12. VECTOR

1. Given that O, P and Q are three points in a plane and λ, μ are two real numbers ($\lambda + \mu \neq 0$), show that there exists a point R on the line joining P and Q such that $\lambda\mathbf{OP} + \mu\mathbf{OQ} = (\lambda + \mu)\,\mathbf{OR}$.
If D, E, F are the mid-points of the sides of the triangle ABC and X is any point, not necessarily in the plane of the triangle ABC, show that
$$\mathbf{XA} + \mathbf{XB} + \mathbf{XC} = \mathbf{XD} + \mathbf{XE} + \mathbf{XF}.$$
(The notation $\overrightarrow{OP}$ may be used instead of $\mathbf{OP}$.]. (J72/I/4)

2. Show that the equation of the plane through the point A with position vector $\mathbf{a}$, relative to an origin O, and parallel to the vectors $\mathbf{s}$ and $\mathbf{t}$ ($\mathbf{s} \wedge \mathbf{t} \neq \mathbf{0}$) may be written $\mathbf{r} = \mathbf{a} + \lambda\mathbf{s} + \mu\mathbf{t}$. Explain the significance of the condition $\mathbf{s} \wedge \mathbf{t} \neq \mathbf{0}$.
Hence or otherwise show that the equation of the plane may be written in the form $\mathbf{r.p} = \mathbf{a.p}$ where $\mathbf{p} = \mathbf{s} \wedge \mathbf{t}$.
Show that the equation of the plane through the distinct, non-collinear points A, B, C, with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively, may be written in the form $\mathbf{r.q} = \mathbf{a.q}$, where $\mathbf{q} = \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b}$ and hence or otherwise deduce that $\mathbf{a.q} = \mathbf{b.q} = \mathbf{c.q}$. (J72 / I I /4)

3. The position vectors of four points A, B, C, D relative to an origin O are given below. The vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are mutually perpendicular unit vectors.
A: $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, B: $\mathbf{i} + \mathbf{j} - \mathbf{k}$, C: $\mathbf{i} + \mathbf{k}$, D: $3\mathbf{j}$.
Find (a) the equation (in any form) of the line AB,
(b) the shortest distance between AB and CD,
(c) the equation (in any form) of the plane ABC,
(d) the angle between the planes ABC and ABD. (J73/I/5)

4. The position vectors of four points A, B, C, D relative to an origin O are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ respectively. Obtain the equation of the line l joining the midpoints of AB and CD. Show that l intersects the line joining the midpoints of AC and BD and find the position vector $\mathbf{p}$ of the point P of intersection.
Deduce that the line joining the midpoints of AD and BC also passes through P. (J73/II/4)

5. The unit vector normal to a given plane is $\mathbf{n}$ and p is the length of the perpendicular from the origin to the plane. Show tha the equation of the plane may be expressed as $\mathbf{r.n} = p$.
A, B, C are the points $\mathbf{i} + 2\mathbf{j}$, $2\mathbf{i} + 6\mathbf{j}$, $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ respectively and π is the plane $\mathbf{r}.(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 1$. Find
(a) the length of the perpendicular from B to π,
(b) the cosine of the acute angle between the planes π and ABC. (N73/I/4)

6. In the tetrahedron $LMNO$, the line OL is perpendicular to MN and the line OM is perpendicular to NL. By taking O as origin and letting L, M, N have position vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, or otherwise, prove that
(a) ON is perpendicular to LM,
(b) $OL^2 + MN^2 = OM^2 + NL^2 = ON^2 + LM^2$. (N73/II/4)

7. Given that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and $\mathbf{a} \neq \pm \mathbf{b}$, show that the lines l and m intersect, where l and m are given by $l : \mathbf{r} = \mathbf{a} + 3\mathbf{b} + \lambda \mathbf{a}$, $m : \mathbf{r} = 3\mathbf{a} - 4\mathbf{b} + \mu \mathbf{b}$, λ and μ being parameters, and find the position vector of the point of intersection C.
Find
   (a) the equation (in any form) of the straight line through C, perpendicular to l and m,
   (b) the equation (in any form) of the plane containing l and m,
   (c) the equation (in any form) of the bisectors of the angles between l and m at C. (J74/I/4)

8. (a) If $\mathbf{a}$ is a given non-zero vector, state the geometrical significance of the equation $\mathbf{r}.\mathbf{a} = 0$ and hence obtain the general solution for $\mathbf{r}$.
   (b) If $\mathbf{a}$ and $\mathbf{b}$ are given non-zero vectors, state the geometrical significance (in three dimensions) of the pair of equations $\mathbf{r}.\mathbf{a} = 0$, $\mathbf{r}.\ \mathbf{b} = 0$ and hence or otherwise show that, provided $\mathbf{a}$ and $\mathbf{b}$ are not in special relationship, which should be stated, the general solution may be written in the form $\mathbf{r} - A\mathbf{c}$, where A is a parameter and $\mathbf{c}$ is to be determined. (J74/II/4)

9. The point P divides the line segment AB internally in the ratio $\lambda : \mu$. If O is the origin show that $\mathbf{OP} = \dfrac{\lambda \mathbf{OB} + \mu \mathbf{OA}}{\lambda + \mu}$
Show that three points A, B and C are coollinear if and only if there exist α, β, γ, not all zero, such that $\alpha \mathbf{OA} + \beta \mathbf{OB} + \gamma \mathbf{OC} = 0$, with $\alpha + \beta + \gamma = 0$.
Are the three points $(1, 2, 3)$, $(-2, 1, 4)$ and $(3, \frac{8}{3}, \frac{7}{3},)$ collinear ? Give a reason.
(N74/I/6)

10. Find a unit normal to each of the planes
π_1: $\mathbf{r} = (3, 4, 5) + \lambda(2, 1, -1) + \mu(3, 1, 4)$,
π_2: $\mathbf{r}.(-2, 4, 1) = 9$.
Find the cosine of the angle between these two planes and obtain also the equation of their line of intersection in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. (N74/II/6)

11. If P is the mid-point of the line segment AB and O is any point, show that $\mathbf{OP} = \frac{1}{2}(\mathbf{OA} + \mathbf{OB})$.
Hence, or otherwise, prove that the lines joining the mid-points of the opposite sides of a tetrahedron are concurrent. (J75/I/5)

