

OCTOBER / NOVEMBER 2003

1.  $C: r = \theta^{\frac{1}{2}} e^{\frac{\theta^2}{\pi}}, 0 \leq \theta \leq \pi$

If the area of the finite region bounded by  
C and the line  $\theta = \beta$  is  $\pi$ ,

$$\pi = \int_0^\beta \frac{r^2}{2} d\theta$$

$$= \int_0^\beta \frac{\theta e^{\frac{2\theta^2}{\pi}}}{2} d\theta$$

$$u = 2\theta^2$$

$$du = 4\theta d\theta$$

$$\theta = 0 \quad u = 0$$

$$\theta = \beta \quad u = 2\beta^2$$

$$= \int_0^{2\beta^2} \frac{e^{\frac{u}{\pi}}}{4(u)} du$$

$$= \left[ \frac{\pi e^{\frac{u}{\pi}}}{8} \right]_0^{2\beta^2}$$

$$= \frac{\pi}{8} \left( e^{\frac{2\beta^2}{\pi}} - 1 \right)$$

$$e^{\frac{2\beta^2}{\pi}} - 1 = 8$$

$$e^{\frac{2\beta^2}{\pi}} = 9$$

$$\frac{2\beta^2}{\pi} = \ln 9$$

$$= 2 \ln 3$$

$$\beta^2 = \pi \ln 3$$

$$\beta = (\pi \ln 3)^{\frac{1}{2}}$$

$$2. \quad u_n = \frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1}$$

$$\sum_{n=1}^N u_n = \sum_{n=1}^N \frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1}$$

$$= \frac{1}{1} - \frac{1}{3}$$

$$+ \frac{1}{3} - \frac{1}{7}$$

$$+ \frac{1}{7} - \frac{1}{13}$$

$$\vdots$$

$$+ \frac{1}{N^2 - 5N + 7} - \frac{1}{N^2 - 3N + 3}$$

$$+ \frac{1}{N^2 - 3N + 3} - \frac{1}{N^2 - N + 1}$$

$$+ \frac{1}{N^2 - N + 1} - \frac{1}{N^2 + N + 1}$$

$$= 1 - \frac{1}{N^2 + N + 1}$$

$$S_N = \sum_{n=N+1}^{2N} u_n$$

$$= \sum_{n=1}^{2N} u_n - \sum_{n=1}^N u_n$$

$$= 1 - \frac{1}{4N^2 + 2N + 1} - \left( 1 - \frac{1}{N^2 + N + 1} \right)$$

$$= \frac{1}{N^2 + N + 1} - \frac{1}{4N^2 + 2N + 1}$$

$$S_N = \frac{1}{N^2 + N + 1} - \frac{1}{4N^2 + 2N + 1}$$

$$< \frac{1}{N^2 + N + 1}$$

$$< \frac{1}{N^2}$$

If  $M = 10^{10}$   
and  $N > M$   
 $N^2 > 10^{20}$

$$\frac{1}{N^2} < 10^{-20}$$

$$S_N < 10^{-20} \text{ for all } N > M$$

3. If  $\underline{x}_1, \underline{x}_2, \underline{x}_3$  are linearly dependent, there exist  $k_1, k_2, k_3$  not all zero such that  $k_1\underline{x}_1 + k_2\underline{x}_2 + k_3\underline{x}_3 = \underline{0}$ .

$$M(k_1\underline{x}_1 + k_2\underline{x}_2 + k_3\underline{x}_3) = M\underline{0}$$

$$M(k_1\underline{x}_1) + M(k_2\underline{x}_2) + M(k_3\underline{x}_3) = \underline{0}$$

$$k_1(M\underline{x}_1) + k_2(M\underline{x}_2) + k_3(M\underline{x}_3) = \underline{0}$$

Since  $k_1, k_2, k_3$  are not all zero, the vectors  $M\underline{x}_1, M\underline{x}_2, M\underline{x}_3$  are also linearly dependent.

$$\underline{y}_1 = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \underline{y}_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \underline{y}_3 = \begin{pmatrix} 5 \\ 51 \\ 55 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -7 \end{pmatrix}$$

i) If  $k_1\underline{y}_1 + k_2\underline{y}_2 + k_3\underline{y}_3 = \underline{0}$ ,

$$k_1 \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + k_3 \begin{pmatrix} 5 \\ 51 \\ 55 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} k_1 + 2k_2 + 5k_3 \\ 5k_1 - 3k_2 + 51k_3 \\ 7k_1 + 4k_2 + 55k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 5 \\ 5 & -3 & 51 \\ 7 & 4 & 55 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 5 & | & 0 \\ 5 & -3 & 51 & | & 0 \\ 7 & 4 & 55 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-5r_1 + r_2} \left( \begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -13 & 26 & 0 \\ 0 & -10 & 20 & 0 \end{array} \right)$$

$$\xrightarrow{\frac{r_2}{-13}, \frac{r_3}{-10}} \left( \begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{-r_2 + r_3} \left( \begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Let  $k_3 = s, s \in R$

