1. Let $f(n) = 169^{2n-1} + 239^n$

when $n=1$: $f(1) = 169^{2(1)-1} + 239^1$

$= 169^{2-1} + 239$

$= 169^1 + 239$

$= 169 + 239$

$= 408$

$= 12(34)$

$\therefore 34 \mid f(1)$

Assume the statement is true when $n=k$.

$n=k$: $f(k) = 169^{2k-1} + 239^k$

$34 \mid f(k)$

$\therefore f(k) = 34s$, s is an integer.

$169^{2k-1} + 239^k = 34s$

when $n=k+1$: $f(k+1) = 169^{2(k+1)-1} + 239^{k+1}$

$= 169^{2k+2-1} + 239^{k+1}$

$= 169^{2k-1}169^2 + 239^k 239$

$= 169^{2k-1}(28561) + 239^k 239$

$= 169^{2k-1}(28322 + 239)$

$+ 239^k 239$

$$= 169^{2k-1}28322 + 169^{2k-1}239 + 239^{k}239$$

$$= 169^{2k-1}34(833) + 239(169^{2k-1} + 239^{k})$$

$$= 169^{2k-1}34(833) + 239(34s)$$

$$= 34(169^{2k-1}833 + 239s)$$

Since s is an integer and k is an integer, $169^{2k-1}833 + 239s$ is an integer.

$\therefore 34 \mid f(k+1)$

$\therefore 169^{2n-1} + 239^{n}$ is divisible by 34 for every positive integer n.

2. $3x^2 + 5xy + 4y^2 = 12 \quad (1,1)$

$$\frac{d}{dx}(3x^2 + 5xy + 4y^2) = \frac{d}{dx}(12)$$

$$6x + 10xy\frac{dy}{dx} + 5y^2 + 8y\frac{dy}{dx} = 0$$

At $(1,1)$ $\quad 6 + 10\frac{dy}{dx} + 5 + 8\frac{dy}{dx} = 0$

$$18\frac{dy}{dx} = -11$$

$$\frac{dy}{dx} = \frac{-11}{18}$$

$$\frac{d}{dx}\left(6x + 10xy\frac{dy}{dx} + 5y^2 + 8y\frac{dy}{dx}\right) = 0$$

$$6 + 10y\frac{dy}{dx} + 10x\left(\frac{dy}{dx}\right)^2 + 10xy\frac{d^2y}{dx^2} + 10\frac{dy}{dx}$$

$$+ 8y\frac{d^2y}{dx^2} + 8\left(\frac{dy}{dx}\right)^2 = 0.$$

At $(1,1)$, $\frac{dy}{dx} = \frac{-11}{18}$:

$$6 - \frac{55}{9} + 10\left(\frac{-11}{18}\right)^2 + 10\frac{d^2y}{dx^2} - \frac{55}{9} + 8\frac{d^2y}{dx^2} + 8\left(\frac{-11}{18}\right)^2 = 0$$

$$18\frac{d^2y}{dx^2} = \frac{110}{9} - 6 - 18\left(\frac{-11}{18}\right)^2$$

$$= \frac{-1}{2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-1}{36}$$

3. $z + z^3 + z^5 + \cdots + z^{2N+1}$, $z = e^{i\theta}$

$$= \frac{z(1 - z^{2N+2})}{1 - z^2}$$

$$\sum_{r=0}^{N} e^{(2r+1)i\theta} = \frac{e^{i\theta}(1 - e^{(2N+2)i\theta})}{1 - e^{2i\theta}}$$

$$\sum_{r=0}^{N} \cos(2r+1)\theta + i\sin(2r+1)\theta$$

$$= \frac{e^{(N+1)i\theta}\left(e^{-(N+1)i\theta} - e^{(N+1)i\theta}\right)}{e^{-i\theta} - e^{i\theta}}$$

$$\sum_{r=0}^{N} \cos(2r+1)\theta + i\sum_{r=0}^{N} \sin(2r+1)\theta$$

$$= \frac{\dfrac{e^{(N+1)i\theta}\left(e^{(N+1)i\theta} - e^{-(N+1)i\theta}\right)}{2i}}{\dfrac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$= \left(\cos(N+1)\theta + i\sin(N+1)\theta\right)\frac{\sin(N+1)\theta}{\sin\theta}$$

$$= \frac{\cos(N+1)\theta\sin(N+1)\theta}{\sin\theta} + \frac{i\sin^2(N+1)\theta}{\sin\theta}$$

$$\therefore \sum_{r=0}^{N} \sin(2r+1)\theta = \frac{\sin^2(N+1)\theta}{\sin\theta}$$

4. $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ bcd & acd & abd & abc \end{pmatrix}$, $a > b > c > d > 0$

$$\xrightarrow[-bcdr_1 + r_3]{-ar_1 + r_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & acd-bcd & abd-bcd & abc-bcd \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & (a-b)cd & (a-c)bd & (a-d)bc \end{pmatrix}$$

$$\xrightarrow{-cdr_2 + r_3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & 0 & (a-c)bd-(c-a)cd & (a-d)bc-(d-a)cd \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & 0 & (a-c)(b+c)d & (a-d)(b+d)c \end{pmatrix}$$

