3. a) $$x\frac{dy}{dx} + 2y = \frac{2\sin x}{x\cos^3 x}$$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{2\sin x}{x^2\cos^3 x}$$

$$e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$x^2\frac{dy}{dx} + 2xy = \frac{2\sin x}{\cos^3 x}$$

$$\frac{d}{dx}(x^2y) = \frac{2\sin x}{\cos^3 x}$$

$$x^2y = \int \frac{2\sin x}{\cos^3 x}\,dx$$

$$= \int 2\tan x\sec^2 x\,dx$$

$$= \sec^2 x + c$$

$$y = \frac{\sec^2 x}{x^2} + \frac{c}{x^2}$$

b) $$x\frac{dy}{dx} - y = \frac{1}{4}x^2 - y^2 \quad , \; y = 0 \;\; x = 1$$

$$y = ux$$

$$\frac{dy}{dx} = u + x\frac{dy}{dx}$$

$$x\left(u + x\frac{dy}{dx}\right) - ux = \frac{1}{4}x^2 - u^2x^2$$

$$ux + x^2\frac{du}{dx} - ux = \left(\frac{1}{4} - u^2\right)x^2$$

$$x^2\frac{du}{dx} = \left(\frac{1}{4} - u^2\right)x^2$$

$$\frac{du}{dx} = \frac{1}{4} - u^2$$

$$= \frac{1 - 4u^2}{4}$$

$$\int \frac{4}{1 - 4u^2}\,du = \int 1\,dx$$

$$2\int \frac{1}{1 - 2u} + \frac{1}{1 + 2u}\,du = x + c$$

$$\ln|1 + 2u| - \ln|1 - 2u| = x + c$$

$$\ln\left|\frac{1 + 2u}{1 - 2u}\right| = x + c$$

$$\ln\left|\frac{x + 2y}{x - 2y}\right| = x + c$$

$$y = 0 \quad x = 1 \quad 2\ln 1 = 1 + c$$

$$c = -1$$

$$\ln\left|\frac{x + 2y}{x - 2y}\right| = x - 1$$

$$\ln\left|\frac{x + 2y}{x - 2y}\right| = x - 1$$

$$\frac{x + 2y}{x - 2y} = e^{x-1}$$

5. $\dfrac{dy}{dx} + 2y = \sin 3x$

$$e^{\int 2\,dx} = e^{2x}$$

$$e^{2x}\frac{dy}{dx} + 2ye^{2x} = e^{2x}\sin 3x$$

$$\frac{d}{dx}(ye^{2x}) = e^{2x}\sin 3x$$

$$ye^{2x} = \int e^{2x}\sin 3x\,dx$$

$u = e^{2x}$ $\quad$ $dv = \sin 3x\,dx$

$du = 2e^{2x}dx$ $\quad$ $v = \dfrac{-\cos 3x}{3}$

$$= \frac{-e^{2x}\cos 3x}{3} + \frac{2}{3}\int e^{2x}\cos 3x\,dx$$

$u = e^{2x}$ $\quad$ $dv = \cos 3x\,dx$

$du = 2e^{2x}dx$ $\quad$ $v = \dfrac{\sin 3x}{3}$

$$= \frac{-e^{2x}\cos 3x}{3} + \frac{2}{3}\left(\frac{e^{2x}\sin 3x}{3} - 2\int \frac{e^{2x}\sin 3x\,dx}{3}\right)$$

$$= \frac{2e^{2x}\sin 3x}{9} - \frac{e^{2x}\cos 3x}{3} - \frac{4}{9}\int e^{2x}\sin 3x\,dx$$

$$\frac{13}{9}\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{9}(2\sin 3x - 3\cos 3x)$$

$$\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

$$ye^{2x} = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

$$y = \frac{2\sin 3x - 3\cos 3x}{13} + ce^{-2x}$$

$$\frac{d^2z}{dx^2} + \frac{2dz}{dx} = 3\cos 3x$$

$$\frac{d}{dx}\left(\frac{dz}{dx} + 2z\right) = \frac{d}{dx}(\sin 3x)$$

$$\frac{dz}{dx} + 2z = \sin 3x$$

$$z = \frac{2\sin 3x - 3\cos 3x}{13} + k + ce^{-2x}$$

$$\frac{dz}{dx} = \frac{6\cos 3x + 9\sin 3x}{13} - 2ce^{-2x}$$

$x = 0 \quad z = 1 \quad 1 = \frac{-3}{13} + c + k$

$x = 0 \quad \frac{dz}{dx} = 2: \quad 2 = \frac{6}{13} - 2c \qquad c = \frac{-10}{13} \quad k = 2$

