1

$$\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}$$

$$= \frac{(n+3) - n}{n(n+1)(n+2)(n+3)}$$

$$= \frac{3}{n(n+1)(n+2)(n+3)}$$

$$\sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)(n+3)}$$

$$= \frac{1}{3}\sum_{n=1}^{N}\left[\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right]$$

$$= \frac{1}{3}\left[\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots + \frac{1}{N(N+1)(N+2)} - \frac{1}{2\cdot3\cdot4} - \frac{1}{3\cdot4\cdot5} - \cdots - \frac{1}{N(N+1)(N+2)} - \frac{1}{(N+1)(N+2)(N+3)}\right]$$

$$= \frac{1}{3}\left[\frac{1}{6} - \frac{1}{(N+1)(N+2)(N+3)}\right]$$

$$= \frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}$$

2. For $N=1$,

$$\text{LHS} = \frac{2\cdot 5}{3\cdot 4} = \frac{5}{6}$$

$$\text{RHS} = \frac{2\cdot 3}{4} - \frac{2}{3} = \frac{5}{6}$$

$\therefore$ It is true for $N=1$

Assume it is true for $N=k$,

$$\therefore \sum_{n=1}^{k} \frac{(n+1)(n+4)}{(n+2)(n+3)} = \frac{(k+1)(k+2)}{k+3} - \frac{2}{3}$$

For $N=k+1$,

$$\sum_{n=1}^{k+1} \frac{(n+1)(n+4)}{(n+2)(n+3)} = \sum_{n=1}^{k} \frac{(n+1)(n+4)}{(n+2)(n+3)} + T_{k+1}$$

$$= \frac{(k+1)(k+2)}{k+3} - \frac{2}{3} + \frac{(k+2)(k+5)}{(k+3)(k+4)}$$

$$= \frac{(k+1)(k+2)(k+4) + (k+2)(k+5)}{(k+3)(k+4)} - \frac{2}{3}$$

$$= \frac{(k+2)(k^2+6k+9)}{(k+3)(k+4)} - \frac{2}{3}$$

$$= \frac{(k+2)(k+3)(k+3)}{(k+3)(k+4)} - \frac{2}{3}$$

is true for $N=k+1$.

$\therefore$ By induction, $\displaystyle\sum_{n=1}^{N} \frac{(n+1)(n+4)}{(n+2)(n+3)} = \frac{(N+1)(N+2)}{N+3} - \frac{2}{3}$

is true.

3. $|C - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 4 \\ 1 & 2-\lambda & -1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 4\lambda^2 - 4\lambda + 16 = 0$

$\lambda = -2, 2, 4$

when $\lambda = -2$,

$$\begin{pmatrix} 2 & 1 & 4 \\ 1 & 4 & -1 \\ 2 & 1 & 4 \end{pmatrix} \underset{\sim}{x} = \underset{\sim}{e}$$

eigenvector $= \begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$

when $\lambda = 2$, eigenvector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

When $\lambda = 4$, eigenvector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

4. $I_n = \int \sec^n x \, dx$

$= \int \sec^{n-2} x \sec^2 x \, dx$

Let $u = \sec^{n-2} x$

$du = (n-2)\sec^{n-3} x \cdot \sec x \tan x \, dx$

$dv = \sec^2 x \, dx$

$v = \tan x$

$I_n = uv - \int v \, du$

$= \sec^{n-2} x \tan x - \int \tan^2 x (n-2) \sec^{n-2} x \, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$

$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$\therefore (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$\int \sec^4 x \, dx = I_4$

$= \frac{1}{3}\left[\sec^2 x \tan x + 2 I_2\right]$

$= \frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x + C$

5. a) $S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^1 \sqrt{2^2(t+1)^2}\, dt$

$\frac{dx}{dt} = 4t^{\frac{1}{2}}, \quad \frac{dy}{dt} = 2t - 2$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (4t^{\frac{1}{2}})^2 + (2t-2)^2$