$k_2 = 2s$

$$k_1 + 2(2s) + 5s = 0$$

$$k_1 + 4s + 5s = 0$$

$$k_1 = -9s$$

$\therefore y_1, y_2, y_3$  are linearly dependent.

ii)  $P_{y_1} = \begin{pmatrix} 2 \\ 45 \\ -49 \end{pmatrix}$   $P_{y_2} = \begin{pmatrix} 26 \\ 14 \\ -28 \end{pmatrix}$   $P_{y_3} = \begin{pmatrix} -34 \\ 377 \\ -385 \end{pmatrix}$

A basis for the linear space spanned by  $P_{y_1}, P_{y_2}, P_{y_3}$  is  $\left\{ \begin{pmatrix} 2 \\ 45 \\ -49 \end{pmatrix}, \begin{pmatrix} 26 \\ 14 \\ -28 \end{pmatrix} \right\}$ .

4.  $y = x \sin x$

$$\frac{dy}{dx} = x \cos x + \sin x$$

$$\frac{d^2y}{dx^2} = \cos x - x \sin x + \cos x$$

$$= 2 \cos x - x \sin x$$

$$\frac{d^3y}{dx^3} = -2 \sin x - \sin x + x \cos x$$

$$= -3 \sin x - x \cos x$$

$$\frac{d^4y}{dx^4} = -3 \cos x - \cos x + x \sin x$$

$$= x \sin x - 4 \cos x$$

$$= 2 \cos x - \frac{d^2y}{dx^2} - 4 \cos x$$

$$= -2 \cos x - \frac{d^2y}{dx^2}$$

$$\frac{d^5y}{dx^5} = 2 \sin x - \frac{d^3y}{dx^3}$$

$$\frac{d^6y}{dx^6} = 2 \cos x - \frac{d^4y}{dx^4}$$

$$= 2 \cos x + 4 \cos x - x \sin x$$

$$= -x \sin x + 6 \cos x$$

$$\frac{d^{2n}y}{dx^{2n}} = (-1)^n x \sin x + 2(-1)^{n+1} n \cos x$$

when  $n=1$ :

$$\frac{d^{2(1)}y}{dx^{2(1)}} = \frac{d^2y}{dx^2}$$

$$= -x \sin x + 2 \cos x$$

$$= (-1)^1 x \sin x + 2(-1)^{1+1} 1 \cos x$$

Assume the statement is true when  $n=k$ .

$$n=k: \frac{d^{2k}y}{dx^{2k}} = (-1)^k x \sin x + 2(-1)^{k+1} k \cos x$$

when  $n=k+1$ :

$$\frac{d^{2(k+1)}y}{dx^{2(k+1)}} = (-1)^{k+1} x \sin x + 2(-1)^{k+2} (k+1) \cos x$$

$$\frac{d^{2k}y}{dx^{2k}} = (-1)^k x \sin x + 2(-1)^{k+1} k \cos x$$

$$\frac{d}{dx} \left( \frac{d^{2k}y}{dx^{2k}} \right) = \frac{d}{dx} ((-1)^k x \sin x + 2(-1)^{k+1} k \cos x)$$

$$\frac{d^{2k+1}y}{dx^{2k+1}} = (-1)^k \frac{d}{dx}(x \sin x) + 2(-1)^{k+1} k \frac{d}{dx}(\cos x)$$

$$= (-1)^k x \cos x + (-1)^k \sin x - 2(-1)^{k+1} k \sin x$$

$$\frac{d}{dx} \left( \frac{d^{2k+1}y}{dx^{2k+1}} \right) = \frac{d}{dx} ((-1)^k x \cos x + (-1)^k \sin x - 2(-1)^{k+1} k \sin x)$$

$$\frac{d^{2k+2}y}{dx^{2k+2}} = (-1)^k \cos x - (-1)^k \sin x + (-1)^k \cos x - 2(-1)^{k+1} k \cos x$$

$$= 2(-1)^k \cos x + (-1)^{k+1} x \sin x + 2(-1)^{k+2} k \cos x$$

$$= 2(-1)^{k+2} k \cos x + 2(-1)^{k+2} \cos x + (-1)^{k+1} x \sin x$$

$$= 2(-1)^{k+2} (k+1) \cos x + (-1)^{k+1} x \sin x$$

$$\frac{d^{2n}y}{dx^{2n}} = (-1)^n x \sin x + 2(-1)^{n+1} n \cos x$$

for every positive integer  $n$

$$5. I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$$

$$\begin{aligned} \frac{d}{dx} (\tan x \sec^n x) &= \sec^2 x \sec^n x + n \tan x \sec^{n-1} x \sec x \tan x \\ &= \sec^{n+2} x + n \tan^2 x \sec^n x \\ &= \sec^{n+2} x + n(\sec^2 x - 1) \sec^n x \\ &= \sec^{n+2} x + n \sec^n x \sec^2 x - n \sec^n x \\ &= \sec^{n+2} x + n \sec^{n+2} x - n \sec^n x \\ &= (n+1) \sec^{n+2} x - n \sec^n x \end{aligned}$$