$$z = \frac{2\sin 3x - 3\cos 3x}{13} + 2 - \frac{10}{13}e^{-2x}$$

$$\frac{d^2z}{dx^2} + \frac{2dz}{dx} = 0$$

$m^2 + 2m = 0$

$m(m + 2) = 0$

$m = 0, -2$

$z_c = A + Be^{-2x}$

$z_p = C\cos 3x + D\sin 3x$

$$\frac{dz_p}{dx} = -3C\sin 3x + 3D\cos 3x$$

$$\frac{d^2z_p}{dx^2} = -9C\cos 3x - 9D\sin 3x$$

$$\frac{d^2 z_p}{dx^2} + \frac{2 d z_p}{dx} = -9C\cos 3x - 9D\sin 3x$$

$$-6C\sin 3x + 6D\cos 3x$$

$$= (-6C - 9D)\sin 3x + (-9C + 6D)\cos 3x$$

$$= 3\cos 3x$$

$-6C - 9D = 0$ $\quad$ $-9C + 6D = 3$

$C = \frac{-3D}{2}$ $\quad$ $-3C + 2D = 1$

$$\frac{9D}{2} + 2D = 1$$

$$9D + 4D = 2$$

$$13D = 2$$

$$D = \frac{2}{13}$$

$$C = \frac{-3}{13}$$

$$z_p = \frac{2\sin 3x - 3\cos 3x}{13}$$

$$z = z_c + z_p$$

$$= A + Be^{-2x} + \frac{2\sin 3x - 3\cos 3x}{13}$$

$$\frac{dz}{dx} = -2Be^{-2x} + \frac{6\cos 3x + 9\sin 3x}{13}$$

$x = 0$ $z = 1$: $1 = A + B - \frac{3}{13}$

$x = 0$ $\frac{dz}{dx} = 2$: $2 = -2B + \frac{6}{13}$

$$2B = \frac{-20}{13}$$

$$B = \frac{-10}{13}$$

$$A = 2$$

$$z = 2 - \frac{10e^{-2x}}{13} + \frac{2\sin 3x - 3\cos 3x}{13}$$

10. $$\frac{d^2x}{dt^2} + \frac{2dx}{dt} + 5x = 13\sin 3t$$

$$\frac{d^2x}{dt^2} + \frac{2dx}{dt} + 5x = 0$$

$$m^2 + 2m + 5 = 0$$

$$(m+1)^2 + 1 = 0$$

$$(m+1)^2 = -1$$

$$m + 1 = \pm i$$

$$m = -1 \pm i$$

$$x_c = e^{-t}(A\cos t + B\sin t)$$

$$x_p = C\cos 3t + D\sin 3t$$

$$\frac{dx_p}{dt} = -3C\sin 3t + 3D\cos 3t$$

$$\frac{d^2x_p}{dt^2} = -9C\cos 3t - 9D\sin 3t$$

$$\frac{d^2x_p}{dt^2} + \frac{2dx_p}{dt} + 5x_p$$

$$= -9C\cos 3t - 9D\sin 3t$$

$$+ 2(-3C\sin 3t + 3D\cos 3t)$$

$$+ 5(C\cos 3t + D\sin 3t)$$

$$= -9C\cos 3t - 9D\sin 3t$$

$$-6C\sin 3t + 6D\cos 3t$$

$$+ 5C\cos 3t + 5D\sin 3t$$

$$= (-4C + 6D)\cos 3t + (-6C - 4D)\sin 3t$$

$$= 13\sin 3t$$

$$-4C + 6D = 0 \qquad -6C - 4D = 13$$

$$C = \frac{3D}{2}$$

$$-6\left(\frac{3D}{2}\right) - 4D = 13$$

$$-9D - 4D = 13$$

$$-13D = 13$$

$$D = -1$$

$$C = \frac{-3}{2}$$

$$x_p = \frac{-3\cos 3t}{2} - \sin 3t$$

$$x = x_c + x_p$$

$$= e^{-t}(A\cos t + B\sin t)$$

$$\frac{-3\cos 3t}{2} - \sin 3t$$

$$\frac{dx}{dt} = e^{-t}(-A\sin t + B\cos t)$$

$$-e^{-t}(A\cos t + B\sin t)$$

$$+\frac{9\sin 3t}{2} - 3\cos 3t$$

$x = -\frac{3}{2}$ and $\frac{dx}{dt} = -4$ when $t = 0$

$t = 0 \quad x = \frac{-3}{2} \qquad \frac{-3}{2} = A - \frac{3}{2}$

$t = 0 \quad \frac{dx}{dt} = -4 \qquad -4 = B - A - 3$

$$A = 0$$

$$B = -1$$

$$x = -e^{-t}\sin 3t - \frac{3\cos 3t}{2} - \sin 3t$$

11. $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3\cos x$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

$$m^2 - m - 6 = 0$$

$$(m - 3)(m + 2) = 0$$

$$m = 3, -2$$

$$y_c = Ae^{3x} + Be^{-2x}$$

$$y_p = C\cos x + D\sin x$$

$$\frac{dy_p}{dx} = -C\sin x + D\cos x$$

$$\frac{d^2y_p}{dx^2} = -C\cos x - D\sin x$$

$$\frac{d^2y_p}{dx^2} - \frac{dy_p}{dx} - 6y_p$$

$$= -C\cos x - D\sin x$$

$$-(-C\sin x + D\cos x)$$

$$-6(C\cos x + D\sin x)$$

$= -C\cos x - D\sin x$

$+ C\sin x - D\cos x$

$- 6C\cos x - 6D\sin x$

$= (-7C - D)\cos x + (C - 7D)\sin x$

$= 3\cos x$

$-7C - D = 3 \qquad C - 7D = 0$

$C = 7D$

$-7(7D) - D = 3$

$-49D - D = 3$

$-50D = 3$

$D = \frac{-3}{50}$

$C = \frac{-21}{50}$

$y_p = \frac{-21\cos x}{50} - \frac{3\sin x}{50}$

$y = y_c + y_p$

$= Ae^{3x} + Be^{-2x}$

$\frac{-21\cos x}{50} - \frac{3\sin x}{50}$

$$\frac{dy}{dx} = 3Ae^{3x} - 2Be^{-2x} + \frac{21\sin x}{50} - \frac{3\cos x}{50}$$

$$y = \frac{29}{50} \quad \text{and} \quad \frac{dy}{dx} = \frac{191}{150} \quad \text{at} \quad x = 0$$

$$x = 0 \quad y = \frac{29}{50} \cdot \frac{29}{50} = A + B - \frac{21}{50}$$

$$x = 0 \quad \frac{dy}{dx} = \frac{191}{150} \cdot \frac{191}{150} = 3A - 2B - \frac{3}{50}$$

$$A + B = 1$$

$$3A - 2B = \frac{4}{3}$$

$$2A + 2B = 2$$

$$5A = \frac{10}{3}$$

$$A = \frac{2}{3}$$

$$B = \frac{1}{3}$$

$$y = \frac{2e^{3x}}{3} + \frac{e^{-2x}}{3} - \frac{21\cos x}{50} - \frac{3\sin x}{50}$$

13. $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = 10$ , $x = 2$ $\dfrac{dx}{dt} = 2$ $t = 0$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$

$$m^2 + 2m + 5 = 0$$

$$(m+1)^2 + 4 = 0$$

$$(m+1)^2 = -4$$

$$m + 1 = \pm 2i$$

$$m = -1 \pm 2i$$

$$x_c = e^{-t}(A\cos 2t + B\sin 2t)$$

$$x_p = C$$

$$\frac{dx_p}{dt} = 0$$

$$\frac{d^2x_p}{dt^2} = 0$$

$$\frac{d^2x_p}{dt^2} + 2\frac{dx_p}{dt} + 5x_p = 0 + 0 + 5C = 10$$

$$C = 2$$

$$x_p = 2$$

$$x = x_c + x_p$$

$$= e^{-t}(A\cos 2t + B\sin 2t) + 2$$

$$\frac{dx}{dt} = e^{-t}(-2A\sin 2t + 2B\cos 2t)$$

$$-e^{-t}(A\cos 2t + B\sin 2t)$$

$t = 0 \quad x = 2 \; : \; 2 = A + 2$

$t = 0 \quad \frac{dx}{dt} = 2 : \; 2 = 2B - A$

$$A = 0$$

$$B = 1$$

$$x = e^{-t} \sin 2t + 2$$

As $t \to \infty \quad x \to 2$

17 - a) $\dfrac{dy}{dx} = y\cot x$ , $y = 1$ $x = \dfrac{\pi}{6}$

$$\int \frac{1}{y}\,dy = \int \frac{\cos x}{\sin x}\,dx$$

$$\ln y = \ln|\sin x| + c$$

$$y = 1\ x = \frac{\pi}{6}:\quad 0 = \ln\frac{1}{2} + c$$

$$c = -\ln 2$$

$$\ln y = \ln|\sin x| - \ln 2$$
$$= \ln\left|\frac{\sin x}{2}\right|$$

$$y = \frac{\sin x}{2}$$

b) $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - 2x = 3e^{-t}$ , $x = 1$ $t = 0$

$\lim_{t \to \infty} x = 0$

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$$

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$$m = 2, -1$$

$$x_c = Ae^{2t} + Be^{-t}$$

$$x_p = Cte^{-t}$$

$$\frac{dx_p}{dt} = Ce^{-t} - cte^{-t}$$

$$\frac{d^2x_p}{dt^2} = -Ce^{-t} - Ce^{-t} + Cte^{-t}$$
$$= Cte^{-t} - 2Ce^{-t}$$

$$\frac{d^2x_p}{dt^2} - \frac{dx_p}{dt} - 2x_p = Cte^{-t} - 2Ce^{-t} - Ce^{-t} + Cte^{-t} - 2Cte^{-t}$$

$$= -3Ce^{-t}$$

$$= 3e^{-t}$$

$$-3C = 3$$

$$C = -1$$

$$x_p = -te^{-t}$$

$$x = x_c + x_p$$

$$= Ae^{2t} + Be^{-t} - te^{-t}$$

$t = 0 \quad x = 1: \quad 1 = A + B$

$t \to \infty \quad x \to 0: \quad 0 = A$

$$B = 1$$

$$x = (1 - t)e^{-t}$$

18. $\frac{dx}{dt} = 3x - 5y \quad \frac{dy}{dt} = 2x - 4y$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(3x - 5y)$$

$$\frac{d^2x}{dt^2} = \frac{3dx}{dt} - \frac{5dy}{dt}$$

$$= -\frac{dx}{dt} + \frac{4dx}{dt} - 5(2x - 4y)$$

$$= -\frac{dx}{dt} + 4(3x - 5y) - 5(2x - 4y)$$

$$= -\frac{dx}{dt} + 12x - 20y - 10x + 20y$$

$$= -\frac{dx}{dt} + 2x$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 0$$

$$m^2 + m - 2 = 0$$

$$(m + 2)(m - 1) = 0$$

$$m = 1, -2$$

$$x = Ae^{t} + Be^{-2t}$$

$$\frac{dx}{dt} = Ae^{t} - 2Be^{-2t}$$

when $t = 0$, $x = 7$ and $\frac{dy}{dx} = 1$.