12. The non-collinear points A, B and C have position vectors A, B and C respectively relative to an origin O. Show that the vector $(\mathbf{a} - \mathbf{b}) \wedge (\mathbf{a} - \mathbf{c})$ is normal to the plane through the points A, B and C.
Show that the points having position vectors $5\mathbf{j} - 4\mathbf{k}$, $\mathbf{i} + \mathbf{j} - \mathbf{k}$, $2\mathbf{i} - \mathbf{k}$, $3\mathbf{j} - 2\mathbf{k}$ are coplanar.
Find, in any form, the equation of the plane containing these four points.
(J75/II/4)

13. Find, to the nearest minute, the acute angle between the lines given by $\mathbf{r} = (1, 2, 3) + \lambda(1, 1, 2)$ and $\mathbf{r} = (1, 2, 3) + \mu(2, -2, 2)$. (N75/I/4)

14. (a) Show that exactly one of the following sets contains the position vectors of three collinear points:

$X = \{3\mathbf{i} + \mathbf{j} - \mathbf{k}, 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} - \mathbf{j} + \mathbf{k}\}$,

$Y = \{\mathbf{i} - \mathbf{j}, 3\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}, 5\mathbf{i} + 3\mathbf{j} + \mathbf{k}\}$,

$C = \{\mathbf{i} + 3\mathbf{j} - \frac{1}{2}\mathbf{k}, \mathbf{i} + 4\mathbf{k}, 3\mathbf{i} - \mathbf{j} + \mathbf{k}\}$.

(b) The position vectors of the vertices A, B and C of a triangle are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. If O is the origin, show that the area of triangle OAB is $\frac{1}{2}|\mathbf{a} \wedge \mathbf{b}|$ and deduce an expression for the area of the triangle ABC.

Hence, or otherwise, show that the perpendicular distance from B to AC is

$$\frac{|\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a}|}{|\mathbf{c} - \mathbf{a}|}.$$

(N75/I/5)

15. Given $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, and that $\mathbf{n} = \mathbf{b} \wedge \mathbf{c}$, show that $\mathbf{a}.\,\mathbf{n} = \Delta$, where

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors, relative to the origin O, of points A, B and C respectively. The four points O, A, B and C are coplanar. Show that $\Delta = 0$.
If $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \alpha\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, find
(a) the value of α,
(b) a unit vector, lying in the plane OAB, which is perpendicular to $\mathbf{a}$. (N75/II/3)

16. If O, P and Q are three points and R is the point dividing the line segment PQ internally in the ratio $m : n$, write down an equation connecting the vectors $\mathbf{OR}$, $\mathbf{OP}$ and $\mathbf{OQ}$.
Show that the three lines, obtained by joining the vertices of a triangle ABC to the mid-points of the opposite sides, concur at a point G (the centroid of ΔABC) given by $3\mathbf{OG} = \mathbf{OA} + \mathbf{OB} + \mathbf{OC}$.
Deduce that the four lines, obtained by joining the vertices of a tetrahedron to the centroids of the opposite faces, are concurrent. (J76/I/5)

17. The position vectors of the four non-coplanar points A, B, C and D relative to an origin O are respectively $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Find the following in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$:
(a) the equation of the plane ABC,
(b) the shortest distance from D to the plane ABC,
(c) the equation of the plane through AB parallel to CD,
(d) the shortest distance between the lines AB and CD. (J76/II/2)

18. The vector $\mathbf{OP}$ makes angles α, β, γ with the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
Find the vector $\mathbf{OP}$ given that $|\mathbf{OP}| = 18$, $\alpha = \cos^{-1}(\frac{1}{3})$, $\beta = \cos^{-1}(-\frac{2}{3})$ and that the angle between $\mathbf{OP}$ and the vector $\mathbf{OQ} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is acute.
Find the unit vectors along the internal bisectors of the angles POQ and $P'OQ$, where P' is a point on PO produced. (N76/I/4)

19. (a) Let O, P_1, P_2, P_3, be four points and $\lambda_1, \lambda_2, \lambda_3$ be three real numbers. Show that $\lambda_1\mathbf{OP}_1 + \lambda_2\mathbf{OP}_2 + \lambda_3\mathbf{OP}_3 = (\lambda_1 + \lambda_2 + \lambda_3)\ \mathbf{OS}$, where S is a point such that $\lambda_1\mathbf{SP}_1 + \lambda_2\mathbf{SP}_2 + \lambda_3\mathbf{SP} = \mathbf{0}$.

(b) Three points A, B, C have position vectors $a\mathbf{i}$, $b\mathbf{j}$, $c\mathbf{k}$ respectively, relative to the origin O, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard orthogonal unit vectors. Find an expression for a vector normal to the plane ABC and find the position vector of the foot of the perpendicular from O to the plane ABC.
Determine the area of the triangle ABC. (N76/II/3)

20. (a) By considering the scalar product $\mathbf{a}.\ \mathbf{b}$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, or by any other method, prove that $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leqslant (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

(b) The two lines l, m have equations $\mathbf{r} = (1, 0, -5) + t(1, 1, 1)$ and $\mathbf{r} = (0, -2, -2) + s(3, 2, 1)$ respectively, where $s, t \in \mathbf{R}$ (the set of real numbers). P is a point on l and Q is a point on m. If PQ is perpendicular to both l and m find the coordinates of P and Q, and show that the length of PQ is $\sqrt{6}$.
(J77/I/5)

21. Find, in any form, the equation of the plane containing the points A, B and C whose position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively are given by $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.
Find, in any form, the image by reflection in this plane of the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, where $\mathbf{d} = \mathbf{j} - 5\mathbf{k}$. (J77/II/4)

22. A, B, C are points whose position vectors are $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} - 10\mathbf{k}$ respectively. Show that any vector normal to the plane ABC is of the form $\lambda(5\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, and give, in any form, the equation of the plane ABC.
The point P has position vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, and Q is the image of P after reflection in the plane ABC. Find the position vector of Q. (N77/I/5)

