MAY / JUNE 2013

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1. $f(r) = r!(r-1)$

$$f(r+1) - f(r) = (r+1)!r - r!(r-1)$$
$$= (r+1)r!r - r!(r-1)$$
$$= r!(r^2+r-r+1)$$
$$= r!(r^2+1)$$

$$\sum_{r=1}^{n} r!(r^2+1) = \sum_{r=1}^{n} f(r+1) - f(r)$$

$$\begin{aligned} &= f(n+1) - f(n) \\ &\quad + f(n) - f(n-1) \\ &\quad + f(n-1) - f(n-2) \\ &\quad \vdots \\ &\quad + f(4) - f(3) \\ &\quad + f(3) - f(2) \\ &\quad + f(2) - f(1) \end{aligned}$$

$$= f(n+1) - f(1)$$
$$= (n+1)!n - 0$$
$$= (n+1)!n$$

$$\sum_{r=n+1}^{2n} r!(r^2+1) = \sum_{r=1}^{2n} r!(r^2+1) - \sum_{r=1}^{n} r!(r^2+1)$$

$$= (2n+1)!2n - (n+1)!n$$

2. $x^4 - 4x^2 + 3x - 2 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$

$\alpha^2, \beta^2, \gamma^2, \delta^2$

Let $y = \alpha^2$

$\alpha = \pm\sqrt{y}$

α is a root

$\therefore \alpha^4 - 4\alpha^2 + 3\alpha - 2 = 0$

$(\pm\sqrt{y})^4 - 4(\pm\sqrt{y})^2 + 3(\pm\sqrt{y}) - 2 = 0$

$y^2 - 4y \pm 3\sqrt{y} - 2 = 0$

$\pm 3\sqrt{y} = -(y^2 - 4y - 2)$

$9y = y^4 - 4y^3 - 2y^2 - 4y^3 + 16y^2 + 8y - 2y^2 + 8y + 4$

$= y^4 - 8y^3 + 12y^2 + 16y + 4$

$y^4 - 8y^3 + 12y^2 + 7y + 4 = 0$

$\therefore$ The equation $y^4 - 8y^3 + 12y^2 + 7y + 4 = 0$ has roots $\alpha^2, \beta^2, \gamma^2, \delta^2$.

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 8$

$\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2 = 12$

$\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2 = -7$

$\alpha^2\beta^2\gamma^2\delta^2 = 4.$

$\therefore S_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 8$

Since $y^4 - 8y^3 + 12y^2 + 7y + 4 = 0$ has roots

$\alpha^2, \beta^2, \gamma^2, \delta$

$$(\alpha^2)^4 - 8(\alpha^2)^3 + 12(\alpha^2)^2 + 7\alpha^2 + 4 = 0$$

$$(\beta^2)^4 - 8(\beta^2)^3 + 12(\beta^2)^2 + 7\beta^2 + 4 = 0$$

$$(\gamma^2)^4 - 8(\gamma^2)^3 + 12(\gamma^2)^2 + 7\gamma^2 + 4 = 0$$

$$(\delta^2)^4 - 8(\delta^2)^3 + 12(\delta^2)^2 + 7\delta^2 + 4 = 0$$

$$\alpha^8 - 8\alpha^6 + 12\alpha^4 + 7\alpha^2 + 4 = 0 \quad - (1)$$

$$\beta^8 - 8\beta^6 + 12\beta^4 + 7\beta^2 + 4 = 0 \quad - (2)$$

$$\gamma^8 - 8\gamma^6 + 12\gamma^4 + 7\gamma^2 + 4 = 0 \quad - (3)$$

$$\delta^8 - 8\delta^6 + 12\delta^4 + 7\delta^2 + 4 = 0 \quad - (4)$$

$(1) + (2) + (3) + (4)$:

$$\alpha^8 + \beta^8 + \gamma^8 + \delta^8 - 8(\alpha^6 + \beta^6 + \gamma^6 + \delta^6)$$
$$+ 12(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) + 7(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) + 16 = 0$$

$$S_8 - 8S_6 + 12S_4 + 7S_2 + 16 = 0$$

$$S_8 - 8S_6 + 12S_4 + 7(8) + 16 = 0$$

$$S_8 - 8S_6 + 12S_4 + 72 = 0$$

$$\therefore S_8 = 8S_6 - 12S_4 - 72$$

3. $$\frac{d^n}{dx^n}(e^x \sin x) = (\sqrt{2})^n e^x \sin\left(x + \frac{n\pi}{4}\right), \quad n \geq 1.$$

when $n=1$: $$\frac{d^1}{dx^1}(e^x \sin x) = \frac{d}{dx}(e^x \sin x)$$
$$= e^x \sin x + e^x \cos x$$
$$= \sqrt{2}e^x\left(\frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}\right)$$
$$= \sqrt{2}e^x\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right)$$
$$= \sqrt{2}e^x \sin\left(x + \frac{\pi}{4}\right)$$
$$= (\sqrt{2})^1 e^x \sin\left(x + \frac{1\pi}{4}\right)$$

Assume the statement is true when $n=k$.

