OCTOBER / NOVEMBER 2006

1. $A = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{pmatrix}$.

$$A - \lambda I = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -1 & -2 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 0 & -3-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 0 & 1 \\ 0 & -3-\lambda \end{vmatrix} - 2\begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(-3-\lambda) + 1\cdot 0 - 2\cdot 0$$

$$= (1-\lambda)(\lambda-2)(\lambda+3)$$

$|A - \lambda I| = 0$

$(1-\lambda)(\lambda-2)(\lambda+3) = 0$

$\lambda = 1, 2, -3$

When $\lambda = 1$: $\begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 + r_2} \left(\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{-4r_2 + r_3} \left(\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-z = 0$

$-y - 2z = 0$

$z = 0$

$y = 0$

Let $x = s$, $s \in R$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = 2$: $\begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right)$$

$$\xrightarrow{5r_2 + r_3} \left(\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = 0$

$-x - y - 2z = 0$

$-x - y = 0$

Let $y = s$, $s \in R$

$x = -s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ s \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

when $\lambda = -3$: $\begin{pmatrix} 4 & -1 & -2 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 4 & -1 & -2 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$5y + z = 0$

Let $z = 20s$, $s \in \mathbb{R}$

$y = -4s$

$4x - y - 2z = 0$

$4x - (-4s) - 2(20s) = 0$

$4x + 4s - 40s = 0$

$4x = 36s$

$x = 9s$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9s \\ -4s \\ 20s \end{pmatrix}$

$= s\begin{pmatrix} 9 \\ -4 \\ 20 \end{pmatrix}$

$\therefore$ The eigenvalues of A are $1, 2, -3$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ -4 \\ 20 \end{pmatrix}$.

2. $I_n = \int_0^1 x^n e^{-x^3}\,dx$

$$\frac{d}{dx}(x^{n+1}e^{-x^3}) = x^{n+1}\frac{d}{dx}(e^{-x^3}) + e^{-x^3}\frac{d}{dx}(x^{n+1})$$

$$= x^{n+1}(-3x^2e^{-x^3}) + e^{-x^3}(n+1)x^n$$

$$= -3x^{n+3}e^{-x^3} + (n+1)x^ne^{-x^3}$$

$$\therefore [x^{n+1}e^{-x^3}]_0^1 = \int_0^1 -3x^{n+3}e^{-x^3}\,dx + \int_0^1 (n+1)x^ne^{-x^3}\,dx$$

$$1^{n+1}e^{-1} - 0^{n+1}e^0 = -3\int_0^1 x^{n+3}e^{-x^3}\,dx + (n+1)\int_0^1 x^ne^{-x^3}\,dx$$

$$e^{-1} = -3I_{n+3} + (n+1)I_n$$

$$\therefore 3I_{n+3} = (n+1)I_n - e^{-1}$$

When $n = 3$: $3I_6 = 4I_3 - e^{-1}$

When $n = 0$: $3I_3 = I_0 - e^{-1}$

$$I_3 = \frac{I_0}{3} - \frac{1}{3e}$$

$$\therefore 3I_6 = 4\left(\frac{I_0}{3} - \frac{1}{3e}\right) - \frac{1}{e}$$

$$= \frac{4I_0}{3} - \frac{4}{3e} - \frac{1}{e}$$

$$= \frac{4I_0}{3} - \frac{7}{3e}$$

$$\therefore I_6 = \frac{4I_0}{9} - \frac{7e^{-1}}{9}$$

3. If $v_n = n(n+1)(n+2)\dots(n+m)$,

$$v_{n+1} = (n+1)(n+2)(n+3)\cdots(n+1+m)$$

$$\therefore v_{n+1} - v_n = (n+1)(n+2)(n+3)\dots(n+m+1)$$
$$- n(n+1)(n+2)\dots(n+m)$$
$$= (n+1)(n+2)(n+3)\dots(n+m)(n+m+1)$$
$$- n(n+1)(n+2)\dots(n+m)$$
$$= (n+1)(n+2)(n+3)\dots(n+m)(n+m+1-n)$$
$$= (m+1)(n+1)(n+2)\dots(n+m)$$

If $u_n = (n+1)(n+2)\dots(n+m)$,

since $v_{n+1} - v_n = (m+1)(n+1)(n+2)\dots(n+m)$
$$= (m+1)u_n$$

$$u_n = \frac{v_{n+1} - v_n}{m+1}$$

$$\therefore \sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \frac{v_{n+1} - v_n}{m+1}$$

$$= \frac{1}{m+1} \sum_{n=1}^{N} v_{n+1} - v_n$$

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$$= \frac{1}{m+1}\Big(v_{N+1} - v_N$$
$$+ v_N - v_{N-1}$$
$$+ v_{N-1} - v_{N-2}$$
$$\vdots$$
$$+ v_4 - v_3$$
$$+ v_3 - v_2$$
$$+ v_2 - v_1 \Big)$$

$$= \frac{1}{m+1}(v_{N+1} - v_1)$$

$$= \frac{(N+1)(N+2)\cdots(N+m+1) - 1\cdot 2\cdot 3 \cdots m(m+1)}{m+1}$$

$$= \frac{(N+1)(N+2)\cdots(N+m+1) - (m+1)!}{m+1}$$

$$= \frac{(N+1)(N+2)\cdots(N+m+1)}{m+1} - \frac{(m+1)!}{m+1}$$

$$= \frac{(N+1)(N+2)\cdots(N+m+1)}{m+1} - m!$$

4. Let $f(n) = 10^{3n} + 13^{n+1}$

When $n=1$: $f(1) = 10^{3(1)} + 13^{1+1}$
$$= 10^3 + 13^2$$
$$= 1000 + 169$$
$$= 1169$$
$$= 7(167)$$

$\therefore 7 \mid f(1)$

Assume the statement is true when $n=k$.

$n=k$: $7 \mid f(k)$

$f(k) = 7s$, s is an integer.