$t = 0 \quad x = 7 : \quad 7 = A + B$

$t = 0 \quad \frac{dy}{dx} = 1 : \quad 1 = A - 2B$

$$3B = 6$$

$$B = 2$$

$$A = 5$$

$$x = 5e^{t} + 2e^{-2t}$$

$$\frac{dx}{dt} = 5e^{t} - 4e^{-2t}$$

$$\frac{dx}{dt} = 3x - 5y$$

$$5e^{t} - 4e^{-2t} = 3(5e^{t} + 2e^{-2t}) - 5y$$

$$= 15e^{t} + 6e^{-2t} - 5y$$

$$5y = 10e^{t} + 10e^{-2t}$$

$$y = 2e^{t} + 2e^{-2t}$$

$$\frac{y}{x} = \frac{2e^{t} + 2e^{-2t}}{5e^{t} + 2e^{-2t}}$$

As $t \longrightarrow \infty$, $\frac{2e^{t} + 2e^{-2t}}{5e^{t} + 2e^{-2t}} \quad \frac{2e^{t}}{5e^{t}} = \frac{2}{5}$

$\therefore \frac{y}{x} \rightarrow \frac{2}{5}$

or $2x \sim 5y$

20. $\frac{d^2y}{dx^2} + 4y = 3\cos x$ , $y = \frac{dy}{dx} = 0$ when $x = 0$

$$\frac{d^2y}{dx^2} + 4y = 0$$

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$y_c = A\cos 2x + B\sin 2x$$

$$y_p = C\cos x + D\sin x$$

$$\frac{dy_p}{dx} = -C\sin x + D\cos x$$

$$\frac{d^2y_p}{dx^2} = -C\cos x - D\sin x$$

$$\frac{d^2y_p}{dx^2} + 4y_p = -C\cos x - D\sin x$$

$$+ 4(C\cos x + D\sin x)$$

$$= -C\cos x - D\sin x$$

$$+ 4C\cos x + 4D\sin x$$

$$= 3C\cos x + 3D\sin x$$

$$= 3\cos x$$

$3C = 3 \quad 3D = 0$

$C = 1 \quad D = 0$

$y_p = \cos x$

$y = y_c + y_p$

$= A\cos 2x + B\sin 2x + \cos x$

$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x - \sin x$

$x = 0 \quad y = 0 : \quad 0 = A + 1$

$x = 0 \quad \frac{dy}{dx} = 0 : \quad 0 = 2B$

$A = -1$

$B = 0$

$y = -\cos 2x + \cos x$

when $y = 0 : \quad -\cos 2x + \cos x = 0$

$1 - 2\cos^2 x + \cos x = 0$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2}, 1$

$x = 2k\pi, 2k\pi \pm \frac{2\pi}{3}, \quad k \in \mathbb{Z}$

$\frac{dy}{dx} = 2\sin 2x - \sin x$

When $\frac{dy}{dx} = 0$ : $2\sin 2x - \sin x = 0$

$4\sin x \cos x - \sin x = 0$

$\sin x (4\cos x - 1) = 0$

$\sin x = 0, \ \cos x = \frac{1}{4}$

$x = 0, \pi, 2\pi$

$\cos x = \pm 1$

$$y = \cos x - \cos 2x$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots - \left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \dots\right)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots - \left(1 - 2x^2 + \frac{2x^4}{3} - \dots\right)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots - 1 + 2x^2 - \frac{2x^4}{3} + \dots$$

$$= \frac{3x^2}{2} - \frac{5x^4}{8} + \dots$$

$$= \frac{x^2}{2}\left(3 - \frac{5x^2}{4}\right) + \dots$$

$$\approx \frac{x^2}{2}\left(3 - \frac{5x^2}{4}\right), \text{ if } x \text{ is small}$$

21. a) $x\frac{dy}{dx} - 2y = x^4 \cos 2x$

$$\frac{dy}{dx} - \frac{2y}{x} = x^3 \cos 2x$$

$$e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = x^{-2}$$

$$x^{-2}\frac{dy}{dx} - 2x^{-3}y = x\cos 2x$$

$$\frac{d}{dx}(x^{-2}y) = x\cos 2x$$

$$x^{-2}y = \int x\cos 2x\,dx$$

$u = x \qquad dv = \cos 2x\,dx$

$du = dx \qquad v = \frac{\sin 2x}{2}$

$$= \frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}\,dx$$

$$= \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + c$$

$$y = \frac{x^3\sin 2x}{2} + \frac{x^2\cos 2x}{4} + cx^2.$$

b) $(x^2 + y^2)\frac{dy}{dx} = xy$

$y = xu$

$$\frac{dy}{dx} = u + x\frac{dy}{dx}$$

$$(x^2 + x^2u^2)\left(u + x\frac{dy}{dx}\right) = x^2u$$

$$x^2(1 + u^2)\left(u + x\frac{dy}{dx}\right) = x^2u$$

$$(1+u^2)\left(u+x\frac{du}{dx}\right)=u$$

$$u+x\frac{du}{dx}+u^3+u^2x\frac{du}{dx}=u$$

$$x(1+u^2)\frac{du}{dx}+u^3=0$$

$$\int\left(\frac{1}{u^3}+\frac{1}{u}\right)du+\int\frac{dx}{x}=0$$

$$\frac{-1}{2u^2}+\ln u+\ln x=C$$

$$\ln ux=C+\frac{1}{2u^2}$$

$$\ln y=C+\frac{1}{2u^2}$$

$$=c+\frac{x^2}{2y^2}$$

$$y=e^{c+\frac{x^2}{2y^2}}$$

$x=1 \quad y=1 \qquad 1=e^{c+\frac{1}{2}}$

$$c+\frac{1}{2}=0$$

$$c=-\frac{1}{2}$$

$$\ln y=\frac{x^2}{2y^2}-\frac{1}{2}$$

$$2y^2\ln y=x^2-y^2$$

$$x^2=y^2(1+2\ln y)$$

$$x=y(1+2\ln y)^{\frac{1}{2}}$$

22. $$\frac{d^2x}{dt^2} + \frac{4dx}{dt} + 13x = 80\sin 3t$$

$$\frac{d^2x}{dt^2} + \frac{4dx}{dt} + 13x = 0$$

$$m^2 + 4m + 13 = 0$$

$$(m+2)^2 + 9 = 0$$

$$(m+2)^2 = -9$$

$$m + 2 = \pm 3i$$

$$m = -2 \pm 3i$$

$$x_c = e^{-2t}(A\cos 3t + B\sin 3t)$$

$$x_p = C\cos 3t + D\sin 3t$$

$$\frac{dx_p}{dt} = -3C\sin 3t + 3D\cos 3t$$

$$\frac{d^2x_p}{dt^2} = -9C\cos 3t - 9D\sin 3t$$

$$\frac{d^2x_p}{dt^2} + \frac{4dx_p}{dt} + 13x_p$$

$$= -9C\cos 3t - 9D\sin 3t$$

$$+ 4(-3C\sin 3t + 3D\cos 3t)$$

$$+ 13(C\cos 3t + D\sin 3t)$$

$$= -9C\cos 3t - 9D\sin 3t$$
$$-12C\sin 3t + 12D\cos 3t$$
$$+13C\cos 3t + 13D\sin 3t$$