$n=k$: $$\frac{d^k}{dx^k}(e^x \sin x) = (\sqrt{2})^k e^x \sin\left(x + \frac{k\pi}{4}\right)$$

when $n=k+1$: $$\frac{d^{k+1}}{dx^{k+1}}(e^x \sin x) = (\sqrt{2})^{k+1} e^x \sin\left(x + \frac{(k+1)\pi}{4}\right)$$

(what needs to be proved)

$$\frac{d^{k+1}}{dx^{k+1}}(e^x \sin x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(e^x \sin x)\right)$$
$$= \frac{d}{dx}\left((\sqrt{2})^k e^x \sin\left(x + \frac{k\pi}{4}\right)\right)$$

$$= (\sqrt{2})^k \left(e^x \sin\left(x + \frac{k\pi}{4}\right) + e^x \cos\left(x + \frac{k\pi}{4}\right)\right)$$

$$= (\sqrt{2})^k \sqrt{2} e^x \left(\frac{\sin\left(x + \frac{k\pi}{4}\right)}{\sqrt{2}} + \frac{\cos\left(x + \frac{k\pi}{4}\right)}{\sqrt{2}}\right)$$

$$= (\sqrt{2})^{k+1} e^x \left(\sin\left(x + \frac{k\pi}{4}\right) \cos\frac{\pi}{4} + \cos\left(x + \frac{k\pi}{4}\right) \sin\frac{\pi}{4}\right)$$

$$= (\sqrt{2})^{k+1} e^x \sin\left(x + (k+1)\frac{\pi}{4}\right)$$

$$\therefore \frac{d^n}{dx^n}(e^x \sin x) = (\sqrt{2})^n e^x \sin\left(x + \frac{n\pi}{4}\right)$$

for every integer $n \geq 1$.

4. $y^3 - 3x^2y + 2 = 0 \quad A(1,-2)$

$$\frac{d}{dx}(y^3 - 3x^2y + 2) = 0$$

$$3y^2\frac{dy}{dx} - 3x^2\frac{dy}{dx} - 6xy = 0$$

$$(y^2 - x^2)\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} = \frac{2xy}{y^2 - x^2}$$

At $A(1,-2)$: $\frac{dy}{dx} = -\frac{4}{3}$

$$\frac{d}{dx}\left[(y^2-x^2)\frac{dy}{dx}\right] = \frac{d}{dx}(2xy)$$

$$(y^2-x^2)\frac{d^2y}{dx^2} + \left(2y\frac{dy}{dx} - 2x\right)\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y$$

At $A(1,-2)$, $\frac{dy}{dx} = -\frac{4}{3}$:

$$3\frac{d^2y}{dx^2} + \left[-4\left(-\frac{4}{3}\right) - 2\right]\left(-\frac{4}{3}\right) = -\frac{8}{3} - 4$$

$$3\frac{d^2y}{dx^2} - \frac{40}{9} = -\frac{20}{3}$$

$$3\frac{d^2y}{dx^2} = -\frac{20}{9} \qquad \therefore \frac{d^2y}{dx^2} = -\frac{20}{27}$$

5. $\int_0^1 xe^{-x^2}\,dx$

$s = -x^2$ $\quad$ $x = 0 : s = 0$

$ds = -2x\,dx$ $\quad$ $x = 1 : s = -1$

$$\therefore \int_0^1 xe^{-x^2}\,dx = \int_0^{-1} \frac{e^s}{-2}\,ds$$

$$= \int_{-1}^{0} \frac{e^s}{2}\,ds = \left[\frac{e^s}{2}\right]_{-1}^{0} = \frac{1}{2} - \frac{1}{2e}$$

