MAY / JUNE 2003

1. $r = a\sin 2\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad a > 0$

$$\text{Area} = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{a^2\sin^2 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{\theta}{2} - \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi a^2}{8}$$

2. $$\sum_{n=1}^{N} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}, \quad N \geq 1$$

When $n = 1$:
$$\sum_{n=1}^{1} \frac{n+2}{n(n+1)2^n}$$
$$= \frac{1+2}{1(1+1)2^1}$$
$$= \frac{3}{1 \cdot 2 \cdot 2}$$
$$= \frac{3}{4}$$
$$= 1 - \frac{1}{4}$$
$$= 1 - \frac{1}{2 \cdot 2}$$
$$= 1 - \frac{1}{(1+1)2^1}$$

Assume the statement is true when $N = k$.
$$\sum_{n=1}^{k} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(k+1)2^k}.$$

When $N = k+1$:
$$\sum_{n=1}^{k+1} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(k+2)2^{k+1}}.$$
(what needs to be proved).

$$\sum_{n=1}^{k+1} \frac{n+2}{n(n+1)2^n} = \frac{k+1+2}{(k+1)(k+1+1)2^{k+1}} + \sum_{n=1}^{k} \frac{n+2}{n(n+1)2^n}$$
$$= \frac{k+3}{(k+1)(k+2)2^{k+1}} + 1 - \frac{1}{(k+1)2^k}$$
$$= 1 + \frac{k+3}{(k+1)(k+2)2^{k+1}} - \frac{1}{(k+1)2^k}$$
$$= 1 + \frac{1}{(k+1)2^k}\left(\frac{k+3}{2(k+2)} - 1\right)$$
$$= 1 + \frac{1}{(k+1)2^k}\left(\frac{k+3-2k-4}{2(k+2)}\right)$$
$$= 1 + \frac{(-k-1)}{2^k 2(k+1)(k+2)}$$
$$= 1 - \frac{1}{(k+2)2^{k+1}}$$

$$\therefore \sum_{n=1}^{N} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}$$
for every positive integer N.

3. $v_1, v_2, v_3, \ldots$

$$u_n = nv_n - (n+1)v_{n+1}, \quad n = 1, 2, 3, \ldots$$

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} nv_n - (n+1)v_{n+1}$$

$$\begin{aligned}
&= v_1 - 2v_2 \\
&\quad + 2v_2 - 3v_3 \\
&\quad + 3v_3 - 4v_4 \\
&\quad \vdots \\
&\quad + (N-2)v_{N-2} - (N-1)v_{N-1} \\
&\quad + (N-1)v_{N-1} - Nv_N \\
&\quad + Nv_N - (N+1)v_{N+1} \\
&= v_1 - (N+1)v_{N+1}
\end{aligned}$$

i) $v_n = n^{-\frac{1}{2}}$

$$\begin{aligned}
\sum_{n=1}^{N} u_n &= v_1 - (N+1)v_{N+1} \\
&= 1 - (N+1)(N+1)^{-\frac{1}{2}} \\
&= 1 - (N+1)^{\frac{1}{2}}
\end{aligned}$$

Since $(N+1)^{\frac{1}{2}} \longrightarrow \infty$ as $N \longrightarrow \infty$,

the series $\sum_{n=1}^{\infty} u_n$ is not convergent.

ii) $v_n = n^{-\frac{3}{2}}$

$$\begin{aligned}
\sum_{n=1}^{N} u_n &= v_1 - (N+1)v_{N+1} \\
&= 1 - (N+1)(N+1)^{-\frac{3}{2}} \\
&= 1 - (N+1)^{-\frac{1}{2}}
\end{aligned}$$

Since $(N+1)^{-\frac{1}{2}} \longrightarrow 0$ as $N \longrightarrow \infty$,

$$\begin{aligned}
\sum_{n=1}^{\infty} u_n &= \lim_{N \to \infty} \left(1 - (N+1)^{-\frac{1}{2}}\right) \\
&= 1
\end{aligned}$$

