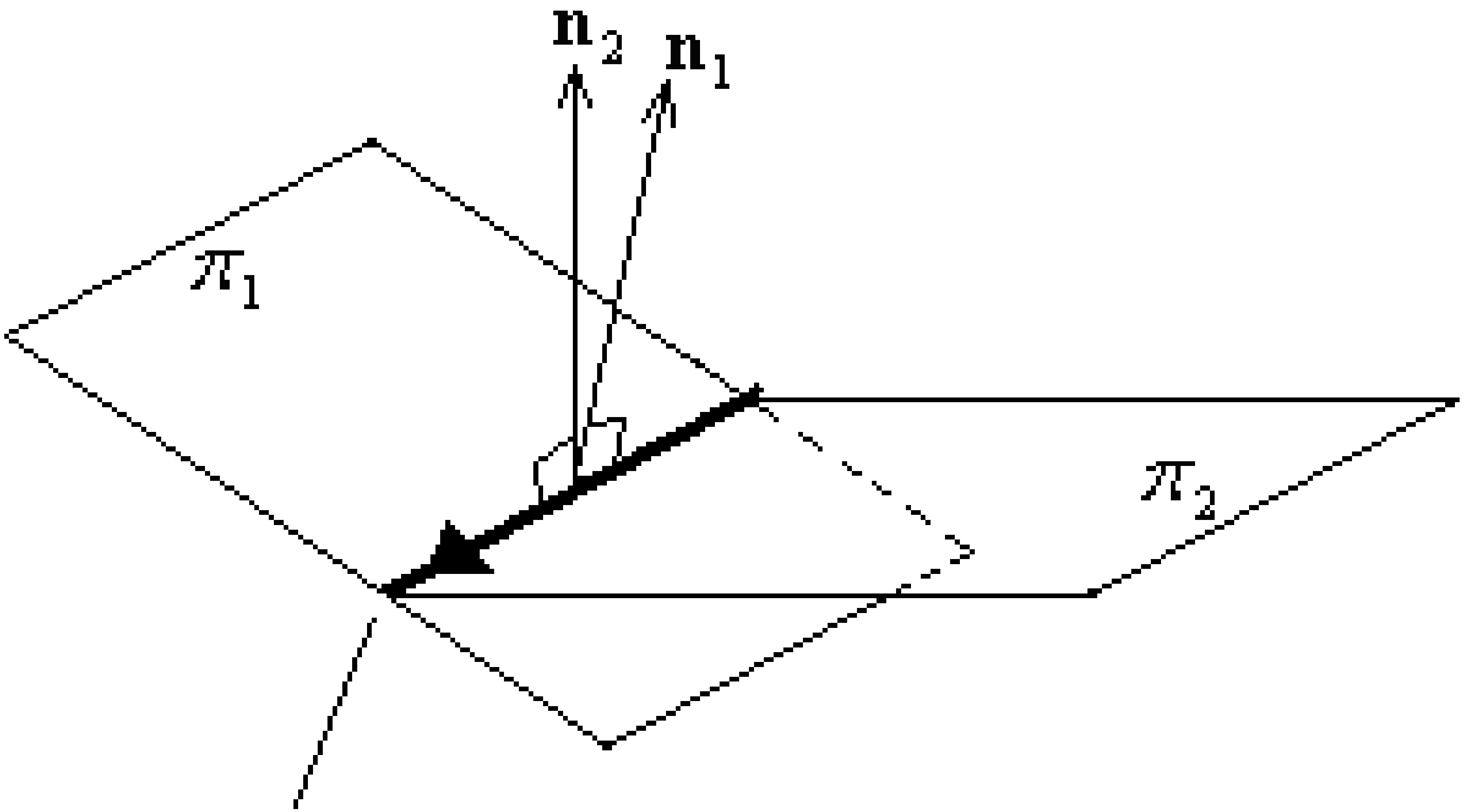
Common Perpendicular between Two Planes

The direction of the common perpendicular to the planes $\pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = p_1$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = p_2$ is given by $\mathbf{n}_1 \times \mathbf{n}_2$, which is parallel to the line of intersection between π_1 and π_2.



direction of the common perpendicular between two planes = direction of $\mathbf{n}_1 \times \mathbf{n}_2$

Example:

Find the vector parallel to the common perpendicular of the planes $\pi_1 : \mathbf{r} \cdot \left(-5\mathbf{i} + 11\mathbf{j} + \mathbf{k}\right) = 34$ and $\pi_2 : \mathbf{r} \cdot \left(-2\mathbf{i} + 4\mathbf{j} + \mathbf{k}\right) = 9.$

Example:

