1. Let $P(n) = 2^n \cdot 3^{n+2} + 2 \cdot 5^{2n+1}$

$n=0$, $2^0 3^2 + 2 \cdot 5^1 = 9 + 10 = 19$

$n=1$, $2^1 3^3 + 2 \cdot 5^3 = 54 + 250 = 304 = 19(16)$

$n=2$, $2^2 \cdot 3^4 + 2\ 5^5 = 324 + 6250 = 6574$

$= 19(346)$

Conjecture : To prove $P(n)$ is divisible by 19.

Assume that $P(k)$ is true :

$2^k \cdot 3^{k+2} + 2 \cdot 5^{2k+1}$ is divisible by 19.

Consider $P(k+1) = 2^{k+1} \cdot 3^{k+3} + 2 \cdot 5^{2k+3}$

$= 6(2^k \cdot 3^{k+2}) + 25(2 \cdot 5^{2k+1})$

$= 6(2^k \cdot 3^{k+2} + 2 \cdot 5^{2k+1}) + 19(2 \cdot 5^{2k+1})$

$= 6P(k) + 19(2 \cdot 5^{2k+1})$

$\therefore P(k+1)$ is true if $P(k)$ is true

$\Rightarrow P(n)$ is divisible by 19 for all positive integers.

2 Let $\dfrac{1}{(r+1)(r+3)} = \dfrac{A}{r+1} + \dfrac{B}{r+3}$

$$= \frac{A(r+3) + B(r+1)}{(r+1)(r+3)}$$

$\therefore \quad 1 = A(r+3) + B(r+1)$

$\quad = (A+B)r + 3A + B$

Equating coefficients,

$0 = A + B \qquad 1 = 3A + B$

$2A = 1$

$A = \dfrac{1}{2}$

$B = \dfrac{-1}{2}$

$$\therefore \frac{1}{(r+1)(r+3)} = \frac{1}{2(r+1)} - \frac{1}{2(r+3)}$$

Since $\dfrac{1}{(r+1)(r+3)} \equiv \dfrac{1}{2(r+1)} - \dfrac{1}{2(r+3)}$,

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \sum_{r=1}^{n} 2\left(\frac{1}{2(r+1)} - \frac{1}{2(r+3)}\right)$$

$$= \sum_{r=1}^{n} \frac{1}{r+1} - \frac{1}{r+3}$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$+ \frac{1}{3} - \frac{1}{5}$$

$$+ \frac{1}{4} - \frac{1}{6}$$

$$\vdots$$

$$+ \frac{1}{n-1} - \frac{1}{n+1}$$

$$+ \frac{1}{n} - \frac{1}{n+2}$$

$$+ \frac{1}{n+1} - \frac{1}{n+3}$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$$

3. $5x^3 - x^2 + 4x - 3 = 0$

α, β, γ are the roots.

$\alpha^2 + 2, \beta^2 + 2, \gamma^2 + 2$

Let $u = \alpha^2 + 2$

$\therefore \alpha = \pm\sqrt{u - 2}$

α is a root

$\therefore 5\alpha^3 - \alpha^2 + 4\alpha - 3 = 0$

$5(\pm\sqrt{u-2})^4 - (\pm\sqrt{u-2})^2$

$+ 4(\pm\sqrt{u-2}) - 3 = 0$

$\pm 5\sqrt{u-2}(u-2) - (u-2)$

$\pm 4\sqrt{u-2} - 3 = 0$

$\pm 5\sqrt{u-2}(u-2) \pm 4\sqrt{u-2}(u-2)$

$= u - 2 + 3$

$\pm\sqrt{u-2}(5(u-2) + 4) = u + 1$

$\pm\sqrt{u-2}(5u - 6) = u + 1$

$(\pm\sqrt{u-2})^2(5u - 6)^2 = (u+1)^2$

$(u-2)(25u^2 - 60u + 36) = u^2 + 2u + 1$

$25u^3 - 60u^2 + 36u - 50u^2 + 120u - 72$

$= u^2 + 2u + 1$

$25u^3 - 111u^2 + 154u - 73 = 0$

$\therefore$ The equation having roots

$\alpha^2 + 2, \beta^2 + 2, \gamma^2 + 2$ is

$25u^3 - 111u^2 + 154u - 73 = 0$

4. $I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$

$u = x^{n-1}$ $\qquad dv = x\sqrt{1-x^2}\, dx$

$du = (n-1)x^{n-2}\, dx$ $\qquad v = \frac{1}{3}(1-x^2)^{\frac{3}{2}}$

$$= \left[-\frac{1}{3} x^{n-1} (1-x^2)^{\frac{3}{2}} \right]_0^1 + \frac{1}{3}(n-1) \int x^{n-2} (1-x^2)^{\frac{3}{2}} dx$$

