1. $\left(\frac{x+y}{2}\right)^n \le x^n + y^n$, $x, y > 0$, $n \geqslant 2$

when $n=2$: $\left(\frac{x+y}{2}\right)^2 = \frac{x^2+2xy+y^2}{4}$

$$= \frac{x^2+y^2}{4} + \frac{x^2+y^2}{4} - \frac{(x^2+y^2)}{4} + \frac{2xy}{4}$$

$$= \frac{x^2+y^2}{2} - \frac{(x^2-2xy+y^2)}{4}$$

$$= \frac{x^2+y^2}{2} - \left(\frac{x-y}{2}\right)^2$$

$$\le \frac{x^2+y^2}{2}, \text{ since } \left(\frac{x-y}{2}\right)^2 \geqslant 0$$

suppose $\left(\frac{x+y}{2}\right)^k \le \frac{x^k+y^k}{2}$

when $n=k+1$: $\left(\frac{x+y}{2}\right)^k\left(\frac{x+y}{2}\right)$

$$\le \left(\frac{x^k+y^k}{2}\right)\left(\frac{x+y}{2}\right)$$

$$= \frac{x^{k+1}+y^{k+1}+x^ky+xy^k}{4}$$

$$= \frac{x^{k+1}+y^{k+1}}{4} + \frac{x^{k+1}+y^{k+1}}{4}$$

$$-\left(\frac{x^{k+1}+y^{k+1}}{4}\right) + \frac{x^ky+xy^k}{4}$$

$$= \frac{x^{k+1} + y^{k+1}}{2} - \frac{1}{4}(x^{k+1} + y^{k+1} - x^k y - x y^k)$$

$$= \frac{x^{k+1} + y^{k+1}}{2} - \frac{1}{4}\left[x^k(x-y) + y^k(y-x)\right]$$

$$= \frac{x^{k+1} + y^{k+1}}{2} - \frac{1}{4}(x^k - y^k)(x-y)$$

$$\leq \frac{x^{k+1} + y^{k+1}}{2} \quad \text{since} \quad \frac{(x^k - y^k)(x-y)}{4} \geqslant 0$$

2. $x^3 - 2x - 3 = 0$

Roots : α, β, γ

$\alpha + \beta + \gamma = 0$ $\quad$ $\alpha\beta + \alpha\gamma + \beta\gamma = -2$ $\quad$ $\alpha\beta\gamma = 3$

$\alpha + \beta = -\gamma$

New Roots : $\dfrac{2}{1-\alpha-\beta}, \dfrac{2}{1-\alpha-\gamma}, \dfrac{2}{1-\beta-\gamma}$

Let $y = \dfrac{2}{1-\alpha-\beta}$

$= \dfrac{2}{1-(-\gamma)}$

$1+\gamma = \dfrac{2}{y}$

$\gamma = \dfrac{2}{y} - 1$

since γ is one of the roots for $x^3 - 2x - 3 = 0$

$\gamma^3 - 2\gamma - 3 = 0$

$\left(\frac{2}{y} - 1\right)^3 - 2\left(\frac{2}{y} - 1\right) - 3 = 0$

$\frac{8}{y^3} - 3\left(\frac{4}{y^2}\right) + 3\left(\frac{2}{y}\right) - 1 - \frac{4}{y} + 2 - 3 = 0$

$\frac{8}{y^3} - \frac{12}{y^2} + \frac{2}{y} - 2 = 0$

$4 - 6y + y^2 - y^3 = 0$

$y^3 - y^2 + 6y - 4 = 0$

$\therefore y^3 - y^2 + 6y - 4 = 0$ is the cubic equation having roots $\dfrac{2}{1-\alpha-\beta}, \dfrac{2}{1-\alpha-\gamma}, \dfrac{2}{1-\beta-\gamma}$

in the quad. equation
having roots

3. $C: 3y^2 = 5x^3 , \quad 0 \le x \le 3$

$$6y \frac{dy}{dx} = 15x^2$$

$$\frac{dy}{dx} = \frac{5x^2}{2y}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{25x^4}{4y^2}$$

$$= 1 + \frac{25x^4}{4\left(\frac{5x^3}{3}\right)}$$

$$= 1 + \frac{15x}{4}$$

$$= \frac{4 + 15x}{4}$$

$$\text{Arc length}, s = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^3 \sqrt{\frac{15x + 4}{4}}\, dx$$