4. $C: \; y = \dfrac{x^2 - 4}{x - 3}$

i)

$$\begin{array}{r}
x + 3 \\
x - 3 \,\overline{\big)\, x^2 - 4} \\
\underline{x^2 - 3x} \\
3x - 4 \\
\underline{3x - 9} \\
5
\end{array}$$

$$y = x + 3 + \frac{5}{x - 3}$$

As $x \to \pm\infty$ $y \to x + 3$

As $x \to 3$ $y \to \pm\infty$

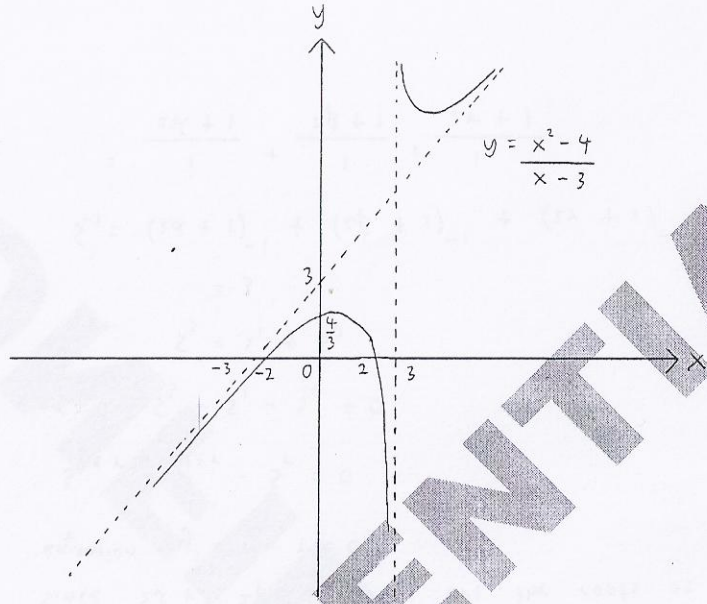
The equations of the asymptotes of C are $y = x + 3$ and $x = 3$.

ii) When $x = 0$: $y = \frac{4}{3}$

When $y = 0$: $\dfrac{x^2 - 4}{x - 3} = 0$

$x^2 - 4 = 0$

$x = \pm 2$



5. $8x^3 + 12x^2 + 4x - 1 = 0$

α, β, γ are the roots.

$2\alpha + 1,\ 2\beta + 1,\ 2\gamma + 1$

Let $y = 2\alpha + 1$

$\alpha = \dfrac{y-1}{2}$

α is a root

$\therefore 8\alpha^3 + 12\alpha^2 + 4\alpha - 1 = 0$

$8\left(\dfrac{y-1}{2}\right)^3 + 12\left(\dfrac{y-1}{2}\right)^2 + 4\left(\dfrac{y-1}{2}\right) - 1 = 0$

$\dfrac{8(y^3 - 3y^2 + 3y - 1)}{8} + 12\left(\dfrac{y^2 - 2y + 1}{4}\right)$

$+ 2y - 2 - 1 = 0$

$y^3 - 3y^2 + 3y - 1 + 3y^2 - 6y + 3 + 2y - 3 = 0$

$y^3 - y - 1 = 0$

$\therefore$ The equation with roots, $2\alpha + 1, 2\beta + 1, 2\gamma + 1$ is $y^3 - y - 1 = 0$.

$S_n = (2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$

$2\alpha + 1 + 2\beta + 1 + 2\gamma + 1 = 0$

$(2\alpha + 1)(2\beta + 1) + (2\alpha + 1)(2\gamma + 1) + (2\beta + 1)(2\gamma + 1) = -1$

