MAY / JUNE 2011

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1. $$\frac{1}{(2r+1)(2r+3)} = \frac{A}{2r+1} + \frac{B}{2r+3}$$

$$= \frac{A(2r+3)+B(2r+1)}{(2r+1)(2r+3)}$$

$$1 = A(2r+3) + B(2r+1)$$

$$= (2A+2B)r + 3A + B$$

$$2A + 2B = 0 \qquad 3A + B = 1$$

$$A = -B$$

$$-3B + B = 1$$

$$-2B = 1$$

$$B = \frac{-1}{2}$$

$$A = \frac{1}{2}$$

$$\therefore \frac{1}{(2r+1)(2r+3)} = \frac{1}{2(2r+1)} - \frac{1}{2(2r+3)}$$

$$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)} = \sum_{r=1}^{n} \frac{1}{2(2r+1)} - \frac{1}{2(2r+3)}$$

$$= \frac{1}{2\cdot 3} - \frac{1}{2\cdot 5}$$

$$+ \frac{1}{2\cdot 5} - \frac{1}{2\cdot 7}$$

$$+ \frac{1}{2\cdot 7} - \frac{1}{2\cdot 9}$$

$$\vdots$$

$$+\frac{1}{2(2r-3)}-\frac{1}{2(2r-1)}$$

$$+\frac{1}{2(2r-1)}-\frac{1}{2(2r+1)}$$

$$+\frac{1}{2(2r+1)}-\frac{1}{2(2r+3)}$$

$$=\frac{1}{6}-\frac{1}{2(2r+3)}$$

$$\sum_{r=1}^{\infty}\frac{1}{(2r+1)(2r+3)}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}$$

$$=\lim_{n\to\infty}\frac{1}{6}-\frac{1}{2(2r+3)}$$

$$=\frac{1}{6}-0$$

$$=\frac{1}{6}$$

2. $x^3+px^2+qx+r=0$

$\frac{\beta}{k},\beta,k\beta$ are the roots, $p,q,r,k,\beta\neq 0$

$\frac{\beta}{k}+\beta+k\beta=-p$ $\quad$ $\frac{\beta(\beta)}{k}+\frac{\beta}{k}(k\beta)+\beta(k\beta)=q$ $\quad$ $\frac{\beta(\beta)k\beta}{k}=-r$

$\beta(\frac{1}{k}+1+k)=-p$ $\quad$ $\beta^2(\frac{1}{k}+1+k)=q$ $\quad$ $\beta^3=-r$

$$\frac{\beta^2(\frac{1}{k}+1+k)}{\beta(\frac{1}{k}+1+k)}=\frac{q}{-p}$$

$$\therefore\beta=-\frac{q}{p}$$

since $\beta^3=-r$,

$$\left(\frac{-q}{p}\right)^3=-r$$

$$\frac{-q^3}{p^3}=-r$$

$$p^3r=q^3$$

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3. $T: R^4 \to R^4$

$$M = \begin{pmatrix} 1 & 3 & -2 & 4 \\ 5 & 15 & -9 & 19 \\ -2 & -6 & 3 & -7 \\ 3 & 9 & -5 & 11 \end{pmatrix}$$

i) $$\xrightarrow[-3r_1 + r_4]{-5r_1 + r_2,\ 2r_1 + r_3} \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow[-r_2 + r_4]{r_2 + r_3} \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank $(M) = 2$

ii) If $$\begin{pmatrix} 1 & 3 & -2 & 4 \\ 5 & 15 & -9 & 19 \\ -2 & -6 & 3 & -7 \\ 3 & 9 & -5 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 0 \\ 5 & 15 & -9 & 19 & 0 \\ -2 & -6 & 3 & -7 & 0 \\ 3 & 9 & -5 & 11 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x + 3y - 2z + 4w = 0$.