$$= (4C + 12D)\cos 3t + (-12C + 4D)\sin 3t$$

$$= 80\sin 3t$$

$$4C + 12D = 0 \qquad -12C + 4D = 80$$
$$C = -3D \qquad -12(-3D) + 4D = 80$$
$$36D + 4D = 80$$
$$40D = 80$$
$$D = 2$$
$$C = -6$$

$$x_p = 2\sin 3t - 6\cos 3t$$

$$x = x_c + x_p$$
$$= e^{-2t}(A\cos 3t + B\sin 3t)$$
$$+ 2\sin 3t - 6\cos 3t$$

$$\frac{dx}{dt} = e^{-2t}(-3A\sin 3t + 3B\cos 3t)$$
$$-e^{-2t}(A\cos 3t + B\sin 3t)$$
$$+ 6\cos 3t + 18\sin 3t$$

when $t = 0$, $x = 0$ and $\frac{dx}{dt} = -6$

$t = 0 \quad x = 0 \qquad 0 = A - 6$

$t = 0 \quad \frac{dx}{dt} = -6 : \quad -6 = 3B - A + 6$

$A = 6$

$B = -2$

$$x = e^{-2t}(6\cos 3t - 2\sin 3t)$$

$$+ 2\sin 3t - 6\cos 3t.$$

28 a) $y = \tan(3\tan^{-1}x)$

$u = 3\tan^{-1}x$

$x = \tan \frac{u}{3}$

$\frac{dx}{du} = \frac{1}{3}\sec^2\frac{u}{3}$

$= \frac{1}{3}\left(1 + \tan^2\frac{u}{3}\right)$

$= \frac{1}{3}(1+x^2)$

$\frac{du}{dx} = \frac{3}{1+x^2}$

$y = \tan u$

$\frac{dy}{du} = \sec^2 u$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \sec^2 u \left(\frac{3}{1+x^2}\right)$

$= \frac{3\sec^2 u}{1+x^2}$

$= \frac{3\sec^2(3\tan^{-1}x)}{1+x^2}$

$(1+x^2)\frac{dy}{dx} = 3\sec^2(3\tan^{-1}x)$

$= 3(1 + \tan^2(3\tan^{-1}x))$

$= 3(1+y^2)$

b) $x\frac{dy}{dx} = 2y + x^2 \ln x$ , $y = 2$ $x = 1$

$$\frac{dy}{dx} - \frac{2y}{x} = x\ln x$$

$$e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

$$x^{-2}\frac{dy}{dx} - 2yx^{-3} = x^{-1}\ln x$$

$$\frac{d}{dx}(yx^{-2}) = \frac{\ln x}{x}$$

$$\frac{y}{x^2} = \int \frac{\ln x}{x}\,dx$$

$$u = \ln x \qquad dv = \frac{1}{x}dx$$

$$du = \frac{dx}{x} \qquad v = \ln x$$

$$= \ln^2 x - \int \frac{\ln x}{x}\,dx$$

$$2\int \frac{\ln x}{x}\,dx = \ln^2 x$$

$$\int \frac{\ln x}{x}\,dx = \frac{\ln^2 x}{2} + c$$

$$\frac{y}{x^2} = \frac{\ln^2 x}{2} + c$$

$$y = \frac{x^2 \ln^2 x}{2} + cx^2$$

$x = 1$ $y = 2$: $2 = c$

$$y = \frac{x^2 \ln^2 x}{2} + 2x^2$$

29. $y = f(x),\ x > 0$

a) $x = \sqrt{t}$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{dt}{dx} = 2\sqrt{t}$$

$$\frac{dy}{dx}\frac{dt}{dy} = 2\sqrt{t}$$

$$\frac{dy}{dx} = 2\sqrt{t}\frac{dy}{dt}$$

b) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(2\sqrt{t}\frac{dy}{dt}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left(2\sqrt{t}\frac{dy}{dt}\right)$$

$$= 2\sqrt{t}\frac{d}{dt}\left(2\sqrt{t}\frac{dy}{dt}\right)$$

$$= 2\sqrt{t}\left(2\sqrt{t}\frac{d}{dt}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dt}(2\sqrt{t})\right)$$

$$= 2\sqrt{t}\left(2\sqrt{t}\frac{d^2y}{dt^2} + \frac{dy}{dt}\left(\frac{1}{\sqrt{t}}\right)\right)$$

$$= 4t\frac{d^2y}{dt^2} + \frac{2dy}{dt}$$

$$\frac{d^2y}{dx^2} - \frac{1}{x}\left(\frac{dy}{dx}\right) + 4x^2(9y + 6) = 0$$

$$x = \sqrt{t}$$

$$\frac{dy}{dx} = 2\sqrt{t}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = 4t\frac{d^2y}{d\ ^2} + \frac{2dy}{dt}$$

$$4t\frac{d^2y}{dt^2} + \frac{2dy}{dt} - \frac{1}{\sqrt{t}}\left(2\sqrt{t}\frac{dy}{dt}\right) + 4t(9y + 6) = 0$$

$$4t\frac{d^2y}{dt^2} + \frac{2dy}{dt} - \frac{2dy}{dt} + 4t(9y + 6) = 0$$

$$4t\frac{d^2y}{dt^2} + 4t(9y + 6) = 0$$

$$4t\left(\frac{d^2y}{dt^2} + 9y + 6\right) = 0$$

$$\frac{d^2y}{dt^2} + 9y + 6 = 0$$

$$\frac{d^2y}{dt^2} + 9y = -6$$

$$\frac{d^2y}{dt^2} + 9y = 0$$

$$m^2 + 9 = 0$$

$$m = \pm 3i$$

$$y_c = A\cos 3t + B\sin 3t$$

$$y_p = C$$

$$\frac{dy_p}{dt} = 0$$

$$\frac{d^2y_p}{dt^2} = 0$$

$$\frac{d^2y_p}{dt^2} + 9y_p = 0 + 9C$$

$$= 9C$$

$$= 6$$

$$C = \frac{2}{3}$$

$$y_p = \frac{2}{3}$$

$$y = y_c + y_p$$

$$= A\cos 3t + B\sin 3t + \frac{2}{3}$$

$$= A\cos 3x^2 + B\sin 3x^2 + \frac{2}{3}$$

30. a) $\frac{d^2x}{dt^2} + \frac{5dx}{dt} + 4x = 4t + 5$, $t=0$ $x=0$ $\frac{dx}{dt}=1$

