Simple Harmonic Motion

Situations described in this chapter may be different but they have much in common.

In each case the object starts in an equilibrium position. It is then displaced from the equilibrium position and released from rest. In all cases the object will travel backwards and forwards (or downwards and upwards) along a path through the equilibrium position. In practice, resistance to motion will eventually bring the object to rest, but if there were no resistance the object would continue to oscillate forever.

This type of repetitive motion is called simple harmonic motion, often abbreviated to SHM. In SHM, an object travels between two extreme positions at either side of a central (equilibrium position). The distance between the centre of the motion and one end of the path is called the amplitude, denoted by a.

Since the letter a now represents the amplitude of the motion, the symbol $\ddot{x}$ will be used for acceleration.

In SHM the resultant force acts towards the centre of motion. The further the object is from the centre, the larger is this force and hence the acceleration.

Linear Simple Harmonic Motion

Linear SHM means simple harmonic motion in a straight line.
Simple harmonic motion (S.H.M.) is defined by the equation of motion

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x \qquad [\text{ or } \ddot{x} = -\omega^2 x\,]}$$

where x is some measure of **displacement from equilibrium position,** and ω is a constant. As we can see from this equation of motion, the acceleration of the body moving with S.H.M. is proportional to its displacement from some fixed point and is directed towards the fixed point. (We use ω^2 and not ω as the constant, to keep the motion directed towards the fixed point, since $\omega^2 \geq 0$).

If we were considering motion in the straight line Ox, then the equation of motion for simple harmonic motion would be $\frac{d^2x}{dt^2} = -\omega^2 x$ where x is the linear displacement from O.

If you are asked to prove that motion is simple harmonic, you must show that the acceleration of the object can be written in the form $\frac{d^2x}{dt^2} = -\omega^2 x$.

In some of the oscillation problems where there is ***no obvious equilibrium position*** and x is not measured from the centre (equilibrium position) but from some other point then the equation of motion will be of the form $\ddot{x} = -\omega^2(x-c)$ where c is a constant. This differential equation can reduced to the standard simple harmonic equation $\ddot{y} = -\omega^2 y$ by means of the substitution $y = x - c$. **Here the centre of oscillation (equilibrium position) is the point with displacement** c (i.e. where $x = c$).

SHM Formulae

Let us derive further equations for S.H.M. from the basic one, $\dfrac{d^2x}{dt^2} = -\omega^2 x$.

(1) Acceleration $= \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = \dfrac{dx}{dt}.\dfrac{dv}{dx} = v\dfrac{dv}{dx} = -\omega^2 x,$

$$\Rightarrow \quad \int v\,dv = -\omega^2 \int x\,dx$$

$$\Rightarrow \quad v^2 = -\omega^2 x^2 + k$$ where k is the constant of integration.

If we have the end-point condition that when $v = 0$, $x = a$, then $k = \omega^2 a^2$, and

$$\boxed{v^2 = \omega^2(a^2 - x^2).}$$

This equation tells us that $a^2 \geq x^2$ in order to keep v^2 positive or zero.

When $v = 0$, $x = \pm a$, therefore SH.M. is oscillatory between $x = +a$ and $x = -a$.

(2) From $v^2 = \omega^2(a^2 - x^2)$ we see that the **maximum speed** ($= v_{max} = \omega a$) **occurs when** $x = 0$.