OCTOBER / NOVEMBER 2007

1. $x = at^2 \qquad y = at$

$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = a$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4a^2t^2 + a^2$$

$$= a^2(4t^2 + 1)$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = a\sqrt{4t^2 + 1}$$

The area of the surface generated when the curve is rotated through one complete revolution about the x - axis from $t = 0$ to $t = \sqrt{2}$ is

$$\int_0^{\sqrt{2}} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\sqrt{2}} 2\pi(at)a\sqrt{4t^2 + 1}\, dt$$

$$= 2\pi a^2 \int_0^{\sqrt{2}} t\sqrt{4t^2 + 1}\, dt$$

Let $u = 4t^2 + 1$

$$\frac{du}{dt} = 8t$$

$$du = 8t\ dt$$

$$\frac{du}{8} = t\ dt$$

When $t = \sqrt{2}$, $u = 9$

when $t = 0$, $u = 1$

$$= 2\pi a^2 \int_1^9 \frac{\sqrt{u}}{8}\ du$$

$$= \frac{\pi a^2}{4} \int_1^9 u^{\frac{1}{2}}\ du$$

$$= \frac{\pi a^2}{4} \frac{2u^{\frac{3}{2}}}{3}$$

$$= \frac{\pi a^2}{4}\left(\frac{2}{3}(27) - \frac{2}{3}(1)\right)$$

$$= \frac{\pi a^2}{4}\left(\frac{2}{3}\right)26$$

$$= \frac{13\pi a^2}{3}$$

2. $$\frac{2n+3}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$= \frac{A(n+1) + Bn}{n(n+1)}$$

$$2n + 3 = A(n+1) + n$$

$$= (A+B)n + A$$

$$A + B = 2 \qquad A = 3$$

$$B = -1$$

$$\therefore \frac{2n+3}{n(n+1)} = \frac{3}{n} - \frac{1}{n+1}$$

Since $\dfrac{2n+3}{n(n+1)} = \dfrac{3}{n} - \dfrac{1}{n+1}$,

$$\sum_{n=1}^{N} \frac{2n+1}{n(n+1)} \left(\frac{1}{3}\right)^{n+1}$$

$$= \sum_{n=1}^{N} \left(\frac{3}{n} - \frac{1}{n+1}\right)\left(\frac{1}{3}\right)^{n+1}$$

$$= \sum_{n=1}^{N} \frac{3}{n}\left(\frac{1}{3}\right)^{n+1} - \frac{1}{n+1}\left(\frac{1}{3}\right)^{n+1}$$

$$= \sum_{n=1}^{N} \frac{1}{n}\left(\frac{1}{3}\right)^{n} - \frac{1}{n+1}\left(\frac{1}{3}\right)^{n+1}$$

$$= 1\left(\frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{3}\right)^2$$

$$+ \frac{1}{2}\left(\frac{1}{3}\right)^2 - \frac{1}{3}\left(\frac{1}{3}\right)^3$$

$$+ \frac{1}{3}\left(\frac{1}{3}\right)^3 - \frac{1}{4}\left(\frac{1}{3}\right)^4$$

$$\vdots$$

$$+ \frac{1}{N-2}\left(\frac{1}{3}\right)^{N-2} - \frac{1}{N-1}\left(\frac{1}{3}\right)^{N-1}$$

$$+ \frac{1}{N-1}\left(\frac{1}{3}\right)^{N-1} - \frac{1}{N}\left(\frac{1}{3}\right)^{N}$$

$$+ \frac{1}{N}\left(\frac{1}{3}\right)^{N} - \frac{1}{N+1}\left(\frac{1}{3}\right)^{N+1}$$

$$= \frac{1}{3} - \frac{1}{N+1}\left(\frac{1}{3}\right)^{N+1}$$

$$\sum_{n=1}^{N} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1}$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1}$$

$$= \lim_{N\to\infty}\left(\frac{1}{3} - \frac{1}{N+1}\left(\frac{1}{3}\right)^{N+1}\right)$$

$$= \frac{1}{3}$$

3. $$\frac{d^n}{dx^n}(e^{x^2}) = P_n(x)e^{x^2}, \quad P_n(x) = 2^n x^n + \sum_{r=0}^{n-1} a_r x^r$$

for all $n \geq 1$.

When $n = 1$: $$\frac{d^1}{dx^1}(e^{x^2}) = \frac{d}{dx}(e^{x^2})$$

$$= 2xe^{x^2}$$

$$= 2^1 x^1 e^{x^2}$$

Assume the statement is true when $n = k$.

$n = k$: $$\frac{d^k}{dx^k}(e^{x^2}) = P_k(x)e^{x^2},$$

$$P_k(x) = 2^k x^k + \sum_{r=0}^{k-1} a_r x^r$$

When $n = k + 1$:

$$\frac{d^{k+1}}{dx^{k+1}}(e^{x^2}) = \bar{P}_{k+1}(x)e^{x^2},$$

$$\bar{P}_{k+1}(x) = 2^{k+1} x^{k+1} + \sum_{r=0}^{k} \bar{a}_r x^r.$$

(what needs to be proved).

$$\frac{d^{k+1}}{dx^{k+1}}(e^{x^2}) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(e^{x^2})\right)$$

$$= \frac{d}{dx}\left(P_k(x)e^{x^2}\right)$$

$$= \frac{d}{dx}\left(\left(2^k x^k + \sum_{r=0}^{k-1} a_r x^r\right)e^{x^2}\right)$$