$$\frac{d^2x}{dt^2} + \frac{5dx}{dt} + 4x = 0$$

$$m^2 + 5m + 4 = 0$$

$$(m+1)(m+4) = 0$$

$$m = -1, -4$$

$$x_c = Ae^{-t} + Be^{-4t}$$

$$x_p = Ct + D$$

$$\frac{dx_p}{dt} = C$$

$$\frac{d^2x_p}{dt^2} = 0$$

$$\frac{d^2x_p}{dt^2} + \frac{5dx_p}{dt} + 4x_p = 0 + 5C + 4(Ct + D)$$

$$= 4Ct + 5C + 4D$$

$$= 4t + 5$$

$4C = 4$ $\quad$ $5C + 4D = 5$

$C = 1$ $\quad$ $D = 0$

$$x_p = t$$

$$x = x_c + x_p$$

$$= Ae^{-t} + Be^{-4t} + t$$

$$\frac{dx}{dt} = -Ae^{-t} - 4Be^{-4t} + 1$$

$t=0$ $x=0$: $0 = A + B$

$t=0$ $\frac{dx}{dt}=1$: $1 = -A - 4B + 1$

$$A + B = 0$$
$$-A - 4B = 0$$
$$-3B = 0$$
$$B = 0$$
$$A = 0$$

$$x = t$$

b) $(x^2+1)\frac{dy}{dx} + y = \tan^{-1}x$

$$\frac{dy}{dx} + \frac{y}{x^2+1} = \frac{\tan^{-1}x}{x^2+1}$$

$$e^{\int \frac{1}{x^2+1}dx} = e^{\tan^{-1}x}$$

$$e^{\tan^{-1}x}\frac{dy}{dx} + \frac{ye^{\tan^{-1}x}}{x^2+1} = \frac{e^{\tan^{-1}x}\tan^{-1}x}{x^2+1}$$

$$\frac{d}{dx}(ye^{\tan^{-1}x}) = \frac{e^{\tan^{-1}x}\tan^{-1}x}{x^2+1}$$

$$ye^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}\tan^{-1}x}{x^2+1}dx$$

$$s = \tan^{-1}x$$
$$ds = \frac{1}{1+x^2}dx$$

$$= \int se^s\,ds$$

$u = s \qquad dv = e^s ds$

$du = ds \qquad v = e^s$

$$= se^s - \int e^s \, ds$$

$$= se^s - e^s + c$$

$$= \tan^{-1}x \, e^{\tan^{-1}x} - e^{\tan^{-1}x} + c$$

$$y = \tan^{-1}x - 1 + ce^{-\tan^{-1}x}$$

31. $\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + x = t$

$\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + x = 0$

$m^2 + 2km + 1 = 0$

$m^2 + 2km + k^2 = k^2 - 1$

$(m + k)^2 = k^2 - 1$

$m + k = \pm\sqrt{k^2 - 1}$

$m = -k \pm \sqrt{k^2 - 1}$

i) $k > 1$: $m = -k + \sqrt{k^2 - 1}, -k - \sqrt{k^2 - 1}$

$x_c = Ae^{(-k + \sqrt{k^2 - 1})t} + Be^{(-k - \sqrt{k^2 - 1})t}$

$x_p = Ct + D$

$\dfrac{dx_p}{dt} = C$

$\dfrac{d^2x_p}{dt^2} = 0$

$\dfrac{d^2x_p}{dt^2} + 2k\dfrac{dx_p}{dt} + x_p$

$= 0 + 2kC + Ct + D$

$= t$

$C = 1 \quad 2kC + D = 0$

$2k + D = 0$

$$D = -2k$$

$$x_p = t - 2k$$

$$x = x_c + x_p$$
$$= Ae^{(-k + \sqrt{k^2-1})t} + Be^{(-k - \sqrt{k^2-1})t} + t - 2k$$

ii) $0 < k < 1$ : $m = -k + \sqrt{1-k^2}\,i, -k - \sqrt{1-k^2}\,i$

$$x_c = e^{-kt}(A\cos\sqrt{1-k^2}\,t + B\sin\sqrt{1-k^2}\,t)$$

$$x_p = Ct + D$$

$$\frac{dx_p}{dt} = C$$

$$\frac{d^2x_p}{dt^2} = 0$$

$$\frac{d^2x_p}{dt^2} + 2k\frac{dx_p}{dt} + x_p = 0 + 2kC + Ct + D$$
$$= Ct + 2kC + D$$
$$= t$$

$$C = 1 \quad 2kC + D = 0$$
$$2k + D = 0$$
$$D = -2k$$

$$x_p = t - 2k$$

$$x = x_c + x_p$$
$$= e^{-kt}(A\cos\sqrt{1-k^2}\,t + B\sin\sqrt{1-k^2}\,t) + t - 2k$$

$k = 1: \quad m = -1$

$$x_c = (At + B)e^{-t}$$

$$x_p = Ct + D$$

$$\frac{dx_p}{dt} = C$$

$$\frac{d^2x_p}{dt^2} = 0$$

$$\frac{d^2x_p}{dt^2} + \frac{2dx_p}{dt} + x_p$$

$$= 0 + 2C + Ct + D$$

$$= Ct + 2C + D$$

$$= t$$

$$C = 1 \quad 2C + D = 0$$

$$D = -2$$

$$x_p = t - 2$$

$$x = x_c + x_p$$

$$= (At + B)e^{-t} + t - 2$$

$$\frac{dx}{dt} = Ae^{-t} - (At + B)e^{-t} + 1$$

$x = 0$ and $\frac{dx}{dt} = 1$ when $t = 0$

$t = 0 \quad x = 0 : \quad 0 = B - 2$

$t = 0 \quad \frac{dx}{dt} = 1 : \quad 1 = A - B + 1$

$B = 2$

$A = 2$

$x = (2t + 2)e^{-t} + t - 2$

32-a) $y = Axe^{Bx}$

$$\frac{dy}{dx} = Ae^{Bx} + ABxe^{Bx} = A(1 + Bx)e^{Bx}$$

$$\frac{d^2y}{dx^2} = ABe^{Bx} + ABe^{Bx} + AB^2xe^{Bx}$$

$$= AB(2 + Bx)e^{Bx}$$

$$x^2y\frac{d^2y}{dx^2} - x^2\left(\frac{dy}{dx}\right)^2 + y^2$$

$$= x^2(Axe^{Bx})AB(2 + Bx)e^{Bx}$$
$$- x^2A^2(1 + Bx)^2e^{2Bx} + A^2x^2e^{2Bx}$$

$$= \left(A^2B(2x + Bx^2) - A^2(1 + Bx)^2 + A^2\right)x^2e^{2Bx}$$

$$= A^2(2Bx + B^2x^2 - 1 - 2Bx - B^2x^2 + 1)x^2e^{2Bx}$$

$$= 0$$

b) $\frac{dy}{dx} + 2y\tan x = \sin x$ , $y = 0 \quad x = 0$

$$e^{\int 2\tan x\, dx} = e^{\int \frac{2\sin x}{\cos x} dx} = e^{-2\ln|\cos x|} = \sec^2 x$$

$$\sec^2 x\frac{dy}{dx} + 2y\tan x \sec^2 x = \sin x \sec^2 x$$

$$\frac{d}{dx}(y\sec^2 x) = \tan x \sec x$$

$$y\sec^2 x = \int \tan x \sec x \, dx$$

$$= \sec x + c$$

$$y = \cos x + c\cos^2 x$$

$x=0 \quad y=0 : \quad 0 = 1 + c$

$c = -1$

$y = \cos x - \cos^2 x$

$\frac{dy}{dx} = -\sin x + 2\cos x \sin x$

$\frac{dy}{dx} = 0 : \quad -\sin x + 2\cos x \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0 \ , \ \cos x = \frac{1}{2}$

$x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

$\frac{d^2y}{dx^2} = -\cos x + 2\cos^2 x - 2\sin^2 x$

$x=0 : \quad \frac{d^2y}{dx^2} = 1 > 0$

$x = \frac{\pi}{3} \quad \frac{d^2y}{dx^2} = \frac{-3}{2} < 0$

$x = \pi : \quad \frac{d^2y}{dx^2} = 3 > 0$

$x = \frac{5\pi}{3} : \quad \frac{d^2y}{dx^2} = -\frac{3}{2} < 0$

$x = 2\pi : \quad \frac{d^2y}{dx^2} = 1 > 0$

$y_{max} = \frac{1}{4}$ when $x = \frac{\pi}{3}$.

