OCTOBER / NOVEMBER 2011

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1. $x^3 + px + q = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \alpha\gamma + \beta\gamma = p \quad \alpha\beta\gamma = -q$

If $\alpha = \beta$,

$\beta + \beta + \gamma = 0 \quad \beta^2 + \beta\gamma + \beta\gamma = p \quad \beta^2\gamma = -q$

$2\beta + \gamma = 0 \quad \beta^2 + 2\beta\gamma = p \quad \beta^2\gamma = -q$

$\gamma = -2\beta \quad \beta^2 + 2\beta(-2\beta) = p \quad \beta^2(-2\beta) = -q$

$\beta^2 - 4\beta^2 = p \quad -2\beta^3 = -q$

$-3\beta^2 = p \quad \beta^3 = \dfrac{q}{2}$

$\beta^2 = -\dfrac{p}{3}$

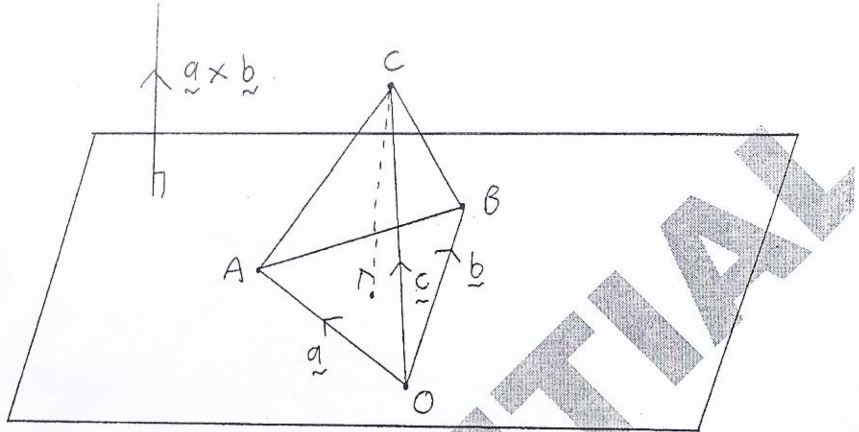
$$\left(-\frac{p}{3}\right)^3 = \left(\frac{q}{2}\right)^2$$

$$-\frac{p^3}{27} = \frac{q^2}{4}$$

$$-4p^3 = 27q^2$$

$$4p^3 + 27q^2 = 0$$

2.



$OA = \underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} \qquad OB = \underset{\sim}{b} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k} \qquad OC = \underset{\sim}{c} = 3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 2 & -1 \\ 4 & -3 & 2 \end{vmatrix} = \underset{\sim}{i} - 10\underset{\sim}{j} - 17\underset{\sim}{k}$$

Since $|\underset{\sim}{a} \times \underset{\sim}{b}| = \left| |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \, \underset{\sim}{n} \right|$

$= |\underset{\sim}{a}||\underset{\sim}{b}||\sin\theta||\underset{\sim}{n}|$

$= |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \, (1)$

$= |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$,

$$\frac{|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta}{2} = \frac{|\underset{\sim}{a} \times \underset{\sim}{b}|}{2} = \frac{|\underset{\sim}{i} - 10\underset{\sim}{j} - 17\underset{\sim}{k}|}{2} = \frac{\sqrt{390}}{2}$$

$\therefore$ The area of the triangle OAB is $\dfrac{\sqrt{390}}{2}$.

Since $\underset{\sim}{a} \times \underset{\sim}{b}$ is perpendicular to the plane, the equation of the line perpendicular to the plane through C, $\underset{\sim}{c}$, is

$$\underset{\sim}{r} = \underset{\sim}{c} + s(\underset{\sim}{a} \times \underset{\sim}{b})$$

$$= \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -10 \\ -17 \end{pmatrix}$$

Since $\underset{\sim}{i} - 10\underset{\sim}{j} - 17\underset{\sim}{k}$ is perpendicular to the plane and O is a point on the plane, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is any point on the plane, the equation of the plane is $\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ -10 \\ -17 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -10 \\ -17 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -10 \\ -17 \end{pmatrix} = 0 + 0 + 0$$

$$x - 10y - 17z = 0$$

When the line perpendicular to the plane through C meets the plane,

$x = 3 + s,\ y = -1 - 10s,\ z = -1 - 17s$

$3 + s - 10(-1 - 10s) - 17(-1 - 17s) = 0$

$3 + s + 10 + 100s + 17 + 289s = 0$

$$390s = -30$$

$$s = -\frac{1}{13}.$$

$\therefore$ The line meets the plane at $\left(\frac{38}{13}, \frac{-3}{13}, \frac{4}{13}\right)$.