$(2\alpha + 1)(2\beta + 1)(2\gamma + 1) = 1$

$S_0 = (2\alpha + 1)^0 + (2\beta + 1)^0 + (2\beta + 1)^0$

$= 1 + 1 + 1$

$= 3$

$S_1 = (2\alpha + 1)^1 + (2\beta + 1)^1 + (2\gamma + 1)^1$

$= 2\alpha + 1 + 2\beta + 1 + 2\gamma + 1$

$= 0$

$S_2 = (2\alpha + 1)^2 + (2\beta + 1)^2 + (2\gamma + 1)^2$

$= (2\alpha + 1 + 2\beta + 1 + 2\gamma + 1)^2$

$- 2\left[(2\alpha + 1)(2\beta + 1) + (2\alpha + 1)(2\gamma + 1) + (2\beta + 1)(2\gamma + 1)\right]$

$= 0^2 - 2(-1)$

$= 2$

Since $2\alpha + 1, 2\beta + 1, 2\gamma + 1$ are the roots of the equation $y^3 - y - 1 = 0$,

$S_{3+r} - S_{1+r} - S_r = 0$

$r = 0: \quad S_3 - S_1 - S_0 = 0$

$S_3 = S_1 + S_0$

$= 3$

$S_{-1} = (2\alpha + 1)^{-1} + (2\beta + 1)^{-1} + (2\gamma + 1)^{-1}$

$= \dfrac{1}{2\alpha + 1} + \dfrac{1}{2\beta + 1} + \dfrac{1}{2\gamma + 1}$

$$= \frac{(2\alpha + 1)(2\beta + 1) + (2\alpha + 1)(2\gamma + 1) + (2\beta + 1)(2\gamma + 1)}{(2\alpha + 1)(2\beta + 1)(2\gamma + 1)}$$

$$= \frac{-1}{1}$$

$$= -1$$

$r = -2: \quad S_1 - S_{-1} - S_{-2} = 0$

$$S_{-2} = S_1 - S_{-1}$$
$$= 0 - (-1)$$
$$= 1.$$

6. $$(\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6\cos^5\theta(i\sin\theta) + 15\cos^4\theta(-\sin^2\theta) + 20\cos^3\theta(-i\sin^3\theta) + 15\cos^2\theta(\sin^4\theta) + 6\cos\theta(i\sin^5\theta) - \sin^6\theta$$

$$\cos 6\theta + i\sin 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta + i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta)$$

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta(1 - \cos^2\theta) + 15\cos^2\theta(1 - \cos^2\theta)^2 - (1 - \cos^2\theta)^3$$

$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta) - (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$

$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$$

$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

$$64x^6 - 96x^4 + 36x^2 - 1 = 0$$

$$64x^6 - 96x^4 + 36x^2 - 2 = -1$$

$$32x^6 - 48x^4 + 18x^2 - 1 = \frac{-1}{2}$$

Let $x = \cos\theta$

$$\therefore 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = \frac{-1}{2}$$

$$\cos 6\theta = -\frac{1}{2}$$

$$6\theta = \frac{2\pi}{3} + 2k\pi, \ \frac{4\pi}{3} + 2k\pi, \ k \in \mathbb{Z}$$

$$\theta = \left(\frac{k}{3} + \frac{1}{9}\right)\pi, \ \left(\frac{k}{3} + \frac{2}{9}\right)\pi, \ k \in \mathbb{Z}$$

$$= \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$$

$$\therefore x = \cos\frac{\pi}{9}, \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}, \cos\frac{8\pi}{9}$$

$$= \pm\cos\frac{\pi}{9}, \ \pm\cos\frac{2\pi}{9}, \ \pm\cos\frac{4\pi}{9}$$

7. $x^4 + y^4 = 1, \ 0 < x < 1, \ 0 < y < 1.$

i) $$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(1)$$

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$x^3 + y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x^3}{y^3}$$

$$\frac{d}{dx}\left(x^3 + y^3\frac{dy}{dx}\right) = 0$$

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right)\frac{dy}{dx} + y^3\frac{d^2y}{dx^2} = 0$$

