1. Let $f(n) = 7^{2n-1} + 3^{2n}$

$n=1$: $f(1) = 7 + 3^2 = 16 = 8(2)$

$\therefore f(1)$ is divisible by 8.

$n=k$: Assume $f(k)$ is divisible by 8.

$n=k+1$: $f(k+1) - f(k) = 7^{2k+2-1} + 3^{2k+2} - (7^{2k-1} + 3^{2k})$

$= (49-1)7^{2k-1} + (9-1)3^{2k}$

$= 48(7^{2k-1}) + 8(3^{2k})$

$= 8(7^{2k-1} + 3^{2k}) + 40(7^{2k-1})$

$f(k+1) = 9f(k) + 40(7^{2k-1})$

Since 40 is divisible by 8 and 7^{2k-1} is an integer, if $f(k)$ is divisible by 8 then $f(k+1)$ is divisible by 8.

$\therefore$ By induction $7^{2n-1} + 3^{2n}$ is divisible by 8 for integers $n \geq 1$.

2. $$\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$$

$$S_N = \sum_{n=1}^{N} \frac{2}{(n+1)(n+3)}$$

$$= \sum_{n=1}^{N} \left[\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} \right]$$

$$= \frac{1}{1+1} - \frac{1}{N+2} + \frac{1}{1+2} - \frac{1}{N+3}$$

$$= \frac{5}{6} - \frac{1}{N+2} - \frac{1}{N+3}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \lim_{N \to \infty} \frac{1}{2} S_N$$

$$= \lim_{N \to \infty} \frac{5}{12} - \frac{1}{2} \left[\frac{1}{N+2} - \frac{1}{N+3} \right]$$

$$= \frac{5}{12}$$

3. $$\sum_{n=1}^{N} n(n+1)\left(\frac{1}{2}\right)^{n-1} = 16 - (N^2 + 5N + 8)\left(\frac{1}{2}\right)^{N-1}$$

$N=1$: LHS $= 1 \times 2 \times \left(\frac{1}{2}\right)^0 = 2$

RHS $= 16 - (1+5+8)\left(\frac{1}{2}\right)^0 = 16 - 14 = 2$

$\therefore$ Statement is true for $N=1$.

$N=k$: Assume the statement is true for $N=k$.

$$\sum_{n=1}^{k} n(n+1)\left(\frac{1}{2}\right)^{n-1} = 16 - (k^2+5k+8)\left(\frac{1}{2}\right)^{k-1}$$

$N=k+1$: Need to show that

$$\sum_{n=1}^{k+1} n(n+1)\left(\frac{1}{2}\right)^{n-1}$$

$$= 16 - [(k+1)^2 + 5(k+1) + 8]\left(\frac{1}{2}\right)^{k+1-1}$$

$$= 16 - (k^2 + 7k + 14)\left(\frac{1}{2}\right)^k$$

$$\therefore \sum_{n=1}^{k+1} n(n+1)\left(\frac{1}{2}\right)^{n-1}$$

$$= 16 - (k^2 + 5k + 8)\left(\frac{1}{2}\right)^{k-1} + (k+1)(k+2)\left(\frac{1}{2}\right)^k$$

$$= 16 - 2(k^2 + 5k + 8)\left(\frac{1}{2}\right)^k + (k^2 + 3k + 2)\left(\frac{1}{2}\right)^k$$

$$= 16 - (k^2 + 7k + 14)\left(\frac{1}{2}\right)^k$$

If the statement is true for $N=k$ then it is true for $N=k+1$.

4. $$\left.\begin{aligned} x+3y+2z&=1\\ x-y-z&=0\\ 2x+2y+\theta z&=3\theta+\phi-2 \end{aligned}\right\}$$

$-①+②$:
$-2\times①+③$:
$$\left.\begin{aligned} x+3y+2z&=1\\ -4y-3z&=-1\\ -4y+(\theta-4)z&=3\theta+\phi-4 \end{aligned}\right\}$$