The distance from C to the point the line meets the plane is

$$\sqrt{\left(3 - \frac{38}{13}\right)^2 + \left(-1 + \frac{3}{13}\right)^2 + \left(-1 - \frac{4}{13}\right)^2}$$

$$= \sqrt{\frac{1}{169} + \frac{100}{169} + \frac{289}{169}}$$

$$= \sqrt{\frac{390}{169}}$$

$$= \frac{\sqrt{390}}{13}$$

$\therefore$ The volume of the tetrahedron OABC is

$$\frac{1}{3} \times \frac{\sqrt{390}}{2} \times \frac{\sqrt{390}}{13} = 5.$$

3. $$\frac{d^n}{dx^n}(e^x \sin x) = 2^{\frac{n}{2}} e^x \sin\left(x + \frac{n\pi}{4}\right).$$

when $n=1$: $$\frac{d^1}{dx^1}(e^x \sin x) = \frac{d}{dx}(e^x \sin x)$$

$$= e^x \sin x + e^x \cos x$$

$$= e^x (\sin x + \cos x)$$

$$= e^x \sqrt{2}\left(\sin x\left(\frac{1}{\sqrt{2}}\right) + \cos x\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= \sqrt{2} e^x \sin\left(x + \frac{\pi}{4}\right)$$

$$= \sqrt{2} e^x \sin\left(x + (1)\frac{\pi}{4}\right)$$

Assume the statement is true when $n=k$.

$n=k$: $$\frac{d^k}{dx^k}(e^x \sin x) = 2^{\frac{n}{2}} e^x \sin\left(x + \frac{k\pi}{4}\right).$$

when $n=k+1$:

$$\frac{d^{k+1}}{dx^{k+1}}(e^x \sin x) = 2^{\frac{n}{2}} e^x \sin\left(x + (k+1)\frac{\pi}{4}\right).$$

(what needs to be proved)

$$\frac{d^{k+1}}{dx^{k+1}}(e^x \sin x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(e^x \sin x)\right)$$

$$= \frac{d}{dx}\left(2^{\frac{n}{2}} e^x \sin\left(x + \frac{k\pi}{4}\right)\right)$$

$$= 2^{\frac{n}{2}} \frac{d^k}{dx^k}\left(e^x \sin\left(x + \frac{k\pi}{4}\right)\right)$$

$$= 2^{\frac{n}{2}}\left(e^x \sin\left(x + \frac{k\pi}{4}\right) + e^x \cos\left(x + \frac{k\pi}{4}\right)\right)$$

$$= 2^{\frac{n}{2}} \sqrt{2} e^x \left(\frac{\sin\left(x + \frac{k\pi}{4}\right)}{\sqrt{2}} + \frac{\cos\left(x + \frac{k\pi}{4}\right)}{\sqrt{2}}\right)$$

$$= 2^{\frac{n+1}{2}} e^x \left(\sin\left(x + \frac{k\pi}{4}\right)\cos\frac{\pi}{4} + \cos\left(x + \frac{k\pi}{4}\right)\sin\frac{\pi}{4}\right)$$

$$= 2^{\frac{n+1}{2}} e^x \sin\left(x + \frac{k\pi}{4} + \frac{\pi}{4}\right)$$

$$= 2^{\frac{n+1}{2}} e^x \sin\left(x + (k+1)\frac{\pi}{4}\right)$$

$$\therefore \frac{d^n}{dx^n}(e^x \sin x) = 2^{\frac{n}{2}} e^x \sin\left(x + \frac{n\pi}{4}\right)$$

for every positive integer n.