$$= \frac{1}{2} \int_0^3 \sqrt{15x + 4}\, dx$$

$$= \frac{1}{2} \left[\frac{(15x + 4)^{\frac{3}{2}}}{\frac{3}{2}(15)} \right]_0^3$$

$$= \frac{1}{45} \left[(15(3) + 4)^{\frac{3}{2}} - 4^{\frac{3}{2}} \right]$$

$$= \frac{1}{45} (7^3 - 2^3)$$

$$= \frac{67}{9} \text{ units.}$$

4. $$\frac{4}{(2r+1)(2r+3)(2r+5)} = \frac{A}{(2r+1)(2r+3)} + \frac{B}{(2r+3)(2r+5)}$$

$$4 = A(2r+5) + B(2r+1)$$

$$r = -\frac{1}{2} : 4 = 4A \Rightarrow A = 1$$

$$r = -\frac{5}{2} : 4 = -4B \Rightarrow B = -1$$

$$\therefore \frac{4}{(2r+1)(2r+3)(2r+5)} = \frac{1}{(2r+1)(2r+3)} - \frac{1}{(2r+3)(2r+5)}$$

$$\frac{1}{5\cdot7\cdot9} + \frac{1}{7\cdot9\cdot11} + \frac{1}{9\cdot11\cdot13} + \cdots$$

$$= \lim_{n\to\infty} \sum_{r=2}^{n} \frac{1}{(2r+1)(2r+3)(2r+5)}$$

$$= \frac{1}{4} \lim_{n\to\infty} \sum_{r=2}^{n} \frac{4}{(2r+1)(2r+3)(2r+5)}$$

$$= \frac{1}{4} \lim_{n\to\infty} \sum_{r=2}^{n} \frac{1}{(2r+1)(2r+3)} - \frac{1}{(2r+3)(2r+5)}$$

$$= \frac{1}{4} \lim_{n\to\infty} \sum_{r=2}^{n} f(r) - f(r+1), \quad f(r) = \frac{1}{(2r+1)(2r+3)}$$

$$= \frac{1}{4} \lim_{n\to\infty} f(2) - f(n+1) \qquad f(r+1) = \frac{1}{(2r+3)(2r+5)}$$

$$= \frac{1}{4} \lim_{n\to\infty} \frac{1}{5\cdot7} - \frac{1}{(2n+3)(2n+5)}$$

$$= \frac{1}{4}\left(\frac{1}{35}\right)$$

$$= \frac{1}{140}$$

5. $x^3 - axy + 3ay^2 = 3a^3, \quad a>0$

$$3x^2 - a\left(x\frac{dy}{dx} + y\right) + 6ay\frac{dy}{dx} = 0$$

$$6x - a\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx}\right) + 6a\left(y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)\frac{dy}{dx}\right) = 0$$

$$6x - a\left(x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right) + 6a\left(y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) = 0$$

i) At $P(a, a)$:

$$3a^2 - a\left(a\frac{dy}{dx} + a\right) + 6a(a)\frac{dy}{dx} = 0$$

$$3a^2 + 5a^2\frac{dy}{dx} - a^2 = 0$$

$$\frac{dy}{dx} = \frac{-2a^2}{5a^2} = -\frac{2}{5}$$

ii) At $P(a,a)$ and $\frac{dy}{dx} = -\frac{2}{5}$:

$$6a - a\left(a\frac{d^2y}{dx^2} + 2\left(-\frac{2}{5}\right)\right) + 6a\left(a\frac{d^2y}{dx^2} + \left(-\frac{2}{5}\right)^2\right) = 0$$

$$6a - a^2\frac{d^2y}{dx^2} + \frac{4a}{5} + 6a^2\frac{d^2y}{dx^2} + \frac{24a}{25} = 0$$

$$5a^2\frac{d^2y}{dx^2} = \frac{-194a}{25}$$

$$\frac{d^2y}{dx^2} = \frac{-194}{125a}$$

6. $y = 1 + \frac{1}{1+x} + \frac{2}{(1+x)^2}$

i) $(x+1)^2 y = (x+1)^2 + (x+1) + 2$

$(x^2 + 2x + 1)y = x^2 + 2x + 1 + x + 3$

$(y-1)x^2 + (2y-3)x + y - 4 = 0$

$$\begin{aligned} B^2 - 4AC &= (2y-3)^2 - 4(y-1)(y-4) \\ &= 4y^2 - 12y + 9 - 4(y^2 - 5y + 4) \\ &= 4y^2 - 12y + 9 - 4y^2 + 20y - 16 \\ &= 8y - 7 \end{aligned}$$

