1. $$\begin{vmatrix} 5 & 3 & 4 \\ \sin\theta & \cos\theta & \sin\theta \\ \cos\theta & -\sin\theta & \cos\theta \end{vmatrix}$$

$$= 5(\cos^2\theta + \sin^2\theta) - 3(\sin\theta\cos\theta - \sin\theta\cos\theta)$$

$$+ 4(-\sin^2\theta - \cos^2\theta)$$

$$= 5(1) - 3(0) + 4(-1)$$

$$= 5 - 0 - 4$$

$$= 1$$

2. $\underset{\sim}{p} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}, \quad \underset{\sim}{q} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$

i) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ or equivalent

ii) $$\left.\begin{aligned} x &= 1 + s + 2t \\ y &= 2 - 3s - t \\ z &= 3 - s - 4t \end{aligned}\right\}$$

$$x + z = 4 - 2t$$

$$3x + y = 3 + 3s + 6t + 2 - 3s - t$$

$$3x + y = 5 + 5t$$

$$5x + 5z = 20 - 10t$$

$$6x + 2y = 10 + 10t$$

$$\overline{11x + 2y + 5z = 30}$$

3. i) $a_n = 5^{2n+1} - 12n + 7$

When $n = 1$, $a_1 = 5^3 - 12 + 7 = 120 = 12 \times 10$

a_1 is divisible by 12

Suppose it is true when $n = k$,

$a_k = 5^{2k+1} - 12k + 7$ is divisible by 12

Consider $a_{k+1} - 25a_k$

$= 5^{2k+3} - 12(k+1) + 7 - 25(5^{2k+1} - 12k + 7)$

$= 25 \times 5^{2k+1} - 12k - 12 + 7 - 25 \times 5^{2k+1}$

$+ 25 \times 12k - 25 \times 7$

$= 24 \times 12k - 180$

$= 12(24k - 15)$

$a_{k+1} = 25a_k + 12(24k - 15)$ is divisible by 12.

a_n is divisible by 12 for all +ve integers n.

ii) Let $P(n)$ be $\sum_{r=1}^{n} \frac{1}{2r-1} - \frac{1}{2r+1} = \frac{2n}{2n+1}$

when $n = 1$ $LHS = 1 - \frac{1}{3} = \frac{2}{3}$

$RHS = \frac{2}{2+1} = \frac{2}{3}$

$P(1)$ is true

Suppose $P(k)$ is true $\Rightarrow \sum_{r=1}^{k} \frac{1}{2r-1} - \frac{1}{2r+1} = \frac{2k}{2k+1}$

when $n = k+1$, $P(k+1) = \frac{2k}{2k+1} + \frac{1}{2(k+1)-1} - \frac{1}{2(k+1)+1}$

$$= \frac{2k+1}{2k+1} - \frac{1}{(2k+3)}$$

$$= 1 - \frac{1}{2k+3}$$

$$= \frac{2k+2}{2k+3}$$

$$= \frac{2(k+1)}{2(k+1)+1}$$

$\therefore$ $P(k+1)$ is also true if $P(k)$ is true

$\therefore$ $P(n)$ is true for all +ve integers n.