$$\begin{aligned} \therefore \tan x \sec^n x &= \int (n+1) \sec^{n+2} x \, dx - \int n \sec^n x \, dx \\ \left[ \tan x \sec^n x \right]_0^{\frac{\pi}{4}} &= \int_0^{\frac{\pi}{4}} (n+1) \sec^{n+2} x \, dx - \int_0^{\frac{\pi}{4}} n \sec^n x \, dx \\ \frac{1}{(\sqrt{2})^n} - 0 &= (n+1) \int_0^{\frac{\pi}{4}} \sec^{n+2} x \, dx - n \int_0^{\frac{\pi}{4}} \sec^n x \, dx \end{aligned}$$

$$\frac{1}{2} = (n+1) I_{n+2} - n I_n$$

$$(n+1) I_{n+2} = \frac{1}{2} + n I_n$$

$$n=4; SI_6 = 2^2 + 4I_4$$

$$3I_4 = 2^1 + 2I_2$$

$$I_2 = \int_0^{\frac{\pi}{4}} \sec^2 x \, dx$$

$$= \left[ \tan x \right]_0^{\frac{\pi}{4}}$$

$$= 1 - 0$$

$$= 1$$

$$3I_4 = 2 + 2$$

$$= 4$$

$$I_4 = \frac{4}{3}$$

$$SI_6 = 4 + 4\left(\frac{4}{3}\right)$$

$$= 4 + \frac{16}{3}$$

$$= \frac{28}{3}$$

$$I_6 = \frac{28}{15}$$

$$6. \quad x^3 + x + 12 = 0$$

$\alpha, \beta, \gamma$  are the roots.

$$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \alpha\gamma + \beta\gamma = 1 \quad \alpha\beta\gamma = -12$$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= 0^2 - 2(1)$$

$$= 0 - 2$$

$$= -2 < 0$$

Since  $\alpha^2 + \beta^2 + \gamma^2 < 0$  and the equation  $x^3 + x + 12 = 0$  has real coefficients, only one of the roots is real.

$$\text{If } \beta = p + qj \text{ and } \gamma = p - qj$$

$$\therefore \beta\gamma = p^2 + q^2$$

$$= |\beta|^2$$

$$= |\gamma|^2$$

$$\text{Let } f(x) = x^3 + x + 12$$

$$f(-3) = -27 - 3 + 12 = -18 < 0$$

$$f(-2) = -8 - 2 + 12 = 2 > 0$$

$$-3 < \alpha < -2$$

$$\frac{-1}{2} < \frac{1}{\alpha} < \frac{-1}{3}$$

$$\text{Since } \alpha = \frac{-12}{\beta\gamma} \quad (\text{from } \alpha\beta\gamma = -12)$$

$$-\frac{1}{2} < \frac{-\beta\gamma}{12} < \frac{-1}{3}$$

$$\frac{1}{3} < \frac{\beta\gamma}{12} < \frac{1}{2}$$

$$4 < \beta\gamma < 6$$

$$2 < \sqrt{\beta\gamma} < \sqrt{6}$$

$$2 < |\beta| = |\gamma| < \sqrt{6}$$

$\therefore$  If  $-3 < \alpha < -2$ , the modulus of each of the other roots lies between 2 and  $\sqrt{6}$ .

$$7. \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-\alpha t}, \alpha \neq 2$$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$$

$$m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0$$

$$m = -2$$

$\therefore$  The complementary function,  $y_c$ , is

$$y_c = (At + B)e^{-2t}$$

The particular integral,  $y_p$ , is given by

$$y_p = Ce^{-\alpha t} \text{ since } \alpha \neq -2.$$

$$\frac{dy_p}{dt} = -\alpha(Ce^{-\alpha t})$$

$$\frac{d^2y_p}{dt^2} = \alpha^2 Ce^{-\alpha t}$$

$$\frac{d^2y_p}{dt^2} + 4\frac{dy_p}{dt} + 4y_p = \alpha^2 Ce^{-\alpha t} - 4\alpha Ce^{-\alpha t} + 4Ce^{-\alpha t}$$