$z - w = 0$

Let $w = s, s \in R$ and $y = t, t \in R$

$\therefore z = s$

$x + 3t - 2s + 4s = 0$

$x = -3t - 2s$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3t - 2s \\ t \\ s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ A basis for the null space of T is

$$\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

4. $f(n) = 3^{3n} + 6^{n-1}$

i) $f(n+1) + f(n) = 3^{3(n+1)} + 6^{n+1-1} + 3^{3n} + 6^{n-1}$

$= 3^{3n+3} + 6^{n-1+1} + 3^{3n} + 6^{n-1}$

$= 3^{3n}3^3 + 6^{n-1}6 + 3^{3n} + 6^{n-1}$

$= 3^{3n}27 + 6^{n-1}6 + 3^{3n} + 6^{n-1}$

$= 3^{3n}(27+1) + 6^{n-1}(6+1)$

$= 3^{3n}28 + 6^{n-1}7$

ii) when $n=1$: $f(1) = 3^{3(1)} + 6^{1-1}$

$= 3^3 + 6^0$

$= 27 + 1$

$= 28$

$= 7(4)$

$\therefore 7 \mid f(1)$

Assume the statement is true when $n=k$.

$n=k$: $7 \mid f(k)$

$f(k) = 7s$, s is an integer

when $n=k+1$: $f(k+1) + f(k) = 3^{3k}28 + 6^{k-1}7$

$f(k+1) + 7s = 3^{3k}28 + 6^{k-1}7$

$f(k+1) = 3^{3k}28 + 6^{k-1}7 - 7s$

$= 7(3^{3k}4 + 6^{k-1} - s)$

since s is an integer and k is an integer,

$3^{3k}4 + 6^{k-1} - s$ is an integer.

$\therefore 7 \mid f(k+1)$

$\therefore 3^{3n} + 6^{n-1}$ is divisible by 7 for every positive integer n.

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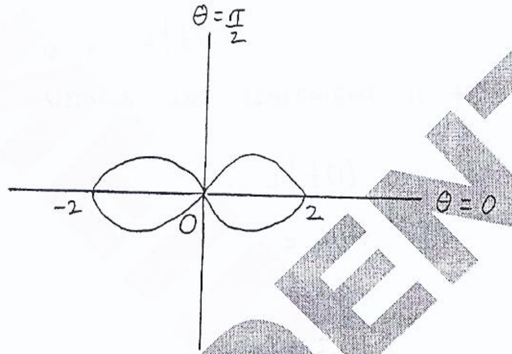
5. $C: r = 2\cos 2\theta, \quad 0 \leq \theta < 2\pi$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	2	1	0	-1	-2	-1	0	1	2

$4\cos^2(-\theta) - 2 = 4\cos^2\theta - 2$



The area of one loop of the curve is

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{r^2}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2 2\theta \; d\theta$$

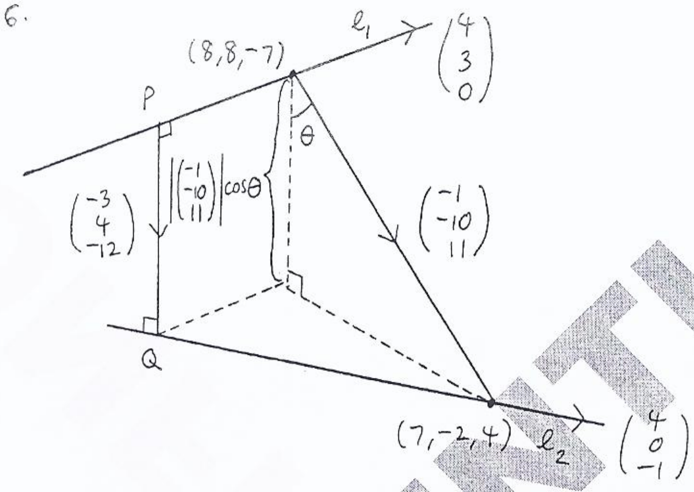
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 4\theta + 1 \; d\theta$$

$$= \left[\frac{\sin 4\theta}{4} + \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - \left(\frac{-\pi}{4}\right)$$

$$= \frac{\pi}{2}$$

6.



$$l_1: \underset{\sim}{r} = \begin{pmatrix}8\\8\\-7\end{pmatrix} + s\begin{pmatrix}4\\3\\0\end{pmatrix}$$

$$l_2: \underset{\sim}{r} = \begin{pmatrix}7\\-2\\4\end{pmatrix} + t\begin{pmatrix}4\\0\\-1\end{pmatrix}$$

i) Since PQ is perpendicular to both l_1 and l_2, $\overrightarrow{PQ}$ is perpendicular to both $\begin{pmatrix}4\\3\\0\end{pmatrix}$ and $\begin{pmatrix}4\\0\\-1\end{pmatrix}$.