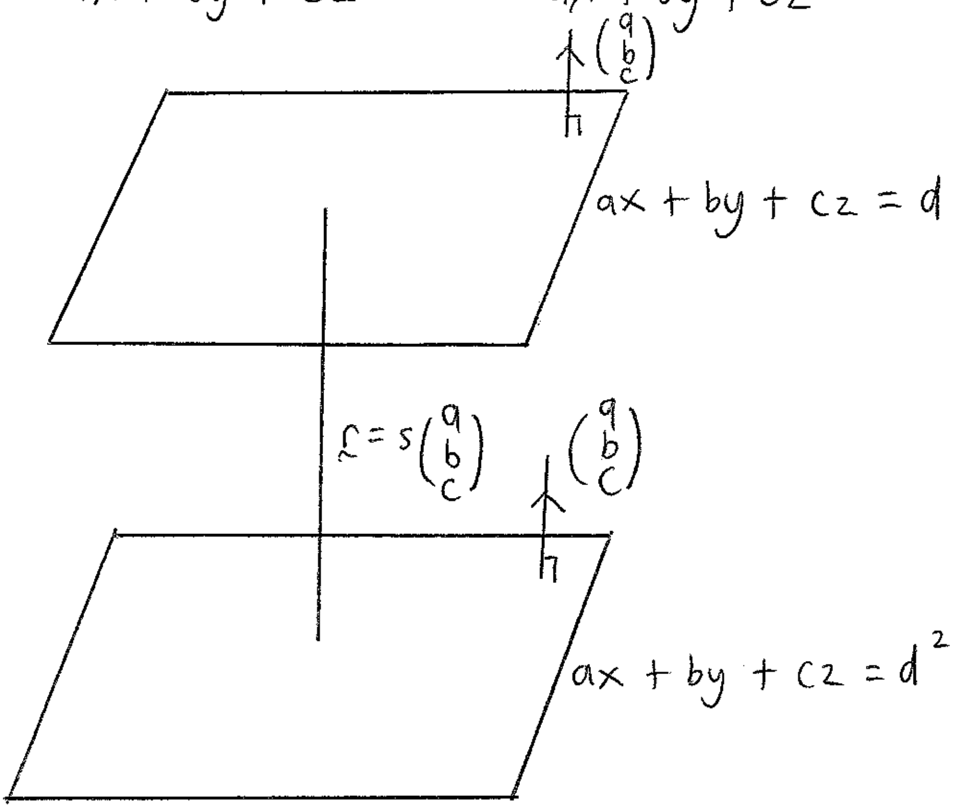
$\therefore$ rank $(A) = 3$

A basis for the range space of A is

$$\left\{ \begin{pmatrix} 1 \\ a \\ bcd \end{pmatrix}, \begin{pmatrix} 1 \\ b \\ acd \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ abd \end{pmatrix} \right\}$$

or $\left\{ \begin{pmatrix} 1 \\ a \\ bcd \end{pmatrix}, \begin{pmatrix} 1 \\ b \\ acd \end{pmatrix}, \begin{pmatrix} 1 \\ d \\ abc \end{pmatrix} \right\}$

5. $ax + by + cz = d \qquad ax + by + cz = d^2$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d^2$$

Since $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is normal to both planes,

the line passing through the origin and perpendicular to both planes has equation

$$\underset{\sim}{r} = s\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

when the line meets the plane $ax + by + cz = d$,

$$a(as) + b(bs) + c(cs) = d$$

$$(a^2 + b^2 + c^2)s = d$$

$$s = \frac{d}{a^2 + b^2 + c^2}$$

$\therefore$ The line meets the plane $ax + by + cz = d$ at the point $\left(\frac{ad}{a^2+b^2+c^2}, \frac{bd}{a^2+b^2+c^2}, \frac{cd}{a^2+b^2+c^2}\right)$

when the line meets the plane $ax + by + cz = d^2$,

$$a(as) + b(bs) + c(cs) = d^2$$

$$(a^2+b^2+c^2)s = d^2$$

$$s = \frac{d^2}{a^2+b^2+c^2}$$

$\therefore$ The line meets the plane $ax + by + cz = d^2$ at the point $\left(\frac{ad^2}{a^2+b^2+c^2}, \frac{bd^2}{a^2+b^2+c^2}, \frac{cd^2}{a^2+b^2+c^2}\right)$

