1. $\left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ -1 & 3 & -1 & k \\ 2 & -5 & k & -2 \end{array}\right)$

$$\xrightarrow[R_3 : R_3 - 2R_1]{R_2 : R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 1 & 1 & k+1 \\ 0 & -1 & k-4 & -4 \end{array}\right)$$

$$\xrightarrow[R_3 : R_3 + R_2]{R_1 : R_1 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 2k+3 \\ 0 & 1 & 1 & k+1 \\ 0 & 0 & k-3 & k-3 \end{array}\right)$$

The common line exists when $k=3$, and the augmented matrix becomes $\left(\begin{array}{ccc|c} 1 & 0 & 4 & 9 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Let, $z = t$

From R_2 : $y = 4 - t$ and R_1 : $x = 9 - 4t$

Common Line of intersection is

$$\underset{\sim}{r} = \begin{pmatrix} 9 \\ 4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}$$

2. $105^3 + 112^3 + 119^3 + \dots + 196^3$

$a = 105 \quad d = 7$

$$T_n = a + (n-1)d$$
$$= 105 + 7(n-1)$$
$$= 105 + 7n - 7$$
$$= 7n + 98$$

when $T_n = 196$,

$$7n + 98 = 196$$
$$7n = 98$$
$$n = 14$$

$$105^3 + 112^3 + 119^3 + \dots + 196^3$$
$$= \sum_{r=1}^{14} (7r + 98)^3$$
$$= \sum_{r=1}^{14} 343r^3 + 14406r^2 + 201684r + 941192$$
$$= 343\sum_{r=1}^{14} r^3 + 14406\sum_{r=1}^{14} r^2 + 201684\sum_{r=1}^{14} r$$
$$+ 941192\sum_{r=1}^{14} 1$$

$$= \frac{343(14^2)(14+1)^2}{4} + \frac{14406(14)(14+1)(2(14)+1)}{6}$$

$$+ \frac{201684(14)(14+1)}{2} + 941192(14)$$

$$= \frac{343(14^2)15^2}{4} + \frac{14406(14)15(29)}{6}$$

$$+ \frac{201684(14)15}{2} + 941192(14)$$

$$= 3781575 + 14622090 + 21176820 + 13176688$$

$$= 52757173$$

3. $n = 1$ $P_1 = 1(1+5) = 6 \Rightarrow P_n$ is divisible by 6 for $n = 1$.

$n = k$: Assume P_n is true for $n = k$:
i.e. $k(k^2 + 5)$ is divisible by 6.

$n = k+1$: $P_{k+1} = (k+1)\left[(k+1)^2 + 5\right]$

$$= (k+1)(k^2 + 2k + 6)$$

$$= k^3 + 3k^2 + 8k + 6$$

$$= (k^3 + 5k) + 3k^2 + 3k + 6$$

$$= k(k^2 + 5) + 3k(k+1) + 6$$

$k(k^2 + 5)$ is divisible by 6 from Step 3 above.

$3k(k+1)$ is divisible by 6 because k and $(k+1)$ are successive integers, therefore $k(k+1)$ is a multiple of 2.

6 is divisible by 6.

If P_k is divisible by 6 then P_{k+1} is also divisible by 6.

By the principle of mathematical induction $n(n^2 + 5)$ is divisible by 6 for all positive integral values of n.

4 a) $3x + 9y - 7z = 4$ $\quad$ $2x - 6y + 5z = 1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 9 \\ -7 \end{pmatrix} = 4 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -6 \\ 5 \end{pmatrix} = 1$$

Since the direction of the line of intersection of the two planes is perpendicular to both $\begin{pmatrix} 3 \\ 9 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -6 \\ 5 \end{pmatrix}$, it is parallel to $\begin{pmatrix} 3 \\ 9 \\ -7 \end{pmatrix} \times \begin{pmatrix} 2 \\ -6 \\ 5 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 9 \\ -7 \end{pmatrix} \times \begin{pmatrix} 2 \\ -6 \\ 5 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 9 & -7 \\ 2 & -6 & 5 \end{vmatrix} = \begin{pmatrix} 3 \\ -29 \\ -36 \end{pmatrix}$$

