1. $\dfrac{x+1}{3} = \lambda \Rightarrow x = -1 + 3\lambda$

$\dfrac{y-2}{4} = \lambda \Rightarrow y = 2 + 4\lambda$

$\dfrac{z-2}{12} = \lambda \Rightarrow z = 3 - 12\lambda$

$$\left.\begin{aligned} x &= -1+3\lambda \\ y &= 2+4\lambda \\ z &= 3-12\lambda \end{aligned}\right\} \underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix}$$

Let θ = the acute angle between the line and the plane

$$\cos\theta = \left| \frac{\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix}}{\sqrt{4+1+4}\sqrt{9+16+144}} \right| = \frac{6+4+24}{3 \times 13} = \frac{34}{39}$$

$$\Rightarrow \theta = 61^\circ.$$

2. $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -9 \end{pmatrix}$

$$\text{Area} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \left| \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} \times \begin{pmatrix} 1 \\ -5 \\ -9 \end{pmatrix} \right|$$

$$= \frac{1}{2} \left| \begin{pmatrix} -57 \\ 12 \\ -13 \end{pmatrix} \right| = \frac{1}{2}\left(\sqrt{3249 + 144 + 169}\right)$$

$$= \frac{\sqrt{3562}}{2}$$

$$\approx 29.8$$

3. i) $1^3 - 2^3 + 3^3 - 4^3 + \cdots - (n-2)^3 + (n-1)^3 - n^3$, n even

$a = 2$ $\quad\quad$ $a = 1$

$d = 2$ $\quad\quad$ $d = 2$

$$T_n = a + (n-1)d \qquad T_n = a + (n-1)d$$
$$= 2 + 2(n-1) \qquad = 1 + 2(n-1)$$
$$= 2 + 2n - 2 \qquad = 1 + 2n - 2$$
$$= 2n \qquad = 2n - 1$$

$$\sum_{r=1}^{\frac{n}{2}} \left[(2r-1)^3 - (2r)^3 \right]$$

$$= \sum_{r=1}^{\frac{n}{2}} \left(8r^3 - 12r^2 + 6r - 1 - 8r^3 \right)$$

$$= \sum_{r=1}^{\frac{n}{2}} \left(-12r^2 + 6r - 1 \right)$$

$$= -12 \sum_{r=1}^{\frac{n}{2}} r^2 + 6 \sum_{r=1}^{\frac{n}{2}} r - \sum_{r=1}^{\frac{n}{2}} 1$$

$$= \frac{-12}{6}\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)\left(2\left(\frac{n}{2}\right)+1\right) + \frac{6}{2}\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) - \frac{n}{2}$$

$$= \frac{-n(n+1)(n+2)}{2} + \frac{3n(n+2)}{4} - \frac{n}{2}$$

$$= \frac{3n^2 + 6n - 2n^3 - 6n^2 - 4n - 2n}{4}$$

$$= \frac{-2n^3 - 3n^2}{4} = \frac{-n^2(2n+3)}{4}$$

ii) $1^3 - 2^3 + 3^3 - 4^3 + \dots + (n-2)^3 - (n-1)^3 + n^3$, n odd

$$= n^3 + \sum_{r=1}^{\frac{n-1}{2}} \left[(2r-1)^3 - (2r)^3 \right]$$

$$= n^3 + \sum_{r=1}^{\frac{n-1}{2}} (-12r^2 + 6r - 1)$$

$$= n^3 - 12\sum_{r=1}^{\frac{n-1}{2}} r^2 + 6\sum_{r=1}^{\frac{n-1}{2}} r - \sum_{r=1}^{\frac{n-1}{2}} 1$$

$$= n^3 - \frac{12}{6}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\left(2\left(\frac{n-1}{2}\right)+1\right)$$
$$+ \frac{6}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right) - \left(\frac{n-1}{2}\right)$$

$$= n^3 - \frac{(n-1)n(n+1)}{2} + \frac{3(n-1)(n+1)}{4} - \frac{(n-1)}{2}$$

$$= \frac{4n^3 - 2n(n^2-1) + 3(n^2-1) - 2n + 2}{4}$$

$$= \frac{4n^3 - 2n^3 + 2n + 3n^2 - 3 - 2n + 2}{4}$$

$$= \frac{2n^3 + 3n^2 - 1}{4}$$

4. i) $\frac{a^{n-1}}{b^{n-1}} + \frac{a^{n-2}}{b^{n-2}} + \cdots + \frac{a^2}{b^2} + \frac{a}{b} + 1 + \frac{b}{a} + \frac{b^2}{a^2} + \cdots + \frac{b^{n-2}}{a^{n-2}} + \frac{b^{n-1}}{a^{n-1}}$

$$= \frac{a^{2n-1} - b^{2n-1}}{a^{n-1}b^{n-1}(a-b)}, \quad n \geqslant 1.$$

when $n=1$:

$$1 = \frac{a-b}{a-b} = \frac{a^{2-1} - b^{2-1}}{a^0 b^0 (a-b)} = \frac{a^{2(1)-1} - b^{2(1)-1}}{a^{1-1}b^{1-1}(a-b)}$$

Assume the statement is true when $n=k$.