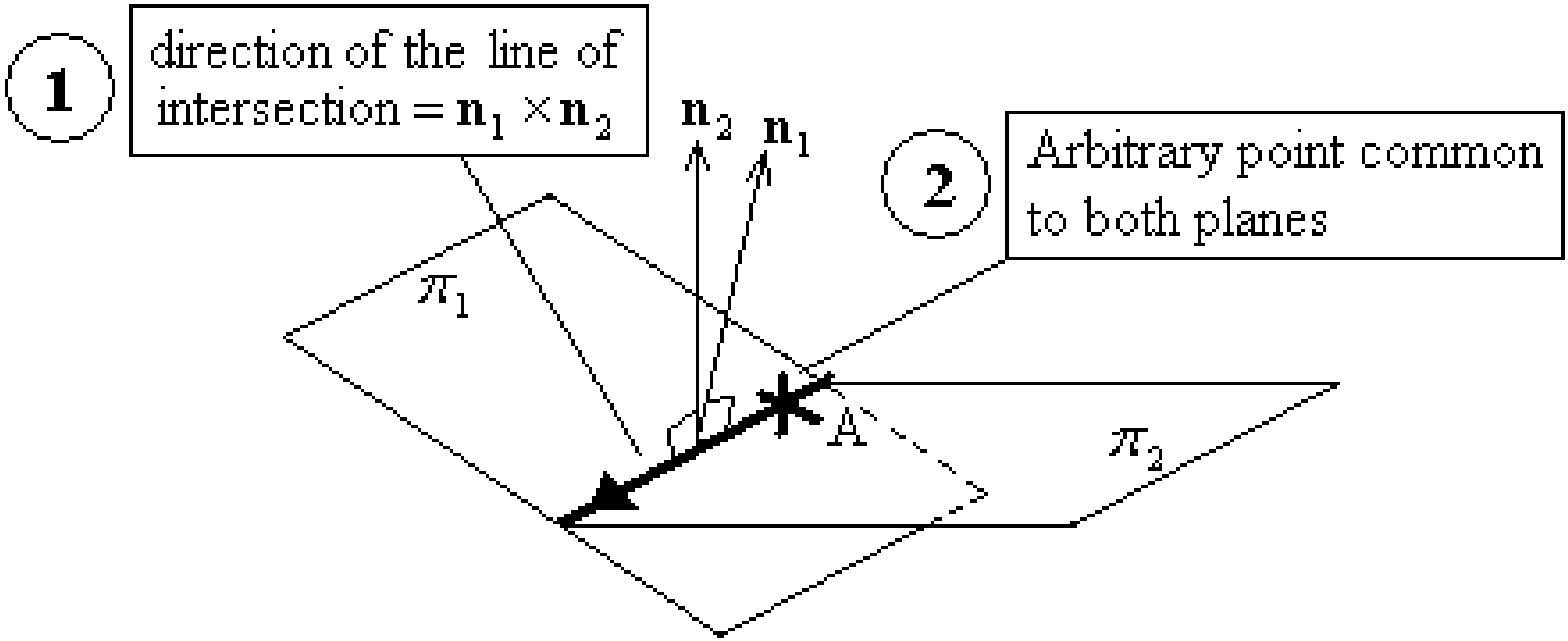
Find the unit vector of the common perpendicular of the planes $\pi_1 : \mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) = 10$ and $\pi_2 : \mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = -4$.

Example:

Show that the line of intersection of the planes $\pi_1 : \mathbf{r} \cdot \left(\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}\right) = 10$ and $\pi_2 : \mathbf{r} \cdot \left(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}\right) = -4$ is parallel to the vector $-\mathbf{i} + 17\mathbf{j} + 5\mathbf{k}$.

Line of Intersection between Two Planes

The equation of the line intersection between two planes $\pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = p_1$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = p_2$, which contain a common point A, is given by $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{n}_1 \times \mathbf{n}_2)$



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Equation of the line of intersection

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{n}_1 \times \mathbf{n}_2)$$

Example:

Find the equation of the line of intersection of the planes $\pi_1 : \mathbf{r} \cdot \left(-5\mathbf{i} + 11\mathbf{j} + \mathbf{k}\right) = 34$ and $\pi_2 : \mathbf{r} \cdot \left(-2\mathbf{i} + 4\mathbf{j} + \mathbf{k}\right) = 9$.

Example:

Find the equation of the line of intersection of the planes $\pi_1 : \mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) = 10$ and $\pi_2 : \mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = -4$.

Example:

Find the equation of the line of intersection of the planes $\pi_1 : \mathbf{r}\cdot\left(\mathbf{i}-2\mathbf{j}+7\mathbf{k}\right)=10$ and $\pi_2 : \mathbf{r}\cdot\left(2\mathbf{i}+\mathbf{j}-3\mathbf{k}\right)=-4.$

Intersection of Three Planes

Let three planes be $\pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$, $\pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$, and $\pi_3 : \mathbf{r} \cdot \mathbf{n}_3 = d_3$. There are three possibility to the nature of the solution of the system of equations.

