MAY / JUNE 2004

1 $$S_N = \sum_{n=1}^{N} 8n^3 - 6n^2$$

Since $\displaystyle\sum_{n=1}^{N} n^3 = \frac{N^2(N+1)^2}{4}$ and $\displaystyle\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$,

$$S_N = \sum_{n=1}^{N} 8n^3 - 6n^2$$

$$= 8\sum_{n=1}^{N} n^3 - 6\sum_{n=1}^{N} n^2$$

$$= \frac{8N^2(N+1)^2}{4} - \frac{6N(N+1)(2N+1)}{6}$$

$$= 2N^2(N+1)^2 - N(N+1)(2N+1)$$

$$= N(N+1)(2N(N+1) - (2N+1))$$

$$= N(N+1)(2N^2 + 2N - 2N - 1)$$

$$= N(N+1)(2N^2 - 1)$$

$$\sum_{n=N+1}^{2N} 8n^3 - 6n^2 = \sum_{n=1}^{2N} 8n^3 - 6n^2 - \sum_{n=1}^{N} 8n^3 - 6n^2$$

$$= 2N(2N+1)(2(2N)^2 - 1)$$
$$- N(N+1)(2N^2 - 1)$$

$$= 2N(2N+1)(8N^2 - 1)$$
$$- N(N+1)(2N^2 - 1)$$

$$= 32N^4 + 16N^3 - 4N^2 - 2N$$

$$- (2N^4 + 2N^3 - N^2 - N)$$

$$= 32N^4 + 16N^3 - 4N^2 - 2N$$

$$- 2N^4 - 2N^3 + N^2 + N$$

$$= 30N^4 + 14N^3 - 3N^2 - N$$

$$= N(30N^3 + 14N^2 - 3N - 1)$$

2. $C: \quad y = \dfrac{x - ax^2}{x - 1}, \quad a > 1$

i)

$$
\begin{array}{r|l}
 & -ax + 1 - a \\
\hline
x - 1 & -ax^2 + x \\
 & -ax^2 + ax \\
\hline
 & (1-a)x \\
 & (1-a)x + a - 1 \\
\hline
 & 1 - a
\end{array}
$$

$$y = -ax + 1 - a + \frac{1 - a}{x - 1}$$

As $x \longrightarrow \pm\infty$ $y \longrightarrow -ax + 1 - a$

As $x \longrightarrow 1$ $y \longrightarrow \pm\infty$

$\therefore$ The asymptotes of C are

$y = -ax + 1 - a$ and $x = 1$.

ii) $\dfrac{dy}{dx} = -a + \dfrac{a - 1}{(x-1)^2}$

when $\dfrac{dy}{dx} = 0$: $\quad -a + \dfrac{a-1}{(x-1)^2} = 0$

$$\frac{a - 1}{(x-1)^2} = a$$

$$(x - 1)^2 = \frac{a - 1}{a}$$

$$x - 1 = \pm\sqrt{\frac{a - 1}{a}}$$

$$x = 1 \pm \sqrt{\frac{a-1}{a}}$$

since $a > 1$,

$$a - 1 > 0$$

$$\frac{a-1}{a} > 0$$

Also, $a > a - 1$

$$1 > \frac{a-1}{a}$$

$$1 > \pm\sqrt{\frac{a-1}{a}}$$

$$1 \pm \sqrt{\frac{a-1}{a}} > 0.$$

$\therefore$ The x-coordinates of both the turning points of C are positive.

3. c. $(x^2 + y^2)^2 = 4xy$

i) $x = r\cos\theta \quad y = r\sin\theta$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2$