$$I_n = \int_0^1 x^n e^{-x^2}\,dx$$

$$I_{2n+1} = \int_0^1 x^{2n+1} e^{-x^2}\,dx$$

$$= \int_0^1 x^{2n}\, xe^{-x^2}\,dx$$

$u = x^{2n}$ $\quad$ $dv = xe^{-x^2}\,dx$

$du = 2n\,x^{2n-1}\,dx$ $\quad$ $v = \frac{e^{-x^2}}{-2}$

$$= \left[\frac{x^{2n}e^{-x^2}}{-2}\right]_0^1 - \int_0^1 2n x^{2n-1} \frac{e^{-x^2}}{-2}\,dx$$

$$= \frac{e^{-1}}{-2} - 0 + n\int_0^1 x^{2n-1} e^{-x^2}\,dx$$

$$= nI_{2n-1} - \frac{1}{2e}, \quad n \geq 1$$

$n=3: I_7 = 3I_5 - \frac{1}{2e}$

$n=2: I_5 = 2I_3 - \frac{1}{2e}$

$n=1: I_3 = I_1 - \frac{1}{2e}$

Since $I_1 = \int_0^1 x^n e^{-x^2} dx = \frac{1}{2} - \frac{1}{2e}$,

$$I_3 = \frac{1}{2} - \frac{1}{2e} - \frac{1}{2e}$$
$$= \frac{1}{2} - \frac{1}{e}$$

$$I_5 = 2\left(\frac{1}{2} - \frac{1}{e}\right) - \frac{1}{2e}$$
$$= 1 - \frac{5}{2e}$$

$$I_7 = 3\left(1 - \frac{5}{2e}\right) - \frac{1}{2e}$$
$$= 3 - \frac{8}{e}$$

6. $T: R^4 \to R^4$

$$M = \begin{pmatrix} -2 & 5 & 3 & -1 \\ 0 & 1 & -4 & -2 \\ 6 & -14 & -13 & 1 \\ \alpha & \alpha & -2\alpha & -11\alpha \end{pmatrix} \quad \text{If} \left(\begin{array}{cccc|c} -2 & 5 & 3 & -1 & 0 \\ 0 & 1 & -4 & -2 & 0 \\ 6 & -14 & -13 & 1 & 0 \\ \alpha & \alpha & -2\alpha & -11\alpha & 0 \end{array}\right)$$

$$2 \times r_4 \left(\begin{array}{cccc|c} -2 & 5 & 3 & -1 & 0 \\ 0 & 1 & -4 & -2 & 0 \\ 6 & -14 & -13 & 1 & 0 \\ 2\alpha & 2\alpha & -4\alpha & -22\alpha & 0 \end{array}\right)$$

$$\begin{array}{l} 3r_1 + r_3 \\ \alpha r_1 + r_4 \end{array} \left(\begin{array}{cccc|c} -2 & 5 & 3 & -1 & 0 \\ 0 & 1 & -4 & -2 & 0 \\ 0 & 1 & -4 & -2 & 0 \\ 0 & 7\alpha & -\alpha & -23\alpha & 0 \end{array}\right)$$

$$\begin{array}{l} -r_2 + r_3 \\ -7\alpha r_2 + r_4 \end{array} \left(\begin{array}{cccc|c} -2 & 5 & 3 & -1 & 0 \\ 0 & 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 27\alpha & -9\alpha & 0 \end{array}\right)$$

$$r_3 \leftrightarrow r_4 \left(\begin{array}{cccc|c} -2 & 5 & 3 & -1 & 0 \\ 0 & 1 & -4 & -2 & 0 \\ 0 & 0 & 27\alpha & -9\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{If} \begin{pmatrix} -2 & 5 & 3 & -1 \\ 0 & 1 & -4 & -2 \\ 6 & -14 & -13 & 1 \\ \alpha & \alpha & -2\alpha & -11\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} -2 & 5 & 3 & -1 & 0 \\ 0 & 1 & -4 & -2 & 0 \\ 0 & 0 & 27\alpha & -9\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

$$\text{If } \alpha = 0: \left(\begin{array}{cccc|c} -2 & 5 & 3 & -1 & 0 \\ 0 & 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 2s, s \in \mathbb{R}$ and $w = 2t, t \in \mathbb{R}$

$\therefore y = 8s + 4t$

$-2x + 5y + 3z - w = 0$

$-2x + 40s + 20t + 6s - 2t = 0$

$2x = 46s + 18t$

$x = 23s + 9t$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 23s + 9t \\ 8s + 4t \\ 2s \\ 2t \end{pmatrix} = s\begin{pmatrix} 23 \\ 8 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 9 \\ 4 \\ 0 \\ 2 \end{pmatrix}$$

