Hypothesis Tests

1. General procedures (Part I)
2. General procedures (Part 2)
3. On single population mean where the population is normally distributed with known variance or where a large sample is used (use z-test).
4. On single population mean where a small sample is drawn from a normal population of unknown variance, using t-test.
5. On single population mean (mixed tests)

Hypothesis Tests

6. General procedure for testing two population means
7. On two population means where a test using a normal distribution is applied.
8. On two population means using a 2-independent-sample *t*-test.
9. On two population means using a paired sample *t*-test.
10. On two population means (mixed tests)

Learning Outcome

1 & 2 : General procedures (Part I) and General Procedures Part 2)

Students should be able to:

1. To identify whether the given problem is a one-tailed or two tailed test.
2. To formulate the null and alternative hypotheses correctly.
3. To realize that the use of test statistics depends on the problem given.
4. To describe the meaning of significance level.
5. To identify the critical region correctly based on the hypotheses formulated
6. To identify and use the criteria for null hypothesis rejection.
7. To draw conclusion correctly.

Learning Outcome

3 On single population mean where the population is normally distributed with known variance or where a large sample is used (use *z*-test).

Students should be able to:

8. To formulate the null and alternative hypotheses correctly
9. To use the test statistics based on normal distribution.
10. To carry out the hypothesis test correctly.
11. To draw conclusion correctly.
12. To write down the confidence interval for a population mean using a normal distribution with a given level of significance.

Learning Outcome

4 On single population mean where a small sample is drawn from a normal population of unknown variance, using *t*-test.

Students should be able to:

1. To formulate the null and alternative hypotheses correctly
2. To use the test statistics based on *t*-distribution.
3. To carry out the hypothesis test correctly.
4. To draw conclusion correctly.
5. To write down the confidence interval for a population mean using a *t*-distribution with a given level of significance.

Learning Outcome

5 On single population mean (mixed tests)

Students should be able to:

1. To identify the correct hypothesis test
2. formulate the null and alternative hypotheses correctly
3. To use the test statistics based on t-distribution.
4. To carry out the hypothesis test correctly.
5. To draw conclusion correctly.
6. To write down the confidence interval for a population mean with a given level of significance.

Learning Outcome

6 General procedure for testing two population means

Students should be able to:

1. To formulate the null and alternative hypotheses correctly
2. To realize the use of different test statistics depending on assumptions/situations.
3. To illustrate the steps in carrying out the hypothesis test.

Learning Outcome

7 On two population means where a test using a normal distribution is applied.

Students should be able to:

1. To select the test appropriate the circumstances of a problem.
2. To formulate the null and alternative hypotheses correctly
3. To use the test statistics based on normal distribution.
4. To carry out the hypothesis test correctly.
5. To draw conclusion correctly.
6. To write down the confidence interval for a difference of population means using a normal distribution, at a given level of significance.

Learning Outcome

8 On two population means using a 2-independent-sample *t*-test.

Students should be able to:

1. To select the test appropriate the circumstances of a problem.
2. To formulate the null and alternative hypotheses correctly
3. To use the test statistics based on *t*-distribution.
4. To carry out the hypothesis test correctly.
5. To draw conclusion correctly.
6. To write down the confidence interval for a difference of population means using a *t*- distribution, at a given level of significance.

Learning Outcome

9 On two population means using a paired sample *t*-test.

Students should be able to:

1. To select the test appropriate the circumstances of a problem.
2. To formulate the null and alternative hypotheses correctly
3. To use the test statistics based on *t*-distribution.
4. To carry out the hypothesis test correctly.
5. To draw conclusion correctly.
6. To write down the confidence interval for a difference of population means using a *t*- distribution, at a given level of significance.

Learning Outcome

10 On two population means (mixed tests)

Students should be able to:

1. To select the test appropriate the circumstances of a problem.
2. To formulate the null and alternative hypotheses correctly To use the test statistics.
3. To carry out the hypothesis test correctly.
4. To draw conclusion correctly.
5. To write down the confidence interval for a difference of population means at a given level of significance.

Hypothesis Testing:

The Nature of Hypothesis testing

Hypothesis

A statement that something is true.

Statistical hypothesis test

A process by which a decision is made between two opposing hypotheses. The two opposing hypotheses are formulated so that each hypothesis is the negation of the other. (That way one of them is always true, and the other one is always false.) Then one hypothesis is tested in hopes that it can be shown to be a very improbable occurrence, thereby implying the other hypothesis is likely the truth.