$$= (\alpha^2 - 4\alpha + 4)Ce^{-\alpha t}$$

$$= e^{-\alpha t}$$

$$(\alpha^2 - 4\alpha + 4)C = 1$$

$$C = \frac{1}{\alpha^2 - 4\alpha + 4}$$

$$= \frac{1}{(\alpha - 2)^2}$$

$$y_p = \frac{e^{-\alpha t}}{(\alpha - 2)^2}$$

$$y = y_c + y_p$$

$$= (At + B)e^{-2t} + \frac{e^{-\alpha t}}{(\alpha - 2)^2}$$

$\therefore$  The general solution of the differential equation

$$\text{is } y = (At + B)e^{-2t} + \frac{e^{-\alpha t}}{(\alpha - 2)^2}.$$

$$ye^{\alpha t} = (At + B)e^{(\alpha - 2)t} + \frac{1}{(\alpha - 2)^2}$$

If  $\alpha < 2$ ,

$$\lim_{t \rightarrow \infty} ye^{\alpha t} = \lim_{t \rightarrow \infty} ((At + B)e^{(\alpha - 2)t} + \frac{1}{(\alpha - 2)^2})$$

$$= \lim_{t \rightarrow \infty} (At + B)e^{(\alpha - 2)t} + \frac{1}{(\alpha - 2)^2}$$

$$= 0 + \frac{1}{(\alpha - 2)^2}$$

$$= \frac{1}{(\alpha - 2)^2}$$

$\therefore$  If  $\alpha < 2$ , then  $ye^{\alpha t} \rightarrow \frac{1}{(2 - \alpha)^2}$  as  $t \rightarrow \infty$

$$8. z = e^{i\theta}$$

$$= \cos \theta + i \sin \theta$$

$$z^n = (\cos \theta + i \sin \theta)^n$$

$$= \cos n\theta + i \sin n\theta$$

$$z^{-n} = (\cos \theta + i \sin \theta)^{-n}$$

$$= \cos(-n\theta) + i \sin(-n\theta)$$

$$= \cos n\theta - i \sin n\theta$$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta, \quad z^n - \frac{1}{z^n} = 2i \sin n\theta$$

$$n=1: z + \frac{1}{z} = 2 \cos \theta, \quad z - \frac{1}{z} = 2i \sin \theta$$

$$(2i \sin \theta) = \left(z - \frac{1}{z}\right)^6$$

$$-64 \sin^6 \theta = z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$$

$$= z^6 + \frac{1}{z^6} - 6(z^4 + \frac{1}{z^4}) + 15(z^2 + \frac{1}{z^2}) - 20$$

$$= 2 \cos 6\theta - 6(2 \cos 4\theta) + 15(2 \cos 2\theta) - 20$$

$$\sin^6 \theta = \frac{-\cos 6\theta}{32} + \frac{3 \cos 4\theta}{16} - \frac{15 \cos 2\theta}{32} + \frac{5}{16}$$

$$= p \cos 6\theta + q \sin 4\theta + r \sin 2\theta + s,$$

$$p = \frac{-1}{32}, \quad q = \frac{3}{16}, \quad r = \frac{-15}{32}, \quad s = \frac{5}{16}$$

The mean value of  $\sin^6 \theta$  over the interval

$$0 \leq \theta \leq \frac{\pi}{4} \text{ is } \frac{1}{\frac{\pi}{4} - 0} \int_0^{\frac{\pi}{4}} \sin^6 \theta \, d\theta$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{-\cos 6\theta}{32} + \frac{3 \cos 4\theta}{16} - \frac{15 \cos 2\theta}{32} + \frac{5}{16} \, d\theta$$

$$= \frac{1}{8\pi} \int_0^{\frac{\pi}{4}} -\cos 6\theta + 6 \cos 4\theta - 15 \cos 2\theta + 10 \, d\theta$$

$$= \frac{1}{8\pi} \left[ \frac{-\sin 6\theta}{6} + \frac{3 \sin 4\theta}{2} - \frac{15 \sin 2\theta}{2} + 10\theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8\pi} \left( \frac{1}{6} - \frac{15}{2} + \frac{5\pi}{2} - 0 \right)$$