$$= 0 + \frac{1}{3}(n-1) \int x^{n-2}(1-x^2)\sqrt{1-x^2}\, dx$$

$$= \frac{1}{3}(n-1) \int x^{n-2}\sqrt{1-x^2}\, dx - \frac{1}{3}(n-1)\int x^n \sqrt{1-x^2}\, dx$$

$$= \frac{1}{3}(n-1) I_{n-2} - \frac{1}{3}(n-1) I_n$$

$$\therefore \left[\frac{1}{3}(n-1) + 1 \right] I_n = \frac{1}{3}(n-1) I_{n-2}$$

$$(n+2) I_n = (n-1) I_{n-2}$$

$$I_1 = \int_0^1 x\sqrt{1-x^2}\, dx = \left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}} \right]_0^1 = \frac{1}{3}$$

$$5 I_3 = 2 I_1 \Rightarrow I_3 = \frac{2}{5}\left(\frac{1}{3}\right) = \frac{2}{15}$$

$$7 I_5 = 4 I_3 \Rightarrow I_5 = \frac{4}{7}\left(\frac{2}{15}\right) = \frac{8}{105}$$

5. $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 10\cos x + 15\sin x$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

The auxillary equation is

$$m^2 + 5m + 6 = 0$$

$$(m + 2)(m + 3) = 0$$

$$m = -2, -3$$

The complementary function, y_c, is

$$y_c = Ae^{-2x} + Be^{-3x}$$

If y_p is the particular integral,

y_p is given by

$$y_p = C\cos x + D\sin x$$

$$\frac{dy_p}{dx} = -C\sin x + D\cos x$$

$$\frac{d^2y_p}{dx^2} = -C\cos x - D\sin x$$

$$\frac{d^2 y_p}{dx^2} + \frac{5dy_p}{dx} + 6y_p$$

$$= -C\cos x - D\sin x$$

$$+ 5(-C\sin x + D\cos x)$$

$$+ 6(C\cos x + D\sin x)$$

$$= (5C + 5D)\cos x + (-5C + 5D)\sin x$$

$$= 10\cos x + 15\sin x$$

$5C + 5D = 10$ $\quad$ $-5C + 5D = 15$

$C + D = 2$ $\quad$ $-C + D = 3$

$$2D = 5$$

$$D = \frac{5}{2}$$

$$C = \frac{-1}{2}$$

$$y_p = \frac{-\cos x}{2} + \frac{5\sin x}{2}$$

$$y = y_c + y_p$$

$$= Ae^{-2x} + Be^{-3x} - \frac{\cos x}{2} + \frac{5\sin x}{2}$$

$\therefore$ The general solution of the differential equation is $y = Ae^{-2x} + Be^{-3x} - \frac{\cos x}{2} + \frac{5\sin x}{2}$

6. $z^4 = -16$ Let $z = r(\cos\theta + i\sin\theta)$

$$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$$

$$r = 2, \quad \cos 4\theta = -1 \quad \sin 4\theta = 0$$

$$4\theta = \pi + 2\pi r$$

$$\therefore \theta = \frac{\pi}{4} + \frac{2\pi r}{4}$$

$r=0$ $\quad z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2} + i\sqrt{2}$

$r=1$ $\quad z_2 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\sqrt{2} + i\sqrt{2}$

$r=2$ $\quad z_3 = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\sqrt{2} - i\sqrt{2}$

