$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

$n=1$: $\displaystyle\sum_{r=1}^{1} \frac{1}{r(r+1)(r+2)} = \frac{1}{1\cdot 2\cdot 3} = \frac{1}{6} = \frac{1}{4} - \frac{1}{2\cdot 2\cdot 3} = \frac{1}{4} - \frac{1}{12}$

Assume the equation is true when $n=k$, that is

$n=k$: $\displaystyle\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$

when $n=k+1$:

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \frac{1}{(k+1)(k+2)(k+3)} + \sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)}$$

$$= \frac{1}{(k+1)(k+2)(k+3)} + \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$$

$$= \frac{1}{4} + \frac{2}{2(k+1)(k+2)(k+3)} \quad \frac{-k-3}{2(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} + \frac{2-k-3}{2(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{k+1}{2(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{1}{2(k+2)(k+3)}$$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

for every positive integer n.

$$\frac{1}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$$= \frac{A(r+1)(r+2) + Br(r+2) + Cr(r+1)}{r(r+1)(r+2)}$$

$$1 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$$

$$= A(r^2+3r+2) + B(r^2+2r) + C(r^2+r)$$

$$= (A+B+C)r^2 + (3A+2B+C)r + 2A$$

$A+B+C = 0 \qquad 3A+2B+C = 0 \qquad 2A = 1$

$2A+B = 0 \qquad\qquad A = \frac{1}{2}$

$B = -1 \qquad C = \frac{1}{2}$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{1}{2r} - \frac{1}{r+1} + \frac{1}{2(r+2)} \right)$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6}$$

$$+ \frac{1}{4} - \frac{1}{3} + \frac{1}{8}$$

$$+ \frac{1}{6} - \frac{1}{4} + \frac{1}{10}$$

$$+ \frac{1}{8} - \frac{1}{5} + \frac{1}{12}$$

$$\vdots$$

$$+ \frac{1}{2(n-2)} - \frac{1}{n-1} + \frac{1}{2n}$$

$$+ \frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}$$

$$+ \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$= \frac{1}{4} + \frac{1}{2(n+2)} - \frac{1}{2(n+1)}$$

$$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

2. $x^3 - 4x - 1 = 0$

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = -4$

$\alpha\beta\gamma = 1$

$a = 1 \quad b = 0 \quad c = -4 \quad d = -1$

$aS_{r+3} + bS_{r+2} + cS_{r+1} + dS_r = 0$

$S_{r+3} - 4S_{r+1} - S_r = 0$

$S_1 = \alpha + \beta + \gamma = 0$

$$S_2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$
$$= 0^2 - 2(-4)$$
$$= 8$$

$n = 1: \quad S_4 - 4S_2 - S_1 = 0$

$$S_4 = 4S_2 + S_1$$
$$= 32 + 0$$
$$= 32$$

$$\frac{S_4}{S_2} = \frac{32}{8}$$
$$= 4$$

3. $f(n) = 3^{4n-2} + 17^n + 22$

$n=1$. $f(1) = 3^{4-2} + 17^1 + 22$

$= 3^2 + 17 + 22$

$= 9 + 39$

$= 48$

$= 3 \cdot 16$

$\therefore$ 16 divides $f(1)$

$n=k$: Assume that 16 divides $f(n)$ when $n=k$,

that is $\frac{f(k)}{16} = s$, $s \in \mathbb{N}$

$f(k) = 16s$

$3^{4k-2} + 17^k + 22 = 16s$

$n=k+1$: $f(k+1) = 3^{4(k+1)-2} + 17^{k+1} + 22$

$= 3^{4k+4-2} + 17^{k+1} + 22$

$= 3^{4k-2}3^4 + 17^k 17 + 22$

$= 3^{4k-2}81 + 17^k 17 + 22$

$= 3^{4k-2}(80+1) + 17^k(16+1) + 22$

$= 3^{4k-2}80 + 3^{4k-2} + 17^k 16 + 17^k + 22$

$= 3^{4k-2}80 + 17^k 16 + 3^{4k-2} + 17^k + 22$

$= 3^{4k-2}(16 \cdot 5) + 17^k 16 + 16s$

$= 16(3^{4k-2}5 + 17^k + s)$

$\frac{f(k+1)}{16} = 3^{4k-2}5 + 17^k + s$

Since $3^{4k-2}5 + 17^k \in N$ and $s \in N$

$\therefore 3^{4k-2}5 + 17^k + s \in N$

$f(k+1)$ is divisible by 16

Since $f(k+1)$ is divisible by 16 if $f(k)$ is divisible by 16 for all $k \in N$ and $f(1)$ is divisible by 16 $f(n) = 3^{4n-2} + 17^n + 22$ is divisible by 16 for every positive integer n.

