1. Let $f(n) = 89^{2n+1} + 316^n$

when $n=1$: $f(1) = 89^{2(1)+1} + 316^1$

$= 89^{2+1} + 316$

$= 89^3 + 316$

$= 705285 + 316$

$= 45(15673)$

$\therefore 45 \mid f(1)$

Assume the statement is true when $n=k$.

$n=k$: $45 \mid f(k)$

$f(k) = 45s$, s is an integer.

$89^{2k+1} + 316^k = 45s$

when $n=k+1$:

$f(k+1) = 89^{2(k+1)+1} + 316^{k+1}$

$= 89^{2k+2+1} + 316^{k+1}$

$= 89^{2k+1} 89^2 + 316^k 316$

$= 89^{2k+1} 7921 + 316^k 316$

$= 89^{2k+1}(7605 + 316) + 316^k 316$

$$= 89^{2k+1}7605 + 89^{2k+1}316 + 316^{k}316$$

$$= 89^{2k+1}45(169) + 316(89^{2k+1} + 316^{k})$$

$$= 89^{2k+1}45(169) + 316(45s)$$

$$= 45(89^{2k+1}169 + 316s)$$

Since s is an integer and k is an integer,

$89^{2k+1}169 + 316s$ is an integer.

$\therefore 45 \mid f(k+1)$

$\therefore 89^{2n+1} + 316^{n}$ is divisible by 45

for every positive integer n.

2. $11^2 + 14^2 + 17^2 + \cdots + (3n+8)^2$

$$= \sum_{r=1}^{n} (3r+8)^2$$

$$= \sum_{r=1}^{n} 9r^2 + 48r + 64$$

$$= 9\sum_{r=1}^{n} r^2 + 48\sum_{r=1}^{n} r + 64\sum_{r=1}^{n} 1$$

$$= \frac{9n(n+1)(2n+1)}{6} + \frac{48n(n+1)}{2} + 64n$$

$$= \frac{3n(n+1)(2n+1)}{2} + 24n(n+1) + 64n$$

$$= \frac{3n(2n^2+3n+1) + 48n^2 + 48n + 128n}{2}$$

$$= \frac{6n^3 + 9n^2 + 3n + 48n^2 + 48n + 128n}{2}$$

$$= \frac{6n^3 + 57n^2 + 179n}{2}$$

$$= \frac{n(6n^2 + 57n + 179)}{2}.$$

$$\sum_{r=4n+1}^{6n} (3r+8)^2 = \sum_{r=1}^{6n} (3r+8)^2 - \sum_{r=1}^{4n} (3r+8)^2$$

$$= \frac{6n\left(6(6n)^2 + 57(6n) + 179\right)}{2}$$

$$\frac{-4n\left(6(4n)^2 + 57(4n) + 179\right)}{2}$$

$$= 3n^2(216n^2 + 342n + 179)$$

$$-2n(96n^2 + 228n + 179)$$

$$= 648n^3 + 1026n^2 + 537n$$

$$-192n^3 - 456n^2 - 358n$$

$$= 456n^3 + 570n^2 + 179n$$

$$= n(456n^2 + 570n + 179)$$

If $\sum_{r=4n+1}^{6n} (3r+8)^2 > 456n^3 + 570n^2 + 1035$,

$$456n^3 + 570n^2 + 179n > 456n^3 + 570n^2 + 1035$$

$$179n > 1035$$

$$n > \frac{1035}{179}$$

$$n_{min} = 6$$

3. $C: y = 7 + \dfrac{9}{4x+3} + \dfrac{5}{x+6}$

i) As $x \to \pm\infty$ $\quad y \to 7$

As $x \to -\dfrac{3}{4}$ $\quad y \to \pm\infty$

As $x \to -6$ $\quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = 7$, $x = -\dfrac{3}{4}$

and $x = -6$.

ii) $$\frac{dy}{dx} = \frac{-36}{(4x+3)^2} - \frac{5}{(x+6)^2}$$

$$= \underbrace{-\underbrace{\left(\underbrace{\frac{36}{(4x+3)^2}}_{>0} + \underbrace{\frac{5}{(x+6)^2}}_{>0}\right)}_{>0}}_{<0}$$

$\dfrac{dy}{dx} < 0$

$\therefore$ no critical points.

