Roots of a Cubic Equation

Theorem :

A cubic equation $ax^3 + bx^2 + cx + d = 0$ always has three roots.

If α, β and γ are the roots of the equation, then

$$\underline{\underline{\alpha + \beta + \gamma = -\frac{b}{a}}}, \quad \underline{\underline{\beta\gamma + \alpha\gamma + \alpha\beta = \frac{c}{a}}}, \text{ and } \underline{\underline{\alpha\beta\gamma = -\frac{d}{a}}}.$$

Example:

Let α, β and γ be the roots of the following equations.

Determine the values of $\alpha+\beta+\gamma$, $\beta\gamma+\alpha\gamma+\alpha\beta$ and $\alpha\beta\gamma$.

(a) $x^3-2x^2+x-3=0$

(b) $2x^3+4x^2-1=0$

(c) $-x^3-x^2+3x+1=0$

Example:

Find the integer c such that the equation $4x^3 + cx - 27 = 0$ has a double root.

Example: (NEAB June 1998)

The equation $x^3 - 3x^2 + px + 4 = 0$, where p is a constant, has roots $\alpha - \beta$, α and $\alpha + \beta$, where $\beta > 0$.

(a) Find the values of α and β.

(b) Find the value of p.

Example:

The roots of a cubic equation $x^3 + px^2 + qx + r = 0$, where $p, q, r \in \mathrm{R}$, are α, β and γ. Given that $\alpha + \beta + \gamma = 4$, $\alpha^2 + \beta^2 + \gamma^2 = 20$ and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -2$, find p, q and r.

Example:

Let the roots of the equation $x^3 - 6x + 2 = 0$ be α, β and γ.

Without evaluating α, β and γ, find the values of

(a) $4\alpha + 4\beta + 4\gamma$

(b) $\alpha^2 + \beta^2 + \gamma^2$

(c) $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$

(d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

Example:

Given that the roots of the equation $x^3 - 6x^2 + 11x + 6 = 0$ are in arithmetic progression, solve the equation.

Formation of a Cubic Equation

Theorem :

If the roots of a cubic equation are R_1, R_2 and R_3,

then its equation is given by

$$x^3 - (R_1 + R_2 + R_3)x^2 + (R_2 R_3 + R_1 R_3 + R_1 R_2)x + R_1 R_2 R_3 = 0.$$

Example:

α, β and γ are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, where $a:b:c:d$ is in the simplest ratio. Find a, b, c and d if

(a) $\alpha + \beta + \gamma = -3$, $\beta\gamma + \alpha\gamma + \alpha\beta = 1$ and $\alpha\beta\gamma = -2$.

(b) $\alpha + \beta + \gamma = 1$, $\beta\gamma + \alpha\gamma + \alpha\beta = 0$ and $\alpha\beta\gamma = 1$

(c) $\alpha + \beta + \gamma = \dfrac{1}{3}$, $\beta\gamma + \alpha\gamma + \alpha\beta = -\dfrac{1}{2}$ and $\alpha\beta\gamma = \dfrac{1}{2}$.

Example:

Each of the following are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, where $a : b : c : d$ is in the simplest ratio. Use the sum and product of roots to find a, b, c and d.

(a) $2, 3, 4$

(b) $-1, 0, 2$

(c) $0, \frac{1}{3}, -\frac{1}{3}$

(d) $1, -\frac{1}{2}, -\frac{2}{3}$

Example:

The roots of the cubic equation $2x^3 + 3x^2 + 4x + 5 = 0$ are α, β and γ. Without evaluating α, β and γ, find the cubic equations with roots

(i) $\alpha - 2$, $\beta - 2$ and $\gamma - 2$ (ii) 3α, 3β and 3γ

Example:

The roots of the cubic equation $x^3 + 2x^2 + 3x + 4 = 0$ are α, β and γ. Without evaluating α, β and γ, find the cubic equations with roots

(i) α^2, β^2 and γ^2

(ii) $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$.

Example: (AQA March 2000)

The roots of the cubic equation $2x^3 + 3x^2 + 4 = 0$ are α, β and γ.

(a) State the values of $\alpha + \beta + \gamma$, $\beta\gamma + \alpha\gamma + \alpha\beta$ and $\alpha\beta\gamma$.

(b) Hence, find the cubic equation with integer coefficients, having roots $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$.

Example:

Solve the simultaneous equations

$$\begin{cases} p+q+r=6 \\ qr+rp+pq=11. \\ pqr=6 \end{cases}$$

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example:

The roots of the equation $x^3 - ax^2 + bx - c = 0$ are α, β and γ. Prove that if $\alpha\beta = \alpha + \beta$, then $\gamma = a + c - b$.

Example:

The cubic equation $ax^3 + bx^2 + cx + d = 0$ has the property that two of its roots are the reciprocal of each other. Prove that $a^2 - d^2 = ac - bd$. Hence, solve the equation $9x^3 + 24x - 11x - 6 = 0$.

Example:

The roots of the equation $x^3 + ax + b = 0$ are α, β and γ.

Find the equation with roots $\dfrac{\beta}{\gamma} + \dfrac{\gamma}{\beta}, \dfrac{\gamma}{\alpha} + \dfrac{\alpha}{\gamma}$ and $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$.

Example :

Solve the simultaneous equations

$$\begin{cases} p+q+r=6=11 \\ p^2+q^2+r^2=26. \\ pqr=-12 \end{cases}$$