$\tan\theta = \dfrac{y}{x}$

$(r^2)^2 = 4(r\cos\theta)r\sin\theta$

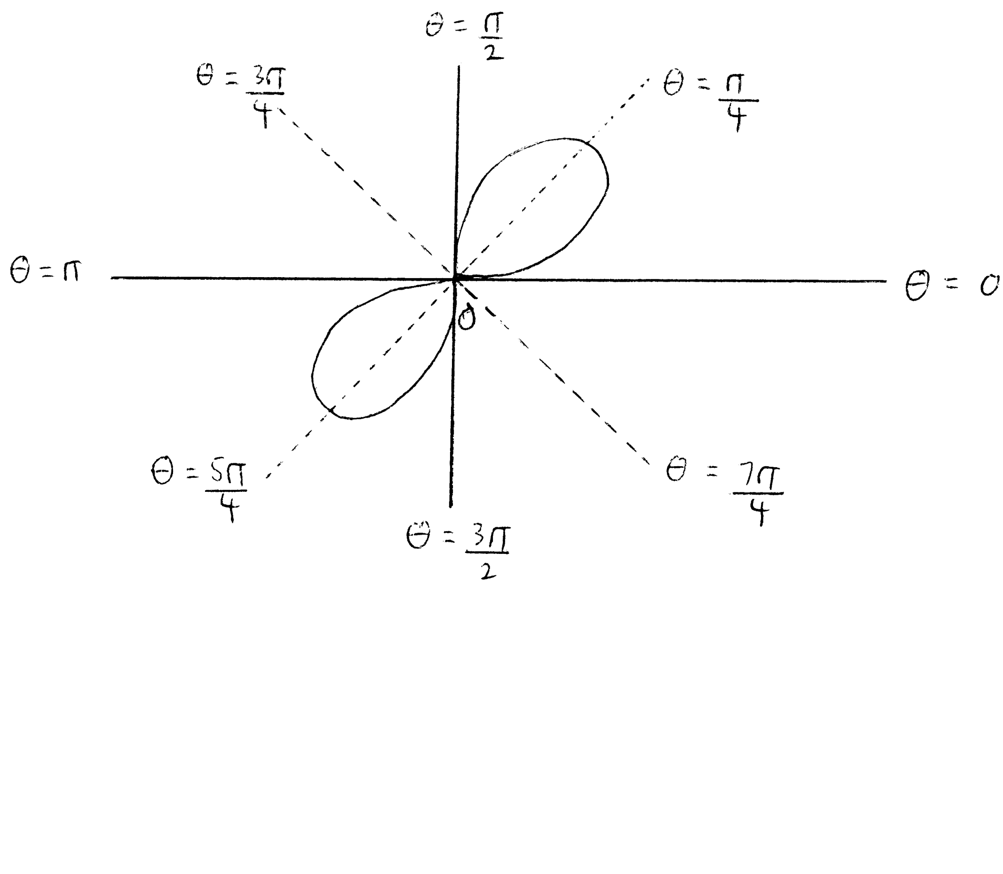
$r^4 = 4r^2\sin\theta\cos\theta$

$r^2 = 4\sin\theta\cos\theta$

$= 2\sin 2\theta$

ii)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r	0	$3^{\frac{1}{4}}$	$2^{\frac{1}{2}}$	$3^{\frac{1}{4}}$	0	0	$3^{\frac{1}{4}}$	$2^{\frac{1}{2}}$	$3^{\frac{1}{4}}$	0



$f(\theta) = \sqrt{2\sin 2\theta}$

$f(2\alpha - \theta) = f(\theta)$

$\sqrt{2\sin 2(2\alpha - \theta)} = \sqrt{2\sin 2\theta}$

$2\sin 2(2\alpha - \theta) = 2\sin 2\theta$

$\sin(4\alpha - 2\theta) = \sin 2\theta$

$\sin 4\alpha \cos 2\theta - \cos 4\alpha \sin 2\theta = \sin 2\theta$

$-\cos 4\alpha = 1$

$\cos 4\alpha = -1$

$4\alpha = \pi, 3\pi, 5\pi, 7\pi$

$\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\therefore$ The lines of symmetry are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

iii) $r^2 = 2\sin 2\theta$

$2r\frac{dr}{d\theta} = 4\cos 2\theta$

when $\frac{dr}{d\theta} = 0$ $\quad 4\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}, \frac{5\pi}{2}$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$2r\frac{d^2r}{d\theta^2} + 2\left(\frac{dr}{d\theta}\right)^2 = -8\sin 2\theta$

when $\theta = \frac{\pi}{4}$: $\frac{d^2r}{d\theta^2} = -2\sqrt{2} < 0$

$\therefore$ The maximum possible distance of a point from the pole is $\sqrt{2}$.

4. $$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{a_n \ln x + b_n}{x^{n+1}},$$ a_n, b_n are functions of n.

i) $$\frac{d^1}{dx^1}\left(\frac{\ln x}{x}\right) = \frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2}$$

$$\frac{d^2}{dx^2}\left(\frac{\ln x}{x}\right) = \frac{-2}{x^3} - \frac{1}{x^3} + \frac{2\ln x}{x^3}$$

$$= \frac{2\ln x - 3}{x^3}$$

$$\frac{d^3}{dx^3}\left(\frac{\ln x}{x}\right) = \frac{2}{x^4} - \frac{6\ln x}{x^4} + \frac{9}{x^4}$$

$$= \frac{11 - 6\ln x}{x^4}$$

$a = -1$, $a_2 = 2$, $a_3 = -6$

ii) $a_n = (-1)^n n!$

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{(-1)^n n! \ln x + b_n}{x^{n+1}}$$

when $n=1$: $$\frac{d^1}{dx^1}\left(\frac{\ln x}{x}\right) = \frac{d}{dx}\left(\frac{\ln x}{x}\right)$$

$$= \frac{1-\ln x}{x^2}$$

$$= \frac{(-1)^1 \ln x + 1}{x^2}$$

$$= \frac{(-1)^1 1! \ln x + 1}{x^2}$$

Assume the statement is true when $n = k$.

$n = k$: $$\frac{d^k}{dx^k}\left(\frac{\ln x}{x}\right) = \frac{(-1)^k k! \ln x + b_k}{x^{k+1}}$$

when $n = k+1$. $$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{\ln x}{x}\right) = \frac{(-1)^{k+1}(k+1)! \ln x + b_{k+1}}{x^{k+1}}.$$

$$\frac{d^k}{dx^k}\left(\frac{\ln x}{x}\right) = \frac{(-1)^k k! \ln x + b_k}{x^{k+1}}$$

$$\frac{d}{dx}\left(\frac{d^k}{dx^k}\left(\frac{\ln x}{x}\right)\right) = \frac{d}{dx}\left(\frac{(-1)^k k! \ln x + b_k}{x^{k+1}}\right)$$

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{\ln x}{x}\right) = \frac{(-1)^k k!}{x^{k+2}} - \frac{(-1)^k (k+1) k! \ln x}{x^{k+2}} - \frac{(k+1) b_k}{x^{k+2}}$$

$$= \frac{(-1)^{k+1}(k+1)! \ln x + (-1)^k k! - (k+1) b_k}{x^{k+2}}$$

$$= \frac{a_{k+1} \ln x + b_{k+1}}{x^{k+1}},$$

$$a_{k+1} = (-1)^k (k+1)!, \quad b_{k+1} = (-1)^k k! - (k+1) b_k$$

$\therefore$ $a_n = (-1)^n n!$ for every positive integer n.

