1. $$S_N = \sum_{n=N}^{N^2} \frac{1}{n(n+1)}$$

Expressing $\frac{1}{n(n+1)}$ as partial fractions,

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$= \frac{A(n+1) + Bn}{n(n+1)}$$

$$\therefore 1 = A(n+1) + Bn$$

$$= (A+B)n + A$$

$$\therefore A = 1 \qquad A + B = 0$$

$$B = -1$$

Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\therefore S_N = \sum_{n=N}^{N^2} \frac{1}{n(n+1)}$$

$$= \sum_{n=N}^{N^2} \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{1}{N} - \frac{1}{N+1}$$

$$+ \frac{1}{N+1} - \frac{1}{N+2}$$

$$+ \frac{1}{N+2} - \frac{1}{N+3}$$

$$\vdots$$

$$+ \frac{1}{N^2-1} - \frac{1}{N^2}$$

$$+ \frac{1}{N^2} - \frac{1}{N^2+1}$$

$$= \frac{1}{N} - \frac{1}{N^2+1}$$

$$S_N = \frac{1}{N} - \frac{1}{N^2+1}$$

$$\therefore \lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{1}{N} - \frac{1}{N^2+1}$$

$$= 0$$

2. $x = t - \sin t$ $\qquad y = 1 - \cos t$, $0 < t < 2\pi$

$\frac{dx}{dt} = 1 - \cos t \qquad \frac{dy}{dt} = \sin t$

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt}$$

$$= \frac{1}{\frac{dx}{dt}}\frac{dy}{dt}$$

$$= \frac{\sin t}{1 - \cos t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{\sin t}{1 - \cos t}\right)$$

$$= \frac{dt}{dx}\frac{d}{dt}\left(\frac{\sin t}{1 - \cos t}\right)$$

$$= \frac{1}{1 - \cos t}\left[\frac{(1-\cos t)\frac{d}{dt}(\sin t) - \sin t\frac{d}{dt}(1-\cos t)}{(1-\cos t)^2}\right]$$

$$= \frac{1}{1 - \cos t}\left(\frac{1}{1-\cos t}\right)^2\left[(1-\cos t)\frac{d}{dt}(\sin t) - \sin t\frac{d}{dt}(1-\cos t)\right]$$

$$= \frac{1}{(1-\cos t)^3}\left[(1-\cos t)\cos t - \sin t(\sin t)\right]$$

$$= \frac{1}{(1-\cos t)^3}\left[\cos t - \cos^2 t - \sin^2 t\right]$$

$$= \frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^3}$$

$$= \frac{\cos t - (\cos^2 t + \sin^2 t)}{(1-\cos t)^3}$$

$$= \frac{\cos t - 1}{(1-\cos t)^3}$$

$$= \frac{-(1-\cos t)}{(1-\cos t)^3}$$

$$= \frac{-1}{(1-\cos t)^2}$$

$$= \frac{-1}{\left[1-\left(1-2\sin^2\frac{t}{2}\right)\right]^2}$$

$$= \frac{-1}{\left(1-1+2\sin^2\frac{t}{2}\right)^2}$$

$$= \frac{-1}{\left(2\sin^2\frac{t}{2}\right)^2}$$

$$= \frac{-1}{4\sin^4\frac{t}{2}}$$

$$= -\frac{1}{4}\csc^4\frac{t}{2}$$

3. $\vec{OA} = a\underset{\sim}{i} \quad \vec{OB} = b\underset{\sim}{j} \quad \vec{OC} = c\underset{\sim}{k} \quad , \quad a, b, c > 0$

i) $$\vec{AB} = \vec{OB} - \vec{OA}$$
$$= b\underset{\sim}{j} - a\underset{\sim}{i}$$
$$= -a\underset{\sim}{i} + b\underset{\sim}{j}$$

$$\vec{AC} = \vec{OC} - \vec{OA}$$
$$= c\underset{\sim}{k} - a\underset{\sim}{i}$$
$$= -a\underset{\sim}{i} + c\underset{\sim}{k}$$

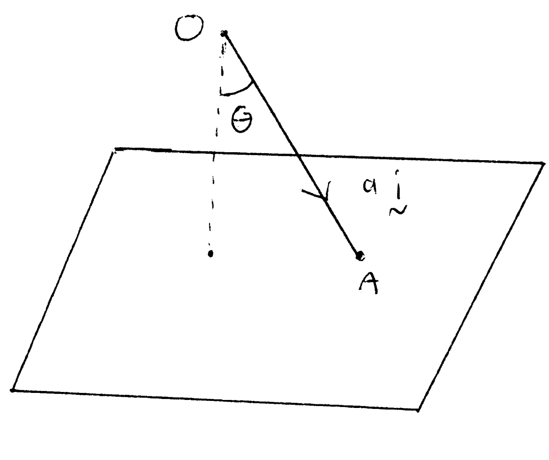
Since $\vec{AB}$ and $\vec{AC}$ lie in Π, $\vec{AB} \times \vec{AC}$ is perpendicular to Π

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$

$$= bc\underset{\sim}{i} + ac\underset{\sim}{j} + ab\underset{\sim}{k}$$

$\therefore$ $bc\underset{\sim}{i} + ac\underset{\sim}{j} + ab\underset{\sim}{k}$ is a vector perpendicular to Π

ii)



Since the line perpendicular to Π and passing through the origin has direction $bc\underset{\sim}{i} + ac\underset{\sim}{j} + ab\underset{\sim}{k}$, the perpendicular distance from the origin to Π is $\overrightarrow{OA} \cdot \underset{\sim}{\hat{n}}$, where $\underset{\sim}{\hat{n}}$ is a unit vector in the direction $bc\underset{\sim}{i} + ac\underset{\sim}{j} + ab\underset{\sim}{k}$.

$$\underset{\sim}{\hat{n}} = \frac{bc\underset{\sim}{i} + ac\underset{\sim}{j} + ab\underset{\sim}{k}}{|bc\underset{\sim}{i} + ac\underset{\sim}{j} + ab\underset{\sim}{k}|}$$

$$= \frac{bc\underset{\sim}{i} + ac\underset{\sim}{j} + ab\underset{\sim}{k}}{\sqrt{(bc)^2 + (ac)^2 + (ab)^2}}$$

$$= \frac{1}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}} \begin{pmatrix} bc \\ ac \\ ab \end{pmatrix}$$

$$\therefore \overrightarrow{OA} \cdot \underset{\sim}{\hat{n}} = |\overrightarrow{OA}| \, 1 \cdot \cos\theta$$

$$= \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}} \begin{pmatrix} bc \\ ac \\ ab \end{pmatrix}$$

$$= \frac{1}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}} \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} bc \\ ac \\ ab \end{pmatrix}$$

$$= \frac{abc + 0 + 0}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$$

$$= \frac{abc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$$

4. $x^3 + \lambda x + 1 = 0$

If α, β and γ are the roots of the equation $x^3 + \lambda x + 1 = 0$,

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = \lambda$

$\alpha\beta\gamma = -1$

If $S_n = \alpha^n + \beta^n + \gamma^n$

$\therefore S_0 = \alpha^0 + \beta^0 + \gamma^0 = 1 + 1 + 1 = 3,$

$S_1 = \alpha^1 + \beta^1 + \gamma^1 = \alpha + \beta + \gamma = 0$

and $S_2 = \alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 0^2 - 2\lambda$

$= -2\lambda$

$x^3 + \lambda x + 1 = 0$

$a = 1 \quad b = 0 \quad c = \lambda \quad d = 1$

$aS_{3+r} + bS_{2+r} + cS_{1+r} + dS_r = 0$

$S_{3+r} + \lambda S_{1+r} + S_r = 0$

when $r = 0$ $\quad S_3 + \lambda S_1 + S_0 = 0$

$S_3 + \lambda(0) + 3 = 0$

$S_3 = -3$

when $r = 1$: $S_4 + \lambda S_2 + S_1 = 0$

$$S_4 + \lambda(-2\lambda) + 0 = 0$$

$$S_4 - 2\lambda^2 = 0$$

$$\therefore S_4 = 2\lambda^2.$$

If $\lambda \in R$

$\lambda^2 \geqslant 0$

$\therefore S_4 = 2\lambda^2 \geqslant 0$

$\therefore$ There is no real value of λ for which the sum of the fourth powers of the roots is negative.

