1. $\sum_{n=1}^{50} n^3 + 3n = \frac{n}{2}(n+1)^2 + \frac{3n(n+1)}{2}$

$= \frac{50}{2}(51^2) + 3\frac{(50)}{2}51$

$= 1629450$

2. $z^5 = 32\left(\cos\left(-\frac{\pi}{2} + 2n\pi\right) + i\sin\left(-\frac{\pi}{2} + 2n\pi\right)\right)$

$z = 2\left(\cos\left(\frac{-\pi}{10} + \frac{2n\pi}{5}\right) + i\sin\left(-\frac{\pi}{2} + \frac{2n\pi}{5}\right)\right)$

when $n = 0$, $z_1 = 2\left(\cos\left(\frac{-\pi}{10}\right) + i\sin\left(\frac{-\pi}{10}\right)\right)$

$z_2 = 2\left(\cos\left(\frac{3\pi}{10}\right) + i\sin\left(\frac{3\pi}{10}\right)\right)$

$z_3 = 2\left(\cos\left(\frac{7\pi}{10}\right) + i\sin\left(\frac{7\pi}{10}\right)\right)$

$z_4 = 2\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$

$z_5 = 2\left(\cos\left(\frac{-9\pi}{10}\right) + i\sin\left(\frac{-9\pi}{10}\right)\right.$

3. $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Compare Imaginary part

$5\cos^4\theta\,(i\sin\theta) + 10\cos^2\theta\,(i\sin\theta)^3 + (i\sin\theta)^5 = i\sin 5\theta$

$5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta = \sin 5\theta$

$5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta = \sin 5\theta$

$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = \sin 5\theta$

$(\times 2\sqrt{3})\quad 32\sqrt{3}\sin^5\theta - 40\sqrt{3}s^3 + 10\sqrt{3}s = 2\sqrt{3}\sin 5\theta$

Let $\sin\theta = x$

$\therefore\ 2\sqrt{3}\sin 5\theta = 3$

$\sin 5\theta = \frac{\sqrt{3}}{2}$

$5\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\theta = \frac{\pi}{15}, \frac{2\pi}{15}, \frac{4\pi}{15}, \frac{5\pi}{15}, \frac{7\pi}{15}$

$\therefore\ x = \sin\frac{k\pi}{15}$ where $k = 1, 2, 4, 5, 7$

4. $y = xe^x$

i) $\frac{dy}{dx} = xe^x + e^x = e^x(x+1)$

$\frac{d^2y}{dx^2} = e^x(x+1) + e^x = e^x(x+2)$

$\frac{d^3y}{dx^3} = e^x(x+2) + e^x(1) = e^x(x+3)$

ii) Proposition: $\frac{d^ny}{dx^n} = e^x(x+n)$

Basis case: $\frac{dy}{dx} = e^x(x+1)$ RHS: $e^x(x+1)$

$\therefore$ True for $n=1$

Inductive: Assume true when $n=k$

$\frac{d^ky}{dx^k} = e^x(x+k)$

when $n = k+1$,

$\frac{d^{k+1}y}{dx^{k+1}} = e^x(x+k) + e^x(1)$

$= e^x(x+k+1)$

$\therefore$ True for $n=k+1$

Conclusion: $\frac{d^ny}{dx^n} = e^x(x+n)$ is true for $n \in \mathbb{Z}^+$

5. i) $\det A = -a$

$$\operatorname{adj} A = \begin{pmatrix} -a & 0 & 0 \\ -a & a & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{a} & 0 & \frac{1}{a} \end{pmatrix}$$

ii) $ACB = I$

$$\therefore C = A^{-1}B^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{a} & 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & -1 \\ 1 & -2 & -\frac{3}{2} \\ -\frac{1}{a} & \frac{1}{a} & \frac{3}{2a} \end{pmatrix}$$

