Example:

A sequence of numbers is given by $u_0 = 1, u_1 = 3$ and $u_n = 2u_{n-1} - 2u_{n-2}$ for $n \geq 2$. Prove by mathematical induction that $u_n = \sqrt{2}^k \left( \cos \frac{\pi}{4} k + 2 \sin \frac{\pi}{4} k \right)$.

Basic case: $u_0 = \sqrt{2}^0 \left[\cos\frac{\pi}{4}(0) + 2\sin\frac{\pi}{4}(0)\right] = 1$

$u_1 = \sqrt{2}^1 \left[\cos\frac{\pi}{4}(1) + 2\sin\frac{\pi}{4}(1)\right] = 3$

$\therefore$ $u_k$ is true for $n=0$ & $n=1$.

Inductive: To prove that $u_{k+2} = \sqrt{2}^{k+2}\left[\cos\frac{\pi}{4}(k+2) + 2\sin\frac{\pi}{4}(k+2)\right]$ if

$u_{k+1} = \sqrt{2}^{k+1}\left[\cos\frac{\pi}{4}(k+1) + 2\sin\frac{\pi}{4}(k+1)\right]$ & $u_k = \sqrt{2}^k\left[\cos\frac{\pi}{4}k + 2\sin\frac{\pi}{4}k\right]$

are true.

$$u_{k+2}$$

$$= 2u_{k+1} - 2u_k$$

$$= 2\sqrt{2}^{k+1}\left[\cos\frac{\pi}{4}(k+1) + 2\sin\frac{\pi}{4}(k+1)\right] - 2\sqrt{2}^k\left[\cos\frac{\pi}{4}(k) + 2\sin\frac{\pi}{4}k\right]$$

$$= 2\sqrt{2}^{k+1}\left[\cos\frac{\pi}{4}k\cos\frac{\pi}{4} - \sin\frac{\pi}{4}k\sin\frac{\pi}{4} + 2\sin\frac{\pi}{4}k\cos\frac{\pi}{4} + 2\cos\frac{\pi}{4}k\sin\frac{\pi}{4}\right] - 2\sqrt{2}^k\left[\cos\frac{\pi}{4}k + 2\sin\frac{\pi}{4}k\right]$$

$$= 2\sqrt{2}^{k+1}\left[\frac{1}{\sqrt{2}}\cos\frac{\pi}{4}k - \frac{1}{\sqrt{2}}\sin\frac{\pi}{4}k + \frac{2}{\sqrt{2}}\sin\frac{\pi}{4}k + \frac{2}{\sqrt{2}}\cos\frac{\pi}{4}k\right] - 2\sqrt{2}^k\left[\cos\frac{\pi}{4}k + 2\sin\frac{\pi}{4}k\right]$$

$$= 6\sqrt{2}^k\cos\frac{\pi}{4}k + 2\sqrt{2}^k\sin\frac{\pi}{4}k - 2\sqrt{2}^k\cos\frac{\pi}{4}k - 4\sqrt{2}^k\sin\frac{\pi}{4}k$$

$$= 4\sqrt{2}^k\cos\frac{\pi}{4}k - 2\sqrt{2}^k\sin\frac{\pi}{4}k$$

$$= \sqrt{2}^{k+2}\left[2\cos\frac{\pi}{4}k - \sin\frac{\pi}{4}k\right]$$

$$= \sqrt{2}^{k+2}\left[2\cos\frac{\pi}{4}k\sin\frac{\pi}{2} + 2\sin\frac{\pi}{4}k\cos\frac{\pi}{2} + \cos\frac{\pi}{4}k\cos\frac{\pi}{2}k - \sin\frac{\pi}{4}k\sin\frac{\pi}{2}\right]$$

$$= \sqrt{2}^{k+2}\left[2\sin\frac{\pi}{4}(k+2) + \cos\frac{\pi}{4}(k+2)\right]$$

$\therefore$ by induction, $u_k$ is true for $n = 0, 1, 2, 3, \ldots$