4. $T: R^4 \to R^4$

$$M = \begin{pmatrix} 3 & 4 & 2 & 5 \\ 6 & 7 & 5 & 8 \\ 9 & 9 & 9 & 9 \\ 15 & 16 & 14 & 17 \end{pmatrix}$$

i) $$\xrightarrow[-5r_1+r_4]{-2r_1+r_2,\ -3r_1+r_3} \begin{pmatrix} 3 & 4 & 2 & 5 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & 3 & -6 \\ 0 & -4 & 4 & -8 \end{pmatrix}$$

$$\xrightarrow[-4r_2+r_4]{-3r_2+r_3} \begin{pmatrix} 3 & 4 & 2 & 5 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the range space of T is $\left\{ \begin{pmatrix} 3 \\ 6 \\ 9 \\ 15 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 9 \\ 16 \end{pmatrix} \right\}$.

$\therefore$ rank $(M) = 2$

ii) If $$\begin{pmatrix} 3 & 4 & 2 & 5 \\ 6 & 7 & 5 & 8 \\ 9 & 9 & 9 & 9 \\ 15 & 16 & 14 & 17 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 4 & 2 & 5 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $w = s,\ s \in R$ and $z = t,\ t \in R$

$y = t - 2s$

$3x + 4y + 2z + 5w = 0$

$3x + 4t - 8s + 2t + 5s = 0$

$3x = 3s - 6t$

$x = s - 2t$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} s-2t \\ t-2s \\ t \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ A basis for the null space of T is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

5. $x^3 - 2y^3 = 3xy \qquad P(2,1)$

i) $\frac{d}{dx}(x^3 - 2y^3) = \frac{d}{dx}(3xy)$

$3x^2 - 6y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$

At $P(2,1)$: $12 - 6\frac{dy}{dx} = 3 + 6\frac{dy}{dx}$

$12\frac{dy}{dx} = 9$

$\frac{dy}{dx} = \frac{3}{4}$

ii) $\frac{d}{dx}\left(3x^2 - 6y^2\frac{dy}{dx}\right) = \frac{d}{dx}\left(3y + 3x\frac{dy}{dx}\right)$

$6x - \frac{d}{dx}\left(6y^2\frac{dy}{dx}\right) = \frac{3dy}{dx} + \frac{d}{dx}\left(3x\frac{dy}{dx}\right)$

$6x - 12y\left(\frac{dy}{dx}\right)^2 - 6y^2\frac{d^2y}{dx^2} = \frac{3dy}{dx} + \frac{3dy}{dx} + \frac{3xd^2y}{dx^2}$

$6x - 12y\left(\frac{dy}{dx}\right)^2 - 6y^2\frac{d^2y}{dx^2} = \frac{6dy}{dx} + 3x\frac{d^2y}{dx^2}$

$2x - 4y\left(\frac{dy}{dx}\right)^2 - 2y^2\frac{d^2y}{dx^2} = \frac{2dy}{dx} + x\frac{d^2y}{dx^2}$

At $P(2,1)$, $\frac{dy}{dx} = \frac{3}{4}$: $4 - 4\left(\frac{9}{16}\right) - \frac{2d^2y}{dx^2} = \frac{3}{2} + \frac{2d^2y}{dx^2}$

$\therefore \frac{d^2y}{dx^2} = \frac{1}{16}$

6. $I_n = \int_0^1 x^n(1-x)^{\frac{1}{2}}\,dx, \quad n \geq 0$

$u = x^n \qquad dv = (1-x)^{\frac{1}{2}}dx$

$du = nx^{n-1}dx \qquad v = -\frac{2}{3}(1-x)^{\frac{3}{2}}$

$= \left[\frac{-2x^n}{3}(1-x)^{\frac{3}{2}}\right]_0^1 - \int_0^1 \frac{-2nx^{n-1}}{3}(1-x)^{\frac{3}{2}}\,dx$

$= 0 + \frac{2n}{3}\int_0^1 x^{n-1}(1-x)^{\frac{3}{2}}\,dx$

$= \frac{2n}{3}\int_0^1 x^{n-1}(1-x)^{\frac{1}{2}}(1-x)\,dx$

$= \frac{2n}{3}\int_0^1 x^{n-1}(1-x)^{\frac{1}{2}} - x^{n-1}x(1-x)^{\frac{1}{2}}\,dx$

$= \frac{2n}{3}\int_0^1 x^{n-1}(1-x)^{\frac{1}{2}}dx - \frac{2n}{3}\int_0^1 x^n(1-x)^{\frac{1}{2}}dx$

