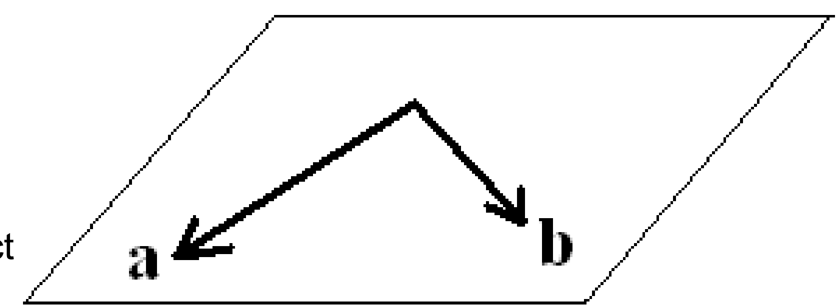
What is a common perpendicular? Try this to find out:

The diagram shows two coplanar vectors $\mathbf{a} = (1 \quad 1 \quad 1)$ and $\mathbf{b} = (-1 \quad 1 \quad -1)$. Another vector is $\mathbf{c} = (-1 \quad 0 \quad 1)$.

(a) Find $\mathbf{a} \cdot \mathbf{c}$. What is the relationship between $\mathbf{a}$ and $\mathbf{c}$.

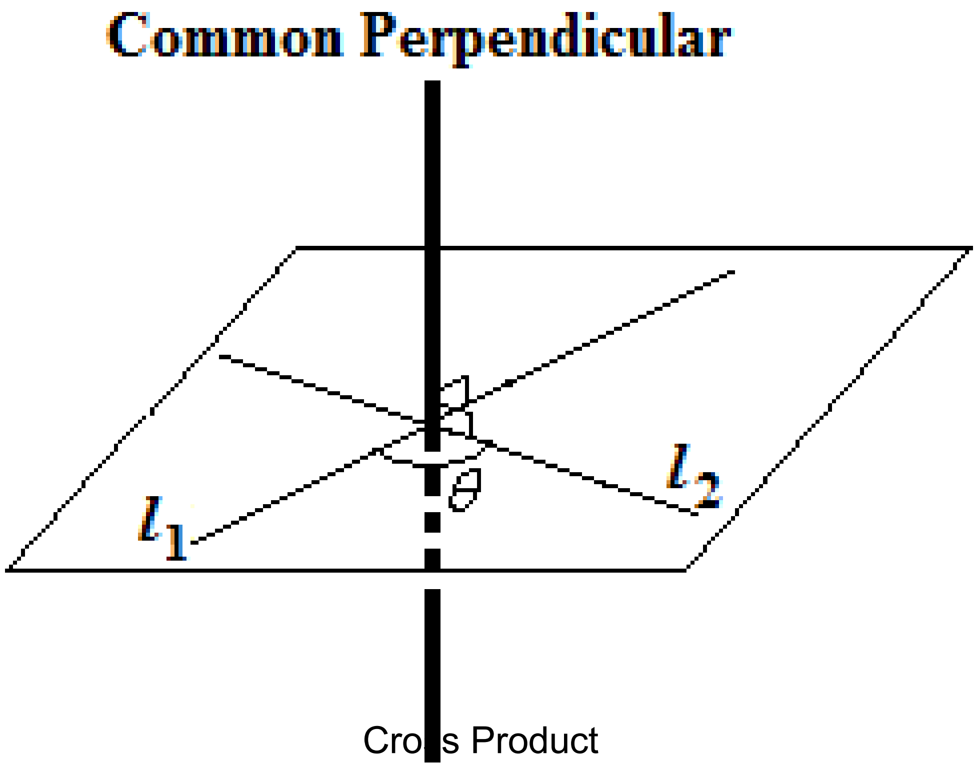
(b) Find $\mathbf{b} \cdot \mathbf{c}$. What is the relationship between $\mathbf{b}$ and $\mathbf{c}$.

(c) Draw $\mathbf{c}$ on the diagram. Explain.



Common Perpendicular

Common perpendicular of two straight lines is defined by a non - coplanar straight line that meet both lines and perpendicular to both.