$10^{3k} + 13^{k+1} = 7s$

when $n = k+1$:

$$f(k+1) = 10^{3(k+1)} + 13^{k+1+1}$$
$$= 10^{3k+3} + 13^{k+1}13$$
$$= 10^{3k}10^3 + 13^{k+1}(6+7)$$
$$= 10^{3k}1000 + 13^{k+1}(6+7)$$
$$= 10^{3k}(994+6) + 13^{k+1}6 + 13^{k+1}7$$
$$= 10^{3k}994 + 10^{3k}6 + 13^{k+1}6 + 13^{k+1}7$$
$$= 10^{3k}994 + 13^{k+1}7 + 10^{3k}6 + 13^{k+1}6$$
$$= 10^{3k}7(142) + 13^{k+1}7 + 6(10^{3k} + 13^{k+1})$$

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$$= 7(10^{3K}142 + 13^{k+1}) + 6(7s)$$

$$= 7(10^{3K}142 + 13^{k+1} + 6s)$$

Since s is an integer and k is an integer, $10^{3K}142 + 13^{K+1} + 6s$ is an integer.

$$\therefore 7 \mid f(K+1)$$

$\therefore 10^{3n} + 13^{n+1}$ is divisible by 7 for every positive integer n.

5. $$\left.\begin{aligned} 2x + 3y + 4z &= -5 \\ 4x + 5y - z &= 5a + 15 \\ 6x + 8y + az &= b - 2a + 21 \end{aligned}\right\}$$

$-2 \times ① + ②$: $-3 \times ① + ③$:

$$\left.\begin{aligned} 2x + 3y + 4z &= -5 \\ -y - 9z &= 5a + 25 \\ -y + (a-12)z &= b - 2a + 36 \end{aligned}\right\}$$

$-② + ③$:

$$\left.\begin{aligned} 2x + 3y + 4z &= -5 \\ -y - 9z &= 5a + 25 \\ (a-3)z &= b - 7a + 11 \end{aligned}\right\}$$

$$(a-3)z = b - 7a + 11$$

If $a \neq 3$: $z = \dfrac{b - 7a + 11}{a - 3}$

$\therefore$ The system of equations has a unique solution if $a \neq 3$.

If $a = 3$: $0z = b - 10$

If $b = 10$: $0z = 0$

Let $z = s, s \in R$

$\therefore$ when $a = 3$, if $b = 10$, the equations are consistent.

6. $x^3+x+1=0$

α, β, γ are the roots

$\frac{4\alpha+1}{\alpha+1}, \frac{4\beta+1}{\beta+1}, \frac{4\gamma+1}{\gamma+1}$

Let $y=\frac{4\alpha+1}{\alpha+1}$

$(\alpha+1)y = 4\alpha+1$

$\alpha y + y = 4\alpha+1$

$\alpha y - 4\alpha = 1-y$

$\alpha(y-4) = 1-y$

$\alpha = \frac{1-y}{y-4}$

α is a root

$\therefore \alpha^3+\alpha+1=0$

$\left(\frac{1-y}{y-4}\right)^3 + \frac{1-y}{y-4}+1=0$

$\frac{(1-y)^3}{(y-4)^3} + \frac{1-y}{y-4}+1=0$

$(1-y)^3 + (1-y)(y-4)^2 + (y-4)^3 = 0$

$1-3y+3y^2-y^3+(1-y)(y^2-8y+16)$
$+y^3-12y^2+48y-64=0$

$1-3y+3y^2-y^3+y^2-8y+16-y^3+8y^2-16y$
$+y^3-12y^2+48y-64=0$

$-y^3+21y-47=0$

$y^3-21y+47=0$

$y^3+py+q=0$

$p=-21 \quad q=47$

$\therefore$ The equation $y^3-21y+47=0$ has roots $\frac{4\alpha+1}{\alpha+1}, \frac{4\beta+1}{\beta+1}, \frac{4\gamma+1}{\gamma+1}$.

$\frac{4\alpha+1}{\alpha+1}+\frac{4\beta+1}{\beta+1}+\frac{4\gamma+1}{\gamma+1}=0$

$\left(\frac{4\alpha+1}{\alpha+1}\right)\left(\frac{4\beta+1}{\beta+1}\right)+\left(\frac{4\alpha+1}{\alpha+1}\right)\left(\frac{4\gamma+1}{\gamma+1}\right)$
$+\left(\frac{4\beta+1}{\beta+1}\right)\left(\frac{4\gamma+1}{\gamma+1}\right)=-21$

$\left(\frac{4\alpha+1}{\alpha+1}\right)\left(\frac{4\beta+1}{\beta+1}\right)\left(\frac{4\gamma+1}{\gamma+1}\right)=-47$

Let $S_n = \left(\frac{4\alpha+1}{\alpha+1}\right)^n+\left(\frac{4\beta+1}{\beta+1}\right)^n+\left(\frac{4\gamma+1}{\gamma+1}\right)^n$

$S_0 = \left(\frac{4\alpha+1}{\alpha+1}\right)^0+\left(\frac{4\beta+1}{\beta+1}\right)^0+\left(\frac{4\gamma+1}{\gamma+1}\right)^0$

$=1+1+1$

$=3$

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$$S_1 = \left(\frac{4\alpha+1}{\alpha+1}\right)^1 + \left(\frac{4\beta+1}{\beta+1}\right)^1 + \left(\frac{4\gamma+1}{\gamma+1}\right)^1$$

$$= \frac{4\alpha+1}{\alpha+1} + \frac{4\beta+1}{\beta+1} + \frac{4\gamma+1}{\gamma+1}$$

$$= 0$$

$$S_2 = \left(\frac{4\alpha+1}{\alpha+1}\right)^2 + \left(\frac{4\beta+1}{\beta+1}\right)^2 + \left(\frac{4\gamma+1}{\gamma+1}\right)^2$$

$$= \left(\frac{4\alpha+1}{\alpha+1} + \frac{4\beta+1}{\beta+1} + \frac{4\gamma+1}{\gamma+1}\right)^2$$

$$-2\left[\left(\frac{4\alpha+1}{\alpha+1}\right)\left(\frac{4\beta+1}{\beta+1}\right) + \left(\frac{4\alpha+1}{\alpha+1}\right)\left(\frac{4\gamma+1}{\gamma+1}\right) + \left(\frac{4\beta+1}{\beta+1}\right)\left(\frac{4\gamma+1}{\gamma+1}\right)\right]$$

$$= 0^2 - 2(-21)$$

$$= 42$$

$$S_3 = \left(\frac{4\alpha+1}{\alpha+1}\right)^3 + \left(\frac{4\beta+1}{\beta+1}\right)^3 + \left(\frac{4\gamma+1}{\gamma+1}\right)^3$$

