MAY / JUNE 2013

11

1. $r = 2(1+\cos\theta)$, $0 \leq \theta \leq 2\pi$

The area enclosed by the curve $r = 2(1+\cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$ is

$$\int_0^{2\pi} \frac{r^2}{2}\, d\theta = \int_0^{2\pi} \frac{2^2(1+\cos\theta)^2}{2}\, d\theta$$

$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} 2\left(1 + 2\cos\theta + \frac{\cos 2\theta + 1}{2}\right) d\theta$$

$$= \int_0^{2\pi} 2 + 4\cos\theta + \cos 2\theta + 1 \; d\theta$$

$$= \int_0^{2\pi} 3 + 4\cos\theta + \cos 2\theta \; d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 6\pi.$$

2. $5^{2n}-1$ is divisible by 8.

Let $f(n) = 5^{2n}-1$

when $n=1$: $f(1) = 5^{2(1)}-1 = 5^2-1 = 25-1 = 24 = 8(3)$

Assume the statement is true when $n=k$.

$n=k$: $f(k) = 5^{2k}-1$

$8 \mid f(k)$

$\therefore f(k) = 8s$, s is an integer

$5^{2k}-1 = 8s$

when $n=k+1$: $f(k+1) = 5^{2(k+1)}-1$

$= 5^{2k+2}-1$

$= 5^{2k}5^2-1$

$= 5^{2k}25-1$

$= (8s+1)25-1$

$= 200s+25-1$

$= 200s+24$

$= 8(25s+3)$

Since s is an integer, $8 \mid f(k+1)$

$\therefore f(n) = 5^{2n}-1$ is divisible by 8 for every positive integer n.

3. $x^3-2x^2-3x+4=0$

α, β, r are the roots

$\alpha+\beta+r=2 \quad \alpha\beta+\alpha r+\beta r=-3 \quad \alpha\beta r=-4$

$c=\alpha+\beta+r$

$\therefore c=2$

$\alpha+\beta, \alpha+r, \beta+r$

Let $y=\alpha+\beta$

$y+r=\alpha+\beta+r$

$=2$

$y=2-r$

$r=2-y$

r is a root

$\therefore r^3-2r^2-3r+4=0$

$(2-y)^3-2(2-y)^2-3(2-y)+4=0$

$8-12y+6y^2-y^3-2(4-4y+y^2)-6+3y+4=0$

$8-12y+6y^2-y^3-8+8y-2y^2-6+3y+4=0$

$-y^3+4y^2-y-2=0$

$y^3-4y^2+y+2=0$

$\therefore$ The equation y^3-4y^2+y+2 has roots $\alpha+\beta, \alpha+r, \beta+r$

$\frac{1}{\alpha+\beta}, \frac{1}{\alpha+r}, \frac{1}{\beta+r}$

let $u=\frac{1}{\alpha+\beta}$

$\alpha + \beta = \frac{1}{u}$

$\alpha + \beta$ is a root

$\therefore (\alpha + \beta)^3 - 4(\alpha + \beta)^2 + (\alpha + \beta) + 2 = 0$

$\frac{1}{u^3} - \frac{4}{u^2} + \frac{1}{u} + 2 = 0$

$1 - 4u + u^2 + 2u^3 = 0$

$2u^3 + u^2 - 4u + 1 = 0$

$\therefore$ The equation $2u^3 + u^2 - 4u + 1 = 0$

has roots $\frac{1}{\alpha + \beta}, \frac{1}{\alpha + \gamma}, \frac{1}{\beta + \gamma}$

$$\frac{1}{\alpha+\beta} + \frac{1}{\alpha+\gamma} + \frac{1}{\beta+\gamma} = \frac{-1}{2}$$

$$\frac{1}{(\alpha+\beta)(\alpha+\gamma)} + \frac{1}{(\alpha+\gamma)(\beta+\gamma)} + \frac{1}{(\alpha+\gamma)(\beta+\gamma)} = -2$$

$$\frac{1}{(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma)} = -\frac{1}{2}$$

$$\frac{1}{(\alpha+\beta)^2} + \frac{1}{(\alpha+\gamma)^2} + \frac{1}{(\beta+\gamma)^2}$$