4. i) $f(x) = \dfrac{x^2 - 2x + 1}{x + 2} = (x - 4) + \dfrac{9}{x + 2}$

a) Asymptotes : $y = x - 4$

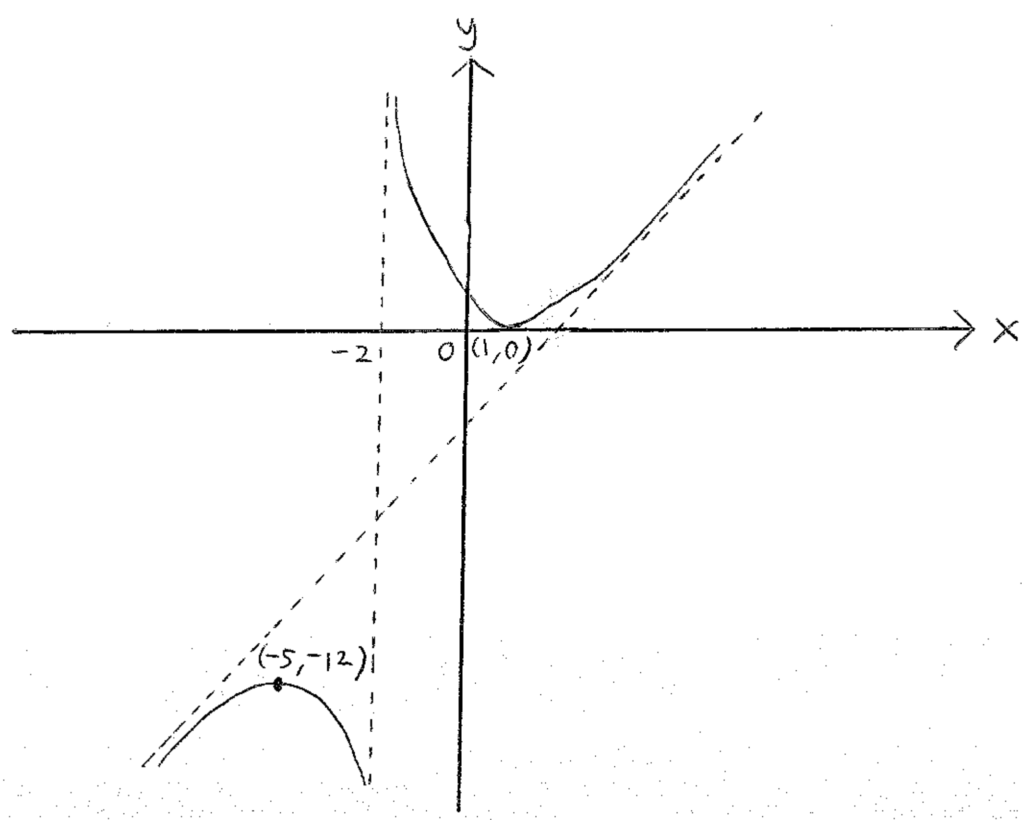
$x = -2$

b) $f'(x) = 1 - \dfrac{9}{(x+2)^2} = \dfrac{x^2 + 4x - 5}{(x+2)^2}$

$= \dfrac{(x-1)(x+5)}{(x+2)^2}$

Turning points are $(-5, -12)$, $(1, 0)$

c) when $x = 0$ $y = \frac{1}{2}$



ii) $k(x+2) = x^2 - 2x - 1$

$\therefore x^2 - 2x + 1 - kx - 2k = 0$

$x^2 - (k+2)x + (1-2k) = 0$

Real roots $\Rightarrow (2+k)^2 - 4(1-2k) \geq 0$

$k^2 + 4k + 4 - 4 + 8k \geq 0$

$k^2 + 12k \geq 0$

$k(k+12) \geq 0$

$\therefore k \leq -12$ or $k \geq 0$

5. l_1: $\underset{\sim}{r} = \begin{pmatrix}1\\3\\1\end{pmatrix} + s\begin{pmatrix}-2\\-1\\2\end{pmatrix}$ l_2: $\underset{\sim}{r} = \begin{pmatrix}0\\-2\\8\end{pmatrix} + t\begin{pmatrix}1\\-1\\1\end{pmatrix}$

i) $$\begin{pmatrix}-2\\-1\\2\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\1\end{pmatrix} = \sqrt{9}\sqrt{3}\cos\theta$$

$$\cos\theta = \frac{1}{3\sqrt{3}} \Rightarrow \theta = 78.9^\circ$$

ii) $$\underset{\sim}{n} = \begin{pmatrix}-2\\-1\\2\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\4\\3\end{pmatrix}$$

