1. $A = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -1 & -2 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 0 & -3-\lambda \end{vmatrix}$$

$$-(-1) \begin{vmatrix} 0 & 1 \\ 0 & -3-\lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(-3-\lambda) + 1 \cdot 0 - 2 \cdot 0$$

$$= (1-\lambda)(\lambda-2)(\lambda+3)$$

$$|A - \lambda I| = 0$$

$$(1-\lambda)(\lambda-2)(\lambda+3) = 0$$

$$\lambda = 1, 2, -3$$

when $\lambda = 1$.

$$\begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 + r_2} \left(\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{-4r_2 + r_3} \left(\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-z = 0$

$-y - 2z = 0$

$y = 0$

Let $x = s$, $s \in R$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

when $\lambda = 2$: $\begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right)$$

$$\xrightarrow{5r_2 + r_3} \left(\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$z = 0$$

$$-x - y - 2z = 0$$

$$-x - y = 0$$

Let $y = s,\ s \in R$

$\therefore\ x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ s \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

When $\lambda = -3$: $\begin{pmatrix} 4 & -1 & -2 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 4 & -1 & -2 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$5y + z = 0$

Let $z = 20s,\ s \in R$

$\therefore y = -4s$

$4x - y - 2z = 0$

$4x - (-4s) - 2(20s) = 0$

$4x + 4s - 40s = 0$

$4x = 36s$

$x = 9s$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9s \\ -4s \\ 20s \end{pmatrix}$

$= s \begin{pmatrix} 9 \\ -4 \\ 20 \end{pmatrix}$

The eigenvalues of A are 1, 2 and -3 and the corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 9 \\ -4 \\ 20 \end{pmatrix}$$

2. $I_n = \int_0^1 x^n e^{-x^3}\,dx$

$$\frac{d}{dx}(x^{n+1}e^{-x^3}) = x^{n+1}\frac{d}{dx}(e^{-x^3}) + e^{-x^3}\frac{d}{dx}(x^{n+1})$$

$$= x^{n+1}(-3x^2e^{-x^3}) + e^{-x^3}(n+1)x^n$$

$$= -3x^{n+3}e^{-x^3} + (n+1)x^n e^{-x^3}$$

$$\therefore \left[x^{n+1}e^{-x^3}\right]_0^1 = \int_0^1 -3x^{n+3}e^{-x^3}\,dx$$

$$+ \int_0^1 (n+1)x^n e^{-x^3}\,dx$$

$$1^{n+1}e^{-1} - 0^{n+1}e^0 = -3\int_0^1 x^{n+3}e^{-x^3}\,dx$$

$$+ (n+1)\int_0^1 x^n e^{-x^3}\,dx$$

$$e^{-1} = -3I_{n+3} + (n+1)I_n$$

$$3I_{n+3} = (n+1)I_n - e^{-1}$$

when $n = 3$: $3I_6 = 4I_3 - e^{-1}$

when $n = 0$: $3I_3 = I_0 - e^{-1}$

$$I_3 = \frac{I_0}{3} - \frac{e^{-1}}{3}$$

$$\therefore 3I_6 = 4\left(\frac{I_0}{3} - \frac{e^{-1}}{3}\right) - e^{-1}$$

$$= \frac{4I_0}{3} - \frac{4e^{-1}}{3} - e^{-1}$$

$$= \frac{4I_0}{3} - \frac{7e^{-1}}{3}$$

$$\therefore \quad I_6 = \frac{4I_0}{9} - \frac{7e^{-1}}{9}$$

3. If $v_n = n(n+1)(n+2)\cdots(n+m)$,

$$v_{n+1} = (n+1)(n+1+1)(n+1+1)\cdots(n+1+m)$$

$$= (n+1)(n+2)(n+3)\cdots(n+1+m)$$

$$\therefore v_{n+1} - v_n = (n+1)(n+2)(n+3)\cdots(n+m+1)$$

$$- n(n+1)(n+2)\cdots(n+m)$$

$$= (n+1)(n+2)\cdots(n+m)(n+1+m-n)$$

$$= (m+1)(n+1)(n+2)\cdots(n+m)$$

If $u_n = (n+1)(n+2)\cdots(n+m)$,

since $v_{n+1} - v_n = (m+1)(n+1)(n+2)\cdots(n+m)$

$$u_n = \frac{v_{n+1} - v_n}{m+1}$$

$$\therefore \sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \frac{v_{n+1} - v_n}{m+1}$$

$$= \frac{1}{m+1} \sum_{n=1}^{N} v_{n+1} - v_n$$

$$= \frac{1}{m+1}\Big(v_{N+1} - v_N$$

$$+ v_N - v_{N-1}$$

$$+ v_{N-1} - v_{N-2}$$

$$\vdots$$

$$+ v_4 - v_3$$

$$+ v_3 - v_2$$

$$+ v_2 - v_1 \Big)$$

$$= \frac{1}{m+1}\left(v_{N+1} - v_1 \right)$$

$$= \frac{1}{m+1}\Big(N(N+1)(N+2)\cdots(N+m)$$

$$- 1\cdot 2\cdot 3\cdots m(m+1) \Big)$$

$$= \frac{(N+1)(N+2)\cdots(N+1+m) - (m+1)!}{m+1}$$

$$= \frac{(N+1)(N+2)\cdots(N+1+m)}{m+1} - \frac{(m+1)!}{(m+1)}$$

$$= \frac{(N+1)(N+2)\cdots(N+1+m)}{m+1} - m!$$

4. Let $f(n) = 10^{3n} + 13^{n+1}$

when $n = 1$: $f(1) = 10^{3(1)} + 13^{1+1}$

$= 10^3 + 13^2$

$= 1000 + 169$

$= 1169$

$= 7(167)$

$\therefore 10^{3n} + 13^{n+1}$ is divisible by 7 when $n = 1$.

Assume that $10^{3n} + 13^{n+1}$ is divisible by 7 when $n = k$.

$n = k$: $f(k) = 10^{3k} + 13^{k+1}$

$7 \mid f(k)$

$\therefore f(k) = 7s, \; s \in \mathbb{Z}$

$\therefore 10^{3k} + 13^{k+1} = 7s$

when $n = k + 1$:

$f(k+1) = 10^{3(k+1)} + 13^{k+1+1}$

$= 10^{3k+3} + 13^{k+1}\,13$

$= 10^{3k}\,10^{3} + 13^{k+1}(6 + 7)$

$= 10^{3k}\,1000 + 13^{k+1}(6 + 7)$

$$= 10^{3k}(994 + 6) + 13^{k+1}(6 + 7)$$

$$= 10^{3k}994 + 10^{3k}6 + 13^{k+1}6 + 13^{k+1}7$$

$$= 10^{3k}994 + 13^{k+1}7 + 10^{3k}6 + 13^{k+1}6$$

$$= 10^{3k}(7 \cdot 142) + 13^{k+1}7 + 6(10^{3k} + 13^{k+1})$$

$$= 7(10^{3k}142 + 13^{k+1}) + 6(7s)$$

$$= 7(10^{3k}142 + 13^{k+1} + 6s)$$

Since s is an integer and k is an integer, $10^{3k}142 + 13^{k+1} + 6s$ is an integer.

$\therefore 7 \mid f(k+1)$.

Since $f(1)$ is divisible by 7 and $f(k+1)$ is divisible by 7 if $f(n)$ is divisible by 7, $f(n)$ is divisible by 7 for every positive integer n.

$\therefore 10^{3n} + 13^{n+1}$ is divisible by 7 for every positive integer n.