23. The points A, B, C have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. The points P, Q, R divide BC, CA, AB respectively, each in the ratio 1 : 2. Points X, Y, Z are defined by $X \equiv (BQ \cap CR)$, $Y \equiv (CR \cap AP)$, $Z \equiv (AP \cap BQ)$. Prove that the position vector of X is $\frac{1}{7}(2\mathbf{a} + \mathbf{b} + 4\mathbf{c})$. Write down the position vectors of Y and Z in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
Hence prove that the vector products $\mathbf{XY} \wedge \mathbf{XZ}$ and $\mathbf{AB} \wedge \mathbf{AC}$ are related by $\mathbf{XY} \wedge \mathbf{XZ} = \frac{1}{7}(\mathbf{AB} \wedge \mathbf{AC})$. (N77/II/4)

24. The points A, B, C and D have position vectors given by: $\mathbf{a} = \mathbf{i} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = -2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{d} = 2\mathbf{i} + \mathbf{k}$ respectively. Find
(a) a unit vector perpendicular tot he plane ABC,
(b) the length of the perpendicular from D to the plane ABC,
(c) the cosine of the angle between the line AD and the plane ABC. (J78/I/3)

25. The position vector $\mathbf{p}$, at time t, of a particle P of mass $2m$ is given by $\mathbf{p} = (t^3 - t - 6)\mathbf{i} + (t + 6)\mathbf{j}$. The position vector $\mathbf{q}$ of a second particle Q, of mass m, is given by $\mathbf{q} = (4 - t^2)\mathbf{i} + 4t\mathbf{j}$. Show that Q is moving with constant acceleration, and find the magnitude and direction of the velocity of P relative to Q when $t = 1$.
Show also that the particles collide, and find the value of t when the collision takes place. If the collision halves the speed of Q and reverses its direction, find the speed of P immediately after the collision. (J78/I/9)

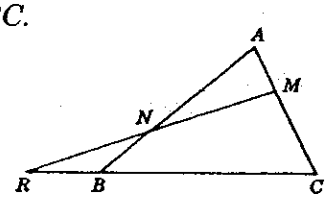
26. For each of the following assertions, state whether it is true or false, justifying your answer.
    (a) $(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \wedge (-\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = 3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$.
    (b) If $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$ then either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.
    (c) The area of triangle ABC is $\frac{1}{2}|\mathbf{AB} \wedge \mathbf{AC}|$.
    (d) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are any vectors then $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$.
    (e) The equation of the line passing through a fixed point with position vector $\mathbf{a}$, and parallel to the non-zero vector $\mathbf{u}$, can be written in the form $(\mathbf{r} - \mathbf{a}) \wedge \mathbf{u} = \mathbf{0}$, where $\mathbf{r}$ is the position vector of any point on the line. (J78/II/3)

27. (a) The vertices A, B, C, D of a plane convex quadriateral have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ respectively. Prove that the area of the triangle ABC is given by
    $\frac{1}{2}|(\mathbf{b} \wedge \mathbf{c}) + (\mathbf{c} \wedge \mathbf{a}) + (\mathbf{a} \wedge \mathbf{b})|$.
    Prove also that the area of the quadrilateral $ABCD$ is given by
    $\frac{1}{2}|(\mathbf{a} \wedge \mathbf{b}) + (\mathbf{b} \wedge \mathbf{c}) + (\mathbf{c} \wedge \mathbf{d}) + (\mathbf{d} \wedge \mathbf{a})|$.
    (b) The points P, Q, R have position vectors $6\mathbf{i} + 2\mathbf{k}$, $3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $-2\mathbf{j} + 3\mathbf{k}$ respectively relative to the origin O. Find the equation of the plane PQR in the form $\mathbf{r} \cdot \mathbf{n} = p$.
    Obtain the volume of the tetrahedron $OPQR$. (N78/I/4)

28. The position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ of the points A, B, C, D respectively are given by $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -\mathbf{j} + \mathbf{k}$, $\mathbf{c} = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{d} = 5\mathbf{i} - \mathbf{j} - 6\mathbf{k}$. Find
    (a) a unit vector perpendicular to the plane ABC,
    (b) the length of the perpendicular from D to the plane ABC,
    (c) the position vector of the point D', where D' is the reflection of D in the plane ABC. (J79/I/4)

29. (a) Find, in any form, the equation of the plane passing through the points with coordinates $(1, -1, 1)$, $(2, 1, 2)$, $(3, 1, -1)$. Show that the acute angle between this plane and the plane with equation $2x - y + 3z = 0$ is $\cos^{-1}\left(\frac{11}{14}\right)$.
    (b)

    The diagram shows points M and N on the sides AC, AB of the triangle ABC, such that $AM : MC = BN : NA = 1 : 2$. The line MN produced meets the line CB produced at R. Find $RB : BC$. (J79/II/4)



30. (a) By considering the vector product $\mathbf{a} \wedge \mathbf{b}$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, prove that $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geqslant (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$.

(b) For each of the following assertions, state whether it is true or false and justify your answer.

(i) if $\mathbf{a} \wedge \mathbf{b} = \mathbf{b} \wedge \mathbf{a}$ then $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$;

(ii) $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are any vectors;

(iii) $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{a} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{a})$, where $\mathbf{a}, \mathbf{b}$ are any vectors;

[No credit will be given for answers unsupported by valid reasons.] (N79/I/3)

31. The position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ of the points A, B, C, D respectively are given by $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = -7\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{c} = 7\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{d} = -8\mathbf{i} - 2\mathbf{j}$. Find, in the form $\mathbf{r}.\mathbf{n} = p$, the equation of the plane containing A and B and parallel to the line CD.

Show that the shortest distance between the lines AB and CD is $1\frac{1}{7}$ units.

Find the cosine of the acute angle between the directions of the lines AC and BD. (N79/II/3)

32. Two straight lines l_1 and l_2 have equations given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} \qquad l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 6 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix}$$

(a) The common perpendicular of the two lines meets them in P_1 and P_2. Find the position vectors of P_1 and P_2 and show that the length of P_1P_2 is 6.