$r=3$ $\quad z_4 = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2} - i\sqrt{2}$

$$\left(\frac{z+1}{z}\right)^4 = -16 \Rightarrow \left(1 + \frac{1}{z}\right)^4 = -16$$

$$1 + \frac{1}{z} = \sqrt{2} + i\sqrt{2}$$

$$\frac{1}{z} = (\sqrt{2} - 1) + i\sqrt{2}$$

$$z = \frac{1}{(\sqrt{2}-1) + i\sqrt{2}} \times \frac{(\sqrt{2}-1) - i\sqrt{2}}{(\sqrt{2}-1) - i\sqrt{2}}$$

$$= \frac{(\sqrt{2}-1) - i\sqrt{2}}{5 - 2\sqrt{2}} \qquad z^* = \frac{(\sqrt{2}-1) + i\sqrt{2}}{5 - 2\sqrt{2}}$$

$$1 + \frac{1}{2} = -\sqrt{2} + i\sqrt{2}$$

$$z = \frac{1}{-(\sqrt{2}+1) + i\sqrt{2}}$$

$$= \frac{-(\sqrt{2}+1) - i\sqrt{2}}{5 + 2\sqrt{2}} \qquad z^{*} = \frac{-(\sqrt{2}+1) + i\sqrt{2}}{5 + 2\sqrt{2}}$$

7 a) $\underset{\sim}{r} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k} + s(\underset{\sim}{i} + 9\underset{\sim}{j} - 3\underset{\sim}{k})$

$(4, 8, -3)$

$$\begin{pmatrix} 4 \\ 8 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

Since $\begin{pmatrix} 1 \\ 9 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ are vectors parallel to the plane, $\begin{pmatrix} 1 \\ 9 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ is a vector perpendicular to the plane.

$$\begin{pmatrix} 1 \\ 9 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 9 & -3 \\ 2 & 5 & 1 \end{vmatrix}$$

$$= 24\underset{\sim}{i} - 7\underset{\sim}{j} + 13\underset{\sim}{k}.$$

Since $24\underset{\sim}{i} - 7\underset{\sim}{j} - 13\underset{\sim}{k}$ is normal to the plane and $(4, 8, -3)$ is a point on the plane, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a point on the plane

$$\underset{\sim}{r} \cdot \begin{pmatrix} 24 \\ -7 \\ -13 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 24 \\ -7 \\ -13 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 24 \\ -7 \\ -13 \end{pmatrix} = 96 - 56 + 39$$

$$24x - 7y - 13z = 79$$

The equation of the plane containing the line

$\underset{\sim}{r} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k} + s(\underset{\sim}{i} + 9\underset{\sim}{j} - 3\underset{\sim}{k})$

and the point $(4, 8, -3)$ is

$24x - 7y - 13z = 79$.

b) $\underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k} + s(-3\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k})$

$\underset{\sim}{r} = 7\underset{\sim}{i} + 6\underset{\sim}{j} - 5\underset{\sim}{k} + t(6\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$

Since $\begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix}$ are vectors

parallel to the plane, $\begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix}$

is a vector perpendicular to the plane.

$$\begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -3 & 1 & 5 \\ 6 & 4 & -1 \end{vmatrix}$$

$$= -21\underset{\sim}{i} + 27\underset{\sim}{j} - 18\underset{\sim}{k}$$

$$= -3\left(7\underset{\sim}{i} - 9\underset{\sim}{j} + 6\underset{\sim}{k}\right)$$

$$\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + s\begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ -5 \end{pmatrix} + t\begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 - 3s \\ 2 + s \\ -4 + 5s \end{pmatrix} = \begin{pmatrix} 7 + 6t \\ 6 + 4t \\ -5 - t \end{pmatrix}$$

$$\left.\begin{aligned} 1 - 3s &= 7 + 6t \\ 2 + s &= 6 + 4t \\ -4 + 5s &= -5 - t \end{aligned}\right\}$$

$$\left.\begin{aligned} s + 2t &= -2 \\ s - 4t &= 4 \\ 5s + t &= -1 \end{aligned}\right\}$$

$$\begin{aligned} &-1 + 2 : \\ &-5 \times 1 + 3 : \end{aligned} \qquad \left.\begin{aligned} s + 2t &= -2 \\ -6t &= 6 \\ -9t &= 9 \end{aligned}\right\}$$

