1. $$\frac{1}{n^2+3n+2} - \frac{1}{n^2+5n+6}$$

$$= \frac{n^2+5n+6-n^2-3n-2}{(n^2+3n+2)(n^2+5n+6)}$$

$$= \frac{2n+4}{(n^2+3n+2)(n^2+5n+6)}$$

$$= \frac{2(n+2)}{(n+1)(n+2)(n+2)(n+3)}$$

$$= \frac{2(n+2)}{(n+1)(n+2)^2(n+3)}$$

$$= \frac{2}{(n+1)(n+2)(n+3)}$$

Since
$$\frac{1}{n^2 + 3n + 2} - \frac{1}{n^2 + 5n + 6}$$

$$= \frac{2}{(n+1)(n+2)(n+3)},$$

$$S_n = \sum_{r=1}^{n} \frac{4}{(r+1)(r+2)(r+3)}$$

$$= \sum_{r=1}^{n} \frac{2}{r^2 + 3r + 2} - \frac{2}{r^2 + 5r + 6}$$

$$= \frac{2}{2\;3} - \frac{2}{3\;4}$$

$$+ \frac{2}{3\;4} - \frac{2}{4\;5}$$

$$+ \frac{2}{4\;5} - \frac{2}{5\;6}$$

$$\vdots$$

$$+ \frac{2}{(n-1)n} - \frac{2}{n(n+1)}$$

$$+ \frac{2}{n(n+1)} - \frac{2}{(n+1)(n+2)}$$

$$+ \frac{2}{(n+1)(n+2)} - \frac{2}{(n+2)(n+3)}$$

$$= \frac{1}{3} - \frac{2}{(n+2)(n+3)}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(\frac{1}{3} - \frac{2}{(n+2)(n+3)} \right)$$

$$= \frac{1}{3} - 0$$

$$= \frac{1}{3}$$

2. Let $f(n) = 9^n + 7^{2n+3}$

When $n = 1$: $f(1) = 9^1 + 7^{2(1)+3}$

$= 9 + 7^{2+3}$

$= 9 + 7^5$

$= 9 + 16807$

$= 16816$

$= 8(2102)$

$\therefore 8 \mid f(1)$

Assume the statement is true when $n = k$.

$n = k$:

$8 \mid f(k)$

$f(k) = 8s, \; s \in \mathbb{Z}$

$9^k + 7^{2k+3} = 8s$

When $n = k + 1$:

$f(k+1) = 9^{k+1} + 7^{2(k+1)+3}$

$= 9^k 9 + 7^{2k+2+3}$

$= 9^k 9 + 7^{2k+3} 7^2$

$$= 9^k 9 + 7^{2k+3} 49$$

$$= 9^k 9 + 7^{2k+3}(40 + 9)$$

$$= 9^k 9 + 7^{2k+3} 40 + 7^{2k+3} 9$$

$$= 9^k 9 + 7^{2k+3} 9 + 7^{2k+3} 40$$

$$= 9(9^k + 7^{2k+3}) + 7^{2k+3} 40$$

$$= 9(8s) + 7^{2k+3} 40$$

$$= 8(9s + 7^{2k+3} 5)$$

Since s is an integer and k is an integer, $9s + 7^{2k+3}5$ is an integer

$$\therefore 8 \mid f(k+1)$$

Since $f(k+1)$ is divisible by 8 if $f(k)$ is divisible by 8 and $f(1)$ is divisible by 8, $f(n)$ is divisible by 8 for every positive integer n.

$\therefore 9^n + 7^{2n+3}$ is divisible by 8 for every positive integer n.

3. $5x^4 + 3x^2 - x - 1 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots.

