TAYLOR'S COLLEGE

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CAMBRIDGE A LEVEL PROGRAMME

SEMESTER ONE EXAMINATION JUNE 2011

(March 2011 Intake)

Wednesday **8 June 2011** **12.30 pm – 2.30 pm**

FURTHER MATHEMATICS 9231

PAPER 1 & 2 **2 hours**

Additional materials: Answer Booklet/Paper
List of formulae (MF 10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total marks for this paper is **50**.

This document consists of **4** printed pages.

 [Turn over

Section A: Pure Mathematics (50%)

1 Prove that $89^{2n+1} + 316^n$ is divisible by 45 for every positive integer n. [4]

2 Find the sum $11^2 + 14^2 + 17^2 + \cdots + (3n+8)^2$. [2]

Hence find $\sum_{r=4n+1}^{6n} (3r+8)^2$ and the minimum value of n such that

$\sum_{r=4n+1}^{6n} (3r+8)^2 > 456n^3 + 570n^2 + 1035$. [3]

3 The curve C has equation $y = 7 + \dfrac{9}{4x+3} + \dfrac{5}{x+6}$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Sketch C, indicating any intersection points with the coordinate axes.

[5]

4 If the equation $9x^4 - 2x^3 + 5x^2 - 8x + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$, find the equation having roots $\alpha^2, \beta^2, \gamma^2, \delta^2$. [5]

Find also $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$ and $\alpha^{-2} + \beta^{-2} + \gamma^{-2} + \delta^{-2}$. [3]

Students are required to answer questions in Section B using a fresh sheet of answer paper. Answers for Section A and Section B need to be handed in separately.

Section B: Mechanics (50%)

1 A snooker ball of mass m is moving with constant speed U in a straight line on a smooth horizontal table. It moves towards a straight smooth cushion, in a direction making an angle θ with the cushion. The coefficient of restitution between the ball and the cushion is e. Find the components, parallel and perpendicular to the cushion, of the velocity of the ball as it leaves the cushion. After colliding with the cushion it rebounds in a direction making an angle ϕ with the cushion. Show that

$$\tan \phi = e \tan \theta.$$ [3]

Show that the kinetic energy lost in the impact is

$$\frac{1}{2}mU^2 \sin^2 \theta(1-e^2).$$ [3]

2 A particle A of mass $2m$ is moving with speed $2u$ on a smooth horizontal table. The particle collides directly with a particle B of mass $4m$ moving with speed u in the same direction as A. The coefficient of restitution between A and B is $\frac{1}{2}$

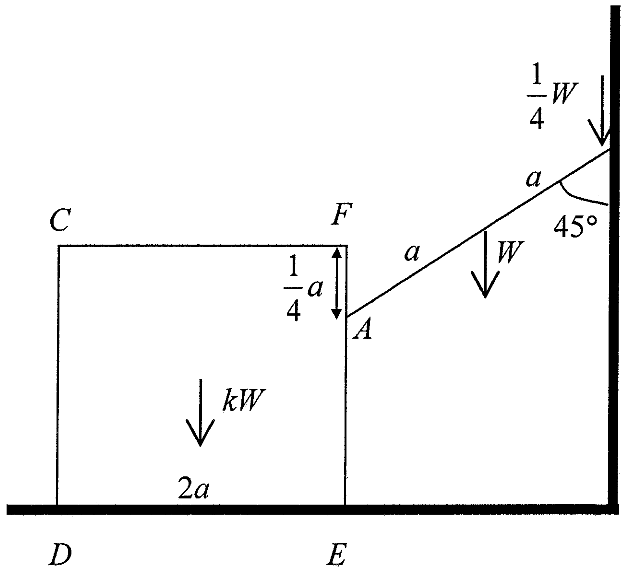
(i) Show that the speed of B after the collision is $\frac{3}{2}u$. [3]

(ii) Find the speed of A after the collision. [1]

Subsequently B collides directly with a particle C of mass m which is at rest on the table. The coefficient of restitution between B and C is e. Given that there are no further collisions,

(iii) find the range of possible values for e. [4]