5.
$$A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & 2 & -3 \\ 0 & 3-\lambda & -1 \\ 0 & 0 & 4-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda)\left[(3-\lambda)(4-\lambda) - 0\right] - 2 \cdot 0 - 3 \cdot 0$$

$$= (1-\lambda)(3-\lambda)(4-\lambda)$$

when $|A - \lambda I| = 0$,

$$(1-\lambda)(3-\lambda)(4-\lambda) = 0$$

$$\lambda = 1, 3, 4$$

when $\lambda = 1$: $\begin{pmatrix} 0 & 2 & -3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 0 & 2 & -3 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-r_1 + r_2,\ \frac{r_3}{3}} \left(\begin{array}{ccc|c} 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$z = 0$

$y = 0$

Let $x = s$, $s \in R$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$$

$$= s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

when $\lambda = 3$: $\begin{pmatrix} -2 & 2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -2 & 2 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$z = 0$

Let $y = s$, $s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix}$$

$$= s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

when $\lambda = 4$: $\begin{pmatrix} -3 & 2 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -3 & 2 & -3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 3s$ $s \in R$

$y = -3s$

$-3x - 6s - 9s = 0$

$x = -5s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5s \\ -3s \\ 3s \end{pmatrix}$$

$$= s\begin{pmatrix} -5 \\ -3 \\ 3 \end{pmatrix}$$

$\therefore$ The corresponding eigenvalues of A are $1, 3, 4$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 3 \end{pmatrix}$.

If $\underset{\sim}{e}$ is an eigenvector of A with corresponding eigenvalue λ,

$$\begin{aligned} A^2\underset{\sim}{e} &= A(A\underset{\sim}{e}) \\ &= A(\lambda\underset{\sim}{e}) \\ &= \lambda(A\underset{\sim}{e}) \\ &= \lambda(\lambda\underset{\sim}{e}) \\ &= \lambda^2\underset{\sim}{e} \end{aligned}$$

$$\begin{aligned} A^3\underset{\sim}{e} &= A(A^2\underset{\sim}{e}) \\ &= A(\lambda^2\underset{\sim}{e}) \\ &= \lambda^2(A\underset{\sim}{e}) \\ &= \lambda^2(\lambda\underset{\sim}{e}) \\ &= \lambda^3\underset{\sim}{e} \end{aligned}$$

$$\begin{aligned} A^4\underset{\sim}{e} &= A(A^3\underset{\sim}{e}) \\ &= A(\lambda^3\underset{\sim}{e}) \\ &= \lambda^3(A\underset{\sim}{e}) \\ &= \lambda^3(\lambda\underset{\sim}{e}) \\ &= \lambda^4\underset{\sim}{e} \end{aligned}$$

$$A^5 \underset{\sim}{e} = A(A^4 \underset{\sim}{e})$$
$$= A(\lambda^4 \underset{\sim}{e})$$
$$= \lambda^4 (A\underset{\sim}{e})$$
$$= \lambda^4 (\lambda \underset{\sim}{e})$$
$$= \lambda^5 \underset{\sim}{e}$$

If P is a non-singular matrix and D is a diagonal matrix such that $A^5 = PDP^{-1}$

let $P = \begin{pmatrix} 1 & 1 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{pmatrix}$

and $D = \begin{pmatrix} 1^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 4^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 243 & 0 \\ 0 & 0 & 1024 \end{pmatrix}.$

6. $I_n = \int_e^{e^2} (\ln x)^n \, dx, \quad n \geqslant 0$

$$\frac{d}{dx}\left[x(\ln x)^{n+1}\right] = (\ln x)^{n+1} + \frac{(n+1)\,x(\ln x)^n}{x}$$

$$= (\ln x)^{n+1} + (n+1)(\ln x)^n$$

$$x(\ln x)^{n+1} = \int (\ln x)^{n+1} + (n+1)(\ln x)^n \, dx$$

$$= \int (\ln x)^{n+1} \, dx + (n+1)\int (\ln x)^n \, dx$$

$$\left[x(\ln x)^{n+1}\right]_e^{e^2} = \int_e^{e^2} (\ln x)^{n+1} \, dx + (n+1)\int_e^{e^2} (\ln x)^n \, dx$$

$$e^2(\ln e^2)^{n+1} - e(\ln e)^{n+1} = I_{n+1} + (n+1) I_n$$

$$2^{n+1}e^2 - e = I_{n+1} + (n+1) I_n$$

$$I_{n+1} = 2^{n+1}e^2 - e - (n+1) I_n$$

$n = 2 \quad I_3 = 2^3 e^2 - e - 3I_2$

$$I_2 = 2^2 e^2 - e - 2I_1$$

$$I_1 = 2^1 e^2 - e - 1 I_0$$

$$I_0 = \int_e^{e^2} (\ln x)^0 \, dx$$

$$= \int_e^{e^2} 1 \, dx$$

$$= \left[x\right]_e^{e^2}$$

$$= e^2 - e$$

$$I_1 = 2e^2 - e - (e^2 - e)$$

$$= e^2$$

$$I_2 = 4e^2 - e - 2e^2$$

$$= 2e^2 - e$$

$$I_3 = 8e^2 - e - 3(2e^2 - e)$$

$$= 8e^2 - e - 6e^2 + 3e$$

$$= 2e^2 + 2e$$

The mean value of $(\ln x)^3$ over the interval $e \leq x \leq e^2$ is $\frac{1}{e^2 - e}\int_e^{e^2} (\ln x)^3 \, dx$