(a) Unique solution,

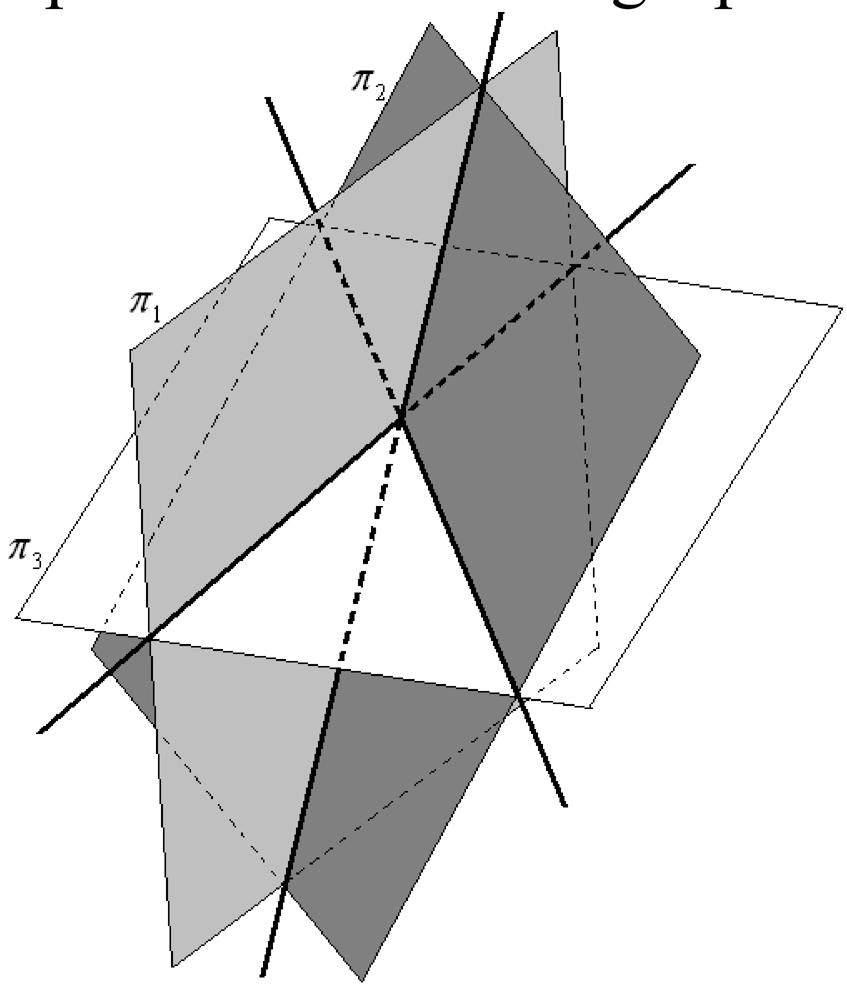
(b) infinite number of solutions (three types), &

(c) no solution (four types)

(a) Unique solution:

~ If $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are <u>not coplanar</u> i.e. $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) \neq 0$,

~ then, all three planes meet at a single point.



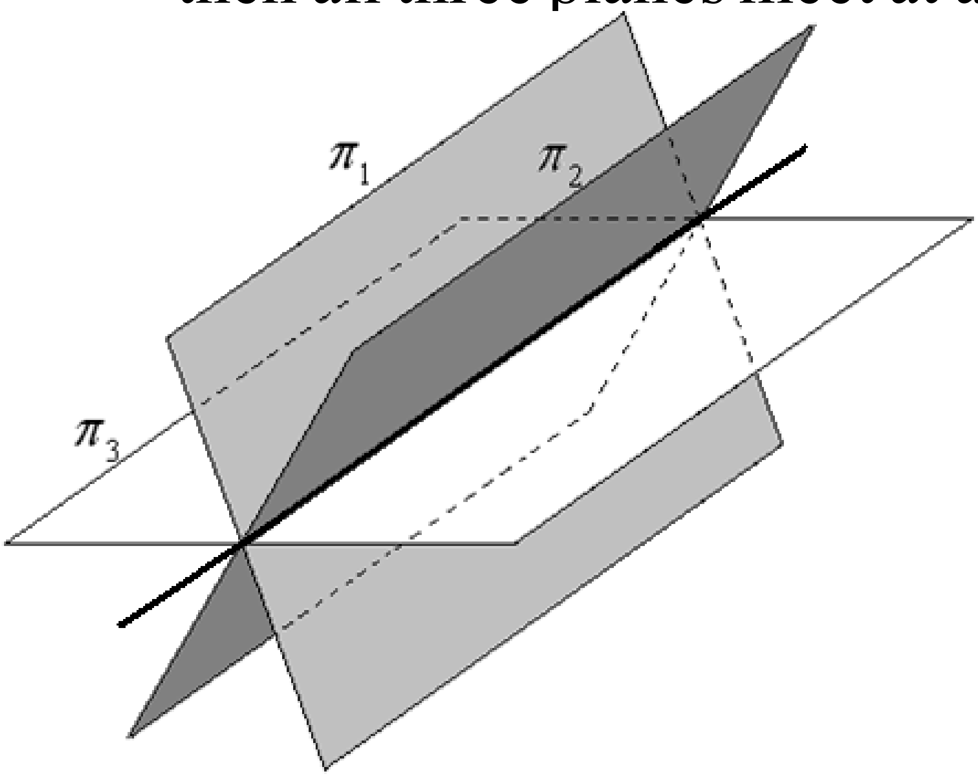
Example:

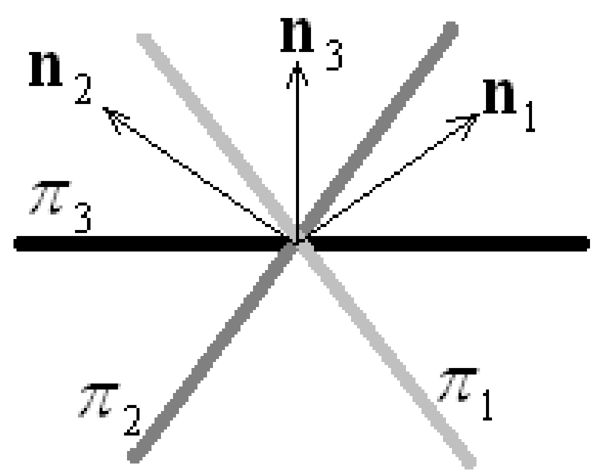
Let three planes be $\pi_1 : \mathbf{r} \cdot (\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = -9$,
$\pi_2 : \mathbf{r} \cdot (2\mathbf{i} - 5\mathbf{j} + \mathbf{k}) = 3$ and $\pi_3 : \mathbf{r} \cdot (-3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) = 8$.
Explain, in terms of normal vectors, the nature of intersection of all planes. Find the intersection if there is any.

$(b)(i)$ Infinite number of solutions (Type 1)

~ If all planes are <u>not parallel</u>, $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are <u>coplanar</u> i.e. $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) = 0$, and <u>one scalar equation</u> can be obtained by combining two equations,

~ then all three planes meet at a line and form a <u>sheaf</u>.





Example:

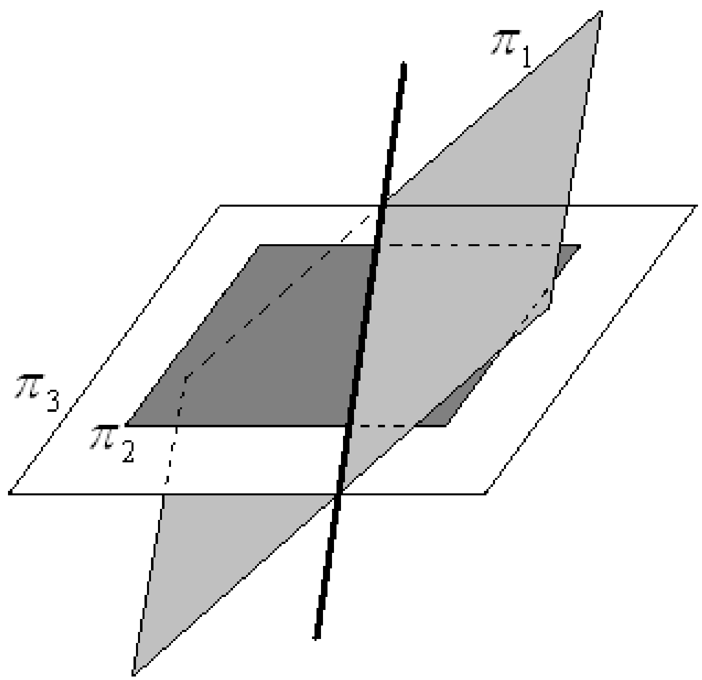
Let three planes be $\pi_1 : x + y + 2z = -2$, $\pi_2 : 3x - y + 14z = 6$ and $\pi_3 : x + 2y = -5$.

