CHALLENGING QUESTIONS ON INDUCTION

1.

(i) Prove by mathematical induction that $\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$. [5]

(ii) **(a)** Prove by the method of differences that $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$. [4]

(b) Explain why $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$ is a convergent series, and state the value of the sum to infinity. [2]

2. **(i)** Show that $\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} = \frac{A}{n^3-n}$, where A is a constant to be found. [2]

(ii) Hence find $\sum_{r=2}^{n} \frac{1}{r^3-r}$. (There is no need to express your answer as a single algebraic fraction.) [3]

(iii) Give a reason why the series $\sum_{r=2}^{\infty} \frac{1}{r^3-r}$ converges, and write down its value. [2]

3. **The nth term of a sequence is given by**

$$u_n = n(2n+1),$$

for $n \geqslant 1$. The sum of the first n terms is denoted by S_n. Use the method of mathematical induction to show that

$$S_n = \tfrac{1}{6}n(n+1)(4n+5)$$

for all positive integers n. **[5]**

4. A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and

$$u_{n+1} = u_n - \frac{2n+1}{n^2(n+1)^2}, \quad \text{for all } n \geqslant 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \frac{1}{n^2}$. [4]

(ii) Hence find $\sum_{n=1}^{N} \frac{2n+1}{n^2(n+1)^2}$. [2]

(iii) Give a reason why the series in part (ii) is convergent and state the sum to infinity. [2]

(iv) Use your answer to part (ii) to find $\sum_{n=2}^{N} \frac{2n-1}{n^2(n-1)^2}$. [2]

5. Use the method of mathematical induction to prove that the sum of the first n terms of the series

$$(1\times3\times5)+(2\times4\times6)+(3\times5\times7)+\cdots$$

is $\frac{1}{4}n(n+1)(n+4)(n+5)$. [5]

6. i) Show that

$$\frac{3}{2r-1}-\frac{2}{2r+1}-\frac{1}{2r+3}=\frac{16r+16}{(2r-1)(2r+1)(2r+3)}$$ [2]

ii) Hence find

$$\sum_{r=1}^{n}\frac{r+1}{(2r-1)(2r+1)(2r+3)},$$

giving your answer in the form $k-\mathrm{f}(n)$, where k is a constant. [4]

iii) State the sum to infinity of the series in which the rth term is $\frac{r+1}{(2r-1)(2r+1)(2r+3)}$. [1]