5. $x = t - 8t^{\frac{1}{2}}$ $\qquad y = \frac{16}{3}t^{\frac{3}{4}}$

i) $\frac{dx}{dt} = 1 - 4t^{-\frac{1}{2}}$ $\qquad \frac{dy}{dt} = 4t^{-\frac{1}{4}}$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 - 4t^{-\frac{1}{2}}\right)^2 + \left(4t^{-\frac{1}{4}}\right)^2$$

$$= 1 - 8t^{-1} + 16t^{-1} + 16t^{-\frac{1}{2}}$$

$$= 1 + 8t^{-1} + 16t^{-\frac{1}{2}}$$

$$= \left(1 + 4t^{-\frac{1}{2}}\right)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(1 + 4t^{-\frac{1}{2}}\right)^2}$$

$$= 1 + 4t^{-\frac{1}{2}}$$

$\therefore$ The length of C is $\int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_1^4 1 + 4t^{-\frac{1}{2}}\, dt$$

$$= \left[t + 8t^{\frac{1}{2}}\right]_1^4$$

$$= 4 + 8\sqrt{4} - \left(1 + 8\sqrt{1}\right)$$

$$= 4 + 16 - 1 - 8$$

$$= 11$$

11) The area of the surface generated when C is rotated through one complete revolution about the x-axis is $\int_1^4 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_1^4 2\pi \left(\frac{16t^{\frac{3}{4}}}{3}\right)\left(1 + 4t^{-\frac{1}{2}}\right) dt$$

$$= \int_1^4 \frac{32\pi}{3} t^{\frac{3}{4}}\left(1 + 4t^{-\frac{1}{2}}\right) dt$$

$$= \frac{32\pi}{3} \int_1^4 t^{\frac{3}{4}}\left(1 + 4t^{-\frac{1}{2}}\right) dt$$

$$= \frac{32\pi}{3} \int_1^4 t^{\frac{3}{4}} + 4t^{\frac{1}{4}}\, dt$$

$$= \frac{32\pi}{3} \left[\frac{4t^{\frac{7}{4}}}{7} + \frac{4(4)t^{\frac{5}{4}}}{5}\right]_1^4$$

$$= \frac{32\pi}{3} \left[\frac{4t^{\frac{7}{4}}}{7} + \frac{16t^{\frac{5}{4}}}{5}\right]_1^4$$

$$= \frac{32\pi}{3} \left(\frac{4}{7}\left(4^{\frac{7}{4}}\right) + \frac{16}{5}\left(4^{\frac{5}{4}}\right) - \left(\frac{4}{7} + \frac{16}{5}\right)\right)$$

$$= \frac{32\pi}{3} \left(\frac{4}{7}\left(2^{\frac{7}{2}}\right) + \frac{16}{5}\left(2^{\frac{5}{2}}\right) - \frac{4}{7} - \frac{16}{5}\right)$$

$$= \frac{32\pi}{3} \left(\frac{4}{7}\left(8\sqrt{2}\right) + \frac{16}{5}\left(4\sqrt{2}\right) - \frac{4}{7} - \frac{16}{5}\right)$$

$$= \frac{32\pi}{3}\left(\frac{32}{7}\sqrt{2} + \frac{64}{5}\sqrt{2} - \frac{4}{7} - \frac{16}{5}\right)$$

$$= \frac{32\pi}{3}\left(\frac{608\sqrt{2}}{35} - \frac{132}{35}\right)$$

$$= \frac{32\pi}{105}(608\sqrt{2} - 132)$$

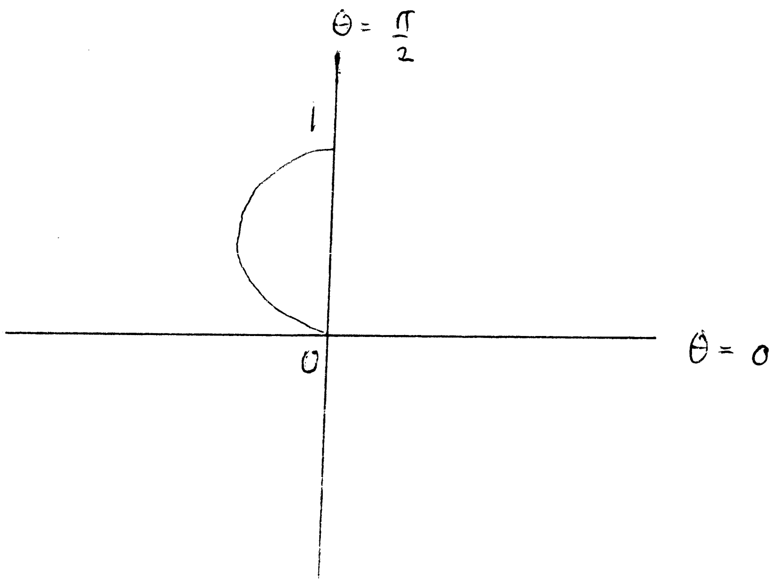
$$= 696.87$$

$$\approx 697.$$

6 C: $r = \dfrac{\pi - \theta}{\theta}$, $\dfrac{\pi}{2} \leq \theta \leq \pi$

i)

θ	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	0



ii) The area of the region bounded by the line $\theta = \frac{\pi}{2}$ and C is $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{r^2}{2}\, d\theta$

$$= \int_{\frac{\pi}{2}}^{\pi} \frac{(\pi - \theta)^2}{\theta^2}\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \frac{\pi^2}{\theta^2} - \frac{2\pi\theta}{\theta^2} + \frac{\theta^2}{\theta^2}\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \frac{\pi^2}{\theta^2} - \frac{2\pi}{\theta} + 1\, d\theta$$

$$= \left[-\frac{\pi^2}{\theta} - 2\pi \ln \theta + \theta \right]_{\frac{\pi}{2}}^{\pi}$$

$$= \left(-\frac{\pi^2}{\pi} - 2\pi \ln \pi + \pi \right) - \left(-\pi^2 \left(\frac{2}{\pi}\right) - 2\pi \ln\left(\frac{\pi}{2}\right) + \frac{\pi}{2} \right)$$

$$= \left(-\pi - 2\pi \ln \pi + \pi \right) - \left(-2\pi - 2\pi \ln \frac{\pi}{2} + \frac{\pi}{2} \right)$$

$$= -2\pi \ln \pi - \left(-\frac{3\pi}{4} - 2\pi \ln \frac{\pi}{2} \right)$$

$$= -2\pi \ln \pi + \frac{3\pi}{4} + 2\pi \ln \frac{\pi}{2}$$

$$= \frac{3\pi}{4} + 2\pi \left(\ln \frac{\pi}{2} - \ln \pi \right)$$

$$= \frac{3\pi}{4} + 2\pi \ln \left(\frac{\pi}{2\pi} \right)$$

$$= \frac{3\pi}{4} + 2\pi \ln \frac{1}{2}$$

$$= \frac{3\pi}{4} - 2\pi \ln 2$$

$$= \pi \left(\frac{3}{4} - 2 \ln 2 \right)$$

7 $\Pi_1 : x + 2y - 3z + 4 = 0$ $\Pi_2 : 2x + y - 4z - 3 = 0$

If a point (u, v, w) lies in both Π_1 and Π_2, it satisfies both equations $x + 2y - 3z = 4$ and $2x + y - 4z - 3 = 0$

$\therefore$ $u + 2v - 3w + 4 = 0$ —— ①

$2u + v - 4w - 3 = 0$ —— ②

① + $\lambda \times$ ② : $u + 2v - 3w + 4 + \lambda(2u + v - 4w - 3) = 0$

$\therefore$ (u, v, w) lies in the plane

$x + 2y - 3z + 4 + \lambda(2x + y - 4z - 3) = 0.$

$\therefore$ for all values of λ, every point which is in both Π_1 and Π_2 is also in the plane

$x + 2y - 3z + 4 + \lambda(2x + y - 4z - 3) = 0.$

i) Since the planes intersect in the line ℓ, ℓ must be perpendicular to $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$, where $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ are the normal vectors to Π_1 and Π_2. $\therefore$ The direction vector of ℓ is parallel to $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$ since ℓ lies in both planes.