5. $$\left.\begin{aligned} 2x + 3y + 4z &= -5 \\ 4x + 5y - z &= 5a + 15 \\ 6x + 8y + az &= b - 2a + 21 \end{aligned}\right\}$$

$-2 \times (1) + (2)$

$-3 \times (1) + (3)$:

$$\left.\begin{aligned} 2x + 3y + 4z &= -5 \\ -y - 9z &= 5a + 25 \\ -y + (a-12)z &= b - 2a + 36 \end{aligned}\right\}$$

$-(2) + (3)$:

$$\left.\begin{aligned} 2x + 3y + 4z &= -5 \\ -y - 9z &= 5a + 25 \\ (a-3)z &= b - 7a + 11 \end{aligned}\right\}$$

$$(a-3)z = b - 7a + 11$$

If $a \neq 3$: $z = \dfrac{b - 7a + 11}{a - 3}$

$\therefore$ The system of equations has a unique solution if $a \neq 3$.

If $a = 3$: $cz = b - 10$

$\therefore$ when $a = 3$, if $b = 10$ the equations are consistent.

6. $x^3 + x + 1 = 0$

α, β, γ are the roots

$$\frac{4\alpha + 1}{\alpha + 1}, \quad \frac{4\beta + 1}{\beta + 1}, \quad \frac{4\gamma + 1}{\gamma + 1}$$

Let $y = \dfrac{4\alpha + 1}{\alpha + 1}$

$$\therefore y(\alpha + 1) = 4\alpha + 1$$

$$\alpha y + y = 4\alpha + 1$$

$$\alpha y - 4\alpha = 1 - y$$

$$(y - 4)\alpha = 1 - y$$

$$\alpha = \frac{1 - y}{y - 4}$$

α is a root

$$\therefore \alpha^3 + \alpha + 1 = 0$$

$$\left(\frac{1 - y}{y - 4}\right)^3 + \frac{1 - y}{y - 4} + 1 = 0$$

$$\frac{(1 - y)^3}{(y - 4)^3} + \frac{1 - y}{y - 4} + 1 = 0$$

$$(1-y)^3 + (1-y)(y-4)^2 + (y-4)^3 = 0$$

$$1 - 3y + 3y^2 - y^3 + (1-y)(y^2 - 8y + 16)$$

$$+ y^3 - 12y^2 + 48y - 64 = 0$$

$$1 - 3y + 3y^2 - y^3 + y^2 - 8y + 16$$

$$-y^3 + 8y^2 - 16y + y^3 - 12y^2 + 48y - 64 = 0$$

$$-y^3 + 21y - 47 = 0$$

$$y^3 - 21y + 47 = 0$$

$$y^3 + py + q = 0,$$

$$p = -21 \quad q = 47.$$

$\therefore$ The equation having roots

$$\frac{4\alpha + 1}{\alpha + 1}, \quad \frac{4\beta + 1}{\beta + 1}, \quad \frac{4\gamma + 1}{\gamma + 1} \quad \text{is}$$

$$y^3 - 21y + 47 = 0$$

$$\frac{4\alpha+1}{\alpha+1} + \frac{4\beta+1}{\beta+1} + \frac{4\gamma+1}{\gamma+1} = 0$$

$$\left(\frac{4\alpha+1}{\alpha+1}\right)\left(\frac{4\beta+1}{\beta+1}\right) + \left(\frac{4\alpha+1}{\alpha+1}\right)\left(\frac{4\gamma+1}{\gamma+1}\right)$$

$$+ \left(\frac{4\beta+1}{\beta+1}\right)\left(\frac{4\gamma+1}{\gamma+1}\right) = -21$$

$$\left(\frac{4\alpha+1}{\alpha+1}\right)\left(\frac{4\beta+1}{\beta+1}\right)\left(\frac{4\gamma+1}{\gamma+1}\right) = -47$$

Let $S_n = \left(\frac{4\alpha+1}{\alpha+1}\right)^n + \left(\frac{4\beta+1}{\beta+1}\right)^n + \left(\frac{4\gamma+1}{\gamma+1}\right)^n$

$$S_0 = \left(\frac{4\alpha+1}{\alpha+1}\right)^0 + \left(\frac{4\beta+1}{\beta+1}\right)^0 + \left(\frac{4\gamma+1}{\gamma+1}\right)^0$$

$$= 1 + 1 + 1$$

$$= 3$$

$$S_1 = \left(\frac{4\alpha+1}{\alpha+1}\right)^1 + \left(\frac{4\beta+1}{\beta+1}\right)^1 + \left(\frac{4\gamma+1}{\gamma+1}\right)^1$$

$$= \frac{4\alpha+1}{\alpha+1} + \frac{4\beta+1}{\beta+1} + \frac{4\gamma+1}{\gamma+1}$$

$= 0$

$$S_2 = \left(\frac{4\alpha + 1}{\alpha + 1}\right)^2 + \left(\frac{4\beta + 1}{\beta + 1}\right)^2 + \left(\frac{4\gamma + 1}{\gamma + 1}\right)^2$$

$$= \left(\frac{4\alpha + 1}{\alpha + 1} + \frac{4\beta + 1}{\beta + 1} + \frac{4\gamma + 1}{\gamma + 1}\right)^2$$

$$- 2\left[\left(\frac{4\alpha + 1}{\alpha + 1}\right)\left(\frac{4\beta + 1}{\beta + 1}\right) + \left(\frac{4\alpha + 1}{\alpha + 1}\right)\left(\frac{4\gamma + 1}{\gamma + 1}\right)\right.$$

$$\left. + \left(\frac{4\beta + 1}{\beta + 1}\right)\left(\frac{4\gamma + 1}{\gamma + 1}\right)\right]$$

$$= 0^2 - 2(-21)$$

$$= 0 + 42$$

$$= 42$$

$$S_3 = \left(\frac{4\alpha + 1}{\alpha + 1}\right)^3 + \left(\frac{4\beta + 1}{\beta + 1}\right)^3 + \left(\frac{4\gamma + 1}{\gamma + 1}\right)^3$$

Since $1 \cdot S_3 + 0 S_2 + p S_1 + q S_0 = 0$

$$S_3 + 0 + p \cdot 0 + 3q = 0$$

$$S_3 + 3 \cdot 47 = 0$$

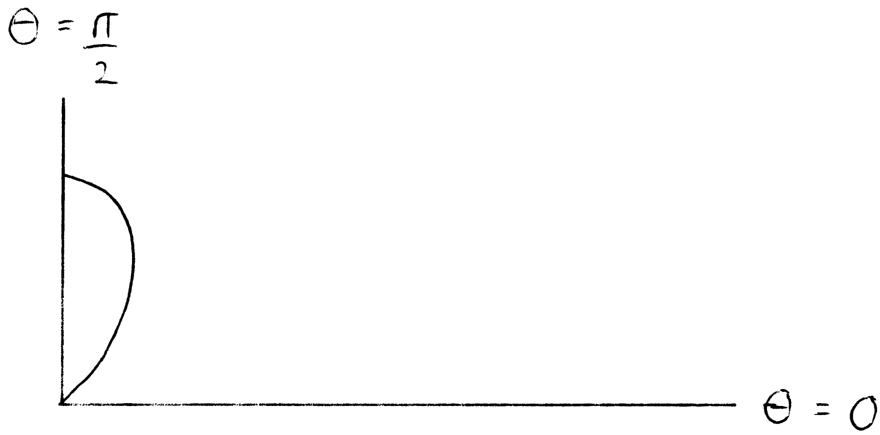
$$S_3 + 141 = 0$$

$$S_3 = -141$$

7. $C: r = 10\ln(1+\theta), \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	0	$10\ln\left(1+\frac{\pi}{6}\right)$	$10\ln\left(1+\frac{\pi}{4}\right)$

θ	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	$10\ln\left(1+\frac{\pi}{3}\right)$	$10\ln\left(1+\frac{\pi}{2}\right)$



The area of the sector bounded by the line $\theta = \frac{\pi}{2}$ and the arc of C from the origin to the point where $\theta = \frac{\pi}{2}$ is