$$= \left(\frac{1}{\alpha+\beta} + \frac{1}{\alpha+\gamma} + \frac{1}{\beta+\gamma}\right)^2$$

$$-2\left(\frac{1}{(\alpha+\beta)(\alpha+\gamma)} + \frac{1}{(\alpha+\gamma)(\beta+\gamma)} + \frac{1}{(\alpha+\gamma)(\beta+\gamma)}\right)$$

$$= \left(-\frac{1}{2}\right)^2 - 2(-2) = \frac{1}{4} + 4 = \frac{17}{4}.$$

4. $$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$$

$$= \int_0^1 (1+x^2)^{-n}\,dx$$

$$u = (1+x^2)^{-n} \qquad dv = dx$$

$$du = -2xn(1+x^2)^{-n-1} \qquad v = x$$

$$= \left[\frac{x}{(1+x^2)^n}\right]_0^1 - \int_0^1 \frac{-2x^2n}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} - 0 + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{1}{(1+x^2)^n}\,dx - 2n\int_0^1 \frac{1}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$$

$$\therefore\ 2nI_{n+1} = 2nI_n + I_n + 2^{-n}$$

$$= 2^{-n} + (2n+1)I_n$$

$$I_1 = \frac{\pi}{4}$$

$$2n I_{n+1} = 2^{-n} + (2n-1) I_n$$

$$n=2: \; 4I_3 = 2^{-2} + 3I_2$$

$$n=1: \; 2I_2 = 2^{-1} + I_1$$

$$I_1 = \frac{\pi}{4} \therefore 2I_2 = \frac{1}{2} + \frac{\pi}{4}$$

$$I_2 = \frac{1}{4} + \frac{\pi}{8}$$

$$4I_3 = \frac{1}{4} + 3\left(\frac{1}{4} + \frac{\pi}{8}\right)$$

$$= 1 + \frac{3\pi}{8}$$

$$\therefore I_3 = \frac{1}{4} + \frac{3\pi}{32}$$

5. $$\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)} = \sum_{r=1}^{N} \frac{1}{2(2r+1)} - \frac{1}{2(2r+3)}$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5} \right.$$
$$+ \frac{1}{5} - \frac{1}{7}$$
$$+ \frac{1}{7} - \frac{1}{9}$$
$$\vdots$$
$$+ \frac{1}{2N-3} - \frac{1}{2N-1}$$
$$+ \frac{1}{2N-1} - \frac{1}{2N+1}$$
$$\left. + \frac{1}{2N+1} - \frac{1}{2N+3} \right)$$
$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2N+3} \right)$$
$$= \frac{1}{2}\left(\frac{2N+3-3}{3(2N+3)} \right)$$
$$= \frac{N}{3(2N+3)}$$
$$= \frac{1}{6} - \frac{1}{2(2N+3)}.$$

$$\sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)} = \sum_{r=1}^{2N} \frac{1}{(2r+1)(2r+3)}$$
$$- \sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)}$$

$$= \frac{1}{6} - \frac{1}{2(4N+3)} - \left(\frac{1}{6} - \frac{1}{2(2N+3)}\right)$$

$$= \frac{1}{2(2N+3)} - \frac{1}{2(4N+3)}$$

$$= \frac{1}{2}\left(\frac{4N+3-2N-3}{(2N+3)(4N+3)}\right)$$

$$= \frac{N}{(2N+3)(4N+3)}$$

since $N \geqslant 1$, $18N+9 > 0$

$$8N^2 + 18N + 9 > 8N^2$$

$$(2N+3)(4N+3) > 8N^2$$

$$\frac{N}{(2N+3)(4N+3)} < \frac{1}{8N}$$

$$\therefore \sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)} < \frac{1}{8N}$$

6. $A = \begin{pmatrix} 4 & -5 & 3 \\ 3 & -4 & 3 \\ 1 & -1 & 2 \end{pmatrix}$ $\underset{\sim}{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$A\underset{\sim}{e} = \begin{pmatrix} 4 & -5 & 3 \\ 3 & -4 & 3 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2\underset{\sim}{e}$$