$$= \frac{1}{8\pi} \left( \frac{5\pi}{2} - \frac{22}{3} \right).$$

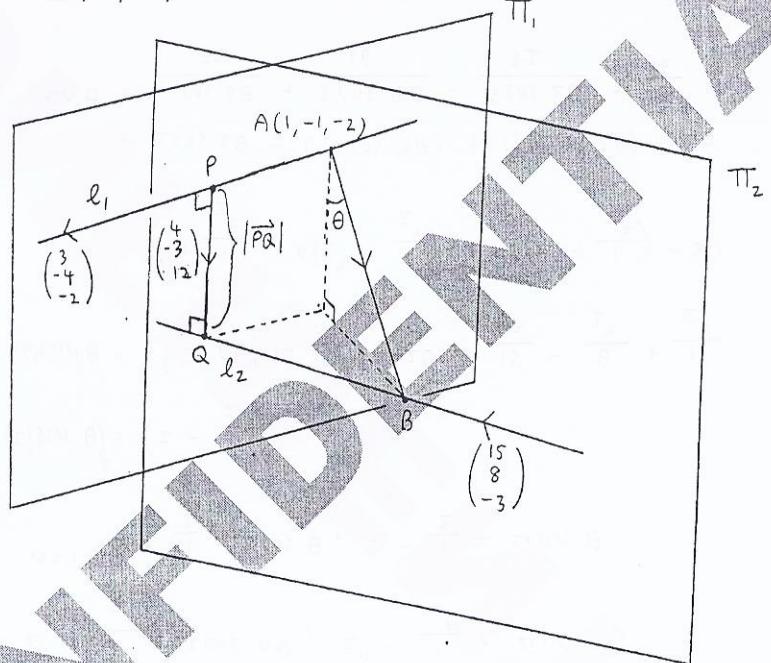
$$= \frac{5}{16} - \frac{11}{12\pi}.$$

$$9. l_1: \underline{z} = \underline{i} - \underline{j} - 2\underline{k} + s(3\underline{i} - 4\underline{j} - 2\underline{k})$$

$$l_2: \underline{z} = (1+5\cos t)\underline{i} - (1+5\sin t)\underline{j} - 14\underline{k} + t(15\underline{i} + 8\underline{j} - 3\underline{k}),$$

$$0 \leq t \leq 2\pi$$

$A(1, -1, -2)$



i) Let  $B$  denote the point  $(1+5\cos t, -(1+5\sin t), -14)$ .

Since  $PQ$  is perpendicular to both  $l_1$  and  $l_2$ ,

the direction of  $\overrightarrow{PQ}$  is parallel to  $\begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 15 \\ 8 \\ -3 \end{pmatrix}$ .

$$\begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 15 \\ 8 \\ -3 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & -2 \\ 15 & 8 & -3 \end{vmatrix} = 28\underline{i} - 21\underline{j} + 84\underline{k}$$

$$= 7(4\underline{i} - 3\underline{j} + 12\underline{k})$$

$$\overrightarrow{AB} \cdot \overrightarrow{PQ} = |\overrightarrow{AB}| |\overrightarrow{PQ}| \cos \theta$$

$$\left[ \begin{pmatrix} 1+5\cos t \\ -(1+5\sin t) \\ -14 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right] \cdot \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = |\overrightarrow{AB}| \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \cos \theta$$

$$\begin{pmatrix} 5\cos t \\ -5\sin t \\ -12 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = |\overrightarrow{AB}| \sqrt{169} \cos \theta$$

$$|20\cos t + 15\sin t - 144| = 13|\overrightarrow{AB}| \cos \theta$$

$$|\overrightarrow{AB}| \cos \theta = \frac{|20\cos t + 15\sin t - 144|}{13}$$

since  $|\overrightarrow{PQ}| = |\overrightarrow{AB}| \cos \theta$ ,

$$|\overrightarrow{PQ}| = \frac{|20\cos t + 15\sin t - 144|}{13}$$

ii) since  $20\cos t + 15\sin t = 25\sin(t + \tan^{-1}\frac{4}{3})$

$$|\overrightarrow{PQ}| = \frac{\left| 25\sin\left(t + \tan^{-1}\frac{4}{3}\right) - 144 \right|}{13}$$

$$-25 \leq 25 \sin(t + \tan^{-1} \frac{4}{3}) \leq 25$$

$$-169 \leq 25 \sin(t + \tan^{-1} \frac{4}{3}) - 144 \leq -119$$

$$119 \leq |25 \sin(t + \tan^{-1} \frac{4}{3}) - 144| \leq 169$$

$$119 \leq 13|PQ| \leq 169$$

$$\frac{119}{13} \leq |PQ| \leq 13$$

Since the minimum distance between the lines  $\ell_1$  and  $\ell_2$  is  $\frac{119}{13}$ , the lines do not intersect.  
The maximum length of  $PQ$  as  $t$  varies is 13.

- iii) Since  $\Pi_1$  contains  $\ell_1$  and  $PQ$ , the normal of  $\Pi_1$  is perpendicular to both  $\ell_1$  and  $PQ$ .

$$\begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = \begin{vmatrix} i & j & k \\ \frac{1}{3} & -4 & -2 \\ 4 & -3 & 12 \end{vmatrix} = \begin{pmatrix} -54 \\ -44 \\ 7 \end{pmatrix}$$

Also, since  $\Pi_2$  contains  $\ell_2$  and  $PQ$ , the normal of  $\Pi_2$  is perpendicular to both  $\ell_2$  and  $PQ$ .

$$\begin{pmatrix} 15 \\ 8 \\ -3 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 15 & 8 & -3 \\ 4 & -3 & 12 \end{vmatrix} = \begin{pmatrix} 87 \\ -192 \\ -77 \end{pmatrix}$$