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(10 \ln(1 + \theta))^2}{2} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{100}{2} (\ln(1 + \theta))^2 \, d\theta$$

$$w = \ln(1 + \theta)$$

$$dw = \frac{d\theta}{1 + \theta}$$

$$d\theta = (1 + \theta) \, dw$$

$$= (1 + e^w - 1) \, dw$$

$$= e^w \, dw$$

$$\theta = 0 \qquad w = 0$$

$$\theta = \frac{\pi}{2} \qquad w = \ln\left(1 + \frac{\pi}{2}\right)$$

$$= \int_0^{\ln(1+\frac{\pi}{2})} 50w^2e^w \, dw$$

$$= 50 \int_0^{\ln(1+\frac{\pi}{2})} w^2e^w \, dw$$

$$u = w^2 \qquad dv = e^w \, dw$$

$$du = 2w \, dw \qquad v = e^w$$

$$= 50\left(\left[w^2e^w \right]_0^{\ln(1+\frac{\pi}{2})} - \int_0^{\ln(1+\frac{\pi}{2})} 2we^w \, dw \right)$$

$$= 50\left(\left(\ln\left(1+\frac{\pi}{2}\right)\right)^2 e^{\ln(1+\frac{\pi}{2})} - 0 - 2\int_0^{\ln(1+\frac{\pi}{2})} we^w \, dw \right)$$

$$= 50\left(\ln\left(1+\frac{\pi}{2}\right)\right)^2 e^{\ln(1+\frac{\pi}{2})} - 100 \int_0^{\ln(1+\frac{\pi}{2})} we^w \, dw$$

$$u = w \qquad dv = e^{w}\,dw$$
$$du = dw \qquad v = e^{w}$$

$$= 50\left(\ln\left(1+\frac{\pi}{2}\right)\right)^{2} e^{\ln\left(1+\frac{\pi}{2}\right)}$$
$$-100\left(\left[we^{w}\right]_{0}^{\ln\left(1+\frac{\pi}{2}\right)} - \int_{0}^{\ln\left(1+\frac{\pi}{2}\right)} e^{w}\,dw\right)$$

$$= 50\left(\ln\left(1+\frac{\pi}{2}\right)\right)^{2} e^{\ln\left(1+\frac{\pi}{2}\right)}$$
$$-100\left(\ln\left(1+\frac{\pi}{2}\right) e^{\ln\left(1+\frac{\pi}{2}\right)} - 0\right.$$
$$\left.- \int_{0}^{\ln\left(1+\frac{\pi}{2}\right)} e^{w}\,dw\right)$$

$$= 50\left(\ln\left(1+\frac{\pi}{2}\right)\right)^{2} e^{\ln\left(1+\frac{\pi}{2}\right)}$$
$$-100\left(\ln\left(1+\frac{\pi}{2}\right) e^{\ln\left(1+\frac{\pi}{2}\right)}\right.$$
$$\left.- \left[e^{w}\right]_{0}^{\ln\left(1+\frac{\pi}{2}\right)}\right)$$

$$= 50\left(\ln\left(1+\frac{\pi}{2}\right)\right)^2 e^{\ln\left(1+\frac{\pi}{2}\right)}$$

$$-100\left(\ln\left(1+\frac{\pi}{2}\right) e^{\ln\left(1+\frac{\pi}{2}\right)}\right.$$

$$\left. - \left(e^{\ln\left(1+\frac{\pi}{2}\right)} - 1\right)\right)$$

$$= 50\left(\ln\left(1+\frac{\pi}{2}\right)\right)^2 e^{\ln\left(1+\frac{\pi}{2}\right)}$$

$$-100\left(\ln\left(1+\frac{\pi}{2}\right) e^{\ln\left(1+\frac{\pi}{2}\right)}\right.$$

$$\left. +100 e^{\ln\left(1+\frac{\pi}{2}\right)} - 100\right)$$

$$= 50\left(\ln\left(1+\frac{\pi}{2}\right)\right)^2 e^{\ln\left(1+\frac{\pi}{2}\right)}$$

$$-100 \ln\left(1+\frac{\pi}{2}\right) e^{\ln\left(1+\frac{\pi}{2}\right)}$$

$$+100 e^{\ln\left(1+\frac{\pi}{2}\right)} - 100$$

$$= \left(50\left(\ln\left(1+\frac{\pi}{2}\right)\right)^2 - 100 \ln\left(1+\frac{\pi}{2}\right)\right.$$

$$\left. + 100\right) e^{\ln\left(1+\frac{\pi}{2}\right)} - 100$$

$$= 50\left(\ln\left(1+\frac{\pi}{2}\right)^2 - 2\ln\left(1+\frac{\pi}{2}\right) + 2\right)e^{\ln\left(1+\frac{\pi}{2}\right)}$$

$$-100$$

$$= 50\left(b^2 - 2b + 2\right)e^{b} - 100, \text{ where } b = \ln\left(1+\frac{\pi}{2}\right)$$

8. $$2y^3 \frac{d^2 y}{dx^2} + 12y^3 \frac{dy}{dx} + 6y^2\left(\frac{dy}{dx}\right)^2 + 17y^4 = 13e^{-4x}$$

$$v = y^4$$

$$\frac{dv}{dx} = \frac{d}{dx}(y^4)$$

$$= \frac{dy}{dx}\frac{d}{dy}(y^4)$$

$$= 4y^3 \frac{dy}{dx}$$

$$\frac{d^2 v}{dx^2} = \frac{d}{dx}\left(4y^3 \frac{dy}{dx}\right)$$

$$= 4y^3 \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(4y^3)$$

$$= 4y^3 \frac{d^2 y}{dx^2} + \frac{dy}{dx}\left(\frac{dy}{dx}\right)\frac{d}{dy}(4y^3)$$

$$= 4y^3 \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 (12y^2)$$

$$= 4y^3 \frac{d^2 y}{dx^2} + 12y^2\left(\frac{dy}{dx}\right)^2$$

$$\therefore \frac{d^2v}{dx^2} + \frac{6dv}{dx} + 34v$$

$$= 4y^3\frac{d^2y}{dx^2} + 12y^2\left(\frac{dy}{dx}\right)^2 + 6\left(4y^3\frac{dy}{dx}\right)$$

$$+ 34y^4$$

$$= 4y^3\frac{d^2y}{dx^2} + 12y^2\left(\frac{dy}{dx}\right)^2 + 24y^3\frac{dy}{dx} + 34y^4$$

$$= 2\left(2y^3\frac{d^2y}{dx^2} + 6y^2\left(\frac{dy}{dx}\right)^2 + 12y^3\frac{dy}{dx} + 17y^4\right)$$