$= \frac{2n}{3}I_{n-1} - \frac{2n}{3}I_n$

$3I_n = 2nI_{n-1} - 2nI_n$

$3I_n + 2nI_n = 2nI_{n-1}$

$(2n+3)I_n = 2nI_{n-1}, \quad n \geq 1.$

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$n=3: 9I_3 = 6I_2$

$7I_2 = 4I_1$

$5I_1 = 2I_0$

$I_0 = \int_0^1 x^0(1-x)^{\frac{1}{2}}\,dx$

$= \int_0^1 (1-x)^{\frac{1}{2}}\,dx$

$= \left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1$

$= 0 - \left(-\frac{2}{3}\right)$

$= \frac{2}{3}$

$I_1 = \frac{4}{15}$

$I_2 = \frac{16}{105}$

$I_3 = \frac{32}{315}$

7. $C: y = \dfrac{x^2+px+1}{x-2}$

$$\begin{array}{r}
x + p + 2 \\
x-2 \,\big)\, \overline{x^2 + px + 1} \\
\underline{x^2 - 2x} \\
(p+2)x + 1 \\
\underline{(p+2)x - 2p - 4} \\
2p+5
\end{array}$$

$y = x + p + 2 + \dfrac{2p+5}{x-2}$

As $x \to \pm\infty$ $y \to x+p+2$.

As $x \to 2$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = x + p + 2$ and $x = 2$.

$\dfrac{dy}{dx} = 1 - \dfrac{(2p+5)}{(x-2)^2}$

When $\dfrac{dy}{dx} = 0$: $1 - \dfrac{(2p+5)}{(x-2)^2} = 0$

$\dfrac{2p+5}{(x-2)^2} = 1$

$(x-2)^2 = 2p+5$

$x = 2 \pm \sqrt{2p+5}$

If C has 2 turning points, $2p+5 > 0$

$\therefore p > -\frac{5}{2}$.

when $p=-1$ and $\frac{dy}{dx}=0$: $x = 2 \pm \sqrt{3}$

$$y = 2 \pm \sqrt{3} + 1 + \frac{3}{2 \pm \sqrt{3} - 2}$$

$$= 3 \pm \sqrt{3} \pm \sqrt{3}$$

$$= 3 \pm 2\sqrt{3}$$

$$\frac{d^2y}{dx^2} = \frac{2(2p+5)}{(x-2)^3} = \frac{6}{(x-2)^3}.$$

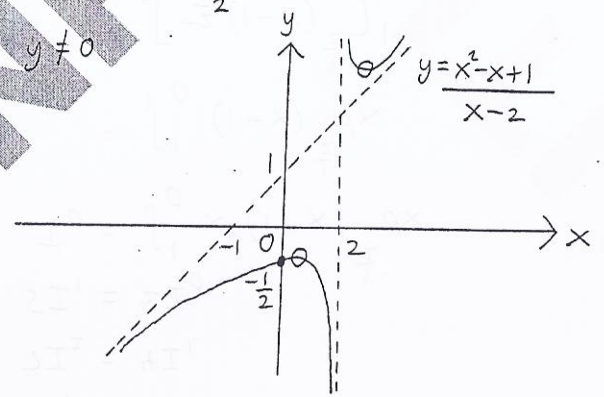
when $x = 2+\sqrt{3}$: $\frac{d^2y}{dx^2} = \frac{6}{3\sqrt{3}} = \frac{2}{\sqrt{3}} > 0$

when $x = 2-\sqrt{3}$: $\frac{d^2y}{dx^2} = \frac{6}{-3\sqrt{3}} = \frac{-2}{\sqrt{3}} < 0$

$\therefore (2+\sqrt{3}, 3+2\sqrt{3})$ is a minimum point and $(2-\sqrt{3}, 3-2\sqrt{3})$ is a maximum point.

when $x=0$: $y = \frac{-1}{2}$

$y \neq 0$



o: Critical point
•: Intersection point.