$t = -1 \quad s = 0$

$\therefore (1, 2, -4)$ is the point of intersection of the two lines

Since $7\underset{\sim}{i} - 9\underset{\sim}{j} + 6\underset{\sim}{k}$ is normal to the plane and $(1, 2, -4)$ is a point on the plane, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a point on the plane

$$\underset{\sim}{r} \cdot \begin{pmatrix} 7 \\ -9 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -9 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -9 \\ 6 \end{pmatrix} = 7 - 18 - 24$$

$$7x - 9y + 6z = -35.$$

$\therefore$ The equation of the plane containing the lines

$\underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k} + s(-3\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k})$

and $\underset{\sim}{r} = 7\underset{\sim}{i} + 6\underset{\sim}{j} - 5\underset{\sim}{k} + t(6\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$

is $7x - 9y + 6z = -35$.

c) $24x - 7y - 13z = 79$
$7x - 9y + 6z = -35$

$-24 \times ②$. $24x - 7y - 13z = 79$
$-168x + 216y - 144z = 840$

$7 \times ① + ②$. $24x - 7y - 13z = 79$
$167y - 235z = 1393$

Let $z = 167t,\ t \in R$

$$y = \frac{1393}{167} + 235t$$

$$24x - 7\left(\frac{1393}{167} + 235t\right) - 13(167t) = 79$$

$$24x - \frac{9751}{167} - 1645t = 2171t = 79$$

$$24x - \frac{22944}{167} = 3816t$$

$$x = 159t + \frac{956}{167}$$

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{956}{167} + 159t \\ \frac{1393}{167} + 235t \\ 167t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{956}{167} \\ \frac{1393}{167} \\ 0 \end{pmatrix} + t \begin{pmatrix} 159 \\ 235 \\ 167 \end{pmatrix}$$

$\therefore$ The equation of the line of intersection of the two planes is

$$\underset{\sim}{r} = \begin{pmatrix} \frac{956}{167} \\ \frac{1393}{167} \\ 0 \end{pmatrix} + t \begin{pmatrix} 159 \\ 235 \\ 167 \end{pmatrix}.$$

8. $y = \dfrac{2(x^2 - x - 1)}{x^2 + 2x + 2} = 2 - \dfrac{6(x+1)}{x^2 + 2x + 2}$

(a) $x = 0, \ y = -1$

$y = 0, \ x^2 - x - 1 = 0 \Rightarrow x = \dfrac{1}{2} \pm \dfrac{\sqrt{5}}{2}$

$= 1.62$ or -0.62

$(0, -1), \ (-0.62, 0), \ (1.62, 0)$

(b) Asymptote : $y = 2$

(c) $\dfrac{dy}{dx} = \dfrac{-(x^2 + 2x + 2)(6) - 6(x+1)(2x+2)}{(x^2 + 2x + 2)^2}$