$2\alpha - 1,\ 2\beta - 1,\ 2\gamma - 1,\ 2\delta - 1$

Let $u = 2\alpha - 1$

$$\alpha = \frac{u+1}{2}$$

α is a root

$$5\alpha^4 + 3\alpha^2 - \alpha - 1 = 0$$

$$5\left(\frac{u+1}{2}\right)^4 + 3\left(\frac{u+1}{2}\right)^2 - \left(\frac{u+1}{2}\right) - 1 = 0$$

$$5\left(\frac{u^4 + 3u^3 + 6u^2 + 4u + 1}{16}\right)$$

$$+ 3\left(\frac{u^2 + 2u + 1}{4}\right) - \frac{u}{2} - \frac{1}{2} - 1 = 0$$

$$5u^4 + 20u^3 + 30u^2 + 20u + 5$$

$$+ 12u^2 + 24u + 12 - 8u - 8 - 16 = 0$$

$$5u^4 + 20u^3 + 42u^2 + 36u - 7 = 0$$

The equation having roots $2\alpha - 1$, $2\beta - 1$, $2\gamma - 1$, $2\delta - 1$ is

$$5u^4 + 20u^3 + 42u^2 + 36u - 7 = 0.$$

$$2\alpha - 1 + 2\beta - 1 + 2\gamma - 1 + 2\delta - 1 = -4$$

$$(2\alpha - 1)(2\beta - 1) + (2\alpha - 1)(2\gamma - 1)$$
$$+ (2\alpha - 1)(2\delta - 1) + (2\beta - 1)(2\gamma - 1)$$
$$+ (2\beta - 1)(2\delta - 1) + (2\gamma - 1)(2\delta - 1) = \frac{42}{5}$$

$$(2\alpha - 1)(2\beta - 1)(2\gamma - 1)$$
$$+ (2\alpha - 1)(2\beta - 1)(2\delta - 1)$$
$$+ (2\alpha - 1)(2\gamma - 1)(2\delta - 1)$$
$$+ (2\beta - 1)(2\gamma - 1)(2\delta - 1) = \frac{-36}{5}$$

$$(2\alpha - 1)(2\beta - 1)(2\gamma - 1)(2\delta - 1) = \frac{-7}{5}$$

If $S_n = (2\alpha - 1)^n + (2\beta - 1)^n$

$+ (2\gamma - 1)^n + (2\delta - 1)^n,$

$S_0 = (2\alpha - 1)^0 + (2\beta - 1)^0$

$+ (2\gamma - 1)^0 + (2\delta - 1)^0$

$= 1 + 1 + 1 + 1$

$= 4$

$S_1 = (2\alpha - 1)^1 + (2\beta - 1)^1$

$+ (2\gamma - 1)^1 + (2\delta - 1)^1$

$= 2\alpha - 1 + 2\beta - 1 + 2\gamma - 1 + 2\delta - 1$

$= -4$

$S_2 = (2\alpha - 1)^2 + (2\beta - 1)^2$

$+ (2\gamma - 1)^2 + (2\delta - 1)^2$

$$= (2\alpha - 1 + 2\beta - 1 + 2\gamma - 1 + 2\delta - 1)^2$$

$$- 2[(2\alpha - 1)(2\beta - 1) + (2\alpha - 1)(2\gamma - 1)$$

$$+ (2\alpha - 1)(2\delta - 1) + (2\beta - 1)(2\gamma - 1)$$

$$+ (2\beta - 1)(2\delta - 1) + (2\gamma - 1)(2\delta - 1)]$$

$$= (-4)^2 - 2\left(\frac{42}{5}\right)$$

$$= 16 - \frac{84}{5}$$

$$= \frac{-4}{5}$$

$$S_{-1} = (2\alpha - 1)^{-1} + (2\beta - 1)^{-1}$$

$$+ (2\gamma - 1)^{-1} + (2\delta - 1)^{-1}$$

$$= \frac{1}{2\alpha - 1} + \frac{1}{2\beta - 1} + \frac{1}{2\gamma - 1} + \frac{1}{2\delta - 1}$$

$$= \frac{\begin{array}{l}(2\alpha - 1)(2\beta - 1)(2\gamma - 1) \\ + (2\alpha - 1)(2\beta - 1)(2\delta - 1) \\ + (2\alpha - 1)(2\gamma - 1)(2\delta - 1) \\ + (2\beta - 1)(2\gamma - 1)(2\delta - 1)\end{array}}{(2\alpha - 1)(2\beta - 1)(2\gamma - 1)(2\delta - 1)}$$