$$= e^{x^2}\frac{d}{dx}\left(2^k x^k + \sum_{r=0}^{k-1} a_r x^r\right)$$

$$+ \left(2^k x^k + \sum_{r=0}^{k-1} a_r x^r\right)\frac{d}{dx}(e^{x^2})$$

$$= e^{x^2}\left(2^k k x^{k-1} + \sum_{r=1}^{k-1} r a_r x^{r-1}\right)$$

$$+ \left(2^k x^k + \sum_{r=0}^{k-1} a_r x^r\right) 2xe^{x^2}$$

$$= e^{x^2}\left(2^k k x^{k-1} + \sum_{r=1}^{k-1} r a_r x^{r-1}\right)$$

$$+ \left(2^{k+1} x^{k+1} + \sum_{r=0}^{k-1} 2a_r x^{r+1}\right)e^{x^2}$$

$$= \left(2^k k x^{k-1} + \sum_{r=1}^{k-1} r a_r x^{r-1}\right) e^{x^2}$$

$$+ \left(2^{k+1} x^{k+1} + \sum_{r=0}^{k-1} 2 a_r x^{r+1}\right) e^{x^2}$$

$$= 2^{k+1} x^{k+1} e^{x^2}$$

$$+ \left(2^k k x^{k-1} + \sum_{r=1}^{k-1} r a_r x^{r-1} + \sum_{r=0}^{k-1} 2 a_r x^{r+1}\right) e^{x^2}$$

$$= 2^{k+1} x^{k+1} e^{x^2} + \sum_{r=0}^{k} \bar{a}_r x^r e^{x^2}$$

$$\therefore \frac{d^n}{dx^n}\left(e^{x^2}\right) = P_n(x) e^{x^2}, \quad P_n(x) = 2^n x^n + \sum_{r=0}^{n-1} a_r x^r$$

for all $n \geqslant 1$.

4. $x^3 - 8x^2 + 5 = 0$

α, β, γ ar the roots.

$\alpha + \beta + \gamma = 8,$

$\alpha\beta + \alpha\gamma + \beta\gamma = 0$

$\alpha\beta\gamma = -5$

Since $\beta\gamma = \frac{-5}{\alpha},$

$\alpha\beta + \alpha\gamma - \frac{5}{\alpha} = 0$

$\alpha\beta + \alpha\gamma = \frac{5}{\alpha}$

$\alpha(\beta + \gamma) = \frac{5}{\alpha}$

$\alpha^2 = \frac{5}{\beta + \gamma}$

If $\alpha, \beta, \gamma \in R,$

$\alpha^2 > 0$

$\frac{5}{\beta + \gamma} > 0$

$\therefore \beta + \gamma > 0$

Since $\alpha\beta\gamma = -5,$ if $\alpha, \beta, \gamma < 0,$

$\beta + \gamma < 0.$

$\therefore$ One of the roots is negative and the other two roots are positive.

5. $y = x^2 + 2\ln(xy), \quad x, y > 0$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + 2\ln(xy)\right)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(2\ln(xy))$$

$$= 2x + \frac{2\frac{d}{dx}(xy)}{xy}$$

$$= 2x + \frac{2\left(x\frac{dy}{dx} + y\frac{d}{dx}(x)\right)}{xy}$$

$$= 2x + \frac{2}{xy}\left(x\frac{dy}{dx} + y\right)$$

$$= 2x + \frac{2}{y}\frac{dy}{dx} + \frac{2}{x}$$

when $x = y = 1$:

$$\frac{dy}{dx} = 2 + 2\frac{dy}{dx} + 2$$

$$\therefore \frac{dy}{dx} = -4$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x + \frac{2}{y}\frac{dy}{dx} + \frac{2}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x) + \frac{d}{dx}\left(\frac{2}{y}\frac{dy}{dx}\right) + \frac{d}{dx}\left(\frac{2}{x}\right)$$

$$= 2 + \frac{2}{y}\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}\left(\frac{2}{y}\right) - \frac{2}{x^2}$$

$$= 2 + \frac{2}{y}\frac{d^2y}{dx^2} - \frac{2}{y^2}\left(\frac{dy}{dx}\right)^2 - \frac{2}{x^2}$$

When $x = y = 1$, $\frac{dy}{dx} = -4$

$$\frac{d^2y}{dx^2} = 2 + \frac{2d^2y}{dx^2} - 2(-4)^2 - 2$$

$$= 2 + \frac{2d^2y}{dx^2} - 32 - 2$$

$$\frac{d^2y}{dx^2} = 32$$

6. $\overrightarrow{OA} = 2\underset{\sim}{i}$ $\quad \overrightarrow{OB} = 3\underset{\sim}{j}$ $\quad \overrightarrow{OC} = 4\underset{\sim}{k}$

Since the plane Π_1 contains A, B and C, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel to the plane and therefore $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the plane.

$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
&= 3\underset{\sim}{j} - 2\underset{\sim}{i} \\
&= -2\underset{\sim}{i} + 3\underset{\sim}{j}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} \\
&= 4\underset{\sim}{k} - 2\underset{\sim}{i} \\
&= -2\underset{\sim}{i} + 4\underset{\sim}{k}
\end{aligned}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & 3 & 0 \\ -2 & 0 & 4 \end{vmatrix}$$

$$\begin{aligned}
&= 12\underset{\sim}{i} + 8\underset{\sim}{j} + 6\underset{\sim}{k} \\
&= 2(6\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k})
\end{aligned}$$

$\therefore$ A vector perpendicular to the plane containing A, B and C is $6\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$.

$\Pi_2 : \underset{\sim}{r} = \underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} - \underset{\sim}{j}) + \mu(\underset{\sim}{j} - \underset{\sim}{k})$

Expressing Π_2 in Cartesian form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \lambda \\ 4 - \lambda + \mu \\ 2 - \mu \end{pmatrix}$$

$$\left.\begin{aligned} x &= 1 + \lambda \\ y &= 4 - \lambda + \mu \\ z &= 2 - \mu \end{aligned}\right\}$$

$$\left.\begin{aligned} \lambda &= x - 1 \\ -\lambda + \mu &= y - 4 \\ -\mu &= z - 2 \end{aligned}\right\}$$

$$① + ② : \left.\begin{aligned} \lambda &= x - 1 \\ \mu &= x + y - 5 \\ -\mu &= z - 2 \end{aligned}\right\}$$

$$② + ③ : \left.\begin{aligned} \lambda &= x - 1 \\ \mu &= x + y - 5 \\ 0 &= x + y + z - 7 \end{aligned}\right\}$$

