sin\theta$

$$z^{-1} = (\cos\theta + i\sin\theta)^{-1}$$
$$= \cos(-\theta) + i\sin(-\theta)$$
$$= \cos\theta - i\sin\theta$$
$$z^{n} = (\cos\theta + i\sin\theta)^{n}$$
$$= \cos n\theta + i\sin n\theta$$
$$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

$$z + \frac{1}{z} = 2\cos\theta, \quad z - \frac{1}{z} = 2i\sin\theta$$

$$(2i\sin\theta)^6 = \left(z - \frac{1}{z}\right)^6$$

$$-64\sin^6\theta = z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$$

$$= z^6 + \frac{1}{z^6} - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$= 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$\sin^6\theta = \frac{5}{16} - \frac{15}{32}\cos 2\theta + \frac{3}{16}\cos 4\theta - \frac{\cos 6\theta}{32}$$

$$= a + b\cos 2\theta + c\cos 4\theta + d\cos 6\theta,$$

$$a = \frac{5}{16}, \quad b = \frac{-15}{32}, \quad c = \frac{3}{16}, \quad d = \frac{-1}{32}$$

When $\theta = 2x$:

$$\sin^6 2x = \frac{5}{16} - \frac{15}{32}\cos 4x + \frac{3}{16}\cos 8x - \frac{1}{32}\cos 12x$$

$$\int_0^{\frac{\pi}{4}} \sin^6 2x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{5}{16} - \frac{15}{32}\cos 4x + \frac{3}{16}\cos 8x - \frac{1}{32}\cos 12x \, dx$$

$$= \left[\frac{5x}{16} - \frac{15}{128}\sin 4x + \frac{3}{128}\sin 8x - \frac{\sin 12x}{384} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{5\pi}{64} - 0$$

$$= \frac{5\pi}{64}$$

8. a) C_1 : $y = -\ln(\cos x)$

The arc length of C_1 from $x = 0$ to $x = \frac{\pi}{3}$ is $\int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{-(-\sin x)}{\cos x}\right)^2}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec x\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\frac{\pi}{3}}$$

$$= \left[\ln\left|\frac{1 + \sin x}{\cos x}\right|\right]_0^{\frac{\pi}{3}}$$

$$= \ln\left(\frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) - \ln\left(\frac{1+0}{1}\right)$$

$$= \ln(2+\sqrt{3}) - \ln 1$$

$$= \ln(2+\sqrt{3}) - 0$$

$$= \ln(2+\sqrt{3})$$

b) $C_2 : y = 2\sqrt{x+3}$

The surface area of revolution of C_2 about the x-axis from $x = 0$ to $x = 1$ is

$$\int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 2\pi (2\sqrt{x+3}) \sqrt{1 + \left(\frac{1}{\sqrt{x+3}}\right)^2}\, dx$$

$$= 4\pi \int_0^1 \sqrt{x+3} \sqrt{1 + \frac{1}{x+3}}\, dx$$

$$= 4\pi \int_0^1 \sqrt{x+3} \sqrt{\frac{x+4}{x+3}}\, dx$$

$$= 4\pi \int_0^1 \sqrt{x+4}\, dx$$

$$= 4\pi \left[\frac{2(x+4)^{\frac{3}{2}}}{3}\right]_0^1 dx$$

$$= \frac{8\pi}{3}(5\sqrt{5} - 8)$$

9. $y = f(x)$, $x = e^u$

$$\frac{dx}{du} = e^u$$

$$\frac{dx}{dy}\frac{dy}{du} = e^u$$

$$\frac{dy}{du} = e^u \frac{dy}{dx}$$

$$= x\frac{dy}{dx}$$

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\right)$$

$$x\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(x) = \frac{du}{dx}\frac{d}{du}\left(\frac{dy}{du}\right)$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{du}{dx}\frac{d^2y}{du^2}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x\frac{du}{dx}\frac{d^2y}{du^2}$$

$$x^2\frac{d^2y}{dx^2} + \frac{dy}{du} = e^u\left(\frac{1}{e^u}\right)\frac{d^2y}{du^2}$$

$$= \frac{d^2y}{du^2}$$

$$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$$

if $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 3y = 30x^2$ and $x = e^u$

