(4)

Ex13A

7. Find the speed in $m\,s^{-1}$ of a particle which is moving on a circular path of radius 50 cm with an angular speed of:
   (a) $10\,rad\,s^{-1}$
   (b) $20\,rev\,s^{-1}$
8. A particle moving in a circular path of radius 50 cm has a speed of $12\,m\,s^{-1}$. Find the angular speed of the particle:
   (a) in $rad\,s^{-1}$ (b) in $rev\,min^{-1}$.
9. A particle moving in a circular path of radius 20 cm has a speed of $20\,cm\,s^{-1}$. Find the angular speed of the particle:
   (a) in $rad\,s^{-1}$ (b) in $rev\,min^{-1}$.
10. Sue and Pam stand on a playground roundabout at distances of 1 m and 1·5 m respectively from the centre of rotation. If the roundabout revolves at $12\,rev\,min^{-1}$, find the speeds with which Sue and Pam are moving.
11. Tom and John stand on a playground roundabout respectively 50 cm and 175 cm from the centre. The roundabout moves with constant angular velocity and Tom's speed is found to be $1\,m\,s^{-1}$.
    Find:
    (a) the angular speed of the roundabout in $rad\,s^{-1}$
    (b) the time taken for the roundabout to complete ten revolutions
    (c) John's speed.

Answer

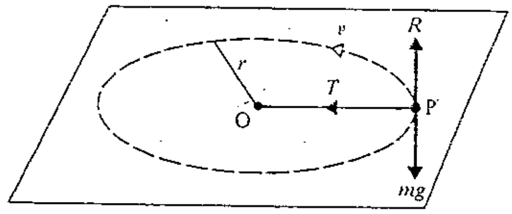
7. (a) $5\,m\,s^{-1}$ (b) $20\pi\,m\,s^{-1}$
8. (a) $24\,rad\,s^{-1}$ (b) $\frac{720}{\pi}\,rev\,min^{-1}$
9. (a) $1\,rad\,s^{-1}$ (b) $\frac{30}{\pi}\,rev\,min^{-1}$
10. $\frac{2\pi}{5}\,m\,s^{-1}$, $\frac{3\pi}{5}\,m\,s^{-1}$
11. (a) $2\,rad\,s^{-1}$ (b) $10\pi\,s$ (c) $3{\cdot}5\,m\,s^{-1}$

hopiko

The force which acts upon a body so that it follows a circular path may be provided in various ways.

(i) Particle on a string

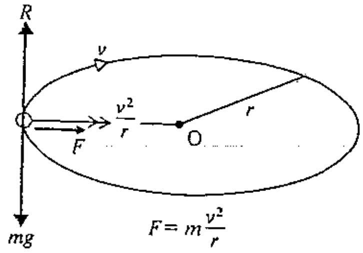
A particle is attached to one end of a string, the other end of the string being attached at a point O on a smooth horizontal surface. If the particle describes circles on the surface, the necessary force towards the centre of the circle is provided by the tension in the string.



$T = m\frac{v^2}{r}$

(ii) Bead on a circular wire

If a bead is threaded on a smooth horizontal circular wire and moves at a speed v, the necessary force towards the centre of the circular wire is provided by the force between the bead and the wire. In addition, the wire supports the weight of the bead, and the vertical reaction R equals mg.

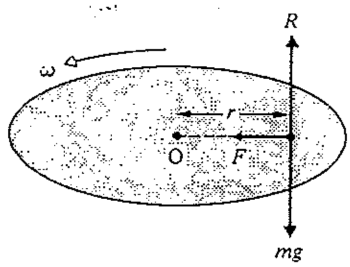


$F = m\frac{v^2}{r}$

(iii) Particle on a rotating disc

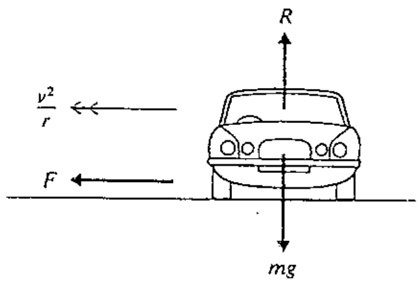
If a particle rests on the surface of a rotating horizontal disc, the only horizontal force acting on the particle is the frictional force between the particle and the surface of the disc. This frictional force provides the necessary force towards the centre of rotation.

For any particular surface, there will be a maximum value of F, i.e. $F_{max} = \mu R$, and then the particle will be on the point of slipping.



(iv) Car on circular path

Again, the necessary force towards the centre of the circular path is provided by the frictional force between the tyres of the car and the surface of the road.



$F = m\frac{v^2}{r}$

CIRCULAR MOTION

(A) Circular Motion with Constant Speed (i.e. Horizontal Circular Motion)

If ω is constant (constant angular speed, hence constant speed since $v = r\omega$), the particle is said to be moving in **uniform circular motion**.

From (5) in the Mathematical Notes,

(1) **the transverse (or tangential) component of acceleration is zero and**

(2) **the radial component of acceleration** $= r\omega^2 = \dfrac{v^2}{r}$

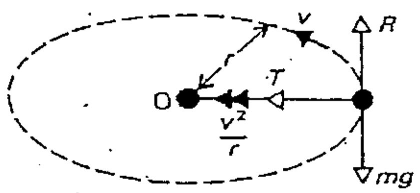
Hence, for horizontal circular motion:

(1) Resultant force in the vertical component is zero, i.e upward force = downward force.

(2) Resultant force acts towards the centre (to keep a body moving in a circle at constant speed), $F = m(\dfrac{v^2}{r}) = mr\omega^2$.

Examples:

(i) Particle on a string moving on a smooth horizontal surface.

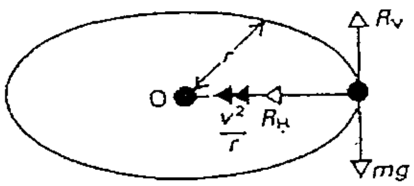


The tension in the string causes the radial acceleration.

$$T = \frac{mv^2}{r}$$

If the string breaks, the particle moves off tangentially.

(ii) Bead on a smooth horizontal circular wire.

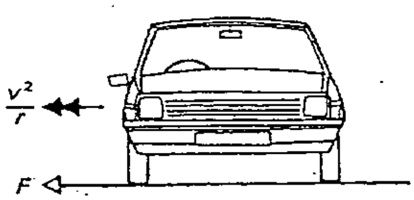


The normal reaction between the bead and the wire has two components: R_V vertically upwards to balance mg and R_H horizontally, where

$$R_H = \frac{mv^2}{r}$$

> Note: F is the maximum frictional force resulting from contact of two surfaces. If the maximum friction force is not sufficient to produce the required acceleration, the car will slip as its attempt to round the bend.

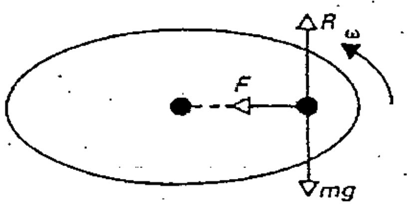
(iii) Car rounding a bend on a level road.



The central force is provided by the friction between tyres and road,

$$F = \frac{mv^2}{r}$$

(iv) Particle resting on a rotating disc.



The central force is provided by the friction between the particle and the rotating disc.

> Note that, if ω varied, a second frictional force, normal to the radial friction, would be induced to keep the particle in place on the disc.

Example 1:

A particle of mass 5 kg is attached to a point on a smooth horizontal table by a rope 1.2 m long. If it describes circles at 150 rev/min, find the tension in the rope. (*1480 N*).

