Polar Coordinates Questions

June 2013

8.

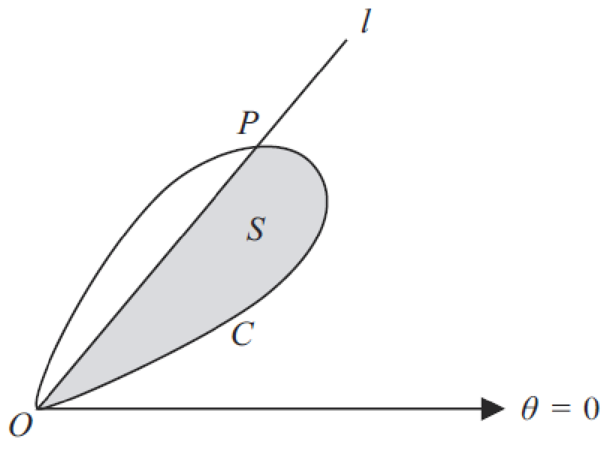

Figure 1

Figure 1 shows a curve C with polar equation $r = a\sin 2\theta,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}$, and a half-line l.

The half-line l meets C at the pole O and at the point P. The tangent to C at P is parallel to the initial line. The polar coordinates of P are (R, ϕ).

(a) Show that $\cos\phi = \frac{1}{\sqrt{3}}$

(6)

(b) Find the exact value of R.

(2)

The region S, shown shaded in Figure 1, is bounded by C and l.

(c) Use calculus to show that the exact area of S is

$$\frac{1}{36}a^2\left(9\arccos\left(\frac{1}{\sqrt{3}}\right) + \sqrt{2}\right)$$

(7)

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2. The curve C has polar equation

$$r = 1 + 2\cos\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point P on C, the tangent to C is parallel to the initial line.

Given that O is the pole, find the exact length of the line OP.

(7)

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6.

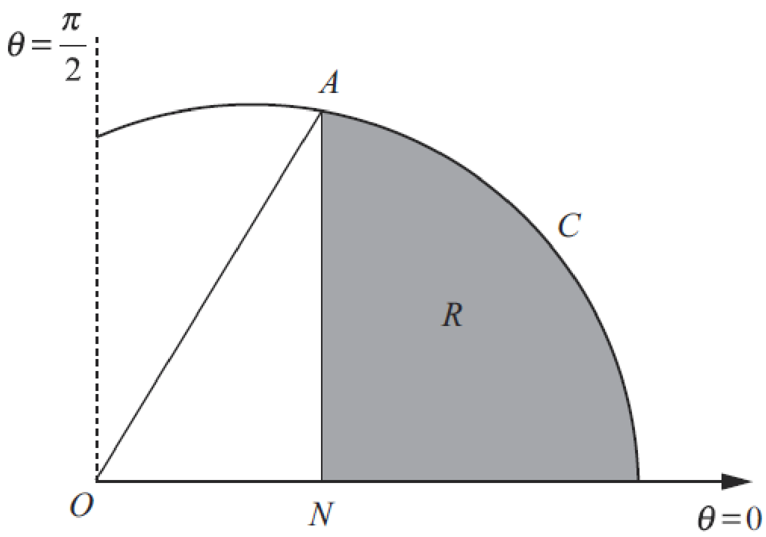

Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 2 + \cos\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point A on C, the value of r is $\frac{5}{2}$.

The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line AN.

Find the exact area of the shaded region R.

(9)

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5.

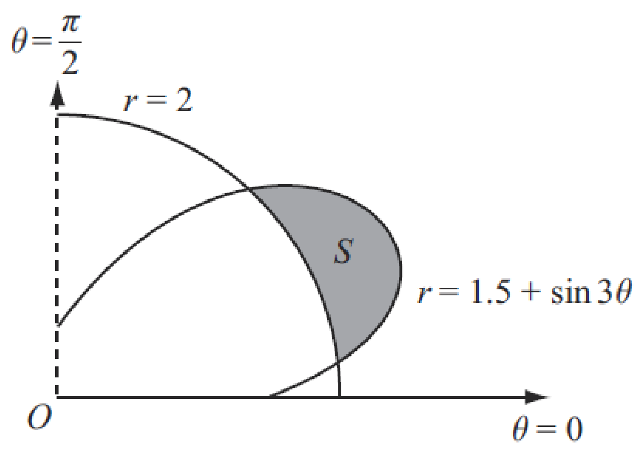

Figure 1

Figure 1 shows the curves given by the polar equations

$$r = 2, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

$$\text{and} \qquad r = 1.5 + \sin 3\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

(a) Find the coordinates of the points where the curves intersect.

(3)

The region S, between the curves, for which $r > 2$ and for which $r < (1.5 + \sin 3\theta)$, is shown shaded in Figure 1.