$$\frac{d^2y}{du^2} - \frac{dy}{du} + 5\frac{dy}{du} + 3y = 30e^{2u},$$

since $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

$$\therefore \frac{d^2y}{du^2} + 4\frac{dy}{du} + 3y = 30e^{2u}$$

$$\frac{d^2y}{du^2} + 4\frac{dy}{du} + 3y = 0$$

The auxillary equation is

$$m^2 + 4m + 3 = 0$$

$$(m+1)(m+3) = 0$$

$$m = -1, -3$$

$\therefore$ The complementary function, y_c, is

$$y_c = Ae^{-u} + Be^{-3u}$$

The particular integral, y_p, is given by

$$y_p = Ce^{2u}$$

$$\frac{dy_p}{du} = 2Ce^{2u}$$

$$\frac{d^2y_p}{du^2} = 4Ce^{2u}$$

$$\frac{d^2y_p}{du^2} + \frac{4dy_p}{du} + 3y_p = 4Ce^{2u} + 4(2Ce^{2u}) + 3Ce^{2u}$$

$$= 4Ce^{2u} + 8Ce^{2u} + 3Ce^{2u}$$

$$= 15Ce^{2u}$$

$$= 30e^{2u}$$

$$15C = 30$$

$$C = 2$$

$$y_p = 2e^{2u}$$

$$y = y_c + y_p$$

$$= Ae^{-u} + Be^{-3u} + 2e^{2u}$$

$$= \frac{A}{x} + \frac{B}{x^3} + 2x^2$$

$\therefore$ The general solution for y is

$$y = \frac{A}{x} + \frac{B}{x^3} + 2x^2$$

10. $C: \quad r = a\sin 3\theta, \quad 0 \leq \theta \leq \frac{\pi}{3}$

i) The area enclosed by C is

$$\int_0^{\frac{\pi}{3}} \frac{r^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{a^2\sin^2 3\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{a^2}{2}\left(\frac{1-\cos 6\theta}{2}\right) d\theta$$

$$= \frac{a^2}{4}\int_0^{\frac{\pi}{3}} 1 - \cos 6\theta \; d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{4}\left(\frac{\pi}{3} - 0\right)$$

$$= \frac{\pi a^2}{12}$$

ii) The distance of a point on C from the initial line $\theta = 0$ is $r\sin\theta$.

$$r\sin\theta = a\sin 3\theta \sin\theta$$

$$\frac{d(r\sin\theta)}{d\theta} = a\sin 3\theta\cos\theta + 3a\cos 3\theta\sin\theta$$

when $\frac{d(r\sin\theta)}{d\theta} = 0$: $a\sin 3\theta\cos\theta + 3a\cos 3\theta\sin\theta = 0$

$$\sin 3\theta\cos\theta + 3\cos 3\theta\sin\theta = 0$$

$$\frac{\sin 3\theta\cos\theta + 3\cos 3\theta\sin\theta}{\cos 3\theta\cos\theta} = 0$$

$$\tan 3\theta + 3\tan\theta = 0.$$

$\therefore$ At the point on C which has maximum distance from the initial line, $\tan 3\theta + 3\tan\theta = 0$

iii) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} + 3\tan\theta = 0$$

$$\frac{3\tan\theta - \tan^3\theta + 3\tan\theta - 9\tan^3\theta}{1 - 3\tan^2\theta} = 0$$

$$\frac{6\tan\theta - 10\tan^3\theta}{1 - 3\tan^2\theta} = 0$$

$$6\tan\theta - 10\tan^3\theta = 0$$

$$5\tan^3\theta - 3\tan\theta = 0$$

$$\tan\theta(5\tan^2\theta - 3) = 0$$

$\tan\theta \neq 0$ $\therefore$ $5\tan^2\theta - 3 = 0$

$$\tan^2\theta = \frac{3}{5}$$

$$\tan\theta = \sqrt{\frac{3}{5}}$$

$$\theta = \tan^{-1}\sqrt{\frac{3}{5}}$$

$$r\sin\theta = a\sin 3\theta \sin\theta$$
$$= a(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)\sin\theta$$
$$= a(2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta)\sin\theta$$
$$= a(2\sin^2\theta\cos^2\theta + (2\cos^2\theta - 1)\sin^2\theta)$$