For $B^2 - 4AC \geq 0$,

$8y - 7 \geq 0$

$y \geq \frac{7}{8}$

$\therefore$ The range of y is $\{y : y \geq \frac{7}{8}\}$

ii) $\frac{dy}{dx} = \frac{-1}{(x+1)^2} - \frac{4}{(x+1)^3}$

when $\frac{dy}{dx} < 0$,

$\frac{1}{(x+1)^2} + \frac{4}{(x+1)^3} > 0$

$\frac{x+1+4}{(x+1)^3} > 0$

$\frac{x+5}{(x+1)^3} > 0$

$(x+1)^3 < 0 \quad x+5 < 0$

$x < -1 \quad x < -5$

$\therefore x < -5$

$(x+1)^3 > 0 \quad x+5 > 0$

$x > -1 \quad x > -5$

$\therefore x > -1$

$\therefore$ The set of values for $\frac{dy}{dx} < 0$ is $\{x : x < -5\}$
$\cup \{x : x > -1\}$

7. $I_n = \int_0^1 (1+x^2)^{\frac{n}{2}-1} dx$

$x = \tan\theta \quad dx = \sec^2\theta \, d\theta$

$x=0 \quad \tan\theta = 0 \quad \theta = 0$

$x=1 \quad \tan\theta = 1 \quad \theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} (1+\tan^2\theta)^{\frac{n-2}{2}} \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2\theta)^{\frac{n-2}{2}} \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^{n-2}\theta \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^n\theta \, d\theta$$

$$= \left[\sec^{n-2}\theta \int \sec^2\theta \, d\theta \right]_0^{\frac{\pi}{4}}$$

$$- \int_0^{\frac{\pi}{4}} \int \sec^2\theta \, d\theta \, \frac{d}{d\theta} \sec^{n-2}\theta \, d\theta$$

$$= \left[\sec^{n-2}\theta \tan\theta \right]_0^{\frac{\pi}{4}}$$

$$- \int_0^{\frac{\pi}{4}} (n-2)\tan\theta \sec^{n-3}\theta \sec\theta \tan\theta \, d\theta$$

$$= \left(\frac{1}{\cos\frac{\pi}{4}}\right)^{n-2} \tan\frac{\pi}{4} - \frac{1}{\cos 0} \tan 0$$

$$- (n-2) \int_0^{\frac{\pi}{4}} \tan^2\theta \sec^{n-2}\theta \, d\theta$$

$$= (\sqrt{2})^{n-2} - (n-2) \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1) \sec^{n-2}\theta \, d\theta$$

$$= (\sqrt{2})^{n-2} - (n-2) \int_0^{\frac{\pi}{4}} \sec^n\theta \, d\theta + (n-2) \int_0^{\frac{\pi}{4}} \sec^{n-2}\theta \, d\theta$$

$$= (\sqrt{2})^{n-2} - (n-2) I_n + (n-2) I_{n-2}$$

$$(1 + n-2) I_n = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$$

$$(n-1) I_n = (\sqrt{2})^{n-2} + (n-2) I_{n-2}, \quad n \geq 2$$

$$I_1 = \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

$$= \left[\ln(\sec\theta + \tan\theta) \right]_0^{\frac{\pi}{4}}$$

$$= \ln\left(\frac{1}{\cos\frac{\pi}{4}} + \tan\frac{\pi}{4} \right)$$

$$= \ln(\sqrt{2} + 1)$$

$$2 I_3 = \sqrt{2} + I_1$$

$$= \sqrt{2} + \ln(\sqrt{2} + 1)$$

$$\therefore I_3 = \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2} + 1)\right)$$