Let $x = 0$: $\quad 9y - 7z = 4 \qquad -6y + 5z = 1$

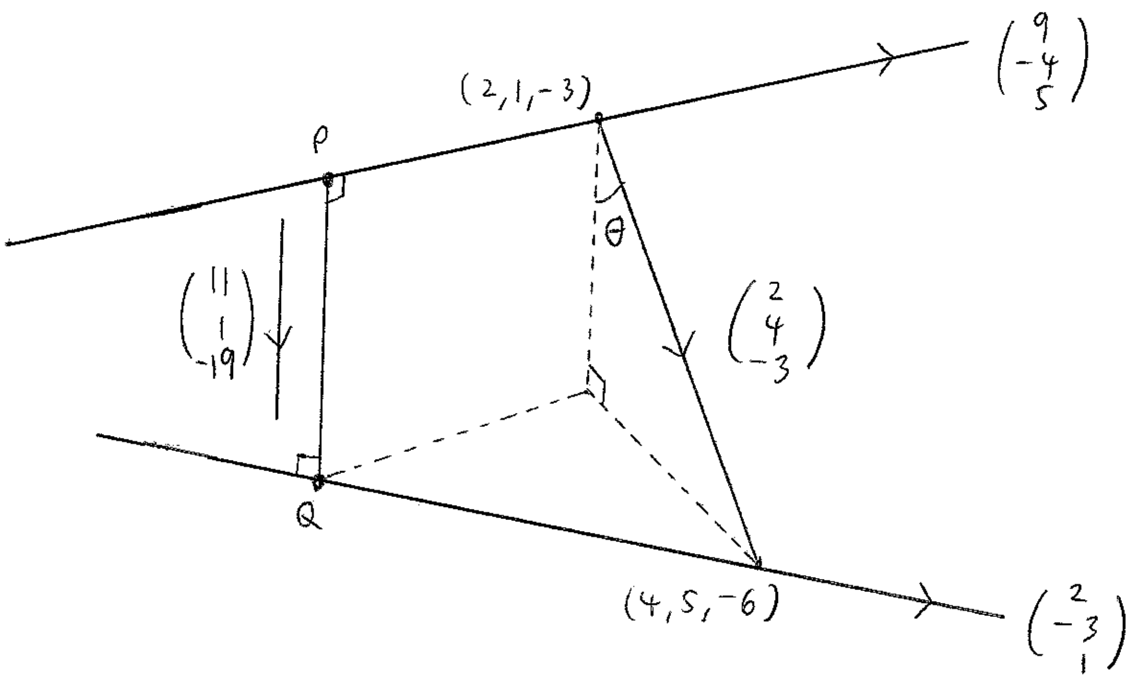
$$18y - 14z = 8 \qquad -18y + 15z = 3$$

$$z = 11$$

$$y = 9$$

$\therefore$ The equation of the line of intersection of the planes $3x + 9y - 7z = 4$ and $2x - 6y + 5z = 1$ is $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 9 \\ 11 \end{pmatrix} + s \begin{pmatrix} 3 \\ -29 \\ -36 \end{pmatrix}$

b) $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + s \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \qquad \underset{\sim}{r} = \begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$



If P and Q are the points on the lines with the minimum distance, since PQ is perpendicular to both lines, $\overrightarrow{PQ}$ is parallel to $\begin{pmatrix}9\\-4\\5\end{pmatrix} \times \begin{pmatrix}2\\-3\\1\end{pmatrix}$.

$$\begin{pmatrix}9\\-4\\5\end{pmatrix} \times \begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 9 & -4 & 5 \\ 2 & -3 & 1 \end{vmatrix} = \begin{pmatrix}11\\1\\-19\end{pmatrix}$$

Since
$$|\overrightarrow{PQ}| = \left| \begin{pmatrix}4\\5\\-6\end{pmatrix} - \begin{pmatrix}2\\1\\-3\end{pmatrix} \right| \cos\theta$$
$$= \left| \begin{pmatrix}2\\4\\-3\end{pmatrix} \right| \cos\theta$$

and
$$\begin{pmatrix}11\\1\\-19\end{pmatrix} \cdot \begin{pmatrix}2\\4\\-3\end{pmatrix} = \left| \begin{pmatrix}11\\1\\-19\end{pmatrix} \right| \left| \begin{pmatrix}2\\4\\-3\end{pmatrix} \right| \cos\theta$$

$\therefore \; |\overrightarrow{PQ}| = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \cos\theta$

$$= \frac{\begin{pmatrix} 11 \\ 1 \\ -19 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}}{\left| \begin{pmatrix} 11 \\ 1 \\ -19 \end{pmatrix} \right|}$$

$$= \frac{22 + 4 + 57}{\sqrt{121 + 1 + 361}}$$

$$= \frac{83}{\sqrt{483}}$$

$\therefore$ The minimum distance between the two lines is $\dfrac{83}{\sqrt{483}}$

c) Since $5x + 2y - 8z = 3$ is a plane parallel to the plane and $(9,-6,4)$ is a point on the plane,

$$\underset{\sim}{r} \cdot \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix} = 45 - 12 - 32$$

$$5x + 2y - 8z = 1$$

$\therefore$ The equation of the plane parallel to the plane $5x + 2y - 8z = 3$ and containing the point $(9,-6,4)$ is $5x + 2y - 8z = 1$.