3



The diagram shows a central cross-section *CDEF* of a uniform solid cube of weight kW with edges of length $2a$. The cube rests on a rough horizontal floor. One of the vertical faces of the cube is parallel to a smooth vertical wall. A uniform ladder, of length $2a$ and weight W, is represented by AB. The ladder rests in equilibrium with A in contact with a rough vertical surface of the cube and B in contact with the wall and makes an angle of 45° with the wall. The distance AF is $\frac{1}{4}a$ and the vertical plane containing AB is perpendicular to the wall. The coefficients of friction between the ladder and the cube is equal to μ. A vertical force of $\frac{1}{4}W$ is applied at the top of the ladder. Show that $\mu \geq \frac{5}{3}$.

[5]

Given that the coefficient of friction between the cube and the floor is equal $\frac{2}{5}$, find the largest value of k for which the cube does not slide. [3]

Show that the cube will not turn about the edge D for any value of k. [3]

TAYLOR'S UNIVERSITY COLLEGE

Cambridge A Level Programme

Mark Scheme : **Further Mathematics Paper 1**

Exam : **Semester 1 Exam** (June 2011)

Intake : **March 2011**

Examiner : Mr. Peter Suresh

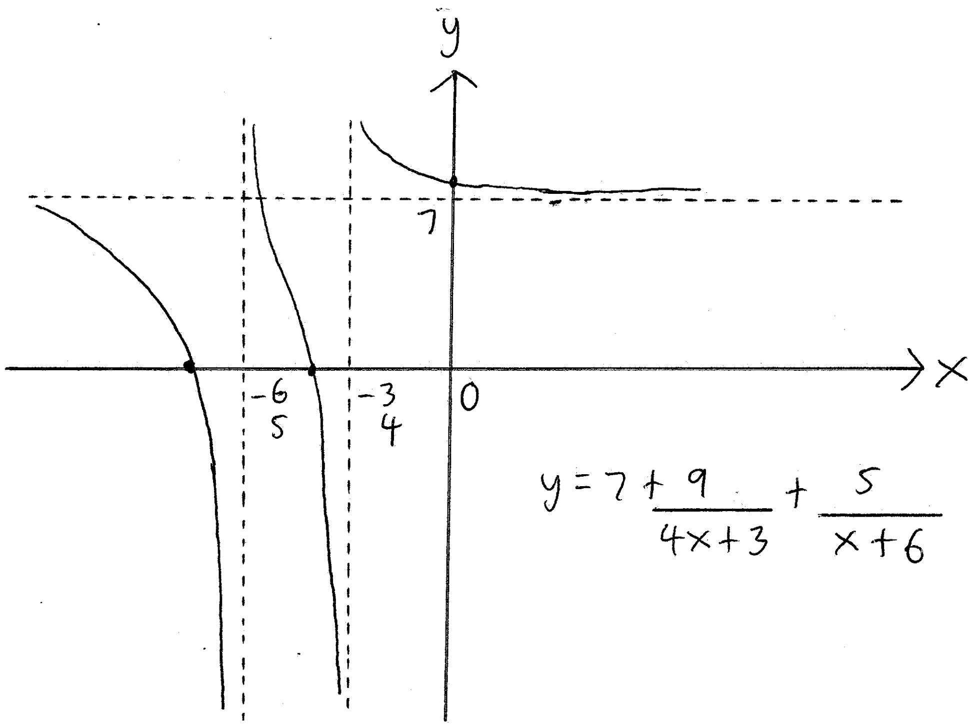
No	Solution	Mark
1.	Let $f(n) = 89^{2n+1} + 316^n$	
	when $n=1$: $f(1) = 89^{2(1)+1} + 316^1$	
	$= 89^{2+1} + 316$	
	$= 89^3 + 316$	
	$= 705285$	
	$= 45(15673)$	
	$\therefore 45 \mid f(1)$	1
	Assume the statement is true when $n=k$.	
	$n=k$: $45 \mid f(k)$	
	$f(k) = 45s$, s is an integer.	
	$89^{2k+1} + 316^k = 45s$	1
	when $n=k+1$:	
	$f(k+1) = 89^{2(k+1)+1} + 316^{k+1}$	
	$= 89^{2k+2+1} + 316^{k+1}$	
	$= 89^{2k+1}89^2 + 316^k 316$	
	$= 89^{2k+1}7921 + 316^k 316$	
	$= 89^{2k+1}(7605 + 316) + 316^k 316$	1