$\therefore$ $\underset{\sim}{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 2.

$$A - \lambda I = \begin{pmatrix} 4 & -5 & 3 \\ 3 & -4 & 3 \\ 1 & -1 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & -5 & 3 \\ 3 & -4-\lambda & 3 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$

$$|A - \lambda I| = [(-4-\lambda)(2-\lambda)+3](4-\lambda)$$
$$- -5(3(2-\lambda)-3) + 3(-3-(-4-\lambda))$$
$$= [(\lambda-2)(\lambda+4)+3](4-\lambda)$$
$$+5(3-3\lambda) + 3(1+\lambda)$$
$$= (\lambda^2+2\lambda-5)(4-\lambda) + 15 - 15\lambda + 3 + 3\lambda$$
$$= 4\lambda^2 + 8\lambda - 20 - \lambda^3 - 2\lambda^2 + 5\lambda + 18 - 12\lambda$$
$$= -\lambda^3 + 2\lambda^2 + \lambda - 2$$
$$= -\lambda^2(\lambda-2) + \lambda - 2$$
$$= (\lambda-2)(1-\lambda^2)$$
$$= (\lambda-2)(1-\lambda)(1+\lambda)$$

$\therefore$ The other two eigenvalues of A are 1 and -1.

$$B = \begin{pmatrix} -1 & 4 & 0 \\ -1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}$$

$$B\underset{\sim}{e} = \begin{pmatrix} -1 & 4 & 0 \\ -1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3\underset{\sim}{e}$$

$\therefore \underset{\sim}{e}$ is an eigenvector of B with eigenvalue 3.

If A has eigenvalue λ with eigenvector $\underset{\sim}{x}$ and B has eigenvalue μ with eigenvector $\underset{\sim}{x}$,

$$A\underset{\sim}{x} = \lambda\underset{\sim}{x}, \quad B\underset{\sim}{x} = \mu\underset{\sim}{x}$$

$$\begin{aligned} \therefore (AB)\underset{\sim}{x} &= A(B\underset{\sim}{x}) \\ &= A(\mu\underset{\sim}{x}) \\ &= \mu(A\underset{\sim}{x}) \\ &= \mu(\lambda\underset{\sim}{x}) \\ &= (\lambda\mu)\underset{\sim}{x} \end{aligned}$$

$\therefore AB$ has eigenvalue $\lambda\mu$ with eigenvector $\underset{\sim}{x}$.

$\therefore$ If $A = \begin{pmatrix} 4 & -5 & 3 \\ 3 & -4 & 3 \\ 1 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 4 & 0 \\ -1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}$

AB has eigenvector $\underset{\sim}{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with eigenvalue $3(2)=6$.

7. $\left(z-\frac{1}{z}\right)^6 = z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$

$z = e^{i\theta} = \cos\theta + i\sin\theta \qquad z^n = \cos n\theta + i\sin n\theta$

$(2i\sin\theta)^6 = \left(z-\frac{1}{z}\right)^6 \qquad z^{-n} = \cos n\theta - i\sin n\theta$

$\therefore z^n + \frac{1}{z^n} = 2i\sin n\theta$

$-64\sin^6\theta = z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$

$z = e^{i\theta} = \cos\theta + i\sin\theta \qquad z^n = \cos n\theta + i\sin n\theta$

$z^{-1} = \cos\theta - i\sin\theta \qquad z^{-n} = \cos n\theta - i\sin n\theta$

$z - \frac{1}{z} = 2i\sin\theta \qquad z^n + \frac{1}{z^n} = 2\cos n\theta$

$$-64\sin^6\theta = z^6 + \frac{1}{z^6} - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$= 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$32\sin^6\theta = 10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta$$

$$\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$$

$$= \frac{1}{32}(p + q\cos 2\theta + r\cos 4\theta + s\cos 6\theta)$$

$p = 10 \quad q = -15 \quad r = 6 \quad s = -1$

$$\int_0^{\frac{\pi}{4}} \sin^6\theta \, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)\,d\theta$$

$$= \frac{1}{32}\left[10\theta - \frac{15\sin 2\theta}{2} + \frac{3\sin 4\theta}{2} - \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{32}\left(\frac{5\pi}{2} - \frac{15}{2} + \frac{1}{6} - 0\right) = \frac{5\pi}{64} - \frac{11}{48}$$