The Nature of Hypothesis testing

Null Hypothesis, H_o

The hypothesis that we will test. Generally this is a statement that a population parameter has a specific value. The null hypothesis is so named because it is the “starting point” for the investigation.

The Nature of Hypothesis testing

Alternative Hypothesis, H_1

The statement about the same population parameter that is use in the null hypothesis. Generally this is the statement that specifies the population parameter has a value different, in some way, from the value given in the null hypothesis. The rejection of the null hypothesis will imply the likely truth of this alternative hypothesis.

The Nature of Hypothesis testing

Writing hypotheses

You suspect that a brand-name detergent outperforms the store's brand of detergent, and you wish to test the two detergents because you would prefer to buy the cheaper store brand. State the null and alternative hypotheses.

H_o : "There is no difference in detergent performance."

H_1 : "The brand-name detergent performs better than the Store brand."

- The **test statistic** is calculated from the sample. Its value is used to decide whether the null hypothesis, H_o , should be rejected.

- The **rejection** (or **critical**) **region** gives the values of the test statistic for which the null hypothesis, H_o , is rejected.

- The **acceptance region** gives the values of the test statistic for which the null hypothesis, H_o , is not rejected.

- The boundary value (s) of the rejection region is (are) called the critical value (s).

- The **significance level** of a test gives the probability of the test statistic falling in the rejection region when H_o is true.

The value of the calculated test statistics is used in conjunction with decision rule to determine either "reject H_0 " or "fail to reject H_0 " .

The conclusion

a. If the decision is "reject H_0 " then the conclusion should be worded " There is sufficient evidence at the α level of significance to show that…… (the meaning of the alternative hypothesis)."

b. If the decision is "fail to reject H_0 " then the conclusion should be worded " There is not sufficient evidence at the α level of significance to show that…… (the meaning of the alternative hypothesis)."

Testing μ (the mean of a population)

For a hypothesis test on the population mean, μ , the null hypothesis, H_o , proposes a value, μ_o for μ,

$$H_o : \mu = \mu_o$$

for a significance level of α% and the test value x,

The alternative hypothesis, H_1, suggests the way in which μ might differ from μ_o. H_1 can take three forms:

$H_1 : \mu < \mu_o$, a one-tailed (lower tail) for a decrease in μ;

$H_1 : \mu > \mu_o$, a one-tailed (upper tail) for a increase in μ;

$H_1 : \mu \neq \mu_o$, a two-tailed for a difference/change in μ;

Procedure for carrying out a hypothesis test

(1) Define the variable

(2) State H_o and H_1

$H_o : \mu = \mu_o$ OR $H_o : \mu = \mu_o$ OR $H_o : \mu = \mu_o$

$H_1 : \mu > \mu_o$ $\quad$ $H_1 : \mu < \mu_o$ $\quad$ $H_1 : \mu \neq \mu_o$

(3) State the distribution according to H_o

If H_o is true $\Rightarrow \bar{X} \sim N(\mu_o , \frac{variance}{n})$

(4) State level and type of test

Use a one-tailed (upper tail/lower tail) or two-tailed test, at the α% level.

(5) State the criterion for rejection of H_o.

(6) Obtain the test statistic.

(7) Make you conclusion by rejecting H_o or not.

Hypothesis Test 1:

Testing μ when the population X is normal and the variance σ^2 is known (any size sample)

When testing the mean of a normal population X with known variance σ^2 for samples of size n, the test statistic is

$$\bar{X}, \text{ where } \bar{X} \sim N\left(\mu_o, \frac{\sigma^2}{n}\right).$$

In standardized form, the test statistic is

$$Z = \frac{\bar{X} - \mu_o}{\sigma / \sqrt{n}} \text{ where } Z \sim N(0,1).$$

Example 1: *

Each year trainees throughout the country sit a test. Over a period of time it has been established that the marks can be modelled by a normal distribution with mean 70 and standard deviation 6.

This year it was thought that trainees from a particular county did not perform as well as expected. The marks of a random sample of 25 trainees from the county were scrutinized and it was found that their mean mark was 67.3. Does this provide evidence, at the 5% significance level, that trainees from this county did not perform well as expected?

Example 2:

A test of mental ability has been constructed so that, for adults in Great Britain, the test score is normally distributed with mean 100 and standard deviation 15. A doctor wishes to test whether sufferers from a particular disease differ in mean from the general population in their performance on this test. She chooses a random sample of 10 sufferers. Their scores on the test are

119 131 95 107 125 90 123 89 103 103

Carry out a test at the 5% significance level to test whether sufferers from the disease differ from the general population in the way in which they perform at this test.