$= 16t + 4t^2 - 8t + 4$

$= 4t^2 + 8t + 4$

$= 4(t+1)^2$

$S = 2\int_0^1 t + 1 \, dt$

$= 2\left[\frac{t^2}{2} + t\right]_0^1$

$= 3$

b) $A = \int 2\pi x \, ds$

$= 2\pi \int_{t_1}^{t_2} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= 2\pi \int_0^1 \frac{8}{3}t^{\frac{3}{2}} \, 2(t+1)\, dt$

$= \frac{32\pi}{3}\int_0^1 t^{\frac{5}{2}} + t^{\frac{3}{2}}\, dt$

$= \frac{32\pi}{3}\left[\frac{2t^{\frac{7}{2}}}{7} + \frac{2t^{\frac{5}{2}}}{5}\right]_0^1$

$= \frac{256\pi}{35}$

6. $x = r\cos\theta$, $y = r\sin\theta$

substitute into

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

$$(r^2\cos^2\theta + r^2\sin^2\theta)^2 = a^2(r^2\cos^2\theta - r^2\sin^2\theta$$

$$r^4 = a^2r^2(\cos^2\theta - \sin^2\theta)$$

$$r^2 = a^2\cos 2\theta$$

tangent happened when $r=0$,

$$\therefore \cos 2\theta = 0$$

$$2\theta = 90^\circ, 270^\circ$$

$$\theta = 45^\circ, 135^\circ$$

$\therefore$ $y = x$ is tangential to the curve at the pole

$$\text{Area} = \frac{1}{2}\int r^2\, d\theta$$

$$= \frac{1}{2}\int a^2\cos 2\theta\, d\theta$$

$$= \frac{1}{2}\cdot a^2 \cdot \frac{\sin 2\theta}{2}$$

$$= \frac{a^2}{4}\sin 2\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{a^2}{2}$$

7. $(z+1)^6 = z^6$

$\left(\frac{z+1}{z}\right)^6 = 1$

Let $\frac{z+1}{z} = w$

$w^6 = 1$

$w^6 = e^{2k\pi i}$

$w = (e^{2k\pi i})^{\frac{1}{6}}$

$= e^{\frac{1}{3}k\pi i}$

$\frac{z+1}{z} = e^{\frac{1}{3}k\pi i}$

$z+1 = ze^{\frac{1}{3}k\pi i}$

$z(e^{\frac{1}{3}k\pi i} - 1) = 1$

$z = \frac{1}{e^{\frac{1}{3}k\pi i} - 1}$

where $k = 1, 2, 3, 4, 5$

when $k = 3$,

$z = \frac{1}{e^{\pi i} - 1} = \frac{1}{\cos\pi - i\sin\pi - 1}$

$z = \frac{-1}{2}$

When $k=1$,

$$z = \frac{1}{e^{\frac{1}{3}\pi i} - 1} = \frac{1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - 1}$$

$$= \frac{1}{\frac{1}{2} + \frac{\sqrt{3}i}{2} - 1} = \frac{1}{-\frac{1}{2} + \frac{\sqrt{3}i}{2}}$$

$$= \frac{1}{-\frac{1}{2} + \frac{\sqrt{3}i}{2}} \times \frac{-\frac{1}{2} - \frac{\sqrt{3}i}{2}}{-\frac{1}{2} - \frac{\sqrt{3}i}{2}}$$