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right)^2 + y^3\frac{d^2y}{dx^2} = 0$$

$$3x^2 + 3y^2\left(\frac{-x^3}{y^3}\right)^2 + y^3\frac{d^2y}{dx^2} = 0$$

$$3x^2 + \frac{3x^6y^2}{y^6} + y^3\frac{d^2y}{dx^2} = 0$$

$$y^3\frac{d^2y}{dx^2} = -3x^2 - \frac{3x^6}{y^4}$$

$$= \frac{-3x^2y^4 - 3x^6}{y^4}$$

$$= \frac{-3x^2(x^4 + y^4)}{y^4}$$

$$= \frac{-3x^2}{y^4}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-3x^2}{y^7}$$

ii) The mean value of $\frac{d^3y}{dx^3}$ over the interval

$a_1 \leq x \leq a_2$ is $\displaystyle \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \frac{d^3y}{dx^3}\,dx$

$$= \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) dx$$

$$= \frac{1}{a_2 - a_1}\left[\frac{d^2y}{dx^2}\right]_{a_1}^{a_2}$$

$$= \frac{1}{a_2 - a_1}\left[\frac{-3x^2}{y^7}\right]_{(a_1, b_1)}^{(a_2, b_2)}$$

$$= \frac{1}{a_2 - a_1}\left(\frac{-3a_2^2}{b_2^7} - \left(\frac{-3a_1^2}{b_1^7}\right)\right)$$

$$= \frac{1}{a_2 - a_1}\left(\frac{3a_1^2}{b_1^7} - \frac{3a_2^2}{b_2^7}\right)$$

$$= \frac{1}{a_2 - a_1}\left(\frac{3a_1^2 b_2^7 - 3a_2^2 b_1^7}{b_1^7 b_2^7}\right)$$

$$= \frac{3(a_1^2 b_2^7 - a_2^2 b_1^7)}{b_1^7 b_2^7 (a_2 - a_1)}$$

8. $T : R^4 \longrightarrow R^4$

$$A = \begin{pmatrix} 1 & -1 & -2 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & -2 & -3 & 14 \\ 4 & -3 & -5 & 17 \end{pmatrix}$$

$$\xrightarrow[-4r_1 + r_4]{-2r_1 + r_2,\ -3r_1 + r_3} \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \end{pmatrix}$$

$$\xrightarrow[-r_2 + r_4]{-r_2 + r_3} \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $(A) = 2$

$$\begin{pmatrix} 1 & -1 & -2 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & -2 & -3 & 14 \\ 4 & -3 & -5 & 17 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & 0 \\ 2 & -1 & -1 & 11 & 0 \\ 3 & -2 & -3 & 14 & 0 \\ 4 & -3 & -5 & 17 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & 0 \\ 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_4 = s$ and $x_3 = t$, $s, t \in R$

$x_2 = -3t - 5s$

$x_1 - (-3t - 5s) - 2t + 3s = 0$

$x_1 + 3t + 5s - 2t + 3s = 0$

$x_1 = -8s - t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -8s - t \\ -3t - 5s \\ t \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ A basis for the null space of T is

$$\left\{ \begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$$\underset{\sim}{e} = \begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}$$

Since $\underset{\sim}{e}$ is a solution of the equation $A\underset{\sim}{x} = A\underset{\sim}{e}$ and $\left\{ \begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for the null space of T, the general solution of the equation $A\underset{\sim}{x} = A\underset{\sim}{e}$ is $\begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$.

if $\begin{pmatrix} p \\ q \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 - 8s - t \\ -2 - 5s - 3t \\ -1 + t \\ -1 + s \end{pmatrix}$$

$p = 1 - 8s - t$

$q = -2 - 5s - 3t$

$1 = -1 + t$

$1 = -1 + s$

$s = t = 2$, $p = -17$, $q = -18$

9. $$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 5x\frac{dx}{dt} + 3x^2 = 3\sin 2t + 15\cos 2t,$$

$x > 0,$
$0 \le t \le \frac{\pi}{2}$

$y = x^2$

$\frac{dy}{dx} = 2x$

$\frac{dy}{dt}\frac{dt}{dx} = 2x$

$\frac{dy}{dt} = 2x\frac{dx}{dt}$

$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(2x\frac{dx}{dt}\right)$

$= 2x\frac{d^2x}{dt^2} + 2\left(\frac{dx}{dt}\right)^2$

$2x\frac{d^2x}{dt^2} + 2\left(\frac{dx}{dt}\right)^2 + 10x\frac{dx}{dt} + 6x^2 = 6\sin 2t + 30\cos 2t$