Example 3:*

A sample of size 16 is taken from the distribution of $X \sim N(\mu, 3^2)$ and a hypothesis test is carried out at the 1% level of significance. On the basis of the value of the sample mean $\boxed{\overline{x}}$, the null hypothesis $\mu = 100$ is rejected in favour of the alternative hypothesis $\mu > 100$.

What can be said about the value of $\boxed{\overline{x}}$?

Hypothesis Test 2:

Testing μ when the population *X* is not normal, the variance σ^2 is known (sample size *n* is large)

When testing the mean of a non-normal population *X* with known variance σ^2, provided that the sample size *n* is large, the test statistic is

$$\bar{X}, \text{ where } \bar{X} \text{ is approximately normal}, \ \bar{X} \sim N\left(\mu_o, \frac{\sigma^2}{n}\right).$$

In standardized form, the test statistic is

$$Z = \frac{\bar{X} - \mu_o}{\sigma / \sqrt{n}} \quad \text{where } Z \sim N(0,1).$$

Example 4:*

The management of a large hospital states that the mean age of its patients is 45 years. Records of a random sample of 100 patients give a mean age of 48.4 years. Using a population standard deviation of 18 years, test at the 5% significance level whether there is evidence that the management's statement is incorrect. State clearly your null and alternative hypothesis.

Hypothesis Test 3:

Testing μ of a population X where the variance σ^2 is unknown and the sample size n is large

When testing the mean of a population X with unknown variance σ^2, provided that the sample size n is large, the test statistic is

$$\overline{X}, \text{ where } \overline{X} \text{ is approximately normal}, \ \overline{X} \sim N\left(\mu, \frac{\hat{\sigma}^2}{n}\right).$$

In standardized form, the test statistic is

$$Z = \frac{\overline{X} - \mu}{\hat{\sigma}/\sqrt{n}} \quad \text{where } Z \sim N(0,1).$$

$$\hat{\sigma}^2 = \frac{n}{n-1} s^2$$

Example 5:*

The packaging on an electric light bulb states that the average length of life of bulbs is 1000 hours. A consumer association thinks that this is an overestimate and tests a random sample of 64 bulbs, recording the life x hours, of each bulb.

The results are summarized as follows:

$$\sum x = 63910.4, \qquad \sum x^2 = 63824061.$$

(a) Calculate the sample mean.

(b) Calculate an unbiased estimate for the standard deviation of the length of life of all light bulbs of this type.

Example 5:*

The packaging on an electric light bulb states that the average length of life of bulbs is 1000 hours. A consumer association thinks that this is an overestimate and tests a random sample of 64 bulbs, recording the life x hours, of each bulb.

The results are summarized as follows:

$$\sum x = 63910.4, \qquad \sum x^2 = 63824061.$$

(c) Is there evidence, at the 10% significance level, that the statement on the packaging is overestimating the length of life of this type of light bulb?

Hypothesis Test 4:

Testing μ when the population X is normal but the variance σ^2 is unknown and the sample size n is small

When testing the mean of a population X with unknown variance σ^2, when the sample size n is small, the test statistic is T where

$$T = \frac{\overline{X} - \mu}{\sigma\boxed{?}/\sqrt{n}} \quad \text{and } \mathrm{T} \sim t(n-1).$$

Example 6:*

Five readings of the resistance X, in ohms, of a piece of wire gave the following results: 1.51 , 1.49 , 1.54 , 1.52 , 1.54

There are summarized by

$$\sum x = 7.6, \qquad \sum x^2 = 11.5538.$$

If the wire is impure, its resistance is higher than 1.50 ohms. Assuming that the resistance can be modeled by a normal variable with mean μ, and standard deviation σ,

(a) Calculate the sample mean.

(b) Calculate an unbiased estimate of σ,

Is there evidence, at the 5% level of significance, that the wire is impure?

Example 7:*

A machine is supposed to produce paper with a mean thickness of 0.05mm. Eight random measurements of the paper gave a mean of 0.047mm with a standard deviation of 0.002mm. Assuming that the thickness of the paper produced by the machine is normally distributed, test at the 1% level whether the output from the machine is different from expected.