150 rev/min
$= 2.5$ rev/s
$= 5\pi$ rad/s

$T = mr\omega^2$
$= (5)(1.2)(5\pi)^2$
$= 1480N$

Example 2:

A particle of mass 200 grams lies on a horizontal turntable at a distance of 30 cm from the axis of revolution. The particle is on the point of slipping when travelling at 1.5 ms^{-1}. Find the coefficient of friction. ($\mu = 0.75$)

$g = 10 m/s^2$

$F = uR$
$R = (0.2)g$
$= (0.2)(10) = 2$

$F = \frac{mv^2}{r}$
$= \frac{(0.2)(1.5)^2}{0.3}$
$= 1.5$

$u = \frac{F}{R} = \frac{1.5}{2} = 0.75$

Example 3:

A particle P of mass 0.25 kg is attached to one end of a light elastic string of modulus 5N and natural length 0.75m. The other end of the string is fixed to a point O on a smooth horizontal table. P moves in a horizontal circle of radius 1m.

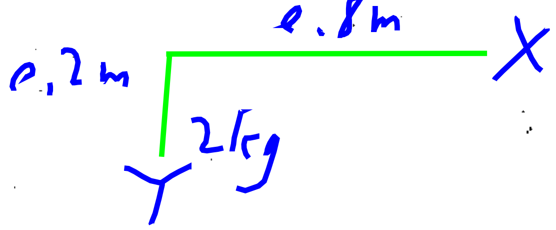
Calculate the constant linear speed of P. (2.58 ms^{-1}, recall Hooke's Law: $T = \frac{\lambda x}{l}$).

$$T = \frac{\lambda x}{l} = \frac{5(0.25)}{0.75} = \frac{5}{3} \qquad T = \frac{mv^2}{r} \qquad v^2 = \frac{5}{3}\left(\frac{1}{0.25}\right) = \frac{20}{3}$$

$$v = 2.58 \, ms^{-1}$$

Example 4:

Two particles, X and Y, both of mass 2 kg are connected by a light inextensible string which passes through a small smooth hole in a smooth horizontal table. The strip is 1 m long and Y hangs freely 20 cm below the table when X makes horizontal circles on top of the table. Using g as 9.8 ms^{-2}, how many revolutions per minute does X make for Y to hang in this position? (*33.4 rev/min*)



$$2g = mr\omega^2$$
$$2(9.8) = 2(0.8)\omega^2$$
$$\omega^2 = 12.25$$
$$\omega = 3.5 \, rad\,s^{-1}$$
$$= 0.557 \, rev\,s^{-1}$$
$$= 33.4 \, rev\,min^{-1}$$

Example 5:

A car of mass M is turning a corner of radius r. The coefficient of friction between the wheels and the horizontal road surface is μ. What is the maximum speed at which the car can turn the corner without skidding? ($\sqrt{\mu rg}$)

$$F = \mu R \qquad \frac{Mv^2}{r} = \mu R$$
$$F = \frac{Mv^2}{r} \qquad Mv^2 = \mu r M g$$
$$v^2 = \mu r g$$
$$v = \sqrt{\mu r g}$$

Homework: Exercise 1.

Exercise 13B

1. A particle moves with a constant angular speed of 2 rad s^{-1} around a circular path of radius $1{\cdot}5$ m.
Find the acceleration of the particle.

2. A particle moves with a constant speed of 3 m s^{-1} around a circular path of radius 50 cm.
Find the acceleration of the particle.

3. A record is revolving at 45 rev min^{-1}. Points A, B and C lie on the record at 5 cm, 10 cm and 20 cm from the centre respectively.
Find the accelerations of A, B and C.

4. A body of mass 2 kg moves with a constant angular speed of 5 rad s^{-1} around a horizontal circle of radius 10 cm.
Find the magnitude of the horizontal force that must be acting on the body towards the centre of the circle.

5. A body of mass $0{\cdot}5$ kg moves with a constant speed of 4 m s^{-1} around a horizontal circle of radius 1 m.
Find the magnitude of the horizontal force that must be acting on the body towards the centre of the circle.

6. A body of mass $1{\cdot}5$ kg moves with a constant angular speed of 600 rev min^{-1} around a horizontal circle of radius 10 cm.
Find the magnitude of the horizontal force that must be acting on the body towards the centre of the circle.

7. A body of mass 100 g moves with a constant angular speed around a horizontal circle of radius 50 cm.
If the body completes 50 revolutions in 3 minutes, find the magnitude of the horizontal force that must be acting on the body towards the centre of the circle.

8. A particle of mass 250 g lies on a smooth horizontal surface and is connected to a point O on the surface by a light inextensible string of length 20 cm. With the string taut, the particle describes a horizontal circle, centre O, with constant angular speed ω.
Find the tension in the string when ω is
(a) 10 rad s^{-1}
(b) 20 rad s^{-1}
(c) 100 rev min^{-1}.

9. A particle of mass 125 g lies on a smooth horizontal surface and is connected to a point O on the surface by a light inextensible string of length 50 cm. With the string taut, the particle describes a horizontal circle, centre O, with constant speed v.
Find the tension in the string when v is
(a) 2 m s^{-1} (b) 10 m s^{-1} (c) 20 m s^{-1}

10. A body of mass 2 kg lies on a smooth horizontal surface and is connected to a point O on the surface by a light inextensible string of length 25 cm. With the string taut, the body describes a horizontal circle with centre O. If the tension in the string is 18 N, find the angular speed of the body.

11. A body of mass 3 kg lies on a smooth horizontal surface and is connected to a point O on the surface by a light inextensible string of length $1{\cdot}5$ m.
With the string taut, the body describes a horizontal circle with centre O.
If the tension in the string is 32 N, find the speed of the body. Find the greatest possible speed of the body around the circle given that the string will break if the tension exceeds 98N.

12. A smooth wire is in the form of a circle of radius 20 cm and centre O. The wire has a bead of mass 100 g threaded on it. The wire is held horizontally and the bead moves along the wire with a constant speed of 3 m s^{-1}. Find:
(a) the vertical force experienced by the wire due to the bead (state up or down)
(b) the vertical force experienced by the bead due to the wire (state up or down)
(c) the horizontal force experienced by the wire due the bead (state whether towards or away from O)
(d) the horizontal force experienced by the bead due to the wire (state whether towards or away from O).

13. A bend in a level road forms a circular arc of radius 50 m. A car travels around the bend at a speed of 14 m s^{-1} without slipping occurring. What information does this give about the value of the coefficient of friction between the tyres of the car and the road surface?

14. A bend in a level road forms a circular arc of radius 75 m. The greatest speed at which a car can travel around the bend without slipping occurring is 63 km h^{-1}.
Find the coefficient of friction between the tyres of the car and the road surface.

15. A bend in a level road forms a circular arc of radius 54 m.
Find the greatest speed at which a car can travel around the bend without slipping occurring if the coefficient of friction between the tyres of the car and the road surface is $0{\cdot}3$.

16. A body of mass 200 g rests without slipping on a horizontal disc which is rotating at $5{\cdot}6\ \text{rad s}^{-1}$. The coefficient of friction between the body and the disc is 0·4. Find the greatest possible distance between the body and the centre of rotation. Repeat the question for a body of mass 400 g.

17. A body is placed on a horizontal disc at a point which is 15 cm from the centre of the disc. When the disc rotates at $30\ \text{rev min}^{-1}$, the body is just on the point of slipping. Find the coefficient of friction between the body and the surface of the disc.

18. A level road is to be constructed; one design requirement is that any vehicle travelling around any bend in the road, can travel at any speed up to $21\ \text{m s}^{-1}$ without slipping occurring. It is calculated that under the worst conditions, the coefficient of friction between a vehicle's tyres and the surface of the road could be as low as 0·125. Find the minimum radius that any bend in the road may have.

