1. $y = x\cos x$

$$\frac{d^{2n}y}{dx^{2n}} = 2(-1)^n n \sin x + (-1)^n x\cos x$$

When $n = 1$:

$$\frac{d^{2(1)}y}{dx^{2(1)}} = \frac{d^2y}{dx^2}$$

$$= \frac{d}{dx}\left(\frac{d}{dx}(x\cos x)\right)$$

$$= \frac{d}{dx}(\cos x - x\sin x)$$

$$= -\sin x - \sin x - x\cos x$$

$$= -2\sin x - x\cos x$$

$$= 2(-1)^1 \sin x + (-1)^1 x\cos x$$

Assume the statement is true when $n = k$

$n = k$: $$\frac{d^{2k}y}{dx^{2k}} = 2(-1)^k k\sin x + (-1)^k x\sin x$$

$n = k+1$: $$\frac{d^{2(k+1)}y}{dx^{2(k+1)}} = 2(-1)^{k+1}(k+1)\sin x + (-1)^{k+1} x\cos x$$

(What needs to be proved)

$$\frac{d^{2k+1}y}{dx^{2k+1}} = \frac{d}{dx}\left(\frac{d^{2k}y}{dx^{2k}}\right)$$

$$= \frac{d}{dx}\left(2(-1)^k k\sin x + (-1)^k x\cos x\right)$$

$$= 2(-1)^k k \cos x + (-1)^k \cos x - (-1)^k x \sin x$$

$$= (2k+1)(-1)^k \cos x + (-1)^{k+1} x \sin x$$

$$\frac{d^{2k+2}y}{dx^{2k+2}} = \frac{d}{dx}\left(\frac{d^{2k+1}y}{dx^{2k+1}}\right)$$

$$= \frac{d}{dx}\left((2k+1)(-1)^k \cos x + (-1)^{k+1} x \sin x\right)$$

$$= (2k+1)(-1)^k(-\sin x) + (-1)^{k+1} \sin x$$

$$+ (-1)^{k+1} x \cos x$$

$$= (2k+1)(-1)^{k+1} \sin x + (-1)^{k+1} \sin x$$

$$+ (-1)^{k+1} x \cos x$$

$$= 2(k+1)(-1)^{k+1} \sin x + (-1)^{k+1} x \cos x$$

$$\therefore \frac{d^{2n}y}{dx^{2n}} = 2(-1)^n n \sin x + (-1)^n x \cos x$$

for every positive integer n.

2 $x = \sec\theta + \csc\theta$ $\qquad$ $y = \tan\theta + \cot\theta$

a) $\frac{dx}{d\theta} = \sec\theta\tan\theta - \csc\theta\cot\theta$ $\qquad$ $\frac{dy}{d\theta} = \sec^2\theta - \csc^2\theta$

$= \frac{\sin\theta}{\cos^2\theta} - \frac{\cos\theta}{\sin^2\theta}$ $\qquad$ $= \frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta}$

$= \frac{\sin^3\theta - \cos^3\theta}{\sin^2\theta\cos^2\theta}$ $\qquad$ $= \frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta\cos^2\theta}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{\sin^3\theta - \cos^3\theta}$$

$$= \frac{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}{(\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta)}$$

$$= \frac{\sin\theta + \cos\theta}{1 + \sin\theta\cos\theta}$$

b) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d\theta}{dx}\frac{d}{d\theta}\left(\frac{dy}{dx}\right)$$

$$= \frac{\sin^2\theta\cos^2\theta}{\sin^3\theta - \cos^3\theta}\frac{d}{d\theta}\left(\frac{\sin\theta + \cos\theta}{1 + \sin\theta\cos\theta}\right)$$

$$= \frac{\dfrac{\sin^2\theta\cos^2\theta}{\sin^3\theta - \cos^3\theta}\left[\begin{array}{l}(1 + \sin\theta\cos\theta)(\cos\theta - \sin\theta) \\ -(\sin\theta + \cos\theta)(-\sin^2\theta + \cos^2\theta)\end{array}\right]}{(1 + \sin\theta\cos\theta)^2}$$

$$= \frac{\sin^2\theta\cos^2\theta\left[\begin{array}{l}\cos\theta - \sin\theta + \sin\theta\cos^2\theta - \sin^2\theta\cos\theta \\ + \sin^3\theta - \sin\theta\cos^2\theta + \sin^2\theta\cos\theta - \cos^3\theta\end{array}\right]}{(\sin^3\theta - \cos^3\theta)(1 + \sin\theta\cos\theta)^2}$$