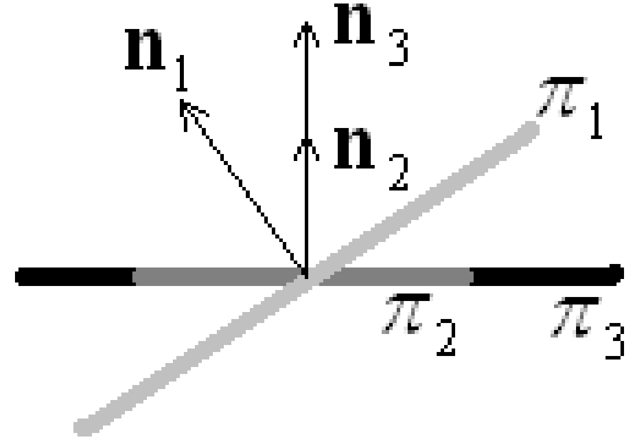
Explain, in terms of normal vectors, the nature of intersection of all planes. Find the intersection if there is any.

(b)(ii) Infinite number of solutions (Type 2)

~ If two planes are <u>coincident</u> i.e two scalar equation are equivalent, and the third plane is <u>not parallel</u>,

~ then all planes meet at a line.





Example:

Let three planes be $\pi_1 : x + y - z = 2$,

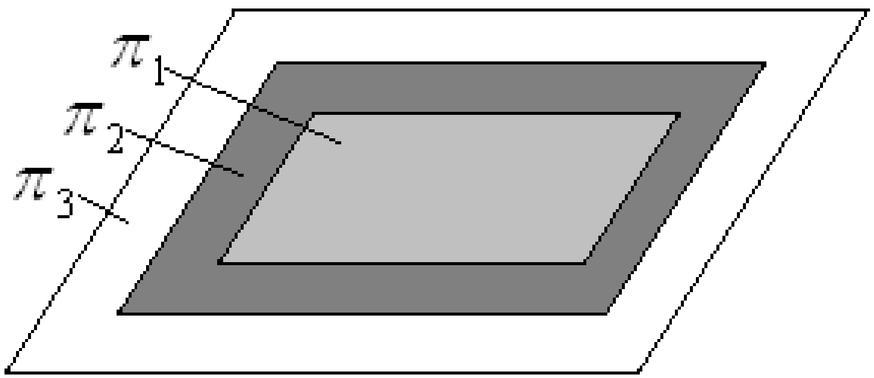
$\pi_2 : x - 2y + z = 4$ and $\pi_3 : 2x - 4y + 2z = 8$.

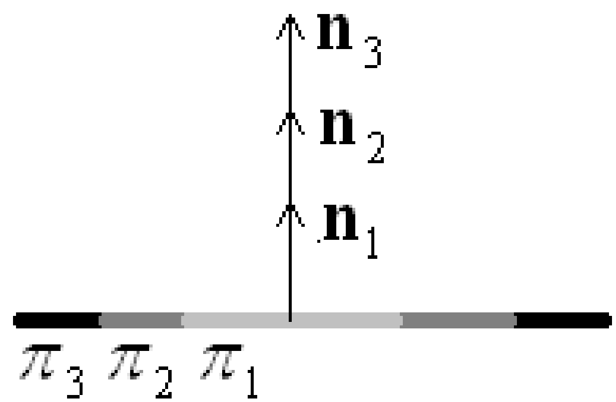
Explain the nature of the intersection without solving the equations. Find the intersection if there is any.

$\left(\text{b}\right)\left(\text{iii}\right)$ Infinite number of solutions $\left(\text{Type } 3\right)$

~ If all three planes are <u>coincident</u>,

~ then all planes meet at a plane.





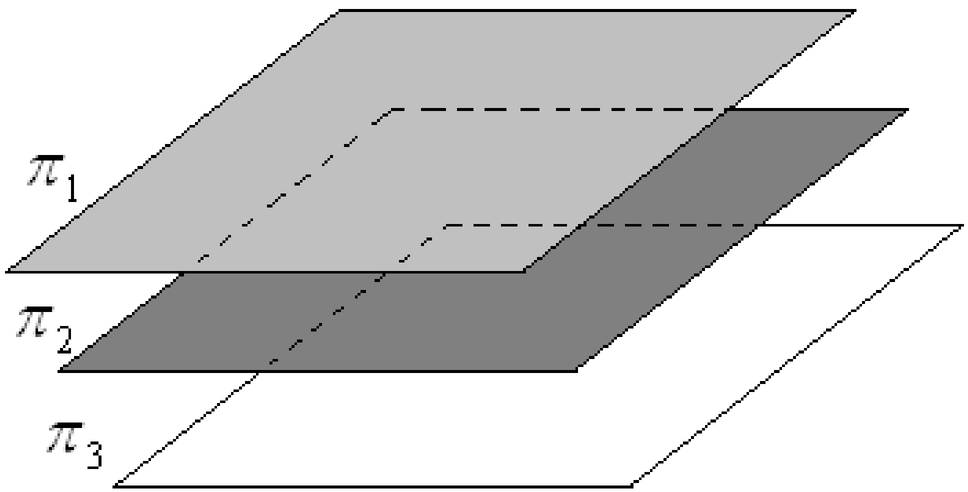
Example:

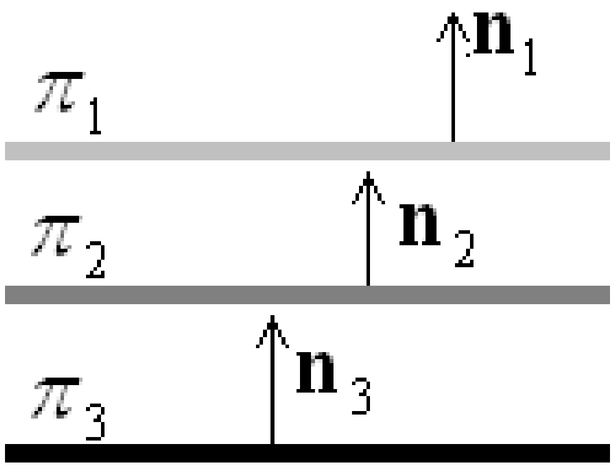
Let three planes be $\pi_1 : \mathbf{r} \cdot (\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 1$,
$\pi_2 : \mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}) = 2$ and $\pi_3 : \mathbf{r} \cdot (-4\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}) = -4$.
Explain the nature of the intersection without solving the equations. Find the intersection if there is any.

(c)(i) No solution (Type 1)

~ If all three planes are <u>parallel</u> and <u>non - coincident</u>,

~ then all planes do not intersect.





Example:

Let three planes be $\pi_1 : x + 2y + 3z = 1$,

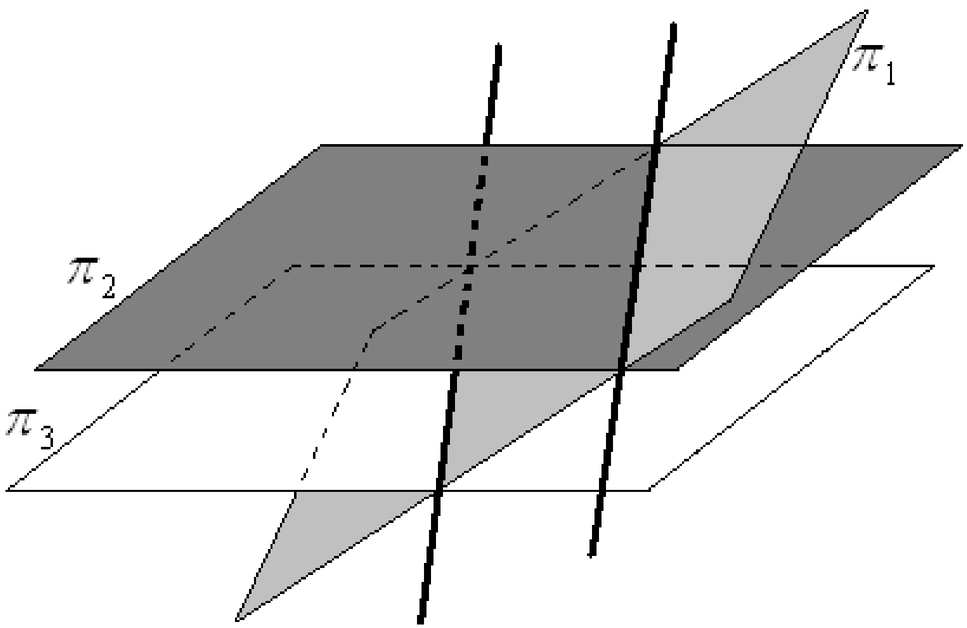
$\pi_2 : 2x + 4y + 6z = -1$ and $\pi_3 : -x - 2y - 3z = 3$.

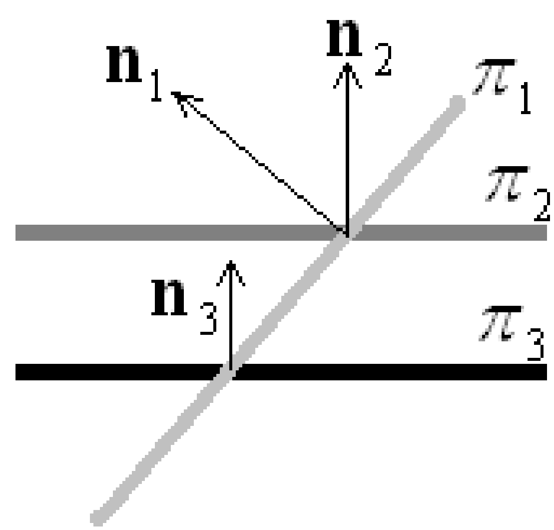
Explain the nature of the intersection without solving the equations. Find the intersection if there is any.