$\underset{\sim}{n}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\underset{\sim}{n}_2 = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$

$$\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & -3 \\ 2 & 1 & -4 \end{vmatrix}$$

$$= -5\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$$

Since ℓ lies in both planes, a point on the line ℓ satisfies the equations $x + 2y - 3z + 4 = 0$ and $2x + y - 4z - 3 = 0$.

If $x = 0$:

$$x + 2y - 3z + 4 = 0 \qquad\qquad 2x + y - 4z - 3 = 0$$

$$2y - 3z \quad 4 = 0 \qquad\qquad y - 4z - 3 = 0$$

$$2y = 3z - 4 \quad \text{— (1)} \qquad\qquad y = 4z + 3 \quad \text{— (2)}.$$

$$\therefore \; 2(4z+3) = 3z - 4$$

$$8z + 6 = 3z - 4$$

$$5z = -10$$

$$z = -2$$

$$y = -5$$

$\therefore$ $(0, -5, -2)$ is a point in ℓ.

Since ℓ is parallel to $-5\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$ and $(0, -5, -2)$ is a point on ℓ, a vector equation for ℓ is

$$\underset{\sim}{r} = -5\underset{\sim}{j} - 2\underset{\sim}{k} + s(-5\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k})$$

If the plane Π_3 passes through ℓ and the point $(0, 0, a)$, the vectors $-5\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ -2 \end{pmatrix}$ are in the direction of the plane.

$\therefore$ The vector $\begin{pmatrix} -5 \\ -2 \\ -3 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ -2 \end{pmatrix} \right]$ is parallel to the normal of Π_3 since it is perpendicular to the plane.

$$\begin{pmatrix} -5 \\ -2 \\ -3 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ -2 \end{pmatrix} \right] = \begin{pmatrix} -5 \\ -2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 5 \\ a+2 \end{pmatrix}$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -5 & -2 & -3 \\ 0 & 5 & a+2 \end{vmatrix}$$

$$= \begin{pmatrix} -2(a+2) + 15 \\ 5(a+2) \\ -25 \end{pmatrix}$$

$$= \begin{pmatrix} -2a + 11 \\ 5a + 10 \\ -25 \end{pmatrix}$$

Since $\begin{pmatrix} -2a + 11 \\ 5a + 10 \\ -25 \end{pmatrix}$ is a normal to Π_3 and $(0, 0, a)$ is a point in Π_3

the equation of Π_3 can be expressed as

$$\underset{\sim}{r} \cdot \begin{pmatrix} -2a+11 \\ 5a+10 \\ -25 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \cdot \begin{pmatrix} -2a+11 \\ 5a+10 \\ -25 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2a+11 \\ 5a+10 \\ -25 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \cdot \begin{pmatrix} -2a+11 \\ 5a+10 \\ -25 \end{pmatrix}$$

$$(11-2a)x + (5a+10)y - 25z = 0 + 0 - 25a$$

$$= -25a$$

The plane Π_3 which passes through l and the point $(0, 0, a)$ has equation

$$(11-2a)x + (5a+10)y - 25z = -25a$$

ii) If Π_2 is perpendicular to Π_3, $\underset{\sim}{n}_2 \cdot \begin{pmatrix} 11-2a \\ 5a+10 \\ -25 \end{pmatrix} = 0$

since the normal of Π_2 is perpendicular to the normal of Π_3.

$$\underset{\sim}{n}_2 \cdot \begin{pmatrix} 11-2a \\ 5a+10 \\ -25 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 11-2a \\ 5a+10 \\ -25 \end{pmatrix} = 0$$

$$2(11-2a) + 1(5a+10) + (-4)(-25) = 0$$

$$22 - 4a + 5a + 10 + 100 = 0$$

$$a + 132 = 0$$

$$a = -132.$$

8. $I_n = \int_0^1 e^{-x}(1-x)^n \, dx$

i) Using integration by parts

$u = e^{-x}$ $\qquad dv = (1-x)^n \, dx$

$du = -e^{-x} \, dx$ $\qquad v = \dfrac{-(1-x)^{n+1}}{n+1}$

$$\therefore I_n = \left[\frac{-e^{-x}(1-x)^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{(-e^{-x})(-(1-x)^{n+1})}{n+1} \, dx$$

$$= \frac{-e^{-1}(1-1)^{n+1}}{n+1} - \left(\frac{-e^{-0}(1-0)^{n+1}}{n+1}\right) - \frac{1}{n+1}\int_0^1 e^{-x}(1-x)^{n+1} \, dx$$

$$= 0 + \frac{1}{n+1} - \frac{1}{n+1} I_{n+1}$$

$$= \frac{1}{n+1} - \frac{1}{n+1} I_{n+1}$$

$$\therefore (n+1) I_n = 1 - I_{n+1}$$

$$I_{n+1} = 1 - (n+1) I_n$$

ii) $I_0 = \int_0^1 e^{-x}(1-x)^0 \, dx$

$$= \int_0^1 e^{-x} \, dx$$

$$= \left[-e^{-x}\right]_0^1$$

$$= -e^{-1} - (-e^{-0})$$

$$= -e^{-1} + 1$$

$$= 1 - e^{-1}$$

$I_n = A_n + B_n e^{-1}$

when $n=0$. $I_0 = A_0 + B_0 e^{-1}$

$= 1 - e^{-1}$

$A_0 = 1 \quad B_0 = -1$

$\therefore I_0$ is of the form $A_0 + B_0 e^{-1}$ where A_0 and B_0 are integers.

Assume I_n is of the form $A_n + B_n e^{-1}$ where A_n and B_n are integers when $n=k$: $I_k = A_k + B_k e^{-1}$

when $n = k+1$ $\quad I_{k+1} = 1-(k+1) I_k$

$= 1 - (k+1)(A_k + B_k e^{-1})$

$= 1 - (k+1) A_k - (k+1) B_k e^{-1}$

$= A_{k+1} + B_{k+1} e^{-1}$

where $A_{k+1} = 1-(k+1)A_k$ and $B_{k+1} = -(k+1)B_k$.

$\therefore I_n$ is $A_n + B_n e^{-1}$ where A_n and B_n are integers.