If $\alpha \neq 0$: Let $w = 3s$, $s \in \mathbb{R}$

$\therefore z = s$

$y = 4s + 6s$
$= 10s$

$-2x + 5y + 3z - w = 0$

$2x = 50s + 3s - 3s$

$x = 25s$

$$\xrightarrow[9\alpha]{r_3} \left(\begin{array}{cccc|c} -2 & 5 & 3 & -1 & 0 \\ 0 & 1 & -4 & -2 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 25s \\ 10s \\ s \\ 3s \end{pmatrix} = s\begin{pmatrix} 25 \\ 10 \\ 1 \\ 3 \end{pmatrix}$$

$\therefore$ A basis for K_1 is $\left\{\begin{pmatrix} 25 \\ 10 \\ 1 \\ 3 \end{pmatrix}\right\}$ and a basis for K_2 is $\left\{\begin{pmatrix} 23 \\ 8 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 4 \\ 0 \\ 2 \end{pmatrix}\right\}$.

7. $\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 4y = 6e^{-x}$

If $\lambda x e^{-x}$ is a particular integral,

$$\frac{d^2}{dx^2}(\lambda x e^{-x}) + \frac{5d}{dx}(\lambda x e^{-x}) + 4\lambda x e^{-x} = 6e^{-x}$$

$$\frac{d}{dx}(\lambda e^{-x} - \lambda x e^{-x}) + 5(\lambda e^{-x} - \lambda x e^{-x}) + 4\lambda x e^{-x} = 6e^{-x}$$

$$-\lambda e^{-x} - \lambda e^{-x} + \lambda x e^{-x} + 5\lambda e^{-x} - 5\lambda x e^{-x} + 4\lambda x e^{-x} = 6e^{-x}$$

$$3\lambda e^{-x} = 6e^{-x}$$

$$\therefore \lambda = 2$$

$m^2 + 5m + 4 = 0$

$(m+1)(m+4) = 0$

$m = -1, -4$

$\therefore$ The complementary function, y_c, is

$y_c = Ae^{-x} + Be^{-4x}$

The particular integral, y_p, is given by

$y_p = \lambda x e^{-x}$

$\quad = 2xe^{-x}$

$\therefore y = y_c + y_p$

$\quad = Ae^{-x} + Be^{-4x} + 2xe^{-x}$

$\frac{dy}{dx} = -Ae^{-x} - 4Be^{-4x} + 2e^{-x} - 2xe^{-x}$

$y=2 \quad \frac{dy}{dx}=3 \quad x=0$

$x=0 \quad y=2: \quad 2=A+B$

$x=0 \quad \frac{dy}{dx}=3: \quad 3=-A-4B+2$

$$A+B=2$$
$$-A-4B=1$$
$$-3B=3$$
$$B=-1$$
$$A=3$$

$\therefore y = 3e^{-x} - e^{-4x} + 2xe^{-x}$

8. $C: y = t^3 \quad x = \frac{3t^2}{2}, \quad 0 \le t \le 2.$

$$\frac{dy}{dt} = 3t^2 \quad \frac{dx}{dt} = 3t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9t^2 + 9t^4$$

$$= 9t^2(t^2+1)$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3t\sqrt{t^2+1}$$

The arc length of C is

$$\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^2 3t\sqrt{t^2+1}\, dt$$

$$s = t^2+1 \qquad t=0 \quad s=1$$

$$ds = 2t\,dt \qquad t=2 \quad s=5$$

$$= \int_1^5 \frac{3\sqrt{s}}{2}\, ds$$

$$= \left[\frac{3s^{\frac{3}{2}}}{3}\right]_1^5$$

$$= 5\sqrt{5} - 1$$

If the centroid of the region bounded by C, the x-axis and the line $x=6$ has coordinates $(\bar{x}, \bar{y})$, $x = 6 = \frac{3t^2}{2}$

$$\therefore t = 2$$

The area, A, of the region bounded by C, the x-axis and the line $x=6$ is

$$A = \int_0^6 y\,dx = \int_0^2 y\frac{dx}{dt}\,dt = \int_0^2 t^3 3t\,dt$$

$$= \int_0^2 3t^4\,dt = \left[\frac{3t^5}{5}\right]_0^2 = \frac{96}{5} - 0 = \frac{96}{5}$$