4. i) $y = \dfrac{2x^2+3x+1}{x-1} = Ax + B + \dfrac{C}{x-1}$

$$= \frac{(Ax+B)(x-1)+C}{x-1}$$

$$2x^2+3x+1 = (Ax+B)(x-1)+C$$
$$= Ax^2 + Bx - Ax - B + C$$
$$= Ax^2 + (B-A)x + C - B$$

$A = 2$ $\quad B - A = 3 \quad C - B = 1$

$B - 2 = 3 \quad C = B + 1$

$B = 5 \quad = 6$

$\therefore y = 2x + 5 + \dfrac{6}{x-1}$

$x \to 1 \quad y \to \pm\infty$

$x \to \pm\infty \quad y \to 2x+5$

$x = 1$

$y = 2x + 5$

ii) $y = 2x + 5 + \dfrac{6}{x-1}$

$$\frac{dy}{dx} = 2 - \frac{6}{(x-1)^2}$$

$$\frac{dy}{dx} = 0 \qquad \therefore \quad 2 - \frac{6}{(x-1)^2} = 0$$

$$\frac{6}{(x-1)^2} = 2$$

$$(x-1)^2 = 3$$

$$x - 1 = -\sqrt{3}, \sqrt{3}$$

$$x = 1 - \sqrt{3}, \ 1 + \sqrt{3}$$

$$x = 1 - \sqrt{3} \qquad y = 2(1 - \sqrt{3}) + 5 + \frac{6}{-\sqrt{3}}$$

$$= 2 - 2\sqrt{3} + 5 - 2\sqrt{3}$$

$$= 7 - 4\sqrt{3}$$

$$x = 1 + \sqrt{3} \qquad y = 2(1 + \sqrt{3}) + 5 + \frac{6}{\sqrt{3}}$$

$$= 2 + 2\sqrt{3} + 5 + 2\sqrt{3}$$

$$= 7 + 4\sqrt{3}$$

$$\frac{d^2y}{dx^2} = \frac{12}{(x-1)^3}$$

$$x = 1 - \sqrt{3} \qquad \frac{d^2y}{dx^2} = \frac{12}{-3\sqrt{3}} = \frac{-4}{\sqrt{3}} < 0$$

$$x = 1 + \sqrt{3} \qquad \frac{d^2y}{dx^2} = \frac{12}{3\sqrt{3}} = \frac{4}{\sqrt{3}} > 0$$