Testing $\mu_1 - \mu_2$, the difference between means of two populations

Consider $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$.

$$H_o : \mu_1 - \mu_2 = 0$$

$$H_1 : \mu_1 - \mu_2 > 0$$

$$\text{or } H_1 : \mu_1 - \mu_2 < 0$$

$$\text{or } H_1 : \mu_1 - \mu_2 \neq 0$$

The test statistics, $\overline{X}_1 - \overline{X}_2$,

$$E(\overline{X}_1 - \overline{X}_2) = E(\overline{X}_1) - E(\overline{X}_2) = \mu_1 - \mu_2$$

$$Var(\overline{X}_1 - \overline{X}_2) = Var(\overline{X}_1) + Var(\overline{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Hypothesis Test 1:
The population variances σ_1^2 and σ_2^2 are known

If the population variances σ_1^2 and σ_2^2 are known, the test statistics is $\bar{X}_1 - \bar{X}_2$, where

$$\bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2 , \frac{\sigma_1^{\,2}}{n_1} + \frac{\sigma_2^{\,2}}{n_2}\right)$$

In standardized form, the test statistics is

$$z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^{\,2}}{n_1} + \frac{\sigma_2^{\,2}}{n_2}}} \quad \text{where } Z \sim N(0,1)$$

Example 8:

For a test whether the means of two normally distributed populations are equal or not, a random sample is taken from each population. The sample from the first population has size 8 and the sample mean is 19.9. The sample mean from the second population has size 10 and the sample mean 16.7. The population variances are known to be 2.8 and 2.9 respectively. Calculate an appropriate test statistic and decide, at the 1% significance level, whether to reject the hypothesis that the means of the two population are equal.

Hypothesis Test 2:

The population variances σ_1^2 and σ_2^2 are unknown

If the population variances σ_1^2 and σ_2^2 are known, the test statistics is $\overline{X}_1 - \overline{X}_2,$ where

$$\overline{X}_1 - \overline{X}_2 \sim N\left(\mu_1 - \mu_2\,,\, \frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_2^2}{n_2}\right)$$

$$\hat{\sigma}^2 = \frac{n}{n-1} s^2$$

In standardized form, the test statistics is

$$z = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_2^2}{n_2}}} \quad \text{where } Z \sim N(0,1)$$

Example 9:

Manufacturer X claims that a certain type of electronic component, which it produces, lasts at least 10 hours longer than the corresponding component produced by manufacturer *Y*. A customer tests samples of size 50 of each type of component. The results can be summarized as follows.

$$\sum x = 2332, \quad \sum x^2 = 206389, \sum y = 2091, \quad \sum y^2 = 221829$$

Assuming that the customer's samples are random, test, at the 5% significance level, whether these results give support to manufacturer X's claim.

Hypothesis Test 3:

The population have common variance σ^2 which is known

If there is a common population variance σ^2 ($\sigma_1^2 = \sigma_2^2$) and σ^2 is known, then the test statistic is $\bar{X}_1 - \bar{X}_2$, where

$$\bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2 , \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$$

In standardized form, the test statistics is

$$z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \quad \text{where } Z \sim N(0,1)$$

Example 10:*

The same physical fitness test given to a group of 100 scouts and to a group of 144 guides. The maximum score was 30. The guides obtained a mean score of 26.81 and the scouts obtained a mean score of 27.53. Assuming that the fitness scores are normally distributed with a common population standard deviation of 3.48, test at the 5% level of significance whether the guides did not do as well as the scouts in the fitness test.

Hypothesis Test 4:

The population have common population variance σ^2 which is unknown

If the common population variance σ^2 is known, then an unbiased estimate $\hat{\sigma}^2$, is used.

$$\text{Pooled two-sample estimste: } \hat{\sigma}^2 = \frac{n_1 s_1^{\ 2} + n_2 s_2^{\ 2}}{n_1 + n_2 - 2}$$

$$OR$$

$$\hat{\sigma}^2 = \frac{\sum (x_1 - \bar{x}_1)^2 + \sum (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

($s_1^{\ 2}$ and $s_1^{\ 2}$ are the sample variances)

Hypothesis Test 4(i):

The population have common population variance σ^2 which is unknown, for large samples.

The distribution of $\bar{X}_1 - \bar{X}_2$ is approximately normal.

The test statistics is $\bar{X}_1 - \bar{X}_2$ where

$$\bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2\,,\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$$

In standardized form, the test statistics is

$$z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad \text{where } Z \sim N(0,1)$$

Example 11:*

An investigation was carried out to assess the effects of adding certain vitamins to the diet. A group of two-week old rats was given a vitamin supplement in their diet for a period of one month, after which time their masses were noted. A control group of rats of the same age was fed on an ordinary diet and their masses were also noted after one month.