$$= \frac{\frac{-36}{5}}{\frac{-7}{5}}$$

$$= \frac{36}{7}$$

$$5S_{4+r} + 20S_{3+r} + 42S_{2+r}$$

$$+ 36S_{1+r} - 7S_r = 0.$$

$$r = -1: \quad 5S_3 + 20S_2 + 42S_1 + 36S_0 - 7S_{-1} = 0$$

$$5S_3 + 20\left(\frac{-4}{5}\right) + 42(-4)$$

$$+ 36(4) - 7\left(\frac{36}{7}\right) = 0$$

$$5S_3 - 16 - 168 + 144 - 36 = 0$$

$$5S_3 = 76$$

$$S_3 = \frac{76}{5}$$

$$r = 0 \quad 5S_4 + 20S_3 + 42S_2 + 36S_1 - 7S_0 = 0$$

$$5S_4 + 20\left(\frac{76}{5}\right) + 42\left(\frac{-4}{5}\right)$$

$$+ 36(-4) - 7(4) = 0$$

$$5S_4 + 304 - \frac{168}{5} - 144 - 28 = 0$$

$$5S_4 = \frac{-492}{5}$$

$$S_4 = \frac{-492}{25}$$

4 C

$$y = \frac{2x^2 - 7x - 13}{(x+1)(x-3)}$$

$$\begin{array}{r|l} & \quad\quad 2 \\ \cline{2-2} x^2 - 2x - 3 & 2x^2 - 7x - 3 \\ & 2x^2 - 4x - 6 \\ \cline{2-2} & \quad\quad -3x - 7 \end{array}$$

$$y = 2 + \frac{-3x - 7}{(x+1)(x-3)}$$

$$\frac{-3x-7}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$= \frac{A(x-3) + B(x+1)}{(x+1)(x-3)}$$

$$-3x - 7 = A(x-3) + B(x+1)$$

$$= (A+B)x - 3A + B$$

$$\therefore A + B = -3 \quad -3A + B = -7$$

$$4A = 4$$

$$A = 1$$

$$B = -4$$

$\therefore y = 2 + \dfrac{1}{x+1} - \dfrac{4}{x-3}$

Asymptotes:

As $x \to \pm\infty$ $y \to 2$

As $x \to -1$ $y \to \pm\infty$

As $x \to 3$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = 2$, $x = -1$ and $x = 3$.

Intersection points:

When $x = 0$: $y = \dfrac{13}{3}$

$y = 0$: $\dfrac{2x^2 - 7x - 13}{(x+1)(x-3)} = 0$

$$2x^2 - 7x - 13 = 0$$

$$x = \frac{7 \pm \sqrt{153}}{4}$$

$\therefore$ The intersection points of C are $\left(0, \frac{13}{3}\right)$, $\left(\frac{7+\sqrt{153}}{4}, 0\right)$ and $\left(\frac{7-\sqrt{153}}{4}, 0\right)$

Critical points :

$$\frac{dy}{dx} = \frac{1}{(x+1)^2} + \frac{4}{(x-3)^2}$$

when $\frac{dy}{dx} = 0$

$$\frac{-1}{(x+1)^2} + \frac{4}{(x-3)^2} = 0$$

$$\frac{1}{(x+1)^2} = \frac{4}{(x-3)^2}$$

$$(x-3)^2 = 4(x+1)^2$$

$$x - 3 = 2(x+1),\ -2(x+1)$$

$$= 2x + 2,\ -2x - 2$$

$$x = -5,\ 3x = 1$$

$$x = \frac{1}{3}$$

$$y = \frac{9}{4},\ y = \frac{17}{4}$$

$\therefore$ The critical points of C are

$\left(-5, \frac{9}{4}\right)$ and $\left(\frac{1}{3}, \frac{17}{4}\right)$.