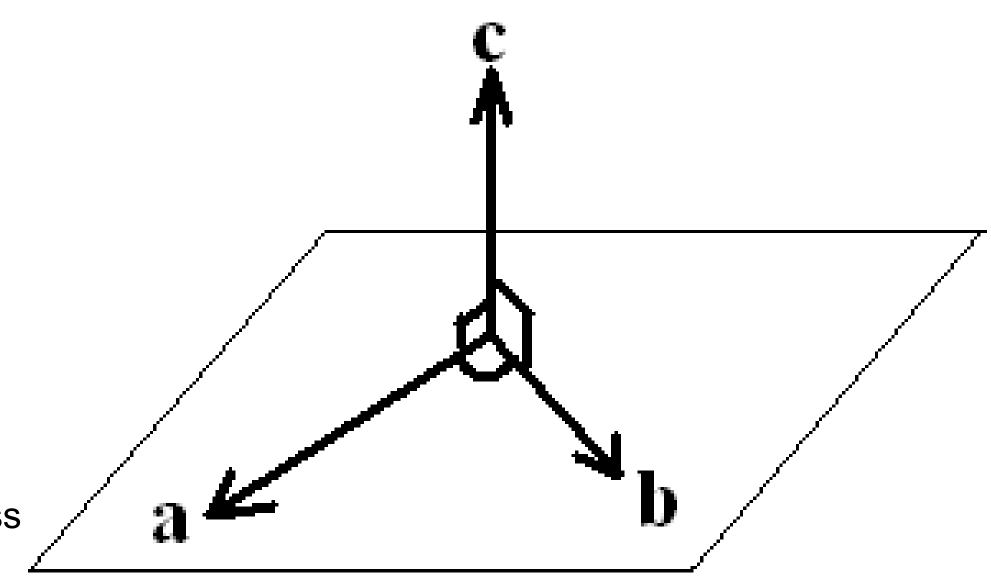


Is there ONE &ONLY ONE vector of common perpendicular between two vectors? Try this to find out :

Let $\mathbf{a} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 & 1 & -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} x & y & z \end{pmatrix}$.

(a) From the diagram, write an equation involving **a** and **c**.

(b) Write an equation involving **b** and **c**.



(c) Solve the equations (a) and (b). State any assumption you made.

(d) what can you deduce from your result?

(e) Use the similar method to prove that the vector of a common perpendicular of the vectors $\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$ is

$$\begin{pmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \end{pmatrix}.$$

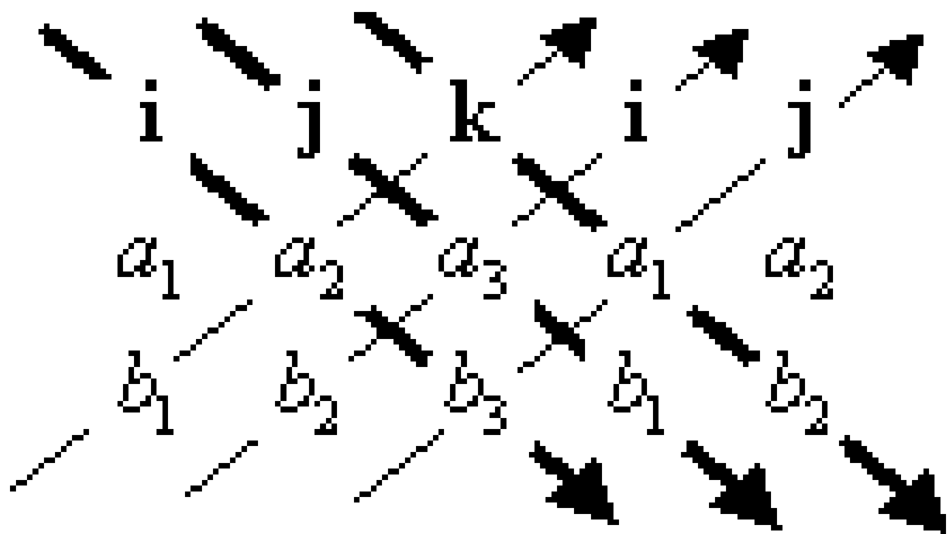
Cross Product

Definition

Cross product of two vectors $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \mathrm{b}_1 & b_2 & b_3 \end{pmatrix}$, $\mathbf{a} \times \mathbf{b}$, is defined by the VECTOR perpendicular to both **a** and **b** and is parallel to $\begin{pmatrix} a_2 b_3 - a_3 b_2 & a_3 b_1 - a_1 b_3 & a_1 b_2 - a_2 b_1 \end{pmatrix}$.