when $x = 0$: $y = \dfrac{65}{6}$

when $y=0$: $7+\dfrac{9}{4x+3}+\dfrac{5}{x+6}=0$

$$\frac{9}{4x+3}+\frac{5}{x+6}=-7$$

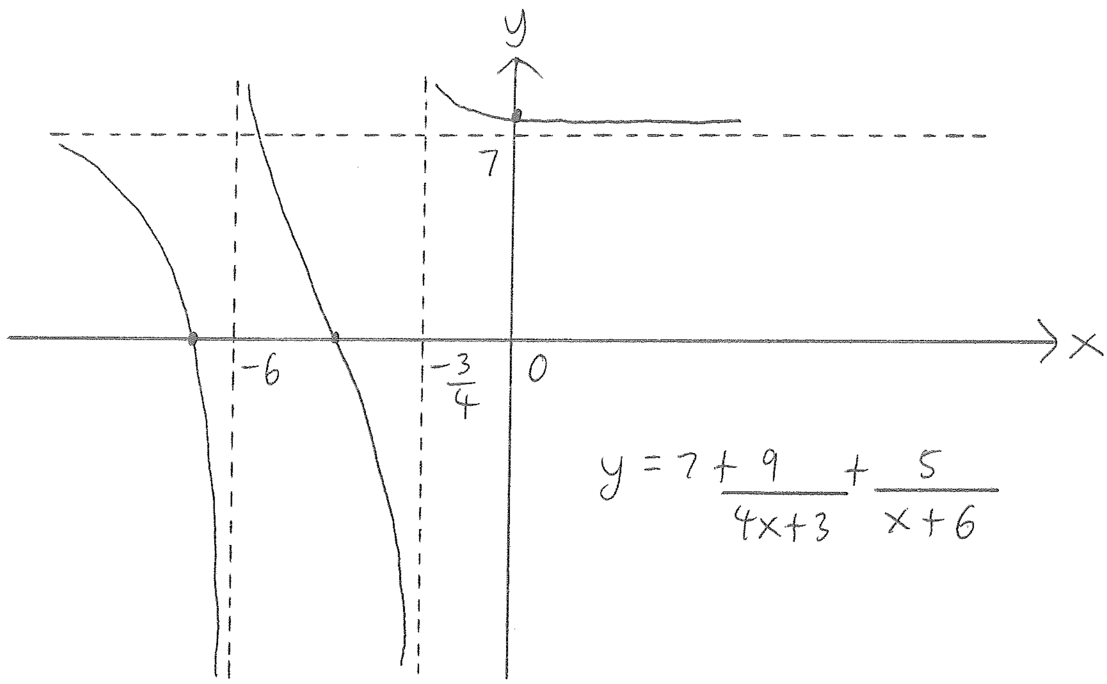
$$9(x+6)+5(4x+3)=-7(x+6)(4x+3)$$

$$9x+54+20x+15=-7(4x^2+27x+18)$$

$$29x+69=-28x^2-189x-126$$

$$28x^2+218x+195=0$$

$$x=\frac{-218\pm\sqrt{25684}}{56}$$



4. $9x^4 - 2x^3 + 5x^2 - 8x + 1 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots.

$\alpha^2, \beta^2, \gamma^2, \delta^2$

Let $u = \alpha^2$

$\alpha = \pm\sqrt{u}$

α is a root

$\therefore 9\alpha^4 - 2\alpha^3 + 5\alpha^2 - 8\alpha + 1 = 0$

$9(\pm\sqrt{u})^4 - 2(\pm\sqrt{u})^3 + 5(\pm\sqrt{u})^2 - 8(\pm\sqrt{u}) + 1 = 0$

$9u^2 - 2(\pm u\sqrt{u}) + 5u - 8(\pm\sqrt{u}) + 1 = 0$

$\pm\sqrt{u}\,(2u + 8) = 9u^2 + 5u + 1$

$(\pm\sqrt{u}(2u + 8))^2 = (9u^2 + 5u + 1)^2$

$u(2u + 8)^2 = (9u^2 + 5u + 1)^2$

$u(4u^2 + 32u + 64) = 81u^4 + 45u^3 + 9u^2$
$+ 45u^3 + 25u^2 + 5u$
$+ 9u^2 + 5u + 1$

$4u^3 + 32u^2 + 64u = 81u^4 + 90u^3 + 43u^2 + 10u + 1$

$81u^4 + 86u^3 + 11u^2 - 54u + 1 = 0$

$\therefore$ The equation $81u^4 + 86u^3 + 11u^2 - 54u + 1 = 0$

has roots $\alpha^2, \beta^2, \gamma^2, \delta^2$.

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{-86}{81}$$

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2 = \frac{11}{81}$$

$$\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2 = \frac{2}{3}$$

$$\alpha^2\beta^2\gamma^2\delta^2 = \frac{1}{81}$$

$$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2$$

$$-2(\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2)$$

$$= \left(\frac{-86}{81}\right)^2 - 2\left(\frac{11}{81}\right)$$

$$= \frac{5614}{6561}$$

$$\alpha^{-2} + \beta^{-2} + \gamma^{-2} + \delta^{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$$

$$= \frac{\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2}{\alpha^2\beta^2\gamma^2\delta^2}$$

$$= \frac{\frac{2}{3}}{\frac{1}{81}} = 54.$$