8. Curve: $x = 2t^2 \quad y = 4t \quad 0 \le t \le 1$

$$\frac{dx}{dt} = 4t \quad \frac{dy}{dt} = 4$$

i) Surface Area, $A = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^1 2\pi (4t) \sqrt{(4t)^2 + 4^2}\, dt$$

$$= 8\pi \int_0^1 t \sqrt{16(t^2+1)}\, dt$$

$$= 8\pi \int_0^1 4t \sqrt{t^2+1}\, dt$$

$$= 16\pi \int_0^1 2t \sqrt{t^2+1}\, dt$$

$$= 16\pi \left[\frac{(t^2+1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1$$

$$= \frac{32\pi}{3} \left[(1+1)^{\frac{3}{2}} - 1^{\frac{3}{2}} \right]$$

$$= \frac{32\pi}{3} (2\sqrt{2} - 1) \text{ unit}^2$$

ii) $\int_{t=0}^{1} xy^2\,dx = \int_0^1 2t^2(4t)^2\,4t\,dt$

$= \int_0^1 128t^5\,dt$

$= \left[\frac{128t^6}{6}\right]_0^1$

$= \frac{64}{3} - 0$

$= \frac{64}{3}$

$\int_{t=0}^{1} y^2\,dx = \int_0^1 (4t)^2(4t\,dt)$

$= 64\int_0^1 t^3\,dt$

$= 64\left[\frac{t^4}{4}\right]_0^1$

$= 64\left(\frac{1}{4} - 0\right)$

$= 16$

Centroid of solid, $\bar{x} = \dfrac{\int_0^1 xy^2\,dx}{\int_0^1 y^2\,dx}$

$= \dfrac{\frac{64}{3}}{16}$

$= \frac{4}{3}$

9. Π_1: $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

$$\underset{\sim}{n}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 1 + 0 - 4 = -3$$

$$x + 3y - 2z = -3 \quad — \text{(I)}$$

Π_2: $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

$$\underset{\sim}{n}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = 12 + 0 + 0 = 12$$

$$4x + z = 12 \quad — \text{(II)}$$

i) Let θ denotes the angle between Π_1 & Π_2

$$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 = |\underset{\sim}{n}_1||\underset{\sim}{n}_2| \cos\theta$$

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \sqrt{1^2 + 3^2 + (-2)^2} \sqrt{4^2 + 0^2 + 1^2} \cos\theta$$

$$4 + 0 - 2 = \sqrt{14}\sqrt{17} \cos\theta$$

$$\cos\theta = \frac{2}{\sqrt{238}}$$

ii) l is the line of intersection for Π_1 and Π_2, it is in the direction $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$. Let the direction of l denoted by $\underset{\sim}{b}$.

$$\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ -12 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$$

$\therefore \underset{\sim}{b} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

substitute $z = 0$ in ① & ②:

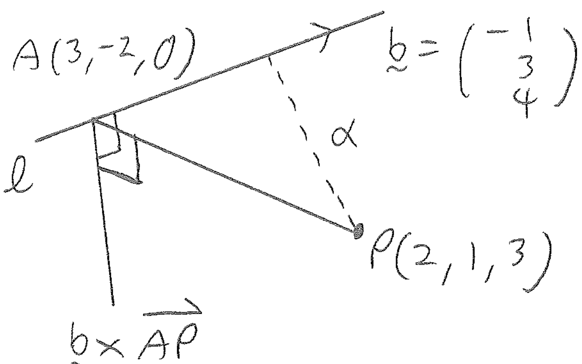
$$\left.\begin{aligned} x + 3y &= -3 \\ 4x &= 12 \end{aligned}\right\}$$

$$x = 3 \quad y = -2$$

$\therefore (3, -2, 0)$ is a point on line ℓ.

$\therefore$ Equation of ℓ is $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$.

iii)



$$\overrightarrow{AP} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$$

$$\underset{\sim}{b} \times \overrightarrow{AP} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$$

Perpendicular distance from point $P(2,1,3)$ to line ℓ, $d = \dfrac{|\underset{\sim}{b} \times \overrightarrow{AP}|}{|\underset{\sim}{b}|}$

$$= \frac{\sqrt{9+1+0}}{\sqrt{1+9+16}}$$

$$= \sqrt{\frac{5}{13}} \text{ units}$$

10. $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$ $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(-\lambda)(5-\lambda) + 8 + 4 - 4\lambda - 2(5-\lambda) + 4(1-\lambda) = 0$

$-\lambda(5 - 6\lambda + \lambda^2) + 12 - 4\lambda - 10 + 2\lambda + 4 - 4\lambda = 0$

$-\lambda^3 + 6\lambda^2 - 5\lambda + 6 - 6\lambda = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 1, 2, 3$