$(1-\sqrt{3},\ 7-4\sqrt{3})$ is a maximum point

$(1+\sqrt{3},\ 7+4\sqrt{3})$ is a minimum point

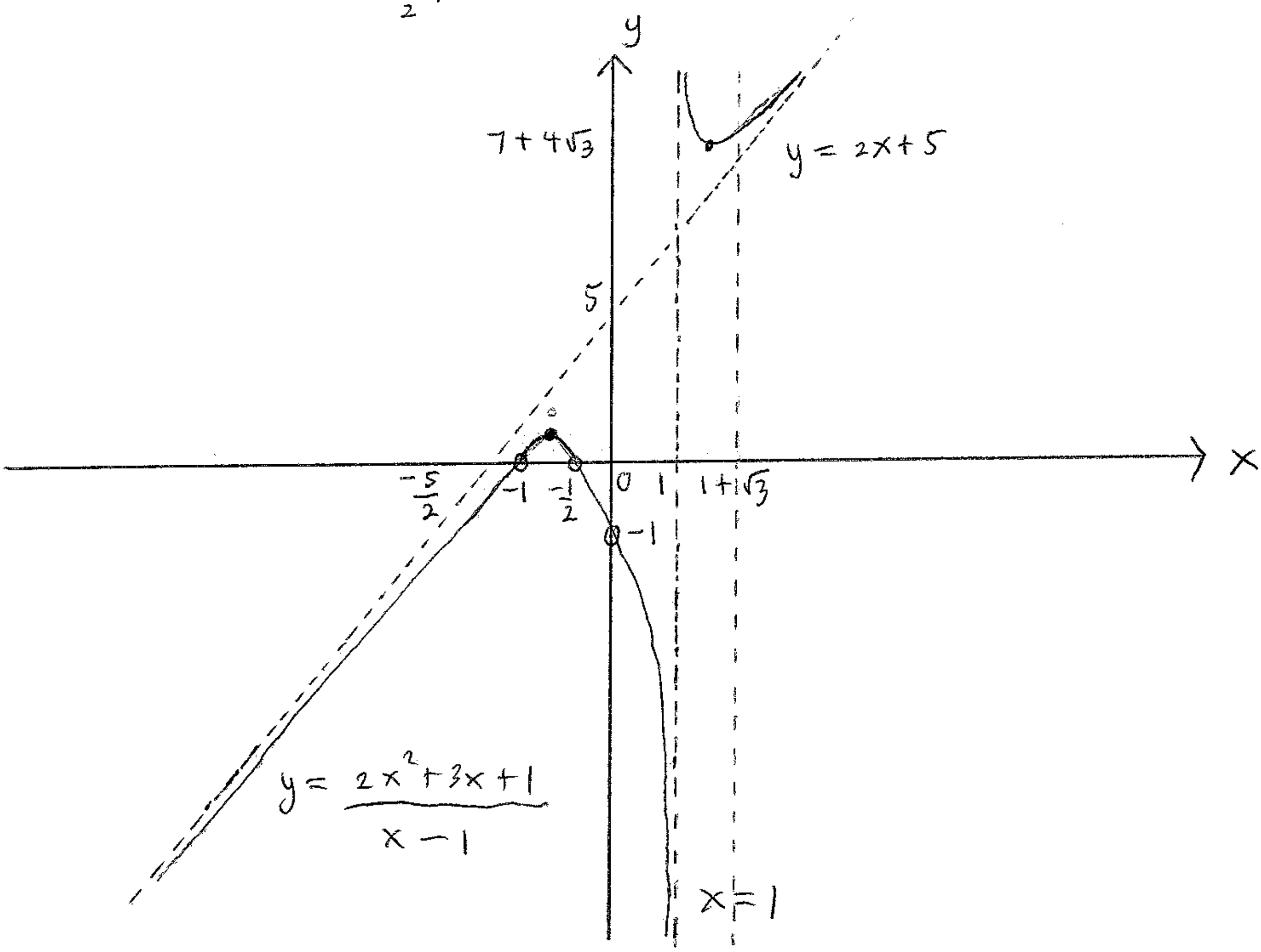
iii) $x=0 \quad y=-1$

$y=0 \quad \dfrac{2x^2+3x+1}{x-1}=0$

$2x^2+3x+1=0$

$(2x+1)(x+1)=0$

$x=-\frac{1}{2},\ -1$



o : intersection point

• : critical point

5. $$\sum_{j=1}^{n} \frac{j2^{j-1}}{(j+1)(j+2)} = \frac{2^n}{n+2} - \frac{1}{2}$$

$$n=1: \sum_{j=1}^{1} \frac{j2^{j-1}}{(j+1)(j+2)} = \frac{1\cdot 2^{1-1}}{(1+1)(1+2)} = \frac{1\cdot 2^0}{2\cdot 3}$$

$$= \frac{1\cdot 1}{6} = \frac{1}{6} = \frac{2^1}{3} - \frac{1}{2} = \frac{2^1}{1+2} - \frac{1}{2}$$