$= - \dfrac{6x^2 + 12x + 12 - (12x^2 + 24x + 12)}{(x^2 + 2x + 2)^2}$

$= \dfrac{6x(x+2)}{(x^2 + 2x + 2)^2}$

For $\dfrac{dy}{dx} = 0 \Rightarrow x(x+2) = 0$

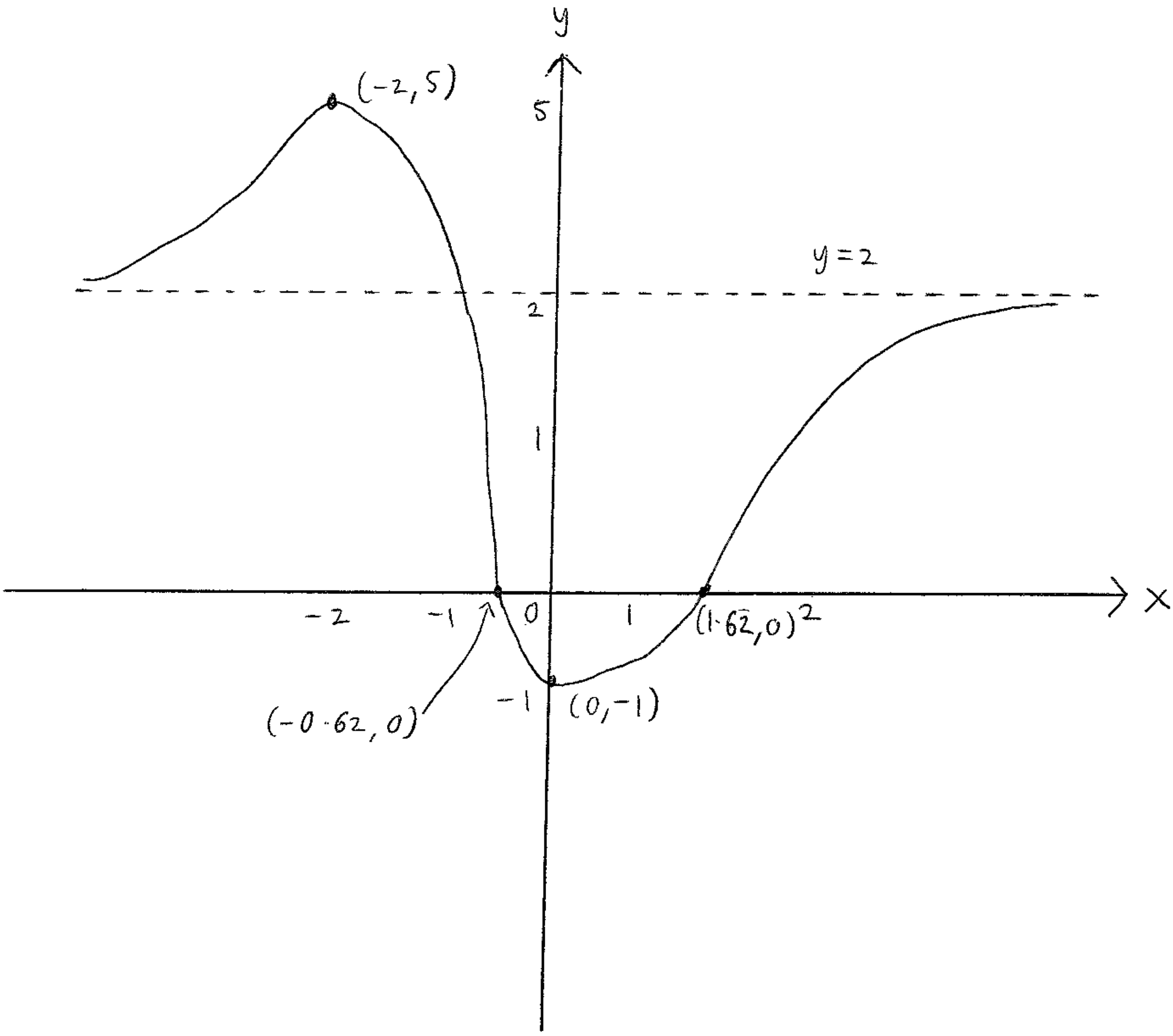
$x = 0$ or $x = -2$

$y = -1$ $\quad$ $y = 5$

$\therefore$ Coordinates of turning points are $(0,-1), (-2,5)$

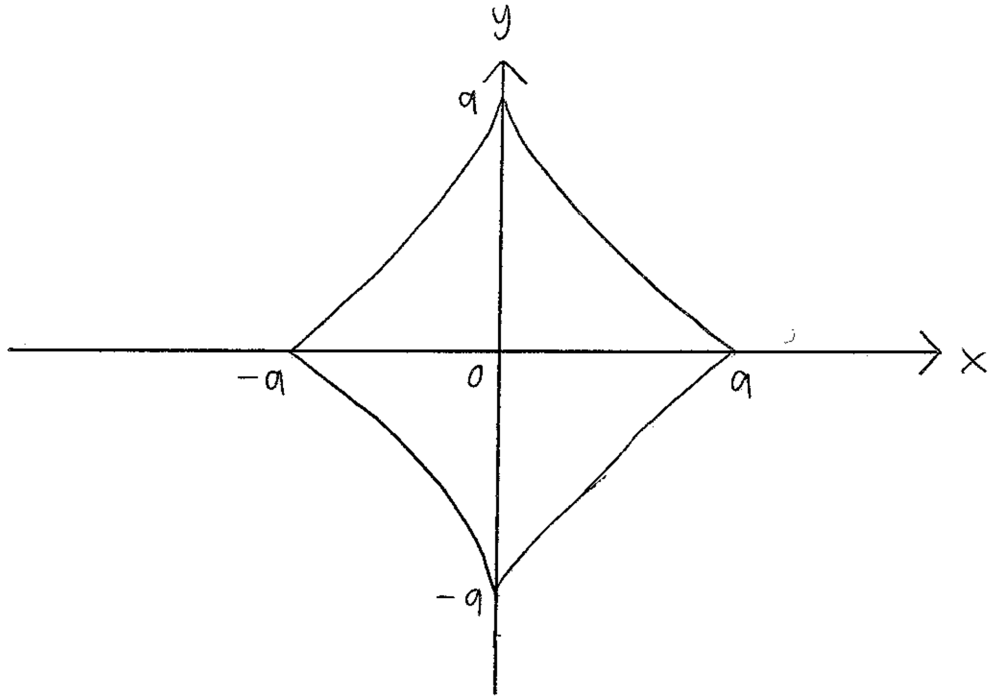
As $x \to \infty, \ y \to 2^-$

As $x \to -\infty, \ y \to 2^+$



y
x
(-2, 5)
5
y = 2
2
1
-2
-1
0
1
(1.62, 0)
2
-1
(0, -1)
(-0.62, 0)

9. $x = a\cos^3 t$, $y = a\sin^3 t$



$$\frac{dx}{dt} = -3a\sin t\cos^2 t \qquad \frac{dy}{dt} = 3a\cos t\sin^2 t$$

$$\left(\frac{dx}{dt}\right)^2 = 9a^2\sin^2 t\cos^4 t \qquad \left(\frac{dy}{dt}\right)^2 = 9a^2\cos^2 t\sin^4 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9a^2\sin^2 t\cos^2 t}$$

$$= 3a\sin t\cos t$$

$$\text{surface area} = 2\pi \int y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int 3a^2\sin^4 t\cos t\ dt$$

$$= 6a^2\pi \left[\frac{\sin^5 t}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{6}{5}\pi a^2$$

Moment about $Oy = 2\pi \int xy \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= 2\pi \int 3a^3 \sin^4 t \cos^4 t \, dt$$

$$= 6\pi a^3 \int \sin^4 t \left(\sin^4 t - 2\sin^2 t + 1\right) dt$$

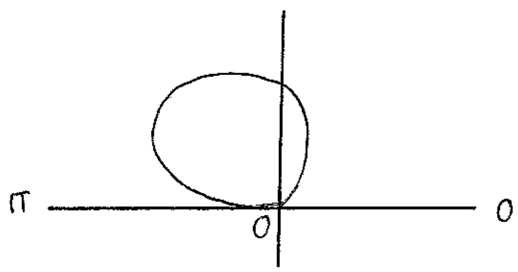