$-②+③$:
$$\left.\begin{aligned} x+3y+2z&=1\\ -4y-3z&=-1\\ (\theta-1)z&=3\theta+\phi-3 \end{aligned}\right\}$$

i) $(\theta-1)z = 3\theta+\phi-3$

If $\theta\neq 1$: $z = \dfrac{3\theta+\phi-3}{\theta-1}$

$\phi=0$: $z = \dfrac{3\theta-3}{\theta-1}$

$= 3$

$$\begin{aligned} -4y-3z&=-1\\ -4y-3(3)&=-1\\ -4y-9&=-1\\ -4y&=8\\ y&=-2 \end{aligned}$$

$$x + 3y + 2z = 1$$
$$x + 3(-2) + 2(3) = 1$$
$$x - 6 + 6 = 1$$
$$x = 1$$

If $\theta \neq 1$ and $\phi \neq 0$ $\therefore$ $x = 1, y = -2, z = 3$

ii) $(\theta - 1)z = 3\theta + \phi - 3$

If $\theta = 1$ and $\phi = 0$

$$0z = 0$$

Let $z = s, s \in R$

$$-4y - 3s = -1$$
$$y = \frac{1 - 3s}{4}$$

$$x + 3y + 2z = 1$$
$$x + 3\left(\frac{1 - 3s}{4}\right) + 2s = 1$$
$$x + \frac{3}{4} - \frac{9s}{4} + 2s = 1$$
$$x = \frac{s + 1}{4}$$

iii) $(\theta - 1)z = 3\theta + \phi - 3$

If $\theta = 1$ and $\phi \neq 0$ $\therefore$ $0z = \phi$

since $\phi \neq 0$ $\therefore$ S has no solution.

5. $4x^3 + 8x + 5 = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \alpha\gamma + \beta\gamma = 2 \quad \alpha\beta\gamma = \frac{-5}{4}$

$\alpha\beta + \gamma, \ \alpha\gamma + \beta, \ \beta\gamma + \alpha.$

Let $u = \alpha\beta + \gamma$

$= \frac{\alpha\beta\gamma + \gamma^2}{\gamma}$

$u\gamma = \alpha\beta\gamma + \gamma^2$

$= \gamma^2 - \frac{5}{4}$

$\gamma^2 - u\gamma = \frac{5}{4}$

$\gamma^2 - u\gamma + \frac{u^2}{4} = \frac{u^2 + 5}{4}$

$\left(\gamma - \frac{u}{2}\right)^2 = \frac{u^2 + 5}{4}$

$\gamma - \frac{u}{2} = \pm \sqrt{\frac{u^2 + 5}{4}}$

$\gamma = \frac{u}{2} \pm \frac{\sqrt{u^2 + 5}}{2}$

r is a root

$\therefore 4r^3 + 8r + 5 = 0$

$$4\left(\frac{u}{2} \pm \frac{\sqrt{u^2+5}}{2}\right)^3 + 8\left(\frac{u}{2} \pm \frac{\sqrt{u^2+5}}{2}\right) + 5 = 0$$

$$\frac{4\left(u^3 \pm 3u^2\sqrt{u^2+5} + 3u(u^2+5) \pm (u^2+5)\sqrt{u^2+5}\right)}{8}$$

$$+ 4(u \pm \sqrt{u^2+5}) + 5 = 0$$

$$u^3 \pm 3u^2\sqrt{u^2+5} + 3u^3 + 15u \pm (u^2+5)\sqrt{u^2+5}$$

$$+ 8u \pm 8\sqrt{u^2+5} + 10 = 0$$

$$\pm\sqrt{u^2+5}\,(3u^2 + u^2 + 5 + 8) = -4u^3 - 23u - 10$$

$$\pm\sqrt{u^2+5}\,(4u^2+13) = -4u^3 - 23u - 10$$

$$\left(\pm\sqrt{u^2+5}\,(4u^2+13)\right)^2 = (-4u^3 - 23u - 10)^2$$

$$(u^2+5)(4u^2+13)^2 = 16u^6 + 92u^4 + 40u^3$$

$$+ 92u^4 + 529u^2 + 230u$$

$$+ 40u^3 + 230u + 100$$

$(u^2+5)(16u^4+104u^2+169)$

$= 16u^6 + 184u^4 + 80u + 529u^2 + 460u + 100$

$16u^6 + 104u^4 + 169u^2 + 80u^4 + 520u^2 + 845$

$= 16u^6 + 184u^4 + 80u^3 + 529u^2 + 460u + 100$

$80u^3 - 160u^2 + 460u - 745 = 0$

$16u^3 - 32u^2 + 92u - 149 = 0$

$\therefore$ The equation $16u^3 - 32u^2 + 92u - 149 = 0$

has roots $\alpha\beta + \gamma$, $\alpha\gamma + \beta$, $\beta\gamma + \alpha$.