$\therefore$ The perpendicular distance between the planes $ax + by + cz = d$ and $ax + by + cz = d^2$ is

$$\sqrt{\left(\frac{ad^2-ad}{a^2+b^2+c^2}\right)^2 + \left(\frac{bd^2-bd}{a^2+b^2+c^2}\right)^2 + \left(\frac{cd^2-cd}{a^2+b^2+c^2}\right)^2}$$

$$= \frac{\sqrt{a^2(d^2-d)^2 + b^2(d^2-d)^2 + c^2(d^2-d)^2}}{a^2+b^2+c^2}$$

$$= \frac{\sqrt{(a^2+b^2+c^2)(d^2-d)^2}}{a^2+b^2+c^2}$$

$$= \frac{|d^2-d|}{\sqrt{a^2+b^2+c^2}}$$

6. $5x^4 + 8x^3 + 2 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$\alpha + \beta + \gamma + \delta = \frac{-8}{5}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 0$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 0$

$\alpha\beta\gamma\delta = \frac{2}{5}$

$\alpha^3, \beta^3, \gamma^3, \delta^3$

Let $u = \alpha^3$

$\alpha = u^{\frac{1}{3}}$

α is a root

$\therefore 5\alpha^4 + 8\alpha^3 + 2 = 0$

$5(u^{\frac{1}{3}})^4 + 8(u^{\frac{1}{3}})^3 + 2 = 0$

$5u^{\frac{4}{3}} + 8u + 2 = 0$

$5u^{\frac{4}{3}} = -8u - 2$

$(5u^{\frac{4}{3}})^3 = (-8u - 2)^3$

$125u^4 = -512u^3 - 384u^2 - 96u - 8$

$125u^4 + 512u^3 + 384u^2 + 96u + 8 = 0$

$\therefore$ The equation having roots $\alpha^3, \beta^3, \gamma^3$ and δ^3 is $125u^4 + 512u^3 + 384u^2 + 96u + 8 = 0$.

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = -\frac{512}{125}$$

$$\alpha^3\beta^3 + \alpha^3\gamma^3 + \alpha^3\delta^3 + \beta^3\gamma^3 + \beta^3\delta^3 + \gamma^3\delta^3 = \frac{384}{125}$$

$$\alpha^3\beta^3\gamma^3 + \alpha^3\beta^3\delta^3 + \alpha^3\gamma^3\delta^3 + \beta^3\gamma^3\delta^3 = -\frac{96}{125}$$

$$\alpha^3\beta^3\gamma^3\delta^3 = \frac{8}{125}$$

$$\alpha^6 + \beta^6 + \gamma^6 + \delta^6$$

$$= (\alpha^3 + \beta^3 + \gamma^3 + \delta^3)$$

$$-2(\alpha^3\beta^3 + \alpha^3\gamma^3 + \alpha^3\delta^3 + \beta^3\gamma^3 + \beta^3\delta^3 + \gamma^3\delta^3)$$

$$= \left(\frac{-512}{125}\right)^2 - 2\left(\frac{384}{125}\right)$$

$$= \frac{166144}{15625}$$

7. $$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 90y = 4\cos 2x + 7\sin 2x$$

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 90y = 0$$

$$m^2 + 6m + 90 = 0$$

$$(m+3)^2 + 81 = 0$$

$$(m+3)^2 = -81$$

$$m + 3 = \pm 9i$$

$$m = -3 \pm 9i$$

$\therefore$ The complementary function, y_c, is

$$y_c = e^{-3x}(A\cos 9x + B\sin 9x)$$

The particular integral, y_p, is given by

$$y_p = C\cos 2x + D\sin 2x$$

$$\frac{dy_p}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\frac{d^2y_p}{dx^2} = -4C\cos 2x - 4D\sin 2x$$

$$\begin{aligned}\frac{d^2y_p}{dx^2} + 6\frac{dy_p}{dx} + 90y_p &= -4C\cos 2x - 4D\sin 2x \\ &\quad + 6(-2C\sin 2x + 2D\cos 2x) \\ &\quad + 90(C\cos 2x + D\sin 2x) \\ &= (86C + 12D)\cos 2x \\ &\quad + (-12C + 86D)\sin 2x\end{aligned}$$

$$= 4\cos 2x + 7\sin 2x$$

$$86C + 12D = 4 \qquad -12C + 86D = 7$$

$$43C + 6D = 2$$

$$D = \frac{-43C + 2}{6}$$

$$-12C + 86\left(\frac{-43C+2}{6}\right) = 7$$

$$-72C - 3698C + 172 = 42$$

$$3770C = 130$$

$$C = \frac{1}{29}$$

$$D = \frac{5}{58}$$

$$y_p = \frac{\cos 2x}{29} + \frac{5\sin 2x}{58}$$

$$y = y_c + y_p$$

$$= e^{-3x}(A\cos 9x + B\sin 9x)$$

$$+ \frac{\cos 2x}{29} + \frac{5\sin 2x}{58}$$

The general solution of the equation

$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 90y = 4\cos 2x + 7\sin 2x$ is

