1. $$A = \begin{pmatrix} 1 & 4 & -5 & 8 \\ 2 & 9 & -8 & 13 \\ -3 & -14 & 11 & -18 \\ 5 & 23 & -19 & 31 \end{pmatrix}$$

$$\xrightarrow[\substack{-2r_1 + r_2 \\ 3r_1 + r_3 \\ -5r_1 + r_4}]{} \begin{pmatrix} 1 & 4 & -5 & 8 \\ 0 & 1 & 2 & -3 \\ 0 & -2 & -4 & 6 \\ 0 & 3 & 6 & -9 \end{pmatrix}$$

$$\xrightarrow[\substack{2r_2 + r_3 \\ -3r_2 + r_4}]{} \begin{pmatrix} 1 & 4 & -5 & 8 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $(A) = 2$

Let $w = s, s \in R$ and $z = t, t \in R$

$y = 3s - 2t$

$x + 4(3s - 2t) - 5t + 8s = 0$

$x + 12s - 8t - 5t + 8s = 0$

$x = 13t - 20s$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 13t - 20s \\ 3s - 2t \\ t \\ s \end{pmatrix}$$

$$= t\begin{pmatrix} 13 \\ -2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -20 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

A basis for the range space is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ -14 \\ 23 \end{pmatrix} \right\}$$

2. $y = x^2 e^x$

$$\frac{d^n y}{dx^n} = x^2e^x + 2nxe^x + n(n-1)e^x$$

when $n = 1$: $$\frac{d^1 y}{dx^1} = \frac{dy}{dx}$$

$$= x^2e^x + 2xe^x$$

$$= x^2e^x + 2(1)xe^x + 0e^x$$

$$= x^2e^x + 2(1)xe^x + 1(1-1)e^x$$

Assume the statement is true when $n = k$.

$n = k$: $$\frac{d^k y}{dx^k} = x^2e^x + 2kxe^x + k(k-1)e^x$$

when $n = k+1$:

$$\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left(x^2e^x + 2kxe^x + k(k-1)e^x\right)$$

$$= x^2e^x + 2xe^x + 2kxe^x + 2ke^x + k(k-1)e^x$$

$$= x^2e^x + 2(k+1)xe^x + (k^2 - k + 2k)e^x$$

$$= x^2e^x + 2(k+1)xe^x + (k^2 + k)e^x$$

$$= x^2e^x + 2(k+1)xe^x + k(k+1)e^x$$

$$\frac{d^n y}{dx^n} = x^2e^x + 2nxe^x + n(n-1)e^x$$

for every positive integer n.

3. $A\underset{\sim}{x} = \lambda_A \underset{\sim}{x}$, $B\underset{\sim}{x} = \lambda_B \underset{\sim}{x}$, $C\underset{\sim}{x} = \lambda_C \underset{\sim}{x}$

i) $(AB + BC + CA)\underset{\sim}{x}$

$$= (AB)\underset{\sim}{x} + (BC)\underset{\sim}{x} + (CA)\underset{\sim}{x}$$

$$= A(B\underset{\sim}{x}) + B(C\underset{\sim}{x}) + C(A\underset{\sim}{x})$$

$$= A(\lambda_B \underset{\sim}{x}) + B(\lambda_C \underset{\sim}{x}) + C(\lambda_A \underset{\sim}{x})$$

$$= \lambda_B (A\underset{\sim}{x}) + \lambda_C (B\underset{\sim}{x}) + \lambda_A (C\underset{\sim}{x})$$

$$= \lambda_B (\lambda_A \underset{\sim}{x}) + \lambda_C (\lambda_B \underset{\sim}{x}) + \lambda_A (\lambda_C \underset{\sim}{x})$$

$$= \lambda_A \lambda_B \underset{\sim}{x} + \lambda_B \lambda_C \underset{\sim}{x} + \lambda_A \lambda_C \underset{\sim}{x}$$

$$= (\lambda_A \lambda_B + \lambda_B \lambda_C + \lambda_A \lambda_C)\underset{\sim}{x}$$

The matrix $AB + BC + CA$ has an eigenvalue $\lambda_A\lambda_B + \lambda_B\lambda_C + \lambda_A\lambda_C$ with eigenvector $\underset{\sim}{x}$.