8. $M_1 = \begin{pmatrix} 1 & -2 & 3 & 5 \\ 3 & -4 & 17 & 33 \\ 5 & -9 & 20 & 36 \\ 4 & -7 & 16 & 29 \end{pmatrix}$ $M_2 = \begin{pmatrix} 1 & -2 & 0 & -3 \\ 2 & -1 & 0 & 0 \\ 4 & -7 & 1 & -9 \\ 6 & -10 & 0 & -14 \end{pmatrix}$

$M_1 \underset{\sim}{x} = \underset{\sim}{0}$, $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ $M_2 \underset{\sim}{x} = \underset{\sim}{0}$, $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 3 & -4 & 17 & 33 & 0 \\ 5 & -9 & 20 & 36 & 0 \\ 4 & -7 & 16 & 29 & 0 \end{array}\right)$$

$$\xrightarrow[-4r_1 + r_4]{\substack{-3r_1 + r_2 \\ -5r_1 + r_3}} \left(\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 0 & 2 & 8 & 18 & 0 \\ 0 & 1 & 5 & 11 & 0 \\ 0 & 1 & 4 & 9 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{2}} \left(\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 0 & 1 & 4 & 9 & 0 \\ 0 & 1 & 5 & 11 & 0 \\ 0 & 1 & 4 & 9 & 0 \end{array}\right)$$

$$\xrightarrow[-r_2 + r_4]{-r_2 + r_3} \left(\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 0 & 1 & 4 & 9 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = s$, $s \in \mathbb{R}$

$z = -2s$

$y = -s$

$x = -s$

$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s \begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -3 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 4 & -7 & 1 & -9 & 0 \\ 6 & -10 & 0 & -14 & 0 \end{array}\right)$$

$$\xrightarrow[-6r_1+r_4]{\substack{-2r_1+r_2 \\ -4r_1+r_3}} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -3 & 0 \\ 0 & 3 & 0 & 6 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 2 & 0 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{3},\ \frac{r_4}{2}} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[-r_2+r_4]{-r_2+r_3} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

let $w = s,\ s \in \mathbb{R}$

$z = -s$

$y = -2s$

$x = -s$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s\begin{pmatrix} -1 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

If $M_1 \underset{\sim}{x}_1 = M_1 \underset{\sim}{a}$ and $M_2 \underset{\sim}{x}_2 = M_2 \underset{\sim}{a}$, $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$M_1(\underset{\sim}{x}_1 - \underset{\sim}{a}) = \underset{\sim}{0}$ $\qquad$ $M_2(\underset{\sim}{x}_2 - \underset{\sim}{a}) = \underset{\sim}{0}$

$\therefore \underset{\sim}{x}_1 - \underset{\sim}{a} = s\begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}$ $\qquad$ $\therefore \underset{\sim}{x}_2 - \underset{\sim}{a} = t\begin{pmatrix} -1 \\ -2 \\ -1 \\ 1 \end{pmatrix}$

If $\underset{\sim}{x}_1 - \underset{\sim}{x}_2 = \begin{pmatrix} p \\ s \\ 7 \\ q \end{pmatrix}$

$\underset{\sim}{x}_1 = \underset{\sim}{a} + s\begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}$

$\underset{\sim}{x}_2 = \underset{\sim}{a} + t\begin{pmatrix} -1 \\ -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ - \\ 1 \end{pmatrix}$

$$\underset{\sim}{x}_1 - \underset{\sim}{a} - \underset{\sim}{x}_2 + \underset{\sim}{a} = \begin{pmatrix} p \\ 5 \\ 7 \\ q \end{pmatrix}$$

$$\underset{\sim}{x}_1 - \underset{\sim}{a} - (\underset{\sim}{x}_2 - \underset{\sim}{a}) = \begin{pmatrix} p \\ 5 \\ 7 \\ q \end{pmatrix}$$

$$s\begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix} - t\begin{pmatrix} -1 \\ -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} p \\ 5 \\ 7 \\ q \end{pmatrix}$$

$$\begin{pmatrix} t-s \\ 2t-s \\ t-2s \\ s-t \end{pmatrix} = \begin{pmatrix} p \\ 5 \\ 7 \\ q \end{pmatrix}$$