$$y = e^{-3x}(A\cos 9x + B\sin 9x) + \frac{\cos 2x}{29} + \frac{5\sin 2x}{58}$$

8. $(5n+6)(8n+3) + (5n+7)(8n+2) + (5n+8)(8n+1) + \dots$

$+ (9n+8)(4n+1)$

$$= \sum_{r=5n+6}^{9n+8} r(13n+9-r) = \sum_{r=5n+6}^{9n+8} r(an+b+cr)$$

$$= \sum_{r=1}^{9n+8} r(13n+9-r) - \sum_{r=1}^{5n+5} r(13n+9-r)$$

$$= \sum_{r=1}^{9n+8} (13n+9)r - r^2 - \sum_{r=1}^{5n+5} (13n+9)r - r^2$$

$$= (13n+9)\sum_{r=1}^{9n+8} r - \sum_{r=1}^{9n+8} r^2$$

$$- (13n+9)\sum_{r=1}^{5n+5} r + \sum_{r=1}^{5n+5} r^2$$

$$= \frac{(13n+9)(9n+8)(9n+9)}{2} - \frac{(9n+8)(9n+9)(18n+17)}{6}$$

$$- \frac{(13n+9)(5n+5)(5n+6)}{2} + \frac{(5n+5)(5n+6)(10n+11)}{6}$$

$$= \frac{(13n+9)(n+1)}{2}(9(9n+8) - 5(5n+6))$$

$$+ \frac{(n+1)}{6}(5(5n+6)(10n+11) - 9(9n+8)(18n+17))$$

$$= \frac{(13n+9)(n+1)}{2}(81n + 72 - 25n - 30)$$

$$+ \frac{(n+1)}{6}\left(5(50n^2 + 115n + 66) - 9(162n^2 + 297n + 136)\right)$$

$$= \frac{(13n+9)(n+1)(56n+42)}{2}$$

$$+ \frac{(n+1)}{6}(-1208n^2 - 2098n - 894)$$

$$= (13n+9)(n+1)(28n+21)$$

$$+ (n+1)\left(-\frac{604n^2}{3} - \frac{1049n}{3} - 149\right)$$

$$= \frac{(n+1)}{3}\left[(13n+9)3(28n+21) - (604n^2 + 1049n + 447)\right]$$

$$= \frac{(n+1)}{3}(1092n^2 + 1575n + 567 - 604n^2 - 1049n - 447)$$

$$= \frac{(n+1)}{3}(488n^2 + 526n + 120)$$

$$= \frac{2(n+1)(61n+20)(4n+3)}{3}$$

9. $A = \begin{pmatrix} 5 & 2 & -4 \\ 7 & -8 & -7 \\ -4 & 12 & 5 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 5 & 2 & -4 \\ 7 & -8 & -7 \\ -4 & 12 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5-\lambda & 2 & -4 \\ 7 & -8-\lambda & -7 \\ -4 & 12 & 5-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (5-\lambda) \begin{vmatrix} -8-\lambda & -7 \\ 12 & 5-\lambda \end{vmatrix} - 2 \begin{vmatrix} 7 & -7 \\ -4 & 5-\lambda \end{vmatrix}$$

$$-4 \begin{vmatrix} 7 & -8-\lambda \\ -4 & 12 \end{vmatrix}$$

$$= (5-\lambda)\left[(\lambda+8)(\lambda-5) + 84\right]$$

$$-2(7(5-\lambda) - 28) - 4(84 + 4(-8-\lambda))$$

$$= (5-\lambda)(\lambda^2 + 3\lambda - 40 + 84)$$

$$-2(35 - 7\lambda - 28) - 4(84 - 32 - 4\lambda)$$

$$= (5-\lambda)(\lambda^2 + 3\lambda + 44)$$

$$-2(-7\lambda + 7) - 4(52 - 4\lambda)$$

$$= 5\lambda^2 + 15\lambda + 220 - \lambda^3 - 3\lambda^2 - 44\lambda$$

$$+ 14\lambda - 14 - 208 + 16\lambda$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2$$

$$= -(\lambda^3 - 2\lambda^2 - \lambda + 2)$$

$$= -\left[\lambda^2(\lambda-2) - (\lambda-2)\right]$$

$$= (\lambda-2)(1-\lambda^2)$$

$$= (\lambda - 2)(1 - \lambda)(1 + \lambda)$$

If $|A - \lambda I| = 0$,

$$(\lambda - 2)(1 - \lambda)(1 + \lambda) = 0$$

$$\lambda = 1, 2, -1$$

$\lambda = -1$:

$$\begin{pmatrix} 6 & 2 & -4 \\ 7 & -7 & -7 \\ -4 & 12 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 6 & 2 & -4 & 0 \\ 7 & -7 & -7 & 0 \\ -4 & 12 & 6 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2},\ \frac{r_2}{7},\ \frac{r_3}{2}} \left(\begin{array}{ccc|c} 3 & 1 & -2 & 0 \\ 1 & -1 & -1 & 0 \\ -2 & 6 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & 1 & -2 & 0 \\ -2 & 6 & 3 & 0 \end{array}\right)$$

$$\xrightarrow[2r_1 + r_3]{-3r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 4s$, $s \in R$