19. A level railway track is in the form of a circular arc of radius 200 m. Find the horizontal force acting on the rails when a train of mass 15 000 kg travels around the bend with a speed of $20\ \text{m s}^{-1}$.

20. A hollow circular cylinder of radius 4 m rotates at $3{\cdot}5\ \text{rad s}^{-1}$ about its axis of symmetry which is vertical. A body rotates, without slipping, on the inner surface of the cylinder.

Given that the body is on the point of sliding down the cylinder find the coefficient of friction between the body and the cylinder.

21. A hollow circular cylinder of radius $\left(\dfrac{50}{\pi^2}\right)$ m rotates at 14 revolutions per minute about its axis of symmetry which is vertical. A body rotates, without slipping, on the inner surface of the cylinder. Given that the body is on the point of sliding down the cylinder find the coefficient of friction between the body and the cylinder.

22. A ride at a fair consists of a hollow circular cylinder of radius 3 m which rotates about its axis of symmetry which is vertical. People remain, without slipping, in contact with the inner surface of the cylinder. Given that the coefficient of friction between a person and the inside of the cylinder is 0·6, find the least angular velocity with which the cylinder can rotate.

23. Two particles A and B, of masses 200 g and 50 g respectively, are connected by a light inextensible string. Particle A, at one end of the string, lies on a smooth horizontal table. The string passes smoothly through a small hole O in the table, and particle B hangs freely at the other end of the string. When particle A describes a horizontal circle about O with angular speed $3{\cdot}5\ \text{rad s}^{-1}$, particle B hangs 30 cm below the level of the surface of the table. Find the length of the string.

24. Two particles A and B, of masses 100 g and 250 g respectively, are connected by a light inextensible string of length 20 cm. Particle A at one end of the string lies on a smooth horizontal table. The string passes smoothly through a small hole O in the table and particle B hangs freely at the other end of the string. Find the speed (in m s^{-1}) with which A must be made to perform horizontal circles about O in order to keep B in equilibrium at a position 7·5 cm below the level of the surface of the table.

25. A light inextensible string of length l has one end fixed at a point on a smooth horizontal surface, and the other end attached to a body of mass m lying on the surface. With the string taut, the body is given an initial speed of $2\sqrt{(gl)}$ in a direction parallel to the plane of the surface and perpendicular to the string. Show that the tension in the string during the ensuing circular motion will be four times the weight of the particle.

26. A car is just on the point of slipping when travelling on level ground at a speed v around a bend of radius r. Under the same road surface conditions, the car is just on the point of slipping when travelling on level ground at a speed $2v$ around a bend of radius R. Show that $R = 4r$.

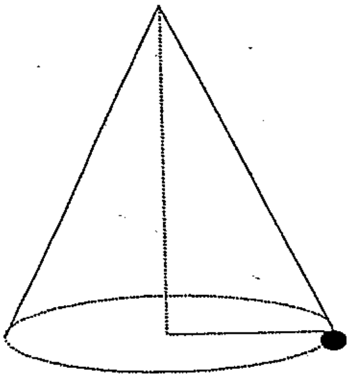
27. Two particles A and B of masses m_1 and m_2 respectively are connected by a light inextensible string of length l. Particle A, at one end of the string, lies on a smooth horizontal table. The string passes smoothly through a small hole O in the table and particle B hangs freely at the other end of the string.

If A follows a horizontal circular path with centre O and angular speed ω, show that particle B will rest in equilibrium at a point which is a distance x below the level of the surface of the table, where

$$x = \frac{lm_1\omega^2 - m_2 g}{m_1\omega^2}$$

(L3.)

(B) The Conical Pendulum



A light inextensible string has one end attached to a fixed point. The other end is fastened to a particle which describes a **horizontal circle**. Since the moving string describes the curved surface of a cone, this system is known as a <u>**conical pendulum**</u>.

<u>Other examples:</u>

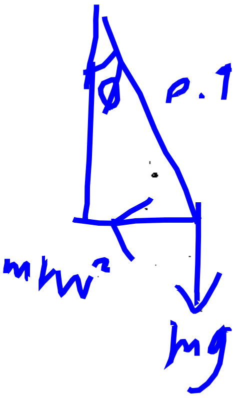
Like the horizontal circular motion, for conical pendulum,

(1) the resultant force in the vertical component is zero.

(2) the resultant force towards the centre provides the central force needed to keep the particle rotating, i.e. $F = mr\omega^2$

<u>Example 6:</u>

A bob of mass 5 kg moves as a conical pendulum at the end of a string of length 90 cm. If the bob describes a circle with a constant angular velocity of 4 rads^{-1}, find the tension in the string and the angle the string makes with the vertical. (*72N, 47.1°*)



$T \sin\phi = (5) r (4)^2$

$T \sin\phi = 5(0.9 \sin\phi)(16)$

$T = 72N$

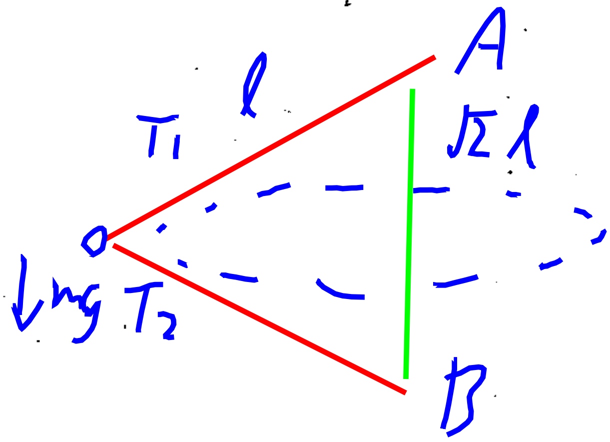
$T\cos\phi = 5(9.8)$

$\cos\phi = \frac{49}{72}$

$\phi = 47.1$

Example 7:

Two light inextensible strings, each of length l, are each *attached* at one end to a particle of mass m. The other ends of the strings are attached to two fixed points A and B, where B is a distance $\sqrt{2}\,l$ vertically below A. When both strings are taut, the particle is given an angular velocity ω and then moves in a horizontal circle with constant speed. Find the tensions in the two strings and show that this motion is only possible if $\omega^2 > \dfrac{g\sqrt{2}}{l}$. $(T_U = \frac{1}{2}m[l\omega^2 + g\sqrt{2}],\ T_L = \frac{1}{2}m[l\omega^2 - g\sqrt{2}])$



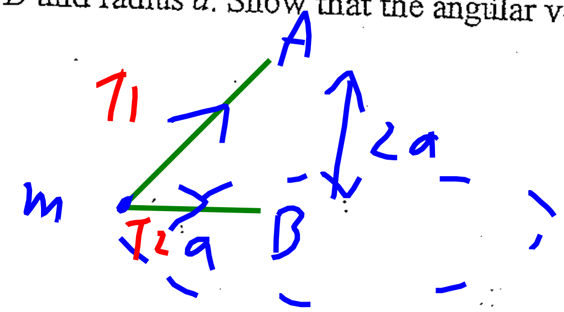
$l^2 - \left(\frac{\sqrt{2}}{2}l\right)^2 = \frac{1}{2}l^2$

$\phi = 45°$

T1 cos45 = T2 cos45 + mg
T1 cos45 + T2 cos45 = mrw2

Example 8:

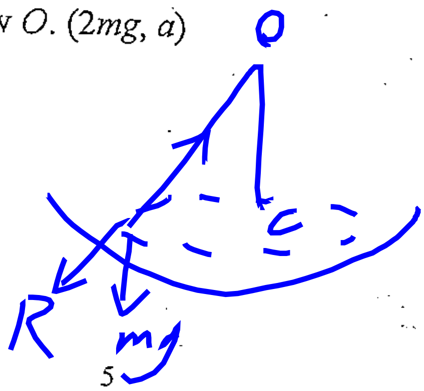
A bead of mass m is $\boxed{\textit{threaded}}$ onto a string whose ends are fastened to two points A and B with B a distance $2a$ vertically below A. The bead rotates in a horizontal circle with centre B and radius a. Show that the angular velocity of the bead is $\sqrt{\dfrac{g(\sqrt{5}+1)}{2a}}$.