Since  $\begin{pmatrix} -54 \\ -44 \\ 7 \end{pmatrix}$  is a vector perpendicular to  $\Pi_1$ ,

and  $\begin{pmatrix} 87 \\ -192 \\ -77 \end{pmatrix}$  is a vector perpendicular to  $\Pi_2$ ,

if  $\theta$  is the angle between  $\begin{pmatrix} -54 \\ -44 \\ 7 \end{pmatrix}$  and  $\begin{pmatrix} 87 \\ -192 \\ -77 \end{pmatrix}$

$$\begin{pmatrix} -54 \\ -44 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 87 \\ -192 \\ -77 \end{pmatrix} = \left| \begin{pmatrix} -54 \\ -44 \\ 7 \end{pmatrix} \right| \left| \begin{pmatrix} 87 \\ -192 \\ -77 \end{pmatrix} \right| \cos \theta$$

$$-4698 + 8448 - 539 = \sqrt{4901} \sqrt{50362} \cos \theta$$

$$\cos \theta = \frac{3211}{\sqrt{4901} \sqrt{50362}}$$

$$= 0.204$$

$$\theta = 78.2^\circ$$

$$10. \quad A = \begin{pmatrix} 6 & 4 & 1 \\ -6 & -1 & 3 \\ 8 & 8 & 4 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 6 & 4 & 1 \\ -6 & -1 & 3 \\ 8 & 8 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 - \lambda & 4 & 1 \\ -6 & -1 - \lambda & 3 \\ 8 & 8 & 4 - \lambda \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (6 - \lambda)[(-1 - \lambda)(4 - \lambda) - 24] \\ &\quad - 4(-6(4 - \lambda) - 24) + 1(-48 - 8(-1 - \lambda)) \\ &= (6 - \lambda)(-4 + \lambda - 4\lambda + \lambda^2 - 24) \\ &\quad - 4(-24 + 6\lambda - 24) - 48 + 8 + 8\lambda \\ &= (6 - \lambda)(\lambda^2 - 3\lambda - 28) - 4(6\lambda - 48) + 8\lambda - 40 \\ &= 6\lambda^2 - 18\lambda - 168 - \lambda^3 + 3\lambda^2 + 28\lambda - 24\lambda + 192 \\ &\quad + 8\lambda - 40 \\ &= -\lambda^3 + 9\lambda^2 - 6\lambda - 16 \end{aligned}$$

$$\begin{array}{r} -\lambda^2 + 10\lambda - 16 \\ \lambda + 1 \left[ \begin{array}{r} -\lambda^3 + 9\lambda^2 - 6\lambda - 16 \\ -\lambda^3 - \lambda^2 \\ \hline 10\lambda^2 - 6\lambda \\ 10\lambda^2 + 10\lambda \\ \hline -16\lambda - 16 \\ -16\lambda - 16 \\ \hline 0 \end{array} \right] \end{array}$$

$$= (\lambda + 1)(-\lambda^2 + 10\lambda - 16)$$

$$= -(\lambda + 1)(\lambda - 2)(\lambda - 8)$$

when  $|A - \lambda I| = 0$ ,

$$-(\lambda + 1)(\lambda - 2)(\lambda - 8) = 0$$

$$\lambda = 2, 8, -1$$

$$\text{when } \lambda = 2: \quad \begin{pmatrix} 4 & 4 & 1 \\ -6 & -3 & 3 \\ 8 & 8 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 4 & 1 & | & 0 \\ -6 & -3 & 3 & | & 0 \\ 8 & 8 & 2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{r_2}{-3}, \frac{r_3}{2}} \begin{pmatrix} 4 & 4 & 1 & | & 0 \\ 2 & 1 & -1 & | & 0 \\ 4 & 4 & 1 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 2 & 1 & -1 & | & 0 \\ 4 & 4 & 1 & | & 0 \\ 4 & 4 & 1 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-2r_1 + r_2} \begin{pmatrix} 2 & 1 & -1 & | & 0 \\ 0 & 2 & 3 & | & 0 \\ 4 & 4 & 1 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-2r_1 + r_3} \begin{pmatrix} 2 & 1 & -1 & | & 0 \\ 0 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-r_2 + r_3} \begin{pmatrix} 2 & 1 & -1 & | & 0 \\ 0 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Let  $z = 4s, s \in \mathbb{R}$

$$y = -6s$$

$$2x - 6s - 4s = 0$$

$$x = 5s$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5s \\ -6s \\ 4s \end{pmatrix}$$

$$= s \begin{pmatrix} 5 \\ -6 \\ 4 \end{pmatrix}$$

when  $\lambda = 8$ :

$$\begin{pmatrix} -2 & 4 & 1 \\ -6 & -9 & 3 \\ 8 & 8 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 4 & 1 & | & 0 \\ -6 & -9 & 3 & | & 0 \\ 8 & 8 & -4 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-3r_1 + r_2} \begin{pmatrix} -2 & 4 & 1 & | & 0 \\ 0 & -21 & 0 & | & 0 \\ 0 & 24 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{r_2}{-21}, \frac{r_3}{24}} \begin{pmatrix} -2 & 4 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-r_2 + r_3} \begin{pmatrix} -2 & 4 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$y = 0$$