$$= 2(13e^{-4x})$$

$$= 26e^{-4x}.$$

$$\frac{d^2v}{dx^2} + \frac{6dv}{dx} + 34v = 0$$

$$m^2 + 6m + 34 = 0$$

$$(m+3)^2 + 25 = 0$$

$$(m+3)^2 = -25$$

$$m + 3 = \pm 5i$$

$m = -3 \pm 5i$

$v_c = e^{-3x}(A\cos 5x + B\sin 5x)$

Let $v_p = Ce^{-4x}$

$$\frac{dv_p}{dx} = -4Ce^{-4x}$$

$$\frac{d^2v_p}{dx^2} = 16Ce^{-4x}$$

$$\frac{d^2v_p}{dx^2} + \frac{6dv_p}{dx} + 34v_p = 16Ce^{-4x}$$

$$+6(-4Ce^{-4x})$$

$$+34Ce^{-4x}$$

$$= (16-24+34)Ce^{-4x}$$

$$= 26Ce^{-4x}$$

$$= 26e^{-4x}$$

$$26C = 26$$

$$C = 1$$

$$v_p = e^{-4x}$$

$$v = v_c + v_p$$

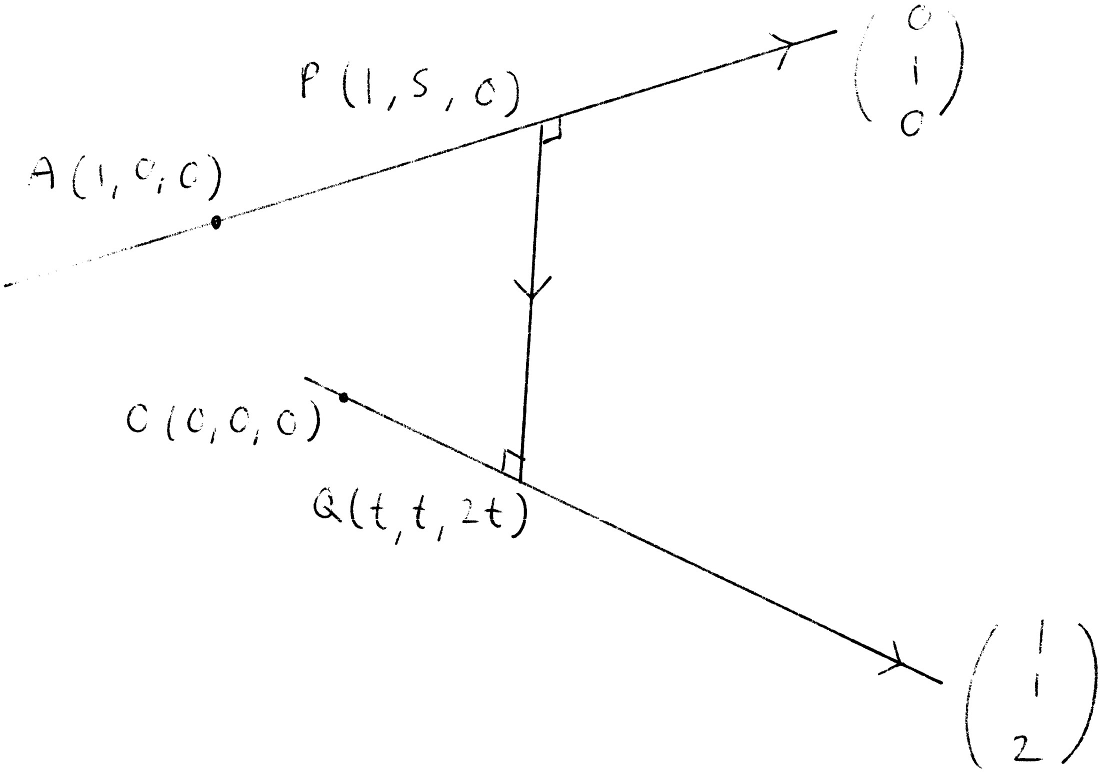
$$= e^{-3x}(A\cos 5x + B\sin 5x) + e^{-4x}$$

Since $v = y^4$,

$$y^4 = e^{-3x}(A\cos 5x + B\sin 5x) + e^{-4x}$$

$$y = \left(e^{-3x}(A\cos 5x + B\sin 5x) + e^{-4x}\right)^{\frac{1}{4}}$$

9



$\overrightarrow{OA} = \underset{\sim}{i} \qquad \overrightarrow{OB} = \underset{\sim}{i} + \underset{\sim}{j} \qquad \overrightarrow{OC} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Since the common perpendicular of the $\overrightarrow{AB}$ and $\overrightarrow{OC}$ is perpendicular to both $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ it is parallel to

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\1\\2\end{pmatrix}.$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2\underset{\sim}{i} - \underset{\sim}{k}$$

Also, if P and Q are the points on the lines AB and OC such that PQ is the common perpendicular of the lines AB and OC, P and Q have the form

$P(0, s, 0)$ and $Q(t, t, 2t)$.

Since PQ is the common perpendicular of the lines AB and OC and $\begin{pmatrix}2\\0\\-1\end{pmatrix}$

is perpendicular to the directions of both AB and OC,

$$\overrightarrow{PQ} \parallel \begin{pmatrix}2\\0\\-1\end{pmatrix}$$

$$\overrightarrow{PQ} = \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \quad \lambda \in R$$

$$\begin{pmatrix} t \\ t \\ 2t \end{pmatrix} - \begin{pmatrix} 1 \\ s \\ 0 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ 0 \\ -\lambda \end{pmatrix}$$

$$\begin{pmatrix} t-1 \\ t-s \\ 2t \end{pmatrix} = \begin{pmatrix} 2\lambda \\ 0 \\ -\lambda \end{pmatrix}$$

$$\left.\begin{aligned} t - 1 &= 2\lambda \\ t - s &= 0 \\ 2t &= -\lambda \end{aligned}\right\}$$

$$\left.\begin{aligned} t &= 2\lambda + 1 \\ t - s &= 0 \\ 2t &= -\lambda \end{aligned}\right\}$$

$$\begin{aligned} &-① + ② : \\ &-2 \times ① + ③ : \end{aligned} \qquad \left.\begin{aligned} t \quad &= 2\lambda + 1 \\ -s &= -2\lambda - 1 \\ 0 &= -5\lambda - 2 \end{aligned}\right\}$$

$$-5\lambda - 2 = 0$$

$$5\lambda + 2 = 0$$

$$\lambda = \frac{-2}{5}$$

$$t = 2\lambda + 1$$

$$= \frac{1}{5}$$

$$s = 2\lambda + 1$$

$$= \frac{1}{5}$$

$\therefore$ $P\left(1, \frac{1}{5}, 0\right)$ $Q\left(\frac{1}{5}, \frac{1}{5}, \frac{2}{5}\right)$

$\therefore$ A vector equation for the common perpendicular of the lines AB and OC is $\underset{\sim}{r} = \underset{\sim}{i} + \frac{\underset{\sim}{j}}{5} + \mu(2\underset{\sim}{i} - \underset{\sim}{k})$

The shortest distance between the lines AB and OC is $|\overrightarrow{PQ}|$.

$$\therefore |\overrightarrow{PQ}| = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{pmatrix} - \begin{pmatrix} 1 \\ \frac{1}{5} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{5} \\ 0 \\ \frac{2}{5} \end{pmatrix}$$

$$= \sqrt{\left(-\frac{4}{5}\right)^2 + 0^2 + \left(\frac{2}{5}\right)^2}$$

$$= \sqrt{\frac{16}{25} + 0 + \frac{4}{25}}$$

$$= \sqrt{\frac{20}{25}}$$

$$= \sqrt{\frac{4}{5}}$$

$$= \frac{\sqrt{4}}{\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{\sqrt{5}\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{5}$$

The normal of the plan containing AB and PQ is parallel to $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -\frac{4}{5} \\ 0 \\ \frac{2}{5} \end{pmatrix}$, since

AB and PQ have directions $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and

$$\begin{pmatrix} -\frac{4}{5} \\ 0 \\ \frac{2}{5} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -\frac{4}{5} \\ 0 \\ \frac{2}{5} \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{2}{5} \end{vmatrix}$$

$$= \frac{2}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{k}$$

$\therefore$ Since $A(1, 0, 0)$ is a point on the plane and $\frac{2}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{k}$ is the direction of the normal of the plane, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a point on the plane,

$$\underset{\sim}{r} \cdot \begin{pmatrix} \frac{2}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix}$$

$$\frac{2x}{5} + 0 + \frac{4z}{5} = \frac{2}{5} + 0 + 0$$

$$\frac{2x}{5} + \frac{4z}{5} = \frac{2}{5}$$

$$x + 2z = 1$$

$\therefore$ The equation of the plane containing AB and the common perpendicular of the lines AB and OC is

$x + 2z = 1.$

10. $C: y = x^2 + \lambda \sin(x + y)$, $A\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