(c)(ii) No solution (Type 2)

~ If two planes are parallel and non-coincident and the third one is not parallel.

~ then all planes do not intersect, but form a H configuration.





Example:

Let three planes be $\pi_1 : \mathbf{r} \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k}) = 1$,

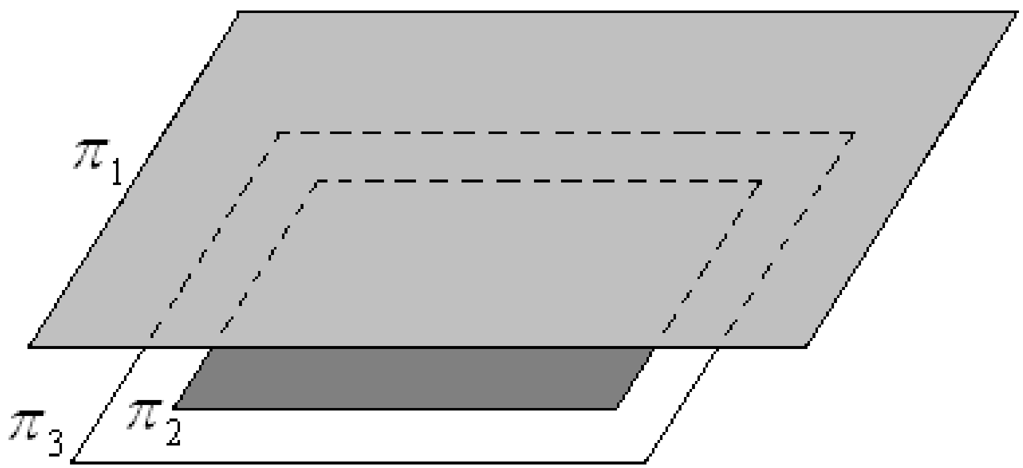
$\pi_2 : \mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$ and $\pi_3 : \mathbf{r} \cdot (-2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 3$.

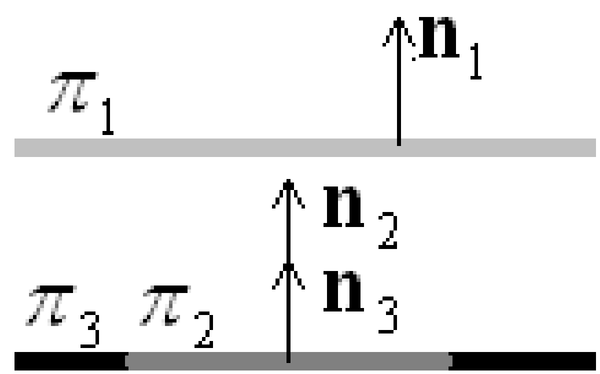
Explain the nature of the intersection without solving the equations. Find the intersection if there is any.

$(c)(iii)$ No solution (Type 3)

$\sim$ If three planes are parallel but only two are coincident.

$\sim$ then all planes do not intersect, but the the coincident planes intersect at a plane.





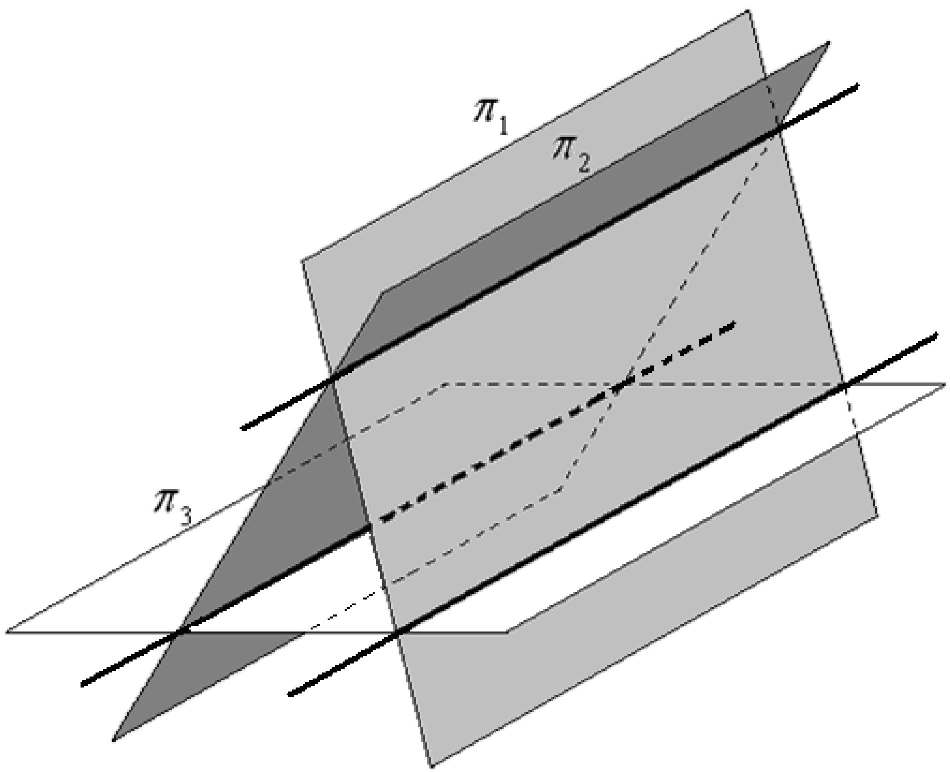
Example:

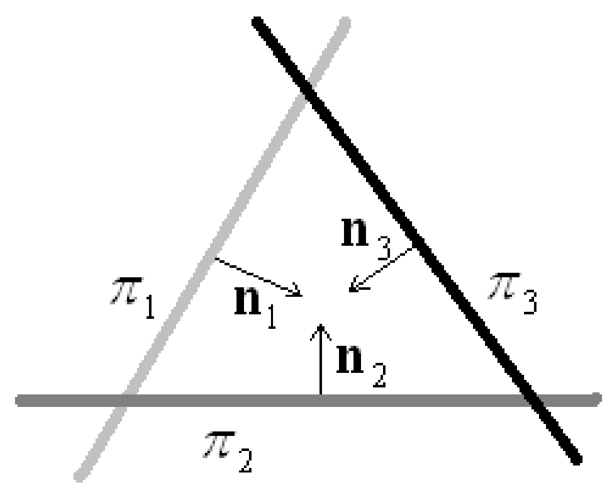
Let three planes be $\pi_1 : \mathbf{r} \cdot \left(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \right) = 1$,
$\pi_2 : \mathbf{r} \cdot \left(3\mathbf{i} + 6\mathbf{j} + 9\mathbf{k} \right) = 3$ and $\pi_3 : \mathbf{r} \cdot \left(-2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k} \right) = 2$.
Explain the nature of the intersection without solving the equations. Find the intersection if there is any.

$(c)(iv)$ No solution (Type 4)

~ If three planes are not <u>parallel</u>, $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are <u>coplanar</u> i.e. $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) = 0$ but <u>no scalar equation</u> is the <u>combination</u> of the other two,

~ then all planes do not intersect, but form a Δ configuration.





Example:

Let three planes be $\pi_1 : 2x + y + z = 1$,

$\pi_2 : -x + y + z = -1$ and $\pi_3 : x + y + z = 0$.

Explain the nature of the intersection without solving the equations. Find the intersection if there is any.

Summary of nature of intersection of three planes.

(1) **Not Coplanar** => point.

(2) **Coplanar** &

(a) **all not parallel** &

(i) **no unique scalar equation** => no intersection (Δ).

(ii) **unique scalar equation** => line (sheaf).

(b) **two parallel** &

(i) **not coincident** => no intersection (H).

(ii) **both coincident** => line.

(c) **all parallel** &

(i) **not coincident** => no intersection.

(ii) **two coincident** => no intersection.

(iii) **all coincident** => plane.

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example:

Find the vector equations of the lines of intersection of the planes $x - y + z = 7$ and $x + y + 3z = 5$.

Example :

Find the Cartesian equation of the line of intersection of the planes

$x - 2y + z = 3$ and $3x + y - 2z = 4.$

Example:

Find the cartesian equation of the plane which passes through the point $(1,2,3)$ and contains the line of intersection of the planes $2x - y + z = 4$ and $x + y + z = 4$.

Example:

Two planes are defined by the equations $x+2y+z=4$ and $2x-3y=6$. Find

(a) the acute angle between the planes.

(b) a vector equation of the line of intersection.

Example:

Determine the value of the constant k for which the planes
$\pi_1 : \mathbf{r}\cdot\left(2\mathbf{i}-\mathbf{j}-\mathbf{k}\right)=3$, $\pi_2 : \mathbf{r}\cdot\left(-4\mathbf{i}+7\mathbf{j}+3\mathbf{k}\right)=-5$ and
$\pi_3 : \mathbf{r}\cdot\left(k\mathbf{i}+\mathbf{j}-\mathbf{z}\right)=5$ do not intersect at a single point.

For this value of k, determine the nature of the intersection of the planes.

Example: (Try this question using geometry)

Determine the value of the constant a for which the system of

equations $\begin{cases} 3x+2y-z=10 \\ 5x-y-4z=17 \\ x+5y+az=b \end{cases}$ do not have a unique solution. Show

that, for this value of a, the system is consistent only when $b=3$.

For the case where the system represents three planes intersecting at a line l, find the Cartesian equation of l.