$t - s = p$

$2t - s = 5$

$t - 2s = 7$

$s - t = q$

$2t - s = 5$

$t - 2s = 7$

$2t - 4s = 14$

$3s = -9$

$s = -3$

$t = 1$

$\therefore p = 4$

$q = -4$

9. $4\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 6e^{-2t}$, $t=0$ $x=\frac{5}{3}$ $\frac{dx}{dt}=\frac{7}{6}$.

$4m^2 + 4m + 1 = 0$

$(2m+1)^2 = 0$

$m = -\frac{1}{2}$

$\therefore$ The complementary function, x_c, is

$x_c = (At+B)e^{-\frac{t}{2}}$

The particular integral, x_p, is given by

$x_p = Ce^{-2t}$

$\frac{dx_p}{dt} = -2Ce^{-2t}$

$\frac{d^2x_p}{dt^2} = 4Ce^{-2t}$

$4\frac{d^2x_p}{dt^2} + 4\frac{dx_p}{dt} + x_p = 16Ce^{-2t} - 8Ce^{-2t} + Ce^{-2t}$

$= 9Ce^{-2t}$

$= 6e^{-2t}$

$9C = 6$

$C = \frac{2}{3}$

$x_p = \frac{2e^{-2t}}{3}$

$x = x_c + x_p$

$= (At+B)e^{-\frac{t}{2}} + \frac{2e^{-2t}}{3}$

$$\frac{dx}{dt} = Ae^{-\frac{t}{2}} - (At+B)\frac{e^{-\frac{t}{2}}}{2} - \frac{4e^{-2t}}{3}$$

$t=0 \quad x = \frac{5}{3} : \quad \frac{5}{3} = B + \frac{2}{3}$

$t=0 \quad \frac{dx}{dt} = \frac{7}{6} : \quad \frac{7}{6} = A - \frac{B}{2} - \frac{4}{3}$

$\therefore B = 1 \quad A = 3$

$$\therefore x = (3t+1)e^{-\frac{t}{2}} + \frac{2e^{-2t}}{3}$$

$$\lim_{t\to\infty} x = \lim_{t\to\infty} (3t+1)e^{-\frac{t}{2}} + \frac{2e^{-2t}}{3}$$

$$= 0$$

10. C: $y = \dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1}$

As $x \to \pm\infty$ $y \to 2$
As $x \to 1$ $y \to \pm\infty$
$\therefore$ The asymptotes of C are $x = 1$ and $y = 2$.
when $x = 0$: $y = -2$
when $y = 0$: $\dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1} = 0$

$2x^2 - 3x - 2 = 0$

$(2x+1)(x-2) = 0$

$x = 2, -\frac{1}{2}$

since $(x-7)^2 \geq 0$

$x^2 - 14x + 49 \geq 0$

$24x^2 - 36x - 24 \leq 25x^2 - 50x + 25$

or $(x^2-2x+1)y = 2x^2-3x-2$

$(y-2)x^2 - (2y-3)x + y + 2 = 0$

$a = y-2$ $b = -(2y-3)$ $c = y+2$

$b^2 - 4ac = 4y^2 - 12y + 9 - 4y^2 + 16$

$= -12y + 25$

$b^2 - 4ac \geq 0 \therefore -12y + 25 \geq 0$

$y \leq \frac{25}{12}$

$\dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1} \leq \dfrac{25}{12}$

$\therefore y \leq \frac{25}{12}$

$$\begin{array}{r} 2 \\ x^2-2x+1 \overline{\big) 2x^2 - 3x - 2} \\ \underline{2x^2 - 4x + 2} \\ x - 4 \end{array}$$