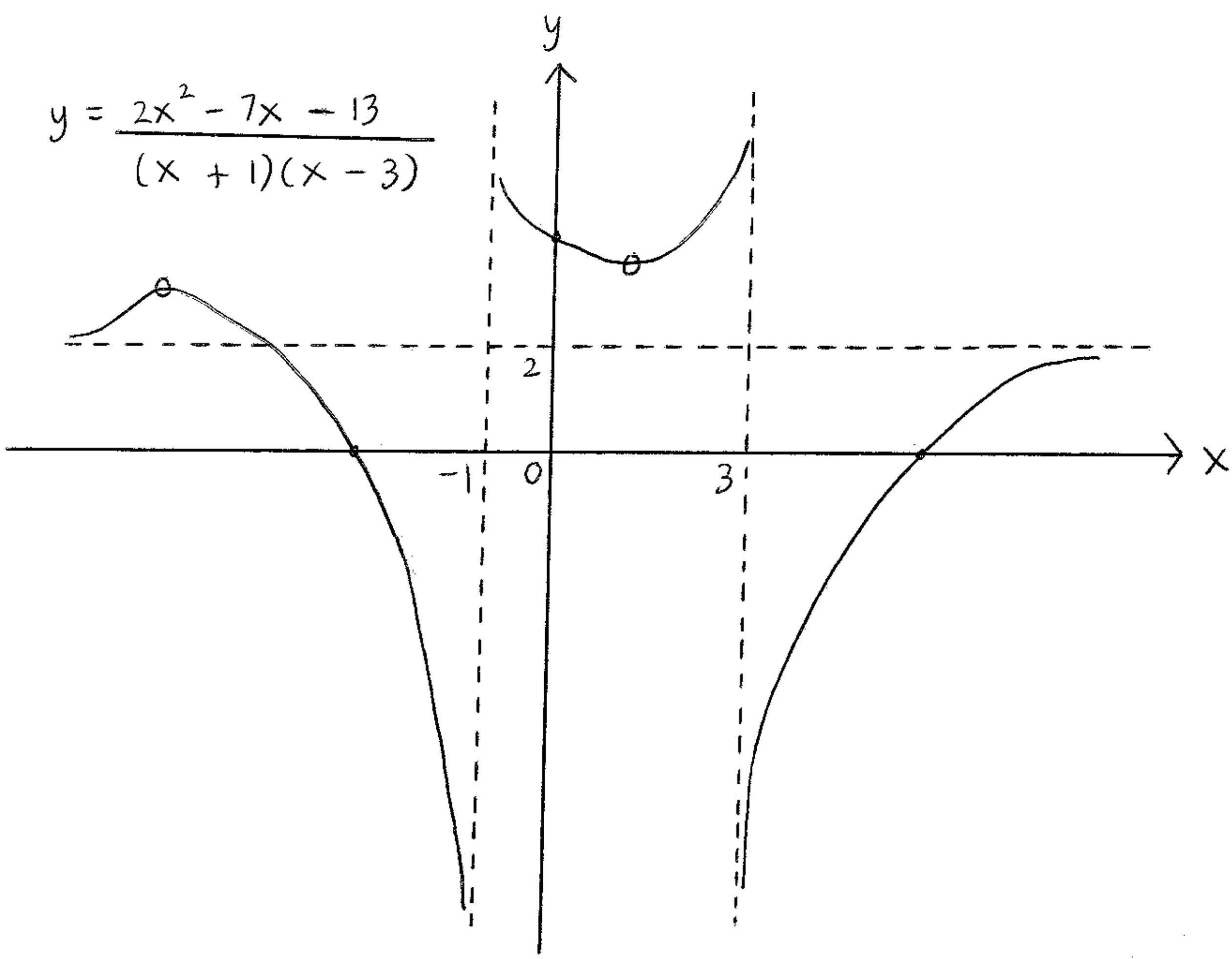
$$\frac{d^2y}{dx^2} = \frac{2}{(x+1)^3} - \frac{8}{(x-3)^3}$$

When $x = -5$: $\dfrac{d^2y}{dx^2} = \dfrac{-1}{64} < 0$

When $x = \dfrac{1}{3}$: $\dfrac{d^2y}{dx^2} = \dfrac{81}{64} > 0$

$\therefore$ $\left(-5, \frac{9}{4}\right)$ is a maximum point

and $\left(\frac{1}{3}, \frac{17}{4}\right)$ is a minimum point.