$$= \frac{\sin^2\theta\cos^2\theta\,(\sin^3\theta - \cos^3\theta - \sin\theta + \cos\theta)}{(\sin^3\theta - \cos^3\theta)(1 + \sin\theta\cos\theta)^2}$$

$$= \frac{\sin^2\theta\cos^2\theta\left[(\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta) - (\sin\theta - \cos\theta)\right]}{(\sin^3\theta - \cos^3\theta)(1 + \sin\theta\cos\theta)^2}$$

$$= \frac{\sin^2\theta\cos^2\theta\,(\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta - 1)}{(\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta)(1 + \sin\theta\cos\theta)^2}$$

$$= \frac{\sin^2\theta\cos^2\theta\,(\sin\theta - \cos\theta)\sin\theta\cos\theta}{(\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)(1 + \sin\theta\cos\theta)^2}$$

$$= \frac{\sin^3\theta\cos^3\theta}{(1 + \sin\theta\cos\theta)^3}$$

3 $$M = \begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & -4 & -5 \\ 3 & a^2+5 & 2a-7 & 3a-9 \\ 6 & a^2+12 & 2a-14 & 3a-18 \end{pmatrix}$$

$$\xrightarrow[R_4 : R_4 - 6R_1]{\substack{R_2 : R_2 - 2R_1 \\ R_3 : R_3 - 3R_1}} \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & a^2-1 & 2a+2 & 3a+3 \\ 0 & a^2 & 2a+4 & 3a+6 \end{pmatrix}$$

$$\xrightarrow{R_4 : R_4 - R_2} \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & a^2-1 & 2a+2 & 3a+3 \\ 0 & a^2-1 & 2a+2 & 3a+3 \end{pmatrix}$$

$$\xrightarrow[R_4 : R_4 - R_3]{R_4 : R_3 - (a^2-1)R_2} \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2(a+1)(a-2) & 3(a+1)(a-2) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The dimension of the range space of T is 3 provided

$$(a+1)(a-2) \neq 0$$

$$a \neq -1, 2.$$

Given that $a = 2$,

$$M = \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 : R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 - 7x_3 - 10x_4 = 0 \Rightarrow x_1 = 7x_3 + 10x_4$

$x_2 + 2x_3 + 3x_4 = 0 \Rightarrow x_2 = -2x_3 - 3x_4$

Let $x_3 = s$ and $x_4 = t$

$x_2 = -2x_3 - 3x_4$

$= -2s - 3t$

$x_1 = 7x_3 + 10x_4$

$= 7s + 10t$

Basis for the null space of T is $\left\{ \begin{pmatrix} 7 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

4 The matrix A is given by $A = PDP^{-1}$

where $P = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

The inverse matrix of P is $P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ 0 & 3 & 0 \end{pmatrix}$

$$A = PDP^{-1}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ 0 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -4 & 2 \\ 0 & 3 & 0 \\ -2 & -5 & 10 \end{pmatrix}$$

3 $x = \sec t + \tan t$

$y = \tan t + \cot t$

$$\frac{dx}{dt} = \sec t \tan t - \csc t \cot t$$

$$= \frac{\sin t}{\cos^2 t} - \frac{\cos t}{\sin^2 t}$$

$$= \frac{\sin^3 t - \cos^3 t}{\sin^2 t \cos^2 t}$$

$$\frac{dy}{dt} = \sec^2 t - \csc^2 t$$

$$= \frac{1}{\cos^2 t} - \frac{1}{\sin^2 t}$$

$$= \frac{\sin^2 t - \cos^2 t}{\sin^2 t \cos^2 t}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\sin^2 t - \cos^2 t}{\sin^3 t - \cos^3 t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{\sin^2 t \cos^2 t}{\sin^3 t - \cos^3 t}\frac{d}{dt}\left(\frac{\sin^2 t - \cos^2 t}{\sin^3 t - \cos^3 t}\right)$$

$$= \frac{\sin^2 t \cos^2 t}{\sin^3 t - \cos^3 t}\frac{\begin{pmatrix}(\sin^3 t - \cos^3 t)(2\sin t \cos t + 2\sin t \cos t) \\ -(\sin^2 t - \cos^2 t)(3\sin^2 t \cos t + 3\cos^2 t \sin t)\end{pmatrix}}{(\sin^3 t - \cos^3 t)^2}$$

$$= \frac{\sin^2 t \cos^2 t}{(\sin^3 t - \cos^3 t)^3}\begin{pmatrix}4\sin t \cos t(\sin^3 t - \cos^3 t) \\ -3\sin t \cos t(\sin^2 t - \cos^2 t)(\sin t + \cos t)\end{pmatrix}$$

