1. $x = a\cos t \quad y = a\sin t \quad t = 0, 2\pi$

$$\frac{dx}{dt} = -a\sin t \quad \frac{dy}{dt} = a\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-a\sin t)^2 + (a\cos t)^2$$

$$= a^2\sin^2 t + a^2\cos^2 t$$

$$= a^2(\sin^2 t + \cos^2 t)$$

$$= a^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = a$$

The area of the surface generated by rotating the curve defined parametrically by the equations $x = a\cos t$, $y = a\sin t$ from $t=0$ to $t = 2\pi$ about the x-axis is

$$\int_0^{\pi} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\pi} 2\pi a\sin t\,(a)\,dt$$

$$= \int_0^{\pi} 2\pi a^2 \sin t\,dt$$

$$= 2\pi a^2 \int_0^{\pi} \sin t\,dt$$

$$= 2\pi a^2 \left[-\cos t\right]_0^{\pi}$$

$$= 2\pi a^2(-\cos\pi - -\cos 0)$$

$$= 2\pi a^2(1+1)$$

$$= 2\pi a^2(2)$$

$$= 4\pi a^2.$$

2. Expressing $\dfrac{r+6}{r(r+2)(r+3)}$ as partial fractions,

$$\frac{r+6}{r(r+2)(r+3)} = \frac{A}{r} + \frac{B}{r+2} + \frac{C}{r+3}$$

$$= \frac{A(r+2)(r+3) + Br(r+3) + Cr(r+2)}{r(r+2)(r+3)}$$

$$\therefore r+6 = A(r+2)(r+3) + Br(r+3) + Cr(r+2)$$

$$= A(r^2+5r+6) + B(r^2+3r) + C(r^2+2r)$$

$$= (A+B+C)r^2 + (5A+3B+2C)r + 6A$$

Equating coefficients of powers of r:

$A+B+C=0$ — (1) $\quad 5A+3B+2C=1$ — (2) $\quad 6A=6$ — (3)

From (3): $6A=6$

$A=1$

Substituting $A=1$ into equations (1) and (2):

$B+C=-1$ — (4) $\quad 3B+2C=-4$ — (5)

From (4): $B=-1-C$

Substituting $B=-1-C$ into $3B+2C=-4$:

$3(-1-C)+2C=-4$

$-3-3C+2C=-4$

$-C=-1$

$C=1$

Substituting $C=1$ into equation (4): $B=-2$

$$\therefore \frac{r+6}{r(r+2)(r+3)} = \frac{1}{r} - \frac{2}{r+2} + \frac{1}{r+3}$$

$$S_n = \sum_{r=1}^{n} \frac{r+6}{r(r+2)(r+3)}$$

since $\dfrac{r+6}{r(r+2)(r+3)} = \dfrac{1}{r} - \dfrac{2}{r+2} + \dfrac{1}{r+3}$

$$\therefore S_n = \sum_{r=1}^{n} \frac{r+6}{r(r+2)(r+3)} = \sum_{r=1}^{n} \frac{1}{r} - \frac{2}{r+2} + \frac{1}{r+3}$$

$$= \frac{1}{1} - \frac{2}{3} + \frac{1}{4}$$

$$+ \frac{1}{2} - \frac{2}{4} + \frac{1}{5}$$

$$+ \frac{1}{3} - \frac{2}{5} + \frac{1}{6}$$

$$+ \frac{1}{4} - \frac{2}{6} + \frac{1}{7}$$

$$+ \frac{1}{5} - \frac{2}{7} + \frac{1}{8}$$

$$\vdots$$

$$+ \frac{1}{n-3} - \frac{2}{n-1} + \frac{1}{n}$$

$$+ \frac{1}{n-2} - \frac{2}{n} + \frac{1}{n+1}$$

$$+ \frac{1}{n-1} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$+ \frac{1}{n} - \frac{2}{n+2} + \frac{1}{n+3}$$

$$= \frac{1}{1} - \frac{2}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3}$$

$$= \frac{7}{6} - \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3}$$

3 i) $\cos^5 x = \cos x \cos^4 x$

$= \cos x (1 - \sin^2 x)^2$, since $\cos^2 x = 1 - \sin^2 x$

$= \cos x (1 - 2\sin^2 x + \sin^4 x)$

$= \cos x - 2\sin^2 x \cos x + \sin^4 x \cos x$

ii) The mean value of $\cos^5 x$ over the interval

$0 \leq x \leq \frac{\pi}{2}$ is $\dfrac{1}{\frac{\pi}{2} - 0} \displaystyle\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$.