$$\text{let } z = 2s, s \in \mathbb{R}$$

$$x = s$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 2s \end{pmatrix}$$

$$= s \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

when  $\lambda = -1$ :

$$\begin{pmatrix} 7 & 4 & 1 \\ -6 & 0 & 3 \\ 8 & 8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 4 & 1 & | & 0 \\ -6 & 0 & 3 & | & 0 \\ 8 & 8 & 5 & | & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{r_2}{-3}} \begin{pmatrix} 7 & 4 & 1 & | & 0 \\ 2 & 0 & -1 & | & 0 \\ 8 & 8 & 5 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 7 & 4 & 1 & | & 0 \\ 8 & 8 & 5 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_2 \times 2} \begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 14 & 8 & 2 & | & 0 \\ 8 & 8 & 5 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-7r_1 + r_2} \begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 0 & 8 & 9 & | & 0 \\ 0 & 8 & 9 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-r_2 + r_3} \begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 0 & 8 & 9 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\text{Let } z = 8s, s \in \mathbb{R}$$

$$y = -9s$$

$$x = 4s$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4s \\ -9s \\ 8s \end{pmatrix}$$

$$= s \begin{pmatrix} 4 \\ -9 \\ 8 \end{pmatrix}$$

The eigenvalues of  $A$  are  $2, 8, -1$  with corresponding eigenvectors  $\begin{pmatrix} 5 \\ -6 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -9 \\ 8 \end{pmatrix}$ .

If  $\tilde{e}$  is an eigenvector of  $A$  with corresponding eigenvalue  $\lambda$ ,

$$\begin{aligned} A^2\tilde{e} &= A(A\tilde{e}) \\ &= A(\lambda\tilde{e}) \\ &= \lambda(A\tilde{e}) \\ &= \lambda(\lambda\tilde{e}) \\ &= \lambda^2\tilde{e} \end{aligned}$$

$$\begin{aligned} \text{and } A^3\tilde{e} &= A(A^2\tilde{e}) \\ &= A(\lambda^2\tilde{e}) \\ &= \lambda^3(A\tilde{e}) \\ &= \lambda^3(\lambda\tilde{e}) \\ &= \lambda^4\tilde{e}. \end{aligned}$$

$$\begin{aligned} (A + A^2 + A^3)\tilde{e} &= A\tilde{e} + A^2\tilde{e} + A^3\tilde{e} \\ &= \lambda\tilde{e} + \lambda^2\tilde{e} + \lambda^3\tilde{e} \\ &= (\lambda + \lambda^2 + \lambda^3)\tilde{e}. \end{aligned}$$

$\therefore$  The matrix  $A + A^2 + A^3$  has eigenvalue  $\lambda + \lambda^2 + \lambda^3$  with corresponding eigenvector  $\tilde{e}$ .

If  $P$  is a non-singular matrix and  $D$  is a diagonal matrix such that  $A + A^2 + A^3 = PDP^{-1}$

$$\text{let } P = \begin{pmatrix} 5 & 1 & 4 \\ -6 & 0 & -9 \\ 4 & 2 & 8 \end{pmatrix}$$

$$\begin{aligned} \text{and } D &= \begin{pmatrix} 2 + 2^2 + 2^3 & 0 & 0 \\ 0 & 8 + 8^2 + 8^3 & 0 \\ 0 & 0 & -1 + (-1)^2 + (-1)^3 \end{pmatrix} \\ &= \begin{pmatrix} 14 & 0 & 0 \\ 0 & 584 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

ii. EITHER

$$C: y = \frac{5(x-1)(x+2)}{(x-2)(x+3)}$$

(i)

$$\begin{aligned} & x^2 + x - 6 \overline{) 5x^2 + 5x - 10} \\ & \quad 5x^2 + 5x - 30 \\ & \quad \hline 20 \\ &= 5 + \frac{20}{(x-2)(x+3)} \\ \frac{20}{(x-2)(x+3)} &= \frac{A}{x-2} + \frac{B}{x+3} \\ &= \frac{A(x+3) + B(x-2)}{(x-2)(x+3)} \\ 20 &= A(x+3) + B(x-2) \\ &= (A+B)x + 3A - 2B \\ A+B=0 & \quad 3A - 2B = 20 \\ 2A + 2B = 0 & \\ 5A = 20 & \\ A = 4 & \\ B = -4 & \end{aligned}$$

$$y = 5 + \frac{4}{x-2} - \frac{4}{x+3}$$

$$\text{i)} \quad \frac{dy}{dx} = \frac{-4}{(x-2)^2} + \frac{4}{(x+3)^2}$$

$$\text{when } \frac{dy}{dx} = 0 : \quad \frac{-4}{(x-2)^2} + \frac{4}{(x+3)^2} = 0$$