$\therefore$ The eigenvalues are 1, 2, 3.

when $\lambda = 1$: $A - \lambda I = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ $\therefore \underset{\sim}{e}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

when $\lambda = 2$, $A - \lambda I = \begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ $\therefore \underset{\sim}{e}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

when $\lambda = 3$: $A - \lambda I = \begin{pmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{pmatrix}$

$\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ $\therefore$ $\underset{\sim}{e}_3 = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

Let $B = A + 3I$

$$\begin{aligned} B\underset{\sim}{x} &= (A+3I)\underset{\sim}{x} \\ &= A\underset{\sim}{x} + 3I\underset{\sim}{x} \\ &= \lambda\underset{\sim}{x} + 3\underset{\sim}{x} \\ &= (\lambda+3)\underset{\sim}{x} \end{aligned}$$

If A has eigenvalue λ with corresponding eigenvector $\underset{\sim}{e}$, B has an eigenvalue $\lambda+3$ with corresponding eigenvector $\underset{\sim}{e}$.

$\therefore$ B has eigenvalues $1+3=4$, $2+3=5$, $3+3=6$ with corresponding eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

If $(A+3I)^2 = MDM^{-1}$, that is $B^2 = MDM^{-1}$,

$\therefore$ $M = \begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}^2$

$$= \begin{pmatrix} 16 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 36 \end{pmatrix}$$

11. $$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=6x$$

$$x=e^t \qquad \frac{dx}{dt}=e^t=x$$

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}$$
$$=e^{-t}\frac{dy}{dt}$$
$$=\frac{1}{x}\frac{dy}{dt}$$

$$x\frac{dy}{dx}=\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$
$$=\left[\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\right]e^{-t}$$
$$=\left(e^{-t}\frac{d^2y}{dt^2}-e^{-t}\frac{dy}{dt}\right)e^{-t}$$
$$=e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right), \quad x=e^t$$
$$=\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

$$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$$
$$\frac{d^2y}{dt^2}-\frac{dy}{dt}+4\frac{dy}{dt}+2y=6e^t$$
$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=6e^t$$

$m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

$\therefore y_n = Ae^{-t} + Be^{-2t}$

$f(t) = 6e^t$

let $y_p = ke^t$

$y_p' = ke^t$

$y_p'' = ke^t$

$ke^t + 3ke^t + 2ke^t = 6e^t$

$6ke^t = 6e^t$

$\therefore k = 1$

$y_p = e^t$

$\therefore y = y_n + y_p$

$= Ae^{-t} + Be^{-2t} + e^t$

$= \frac{A}{x} + \frac{B}{x^2} + x$

12. EITHER

$T: R^3 \to R^3$

$$M = \begin{pmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{pmatrix}, \quad a \neq b, \ b = c$$

i) R is the range space of T.

a) $$\begin{pmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{pmatrix} \xrightarrow{\frac{r_1}{a}, \frac{r_2}{b}, \frac{r_3}{c}} \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

$$\xrightarrow[-r_2+r_3]{-r_1+r_2} \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} \xrightarrow{\frac{r_2}{b-a}, \frac{r_3}{c-a}} \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{pmatrix}$$

$$\xrightarrow{-r_2+r_3} \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{pmatrix}$$ since $b \neq a$, $b = c$

$$\therefore \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Dim of R is 2

b) Basis of $R = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \right\}$

c) $$\left(\begin{array}{ccc|c} 1 & a & a^2 & 0 \\ 0 & 1 & a+b & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x + ay + a^2 z = 0$

$y + (a+b)z = 0$

Let $z = t, t \in R$

$y = -(a+b)t$

$x = -a(-(a+b)t) - a^2 t$

$= a^2 t + abt - a^2 t$

$= abt$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} ab \\ -a-b \\ 1 \end{pmatrix}$$

$\therefore$ Basis for the nullspace of $T = \left\{ \begin{pmatrix} ab \\ -a-b \\ 1 \end{pmatrix} \right\}$

ii) If $a=-1$, $b=1$ $\therefore c=1$

$M\underset{\sim}{x} = \underset{\sim}{v}$ becomes

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & -6 \\ 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 4 \end{array}\right) \xrightarrow[r_1+r_3]{r_1+r_2} \left(\begin{array}{ccc|c} -1 & 1 & -1 & -6 \\ 0 & 2 & 0 & -2 \\ 0 & 2 & 0 & -2 \end{array}\right)$$

$$\xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|c} -1 & 1 & -1 & -6 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$-x+y-z=-6$$
$$2y=-2$$
$$y=-1$$
$$z=\lambda$$
$$-x-1-\lambda=-6$$
$$x=5-\lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

is the general solution for $M\underset{\sim}{x} = \underset{\sim}{v}$

iii) If $a=b=c$

$$\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & a & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ yes dim of R changes to 1.