Since $\cos^5 x = \cos x - 2\sin^2 x \cos x + \sin^4 x \cos x$

$$\therefore \frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} \cos^5 x \, dx$$

$$= \frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} \cos x - 2\sin^2 x \cos x + \sin^4 x \cos x \, dx$$

$$= \frac{2}{\pi} \left[\sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi} \left(\sin\frac{\pi}{2} - \frac{2}{3}\sin^3\frac{\pi}{2} + \frac{\sin^5\frac{\pi}{2}}{5} - \sin 0 - \frac{2}{3}\sin^3 0 - \frac{\sin^5 0}{5} \right)$$

$$= \frac{2}{\pi} \left(1 - \frac{2}{3} + \frac{1}{5} - 0 \right)$$

$$= \frac{2}{\pi} \left(\frac{8}{15} \right)$$

$$= \frac{16}{15\pi}$$

4 $\displaystyle\sum_{r=1}^{n} \sin(2r+1)\theta = \sin(n+2)\theta \sin n\theta \csc\theta$

when $n=1$: $\displaystyle\sum_{r=1}^{1} \sin(2r+1)\theta = \sin(2(1)+1)\theta$

$$= \sin(2+1)\theta = \sin 3\theta = \sin 3\theta \left(\frac{\sin\theta}{\sin\theta}\right)$$

$$= \sin(1+2)\theta \sin(1)\theta \csc\theta$$

Assume the equation is true when $n=k$:

$n=k$: $\displaystyle\sum_{r=1}^{k} \sin(2r+1)\theta \doteq \sin(k+2)\theta \sin k\theta \csc\theta$

when $n=k+1$:

$$\sum_{r=1}^{k+1} \sin(2r+1)\theta = \sin(2(k+1)+1)\theta + \sum_{r=1}^{k} \sin(2r+1)\theta$$

$$= \sin(2k+3)\theta + \sin(k+2)\theta \sin k\theta \csc\theta$$

$$= (\sin(2k+3)\theta \sin\theta + \sin(k+2)\theta \sin k\theta) \csc\theta$$

$$= \left(\frac{1}{2}\left[\cos((2k+3)\theta-\theta) - \cos((2k+3)\theta+\theta)\right]\right.$$

$$\left.+\frac{1}{2}\left[\cos((k+2)\theta-k\theta) - \cos((k+2)\theta+k\theta)\right]\right)\csc\theta$$

$$= \left(\frac{1}{2}\left[\cos(2k+2)\theta - \cos(2k+4)\theta\right]\right.$$

$$\left.+\frac{1}{2}\left[\cos 2\theta - \cos(2k+2)\theta\right]\right)\csc\theta$$

$$= \frac{1}{2}\left(\cos 2\theta - \cos(2k+4)\theta \right) \csc\theta$$

$$= \frac{1}{2}\left(\cos[k+3-(k+1)]\theta - \cos(k+3+k+1)\theta \right) \csc\theta$$

$$= \sin(k+3)\theta \sin(k+1)\theta \csc\theta$$

$\therefore \sum_{r=1}^{n} \sin(2r+1)\theta = \sin(n+2)\theta \sin n\theta \csc\theta$ is

true for $n = k+1$ if it is true for $n = k$ and since it is true when $n = 1$, it is true for every positive integer n.

5 $x^2 + 9y^2 - 3xy + 11x^2y^2 = 36$

The mean value of $\frac{d^2y}{dx^2}$ over the interval

$0 \leq x \leq 6$ is $\frac{1}{6-0}\int_0^6 \frac{d^2y}{dx^2}\,dx = \frac{1}{6}\int_0^6 \frac{d}{dx}\left(\frac{dy}{dx}\right)dx$

$$= \frac{1}{6}\left[\frac{dy}{dx}\right]_0^6$$

$$\frac{d}{dx}(x^2 + 9y^2 - 3xy + 11x^2y^2) = \frac{d}{dx}(36)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(9y^2) - \frac{d}{dx}(3xy) + \frac{d}{dx}(11x^2y^2) = 0$$

$$2x + 18y\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y + 22xy^2 + 22x^2y\frac{dy}{dx} = 0$$

$$2x - 3y + 22xy^2 + (18y - 3x + 22x^2y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3y - 2x - 22xy^2}{18y - 3x + 22x^2y}$$

Since $y = 2$ when $x = 0$ and $y = 0$ when $x = 6$,

$$\frac{1}{6}\int_0^6 \frac{d}{dx}\left(\frac{dy}{dx}\right)dx = \frac{1}{6}\int_{(0,2)}^{(6,0)} \frac{d}{dx}\left(\frac{dy}{dx}\right)dx$$

$$= \frac{1}{6}\left[\frac{dy}{dx}\right]_{(0,2)}^{(6,0)}$$

$$= \frac{1}{6}\left[\frac{3y - 2x - 22xy^2}{18y - 3x + 22x^2y}\right]_{(0,2)}^{(6,0)}$$

$$= \frac{1}{6}\left(\frac{-12}{-18} - \frac{6}{36}\right)$$

$$= \frac{1}{6}\left(\frac{2}{3} - \frac{1}{6}\right)$$

$$= \frac{1}{6}\left(\frac{1}{2}\right)$$

$$= \frac{1}{12}$$

6. If $z = \cos\theta + i\sin\theta$

$$z^n = \cos n\theta + i\sin n\theta \quad \text{and} \quad z^{-n} = \cos n\theta - i\sin n\theta$$