iii) Since $B_{n+1} = -(n+1) B_n$

$\therefore B_n = -n B_{n-1}$

$= -n(-(n-1) B_{n-2})$

$= (-1)^2 n(n-1) B_{n-2}$

$= (-1)^2 n(n-1)(-(n-2)B_{n-3})$

$= (-1)^3 n(n-1)(n-2) B_{n-3}$

$\vdots$

$$= (-1)^r n(n-1)(n-2)\dots(n-r+1)\, B_{n-r}$$

$$\vdots$$

$$= (-1)^{n-2} n(n-1)(n-2)\dots 3\, B_2$$

$$= (-1)^{n-2} n(n-1)(n-2)\dots 3(-2B_1)$$

$$= (-1)^{n-1} n(n-1)(n-2)\dots 3.2\, B_1$$

$$= (-1)^{n-1} n(n-1)(n-2)\dots 3.2(-B_0)$$

$$= (-1)^{n} n(n-1)(n-2)\dots 3.2.1\, B_0$$

$$= (-1)^n n!\, B_0$$

$$= (-1)^n n!\,(-1)$$

$$= (-1)^{n+1} n!$$

9. $$m = \begin{pmatrix} a & 2 & 1 \\ 0 & b & -1 \\ 0 & 0 & c \end{pmatrix}$$

$$m - \lambda I = \begin{pmatrix} a & 2 & 1 \\ 0 & b & -1 \\ 0 & 0 & c \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a-\lambda & 2 & 1 \\ 0 & b-\lambda & -1 \\ 0 & 0 & c-\lambda \end{pmatrix}$$

$$\therefore |m - \lambda I| = \begin{vmatrix} a-\lambda & 2 & 1 \\ 0 & b-\lambda & -1 \\ 0 & 0 & c-\lambda \end{vmatrix}$$

$$= (a-\lambda)\begin{vmatrix} b-\lambda & -1 \\ 0 & c-\lambda \end{vmatrix} - 2\begin{vmatrix} 0 & -1 \\ 0 & c- \end{vmatrix} + 1\begin{vmatrix} 0 & b-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (a-\lambda)\ (b-\lambda)(c-\lambda) - 0 + 0$$

$$= (a-\lambda)\ (b-\lambda)(c-\lambda)$$

$$|m - \lambda I| = 0$$

$$(a-\lambda)(b-\lambda)(c-\lambda) = 0$$

$$\lambda = a, b, c$$

$\therefore$ The eigenvalues of m are a, b and c.

when $\lambda = a$:
$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & b-a & -1 \\ 0 & 0 & c-a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 0x + 2y + z &= 0 \\ 0x + (b-a)y - z &= 0 \\ 0x + 0y + (c-a)z &= 0 \end{aligned}\right\}$$

$$(c-a)z = 0$$

$$c \neq a \quad \therefore \quad z = 0$$

$$(b-a)y - z = 0$$

$$(b-a)y = 0$$

$$b \neq a \quad \therefore \quad y = 0$$

$$0x + 2y + z = 0$$

$$0x = 0$$

Let $x = s, \; s \in R$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

when $\lambda = b$: $\begin{pmatrix} a-b & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & c-b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} (a-b)x + 2y + z &= 0 \\ 0x + 0y - z &= 0 \\ 0x + 0y + (c-b)z &= 0 \end{aligned}\right\}$$

$$(c-b)z = 0$$

$$c \neq b \quad \therefore \quad z = 0$$

$$0x + 0y - z = 0$$

$$0y = 0$$

Let $y = s$, $s \in R$

$$(a-b)x + 2y + z = 0$$

$$(a-b)x + 2s = 0$$

$a \neq b$ $\quad x = \dfrac{-2s}{a-b}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \dfrac{-2s}{a-b} \\ s \\ 0 \end{pmatrix}$$

$$= s \begin{pmatrix} \dfrac{-2}{a-b} \\ 1 \\ 0 \end{pmatrix}$$

when $\lambda = c$: $\begin{pmatrix} a-c & 2 & 1 \\ 0 & b-c & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} (a-c)x + 2y + z &= 0 \\ 0x + (b-c)y - z &= 0 \\ 0x + 0y + 0z &= 0 \end{aligned}\right\}$$

$$(b-c)y - z = 0$$

Let $z = s$, $s \in R$

$b \neq c$ $\quad y = \dfrac{s}{b-c}$

$$(a-c)x + 2y + z = 0$$

$$(a-c)x + \frac{2s}{b-c} + s = 0$$

$$(a-c)x = \frac{(c-b-2)s}{b-c}$$

$$a \neq c \quad x = \frac{(c-b-2)s}{(a-c)(b-c)}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{(c-b-2)s}{(a-c)(b-c)} \\ \frac{s}{b-c} \\ s \end{pmatrix} = s \begin{pmatrix} \frac{c-b-2}{(a-c)(b-c)} \\ \frac{1}{b-c} \\ 1 \end{pmatrix}$$

The set of eigenvectors corresponding to the eigenvalues a, b and c are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \frac{2}{b-a} \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{c-b-2}{(a-c)(b-c)} \\ \frac{1}{b-c} \\ 1 \end{pmatrix}$

$$M - kI = \begin{pmatrix} a & 2 & 1 \\ 0 & b & -1 \\ 0 & 0 & c \end{pmatrix} - k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a-k & 2 & 1 \\ 0 & b-k & -1 \\ 0 & 0 & c-k \end{pmatrix}$$

$$M - kI - \lambda I = \begin{pmatrix} a-k & 2 & 1 \\ 0 & b-k & -1 \\ 0 & 0 & c-k \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a-k-\lambda & 2 & 1 \\ 0 & b-k-\lambda & -1 \\ 0 & 0 & c-k-\lambda \end{pmatrix}$$

$$\therefore |M - kI - \lambda I| = \begin{vmatrix} a-k-\lambda & 2 & 1 \\ 0 & b-k-\lambda & -1 \\ 0 & 0 & c-k-\lambda \end{vmatrix}$$

$$= (a-k-\lambda) \begin{vmatrix} b-k-\lambda & -1 \\ 0 & c-k-\lambda \end{vmatrix}$$

$$- 2 \begin{vmatrix} 0 & -1 \\ 0 & c-k-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & b-k-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (a-k-\lambda)(b-k-\lambda)(c-k-\lambda)$$

$$|M - kI - \lambda I| = 0$$

$$\therefore (a-k-\lambda)(b-k-\lambda)(c-k-\lambda) = 0$$

$$\lambda = a-k,\ b-k,\ c-k.$$

$\therefore$ The eigenvalues of $M - kI$ are $a-k$, $b-k$ and $c-k$.

when $\lambda = a-k$:

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & b-a & -1 \\ 0 & 0 & c-a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 0x + 2y + z &= 0 \\ 0x + (b-a)y - z &= 0 \\ 0x + 0y + (c-a)z &= 0 \end{aligned}\right\}$$

$$(c-a)z = 0$$

$$c \neq a \quad \therefore \quad z = 0$$

$$(b-a)y - z = 0$$

$$(b-a)y = 0$$

$$b \neq a \quad \therefore \quad y = 0$$

$$0x + 2y + z = 0$$

$$0x = 0$$

$$x = s, \quad s \in \mathbb{R}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

when $\lambda = b-k$:

$$\begin{pmatrix} a-b & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & c-b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (a-b)x + 2y + z &= 0 \\ 0x + 0y - z &= 0 \\ 0x + 0y + (c-b)z &= 0 \end{aligned}\right\}$$

$$(c-b)z = 0$$

$$b \neq c \quad \therefore \quad z = 0$$

$$0x + 0y - z = 0$$

$$0y = 0$$

Let $y = (a-b)s, \; s \in R$

$$(a-b)x + 2y + z = 0$$

$$(a-b)x + 2(a-b)s = 0$$

$$a \neq b \qquad x = -2s$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s \\ (a-b)s \\ 0 \end{pmatrix} = s \begin{pmatrix} -2 \\ a-b \\ 0 \end{pmatrix}$$

when $\lambda = c-k$: $\begin{pmatrix} a-c & 2 & 1 \\ 0 & b-c & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} (a-c)x + 2y + z &= 0 \\ 0x + (b-c)y - z &= 0 \\ 0x + 0y + 0z &= 0 \end{aligned}\right\}$$

$$(b-c)y - z = 0$$

Let $z = (b-c)(a-c)s, \; s \in R$

$$(b-c)y - (b-c)(a-c)s = 0$$

$$b \neq c \quad \therefore \quad y = (a-c)s$$

$$(a-c)x + 2y + z = 0$$

$$(a-c)x + 2(a-c)s + (a-c)(b-c)s = 0$$

$$a \neq c \therefore x = -2s - (b-c)s$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s-(b-c)s \\ (a-c)s \\ (a-c)(b-c)s \end{pmatrix} = s\begin{pmatrix} c-b-2 \\ a-c \\ (a-c)(b-c) \end{pmatrix}$$