A is a point on C,

$$\frac{\pi}{4} = \left(\frac{\pi}{4}\right)^2 + \lambda \sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{16} + \lambda \sin \frac{\pi}{2}$$

$$= \frac{\pi^2}{16} + \lambda$$

$$\lambda = \frac{\pi}{4} - \frac{\pi^2}{16}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(\lambda \sin(x + y))$$

$$= 2x + \lambda \cos(x + y) \frac{d}{dx}(x + y)$$

$$= 2x + \lambda \cos(x + y)\left(1 + \frac{dy}{dx}\right)$$

$$= 2x + \lambda\left(1 + \frac{dy}{dx}\right) \cos(x + y)$$

$$= 2x + \lambda\cos(x + y) + \lambda\cos(x + y)\frac{dy}{dx}$$

$$\frac{dy}{dx} - \lambda\cos(x + y)\frac{dy}{dx} = 2x + \lambda\cos(x + y)$$

$$(1 - \lambda\cos(x + y))\frac{dy}{dx} = 2x + \lambda\cos(x + y)$$

$$\frac{dy}{dx} = \frac{2x + \lambda\cos(x + y)}{1 - \lambda\cos(x + y)}$$

If C has a tangent which is parallel to the y-axis,

$$1 - \lambda\cos(x + y) = 0$$

$$\lambda = \frac{\pi}{4} - \frac{\pi^2}{16}$$

$$= \frac{4\pi - \pi^2}{16}$$

$$= \frac{\pi(4 - \pi)}{16}$$

$$\frac{1}{\lambda} = \frac{16}{\pi(4 - \pi)}$$

Since $\frac{1}{\lambda} > 1$,

$$1 - \lambda\cos(x+y) \neq 0$$

$\therefore$ C has no tangent which is parallel to the y-axis.

At $A\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$:

$$\frac{dy}{dx} = \frac{2\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right)}{1 - \lambda\cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right)}$$

$$= \frac{\frac{\pi}{2} + \cos\frac{\pi}{2}}{1 - \lambda\cos\frac{\pi}{2}}$$

$$= \frac{\frac{\pi}{2} + 0}{1 - 0}$$

$$= \frac{\pi}{2}.$$

$$(1 - \lambda\cos(x + y))\frac{dy}{dx} = 2x + \lambda\cos(x + y)$$

$$\frac{d}{dx}\left[(1 - \lambda\cos(x + y))\frac{dy}{dx}\right] = \frac{d}{dx}\left[2x + \lambda\cos(x + y)\right]$$

$$(1 - \lambda\cos(x + y))\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(1 - \lambda\cos(x + y))$$

$$= \frac{d}{dx}(2x) + \lambda\frac{d}{dx}(\cos(x + y))$$

$$(1 - \lambda\cos(x + y))\frac{d^2y}{dx^2} + \frac{dy}{dx}(-\lambda)\frac{d}{dx}(\cos(x + y)$$

$$= 2 - \lambda\sin(x + y)\frac{d}{dx}(x + y)$$

$$(1 - \lambda\cos(x + y))\frac{d^2y}{dx^2} - \lambda\frac{dy}{dx}(-\sin(x + y)\frac{d}{dx}(x + y)$$

$$= 2 - \lambda\sin(x + y)(1 + \frac{dy}{dx})$$

$$(1 - \lambda\cos(x + y))\frac{d^2y}{dx^2} + \lambda\sin(x + y)\left(1 + \frac{dy}{dx}\right)\frac{dy}{dx}$$

$$= 2 - \lambda\sin(x + y)\left(1 + \frac{dy}{dx}\right)$$

At $A\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$: $\frac{dy}{dx} = \frac{\pi}{2}$

$$\left(1 - \lambda\cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right)\right)\frac{d^2y}{dx^2} + \lambda\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right)\left(1 + \frac{dy}{dx}\right)\frac{dy}{dx}$$

$$= 2 - \lambda\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right)\left(1 + \frac{dy}{dx}\right)$$

$$\left(1 - \lambda\cos\frac{\pi}{2}\right)\frac{d^2y}{dx^2} + \lambda\sin\frac{\pi}{2}\left(1 + \frac{\pi}{2}\right)\frac{\pi}{2}$$

$$= 2 - \lambda\sin\frac{\pi}{2}\left(1 + \frac{\pi}{2}\right)$$

$$(1 - 0)\frac{d^2y}{dx^2} + \lambda\cdot 1\left(1 + \frac{\pi}{2}\right)\frac{\pi}{2} = 2 - \lambda\cdot 1\left(1 + \frac{\pi}{2}\right)$$

$$\frac{d^2y}{dx^2} + \frac{\lambda\pi}{2}\left(1 + \frac{\pi}{2}\right) = 2 - \lambda\left(1 + \frac{\pi}{2}\right)$$

$$= 2 - \lambda - \frac{\lambda\pi}{2}$$

$$= 2 - \lambda\left(1 + \frac{\pi}{2}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{\lambda\pi}{2}\left(1 + \frac{\pi}{2}\right) + 2 - \lambda\left(1 + \frac{\pi}{2}\right)$$

$$= -\frac{\lambda\pi}{2} - \frac{\lambda\pi^2}{4} + 2 - \lambda - \frac{\lambda\pi}{2}$$

$$= 2 - \lambda\pi - \frac{\lambda\pi^2}{4} - \lambda$$

$$= 2 - \lambda - \lambda\pi - \frac{\lambda\pi^2}{4}$$

$$= 2 - \lambda\left(1 + \pi + \frac{\pi^2}{4}\right)$$

$$= 2 - \left(\frac{\pi}{4} - \frac{\pi^2}{16}\right)\left(1 + \pi + \frac{\pi^2}{4}\right)$$

$$= 2 - \left(\frac{4\pi - \pi^2}{16}\right)\frac{(\pi^2 + 4\pi + 4)}{4}$$

$$= 2 - \frac{\pi(4 - \pi)(\pi + 2)^2}{64}$$

11. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,\ n \in N$

when $n = 1$:

$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

$= \cos 1\theta + i\sin 1\theta.$

Assume the statement is true when $n = k$

$n = k$: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

when $n = k + 1$:

$(\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\sin k\theta\cos\theta$
$+ i\sin\theta\cos k\theta - \sin k\theta\sin\theta$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta$
$+ i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

Since $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

is true when $n=1$ and is true when $n=k+1$

if it is true when $n=k$,

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

for every positive integer n.

$$(\cos\theta + i\sin\theta)^7 = \cos^7\theta + \binom{7}{1}\cos^6\theta\,(i\sin\theta)$$

$$+ \binom{7}{2}\cos^5\theta\,(i\sin\theta)^2$$

$$+ \binom{7}{3}\cos^4\theta\,(i\sin\theta)^3$$

$$+ \binom{7}{4}\cos^3\theta\,(i\sin\theta)^4$$

$$+ \binom{7}{5}\cos^2\theta\,(i\sin\theta)^5$$

$$+ \binom{7}{6}\cos\theta\,(i\sin\theta)^6$$

$$+ (i\sin\theta)^7$$

$$\cos 7\theta + i\sin 7\theta = \cos^7\theta + 7i\cos^6\theta\sin\theta$$

$$-21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta$$

$$+35\cos^3\theta\sin^4\theta + 21i\cos^2\theta\sin^5\theta$$

$$-7\cos\theta\sin^6\theta - i\sin^7\theta$$

$$= \cos^7\theta - 21\cos^5\theta\sin^2\theta$$

$$+35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$$

$$+i(7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta$$

$$+21\cos^2\theta\sin^5\theta - \sin^7\theta)$$

Equating real parts,

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta$$

$$-7\cos\theta\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta)$$

$$+35\cos^3\theta(1-\cos^2\theta)^2$$

$$-7\cos\theta(1-\cos^2\theta)^3$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta$$

$$+ 35\cos^3\theta(1 - 2\cos^2\theta + \cos^4\theta)$$

$$-7\cos\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta$$

$$+ 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta$$

$$-7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

$$64x^7 - 112x^5 + 56x^3 - 7x + \frac{1}{2} = 0$$

$$64x^7 - 112x^5 + 56x^3 - 7x = \frac{-1}{2}$$

Let $x = \cos\theta$

$$64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta = \frac{-1}{2}$$

$$\cos 7\theta = \frac{-1}{2}$$

$$7\theta = \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}, \frac{20\pi}{3}, \frac{26\pi}{3}, \frac{32\pi}{3}, \frac{38\pi}{3}$$

$$\theta = \frac{2\pi}{21}, \frac{8\pi}{21}, \frac{14\pi}{21}, \frac{20\pi}{21}, \frac{26\pi}{21}, \frac{32\pi}{21}, \frac{38\pi}{21}$$

$$x = \cos\theta$$

$$x = \cos\frac{2\pi}{21}, \cos\frac{8\pi}{21}, \cos\frac{14\pi}{21}, \cos\frac{20\pi}{21},$$

$$\cos\frac{26\pi}{21}, \cos\frac{32\pi}{21}, \cos\frac{38\pi}{21}.$$

The roots of the equation

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

are $\cos\frac{2\pi}{21}$, $\cos\frac{8\pi}{21}$, $\cos\frac{14\pi}{21}$, $\cos\frac{20\pi}{21}$,

$\cos\frac{26\pi}{21}$, $\cos\frac{32\pi}{21}$ and $\cos\frac{38\pi}{21}$.