$$\therefore z^n + z^{-n} = 2\cos n\theta \quad \text{and} \quad z^n - z^{-n} = 2i\sin n\theta$$

when $n = 1$: $2\cos\theta = z + \frac{1}{z}$

$$(2\cos\theta)^6 = \left(z + \frac{1}{z}\right)^6$$

$$64\cos^6\theta = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$$

$$= z^6 + \frac{1}{z^6} + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$$\therefore \cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

since $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

$$\therefore \int_{-\pi}^{\pi} \cos^6\theta \, d\theta = \int_{-\pi}^{\pi} \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\, d\theta$$

$$= \frac{1}{32}\left[\frac{\sin 6\theta}{6} + \frac{3\sin 4\theta}{2} + \frac{15\sin 2\theta}{2} + 10\theta\right]_{-\pi}^{\pi}$$

$$= \frac{1}{32}\left(\frac{\sin 6\pi}{6} + \frac{3\sin 4\pi}{2} + \frac{15\sin 2\pi}{2} + 10\pi - \frac{\sin(-6\pi)}{6} - \frac{3\sin(-4\pi)}{2} - \frac{15\sin(-2\pi)}{2} - 10(-\pi)\right)$$

$$= \frac{1}{32}(10\pi - -10\pi)$$

$$= \frac{1}{32}(10\pi + 10\pi)$$

$$= \frac{20\pi}{32}$$

$$= \frac{5\pi}{8}$$

7. i) $\frac{d^2y}{dx^2} - 3a\frac{dy}{dx} + 2a^2y = 0$, $a \neq 0$, $y=4$ and $\frac{dy}{dx} = 5$ when $x=0$

$\lambda^2 - 3a\lambda + 2a^2 = 0$

$(\lambda - a)(\lambda - 2a) = 0$

$\lambda = a, 2a$

$\therefore y = Ae^{ax} + Be^{2ax}$

$\frac{dy}{dx} = Aae^{ax} + 2Bae^{2ax}$

$x=0 \quad y=4 : \quad 4 = (A+B)a$ — ①

$x=0 \quad \frac{dy}{dx} = 5 : \quad 5 = (A+2B)a$ — ②

② − ① : $(A+2B)a - (A+B)a = 5-4$

$$Ba = 1$$

$$B = \frac{1}{a}$$

Substituting $B = \frac{1}{a}$ into ①:

$$A = \frac{3}{a}$$

$$y = \frac{3e^{ax}}{a} + \frac{e^{2ax}}{2a}$$

ii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x$

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda-1)(\lambda-2)=0$

$\lambda=1,2$

$\therefore y_c = Ae^{x} + Be^{2x}$

Let $y_p = C\cos x + D\sin x$

$$\frac{dy_p}{dx} = -C\sin x + D\cos x$$

$$\frac{d^2y_p}{dx^2} = -C\cos x - D\sin x$$

$$\frac{d^2y_p}{dx^2} - \frac{3dy_p}{dx} + 2y_p = (-C-3D+2C)\cos x + (-D+3C+2D)\sin x$$

$$= (C-3D)\cos x + (D+3C)\sin x$$

$$= 10\sin x$$

Equating coefficients,

$C-3D=0$ — (3) $\quad D+3C=10$ — (4)

From (3) : $C=3D$

substituting $C=3D$ into (4) : $D+3(3D)=10$

$D+9D=10$

$10D=10$

$D=1$

$C=3$

$\therefore y_p = 3\cos x + \sin x$

$y = y_c + y_p$

$= Ae^{x} + Be^{2x} + 3\cos x + \sin x$

8. i) $x^3 + 3x^2 + 2x + 1 = 0$

$\alpha + \beta + \gamma = -3 \quad \alpha\beta + \alpha\gamma + \beta\gamma = 2 \quad \alpha\beta\gamma = -1$