$\frac{d^2y}{dt^2} + \frac{5dy}{dt} + 6y = 6\sin 2t + 30\cos 2t$

$\frac{d^2y}{dt^2} + \frac{5dy}{dt} + 6y = 0$

$m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$m = -2, -3$

The complementary function, y_c, is

$y_c = Ae^{-2t} + Be^{-3t}$

The particular integral, y_p, is given by

$y_p = C\cos 2t + D\sin 2t$

$\frac{dy_p}{dt} = -2C\sin 2t + 2D\cos 2t$

$\frac{d^2y_p}{dt^2} = -4C\cos 2t - 4D\sin 2t$

$\frac{d^2y_p}{dt^2} + \frac{5dy_p}{dt} + 6y_p = -4C\cos 2t - 4D\sin 2t$

$+ 5(-2C\sin 2t + 2D\cos 2t)$

$+ 6(C\cos 2t + D\sin 2t)$

$= -4C\cos 2t - 4D\sin 2t$

$-10C\sin 2t + 10D\cos 2t$

$+ 6C\cos 2t + 6D\sin 2t$

$= (2C + 10D)\cos 2t$

$+ (-10C + 2D)\sin 2t$

$= 6\sin 2t + 30\cos 2t$

$2C + 10D = 30 \qquad -10C + 2D = 6$

$C + 5D = 15 \qquad -5C + D = 3$

$$-5(15 - 50) + 0 = 3$$

$$-75 + 250 + 0 = 3$$

$$260 = 78$$

$$0 = 3$$

$$C = 0$$

$$y_p = 3\sin 2t$$

$$y = y_c + y_p$$

$$= Ae^{-2t} + Be^{-3t} + 3\sin 2t$$

$$x^2 = Ae^{-2t} + Be^{-3t} + 3\sin 2t$$

$x = 2$ and $\frac{dx}{dt} = -\frac{3}{2}$ when $t = 0$

$$2x\frac{dx}{dt} = -2Ae^{-2t} - 3Be^{-3t} + 6\cos 2t$$

$t = 0$ $x = 2$: $4 = A + B$

$t = 0$ $x = 2$ $\frac{dx}{dt} = -\frac{3}{2}$: $-6 = -2A - 3B + 6$

$$A + B = 4$$

$$2A + 3B = 12$$

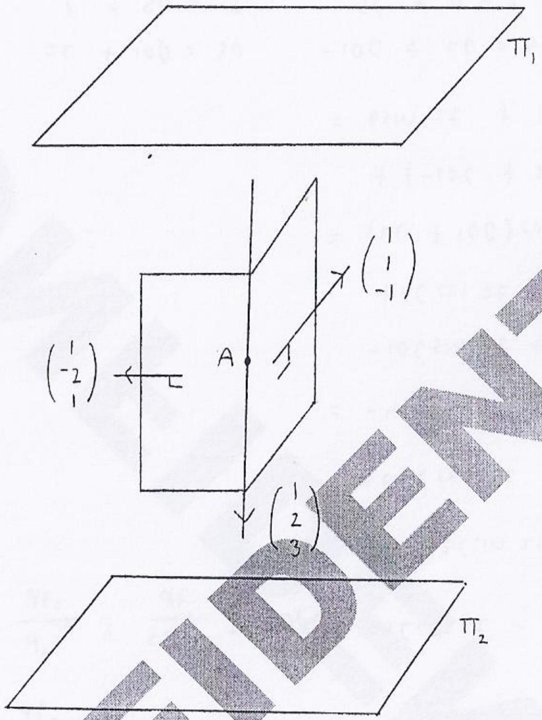
$$B = 4$$

$$A = 0$$

$$\therefore x^2 = 4e^{-3t} + 3\sin 2t$$

$$x = \sqrt{4e^{-3t} + 3\sin 2t}$$

10. $x - 2y + z - 9 = 0$ $\quad x + y - z + 2 = 0$ $\quad \vec{OA} = p\underset{\sim}{i} + q\underset{\sim}{j} + k$