$$= 6\pi a^3 \int \left(\sin^8 t - 2\sin^6 t + \sin^4 t\right) dt$$

$$= 6\pi a^3 \left(\frac{35\pi}{256} - \frac{10\pi}{32} + \frac{3\pi}{16}\right) = \frac{9\pi^2 a^3}{128}$$

$$\therefore \bar{x} = \frac{\frac{9\pi^2 a^3}{128}}{\frac{6\pi a^2}{5}}$$

$$= \frac{15\pi a}{256}$$

10. $r = \sin\theta(1-\cos\theta)$



a) $\text{Area} = \frac{1}{2}\int r^2 \, d\theta$

$= \frac{1}{2}\int \sin^2\theta(\cos^2\theta - 2\cos\theta + 1)\, d\theta$

$= \frac{1}{2}\int (\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos\theta + \sin^2\theta)\, d\theta$

$= \frac{1}{8}\int \sin^2 2\theta \, d\theta - \int \sin^2\theta\cos\theta\, d\theta$

$+ \frac{1}{4}\int (1-\cos 2\theta)\, d\theta$

$= \frac{1}{16}\int (1-\cos 4\theta)\, d\theta - \left[\frac{\sin^3\theta}{3}\right]_0^{\pi}$

$+ \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$

$= \frac{1}{16}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi} - 0 + \frac{\pi}{4}$

$= \frac{\pi}{16} + \frac{\pi}{4}$

$= \frac{5\pi}{16}$

(b) $r = \sin\theta - \sin\theta\cos\theta$

$= \sin\theta - \frac{1}{2}\sin 2\theta$

$\frac{dr}{d\theta} = \cos\theta - \cos 2\theta$

$= \cos\theta - 2\cos^2\theta + 1$

$\therefore 2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -\frac{1}{2} \quad \cos\theta = 1 \quad \therefore \left(\frac{3\sqrt{3}}{4}, \frac{2\pi}{3}\right)$