12. $y = \dfrac{x^2 + qx + 1}{2x + 3}, \quad q > 0$

i)

$$\begin{array}{r} \frac{x}{2} + \frac{q}{2} - \frac{3}{4} \\ 2x+3 \,\big)\, \overline{x^2 + qx + 1} \\ x^2 + \frac{3x}{2} \\ \hline (q - \frac{3}{2})x + 1 \\ (q - \frac{3}{2})x + \frac{3q}{2} - \frac{9}{4} \\ \hline -\frac{3q}{2} + \frac{13}{4} \end{array}$$

$$y = \frac{x}{2} + \frac{q}{2} - \frac{3}{4} + \frac{-\frac{3q}{2} + \frac{13}{4}}{2x + 3}$$

As $x \to \pm\infty$ $\quad y \to \dfrac{x}{2} + \dfrac{q}{2} - \dfrac{3}{4}$

As $x \to -\dfrac{3}{2}$ $\quad y \to \pm\infty$

$\therefore$ The asymptotes of C are

$y = \dfrac{x}{2} + \dfrac{q}{2} - \dfrac{3}{4}$ and $x = -\dfrac{3}{2}$

ii) when $y = 0$ $\dfrac{x^2 + qx + 1}{2x + 3} = 0$

$$x^2 + qx + 1 = 0$$

$$x^2 + qx + \frac{q^2}{4} = \frac{q^2}{4} - 1$$

$$\left(x + \frac{q}{2}\right)^2 = \frac{q^2 - 4}{4}$$

$$x + \frac{q}{2} = \pm \frac{\sqrt{q^2 - 4}}{2}$$

$$x = -\frac{q}{2} \pm \frac{\sqrt{q^2 - 4}}{2}$$

If the x - axis is a tangent to C, the curve intersects the x - axis at 1 point

$$q^2 - 4 = 0$$

$$q^2 = 4$$

$$q = 2.$$

$\therefore$ If the x - axis is a tangent to C, $q = 2$.

$q = 2 \qquad y = \dfrac{x^2 + 2x + 1}{2x + 3}$

$$
\begin{array}{r|l}
 & \frac{x}{2} + \frac{1}{4} \\
\hline
2x + 3 & x^2 + 2x + 1 \\
 & x^2 + \frac{3x}{2} \\
\hline
 & \frac{x}{2} + 1 \\
 & \frac{x}{2} + \frac{3}{4} \\
\hline
 & \frac{1}{4}
\end{array}
$$

$y = \dfrac{x}{2} + \dfrac{1}{4} + \dfrac{1}{4(2x + 3)}$

As $x \to \pm\infty \quad y \to \dfrac{x}{2} + \dfrac{1}{4}$

As $x \to -\dfrac{3}{2} \quad y \to \pm\infty$

$\therefore$ The asymptotes of y are the lines

$y = \dfrac{x}{2} + \dfrac{1}{4}$ and $x = -\dfrac{3}{2}$.

When $x = 0 \quad y = \dfrac{1}{3}$

When $y = 0$, $\dfrac{x^2 + 2x + 1}{2x + 3} = 0$

$$x^2 + 2x + 1 = 0$$

$$(x + 1)^2 = 0$$

$$x = -1$$

$\therefore$ The intersection points of y are

$(0, \frac{1}{3})$ and $(-1, 0)$.

$$\frac{dy}{dx} = \frac{1}{2} - \frac{1}{2(2x + 3)^2}$$

When $\dfrac{dy}{dx} = 0$, $\dfrac{1}{2} - \dfrac{1}{2(2x + 3)^2} = 0$

$$\frac{1}{(2x + 3)^2} = 1$$

$$(2x + 3)^2 = 1$$

$$2x + 3 = \pm 1$$

$$x = -2, -1$$

$$y = -1, 0$$

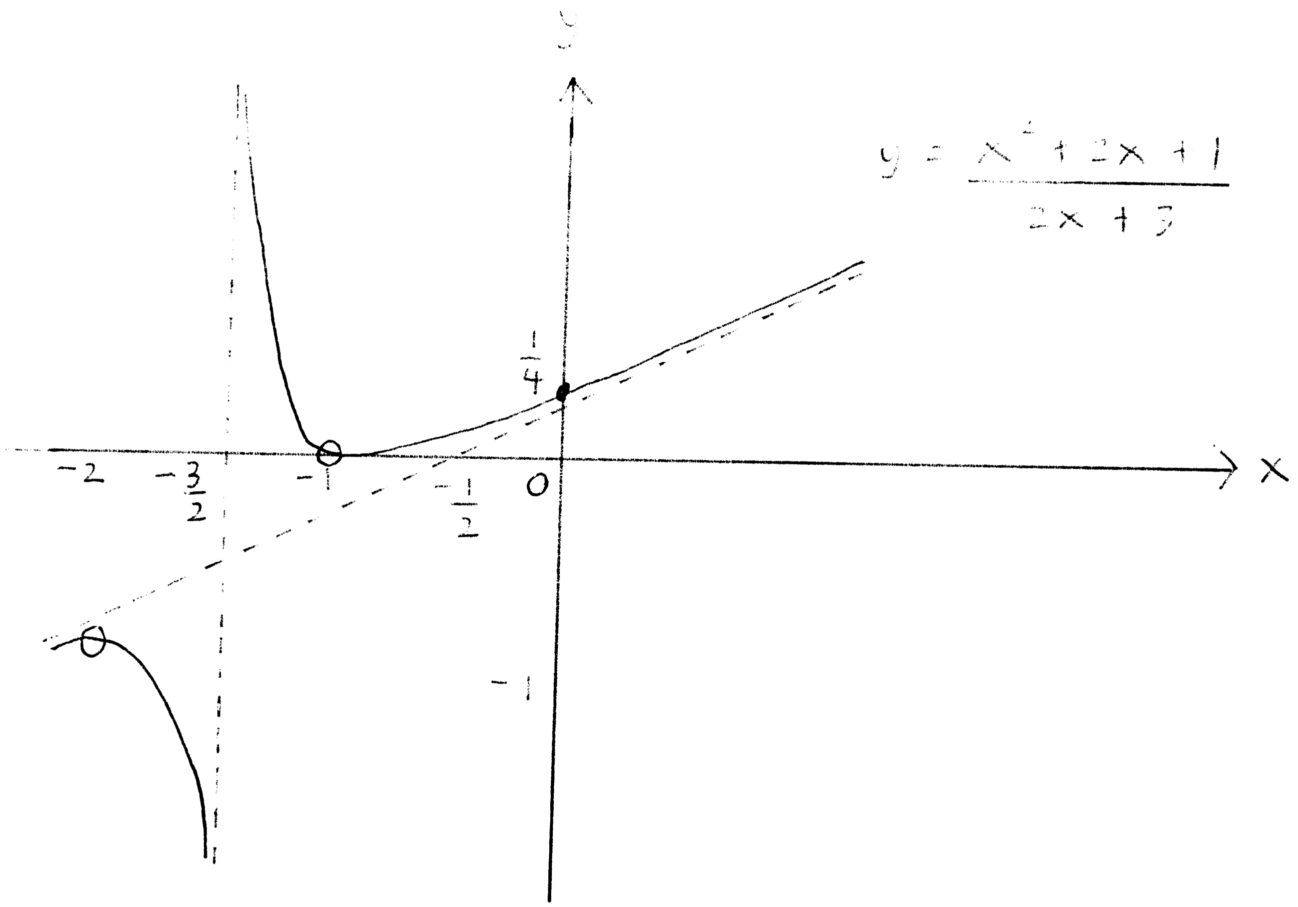
$$\frac{d^2y}{dx^2} = \frac{2}{(2x+3)^3}$$

when $x = -2$: $\frac{d^2y}{dx^2} = -2 < 0$

when $x = -1$: $\frac{d^2y}{dx^2} = 2 > 0$

$(-2, -1)$ is a maximum point and

$(-1, 0)$ is a minimum point.