5. $y = \dfrac{x^3}{1+3x^4} \Rightarrow \dfrac{dy}{dx} = \dfrac{(1+3x^4)(3x^2) - (x^3)(12x^3)}{(1+3x^4)^2}$

$$= \frac{3x^2(1-x^4)}{(1+3x^4)^2}$$

At the stationary points, $\dfrac{dy}{dx} = 0 \Rightarrow x = -1, 0, 1$

$y = \frac{-1}{4}, 0, \frac{1}{4}$

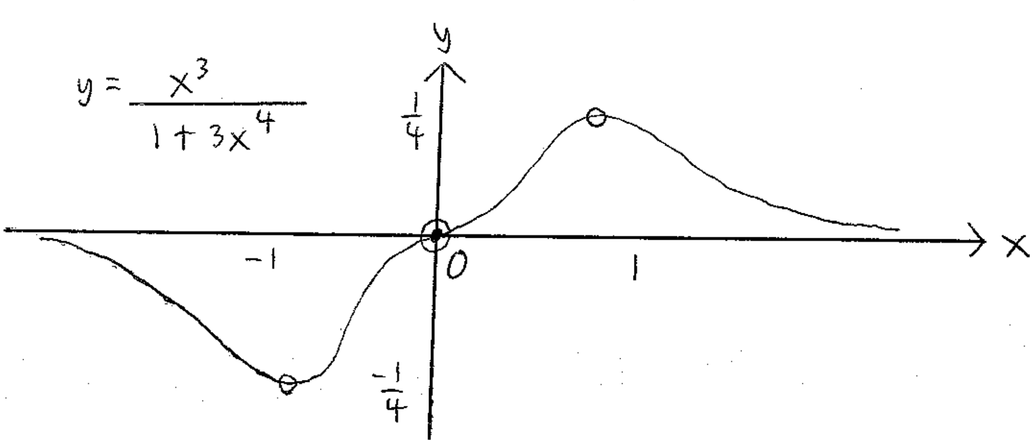
$$\frac{d^2y}{dx^2} = \frac{(1+3x^4)^2(6x - 18x^5) - (3x^2 - 3x^6)2(1+3x^4)12x^3}{(1+3x^4)^4}$$

when $x = -1$: $\dfrac{d^2y}{dx^2} = \dfrac{3}{64} > 0$

when $x = 0$: $\dfrac{d^2y}{dx^2} = 0$

when $x = 1$: $\dfrac{d^2y}{dx^2} = \dfrac{-3}{64} < 0$

$\therefore$ $(-1, \frac{-1}{4})$ is a minimum point, $(0,0)$ is an inflexion point and $(1, \frac{1}{4})$ is a maximum point.



o : critical point

• : intersection point.

6. $2x^3 - 4x^2 + 3x - 9 = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = 2 \quad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{3}{2} \quad \alpha\beta\gamma = \frac{9}{2}$

$S_n = \alpha^n + \beta^n + \gamma^n$

$S_2 = \alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 2^2 - 2\left(\frac{3}{2}\right)$

$= 4 - 3$

$= 1$

$\alpha^3, \beta^3, \gamma^3$

Let $u = \alpha^3$

$\alpha = u^{\frac{1}{3}}$

α is a root

$\therefore 2\alpha^3 - 4\alpha^2 + 3\alpha - 9 = 0$

$2(u^{\frac{1}{3}})^3 - 4(u^{\frac{1}{3}})^2 + 3u^{\frac{1}{3}} - 9 = 0$

$2u - 4u^{\frac{2}{3}} + 3u^{\frac{1}{3}} - 9 = 0$

$2u - 9 = 4u^{\frac{2}{3}} - 3u^{\frac{1}{3}}$

$= u^{\frac{1}{3}}(4u^{\frac{1}{3}} - 3)$

$(2u - 9)^3 = \left[u^{\frac{1}{3}}(4u^{\frac{1}{3}} - 3)\right]^3$

$8u^3 - 108u^2 + 486u - 729$

$= u(4u^{\frac{1}{3}} - 3)^3$

$= u(64u - 144u^{\frac{2}{3}} + 108u^{\frac{1}{3}} - 27)$

$= u(64u - 36(4u^{\frac{2}{3}} - 3u^{\frac{1}{3}}) - 27)$

$= u(64u - 36(2u - 9) - 27)$

$= u(64u - 72u + 324 - 27)$

$= u(-8u + 297)$

$= -8u^2 + 297u$

$8u^3 - 100u^2 + 189u - 729 = 0$

The equation having roots α^3, β^3, r^3 is $8u^3 - 100u^2 + 189u - 729 = 0$.

$\alpha^3 + \beta^3 + r^3 = \frac{25}{2}$

$\alpha^3\beta^3 + \alpha^3 r^3 + \beta^3 r^3 = \frac{189}{8}$

$\alpha^3\beta^3 r^3 = \frac{729}{8}$

$S_3 = \alpha^3 + \beta^3 + r^3$

$= \frac{25}{2}$

$$S_6 = \alpha^6 + \beta^6 + r^6$$

$$= (\alpha^3 + \beta^3 + r^3)^2 - 2(\alpha^3\beta^3 + \alpha^3 r^3 + \beta^3 r^3)$$

$$= \left(\frac{25}{2}\right)^2 - 2\left(\frac{189}{8}\right)$$

$$= \frac{625}{4} - \frac{189}{4}$$

$$= \frac{436}{4}$$

$$= 109$$

$$S_{-3} = \alpha^{-3} + \beta^{-3} + r^{-3}$$

$$= \frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{r^3}$$

$$= \frac{\alpha^3\beta^3 + \alpha^3 r^3 + \beta^3 r^3}{\alpha^3\beta^3 r^3}$$

$$= \frac{\frac{189}{8}}{\frac{729}{8}}$$

$$= \frac{189}{729}$$

$$= \frac{7}{27}$$