Threaded
$T_1 = T_2$
Fastened / Fixed
$T_1 \neq T_2$

Example 9:

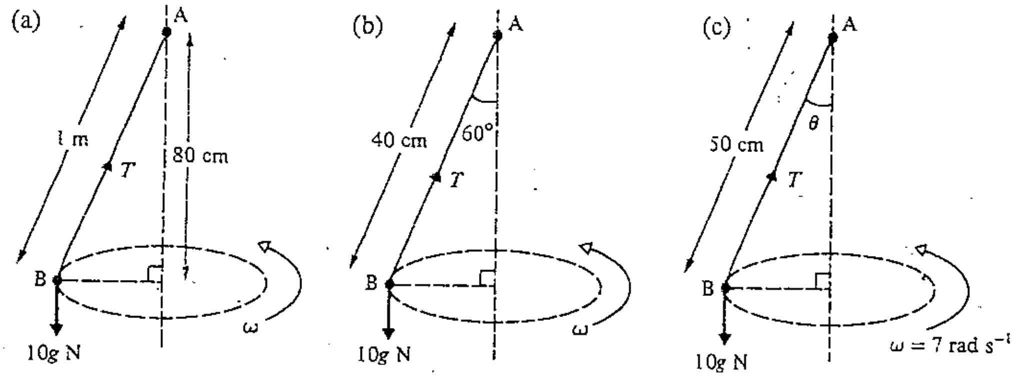
A particle, P is moving on the inner surface of a smooth hemispherical bowl with centre O and radius $2a$. The particle is describing a horizontal circle, centre C with angular speed $\sqrt{\dfrac{g}{a}}$. Find the magnitude of the force exerted on P by the surface of the bowl and the depth of C below O. $(2mg,\ a)$



Homework: Exercise 2.

Exercise 13C

1. Each of the following diagrams shows a conical pendulum consisting of a light inextensible string AB fixed at A and carrying a bob of mass 10 kg at B. The bob moves in a horizontal circle, centre vertically below A, with a constant angular speed ω. The tension in the string is T.



Find T and ω Find T and ω Find T and θ

a) 122.5 N, 3.5 rad/s
b) 196 N, 7 rad/s
c) 245 N, 66.4°

2. A conical pendulum consists of a light inextensible string AB of length 50 cm fixed at A and carrying a bob of mass 2 kg at B. The bob describes a horizontal circle about the vertical through A with a constant angular speed of 5 rad s^{-1}.
Find the tension in the string.

25 N

3. A particle is attached to one end of a light inextensible string which has its other end attached to a fixed point A. With the string taut, the particle describes a horizontal circle with constant angular speed 2·8 rad s^{-1}, the centre of the circle being at a point O vertically below A. Find the distance OA.

1.25 m

4. A light inextensible string AB has a particle attached at end B and A is fixed. With the string taut, the particle describes a horizontal circle with constant angular speed ω.
If the centre of the circle is at a point which is a distance x vertically below A, show that $\omega^2 x = g$.

5. A light inextensible string AB of length l has end A fixed and carries a particle of mass m at B.
With the string taut, the particle describes a horizontal circle about the vertical axis through A, with constant angular speed ω. Show that the tension T in the string is given by $T = ml\omega^2$.

6. A point A lies 25 cm above a smooth horizontal surface. A light inextensible string of length 50 cm has one end fixed at A and carries a body B of mass 0·5 kg at its other end. With the string taut, B moves in a circular path on the surface, the centre of the circle being vertically below A.
If B moves with constant angular speed of 4 rad s^{-1}, find the magnitude of:

(a) the tension in the string
(b) the reaction force between B and the surface.

a) 4 N
b) 2.9 N

7. A smooth bead of mass 100 g is threaded on a light inextensible string of length 70 cm. The string has one end attached to a fixed point A and the other to a fixed point B 50 cm vertically below A. The bead moves in a horizontal circle about the line AB with a constant angular speed of ω rad s^{-1}, and the string taut. If the bead is at a point C on the string with AC = 40 cm, find the value of ω and the tension in the string.

16.9, 4.9N

8. A smooth hemispherical shell of radius r is fixed with its rim horizontal. A small ball bearing of mass m lies inside the shell and performs horizontal circles on the inner surface of the shell, the plane of these circles lying 5 cm below the level of the rim.
Find the angular speed of the ball bearing in rad s^{-1}.

14 rad/s

9. A light inextensible string AB of length 90 cm has a particle of mass 600 g fastened to it at a point C. The ends A and B are attached to two fixed points in the same vertical line as each other, with A 60 cm above B. The particle moves on a horizontal circle at a constant angular speed of 5 rad s^{-1} with both parts of the string taut and CB horizontal.
Find the tensions in the two parts of the string.

1.3N, 6.37N.

10. A light inextensible string AB of length $2l$ has a particle attached to its mid-point C. The ends A and B of the string are fastened to two fixed points with A distance l vertically above B. With both parts of the string taut, the particle describes a horizontal circle about the line AB with constant angular speed ω.
If the tension in CA is three times that in CB, show that $\omega = 2\sqrt{\frac{g}{l}}$.

ANSWERS

Exercise 13B page 313

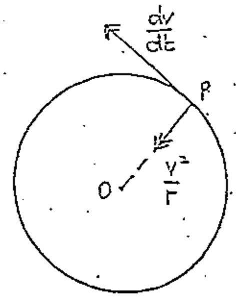
1. $6\,\text{m s}^{-2}$
2. $18\,\text{m s}^{-2}$
3. $1{\cdot}11\,\text{m s}^{-2}$, $2{\cdot}22\,\text{m s}^{-2}$, $4{\cdot}44\,\text{m s}^{-2}$
4. 5 N
5. 8 N
6. 592 N
7. 0·152 N
8. (a) 5 N (b) 20 N (c) 5·48 N
9. (a) 1 N (b) 25 N (c) 100 N
10. $6\,\text{rad s}^{-1}$
11. $4\,\text{m s}^{-1}$, $7\,\text{m s}^{-1}$
12. (a) 0·98 N down (b) 0·98 N up (c) 4·5 N away from O (d) 4·5 N towards O
13. $\mu \geqslant 0{\cdot}4$
14. 0·417
15. $12{\cdot}6\,\text{m s}^{-1}$
16. 12·5 cm, 12·5 cm
17. 0·151
18. 360 m
19. 30 000 N
20. 0·2
21. 0·9
22. $2\frac{1}{3}\,\text{rad s}^{-1}$
23. 50 cm
24. $1{\cdot}75\,\text{m s}^{-1}$