(b) Find, in the form $\mathbf{r}.\mathbf{n} = p$, the equation of the plane which passes through the point with coordinates $(1, 0, 1)$ and which does not interest either l_1 or l_2. (J80/I/3)

33. A line l with direction cosines $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$ passes through the point A whose position vector is $9\mathbf{j} - 5\mathbf{j} + 2\mathbf{k}$. A plane whose normal has direction cosines $\left(\frac{3}{\sqrt{26}}, \frac{-4}{\sqrt{26}}, \frac{-1}{\sqrt{26}}\right)$ passes through the point whose position vector is $-\mathbf{i} + 4\mathbf{k}$.

(a) Obtain the equation of the plane in the form $\mathbf{r}.\mathbf{n} = p$, where $\mathbf{n}$ and p are to be found.

(b) Calculate the position vector of the foot of the perpendicular from A to the plane.

(c) Calculate the position vector of the point of intersection of l and the plane.

(d) Show that the reflection, in the plane, of the line l has direction cosines $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. (J80/II/3)

34. (a) Given $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$, $\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, verify that $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}$.

Hence show that the points A, B, K with position vectors $\mathbf{a}, \mathbf{b}, (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$ respectively are collinear, and obtain the ratio $AK : KB$.

(b) Given the vectors $\mathbf{p} = p\mathbf{i}$, $\mathbf{q} = q\mathbf{j}$, where $p \neq 0$, show that the general solution of the equation $\mathbf{x} \wedge \mathbf{p} = \mathbf{q}$ is $\mathbf{x} = \frac{\mathbf{p} \wedge \mathbf{q}}{p^2} + \lambda\mathbf{p}$, where λ is an arbitrary scalar. (N80/I/3)

35. The lines l_1 and l_2 have equations $\mathbf{r} = (2, 3, 1) + \lambda(1, 2, 2)$ and $\mathbf{r} = (5, -1, -13) + \mu(-2, 1, 6)$ respectively. Show that these lines intersect and give the coordinates of their point of intersection. Find, in the form $\mathbf{r}.\ \mathbf{n} = p$, the equation of the plane P_1 which contains both l_1 and l_2.
Show that the plane P_2 which contains the l_1 and passes through the origin has equation $\mathbf{r}.\ (4, -3, -1) = 0$. Calculate
(a) the cosine of the acute angle between the planes P_1 and P_2,
(b) the sine of the angle between the line l_2 and the plane P_2. (N80/II/4)

36. Three distinct lines DA, EB, FC intersect at the origin O. With respect to O, the position vectors of the points A, B, C are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively, and the position vectors of the points D, E, F are $3\mathbf{a}$, $2\mathbf{b}$, $5\mathbf{c}$ respectively. The lines AB and DE intersect at Z, the lines AC and DF intersect at Y and the lines CB and FE intersect at X. Show that the position vector of Z is $4\mathbf{b} - 3\mathbf{a}$ and find, in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, the position vectors of the points Y and X.
Show that the points X, Y and Z lie on a straight line. (J81/I/4)

37. The position vectors of three non-collinear points A, B, C are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively relative to the origin O.
(a) Show that the length of the perpendicular from C to the line AB is

$$\frac{|\mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b}|}{|\mathbf{b} - \mathbf{a}|}$$

(b) Given that $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -4\mathbf{j} + 4\mathbf{k}$, $\mathbf{c} = -\mathbf{i} + 5\mathbf{k}$, find the position vector of the reflection of C in the line AB. (J81/II/2)

38. For each of the following assertions, state whether it is true or false and justify your answer.
(a) Given that the position vectors, relative to O, of the points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, then the area of the triangle OAB is $\frac{1}{2}\ |\mathbf{a} \wedge \mathbf{b}|$.
(b) Given that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar position vectors, then $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{0}$.
(c) Given that $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = \mathbf{i} - \mathbf{j} - \mathbf{k}$, then the four points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a} + \mathbf{b} + \mathbf{c}$ are coplanar.
(d) The line with equation $\mathbf{r} = (2 + 3\lambda)\,\mathbf{i} + (1 + \lambda)\,\mathbf{j} + (-1 - 3\lambda)\,\mathbf{k}$ meets the plane with equation $\mathbf{r}.\ (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = \mathbf{0}$ in a single point. (N81/I/3)

39. (a) The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are distinct and each of unit length and are such that $\lambda\mathbf{a} + \mu\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$, where λ, μ, γ are scalars, not all zero. Show that $\mu(\mathbf{a} \wedge \mathbf{b}) = \gamma(\mathbf{c} \wedge \mathbf{a})$.
Hence, or otherwise, prove that $|\lambda|$, $|\mu|$, $|\gamma|$ are proportional to the sines of the angles between $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{c}$ and $\mathbf{a}$, $\mathbf{a}$ and $\mathbf{b}$ respectively.
(b) The vertices O, P, Q, R, S, T, U, V of a cuboid have position vectors $\mathbf{0}$, $a\mathbf{i}$, $a\mathbf{i} + b\mathbf{j}$, $b\mathbf{j}$, $c\mathbf{k}$, $a\mathbf{i} + c\mathbf{k}$, $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, $b\mathbf{j} + c\mathbf{k}$ respectively. Prove that the distance between PT and OU is $ab(a^2 + b^2)^{-\frac{1}{2}}$, and give the distance between RQ and OU, and between SV and OU. (N81/II/4)

40. The parametric equations of two planes are

$$\mathbf{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+t\begin{pmatrix}1\\-2\\1\end{pmatrix}+s\begin{pmatrix}0\\3\\2\end{pmatrix} \text{ and } \mathbf{r}=\begin{pmatrix}2\\1\\-1\end{pmatrix}+u\begin{pmatrix}0\\0\\5\end{pmatrix}+v\begin{pmatrix}-2\\4\\3\end{pmatrix}$$

(a) Find the cosine of the acute angle between the two planes.

(b) The line of intersection of the two planes is l. Find, in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, the equation of l.