$$\text{OR } \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$



Example : Find $\mathbf{a}\times\mathbf{b}$

$$(\mathrm{i})\begin{cases}\mathbf{a}=\mathbf{i}+2\mathbf{j}\\ \mathbf{b}=2\mathbf{i}-\mathbf{j}+\mathbf{k}\end{cases}\qquad(\mathrm{ii})\begin{cases}\mathbf{a}=\begin{pmatrix}1 & 2 & 1\end{pmatrix}\\ \mathbf{b}=\begin{pmatrix}2 & -3 & 2\end{pmatrix}\end{cases}$$

$$(\mathrm{iii})\begin{cases}\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k} \\ \mathbf{b} = 3\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}\end{cases} \qquad (\mathrm{iv})\begin{cases}\mathbf{a} = \begin{pmatrix}5 & 0 & -7\end{pmatrix} \\ \mathbf{b} = \begin{pmatrix}4 & 0 & -1\end{pmatrix}\end{cases}$$

Example :

A plane contains points $A(1,2,4)$, $B(0,1,2)$ and $C(2,0,-1)$. Find the unit vector perpendicular to vectors **AB** and **AC**.

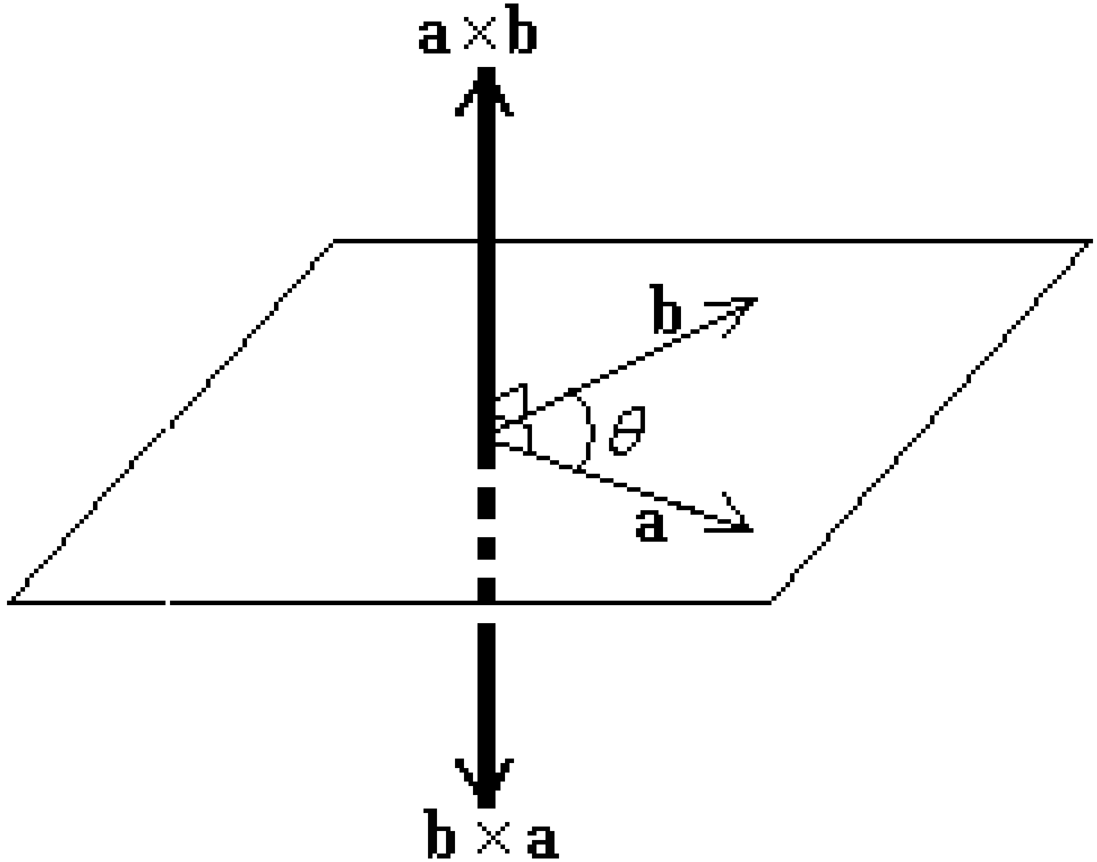
Question : Is $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{a}$?

From definition of the cross product, it is know that

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}a_2b_3-a_3b_2 & a_3b_1-a_1b_3 & a_1b_2-a_2b_1\end{pmatrix}.$$

(a) Find $\mathbf{b}\times\mathbf{a}$.

(b) Compare $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$.