No	Solution	Mark
	$= 89^{2k+1}7605 + 89^{2k+1}316 + 316^{k}316$	
	$= 89^{2k+1}45(169) + 316(89^{2k+1} + 316^{k})$	
	$= 89^{2k+1}45(169) + 316(45s)$	
	$= 45(89^{2k+1}169 + 316s)$	1
	Since s is an integer and k is an integer,	
	$89^{2k+1}169 + 316s$ is an integer.	
	$\therefore 45 \mid f(k+1)$	
	$\therefore 89^{2n+1} + 316^{n}$ is divisible by 45	
	for every positive integer n.	

No	Solution	Mark
2.	$11^2 + 14^2 + 17^2 + \cdots + (3n+8)^2$	
	$= \sum_{r=1}^{n} (3r+8)^2$	1
	$= \sum_{r=1}^{n} 9r^2 + 48r + 64$	
	$= 9\sum_{r=1}^{n} r^2 + 48\sum_{r=1}^{n} r + 64\sum_{r=1}^{n} 1$	
	$= \dfrac{9n(n+1)(2n+1)}{6} + \dfrac{48n(n+1)}{2} + 64n$	
	$= \dfrac{3n(n+1)(2n+1)}{2} + 24n(n+1) + 64n$	
	$= \dfrac{3n(2n^2+3n+1) + 48n^2 + 48n + 64n}{2}$	
	$= \dfrac{6n^3 + 9n^2 + 3n + 48n^2 + 48n + 64n}{2}$	
	$= \dfrac{6n^3 + 57n^2 + 115n}{2}$	
	$= \dfrac{n(6n^2 + 57n + 115)}{2}$	1

No	Solution	Mark
	$\displaystyle\sum_{r=4n+1}^{6n}(3r+8)^2 = \sum_{r=1}^{6n}(3r+8)^2 - \sum_{r=1}^{4n}(3r+8)^2$	1
	$= \dfrac{6n(6(6n)^2+57(6n)+115)}{2} - \dfrac{4n(6(4n)^2+57(4n)+115)}{2}$	
	$= 3n(216n^2+342n+115) - 2n(96n^2+228n+115)$	
	$= 648n^3+1026n^2+345n - 192n^3-456n^2-230n$	
	$= 456n^3+570n^2+115n$	1
	$= n(456n^2+570n+115)$	
	If $\displaystyle\sum_{r=4n+1}^{6n}(3r+8)^2 > 456n^3+570n^2+1035$,	
	$456n^3+570n^2+115n > 456n^3+570n^2+1035$	
	$115n > 1035$	
	$n > 9$	1
	$\therefore n_{min} = 10.$	

No	Solution	Mark
3.	C: $y = 7 + \dfrac{9}{4x+3} + \dfrac{5}{x+6}$	
i)	As $x \to \pm\infty$ $y \to 7$	
	As $x \to -\frac{3}{4}$ $y \to \pm\infty$	
	As $x \to -\frac{6}{5}$ $y \to \pm\infty$	
	$\therefore$ The asymptotes of C are $y = 7$, $x = -\frac{3}{4}$	1,1
	and $x = -\frac{6}{5}$.	1
ii)	$\dfrac{dy}{dx} = \dfrac{-36}{(4x+3)^2} - \dfrac{5}{(x+6)^2}$	
	$= \underbrace{-\left(\underbrace{\underbrace{\dfrac{36}{(4x+3)^2}}_{>0} + \underbrace{\dfrac{5}{(x+6)^2}}_{>0}}\right)}_{<0}$	
	$\dfrac{dy}{dx} < 0$	1
	$\therefore$ no critical points	
	when $x = 0$: $y = \dfrac{65}{6}$	1