(c) Show that the length of the perpendicular from the point $\begin{pmatrix}1\\5\\1\end{pmatrix}$ to the line l is $\sqrt{2}$. (J82/I/4)

41. (a) Show that if it is possible to find a vector $\mathbf{r}$ such that $\mathbf{r}\wedge\mathbf{a}=\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are given vectors, then $\mathbf{a}.\mathbf{b}=\mathbf{0}$.
Find the set of vectors $\mathbf{r}$ which satisfy $\mathbf{r}\wedge\mathbf{a}=\mathbf{b}$ in the following cases:

(i) $\mathbf{a}=\begin{pmatrix}2\\1\\-3\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}3\\-3\\1\end{pmatrix}$; (ii) $\mathbf{a}=\begin{pmatrix}4\\-1\\5\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}8\\0\\7\end{pmatrix}$.

(b) Given that $2\mathbf{x}+(\mathbf{x}.\mathbf{b})\,\mathbf{a}=\mathbf{c}$, where $\mathbf{a}.\mathbf{b}\neq-2$, show that $\mathbf{x}.\mathbf{b}.=\dfrac{\mathbf{c}.\mathbf{b}}{2+\mathbf{a}.\mathbf{b}}$

Deduce an expression for $\mathbf{x}$ in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. (J82/II/4)

42. Find, in the form $\mathbf{r}.\mathbf{n}=p$, the equation of the plane π which contains the points $P(4,-2,1)$ $Q(-5,8,3)$ and $R(4,6,5)$.
The point A is $(9,9,0)$. Find the coordinates of its reflection in the plane π.
Calculate the shortest distance between the lines AQ, PR. (N82/I/3)

43. The vertices A, B of a triangle OAB have position vectors $\mathbf{a}$, $\mathbf{b}$ respectively relative to O. Give a vector expression for the area of the triangle.

Points P, Q are taken on OA, OB respectively such that $\dfrac{OP}{OA}=\dfrac{OQ}{OB}=\lambda$, where $\lambda^2\neq1$. Write down the position vectors of P and Q.
The lines AQ and BP intersect at X. Show that the position vector of X is $\dfrac{\lambda}{1+\lambda}$ $(\mathbf{a}+\mathbf{b})$.

Find, in terms of λ, the ratio of the area of the triangle XPQ to the area of the triangle OAB. (N82/II/4)

44. The unit vectors $\mathbf{u}$ and $\mathbf{v}$ are inclined at an angle $\frac{1}{3}\pi$ to each other. The lines l_1 and l_2 have equations $\mathbf{r}=\mathbf{a}+\lambda\mathbf{u}$ and $\mathbf{r}=\mathbf{a}+\mathbf{b}+\mu\,\mathbf{v}$ respectively. Given that the lines l_1 and l_2 intersect, show that, at the point of intersection, $\lambda=\frac{2}{3}\mathbf{b}.(2\mathbf{u}-\mathbf{v})$ and find the corresponding value of μ. Find, in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{v}$,

(a) the equation (in any form) of the plane containing l_1 and l_2,

(b) a unit vector perpendicular to this plane,

(c) the equation (in any form) of the line joining the origin and the intersection of l_1 and l_2. (J83/I/4)

45. The line l has equation $\mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$

(a) The point A on l is such that OA is perpendicular to l, where O is the origin. Find the position vector of A.

(b) Find a unit vector perpendicular to the plane containing O and l.

(c) Find the position vectors of the two points P on l such that $\cos P\hat{O}A = \frac{1}{\sqrt{7}}$

(J83/II/3)

46. The point O in the plane of a given triangle ABC is equidistant from A, B, and C. With respect to O the position vectors of A, B, C are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively, and H is the point with position vector $\mathbf{a} + \mathbf{b} + \mathbf{c}$. Prove that H is the point of intersection of the altitudes of triangle ABC.
Given that N, P, D are the midpoints of OH, AH, BC respectively, prove that N is also the midpoint of PD.
Given further that $\mathbf{a} = \beta\mathbf{b} + \gamma\mathbf{c}$, where β, γ are real constants, and that AH and BC intersect in a point X with position vector x, show that $2\mathbf{x} = (1 + \beta - \gamma)\,\mathbf{b} + (1 - \beta + \gamma)\,\mathbf{c}$.
Show, by any method, that N is equidistant from X, P and D. (N83/I/4)

47. (a) The plane π passes through the points with position vectors $p\mathbf{i}$, $q\mathbf{j}$, $r\mathbf{k}$, where $pqr \neq \mathbf{0}$. Show that c, the perpendicular distance of the origin from the plane π, satisfies $\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} = \frac{1}{c^2}$

(b) A straight line l through the point A with position vector $\mathbf{a}$ is parallel to a unit vector $\mathbf{e}$. The point R lies on the line l and has position vector $\mathbf{r}$. Show that $(\mathbf{r} - \mathbf{a}) \times \mathbf{e} = \mathbf{0}$.
The point P has position vector $\mathbf{p}$. Show that the perpendicular distance of P from the line l is $|(\mathbf{p} - \mathbf{a}) \times \mathbf{e}|$.
Obtain the perpendicular distance from the origin to the straight line through the points $(5, 1, -2)$ and $(2, -1, 4)$. (N83/II/4)

48. (a) Find the length of the perpendicular from the origin O to the plane through the points with position vectors $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$.

(b) The distinct points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ with respect to the origin O, and the plane ABC does not contain O. Given that OA is perpendicular to BC and that OB is perpendicular to AC, prove that

(i) OC is perpendicular to AB,

(ii) $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$. (J84/I/3)

49. The line l with equation $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ meets, at the point A, the plane π with equation $\mathbf{r} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. The point C on l has position vector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

and B is the foot of the perpendicular from C to the plane π.