$S_n = \alpha^n + \beta^n + \gamma^n$

$\therefore S_{3+r} + 3S_{2+r} + 2S_{1+r} + S_r = 0$

When $r = 0$,

$$S_3 + 3S_2 + 2S_1 + S_0 = 0$$

$S_0 = \alpha^0 + \beta^0 + \gamma^0 = 1 + 1 + 1 = 3$

$S_1 = \alpha + \beta + \gamma = -3$

$S_2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (-3)^2 - 2(2)$

$= 9 - 4$

$= 5$

$\therefore S_3 + 3(5) + 2(-3) + 3 = 0$

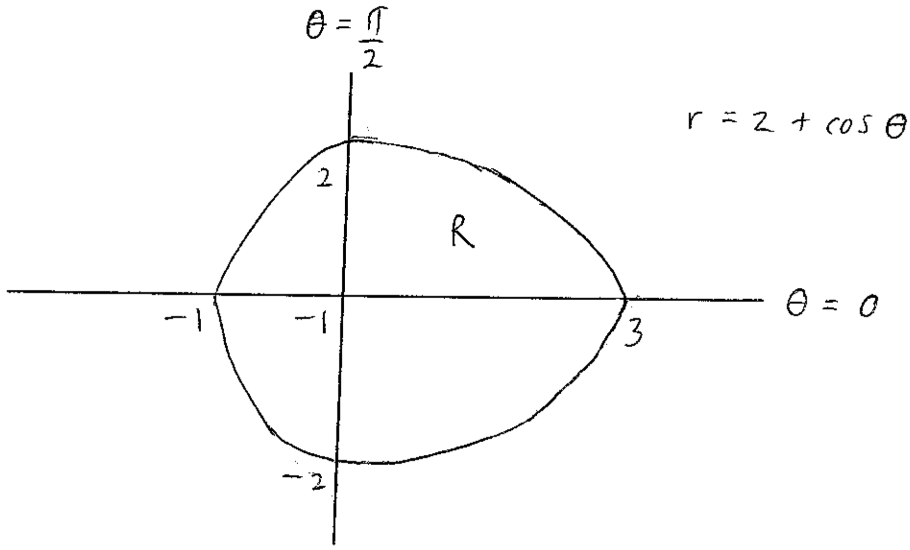
$S_3 + 15 - 6 + 3 = 0$

$S_3 = -12.$

ii) $r = 2 + \cos\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	3	$2 + \frac{\sqrt{3}}{2}$	$2 + \frac{1}{\sqrt{2}}$	$2\frac{1}{2}$	2	$1\frac{1}{2}$	$2 - \frac{1}{\sqrt{2}}$	$2 - \frac{\sqrt{3}}{2}$	1

$2 + \cos(-\theta) = 2 + \cos\theta$



$$\text{Area of } R = 2\int_0^{\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi} r^2 \, d\theta$$

$$= \int_0^{\pi} (2 + \cos\theta)^2 \, d\theta$$

$$= \int_0^{\pi} 4 + 4\cos\theta + \cos^2\theta \, d\theta$$

$$= \int_0^{\pi} 4 + 4\cos\theta + \frac{\cos 2\theta + 1}{2} \, d\theta$$

$$= \int_0^{4} 4\cos\theta + \frac{\cos 2\theta}{2} + \frac{9}{2} \, d\theta$$

$$= \left[4\sin\theta + \frac{\sin 2\theta}{8} + \frac{9\theta}{2} \right]_0^{\pi}$$

$$= 4\sin\pi + \frac{\sin 2\pi}{8} + \frac{9\pi}{2} - \left(4\sin 0 + \frac{\sin 0}{8} + 0 \right)$$

$$= \frac{9\pi}{2}$$

9. i) $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$

$$u = x^n \qquad dv = \cos x \, dx$$
$$du = nx^{n-1} dx \qquad v = \sin x$$

$$I_n = \left[x^n \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} nx^{n-1} \sin x \, dx$$

$$= \left(\frac{\pi}{2}\right)^n \sin\frac{\pi}{2} - 0 - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$= \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$u = x^{n-1} \qquad dv = \sin x \, dx$$
$$du = (n-1)x^{n-2} dx \qquad v = -\cos x$$

$$= \left(\frac{\pi}{2}\right)^n - n\left(\left[-x^{n-1}\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -(n-1)x^{n-2}\cos x \, dx\right)$$

$$= \left(\frac{\pi}{2}\right)^n - n\left(-\left(\frac{\pi}{2}\right)^{n-1}\cos\frac{\pi}{2} - 0 + (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x \, dx\right)$$

$$= \left(\frac{\pi}{2}\right)^n - n\left(0 + (n-1) I_{n-2}\right)$$

$$= \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}, \quad n \geq 2.$$

ii) when $n = 3$:

$$I_3 = \left(\frac{\pi}{2}\right)^3 - 3 \cdot 2 I_1$$

$$= \frac{\pi^3}{8} - 6 I_1$$

when $n=1$:

$$I_1 = \int_0^{\frac{\pi}{2}} x\cos x \, dx$$

$$u = x \qquad dv = \cos x \, dx$$
$$du = dx \qquad v = \sin x$$

$$= \left[x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \frac{\pi}{2}\sin\frac{\pi}{2} - 0 + \left[\cos x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \cos\frac{\pi}{2} - \cos 0$$

$$= \frac{\pi}{2} - 1$$

$$\therefore I_3 = \frac{\pi^3}{8} - 6\left(\frac{\pi}{2} - 1\right)$$

$$= \frac{\pi^3}{8} - 3\pi + 6$$

10. $C: y = 2x+1 + \dfrac{8}{x+2}$

i) $x \to \pm\infty \quad y \to 2x+1$

$x \to -2 \quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = 2x+1$ and $x = -2$.