$\therefore \overrightarrow{PQ}$ is parallel to $\begin{pmatrix}4\\3\\0\end{pmatrix} \times \begin{pmatrix}4\\0\\-1\end{pmatrix}$.

$$\begin{pmatrix}4\\3\\0\end{pmatrix} \times \begin{pmatrix}4\\0\\-1\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & 3 & 0 \\ 4 & 0 & -1 \end{vmatrix} = \begin{pmatrix}-3\\4\\-12\end{pmatrix}$$

$$\therefore \overrightarrow{PQ} = c\begin{pmatrix}-3\\4\\-12\end{pmatrix}, \quad c \in R$$

$$\vec{PQ}\cdot\begin{pmatrix}-1\\-10\\11\end{pmatrix} = |\vec{PQ}|\left|\begin{pmatrix}-1\\-10\\11\end{pmatrix}\right|\cos\theta$$

$$\left|c\begin{pmatrix}-3\\4\\-12\end{pmatrix}\cdot\begin{pmatrix}-1\\-10\\11\end{pmatrix}\right| = \left|c\begin{pmatrix}-3\\4\\-12\end{pmatrix}\right|\left|\begin{pmatrix}-1\\-10\\11\end{pmatrix}\right|\cos\theta$$

$$|3-40-132| = \sqrt{9+16+144}\left|\begin{pmatrix}-1\\-10\\11\end{pmatrix}\right|\cos\theta$$

$$169 = \sqrt{169}\left|\begin{pmatrix}-1\\-10\\11\end{pmatrix}\right|\cos\theta$$

$$\therefore \left|\begin{pmatrix}-1\\-10\\11\end{pmatrix}\right|\cos\theta = \frac{169}{13}$$

$$|\vec{PQ}| = 13$$

ii) since P lies on l_1 and Q lies on l_2,

$P(4s+8, 3s+8, -7)$ and $Q(4t+7, -2, -t+4)$.

$$\therefore \vec{PQ} = \begin{pmatrix}4t+7\\-2\\-t+4\end{pmatrix} - \begin{pmatrix}4s+8\\3s+8\\-7\end{pmatrix} = \begin{pmatrix}4t-4s-1\\-3s-10\\-t+11\end{pmatrix}$$

$$\begin{pmatrix}4t-4s-1\\-3s-10\\-t+11\end{pmatrix} = c\begin{pmatrix}-3\\4\\-12\end{pmatrix}$$

$$= \begin{pmatrix}-3c\\4c\\-12c\end{pmatrix}$$

$$\left.\begin{aligned}4t-4s-1 &= -3c\\ -3s-10 &= 4c\\ -t+11 &= -12c\end{aligned}\right\}$$

$$\left.\begin{aligned}4t-4s+3c &= 1\\ 3s+4c &= -10\\ -t+12c &= -11\end{aligned}\right\}$$

$$\left.\begin{aligned}t-12c &= 11\\ 4t-4s+3c &= 1\\ 3s+4c &= -10\end{aligned}\right\}$$

$$-4\times① + ②:\quad \left.\begin{aligned}t-12c &= 11\\ -4s+51c &= -43\\ 3s+4c &= -10\end{aligned}\right\}$$

$$\begin{aligned}&\\ 3\times②:&\\ 4\times③:&\end{aligned}\quad \left.\begin{aligned}t-12c &= 11\\ -12s+153c &= -129\\ 12s+16c &= -40\end{aligned}\right\}$$

$$②+③:\quad \left.\begin{aligned}t-12c &= 11\\ -12s+153c &= -129\\ 169c &= -169\end{aligned}\right\}$$

$$c = -1$$
$$t = -1$$
$$s = -2$$

$\therefore P(0, 2, -7)\quad Q(3, -2, 5)$

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7. $y^2\frac{d^2y}{dx^2} + 2y^2\frac{dy}{dx} + 2y\left(\frac{dy}{dx}\right)^2 - 5y^3 = 8e^{-x}$.