6. $x = a(\theta - \sin\theta)$ $\qquad$ $y = a(1 - \cos\theta)$

$\frac{dx}{d\theta} = a(1 - \cos\theta)$ $\qquad$ $\frac{dy}{d\theta} = a\sin\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\left[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right]$$

$$= 2a^2(1 - \cos\theta)$$

$$= 2a^2\left(2\sin^2\frac{\theta}{2}\right)$$

$$= 4a^2\sin^2\frac{\theta}{2}$$

Arc length $AB = \int_0^{\pi} \sqrt{4a^2\sin^2\frac{\theta}{2}}\, d\theta$

$$= 2a\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta$$

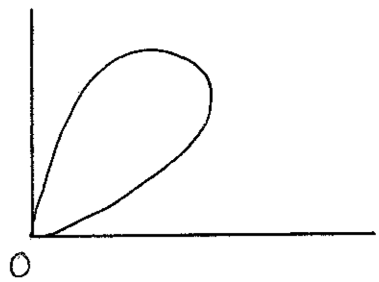
$$= 2a\left[\frac{-\cos\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi}$$

$$= -4a\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= 4a \text{ unit}$$

7. i)

θ	0	10°	20°	30°	45°	60°	70°	80°	90°
r	0	0.168	0.302	0.375	0.354	0.217	0.100	0.0297	0



ii) $r = a\sin\theta\cos^2\theta$

$$\frac{dr}{d\theta} = a\left[\sin\theta\left[-2\cos\theta\sin\theta\right] + \cos^2\theta(\cos\theta)\right]$$

$$= a\left[-2\cos\theta\sin^2\theta + \cos^3\theta\right]$$

$$= a\cos\theta\left[-2\sin^2\theta + \cos^2\theta\right]$$

$$= a\cos\theta\left[3\cos^2\theta - 2\right]$$

$0 = a\cos\theta$ $\qquad$ $\cos\theta = \pm\sqrt{\frac{2}{3}}$

$\theta = 90^\circ$

$\theta = 35.3^\circ$

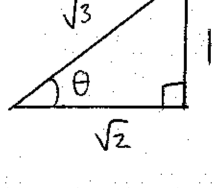
θ	90^{0-}	90^0	90^{0+}
$\frac{dr}{d\theta}$	$-$	0	+
	\	—	/

min

θ	35.3^-	35.3^0	35.3^+
$\frac{dr}{d\theta}$	+	0	$-$
	/	—	\

max

Max value of $r = a\left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{3}\right)$



$$= \frac{2a}{3\sqrt{3}}$$

$$= \frac{2\sqrt{3}a}{9}$$

8. Planes intersect on a line where

$$x + 2y - 3z + 4 = \beta(2x + y - 4z - 3)$$

$$x + 2y - 3z + 4 + \lambda(2x + y - 4z - 3) = 0 \quad \text{where} \quad \beta = -\lambda$$

i) substitute $(0, 0, a)$

$$-3a + 4 + \lambda(-4a - 3) = 0$$

$$\lambda = \frac{4 - 3a}{4a + 3}$$

$$x + 2y - 3z + 4 + \frac{4 - 3a}{4a + 3}(2x + y - 4z - 3) = 0$$

$$(4a + 3)(x + 2y - 3z + 4) + (4 - 3a)(2x + y - 4z - 3) = 0$$

$$(4a + 3 + 8 - 6a)x + (8a + 6 + 4 - 3a)y$$

$$+ (16a + 12 - 12 + 9a)z = 0$$

$$(-2a + 11)x + (5a + 10)y + (-25z) + 25a = 0$$

ii)

$$\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -2a + 11 \\ 5a + 10 \\ -25 \end{pmatrix} = 0$$