ii) $y = 2x+1 + \dfrac{8}{x+2}$

$$\frac{dy}{dx} = 2 - \frac{8}{(x+2)^2}$$

When $\dfrac{dy}{dx} = 0$: $\quad 2 - \dfrac{8}{(x+2)^2} = 0$

$$\frac{8}{(x+2)^2} = 2$$

$$(x+2)^2 = \frac{8}{2}$$

$$= 4$$

$$x+2 = \pm 2$$

$$x = -4, 0$$

$$y = -11, 5$$

$\therefore$ The critical points of C are $(-4,-11)$ and $(0,5)$

$$\frac{d^2y}{dx^2} = \frac{16}{(x+2)^2}$$

when $x = -4$: $\frac{d^2y}{dx^2} = -2 < 0$

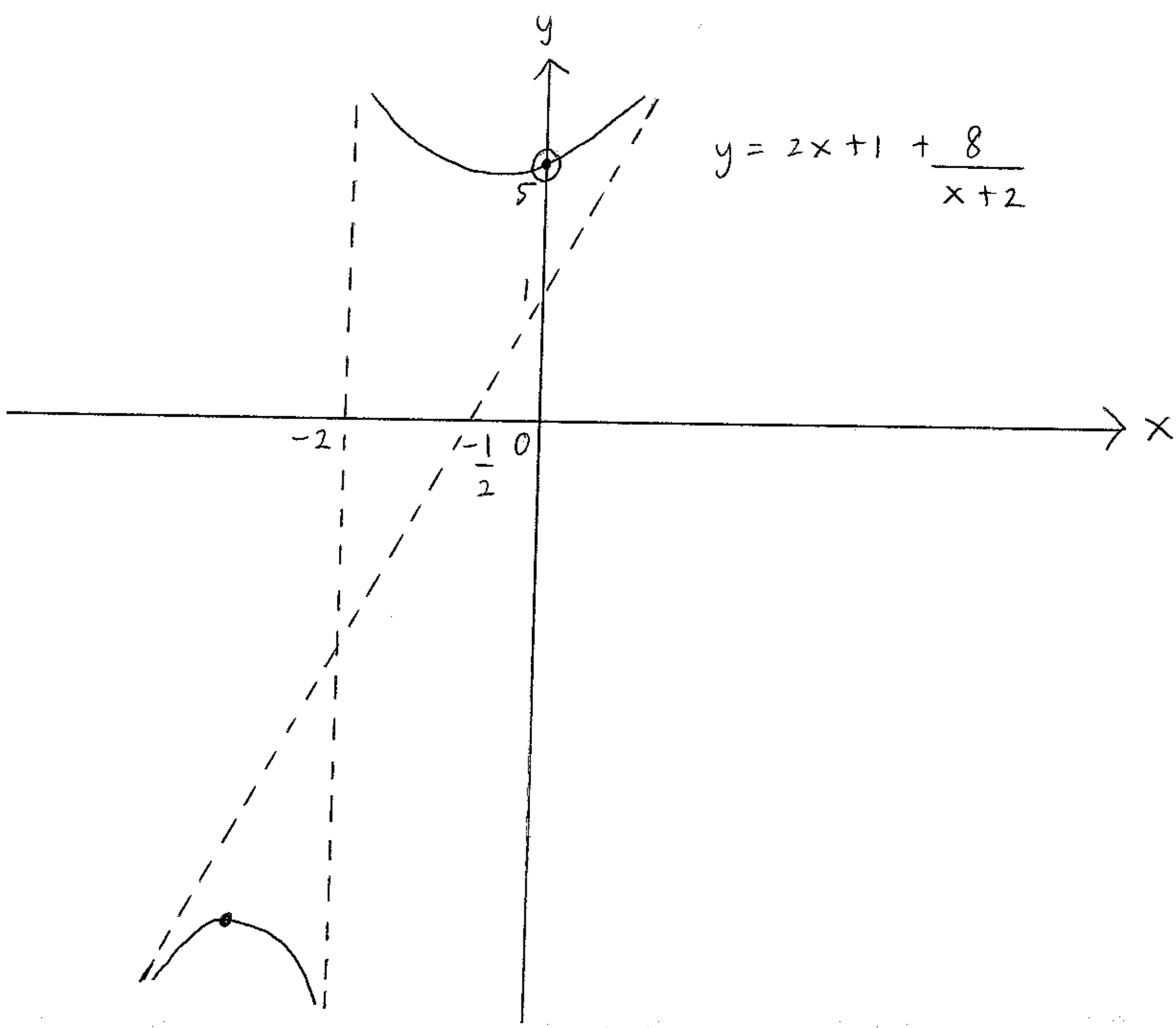
when $x = 0$: $\frac{d^2y}{dx^2} = 2 > 0$