Since $1S_3 + 0S_2 + pS_1 + qS_0 = 0$

$$S_3 + pS_1 + qS_0 = 0$$

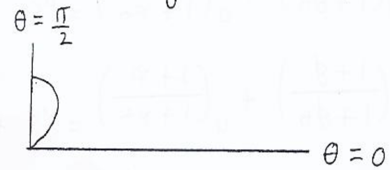
$$S_3 + 3(47) = 0$$

$$S_3 + 141 = 0$$

$$S_3 = -141.$$

7. $C: r = 10\ln(1+\theta)$, $0 \le \theta \le \frac{\pi}{2}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	$10\ln(1+\frac{\pi}{6})$	$10\ln(1+\frac{\pi}{4})$	$10\ln(1+\frac{\pi}{3})$	$10\ln(1+\frac{\pi}{2})$



The area of the sector bounded by the line $\theta = \frac{\pi}{2}$ and the arc of C from the origin to the point where $\theta = \frac{\pi}{2}$ is

$$\int_0^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(10\ln(1+\theta))^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{100}{2}\ln^2(1+\theta)\, d\theta$$

Let $w = \ln(1+\theta)$

$$dw = \frac{d\theta}{1+\theta}$$

$$d\theta = (1+\theta)\,dw$$

$$= (1 + e^w - 1)\, d^w$$

$$= e^w dw$$

$\theta = 0 \quad w = 0$

$\theta = \frac{\pi}{2} \quad w = \ln(1+\frac{\pi}{2})$

$$= \int_0^{\ln(1+\frac{\pi}{2})} 50 w^2 e^w \, dw$$

$$= 50 \int_0^{\ln(1+\frac{\pi}{2})} w^2 e^w \, dw$$

$u = w^2 \qquad dv = e^w dw$

$du = 2w\,dw \qquad v = e^w$

$$= 50\left(\left[w^2 e^w \right]_0^{\ln(1+\frac{\pi}{2})} - \int_0^{\ln(1+\frac{\pi}{2})} 2we^w \, dw \right)$$

$$= 50\left(\left(\ln(1+\tfrac{\pi}{2})\right)^2 e^{\ln(1+\frac{\pi}{2})} - 0 - 2\int_0^{\ln(1+\frac{\pi}{2})} we^w \, dw \right)$$

$$= 50 \cdot \ln(1+\tfrac{\pi}{2})^2 e^{\ln(1+\frac{\pi}{2})} - 100 \int_0^{\ln(1+\frac{\pi}{2})} we^w \, dw$$

$u = w \qquad dv = e^w dw$

$du = dw \qquad v = e^w$

$$= 50(\ln(1+\tfrac{\pi}{2}))^2 e^{\ln(1+\frac{\pi}{2})}$$
$$-100\left(\left[we^w \right]_0^{\ln(1+\frac{\pi}{2})} - \int_0^{\ln(1+\frac{\pi}{2})} e^w \, dw \right)$$

$$= 50(\ln(1+\tfrac{\pi}{2}))^2 e^{\ln(1+\frac{\pi}{2})}$$
$$-100\left(\ln(1+\tfrac{\pi}{2}) e^{\ln(1+\frac{\pi}{2})} - 0 - \int_0^{\ln(1+\frac{\pi}{2})} e^w \, dw \right)$$

$$= 50(\ln(1+\tfrac{\pi}{2}))^2 e^{\ln(1+\frac{\pi}{2})}$$
$$-100\left(\ln(1+\tfrac{\pi}{2}) e^{\ln(1+\frac{\pi}{2})} - \left[e^w \right]_0^{\ln(1+\frac{\pi}{2})} \right)$$

$$= 50(\ln(1+\tfrac{\pi}{2}))^2 e^{\ln(1+\frac{\pi}{2})} - 100(\ln(1+\tfrac{\pi}{2}) e^{\ln(1+\frac{\pi}{2})} - (e^{\ln(1+\frac{\pi}{2})} - 1))$$

$$= 50(\ln(1+\tfrac{\pi}{2}))^2 e^{\ln(1+\frac{\pi}{2})} - 100\ln(1+\tfrac{\pi}{2}) e^{\ln(1+\frac{\pi}{2})}$$
$$+ 100 e^{\ln(1+\frac{\pi}{2})} - 100$$

$$= 50(\ln(1+\tfrac{\pi}{2}))^2 e^{\ln(1+\frac{\pi}{2})} - 100\ln(1+\tfrac{\pi}{2}) e^{\ln(1+\frac{\pi}{2})}$$
$$+ 100 e^{\ln(1+\frac{\pi}{2})} - 100$$

$$= \left(50(\ln(1+\tfrac{\pi}{2}))^2 - 100\ln(1+\tfrac{\pi}{2}) + 100 \right) e^{\ln(1+\frac{\pi}{2})} - 100$$

$$= 50\left[(\ln(1+\tfrac{\pi}{2}))^2 - 2\ln(1+\tfrac{\pi}{2}) + 2 \right] e^{\ln(1+\frac{\pi}{2})} - 100$$

$$= 50(b^2 - 2b + 2)e^b - 100, \quad b = \ln(1+\tfrac{\pi}{2})$$

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8. $$2y^3\frac{d^2y}{dx^2}+12y^3\frac{dy}{dx}+6y^2\left(\frac{dy}{dx}\right)^2+17y^4=13e^{-4x}$$

$$v=y^4$$

$$\frac{dv}{dy}=4y^3$$

$$\frac{dv}{dx}=4y^3\frac{dy}{dx}$$

$$\frac{d^2v}{dx^2}=\frac{d}{dx}\left(4y^3\frac{dy}{dx}\right)$$

$$=4y^3\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\frac{d}{dx}(4y^3)$$

$$=4y^3\frac{d^2y}{dx^2}+12y^2\left(\frac{dy}{dx}\right)^2$$

$$\therefore \frac{d^2v}{dx^2}+\frac{6dv}{dx}+34v$$

$$=4y^3\frac{d^2y}{dx^2}+12y^2\left(\frac{dy}{dx}\right)^2+6\left(4y^3\frac{dy}{dx}\right)+34y^4$$

$$=4y^3\frac{d^2y}{dx^2}+12y^2\left(\frac{dy}{dx}\right)^2+24y^3\frac{dy}{dx}+34y^4$$

$$=2\left(2y^3\frac{d^2y}{dx^2}+6y^2\left(\frac{dy}{dx}\right)^2+12y^3\frac{dy}{dx}+17y^4\right)$$