$$= -\frac{1}{2} - \frac{\sqrt{3}i}{2}$$

similarly,

when $k=2$, $z = -\frac{1}{2} - \frac{\sqrt{3}i}{6}$

when $k=4$, $z = -\frac{1}{2} + \frac{\sqrt{3}i}{6}$

when $k=5$, $z = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$

8. i) Let $x = at + be^{-t}$

$x' = a - be^{-t}$

$x'' = be^{-t}$

substitute into $\frac{d^2x}{dt^2} + \frac{2\,dx}{dt} = 1 - e^{-t}$

$\therefore be^{-t} + 2(a - be^{-t}) = 1 - e^{-t}$

$\therefore 2a = 1$, $b - 2b = -1$

$a = \frac{1}{2}$ $\quad -b = -1$

$b = 1$

ii) C.F. Auxillary equation $m^2 + 2m = 0$

$m = 0, -2$

$\therefore x = A + Be^{-2t}$

G.S. $x = A + Be^{-2t} + \frac{t}{2} + e^{-t}$

When $t = 0$, $x = 0$, $\therefore$ $0 = A + B + 1$

$A + B = -1$

when $t = 0$, $x' = 0$,

$x' = -2Be^{-2t} + \frac{1}{2} - e^{-t}$

$\therefore 0 = -2B + \frac{1}{2} - 1$

$B = \frac{-1}{4}$

$\therefore A = -\frac{3}{4}$

$\therefore x = -\frac{3}{4} - \frac{1}{4}e^{-2t} + \frac{t}{2} + e^{-t}$

iii) $\frac{dx}{dt} = \frac{1}{2}e^{-2t} + \frac{1}{2} - e^{-t}$

When $t \to \infty$, $e^{-2t} \to 0$

$e^{-t} \to 0$

$\therefore \frac{dx}{dt} \to \frac{1}{2}$

9. i) long division:

$$\begin{array}{r} 5 \\ x^2+x-6 \enclose{longdiv}{5x^2 + 5x - 10} \\ \underline{5x^2 + 5x - 30} \\ 20 \end{array}$$

$$\therefore y = 5 + \frac{20}{(x-2)(x+3)}$$

Partial fractions:

$$\frac{20}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$$\therefore A = 4, \quad B = -4$$