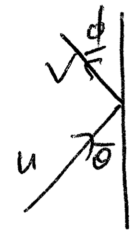
No	Solution	Mark
	when $y = 0$: $7 + \dfrac{9}{4x+3} + \dfrac{5}{x+6} = 0$	
	$\dfrac{9}{4x+3} + \dfrac{5}{x+6} = -7$	
	$9(x+6) + 5(4x+3) = -7(x+6)(4x+3)$	
	$9x + 54 + 20x + 15 = -7(4x^2+27x+18)$	
	$29x + 69 = -28x^2 - 189x - 126$	
	$28x^2 + 218x + 195 = 0$	
	$x = \dfrac{-218 \pm \sqrt{25684}}{56}$	1
		1 1

No	Solution	Mark
4.	$9x^4 - 2x^3 + 5x^2 - 8x + 1 = 0$ $\alpha, \beta, \gamma, \delta$ are the roots $\alpha^2, \beta^2, \gamma^2, \delta^2$ Let $u = \alpha^2$ $\alpha = \pm\sqrt{u}$ α is a root $\therefore 9\alpha^4 - 2\alpha^3 + 5\alpha^2 - 8\alpha + 1 = 0$ $9(\pm\sqrt{u})^4 - 2(\pm\sqrt{u})^3 + 5(\pm\sqrt{u})^2 - 8(\pm\sqrt{u}) + 1 = 0$ $9u^2 - 2(\pm u\sqrt{u}) + 5u - 8(\pm\sqrt{u}) + 1 = 0$ $\pm\sqrt{u}(2u + 8) = 9u^2 + 5u + 1$ $(\pm\sqrt{u}(2u+8))^2 = (9u^2 + 5u + 1)^2$ $u(2u+8)^2 = (9u^2 + 5u + 1)^2$ $u(4u^2 + 32u + 64) = 81u^4 + 45u^3 + 9u^2 + 45u^3 + 25u^2 + 5u + 9u^2 + 5u + 1$ $4u^3 + 32u^2 + 64u = 81u^4 + 90u^3 + 43u^2 + 10u + 1$ $81u^4 + 86u^3 + 11u^2 - 54u + 1 = 0.$	

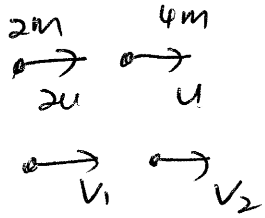
No	Solution	Mark
	$\therefore$ The equation $81u^4 + 86u^3 + 11u^2 - 54u + 1 = 0$ has roots $\alpha^2, \beta^2, \gamma^2, \delta^2$.	1,1,1,1,1
	$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -\frac{86}{81}$	
	$\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2 = \frac{11}{81}$	1
	$\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2 = \frac{2}{3}$	
	$\alpha^2\beta^2\gamma^2\delta^2 = \frac{1}{81}$	
	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2 - 2(\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2)$	
	$= \left(-\frac{86}{81}\right)^2 - 2\left(\frac{11}{81}\right)$	
	$= \frac{5614}{6561}$	1
	$\alpha^{-2} + \beta^{-2} + \gamma^{-2} + \delta^{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$	
	$= \frac{\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2}{\alpha^2\beta^2\gamma^2\delta^2}$	
	$= \frac{\frac{2}{3}}{\frac{1}{81}} = 54$	1

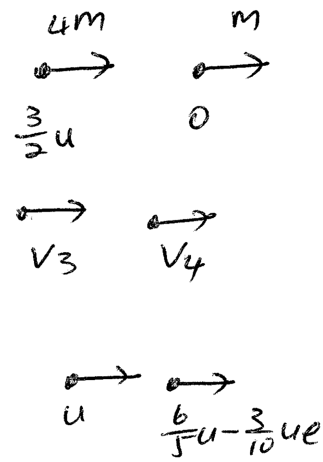
Paper 2 : Mark Scheme for Mar 2011 Intake Semester 1 Exam