$y = -s$

$x + s - 4s = 0$

$x = 3s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3s \\ -s \\ 4s \end{pmatrix}$$

$$= s\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$\lambda = 1$: $\begin{pmatrix} 4 & 2 & -4 \\ 7 & -9 & -7 \\ -4 & 12 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 4 & 2 & -4 & 0 \\ 7 & -9 & -7 & 0 \\ -4 & 12 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2},\ \frac{r_3}{4}} \left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 7 & -9 & -7 & 0 \\ -1 & 3 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} -1 & 3 & 1 & 0 \\ 7 & -9 & -7 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[2r_1 + r_3]{7r_1 + r_2} \left(\begin{array}{ccc|c} -1 & 3 & 1 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s$, $s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 2:$ $\begin{pmatrix} 3 & 2 & -4 \\ 7 & -10 & -7 \\ -4 & 12 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & 2 & -4 & 0 \\ 7 & -10 & -7 & 0 \\ -4 & 12 & 3 & 0 \end{array}\right)$$

$$\xrightarrow[3 \times r_3]{3 \times r_2} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 0 \\ 21 & -30 & -21 & 0 \\ -12 & 36 & 9 & 0 \end{array}\right)$$

$$\xrightarrow[4r_1 + r_3]{-7r_1 + r_2} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 0 \\ 0 & -44 & 7 & 0 \\ 0 & 44 & -7 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 0 \\ 0 & -44 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 44s,\ s \in \mathbb{R}$

$y = 7s$

$3x + 14s - 176s = 0$

$3x = 162s$

$x = 54s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 54s \\ 7s \\ 44s \end{pmatrix}$$

$$= s\begin{pmatrix} 54 \\ 7 \\ 44 \end{pmatrix}$$

The eigenvalues of A are $-1, 1, 2$ with corresponding eigenvectors $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 54 \\ 7 \\ 44 \end{pmatrix}$.

$$A^2 \underset{\sim}{x} = A(A\underset{\sim}{x}) = A(\lambda \underset{\sim}{x}) = \lambda(A\underset{\sim}{x}) = \lambda(\lambda \underset{\sim}{x}) = \lambda^2 \underset{\sim}{x}$$

$$A^3 \underset{\sim}{x} = A(A^2\underset{\sim}{x}) = A(\lambda^2 \underset{\sim}{x}) = \lambda^2(A\underset{\sim}{x}) = \lambda^2(\lambda \underset{\sim}{x}) = \lambda^3 \underset{\sim}{x}$$

$$\begin{aligned}(A + A^2 + A^3)\underset{\sim}{x} &= A\underset{\sim}{x} + A^2\underset{\sim}{x} + A^3\underset{\sim}{x} \\ &= \lambda\underset{\sim}{x} + \lambda^2\underset{\sim}{x} + \lambda^3\underset{\sim}{x} \\ &= (\lambda + \lambda^2 + \lambda^3)\underset{\sim}{x}.\end{aligned}$$

$\therefore$ The matrix $A + A^2 + A^3$ has eigenvalue $\lambda + \lambda^2 + \lambda^3$ with corresponding eigenvector $\underset{\sim}{x}$.

If $A + A^2 + A^3 = PDP^{-1}$, where P is a square matrix and D is a diagonal matrix,

$$P = \begin{pmatrix} 3 & 1 & 54 \\ -1 & 0 & 7 \\ 4 & 1 & 44 \end{pmatrix} \text{ and } D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}.$$

10. $C: y = 3 - \dfrac{4}{5(5x+6)} + \dfrac{9}{7(7x+8)}$

i) As $x \to \pm\infty$ $\quad y \to 3$

As $x \to -\dfrac{6}{5}$ $\quad y \to \pm\infty$

As $x \to \dfrac{-8}{7}$ $\quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = 3$, $x = -\dfrac{6}{5}$ and $x = \dfrac{-8}{7}$

ii) $\dfrac{dy}{dx} = \dfrac{4}{(5x+6)^2} - \dfrac{9}{(7x+8)^2}$

when $\dfrac{dy}{dx} = 0$: $\quad \dfrac{4}{(5x+6)^2} - \dfrac{9}{(7x+8)^2} = 0$

$$\frac{4}{(5x+6)^2} = \frac{9}{(7x+8)^2}$$

$$4(7x+8)^2 = 9(5x+6)^2$$

$$2(7x+8) = \pm 3(5x+6)$$

$$14x + 16 = 15x + 18, \; -15x - 18$$

$$x = -2, \quad 29x = -34$$

$$x = \frac{-34}{29}$$

$$y = \frac{209}{70}, \; \frac{-631}{70}$$

$\therefore$ The critical points of C are $\left(-2, \dfrac{209}{70}\right)$ and $\left(\dfrac{-34}{29}, \dfrac{-631}{70}\right)$