(b) Find, by integration, the area of the shaded region S, giving your answer in the form $a\pi + b\sqrt{3}$, where a and b are simplified fractions.

(7)

FP2 Polar Coordinates Questions ANSWERS (41 marks)

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8 (a)	$(y=)r\sin\theta = a\sin 2\theta \sin\theta$	M1
	$\left(\dfrac{dy}{d\theta}=\right)a\left(2\cos 2\theta\sin\theta+\sin 2\theta\cos\theta\right)$	M1depA1
	$\left(\dfrac{dy}{d\theta}=\right)2a\sin\theta\left(\cos 2\theta+\cos^2\theta\right)$	M1
	At P $\dfrac{dy}{d\theta}=0\Rightarrow\sin\theta=0$ (n/a) or $2\cos^2\theta-1+\cos^2\theta=0$	M1 $\sin\theta=0$ not needed
	$3\cos^2\theta=1$	
	$\cos\theta=\dfrac{1}{\sqrt{3}}$ *	A1cso
		(6)
(b)	$r=a\sin 2\theta=2a\sin\theta\cos\theta$	
	$r=2a\sqrt{\left(1-\dfrac{1}{3}\right)}\sqrt{\dfrac{1}{3}}=2a\dfrac{\sqrt{2}}{3}$	M1A1 (2)
(c)	Area $=\int_0^{\phi}\dfrac{1}{2}r^2 d\theta=\dfrac{1}{2}a^2\int_0^{\phi}\sin^2 2\theta\, d\theta$	M1
	$=\dfrac{1}{2}a^2\int_0^{\phi}\dfrac{1}{2}(1-\cos 4\theta)d\theta$	M1
	$=\dfrac{1}{4}a^2\left[\theta-\dfrac{1}{4}\sin 4\theta\right]_0^{\phi}$	M1A1
	$=\dfrac{1}{4}a^2\left[\phi-\dfrac{1}{4}\left(4\sin\phi\cos\phi\left(2\cos^2\phi-1\right)\right)\right]$	M1dep on 2nd M mark
	$=\dfrac{1}{4}a^2\left[\arccos\left(\dfrac{1}{\sqrt{3}}\right)-\left(\sqrt{\dfrac{2}{3}}\times\sqrt{\dfrac{1}{3}}\times\left(\dfrac{2}{3}-1\right)\right)\right]$	M1 dep (all Ms)
	$\dfrac{1}{36}a^2\left[9\arccos\left(\dfrac{1}{\sqrt{3}}\right)+\sqrt{2}\right]$ *	A1 (7)
		[15]

Notes for Question 8

(a)

M1 for obtaining the y coordinate $y = r\sin\theta = a\sin 2\theta\sin\theta$

M1dep for attempting the differentiation to obtain $\frac{dy}{d\theta}$ Product rule and/or chain rule must be used; sin to become $\pm\cos$ (cos to become $\pm\sin$). The 2 may be omitted. Dependent on the first M mark.

A1 for correct differentiation eg $\frac{dy}{d\theta} = a\left(2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta\right)$ oe

M1 for using $\sin 2\theta = 2\sin\theta\cos\theta$ anywhere in their solution to (a)

M1 for setting $\frac{dy}{d\theta} = 0$ and getting a quadratic factor with no $\sin^2\theta$ included.

Alternative: Obtain a quadratic in $\sin\theta$ or $\tan\theta$ and complete to $\cos\theta =$ later.

A1cso for $\cos\theta = \frac{1}{\sqrt{3}}$ or $\cos\phi = \frac{1}{\sqrt{3}}$ *

Question 8 (a) Variations you may see:

$y = r\sin\theta = a\sin 2\theta\sin\theta$

$y = a\sin 2\theta\sin\theta$	$y = 2a\sin^2\theta\cos\theta$	$y = 2a(\cos\theta - \cos^3\theta)$
$dy/d\theta = a(2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta)$ $= a(2\cos 2\theta\sin\theta + 2\sin\theta\cos^2\theta)$ $= 2a\sin\theta(\cos 2\theta + \cos^2\theta)$ $= 2a\sin\theta(3\cos^2\theta - 1)$ or $= 2a\sin\theta(2\cos^2\theta - \sin^2\theta)$ or $= 2a\sin\theta(2 - 3\sin^2\theta)$	$dy/d\theta = 2a(2\sin\theta\cos^2\theta - \sin^3\theta)$ $= 2a\sin\theta(2\cos^2\theta - \sin^2\theta)$	$dy/d\theta = 2a(-\sin\theta + 3\sin\theta\cos^2\theta)$ $= 2a\sin\theta(3\cos^2\theta - 1)$

At P: $dy/d\theta = 0 \Rightarrow \sin\theta = 0$ or:		
$2\cos^2\theta - \sin^2\theta = 0$	$3\cos^2\theta - 1 = 0$	$2 - 3\sin^2\theta = 0$
$\tan^2\theta = 2$	$\cos^2\theta = 1/3$	$\sin^2\theta = 2/3$
$\tan\theta = \pm\sqrt{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{3}}$	$\cos\theta = \pm\frac{1}{\sqrt{3}}$	$\text{Sin}\,\theta = \pm\frac{\sqrt{2}}{\sqrt{3}} = \pm\frac{\sqrt{6}}{3} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{3}}$

(b)

M1 for using $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos^2\theta + \sin^2\theta = 1$ and $\cos\phi = \frac{1}{\sqrt{3}}$ in $r = a\sin 2\theta$ to obtain a numerical multiple of a for R. Need not be simplified.