Example 11:*

The results are summarized in the table:

	Number in Sample	Mean	Standard deviation
With vitamin supplement	64	89.6g	12.96g
Without vitamin supplement	36	83.5g	11.41g

Treating the samples as large samples from normal distributions with the same variance, σ^2 , test at the 5% level whether the results provide evidence that rats given the vitamin supplement have a greater mass, at age six weeks, than those not given the vitamin supplement.

Two-sample t-test (Independent t-test)

Hypothesis Test 4(ii):

The population have common population variance σ^2 which is unknown, for small samples.

The standardized form of the distribution of $\overline{X}_1 - \overline{X}_2$ follows a t-distribution.

The test statistics is

$$T = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sigma_{\boxed{?}} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad \text{where } T \sim t(n_1 + n_2 - 2).$$

Example 12:

In an experiment 22 mice were divided into two groups, one of which was given a special diet which was designed to give faster mass gain. After an interval the gains in mass (measured in grams) of the mice were measured, with the results given below.

	Sample size	Mean (grams)	Unbiased estimate of Population variance (grams^2)
Special diet	10	20.6	4.51
Usual diet	12	18.2	3.17

Example 12:

	Sample size	Mean (grams)	Unbiased estimate of Population variance (grams^2)
Special diet	10	20.6	4.51
Usual diet	12	18.2	3.17

The mass gains for each group may be assumed to be distributed normally with the same variance. Calculate an estimate of this common variance from the combined results of both groups. Test, at the 5% significance level, whether the special diet produced a greater increase in mass than the usual diet. What additional assumption must be made carry out this test?

Example 13:

Two different types of nylon fibre were tested for the amount of stretching under tension. Ten random samples of each fibre, of the same length and diameter, were stretched by applying a standard load. For fibre 1 the increases in length, x mm, were as follows.

12.84	14.26	13.23	14.75	15.13
14.15	13.37	12.96	15.02	14.38

$$\left(\sum x = 140.9, \quad \sum x^2 = 1969.0513\right)$$

Example 13:

For fibre 2 the increases in length, y mm, were as follows.

14.27	13.25	14.17	13.11	14.92
12.12	14.21	13.68	15.14	14.81

$$\left(\sum y = 139.68, \quad \sum y^2 = 1958.9794\right)$$

Test whether the mean increase in length of the two types of fibre is different, stating any conditions necessary for your test to hold. Use a 10% significance level.

The Paired t-test

If n pairs of data are drawn from two populations with mean μ_x and μ_y respectively, then the null hypothesis

$$H_o : \mu_x - \mu_y = c$$

can be tested using the test statistic T with $n - 1$ degrees of freedom, where

$$T = \frac{\overline{D} - c}{\sqrt{\dfrac{\sigma_d^2}{n}}}$$

where D is equal to $X - Y$, the difference between the pairs of values, $\overline{D}$ is the sample mean of these differences and

σ_d^2 is an unbiased estimator of the variance of the

The Paired t-test

population of the differences. The critical values of T are found in the same way as they are for a one-sample *t*-test.

The test may assumes that differences between the paired values are normally distributed. When *n* is large the test statistic

$$z = \frac{\overline{D} - c}{\sqrt{\frac{\sigma_d^2}{n}}}$$

May be used even if the differences are not distributed normally.

Example 14:

The manufacturer of a food supplement that is meant to aid dieting claims that a person who uses it correctly should lose at least 3 kg over the first two weeks of use. The table shows the masses of a random sample of 11 people before and after they had used the supplement for two weeks.

Mass before (kg)	70.0	71.1	64.1	63.3	73.2	55.1	74.1	71.7	64.5	63.8	69.2
Mass after (kg)	67.3	70.1	60.2	61.4	70.6	53.2	73.1	69.7	63.0	62.0	67.5

Example 14:

Test, at the 5% significance level, whether these data support the manufacturer's claim. You may assume that the difference between the mass before and the mass after is normally distributed.

Example 15:

To investigate the difference in wear on front and rear tyres of motorcycles, 50 motorcycles of the same model were fitted with new tyres of the same brand. After the motorcycles had been driven for 2000 miles the depths of tread on the front and rear tyres were measured in mm. for each motorcycle the value of d = (depth of front tread – depth of rear tread) was calculated. The results can be summarized by $\sum d = 4.7 \ and \ \sum d^2 = 0.79$. Test at the 5% significance level, whether there is a difference in wear on the front and rear tyres.