$y = 2 + \dfrac{x-4}{(x-1)^2}$

$= 2 + \dfrac{x-1-3}{(x-1)^2} = 2 + \dfrac{1}{x-1} - \dfrac{3}{(x-1)^2}$

$\dfrac{dy}{dx} = \dfrac{-1}{(x-1)^2} + \dfrac{6}{(x-1)^3}$

when $\dfrac{dy}{dx} = 0$:

$$\frac{-1}{(x-1)^2}+\frac{6}{(x-1)^3}=0$$

$$\frac{1}{(x-1)^2}=\frac{6}{(x-1)^3}$$

$$x-1=6$$

$$x=7$$

$$y=\frac{25}{12}$$

If $y=2$

$$2+\frac{x-4}{(x-1)^2}=2$$

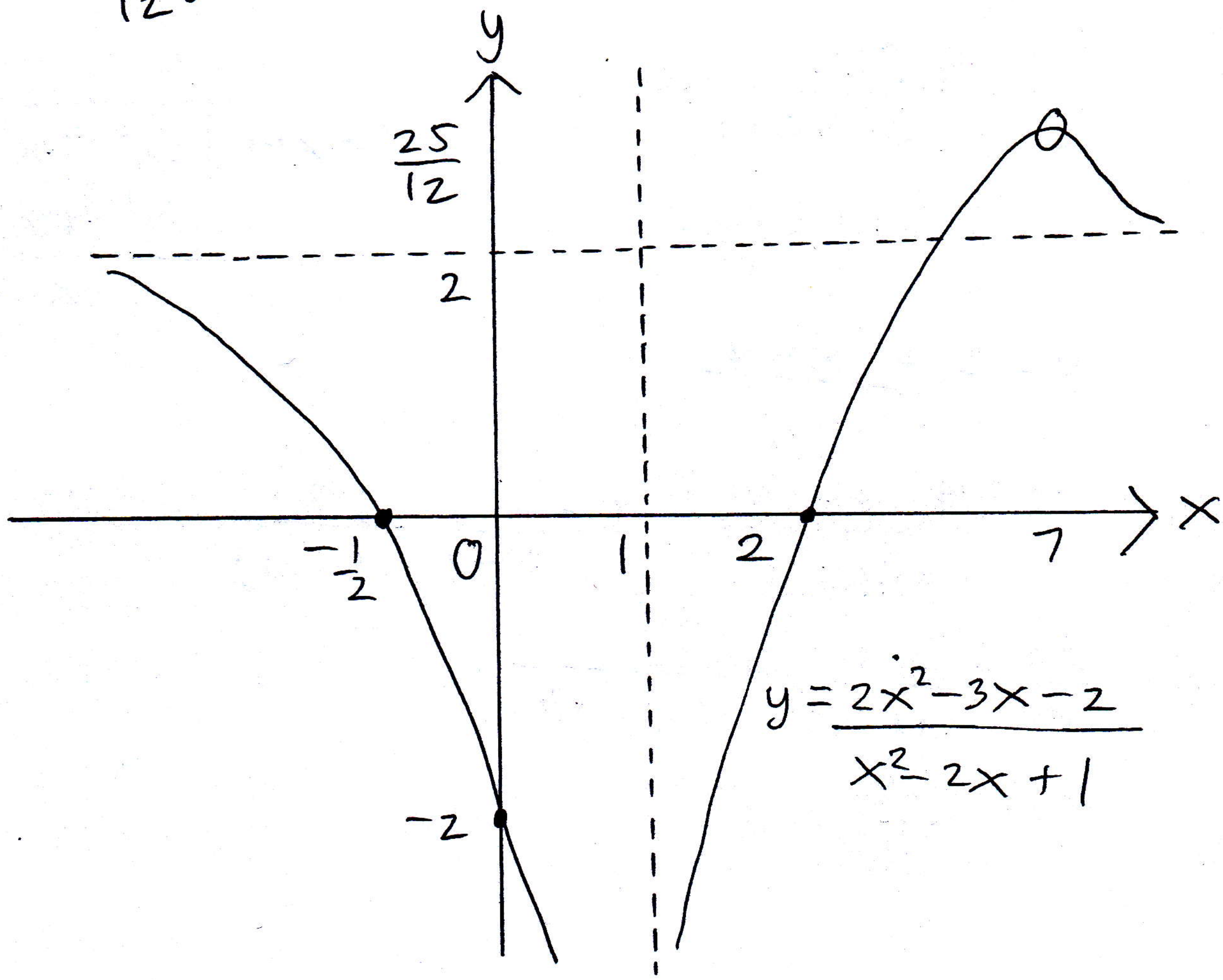
$$\frac{x-4}{(x-1)^2}=0$$

$$x=4$$

$$\frac{d^2y}{dx^2}=\frac{2}{(x-1)^3}-\frac{18}{(x-1)^4}$$

when $x=7$: $\dfrac{d^2y}{dx^2}=\dfrac{1}{108}-\dfrac{1}{72}=\dfrac{-1}{216}<0$

$\therefore \left(7,\frac{25}{12}\right)$ is a maximum point.



11 EITHER

$C: y = 2\sec x, \quad 0 \leq x \leq \frac{\pi}{4}$

$$\frac{dy}{dx} = 2\sec x \tan x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + (2\sec x \tan x)^2$$

$$= 1 + 4\sec^2 x \tan^2 x$$

$$= 1 + 4\sec^2 x(\sec^2 x - 1)$$

$$= 1 + 4\sec^4 x - 4\sec^2 x$$

$$= (2\sec^2 x - 1)^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2\sec^2 x - 1$$

The arc length s is

$$s = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} 2\sec^2 x - 1\, dx$$

$$= \left[2\tan x - x\right]_0^{\frac{\pi}{4}}$$

$$= 2 - \frac{\pi}{4}$$

i) The surface area of revolution about the x-axis of 360° S is

$$S = \int_0^{\frac{\pi}{4}} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} 2\pi(2\sec x)(2\sec^2 x - 1)\, dx$$

$$= 4\pi \int_0^{\frac{\pi}{4}} 2\sec^3 x - \sec x\, dx$$