$$A\bar{x} = \int_0^6 xy\,dx = \int_0^2 xy\frac{dx}{dt}\,dt = \int_0^2 \frac{3t^2}{2}t^3 3t\,dt$$

$$= \int_0^2 \frac{9t^6}{2}\,dt = \left[\frac{9t^7}{14}\right]_0^2 = \frac{9}{14}(2^7 - 0) = \frac{9}{7}(64) = \frac{576}{7}$$

$$\therefore \bar{x} = \frac{30}{7}$$

$$A\bar{y} = \int_0^6 \frac{y^2}{2}\,dx = \int_0^2 \frac{y^2}{2}\frac{dx}{dt}\,dt = \int_0^2 \frac{t^6}{2} 3t\,dt$$

$$= \int_0^2 \frac{3t^7}{2}\,dt = \left[\frac{3t^8}{16}\right]_0^2 = \frac{3}{16}(2^8 - 0) = 48$$

$$\therefore \bar{y} = \frac{5}{2}$$

$\therefore$ The centroid of the region bounded by C, the x-axis and the line $x=6$ is $\left(\frac{30}{7}, \frac{5}{2}\right)$.

9. $A\underset{\sim}{e} = \lambda \underset{\sim}{e}$ $\quad A = [a_{ij}]_{n\times n}$ $\quad M = [m_{ij}]_{n\times n}$

$B = MAM^{-1}$

$$
\begin{aligned}
B(M\underset{\sim}{e}) &= MAM^{-1}(M\underset{\sim}{e}) \\
&= MAM^{-1}M\underset{\sim}{e} \\
&= MAI\underset{\sim}{e} \\
&= MA\underset{\sim}{e} \\
&= M(A\underset{\sim}{e}) \\
&= M(\lambda \underset{\sim}{e}) \\
&= \lambda(M\underset{\sim}{e})
\end{aligned}
$$

$\therefore$ B has eigenvector $M\underset{\sim}{e}$ with corresponding eigenvalue λ.

$A = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad A\underset{\sim}{e} = \lambda\underset{\sim}{e}$, $\underset{\sim}{e} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\therefore$ The eigenvalues of A are $-1, 1, 2$.

$\lambda = -1$: $\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} -a + 2b + c \\ b + 4c \\ 2c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix}$$

$c = 0 \quad b = 0 \quad 0a = 0$

Let $a = s$, $s \in R$

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\lambda=1: \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} -a+2b+c \\ b+4c \\ 2c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$c=0 \quad 0b=0$ $\qquad 2a=2b$

Let $b=s, s\in\mathbb{R}$ $\qquad a=b$

$=s$

$$\lambda=2: \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} -a+2b+c \\ b+4c \\ 2c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

$0c=0$ $\qquad b=4s \quad 3a=9s$

Let $c=s, s\in\mathbb{R}$ $\qquad a=3s$

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3s \\ 4s \\ s \end{pmatrix} = s\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

$\therefore$ A has eigenvalues $-1, 1, 2$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$$

$\therefore$ B has eigenvalues $-1, 1, 2$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$

10. $2\sin P\cos Q = \sin(P+Q) + \sin(P-Q)$

If $P=\theta$, $Q=\theta-\frac{\pi}{4}$:

$$2\sin\theta\cos\left(\theta-\frac{\pi}{4}\right) = \sin\left(\theta+\theta-\frac{\pi}{4}\right)+\sin\left(\theta-\left(\theta-\frac{\pi}{4}\right)\right)$$

$$= \sin\left(2\theta-\frac{\pi}{4}\right)+\sin\frac{\pi}{4}$$

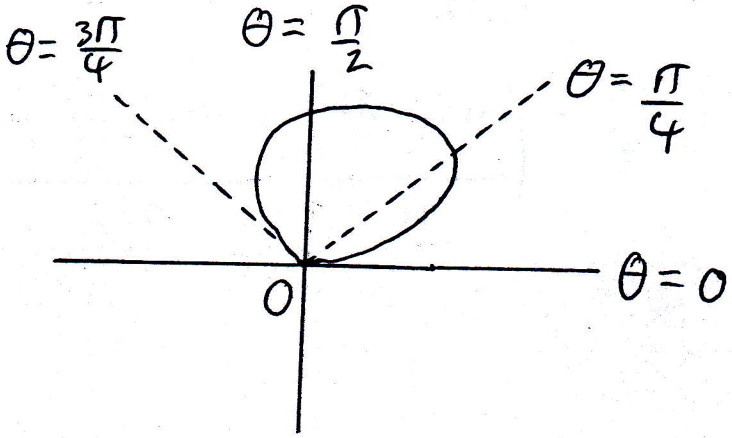
$$= \sin 2\theta\cos\frac{\pi}{4}-\cos 2\theta\sin\frac{\pi}{4}+\frac{1}{\sqrt{2}}$$