$$\frac{d^2y}{dx^2} = \frac{-40}{(5x+6)^3} + \frac{126}{(7x+8)^3}$$

when $x = -2$: $\frac{d^2y}{dx^2} = \frac{1}{24} > 0$

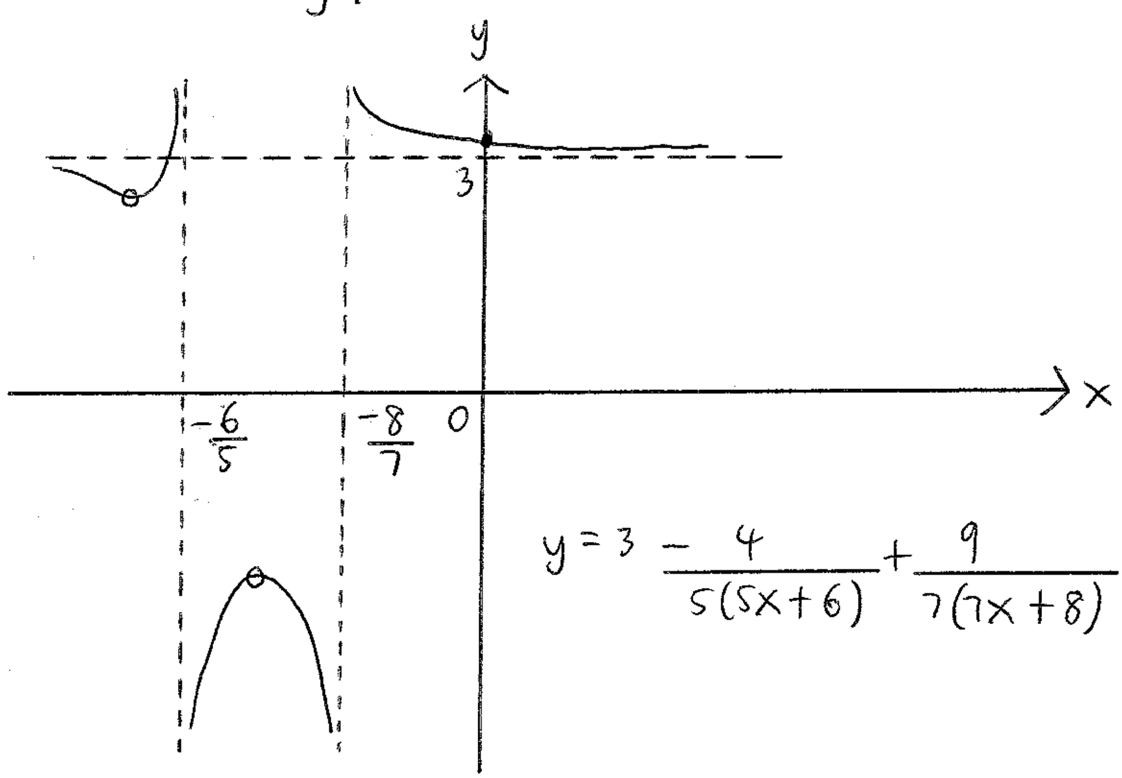
when $x = \frac{-34}{29}$: $\frac{d^2y}{dx^2} = \frac{-707281}{24} < 0$

$\therefore \left(-2, \frac{209}{70}\right)$ is a minimum point

and $\left(\frac{-34}{29}, \frac{-631}{70}\right)$ is a maximum point.

iii) when $x = 0$: $y = \frac{2543}{840}$

$y \neq 0$



o: critical point

•: intersection point

11. $I_n = \int_0^{\frac{\pi}{4}} \tan^n\theta \, d\theta$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta \tan^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta (\sec^2\theta - 1) \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta \sec^2\theta - \tan^{n-2}\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta \sec^2\theta \, d\theta - \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta \sec^2\theta \, d\theta - I_{n-2}$$

$w = \tan\theta$

$dw = \sec^2\theta \, d\theta$

$\theta = 0 \quad w = 0$

$\theta = \frac{\pi}{4} \quad w = 1$

$$= \int_0^1 w^{n-2} \, dw - I_{n-2}$$

$$= \left[\frac{w^{n-1}}{n-1} \right]_0^1 - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

$y = \tan^3 x, \quad x = 0 \quad x = \frac{\pi}{4} \quad y = 0$

The y-coordinate, $\bar{y}$, of the centroid of the region bounded by the curve $y = \tan^3 x$, the x-axis and the line $x = \frac{\pi}{4}$ is

$$\bar{y} = \frac{\int_0^{\frac{\pi}{4}} \frac{y^2}{2}\, dx}{\int_0^{\frac{\pi}{4}} y \, dx}$$

$$= \frac{\int_0^{\frac{\pi}{4}} \tan^6 x \, dx}{2\int_0^{\frac{\pi}{4}} \tan^3 x \, dx}$$

$$= \frac{I_6}{2I_3}$$

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$n = 3$: $I_3 = \frac{1}{2} - I_1$

$$I_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\, dx$$

$$= \left[-\ln|\cos x| \right]_0^{\frac{\pi}{4}}$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right) + \ln 1$$