$x - 2y + z = 9$ $\qquad x + y - z = -2$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = 9 \qquad \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix} = -2$$

$$\begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix} = \left|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|\left|\begin{pmatrix}1\\1\\-1\end{pmatrix}\right|\cos\theta$$

$$1 - 2 - 1 = \sqrt{6}\sqrt{3}\cos\theta$$

$$-2 = 3\sqrt{2}\cos\theta$$

$$\cos\theta = \frac{-\sqrt{2}}{3}$$

$$\theta = 118.1$$

$\therefore$ The acute angle between the planes $x - 2y + z = 9$ and $x + y - z = -2$ is $61 \cdot 9°$.

i) Since ℓ lies in both the planes and A is a point on ℓ,

$$p - 2q + 1 = 9$$
$$p + q - 1 = -2$$

$$p - 2q = 8$$
$$p + q = -1$$

$$3q = -9$$
$$q = -3$$
$$p = 2$$

ii) Since ℓ is perpendicular to both the normals $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, it is parallel to $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Since ℓ has direction $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and A is a point on ℓ, a vector equation for ℓ is

$$\underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k} + s(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$$

Since ℓ is perpendicular to both the planes Π_1 and Π_2, Π_1 and Π_2 have the form

$$\Pi_1 : x + 2y + 3z = d_1$$
$$\Pi_2 : x + 2y + 3z = d_2$$

when ℓ meets the plane Π_1,

$$2 + s + 2(-3 + 2s) + 3(1 + 3s) = d_1$$
$$2 + s - 6 + 4s + 3 + 9s = d_1$$
$$14s = d_1 + 1$$
$$s = \frac{d_1 + 1}{14}$$

$\therefore$ ℓ meets Π_1 at the point $\left(\frac{d_1 + 29}{14}, \frac{2d_1 - 40}{14}, \frac{3d_1 + 17}{14}\right)$.

Also, when ℓ meets the plane Π_2,

$$2 + s + 2(-3 + 2s) + 3(1 + 3s) = d_2$$
$$2 + s - 6 + 4s + 3 + 9s = d_2$$
$$14s = d_2 + 1$$
$$s = \frac{d_2 + 1}{14}$$

$\therefore$ ℓ meets Π_2 at the point $\left(\frac{d_2 + 29}{14}, \frac{2d_2 - 40}{14}, \frac{3d_2 + 17}{14}\right)$.

Since the perpendicular distance from A to both Π_1 and Π_2 is $\sqrt{14}$,

$$\sqrt{\left(\frac{d_1 + 29}{14} - 2\right)^2 + \left(\frac{2d_1 - 40}{14} + 3\right)^2 + \left(\frac{3d_1 + 17}{14} - 1\right)^2} = \sqrt{14}$$

and

$$\sqrt{\left(\frac{d_1 + 29}{14} - 2\right)^2 + \left(\frac{2d_1 - 40}{14} + 3\right)^2 + \left(\frac{3d_1 + 17}{14} - 1\right)^2} = \sqrt{14}$$

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$$\left(\frac{d_1+1}{14}\right)^2 + \left(\frac{2d_2+2}{14}\right)^2 + \left(\frac{3d_1+3}{14}\right)^2 = 14$$

$$\left(\frac{d_1+1}{14}\right)^2 + 4\left(\frac{d_2+1}{14}\right)^2 + 9\left(\frac{d_1+1}{14}\right)^2 = 14$$

$$14\left(\frac{d_1+1}{14}\right)^2 = 14$$

$$\left(\frac{d_1+1}{14}\right)^2 = 1$$

$$\frac{d_1+1}{14} = \pm 1$$

$$d_1 + 1 = \pm 14$$

$$d_1 = 13,\ -15$$

$$d_2 = -15,\ 13$$

$\therefore$ The equations of the planes Π_1 and Π_2 are $x + 2y + 3z = 13$ and $x + 2y + 3z = -15$