$$=2(13e^{-4x})$$

$$=26e^{-4x}$$

$$\therefore \frac{d^2v}{dx^2}+\frac{6dv}{dx}+34v=0$$

$$m^2+6m+34=0$$

$$(m+3)^2+25=0$$

$$(m+3)^2=-25$$

$$m+3=\pm 5i$$

$$m=-3\pm 5i$$

$\therefore$ The complementary function, v_c, is

$$v_c=e^{-3x}(A\cos 5x+B\sin 5x)$$

The particular integral, v_p, is given by

$$v_p=Ce^{-4x}$$

$$\frac{dv_p}{dx}=-4Ce^{-4x}$$

$$\frac{d^2v_p}{dx^2}=16Ce^{-4x}$$

$$\therefore \frac{d^2v_p}{dx^2}+6\frac{dv_p}{dx}+34v_p=16Ce^{-4x}+6(-4Ce^{-4x})+34Ce^{-4x}$$

$$=16Ce^{-4x}-24Ce^{-4x}+34Ce^{-4x}$$

$$=26Ce^{-4x}$$

$$=26e^{-4x}$$

$$26C=26$$

$$C=1$$

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$$v_p = e^{-4x}$$

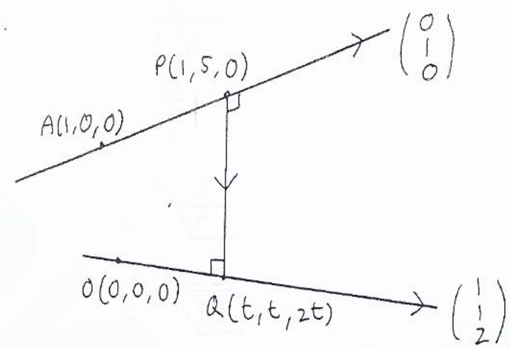
$$v = v_c + v_p$$

$$= e^{-3x}(A\cos 5x + B\sin 5x) + e^{-4x}$$

Since $v = y^4$, $y^4 = e^{-3x}(A\cos 5x + B\sin 5x) + e^{-4x}$

$$\therefore y = (e^{-3x}(A\cos 5x + B\sin 5x) + e^{-4x})^{\frac{1}{4}}.$$

9.



$\overrightarrow{OA} = \underset{\sim}{i} \quad \overrightarrow{OB} = \underset{\sim}{i} + \underset{\sim}{j} \quad \overrightarrow{OC} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\0\end{pmatrix}$$

Since the common perpendicular of the lines AB and OC is perpendicular to both $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\\2\end{pmatrix}$, it is parallel to $\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\1\\2\end{pmatrix}$.

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2\underset{\sim}{i} - \underset{\sim}{k}$$

Also, if P and Q are the points on the lines AB and OC such that PQ is the common perpendicular of the lines AB and OC, P and Q have the form $P(1,s,0)$ and $Q(t,t,2t)$.

Since PQ is the common perpendicular of the lines AB and OC and $\begin{pmatrix}2\\0\\-1\end{pmatrix}$ is perpendicular to both AB and OC, $\overrightarrow{PQ} \parallel \begin{pmatrix}2\\0\\-1\end{pmatrix}$.

$$\overrightarrow{PQ} = \lambda\begin{pmatrix}2\\0\\-1\end{pmatrix}, \lambda \in R$$

$$\begin{pmatrix}t\\t\\2t\end{pmatrix} - \begin{pmatrix}1\\s\\0\end{pmatrix} = \begin{pmatrix}2\lambda\\0\\-\lambda\end{pmatrix}$$

$$\begin{pmatrix}t-1\\t-s\\2t\end{pmatrix} = \begin{pmatrix}2\lambda\\0\\-\lambda\end{pmatrix}$$

$$\left.\begin{aligned} t-1 &= 2\lambda \\ t-s &= 0 \\ 2t &= -\lambda \end{aligned}\right\}$$

$\therefore t = 2\lambda+1,\ t = s,\ 2t = -\lambda$

$2t = 4\lambda + 2$

$4\lambda + 2 = -\lambda$

$5\lambda = -2$

$\lambda = \frac{-2}{5},\ t = \frac{1}{5},\ s = \frac{1}{5}$

$\therefore P(1, \frac{1}{5}, 0) \quad Q(\frac{1}{5}, \frac{1}{5}, \frac{2}{5})$

$\therefore$ A vector equation for the common perpendicular of the lines AB and OC is $\underset{\sim}{r} = \underset{\sim}{i} + \frac{\underset{\sim}{j}}{5} + \mu(2\underset{\sim}{i} - \underset{\sim}{k})$

The shortest distance between the lines AB and OC is $|\overrightarrow{PQ}|$.

$$\therefore |\overrightarrow{PQ}| = \left|\begin{pmatrix}\frac{1}{5}\\\frac{1}{5}\\\frac{2}{5}\end{pmatrix} - \begin{pmatrix}1\\\frac{1}{5}\\0\end{pmatrix}\right|$$

$$= \left|\begin{pmatrix}-\frac{4}{5}\\0\\\frac{2}{5}\end{pmatrix}\right|$$

$$= \sqrt{\left(-\frac{4}{5}\right)^2 + 0^2 + \left(\frac{2}{5}\right)^2}$$

$$= \sqrt{\frac{16}{25} + 0 + \frac{4}{25}}$$

$$= \sqrt{\frac{20}{25}}$$

$$= \sqrt{\frac{4}{5}}$$

$$= \frac{2}{\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{5}$$

The normal of the plane containing $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ is parallel to $\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}-\frac{4}{5}\\0\\\frac{2}{5}\end{pmatrix}$ since $\overrightarrow{AB}$ and $\overrightarrow{PQ}$

have directions $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-\frac{4}{5}\\0\\\frac{2}{5}\end{pmatrix}$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}-\frac{4}{5}\\0\\\frac{2}{5}\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{2}{5}\end{vmatrix} = \begin{pmatrix}\frac{2}{5}\\0\\\frac{4}{5}\end{pmatrix}$$

Since $A(1,0,0)$ is a point on the plane and $\begin{pmatrix}\frac{2}{5}\\0\\\frac{4}{5}\end{pmatrix}$ is perpendicular to the plane, if $\underset{\sim}{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ is a point on the plane, $\underset{\sim}{r} \cdot \begin{pmatrix}\frac{2}{5}\\0\\\frac{4}{5}\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}\frac{2}{5}\\0\\\frac{4}{5}\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}\frac{2}{5}\\0\\\frac{4}{5}\end{pmatrix} = \frac{2}{5} + 0 + 0$$

$$\frac{2x}{5} + \frac{4z}{5} = \frac{2}{5}$$

$$x + 2z = 1$$

$\therefore$ The equation of the plane containing AB and the common perpendicular of the lines AB and OC is $x + 2z = 1$.