34 a) $y = \tan^{-1}\frac{2}{x}$ $\qquad$ $y = \tan^{-1}\frac{x}{2}$

$\frac{2}{x} = \tan y$ $\qquad$ $\frac{x}{2} = \tan y$

$\frac{-2}{x^2} = \sec^2 y \frac{dy}{dx}$ $\qquad$ $\frac{1}{2} = \sec^2 y \frac{dy}{dx}$

$= \left(1 + \frac{4}{x^2}\right)\frac{dy}{dx}$ $\qquad$ $= \left(1 + \frac{x^2}{4}\right)\frac{dy}{dx}$

$-2 = (x^2+4)\frac{dy}{dx}$ $\qquad$ $2 = (4+x^2)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-2}{x^2+4}$ $\qquad$ $\frac{dy}{dx} = \frac{2}{x^2+4}$

$$y = \tan^{-1}\frac{2}{x} + \tan^{-1}\frac{x}{2}$$

$$\frac{dy}{dx} = \frac{-2}{x^2+4} + \frac{2}{x^2+4}$$

$$= 0$$

b) $(x^2+4)\frac{dy}{dx} = 2y + x^2 e^{\tan^{-1}\left(\frac{x}{2}\right)}$

$$\frac{dy}{dx} - \frac{2y}{x^2+4} = \frac{x^2 e^{\tan^{-1}\left(\frac{x}{2}\right)}}{x^2+4}$$

$$e^{\int \frac{-2}{x^2+4}dx} = e^{\tan^{-1}\frac{2}{x}} = e^{-\tan^{-1}\frac{x}{2}}$$

$$e^{-\tan^{-1}\frac{x}{2}}\frac{dy}{dx} - \frac{2ye^{-\tan^{-1}\frac{x}{2}}}{x^2+4} = \frac{x^2}{x^2+4}$$

$$\frac{d}{dx}\left(ye^{-\tan^{-1}\frac{x}{2}}\right) = \frac{x^2}{x^2+4}$$

$$ye^{-\tan^{-1}\frac{x}{2}} = \int 1 - \frac{4}{x^2+4}\,dx$$

$$= x - 2\tan^{-1}\frac{x}{2} + c$$

$$y = xe^{\tan^{-1}\frac{x}{2}} - 2\tan^{-1}\frac{x}{2}e^{\tan^{-1}\frac{x}{2}} + ce^{\tan^{-1}\frac{x}{2}}$$

$x = 2 \quad y = e^{\frac{\pi}{4}}$ :

$$e^{\frac{\pi}{4}}e^{-\tan^{-1}1} = 2 - 2\tan^{-1}1 + c$$

$$e^{\frac{\pi}{4}}e^{\frac{-\pi}{4}} = 2 - 2\left(\frac{\pi}{4}\right) + c$$

$$1 = 2 - \frac{\pi}{2} + c$$

$$c = \frac{\pi}{2} - 1$$

$$y = xe^{\tan^{-1}\frac{x}{2}} - 2\tan^{-1}\frac{x}{2}e^{\tan^{-1}\frac{x}{2}} - \left(\frac{\pi}{2} - 1\right)e^{\tan^{-1}\frac{x}{2}}$$

35. $\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 5y = 4e^{-x}$

$\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 5y = 0$

$m^2 + 2m + 5 = 0$

$(m+1)^2 + 4 = 0$

$(m+1)^2 = -4$

$m + 1 = \pm 2i$

$m = -1 \pm 2i$

$y_c = e^{-x}(A\cos 2x + B\sin 2x)$

$y_p = Ce^{-x}$

$\frac{dy_p}{dx} = -Ce^{-x}$

$\frac{d^2y_p}{dx^2} = Ce^{-x}$

$\frac{d^2y_p}{dx^2} + \frac{2dy_p}{dx} + 5y_p$

$= Ce^{-x} - 2Ce^{-x} + 5Ce^{-x}$

$= 4Ce^{-x}$

$= 4e^{-x}$

$4C = 4$

$C = 1$

$$y_p = e^{-x}$$

$$y = y_c + y_p$$

$$= e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}$$

$$\frac{dy}{dx} = -e^{-x}(A\cos 2x + B\sin 2x)$$

$$+2e^{-x}(-A\sin 2x + B\cos 2x) - e^{-x}$$

$y = 1$ and $\frac{dy}{dx} = 3$ when $x = 0$

$x = 0$ $y = 1$ : $1 = A + 1$

$x = 0$ $\frac{dy}{dx} = 3$ : $3 = -A + 2B - 1$

$$A = 0$$

$$2B = 4$$

$$B = 2$$

$$y = 2e^{-x}\sin 2x + e^{-x}$$

when $x = \frac{7\pi}{12}$ : $y = 0$

when $y = 0$ : $2e^{-x}\sin 2x + e^{-x} = 0$

$$e^{-x}(2\sin 2x + 1) = 0$$

$$2\sin 2x + 1 = 0$$

$$\sin 2x = \frac{-1}{2}$$

$$2x = \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

$$x = \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

37. a) $$\frac{d^2y}{dx^2} - \frac{6dy}{dx} + 9y = 9x$$

$$\frac{d^2y}{dx^2} - \frac{6dy}{dx} + 9y = 0$$

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3$$

$$y_c = (Ax + B)e^{3x}$$

$$y_p = Cx + D$$

$$\frac{dy_p}{dx} = C$$

$$\frac{d^2y_p}{dx^2} = 0$$

$$\frac{d^2y_p}{dx^2} - \frac{6dy_p}{dx} + 9y_p = 0 - 6C + 9(Cx + D)$$

$$= 9Cx - 6C + 9D$$

$$= 9x$$

$$9C = 9 \qquad -6C + 9D = 0$$

$$C = 1 \qquad D = \frac{2}{3}$$

$$y_p = x + \frac{2}{3}$$

$$y = y_c + y_p$$

$$= (Ax + B)e^{3x} + x + \frac{2}{3}$$

b) $(a^2-x^2)\frac{dy}{dx} - xy + a^2 = 0 \ , \ a>0, \ |x|<a$

$$\frac{dy}{dx} - \frac{xy}{a^2-x^2} = \frac{a^2}{x^2-a^2}$$

$$e^{\int \frac{-x}{a^2-x^2}\,dx} = e^{\frac{1}{2}\int \frac{-1}{a-x} + \frac{1}{a+x}\,dx}$$

$$= e^{\frac{1}{2}\ln|(a+x)(a-x)|}$$

$$= \sqrt{(a+x)(a-x)}$$

$$\sqrt{(a+x)(a-x)}\frac{dy}{dx} - \frac{xy}{a^2-x^2}\sqrt{(a+x)(a-x)} = \frac{-a^2\sqrt{(a+x)(a-x)}}{a^2-x^2}$$

$$\frac{d}{dx}\left(y\sqrt{(a+x)(a-x)}\right) = \frac{-a^2}{\sqrt{(a-x)(a+x)}}$$

$$y\sqrt{a^2-x^2} = \int \frac{-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$x = a\sin\theta$$

$$\frac{dx}{d\theta} = a\cos\theta$$

$$= \int \frac{-a^2}{a\cos\theta}\,a\cos\theta\ d\theta$$

$$= \int -a^2\,d\theta$$

$$= -a^2\theta + c$$

$$= c - a^2\sin^{-1}\left(\frac{x}{a}\right)$$

39. a) $\sin x \dfrac{dy}{dx} - (\cos x)y = \sin^3 x$ , $y=1$ $x=\dfrac{\pi}{2}$

$$\frac{dy}{dx} - \left(\frac{\cos x}{\sin x}\right)y = \sin^2 x$$

$$e^{-\int \frac{\cos x}{\sin x}dx} = e^{-\ln|\sin x|} = \frac{1}{\sin x}$$

$$\frac{1}{\sin x}\frac{dy}{dx} - \left(\frac{\cos x}{\sin^2 x}\right)y = \sin x$$