$v = y^3$

$\frac{dv}{dy} = 3y^2$

$\frac{dv}{dx} = 3y^2\frac{dy}{dx}$

$\frac{d^2v}{dx^2} = \frac{d}{dx}\left(3y^2\frac{dy}{dx}\right)$

$= 3y^2\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(3y^2)$

$= 3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2$

$3y^2\frac{d^2y}{dx^2} + 6y^2\frac{dy}{dx} + 6y\left(\frac{dy}{dx}\right)^2 - 15y^3 = 24e^{-x}$

$3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 2\left(3y^2\frac{dy}{dx}\right) - 15y^3 = 24e^{-x}$

$\frac{d^2v}{dx^2} + \frac{2dv}{dx} - 15v = 24e^{-x}$

$\frac{d^2v}{dx^2} + \frac{2dv}{dx} - 15v = 0$

$m^2 + 2m - 15 = 0$

$(m+1)^2 = 16$

$m + 1 = \pm 4$

$m = -5, 3$

$\therefore$ The complementary function, v_c, is

$v_c = Ae^{-5x} + Be^{3x}$

The particular integral, v_p, is given by

$v_p = Ce^{-x}$

$\frac{dv_p}{dx} = -Ce^{-x}$

$\frac{d^2v_p}{dx^2} = Ce^{-x}$

$\frac{d^2v_p}{dx^2} + \frac{2dv_p}{dx} - 15v_p = Ce^{-x} + 2(-Ce^{-x}) - 15Ce^{-x}$

$= Ce^{-x} - 2Ce^{-x} - 15Ce^{-x}$

$= -16Ce^{-x}$

$= 24e^{-x}$

$-16C = 24$

$C = -\frac{3}{2}$

$v_p = \frac{-3e^{-x}}{2}$

$v = v_c + v_p$

$= Ae^{-5x} + Be^{3x} - \frac{3e^{-x}}{2}$

$y^3 = Ae^{-5x} + Be^{3x} - \frac{3e^{-x}}{2}$

$y = \left(Ae^{-5x} + Be^{3x} - \frac{3e^{-x}}{2}\right)^{\frac{1}{3}}$

$\therefore$ The general solution is $y = \left(Ae^{-5x} + Be^{3x} - \frac{3e^{-x}}{2}\right)^{\frac{1}{3}}$

8. $A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 0 & -3 \\ 1 & -3 & 0 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 0 & -3 \\ 1 & -3 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & -1 & 1 \\ -1 & -\lambda & -3 \\ 1 & -3 & -\lambda \end{pmatrix}$$

$$|A - \lambda I| = (4-\lambda)\begin{vmatrix} -\lambda & -3 \\ -3 & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -3 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & -\lambda \\ 1 & -3 \end{vmatrix}$$

$$= (4-\lambda)(\lambda^2 - 9) + 1(\lambda + 3) + 1(3 + \lambda)$$

$$= (4-\lambda)(\lambda-3)(\lambda+3) + \lambda + 3 + \lambda + 3$$

$$= (\lambda+3)(4\lambda - \lambda^2 - 12 + 3\lambda + 1 + 1)$$

$$= (\lambda+3)(-\lambda^2 + 7\lambda - 10)$$

$$= (\lambda+3)(-\lambda+2)(\lambda-5)$$

when $|A - \lambda I| = 0$, $(\lambda+3)(-\lambda+2)(\lambda-5) = 0$

$$\lambda = -3, 2, 5$$

when $\lambda = -3$: $\begin{pmatrix} 7 & -1 & 1 \\ -1 & 3 & -3 \\ 1 & -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 7 & -1 & 1 & 0 \\ -1 & 3 & -3 & 0 \\ 1 & -3 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ -1 & 3 & -3 & 0 \\ 7 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-7r_1 + r_3]{r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 20 & -20 & 0 \end{array}\right)$$