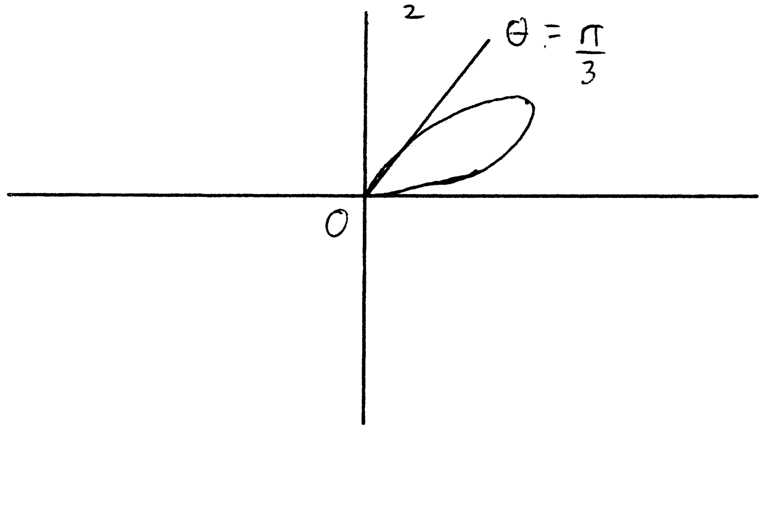
when $\theta = \tan^{-1}\sqrt{\frac{3}{5}}$:

$$r\sin\theta = a\left(2\left(\frac{3}{8}\right)\frac{5}{8} + \left(2\left(\frac{5}{8}\right) - 1\right)\frac{3}{8}\right)$$
$$= a\left(\frac{15}{32} + \frac{3}{32}\right)$$
$$= \frac{9a}{16}$$

The maximum distance of C from the initial line $\theta = 0$ is $\frac{9a}{16}$.

iv)

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
r	0	$\frac{a}{\sqrt{2}}$	a	$\frac{a}{\sqrt{2}}$	0



11 EITHER

$$\sum_{n=1}^{N} n^3 = \frac{N^2}{4}(N+1)^2$$

when $N = 1$:
$$\sum_{n=1}^{1} n^3 = 1^3$$
$$= 1$$
$$= \frac{4}{4}$$
$$= \frac{1 \cdot 4}{4}$$
$$= \frac{1^2 2^2}{4}$$
$$= \frac{1^2(1+1)^2}{4}$$

Assume the statement is true when $N = k$.

$N = k$:
$$\sum_{n=1}^{k} n^3 = \frac{k^2(k+1)^2}{4}$$

when $N = k+1$:
$$\sum_{n=1}^{k+1} n^3 = \frac{(k+1)^2(k+2)^2}{4}$$

(what needs to be proved)

$$\sum_{n=1}^{k+1} n^3 = \sum_{n=1}^{k} n^3 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2\left(\frac{k^2}{4} + k + 1\right)$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$\sum_{n=1}^{N} n^3 = \frac{N^2(N+1)^2}{4}$$ for every positive integer N.