$$= \frac{I_3}{e^2 - e}$$

$$= \frac{2e^2 + 2e}{e^2 - e}$$

$$= \frac{2e(e+1)}{e(e-1)}$$

$$= \frac{2(e+1)}{e-1}$$

7. $z^3 = -4\sqrt{3} + 4i$

$$= 8\left(\frac{-\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= 8\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$= 8\left[\cos\left(\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(\frac{5\pi}{6} + 2k\pi\right)\right]$$

$$z = 8^{\frac{1}{3}}\left[\cos\left(\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(\frac{5\pi}{6} + 2k\pi\right)\right]^{\frac{1}{3}}$$

$$= 2\left(\cos\left(\frac{5\pi}{18} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{5\pi}{18} + \frac{2k\pi}{3}\right)\right), \ k = 0, 1, 2$$

$$= 2\left(\cos\frac{5\pi}{18} + i\sin\frac{5\pi}{18}\right), \ 2\left(\cos\frac{17\pi}{18} + i\sin\frac{17\pi}{18}\right),$$

$$2\left(\cos\frac{29\pi}{18} + i\sin\frac{29\pi}{18}\right).$$

$$= 2e^{\frac{5\pi i}{18}}, \ 2e^{\frac{17\pi i}{18}}, \ 2e^{\frac{29\pi i}{18}}$$

Let $z_1 = 2e^{\frac{5\pi i}{18}}$, $z_2 = 2e^{\frac{17\pi i}{18}}$, $z_3 = 2e^{\frac{29\pi i}{18}}$

$$\therefore z_1^{3k} + z_2^{3k} + z_3^{3k} = \left(2e^{\frac{5\pi i}{18}}\right)^{3k} + \left(2e^{\frac{17\pi i}{18}}\right)^{3k} + \left(2e^{\frac{29\pi i}{18}}\right)^{3k}$$

$$= 2^{3k}e^{\frac{5k\pi i}{6}} + 2^{3k}e^{\frac{17k\pi i}{6}} + 2^{3k}e^{\frac{29k\pi i}{6}}$$

$$= 2^{3k}e^{\frac{5k\pi i}{6}} + 2^{3k}e^{\frac{5k\pi i}{6}} + 2^{3k}e^{\frac{5k\pi i}{6}}$$

$$= 3\left(2^{3k}e^{\frac{5k\pi i}{6}}\right).$$

8. $C: \quad x = t^3 - 3t, \quad y = 3t^2 + 1, \quad t > 1.$

i) $\frac{dx}{dt} = 3t^2 - 3 \qquad \frac{dy}{dt} = 6t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{6t}{3t^2 - 3}$$

$$= \frac{2t}{t^2 - 1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{2t}{t^2-1}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left(\frac{2t}{t^2-1}\right)$$

$$= \frac{1}{3t^2-3}\frac{d}{dt}\left(\frac{2t}{t^2-1}\right)$$

$$= \frac{1}{3t^2-3}\frac{(2(t^2-1) - 2t(2t))}{(t^2-1)^2}$$

$$= \frac{1}{3(t^2-1)}\frac{(2t^2 - 2 - 4t^2)}{(t^2-1)^2}$$

$$= \frac{-2t^2 - 2}{3(t^2-1)^3}$$

$$= \frac{-2(t^2+1)}{3(t^2-1)^3}$$

Since $t > 1$, $\dfrac{t^2+1}{(t^2-1)^3} > 0$

$$\frac{-2(t^2+1)}{3(t^2-1)^3} < 0$$

$$\frac{d^2y}{dx^2} < 0$$

$\dfrac{d^2y}{dx^2}$ is negative at every point of C.

ii) When the arc of C is rotated through one complete revolution about the x - axis from $t = 2$ to $t = 3$, the area of the surface generated is $\displaystyle\int_2^3 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_2^3 2\pi(3t^2+1)\sqrt{(3t^2-3)^2 + (6t)^2}\, dt$$

$$= \int_2^3 2\pi(3t^2+1)\sqrt{9t^4 - 18t^2 + 9 + 36t^2}\, dt$$

$$= \int_2^3 2\pi(3t^2+1)\sqrt{9t^4 + 18t^2 + 9}\, dt$$

$$= \int_2^3 2\pi(3t^2+1)\sqrt{(3t^2+3)^2}\, dt$$

$$= \int_2^3 2\pi (3t^2+1)(3t^2+3) \, dt$$

$$= 6\pi \int_2^3 (3t^2+1)(t^2+1) \, dt$$

$$= 6\pi \int_2^3 3t^4 + 4t^2 + 1 \, dt$$

$$= 6\pi \left[\frac{3t^5}{5} + \frac{4t^3}{3} + t \right]_2^3$$

$$= 6\pi \left(\frac{729}{5} + 36 + 3 - \frac{96}{5} - \frac{32}{3} - 2 \right)$$

$$= 6\pi \left(\frac{2294}{15} \right)$$

$$= \frac{4588\pi}{5}$$

$$= 917\tfrac{3}{5}\pi$$

9. $y = f(x), \quad x = \frac{1}{t}$

$$\frac{dx}{dt} = -\frac{1}{t^2}$$

$$\frac{dt}{dx} = -t^2$$

$$\frac{dt}{dy}\frac{dy}{dx} = -t^2$$

$$\frac{dy}{dx} = -t^2\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(-t^2\frac{dy}{dt}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)$$