Let $z = s$, $s \in R$

$y = s$

$x - 3s + 3s = 0$

$x = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ s \end{pmatrix} = s\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

when $\lambda = 2$: $\begin{pmatrix} 2 & -1 & 1 \\ -1 & -2 & -3 \\ 1 & -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & -3 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-2r_1 + r_3]{r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 + r_3} \left(\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s$, $s \in R$

$y = -s$

$x + 3s - 2s = 0$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ -s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

when $\lambda = 5$: $\begin{pmatrix} -1 & -1 & 1 \\ -1 & -5 & -3 \\ 1 & -3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -1 & -5 & -3 & 0 \\ 1 & -3 & -5 & 0 \end{array}\right)$$

$$\xrightarrow[r_1+r_3]{-r_1+r_2} \left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s,\ s \in R$

$y = -s$

$-x + s + s = 0$

$x = 2s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ -s \\ s \end{pmatrix} = s\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are $-3, 2, 5$ with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

If $A^5 = PDP^{-1}$, where P is a square matrix and D is a diagonal matrix, $P = \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ and

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}^5 = \begin{pmatrix} -243 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 3125 \end{pmatrix}.$$

9. $C: y = x^{\frac{3}{2}}$

The area of the region bounded by C, the lines $x=1$, $x=4$ and the x-axis, A, is

$$\int_1^4 x^{\frac{3}{2}}\,dx$$

$$= \left[\frac{2}{5}x^{\frac{5}{2}}\right]_1^4$$

$$= \frac{2}{5}(4^{\frac{5}{2}}-1)$$

$$= \frac{2}{5}(31)$$

$$= \frac{62}{5}.$$

If $(\bar{x},\bar{y})$ are the coordinates of the centroid of the region bounded by C, the lines $x=1$, $x=4$ and the x-axis,

$$\bar{x} = \frac{\int_1^4 xy\,dx}{A}$$

$$= \frac{\int_1^4 x^{\frac{5}{2}}\,dx}{\frac{62}{5}}$$

$$= \frac{\left[\frac{2x^{\frac{7}{2}}}{7}\right]_1^4}{\frac{62}{5}}$$

$$= \frac{\frac{2}{7}(4^{\frac{7}{2}}-1)}{\frac{62}{5}}$$

$$= \frac{2}{7}(128-1)\frac{5}{62}$$

$$= \frac{2\cdot 127\cdot 5}{7\cdot 62}$$

$$= \frac{635}{217}$$

$$\bar{y} = \frac{\int_1^4 \frac{y^2}{2}\,dx}{A}$$

$$= \frac{\int_1^4 \frac{(x^{\frac{3}{2}})^2}{2}\,dx}{A}$$

$$= \frac{\int_1^4 \frac{x^3}{2}\,dx}{\frac{62}{5}}$$

$$= \frac{\left[\frac{x^4}{8}\right]_1^4}{\frac{62}{5}}$$

$$= \frac{4^4-1}{8}\left(\frac{5}{62}\right)$$

$$= \frac{(256-1)5}{8\cdot 62}$$

$$= \frac{1275}{496}.$$

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∴ The coordinates of the centroid are

$\left(\frac{635}{217}, \frac{1275}{496}\right)$.

The arc length of C from $x=5$ to $x=28$ is

$$\int_5^{28} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_5^{28} \sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$$

$$= \int_5^{28} \sqrt{1+\frac{9x}{4}}\,dx$$

$$= \left[\frac{8}{27}\left(1+\frac{9x}{4}\right)^{\frac{3}{2}}\right]_5^{28}$$

$$= \frac{8}{27}\left(64^{\frac{3}{2}} - \left(\frac{49}{4}\right)^{\frac{3}{2}}\right)$$