$$= \sin 2\theta\sin\frac{3\pi}{4}+\cos 2\theta\cos\frac{3\pi}{4}+\frac{1}{\sqrt{2}}$$

$$= \cos\left(2\theta-\frac{3\pi}{4}\right)+\frac{1}{\sqrt{2}}.$$

$r = 2\sin\theta\cos\left(\theta-\frac{\pi}{4}\right)$, $0\le\theta\le\frac{3\pi}{4}$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
r	0	$\sqrt{2}$	$\sqrt{2}$	0



If $2\theta-\frac{3\pi}{4}=0$

$\theta=\frac{3\pi}{8}$.

The area enclosed by the curve is

$$\int_0^{\frac{3\pi}{4}} \frac{r^2}{2}\, d\theta$$

$$= \int_0^{\frac{3\pi}{4}} \frac{1}{2}\left(\cos\left(2\theta - \frac{3\pi}{4}\right) + \frac{1}{\sqrt{2}}\right)^2 d\theta$$

$$= \int_0^{\frac{3\pi}{4}} \frac{1}{2}\left(\cos^2\left(2\theta - \frac{3\pi}{4}\right) + \sqrt{2}\cos\left(2\theta - \frac{3\pi}{4}\right) + \frac{1}{2}\right) d\theta$$

$$= \int_0^{\frac{3\pi}{4}} \frac{1}{2}\left(\frac{\cos\left(4\theta - \frac{3\pi}{2}\right) + 1}{2} + \sqrt{2}\cos\left(2\theta - \frac{3\pi}{4}\right) + \frac{1}{2}\right) d\theta$$

$$= \int_0^{\frac{3\pi}{4}} \frac{\cos\left(4\theta - \frac{3\pi}{2}\right)}{4} + \frac{\cos\left(2\theta - \frac{3\pi}{4}\right)}{\sqrt{2}} + \frac{1}{2}\, d\theta$$

$$= \left[\frac{\sin\left(4\theta - \frac{3\pi}{2}\right)}{16} + \frac{\sin\left(2\theta - \frac{3\pi}{4}\right)}{2\sqrt{2}} + \frac{\theta}{2}\right]_0^{\frac{3\pi}{4}}$$

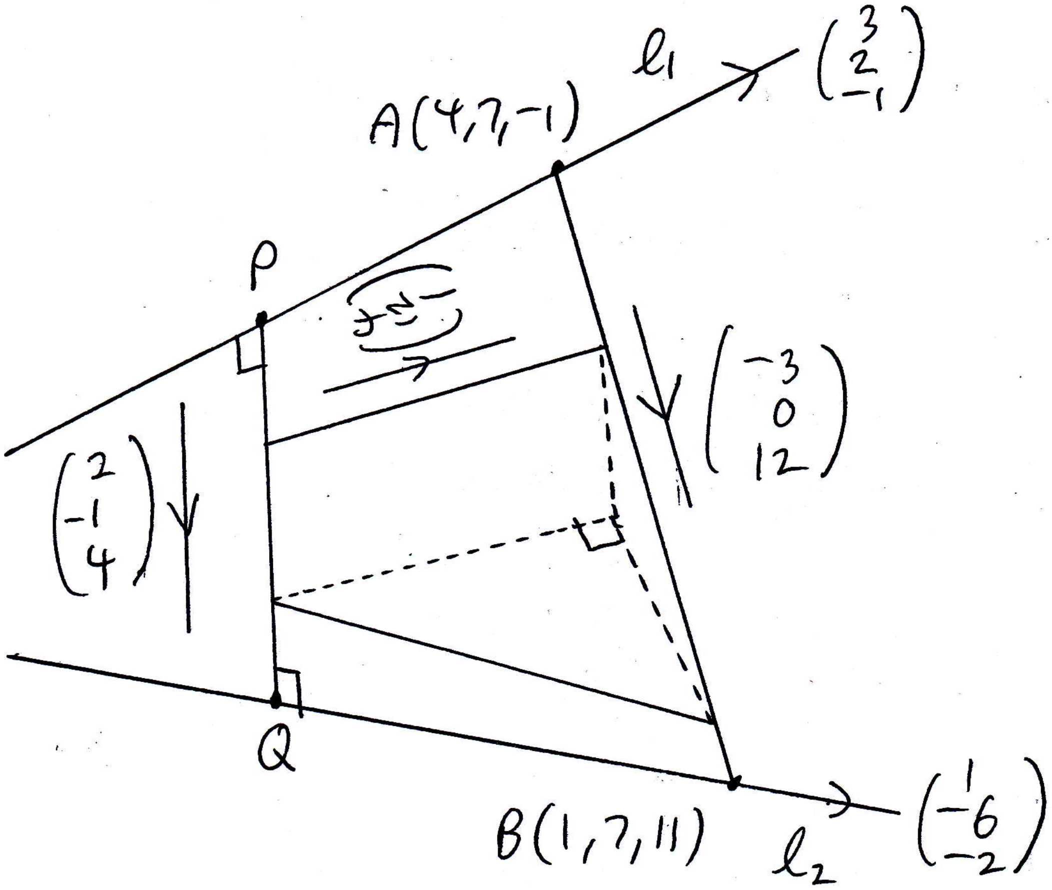
$$= \frac{\sin\frac{3\pi}{2}}{16} + \frac{\sin\frac{3\pi}{4}}{2\sqrt{2}} + \frac{3\pi}{8} - \left(\frac{\sin\left(-\frac{3\pi}{2}\right)}{16} + \frac{\sin\left(-\frac{3\pi}{4}\right)}{2\sqrt{2}} + 0\right)$$