$$= -t^2\left(-t^2\frac{d^2y}{dt^2} + \frac{dy}{dt}\frac{d}{dt}(-t^2)\right)$$

$$= -t^2\left(-t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt}\right)$$

$$= t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}$$

$$x^5 \frac{d^2y}{dx^2} + (2x^4 - 5x^3)\frac{dy}{dx} + 4xy = 14x + 8$$

$$\left(\frac{1}{t}\right)^5\left(t^4 \frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \left(2\left(\frac{1}{t}\right)^4 - 5\left(\frac{1}{t}\right)^3\right)\left(-t^2\frac{dy}{dt}\right)$$

$$+ 4\left(\frac{1}{t}\right)y = \frac{14}{t} + 8$$

$$\frac{1}{t}\frac{d^2y}{dt^2} + \frac{2}{t^2}\frac{dy}{dt} - \frac{2}{t^2}\frac{dy}{dt} + \frac{5}{t}\frac{dy}{dt} + \frac{4y}{t} = \frac{14}{t} + 8$$

$$\frac{1}{t}\frac{d^2y}{dt^2} + \frac{5}{t}\frac{dy}{dt} + \frac{4y}{t} = \frac{14}{t} + 8$$

$$\frac{d^2y}{dt^2} + \frac{5dy}{dt} + 4y = 8t + 14.$$

$$\frac{d^2y}{dt^2} + \frac{5dy}{dt} + 4y = 0$$

$$m^2 + 5m + 4 = 0$$

$$(m+1)(m+4) = 0$$

$$m = -1, -4$$

$\therefore$ The complementary function, y_c, is

$$y_c = Ae^{-t} + Be^{-4t}$$

The particular integral, y_p, is given by

$$y_p = Ct + D$$

$$\frac{dy_p}{dt} = C$$

$$\frac{d^2y_p}{dt^2} = 0$$

$$\frac{d^2y_p}{dt^2} + \frac{5dy_p}{dt} + 4y_p = 0 + 5C + 4(Ct + D)$$

$$= 4Ct + 5C + 4D$$

$$= 8t + 14$$

$4C = 8$ $\quad$ $5C + 4D = 14$

$C = 2$ $\quad$ $4D = 4$

$D = 1$

$$y_p = 2t + 1$$

$$y = y_c + y_p$$

$$= Ae^{-t} + Be^{-4t} + 2t + 1$$

$$= Ae^{-\frac{1}{x}} + Be^{-\frac{4}{x}} + \frac{2}{x} + 1$$

10.

$$A = \begin{pmatrix} 3 & 1 & 3 & -2 \\ 5 & 0 & 7 & -7 \\ 6 & 2 & 6 & \theta+2 \\ 9 & 3 & 9 & \theta \end{pmatrix}$$

i) $\xrightarrow{r_2 \times 3} \begin{pmatrix} 3 & 1 & 3 & -2 \\ 15 & 0 & 21 & -21 \\ 6 & 2 & 6 & \theta+2 \\ 9 & 3 & 9 & \theta \end{pmatrix}$

$$\xrightarrow[\substack{-2r_1 + r_3 \\ -3r_1 + r_4}]{-5r_1 + r_2} \begin{pmatrix} 3 & 1 & 3 & -2 \\ 0 & -5 & 6 & -11 \\ 0 & 0 & 0 & \theta+6 \\ 0 & 0 & 0 & \theta+6 \end{pmatrix}$$

$$\xrightarrow{-r_3 + r_4} \begin{pmatrix} 3 & 1 & 3 & -2 \\ 0 & -5 & 6 & -11 \\ 0 & 0 & 0 & \theta+6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

when $\theta \neq -6$,

$\theta + 6 \neq 0$

$$\xrightarrow{\frac{r_3}{\theta+6}} \begin{pmatrix} 3 & 1 & 3 & -2 \\ 0 & -5 & 6 & -11 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 3$

The dimension of the null space K of T is $4 - 3 = 1$.

when $\theta = -6$,

$$\begin{pmatrix} 3 & 1 & 3 & -2 \\ 0 & -5 & 6 & -11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 2$

The dimension of the null space K of T is $4 - 2 = 2$.

ii) $\theta \neq -6$:

$$\begin{pmatrix} 3 & 1 & 3 & -2 \\ 0 & -5 & 6 & -11 \\ 0 & 0 & 0 & \theta+6 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 3 & 1 & 3 & -2 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & 0 & 0 & \theta+6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_3}{\theta+6}} \left(\begin{array}{cccc|c} 3 & 1 & 3 & -2 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$t = 0$

Let $z = 5s, \ s \in R$

$y = 6s$

$3x + 6s + 3(5s) - 0 = 0$

$x = -7s$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -7s \\ 6s \\ 5s \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} -7 \\ 6 \\ 5 \\ 0 \end{pmatrix}$$

