1. $w_n = (n-1)n(n+1)(n+2)\dots(n+m)$

$w_{n+1} = n(n+1)(n+2)(n+3)\dots(n+m+1)$

$$w_n - w_{n+1} = (n-1)n(n+1)(n+2)\dots(n+m) - n(n+1)(n+2)(n+3)\dots(n+m+1)$$

$$= n(n+1)(n+2)(n+3)\dots(n+m)(n-1-n-m-1)$$

$$= -(m+2)n(n+1)(n+2)\dots(n+m)$$

Given $v_n = 2n(n+1)(n+2)\dots(n+m)$

$$\sum_{n=1}^{N} v_n = \sum_{n=1}^{N} 2n(n+1)(n+2)\dots(n+m)$$

$$= \frac{-2}{m+2}\sum_{n=1}^{N} -(m+2)n(n+1)(n+2)\dots(n+m)$$

$$= \frac{-2}{m+2}\sum_{n=1}^{N} w_n - w_{n+1}$$

$$= \frac{2}{m+2}\sum_{n=1}^{N} w_{n+1} - w_n$$

$$= \frac{2}{m+2}(w_{N+1} - w_1)$$

$$= \frac{2}{m+2}(N(N+1)(N+2)\dots(N+m+1) - 0)$$

$$= \frac{2N(N+1)(N+2)\dots(N+m+1)}{m+2}$$

2. $\left.\begin{aligned} x + b^2y + z &= 0 \\ x + by + z &= 0 \\ x + y - bz &= 0 \end{aligned}\right\}$

$$\begin{aligned} -r_1+r_2: \quad & x + b^2y + z = 0 \\ -r_1+r_3: \quad & (b-b^2)y = 0 \\ & (1-b^2)y - (b+1)z = 0 \end{aligned}\Bigg\}$$

$$\begin{aligned} -(1+b)r_2 + br_3: \quad & x + b^2y + z = 0 \\ & b(1-b)y = 0 \\ & -b(b+1)z = 0 \end{aligned}\Bigg\}$$

when $b=0$: $x + z = 0$

$0y = 0$

$0z = 0$

let $y=s,\ z=t,\ s,t \in R$

$\therefore x = -t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

when $b=-1$: $x + y + z = 0$

$-2y = 0$

$y = 0$

$0z = 0$

let $z = t,\ t \in R$

$x = -t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

when $b=1$: $x+y+z=0$

$$0y = 0$$

$$-2z = 0$$

$$z = 0$$

Let $y=t$, $t \in R$

$$x = -t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

3. $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}$

$$|A| = 2(1-2) - (-1)(-1--2) + 1(1-1)$$
$$= -2+1+0$$
$$= -1$$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & -3 & -1 \end{pmatrix}^T = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & -3 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\therefore A = \frac{1}{|A|} \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & -3 \\ 0 & -1 & -1 \end{pmatrix}$$

$$= \frac{1}{-1} \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & -3 \\ 0 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

4. $\frac{d}{dx}(u_1 u_2 \cdots u_n) = u_1 u_2 \cdots u_n \sum_{r=1}^{n} \frac{1}{u_r}\frac{du_r}{dx}, \quad n \geq 1.$

when $n=1$: $\frac{d}{dx}(u_1) = \frac{du_1}{dx} = u_1\left(\frac{1}{u_1}\right)\frac{du_1}{dx} = u_1 \sum_{r=1}^{1} \frac{1}{u_r}\frac{du_r}{dx}$

Assume the statement is true when $n=k$.

$n=k$: $\frac{d}{dx}(u_1 u_2 \cdots u_k) = u_1 u_2 \cdots u_k \sum_{r=1}^{k} \frac{1}{u_r}\frac{du_r}{dx}$

when $n=k+1$:

$$\frac{d}{dx}(u_1 u_2 \cdots u_k u_{k+1}) = u_1 u_2 \cdots u_k u_{k+1} \sum_{r=1}^{k+1} \frac{1}{u_r}\frac{du_r}{dx}$$

(what needs to be proved)

$$\frac{d}{dx}(u_1 u_2 \cdots u_k u_{k+1}) = \frac{d}{dx}\left[(u_1 u_2 \cdots u_k) u_{k+1}\right]$$

$$= u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx} + u_{k+1}\frac{d}{dx}(u_1 u_2 \cdots u_k)$$

$$= u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx} + u_{k+1}\left(u_1 u_2 \cdots u_k \sum_{r=1}^{k} \frac{1}{u_r}\frac{du_r}{dx}\right)$$

$$= u_1 u_2 \cdots u_k u_{k+1}\left(\frac{1}{u_{k+1}}\right)\frac{du_{k+1}}{dx}$$

$$+ u_1 u_2 \cdots u_k u_{k+1} \sum_{r=1}^{k} \frac{1}{u_r}\frac{du_r}{dx}$$

$$= u_1 u_2 \cdots u_k u_{k+1} \left(\frac{1}{u_{k+1}} \frac{du_{k+1}}{dx} + \sum_{r=1}^{k} \frac{1}{u_r} \frac{du_r}{dx} \right)$$