10. $C: \; y = x^2 + \lambda \sin(x+y)$, $A\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

A is a point on C

$$\therefore \frac{\pi}{4} = \left(\frac{\pi}{4}\right)^2 + \lambda \sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{16} + \lambda \sin\frac{\pi}{2}$$

$$= \frac{\pi^2}{16} + \lambda$$

$$\lambda = \frac{\pi}{4} - \frac{\pi^2}{16}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(\lambda \sin(x+y))$$

$$= 2x + \lambda \cos(x+y)\frac{d}{dx}(x+y)$$

$$= 2x + \lambda \cos(x+y)\left(1 + \frac{dy}{dx}\right)$$

$$= 2x + \lambda\left(1 + \frac{dy}{dx}\right)\cos(x+y).$$

$$= 2x + \lambda\cos(x+y) + \lambda\cos(x+y)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} - \lambda\cos(x+y)\frac{dy}{dx} = 2x + \lambda\cos(x+y)$$

$$(1 - \lambda\cos(x+y))\frac{dy}{dx} = 2x + \lambda\cos(x+y)$$

$$\frac{dy}{dx} = \frac{2x + \lambda\cos(x+y)}{1 - \lambda\cos(x+y)}$$

CONFIDENTIAL

If c has a tangent which is parallel to the y-axis, $1 - \lambda\cos(x+y) = 0$

$$\cos(x+y) = \frac{1}{\lambda}$$

$$\lambda = \frac{\pi}{4} - \frac{\pi^2}{16}$$

$$= \frac{4\pi - \pi^2}{16}$$

$$= \frac{\pi(4-\pi)}{16}$$

$$\frac{1}{\lambda} = \frac{16}{\pi(4-\pi)}$$

since $\frac{1}{\lambda} > 1$,

$$1 - \lambda\cos(x+y) \neq 0$$

$\therefore$ c has no tangent which is parallel to the y-axis.

At $A\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$:

$$\frac{dy}{dx} = \frac{2\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right)}{1 - \lambda\cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right)}$$

$$= \frac{\frac{\pi}{2} + \cos\frac{\pi}{2}}{1 - \lambda\cos\frac{\pi}{2}}$$

$$= \frac{\frac{\pi}{2} + 0}{1 - 0}$$

$$= \frac{\pi}{2}$$

$$(1 - \lambda\cos(x+y))\frac{dy}{dx} = 2x + \lambda\cos(x+y)$$

$$\frac{d}{dx}\left((1 - \lambda\cos(x+y))\frac{dy}{dx}\right) = \frac{d}{dx}(2x + \lambda\cos(x+y))$$

$$(1 - \lambda\cos(x+y))\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(1 - \lambda\cos(x+y))$$

$$= \frac{d}{dx}(2x) + \lambda\frac{d}{dx}(\cos(x+y))$$

$$(1 - \lambda\cos(x+y))\frac{d^2y}{dx^2} + \frac{dy}{dx}(-\lambda)\frac{d}{dx}(\cos(x+y))$$

$$= 2 - \lambda\sin(x+y)\frac{d}{dx}(x+y)$$

$$(1 - \lambda\cos(x+y))\frac{d^2y}{dx^2} - \lambda\frac{dy}{dx}\left(-\sin(x+y)\frac{d}{dx}(x+y)\right)$$

$$= 2 - \lambda\sin(x+y)\left(1 + \frac{dy}{dx}\right)$$

CONFIDENTIAL

$$(1-\lambda\cos(x+y))\frac{d^2y}{dx^2}+\lambda\sin(x+y)\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}$$

$$=2-\lambda\sin(x+y)\left(1+\frac{dy}{dx}\right)$$

At $A\left(\frac{\pi}{4},\frac{\pi}{4}\right)$: $\frac{dy}{dx}=\frac{\pi}{2}$

$$\therefore\left(1-\lambda\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\right)\frac{d^2y}{dx^2}+\lambda\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}$$

$$=2-\lambda\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\left(1+\frac{dy}{dx}\right)$$

$$\left(1-\lambda\cos\frac{\pi}{2}\right)\frac{d^2y}{dx^2}+\lambda\sin\frac{\pi}{2}\left(1+\frac{\pi}{2}\right)\frac{\pi}{2}=2-\lambda\sin\frac{\pi}{2}\left(1+\frac{\pi}{2}\right)$$

$$(1-0)\frac{d^2y}{dx^2}+\lambda(1)\left(1+\frac{\pi}{2}\right)\frac{\pi}{2}=2-\lambda(1)\left(1+\frac{\pi}{2}\right)$$

$$\frac{d^2y}{dx^2}+\frac{\lambda\pi}{2}\left(1+\frac{\pi}{2}\right)=2-\lambda\left(1+\frac{\pi}{2}\right)$$

$$=2-\lambda-\frac{\lambda\pi}{2}$$

$$\frac{d^2y}{dx^2}=2-\lambda-\frac{\lambda\pi}{2}-\frac{\lambda\pi}{2}\left(1+\frac{\pi}{2}\right)$$

$$=2-\lambda-\frac{\lambda\pi}{2}-\frac{\lambda\pi}{2}-\frac{\lambda\pi^2}{4}$$

$$=2-\lambda-\lambda\pi-\frac{\lambda\pi^2}{4}$$

$$=2-\lambda\left(1+\pi+\frac{\pi^2}{4}\right)$$

$$=2-\left(\frac{\pi}{4}-\frac{\pi^2}{16}\right)\left(1+\pi+\frac{\pi^2}{4}\right)$$

$$=2-\left(\frac{4\pi-\pi^2}{16}\right)\frac{(\pi^2+4\pi+4)}{4}$$

$$=2-\frac{\pi(4-\pi)(\pi+2)^2}{64}$$

CONFIDENTIAL

11. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,\ n \in \mathbb{N}$

when $n=1$: $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

$= \cos 1\theta + i\sin 1\theta$

Assume the statement is true when $n=k$.