$\therefore$ When $\theta \neq -6$, a basis vector for K is

$$\underset{\sim}{e}_1 = \begin{pmatrix} -7 \\ 6 \\ 5 \\ 0 \end{pmatrix}$$

iii) $\theta = -6$: $\begin{pmatrix} 3 & 1 & 3 & -2 \\ 0 & -5 & 6 & -11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 3 & 1 & 3 & -2 & 0 \\ 0 & -5 & 6 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $t = 5s,\ s \in \mathbb{R}$

and $z = 5\lambda,\ \lambda \in \mathbb{R}$

$y = 6\lambda - 11s$

$3x + 6\lambda - 11s + 3(5\lambda) - 2(5s) = 0$

$3x + 6\lambda - 11s + 15\lambda - 10s = 0$

$3x = 21s - 21\lambda$

$x = 7s - 7\lambda$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 7s - 7\lambda \\ 6\lambda - 11s \\ 5\lambda \\ 5s \end{pmatrix}$$

$$= s\begin{pmatrix} 7 \\ -11 \\ 0 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} -7 \\ 6 \\ 5 \\ 0 \end{pmatrix}$$

$\therefore$ when $\theta = -6$, if $\underset{\sim}{e}_2 = \begin{pmatrix} 7 \\ -11 \\ 0 \\ 5 \end{pmatrix}$

then $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ is a basis of K.

iv) when $\theta = -6$, $\underset{\sim}{b} = \begin{pmatrix} 5 \\ 5 \\ 10 \\ 15 \end{pmatrix}$, $\underset{\sim}{e}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

if $\underset{\sim}{x} = \underset{\sim}{e}_0 + k_1 \underset{\sim}{e}_1 + k_2 \underset{\sim}{e}_2$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + k_1 \begin{pmatrix} -7 \\ 6 \\ 5 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 7 \\ -11 \\ 0 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 7k_1 + 7k_2 \\ 1 + 6k_1 - 11k_2 \\ 1 + 5k_1 \\ 1 + 5k_2 \end{pmatrix}$$

$$A\underset{\sim}{x} = \begin{pmatrix} 3 & 1 & 3 & -2 \\ 5 & 0 & 7 & -7 \\ 6 & 2 & 6 & -4 \\ 9 & 3 & 9 & -6 \end{pmatrix} \begin{pmatrix} 1 - 7k_1 + 7k_2 \\ 1 + 6k_1 - 11k_2 \\ 1 + 5k_1 \\ 1 + 5k_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 - 21k_1 + 21k_2 + 1 + 6k_1 - 11k_2 + 3 + 15k_1 - 2 - 10k_2 \\ 5 - 35k_1 + 35k_2 + 7 + 35k_1 - 7 - 35k_2 \\ 6 - 42k_1 + 42k_2 + 2 + 12k_1 - 22k_2 + 6 + 30k_1 - 4 - 20k_2 \\ 9 - 63k_1 + 63k_2 + 3 + 18k_1 - 33k_2 + 9 + 45k_1 - 6 - 30k_2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 5 \\ 10 \\ 15 \end{pmatrix}$$

$$= \underset{\sim}{b}$$

$\therefore$ If $\theta = -6$, $\underset{\sim}{b} = \begin{pmatrix} 5 \\ 5 \\ 10 \\ 15 \end{pmatrix}$, $\underset{\sim}{e}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\underset{\sim}{x} = \underset{\sim}{e}_0 + k_1 \underset{\sim}{e}_1 + k_2 \underset{\sim}{e}_2$

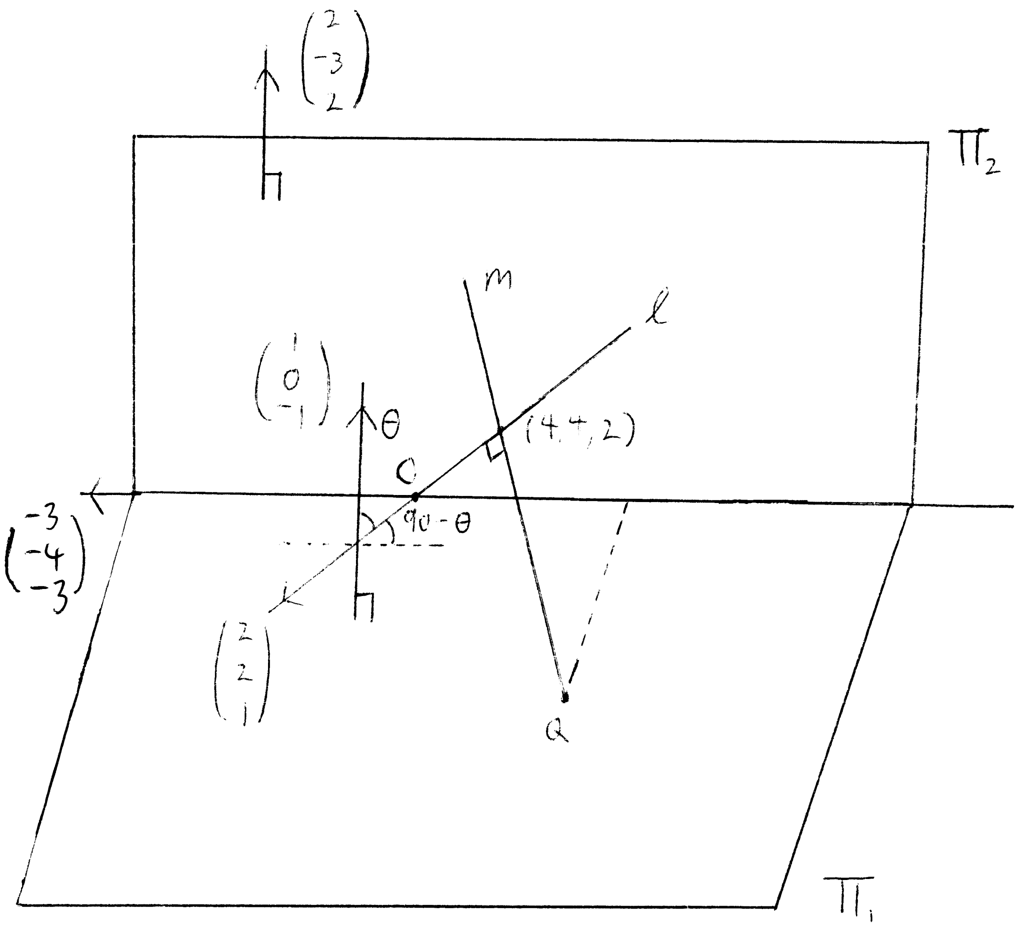
is a solution of $A\underset{\sim}{x} = \underset{\sim}{b}$ for all $k_1, k_2 \in R$.