$$= \frac{8}{27}\left(512 - \frac{343}{8}\right)$$

$$= \frac{8}{27}\left(\frac{3753}{8}\right)$$

$$= 139$$

10. $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx,\ n \geq 0$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2}x \cos^2 x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2}x - \cos^{n-2}x \sin^2 x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2}x\,dx - \int_0^{\frac{\pi}{2}} \cos^{n-2}x \sin^2 x\,dx$$

$$= I_{n-2} - \int_0^{\frac{\pi}{2}} \cos^{n-2}x \sin x \sin x\,dx$$

$u = \sin x$ $\qquad dv = \cos^{n-2}x \sin x\,dx$

$du = \cos x\,dx$ $\qquad v = \int \cos^{n-2}x \sin x\,dx$

$w = \cos x$

$dw = -\sin x\,dx$

$$= \int w^{n-2}(-dw)$$

$$= -\int w^{n-2}\,dw$$

$$= \frac{-w^{n-1}}{n-1}$$

$$= -\frac{\cos^{n-1}x}{n-1}$$

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$$= I_{n-2} - \left(\left[\frac{-\sin x \cos^{n-1} x}{n-1} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-\cos^{n-1} x \cos x \, dx}{n-1} \right)$$

$$= I_{n-2} - \left(0 + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{n-1} dx \right)$$

$$= I_{n-2} - \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{n-1} dx$$

$$= I_{n-2} - \frac{1}{n-1} \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$= I_{n-2} - \frac{1}{n-1} I_n$$

$$\left(1 + \frac{1}{n-1}\right) I_n = I_{n-2}$$

$$\left(\frac{n}{n-1}\right) I_n = I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

$x = a\sin^3 t$, $y = a\cos^3 t$, $0 \le t \le \frac{\pi}{2}$

The mean value of y over the interval $0 \le x \le a$, m, is $\frac{1}{a-0} \int_0^a y \, dx$

$x = a$, $a = a\sin^3 t$
$\sin^3 t = 1$
$\sin t = 1$
$t = \frac{\pi}{2}$

$$= \frac{1}{a} \int_0^{\frac{\pi}{2}} a\cos^3 t \, d(a\sin^3 t)$$

$$= \frac{1}{a} \int_0^{\frac{\pi}{2}} a\cos^3 t \, (3a\sin^2 t \cos t \, dt)$$

$$= \frac{1}{a} \int_0^{\frac{\pi}{2}} 3a^2 \sin^2 t \cos^4 t \, dt$$

$$= 3a \int_0^{\frac{\pi}{2}} (1 - \cos^2 t) \cos^4 t \, dt$$

$$= 3a \int_0^{\frac{\pi}{2}} \cos^4 t - \cos^6 t \, dt$$

$$\therefore m = 3a \int_0^{\frac{\pi}{2}} \cos^4 t - \cos^6 t \, dt$$

$$= 3a \int_0^{\frac{\pi}{2}} \cos^4 t \, dt - 3a \int_0^{\frac{\pi}{2}} \cos^6 t \, dt$$

$$= 3a I_4 - 3a I_6$$

$$= 3a (I_4 - I_6)$$

$$= 3a \left(\frac{3}{4} I_2 - \frac{5}{6} I_4 \right)$$

$$= 3a \left(\frac{3}{4} \cdot \frac{1}{2} I_0 - \frac{5}{6} \cdot \frac{3}{4} \cdot I_2 \right)$$

$$= 3a \left(\frac{3}{4} \cdot \frac{1}{2} I_0 - \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0 \right)$$

$$= 3a \left(\frac{3}{8} I_0 - \frac{15}{48} I_0 \right)$$

$$= 3a \left(\frac{I_0}{16} \right)$$

$$= \frac{3a}{16} I_0$$

CONFIDENTIAL

$$I_0 = \int_0^{\frac{\pi}{2}} \cos^0 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= [x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

$$\therefore m = \frac{39}{16}\left(\frac{\pi}{2}\right)$$

$$= \frac{39\pi}{32}$$

11. EITHER

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$\frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

$$\tan 3\theta = \frac{\dfrac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta}}{\dfrac{\cos^3\theta - 3\cos\theta\sin^2\theta}{\cos^3\theta}}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