$\therefore$ The set of eigenvectors corresponding to the eigenvalues $a-k$, $b-k$ and $c-k$ are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ a-b \\ 0 \end{pmatrix}$ and

$$\begin{pmatrix} c-b-2 \\ a-c \\ (a-c)(b-c) \end{pmatrix}.$$

If $(M-kI)^n = PDP^{-1}$ where D is a diagonal matrix and n is a positive integer

$$\therefore P = \begin{pmatrix} 1 & -2 & c-b-2 \\ 0 & a-b & a-c \\ 0 & 0 & (a-c)(b-c) \end{pmatrix}$$

and $$D = \begin{pmatrix} a-k & 0 & 0 \\ 0 & b-k & 0 \\ 0 & 0 & c-k \end{pmatrix}^n$$

$$= \begin{pmatrix} (a-k)^n & 0 & 0 \\ 0 & (b-k)^n & 0 \\ 0 & 0 & (c-k)^n \end{pmatrix}$$

10. i) $w^{12} = 1$

$= \cos 0 + i\sin 0$

$= \cos(0 + 2k\pi) + i\sin(0 + 2k\pi)$

$= \cos 2k\pi + i\sin 2k\pi, \quad k = 0, 1, 2, \ldots, 11$

$\therefore w = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{12}}$

$= \cos \frac{2k\pi}{12} + i\sin \frac{2k\pi}{12}$

$= \cos \frac{k\pi}{6} + i\sin \frac{k\pi}{6}, \quad k = 0, 1, 2, \ldots, 11$

ii) $(2 + z)^{12} = z^{12}$

$z^{12} + \binom{12}{1}2z^{11} + \binom{12}{2}2^2z^{10} + \binom{12}{3}2^3z^9 + \binom{12}{4}2^4z^8$

$+ \binom{12}{5}2^5z^7 + \binom{12}{6}2^6z^6 + \binom{12}{7}2^7z^5 + \binom{12}{8}2^8z^4$

$+ \binom{12}{9}2^9z^3 + \binom{12}{10}2^{10}z^2 + \binom{12}{11}2^{11}z + 2^{12} = z^{12}$

$\therefore \binom{12}{1}2z^{11} + \binom{12}{2}2^2z^{10} + \binom{12}{3}2^3z^9 + \binom{12}{4}2^4z^8$

$+ \binom{12}{5}2^5z^7 + \binom{12}{6}2^6z^6 + \binom{12}{7}2^7z^5 + \binom{12}{8}2^8z^4$

$+ \binom{12}{9}2^9z^3 + \binom{12}{10}2^{10}z^2 + \binom{12}{11}2^{11}z + 2^{12} = 0$

$\therefore (z+2)^{12} = z^{12}$ has 11 roots since it is a polynomial of degree 11.

-1 is a root since $(-1+2)^{12} = 1^{12} = (-1)^{12}$.

Since $(z+2)^{12} = z^{12}$

$$\therefore \frac{(z+2)^{12}}{z^{12}} = 1$$

$$\left(\frac{z+2}{z}\right)^{12} = 1$$

$$\left(\frac{z+2}{z}\right)^{12} = \cos 0 + i\sin 0$$

$$= \cos(0+2k\pi) + i\sin(0+2k\pi)$$

$$= \cos 2k\pi + i\sin 2k\pi$$

$$\therefore \frac{z+2}{z} = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{12}}$$

$$= \cos\frac{2k\pi}{12} + i\sin\frac{2k\pi}{12}, \quad k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$$

$$= \cos\frac{k\pi}{6} + i\sin\frac{k\pi}{6}$$

$$\frac{z}{z} + \frac{2}{z} = \cos\frac{k\pi}{6} + i\sin\frac{k\pi}{6}$$

$$1 + \frac{2}{z} = \cos\frac{k\pi}{6} + i\sin\frac{k\pi}{6}$$

$$\frac{2}{z} = \cos\frac{k\pi}{6} - 1 + i\sin\frac{k\pi}{6}$$

$$\frac{z}{2} = \frac{1}{\cos\frac{k\pi}{6} - 1 + i\sin\frac{k\pi}{6}}$$

$$= \frac{1}{\left(\cos\frac{k\pi}{6} - 1 + i\sin\frac{k\pi}{6}\right)} \times \frac{\left(\cos\frac{k\pi}{6} - 1 - i\sin\frac{k\pi}{6}\right)}{\left(\cos\frac{k\pi}{6} - 1 - i\sin\frac{k\pi}{6}\right)}$$

$$= \frac{\cos\frac{k\pi}{6} - 1 - i\sin\frac{k\pi}{6}}{\left(\cos\frac{k\pi}{6} - 1 + i\sin\frac{k\pi}{6}\right)\left(\cos\frac{k\pi}{6} - 1 - i\sin\frac{k\pi}{6}\right)}$$

$$= \frac{\cos\frac{k\pi}{6} - 1 - i\sin\frac{k\pi}{6}}{\left(\cos\frac{k\pi}{6} - 1\right)^2 + i\sin\frac{k\pi}{6}\left(\cos\frac{k\pi}{6} - 1\right) - i\sin\frac{k\pi}{6}\left(\cos\frac{k\pi}{6} - 1\right) - i^2\sin^2\frac{k\pi}{6}}$$

$$= \frac{\cos\frac{k\pi}{6} - 1 - i\sin\frac{k\pi}{6}}{\left(\cos\frac{k\pi}{6} - 1\right)^2 + \sin^2\frac{k\pi}{6}}$$

$$= \frac{\cos\frac{k\pi}{6} - 1 - i\sin\frac{k\pi}{6}}{\cos^2\frac{k\pi}{6} - 2\cos\frac{k\pi}{6} + 1 + \sin^2\frac{k\pi}{6}}$$

$$= \frac{\cos\frac{k\pi}{6} - 1 - i\sin\frac{k\pi}{6}}{1 + 1 - 2\cos\frac{k\pi}{6}}$$

$$= \frac{\cos\frac{k\pi}{6} - 1 - i\sin\frac{k\pi}{6}}{2 - 2\cos\frac{k\pi}{6}}$$

$$= \frac{\cos\frac{2k\pi}{12} - 1 - i\sin\frac{2k\pi}{12}}{2 - 2\cos\frac{2k\pi}{12}}$$

$$= \frac{\cos^2\frac{k\pi}{12} - \sin^2\frac{k\pi}{12} - 1 - 2i\sin\frac{k\pi}{12}\cos\frac{k\pi}{12}}{2 - 2\left(1 - 2\sin^2\frac{k\pi}{12}\right)}$$

$$= \frac{\cos^2\frac{k\pi}{12} + \sin^2\frac{k\pi}{12} - 2\sin^2\frac{k\pi}{12} - 1 - 2i\sin\frac{k\pi}{12}\cos\frac{k\pi}{12}}{2 - 2 + 4\sin^2\frac{k\pi}{12}}$$

$$= \frac{1 - 2\sin^2\frac{k\pi}{12} - 1 - 2i\sin\frac{k\pi}{12}\cos\frac{k\pi}{12}}{4\sin^2\frac{k\pi}{12}}$$

$$= \frac{-2\sin^2\frac{k\pi}{12} - 2i\sin\frac{k\pi}{12}\cos\frac{k\pi}{12}}{4\sin^2\frac{k\pi}{12}}$$

$$= \frac{-2\sin\frac{k\pi}{12}\left(\sin\frac{k\pi}{12} + i\cos\frac{k\pi}{12}\right)}{4\sin^2\frac{k\pi}{12}}$$

$$= \frac{-\left(\sin\frac{k\pi}{12} + i\cos\frac{k\pi}{12}\right)}{2\sin\frac{k\pi}{12}}$$

$$= \frac{-\sin\frac{k\pi}{12} - i\cos\frac{k\pi}{12}}{2\sin\frac{k\pi}{12}}$$

$$= \frac{-\sin\frac{k\pi}{12}}{2\sin\frac{k\pi}{12}} - \frac{i\cos\frac{k\pi}{12}}{2\sin\frac{k\pi}{12}}$$

$$= \frac{-1}{2} - \frac{i}{2}\left(\frac{\cos\frac{k\pi}{12}}{\sin\frac{k\pi}{12}}\right)$$

$$= \frac{-1}{2} - \frac{i}{2}\cot\frac{k\pi}{12}$$

$$\therefore z = 2\left(\frac{-1}{2} - \frac{i}{2}\cot\frac{k\pi}{12}\right)$$

$$= -1 - i\cot\frac{k\pi}{12}, \quad k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5.$$