$$-4a + 22 + 5a + 10 + 100 = 0$$

$$a = -132$$

9. $y = \dfrac{3x-5}{(x+1)(x-1)}$

i) $x = \pm 1$, $y = 0$

ii) $\dfrac{dy}{dx} = \dfrac{-(3x^2 - 10x + 3)}{(x^2-1)^2}$

$0 = (3x-1)(x-3)$

$\therefore x = \frac{1}{3}, 3$

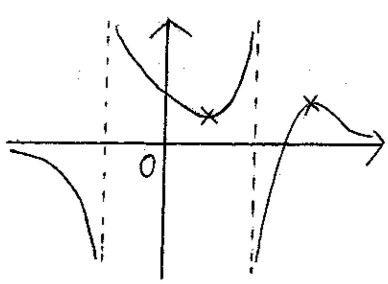
x	$\frac{1}{3}^-$	$\frac{1}{3}$	$\frac{1}{3}^+$
$\frac{dy}{dx}$	$-$	0	$+$

min pt $(\frac{1}{3}, 4\frac{1}{2})$

x	3^-	3	3^+
$\frac{dy}{dx}$	$+$	0	$-$

max $(3, \frac{1}{2})$

iii)



$(0, 5)$

10. i)

$$\left(\begin{array}{cccc|c} 3 & 0 & -2 & 3 & 0 \\ -4 & 2 & 0 & -8 & 0 \\ 1 & -2 & 2 & 5 & 0 \\ 3 & -3 & 2 & 9 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & -1 & 0 \\ 0 & -3 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Dim (range space) = 2

ii) From Dim theorem, Dim (null space) = 4−3 = 1

Let $x_4 = \lambda$

$x_3 = \beta$

$\therefore x_2 = 2\lambda + \frac{4}{3}\beta$

$\therefore x_1 = -\lambda + \frac{2}{3}\beta$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix}$$

Basis of null space $= \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} \right\}$

iii) One particular solution is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

general solution, $\underset{\sim}{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix}$

11. $$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 3-\lambda & -2 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)[(3-\lambda)(2-\lambda)-2] + (-1)[2(2-\lambda)+2] - 1[-2+\lambda-3] = 0$$

$$(\lambda-1)(\lambda-5)(\lambda-3) = 0$$

$$\lambda = 1, 5, 3$$

when $\lambda = 1$,

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 2 & 2 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \therefore \begin{array}{l} x_1 = 0 \\ x_2 = \lambda \\ x_3 = \lambda \end{array} \quad \underset{\sim}{e} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

when $\lambda = 5$,

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & -2 & -2 & 0 \\ 1 & -1 & -3 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \therefore \begin{array}{l} x_1 = \lambda \\ x_2 = \lambda \\ x_3 = 0 \end{array} \quad \underset{\sim}{e} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

when $\lambda = 3$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \therefore \begin{array}{l} x_1 = \lambda \\ x_2 = 0 \\ x_3 = \lambda \end{array} \quad \underset{\sim}{e} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \ \& \ D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \text{for } A = PD_1P^{-1}$$

$\therefore$ For $(A - kI)^n$, eigenvalues are $(\lambda - k)^n$ & eigenvectors are the same.

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \ \& \ D = \begin{pmatrix} (1-k)^n & 0 & 0 \\ 0 & (5-k)^n & 0 \\ 0 & 0 & (-3-k)^n \end{pmatrix}$$

12. EITHER

$x = \sqrt{t}$

i) $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$ $\therefore$ $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times 2\sqrt{t} = 2\sqrt{t}\dfrac{dy}{dt}$

ii) $\dfrac{d^2y}{dx^2} = \dfrac{d\left(\frac{dy}{dx}\right)}{dt} \times \dfrac{dt}{dx} = \left(2\sqrt{t}\dfrac{d^2y}{dt^2} + \dfrac{1}{\sqrt{t}}\dfrac{dy}{dt}\right)\cdot 2\sqrt{t}$