$n=k$: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

when $n=k+1$: $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

(what needs to be proved)

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\ &= \cos k\theta\cos\theta + i\sin k\theta\cos\theta\\ &\quad + i\sin\theta\cos k\theta - \sin k\theta\sin\theta\\ &= \cos k\theta\cos\theta - \sin k\theta\sin\theta\\ &\quad + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\\ &= \cos(k+1)\theta + i\sin(k+1)\theta\end{aligned}$$

$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for every positive integer n.

$$\begin{aligned}(\cos\theta + i\sin\theta)^7 &= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta\\ &\quad - 35i\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta\\ &\quad + 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta\end{aligned}$$

$$\begin{aligned}\cos 7\theta + i\sin 7\theta &= \cos^7\theta - 21\cos^5\theta\sin^2\theta\\ &\quad + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta\\ &\quad + i(7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta\\ &\quad + 21\cos^2\theta\sin^5\theta - \sin^7\theta)\end{aligned}$$

$$\begin{aligned}\cos 7\theta &= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta\\ &= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2\\ &\quad - 7\cos\theta(1-\cos^2\theta)^3\\ &= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta)\\ &\quad - 7\cos\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)\\ &= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta\\ &\quad - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta\\ &= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta\end{aligned}$$

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

$$64x^7 - 112x^5 + 56x^3 - 7x + \frac{1}{2} = 0$$

$$64x^7 - 112x^5 + 56x^3 - 7x = \frac{-1}{2}$$

Let $x = \cos\theta$

$$64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta = \frac{-1}{2}$$

$$\cos 7\theta = \frac{-1}{2}$$

$$7\theta = \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}, \frac{20\pi}{3}, \frac{26\pi}{3}, \frac{32\pi}{3}, \frac{38\pi}{3}$$

$$\theta = \frac{2\pi}{21}, \frac{8\pi}{21}, \frac{14\pi}{21}, \frac{20\pi}{21}, \frac{26\pi}{21}, \frac{32\pi}{21}, \frac{38\pi}{21}$$

$x = \cos\theta$

$$\therefore x = \cos\frac{2\pi}{21}, \cos\frac{8\pi}{21}, \cos\frac{14\pi}{21}, \cos\frac{20\pi}{21},$$

$$\cos\frac{26\pi}{21}, \cos\frac{32\pi}{21}, \cos\frac{38\pi}{21}.$$

$\therefore$ The roots of the equation

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

are $\cos\frac{2\pi}{21}$, $\cos\frac{8\pi}{21}$, $\cos\frac{14\pi}{21}$, $\cos\frac{20\pi}{21}$,

$\cos\frac{26\pi}{21}$, $\cos\frac{32\pi}{21}$ and $\cos\frac{38\pi}{21}$.

12. EITHER

$C: y = \dfrac{x^2 + qx + 1}{2x+3}$, $q > 0$

i)

$$\begin{array}{r|l} & \frac{x}{2} + \frac{q}{2} - \frac{3}{4} \\ \hline 2x+3 & x^2 + qx + 1 \\ & x^2 + \frac{3x}{2} \\ \hline & (q - \frac{3}{2})x + 1 \\ & (q - \frac{3}{2})x + \frac{3q}{2} - \frac{9}{4} \\ \hline & -\frac{3q}{2} + \frac{13}{4} \end{array}$$

$$y = \frac{x}{2} + \frac{q}{2} - \frac{3}{4} + \frac{-\frac{3q}{2} + \frac{13}{4}}{2x+3}$$

As $x \to \pm\infty$ $y \to \frac{x}{2} + \frac{q}{2} - \frac{3}{4}$

As $x \to -\frac{3}{2}$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are

$y = \frac{x}{2} + \frac{q}{2} - \frac{3}{4}$ and $x = -\frac{3}{2}$

ii) when $y=0$: $\dfrac{x^2+qx+1}{2x+3}=0$

$$x^2+qx+1=0$$

$$x^2+qx+\frac{q^2}{4}=\frac{q^2}{4}-1$$

$$\left(x+\frac{q}{2}\right)^2=\frac{q^2-4}{4}$$

$$x+\frac{q}{2}=\frac{\pm\sqrt{q^2-4}}{2}$$

$$x=-\frac{q}{2}\pm\frac{\sqrt{q^2-4}}{2}$$

If the x-axis is a tangent to C, the curve intersects the x-axis at one point.

$$q^2-4=0$$

$$q^2=4$$

$$q=2$$

$\therefore$ If the x-axis is a tangent to C, $q=2$.

$q=2$: $y=\dfrac{x^2+2x+1}{2x+3}$

$$=\frac{x}{2}+\frac{1}{4}+\frac{1}{4(2x+3)}$$

As $x\to\pm\infty$ $y\to\dfrac{x}{2}+\dfrac{1}{4}$

As $x\to-\dfrac{3}{2}$ $y\to\pm\infty$

$\therefore$ The asymptotes of y are $y=\dfrac{x}{2}+\dfrac{1}{4}$ and $x=-\dfrac{3}{2}$

when $x=0$: $y=\dfrac{1}{3}$

when $y=0$:

$$\frac{x^2+2x+1}{2x+3}=0$$

$$x^2+2x+1=0$$

$$(x+1)^2=0$$

$$x=-1$$

$\therefore$ The intersection points of y are $(0,\frac{1}{3})$ and $(-1,0)$.

$$\frac{dy}{dx}=\frac{1}{2}-\frac{1}{2(2x+3)^2}$$

when $\dfrac{dy}{dx}=0$: $\dfrac{1}{2}-\dfrac{1}{2(2x+3)^2}=0$

$$\frac{1}{(2x+3)^2}=1$$

$$(2x+3)^2=1$$

$$2x+3=\pm1$$

$$x=-2,-1$$

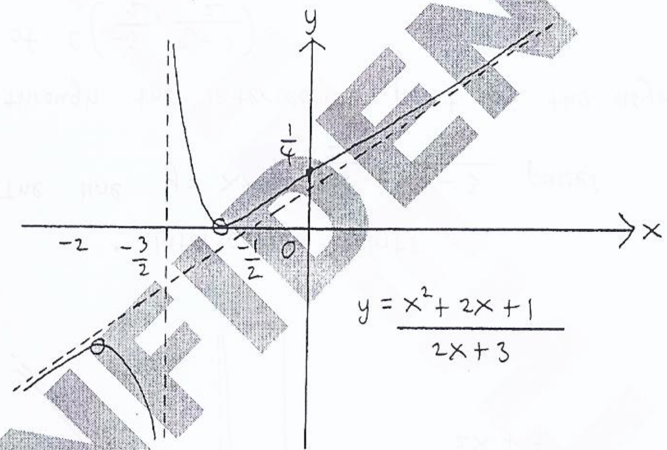
$$y=-1,0$$

$$\frac{d^2y}{dx^2} = \frac{2}{(2x+3)^3}$$

when $x = -2$: $\dfrac{d^2y}{dx^2} = -2 < 0$

when $x = -1$: $\dfrac{d^2y}{dx^2} = 2 > 0$

$\therefore$ $(-2,-1)$ is a maximum point and $(-1,0)$ is a minimum point.



o : Critical point

• : Intersection point.

iii) $q = 3$: $y = \dfrac{x^2+3x+1}{2x+3}$

$$= \frac{x}{2} + \frac{3}{4} - \frac{5}{4(2x+3)}$$

As $x \to \pm\infty$ $y \to \frac{x}{2} + \frac{3}{4}$

As $x \to \pm\infty$ $y \to \pm\infty$

$\therefore$ The asymptotes of y are $y = \frac{x}{2} + \frac{3}{4}$ and $x = -\frac{3}{2}$.