or $\left(1.299, \frac{2\pi}{3}\right)$

11 EITHER

(i) $\frac{z}{2} - \frac{z^2}{4} + \frac{z^3}{8} - \frac{z^4}{16} + \cdots$

$$= \frac{\frac{z}{2}\left(1 - \left(-\frac{z}{2}\right)^n\right)}{1 - \left(-\frac{z}{2}\right)}$$

$$= \frac{\frac{z}{2}\left(1 - \left(\frac{-z}{2}\right)^n\right)}{\frac{z+2}{2}}$$

$$= \frac{z}{z+2} \quad \text{as} \quad \left(-\frac{z}{2}\right)^n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

$$\frac{z}{z+2} = \frac{\cos\theta + i\sin\theta}{(\cos\theta + 2) + i\sin\theta}$$

$$= \frac{(c + is)}{(c+2) + is} \times \frac{(c+2) - is}{(c+2) - is}$$

$$= \frac{c(c+2) - ics + ics + i2s + s^2}{(c+2)^2 + s^2}$$

$$= \frac{c^2 + s^2 + 2c - isc + isc + i2s}{c^2 + 4c + 4 + s^2}$$

$$= \frac{1 + 2(\cos\theta + i\sin\theta)}{5 + 4\cos\theta}$$

$$= \frac{1 + 2e^{i\theta}}{5 + 4\cos\theta}$$

11 (ii) $(c+is)^7 = c^7 + 7c^6(is) + 21c^5(is)^2$

$$+ 35c^4(is)^3 + 35c^3(is)^4 + 21c^2(is)^5$$

$$+ 7c(is)^6 + (is)^7$$

Taking only the imaginary parts:

$$(c+is)^7 = \cos 7\theta + i\sin 7\theta$$

$$\therefore \sin 7\theta = 7sc^6 - 35s^3c^4 + 21s^5c^2 - s^7$$

$$= 7s(1-s^2)^3 - 35s^3(1-s^2)^2$$

$$+ 21s^2(1-s^2) - s^7$$

$$= 7s - 21s^3 + 21s^5 - 7s^7 - 35s^3$$

$$+ 70s^6 - 35s^7 + 21s^5 - 21s^7 - s^7$$

$$\therefore \sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

$$64s^7 - 112s^5 + 56s^3 - 7s - 1 = 0$$

$$\sin 7\theta = -1 \Rightarrow 7\theta = -\frac{\pi}{2} + 2\pi r$$

$$\theta = -\frac{\pi}{14} + \frac{2\pi r}{7}$$

The roots are $\sin\left(\frac{-\pi}{14}\right)$, $\sin\left(\frac{3\pi}{14}\right)$, $\sin\left(\frac{7\pi}{14}\right)$,

$\sin\left(\frac{11\pi}{14}\right)$, $\sin\left(\frac{-5\pi}{14}\right)$, $\sin\left(\frac{-9\pi}{14}\right)$, $\sin\left(\frac{-13\pi}{14}\right)$

But $\sin\left(\frac{-13\pi}{14}\right) = \sin\left(\frac{-\pi}{14}\right) = -\sin\frac{\pi}{14}$

$\sin\left(\frac{-9\pi}{14}\right) = \sin\left(\frac{-5\pi}{14}\right) = -\sin\frac{5\pi}{14}$

$\sin\left(\frac{11\pi}{14}\right) = \sin\left(\frac{3\pi}{14}\right)$ and $\sin\left(\frac{7\pi}{14}\right) = 1$

sum of roots $= 0 \Rightarrow 1 - 2\sin\frac{\pi}{14} + 2\sin\frac{3\pi}{14} - 2\sin\frac{5\pi}{14} = 0$

11 OR

a)

$$A = \begin{pmatrix} 1 & 5 & -3 & 8 \\ -1 & -4 & 1 & -12 \\ 2 & 7 & 0 & 28 \\ -3 & -13 & 5 & -32 \end{pmatrix}$$

$$\xrightarrow[\substack{r_1 + r_2 \\ -2r_1 + r_3 \\ 3r_1 + r_4}]{} \begin{pmatrix} 1 & 5 & -3 & 8 \\ 0 & 1 & -2 & -4 \\ 0 & -3 & 6 & 12 \\ 0 & 2 & -4 & -8 \end{pmatrix}$$

$$\xrightarrow[\substack{3r_2 + r_3 \\ -2r_2 + r_4}]{} \begin{pmatrix} 1 & 5 & -3 & 8 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank $(A) = 2$.