A1cao for $R = 2a\frac{\sqrt{2}}{3}$

Can be done on a calculator. Completely correct answer with no working scores 2/2; incorrect answer with no working scores 0/2

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2.	$y = r\sin\theta = \sin\theta + 2\sin\theta\cos\theta$	B1
	$\dfrac{dy}{d\theta} = \cos\theta + 2\cos 2\theta$	M1
	$4\cos^2\theta + \cos\theta - 2 = 0$	A1oe
	$\cos\theta = \dfrac{-1 \pm \sqrt{1+32}}{8}$	M1 A1
	$OP = r = 1 + \dfrac{-1+\sqrt{1+32}}{4} = \dfrac{3+\sqrt{33}}{4}$	M1 A1
		(7)
		7
	Notes B1 for $\sin\theta + 2\sin\theta\cos\theta$ or $\sin\theta(1 + 2\cos\theta)$ 1st M1 for use of Product Rule or Chain Rule (require 2 or condone ½) 1st A1 equation required 2nd M1 Valid attempt at solving 3 term quadratic (usual rules) to give $\cos\theta = \cdots$ 2nd A1 for exact or 3 dp or better (-0.843.....and 0.593....) 3rd M1 for 1+2x 'their $\cos\theta$' 3rd A1 for any form A0 if negative solution not discounted.	

6.	$2+\cos\theta=\frac{5}{2}\Rightarrow\theta=\frac{\pi}{3}$	B1	
	$\frac{1}{2}\int(2+\cos\theta)^2 d\theta=\frac{1}{2}\int(4+4\cos\theta+\cos^2\theta)d\theta$	M1	
	$=\frac{1}{2}\left[4\theta+4\sin\theta+\frac{\sin 2\theta}{4}+\frac{\theta}{2}\right]$	M1 A1	
	Substituting limits $\left(\frac{1}{2}\left[\frac{9\pi}{6}+4\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{8}\right]=\frac{1}{2}\left(\frac{3\pi}{2}+\frac{17\sqrt{3}}{8}\right)\right)$	M1	
	Area of triangle = $\frac{1}{2}(r\cos\theta)(r\sin\theta)=\frac{1}{2}\times\frac{25}{4}\times\frac{1}{2}\times\frac{\sqrt{3}}{2}\left(=\frac{25\sqrt{3}}{32}\right)$	M1 A1	
	Area of $R=\frac{3\pi}{4}+\frac{17\sqrt{3}}{16}-\frac{25\sqrt{3}}{32}=\frac{3\pi}{4}+\frac{9\sqrt{3}}{32}$	M1 A1	(9) **9**
	1st M1 for use of $\frac{1}{2}\int r^2\, d\theta$ and correct attempt to expand 2nd M1 for use of double angle formula - $\sin 2\theta$ required in square brackets 3rd M1 for substituting their limits 4th M1 for use of $\frac{1}{2}$ base x height 5th M1 area of sector – area of triangle Please note there are no follow through marks on accuracy.		

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5(a)	$1.5+\sin 3\theta = 2 \quad \rightarrow \quad \sin 3\theta = 0.5 \quad \therefore 3\theta = \frac{\pi}{6}\left(\text{or } \frac{5\pi}{6}\right),$ and $\therefore \theta = \frac{\pi}{18}$ or $\frac{5\pi}{18}$	M1 A1, A1 (3)
(b)	Area $= \frac{1}{2}\left[\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} (1.5+\sin 3\theta)^2 \mathrm{d}\theta\right], \ -\frac{1}{9}\pi \times 2^2$ $= \frac{1}{2}\left[\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} (2.25+3\sin 3\theta+\frac{1}{2}(1-\cos 6\theta))\mathrm{d}\theta\right] - \frac{1}{9}\pi \times 2^2$ $= \frac{1}{2}\left[(2.25\theta - \cos 3\theta + \frac{1}{2}(\theta - \frac{1}{6}\sin 6\theta))\right]_{\frac{\pi}{18}}^{\frac{5\pi}{18}} - \frac{1}{9}\pi \times 2^2$ $= \frac{13\sqrt{3}}{24} - \frac{5\pi}{36}$	M1, M1 M1 M1 A1 M1 A1 (7) **10**