If $\tan 3\theta = 1$

$$3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$$

Also, $\dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 1$

$$3\tan\theta - \tan^3\theta = 1 - 3\tan^2\theta$$

$$\tan^3\theta - 3\tan^2\theta - 3\tan\theta + 1 = 0$$

If $t = \tan\theta$, $t^3 - 3t^2 - 3t + 1 = 0$

since $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$,

$t = \tan\theta = \tan\frac{\pi}{12}, \tan\frac{5\pi}{12}, \tan\frac{3\pi}{4}$

Since $t = \tan\frac{3\pi}{4} = -1$ is a root of

$t^3 - 3t^2 - 3t + 1 = 0$

$$
\begin{array}{r l}
 & t^2 - 4t + 1 \\
t+1 & \overline{\big)\, t^3 - 3t^2 - 3t + 1} \\
 & \underline{t^3 + t^2} \\
 & -4t^2 - 3t \\
 & \underline{-4t^2 - 4t} \\
 & t + 1 \\
 & \underline{t + 1} \\
 & 0
\end{array}
$$

$(t+1)(t^2 - 4t + 1) = 0$

$t + 1 = 0, \quad t^2 - 4t + 1 = 0$

$(t-2)^2 = 3$

$t - 2 = \pm\sqrt{3}$

$t = 2 \pm \sqrt{3}$

Since $\tan\frac{5\pi}{12} > \tan\frac{\pi}{12}$,

$\tan\frac{\pi}{12} = 2 - \sqrt{3}, \quad \tan\frac{5\pi}{12} = 2 + \sqrt{3}, \quad \tan\frac{3\pi}{4} = -1$

OR

$C: y = \frac{x^2 + \lambda x - 6\lambda^2}{x+3}, \quad \lambda \neq 1, \lambda \neq -\frac{3}{2}$.

i)

$$
\begin{array}{r l}
 & x + \lambda - 3 \\
x+3 & \overline{\big)\, x^2 + \lambda x - 6\lambda^2} \\
 & \underline{x^2 + 3x} \\
 & (\lambda - 3)x - 6\lambda^2 \\
 & \underline{(\lambda - 3)x + 3\lambda - 9} \\
 & -6\lambda^2 - 3\lambda + 9
\end{array}
$$

$y = x + \lambda - 3 + \frac{-6\lambda^2 - 3\lambda + 9}{x+3}$

$\frac{dy}{dx} = 1 + \frac{6\lambda^2 + 3\lambda - 9}{(x+3)^2}$

$= \frac{x^2 + 6x + 9 + 6\lambda^2 + 3\lambda - 9}{(x+3)^2}$

$= \frac{x^2 + 6x + 6\lambda^2 + 3\lambda}{(x+3)^2}$

when $\frac{dy}{dx} = 0$: $\frac{x^2 + 6x + 6\lambda^2 + 3\lambda}{(x+3)^2} = 0$

$x^2 + 6x + 6\lambda^2 + 3\lambda = 0$

$(x+3)^2 = -6\lambda^2 - 3\lambda + 9$

$= -3(2\lambda^2 + \lambda - 3)$

$= -3(\lambda - 1)(2\lambda + 3)$

If C has two stationary points,

$(\lambda-1)(2\lambda+3)<0$

$\frac{-3}{2}<\lambda<1$.

ii) As $x\to\pm\infty$ $y\to x+\lambda-3$

As $x\to -3$ $y\to\pm\infty$

$\therefore$ The asymptotes of C are $y=x+\lambda-3$ and $x=-3$.