$\therefore$ The other 10 non-real roots of $(z+2)^{12} = z^{12}$ may be expressed as $-1 - i\cot\frac{k\pi}{12}$, $k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.

iii) $\left(-1 - i\cot\frac{k\pi}{12}\right)\left(-1 + i\cot\frac{k\pi}{12}\right)$

$= 1 + i\cot\frac{k\pi}{12} - i\cot\frac{k\pi}{12} - i^2\cot^2\frac{k\pi}{12}$

$= 1 + \cot^2\frac{k\pi}{12}$

$= \operatorname{cosec}^2\frac{k\pi}{12}.$

iv) Since the roots of $(z+2)^{12} = z^{12}$ are expressed in the form $-1 + i\cot\frac{k\pi}{12}$, $k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, 6$ and the product of the roots is $-\frac{512}{3}$

$$\therefore \left(-1 - i\cot\frac{\pi}{12}\right)\left(-1 - i\cot\frac{2\pi}{12}\right)\left(-1 - i\cot\frac{3\pi}{12}\right)\left(-1 - i\cot\frac{4\pi}{12}\right)$$

$$\times\left(-1 - i\cot\frac{5\pi}{12}\right)\left(-1 - i\cot\frac{6\pi}{12}\right)\left(-1 - i\cot\left(-\frac{\pi}{12}\right)\right)\left(-1 - i\cot\left(-\frac{2\pi}{12}\right)\right)$$

$$\times\left(-1 - i\cot\left(-\frac{3\pi}{12}\right)\right)\left(-1 - i\cot\left(-\frac{4\pi}{12}\right)\right)\left(-1 - i\cot\left(-\frac{5\pi}{12}\right)\right) = -\frac{512}{3}$$

$$\left(-1 - i\cot\frac{\pi}{12}\right)\left(-1 - i\cot\frac{2\pi}{12}\right)\left(-1 - i\cot\frac{3\pi}{12}\right)\left(-1 - i\cot\frac{4\pi}{12}\right)$$

$$\times\left(-1 - i\cot\frac{5\pi}{12}\right)\left(-1 - i\cot\frac{6\pi}{12}\right)\left(-1 + i\cot\frac{\pi}{12}\right)\left(-1 + i\cot\frac{2\pi}{12}\right)$$

$$\times\left(-1 + i\cot\frac{3\pi}{12}\right)\left(-1 + i\cot\frac{4\pi}{12}\right)\left(-1 + i\cot\frac{5\pi}{12}\right) = -\frac{512}{3}$$

$$\left(-1-i\cot\frac{\pi}{12}\right)\left(-1+i\cot\frac{\pi}{12}\right)\left(-1-i\cot\frac{2\pi}{12}\right)\left(-1+i\cot\frac{2\pi}{12}\right)$$

$$\times\left(-1-i\cot\frac{3\pi}{12}\right)\left(-1+i\cot\frac{3\pi}{12}\right)\left(-1-i\cot\frac{4\pi}{12}\right)\left(-1+i\cot\frac{4\pi}{12}\right)$$

$$\times\left(-1-i\cot\frac{5\pi}{12}\right)\left(-1+i\cot\frac{5\pi}{12}\right)(-1) = -\frac{512}{3}$$

$$\therefore\left(\operatorname{cosec}^2\frac{\pi}{12}\operatorname{cosec}^2\frac{2\pi}{12}\operatorname{cosec}^2\frac{3\pi}{12}\operatorname{cosec}^2\frac{4\pi}{12}\operatorname{cosec}^2\frac{5\pi}{12}\right)(-1) = \frac{-512}{3}$$

$$\operatorname{cosec}^2\frac{\pi}{12}\operatorname{cosec}^2\frac{2\pi}{12}\operatorname{cosec}^2\frac{3\pi}{12}\operatorname{cosec}^2\frac{4\pi}{12}\operatorname{cosec}^2\frac{5\pi}{12} = \frac{512}{3}$$

$$\left(\frac{1}{\sin^2\frac{\pi}{12}}\right)\left(\frac{1}{\sin^2\frac{2\pi}{12}}\right)\left(\frac{1}{\sin^2\frac{3\pi}{12}}\right)\left(\frac{1}{\sin^2\frac{4\pi}{12}}\right)\left(\frac{1}{\sin^2\frac{5\pi}{12}}\right) = \frac{512}{3}$$

$$\frac{1}{\sin^2\frac{\pi}{12}\sin^2\frac{2\pi}{12}\sin^2\frac{3\pi}{12}\sin^2\frac{4\pi}{12}\sin^2\frac{5\pi}{12}} = \frac{512}{3}$$

$$\therefore\ \sin^2\frac{\pi}{12}\sin^2\frac{2\pi}{12}\sin^2\frac{3\pi}{12}\sin^2\frac{4\pi}{12}\sin^2\frac{5\pi}{12} = \frac{3}{512}$$

11. $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & -1 \\ 2 & 2 & \theta \end{pmatrix}$

$$\xrightarrow[-2r_1 + r_3]{-r_1 + r_2} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -4 & -3 \\ 0 & -4 & \theta - 4 \end{pmatrix}$$

$$\xrightarrow{-r_2 + r_3} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -4 & -3 \\ 0 & 0 & \theta - 1 \end{pmatrix}$$

If $\theta = 1$ $\begin{pmatrix} 1 & 3 & 2 \\ 0 & -4 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore \text{rank}(A) = 2$

If $\theta \neq 1$ $\begin{pmatrix} 1 & 3 & 2 \\ 0 & -4 & -3 \\ 0 & 0 & \theta - 1 \end{pmatrix}$

$$\xrightarrow{\frac{1}{\theta - 1} \times r_3} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -4 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore \text{rank}(A) = 3$

S:
$$\begin{aligned} x + 3y + 2z &= 1 \\ x - y - z &= 0 \\ 2x + 2y + \theta z &= 3\theta + \phi - z \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 1 & -1 & -1 & 0 \\ 2 & 2 & \theta & 3\theta + \phi - z \end{array}\right)$$

$$\xrightarrow[-2r_1+r_3]{-r_1+r_2} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -4 & -3 & -1 \\ 0 & -4 & \theta-4 & 3\theta+\phi-4 \end{array}\right)$$

$$\xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -4 & -3 & -1 \\ 0 & 0 & \theta-1 & 3\theta+\phi-3 \end{array}\right)$$

i) $(\theta-1)z = 3\theta+\phi-3$

If $\theta \neq 1$ $\therefore$ $z = \dfrac{3\theta+\phi-3}{\theta-1}$

$\phi = 0$ $\therefore$ $z = \dfrac{3\theta-3}{\theta-1}$

$= 3$

$-4y-3z = -1$

$-4y-3(3) = -1$

$-4y-9 = -1$

$-4y = 8$

$y = -2$

$x+3y+2z = 1$

$x+3(-2)+2(3) = 1$

$x-6+6 = 1$

$x = 1$

If $\theta \neq 1$ and $\phi = 0$, $\therefore$ $x=1$, $y=-2$, $z=3$.

ii) $(\theta-1)z = 3\theta+\phi-3$

If $\theta = 1$ and $\phi = 0$

$0z = 0$

Let $z = s$, $s \in \mathbb{R}$

$$-4y - 3z = -1$$

$$-4y - 3s = -1$$

$$y = \frac{1-3s}{4}$$

$$x + 3y + 2z = 1$$

$$x + 3\left(\frac{1-3s}{4}\right) + 2s = 1$$

$$x + \frac{3}{4} - \frac{9s}{4} + 2s = 1$$

$$x = \frac{1}{4} + \frac{s}{4}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{s}{4} \\ \frac{1}{4} - \frac{3s}{4} \\ s \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix} + s\begin{pmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ 1 \end{pmatrix}$$

S has a infinite number of solutions if $\theta = 1$ and $\phi = 0$.

iii) $(\theta - 1)z = 3\theta + \phi - 3$

If $\theta = 1$ and $\phi \neq 0$. $0z = \phi$

Since $\phi \neq 0$ S has no solution.