$$\frac{-4}{(x-2)^2} = \frac{4}{(x+3)^2}$$

$$(x-2)^2 = (x+3)^2$$

$$x^2 - 4x + 4 = x^2 + 6x + 9$$

$$10x = -5$$

$$x = -\frac{1}{2}$$

$$y = \frac{9}{5}$$

iii) As  $x \rightarrow \pm\infty$ ,  $y \rightarrow 5$

As  $x \rightarrow 2$ ,  $y \rightarrow \pm\infty$

As  $x \rightarrow -3$ ,  $y \rightarrow \pm\infty$

The asymptotes of  $C$  are  $y = 5$ ,  $x = 2$   
and  $x = -3$

iv) If  $C$  intersects the line  $y = k$ ,

$$\frac{5(x-1)(x+2)}{(x-2)(x+3)} = k$$

$$5(x^2 + x - 2) = k(x^2 + x - 6)$$

$$5x^2 + 5x - 10 = kx^2 + kx - 6k$$

$$(5-k)x^2 + (5-k)x + 6k - 10 = 0$$

$$a = 5-k \quad b = 5-k \quad c = 6k - 10$$

$$b^2 - 4ac = (5-k)^2 - 4(5-k)(6k-10)$$

$$= (5-k)(5-k - 4(6k-10))$$

$$= (5-k)(5-k - 24k + 40)$$

$$= (5-k)(-25k + 45)$$

$$= 5(k-5)(5k-9)$$

when  $b^2 - 4ac < 0$

$$5(k-5)(5k-9) < 0$$

$$\frac{9}{5} < k < 5$$

$\therefore$  The line  $y=k$  does not intersect  $C$   
if  $\frac{9}{5} < k < 5$

OR

$$y = \frac{2x^{\frac{3}{2}}}{3}, \quad x > 0$$

i) Since  $R$  is the arc of the curve joining the origin to the point where  $x=3$ , the length of  $R$  is,

$$\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_0^3 \sqrt{1 + (x^{\frac{1}{2}})^2} dx$$

$$= \int_0^3 \sqrt{1+x} dx$$

$$= \left[ \frac{2}{3}(1+x)^{\frac{3}{2}} \right]_0^3$$

$$= \frac{2}{3} (4^{\frac{3}{2}} - 1)$$

$$= \frac{2}{3} (8 - 1)$$

$$= \frac{2(7)}{3}$$

$$= \frac{14}{3}$$

- ii) The area of the region bounded by the  $x$ -axis, the line  $x=3$  and  $R$ ,  $A$ , is

$$\int_0^3 y \, dx$$

$$= \int_0^3 \frac{2x^{\frac{3}{2}}}{3} \, dx$$

$$= \left[ \frac{2}{3} \left( 2x^{\frac{5}{2}} \right) \right]_0^3$$

$$= \frac{4}{15} \left( 3^{\frac{5}{2}} - 0 \right)$$

$$= \frac{4(9\sqrt{3})}{15}$$

$$= \frac{12\sqrt{3}}{5}$$

The  $y$ -coordinate of the centroid of the region bounded by the  $x$ -axis, the line  $x=3$  and  $R$  is

$$\frac{\int_0^3 \frac{y^2}{2} \, dx}{A}$$

$$= \int_0^3 \frac{4x^3}{9(2)} \, dx$$

$$= \frac{12\sqrt{3}}{5}$$

$$= \frac{4}{18} \left( \frac{5}{12\sqrt{3}} \right) \left[ \frac{x^4}{4} \right]_0^3$$

$$= \frac{4}{18} \left( \frac{5}{12\sqrt{3}} \right) \left( \frac{81}{4} - 0 \right)$$

$$= \frac{5\sqrt{3}}{8}$$

- iii) The area of the surface generated when  $R$  is rotated through one revolution about the  $y$ -axis is

$$\int_0^3 2\pi x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx$$

$$= \int_0^3 2\pi x \sqrt{1 + (x^{\frac{1}{2}})^2} \, dx$$

$$= 2\pi \int_0^3 x \sqrt{1+x} \, dx$$

$$u = 1+x$$

$$du = dx$$

$$x=0 \quad u=1$$

$$x=3 \quad u=4$$

$$= 2\pi \int_1^4 (u-1)u^{\frac{1}{2}} \, du$$

$$= 2\pi \int_1^4 u^{\frac{3}{2}} - u^{\frac{1}{2}} \, du$$

$$= 2\pi \left[ \frac{\frac{5}{2}}{5} - \left[ \frac{0}{\frac{2\pi}{3}} \right] \right]^4$$

$$= 2\pi \left( \frac{64}{5} - \frac{16}{3} - \left( \frac{2}{3} - \frac{2}{3} \right) \right)$$

$$= 2\pi \left( \frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3} \right)$$

$$= 2\pi \left( \frac{116}{15} \right)$$

$$= \frac{232\pi}{15}$$

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