o : critical point

• intersection point.

iii) $q = 3: \quad y = \dfrac{x^2 + 3x + 1}{2x + 3}$

$$
\begin{array}{r|l}
 & \frac{x}{2} + \frac{3}{4} \\
\hline
2x + 3 & x^2 + 3x + 1 \\
 & x^2 + \frac{3x}{2} \\
\cline{2-2}
 & \qquad \frac{3x}{2} + 1 \\
 & \qquad \frac{3x}{2} + \frac{9}{4} \\
\cline{2-2}
 & \qquad\qquad -\frac{5}{4}
\end{array}
$$

$$y = \frac{x}{2} + \frac{3}{4} - \frac{5}{4(2x + 3)}$$

As $x \to \pm\infty \quad y \to \frac{x}{2} + \frac{3}{4}$

As $x \to -\frac{3}{2} \quad y \to \pm\infty$

$\therefore$ The asymptotes of y are the lines

$y = \frac{x}{2} + \frac{3}{4}$ and $x = -\frac{3}{2}$.

when $x = 0$; $y = \frac{1}{3}$

when $y = 0$. $\frac{x^2 + 3x + 1}{2x + 3} = 0$

$$x^2 + 3x + 1 = 0$$

$$x = \frac{-3 \pm \sqrt{5}}{2}$$

∴ The intersection points of y are

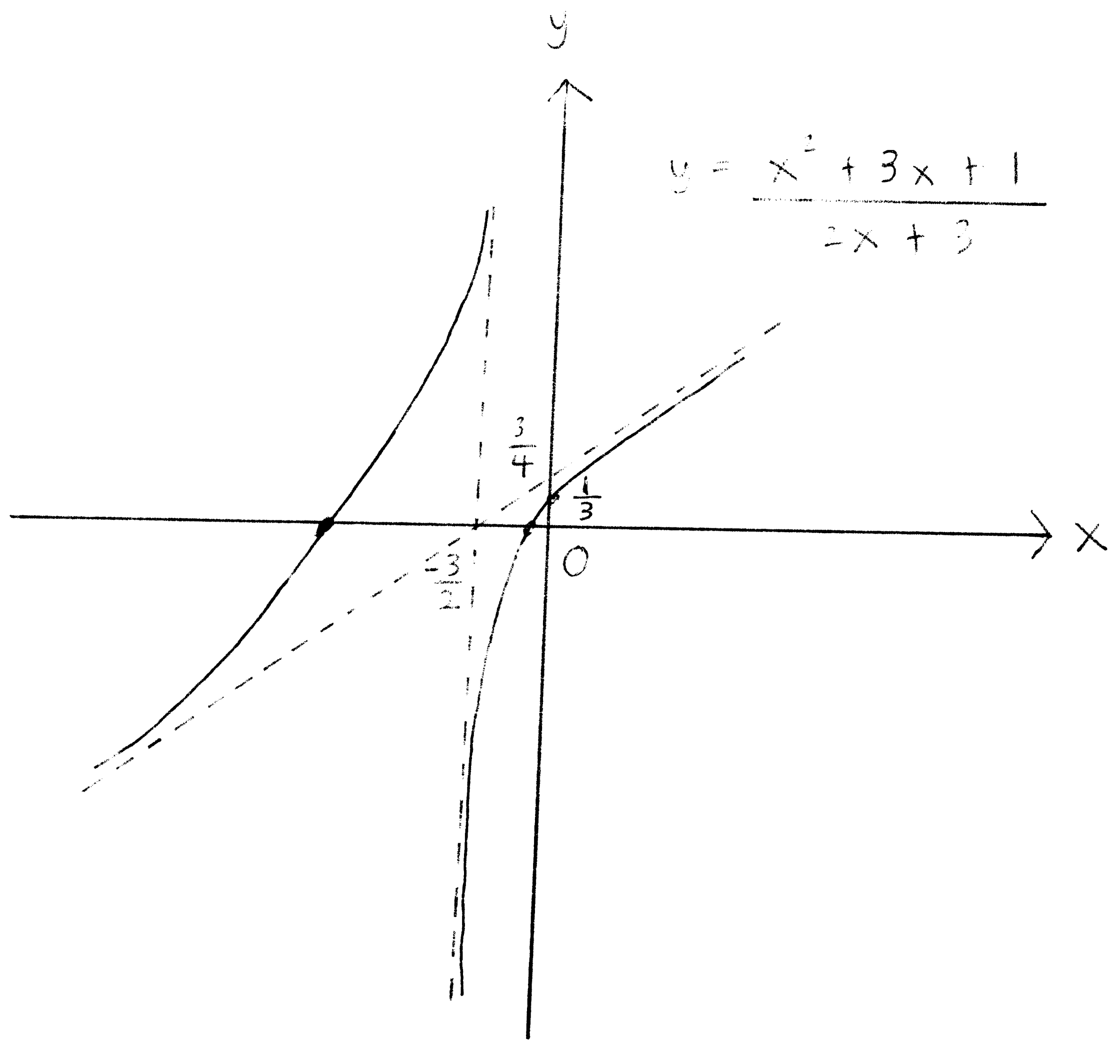
$(0, \frac{1}{3})$, $\left(\frac{-3 + \sqrt{5}}{2}, 0\right)$ and $\left(\frac{-3 - \sqrt{5}}{2}, 0\right)$.

$$\frac{dy}{dx} = \frac{1}{2} + \frac{5}{2(2x + 3)^2}$$

Since $\frac{1}{2} + \frac{5}{2(2x + 3)^2} \geqslant \frac{1}{2}$,

$$\frac{dy}{dx} \neq 0.$$

∴ y has no critical points.



• : intersection point

iv) The line

$$y = \lambda x + \frac{3\lambda}{2} + \frac{1}{2}(q - 3)$$

passes through the intersection point of the asymptotes of C $\left(\frac{-3}{2}, \frac{q-3}{2}\right)$.