$\therefore \Pi_2 : x + y + z = 7.$

Since $\begin{pmatrix} 6 \\ 4 \\ 3 \end{pmatrix}$ is a normal to the plane Π_1

and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a normal to the plane Π_2,

$$\begin{pmatrix} 6 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \left|\begin{pmatrix} 6 \\ 4 \\ 3 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right| \cos \theta$$

$$6 + 4 + 3 = \sqrt{61}\sqrt{3} \cos \theta$$

$$13 = \sqrt{61}\sqrt{3} \cos \theta$$

$$\cos \theta = \frac{13}{\sqrt{61}\sqrt{3}}$$

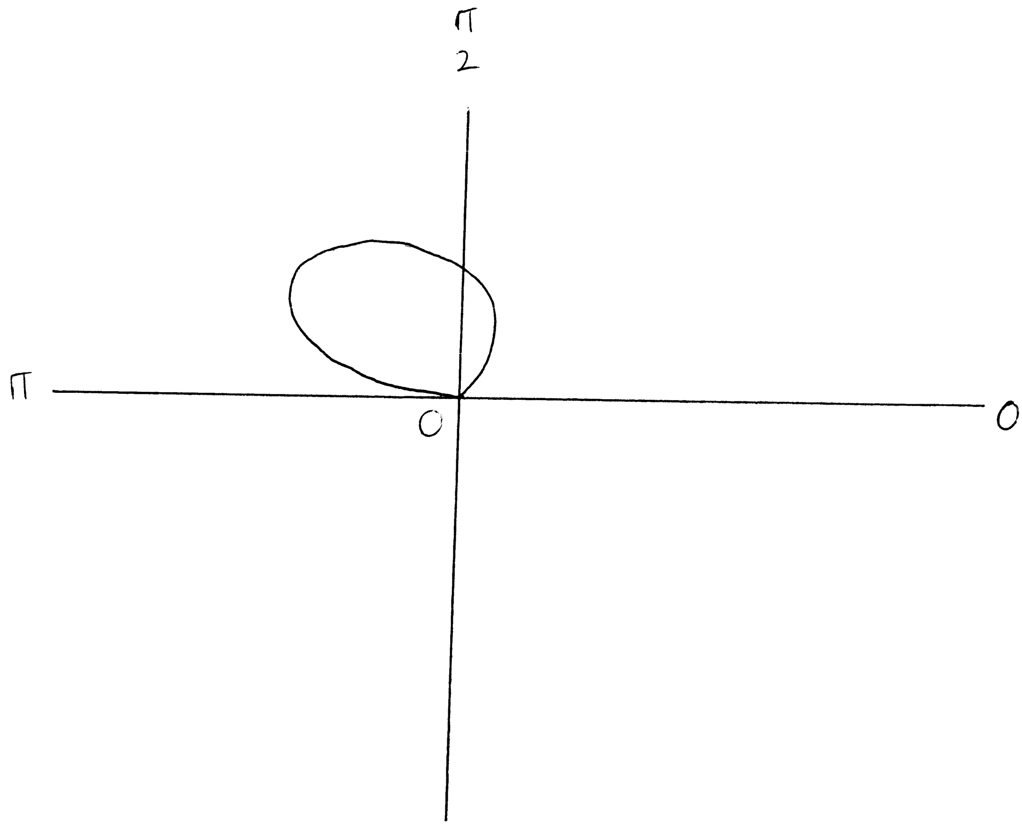
$$\therefore \theta = \cos^{-1}\left(\frac{13}{\sqrt{61}\sqrt{3}}\right)$$

$$= 16.1^\circ$$

$\therefore$ The acute angle between Π_1 and Π_2 is 16.1°

7. $C: \quad r = \theta \sin\theta \,, \quad 0 \leq \theta \leq \pi$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	$\frac{\pi}{12}$	$\frac{\pi}{4\sqrt{2}}$	$\frac{\pi}{2\sqrt{3}}$	$\frac{\pi}{2}$	$\frac{\pi}{\sqrt{3}}$	$\frac{3\pi}{4\sqrt{2}}$	$\frac{5\pi}{12}$	0



The area of the region enclosed by C is given by

$$\int_0^{\pi} \frac{r^2}{2}\, d\theta$$

$$= \int_0^{\pi} \frac{\theta^2 \sin^2\theta}{2}\, d\theta$$

$$= \int_0^{\pi} \frac{\theta^2}{2} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \int_0^{\pi} \frac{\theta^2 (1 - \cos 2\theta)}{4} \, d\theta$$

$$u = \theta^2 \qquad dv = \frac{1 - \cos 2\theta}{4}$$

$$du = 2\theta \, d\theta$$

$$v = \frac{\theta}{4} - \frac{\sin 2\theta}{8}$$

$$= \left[\theta^2 \left(\frac{\theta}{4} - \frac{\sin 2\theta}{8} \right) \right]_0^{\pi}$$

$$- \int_0^{\pi} 2\theta \left(\frac{\theta}{4} - \frac{\sin 2\theta}{8} \right) d\theta$$

$$= \left[\frac{\theta^3}{4} - \frac{\theta^2 \sin 2\theta}{8} \right]_0^{\pi}$$

$$- \int_0^{\pi} \theta \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) d\theta$$

$$= \frac{\pi^3}{4} - \int_0^{\pi} \theta \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) d\theta$$

$u = \theta \qquad dv = \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) d\theta$

