1. $$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n^2-1}+n}$$

$$= \sum_{r=1}^{n^2-1} \frac{1}{\sqrt{r}+\sqrt{r+1}}$$

$$= \sum_{r=1}^{n^2-1} \frac{1}{\sqrt{r}+\sqrt{r+1}} \frac{(\sqrt{r}-\sqrt{r+1})}{(\sqrt{r}-\sqrt{r+1})}$$

$$= \sum_{r=1}^{n^2-1} \frac{\sqrt{r}-\sqrt{r+1}}{r-(r+1)}$$

$$= \sum_{r=1}^{n^2-1} \sqrt{r+1} - \sqrt{r}$$

$$\begin{aligned}
= \; & n - \sqrt{n^2-1} \\
& + \sqrt{n^2-1} - \sqrt{n^2-2} \\
& + \sqrt{n^2-2} - \sqrt{n^2-3} \\
& \quad \vdots \\
& + 2 - \sqrt{3} \\
& + \sqrt{3} - \sqrt{2} \\
& + \sqrt{2} - 1
\end{aligned}$$

$$= n - 1$$

2. Let $f(n) = n^3 + 11n$

when $n=1$: $f(1) = 1^3 + 11(1) = 1 + 11 = 12 = 6(2)$

$\therefore 6 \mid f(1)$

Assume the statement is true when $n=k$.

$n=k$: $f(k) = k^3 + 11k$

$6 \mid f(k)$

$\therefore f(k) = 6s$, s is an integer.

$\therefore k^3 + 11k = 6s$

when $n=k+1$: $f(k+1) = (k+1)^3 + 11(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 11k + 11$

$= k^3 + 11k + 3k^2 + 3k + 12$

$= 6s + 3(k^2 + k + 4)$

$= 6s + 12 + 3k(k+1)$

If k is even, $k=2m$: $f(k+1) = 6s + 12 + 3(2m)(2m+1)$

$= 6(s + 2 + m(2m+1))$

Since s is an integer and m is an integer,
$s + 2 + 2m^2 + m$ is an integer.

$\therefore 6 \mid f(k+1)$

If k is odd, $k = 2m+1$: $f(k+1) = 6s + 12 + 3(2m+1)(2m+2)$

$= 6(s + 2 + (2m+1)(m+1))$

Since s is an integer and m is an integer,
$s + 2 + (2m+1)(m+1)$ is an integer.

$\therefore 6 \mid f(k+1)$

$\therefore n^3 + 11n$ is divisible by 6 for every positive integer n.

3.i) $x^3 + ax^2 + bx + c = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = -a \quad \alpha\beta + \alpha\gamma + \beta\gamma = b \quad \alpha\beta\gamma = -c$

If $\alpha + \beta = \gamma$,

$\gamma + \gamma = -a \qquad \alpha\beta + (\alpha+\beta)\gamma = b$

$2\gamma = -a \qquad \alpha\beta + \gamma^2 = b$

$\gamma = -\frac{a}{2}$

$\alpha\beta\left(-\frac{a}{2}\right) = -c$

$a\alpha\beta = 2c$

$\alpha\beta = \frac{2c}{a}$

$\frac{2c}{a} + \frac{a^2}{4} = b$

$8c + a^3 = 4ab$

ii) $\frac{1}{x} + \frac{1}{a} + \frac{1}{b} = \frac{1}{x+a+b}$

$\frac{ab + (a+b)x}{ab} = \frac{1}{x+a+b}$

$(x+a+b)(ab + (a+b)x) = ab$

$(a+b)x^2 + (a+b)^2x + abx + (a+b)ab = abx$

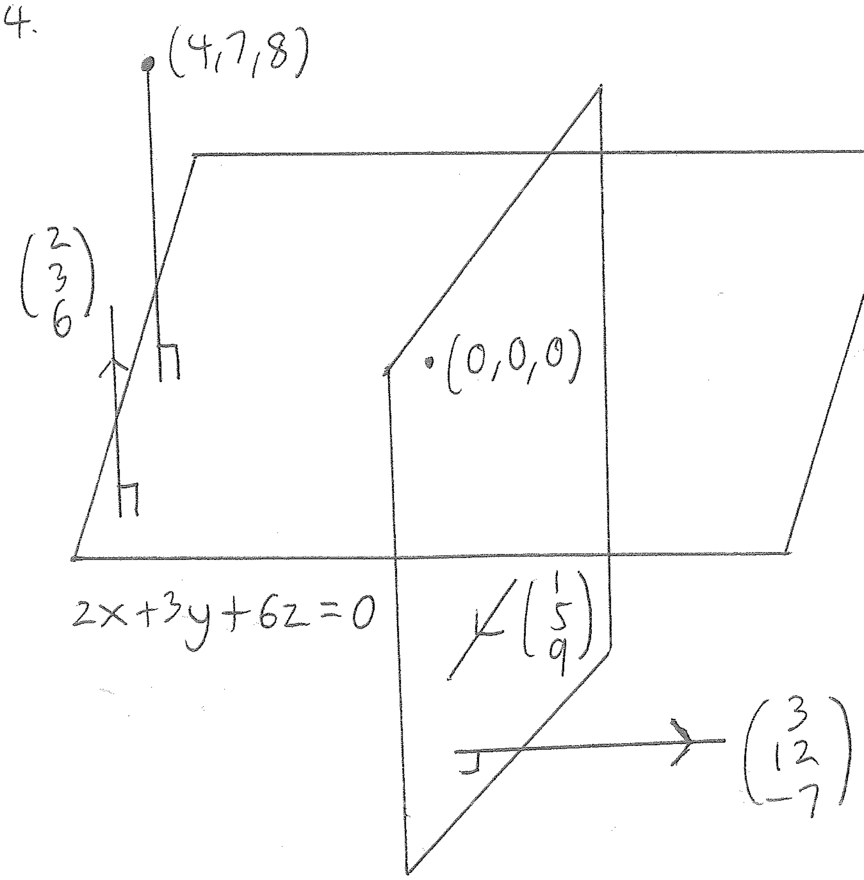
$(a+b)x^2 + (a+b)^2x + (a+b)ab = 0$

$(a+b)(x^2 + (a+b)x + ab) = 0$

$(a+b)(x+a)(x+b) = 0$

$(a+b) \neq 0 \therefore x = -a, -b$

4.