When $x = 0$: $y = \frac{1}{3}$

When $y = 0$: $\dfrac{x^2+3x+1}{2x+3} = 0$

$$x^2+3x+1 = 0$$

$$x = \frac{-3 \pm \sqrt{5}}{2}$$

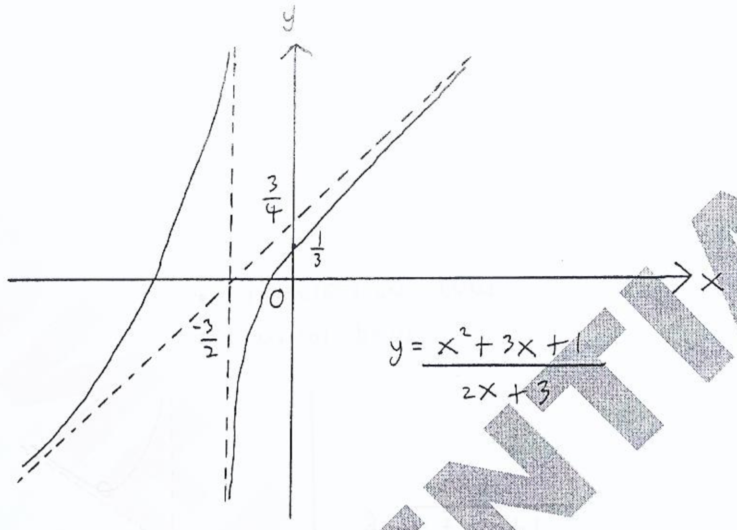
$\therefore$ The intersection points of y are $(0, \frac{1}{3})$, $\left(\frac{-3+\sqrt{5}}{2}, 0\right)$ and $\left(\frac{-3-\sqrt{5}}{2}, 0\right)$.

$$\frac{dy}{dx} = \frac{1}{2} + \frac{5}{2(2x+3)^2}$$

Since $\dfrac{1}{2} + \dfrac{5}{2(2x+3)^2} \geqslant \dfrac{1}{2}$,

$$\frac{dy}{dx} \neq 0$$

$\therefore$ y has no critical points.



• : Intersection Points

iv) The line $y = \lambda x + \frac{3\lambda}{2} + \frac{q-3}{2}$ passes through the intersection point of the asymptotes of $C\left(\frac{-3}{2}, \frac{q-3}{2}\right)$

If $q = 2$ and $\frac{x^2+qx+1}{2x+3} = \lambda x + \frac{3\lambda}{2} + \frac{q-3}{2}$,

$$x^2 + 2x + 1 = (2x+3)\left(\lambda x + \frac{3\lambda}{2} - \frac{1}{2}\right)$$

$$= 2\lambda x^2 + (3\lambda - 1)x + 3\lambda x + \frac{9\lambda}{2} - \frac{3}{2}$$

$$(2\lambda - 1)x^2 + 3(2\lambda - 1)x + \frac{9\lambda}{2} - \frac{5}{2} = 0$$

$a = 2\lambda - 1 \quad b = 3(2\lambda - 1) \quad c = \frac{9\lambda}{2} - \frac{5}{2}$

$$b^2 - 4ac = 9(2\lambda-1)^2 - 4(2\lambda-1)\left(\frac{9\lambda}{2} - \frac{5}{2}\right)$$

$$= 9(2\lambda-1)^2 - 2(2\lambda-1)(9\lambda-5)$$

$$= (2\lambda-1)(9(2\lambda-1) - 2(9\lambda-5))$$

$$= (2\lambda-1)(18\lambda - 9 - 18\lambda + 10)$$

$$= 2\lambda - 1$$

If $\lambda < \frac{1}{2}$

$2\lambda - 1 < 0$

$b^2 - 4ac < 0$

If $q = 3$ and $\frac{x^2+qx+1}{2x+3} = \lambda x + \frac{3\lambda}{2} + \frac{q-3}{2}$,

$$x^2 + 3x + 1 = (2x+3)\left(\lambda x + \frac{3\lambda}{2}\right)$$

$$= 2\lambda x^2 + 3\lambda x + 3\lambda x + \frac{9\lambda}{2}$$

$$= 2\lambda x^2 + 6\lambda x + \frac{9\lambda}{2}$$

$$(2\lambda-1)x^2 + 3(2\lambda-1)x + \frac{9\lambda}{2} - 1 = 0$$

$a = 2\lambda - 1 \quad b = 3(2\lambda - 1) \quad c = \frac{9\lambda}{2} - 1$

$$b^2-4ac = 9(2\lambda-1)^2 - 4(2\lambda-1)\left(\frac{9\lambda}{2}-1\right)$$
$$= 9(2\lambda-1)^2 - 2(2\lambda-1)(9\lambda-2)$$
$$= (2\lambda-1)(9(2\lambda-1)-2(9\lambda-2))$$
$$= (2\lambda-1)(18\lambda-9-18\lambda+4)$$
$$= -5(2\lambda-1)$$

If $\lambda < \frac{1}{2}$

$2\lambda - 1 < 0$

$-5(2\lambda-1) > 0$

$b^2 - 4ac > 0$

$\therefore$ If $q=2$ and $\lambda < \frac{1}{2}$ the equation $\frac{x^2+qx+1}{2x+3} = \lambda x + \frac{3\lambda}{2} + \frac{q-3}{2}$ has no real solutions since the line does not intersect C, but has two real distinct solutions if $q=3$ and $\lambda < \frac{1}{2}$ since the line intersects C at two points.