Exercise 13D page 331

1. $2{\cdot}5\,\text{m s}^{-1}$
2. $3{\cdot}9\,\text{m s}^{-1}$
3. $1{\cdot}8\,\text{m s}^{-1}$
4. 33°
5. 69°
6. 85°
7. $4{\cdot}9\,\text{m s}^{-1}$, 252 N
8. $4{\cdot}8\,\text{m s}^{-1}$, 240 N
9. $4{\cdot}6\,\text{m s}^{-1}$, 211 N
10. $0{\cdot}81\,\text{m s}^{-1}$, 42·5 N, thrust
11. $4{\cdot}05\,\text{m s}^{-1}$, 115 N, tension
12. $6{\cdot}66\,\text{m s}^{-1}$, 395 N, tension
13. (a) $10{\cdot}2\,\text{m s}^{-1}$ (b) 160·5 N (c) $9{\cdot}8\,\text{m s}^{-2}$
14. $\sqrt{\frac{11g}{2}}\,\text{m s}^{-1}$, $\sqrt{\frac{7g}{2}}\,\text{m s}^{-1}$
15. (a) $4\sqrt{\frac{gr}{3}}$ (b) $\frac{mg}{3}$
16. When PO makes 141·7° with downward vertical drawn from O, $3{\cdot}4\,\text{m s}^{-1}$
17. 132°
18. $\sqrt{u^2 - 2gr + 2gr\cos\theta}$, $3mg\cos\theta + \frac{mu^2}{r} - 2mg$
19. (a) $9{\cdot}8\,\text{m s}^{-1}$, $64{\cdot}2\,\text{m s}^{-2}$ (b) $9{\cdot}56\,\text{m s}^{-1}$, $61{\cdot}3\,\text{m s}^{-2}$ (c) $7{\cdot}48\,\text{m s}^{-1}$, $38{\cdot}2\,\text{m s}^{-2}$
20. (a) $1{\cdot}146\,\text{rad s}^{-1}$, 17·58 N (b) $2{\cdot}214\,\text{rad s}^{-1}$, 14·7 N (c) $3{\cdot}13\,\text{rad s}^{-1}$, 58·8 N
21. $|200 - 4g + 6g\cos\theta|$ N (b) $\theta = 48{\cdot}2°$
22. g N
23. 3·71 m, 3·87 m
24. (a) $\frac{r}{5}$ (b) $\sqrt{\frac{2gr}{5}}$ (c) $2\sqrt{\frac{3gr}{5}}$
26. $\sqrt{\frac{g\sqrt{3}}{3a}}\,\text{rad s}^{-1}$

(L4)

(C) VERTICAL CIRCULAR MOTION



Motion in a vertical circle has variable speed.

(1) There is an acceleration component, $\dfrac{v^2}{r}$ in the radial direction.

(2) There is also an acceleration component, $\dfrac{dv}{dt}$ in the direction of motion (tangentially).

We are going to consider cases where a body is moving in a vertical circular motion and has only forces perpendicular to the direction of motion. These forces do no work. Hence, the system of forces is conservative and the Law of Conservation of Energy can be applied.

> To solve vertical circular motion problems, we
> (1) use Conservation of mechanical energy to obtain the velocity at the point of concern and
> (2) apply Newton's second law, $F = ma$ radially.

Vertical circular motion can be classified as follows:

Vertical Circular Motion

Motion restricted to a circular path (i.e. the body <u>cannot</u> leave the circle)	Motion <u>not</u> restricted to a circular path (i.e. the body <u>can</u> leave the circle)
➢ A particle on the end of a light rod. ➢ A small bead threaded onto a circular, vertical wire. ➢ A particle moving in a circular, vertical tube.	➢ A particle on the end of a string ➢ A particle on the inside of a circular arc ➢ A particle on the outside of a circular arc.
<u>Condition for describing complete circles</u>: velocity > 0 at the highest point on circle. For oscillation, velocity = 0 (instantaneous rest) at the highest point reached.	<u>Condition for describing complete circles</u>: the force (at the highest point on circle) which has been restricting it to circular path (e.g. T, R) ≥ 0. For oscillation, velocity = 0 (instantaneous rest) before it goes above the level of the centre of the circle. We are interested to know the conditions for describing complete circles <u>and</u> the point where $T = 0$ or $R = 0$ (i.e. where the particle leaves the circular path)

Case I : Motion restricted to a circular path (i.e. the body cannot leave the circle)

Example 1 (Particle on the end of a light rod.):
One end of a light rod of length a metres is pivoted at a fixed point O and a particle of mass m kg is attached to the other end. When the rod is hanging at rest, the particle is given a blow that makes it begin to move with velocity v ms^{-1}. Give answers in terms of a, m and g.

(a) Find the value of v of the rod first comes to rest when horizontal.

(b) Show that for the particle to perform complete circles, $v > 2\sqrt{ga}$.

(c) When $v = 2\sqrt{ga}$, find the force in the rod when the particle is at the highest point, and say whether it is a tension or a thrust.

(d) Given that $v = \sqrt{3ga}$, find the height above O of the particle when the tension in the rod is zero.

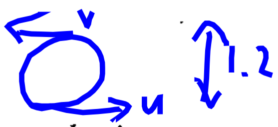
Example 2 (Small bead threaded onto a circular, vertical wire):
Consider a small bead, P, of mass m and is threaded onto a smooth circular wire, radius a and center O, that is fixed in a vertical plane. Suppose that the bead has been set moving around the wire so that it passes through the lowest point A of the wire with speed u and that the speed of the bead is v when it reaches a point B on the wire, where angle AOB is θ.

(a) Prove that $v^2 = u^2 - 2ag(1-\cos\theta)$.

(b) Find the *magnitude* of the normal reaction R between the bead and the wire at B.

(c) Show that the bead can describe complete circles if $u^2 > 4ag$.

$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + 1.2mg$



$u^2 = v^2 + 2.4g$

$v^2 = u^2 - 2.4g$

Example 3 (Particle moving in a circular, vertical tube):
A particle of mass m is free to move in a smooth narrow tube in the form of a circle of center O and radius 60 cm. It is projected from the lowest point with a velocity u. For what values of u does it make complete revolutions? $v > 0$, $v^2 > 0$,

$u^2 - 2.4g > 0$

$u^2 > 2.4g$

$u > 4.9$

Example 4 (More difficult problem - Particles moving in a circular, vertical tube):
A smooth narrow tube is in the form of a circle center O, radius a, which is fixed in a vertical plane. The tube contains two particles, A of mass $4m$ and B of mass m which are connected by a light inextensible string. Initially A and B are on the same horizontal level as O and the system is released from rest. If, after a time t, the line AOB has turned through an angle θ, show that $5a\left(\frac{d\theta}{dt}\right)^2 = 6g\sin\theta$.

(a) Find the reaction between B and the tube in terms of m, g and θ.

(b) Find $a\frac{d^2\theta}{dt^2}$ in terms of g and θ and hence find the tension in the string.

L5

Case II : Motion not restricted to a circular path (i.e. the body can leave the circle)

Example 1 (Particle on the end of a string):
A particle of mass m is attached to one end of a light inelastic string of length a whose other end is fixed to a point O. When the particle is hanging at rest it is given a horizontal blow which causes it to begin to move in a vertical plane with initial speed u. The speed of the particle is v when it has moved angle θ to a point P.

(a) Find v in terms of u, a, g and θ.
(b) Find the tension of the string at P in terms of m, u, a, g and θ.
(c) What are the three possibilities for the motion of the particle? Examine each of them and find the ranges of values of u for each of these cases.

Example 2 (Particle on the inside of a circular arc):
A particle P is given a horizontal velocity u from the lowest point on the smooth inner surface of a hollow spherical shell of radius r and centre O.

(a) Find the range of vales for u so that P will not leave the surface of the shell.
(b) If the particle does leave the surface when OP makes an acute angle α with the upward vertical, show that $3gr\cos\alpha = u^2 - 2gr$. In this case find the reaction between the particle and the surface when OP makes an angle α with the downward vertical.

