TAYLOR'S UNIVERSITY COLLEGE

Wisdom • Integrity • Excellence

CAMBRIDGE A LEVEL PROGRAMME

A2 TRIAL EXAMINATION AUGUST SEPTEMBER 2009

(June 2008 Intake)

Thursday **3 September 2009** **12.30 pm – 3.30 pm**

FURTHER MATHEMATICS 9231/01

PAPER 1 **3 hours**

Additional materials: Answer Booklet/Paper
List of formulae (MF 10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages.

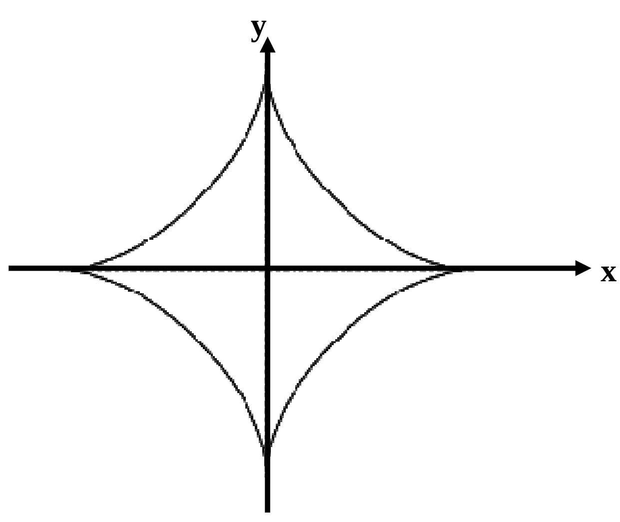
1 Show that $\cos y = x$ satisfies the differential equation

$$\left(x^2 - 1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$ [4]

2 Prove by mathematical induction that, for all positive integers n, $7^{2n-1} + 3^{2n}$ is divisible by 8. [6]

3 One root of the equation $x^3 + px^2 + qx + r = 0$, where p, q and r are constants, is equal to the sum of the other two roots. Show that one root is $-\frac{1}{2}p$ and that $4pq = p^3 + 8r$. Hence, or otherwise, solve $x^3 + 4x^2 + 7x + 6 = 0$. [7]

4 The asteroid in the diagram below is defined parametrically as $x = a\cos^3\theta$ and $y = a\sin^3\theta$, where $a > 0$ and $-\pi < \theta \le \pi$. Find the length of the circumference of the asteroid.



[7]

5 Three planes have equations

$$x + 2y + pz = 1$$
$$2x - y + 5z = 11$$
$$3x + y + 8z = q$$

where p and q are constants.

(i) Find the value of p_1 such that when $p \neq p_1$ the planes have exactly one common point. [4]

(ii) Find the values of p_2 and q_2 such that when $p = p_2$ and $q \neq q_2$ the planes have no common points. [2]

(iii) In the case when $p = p_2$ and $q = q_2$ the planes have a common line of intersection, find this line. [3]

6 **(i)** Verify that $\dfrac{1}{n^2 - n + 1} - \dfrac{1}{n^2 + n + 1} = \dfrac{2n}{n^4 + n^2 + 1}$. [2]

(ii) Express $n^2 + n + 1$ and $n^2 - n + 1$ in completed square form. [2]

(iii) Hence find an expression, in terms of N, for the sum S_N where

$$S_N = \sum_{n=1}^{N} \frac{n}{n^4 + n^2 + 1}$$

Deduce that $S_N < \dfrac{1}{2}$. [5]

7 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 4 & 7 \\ -2 & -3 & 5 & 0 \\ 3 & 2 & 6 & 11 \\ 8 & 3 & 1 & 12 \end{pmatrix}$$

(i) Find the dimension of $\mathbf{K}$, the range space of T. [3]

(ii) Find a set of basis vectors for $\mathbf{K}$. [1]

(iii) Show that $\begin{pmatrix} -1 \\ 45 \\ 9 \\ -15 \end{pmatrix}$ belongs to $\mathbf{K}$. [5]

8 Show that $y = \left(a\cos 2x + b\sin 2x\right)x$, where a and b are constants that need to be found, is the particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4y = \sin 2x\,. \qquad [4]$$

Hence solve the differential equation given that when $x = 0$, $y = \pi$ and $\dfrac{dy}{dx} = \dfrac{-1}{4}$.

[7]

9 Using De Moivre's theorem prove that

$$\tan 5\theta = \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}\,. \qquad [6]$$

Prove that $\tan\dfrac{\pi}{20}$ is a root of the equation

$$t^4 - 4t^3 - 14t^2 - 4t + 1 = 0\,,$$

and find the other roots in the form $\tan\dfrac{n\pi}{20}$. [6]

10 The curve C has polar equation

$$r = 5 - 3\cos 3\theta$$

where $-\pi < \theta \le \pi$.

(i) Draw a sketch of C. Your sketch should show the maximum and minimum values of r and their corresponding value of θ. [5]

(ii) Deduce from your sketch that the area A of the sector for which $0 \le \theta \le \dfrac{\pi}{2}$ satisfies the inequality $\pi < A < 16\pi$. [3]

(iii) Find, by integration, the exact value of A. [4]

11 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation $y = \dfrac{x^2 - a}{(x-a)^2}$, where $a > 1$.

(i) Find the coordinates of the points at which C crosses the coordinate axes. [2]

(ii) Express y in the form $P + \dfrac{Q}{x-a} + \dfrac{R}{(x-a)^2}$ and hence write down the equations of all asymptotes of C. [5]

(iii) Find the coordinates of turning points and determine their nature. [4]

(iv) Sketch the curve C. [3]

OR

Given that $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{2} & \frac{-1}{6} \\ \frac{1}{6} & 0 & \frac{-1}{6} \\ \frac{-1}{6} & \frac{1}{4} & \frac{5}{12} \end{pmatrix}$.

(i) Find the eigenvalue that corresponds to this eigenvector. [2]

(ii) Find a matrix **Q** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{QDQ}^{-1}$. [10]

(iii) Deduce that, as the positive integer n tends to infinity, all the elements of matrix $\mathbf{A}^n$ tend to zero. [2]