iii) $$x + 4y + 3z = d$$

Subst. $(1,3,1)$: $(1) + 4(3) + 3(1) = d$

$$d = 16$$

$$\therefore x + 4y + 3z = 16$$

iv) l_3: $\underset{\sim}{r} = \begin{pmatrix}2\\1\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\4\\3\end{pmatrix}$

v) $$(2+\lambda) + 4(1+4\lambda) + 3(5+3\lambda) = 16$$

$$26\lambda + 21 = 16$$

$$\lambda = \frac{-5}{26}$$

$x = 2 - \frac{5}{26}$, $y = 1 + 4\left(\frac{-5}{26}\right)$, $z = 5 + 3\left(\frac{-5}{26}\right)$

$= \frac{47}{26}$ $\quad = \frac{6}{26}$ $\quad = \frac{115}{26}$

coordinates of the foot of $\perp$ $= \left(\frac{47}{26}, \frac{6}{26}, \frac{115}{26}\right)$

vi) Dist from $(2,1,5)$ to the plane Π :

$$= \sqrt{\left(2 - \frac{47}{20}\right)^2 + \left(1 - \frac{6}{26}\right)^2 + \left(5 - \frac{115}{26}\right)^2}$$

$$= \sqrt{\left(\frac{5}{26}\right)^2 + \left(\frac{20}{26}\right)^2 + \left(\frac{15}{26}\right)^2}$$

$$= \frac{\sqrt{650}}{26}$$

$$= \frac{5\sqrt{26}}{26} \quad (0\ 981)$$

OR: $\text{Dist} = \frac{(2) + 4(1) + 3(5) - 16}{\sqrt{1^2 + 4^2 + 3^2}}$

$$= \frac{2 + 4 + 15 - 16}{\sqrt{26}}$$

$$= \frac{5}{\sqrt{26}}$$

$$= \frac{5\sqrt{26}}{26}$$

6 $$u_n = \frac{1}{n(n+1)\cdots(n+k)}, \quad n \geq 1, \quad k \geq 2$$

$$V_n = (n+k)u_n$$

Since $$u_n = \frac{1}{n(n+1)\cdots(n+k)},$$

$$V_n = \frac{n+k}{n(n+1)\cdots(n+k)}$$

$$= \frac{1}{n(n+1)\cdots(n+k-1)}$$

$$\therefore V_{n+1} = \frac{1}{(n+1)(n+1+1)\cdots(n+1+k-1)}$$

$$= \frac{1}{(n+1)(n+2)\cdots(n+k)}$$

$$V_n - V_{n+1} = \frac{1}{n(n+1)\cdots(n+k-1)}$$

$$- \frac{1}{(n+1)(n+2)\cdots(n+k)}$$

$$= \frac{1}{(n+1)(n+2)\cdots(n+k-1)}\left(\frac{1}{n} - \frac{1}{n+k}\right)$$

$$= \frac{1}{(n+1)(n+2)\cdots(n+k-1)}\left(\frac{n+k-n}{n(n+k)}\right)$$

$$= \frac{1}{(n+1)(n+2)\cdots(n+k-1)}\left(\frac{k}{n(n+k)}\right)$$

$$= \frac{k}{n(n+1)(n+2)\cdots(n+k-1)(n+k)}$$

$$= k\left(\frac{1}{n(n+1)(n+2)\cdots(n+k)}\right)$$

$$= k u_n$$

$$\therefore u_n = \frac{1}{k}(v_n - v_{n+1})$$

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \frac{1}{k}(v_n - v_{n+1})$$

$$= \frac{1}{k}\sum_{n=1}^{N} v_n - v_{n+1}$$

$$= \frac{1}{k}(v_1 - v_2$$

$$+ v_2 - v_3$$

$$+ v_3 - v_4$$

$$\vdots$$

$$+ v_{N-1} - v_N$$

$$+ v_N - v_{N+1} \Big)$$

$$= \frac{1}{k}\left(v_1 - v_{N+1}\right)$$

$$= \frac{1}{k}\left(\frac{k+1}{1(2)\cdots(k+1)} - \frac{k+N+1}{(N+1)(N+2)\cdots(k+N+1)}\right)$$