$$= -\frac{1}{16} + \frac{1}{4} + \frac{3\pi}{8} - \frac{1}{16} + \frac{1}{4}$$

$$= \frac{3\pi}{8} + \frac{3}{8}$$

$$= \frac{3}{8}(\pi + 1)$$

11. EITHER



since PQ is perpendicular to both l_1 and l_2, $\overrightarrow{PQ}$ has direction $\begin{pmatrix}3\\2\\-1\end{pmatrix}\times\begin{pmatrix}1\\-6\\-2\end{pmatrix}=\begin{pmatrix}-10\\5\\-20\end{pmatrix}$

$$=-5\begin{pmatrix}2\\-1\\4\end{pmatrix}$$

$l_1 : \underset{\sim}{r} = \begin{pmatrix}4\\7\\-1\end{pmatrix} + s\begin{pmatrix}3\\2\\-1\end{pmatrix}$ $\quad l_2 : \underset{\sim}{r} = \begin{pmatrix}1\\7\\11\end{pmatrix} + t\begin{pmatrix}1\\-6\\-2\end{pmatrix}$

since P is on l_1 and Q is on l_2,
P has the form $(3s+4, 2s+7, -s-1)$ and
Q has the form $(t+1, -6t+7, -2t+11)$

$$\overrightarrow{PQ} = \begin{pmatrix} t+1 \\ 7-6t \\ 11-2t \end{pmatrix} - \begin{pmatrix} 3s+4 \\ 2s+7 \\ -s-1 \end{pmatrix} = \begin{pmatrix} t-3s-3 \\ -6t-2s \\ -2t+s+12 \end{pmatrix}$$

Since $\overrightarrow{PQ}$ is parallel to $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$\overrightarrow{PQ} = c\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix},\ c \in R,\ c \neq 0.$$

$$\begin{pmatrix} t-3s-3 \\ -6t-2s \\ -2t+s+12 \end{pmatrix} = c\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2c \\ -c \\ 4c \end{pmatrix}$$

$$\left.\begin{array}{l} t-3s-3 = 2c \\ -6t-2s = -c \\ -2t+s+12 = 4c \end{array}\right\}$$

$$\left.\begin{array}{l} t-3s-2c = 3 \\ -6t-2s+c = 0 \\ -2t+s-4c = -12 \end{array}\right\}$$

$$\begin{array}{l} 6\times①+②: \\ 2\times①+③: \end{array} \left.\begin{array}{r} t-3s-2c = 3 \\ -20s-11c = 18 \\ -5s-8c = -6 \end{array}\right\}$$

$$\left.\begin{array}{r} t-3s-2c = 3 \\ -5s-8c = -6 \\ -20s-11c = 18 \end{array}\right\}$$

$$-4\times②+③: \left.\begin{array}{r} t-3s-2c = 3 \\ -5s-8c = -6 \\ 21c = 42 \end{array}\right\}$$