Example 3 (Particle on the outside of a circular arc):
A particle of mass m is placed on top of a fixed smooth vertical hoop, center O and radius a. The hoop is fixed to the horizontal ground. The particle is slightly displaced from rest. It comes away from the hoop at a point P. Calculate:

(a) the angle which OP makes with the vertical,
(b) the velocity of the particle when it leaves the hoop.
(c) the time taken for the particle to reach the ground, assuming that the horizontal displacement from P upon reaching the ground. is in the same plane as the hoop.

A second particle is placed at the lowest part of the hoop and projected along the inside smooth surface. It also leaves the hoop at the same point. Calculate:

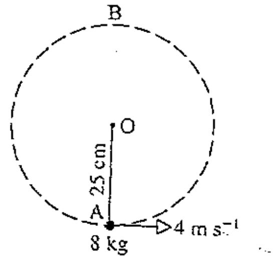
(d) the velocity with which this particle leaves the hoop,
(e) the velocity of projection of this second particle.

Exercise 13D

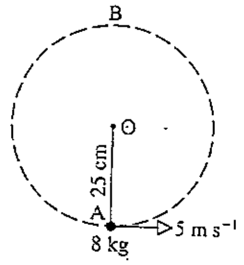
Questions 1, 2 and 3 each involve a particle of mass 8 kg suspended from a fixed point O by a light inextensible string of length 25 cm. The particle is projected from the lowest point, A, with the horizontal speed indicated. In each case the speed of projection is sufficient for the particle to execute complete circles about O.

For each case find the speed of the particle as it passes through its highest position, B.

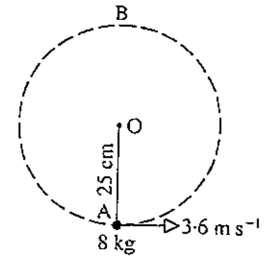
1.



2.



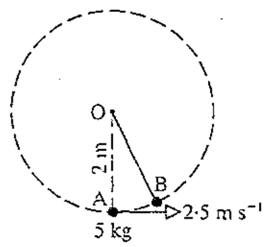
3.



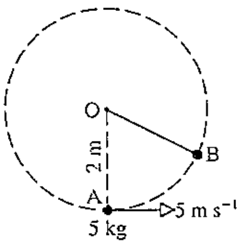
Questions 4, 5 and 6 each involve a particle of mass 5 kg suspended from a fixed point O by a light inextensible string of length 2 m. The particle is projected from the lowest position, A, with the horizontal speed indicated. In each case the speed of projection is such that the particle just reaches point B and then returns through A and oscillates about the lowest position.

For each case determine the size of angle AOB, to the nearest degree.

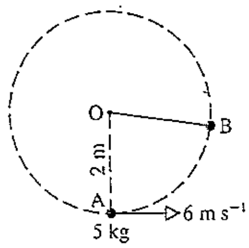
4.



5.



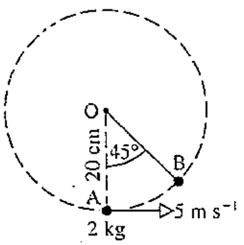
6.



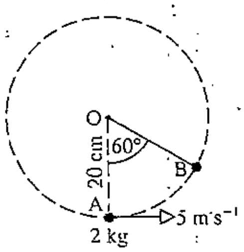
Questions 7, 8 and 9 each involve a particle of mass 2 kg suspended from a fixed point O by a light inextensible string of length 20 cm. The particle is projected from the lowest position, A, with a horizontal speed of $5\ \mathrm{m\,s^{-1}}$.

Determine the speed of the particle and the tension in the string for each of the positions shown below.

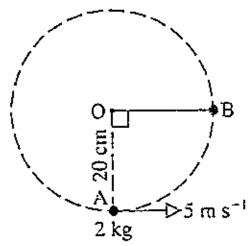
7.



8.



9.



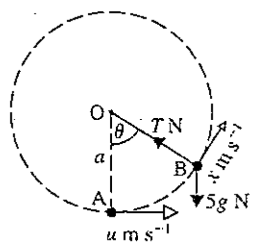
13. A particle of mass 5 kg describes complete vertical circles on the end of a light inextensible string of length 2 m. Given that the speed of the particle is $5\ \mathrm{m\,s^{-1}}$ at its highest point, find:
  (a) its speed at its lowest point
  (b) the tension in the string when the string is horizontal
  (c) the magnitude of the tangential acceleration when the string is horizontal.

14. A particle of mass m describes complete vertical circles on the end of a light inextensible string of length 0·5 metres.
Given that when the particle is at the lowest point of the circle the tension in the string is twice that when the particle is at the highest point on the circle, find the speed of the particle at the lowest and highest points of the circle.

15. A particle of mass m describes complete vertical circles on the end of a light inextensible string of length r.
Given that the speed of the particle at the lowest point is twice the speed at the highest point, find:
(a) the speed of the particle at the lowest point
(b) the tension in the string when the particle is at its highest point.

16. A light rod of length 1·5 m is pivoted at one end O and has a particle P of mass 5 kg attached to the other end. Given that when the rod is vertical with P below O the speed of the particle is $8\,\mathrm{m\,s^{-1}}$, find the location of P when the tension in the rod is zero and determine the speed of the particle at that point.

17. A particle of mass 5 kg is suspended from a fixed point O by a light inextensible string of length a metres. The particle is projected from the lowest point, A, with horizontal speed $u\,\mathrm{m\,s^{-1}}$. At the instant shown in the diagram the particle is at point B, the particle's speed is $x\ \mathrm{m\,s^{-1}}$, the tension in the string is T N and OB makes an angle θ with the downward vertical.

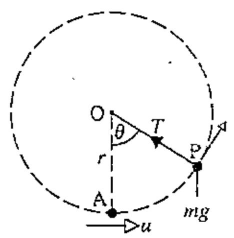


Show that $$x^2 = u^2 - 2ga + 2ga\cos\theta$$

and $$T = 15g\cos\theta + \frac{5u^2}{a} - 10g.$$

Hence determine the value of θ when the string goes slack given that $u = 2{\cdot}8$ and $a = 0{\cdot}2$, giving your answer to the nearest degree.

18. A particle P of mass m is attached to one end of a light inextensible string of length r, the other end being attached to a fixed point O. When the particle is hanging in equilibrium, with P at A, it is given a horizontal velocity of magnitude u. Obtain an expression for the speed of the particle and for the magnitude of the tension in the string at the instant that OP makes an angle θ with the downward vertical. Hence show that if:



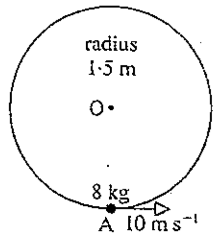
$u^2 < 2gr$ the particle's speed will be zero at some point for which $0 \leqslant \theta < 90°$.

$u^2 = 2gr$ P will just reach a point level with O, at which instant the string will momentarily be slack (i.e. zero tension).

$u^2 > 5gr$ the tension in the string is always positive (i.e. the string is always taut and P will complete vertical circles).

If the string is replaced by a rod, complete circles will be possible provided P still has some forward speed when it reaches the top of the circle.
Show that this will occur provided $u^2 > 4gr$.

19. The diagram shows a particle of mass 8 kg at rest inside a smooth vertical circular rim of radius 1·5 m, centre O. The particle is projected horizontally with speed $10\,\mathrm{m\,s^{-1}}$ from the lowest point, A. Find the speed of the particle and the magnitude of the resultant acceleration when the particle is at the point P where angle POA is:



(a) 30°

(b) 45°

(c) 120°.