$\therefore$ $(-4, -11)$ is a maximum point and $(0, 5)$ is a minimum point.

iii) when $x = 0$: $y = 5$

$$y = \frac{2x^2 + 5x + 10}{x+2}$$

$\neq 0$



• : critical point

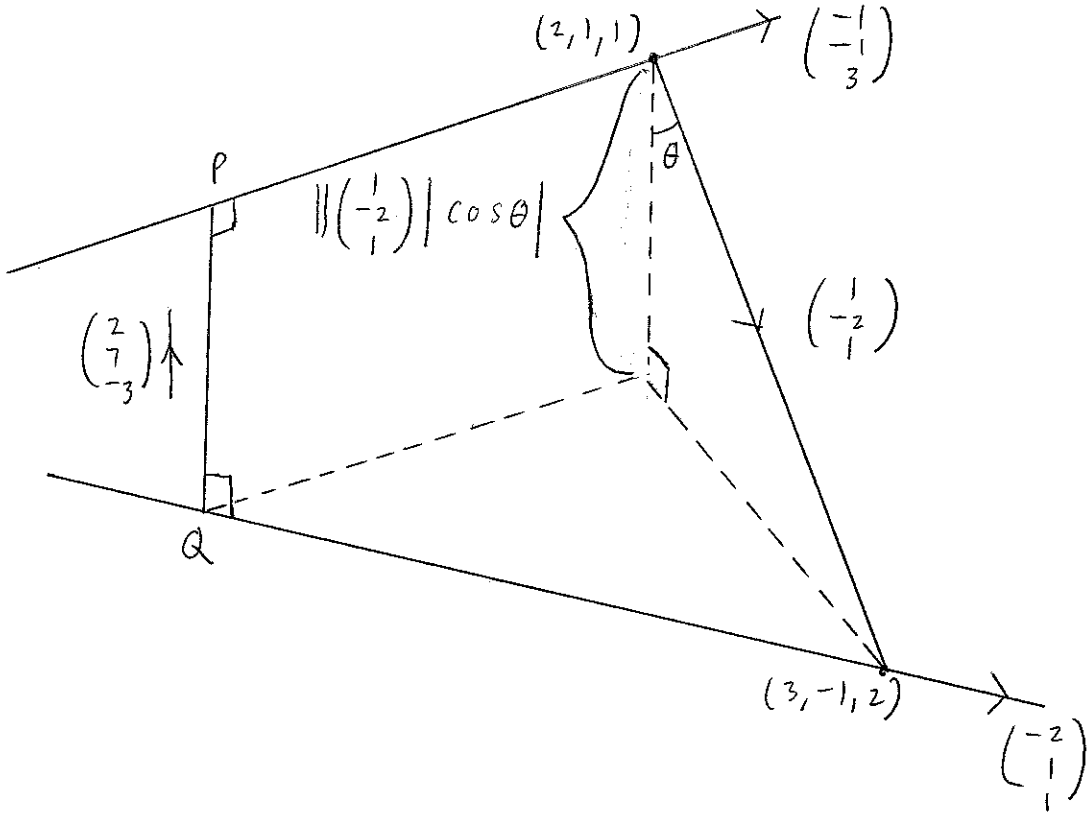
o : intersection point.

11. $\ell_1 : \underset{\sim}{r} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + t(-\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k})$

$\ell_2 : \underset{\sim}{r} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} + s(-2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

$A(3, 4, 5)$

i)



Since the direction of the perpendicular to both lines is normal to $\begin{pmatrix}-1\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}-2\\1\\1\end{pmatrix}$, it is parallel to $\begin{pmatrix}-1\\-1\\3\end{pmatrix} \times \begin{pmatrix}-2\\1\\1\end{pmatrix}$

$$\begin{pmatrix}-1\\-1\\3\end{pmatrix} \times \begin{pmatrix}-2\\1\\1\end{pmatrix} = \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ -1 & -1 & 3\\ -2 & 1 & 1\end{vmatrix} = \begin{pmatrix}2\\7\\-3\end{pmatrix}$$

$$\begin{pmatrix}3\\-1\\2\end{pmatrix} - \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$$

The shortest distance between the two lines is

$$\left|\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|\cos\theta\right|$$

$$\because \begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\7\\-3\end{pmatrix} = \left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}2\\7\\-3\end{pmatrix}\right\|\cos\theta$$

$$2-14-3 = \left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|\sqrt{2^2+7^2+(-3)^2}\cos\theta$$

$$-15 = \sqrt{62}\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|\cos\theta$$

$$\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|\cos\theta = \frac{-15}{\sqrt{62}}$$

$$\therefore \left|\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|\cos\theta\right| = \left|\frac{-15}{\sqrt{62}}\right|$$

$$= \frac{15}{\sqrt{62}}$$

$\therefore$ The shortest distance between the two lines is $\frac{15}{\sqrt{62}}$

ii) If P is the point on l_1 and Q is the point on l_2 where the common perpendicular to the lines meets l_1 and l_2, $\overrightarrow{PQ} \parallel \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$ since $\overrightarrow{PQ}$ is perpendicular to both l_1 and l_2.