(a) Find the position vectors of the points A and B.
(b) Show that $\sin C\hat{A}B = \sqrt{(\frac{7}{15})}$.
(c) Find the position vector of the reflection of C in the plane π. (J84/II/3)

50. The vectors $\mathbf{a}$ and $\mathbf{b}$ are given by $\mathbf{a} = \mathbf{i} + (\sin\theta)\mathbf{j} + (\cos\theta)\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + (\sin\phi)\mathbf{j} + (\cos\phi)\mathbf{k}$, where $0 \leqslant \theta \leqslant \phi \leqslant \pi$.
(a) Find an expression for $|\mathbf{a} \times \mathbf{b}|$ in terms of δ, where $\delta = \frac{1}{2}(\phi - \theta)$, and deduce that the angle α between $\mathbf{a}$ and $\mathbf{b}$ is given by $\sin\alpha = \sin\delta\sqrt{1 + \cos^2\delta}$.
(b) Given that $\mathbf{a} \times \mathbf{b} = -\frac{\sqrt{3}}{2}\mathbf{i} + \mathbf{j}$, find the values of θ and ϕ. (J84/II/4)

51. The position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, relative to the origin O, of the points A, B, C, D, are given by $\mathbf{a} = 2\mathbf{i}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j}$, $\mathbf{c} = 4\mathbf{j}$, $\mathbf{d} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.
(a) Obtain a unit vector normal to the plane BCD.
(b) Calculate the angle between the planes ABD and BCD, leaving your answer in inverse trigonometric form.
(c) Calculate the distance of the point C from the plane ABD.
(d) Calculate the shortest distance between the lines AD and BC. (N84/II/3)

52. The vertices A, B, C of a triangle have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. E is the mid-point of AC, F is the mid-point of AB, and the lines BE and CF intersect at G. Derive the position vector of G and prove that $\overrightarrow{GC} \times \overrightarrow{GB} = \frac{1}{3}(\mathbf{c} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{c})$. Hence, or otherwise, prove that the quadrilateral $AFGE$ and the triangle GBC have equal areas. (J85/I/3)

53. The lines l and m have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} k \\ 4 \\ -5 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \text{ respectively.}$$

(a) Given that l and m intersect find the value of k and the position vector of the point of intersection.
(b) Using the value of k found in (a) find the equation of the image, by reflection, of l in m. (J85/II/3)

54. The vector $\overrightarrow{OP}$ has direction cosines l, m, n. Find $\overrightarrow{OP}$ given that $l = \frac{3}{7}$, $m = \frac{2}{7}$, $|\overrightarrow{OP}| = 21$, and that the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, where $\overrightarrow{OQ} = 6\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}$, is acute.

Find the direction cosines of $\overrightarrow{PQ}$, and show that $|\overrightarrow{PQ}| = \frac{1}{2}|\overrightarrow{OQ}|$.

Given that R is the mid-point of PQ, find the equation of the normal from R to the plane OPQ, and calculate the shortest distance between this normal and the x-axis. (N85/I/4)

55. The points P and Q have position vectors $2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$ and $-\mathbf{i} - 15\mathbf{j} - 6\mathbf{k}$ respectively, and G is the point on the line segment PQ such that $\overrightarrow{PQ} = 3\overrightarrow{PG}$. The plane π, with normal parallel to $\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, passes through G, and N is the foot of the perpendicular from P to π.

(i) Obtain the equation of π in the form $ax + by + cz = d$.

(ii) Find the position vector of N.

(iii) The equation of π may be expressed in the form $\mathbf{r} = \mathbf{g} + \lambda\mathbf{u} + \mu\mathbf{v}$, where $\mathbf{g}$ is the position vector of G, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular vectors, and λ and μ are scalar parameters. Find suitable vectors $\mathbf{u}$ and $\mathbf{v}$. (J86/I/11)

56. The equations of the lines l_1 and l_2 are $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$ respectively, where $\mathbf{a} = -6\mathbf{i} + 3\mathbf{j} + 15\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = 6\mathbf{i} + 15\mathbf{j} + 39\mathbf{k}$, $\mathbf{d} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, and where λ and μ are scalar parameters. The points P and Q are on l_1 and l_2 respectively, and PQ is perpendicular to both l_1 and l_2. In any order:

(i) find a vector which is parallel to PQ, giving your answer in the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$;

(ii) show that $PQ = 10\sqrt{6}$;

(iii) find the position vectors of P and Q. (J86/I/12)

57. With respect to the origin O, points A, B, C, D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ respectively, given by

$$\begin{aligned} \mathbf{a} &= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \\ \mathbf{b} &= -5\mathbf{i} + 8\mathbf{j} - 6\mathbf{k} \\ \mathbf{c} &= \mathbf{i} + \mathbf{j} + \mathbf{k} \\ \mathbf{d} &= 3\mathbf{i} + 2\mathbf{j} - 8\mathbf{k} \end{aligned}$$

The point P on the line segment AB is such that $\overrightarrow{BP} = 2\overrightarrow{PA}$, and the point Q on OP produced is such that $\overrightarrow{OQ} = 4\,\overrightarrow{OP}$. Obtain the position vectors of P and Q.

Find the position vector of the point at which the line through B and Q meets the plane through O, C and D. (N86/I/11)

58. The points A, B, C of the plane π have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Show that the vector $\mathbf{h}$, where $\mathbf{h} = \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}$, is perpendicular to π.

The point P, not in π, has position vector $\mathbf{p}$. Show that the perpendicular distance of P from π is $\dfrac{|\mathbf{p} \cdot \mathbf{h} - \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|h|}$.

The foot of the perpendicular from P to the line AB is Q. Show that the position vector of Q may be expressed as $\mathbf{a} + \theta(\mathbf{b} - \mathbf{a})$, where $\theta = \dfrac{(\mathbf{p} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})}{(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})}$. (N86/I/12)

59. The variable point U which in the plane π_1 has position vector $(3 + 6p)\mathbf{i} + (1 + 4p + q)\mathbf{j} + (6 + 2p - 4q)\mathbf{k}$, and the variable point V which is in the plane π_2 has position vector $(-9 + 3p)\mathbf{i} + (1 + p - 2q)\mathbf{j} + (3 - p + 8q)\mathbf{k}$, where $p, q \in \mathbb{R}$. The point W, which is on the line segment UV, is such that $2UW = WV$.

(i) Find a vector which is perpendicular to π_1 and also a vector which is perpendicular to π_2.

(ii) Find, to the nearest 0·1°, the acute angle between π_1 and π_2.

(iii) Show that the locus of W is a line l, whose vector equation is of the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are to be determined.

(iv) The line $\mathbf{r} = s\mathbf{k}$, where $s \in \mathbb{R}$, meet the plane π_1 in the point A. Find the distance of A from l, giving your answer correct to three significant figures. (J87/I/11)

60. The three vectors $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$ are such that $\mathbf{n} = \mathbf{l} \times \mathbf{m}$. Show that $\mathbf{l}.\,\mathbf{n} = 0$.
With respect to an origin O, the position vectors of three non-collinear points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The scalars λ and μ are such that $\lambda\mathbf{a} + \mu\mathbf{b}$ is the position vector of the projection of C onto the plane containing O, A and B.