$du = d\theta$

$v = \frac{\theta^2}{4} + \frac{\cos 2\theta}{8}$

$$= \frac{\pi^3}{4} - \left(\left[\theta\left(\frac{\theta^2}{4} + \frac{\cos 2\theta}{8}\right)\right]_0^{\pi} - \int_0^{\pi} \left(\frac{\theta^2}{4} + \frac{\cos 2\theta}{8}\right) d\theta\right)$$

$$= \frac{\pi^3}{4} - \left(\left[\frac{\theta^3}{4} + \frac{\theta\cos 2\theta}{8}\right]_0^{\pi} - \left[\frac{\theta^3}{12} + \frac{\sin 2\theta}{16}\right]_0^{\pi}\right)$$

$$= \frac{\pi^3}{4} - \left(\frac{\pi^3}{4} + \frac{\pi}{8} - 0 - \left(\frac{\pi^3}{12} - 0\right)\right)$$

$$= \frac{\pi^3}{4} - \left(\frac{\pi^3}{4} + \frac{\pi}{8} - \frac{\pi^3}{12}\right)$$

$$= \frac{\pi^3}{4} - \frac{\pi^3}{4} - \frac{\pi}{8} + \frac{\pi^3}{12}$$

$$= \frac{\pi^3}{12} - \frac{\pi}{8}$$

8. $I_n = \int_0^{\ln 2} (e^x + e^{-x})^n \, dx$

i) $\frac{d}{dx}\left[(e^x - e^{-x})(e^x + e^{-x})^{n-1}\right]$

$= (e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x})^{n-1}$

$+ (e^x + e^{-x})^{n-1}\frac{d}{dx}(e^x - e^{-x})$

$= (e^x - e^{-x})(n-1)(e^x + e^{-x})^{n-2}\frac{d}{dx}(e^x + e^{-x})$

$+ (e^x + e^{-x})^{n-1}(e^x + e^{-x})$

$= (e^x - e^{-x})(n-1)(e^x + e^{-x})^{n-2}(e^x - e^{-x})$

$+ (e^x + e^{-x})^n$

$= (n-1)(e^x - e^{-x})^2(e^x + e^{-x})^{n-2}$

$+ (e^x + e^{-x})^n$

$= (n-1)(e^{2x} - 2 + e^{-2x})(e^x + e^{-x})^{n-2}$

$+ (e^x + e^{-x})^n$

$= (n-1)(e^{2x} + 2 + e^{-2x} - 4)(e^x + e^{-x})^{n-2}$

$+ (e^x + e^{-x})^n$

$= (n-1)\left[(e^x + e^{-x})^2 - 4\right](e^x + e^{-x})^{n-2}$

$+ (e^x + e^{-x})^n$

$= (n-1)\left[(e^x + e^{-x})^n - 4(e^x + e^{-x})^{n-2}\right]$

$+ (e^x + e^{-x})^n$

$= (n-1)(e^x + e^{-x})^n - 4(n-1)(e^x + e^{-x})^{n-2}$

$+ (e^x + e^{-x})^n$

$= n(e^x + e^{-x})^n - 4(n-1)(e^x + e^{-x})^{n-2}$

ii) $\left[(e^x - e^{-x})(e^x + e^{-x})^{n-1}\right]_0^{\ln 2}$

$= \int_0^{\ln 2} n(e^x + e^{-x})^n \, dx - \int_0^{\ln 2} 4(n-1)(e^x + e^{-x})^{n-2} dx$

$\left(2 - \frac{1}{2}\right)\left(2 + \frac{1}{2}\right)^{n-1} - (1-1)(1+1)^{n-1}$

$= n\int_0^{\ln 2} (e^x + e^{-x})^n \, dx - 4(n-1)\int_0^{\ln 2} (e^x + e^{-x})^{n-2} dx$

$\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)^{n-1} = nI_n - 4(n-1)I_{n-2}$

$\therefore nI_n = 4(n-1)I_{n-2} + \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)^{n-1}.$

iii) The area of the region bounded by the x and y axes, the line $x = \ln 2$ and the curve $y = (e^x + e^{-x})^2$, A, is

$$\int_0^{\ln 2} y \, dx$$

$$= \int_0^{\ln 2} (e^x + e^{-x})^2 \, dx$$

$$= I_2$$

Since $2I_2 = 4(1)I_0 + \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)^1$

$$= 4I_0 + \frac{15}{4}$$

and $I_0 = \int_0^{\ln 2} (e^x + e^{-x})^0 \, dx$

$$= \int_0^{\ln 2} 1 \, dx$$

$$= [x]_0^{\ln 2}$$

$$= \ln 2 - 0$$

$$= \ln 2$$

$$I_2 = 2I_0 + \frac{15}{8}$$

$$= 2\ln 2 + \frac{15}{8}$$

$\therefore\ A = 2\ln 2 + \frac{15}{8}$

The y-coordinate of the centroid of the region bounded by the x and y axes, the line $x = \ln 2$ and the curve $y = (e^x + e^{-x})^2$, $\bar{y}$ is

$$\frac{\int_0^{\ln 2} \frac{y^2}{2}\,dx}{A}$$

$$= \frac{\int_0^{\ln 2} \frac{(e^x + e^{-x})^4}{2}\,dx}{A}$$

$$= \frac{\frac{I_4}{2}}{A}$$

Since $4I_4 = 4(3)I_2 + \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)^3$

$2I_2 = 4(1)I_0 + \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)^1$

$I_0 = \ln 2$

$$2I_2 = 4I_0 + \frac{15}{4}$$

$$= 4(\ln 2) + \frac{15}{4}$$

$$= 4\ln 2 + \frac{15}{4}$$

$$I_2 = 2\ln 2 + \frac{15}{8}$$

$$4I_4 = 12I_2 + \left(\frac{3}{2}\right)\left(\frac{125}{8}\right)$$

$$= 12I_2 + \frac{375}{16}$$

$$= 12\left(2\ln 2 + \frac{15}{8}\right) + \frac{375}{16}$$

$$= 24\ln 2 + \frac{45}{2} + \frac{375}{16}$$

$$= 24\ln 2 + \frac{735}{16}$$

$$I_4 = 6\ln 2 + \frac{735}{64}$$

$$\bar{y} = \frac{\frac{I_4}{2}}{A}$$

$$= \frac{\frac{1}{2}\left(6\ln 2 + \frac{735}{64}\right)}{2\ln 2 + \frac{15}{8}}$$

$$= \frac{3\ln 2 + \frac{735}{128}}{2\ln 2 + \frac{15}{8}}$$

$$= 2.398$$

9. $z^5 - 1 = 0$

$z^5 = 1$

$= \cos 0 + i\sin 0$

$= \cos 2k\pi + i\sin 2k\pi, \quad k \in \mathbb{Z}$

$z = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{5}}$

$= \cos \frac{2k\pi}{5} + i\sin \frac{2k\pi}{5}, \quad k = 0, 1, 2, 3, 4.$

$= 1, \cos \frac{2\pi}{5} + i\sin \frac{2\pi}{5}, \cos \frac{4\pi}{5} + i\sin \frac{4\pi}{5},$