Since P and Q lie on l_1 and l_2, P has the form $(2-t, 1-t, 1-3t)$ and Q has the form $(3-2s, -1+s, 2+s)$.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix} 3-2s \\ -1+s \\ 2+s \end{pmatrix} - \begin{pmatrix} 2-t \\ 1-t \\ 1-3t \end{pmatrix}$$

$$= \begin{pmatrix} t-2s+1 \\ t+s-2 \\ 3t+s+1 \end{pmatrix}$$

$$\overrightarrow{PQ} \parallel \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

$$\overrightarrow{PQ} = c\begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} t-2s+1 \\ t+s-2 \\ 3t+s+1 \end{pmatrix} = c\begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2c \\ 7c \\ -3c \end{pmatrix}$$

Equating components:

$$\left.\begin{aligned} t - 2s + 1 &= 2c \\ t + s - 2 &= 7c \\ 3t + s + 1 &= -3c \end{aligned}\right\}$$

$$t - 2s - 2c = -1 \quad — ①$$

$$t + s - 7c = 2 \quad — ②$$

$$3t + s + 3c = -1 \quad — ③$$

$-1 \times ① + ②$ $\quad t - 2s - 2c = -1$

$-3 \times ① + ③$. $\quad 3s - 5c = 3 \quad — ④$

$$7s + 9c = 2 \quad — ⑤$$

$-7 \times ④$ $\quad t - 2s - 2c = -1$

$3 \times ⑤$. $\quad -21s + 35c = -21 \quad — ⑥$

$$21s + 27c = 6 \quad — ⑦$$

$⑥ + ⑦$: $\quad \left.\begin{aligned} t - 2s - 2c &= -1 \\ -21s + 35c &= -21 \\ 62c &= -15 \end{aligned}\right\}$

$$62c = -15$$

$$c = \frac{-15}{62}$$

$$3s - 5\left(\frac{-15}{62}\right) = 3$$

$$s = \frac{37}{62}$$

$$t = 2s + 2c - 1$$

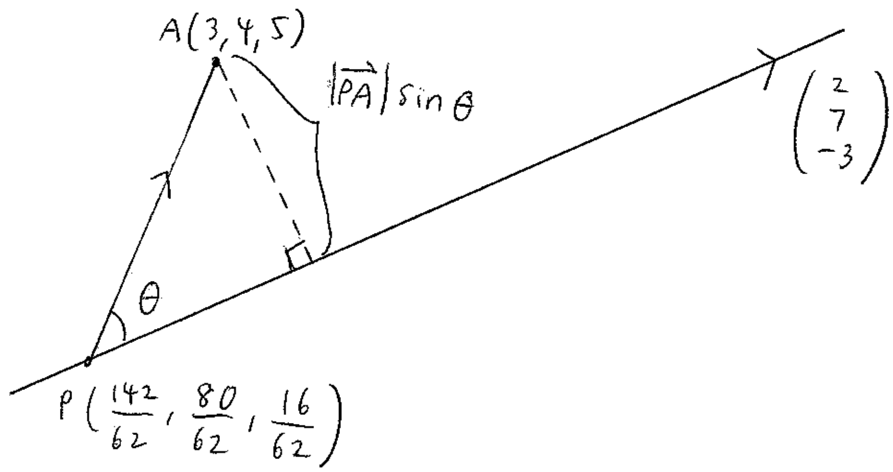
$$= \frac{74}{62} - \frac{30}{62} - 1$$

$$= \frac{-18}{62}$$

$$P\left(\frac{142}{62}, \frac{80}{62}, \frac{16}{62}\right), \quad Q\left(\frac{112}{62}, \frac{-25}{62}, \frac{161}{62}\right)$$

The common perpendicular to the lines has a vector equation $\underset{\sim}{r} = \begin{pmatrix} \frac{142}{62} \\ \frac{80}{62} \\ \frac{16}{62} \end{pmatrix} + r\begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$

iii)



The perpendicular distance from A to the common perpendicular of the lines l_1 and l_2 is $\left| |\overrightarrow{PA}| \sin\theta \right|$.