$2x + 3y + 6z = 0 \quad (4,7,8)$

since $\begin{pmatrix}2\\3\\6\end{pmatrix}$ is normal to the plane, the equation of the line perpendicular to the plane through $(4,7,8)$ is $\underset{\sim}{r} = \begin{pmatrix}4\\7\\8\end{pmatrix} + s\begin{pmatrix}2\\3\\6\end{pmatrix}$.

when the line meets the plane $2x+3y+6z=0$,

$x = 2s+4, \; y = 3s+7, \; z = 6s+8$.

$$2(2s+4) + 3(3s+7) + 6(6s+8) = 0$$

$$4s + 8 + 9s + 21 + 36s + 48 = 0$$

$$49s = -77$$

$$s = \frac{-11}{7}$$

$\therefore$ The line meets the plane at the point

$\left(\frac{6}{7}, \frac{16}{7}, -\frac{10}{7}\right)$.

$\therefore$ The perpendicular distance from $(4,7,8)$ to the plane is

$$\sqrt{\left(4-\frac{6}{7}\right)^2+\left(7-\frac{16}{7}\right)^2+\left(8+\frac{10}{7}\right)^2}$$

$$= \sqrt{\frac{22^2}{7^2}+\frac{33^2}{7^2}+\frac{66^2}{7^2}}$$

$$= \sqrt{\frac{11^2(2^2+3^2+6^2)}{7^2}}$$

$$= \sqrt{\frac{11^2(4+9+36)}{7^2}}$$

$$= 11.$$

Since the plane is perpendicular to the plane $2x+3y+6z=0$ and parallel to the vector $\begin{pmatrix}1\\5\\9\end{pmatrix}$, the normal of the plane is

$$\begin{pmatrix}1\\5\\9\end{pmatrix} \times \begin{pmatrix}2\\3\\6\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 5 & 9 \\ 2 & 3 & 6 \end{vmatrix} = \begin{pmatrix}3\\12\\-7\end{pmatrix}.$$

Since the plane contains O, the Cartesian equation of the plane is

$$\underset{\sim}{r} \cdot \begin{pmatrix}3\\12\\-7\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \cdot \begin{pmatrix}3\\12\\-7\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}3\\12\\-7\end{pmatrix} = 0+0+0$$

$$3x+12y-7z=0.$$

5. $m = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$, $a, b, c > 0$

$$|m| = bc^2 - b^2c - (ac^2 - a^2c) + ab^2 - a^2b$$
$$= ab^2 + bc^2 + a^2c - a^2b - ac^2 - b^2c$$

$$\underset{\sim}{a} = \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix} \quad \underset{\sim}{b} = \begin{pmatrix} 1 \\ b \\ b^2 \end{pmatrix} \quad \underset{\sim}{c} = \begin{pmatrix} 1 \\ c \\ c^2 \end{pmatrix}$$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix} = \begin{pmatrix} ab^2 - a^2b \\ a^2 - b^2 \\ b - a \end{pmatrix}$$

$$\underset{\sim}{b} \times \underset{\sim}{c} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{pmatrix} bc^2 - b^2c \\ b^2 - c^2 \\ c - b \end{pmatrix}$$

$$\underset{\sim}{c} \times \underset{\sim}{a} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & c & c^2 \\ 1 & a & a^2 \end{vmatrix} = \begin{pmatrix} a^2c - ac^2 \\ c^2 - a^2 \\ a - c \end{pmatrix}$$

$$M^{-1} = \frac{1}{|M|}\begin{pmatrix} bc^2 - b^2c & a^2c - ac^2 & ab^2 - a^2b \\ b^2 - c^2 & c^2 - a^2 & a^2 - b^2 \\ c - b & a - c & b - a \end{pmatrix}^T$$

$$= \frac{1}{|M|}\begin{pmatrix} bc^2 - b^2c & b^2 - c^2 & c - b \\ a^2c - ac^2 & c^2 - a^2 & a - c \\ ab^2 - a^2b & a^2 - b^2 & b - a \end{pmatrix}$$

6. C: $y = \frac{3x}{4x+5}$,

$$\begin{array}{r|l} & \frac{3}{4} \\ 4x+5 & 3x \\ & 3x + \frac{15}{4} \\ \hline & -\frac{15}{4} \end{array}$$

$$= \frac{3}{4} - \frac{15}{4(4x+5)}$$

Asymptotes:

As $x \to \pm\infty$ $y \to \frac{3}{4}$

As $x \to -\frac{5}{4}$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = \frac{3}{4}$ and $x = -\frac{5}{4}$.

Intersection Points:

when $x = 0$: $y = 0$

when $y = 0$: $\frac{3x}{4x+5} = 0$

$$x = 0$$

Critical Points:

$$\frac{dy}{dx} = \frac{15}{(4x+5)^2} \neq 0$$

$\therefore$ no critical points.