If $q = 2$ and $\frac{x^2 + qx + 1}{2x+3} = \lambda x + \frac{3\lambda}{2} + \frac{q-3}{2}$,

$$x^2 + 2x + 1 = (2x+3)\left(\lambda x + \frac{3\lambda}{2} - \frac{1}{2}\right)$$

$$= 2\lambda x^2 + (3\lambda - 1)x + 3\lambda x + \frac{9\lambda}{2} - \frac{3}{2}$$

$$(2\lambda - 1)x^2 + 3(2\lambda - 1)x + \frac{9\lambda}{2} - \frac{5}{2} = 0$$

$a = 2\lambda - 1 \quad b = 3(2\lambda - 1) \quad c = \frac{9\lambda}{2} - \frac{5}{2}$

$$b^2 - 4ac = 9(2\lambda - 1)^2 - 4(2\lambda - 1)\left(\frac{9\lambda}{2} - \frac{5}{2}\right)$$

$$= 9(2\lambda - 1)^2 - 2(2\lambda - 1)(9\lambda - 5)$$

$$= (2\lambda - 1)(9(2\lambda - 1) - 2(9\lambda - 5))$$

$$= (2\lambda - 1)(18\lambda - 9 - 18\lambda + 10)$$

$$= 2\lambda - 1$$

if $\lambda < \frac{1}{2}$,

$2\lambda - 1 < 0$

$b^2 - 4ac < 0$

If $q = 3$ and $\dfrac{x^2 + qx + 1}{2x + 3} = \lambda x + \dfrac{3\lambda}{2} + \dfrac{q - 3}{2}$,

$$x^2 + 3x + 1 = (2x + 3)\left(\lambda x + \frac{3\lambda}{2}\right)$$

$$= 2\lambda x^2 + 3\lambda x + 3\lambda x + \frac{9\lambda}{2}$$

$$= 2\lambda x^2 + 6\lambda x + \frac{9\lambda}{2}$$

$$(2\lambda - 1)x^2 + 3(2\lambda - 1)x + \frac{9\lambda}{2} - 1 = 0$$

$a = 2\lambda - 1 \quad b = 3(2\lambda - 1) \quad c = \dfrac{9\lambda}{2} - 1$

$$b^2 - 4ac = 9(2\lambda - 1)^2 - 4(2\lambda - 1)\left(\frac{9\lambda}{2} - 1\right)$$

$$= 9(2\lambda - 1)^2 - 2(2\lambda - 1)(9\lambda - 2)$$

$$= (2\lambda - 1)(9(2\lambda - 1) - 2(9\lambda - 2))$$

$$= (2\lambda - 1)(18\lambda - 9 - 18\lambda + 4)$$

$$= -5(2\lambda - 1)$$

If $\lambda < \frac{1}{2}$

$2\lambda - 1 < 0$

$-5(2\lambda - 1) > 0$

$\therefore b^2 - 4ac > 0$

$\therefore$ If $q = 2$ and $\lambda < \frac{1}{2}$ the equation $\dfrac{x^2 + qx + 1}{2x + 3} = \lambda x + \dfrac{3\lambda}{2} + \dfrac{q - 3}{2}$ has no real solution since the line does not intersect C, but has 2 real distinct solutions if $q = 3$ and $\lambda < \frac{1}{2}$ since the line intersects C at points.

$$C : y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} + \lambda, \quad \lambda > 0, \quad 0 \leq x \leq 3$$

The length of C, s is

$$\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}\right)^2$$

$$= 1 + \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x$$

$$= \frac{1}{4}x^{-1} + \frac{1}{2} + \frac{1}{4}x$$

$$= \left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}\right)^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)} = \sqrt{\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}\right)^2}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}$$

$$\therefore S = \int_0^3 \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}} \, dx$$

$$= \left[x^{\frac{1}{2}} + \frac{1}{3}x^{\frac{3}{2}} \right]_0^3$$

$$= 3^{\frac{1}{2}} + \frac{1}{3}\left(3^{\frac{3}{2}}\right) - 0$$

$$= \sqrt{3} + \frac{1}{3}(3)\sqrt{3}$$

$$= \sqrt{3} + \sqrt{3}$$

$$= -\sqrt{3}$$

The area of the surface generated when C is rotated through one revolution about the x axis, S, is

$$\int_C^3 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_0^3 2\pi\left(x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} + \lambda\right)\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}\right)dx$$

$$= 2\pi\int_0^3 \left(x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} + \lambda\right)\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}\right)dx$$

$$= 2\pi\int_0^3 \frac{1}{2} - \frac{x}{6} + \frac{\lambda}{2}x^{-\frac{1}{2}} + \frac{x}{2} - \frac{1}{6}x^2 + \frac{\lambda x^{\frac{1}{2}}}{2}\,dx$$

$$= 2\pi\int_0^3 \frac{1}{2} + \frac{x}{3} - \frac{x^2}{6} + \frac{\lambda}{2}\left(x^{-\frac{1}{2}} + x^{\frac{1}{2}}\right)dx$$

$$= 2\pi\left[\frac{x}{2} + \frac{x^2}{6} - \frac{x^3}{18} + \frac{\lambda}{2}\left(2x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right)\right]_0^3$$

$$= 2\pi\left(\frac{3}{2} + \frac{9}{6} - \frac{27}{18} + \frac{\lambda}{2}\left(2\sqrt{3} + \frac{2}{3}(3)^{\frac{3}{2}}\right) - 0\right)$$

$$= 2\pi\left(\frac{3}{2} + \frac{\lambda}{2}\left(2\sqrt{3} + \frac{2}{3}(3\sqrt{3})\right)\right)$$

$$= 2\pi\left(\frac{3}{2} + \sqrt{3}\lambda + \sqrt{3}\lambda\right)$$

$$= 2\pi\left(\frac{3}{2} + 2\sqrt{3}\lambda\right)$$

$$= 3\pi + 4\sqrt{3}\lambda\pi$$

The area of the region bounded by C, the x-axis, and the lines $x = 0$ and $x = 3$,

A is $\displaystyle\int_0^3 y \, dx$

$$= \int_0^3 x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} + \lambda \, dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{2}{15}x^{\frac{5}{2}} + \lambda x\right]_0^3$$

$$= \frac{2(3^{\frac{3}{2}})}{3} - \frac{2}{15}(3^{\frac{5}{2}}) + 3\lambda - 0$$

$$= \frac{2(3)\sqrt{3}}{3} - \frac{2(9)\sqrt{3}}{15} + 3\lambda$$

$$= 2\sqrt{3} - \frac{6}{5}\sqrt{3} + 3\lambda$$

$$= \frac{4}{5}\sqrt{3} + 3\lambda$$

The y-coordinate of the centroid of the region bounded by C, the axes and the line $x = 3$, h, is

$$\frac{\int_0^3 \frac{y^2}{2}\,dx}{A}$$

Since $\displaystyle\int_0^3 y^2\,dx = \frac{3}{4} + \frac{3\sqrt{3}\lambda}{5} + 3\lambda^2$

$$h = \frac{\frac{1}{2}\left(\frac{3}{4} + \frac{3\sqrt{3}\lambda}{5} + 3\lambda^2\right)}{\frac{4}{5}\sqrt{3} + 3\lambda}$$

$$= \frac{\frac{3}{8} + \frac{4\sqrt{3}\lambda}{5} + \frac{3\lambda^2}{2}}{\frac{4\sqrt{3}}{5} + 3\lambda}$$

$$\frac{S}{hs} = \frac{3\pi + 4\sqrt{3}\lambda\pi}{\dfrac{\left(\dfrac{3}{8} + \dfrac{4\sqrt{3}\lambda}{5} + \dfrac{3\lambda^2}{2}\right) 2\sqrt{3}}{\dfrac{4\sqrt{3}}{5} + 3\lambda}}$$

$$= \frac{(3\pi + 4\sqrt{3}\lambda\pi)\left(\dfrac{4\sqrt{3}}{5} + 3\lambda\right)}{\left(\dfrac{3\sqrt{3}}{4} + \dfrac{24}{5}\lambda + 3\sqrt{3}\lambda^2\right)}$$

$$= \frac{\left(\dfrac{3\pi}{\lambda} + 4\sqrt{3}\pi\right)\left(\dfrac{4\sqrt{3}}{5\lambda} + 3\right)}{\left(\dfrac{3\sqrt{3}}{4\lambda^2} + \dfrac{24}{5\lambda} + 3\sqrt{3}\right)}$$

$$\lim_{\lambda \to \infty} \frac{S}{hs}$$

$$= \lim_{\lambda \to \infty} \frac{\left(\frac{3\pi}{\lambda} + 4\sqrt{3}\pi\right)\left(\frac{4\sqrt{3}}{5\lambda} + 3\right)}{\left(\frac{3\sqrt{3}}{4\lambda^2} + \frac{24}{5\lambda} + 3\sqrt{3}\right)}$$

$$= \frac{4\sqrt{3}\pi(3)}{3\sqrt{3}}$$

$$= 4\pi.$$