$$\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP}$$

$$= \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} \frac{142}{62} \\ \frac{80}{62} \\ \frac{16}{62} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{44}{62} \\ \frac{168}{62} \\ \frac{194}{62} \end{pmatrix}$$

$$\vec{PA} \times \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} = |\vec{PA}| \left| \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} \right| \sin\theta$$

$$\vec{PA} \times \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{44}{62} \\ \frac{168}{62} \\ \frac{194}{62} \end{pmatrix} \times \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{44}{62} & \frac{168}{62} & \frac{194}{62} \\ 2 & 7 & -3 \end{vmatrix}$$

$$= \frac{1}{62} \begin{pmatrix} -1862 \\ 520 \\ -28 \end{pmatrix}$$

$$\frac{1}{62} \begin{pmatrix} -1862 \\ 520 \\ -28 \end{pmatrix} = |\vec{PA}| \left| \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} \right| \sin\theta$$

$$= |\vec{PA}| \sqrt{2^2 + 7^2 + (-3)^2} \sin\theta$$

$$= \sqrt{62}\, |\vec{PA}| \sin\theta$$

$$\therefore |\overrightarrow{PA}| \sin\theta = \frac{1}{62\sqrt{62}} \begin{pmatrix} -1862 \\ 520 \\ -28 \end{pmatrix}$$

$$||\overrightarrow{PA}| \sin\theta| = \frac{1}{62\sqrt{62}} \sqrt{(-1862)^2 + 520^2 + (-28)^2}$$

$$= \frac{\sqrt{3738228}}{62\sqrt{62}}$$

$$= \frac{\sqrt{60294}}{62}$$

$\therefore$ The distance from A to the common perpendicular of the lines is $\frac{\sqrt{60294}}{62}$.

11. a)

$$A = \begin{pmatrix} 1 & 2 & 3 & 6 \\ -1 & -1 & -2 & -4 \\ 2 & 3 & 6 & 11 \\ 3 & 8 & 10 & 21 \end{pmatrix}$$

$$\xrightarrow[\substack{-2r_1+r_3 \\ -3r_1+r_4}]{r_1+r_2} \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 0 & -1 \\ 0 & 2 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow[-2r_2+r_4]{r_2+r_3} \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\xrightarrow{r_3+r_4} \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore \text{rank}(A) = 3$

If $A\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \underset{\sim}{0}$, $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in R^4$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 6 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = t$, $t \in R$

$\therefore z = -t \quad y = -t \quad x = -t$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ -t \\ t \end{pmatrix}$$

$$= t \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

The nullspace of A is $t \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \; t \in R$.

b) i) Since the square matrices A and B have eigenvalues λ and μ respectively,

$$\begin{aligned}(A + B + kI)\underset{\sim}{x} &= A\underset{\sim}{x} + B\underset{\sim}{x} + kI\underset{\sim}{x} \\ &= A\underset{\sim}{x} + B\underset{\sim}{x} + k\underset{\sim}{x} \\ &= \lambda\underset{\sim}{x} + \mu\underset{\sim}{x} + k\underset{\sim}{x} \\ &= (\lambda + \mu + k)\underset{\sim}{x}\end{aligned}$$

$\therefore \lambda + \mu + k$ is an eigenvalue of the matrix $A + B + kI$.

ii) $C = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 8 \\ 0 & 0 & -3 \end{pmatrix}$

$$C - \lambda I = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 8 \\ 0 & 0 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 8 \\ 0 & 0 & -3-\lambda \end{pmatrix}$$

$$|C - \lambda I| = (1-\lambda) \begin{vmatrix} 2-\lambda & 8 \\ 0 & -3-\lambda \end{vmatrix} - 0 + 0$$

$$= (1-\lambda)(2-\lambda)(-3-\lambda)$$

If $|C - \lambda I| = 0$

$\therefore (1-\lambda)(2-\lambda)(-3-\lambda) = 0$

$\lambda = 1, 2, -3$

$\lambda = 1$: $\begin{pmatrix} 0 & 3 & 4 \\ 0 & 1 & 8 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 0 & 3 & 4 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 0 & 1 & 8 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{-3r_1+r_2} \left(\begin{array}{ccc|c} 0 & 1 & 8 & 0 \\ 0 & 0 & -20 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-20},\ \frac{r_3}{-4}} \left(\begin{array}{ccc|c} 0 & 1 & 8 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|c} 0 & 1 & 8 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = 0 \quad y = 0 \quad 0x = 0$

Let $x = s,\ s \in R$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 2$:

$$\begin{pmatrix} -1 & 3 & 4 \\ 0 & 0 & 8 \\ 0 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 3 & 4 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{r_2}{8},\ \frac{r_3}{-5}]{-1\times r_1,} \left(\begin{array}{ccc|c} 1 & -3 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|c} 1 & -3 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = 0 \quad x - 3y = 0$

Let $y = s,\ s \in R$

$\therefore\ x = 3s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3s \\ s \\ 0 \end{pmatrix}$$

$$= s \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = -3$.
$$\begin{pmatrix} 4 & 3 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 4 & 3 & 4 & 0 \\ 0 & 5 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 5s,\ s \in R$

$\therefore\ y = -8s$

$4x + 3y + 4z = 0$

$4x - 24s + 20s = 0$

$x = s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -8s \\ 5s \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ -8 \\ 5 \end{pmatrix}$$

$\therefore$ The eigenvalues of C are $1, 2, -3$ and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \\ 5 \end{pmatrix}$.