$\cos \frac{6\pi}{5} + i\sin \frac{6\pi}{5}, \cos \frac{8\pi}{5} + i\sin \frac{8\pi}{5}.$

$z^5 - 1 = 0$

$\frac{z^5 - 1}{z - 1} = 0, \quad$ when $z \neq 1$

$z^4 + z^3 + z^2 + z + 1 = 0$

If $z = w - 1$,

$\therefore (w-1)^4 + (w-1)^3 + (w-1)^2 + w - 1 + 1 = 0$

$(w-1)^4 + (w-1)^3 + (w-1)^2 + w = 0$

$\therefore$ The roots of

$(w-1)^4 + (w-1)^3 + (w-1)^2 + w = 0$

are $1 + \cos \frac{2k\pi}{5} + i\sin \frac{2k\pi}{5}, \quad k = 1, 2, 3, 4$

and none of the roots are real since $\sin \frac{2k\pi}{5} \neq 0$

for $k = 1, 2, 3, 4$.

$$\left| 1 + \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} \right|$$

$$= \sqrt{\left(1 + \cos \frac{2k\pi}{5}\right)^2 + \sin^2 \frac{2k\pi}{5}}$$

$$= \sqrt{1 + \cos \frac{2k\pi}{5} + \cos^2 \frac{2k\pi}{5} + \sin^2 \frac{2k\pi}{5}}$$

$$= \sqrt{1 + 2\cos \frac{2k\pi}{5} + 1}$$

$$= \sqrt{2 + 2\cos \frac{2k\pi}{5}}$$

$$= \sqrt{2 + 2\left(2\cos^2 \frac{k\pi}{5} - 1\right)}$$

$$= \sqrt{2 + 4\cos^2 \frac{k\pi}{5} - 2}$$

$$= \sqrt{4\cos^2 \frac{k\pi}{5}}$$

$$= \left| 2\cos \frac{k\pi}{5} \right|$$

$$= 2 \left| \cos \frac{k\pi}{5} \right|$$

$\therefore$ The two roots which have the smaller modulus are $1 + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$ and $1 + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$ since

$$\left|2\cos\frac{\pi}{5}\right| = \left|2\cos\frac{4\pi}{5}\right| > \left|2\cos\frac{2\pi}{5}\right| = \left|2\cos\frac{3\pi}{5}\right|$$

$$\arg\left(1 + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{4\pi}{5}}{1+\cos\frac{4\pi}{5}}\right)$$

$$= \tan^{-1}\left(\frac{2\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}}{1+2\cos^2\frac{2\pi}{5}-1}\right)$$

$$= \tan^{-1}\left(\frac{2\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}}{2\cos^2\frac{2\pi}{5}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{2\pi}{5}}{\cos\frac{2\pi}{5}}\right)$$

$$= \tan^{-1}\left(\tan\frac{2\pi}{5}\right)$$

$$= \frac{2\pi}{5}$$

$$\arg\left(1 + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{6\pi}{5}}{1 + \cos\frac{6\pi}{5}}\right)$$

$$= \tan^{-1}\left(\frac{2\sin\frac{3\pi}{5}\cos\frac{3\pi}{5}}{1 + 2\cos^2\frac{3\pi}{5} - 1}\right)$$

$$= \tan^{-1}\left(\frac{2\sin\frac{3\pi}{5}\cos\frac{3\pi}{5}}{2\cos^2\frac{3\pi}{5}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{3\pi}{5}}{\cos\frac{3\pi}{5}}\right)$$

$$= \tan^{-1}\left(\tan\frac{3\pi}{5}\right)$$

$$= \frac{3\pi}{5}$$

10. $\underset{\sim}{b}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\underset{\sim}{b}_2 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ $\underset{\sim}{b}_3 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}$ $\underset{\sim}{b}_4 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

$$k_1 \underset{\sim}{b}_1 + k_2 \underset{\sim}{b}_2 + k_3 \underset{\sim}{b}_3 = \underset{\sim}{0}$$

$$k_1 \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + k_2 \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + k_3 \begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}k_1 + k_2 + k_3\\ k_2 + k_3\\ k_3\\ 0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$k_1 + k_2 + k_3 = 0$$
$$k_2 + k_3 = 0$$
$$k_3 = 0$$

$$k_1 = k_2 = k_3 = 0$$

$\therefore$ $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$ are linearly independent.

Since V_1 is spanned by $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$ and $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$ are linearly independent, $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ forms a basis of V_1.

$$k_1 \underset{\sim}{b}_1 + k_2 \underset{\sim}{b}_2 + k_4 \underset{\sim}{b}_4 = \underset{\sim}{0}$$

$$k_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} k_1 + k_2 + k_4 \\ k_2 + k_4 \\ k_4 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$k_1 + k_2 + k_4 = 0$$
$$k_2 + k_4 = 0$$
$$k_4 = 0$$
$$k_4 = 0$$

$$k_1 = k_2 = k_4 = 0$$

$\therefore$ $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_4$ are linearly independent.