ii) $ABC(\underset{\sim}{x}) = AB(C\underset{\sim}{x})$

$$= AB(\lambda_C \underset{\sim}{x})$$

$$= \lambda_C (AB\underset{\sim}{x})$$

$$= \lambda_C A(B\underset{\sim}{x})$$

$$= \lambda_C A(\lambda_B \underset{\sim}{x})$$

$$= \lambda_C \lambda_B (A\underset{\sim}{x})$$

$$= \lambda_C \lambda_B \lambda_A \underset{\sim}{x}$$

The matrix ABC has an eigenvalue $\lambda_A \lambda_B \lambda_C$ with eigenvector $\underset{\sim}{x}$.

iii) $(A + I)(B + I)(C + I)\underset{\sim}{x}$

$= (AB + AI + BI + I^2)(C + I)\underset{\sim}{x}$

$= (AB + A + B + I)(C + I)\underset{\sim}{x}$

$= (ABC + AC + BC + CI + ABI + AI + BI + I^2)\underset{\sim}{x}$

$= (ABC + AB + AC + BC + A + B + C + I)\underset{\sim}{x}$

$= (ABC)\underset{\sim}{x} + (AB + AC + BC)\underset{\sim}{x} + A\underset{\sim}{x} + B\underset{\sim}{x} + C\underset{\sim}{x} + I\underset{\sim}{x}$

$= \lambda_A\lambda_B\lambda_C\underset{\sim}{x} + (\lambda_A\lambda_B + \lambda_A\lambda_C + \lambda_B\lambda_C)\underset{\sim}{x}$
$+ \lambda_A\underset{\sim}{x} + \lambda_B\underset{\sim}{x} + \lambda_C\underset{\sim}{x} + \underset{\sim}{x}$

$= (\lambda_A\lambda_B\lambda_C + \lambda_A\lambda_B + \lambda_A\lambda_C + \lambda_B\lambda_C$
$+ \lambda_A + \lambda_B + \lambda_C + 1)\underset{\sim}{x}$

The matrix $(A + I)(B + I)(C + I)$ has an eigenvalue $(\lambda_A\lambda_B\lambda_C + \lambda_A\lambda_B + \lambda_A\lambda_C + \lambda_B\lambda_C + \lambda_A + \lambda_B + \lambda_C + 1)$ with eigenvector $\underset{\sim}{x}$.

4. $$\frac{d^2y}{dx^2} + \frac{2dy}{dx} - 15y = 3e^x \cos x + 4e^x \sin x$$

i) If $y_p = e^x(P\cos x + Q\sin x)$

$$\frac{dy_p}{dx} = e^x(P\cos x + Q\sin x) + e^x(-P\sin x + Q\cos x)$$

$$\frac{d^2y_p}{dx^2} = e^x(P\cos x + Q\sin x) + e^x(-P\sin x + Q\cos x)$$

$$+ e^x(-P\sin x + Q\cos x) + e^x(-P\cos x - Q\sin x)$$

$$= 2e^x(-P\sin x + Q\cos x)$$

$$\frac{d^2y_p}{dx^2} + \frac{2dy_p}{dx} - 15y_p$$

$$= 2e^x(-P\sin x + Q\cos x)$$

$$+ 2e^x(P\cos x + Q\sin x) + e^x(-P\sin x + Q\cos x)$$

$$- 15e^x(P\cos x + Q\sin x)$$

$$= (2Q + 2P + 2Q - 15P)e^x \cos x$$

$$+ (-2P + 2Q - 2P - 15Q)e^x \sin x$$

$$= (4Q - 13P)e^x \cos x + (-13Q - 4P)e^x \sin x$$

$$= 3e^x \cos x + 4e^x \sin x$$

$4Q - 13P = 3$ $\quad$ $-13Q - 4P = 4$

$Q = \dfrac{13P + 3}{4}$ $\quad$ $13Q + 4P = -4$

$$\frac{13(13P + 3)}{4} + 4P = -4$$

$$169P + 39 + 16P = -16$$

$$185P = -55$$

$$P = \frac{-11}{37}$$

$$Q = \frac{13}{4}\left(\frac{-11}{37}\right) + \frac{3}{4}$$

$$= \frac{-8}{37}$$

$$y_p = \frac{e^x}{37}(-11\cos x - 8\sin x)$$

$\therefore$ $y_p = \frac{e^x}{37}(-11\cos x - 8\sin x)$ is a particular integral.