12. Γ: $y = \dfrac{ax^2 + bx + c}{x^2 + px + q}$

Let $S(x) = ax^2 + bx + C$ and $T(x) = x^2 + px + q$

Since $x = 1$ is an asymptote, $T(1) = 0$

$T(1) = 1 + p + q = 0$

$p + q = -1$ — ①

Since $x = 4$ is an asymptote, $T(4) = 0$

$T(4) = 16 + 4p + q = 0$

$4p + q = -16$ — ②

$$\left.\begin{aligned} p + q &= -1 \\ 4p + q &= -16 \end{aligned}\right\}$$

$-① + ②$: $\left.\begin{aligned} p + q &= -1 \\ 3p &= -15 \end{aligned}\right\}$

$3p = -15$

$p = -5 \quad q = 4$

$$y = \frac{ax^2 + bx + c}{x^2 + px + q}$$

$$= \frac{x^2\left(a + \frac{b}{x} + \frac{c}{x^2}\right)}{x^2\left(1 + \frac{p}{x} + \frac{a}{x^2}\right)}$$

$$= \frac{a + \frac{b}{x} + \frac{c}{x^2}}{1 + \frac{p}{x} + \frac{q}{x^2}}$$

Since $y = 2$ is an asymptote, $\lim_{x \to \pm\infty} y = 2$

$$\lim_{x \to \pm\infty} y = \lim_{x \to \pm\infty} \left(\frac{a + \frac{b}{x} + \frac{c}{x^2}}{1 + \frac{p}{x} + \frac{q}{x^2}} \right)$$

$$= \frac{a}{1}$$

$$= a$$

$$= 2$$

$\therefore a = 2 \quad p = -5 \quad q = 4.$

$$\therefore y = \frac{2x^2 + bx + c}{x^2 - 5x + 4}$$

i) $$\frac{dy}{dx} = \frac{(x^2 - 5x + 4)\frac{d}{dx}(2x^2 + bx + c) - (2x^2 + bx + c)\frac{d}{dx}(x^2 - 5x + 4)}{(x^2 - 5x + 4)^2}$$

$$= \frac{(x^2 - 5x + 4)(4x + b) - (2x^2 + bx + c)(2x - 5)}{(x^2 - 5x + 4)^2}$$

$\frac{dy}{dx} = 0$ when $x = 2$

$$0 = \frac{(4-10+4)(b+8)-(8+2b+c)(-1)}{(4-10+4)^2}$$

$$= \frac{-2(b+8)+8+2b+c}{4}$$

$$= \frac{-2b-16+8+2b+c}{4}$$

$$= \frac{c-8}{4}$$

$\therefore c = 8$

ii) $$y = \frac{2x^2+bx+8}{x^2-5x+4}$$

$$\frac{dy}{dx} = \frac{(x^2-5x+4)(4x+b)-(2x^2+bx+8)(2x-5)}{(x^2-5x+4)^2}$$

$$= \frac{4x^3-20x^2+16x+bx^2-5bx+4b-(4x^3+2bx^2+16x-10x^2-5bx-40)}{(x^2-5x+4)^2}$$

$$= \frac{(b-20)x^2+(16-5b)x+4b-(2b-10)x^2-(16-5b)x+40}{(x^2-5x+4)^2}$$

$$= \frac{(-b-10)x^2+4b+40}{(x^2-5x+4)^2}$$

If $\frac{dy}{dx} = 0$ $$\frac{(-b-10)x^2+4b+40}{(x^2-5x+4)^2} = 0$$

$$(-b-10)x^2+4b+40 = 0$$

$$(b+10)x^2 = 4b+40$$

If $b \neq -10$ $\therefore$ $x^2 = \dfrac{4(b+10)}{b+10}$

$$= 4$$

$$x = \pm 2.$$

$\therefore$ If $b \neq -10$ then Γ has exactly 2 stationary points.

iii) $b=-6$: $y = \dfrac{2x^2-6x+8}{(x^2-5x+4}$

$$= \frac{2(x^2-3x+4)}{x^2-5x+4}$$

when $x=0$, $y = 2$

Since $x^2-3x+4 \neq 0$, $\dfrac{2(x^2-3x+4)}{x^2-5x+4} \neq 0$

$\therefore$ $y \neq 0$

$$\frac{dy}{dx} = \frac{2(x^2-5x+4)\frac{d}{dx}(x^2-3x+4) - 2(x^2-3x+4)\frac{d}{dx}(x^2-5x+4)}{(x^2-5x+4)^2}$$

$$= \frac{2(x^2-5x+4)(2x-3) - 2(x^2-3x+4)(2x-5)}{(x^2-5x+4)^2}$$

$$= \frac{2(2x^3-10x^2+8x-3x^2+15x-12-(2x^3-6x^2+8x-5x^2+15x-20)}{(x^2-5x+4)^2}$$