Since V_2 is spanned by $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_4$ and $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_4$ are linearly independent, $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_4\}$ forms a basis of V_2.

i) $V_1 \cup V_2$ is not a linear space since it is not closed under addition.

ii) Since $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ is a basis for V_1 and $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_4\}$ is a basis for V_2, a basis for the linear space $V_1 \cap V_2$ is $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2\}$ and the linear space $V_1 \cap V_2$ has dimension 2.

$$V_3 = \{q\underset{\sim}{b}_2 + r\underset{\sim}{b}_3 + s\underset{\sim}{b}_4 : q, r, s \in R\}$$

If $\underset{\sim}{a}_1 = q_1\underset{\sim}{b}_2 + r_1\underset{\sim}{b}_3 + s_1\underset{\sim}{b}_4$ and

$\underset{\sim}{a}_2 = q_2\underset{\sim}{b}_2 + r_2\underset{\sim}{b}_3 + s_2\underset{\sim}{b}_4$,

$$\underset{\sim}{a}_1 + \underset{\sim}{a}_2 = (q_1 + q_2)\underset{\sim}{b}_2 + (r_1 + r_2)\underset{\sim}{b}_3 + (s_1 + s_2)\underset{\sim}{b}_4 \in V_3$$

If c is a scalar and $\underset{\sim}{a} = q\underset{\sim}{b}_2 + r\underset{\sim}{b}_3 + s\underset{\sim}{b}_4$

$$c\underset{\sim}{a} = cq\underset{\sim}{b}_2 + cr\underset{\sim}{b}_3 + cs\underset{\sim}{b}_4 \in V_3$$

$\therefore$ V_3 is closed under addition and scalar multiplication.

If $k_2\underset{\sim}{b}_2 + k_3\underset{\sim}{b}_3 + k_4\underset{\sim}{b}_4 = \underset{\sim}{0}$

$$k_2\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + k_3\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + k_4\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}k_2 + k_3 + k_4\\k_2 + k_3 + k_4\\k_3 + k_4\\k_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$k_2 + k_3 + k_4 = 0$$
$$k_2 + k_3 + k_4 = 0$$
$$k_3 + k_4 = 0$$
$$k_4 = 0$$

$k_2 = k_3 = k_4 = 0$

$\therefore$ $\underset{\sim}{b}_2, \underset{\sim}{b}_3, \underset{\sim}{b}_4$ are linearly independent.

$\therefore$ V_3 is a linear space and has dimension 3.

$$\begin{pmatrix} 4 \\ 4 \\ 2 \\ 5 \end{pmatrix} = q\underset{\sim}{b}_2 + r\underset{\sim}{b}_3 + s\underset{\sim}{b}_4$$

$$= q\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} q + r + s \\ q + r + s \\ r + s \\ s \end{pmatrix}$$

$\therefore$
$$q + r + s = 4$$
$$q + r + s = 4$$
$$r + s = 2$$
$$s = 5$$

$q = 2 \quad r = -3 \quad s = 5$

$\therefore$ $\begin{pmatrix} 4 \\ 4 \\ 2 \\ 5 \end{pmatrix} \in V_3$

$$\begin{pmatrix} 5 \\ 4 \\ 2 \\ 5 \end{pmatrix} = q\underset{\sim}{b}_2 + r\underset{\sim}{b}_3 + s\underset{\sim}{b}_4$$

$$= q\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} q + r + s \\ q + r + s \\ r + s \\ s \end{pmatrix}$$

$$\begin{aligned} q + r + s &= 5 \\ q + r + s &= 4 \\ r + s &= 2 \\ s &= 5 \end{aligned}$$

no solution

$$\therefore \begin{pmatrix} 5 \\ 4 \\ 2 \\ 5 \end{pmatrix} \notin V_3$$

11. $$A = \begin{pmatrix} -1 & 1 & 4 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 4 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1-\lambda & 1 & 4 \\ 1 & 1-\lambda & -1 \\ 2 & 1 & 1-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (-1-\lambda)\left[(1-\lambda)(1-\lambda) + 1\right]$$
$$-1(1 - \lambda + 2) + 4(1 - 2(1-\lambda))$$
$$= -(\lambda + 1)\left[(\lambda - 1)^2 + 1\right] - (3 - \lambda)$$
$$+ 4(1 - 2 + 2\lambda)$$
$$= -(\lambda + 1)(\lambda^2 - 2\lambda + 2) - (3 - \lambda)$$
$$+ 4(-1 + 2\lambda)$$
$$= -(\lambda^3 - 2\lambda^2 + 2\lambda + \lambda^2 - 2\lambda + 2)$$
$$+ \lambda - 3 - 4 + 8\lambda$$
$$= -(\lambda^3 - \lambda^2 + 2) + 9\lambda - 7$$
$$= -\lambda^3 + \lambda^2 - 2 + 9\lambda - 7$$
$$= -\lambda^3 + \lambda^2 + 9\lambda - 9$$

when $|A - \lambda I| = 0$

$$-\lambda^3 + \lambda^2 + 9\lambda - 9 = 0$$

$$\lambda^3 - \lambda^2 - 9\lambda + 9 = 0$$

$$\lambda^2(\lambda - 1) - 9(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda^2 - 9) = 0$$

$$(\lambda - 1)(\lambda - 3)(\lambda + 3) = 0$$

$$\lambda = 1, 3, -3.$$

$\therefore$ The eigenvalues of A are $1, 3$ and -3.

when $\lambda = 1$.
$$\begin{pmatrix} -2 & 1 & 4 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -2 & 1 & 4 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s,\ s \in R$

$y = -2s$

$x = s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -2s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

When $\lambda = 3$: $\begin{pmatrix} -4 & 1 & 4 \\ 1 & -2 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -4 & 1 & 4 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -4 & 1 & 4 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{4r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-7},\ \frac{r_3}{5}} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s,\ s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

when $\lambda = -3$: $\begin{pmatrix} 2 & 1 & 4 \\ 1 & 4 & -1 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{-2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -7 & 6 & 0 \\ 0 & -7 & 6 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -7 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 7s,\ s \in R$

$y = 6s$

$$x + 4y - z = 0$$

$$x + 4(6s) - 7s = 0$$

$$x + 24s - 7s = 0$$

$$x = -17s$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -17s \\ 6s \\ 7s \end{pmatrix}$$

$$= s \begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$$

The eigenvalues of A are $1, 3$ and -3 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$.

$$B = A - kI$$

$$B\underset{\sim}{x} = (A - kI)\underset{\sim}{x}$$

$$= A\underset{\sim}{x} - kI\underset{\sim}{x}$$

$$= \lambda\underset{\sim}{x} - k\underset{\sim}{x}$$

$$= (\lambda - k)\underset{\sim}{x}$$

$\therefore$ If A has an eigenvalue λ with corresponding eigenvector $\underset{\sim}{x}$, B has an eigenvalue $\lambda - k$ with corresponding eigenvector $\underset{\sim}{x}$.