OR

$C: \; y = x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3} + \lambda, \; \lambda > 0, \; 0 \le x \le 3$

The arc length of C from $x=0$ to $x=3$, is

$$\int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \int_0^3 \sqrt{1+\left(\frac{x^{-\frac{1}{2}}}{2} - \frac{x^{\frac{1}{2}}}{2}\right)^2}\,dx$$
$$= \int_0^3 \sqrt{1+\frac{x^{-1}}{4} - \frac{1}{2} + \frac{x}{4}}\,dx$$
$$= \int_0^3 \sqrt{\frac{x^{-1}}{4} + \frac{1}{2} + \frac{x}{4}}\,dx$$
$$= \int_0^3 \sqrt{\left(\frac{x^{-\frac{1}{2}}}{2} + \frac{x^{\frac{1}{2}}}{2}\right)^2}\,dx$$
$$= \int_0^3 \frac{x^{-\frac{1}{2}}}{2} + \frac{x^{\frac{1}{2}}}{2}\,dx$$
$$= \left[x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{3}\right]_0^3$$
$$= 3^{\frac{1}{2}} + \frac{3^{\frac{3}{2}}}{3} - 0$$
$$= 3^{\frac{1}{2}} + 3^{\frac{1}{2}}$$
$$= 2\sqrt{3}$$

The surface area of revolution when C is rotated through one revolution from x = 0 to x = 3 about the x-axis, S, is

$$\int_0^3 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^3 2\pi\left(\frac{x^{\frac{1}{2}}}{2} - \frac{x^{\frac{3}{2}}}{3} + \lambda\right)\left(\frac{x^{-\frac{1}{2}}}{2} + \frac{x^{\frac{1}{2}}}{2}\right)dx$$

$$= 2\pi\int_0^3 \frac{1}{2} - \frac{x}{6} + \frac{\lambda x^{-\frac{1}{2}}}{2} + \frac{x}{2} - \frac{x^6}{6} + \frac{\lambda x^{\frac{1}{2}}}{2}\,dx$$

$$= 2\pi\int_0^3 \frac{1}{2} + \frac{x}{3} - \frac{x^2}{6} + \frac{\lambda}{2}\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx$$

$$= 2\pi\left[\frac{x}{2} + \frac{x^2}{6} - \frac{x^3}{18} + \frac{\lambda}{2}\left(2x^{\frac{1}{2}} + \frac{2x^{\frac{3}{2}}}{3}\right)\right]_0^3$$

$$= 2\pi\left(\frac{3}{2} + \frac{9}{6} - \frac{27}{18} + \frac{\lambda}{2}\left(2\sqrt{3} + \frac{2(3^{\frac{3}{2}})}{3} - 0\right)\right)$$

$$= 2\pi\left(\frac{3}{2} + \frac{\lambda}{2}\left(2\sqrt{3} + \frac{2}{3}(3\sqrt{3})\right)\right)$$

$$= 2\pi\left(\frac{3}{2} + \sqrt{3}\lambda + \sqrt{3}\lambda\right)$$

$$= 2\pi\left(\frac{3}{2} + 2\sqrt{3}\lambda\right)$$

$$= 3\pi + 4\sqrt{3}\lambda\pi$$

The area of the region bounded by C, the x-axis, and the lines x = 0 and x = 3, A, is

$$\int_0^3 y\,dx$$

$$= \int_0^3 x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3} + \lambda\,dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{2x^{\frac{5}{2}}}{15} + \lambda x\right]_0^3$$

$$= \frac{2(3^{\frac{3}{2}})}{3} - \frac{2(3^{\frac{5}{2}})}{15} + 3\lambda - 0$$

$$= 2\sqrt{3} - \frac{2(3^{\frac{3}{2}})}{5} + 3\lambda$$

$$= 2\sqrt{3} - \frac{6\sqrt{3}}{5} + 3\lambda$$

$$= \frac{4\sqrt{3}}{5} + 3\lambda$$

The y-coordinate of the centroid of the region bounded by C, the axes and the line x = 3, h, is

$$\frac{\int_0^3 \frac{y^2}{2}\,dx}{A}$$

Since $\int_0^3 y^2\,dx = \frac{3}{4} + \frac{8\sqrt{3}\lambda}{5} + 3\lambda^2$

CONFIDENTIAL

$$\therefore h = \frac{\frac{1}{2}\left(\frac{3}{4} + \frac{8\sqrt{3}}{5}\lambda + 3\lambda^2\right)}{\frac{4\sqrt{3}}{5} + 3\lambda}$$

$$= \frac{\frac{3}{8} + \frac{4\sqrt{3}\lambda}{5} + \frac{3\lambda^2}{2}}{\frac{4\sqrt{3}}{5} + 3\lambda}$$

$$\frac{S}{hs} = \frac{3\pi + 4\sqrt{3}\lambda\pi}{\frac{\left(\frac{3}{8} + \frac{4\sqrt{3}}{5}\lambda + \frac{3\lambda^2}{2}\right)2\sqrt{3}}{\frac{4\sqrt{3}}{5} + 3\lambda}}$$

$$= \frac{(3\pi + 4\sqrt{3}\lambda\pi)\left(\frac{4\sqrt{3}}{5} + 3\lambda\right)}{\frac{3\sqrt{3}}{4} + \frac{24\lambda}{5} + 3\sqrt{3}\lambda^2}$$

$$= \frac{\frac{(3\pi + 4\sqrt{3}\lambda\pi)}{\lambda}\frac{\left(\frac{4\sqrt{3}}{5} + 3\lambda\right)}{\lambda}}{\frac{\frac{3\sqrt{3}}{4} + \frac{24\lambda}{5} + 3\sqrt{3}\lambda^2}{\lambda^2}}$$

$$= \frac{\left(\frac{3\pi}{\lambda} + 4\sqrt{3}\pi\right)\left(\frac{4\sqrt{3}}{5\lambda} + 3\right)}{\frac{3\sqrt{3}}{4\lambda^2} + \frac{24}{5\lambda} + 3\sqrt{3}}$$

$$\lim_{\lambda \to \infty} \frac{S}{hs} = \lim_{\lambda \to \infty} \frac{\left(\frac{3\pi}{\lambda} + 4\sqrt{3}\pi\right)\left(\frac{4\sqrt{3}}{5\lambda} + 3\right)}{\frac{3\sqrt{3}}{4\lambda^2} + \frac{24}{5\lambda} + 3\sqrt{3}}$$

$$= \frac{4\sqrt{3}\pi(3)}{3\sqrt{3}}$$

$$= 4\pi$$