$$= \frac{1}{k}\left(\frac{k+1}{(k+1)!} - \frac{k+N+1}{(N+1)(N+2)\cdots(k+N+1)}\right)$$

$$= \frac{1}{k}\left(\frac{k+1}{(k+1)k!} - \frac{1}{(N+1)(N+2)\cdots(k+N)}\right)$$

$$= \frac{1}{k}\left(\frac{1}{k!} - \frac{1}{(N+1)(N+2)\cdots(N+k)}\right)$$

$$= \frac{1}{k}\left(\frac{1}{k!} - \frac{N!}{N!(N+1)(N+2)\cdots(N+k)}\right)$$

$$= \frac{1}{k}\left(\frac{1}{k!} - \frac{N!}{(N+k)!}\right)$$

Since $\displaystyle\sum_{n=1}^{N} u_n = \frac{1}{k}\left(\frac{1}{k!} - \frac{N!}{(N+k)!}\right)$

$$\therefore \lim_{N\to\infty}\left(\sum_{n=1}^{N} u_n\right) = \lim_{N\to\infty} \frac{1}{k}\left(\frac{1}{k!} - \frac{N!}{(N+k)!}\right)$$

$$= \lim_{N\to\infty} \frac{1}{k}\left(\frac{1}{k!}\right) - \lim_{N\to\infty} \frac{1}{k} \frac{N!}{(N+k)!}$$

$$= \frac{1}{k}\left(\frac{1}{k!}\right) - \frac{1}{k} \lim_{N\to\infty} \frac{N!}{(N+k)!}$$

$$= \frac{1}{k}\left(\frac{1}{k!}\right) - \frac{1}{k} \lim_{N\to\infty} \frac{1}{(N+k)(N+k-1)\cdots(N+1)}$$

$$= \frac{1}{k}\left(\frac{1}{k!}\right) - 0$$

$$= \frac{1}{k(k!)}$$

7. $3x^4 + 4x^3 - x^2 - 10x - 6 = 0$

$\alpha + \beta + \gamma + \delta = -\frac{4}{3}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -\frac{1}{3}$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{10}{3}$

$\alpha\beta\gamma\delta = -2$

Since $\alpha\beta = 2$,

$2\gamma\delta = -2$

$\therefore \gamma\delta = -1$

Substituting $\gamma\delta = -1$ and $\alpha\beta = 2$ into the equation

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{10}{3}$

$\therefore 2\gamma + 2\delta - \alpha - \beta = \frac{10}{3}$

Solving the equations $\alpha + \beta + \gamma + \delta = -\frac{4}{3}$ and

$2\gamma + 2\delta - \alpha - \beta = \frac{10}{3}$ simultaneously,

$\alpha + \beta + \gamma + \delta = -\frac{4}{3}$ — (1)

$2\gamma + 2\delta - \alpha - \beta = \frac{10}{3}$ — (2)

(1) + (2) : $2\gamma + 2\delta + \gamma + \delta = \frac{10}{3} - \frac{4}{3}$

$$3\gamma + 3\delta = 2$$

$$3(\gamma + \delta) = 2$$

$$\gamma + \delta = \frac{2}{3}$$

From equation ①: $\alpha + \beta + \frac{2}{3} = -\frac{4}{3}$

$$\therefore \alpha + \beta = -2$$

If $x^4 + \frac{4x^3}{3} - \frac{x^2}{3} - \frac{10x}{3} - 2$

$$= (x^2 - (\alpha + \beta)x + \alpha\beta)(x^2 - (\gamma + \delta)x + \gamma\delta)$$