Since $$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6},$$

$$\sum_{n=1}^{N} 20n^3 + 36n^2 = 20\sum_{n=1}^{N} n^3 + 36\sum_{n=1}^{N} n^2$$

$$= \frac{20N^2(N+1)^2}{4} + \frac{36N(N+1)(2N+1)}{6}$$

$$= 5N^2(N+1)^2 + 6N(N+1)(2N+1)$$

$$= N(N+1)(5N(N+1) + 6(2N+1))$$

$$= N(N+1)(5N^2 + 5N + 12N + 6)$$

$$= N(N+1)(5N^2 + 17N + 6)$$

$$= N(N+1)(N+3)(5N+2)$$

$$S_N = \sum_{n=1}^{N} 20n^3 + 36n^2 + \mu n$$

If $S_N = N^2(N+1)(aN+b)$,

$$\sum_{n=1}^{N} 20n^3 + 36n^2 + \mu n = N^2(N+1)(aN+b)$$

$$\sum_{n=1}^{N} 20n^3 + 36n^2 + \mu \sum_{n=1}^{N} n = N^2(N+1)(aN+b)$$

$$N(N+1)(N+3)(5N+2) + \mu \frac{N(N+1)}{2} = N^2(N+1)(aN+b),$$

since $\displaystyle\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$

$$(N+3)(5N+2) + \frac{\mu}{2} = N(aN+b)$$

$$5N^2 + 17N + 6 + \frac{\mu}{2} = aN^2 + bN$$

$$a = 5, \quad b = 17, \quad \mu = -12$$

When $\mu = -12$, $S_N = N^2(N+1)(5N+17)$

$$N^{-4} S_N = \frac{(N+1)(5N+17)}{N^2}$$

$$= 5 + \frac{22}{N} + \frac{17}{N^2}$$

when $N \geqslant 18$, $\frac{17}{N^2} > 0$

$$5 + \frac{22}{N} < 5 + \frac{22}{N} + \frac{17}{N^2}$$

Also, $\frac{17}{N^2} < \frac{1}{N}$

$$5 + \frac{22}{N} + \frac{17}{N^2} < 5 + \frac{23}{N}$$

$$\therefore \ 5 + \frac{22}{N} < 5 + \frac{22}{N} + \frac{17}{N^2} < 5 + \frac{23}{N}$$

OR

$$A = \begin{pmatrix} 1 & -4 & 6 \\ 2 & -4 & 2 \\ -3 & 4 & a \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1 & -4 & 6 \\ 2 & -4 & 2 \\ -3 & 4 & a \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -4 & 6 \\ 2 & -4-\lambda & 2 \\ -3 & 4 & a-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda) \begin{vmatrix} -4-\lambda & 2 \\ 4 & a-\lambda \end{vmatrix} - (-4) \begin{vmatrix} 2 & 2 \\ -3 & a-\lambda \end{vmatrix}$$

$$+ 6 \begin{vmatrix} 2 & -4-\lambda \\ -3 & 4 \end{vmatrix}$$

$$= (1-\lambda)\left[(\lambda+4)(\lambda-a) - 8\right] + 4(2a - 2\lambda + 6)$$

$$+ 6\left[8 + 3(-4-\lambda)\right]$$

$$= (1-\lambda)(\lambda^2 + 4\lambda - a\lambda - 4a - 8) + 8a - 8\lambda + 24$$

$$+ 6(8 - 12 - 3\lambda)$$

$$= (1-\lambda)(\lambda^2 + 4\lambda - a\lambda - 4a - 8) + 8a - 8\lambda + 24$$

$$+ 6(-4 - 3\lambda)$$

$$= (1-\lambda)(\lambda^2 + 4\lambda - a\lambda - 4a - 8) + 8a - 8\lambda + 24 - 24 - 18\lambda$$

$$= (1-\lambda)(\lambda^2 + 4\lambda - a\lambda - 4a - 8) + 8a - 26\lambda$$

when $|A - \lambda I| = 0$:

$$(1-\lambda)(\lambda^2 + 4\lambda - a\lambda - 4a - 8) + 8a - 26\lambda = 0.$$

If -2 is an eigenvalue, $\lambda = -2$ is a root of the equation $|A - \lambda I| = 0$.

$$3(4 - 8 + 2a - 4a - 8) + 8a + 52 = 0$$

$$3(-2a - 12) + 8a + 52 = 0$$

$$-6a - 36 + 8a + 52 = 0$$

$$2a = -16$$

$$a = -8.$$

$$|A - \lambda I| = (1-\lambda)(\lambda^2 + 4\lambda + 8\lambda + 32 - 8) - 64 - 26\lambda$$

$$= (1-\lambda)(\lambda^2 + 12\lambda + 24) - 64 - 26\lambda$$

$$= \lambda^2 + 12\lambda + 24 - \lambda^3 - 12\lambda^2 - 24\lambda - 64 - 26\lambda$$

$$= -\lambda^3 - 11\lambda^2 - 38\lambda - 40$$

$$\begin{array}{r|l}
 & -\lambda^2 - 9\lambda - 20 \\
\hline
\lambda + 2 & -\lambda^3 - 11\lambda^2 - 38\lambda - 40 \\
 & \underline{-\lambda^3 - 2\lambda^2} \\
 & \quad -9\lambda^2 - 38\lambda \\
 & \quad \underline{-9\lambda^2 - 18\lambda} \\
 & \qquad -20\lambda - 40 \\
 & \qquad \underline{-20\lambda - 40} \\
 & \qquad\qquad 0
\end{array}$$

$$= (\lambda + 2)(-\lambda^2 - 9\lambda - 20)$$

$$= -(\lambda + 2)(\lambda + 4)(\lambda + 5)$$

when $|A - \lambda I| = 0$.