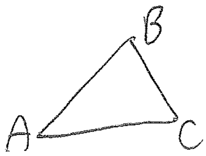
$$= u_1 u_2 \cdots u_k u_{k+1} \sum_{r=1}^{k+1} \frac{1}{u_r} \frac{du_r}{dx}$$

$$\therefore \frac{d}{dx}(u_1 u_2 \cdots u_n) = u_1 u_2 \cdots u_n \sum_{r=1}^{n} \frac{1}{u_r} \frac{du_r}{dx}$$

for every positive integer n.

5. $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\overrightarrow{OC} = \begin{pmatrix} -2 \\ -2 \\ 3 \end{pmatrix}$ $\overrightarrow{OD} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

a)



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$$

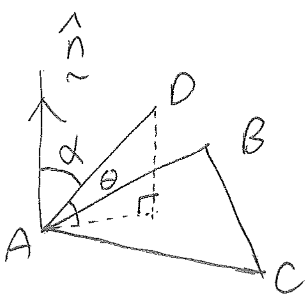
Let $\underset{\sim}{n}$ denote a normdl of the plane with ABC.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\therefore \underset{\sim}{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

The unit normal $\underset{\sim}{\hat{n}} = \dfrac{\underset{\sim}{n}}{|\underset{\sim}{n}|} = \dfrac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

b)



$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{\hat{n}} \cdot \overrightarrow{AD} = |\underset{\sim}{\hat{n}}||\overrightarrow{AD}| \cos\alpha$$

$$\sin\theta = \cos\alpha = \frac{\underset{\sim}{\hat{n}} \cdot \overrightarrow{AD}}{|\underset{\sim}{\hat{n}}||\overrightarrow{AD}|}$$

$$= \frac{\frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{1\sqrt{1^2+0^2+0^2}}$$

$$= \frac{2}{3}$$

$$\cos\theta = \frac{\sqrt{5}}{3}$$

6. a) $x^4 + px^2 + qx + r = 0$

Roots : $\alpha, \beta, \gamma, \delta$

$\alpha + \beta + \gamma + \delta = 0 \quad \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = p$

i) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2$
$-2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$
$= 0^2 - 2p$
$= -2p$

ii) Let $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$

$S_0 = \alpha^0 + \beta^0 + \gamma^0 + \delta^0 = 4$

$S_1 = \alpha + \beta + \gamma + \delta = 0$

$S_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2p$

$S_4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4$

$S_4 + pS_2 + qS_1 + rS_0 = 0$

$S_4 = -p(-2p) - q(0) - r(4)$
$= 2p^2 - 4r$

$\therefore \alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 2p^2 - 4r$

iii) $\beta^5 + p\beta^3 + q\beta^2 + r\beta = \beta(\beta^4 + p\beta^2 + q\beta + r)$
$= 0$ since β is a root.

b) $9x^3 - 9x^2 + mx + 5 = 0$

Roots : $\alpha - h, \alpha, \alpha + h$

$\alpha - h + \alpha + \alpha + h = 1$

$(\alpha - h)\alpha + (\alpha - h)(\alpha + h) + \alpha(\alpha + h) = \frac{m}{9}$

$(\alpha - h)\alpha(\alpha + h) = -\frac{5}{9}$

From 1 : $3\alpha = 1$

$\alpha = \frac{1}{3}$

From 3 : $\left(\frac{1}{3} - h\right)\frac{1}{3}\left(\frac{1}{3} + h\right) = -\frac{5}{9}$

$\frac{1}{3}\left(\frac{1}{9} - h^2\right) = -\frac{5}{9}$

$\frac{1}{9} - h^2 = -\frac{5}{3}$

$h^2 = \frac{16}{9}$

$h = \pm \frac{4}{3}$

From 2 : $\alpha^2 - \alpha h + \alpha^2 - h^2 + \alpha^2 + \alpha h = \frac{m}{9}$

$3\alpha^2 - h^2 = \frac{m}{9}$

$3\left(\frac{1}{3}\right)^2 - \frac{16}{9} = \frac{m}{9}$

$m = -13$

$$x^3 - x^2 + \frac{mx}{9} + \frac{5}{9} = [x-(\alpha-h)](x-\alpha)[x-(\alpha+h)]$$

$$= \left[x-\left(\frac{1}{3}-\frac{4}{3}\right)\right]\left(x-\frac{1}{3}\right)\left[x-\left(\frac{1}{3}+\frac{4}{3}\right)\right]$$

$$= (x+1)\left(x-\frac{1}{3}\right)\left(x-\frac{5}{3}\right)$$

$$9x^3 - 9x^2 + mx + 5 = 0$$

$$\therefore (x+1)(3x-1)(3x-5) = 0$$

$\therefore$ The solutions of the equation are

$x = -1$, $x = \frac{1}{3}$ and $x = \frac{5}{3}$.