12. OR

$C: r = 4\sqrt{3}\cos 2\theta$

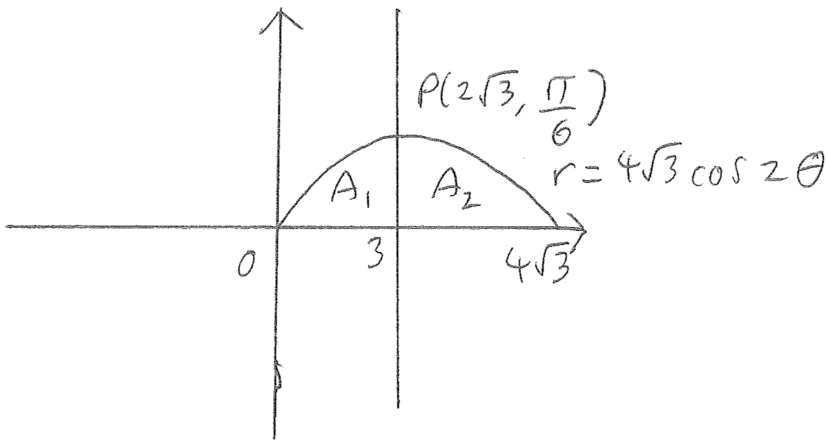
$l: r = 3\sec\theta \qquad , \; 0 \le \theta \le \frac{\pi}{4}$

$r\cos\theta = 3$

$x = 3 \qquad$ Intersect at $P(2\sqrt{3}, \frac{\pi}{6})$

i)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
$4\sqrt{3}\cos 2\theta$	$4\sqrt{3}$	$2\sqrt{3}$	0



ii) Area enclosed by C, $A = \int_0^{\frac{\pi}{4}} \frac{r^2}{2}\, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 16(3)\cos^2 2\theta \, d\theta$

$= 24\int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$

$= 24\int_0^{\frac{\pi}{4}} \frac{\cos 4\theta + 1}{2}\, d\theta$

$= 12\left[\frac{\sin 4\theta}{4} + \theta\right]_0^{\frac{\pi}{4}}$

$= 12\left(\frac{\sin\pi}{4} + \frac{\pi}{4} - 0\right)$

$= 3\pi$

iii) Area of $A_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{r^2}{2}\, d\theta + \frac{3\sqrt{3}}{2}$

$$= 12\left[\frac{\sin 4\theta}{4} + \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \frac{3\sqrt{3}}{2}$$

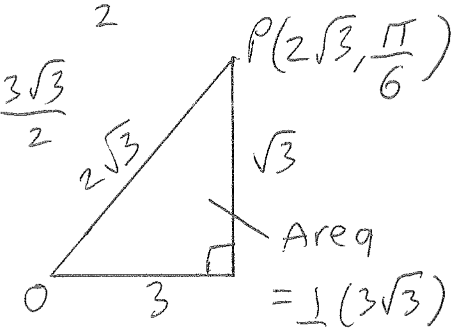
$$= 12\left(\frac{\sin \pi}{4} + \frac{\pi}{4} - \frac{\sin \frac{2\pi}{3}}{4} - \frac{\pi}{6}\right) + \frac{3\sqrt{3}}{2}$$

$$= 12\left(\frac{\pi}{12} - \frac{\frac{\sqrt{3}}{2}}{4}\right) + \frac{3\sqrt{3}}{2}$$

$$= 12\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) + \frac{3\sqrt{3}}{2}$$

$$= \pi - \frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}$$

$$= \pi$$



Area $= \frac{1}{2}(3\sqrt{3})$

$= \frac{3\sqrt{3}}{2}$

Area of $A_2 = 3\pi - \pi = 2\pi$

$\therefore$ Line ℓ divides the area A into

Area A_1 : Area A_2

1 : 2