$c=2 \quad s=-2 \quad t=1$

$$\therefore \overrightarrow{OP} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$$

$\vec{OP} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ $\vec{OQ} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$

The line through P and Q has equation

$$\underset{\sim}{r} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + s\left[\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + s\begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix}$$

The line through A and B has equation

$$\underset{\sim}{r} = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix} + t\left[\begin{pmatrix} 1 \\ 7 \\ 11 \end{pmatrix} - \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix}\right] = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix} \times \begin{pmatrix} -3 \\ 0 \\ 12 \end{pmatrix} = \begin{pmatrix} -24 \\ -72 \\ -6 \end{pmatrix} = 6\begin{pmatrix} -4 \\ -12 \\ -1 \end{pmatrix} = -6\begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}$$

$s=1$: $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$ $t=1$: $\begin{pmatrix} 1 \\ 7 \\ 11 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix}$$

$$\left|\begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}\right| = \left|\begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix}\right| \left|\begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}\right| \cos\theta$$

$$|4-72-2| = \sqrt{161}\left|\begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix}\right| \cos\theta$$

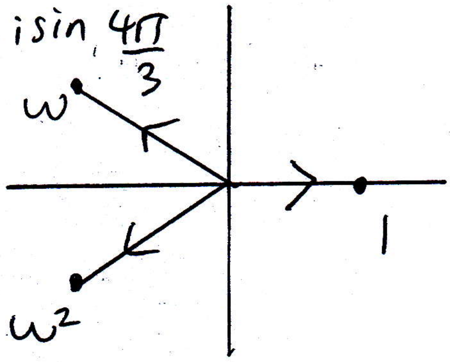
$$70 = \sqrt{161}\left|\begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix}\right| \cos\theta$$

$$\left|\begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix}\right| \cos\theta = \frac{70}{\sqrt{161}}$$

$\therefore$ The shortest distance between the line through A and B and the line through P and Q is $\frac{70}{\sqrt{161}}$.

11. OR

$$1^{\frac{1}{3}} = (\cos 0 + i\sin 0)^{\frac{1}{3}}$$
$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{3}}, \quad k \in \mathbb{Z}$$
$$= \cos \frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}, \quad k = 0,1,2$$
$$= 1, \ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$



$$w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$w^2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2$$
$$= \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$
$$w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad w^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\begin{vmatrix} 1 & 3w & 2w^2 \\ 3w^2 & 2 & w \\ 2w & w^2 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & w \\ w^2 & 3 \end{vmatrix} - 3w\begin{vmatrix} 3w^2 & w \\ 2w & 3 \end{vmatrix}$$
$$+ 2w^2\begin{vmatrix} 3w^2 & 2 \\ 2w & w^2 \end{vmatrix}$$
$$= 6 - w^3 - 3w(9w^2 - 2w^2)$$
$$+ 2w^2(3w^4 - 4w)$$
$$= 6 - w^3 - 21w^3 + 6w^6 - 8w^3$$
$$= 6 - 30w^3 + 6(w^3)^2$$
$$= 6 - 30 + 6$$
$$= -18$$

$$z = 4\sqrt{3}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) - 4\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

$$= 4\sqrt{3}\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right) - 4\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$= -2\sqrt{3} - 6i - 2\sqrt{3} + 2i$$

$$= -4\sqrt{3} - 4i$$

$$= -4(\sqrt{3} + i)$$

$$= 8\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$= 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$= r(\cos\theta + i\sin\theta), \quad r = 8 \quad \theta = \frac{7\pi}{6}.$$

$$z^{\frac{1}{3}} = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)^{\frac{1}{3}}$$

$$= \left[8\left(\cos\left(\frac{7\pi}{6} + 2k\pi\right) + i\sin\left(\frac{7\pi}{6} + 2k\pi\right)\right)\right]^{\frac{1}{3}}, k \in \mathbb{Z}$$

$$= 2\left(\cos\left(\frac{7\pi}{18} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{7\pi}{18} + \frac{2k\pi}{3}\right)\right), k = 0, 1, 2$$

$$= 2\left(\cos\frac{7\pi}{18} + i\sin\frac{7\pi}{18}\right), 2\left(\cos\frac{19\pi}{18} + i\sin\frac{19\pi}{18}\right),$$

$$2\left(\cos\frac{31\pi}{18} + i\sin\frac{31\pi}{18}\right).$$