20. A small bead P of mass 3 kg is threaded on a smooth circular wire of radius 2 m and centre O, fixed in the vertical plane. The bead is gently disturbed from rest at the highest point H of the wire.
Find the angular speed of the bead and the magnitude of the reaction between the bead and the wire when:

(a) angle HOP = 30°

(b) angle HOP = 60°

(c) angle HOP = 90°.

21. A small bead P of mass 2 kg is threaded on a smooth circular wire radius 1 m and centre O, fixed in the vertical plane. The bead is projected from its lowest position A with a horizontal speed of $10\,\mathrm{m\,s^{-1}}$.
Find:

(a) an expression, in terms of θ, for the magnitude of the reaction between the bead and the wire when the bead is at a point such that angle AOP $= \theta$

(b) θ when the reaction between the bead and the wire is 200 N.

22. A light rod is pivoted at one end O and has a particle of mass 2 kg attached to the other end. The system is held at rest with the particle vertically above O and released gently.
Find the tension in the rod when the rod makes an angle of 60° with the upward vertical.

23. A particle of mass m kg is slightly disturbed form rest at the top of a smooth sphere radius 3 metres and centre O. Find, to one decimal place, the horizontal and vertical distances travelled by the particle from the time it was disturbed until 0·5 seconds after it left the surface.

24. A particle of mass m is projected from the top of a smooth sphere of radius r. In the subsequent motion the particle slides down the outside surface of the sphere and leaves the surface of the sphere with a speed of $2\sqrt{\frac{gr}{5}}$. Find:

(a) the vertical distance travelled by the particle while it is in contact with the sphere

(b) the initial speed of projection

(c) the velocity of the particle when it is level with the horizontal diameter of the sphere.

25. A small bead P of mass m is threaded on a smooth circular wire of radius a and centre O, fixed in the vertical plane. Initially the bead is held at rest at a point B, where OB is inclined at an angle of 30° to the downward vertical through O. The bead is projected from B, perpendicular to OB with speed u so that P starts describing a vertical circle about O. A point C is on the wire and OC makes an angle θ with the upward vertical through O. Show that:

(a) the reaction between the wire and the bead at C has magnitude

$$\left|\frac{mu^2}{a} - mg\left(\sqrt{3} + 3\cos\theta\right)\right|$$

(b) the difference between the reaction of the wire on the bead at B and the reaction of the wire on the bead at C is

$$\left|3mg\left(\cos\theta + \frac{\sqrt{3}}{2}\right)\right|.$$

26. A particle P of mass m is released from rest at a point B on the surface of a smooth sphere, centre O and radius a. The line OB is inclined at an angle of 30° to the upward vertical through O. Show that when OP makes an angle θ with the upward vertical:

(a) the reaction between the particle and the sphere is given by $mg\left(3\cos\theta - \sqrt{3}\right)$

(b) the square of the angular speed of P is $\frac{g}{a}\left(\sqrt{3} - 2\cos\theta\right)$.

Find, also, the angular speed of the particle when it leaves the surface of the sphere.

Ex 13E Examination Questions.

5. A child of mass 30 kg keeps herself amused by swinging on a 5 m rope attached to an overhanging tree. She is holding on to the lower end of the rope and "swinging" in a horizontal circle of radius 3 m.

(a) Draw a diagram to show the forces acting on the girl.

(b) Find the tension in the rope.

(c) Show that the time she takes to complete a circle is approximately 4 seconds.

(d) State any assumptions that you have made about the rope.

(e) The girl's older brother then swings, on his own, on the rope in a horizontal circle of the same radius. Show that the tension in the rope is now $\frac{5mg}{4}$ where m is his mass.

Find the time that it takes for him to complete one circle.

(AEB Spec)

6. A particle is attached to the end A of a light inextensible string OA of length 1 m. The end O is fixed at a distance of 1 m above the horizontal ground. The particle describes a horizontal circle at a height of 0·5 m above the ground.

(a) Calculate the steady speed of the particle.

(b) The string is then released at O.

(i) Describe in words the initial direction of the subsequent motion of the particle.

(ii) Calculate the horizontal displacement of the particle from its position when the string is released to where it first strikes the ground. (UODLE)

7. A particle P of mass m is attached to one end of a light inextensible string of length d. The other end of the string is fixed at a height h, where $h < d$, vertically above a point O on a smooth horizontal table. The particle describes a circle, centre O, on the surface of the table, with constant angular speed ω.

Find expressions, in terms of m, d, h, ω and g as appropriate, for

(a) the tension in the string

(b) the magnitude of the force exerted on P by the plane.

(c) Hence show that

$$\omega \leqslant \sqrt{\left(\frac{g}{h}\right)}.$$

The angular speed of P is now increased to $\sqrt{\left(\frac{3g}{h}\right)}$, and P now describes a horizontal circle, centre Q, *above* the surface of the table, with this constant angular speed.

(d) Determine, in terms of h, the distance OQ. (ULEAC)

8. A particle P, of mass M, moves on the smooth inner surface of a fixed hollow spherical bowl, centre O and inner radius r, describing a horizontal circle at constant speed. The centre C of this circle is at a depth $\frac{1}{3}r$ vertically below O. Determine

(a) the magnitude of the force exerted by the surface of the sphere on P

(b) the speed of P. (ULEAC)

10. A smooth loop of wire in the form of a circle, centre O and of radius 0·3 m, is fixed in a vertical plane. A bead of mass 0·5 kg is threaded on the wire and projected with speed $u\,\mathrm{m\,s^{-1}}$ from the lowest point of the wire so that it comes to instantaneous rest at a height of 0·1 m above the level of O. Find

(i) the value of u,

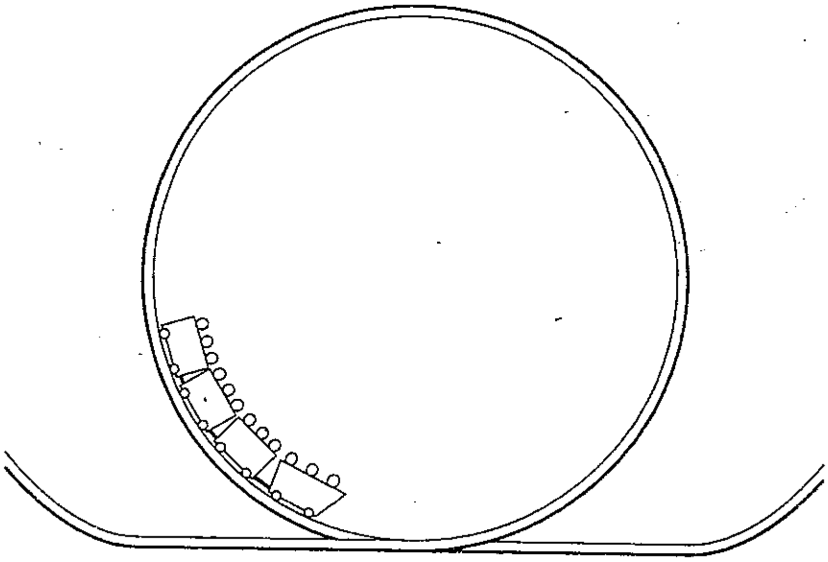
(ii) the reaction of the wire on the particle when the particle is level with O. (WJEC)

11. A particle is free to slide on the smooth outer surface of a fixed sphere of centre O. The particle is released from rest at a point A on the sphere, where OA is inclined at α to the upward vertical, $0 < \alpha < \frac{\pi}{2}$.

Given that the particle leaves the sphere at a point B, where OB is inclined at β to the upward vertical, show that $\frac{\cos\beta}{\cos\alpha} = \frac{2}{3}$.