8. $A\underset{\sim}{e} = \lambda\underset{\sim}{e}$, $B\underset{\sim}{e} = \mu\underset{\sim}{e}$

$$(AB)\underset{\sim}{e} = A(B\underset{\sim}{e}) = A(\mu\underset{\sim}{e}) = \mu(A\underset{\sim}{e}) = \mu(\lambda\underset{\sim}{e}) = (\lambda\mu)\underset{\sim}{e}$$

$\therefore$ The matrix AB has eigenvector $\underset{\sim}{e}$ with corresponding eigenvalue $\lambda\mu$.

$$C = \begin{pmatrix} -1 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$C - \lambda I = \begin{pmatrix} -1 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1-\lambda & -1 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$|C - \lambda I| = (-1-\lambda)[(1-\lambda)(2-\lambda) - 0] - (-1)(0) + 3(0)$$

$$= -(\lambda+1)(\lambda-1)(\lambda-2)$$

when $|C - \lambda I| = 0$: $-(\lambda+1)(\lambda-1)(\lambda-2) = 0$

$$\lambda = 1, 2, -1$$

$\lambda = 1$: $\begin{pmatrix} -2 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$z = 0$

Let $y = 2s, s \in R$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 2s \\ 0 \end{pmatrix} = s\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

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$\lambda = 2$: $\begin{pmatrix} -3 & -1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

let $z = 3s, s \in R$

$y = 6s$

$-3x - 6s + 9s = 0$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 6s \\ 3s \end{pmatrix} = s\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$$

$\lambda = -1$: $\begin{pmatrix} 0 & -1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$z = 0$

$y = 0$

let $x = s, s \in R$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ The eigenvalues of C are $1, 2, -1$ with corresponding eigenvectors $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$D = \begin{pmatrix} 1 & -1 & 1 \\ -6 & -3 & 4 \\ -9 & -3 & 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 1 \\ -6 & -3 & 4 \\ -9 & -3 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -12 \\ -6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ is an eigenvector of D with corresponding eigenvalue -2.

Since $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ is an eigenvector of C and D with corresponding eigenvalues 2 and -2, $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ is an eigenvector of CD with corresponding eigenvalue -4.

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9. $C: y = \dfrac{e^x + e^{-x}}{2}, \quad 0 \le x \le \ln 5.$

i) The mean value of y over $0 \le x \le \ln 5$ is $\dfrac{1}{\ln 5 - 0}\displaystyle\int_0^{\ln 5} y \, dx$

$$= \frac{1}{\ln 5}\int_0^{\ln 5} \frac{e^x + e^{-x}}{2}\, dx$$

$$= \frac{1}{2\ln 5}\Big[e^x - e^{-x}\Big]_0^{\ln 5}$$

$$= \frac{1}{2\ln 5}\left(5 - \frac{1}{5} - 1 + 1\right)$$

$$= \frac{12}{5\ln 5}.$$

ii) $\dfrac{dy}{dx} = \dfrac{e^x - e^{-x}}{2}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{e^x + e^{-x}}{2}$$

The arc length of C is

$$\int_0^{\ln 5} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\ln 5} \frac{e^x + e^{-x}}{2}\, dx$$

$$= \left[\frac{e^x - e^{-x}}{2}\right]_0^{\ln 5}$$

$$= \frac{1}{2}\left(5 - \frac{1}{5} - 1 + 1\right)$$

$$= \frac{12}{5}.$$

iii) The surface area when C is rotated one complete revolution about the x-axis is

$$\int_0^{\ln 5} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\ln 5} 2\pi \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) dx$$

$$= \frac{\pi}{2}\int_0^{\ln 5} e^{2x} + 2 + e^{-2x}\, dx$$

$$= \frac{\pi}{2}\left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_0^{\ln 5}$$

$$= \frac{\pi}{2}\left(\frac{25}{2} + 2\ln 5 - \frac{1}{50} - \frac{1}{2} - 0 + \frac{1}{2}\right) = \pi\left(\frac{156}{25} + \ln 5\right)$$