o : critical point

• : intersection point

a) $2x + 3y - z = 8 \quad 4x + y - 9z = 6$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 8 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -9 \end{pmatrix} = 6$$

Since the line of intersection of the two planes is perpendicular to both $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \\ -9 \end{pmatrix}$, it is parallel to $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 1 \\ -9 \end{pmatrix}$

$$\begin{pmatrix} 4 \\ 1 \\ -9 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & 1 & -9 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= 26\underset{\sim}{i} - 14\underset{\sim}{j} + 10\underset{\sim}{k}$$

$$= 2(13\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k})$$

When $z = 0$: $2x + 3y = 8$, $4x + y = 6$

$$4x + 6y = 16$$

$$5y = 10$$

$$y = 2$$

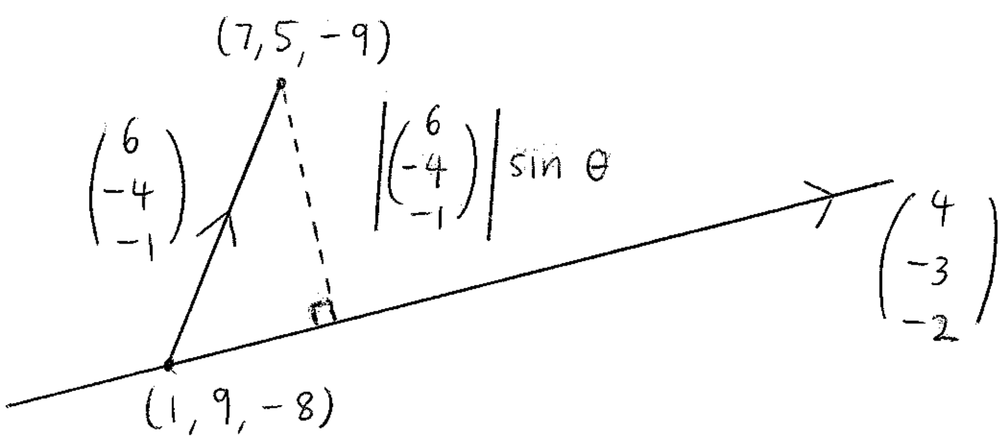
$$x = 1$$

$\therefore$ $(1, 2, 0)$ is a point on both planes.

Since the direction of the line of intersection is $13\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}$ and $(1, 2, 0)$ is a point on the line, the equation of the line of intersection of the planes $2x + 3y - z = 8$ and $4x + y - 9z = 6$ is

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 13 \\ -7 \\ 5 \end{pmatrix}.$$

b) $(7, 5, -9)$ $\underset{\sim}{r} = \underset{\sim}{i} + 9\underset{\sim}{j} - 8\underset{\sim}{k} + s(4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k})$



$$\begin{pmatrix} 7 \\ 5 \\ -9 \end{pmatrix} - \begin{pmatrix} 1 \\ 9 \\ -8 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ -1 \end{pmatrix}$$

Since the direction of the line is $4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$ and $(1, 9, -8)$ is a point on the line, the perpendicular distance of the point $(7, 5, -9)$ from the line is given by

$$\left| \left\| \begin{pmatrix} 6 \\ -4 \\ -1 \end{pmatrix} \right\| \sin\theta \right| = \frac{\left| \begin{pmatrix} 6 \\ -4 \\ -1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix} \right|}$$

$$= \frac{\left\| \begin{matrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 6 & -4 & -1 \\ 4 & -3 & -2 \end{matrix} \right\|}{\sqrt{16 + 9 + 4}}$$

$$= \frac{\left| 5\underset{\sim}{i} + 8\underset{\sim}{j} - 2\underset{\sim}{k} \right|}{\sqrt{29}}$$

$$= \frac{\sqrt{25 + 64 + 4}}{\sqrt{29}}$$

$$= \sqrt{\frac{93}{29}}$$