6. $C: y = 5 - \dfrac{2}{9x+4} - \dfrac{3}{7x+6}$

i) As $x \to \pm\infty$ $\quad y \to 5$

As $x \to -\dfrac{4}{9}$ $\quad y \to \pm\infty$

As $x \to -\dfrac{6}{7}$ $\quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = 5$, $x = -\dfrac{4}{9}$, and $x = -\dfrac{6}{7}$.

ii) $$\frac{dy}{dx} = \underbrace{\underbrace{\frac{18}{(9x+4)^2}}_{>0} + \underbrace{\frac{21}{(7x+6)^2}}_{>0}}_{>0}$$

$\dfrac{dy}{dx} > 0$

$\therefore$ no critical points

iii) when $x = 0$: $y = 4$

when $y = 0$: $5 - \dfrac{2}{9x+4} - \dfrac{3}{7x+6} = 0$

$$\frac{2}{9x+4} + \frac{3}{7x+6} = 5$$

$$2(7x+6) + 3(9x+4) = 5(9x+4)(7x+6)$$

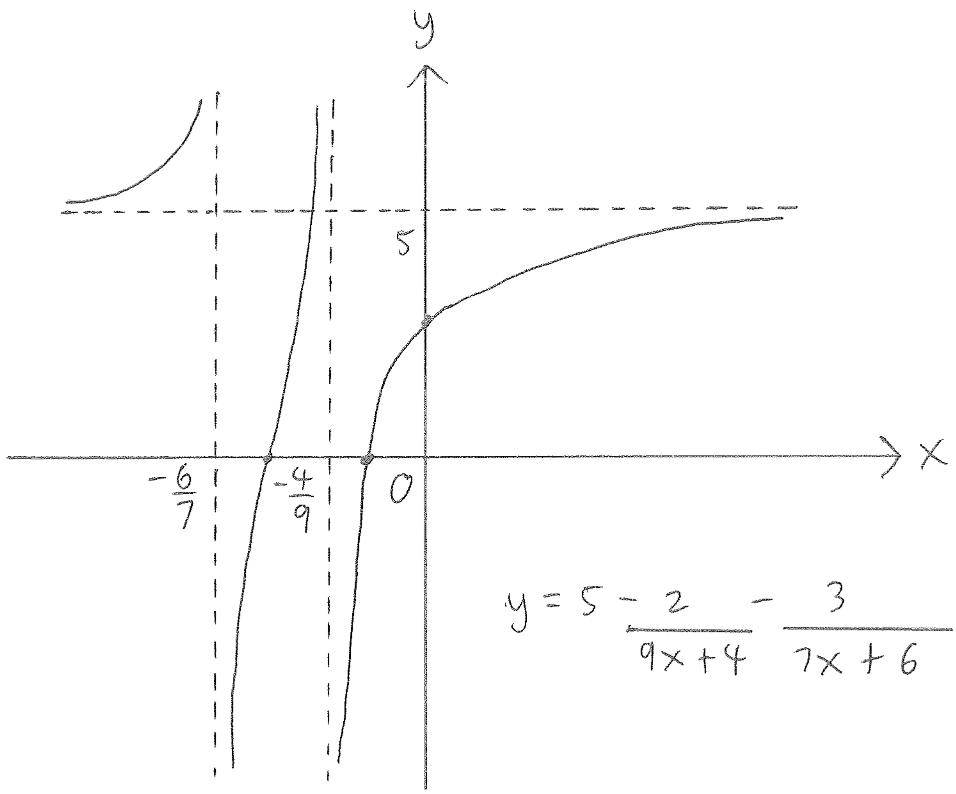
$$14x + 12 + 27x + 12 = 5(63x^2 + 82x + 24)$$

$$41x + 24 = 315x^2 + 410x + 120$$

$$315x^2 + 369x + 96 = 0$$

$$105x^2 + 123x + 32 = 0$$

$$x = \frac{-123 \pm \sqrt{1689}}{210}$$



$$y = 5 - \frac{2}{9x+4} - \frac{3}{7x+6}$$