$\therefore$ B has eigenvalues $1-k$, $3-k$ and $-3-k$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$

If $\quad ^3 = PDP^{-1}$, where P is a non-singular matrix and D is a diagonal matrix,

$$P = \begin{pmatrix} 1 & 1 & -17 \\ -2 & 0 & 6 \\ 1 & 1 & 7 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} (1-k)^3 & 0 & 0 \\ 0 & (3-k)^3 & 0 \\ 0 & 0 & (-3-k)^3 \end{pmatrix}.$$

12. EITHER

$$C : \quad y = \frac{ax^2 + bx + c}{x + 4}$$

i)

$$\begin{array}{r|l} & ax + b - 4a \\ \hline x + 4 & ax^2 + bx + c \\ & \underline{ax^2 + 4ax} \\ & \quad (b - 4a)x + c \\ & \quad \underline{(b - 4a)x + 4b - 16a} \\ & \qquad c - 4b + 16a \end{array}$$

$$\therefore y = ax + b - 4a + \frac{c - 4b + 16a}{x + 4}$$

As $x \longrightarrow \pm\infty$ $y \longrightarrow ax + b - 4a$

$y = ax + b - 4a$ is an asymptote.

If $y = 2x - 5$ is an asymptote of C,

$a = 2$, $b - 4a = -5$

$b - 8 = -5$

$b = 3$

ii) $y = 2x - 5 + \dfrac{c + 20}{x + 4}$

$\dfrac{dy}{dx} = 2 - \dfrac{(c + 20)}{(x + 4)^2}$

Since C has a turning point at $x = -1$,

when $x = -1$, $\dfrac{dy}{dx} = 0$.

$x = -1$: $0 = 2 - \dfrac{(c + 20)}{9}$

$\dfrac{c + 20}{9} = 2$

$c + 20 = 18$

$c = -2$

iii) $y = \dfrac{2x^2 + 3x - 2}{x + 4}$

$(x + 4)y = 2x^2 + 3x - 2$

$xy + 4y = 2x^2 + 3x - 2$

$2x^2 + (3 - y)x - 2 - 4y = 0$

$A = 2 \quad B = 3 - y \quad C = -2 - 4y$

$B^2 - 4AC = (3 - y)^2 - 4(2)(-2 - 4y)$

$= 9 - 6y + y^2 + 16 + 32y$

$= y^2 + 26y + 25$

$= (y + 1)(y + 25)$

when $B^2 - 4AC < 0$,

$$(y+1)(y+25) < 0$$

$$-25 < y < -1.$$

$\therefore$ There are no points on C when

$-25 < y < -1$.

iv) $$y = \frac{2(x-7)^2 + 3(x-7) - 2}{x-3}$$

$$= \frac{2(x-7)^2 + 3(x-7) - 2}{(x-7)+4}$$

if $y = \dfrac{2x^2 + 3x - 2}{x+4}$

when $x = -1$: $y = -1$

when $y = -25$: $-25 = \dfrac{2x^2 + 3x - 2}{x+4}$

$$2x^2 + 3x - 2 = -25x - 100$$

$$2x^2 + 28x + 98 = 0$$

$$x^2 + 14x + 49 = 0$$

$$(x+7)^2 = 0$$

$$x = -7$$

$\therefore$ $(-1, -1)$ is a minimum point

and $(-7, -25)$ is a maximum point.

when $x = 0$. $y = \frac{-1}{2}$

when $y = 0$. $\frac{2x^2 + 3x - 2}{x + 4} = 0$

$$2x^2 + 3x - 2 = 0$$

$$(2x - 1)(x + 2) = 0$$

$$x = \frac{1}{2}, -2.$$

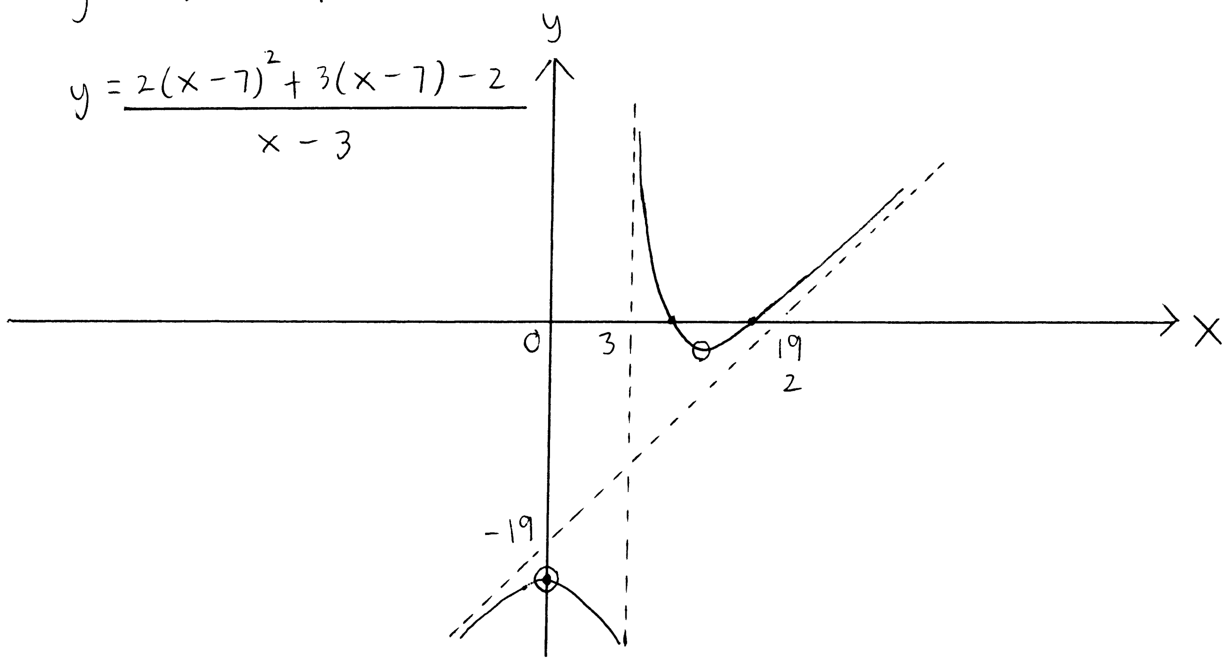
$\therefore$ If $y = \frac{2(x-7)^2 + 3(x-7) - 2}{x - 3}$,