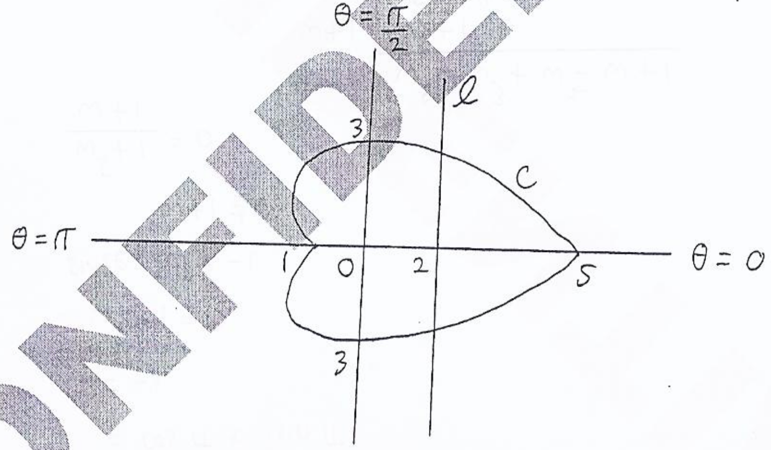
10. $C : r = 3 + 2\cos\theta \ , \ -\pi < \theta \le \pi$

$\ell : r\cos\theta = 2$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{2\pi}{3}$	π
r_C	5	$3+\sqrt{3}$	$3+\sqrt{2}$	4	3	$3-\sqrt{3}$	$3-\sqrt{2}$	2	1
r_ℓ	2	$\frac{4}{\sqrt{3}}$	$2\sqrt{2}$	4		$-\frac{4}{\sqrt{3}}$	$-2\sqrt{2}$	-4	-2

$3 + 2\cos(-\theta) = 3 + 2\cos\theta$

$\frac{2}{\cos(-\theta)} = \frac{2}{\cos\theta}$



$3 + 2\cos\theta = \frac{2}{\cos\theta}$

$3\cos\theta + 2\cos^2\theta = 2$

$2\cos^2\theta + 3\cos\theta - 2 = 0$

$(2\cos\theta - 1)(\cos\theta + 2) = 0$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, -\frac{\pi}{3}$

$r = 4, 4$

$\therefore$ C and ℓ intersect at $\left(4, \frac{\pi}{3}\right)$ and $\left(4, -\frac{\pi}{3}\right)$.

The area of the region enclosed by C and ℓ, R, is

$$2\left(\int_{\frac{\pi}{3}}^{\pi} \frac{(3+2\cos\theta)^2}{2}\,d\theta + \frac{1}{2}(4)2\sin\frac{\pi}{3}\right)$$

$$= \int_{\frac{\pi}{3}}^{\pi} 9 + 12\cos\theta + 4\cos^2\theta\,d\theta + 8\left(\frac{\sqrt{3}}{2}\right)$$

$$= \int_{\frac{\pi}{3}}^{\pi} 9 + 12\cos\theta + 2 + 2\cos 2\theta\,d\theta + 4\sqrt{3}$$

$$= \Big[11\theta + 12\sin\theta + \sin 2\theta\Big]_{\frac{\pi}{3}}^{\pi} + 4\sqrt{3}$$

$$= 11\pi - \left(\frac{11\pi}{3} + 12\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right) + 4\sqrt{3}$$

$$= 11\pi - \frac{11\pi}{3} - \frac{13\sqrt{3}}{2} + 4\sqrt{3}$$

$$= \frac{22\pi}{3} - \frac{5\sqrt{3}}{2}$$

11. EITHER

$$w = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$

$$w^5 = \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^5$$

$$= \cos\pi + i\sin\pi$$

$$= -1$$

$$\therefore w^5 + 1 = 0$$

since $w \neq -1$,

$w + 1 \neq 0$

$$\frac{w^5+1}{w+1} = 0$$

$$\begin{array}{r|l} & w^4 - w^3 + w^2 - w + 1 \\ \hline w+1 & w^5 + 1 \\ & w^5 + w^4 \\ \hline & -w^4 + 1 \\ & -w^4 - w^3 \\ \hline & w^3 + 1 \\ & w^3 + w^2 \\ \hline & -w^2 + 1 \\ & -w^2 - w \\ \hline & w + 1 \\ & w + 1 \\ \hline & 0 \end{array}$$

$$\therefore w^4 - w^3 + w^2 - w + 1 = 0$$

$$w = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$

$$w^2 = \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$$w^3 = \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^3 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$$

$$w^4 = \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^4 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$$

$$w - w^4 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} - \left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$$

$$= \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} - \left(\cos\left(\pi - \frac{\pi}{5}\right) + i\sin\left(\pi - \frac{\pi}{5}\right)\right)$$

$$= \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} - \left(-\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$$

$$= 2\cos\frac{\pi}{5}$$

$$w^3 - w^2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} - \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$$