ii) $$\frac{d^2y}{dx^2} + \frac{2dy}{dx} - 15y = 0$$

$$m^2 + 2m - 15 = 0$$

$$(m + 5)(m - 3) = 0$$

$$m = 3, -5$$

$\therefore$ The complementary function, y_c, is

$y_c = Ae^{3x} + Be^{-5x}$.

$$y = y_c + y_p$$

$$= Ae^{3x} + Be^{-5x} - \frac{e^x}{37}(11\cos x + 8\sin x)$$

$\therefore$ The general solution is

$y = Ae^{3x} + Be^{-5x} - \frac{e^x}{37}(11\cos x + 8\sin x)$

5. i) $x = \cos^4 t + \sin^4 t \qquad y = \cos^4 t - \sin^4 t$

$$\frac{dx}{dt} = -4\cos^3 t \sin t + 4\sin^3 t \cos t$$

$$= -4\cos t \sin t\,(\cos^2 t - \sin^2 t)$$

$$\frac{dy}{dt} = -4\cos^3 t \sin t - 4\sin^3 t \cos t$$

$$= -4\cos t \sin t\,(\cos^2 t + \sin^2 t)$$

$$= -4\cos t \sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{-4\cos t \sin t}{-4\cos t \sin t\,(\cos^2 t - \sin^2 t)}$$

$$= \frac{1}{\cos^2 t - \sin^2 t}$$

$$= \frac{1}{\cos 2t}$$

$$= \sec 2t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{dt}{dx}\,\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{-4\cos t \sin t\,(\cos^2 t - \sin^2 t)}\,\frac{d}{dt}(\sec 2t)$$

$$= \frac{1}{2\sin 2t\,(-\cos 2t)}(2\sec 2t \tan 2t)$$

$$= \frac{\csc 2t\,(-\sec 2t)\,2\sec 2t \tan 2t}{2}$$

$$= -\sec^3 2t$$

ii) $xy^2 + 2x = 3y^2$

$$\frac{d}{dx}(xy^2 + 2x) = \frac{d}{dx}(3y^2)$$

$$\frac{d}{dx}(xy^2) + \frac{d}{dx}(2x) = 6y\frac{dy}{dx}$$

$$x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) + 2 = 6y\frac{dy}{dx}$$

$$2xy\frac{dy}{dx} + y^2 + 2 = 6y\frac{dy}{dx}$$

At $(1,1)$: $$2\frac{dy}{dx} + 3 = 6\frac{dy}{dx}$$

$$4\frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{4}$$

$$\frac{d}{dx}\left(2xy\frac{dy}{dx} + y^2 + 2\right) = \frac{d}{dx}\left(6y\frac{dy}{dx}\right)$$

$$\frac{d}{dx}\left(2xy\frac{dy}{dx}\right) + \frac{d}{dx}(y^2) + \frac{d}{dx}(2)$$

$$= 6y\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(6y)$$

$$2xy\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(2xy) + 2y\frac{dy}{dx}$$

$$= 6y\frac{d^2y}{dx^2} + 6\left(\frac{dy}{dx}\right)^2$$

$$2xy\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(2x\frac{d}{dx}(y) + y\frac{d}{dx}(2x)\right) + 2y\frac{dy}{dx}$$

$$= 6y\frac{d^2y}{dx^2} + 6\left(\frac{dy}{dx}\right)^2$$

$$2xy\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$= 6y\frac{d^2y}{dx^2} + 6\left(\frac{dy}{dx}\right)^2$$

$$2xy\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 + 4y\frac{dy}{dx} = 6y\frac{d^2y}{dx^2} + 6\left(\frac{dy}{dx}\right)^2$$

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} = 3y\frac{d^2y}{dx^2} + 3y\left(\frac{dy}{dx}\right)^2$$

At $(1,1)$: $\frac{dy}{dx} = \frac{3}{4}$

$$\frac{d^2y}{dx^2} + \frac{9}{16} + \frac{3}{2} = \frac{3d^2y}{dx^2} + 3\left(\frac{9}{16}\right)$$

$$\frac{d^2y}{dx^2} + \frac{9}{16} + \frac{3}{2} = \frac{3d^2y}{dx^2} + \frac{27}{16}$$

$$\frac{2d^2y}{dx^2} = \frac{3}{8}$$

$$\frac{d^2y}{dx^2} = \frac{3}{16}$$