$(6, -1)$ is a minimum point and

$(0, -25)$ is a maximum point.

The asymptotes of y are $x = 3$ and

$y = 2x - 19$



o : Critical point

• : Intersection point

OR

$$y\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 - 5y^2 = (5x^2 + 4x + 2)y^3$$

$$y = \frac{1}{w}$$

$$\frac{dy}{dw} = \frac{-1}{w^2}$$

$$\frac{dx}{dw}\frac{dy}{dx} = \frac{-1}{w^2}$$

$$\frac{dy}{dx} = -\frac{1}{w^2}\frac{dw}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{-1}{w^2}\frac{dw}{dx}\right)$$

$$= \frac{-1}{w^2}\frac{d}{dx}\left(\frac{dw}{dx}\right) + \frac{dw}{dx}\frac{d}{dx}\left(\frac{-1}{w^2}\right)$$

$$= \frac{-1}{w^2}\frac{d^2w}{dx^2} + \frac{dw}{dx}\left(\frac{dw}{dx}\right)\frac{d}{dw}\left(\frac{-1}{w^2}\right)$$

$$= \frac{-1}{w^2}\frac{d^2w}{dx^2} + \frac{2}{w^3}\left(\frac{dw}{dx}\right)^2$$

$$\frac{1}{w}\left(\frac{-1}{w^2}\frac{d^2w}{dx^2} + \frac{2}{w^3}\left(\frac{dw}{dx}\right)^2\right)$$

$$+ \frac{2}{w}\left(\frac{-1}{w^2}\frac{dw}{dx}\right) - 2\left(\frac{-1}{w^2}\frac{dw}{dx}\right)^2 - \frac{5}{w^2}$$

$$= (5x^2 + 4x + 2)\left(\frac{1}{w^3}\right)$$

$$\frac{-1}{w^3}\frac{d^2w}{dx^2} - \frac{2}{w^4}\left(\frac{dw}{dx}\right)^2 - \frac{2}{w^3}\frac{dw}{dx} - \frac{2}{w^4}\left(\frac{dw}{dx}\right)^2$$

$$-\frac{5}{w^2} = (5x^2 + 4x + 2)\left(\frac{1}{w^3}\right)$$

$$\frac{-1}{w^3}\frac{d^2w}{dx^2} - \frac{2}{w^3}\frac{dw}{dx} - \frac{5}{w^2} = (5x^2 + 4x + 2)\frac{1}{w^3}$$

$$\frac{d^2w}{dx^2} + \frac{2dw}{dx} + 5w = -5x^2 - 4x - 2$$

$$\frac{d^2w}{dx^2} + \frac{2dw}{dx} + 5 = 0$$

$\therefore$ The auxillary equation has the form

$$m^2 + 2m + 5 = 0$$

$$(m+1)^2 + 4 = 0$$

$$(m+1)^2 = -4$$

$$m + 1 = \pm 2i$$

$$m = -1 \pm 2i$$

$\therefore$ The complementary function is given by

$$w_c = e^{-x}(A\cos 2x + B\sin 2x)$$

The particular integral, w_p has the form

$$w_p = Cx^2 + Dx + E$$

$$\frac{dw_p}{dx} = 2Cx + D$$

$$\frac{d^2w_p}{dx^2} = 2C$$

$$\frac{d^2w_p}{dx^2} + 2\frac{dw_p}{dx} + 5w_p$$

$$= 2C + 2(2Cx + D) + 5(Cx^2 + Dx + E)$$

$$= 2C + 4Cx + 2D + 5Cx^2 + 5Dx + 5E$$

$$= 5Cx^2 + (4C + 5D)x + 2C + 2D + 5E$$

$$= -5x^2 - 4x - 2$$

$5C = -5 \qquad 4C + 5D = -4 \qquad 2C + 2D + 2E = -2$

$C = -1 \qquad\qquad D = 0 \qquad\qquad E = 0$

$$\therefore w_p = -x^2$$

$$w = w_c + w_p$$

$$= e^{-x}(A\cos 2x + B\sin 2x) - x^2$$

$\therefore$ The general solution has the form

$$w = e^{-x}(A\cos 2x + B\sin 2x) - x^2$$

$$\frac{1}{y} = e^{-x}(A\cos 2x + B\sin 2x) - x^2$$

$$= \frac{A\cos 2x + B\sin 2x}{e^x} - x^2$$

$$= \frac{A\cos 2x + B\sin 2x - x^2e^x}{e^x}$$

$$y = \frac{e^x}{A\cos 2x + B\sin 2x - x^2e^x}$$

$$= \frac{1}{e^{-x}(A\cos 2x + B\sin 2x) - x^2}$$

Since $e^{-x}(A\cos 2x + B\sin 2x) \to 0$ as $x \to \infty$,

$$y \to \frac{-1}{x^2}$$

If $f(x) = \frac{-1}{x^2}$

$$\therefore \lim_{x\to\infty} \frac{y}{f(x)} = \lim_{x\to\infty} \frac{y}{\frac{-1}{x^2}}$$

$$= 1$$