(AEB 1994)

9. One section of a "Loop the Loop" ride at an Adventure Park takes passengers round a vertical circle.

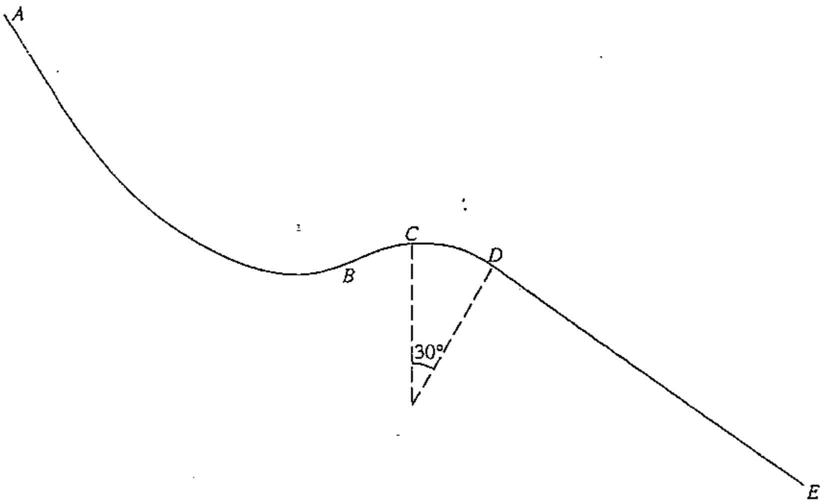


The situation is modelled by considering a small bead P of mass m threaded on a fixed smooth circular wire. The circular wire has centre O and radius a, and its plane is vertical. The bead is projected from the lowest point of the wire with speed $\sqrt{ag}$. When OP makes an angle θ with the downward vertical, find

(a) the speed of the bead,

(b) the reaction of the wire on the bead. (AEB Spec)

12.



The diagram shows the shape of a "slide" for a children's playground. The section DE is straight and BCD is a circular arc of radius 5 m. C is the highest point of the arc and CD subtends an angle of 30° at the centre. For safety reasons, children should not be sliding so fast that they lose contact with the slide at any point. Neglecting any resistances to motion, find:

(i) the child's speed as it passes through C, given that it is on the point of losing contact at C,

(ii) the child's speed as it passes through D, given that it is on the point of losing contact as it reaches D.

Find the greatest possible height of the starting point A of the slide above the level of D, if a child starting from rest at A is not to lose contact with the slide at any point.

Explain briefly whether taking resistances into account would lead to a larger or smaller value for the greatest "safe height" above D. (UCLES)

13. A particle A is free to move on the smooth inner surface of a fixed spherical shell of internal radius a and centre O. Given that A passes through the lowest point of the spherical surface with speed u and that A leaves the surface when OA is inclined at an angle α to the horizontal, and A is above the horizontal through O, show that

$$u^2 = ga(2 + 3 \sin \alpha).$$

Given also that A passes through O before next meeting the surface, show that $\sin^2\alpha = \frac{1}{3}$. (AEB 1993)

14. A light inextensible string of length a has one end attached to a fixed point O and the other end is attached to a particle of mass m.
When the particle is hanging in equilibrium it is given a horizontal velocity of magnitude $2\sqrt{(ga)}$. Obtain an expression for the magnitude of the tension in the string when the particle is still on its circular path and the string has rotated through an angle θ.
Deduce that the tension in the string vanishes when the particle is at a point at a height $\frac{2}{3}a$ above the level of O.
Show that in the subsequent motion the particle passes through a point distant $\frac{9}{16}a$ directly above O. (AEB 1994)

15. A smooth sphere, with centre O and radius a, is fixed with its lowest point on a horizontal table. A particle P of mass m, resting on the sphere at its highest point, is given a horizontal speed $\sqrt{(\frac{1}{4}ga)}$. At the instant when the line OP makes an angle θ with the vertical, the speed of P is v. Show that, while P remains in contact with the sphere,

$$v^2 = \tfrac{1}{4}ga\,(9 - 8 \cos \theta).$$

Find, in terms of m, g and θ, the magnitude of the force exerted by the sphere on P. Hence show that P leaves the surface of the sphere when $\theta = \cos^{-1}\left(\frac{3}{4}\right)$.
After leaving the surface of the sphere, P moves freely under gravity until it strikes the table. Find the magnitude of the horizontal component of the velocity of P just before its strikes the table and show that the magnitude of the vertical component of the velocity of P at this instant is $\frac{7}{8}\sqrt{(5ga)}$. (UCLES)

16. The axis of a smooth circular cylinder of radius a is horizontal. Two particles P of mass m and Q of mass $3m$ are joined together by a light string of length $\frac{1}{2}\pi a$ and placed on the surface of the cylinder in a vertical plane perpendicular to the axis. The point on the axis in the vertical plane through both P and Q is denoted by O. Initially the position of the particles is symmetrical about the vertical plane containing the axis of the cylinder with both OP and OQ inclined at an angle of 45° to the upward vertical through O. The particles are released from rest.
(a) Show, by using the equation of energy or otherwise, that the speed v of the particles when the angle between OQ and the upwards vertical is θ is given by

$$2v^2 = ga\left(2\sqrt{2} - \sin \theta - 3 \cos \theta\right).$$

(b) (i) Show that in this position the reaction of the cylinder on Q is

$$\tfrac{3}{2}mg\left(\sin \theta + 5 \cos \theta - 2\sqrt{2}\right).$$

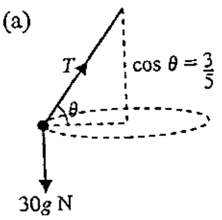
(ii) Find, in terms of m, g, a and θ, the reaction of the cylinder on P and the tension in the string.
(c) Show, assuming that Q leaves the cylinder before P, that it leaves the cylinder before $\theta = 70°$. (UODLE)

ANSWERS

Exercise 13E page 335

1. 0·51 3. (ii) $15{\cdot}7\,\mathrm{m\,s^{-1}}$ 4. $\mu \geqslant 0{\cdot}33$

5. (a)



(b) 367·5 N (d) Light and inextensible (e) $\dfrac{4\pi}{\sqrt{g}}$ s (i.e. again about 4 seconds)

6. (a) $3{\cdot}83\,\mathrm{m\,s^{-1}}$ (b) (i) initially horizontal, at a tangent to the circle (ii) 1·22 m

7. (a) $md\omega^2$ (b) $m(g - h\omega^2)$, $\dfrac{2h}{3}$ 8. (a) $3Mg$ (b) $\sqrt{\dfrac{8gr}{3}}$

9. (a) $\sqrt{ag(2\cos\theta - 1)}$ (b) $mg(3\cos\theta - 1)$ 10. (i) $2{\cdot}8\,\mathrm{m\,s^{-1}}$ (ii) $\dfrac{g}{3}$ N

12. (i) $7\,\mathrm{m\,s^{-1}}$ (ii) $6{\cdot}5\,\mathrm{m\,s^{-1}}$, 2·17 m to nearest cm. The "safe height" is greater if resistances are taken into account. The extra PE lost in descent will go as work against resistance. Thus the speed as BCD is reached can be as before.

14. $mg(3\cos\theta + 2)$

15. $\dfrac{3mg(4\cos\theta - 3)}{4}$, $\dfrac{3\sqrt{3ga}}{8}$ 16. $\dfrac{mg(3\sin\theta + 3\cos\theta - 2\sqrt{2})}{2}$, $\dfrac{3mg(\sin\theta + \cos\theta)}{4}$