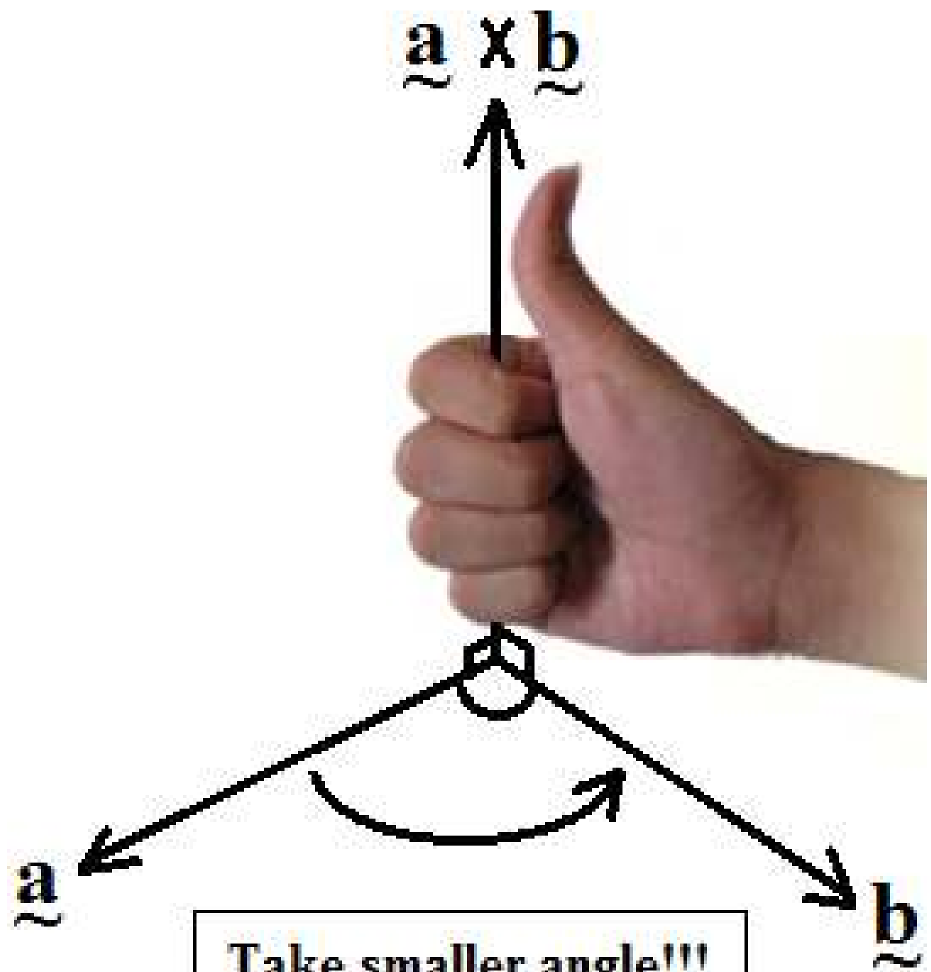


$\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have the same magnitute but opposite direction

$$\boxed{\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})}$$

Standard Direction of Cross Product

Definition :



Take smaller angle!!!

Properties of Cross Product

(i) Cross product is *distributive over addition*, i.e.

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}.$$

(ii) Cross product is *distributive over scalar multiplication*,

i.e. $(r\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(r\mathbf{b})=r(\mathbf{a}\times\mathbf{b}).$

(iii) Cross product is *NOT associative*, i.e.

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$$

Example:

Simplify

$(a)\ \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=$ $\qquad$ $(b)\ (\mathbf{a}-\mathbf{b})\times\mathbf{a}=$

Example:

If $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, show that $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$.

Triple Products

$$(\mathrm{i})\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathrm{ii})\ \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$(\mathrm{iii})\ (\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})=[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{a}$$

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example:

Prove the following:

(a) $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

(b) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$

(c) $\mathbf{j} \times \mathbf{k} = \mathbf{i}$

(d) $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Example:

Find the unit vector perpendicular to the lines

$$\mathbf{r} = (1 + 2\lambda)\mathbf{i} + (1 + \lambda)\mathbf{j} + \mathbf{k} \text{ and}$$

$$\mathbf{r} = (1 + \mu)\mathbf{i} + (1 + 3\mu)\mathbf{k}.$$

Example:

A triangle has vertices $A(1,2,3)$, $B(4,3,2)$ and $C(2,0,1)$. Find the vector unit of the normal of the plane ABC.

Example:

Simplify

$(\mathrm{a})\ (\mathbf{a}+\mathbf{b})\times\mathbf{a}=$

$(\mathrm{b})\ (\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{b})=$

Example:

A plane contains three points A, B and C with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Show that the common perpendicular of the plane is $\lambda(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})$.