$$\therefore y = 5 + \frac{4}{x-2} - \frac{4}{x+3}$$

ii) $\frac{dy}{dx} = 0$

$$\therefore \frac{-4}{(x-2)^2} + \frac{4}{(x+3)^2} = 0$$

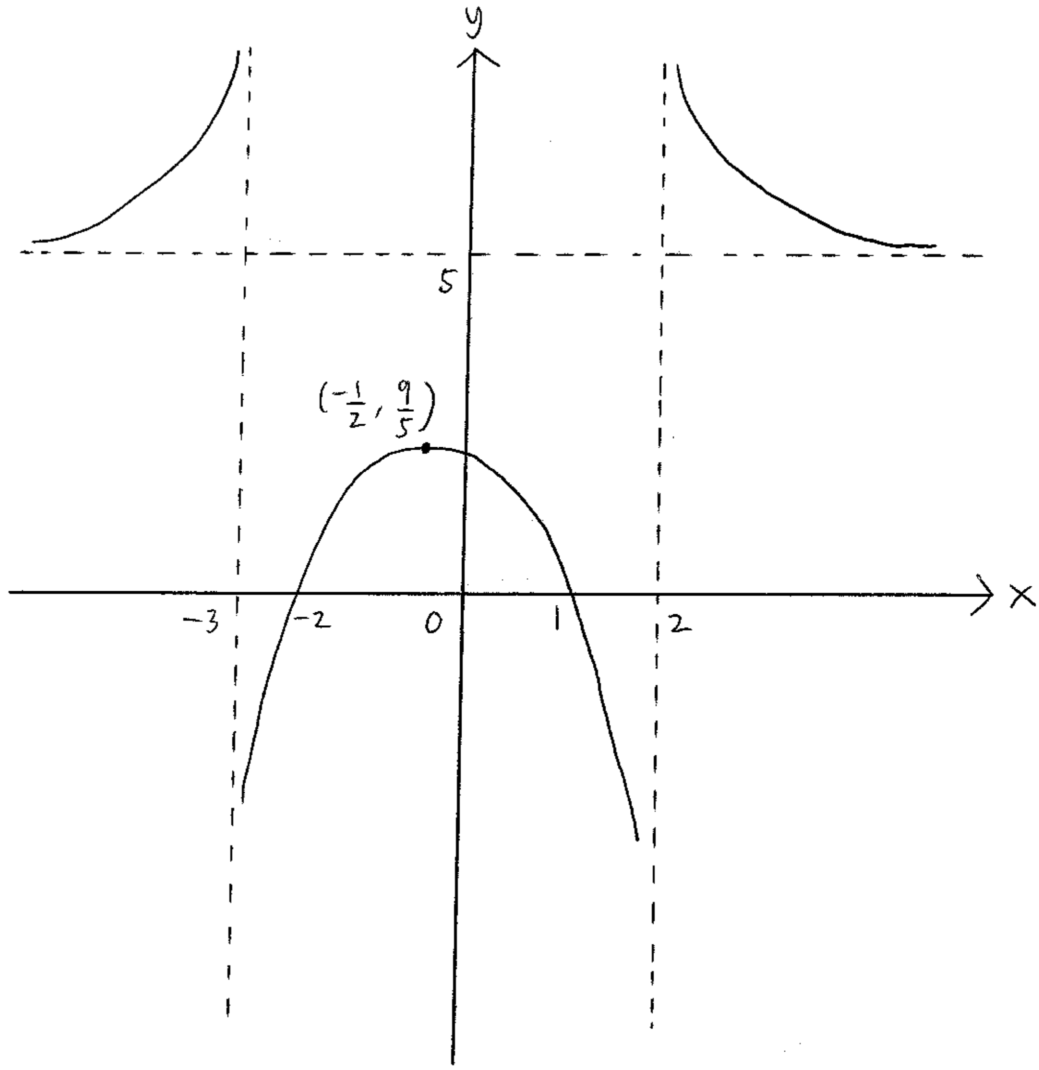
$$\vdots$$

$$x = -\frac{1}{2}$$

$$\therefore y = \frac{9}{5}$$

iii) Asymptotes: $y = 5$,
$x = 2$, $x = -3$

iv)



$\frac{9}{5} < k \leq 5$.

10. i) $x: \quad 1 + \lambda a + \mu = 3 - t$

$$\lambda a + \mu + t = 2$$

$y: \quad 3 + 2\lambda + \mu = a + 1 + t$

$$2\lambda + \mu - t = a - 2$$

$z: \quad 2 - \lambda = 1 + 2t$

$$\lambda + 2t = 1$$

ii)
$$\begin{vmatrix} a & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} \neq 0$$

$$a(2) - (4+1) + (-1) \neq 0$$

$$a \neq 3$$

iii)
$$\underset{\sim}{b} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{n} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$= -\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$= -\underset{\sim}{b}$$

$$\ell \perp p$$

when $a = 0$,

from (i) $\mu + t = 2$

$$2\lambda + 2 - 2t = -2$$

$$\lambda = t - 2$$

$$t - 2 + 2t = 1$$

$$t = 1$$

from ②, $\lambda = -1$,

from ①, $\mu = 1$

$\therefore$ position vector of the point of intersection is

$$\begin{pmatrix}1\\3\\2\end{pmatrix} - \begin{pmatrix}0\\2\\-1\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\3\end{pmatrix}$$

note: can be obtained from either equation of plane or line.

iv) $a = 3$, $|M| = 0$

$\therefore$ from ①,

$$3\lambda + \mu + t = 2 \quad - ①$$

$$2\lambda + \mu - t = 1 \quad - ②$$

$$\lambda \quad + 2t = 1 \quad - ③$$

Since ① $-$ ② $=$ ③

there are infinite set of solutions

$\therefore$ l lies in p.

11. EITHER

$$\begin{pmatrix} 1 & -1 & -2 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & -2 & -3 & 14 \\ 4 & -3 & -5 & 17 \end{pmatrix}$$

$$\begin{array}{l} r_4 - 4r_1 \\ r_3 - 3r_1 \\ r_2 - 2r_1 \end{array} \rightarrow \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank $(A) = 2$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & 0 \\ 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_2 = -3x_3 - 5x_4$$

$$x_1 = x_2 + 3x_3 - 3x_4$$

$$= -x_3 - 8x_4$$

Let $x_3 = s$, $x_4 = t$

$$\therefore \underset{\sim}{x} = \begin{pmatrix} -s - 8t \\ -3s - 5t \\ s \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ Basis for nullspace is

$$\left\{ \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$A\underset{\sim}{x} = A\underset{\sim}{e}$$

$$A(\underset{\sim}{x} - \underset{\sim}{e}) = \underset{\sim}{0}$$

$$\therefore \underset{\sim}{x} - \underset{\sim}{e} = s\begin{pmatrix}-1\\-3\\1\\0\end{pmatrix} + t\begin{pmatrix}-8\\-5\\0\\1\end{pmatrix}$$

$$\underset{\sim}{x} = \underset{\sim}{e} + s\begin{pmatrix}-1\\-3\\1\\0\end{pmatrix} + t\begin{pmatrix}-8\\-5\\0\\1\end{pmatrix}$$