(i) Explain why there is a scalar t such that $t(\mathbf{a} \times \mathbf{b}) = \mathbf{c} - \lambda\mathbf{a} - \mu\mathbf{b}$, and deduce that $(\mathbf{a.a})\lambda + (\mathbf{a.b})\mu = \mathbf{a.c}$.

(ii) Hence, or otherwise, show that the length of the projection of OC onto the plane OAB is $|\,\lambda\mathbf{a} + \mu\mathbf{b}\,|$, where

$$\lambda = \frac{(\mathbf{b.b})(\mathbf{a.c}) - (\mathbf{a.b})(\mathbf{b.c})}{(\mathbf{a.a})(\mathbf{b.b}) - (\mathbf{a.b})^2} \quad \text{and} \quad \mu = \frac{(\mathbf{a.a})(\mathbf{b.c}) - (\mathbf{a.b})(\mathbf{a.c})}{(\mathbf{a.b})(\mathbf{b.b}) - (\mathbf{a.b})^2}.$$

(iii) The points P, Q, R and S have position vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $2\mathbf{i} + \mathbf{j}$, $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ respectively. By using the result of (ii), or otherwise, show that the length of the projection of PS onto the plane containing P, Q and R is $\sqrt{\left(\frac{11}{6}\right)}$. (J87/I/12)

61. The plane π has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ and the point P, not in π, has position vector $\mathbf{p}$. Also Q, the reflection of P in π, has position vector $\mathbf{q}$. (Thus Q is the point such that PQ is perpendicular to π and the midpoint of PQ is in π.)
Show that

(i) $\mathbf{p.b} = \mathbf{q.b}$ and $\mathbf{p.c} = \mathbf{q.c}$,

(ii) there are values of λ and μ for which $\mathbf{p} + \mathbf{q} = 2\mathbf{a} + 2\lambda\mathbf{b} + 2\mu\mathbf{c}$.

Deduce that if $\mathbf{b}$ and $\mathbf{c}$ are perpendicular vectors then
$\mathbf{p.b} + \mathbf{q.b} = 2\mathbf{a.b} + 2\lambda\mathbf{b.b}$, and that in this case

$$\mathbf{q} = 2\mathbf{a} + 2\left(\frac{\mathbf{b.p} - \mathbf{a.b}}{\mathbf{b.b}}\right)\mathbf{b} + 2\left(\frac{\mathbf{c.p} - \mathbf{a.c}}{\mathbf{c.c}}\right)\mathbf{c} - \mathbf{p}.$$

Find the position vector of the reflection of the point with position vector $\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ in the plane $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. (N87/I/11)

62. The points A, B, C and D are given by $\overrightarrow{OA} = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + 5\mathbf{j}$, $\overrightarrow{OC} = -\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OD} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, where O is the origin. Show that AC is perpendicular to AB and to DC.

Hence find the shortest distance between the line through A and B and the line through C and D.

The point E on AC is such that $2\overrightarrow{AE} = \overrightarrow{EC}$ and l is the line through E perpendicular to AB and to AE.

(i) Find a vector parallel to l.

(ii) Find the position vectors of the two points on l which are a distance of $\sqrt{20}$ units from A.

(N87/I/12)

63. With respect to an origin O, the position vectors of the points U, V and W are $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, respectively, and Π is the plane which contains O, U and V. The variable point Z is on the opposite side of Π from W, and for all positions of Z, 2WY = YZ, where Y is the point in which the line WZ meets Π.

(i) Show that the equation of the locus of Z can be expressed in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where $\lambda, \mu \in \mathbb{R}$, and find suitable vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(ii) Hence show that the locus of Z is a plane, Π_1, which is parallel to Π.

(iii) Find a vector which is perpendicular to Π and Π_1.

(iv) Hence, or otherwise, show that the distance between Π and Π_1 is $\frac{4\sqrt{3}}{3}$.

(J88/I/11)

64. The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{a} + s\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + t\mathbf{d}$, where $s, t \in \mathbb{R}$ and $\mathbf{a} = 5\mathbf{i} + 6\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 13\mathbf{j} + 7\mathbf{k}$, $\mathbf{d} = \mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$. Also, the point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. In any order:

(i) show that PQ = $\sqrt{38}$,

(ii) find the position vectors of P and Q,

(iii) obtain the cartesian equation of the plane containing the lines l_1 and PQ.

(J88/I/12)

65. With respect to an origin O the points P and Q have position vectors $(4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3\mathbf{k}$ and $(8\cos t)\mathbf{i} + (8\sin t)\mathbf{j} - \mathbf{k}$, respectively, where $0 \leqslant t \leqslant 2\pi$.

(i) Find, in terms of t, the position vector of the point, R, in which the line PQ meets the plane $z = 0$.

(ii) Show that the cartesian equation of the plane, Π, through R perpendicular to PQ is $(\cos t)x + (\sin t)y - z = 7$.

(iii) The point A has position vector $10\mathbf{i} + 10\mathbf{j} + 40\mathbf{k}$, and the line OA makes an acute angle θ with the direction of the normal to Π. Show that $\cos\theta = \frac{1}{6}[4 - \sqrt{2}\sin(t + \frac{1}{4}\pi)]$.

(iv) Hence find the maximum value of θ as t varies, giving your answer to the nearest to 0.1°.

(N88/I/11)

66. The lines l_1 and l_2 have vector equations $\mathbf{r} = \mathbf{a} + s\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + t\mathbf{d}$, respectively, where $s, t \in R$, $\mathbf{a} = \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ and $\mathbf{d} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(i) Find a vector parallel to PQ.

(ii) Obtain the position vectors of P and Q.