iii) when $x=0$: $y=-2\lambda$

when $y=0$: $\frac{x^2+\lambda x-6\lambda^2}{x+3}=0$

$x^2+\lambda x-6\lambda^2=0$

$(x+\frac{\lambda}{2})^2=6\lambda^2+\frac{\lambda^2}{4}$

$=\frac{25\lambda^2}{4}$

$x+\frac{\lambda}{2}=\pm\frac{5\lambda}{2}$

$x=-3\lambda, 2\lambda$

If $0<\lambda<1$,

$\frac{dy}{dx}=1+\frac{6\lambda^2+3\lambda-9}{(x+3)^2}$

$=1+\frac{3(2\lambda^2+\lambda-3)}{(x+3)^2}$

$=1+\frac{3(2\lambda+3)(\lambda-1)}{(x+3)^2}$

$\frac{d^2y}{dx^2}=\frac{-6(2\lambda+3)(\lambda-1)}{(x+3)^3}$

when $\frac{dy}{dx}=0$: $x=-3\pm\sqrt{3(1-\lambda)(2\lambda+3)}$

$y=\lambda-6\pm\sqrt{3(1-\lambda)(2\lambda+3)}$

$\frac{-3(\lambda-1)(2\lambda+3)}{\pm\sqrt{3(1-\lambda)(2\lambda+3)}}$

$=\lambda-6\pm\sqrt{3(1-\lambda)(2\lambda+3)}$

$\pm\sqrt{3(1-\lambda)(2\lambda+3)}$

$=\lambda-6\pm 2\sqrt{3(1-\lambda)(2\lambda+3)}$

when $x=-3+\sqrt{3(1-\lambda)(2\lambda+3)}$:

$\frac{d^2y}{dx^2}=\frac{-6(2\lambda+3)(\lambda-1)}{-3(\lambda-1)(2\lambda+3)\sqrt{3(1-\lambda)(2\lambda+3)}}$

$=\frac{2}{\sqrt{3(1-\lambda)(2\lambda+3)}}>0$

when $x=-3-\sqrt{3(1-\lambda)(2\lambda+3)}$:

$\frac{d^2y}{dx^2}=\frac{-6(2\lambda+3)(\lambda-1)}{-3(\lambda-1)(2\lambda+3)(-\sqrt{3(1-\lambda)(2\lambda+3)})}$

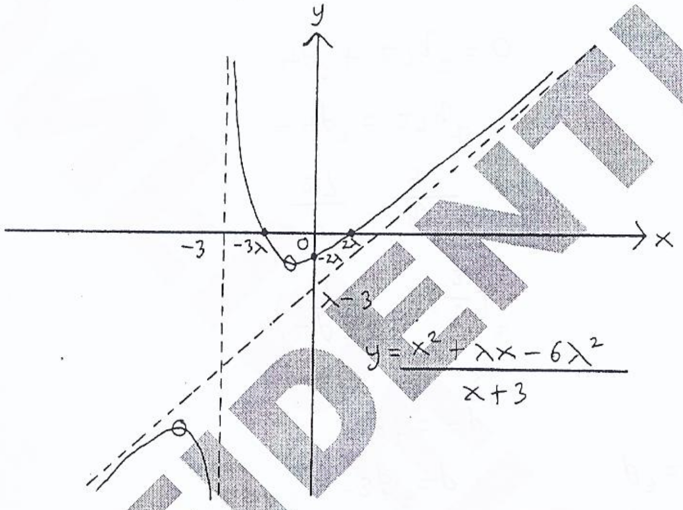
$=\frac{-2}{\sqrt{3(1-\lambda)(2\lambda+3)}}<0$

$\therefore \left(-3+\sqrt{3(1-\lambda)(2\lambda+3)},\ \lambda-6+2\sqrt{3(1-\lambda)(2\lambda+3)}\right)$

is a minimum point and

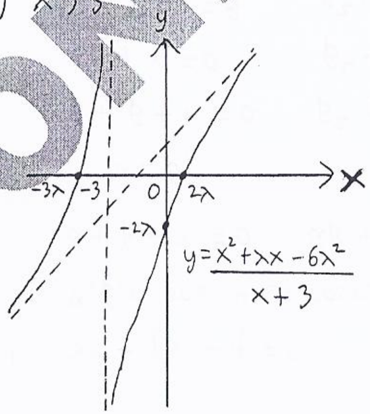
$\left(-3-\sqrt{3(1-\lambda)(2\lambda+3)},\ \lambda-6-2\sqrt{3(1-\lambda)(2\lambda+3)}\right)$

is a maximum point.



o : stationary point
• : intersection point.

iv) $\lambda > 3$



or