No	Solution	Mark
1	component parallel to cushion $= u\cos\theta$	B1
	component perpendicular to cushion $= eu\sin\theta$	B1
	$\therefore\ v\cos\phi = u\cos\theta$	M1
	$v\sin\phi = eu\sin\theta$	
	$\therefore\ \tan\phi = e\tan\theta$	
	KE loss $= \frac{1}{2}mu^2 - \frac{1}{2}mv^2$	
	$= \frac{1}{2}mu^2 - \frac{1}{2}m[u^2\cos^2\theta + e^2u^2\sin^2\theta]$	M1
	$= \frac{1}{2}mu^2 - \frac{1}{2}mu^2(1-\sin^2\theta + e^2\sin^2\theta)$	A1
	$= \frac{1}{2}mu^2 - \frac{1}{2}mu^2 + \frac{1}{2}mu^2\sin^2\theta - \frac{1}{2}mu^2e^2\sin^2\theta$	
	$= \frac{1}{2}mu^2\sin^2\theta(1-e^2)$	A1
		6

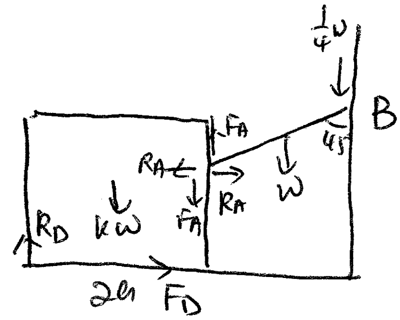


No	Solution	Mark
2	(i) Law of conservation of momentum	
	$2m(2u) + 4mu = 2mv_1 + 4mv_2$	
	$8u = 2v_1 + 4v_2$	M1
	$v_1 + 2v_2 = 4u$ —(1)	
	$\frac{v_2 - v_1}{2u - u} = \frac{1}{2}$	M1
	$2v_2 - 2v_1 = u$ —(2)	
	(1) − (2) $3v_1 = 3u$	
	$v_1 = u$	A1
	$2v_2 = 4u - u$	
	$v_2 = \frac{3}{2}u$	A1
	(ii) $v_1 = u$	
	(iii)	
	$4m(\frac{3}{2}u) = 4mv_3 + mv_4$	M1
	$6u = 4v_3 + v_4$	
	$\frac{v_4 - v_3}{\frac{3}{2}u} = e$	
	$v_4 - v_3 = \frac{3}{2}ue$ —(3)	
	$4v_3 + v_4 = 6u$ —(4)	
	(4) − (3) $5v_3 = 6u - \frac{3}{2}ue$	
	$v_3 = \frac{6}{5}u - \frac{3}{10}ue$	B1
	no further collision	
	$\frac{6}{5}u - \frac{3}{10}ue > u$	m1
	$-\frac{3}{10}ue > -\frac{1}{5}u$	
	$\frac{3}{10}e < \frac{1}{5}$	
	$e < \frac{2}{3}$ $\therefore$ $0 < e < \frac{2}{3}$	A1





(8)

No	Solution	Mark
3	Take moment about A for rod AB	
		
	$w a \cancel{\cos 45°} + \frac{1}{4} w\, 2a \cancel{\cos 45°} = S\, 2a \cancel{\sin 45°}$	M1
	$\frac{3}{2} w = 2S$	A1
	$S = \frac{3}{4} w$	
	$\therefore R_A = \frac{3}{4} w$	A1
	$\Sigma F_y = 0 \quad \therefore F_A = \frac{5}{4} w$	
	$F_A \le \mu R_A$	
	$\mu \ge \dfrac{\frac{5}{4} w}{\frac{3}{4} w}$	M1
	$\mu \ge \frac{5}{3}$	A1
	Take moment about D.	
	$R_A \times \frac{3}{4} a = \frac{3}{4} w \times \frac{3}{4} a = \frac{9}{16} wa$	M1
	$kwa + \frac{5}{4} w \times 2a = kwa + \frac{10}{4} wa$	M1
	$\therefore kwa + \frac{5}{2} wa > \frac{9}{16} wa$ for $k > 0$	A1
	no toppling about D for any k.	
	$F_D = \frac{3}{4} w$	B1
	$\frac{3}{4} w \le \frac{2}{5}(kw + \frac{5}{4} w)$	M1
	$\frac{3}{4} \le \frac{2}{5} k + \frac{1}{2}$	
	$k \ge \frac{5}{8}$	A1

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