b) $\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ -7 \\ 6 \end{pmatrix}$ $\quad \underset{\sim}{v}_2 = \begin{pmatrix} -3 \\ 9 \\ -4 \end{pmatrix}$ $\quad \underset{\sim}{v}_3 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$

If $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in R^3$ and

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = k_1 \underset{\sim}{v}_1 + k_2 \underset{\sim}{v}_2 + k_3 \underset{\sim}{v}_3$$

$$= k_1 \begin{pmatrix} 1 \\ -7 \\ 6 \end{pmatrix} + k_2 \begin{pmatrix} -3 \\ 9 \\ -4 \end{pmatrix} + k_3 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} k_1 - 3k_2 + 2k_3 \\ -7k_1 + 9k_2 + 5k_3 \\ 6k_1 - 4k_2 + 8k_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 2 \\ -7 & 9 & 5 \\ 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -3 & 2 \\ -7 & 9 & 5 \\ 6 & -4 & 8 \end{vmatrix} = 92 - 258 - 52$$

$$= -218$$

$$\neq 0.$$

$$\therefore \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 2 \\ -7 & 9 & 5 \\ 6 & -4 & 8 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Since every vector $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in R^3$ is a

linear combination of $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$

$\therefore \underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ span R^3.

c) $\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 4 \end{pmatrix}$ $\underset{\sim}{v}_2 = \begin{pmatrix} -3 \\ -5 \\ 7 \\ -9 \end{pmatrix}$ $\underset{\sim}{v}_3 = \begin{pmatrix} 4 \\ 10 \\ -16 \\ 22 \end{pmatrix}$ $\underset{\sim}{v}_4 = \begin{pmatrix} 7 \\ 9 \\ -11 \\ 13 \end{pmatrix}$

If $k_1 \underset{\sim}{v}_1 + k_2 \underset{\sim}{v}_2 + k_3 \underset{\sim}{v}_3 + k_4 \underset{\sim}{v}_4 = \underset{\sim}{0}$

$$k_1 \begin{pmatrix} 1 \\ 2 \\ -3 \\ 4 \end{pmatrix} + k_2 \begin{pmatrix} -3 \\ -5 \\ 7 \\ -9 \end{pmatrix} + k_3 \begin{pmatrix} 4 \\ 10 \\ -16 \\ 22 \end{pmatrix} + k_4 \begin{pmatrix} 7 \\ 9 \\ -11 \\ 13 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} k_1 - 3k_2 + 4k_3 + 7k_4 \\ 2k_1 - 5k_2 + 10k_3 + 9k_4 \\ -3k_1 + 7k_2 - 16k_3 - 11k_4 \\ 4k_1 - 9k_2 + 22k_3 + 13k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 4 & 7 \\ 2 & -5 & 10 & 9 \\ -3 & 7 & -16 & -11 \\ 4 & -9 & 22 & 13 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -3 & 4 & 7 & 0 \\ 2 & -5 & 10 & 9 & 0 \\ -3 & 7 & -16 & -11 & 0 \\ 4 & -9 & 22 & 13 & 0 \end{array}\right)$$

$$\xrightarrow[-4r_1 + r_4]{\substack{-2r_1 + r_2 \\ 3r_1 + r_3}} \left(\begin{array}{cccc|c} 1 & -3 & 4 & 7 & 0 \\ 0 & 1 & 2 & -5 & 0 \\ 0 & -2 & -4 & 10 & 0 \\ 0 & 3 & 6 & -15 & 0 \end{array}\right)$$

$$\xrightarrow[-3r_2 + r_4]{2r_2 + r_3} \left(\begin{array}{cccc|c} 1 & -3 & 4 & 7 & 0 \\ 0 & 1 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $k_4 = t,\ t \in R$

$k_3 = s,\ s \in R$

$\therefore\ k_2 = 5t - 2s$

$k_1 - 3k_2 + 4k_3 + 7k_4 = 0$

$k_1 - 3(5t - 2s) + 4s + 7t = 0$

$k_1 - 15t + 6s + 4s + 7t = 0$

$k_1 = 8t - 10s$

$\therefore$ The vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3, \underset{\sim}{v}_4$ are linearly dependent.