Assume the equation is true when $n=k$, that is

$$n=k: \sum_{j=1}^{k} \frac{j2^{j-1}}{(j+1)(j+2)} = \frac{2^k}{k+2} - \frac{1}{2}$$

when $n=k+1$:

$$\sum_{j=1}^{k+1} \frac{j2^{j-1}}{(j+1)(j+2)} = \frac{(k+1)2^{k+1-1}}{(k+1+1)(k+1+2)} + \sum_{j=1}^{k} \frac{j2^{j-1}}{(j+1)(j+2)}$$

$$= \frac{(k+1)2^k}{(k+2)(k+3)} + \frac{2^k}{k+2} - \frac{1}{2}$$

$$= \frac{(k+1)2^k}{(k+2)(k+3)} + \frac{2^k(k+3)}{(k+2)(k+3)} - \frac{1}{2}$$

$$= \frac{(k+1)2^k + (k+3)2^k}{(k+2)(k+3)} - \frac{1}{2}$$

$$= \frac{(k+1+k+3)2^k}{(k+2)(k+3)} - \frac{1}{2}$$

$$= \frac{(2k+4)2^k}{(k+2)(k+3)} - \frac{1}{2}$$

$$= \frac{2(k+2)2^k}{(k+2)(k+3)} - \frac{1}{2}$$

$$= \frac{2^{k+1}}{k+3} - \frac{1}{2}$$

Since $\sum_{j=1}^{n} \frac{j2^{j-1}}{(j+1)(j+2)} = \frac{2^n}{n+2} - \frac{1}{2}$ is true

for $n=k+1$ if it is true for $n=k$ and since it is true for $n=1$ $\therefore$

$$\sum_{j=1}^{n} \frac{j2^{j-1}}{(j+1)(j+2)} = \frac{2^n}{n+2} - \frac{1}{2}$$

for every positive integer n.

Expanding $\frac{j}{(j+1)(j+2)}$ into partial fractions,

$$\frac{j}{(j+1)(j+2)} = \frac{A}{j+1} + \frac{B}{j+2}$$

$$= \frac{A(j+2) + B(j+1)}{(j+1)(j+2)}$$

$$j = A(j+2) + B(j+1)$$

$$= (A+B)j + 2A + B$$

$$A + B = 1 \qquad 2A + B = 0$$
$$A = -1 \qquad B = 2$$

$$\frac{j2^{j-1}}{(j+1)(j+2)} = \frac{-1}{j+1} + \frac{2}{j+2}$$

$$\frac{j2^{j-1}}{(j+1)(j+2)} = \frac{-2^{j-1}}{j+1} + \frac{2^{j}}{j+2}$$

$$\sum_{j=1}^{n} \frac{j2^{j-1}}{(j+1)(j+2)} = \sum_{j=1}^{n} \left(\frac{2^{j}}{j+2} - \frac{2^{j-1}}{j+1}\right)$$

$$= \frac{2}{3} - \frac{1}{2}$$
$$+ \frac{4}{4} - \frac{2}{3}$$
$$+ \frac{8}{5} - \frac{4}{4}$$
$$+ \frac{16}{6} - \frac{8}{5}$$
$$\vdots$$
$$+ \frac{2^{n-1}}{n+1} - \frac{2^{n-2}}{n}$$
$$+ \frac{2^{n}}{n+2} - \frac{2^{n-1}}{n+1}$$

$$= \frac{2^{n}}{n+2} - \frac{1}{2}$$

6 Since U lies on ℓ, it has the form $(4+2\lambda, \lambda, -8)$ and since V lies in Π_1, it has the form $(-8+2\mu+2\nu, 12+\mu, 4-\nu)$. If W has coordinates (x,y,z) $\therefore$ $\overrightarrow{UW} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4+2\lambda \\ \lambda \\ -8 \end{pmatrix} = \begin{pmatrix} x-2\lambda-4 \\ y-\lambda \\ z+8 \end{pmatrix}$

and $\overrightarrow{WV} = \begin{pmatrix} -8+2\mu+2\nu \\ 12+\mu \\ 4-\nu \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8+2\mu+2\nu-x \\ 12+\mu-y \\ 4-\nu-z \end{pmatrix}$