$$= \frac{\sin^3 t \cos^3 t}{(\sin^3 t - \cos^3 t)^3} \left(4\sin^3 t - 4\cos^3 t - 3\sin^3 t + 3\cos^3 t - 3\sin^2 t \cos t + 3\cos^2 t \sin t\right)$$

$$= \frac{\sin^3 t \cos^3 t}{(\sin^3 t - \cos^3 t)^3} \left(\sin^3 t - 3\sin^2 t \cos t + 3\sin t \cos^2 t - \cos^3 t\right)$$

$$= \frac{\sin^3 t \cos^3 t (\sin t - \cos t)^3}{(\sin^3 t - \cos^3 t)^3}$$

5. $6y\left(\frac{dy}{dx}\right)^2 + 3y^2\frac{d^2y}{dx^2} + 15y^2\frac{dy}{dx} + 4y^3$

$= 3\cos 2x - 4\sin 2x$

$u = y^3$

$\frac{du}{dy} = 3y^2$

$\frac{du}{dx}\frac{dx}{dy} = 3y^2$

$\frac{du}{dx} = 3y^2\frac{dy}{dx}$

$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(3y^2\frac{dy}{dx}\right)$

$= 6y\left(\frac{dy}{dx}\right)^2 + 3y^2\frac{d^2y}{dx^2}$

$\frac{d^2u}{dx^2} + 5\frac{du}{dx} + 4u = 3\cos 2x - 4\sin 2x$

$\frac{d^2u}{dx^2} + 5\frac{du}{dx} + 4u = 0$

$m^2 + 5m + 4 = 0$

$(m + 1)(m + 4) = 0$

$m = -1, -4$

$\therefore$ The complementary function, u_c, is

$u_c = Ae^{-x} + Be^{-4x}$

The particular integral, u_p, is given by

$$u_p = C\cos 2x + D\sin 2x$$

$$\frac{du_p}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\frac{d^2u_p}{dx^2} = -4C\cos 2x - 4D\sin 2x$$

$$\frac{d^2u_p}{dx^2} + \frac{5du_p}{dx} + 4u_p$$

$$= -4C\cos 2x - 4D\sin 2x$$
$$+ 5(-2C\sin 2x + 2D\cos 2x)$$
$$+ 4(C\cos 2x + D\sin 2x)$$

$$= -4C\cos 2x - 4D\sin 2x$$
$$-10C\sin 2x + 10D\cos 2x$$
$$+ 4C\cos 2x + 4D\sin 2x$$

$$= -10C\sin 2x + 10D\cos 2x$$

$$= 3\cos 2x - 4\sin 2x$$

$$-10C = -4 \qquad 10D = 3$$
$$C = \frac{2}{5} \qquad D = \frac{3}{10}$$

$$\therefore u_p = \frac{3\sin 2x}{10} - \frac{2\cos 2x}{5}$$

$$u = u_c + u_p$$

$$= Ae^{-x} + Be^{-4x} + \frac{3\sin 2x}{10} - \frac{2\cos 2x}{5}$$

Since $u = y^3$,

$$y^3 = Ae^{-x} + Be^{-4x} + \frac{3\sin 2x}{10} - \frac{2\cos 2x}{5}$$

As $x \to \infty$, $y^3 \longrightarrow \frac{3\sin 2x}{10} - \frac{2\cos 2x}{5}$

$$\frac{3\sin 2x}{10} - \frac{2\cos 2x}{5} = R\sin(2x + \alpha)$$

$$= R\sin 2x \cos\alpha + R\cos 2x \sin\alpha$$

$R\sin\alpha = \frac{-2}{5}$ $\quad$ $R\cos\alpha = \frac{3}{10}$

$$R^2(\sin^2\alpha + \cos^2\alpha) = \frac{4}{25} + \frac{9}{100}$$

$$R^2 = \frac{1}{4}$$

$$R = \frac{1}{2}$$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{\frac{-2}{5}}{\frac{3}{10}}$$

$$\tan\alpha = -\frac{4}{3}$$

$$\alpha = -\tan^{-1}\left(\frac{4}{3}\right)$$

$$y^3 \longrightarrow \frac{\sin\left(2x - \tan^{-1}\frac{4}{3}\right)}{2}$$

$$y \longrightarrow \frac{\sin^{\frac{1}{3}}\left(2x - \tan^{-1}\frac{4}{3}\right)}{2^{\frac{1}{3}}}$$