$= 4t\dfrac{d^2y}{dt^2} + \dfrac{2dy}{dt}$

substitute into DE

$$\frac{2dy}{dt} + 4t\frac{d^2y}{dt^2} - \frac{1}{\sqrt{t}}\left(2\sqrt{t}\frac{dy}{dt}\right) + 4t(9y+6) = 0$$

$$\frac{d^2y}{dt^2} + 9y + 6 = 0$$

$$\frac{d^2y}{dt^2} + 9y = -6$$

AE: $m^2 + 9 = 0$

$m = \pm 3i$

$\therefore y = e^{0t}(A\cos 3t + B\sin 3t)$

$= A\cos 3t + B\sin 3t$

CF: Let $y = \lambda$ $y' = 0$ $y'' = 0$

$\therefore \lambda = -\dfrac{2}{3}$ $\therefore y = -\dfrac{2}{3}$

$\therefore$ GS: $y = A\cos 3t + B\sin 3t - \dfrac{2}{3}$

$= A\cos 3x^2 + B\sin 3x^2 - \dfrac{2}{3}$

where A and B are constant

12. OR

$$u = x^n \qquad \frac{dv}{dx} = e^{-\alpha x}$$

$$u' = nx^{n-1} \qquad v = \frac{e^{-\alpha x}}{-\alpha}$$

$$I_n = \int_0^1 x^n e^{-\alpha x}\, dx$$

$$= \left[\frac{x^n e^{-\alpha x}}{-\alpha}\right]_0^1 + \frac{n}{\alpha}\int_0^1 e^{-\alpha x} x^{n-1}\, dx$$

$$I_n = \frac{e^{-\alpha}}{-\alpha} + \frac{n}{\alpha} I_{n-1}$$

$$\alpha I_n = n I_{n-1} - e^{-\alpha} \qquad n \geqslant 1$$

$$\bar{x} = \frac{\int_0^1 xy\, dx}{\int_0^1 y\, dx} \qquad \bar{y} = \frac{\frac{1}{2}\int_0^1 y^2\, dx}{\int_0^1 y\, dx}$$

$$\int_0^1 y\, dx = \int_0^1 x e^{-x}\, dx \quad \text{where } \alpha = 1$$

$$\therefore I_n = I_0 - e^{-1}$$

$$= \int_0^1 e^{-x}\, dx - \frac{1}{e}$$

$$= \left[-e^{-x}\right]_0^1 - \frac{1}{e}$$

$$= -\frac{1}{e} + 1 - \frac{1}{e}$$

$$= 1 - \frac{2}{e}$$

$$= \frac{e-2}{e}$$

$$\int_0^1 xy\, dx = \int_0^1 x^2 e^{-x}\, dx \qquad \alpha = 1$$

$$I_2 = 2I_1 - e^{-1}$$

$$I_2 = 2\left[\frac{e^{-1}}{-1} + \frac{1}{1} I_0\right] - e^{-1}$$

$$I_2 = -\frac{2}{e} + 2\int_0^1 e^{-x}\, dx - \frac{1}{e}$$

$$= -\frac{3}{e} - 2\left[e^{-x}\right]_0^1$$

$$= -\frac{3}{e} - 2(e^{-1} - 1)$$

$$= -\frac{5}{e} + 2$$

$$= \frac{-5 + 2e}{e}$$

$$\bar{x} = \frac{2e - 5}{e - 2}$$

$$\frac{1}{2}\int_0^1 y^2\, dx = \frac{1}{2}\int_0^1 x^2 e^{-2x}\, dx \qquad \alpha = 2$$

$$= \frac{1}{2}\left[\frac{e^{-2}}{-2} + \frac{2}{2} I_1\right]$$

$$= \frac{-1}{4e^2} + \frac{1}{2}\left[\frac{e^{-2}}{-2} + \frac{1}{2} I_0\right]$$

$$= \frac{-1}{2e^2} + \frac{1}{4}\left[\int_0^1 e^{-2x}\, dx\right]$$

$$= \frac{-1}{2e^2} - \frac{1}{8}\left[e^{-2x}\right]_0^1$$

$$= \frac{-1}{2e^2} - \frac{1}{8}\left[\frac{1}{e^2} - 1\right]$$

$$= \frac{-5}{8e^2} + \frac{1}{8}$$

$$= \frac{-5 + e^2}{8e^2}$$

$$\therefore \bar{y} = \frac{e^2 - 5}{8(e)(e-2)}$$

$$= \frac{e^2 - 5}{8e^2 - 16e}$$