ii) $\frac{d}{dx}(\sec x \tan x) = \sec x \sec^2 x + \tan x \sec x \tan x$

$$= \sec^3 x + \tan^2 x \sec x$$

$$= \sec^3 x + (\sec^2 x - 1)\sec x$$

$$= \sec^3 x + \sec^3 x - \sec x$$

$$= 2\sec^3 x - \sec x$$

$$\therefore S = 4\pi \int_0^{\frac{\pi}{4}} 2\sec^3 x - \sec x\, dx$$

$$= 4\pi \left[\sec x \tan x\right]_0^{\frac{\pi}{4}}$$

$$= 4\pi \left(\frac{1}{\frac{1}{\sqrt{2}}} - 0\right)$$

$$= 4\sqrt{2}\pi$$

11 OR

$A(2,1,-2)$ $B(4,1,-1)$ $C(3,-2,-1)$ $D(3,6,2)$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix}$

since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel to Π_1, $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to Π_1.

since $\begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix}$ is normal to Π_1 and $A(2,1,-2)$ is a point on Π_1, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is any point on Π_1, $\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix} = 6 - 1 + 12$$

$$3x - y - 6z = 17$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = |\overrightarrow{AB}||\overrightarrow{AC}| \sin\theta$$

$$\frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{|\overrightarrow{AB}||\overrightarrow{AC}| \sin\theta}{2}$$

$$= \frac{1}{2}\left|\begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix}\right| = \frac{1}{2}\sqrt{46} = \frac{\sqrt{46}}{2}$$

$\therefore$ The area of triangle ABC is $\frac{\sqrt{46}}{2}$.