$$-(\lambda + 2)(\lambda + 4)(\lambda + 5) = 0$$

$$\lambda = -2, -4, -5.$$

The eigenvalues of A are $-2, -4$ and -5.

when $\lambda = -2$:
$$\begin{pmatrix} 3 & -4 & 6 \\ 2 & -2 & 2 \\ -3 & 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & -4 & 6 & 0 \\ 2 & -2 & 2 & 0 \\ -3 & 4 & -6 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 3 & -4 & 6 & 0 \\ -3 & 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & -4 & 6 & 0 \\ -3 & 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & -4 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-3r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s, s \in \mathbb{R}$

$y = 3s$

$x - 3s + s = 0$

$x = 2s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ 3s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

when $\lambda = -4$: $\begin{pmatrix} 5 & -4 & 6 \\ 2 & 0 & 2 \\ -3 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 5 & -4 & 6 & 0 \\ 2 & 0 & 2 & 0 \\ -3 & 4 & -4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 5 & -4 & 6 & 0 \\ -3 & 4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 5 & -4 & 6 & 0 \\ -3 & 4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow[3r_1 + r_3]{-5r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 4s$, $s \in R$

$y = s$

$x = -4s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4s \\ s \\ 4s \end{pmatrix}$$

$$= s\begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}$$

when $\lambda = -5$: $\begin{pmatrix} 6 & -4 & 6 \\ 2 & 1 & 2 \\ -3 & 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 6 & -4 & 6 & 0 \\ 2 & 1 & 2 & 0 \\ -3 & 4 & -3 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 6 & -4 & 6 & 0 \\ -3 & 4 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{2 \times r_3} \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 6 & -4 & 6 & 0 \\ -6 & 8 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[3r_1 + r_3]{-3r_1 + r_2} \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 11 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-7},\ \frac{r_3}{11}} \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 0$

let $z = s$, $s \in R$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvalues of A are $-2, -4, -5$ with corresponding eigenvectors $\begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}-4\\1\\4\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}$.

$$\mathbf{e}_1 = \begin{pmatrix}2\\3\\1\end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

i) If $\mathbf{x} \in V$, $\mathbf{x} = s\mathbf{e}_1 + t\mathbf{e}_2$, $s, t \in \mathbb{R}$ since V is spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$.

$$\begin{aligned} A\mathbf{x} &= A(s\mathbf{e}_1 + t\mathbf{e}_2) \\ &= A(s\mathbf{e}_1) + A(t\mathbf{e}_2) \\ &= s(A\mathbf{e}_1) + t(A\mathbf{e}_2) \\ &= s(-2\mathbf{e}_1) + t(5\mathbf{e}_2) \\ &= (-2s)\mathbf{e}_1 + (5t)\mathbf{e}_2 \in V. \end{aligned}$$

$\therefore$ If $\mathbf{x}$ is any vector in V, $A\mathbf{x}$ is also in V.

ii) If $\mathbf{e}_3$ is perpendicular to every vector in V, $\mathbf{e}_3$ is to $\mathbf{e}_1$ and $\mathbf{e}_2$. $\therefore$ $\mathbf{e}_3$ is parallel to $\mathbf{e}_1 \times \mathbf{e}_2$.

$$\mathbf{e}_1 \times \mathbf{e}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \begin{pmatrix}3\\-3\\3\end{pmatrix} = 3\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\mathbf{e}_3 = \begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$A\underset{\sim}{e}_3 = \begin{pmatrix} 1 & -4 & 6 \\ 2 & -4 & 2 \\ -3 & 4 & -8 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 \\ 8 \\ -15 \end{pmatrix}$$

$$\neq -4\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore \underset{\sim}{e}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is not an eigenvector of A.