$$\frac{d}{dx}\left(\frac{y}{\sin x}\right) = \sin x$$

$$\frac{y}{\sin x} = \int \sin x$$

$$= -\cos x + c$$

$x=\dfrac{\pi}{2}$ $y=1$: $1 = c$

$$\frac{y}{\sin x} = 1 - \cos x$$

$$y = \sin x(1-\cos x)$$

$x = \dfrac{3\pi}{4}$: $y = \dfrac{1}{\sqrt{2}}\left(1 - \dfrac{-1}{\sqrt{2}}\right)$

$$= \frac{\sqrt{2}+1}{2}$$

b) $y = f(x)$ , $x = e^u$

$$\frac{dx}{du} = e^u$$

$$\frac{dy}{du} = \frac{dy}{dx} e^u$$

$$\frac{dy}{dx} = e^{-u} \frac{dy}{du}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-u}\frac{dy}{du}\right)$$

$$\frac{d^2y}{dx^2} = \frac{du}{dx}\frac{d}{du}\left(e^{-u}\frac{dy}{du}\right)$$

$$= e^{-u}\left(-e^{-u}\frac{dy}{du} + e^{-u}\frac{d^2y}{du^2}\right)$$

$$= \frac{1}{x}\left(-\frac{1}{x}\frac{dy}{du} + \frac{1}{x}\frac{d^2y}{du^2}\right)$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$$

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = x^3 , \quad x > 0$$

$$x = e^u , \quad \frac{dy}{dx} = e^{-u}\frac{dy}{du} , \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$$

$$\frac{d^2y}{du^2} - \frac{dy}{du} + 3xe^{-u}\frac{dy}{du} + 5y = e^{3u}$$

$$\frac{d^2y}{du^2} - \frac{dy}{du} + 3\frac{dy}{du} + 5y = e^{3u}$$

$$\frac{d^2y}{du^2} + 2\frac{dy}{du} + 5y = e^{3u}$$

$$\frac{d^2y}{du^2} + 2\frac{dy}{du} + 5y = 0$$

$$m^2 + 2m + 5 = 0$$

$$(m+1)^2 + 4 = 0$$

$$(m+1)^2 = -4$$

$$m+1 = \pm 2i$$

$$m = -1 \pm 2i$$

$$y_c = e^{-u}(A\cos 2u + B\sin 2u)$$

$$y_p = Ce^{3u}$$

$$\frac{dy_p}{du} = 3Ce^{3u}$$

$$\frac{d^2y_p}{du^2} = 9Ce^{3u}$$

$$\frac{d^2y_p}{du^2} + 2\frac{dy_p}{du} + 5y_p = 9Ce^{3u} + 6Ce^{3u} + 5Ce^{3u}$$

$$= 20Ce^{3u}$$

$$= e^{3u}$$

$$20C = 1$$

$$C = \frac{1}{20}$$

$$y_p = \frac{e^{3u}}{20}$$

$$y = y_c + y_p$$

$$= e^{-u}(A\cos 2u + B\sin 2u) + \frac{e^{3u}}{20}$$

$$= e^{-\ln x}(A\cos(2\ln x) + B\sin(2\ln x)) + \frac{e^{3\ln x}}{20}$$

$$= \frac{A\cos(2\ln x) + B\sin(2\ln x)}{x} + \frac{x^3}{20}$$

40. a) $\frac{dy}{dx} + \left(\frac{e^x}{4+e^x}\right)y = 1, \quad y=1 \quad x=0$

$$e^{\int \frac{e^x}{4+e^x}dx} = e^{\ln|4+e^x|} = 4+e^x$$

$$(e^x+4)\frac{dy}{dx} + e^x y = e^x + 4$$

$$\frac{d}{dx}(y(e^x+4)) = e^x + 4$$

$$y(e^x+4) = \int e^x + 4 \; dx$$

$$= e^x + 4x + c$$

$x=0 \quad y=1: \quad 1(5) = 1 + 0 + c$

$$c = 4$$

$$y(e^x+4) = e^x + 4x + 4$$

$$y = \frac{e^x + 4x + 4}{e^x + 4}$$

b) $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = \sin 2x + 3\cos 2x$

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = 0$$

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

$$y_c = Ae^{-x} + Be^{-2x}$$

$$y_p = C\cos 2x + D\sin 2x$$

$$\frac{dy_p}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\frac{d^2y_p}{dx^2} = -4C\cos 2x - 4D\sin 2x$$

$$\frac{d^2y_p}{dx^2} + 3\frac{dy_p}{dx} + 2y_p = -4C\cos 2x - 4D\sin 2x$$

$$-6C\sin 2x + 6D\cos 2x$$

$$+2C\cos 2x + 2D\sin 2x$$

$$= (-2C + 6D)\cos 2x + (-6C - 2D)\sin 2x$$

$$= \sin 3x + 3\cos 2x$$

$$-2C + 6D = 3 \qquad -6C - 2D = 1$$

$$-18C - 6D = 3$$

$$-20C = 6$$

$$C = \frac{-3}{10}$$

$$D = \frac{2}{5}$$

$$y_p = \frac{2\sin 2x}{5} - \frac{3\cos 2x}{10}$$

$$y = y_c + y_p$$

$$= Ae^{-x} + Be^{-2x} + \frac{2\sin 2x}{5} - \frac{3\cos 2x}{10}$$

As $x \to \infty$, $y \to \dfrac{2\sin 2x}{5} - \dfrac{3\cos 2x}{10}$

$$\frac{2\sin 2x}{5} - \frac{3\cos 2x}{10} = R\sin(2x - \phi), \quad R > 0,\ 0 < \phi < \frac{\pi}{2}$$

$$= R\sin 2x\cos\phi - R\cos 2x\sin\phi$$

$R\cos\phi = \dfrac{2}{5}$ $\quad R\sin\phi = \dfrac{3}{10}$

$$R^2(\cos^2\phi + \sin^2\phi) = \frac{4}{25} + \frac{9}{100}$$

$$R^2 = \frac{25}{100}$$

$$= \frac{1}{4}$$

$$R = \frac{1}{2}$$

$$\frac{R\sin\phi}{R\cos\phi} = \frac{3}{4}$$

$$\tan\phi = \frac{3}{4}$$

$$\phi = \tan^{-1}\frac{3}{4}$$

$$\therefore y \to \frac{\sin\left(2x - \tan^{-1}\frac{3}{4}\right)}{2}$$

42. a) $(1+x^2)y\frac{dy}{dx} + 2xy^2 = 3 \quad (y \geq 0)$, $y=1$ $x=0$

$$z = y^2$$

$$\frac{dz}{dy} = 2y$$

$$\frac{dz}{dx} = 2y\frac{dy}{dx}$$

$$\frac{(1+x^2)}{2}\frac{dz}{dx} + 2xz = 3$$

$$\frac{dz}{dx} + \frac{4xz}{1+x^2} = \frac{6}{1+x^2}$$

$$e^{\int \frac{4x}{1+x^2}dx} = e^{2\ln|1+x^2|} = (1+x^2)^2$$

$$(1+x^2)^2\frac{dz}{dx} + 4xz(1+x^2) = 6(1+x^2)$$

$$\frac{d}{dx}\left(z(1+x^2)^2\right) = 6(1+x^2)$$

$$z(1+x^2)^2 = \int 6(1+x^2)\,dx$$

$$y^2(1+x^2)^2 = 6x + 2x^3 + c$$

$y=1$ $x=0$: $1 = c$

$$y^2 = \frac{2x^3+6x+1}{(x^2+1)^2}$$

$x=1$: $y^2 = \frac{9}{4}$

$$y = \frac{3}{2}.$$

b) $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 20y = 60x + 152$, $y = 8$ $\frac{dy}{dx} = 5$ $x = 0$