The equation of the line through D perpendicular to Π_1 is $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix}$.

when the line meets Π_1, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+3s \\ 6-s \\ 2-6s \end{pmatrix}$

$x = 3+3s \quad y = 6-s \quad z = 2-6s$

$3x - y - 6z = 17$

$3(3+3s) - (6-s) - 6(2-6s) = 17$

$9 + 9s - 6 + s - 12 + 36s = 17$

$46s = 26$

$s = \frac{13}{23}$

$\therefore$ The line meets Π_1 at $\left(\frac{108}{23}, \frac{125}{23}, -\frac{32}{23}\right)$

$\therefore$ The distance D to Π_1 is

$$\sqrt{\left(\frac{108}{23}-3\right)^2 + \left(\frac{125}{23}-6\right)^2 + \left(\frac{-32}{23}-2\right)^2}$$

$$= \sqrt{9\left(\frac{13}{23}\right)^2 + \left(\frac{13}{23}\right)^2 + 36\left(\frac{13}{23}\right)^2}$$

$$= \sqrt{46\left(\frac{13}{23}\right)^2}$$

$$= \frac{13}{23}\sqrt{46} = \frac{26}{46}\sqrt{46} = \frac{26}{\sqrt{46}}$$

$\therefore$ The volume of the tetrahedron ABCO is

$$\frac{1}{3}\left(\frac{\sqrt{46}}{2}\right)\frac{13}{23}\sqrt{46} = \frac{13}{3} \text{ unit}^3.$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -7 \\ 10 \end{pmatrix}$$

since $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are parallel to Π_2, $\overrightarrow{AB} \times \overrightarrow{AD}$ is perpendicular to Π_2.

Since $\begin{pmatrix} -5 \\ -7 \\ 10 \end{pmatrix}$ is normal to Π_2 and $A(2,1,-2)$ is a point on Π_2, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is any point on Π_2,

$$\underset{\sim}{r} \cdot \begin{pmatrix} -5 \\ -7 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -7 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -7 \\ 10 \end{pmatrix} = -10 - 7 - 20$$

$$-5x - 7y + 10z = -37$$

$$5x + 7y - 10z = 37$$

$$\begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 7 \\ -10 \end{pmatrix} = \left| \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix} \right| \left| \begin{pmatrix} 5 \\ 7 \\ -10 \end{pmatrix} \right| \cos\theta$$

$$15 - 7 + 60 = \sqrt{46}\sqrt{174}\cos\theta$$

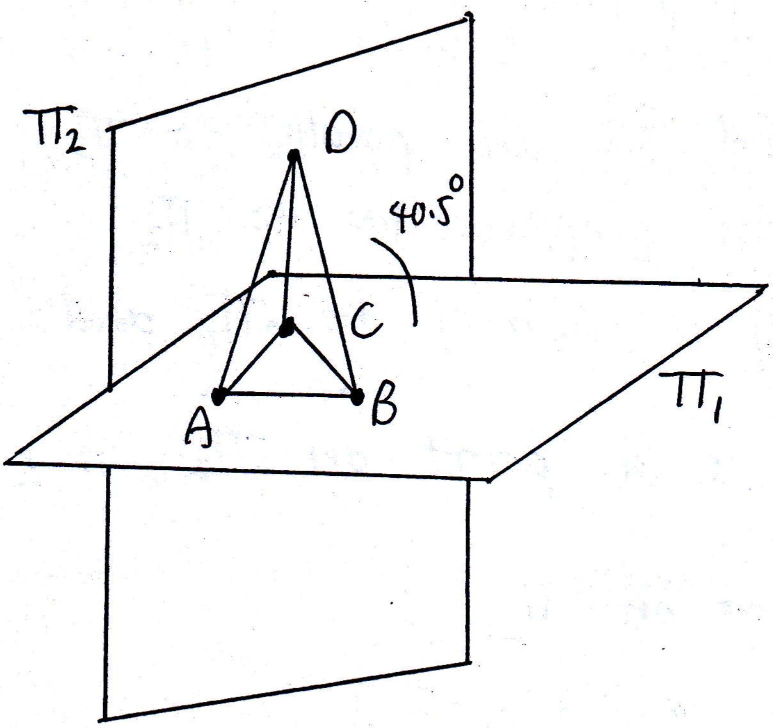
$$\cos\theta = \frac{68}{\sqrt{46}\sqrt{174}}$$

$$\theta = \cos^{-1} \frac{68}{\sqrt{46}\sqrt{174}}$$

$$\approx 40.5^0$$

$$180^0 - 40.5^0 \approx 139.5^0$$

$\therefore$ The angle between Π_1 and Π_2 is 40.5^0.



Π_2
O
40.5°
C
A
B
Π_1