11. EITHER

$$I_n = \int_0^1 x^n e^{-\alpha x}\, dx, \quad \alpha > 0,\ n \geq 0$$

$$u = x^n \qquad dv = e^{-\alpha x}\, dx$$

$$du = nx^{n-1} \qquad v = \frac{e^{-\alpha x}}{-\alpha}$$

$$= \left[\frac{x^n e^{-\alpha x}}{-\alpha}\right]_0^1 - \int_0^1 \frac{nx^{n-1}e^{-\alpha x}}{-\alpha}\, dx$$

$$= \frac{e^{-\alpha}}{-\alpha} - 0 + \frac{n}{\alpha}\int_0^1 x^{n-1}e^{-\alpha x}\, dx$$

$$= \frac{e^{-\alpha}}{-\alpha} + \frac{n}{\alpha} I_{n-1}$$

$$\therefore \alpha I_n = n I_{n-1} - e^{-\alpha}, \quad n \geq 1.$$

The area, A, of the finite region bounded by the x-axis, the line $x = 1$ and the curve $y = xe^{-x}$ is

$$\int_0^1 xe^{-x}\, dx, \quad \alpha = 1$$

$$= I_1$$

$$= I_0 - e^{-1}$$

$$= \int_0^1 e^{-x}\, dx - e^{-1}$$

$$= \left[-e^{-x}\right]_0^1 - e^{-1}$$

$$= -e^{-1} - (-1) - e^{-1}$$

$$= 1 - 2e^{-1}$$

If the centroid has coordinates $(\bar{x}, \bar{y})$,

$$\bar{x} = \frac{\int_0^1 xy \, dx}{A}$$

$$= \frac{\int_0^1 x^2 e^{-x} \, dx}{1 - \frac{2}{e}}, \quad \alpha = 1$$

$$= \frac{I_2}{1 - \frac{2}{e}}$$

$$= \frac{2I_1 - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2(I_0 - e^{-1}) - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2\left(\int_0^1 e^{-x} \, dx - e^{-1}\right) - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2\left(\left[-e^{-x}\right]_0^1 - e^{-1}\right) - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2(-e^{-1} - (-1) - e^{-1}) - e^{-1}}{1 - \frac{2}{e}}$$

$$= \frac{2(1 - \frac{2}{e}) - \frac{1}{e}}{1 - \frac{2}{e}}$$

$$= \frac{2 - \frac{4}{e} - \frac{1}{e}}{1 - \frac{2}{e}}$$

$$= \frac{2 - \frac{5}{e}}{1 - \frac{2}{e}}$$

$$= \frac{2e - 5}{e - 2}$$

$$\bar{y} = \frac{\int_0^1 \frac{y^2}{2} \, dx}{A},$$

$$= \frac{\int_0^1 \frac{x^2 e^{-2x}}{2} \, dx}{1 - \frac{2}{e}}, \quad \alpha = 2$$

$$= \frac{\frac{I_2}{2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{1}{2}\left(\frac{2I_1 - e^{-2}}{2}\right)}{1 - \frac{2}{e}}$$

$$= \frac{\frac{I_1}{2} - \frac{e^{-2}}{4}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{1}{4}(I_0 - e^{-2}) - \frac{e^{-2}}{4}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{I_0}{4} - \frac{e^{-2}}{4} - \frac{e^{-2}}{4}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{\int_0^1 e^{-2x}\,dx - \frac{e^{-2}}{2}}{4}}{1 - \frac{2}{e}}$$

$$= \frac{\left[\frac{e^{-2x}}{-8}\right]_0^1 - \frac{1}{2e^2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{e^{-2}}{-8} - \left(-\frac{1}{8}\right) - \frac{1}{2e^2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{1}{8} - \frac{1}{8e^2} - \frac{1}{2e^2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{1}{8} - \frac{5}{8e^2}}{1 - \frac{2}{e}}$$

$$= \frac{\frac{e^2 - 5}{8e^2}}{\frac{e-2}{e}}$$

$$= \frac{e^2 - 5}{8e(e-2)}$$

$\therefore$ The centroid of the region bounded by the x-axis, the line $x = 1$ and the curve $y = xe^{-x}$ is $\left(\frac{2e - 5}{e(e-2)}, \frac{e^2 - 5}{8e(e-2)}\right)$.