7. $C: y = \dfrac{ax-b}{cx-d}$, $a,b,c,d > 0$

a) $y = \dfrac{a}{c} + \dfrac{bc+ad}{c(cx-d)}$

As $x \to \pm\infty$ $y \to \dfrac{a}{c}$

As $x \to \dfrac{d}{c}$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = \dfrac{a}{c}$ and $x = \dfrac{d}{c}$.

b) $\dfrac{dy}{dx} = \dfrac{-(bc+ad)}{(cx-d)^2} \neq 0$

$\therefore$ no critical points

c) when $x=0$: $y = \dfrac{b}{d}$

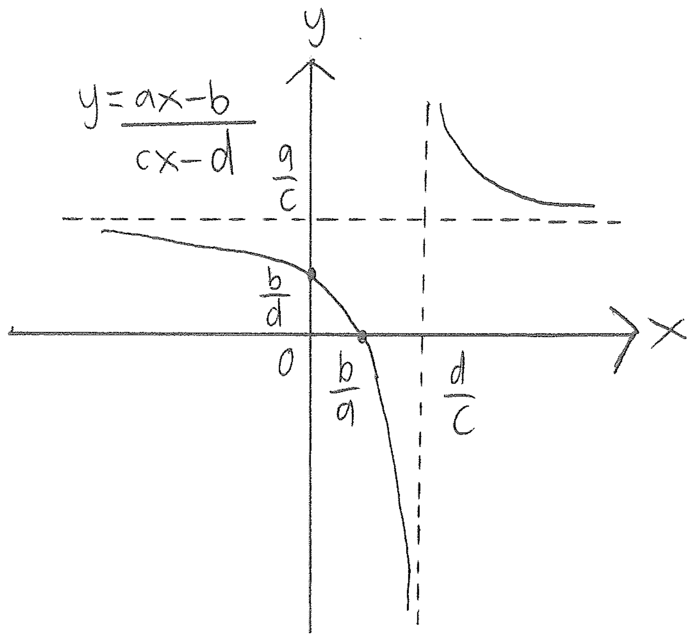
when $y=0$: $\dfrac{ax-b}{cx-d} = 0$

$ax - b = 0$

$x = \dfrac{b}{a}$

$\therefore$ The intersection points of C are $\left(\dfrac{b}{a}, 0\right)$ and $\left(0, \dfrac{b}{d}\right)$.

d)

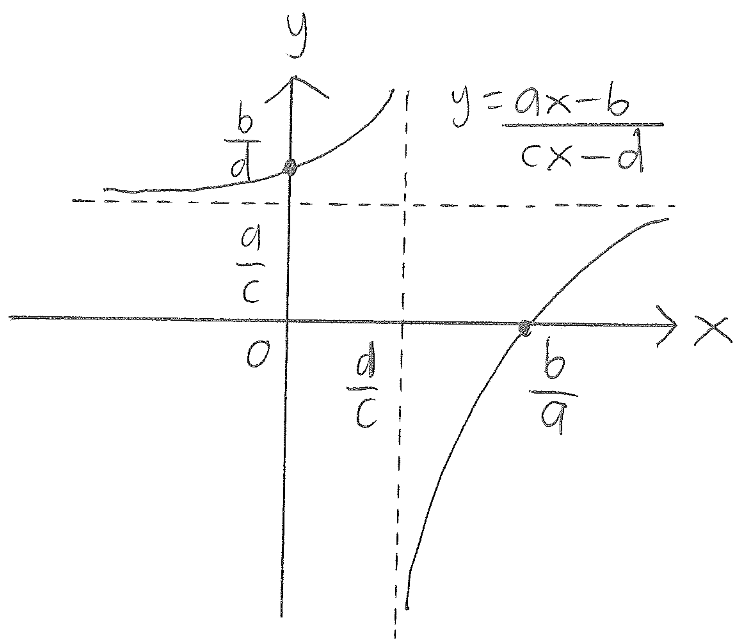


$bc < ad$

$\frac{a}{c} > \frac{b}{d}$

$\frac{d}{c} > \frac{b}{a}$

• : Intersection Point



$bc > ad$

$\frac{a}{c} < \frac{b}{d}$

$\frac{d}{c} < \frac{b}{a}$

• : Intersection Point