$$\therefore \begin{pmatrix}p\\q\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\-1\\-1\end{pmatrix} + s\begin{pmatrix}-1\\-3\\1\\0\end{pmatrix} + t\begin{pmatrix}-8\\-5\\0\\1\end{pmatrix}$$

solve to obtain $s = 2,\ t = 2$

$p = -17$

$q = -18.$

OR

i) a) $\sum \alpha = -a$ — ①

$\sum \alpha\beta = b$ — ②

$\sum \alpha\beta\gamma = -c$ — ③

$\alpha\beta\gamma\delta = d$ — ④

from ④ and $\alpha\beta = 1$, $\gamma\delta = 1$,

$\therefore \quad d = 1 \cdot 1 = 1$

from ③, $\sum \alpha\beta\gamma = -c$

$\Rightarrow \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -e$

$\Rightarrow (1)\gamma + (1)\delta + \alpha(1) + \beta(1) = -c$

$\Rightarrow \alpha + \beta + \gamma + \delta = -c$

$\Rightarrow$ from ①, $\therefore \quad -a = -c$

$a = c$

b) Let roots as x_1 and x_2

$\therefore x_1 = \alpha + \beta$, $x_2 = \gamma + \delta$

$\therefore$ The equation is

$x^2 - (x_1 + x_2)x + x_1 x_2 = 0$

$\Rightarrow x^2 - (\alpha + \beta + \gamma + \delta)x + (\alpha + \beta)(\gamma + \delta) = 0$

$\Rightarrow x^2 - (-a)x + (\alpha + \beta)(\gamma + \delta) = 0$

and from ②, $\sum \alpha\beta = b$

$\Rightarrow \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = b$

$\Rightarrow 1 + (\alpha + \beta)\gamma + (\alpha + \beta)\delta + 1 = b$

$(\alpha + \beta)(\gamma + \delta) = b - 2$

substitute into ⑤,

$\Rightarrow x^2 + ax + b - 2 = 0$

ii) Let $y = \frac{1}{x}$ ∴ $x = \frac{1}{y}$

substitute $x = \frac{1}{y}$ into

$$x^4 + px^3 + qx^2 + px + 1 = 0$$

$$\Rightarrow \left(\frac{1}{y}\right)^4 + p\left(\frac{1}{y}\right)^3 + q\left(\frac{1}{y}\right)^2 + p\left(\frac{1}{y}\right) + 1 = 0$$

$$\Rightarrow \frac{1}{y^4} + \frac{p}{y^3} + \frac{q}{y^2} + \frac{p}{y} + 1 = 0$$

$$\Rightarrow 1 + py + qy^2 + py^3 + y^4 = 0$$

i.e. $y^4 + py^3 + qy^2 + py + 1 = 0$

Compare with

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

from part (i),

$\Rightarrow a = p,\ b = q,\ c = p,\ d = 1$

$\Rightarrow a = c$ & $d = 1$ as part i) a)

iii) $x^4 + 2x^3 - x^2 + 2x + 1 = 0$

∴ $a = 2,\ b = -1,\ c = 2,\ d = 1$

from i) b), $x^2 + ax + b - 2 = 0$

$\Rightarrow x^2 + 2x - 3 = 0$

$(x-1)(x+3) = 0$

$x = 1,\quad x = -3$

$\Rightarrow \alpha + \beta = 1$ & $\gamma + \delta = -3$

$\alpha + \frac{1}{\alpha} = 1$ $\gamma + \frac{1}{\gamma} = -3$

$\alpha^2 - \alpha + 1 = 0$ $\gamma^2 + 3\gamma + 1 = 0$

$\alpha = \frac{1 \pm \sqrt{1-4}}{2}$ $\gamma = \frac{-3 \pm \sqrt{9-4}}{2}$

$= \frac{1}{2}[1 \pm \sqrt{3}i]$ $= \frac{1}{2}[-3 \pm \sqrt{5}]$

∴ roots are $\frac{1}{2}[1 \pm \sqrt{3}i]$, $\frac{1}{2}[-3 \pm \sqrt{5}]$