OR

$A\underset{\sim}{e} = \lambda \underset{\sim}{e}, \quad B\underset{\sim}{e} = \mu \underset{\sim}{e}$

$$
\begin{aligned}
AB\underset{\sim}{e} &= A(B\underset{\sim}{e}) \\
&= A(\mu \underset{\sim}{e}) \\
&= \mu (A\underset{\sim}{e}) \\
&= \mu (\lambda \underset{\sim}{e}) \\
&= (\lambda\mu)\underset{\sim}{e}
\end{aligned}
$$

$\therefore$ If $\underset{\sim}{e}$ is an eigenvector of the $n \times n$ matrices A and B with corresponding eigenvalues λ and μ respectively, $\underset{\sim}{e}$ is an eigenvector of the matrix AB.

$$C = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$
\begin{aligned}
C - \lambda I &= \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -\lambda & 1 & 4 \\ 1 & 2-\lambda & -1 \\ 2 & 1 & 2-\lambda \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
|C - \lambda I| &= -\lambda\left[(2-\lambda)^2 + 1\right] - (2 - \lambda + 2) + 4(1 - 2(2-\lambda)) \\
&= -\lambda(\lambda^2 - 4\lambda + 4 + 1) + \lambda - 4 + 4(2\lambda - 3) \\
&= -\lambda^3 + 4\lambda^2 - 5\lambda + \lambda - 4 + 8\lambda - 12 \\
&= -\lambda^3 + 4\lambda^2 + 4\lambda - 16 \\
&= -\lambda^2(\lambda - 4) + 4(\lambda - 4) \\
&= (\lambda - 4)(4 - \lambda^2) \\
&= (\lambda - 4)(\lambda + 2)(2 - \lambda)
\end{aligned}
$$

when $|C - \lambda I| = 0$,

$$(\lambda - 4)(\lambda + 2)(2 - \lambda) = 0$$

$$\lambda = 2, 4, -2$$

when $\lambda = 2$: $\begin{pmatrix} -2 & 1 & 4 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -2 & 1 & 4 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s, s \in R$

$y = -2s$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -2s \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

CONFIDENTIAL

when $\lambda = 4$: $\begin{pmatrix} -4 & 1 & 4 \\ 1 & -2 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -4 & 1 & 4 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -4 & 1 & 4 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{4r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-7},\ \frac{r_3}{5}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s,\ s \in R$

$x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

when $\lambda = -2$: $\begin{pmatrix} 2 & 1 & 4 \\ 1 & 4 & -1 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 & 4 & 0 \\ 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & 4 & -1 & 0 \\ 2 & 1 & 4 & 0 \\ 2 & 1 & 4 & 0 \end{pmatrix}$$

$$\xrightarrow[-2r_1 + r_3]{-2r_1 + r_2} \begin{pmatrix} 1 & 4 & -1 & 0 \\ 0 & -7 & 6 & 0 \\ 0 & -7 & 6 & 0 \end{pmatrix}$$

$$\xrightarrow{-r_2 + r_3} \begin{pmatrix} 1 & 4 & -1 & 0 \\ 0 & -7 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $z = 7s,\ s \in R$

$y = 6s$

$x + 24s - 7s = 0$

$x = -17s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -17s \\ 6s \\ 7s \end{pmatrix} = s\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$$

$\therefore$ The eigenvalues of C are $2, 4, -2$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}$

$$D = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}$$

$$D\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -4 \end{pmatrix} = -4\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$D\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$$

$$D\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 64 \\ 16 \\ -28 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of D with eigenvalue -4.

Since $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of C and D with eigenvalues 2 and -4 respectively, the matrix CD has an eigenvector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ with eigenvalue -8.