TAYLOR'S COLLEGE

Wisdom - Integrity - Excellence

CAMBRIDGE 'A' LEVEL PROGRAMME

A2 TRIAL EXAMINATION AUGUST/SEPTEMBER 2004

(~~Jan 2004~~ July 2003 Intake)

Friday 3 September 2004 8.30 am – 11.30 pm

FURTHER MATHEMATICS 9231/1

PAPER 1 3 hours

Additional materials: Answer Paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The use of a calculator is expected, where appropriate.
Results obtained solely from graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

This document consists of **4** printed pages.

[Turn over

1. Use the relevant standard results in the List of Formulae to prove that
$$S_N = \sum_{n=1}^{N}\left(8n^3 - 6n^2\right) = N(N+1)\left(2N^2 - 1\right).$$ [2]

Hence show that $\sum_{n=N+1}^{2N}\left(8n^3 - 6n^2\right)$ can be expressed in the form
$$N\left(aN^3 + bN^2 + cN + d\right),$$
where the constants a, b, c, d are to be determined. [2]

2. Find the equations of the asymptotes of the curve whose equation is
$$y = \frac{x^2 + ax + 1}{x+3},$$
where a is a constant. [4]

3. Three $n \times 1$ column vectors are denoted by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, and $\mathbf{M}$ is a $n \times n$ matrix. Show that if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly dependent then the vectors $\mathbf{Mx}_1, \mathbf{Mx}_2, \mathbf{Mx}_3$ are also linearly dependent. [2]

The vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ and the matrix $\mathbf{P}$ are defined as follows:
$$\mathbf{y}_1 = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \quad \mathbf{y}_3 = \begin{pmatrix} 5 \\ 51 \\ 55 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -7 \end{pmatrix}.$$

(a) Show that $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ are linearly dependent. [2]

(b) Find a basis for the linear space spanned by the vectors $\mathbf{Py}_1, \mathbf{Py}_2, \mathbf{Py}_3$. [2]

4. Write down the eigenvalues of the matrix $\mathbf{A}$, given by
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{pmatrix},$$
and obtain a set of corresponding eigenvectors. [6]

Find a non-singular matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^5 = \mathbf{QDQ}^{-1}$. [2]

5. The integral I_n is defined by $I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \, dx$.

By considering $\frac{d}{dx}\left(\tan x \sec^n x\right)$, or otherwise, show that

$$(n+1)I_{n+2} = 2^{\frac{1}{2}n} + nI_n. \qquad [4]$$

Find the value of I_6. [4]

6. Find the sum of the squares of the roots of the equation $x^3 + x + 12 = 0$, and deduce that only one of the roots is real. [4]

The real root of the equation is denoted by α. Prove that $-3 < \alpha < -2$, and hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$. [4]

7. The sequence of positive numbers $u_1, u_2, u_3, \ldots$, is known to converge to the limit m and is such that $u_{n+1} = \dfrac{9u_n + 3}{u_n + 7}$.

(a) Find the value of m. [2]

(b) It is given that $u_1 = 1$.
By first writing u_n in terms of u_{n+1}, and considering $3 - u_n$, prove by induction that $u_n < 3$ for all positive integers n. [6]

8. The curve C has equation $\left(x^2 + y^2\right)^2 = 4xy$.

(a) Show that the polar equation of C is $r^2 = 2\sin 2\theta$. [2]

(b) Sketch the curve C. Indicate clearly any lines of symmetry and the form of C at the pole. [5]

(c) Write down the maximum possible distance of a point of C from the pole. [1]

9. A curve is defined by

$$x = \theta\sin\theta + \cos\theta, \qquad y = \left(\theta^2 + 4\right)\sin\theta,$$

where $0 < \theta < \frac{1}{2}\pi$.

(a) Show that $\dfrac{dy}{dx} = 2\tan\theta + \theta + \dfrac{4}{\theta}$. [4]

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of θ. [2]

(c) Hence show that $\theta^2\left(2\tan^2\theta + 3\right) = 4$ when $\dfrac{d^2y}{dx^2} = 0$. [3]

[Turn Over

10. Given that $z = e^{i\theta}$ and n is a positive integer, show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta. \qquad [2]$$

Hence express $\sin^6\theta$ in the form $p\cos 6\theta + q\cos 4\theta + r\cos 2\theta + s$, where the constants p, q, r, s are to be determined. [4]

Hence find the mean value of $\sin^6\theta$ with respect to θ over the interval $0 \le \theta \le \frac{1}{4}\pi$. [5]

11. The plane Π has equation $\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \theta(3\mathbf{i} + \mathbf{j}) + \phi(\mathbf{j} - 2\mathbf{k})$.
The line l_1, which does not lie in Π, has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$.
The line l_2 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$.

(a) Show that l_1 is parallel to Π. [3]

(b) Find the position vector of the point in which l_2 meets Π. [5]

(c) Find the perpendicular distance from the point with position vector $4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$ to l_1. [4]

12. ANSWER **EITHER**

A curve has equation $y = \frac{2}{3}x^{\frac{3}{2}}$, for $x \ge 0$. The arc of the curve joining the origin to the point where $x = 3$ is denoted by R.

(a) Find the length R. [4]

(b) Find the y-coordinate of the centroid of the region bounded by the x-axis, the line $x = 3$ and R. [5]

(c) Show that the area of the surface generated when R is rotated through one revolution about the y-axis is $\frac{232}{15}\pi$. [5]

OR

The size y of a biological population at time t is modelled by the differential equation

$$\frac{d^2y}{dt^2} + 0.2\frac{dy}{dt} + 1.01y = 10^{10}e^{-\frac{1}{5}t}.$$

Show that the general solution is $y = e^{-\frac{1}{10}t}(A\cos t + B\sin t) + \frac{10^{10}}{1.01}e^{-\frac{1}{5}t}$,

where A and B are arbitrary real constants. [5]

It is given that $y = 5.0 \times 10^9$ and $\frac{dy}{dt} = 10^8$ when $t = 0$. Find the values of A and B correct to 4 significant figures. [6]

Given also that 1 unit of t equals 20 years and that $t = 0$ corresponding to 1 January 1990, find the size of the population, as predicted by the model, on 1 January 2090. [3]