(iii) Show that the plane through PQ perpendicular to l_1 has vector equation $\mathbf{r} = (1 + \lambda + \mu)\mathbf{i} + (-\lambda + \mu)\mathbf{j} + (-1 - 2\lambda)\mathbf{k}$, where $\lambda, \mu \in R$. (N88/I/12)

67. The planes Π_1 and Π_2, which meet in a line l, have vector equations $\mathbf{r} = \mathbf{a} + \lambda_1\mathbf{b}_1 + \mu_1\mathbf{c}_1$ and $\mathbf{r} = \mathbf{a} + \lambda_2\mathbf{b}_2 + \mu_2\mathbf{c}_2$, respectively.

(i) Explain briefly why the point with position vector $\mathbf{a}$ lies on l.

(ii) Show that the vector $\mathbf{h} = (\mathbf{b}_1 \times \mathbf{c}_1) \times (\mathbf{b}_2 \times \mathbf{c}_2)$ is parallel to l.

A third plane, Π_3, has vector equation $\mathbf{r} = \mathbf{d} + \theta\mathbf{e} + \phi\mathbf{f}$. Show that the equation of Π_3 may be expressed in the form $\mathbf{r}.\mathbf{n} = \mathbf{d}.\mathbf{n}$, where $\mathbf{n} = \mathbf{e} \times \mathbf{f}$.

Given that $\mathbf{h}.\mathbf{n} \neq 0$, show that the point of intersection of Π_3 and l has position vector $\mathbf{a} + \dfrac{(\mathbf{d} - \mathbf{a}).\mathbf{n}}{\mathbf{h}.\mathbf{n}}\mathbf{h}$. (J89/I/11)

68. The points A, B, C and D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively, where $\mathbf{a} = -2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = -3\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}$, $\mathbf{d} = 12\mathbf{i} - 10\mathbf{j} + 11\mathbf{k}$.

The points P and Q are on the lines AB and CD respectively, and are such that PQ is perpendicular to both AB and CD.

(i) Find, in either order, the length of PQ and the position vectors of P and Q.

(ii) Show that the perpendicular distance of C from the plane through B, P and Q is $\frac{42}{17}\sqrt{17}$. (J89/I/12)

69. With respect to an origin O, the points A, B and C, which are not coplanar with O, have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The points L, M and N are on the line segments OA, AB and OC respectively, and are such that OL = LA, 2AM = MB and ON = 3NC. The line MN meets the plane LBC at the point Q.

(i) Write down, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, the position vectors of L, M and N.

(ii) Show that the position vector of any point on MN can be expressed in the form $\frac{2}{3}t\mathbf{a} + \frac{1}{3}t\mathbf{b} + \frac{3}{4}(1 - t)\mathbf{c}$.

(iii) Show that the position vector of any point in the plane LBC can be expressed in the form $\frac{1}{2}(1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.

(iv) Hence, or otherwise, find the numerical value of $\dfrac{NQ}{QM}$. (N89/I/11)

70. Prove that the shortest distance, p, between the non-parallel lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$ is given by $p = \frac{|(\mathbf{a} - \mathbf{c}).(\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|}$.

The lines l_1 and l_2, whose vector equations are $\mathbf{r} = h\mathbf{a} + \lambda(k_1\mathbf{b} + k_2\mathbf{d})$ and $\mathbf{r} = h\mathbf{c} + \mu(k_3\mathbf{b} + k_4\mathbf{d})$ respectively, where h, k_1, k_2, k_3 and k_4 are constants, are non-parallel. Show that $k_1k_4 \neq k_2k_3$.

Show also that the shortest distance between l_1 and l_2 is $p|h|$.

Find the shortest distance

(i) between the lines $\mathbf{r} = \begin{pmatrix} 3 \\ 7 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix}$.

(ii) between the lines $\mathbf{r} = \begin{pmatrix} 15 \\ 35 \\ 30 \end{pmatrix} + \lambda\begin{pmatrix} 1002 \\ -3005 \\ 4006 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 10 \\ 15 \\ 20 \end{pmatrix} + \mu\begin{pmatrix} 998 \\ -2995 \\ 3994 \end{pmatrix}$.

[You must show sufficient working to justify your answers.] (N89/I/12)

71. With respect to an origin O the position vectors of the three points A, B and C are **a**, **b** and **c**, respectively. The lines OA, OB and OC are not coplanar. The point D on the line segment BC is such that 2BD = DC, and the point L on the line segment AD is such that AL = 3LD. Find the position vector of L in terms of **a**, **b** and **c**.

The point E is the mid-point of OB, and P is the point on the line AE such that the lines OL and CP intersect. Find, in either order,

(i) the position vector of P in terms of **a** and **b**,

(ii) the position vector of the point of intersection of the lines OL and CP in terms of **a**, **b** and **c**. (J90/I/11)

72. The line l_1, which is parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$, passes through the point A whose position vector is $4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. The line l_2, which is parallel to the vector $2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, passes through the point B whose position vector is $\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q.

Find the shortest distance between the line through A and B and the line through the points with position vectors $3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $5\mathbf{i} + 2\mathbf{k}$. (J90/I/12)

73. The line l has equation $\mathbf{r} = 4\mathbf{i} - 8\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j})$ and the plane Π_1 has equation $\mathbf{r} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j}) + v(2\mathbf{i} - \mathbf{k})$, where $\lambda, \mu, v \in \mathbb{R}$. The points U and V, which vary independently, are on l and in Π_1, respectively. The point W, which is on the line segment UV, is such that UW = 3WV. Find an equation of the locus of W and given a geometrical description of this locus.

The points L and M have position vectors $-4\mathbf{i} + 13\mathbf{j} + 3\mathbf{k}$ and $15\mathbf{j} + 3\mathbf{k}$, respectively.

(i) Show that the points L and M are in Π_1.

(ii) Show that there is a plane, Π_2, which contains the points L and M and also the line l.

(iii) Find an equation of the intersection of Π_2, with the locus of W. (N90/I/11)

74. The position vectors of the points A, B, C, D are $6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, $\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$, $\mathbf{i} + 6\mathbf{j} + m\mathbf{k}$ respectively, where $m > 0$. It is given that the shortest distance between the line AB and the line CD is equal to 3.

(i) Show that the only possible value of m is 3.

(ii) Find the shortest distance of D from the line AC, giving your answer correct to three significant figures.

(iii) Find the acute angle between the planes ACD and BCD, giving your answer to the nearest $0.1°$. (N90/I/12)