$$x^4 + \frac{4x^3}{3} - \frac{x^2}{2} - \frac{10x}{3} - 2$$

$$= (x^2 - (-2)x + 2)(x^2 - \frac{2x}{3} - 1)$$

$$\therefore 3x^4 + 4x^3 - x^2 - 10x - 6 = (x^2 + 2x + 2)(3x^2 - 2x - 3)$$

8 $x^3 + px + q = 0$

$\alpha - \lambda,\ \beta - \lambda,\ \gamma - \lambda$

Let $y = \alpha - \lambda$

$\therefore \alpha = y + \lambda$

Since α is a root of $x^3 + px + q = 0$

$\therefore \alpha^3 + p\alpha + q = 0$

$(y+\lambda)^3 + p(y+\lambda) + q = 0$

$y^3 + 3y^2\lambda + 3y\lambda^2 + \lambda^3 + py + p\lambda + q = 0$

$y^3 + 3\lambda y^2 + (3\lambda^2 + p)y + \lambda^3 + p\lambda + q = 0$

$\therefore$ The equation having roots $\alpha - \lambda,\ \beta - \lambda,\ \gamma - \lambda$

is $y^3 + 3\lambda y^2 + (3\lambda^2 + p)y + \lambda^3 + p\lambda + q = 0$

$S_k = (\alpha - \lambda)^k + (\beta - \lambda)^k + (\gamma - \lambda)^k$

$\therefore AS_3 + BS_2 + CS_1 + DS_0 = 0,$

where $A = 1,\ B = 3\lambda,\ C = 3\lambda^2 + p,\ D = \lambda^3 + p\lambda + q$

since $\alpha - \lambda + \beta - \lambda + \gamma - \lambda = -3\lambda,$

$(\alpha - \lambda)(\beta - \lambda) + (\alpha - \lambda)(\gamma - \lambda) + (\beta - \lambda)(\gamma - \lambda) = 3\lambda^2 + p$

and $(\alpha - \lambda)(\beta - \lambda)(\gamma - \lambda) = -(\lambda^3 + p\lambda + q)$

$\therefore S_2 = (\alpha - \lambda)^2 + (\beta - \lambda)^2 + (\gamma - \lambda)^2$

$$= (\alpha - \lambda + \beta - \lambda + \gamma - \lambda)^2$$
$$-2[(\alpha - \lambda)(\beta - \lambda) + (\alpha - \lambda)(\gamma - \lambda) + (\beta - \lambda)(\gamma - \lambda)]$$
$$= (-3\lambda)^2 - 2(3\lambda^2 + p)$$
$$= 9\lambda^2 - 6\lambda^2 - 2p$$
$$= 3\lambda^2 - 2p.$$

$$S_0 = (\alpha - \lambda)^0 + (\beta - \lambda)^0 + (\gamma - \lambda)^0 = 1 + 1 + 1 = 3$$

$$S_1 = (\alpha - \lambda)^1 + (\beta - \lambda)^1 + (\gamma - \lambda)^1$$
$$= \alpha - \lambda + \beta - \lambda + \gamma - \lambda$$
$$= -3\lambda$$

Since $AS_3 + BS_2 + CS_1 + DS_0 = 0$,

where $A = 1$, $B = 3\lambda$, $C = 3\lambda^2 + p$, $D = \lambda^3 + p\lambda + q$

and $S_0 = 3$, $S_1 = -3\lambda$, $S_2 = 3\lambda^2 - 2p$

$$\therefore 1(S_3) + 3\lambda(3\lambda^2 - 2p) + (3\lambda^2 + p)(-3\lambda) + 3(\lambda^3 + p\lambda + q) = 0$$
$$S_3 + 9\lambda^3 - 6\lambda p - 9\lambda^3 - 3\lambda p + 3\lambda^3 + 3p\lambda + 3q = 0$$
$$S_3 + 3\lambda^3 - 6\lambda p + 3q = 0$$
$$\therefore S_3 = 6\lambda p - 3\lambda^3 - 3q$$