11. EITHER

i) $\ell : \underset{\sim}{r} = s(2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$

$\Pi_1 : x - z = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$$



Since the direction of ℓ is $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is normal to Π_1, if θ is the angle between $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$,

$$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \left|\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right| \cos\theta$$

$$2 + 0 - 1 = \sqrt{9}\sqrt{2}\cos\theta$$

$$\cos\theta = \frac{1}{3\sqrt{2}}$$

$$= 0.2357$$

$$\theta = 76.4^\circ$$

$\therefore$ The acute angle between l and Π_1 is $90^\circ - 76.4^\circ = 13.6^\circ$.

ii) Since the plane Π_2 contains l and is perpendicular to Π_1; $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are parallel to Π_2 and $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is perpendicular to Π_2.

$$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}.$$

Since $\begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$ is perpendicular to Π_2 and $(0,0,0)$ is a point on Π_2,

$$\underset{\sim}{r} \cdot \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} = 0$$

$$-2x + 3y - 2z = 0$$

$$2x - 3y + 2z = 0$$

The equation of the plane Π_2 which contains l and is perpendicular to Π_1 is $2x - 3y + 2z = 0$.

iii) Since the line of intersection of Π_1 and Π_2 is perpendicular to both $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$, it is parallel to $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & -1 \\ 2 & -3 & 2 \end{vmatrix} = \begin{pmatrix} -3 \\ -4 \\ -3 \end{pmatrix}$$

Since the direction of the line of intersection of Π_1 and Π_2 is $\begin{pmatrix} -3 \\ -4 \\ -3 \end{pmatrix}$ and $(0,0,0)$ is a point on both Π_1 and Π_2, a vector equation of the line is $\underset{\sim}{r} = s\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$

If $k_1(\underset{\sim}{i} - \underset{\sim}{k}) + k_2(2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) + k_3(3\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}) = \underset{\sim}{0}$,

$(k_1 + 2k_2 + 3k_3)\underset{\sim}{i} + (2k_2 + 4k_3)\underset{\sim}{j} + (-k_1 + k_2 + 3k_3)\underset{\sim}{k} = \underset{\sim}{0}$

$$\left.\begin{aligned} k_1 + 2k_2 + 3k_3 &= 0 \\ 2k_2 + 4k_3 &= 0 \\ -k_1 + k_2 + 3k_3 &= 0 \end{aligned}\right\}$$

$$(1) + (3): \left.\begin{aligned} k_1 + 2k_2 + 3k_3 &= 0 \\ 2k_2 + 4k_3 &= 0 \\ 3k_2 + 6k_3 &= 0 \end{aligned}\right\}$$

$\frac{1}{2} \times (2)$: $k_1 + 2k_2 + 3k_3 = 0$

$\frac{1}{3} \times (3)$: $k_2 + 2k_3 = 0$

$k_2 + 2k_3 = 0$

$-(2) + (3)$: $k_1 + 2k_2 + 3k_3 = 0$

$k_2 + 2k_3 = 0$

$0k_2 + 0k_3 = 0$

Let $k_3 = s, s \in R$

$k_2 = -2s$

$k_1 + 2(-2s) + 3s = 0$

$k_1 = s$

$\therefore \underset{\sim}{i} - \underset{\sim}{k}$, $2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $3\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$ are linearly dependent.

iv) Since $Q(x, y, z)$ is a point on Π_1 let $z = s, s \in R$ and $y = t, t \in R$.

$\therefore x = z$

$= s$

Since m is perpendicular to l,

$$\left[\begin{pmatrix} s \\ t \\ s \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \left|\left[\begin{pmatrix} s \\ t \\ s \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}\right]\right| \left|\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right| \cos 90^\circ$$

$$\begin{pmatrix} s-4 \\ t-4 \\ s-2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$2s - 8 + 2t - 8 + s - 2 = 0$$

$$3s + 2t = 18$$

$$OQ^2 = (s-0)^2 + (t-0)^2 + (s-0)^2$$

$$= s^2 + t^2 + s^2$$

$$= 2s^2 + t^2$$

$$= 2s^2 + \left(\frac{18 - 3s}{2}\right)^2$$

$$= 2s^2 + 81 - 27s + \frac{9s^2}{4}$$

$$= \frac{17s^2 - 108s + 324}{4}$$

$$= \frac{17}{4}\left(s^2 - \frac{108s}{17} + \frac{324}{17}\right)$$

$$= \frac{17}{4}\left(s^2 - \frac{108s}{17} + \left(\frac{54}{17}\right)^2 - \left(\frac{54}{17}\right)^2 + \frac{324}{17}\right)$$

