1 If $y = x\cos x$, prove by induction that

$$\frac{d^{2n}y}{dx^{2n}} = 2(-1)^n n\sin x + (-1)^n x\cos x$$

for every positive integer n. [9]

2 Given that $x = \sec\theta + \csc\theta$ and $y = \tan\theta + \cot\theta$, show that

(i) $\dfrac{dy}{dx} = \dfrac{\sin\theta + \cos\theta}{1 + \sin\theta\cos\theta}$. [5]

(ii) $\dfrac{d^2y}{dx^2} = \dfrac{\sin^3\theta\cos^3\theta}{(1 + \sin\theta\cos\theta)^3}$ [4]

3 The linear transformation T: $\Re^4 \to \Re^4$ is represented by the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & -4 & -5 \\ 3 & a^2+5 & 2a-7 & 3a-9 \\ 6 & a^2+12 & 2a-14 & 3a-18 \end{pmatrix}$$

and a is a constant.

Show that the dimension of the range space of T is 3 provided $a \neq a_1$ and $a \neq a_2$, where a_1 and a_2 are integers that need to be found and that $a_1 < 0$ and $a_2 > 0$. [5]

In the case where $a = 2$, find a basis for the null space of T. [5]

4 Find a 3×3 matrix $\mathbf{A}$ whose eigenvalues are 6, 9 and 3 with corresponding eigenvectors

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

respectively. [11]

5 Given that

$$3y^2\frac{d^2y}{dx^2}+6y(\frac{dy}{dx})^2+15y^2\frac{dy}{dx}+4y^3=3\cos 2x-4\sin 2x$$

show that with the substitution $u=y^3$,

$$\frac{d^2u}{dx^2}+5\frac{du}{dx}+4u=3\cos 2x-4\sin 2x. \qquad [5]$$

Find the general solution of y in terms of x. [6]

(Please start this section with a fresh sheet of answer paper and fasten your answers separately from the first 5 questions.)

6. The sugar content of a brand of cough medicine was measured in a sample of five bottles. The results as percentages by weight were

52.3 , 55.7 , 51.4 , 56.2 , 54.0.

These may be considered to be a random sample from a normal distribution with mean μ.
(i) Obtain a 95% confidence interval for μ. [5]

A further bottle is analysed. The combined sample of six results gives a smaller variance estimate than that used in part (i).

(ii) State, giving your reasons, whether a 95% confidence interval for μ based on the six results will be wider or narrower than the interval found in part (i). [2]

7. Two types of cloth, Type A and Type B, were tested for flame-resistance. 45 pieces of Type A and 48 pieces of Type B, of equal size, were selected at random. Each was set alight in the same way and the times to extinction of the flames were measured. The results, x_A minutes and x_B minutes, are summarised by

$\Sigma x_A = 276.5$, $\Sigma x_A^2 = 1823.0$, $\Sigma x_B = 267.8$, $\Sigma x_B^2 = 1649.2$.

Test, at the 2% significance level, whether there is a difference in the mean times to extinction for the two types of cloth. [8]

Explain, in the context of the question, the meaning of 'at the 2% significance level'. [1]

[Turn over

1 If $\mathbf{A} = \begin{pmatrix} 1 & 4 & -5 & 8 \\ 2 & 9 & -8 & 13 \\ -3 & -14 & 11 & -18 \\ 5 & 23 & -19 & 31 \end{pmatrix}$, find rank $\mathbf{A}$, the null space of $\mathbf{A}$ and a basis for the range space of $\mathbf{A}$. [7]

2 If $y = x^2e^x$, prove that

$$\frac{d^n y}{dx^n} = x^2e^x + 2nxe^x + n(n-1)e^x,$$

for every positive integer n. [7]

3 If the square matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ of the same size have eigenvalues $\lambda_{\mathbf{A}}$, $\lambda_{\mathbf{B}}$ and $\lambda_{\mathbf{C}}$ respectively with eigenvector $\mathbf{x}$, find an eigenvalue of the matrix

(i) $\mathbf{AB} + \mathbf{BC} + \mathbf{CA}$, [7]

(ii) $\mathbf{ABC}$, [3]

(iii) $(\mathbf{A} + \mathbf{I})(\mathbf{B} + \mathbf{I})(\mathbf{C} + \mathbf{I})$. [2]

4 Given the following differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 3e^x\cos x + 4e^x\sin x.$$

(i) Verify that $e^x\left(P\cos x + Q\sin x\right)$ is a particular integral for this differential equation, where P and Q are constants whose numerical value is to be found. [8]

(ii) Hence, find the general solution to the given differential equation. [4]

5 **(i)** Given that $x = \cos^4 t + \sin^4 t$ and $y = \cos^4 t - \sin^4 t$, show that

$$\frac{d^2y}{dx^2} = -\sec^3 2t.$$ [7]

(ii) If a curve has equation $xy^2 + 2x = 3y^2$, find the value of $\frac{d^2y}{dx^2}$ at the point (1, 1). [7]

1. If $\mathbf{A} = \begin{pmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & -7 & -2 \\ -1 & -3 & 5 & -5 \\ 2 & 5 & -9 & 6 \end{pmatrix}$, find the rank of **A**, the null space of **A** and a basis for the range space of **A**.

[5]

2. If $x^2y^3 + (4x+5y)^6 = 7$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

[9]

3. If $\mathbf{A} = \begin{pmatrix} 7 & 4 & -8 \\ -2 & 1 & 16 \\ 0 & 0 & 9 \end{pmatrix}$ and $\mathbf{A}^n = \mathbf{PDP}^{-1}$, where **P** is a square matrix and **D** is a diagonal matrix, find **P** and **D**.

[9]

4. If $y = e^x \sin x$, prove that $\frac{d^{4n}y}{dx^{4n}} = (-4)^n e^x \sin x$ for every positive integer n.

[9]

5. Given that $9y^2\frac{d^2y}{dx^2} + 18y(\frac{dy}{dx})^2 + 45y^2\frac{dy}{dx} + 12y^3 = 4\cos x + 7\sin x$,

show that if $v = y^3$,

$$\frac{d^2v}{dx^2} + 5\frac{dv}{dx} + 4v = \frac{4\cos x}{3} + \frac{7\sin x}{3}.$$

[5]

Hence, find the general solution of y in terms of x.

[12]

Find also the limit of y as $x \to \infty$.

[1]