$$= \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} - \left(\cos\left(\pi - \frac{3\pi}{5}\right) + i\sin\left(\pi - \frac{3\pi}{5}\right)\right)$$

$$= \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} - \left(-\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$$

$$= 2\cos\frac{3\pi}{5}.$$

$$2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} = w - w^4 + w^3 - w^2$$

$$2\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) = -(w^4 - w^3 + w^2 - w)$$
$$= -(-1)$$
$$= 1$$

$$\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$$

$$\left(2\cos\frac{\pi}{5}\right)\left(2\cos\frac{3\pi}{5}\right) = (w - w^4)(w^3 - w^2)$$

$$4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = w^4 - w^3 - w^7 + w^6$$
$$= w^4 - w^3 - w^2 w^5 + w^5 w$$
$$= w^4 - w^3 - w^2(-1) + w(-1)$$
$$= w^4 - w^3 + w^2 - w$$
$$= -1$$

$$\therefore \cos\frac{\pi}{5}\cos\frac{3\pi}{5} = \frac{-1}{4}$$

Since $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = \frac{-1}{4}$,

a quadratic equation having roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$

is $x^2 - \left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right)x + \cos\frac{\pi}{5}\cos\frac{3\pi}{5} = 0$

$$x^2 - \frac{x}{2} - \frac{1}{4} = 0$$

$$4x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{20}}{8}$$
$$= \frac{2 \pm 2\sqrt{5}}{8}$$
$$= \frac{1 \pm \sqrt{5}}{4}$$

Since $0 < \cos\frac{\pi}{5} < 1$,

$$\therefore \cos\frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}$$

OR

$$x^2\frac{d^2y}{dx^2} + 4x(1+x)\frac{dy}{dx} + 2(1+4x+2x^2)y = 8x^2$$

$$x^2y = z$$

$$\frac{dz}{dx} = \frac{d}{dx}(x^2y)$$

$$= 2xy + x^2\frac{dy}{dx}$$

$$\frac{d}{dx}\left(\frac{dz}{dx}\right) = \frac{d}{dx}\left(2xy + x^2\frac{dy}{dx}\right)$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}(2xy) + \frac{d}{dx}\left(x^2\frac{dy}{dx}\right)$$

$$= 2x\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

$$= x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y.$$

$$x^2\frac{d^2y}{dx^2} + 4x(1+x)\frac{dy}{dx} + 2(1+4x+2x^2)y = 8x^2$$

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 4x^2\frac{dy}{dx} + 2y + 8xy + 4x^2y = 8x^2$$

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y + 4\left(x^2\frac{dy}{dx} + 2xy\right) + 4x^2y = 8x^2$$

$$\therefore \frac{d^2z}{dx^2} + 4\frac{dz}{dx} + 4z = 8x^2.$$

$\therefore$ The auxillary equation is $m^2 + 4m + 4 = 0$

$$(m+2)^2 = 0$$

$$m = -2.$$

$\therefore$ The complementary function, z_c, is

$$z_c = (Ax+B)e^{-2x}$$

The particular integral, z_p, is given by

$$z_p = Cx^2 + Dx + E$$

$$\frac{dz_p}{dx} = 2Cx + D$$

$$\frac{d^2z_p}{dx^2} = 2C$$

$$\frac{d^2z_p}{dx^2} + 4\frac{dz_p}{dx} + 4z_p = 2C + 4(2Cx + D) + 4(Cx^2 + Dx + E)$$

$$= 4Cx^2 + (8C + 4D)x + 2C + 4D + 4E$$

$$= 8x^2$$

$4C = 8$ $\quad 8C + 4D = 0$ $\quad 2C + 4D + 4E = 0$

$C = 2$ $\quad D = -4$ $\quad E = 3$

$\therefore z_p = 2x^2 - 4x + 3$

$z = z_c + z_p$

$= (Ax + B)e^{-2x} + 2x^2 - 4x + 3$

$x^2 y = (Ax + B)e^{-2x} + 2x^2 - 4x + 3$

$y = \left(\frac{A}{x} + \frac{B}{x^2}\right)e^{-2x} + 2 - \frac{4}{x} + \frac{3}{x^2}$

As $x \to \pm\infty$, since $e^{-2x} \to 0$ $y \to 2$.