Since $UW = 3WV$

$$\therefore \begin{pmatrix} x-2\lambda-4 \\ y-\lambda \\ z+8 \end{pmatrix} = 3\begin{pmatrix} -8+2\mu+2\nu-x \\ 12+\mu-y \\ 4-\nu-z \end{pmatrix}$$

$$\left.\begin{aligned} x-2\lambda-4 &= -24+6\mu+6\nu-3x \\ y-\lambda &= 36+3\mu-3y \\ z+8 &= 12-3\nu-3z \end{aligned}\right\}$$

$$\left.\begin{aligned} 2\lambda+6\mu+6\nu &= 4x+20 \\ \lambda+3\mu &= 4y-36 \\ -3\nu &= 4z-4 \end{aligned}\right\}$$

$$\left.\begin{aligned} \lambda+3\mu+3\nu &= 2x+10 \\ \lambda+3\mu &= 4y-36 \\ -3\nu &= 4z-4 \end{aligned}\right\}$$

$① - ② + ③: \quad 2x + 10 - 4y + 36 + 4z - 4 = 0$

$$2x - 4y + 4z + 42 = 0$$

$$x - 2y + 2z + 21 = 0$$

$$\vec{OL} = \begin{pmatrix} -4 \\ 13 \\ 3 \end{pmatrix} \qquad \vec{OM} = \begin{pmatrix} 0 \\ 15 \\ 3 \end{pmatrix}$$

i) writing Π_1 in Cartesian form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 + 2\mu + 2\nu \\ 12 + \mu \\ 4 - \nu \end{pmatrix}$$

$$\left.\begin{aligned} 2\mu + 2\nu - 8 &= x \\ \mu + 12 &= y \\ -\nu + 4 &= z \end{aligned}\right\}$$

$① - 2 \times ② + 2 \times ③: \quad x - 2y + 2z = -8 - 24 + 8$

$$= -24$$

$L: \ -4 - 2(13) + 2(3) = -24$

$M: \ 0 - 2(15) + 2(3) = -24$

$\therefore$ L and M are in Π_1.

ii) $\vec{LM} = \vec{OM} - \vec{OL} = \begin{pmatrix} 0 \\ 15 \\ 3 \end{pmatrix} - \begin{pmatrix} -4 \\ 13 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

If $A(4, 0, -8)$ $\therefore$ $\vec{AM} = \begin{pmatrix} 0 \\ 15 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix} = \begin{pmatrix} -4 \\ 15 \\ 11 \end{pmatrix}$

Since A is on l $\therefore$ $\overrightarrow{AM}$ and $\overrightarrow{LM}$ lie on the plane Π_2. If $\underset{\sim}{n}$ is a perpendicular vector to Π_2 $\therefore$ $\underset{\sim}{n}$ is parallel to $\overrightarrow{AM} \times \overrightarrow{LM}$

$$\overrightarrow{AM} \times \overrightarrow{LM} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -4 & 15 & 11 \\ 4 & 2 & 0 \end{vmatrix} = \begin{pmatrix} -22 \\ 44 \\ -68 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -22 \\ 44 \\ -68 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix} \cdot \begin{pmatrix} -22 \\ 44 \\ -68 \end{pmatrix}$$

since $\begin{pmatrix} -22 \\ 44 \\ -68 \end{pmatrix}$ is a normal to Π_2 and

$(4, 0, -8)$ is a point on Π_2

$$\therefore \quad -22x + 44y - 68z = 456$$

$$\Pi_2 : \quad 11x - 22y + 34z = -228$$

iii) The intersection of Π_2 and the locus of W is the intersection of the planes

$11x - 22y + 34z = -228$ and $x - 2y + 2z = -21$

$$\left.\begin{aligned} 11x - 2y + 2z &= -21 \\ 11x - 22y + 34z &= -228 \end{aligned}\right\}$$

$-11 \times ① + ②$: $\quad 12z = 3$

$$z = \frac{1}{4}$$

$$x - 2y + \frac{1}{2} = -21$$

$$2x - 4y + 1 = -42$$

$$2x - 4y = -43$$

$$y = s \therefore x = \frac{4s - 43}{2}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{4s-43}{2} \\ s \\ \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{43}{2} \\ 0 \\ \frac{1}{4} \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$