$$= \frac{1}{2}\ln 2$$

$$I_3 = \frac{1}{2} - \frac{1}{2}\ln 2$$

$$n = 6: \; I_6 = \frac{1}{5} - I_4$$

$$I_4 = \frac{1}{3} - I_2$$

$$I_2 = \int_0^{\frac{\pi}{4}} \tan^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x - 1 \, dx$$

$$= \left[\tan x - x \right]_0^{\frac{\pi}{4}}$$

$$= 1 - \frac{\pi}{4} - 0$$

$$= 1 - \frac{\pi}{4}$$

$$I_4 = \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

$$I_6 = \frac{1}{5} - \frac{\pi}{4} + \frac{2}{3}$$

$$= \frac{13}{15} - \frac{\pi}{4}$$

$$\therefore \bar{y} = \frac{\frac{13}{15} - \frac{\pi}{4}}{1 - \ln 2}$$

$$= \frac{52 - 15\pi}{60(1 - \ln 2)}$$

12. EITHER

i) $y = ax^2 + c \quad x = 0 \quad x = \frac{\sqrt{3}}{2a}, \quad a, c > 0$

$$\frac{dy}{dx} = 2ax$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 4a^2x^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 4a^2x^2}$$

The arc length of $y = ax^2 + c$ from $x = 0$ to $x = \frac{\sqrt{3}}{2a}$, S_1, is

$$S_1 = \int_0^{\frac{\sqrt{3}}{2a}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\frac{\sqrt{3}}{2a}} \sqrt{1 + 4a^2x^2}\, dx$$

$$x = \frac{1}{2a}\tan\theta$$

$$dx = \frac{1}{2a}\sec^2\theta \, d\theta$$

$x = 0 \quad \theta = 0$

$x = b \quad \theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$$= \int_0^{\frac{\pi}{3}} \sec\theta \left(\frac{1}{2a}\sec^2\theta\right) d\theta$$

$$= \frac{1}{2a}\int_0^{\frac{\pi}{3}} \sec^3\theta \, d\theta$$

$$= \frac{1}{2a}\int_0^{\frac{\pi}{3}} \sec\theta \sec^2\theta \, d\theta$$

$u = \sec\theta$ $\qquad$ $dv = \sec^2\theta \, d\theta$

$du = \sec\theta \tan\theta \, d\theta$ $\qquad$ $v = \tan\theta$

$$= \frac{1}{2a}\left(\left[\sec\theta\tan\theta\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \sec\theta \tan^2\theta \, d\theta\right)$$

$$= \frac{1}{2a}\left(2\sqrt{3} - 0 - \int_0^{\frac{\pi}{3}} \sec\theta(\sec^2\theta - 1) \, d\theta\right)$$

$$= \frac{\sqrt{3}}{a} - \frac{1}{2a}\int_0^{\frac{\pi}{3}} \sec^3\theta \, d\theta + \frac{1}{2a}\int_0^{\frac{\pi}{3}} \sec\theta \, d\theta$$

$$\frac{1}{2a}\int_0^{\frac{\pi}{3}} \sec^3\theta \, d\theta + \frac{1}{2a}\int_0^{\frac{\pi}{3}} \sec^3\theta \, d\theta$$

$$= \frac{\sqrt{3}}{a} + \frac{1}{2a}\int_0^{\frac{\pi}{3}} \sec\theta \, d\theta$$

$$\frac{1}{a}\int_0^{\frac{\pi}{3}} \sec^3\theta \, d\theta = \frac{\sqrt{3}}{a} + \frac{1}{2a}\int_0^{\frac{\pi}{3}} \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta$$

$$= \frac{\sqrt{3}}{a} + \frac{1}{2a}\int_0^{\frac{\pi}{3}} \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} d\theta$$

$$= \frac{\sqrt{3}}{a} + \frac{1}{2a}\left(\left[\ln|\sec\theta + \tan\theta|\right]_0^{\frac{\pi}{3}}\right)$$

$$= \frac{\sqrt{3}}{a} + \frac{1}{2a}\left(\ln(2+\sqrt{3}) - \ln 1\right)$$

$$= \frac{\sqrt{3}}{a} + \frac{1}{2a}\ln(2+\sqrt{3})$$

$\therefore$ The arc length of $y = ax^2 + c$ from $x = 0$ to $x = \frac{\sqrt{3}}{2a}$ is $\frac{\sqrt{3}}{2a} + \frac{1}{4a}\ln(2+\sqrt{3})$.

ii) The surface area of revolution of $y = ax^2 + c$ from $x = 0$ to $x = \frac{\sqrt{3}}{2a}$ about the y-axis, S_2, is

$$S_2 = \int_0^{\frac{\sqrt{3}}{2a}} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\frac{\sqrt{3}}{2a}} 2\pi x \sqrt{1 + 4a^2x^2}\, dx$$