$$= \frac{17}{4}\left[\left(s - \frac{54}{17}\right)^2 + \frac{2592}{289}\right]$$

$$= \frac{17}{4}\left(s - \frac{54}{17}\right)^2 + \frac{2592}{68}$$

$$OQ = \sqrt{\frac{17}{2}\left(s - \frac{54}{17}\right)^2 + \frac{2592}{68}}$$

$$= \sqrt{\frac{2592}{68}} \quad \text{when} \quad s = \frac{54}{17}$$

$$\approx 6.17$$

$\therefore$ The minimum distance of Q from the origin as m varies is 6.17

OR

$x^3 - x - 1 = 0$

α, β, γ are the roots

$S_n = \alpha^n + \beta^n + \gamma^n$

i) $\alpha^2, \beta^2, \gamma^2$

Let $y = x^2$

$x = \pm\sqrt{y}$

x is a root

$\therefore x^3 - x - 1 = 0$

$(\pm\sqrt{y})^3 - (\pm\sqrt{y}) - 1 = 0$

$\pm y\sqrt{y} \mp \sqrt{y} - 1 = 0$

$\pm\sqrt{y}(y - 1) - 1 = 0$

$\pm\sqrt{y}(y - 1) = 1$

$(\pm\sqrt{y}(y - 1))^2 = 1$

$y(y - 1)^2 = 1$

$y(y^2 - 2y + 1) = 1$

$y^3 - 2y^2 + y = 1$

$y^3 - 2y^2 + y - 1 = 0$

$\therefore$ The equation $y^3 - 2y^2 + y - 1 = 0$ has roots $\alpha^2, \beta^2, \gamma^2$.

ii) $y^3 - 2y^2 + y - 1 = 0$

$\alpha^2 + \beta^2 + \gamma^2 = 2 \quad \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = 1 \quad \alpha^2\beta^2\gamma^2 = 1$

$$S_4 = \alpha^4 + \beta^4 + \gamma^4$$

$$= (\alpha^2 + \beta^2 + \gamma^2)^2 - 2(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)$$

$$= 2^2 - 2(1)$$

$$= 4 - 2$$

$$= 2$$

iii) Let $z = y^2$

$y = \pm\sqrt{z}$

y is a root

$$y^3 - 2y^2 + y - 1 = 0$$

$$(\pm\sqrt{z})^3 - 2(\pm\sqrt{z})^2 \pm \sqrt{z} - 1 = 0$$

$$\pm z\sqrt{z} - 2z \pm \sqrt{z} - 1 = 0$$

$$\pm\sqrt{z}(z + 1) = 2z + 1$$

$$(\pm\sqrt{z}(z + 1))^2 = (2z + 1)^2$$

$$z(z + 1)^2 = 4z^2 + 4z + 1$$

$$z(z^2 + 2z + 1) = 4z^2 + 4z + 1$$

$$z^3 + 2z^2 + z = 4z^2 + 4z + 1$$

$$z^3 - 2z^2 - 3z - 1 = 0$$

$\therefore$ The equation $z^3 - 2z^2 - 3z - 1 = 0$ has roots $\alpha^4, \beta^4, \gamma^4$.

$\alpha^4 + \beta^4 + \gamma^4 = 2 \quad \alpha^4\beta^4 + \alpha^4\gamma^4 + \beta^4\gamma^4 = -3 \quad \alpha^4\beta^4\gamma^4 = 1$

Let $T_n = \alpha^{4n} + \beta^{4n} + \gamma^{4n}$

$$T_0 = \alpha^{4(0)} + \beta^{4(0)} + \gamma^{4(0)}$$
$$= \alpha^0 + \beta^0 + \gamma^0$$
$$= 1 + 1 + 1$$
$$= 3$$

$$T_1 = \alpha^{4(1)} + \beta^{4(1)} + \gamma^{4(1)}$$
$$= \alpha^4 + \beta^4 + \gamma^4$$
$$= 2$$

$$T_2 = \alpha^{4(2)} + \beta^{4(2)} + \gamma^{4(2)}$$
$$= \alpha^8 + \beta^8 + \gamma^8$$
$$= (\alpha^4 + \beta^4 + \gamma^4)^2 - 2(\alpha^4\beta^4 + \alpha^4\gamma^4 + \beta^4\gamma^4)$$
$$= 2^2 - 2(-3)$$
$$= 4 + 6$$
$$= 10$$

$$T_{3+r} - 2T_{2+r} - 3T_{1+r} - T_r = 0$$

$r = 0.$ $T_3 - 2T_2 - 3T_1 - T_0 = 0$

$$T_3 - 2(10) - 3(2) - 3 = 0$$
$$T_3 - 20 - 6 - 3 = 0$$
$$T_3 = 29$$

$r=1$: $T_4 - 2T_3 - 3T_2 - T_1 = 0$

$$T_4 - 2(29) - 3(10) - 2 = 0$$

$$T_4 - 58 - 30 - 2 = 0$$

$$T_4 = 90$$

Since $T_2 = \alpha^8 + \beta^8 + \gamma^8 = S_8$,

$T_3 = \alpha^{12} + \beta^{12} + \gamma^{12} = S_{12}$ and $T_4 = \alpha^{16} + \beta^{16} + \gamma^{16} = S_{16}$

$\therefore$ $S_8 = 10$, $S_{12} = 29$ and $S_{16} = 90$.