$$= \frac{2(-2x^2+8)}{(x^2-5x+4)^2}$$

$$= \frac{-4(x^2-4)}{(x^2-5x+4)^2}$$

If $\frac{dy}{dx} = 0$ $\quad \frac{-4(x^2-4)}{(x^2-5x+4)^2} = 0$

$$-4(x^2-4) = 0$$

$$x^2-4 = 0$$

$$x = \pm 2$$

$$y = \frac{14}{9}, -2$$

∴ The stationary points of Γ are $\left(-2, \frac{14}{9}\right)$ and $(2, -2)$

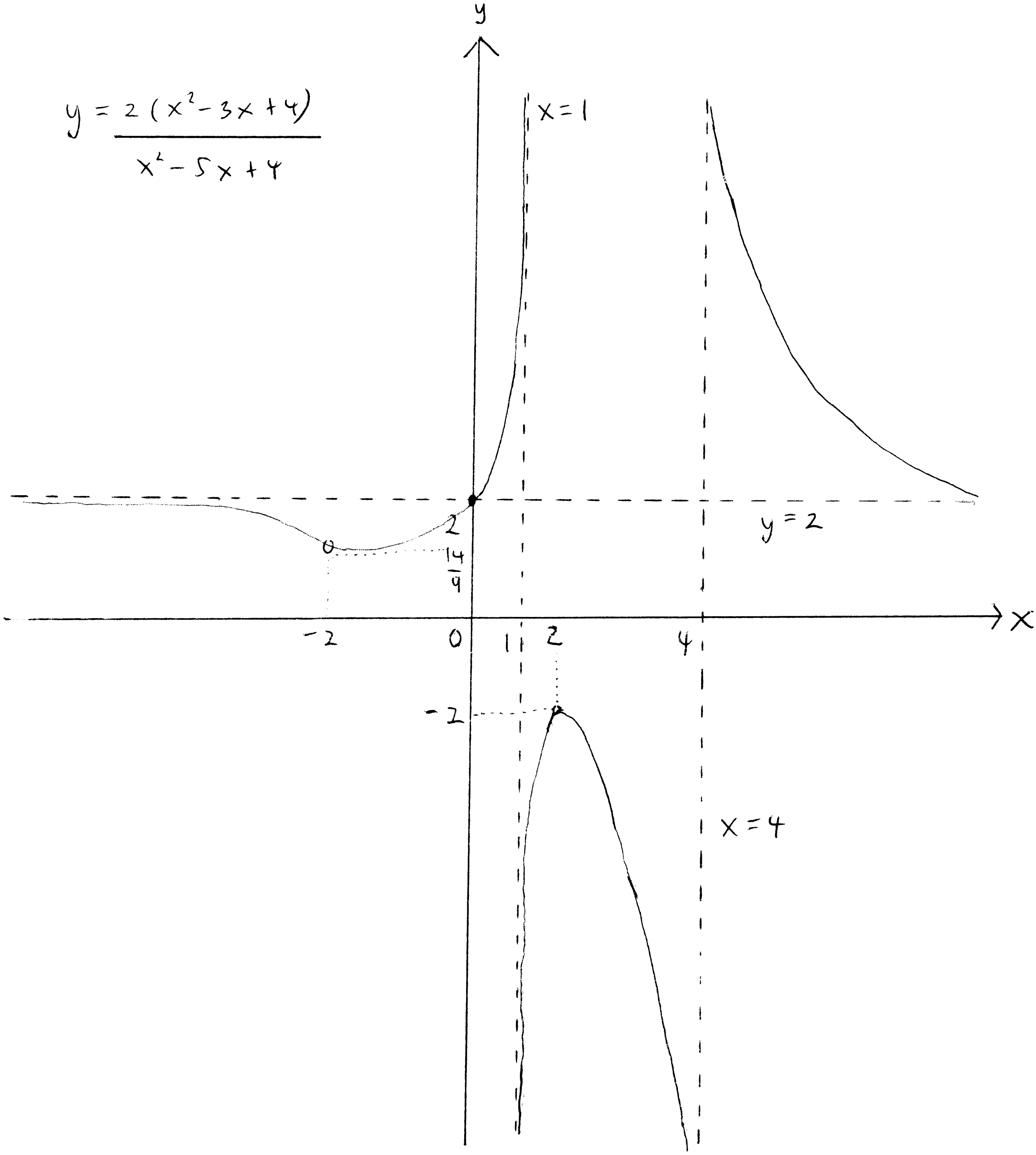
$$\frac{d^2y}{dx^2} = \frac{(x^2-5x+4)^2(-8x) + 4(x^2-4)2(x^2-5x+4)(2x-5)}{(x^2-5x+4)^4}$$

$$= \frac{-8x(x^2-5x+4)^2 + 8(x^2-4)(2x-5)(x^2-5x+4)}{(x^2-5x+4)^4}$$

when $x = -2$: $\frac{d^2y}{dx^2} = \frac{4}{81} > 0$

when $x = 2$: $\frac{d^2y}{dx^2} = -4 < 0$

∴ $\left(-2, \frac{14}{9}\right)$ is a minimum point and $(2, -2)$ is a maximum point.



o : stationary point

• : Intersection point

$$\frac{d^2y}{dx^2} + (2a-1)\frac{dy}{dx} + a(a-1)y = 2a-1 + a(a-1)x.$$

$$\frac{d^2y}{dx^2} + (2a-1)\frac{dy}{dx} + a(a-1)y = 0$$

$$\lambda^2 + (2a-1)\lambda + a(a-1) = 0$$

$$\lambda^2 + a\lambda + (a-1)\lambda + a(a-1) = 0$$

$$\lambda(\lambda + a) + (a-1)(\lambda + a) = 0$$

$$(\lambda + a)(\lambda + a - 1) = 0$$

$$\lambda = -a, 1-a$$

$$\therefore y_c = Ae^{-ax} + Be^{(1-a)x}$$

Let $y_p = Cx + D$

$$\frac{dy_p}{dx} = C$$

$$\frac{d^2y_p}{dx^2} = 0$$

$$\therefore \frac{d^2y_p}{dx^2} + (2a-1)\frac{dy_p}{dx} + a(a-1)y_p$$

$$= 0 + (2a-1)C + a(a-1)(Cx+D)$$

$$(2a-1)C + a(a-1)Cx + a(a-1)D$$

$$= a(a-1)Cx + (2a-1)C + a(a-1)D$$

$$= 2a-1 + a(a-1)x$$

$\therefore a(a-1) = a(a-1)C \quad , \quad 2a-1 = (2a-1)C + a(a-1)D$

$C = 1$

$2a-1 = 2a-1 + a(a-1)D$

$a(a-1)D = 0$

$D = 0$

$\therefore y_p = x$

$y = y_c + y_p$

$= Ae^{-ax} + Be^{(1-a)x} + x$

$\frac{dy}{dx} = -Aae^{-ax} + B(1-a)e^{(1-a)x} + 1$

when $x = 0$ $y = 0$: $0 = A + B$

when $x = 0$ $\frac{dy}{dx} = 0$ $0 = -Aa + B(1-a) + 1$

$A + B = 0$

$A = -B$

$-Aa + B(1-a) + 1 = 0$

$-(-B)a + B(1-a) + 1 = 0$

$Ba + B - Ba + 1 = 0$

$B = -1$

$A = 1$

$\therefore y = e^{-ax} - e^{(1-a)x} + x$

$= e^{-ax} - e^{-(a-1)x} + x$

If $a > 1$, $\lim_{x \to \infty} e^{-ax} = 0$ and $\lim_{x \to \infty} e^{-(a-1)x} = 0$

$\therefore$ As $x \to \infty$ $\quad y \to x$

$$\frac{d^2z}{dx^2} + (2a-1)\frac{dz}{dx} + a(a-1)z = e^x$$

$$\frac{d^2z}{dx^2} + (2a-1)\frac{dz}{dx} + a(a-1)z = 0$$

$$\lambda^2 + (2a-1)\lambda + a(a-1) = 0$$

$$\lambda^2 + a\lambda + a(a-1)\lambda + a(a-1) = 0$$

$$\lambda(\lambda+a) + (a-1)(\lambda+a) = 0$$

$$(\lambda+a)(\lambda+a-1) = 0$$

$$\lambda = -a, 1-a$$

$$\therefore z_c = Ae^{-ax} + Be^{(1-a)x}$$

Let $z_p = Ce^x$

$$\frac{dz_p}{dx} = Ce^x$$

$$\frac{d^2z_p}{dx^2} = Ce^x$$

$$\therefore \frac{d^2z_p}{dx^2} + (2a-1)\frac{dz_p}{dx} + a(a-1)z_p = Ce^x +$$

$$= Ce^x + (2a-1)Ce^x + a(a-1)Ce^x$$

$$= (C + (2a-1)C + a(a-1)C)e^x$$

$$= (C + 2aC - C + a^2C - aC)e^x$$

$$= (a^2C + aC)e^x$$

$$= e^x$$

$$\therefore a^2C + aC =$$

$$(a^2 + a)C = 1$$

$$C = \frac{1}{a + a^2}$$

$$\therefore z_p = \frac{e^x}{a + a^2}$$

$$z = z_c + z_p$$

$$= Ae^{-ax} + Be^{(1-a)x} + \frac{e^x}{a + a^2}$$

$$\therefore e^{-x}z = e^{-x}\left(Ae^{-ax} + Be^{(1-a)x} + \frac{e^x}{a + a^2}\right)$$

$$= Ae^{-ax}e^{-x} + Be^{(1-a)x}e^{-x} + \frac{e^x e^{-x}}{a + a^2}$$

$$= Ae^{-(a+1)x} + Be^{-ax} + \frac{1}{a + a^2}$$

If $a > 0$ $\lim_{x \to \infty} Ae^{-(a+1)x} = \lim_{x \to \infty} Be^{-ax} = 0$

$$\therefore \lim_{x \to \infty} e^{-x}z = \lim_{x \to \infty}\left(Ae^{-(a+1)x} + Be^{-ax} + \frac{1}{a + a^2}\right)$$

$$= \lim_{x \to \infty} Ae^{-(a+1)x} + \lim_{x \to \infty} Be^{-ax} + \lim_{x \to \infty} \frac{1}{a + a^2}$$

$$0 + 0 + \frac{1}{a + a^2}$$

$$= \frac{1}{a + a^2}$$