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 20y = 0$$

$$m^2 + 4m + 20 = 0$$

$$(m+2)^2 = -16$$

$$m + 2 = \pm 4i$$

$$m = -2 \pm 4i$$

$$y_c = e^{-2x}(A\cos 4x + B\sin 4x)$$

$$y_p = Cx + D$$

$$\frac{dy_p}{dx} = C$$

$$\frac{d^2y_p}{dx^2} = 0$$

$$\frac{d^2y_p}{dx^2} + \frac{4dy_p}{dx} + 20y_p = 0 + 4C + 20(Cx + D)$$

$$= 20Cx + 4C + 20D$$

$$= 60x + 152$$

$20C = 60$ $\quad$ $4C + 20D = 152$

$C = 3$ $\quad$ $D = 7$

$$y_p = 3x + 7$$

$$y = y_c + y_p$$

$$= e^{-2x}(A\cos 4x + B\sin 4x) + 3x + 7.$$

$$\frac{dy}{dx} = e^{-2x}(-4A\sin 4x + 4B\cos 4x)$$

$$-2e^{-2x}(A\cos 4x + B\sin 4x) + 3$$

$$x=0 \quad y=8: \; 8 = A+7$$

$$x=0 \quad \frac{dy}{dx}=5: \; 5 = 4B - 2A + 3$$

$$A = 1$$

$$4B = 4$$

$$B = 1$$

$$y = e^{-2x}(\cos 4x + \sin 4x) + 3x + 7.$$

46. a) $$(x\sin x)\frac{dy}{dx} + (3\sin x - x\cos x)y = \sin^2 x$$

$$y = z\sin x$$

$$\frac{dy}{dx} = \frac{dz}{dx}\sin x + z\cos x$$

$$(x\sin x)\left(\frac{dz}{dx}\sin x + z\cos x\right)$$

$$+ (3\sin x - x\cos x)z\sin x = \sin^2 x$$

$$x\sin x\frac{dz}{dx} + xz\cos x + 3z\sin x - xz\cos x = \sin x$$

$$x\sin x\frac{dz}{dx} + 3z\sin x = \sin x$$

$$x\frac{dz}{dx} + 3z = 1$$

$$\frac{dz}{dx} + \frac{3z}{x} = \frac{1}{x}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$x^3\frac{dz}{dx} + 3x^2 z = x^2$$

$$\frac{d}{dx}(x^3 z) = x^2$$

$$x^3 z = \int x^2\,dx$$

$$= \frac{x^3}{3} + c$$

$$z = \frac{1}{3} + \frac{c}{x^3}$$

$$\frac{y}{\sin x} = \frac{1}{3} + \frac{c}{x^3}$$

$$y = \frac{\sin x}{3} + \frac{c \sin x}{x^3}$$

b) $$\frac{d^2y}{dx^2} + \frac{6dy}{dx} + 25y = 21\sin 2x + 12\cos 2x$$

$$\frac{d^2y}{dx^2} + \frac{6dy}{dx} + 25y = 0$$

$$m^2 + 6m + 25 = 0$$

$$(m+3)^2 = -16$$

$$m + 3 = \pm 4i$$

$$m = -3 \pm 4i$$

$$y_c = e^{-3x}(A\cos 4x + B\sin 4x)$$

$$y_p = C\cos 2x + D\sin 2x$$

$$\frac{dy_p}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\frac{d^2y_p}{dx^2} = -4C\cos 2x - 4D\sin 2x$$

$$\frac{d^2y_p}{dx^2} + \frac{6dy_p}{dx} + 25y_p = -4C\cos 2x - 4D\sin 2x$$

$$- 12C\sin 2x + 12D\cos 2x$$

$$+ 25C\cos 2x + 25D\sin 2x$$

$$= (-12C + 21D)\sin 2x$$

$$+ (21C + 12D)\cos 2x$$

$$= 21\sin 2x + 12\cos 2x$$

$$-12C + 21D = 21 \qquad 21C + 12D = 12$$

$$-4C + 7D = 7 \qquad 7C + 4D = 4$$

$$D = \frac{4C+7}{7} \qquad 7C + \frac{16C + 28}{7} = 4$$

$$49C + 16C + 28 = 28$$

$$65C = 0$$

$$C = 0$$

$$D = 1$$

$$y_p = \sin 2x$$

$$y = y_c + y_p$$

$$= e^{-3x}(A\cos 4x + B\sin 4x) + \sin 2x$$

47 a) $(1+x)\dfrac{dy}{dx} - 2y + (1+x)y^2 = 0$

$$y = \frac{1}{z}$$

$$\frac{dy}{dz} = \frac{-1}{z^2}$$

$$\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

$$\frac{-(1+x)}{z^2}\frac{dz}{dx} - \frac{2}{z} + \frac{(1+x)}{z^2} = 0$$

$$-(1+x)\frac{dz}{dx} - 2z + 1 + x = 0$$

$$(1+x)\frac{dz}{dx} + 2z = 1 + x$$

$$\frac{dz}{dx} + \frac{2z}{1+x} = 1$$

$$e^{\int \frac{2}{1+x}dx} = e^{2\ln|1+x|} = (1+x)^2$$

$$(1+x)^2\frac{dz}{dx} + 2z(1+x) = (1+x)^2$$

$$\frac{d}{dx}\left(z(1+x)^2\right) = (1+x)^2$$

$$z(1+x)^2 = \int (1+x)^2\,dx$$

$$= \frac{(1+x)^3}{3} + c$$

$$\frac{(1+x)^2}{y} = \frac{(1+x)^3}{3} + c$$

$y = 1 \quad x = 0: \quad 1 = \frac{1}{3} + c$

$$c = \frac{2}{3}$$

$$\frac{(1+x)^2}{y} = \frac{(1+x)^3}{3} + \frac{2}{3}$$

$$\frac{y}{(1+x)^2} = \frac{3}{(1+x)^3 + 2}$$

$$y = \frac{3(1+x)^2}{(1+x)^3 + 2}$$

b) $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 65 \sin 2x$

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 0$$

$$m^2 + 4m + 3 = 0$$

$$(m+1)(m+3) = 0$$

$$m = -1, -3$$

$$y_c = Ae^{-x} + Be^{-3x}$$

$$y_p = C\cos 2x + D\sin 2x$$

$$\frac{dy_p}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\frac{d^2y_p}{dx^2} = -4C\cos 2x - 4D\sin 2x$$

$$\frac{d^2y_p}{dx^2} + \frac{4dy_p}{dx} + 3y_p = -4C\cos 2x - 4D\sin 2x$$

$$-8C\sin 2x + 8D\cos 2x$$

$$+3C\cos 2x + 3D\sin 2x$$

$$= (-C + 8D)\cos 2x + (-8C - D)\sin x$$

$$= 65\sin 2x$$

$$-C + 8D = 0 \qquad -8C - D = 65$$

$$C = 8D \qquad -64D - D = 65$$

$$-65D = 65$$

$$D = -1 \qquad C = -8$$

$$y_p = -8\cos 2x - \sin 2x$$

$$y = y_c + y_p$$

$$= Ae^{-x} + Be^{-3x} - 8\cos 2x - \sin 2x$$