$w = 1 + 4a^2x^2$
$dw = 8a^2 x\, dx$
$x = 0 \quad w = 1$
$x = \frac{\sqrt{3}}{2a} \quad w = 4$

$$= \int_1^4 2\pi \sqrt{w}\, \frac{dw}{8a^2}$$

$$= \int_1^4 \frac{\pi}{4a^2} \sqrt{w}\, dw$$

$$= \frac{\pi}{4a^2}\left[\frac{2w^{\frac{3}{2}}}{3}\right]_1^4$$

$$= \frac{\pi}{4a^2}\left[\frac{2}{3}(8-1)\right]$$

$$= \frac{7\pi}{6a^2}$$

The surface area of revolution of $y = ax^2 + c$ from $x = 0$ to $x = \frac{\sqrt{3}}{2a}$ about the y-axis is $\frac{7\pi}{6a^2}$.

iii) If $S_1 < S_2$

$$\frac{\sqrt{3}}{2a} + \frac{1}{4a}\ln(2+\sqrt{3}) < \frac{7\pi}{6a^2}$$

$$\left(3\sqrt{3} + \frac{3\ln(2+\sqrt{3})}{2}\right)a < 7\pi$$

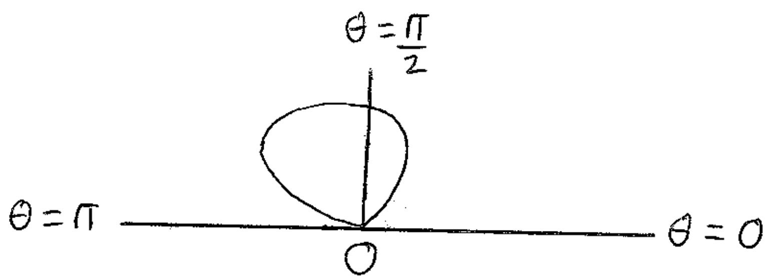
$$a < \frac{14\pi}{3(2\sqrt{3} + \ln(2+\sqrt{3}))}$$

$$< \pi, \text{ since } 14 < 6\sqrt{3} + 3\ln(2+\sqrt{3})$$

OR

i) $r = \sin\theta(1-\cos\theta)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	$\frac{1}{2}-\frac{\sqrt{3}}{4}$ 0.067	$\frac{\sqrt{2}-1}{2}$ 0.207	$\frac{\sqrt{3}}{4}$ 0.433	1	$\frac{3\sqrt{3}}{4}$ 1.30	$\frac{\sqrt{2}+1}{2}$ 1.207	$\frac{1}{2}+\frac{\sqrt{3}}{4}$ 0.933	0



ii) The area enclosed by the curve from $\theta = 0$ to $\theta = \pi$ is $\int_0^\pi \frac{r^2}{2}\,d\theta$

$$= \int_0^\pi \frac{\sin^2\theta(1-\cos\theta)^2}{2}\,d\theta$$

$$= \int_0^\pi \frac{\sin^2\theta(1-2\cos\theta+\cos^2\theta)}{2}\,d\theta$$

$$= \int_0^\pi \frac{\sin^2\theta}{2} - \sin^2\theta\cos\theta + \frac{\sin^2\theta\cos^2\theta}{2}\,d\theta$$

$$= \int_0^\pi \frac{1-\cos 2\theta}{4} - \sin^2\theta\cos\theta + \frac{\sin^2 2\theta}{8}\,d\theta$$

$$= \int_0^\pi \frac{1-\cos 2\theta}{4} - \sin^2\theta\cos\theta + \frac{1-\cos 4\theta}{16}\,d\theta$$

$$= \left[\frac{\theta}{4} - \frac{\sin 2\theta}{8} - \frac{\sin^3\theta}{3} + \frac{\theta}{16} - \frac{\sin 4\theta}{64}\right]_0^{\pi}$$

$$= \frac{\pi}{4} + \frac{\pi}{16} - 0$$

$$= \frac{5\pi}{16}$$

iii) $\frac{dr}{d\theta} = \cos\theta - \cos^2\theta + \sin^2\theta$

$$= \cos\theta - \cos^2\theta + 1 - \cos^2\theta$$

$$= -2\cos^2\theta + \cos\theta + 1$$

when $\frac{dr}{d\theta} = 0$: $-2\cos^2\theta + \cos\theta + 1 = 0$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$\cos\theta = -\frac{1}{2}, 1$$

$$\theta = 0, \frac{2\pi}{3}$$

$$r = 0, \frac{3\sqrt{3}}{4}$$

$$\frac{d^2r}{d\theta^2} = 4\cos\theta\sin\theta - \sin\theta$$

when $\theta = 0$: $\frac{d^2r}{d\theta^2} = 0$

when $\theta = \frac{2\pi}{3}$: $\frac{d^2r}{d\theta^2} = -2\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2} < 0$

$\therefore$ The maximum distance from the pole is $\frac{3\sqrt{3}}{4}$ when $\theta = \frac{2\pi}{3}$.