$n=k$: $\frac{a^{k-1}}{b^{k-1}} + \frac{a^{k-2}}{b^{k-2}} + \cdots + \frac{a^2}{b^2} + \frac{a}{b} + 1 + \frac{b}{a} + \frac{b^2}{a^2} + \cdots$

$$+ \frac{b^{k-2}}{a^{k-2}} + \frac{b^{k-1}}{a^{k-1}} = \frac{a^{2k-1} - b^{2k-1}}{a^{k-1}b^{k-1}(a-b)}$$

when $n=k+1$: $\frac{a^k}{b^k} + \frac{a^{k-1}}{b^{k-1}} + \cdots + \frac{a}{b} + 1 + \frac{b}{a} + \cdots$

$$+ \frac{b^{k-1}}{a^{k-1}} + \frac{b^k}{a^k} = \frac{a^{2k+1} - b^{2k+1}}{a^k b^k (a-b)}$$

(what needs to be proved).

$$\frac{a^{k-1}}{b^{k-1}} + \cdots + \frac{a}{b} + 1 + \frac{b}{a} + \cdots + \frac{b^{k-1}}{a^{k-1}} = \frac{a^{2k-1} - b^{2k-1}}{a^{k-1}b^{k-1}(a-b)}$$

$$\frac{a^k}{b^k}+\frac{a^{k-1}}{b^{k-1}}+\cdots+\frac{a}{b}+1+\frac{b}{a}+\cdots+\frac{b^{k-1}}{a^{k-1}}+\frac{b^k}{a^k}=\frac{a^{2k-1}-b^{2k-1}}{a^{k-1}b^{k-1}(a-b)}+\frac{a^k}{b^k}+\frac{b^k}{a^k}$$

$$=\frac{(a^{2k-1}-b^{2k-1})ab}{a^k b^k(a-b)}+\frac{(a^k+b^k)(a-b)}{a^k b^k(a-b)}$$

$$=\frac{a^{2k}b-ab^{2k}+a^{2k+1}-a^{2k}b+ab^{2k}-b^{2k+1}}{a^k b^k(a-b)}$$

$$=\frac{a^{2k+1}-b^{2k+1}}{a^k b^k(a-b)}$$

$$\therefore\ \frac{a^{n-1}}{b^{n-1}}+\frac{a^{n-2}}{b^{n-2}}+\cdots+\frac{a^2}{b^2}+\frac{a}{b}+1+\frac{b}{a}+\frac{b^2}{a^2}+\cdots$$

$$+\frac{b^{n-2}}{a^{n-2}}+\frac{b^{n-1}}{a^{n-1}}=\frac{a^{2n-1}-b^{2n-1}}{a^{n-1}b^{n-1}(a-b)}$$

for every positive integer n.

ii) $u_{n+3} = 3u_{n+2} + 4u_{n+1} - 12u_n$, $u_1 = 7$ $u_2 = 17$ $u_3 = 43$

$u_n = 3^n + 2^{n+1}$

when $n = 1$: $u_1 = 3^1 + 2^{1+1} = 3 + 2^2 = 3 + 4 = 7$

when $n = 2$: $u_2 = 3^2 + 2^{2+1} = 9 + 2^3 = 9 + 8 = 17$

when $n = 3$: $u_3 = 3^3 + 2^{3+1} = 27 + 2^4 = 27 + 16 = 43$

when $n = 4$: $u_4 = 3^4 + 2^{4+1} = 81 + 2^5 = 81 + 32 = 113$

Assume the statement is true when $n = k$, $n = k+1$ and $n = k+2$.

$n = k$: $u_k = 3^k + 2^{k+1}$

$n = k+1$: $u_{k+1} = 3^{k+1} + 2^{k+2}$

$n = k+2$: $u_{k+2} = 3^{k+2} + 2^{k+3}$

when $n = k+3$: $u_{k+3} = 3^{k+3} + 2^{k+4}$

(what needs to be proved)

$$u_{k+3} = 3u_{k+2} + 4u_{k+1} - 12u_k$$
$$= 3(3^{k+2} + 2^{k+3}) + 4(3^{k+1} + 2^{k+2}) - 12(3^k + 2^{k+1})$$
$$= 3(3^k 9 + 2^k 8) + 4(3^k 3 + 2^k 4) - 12(3^k + 2^k 2)$$
$$= 3^k 27 + 2^k 24 + 3^k 12 + 2^k 16 - 3^k 12 - 2^k 24$$
$$= 3^k 27 + 2^k 16$$
$$= 3^{k+3} + 2^{k+4}$$

$\therefore u_n = 3^n + 2^{n+1}$ for every positive integer n.

5. $y = \dfrac{3x}{x^2 - 5x + 4} = \dfrac{4}{x-4} - \dfrac{1}{x-1}$

a. Asymptotes : $y = 0$; $x = 4$; $x = 1$

b. $\dfrac{dy}{dx} = \dfrac{-4}{(x-4)^2} + \dfrac{1}{(x-1)^2}$; $\dfrac{d^2y}{dx^2} = \dfrac{8}{(x-4)^3} - \dfrac{2}{(x-1)^3}$

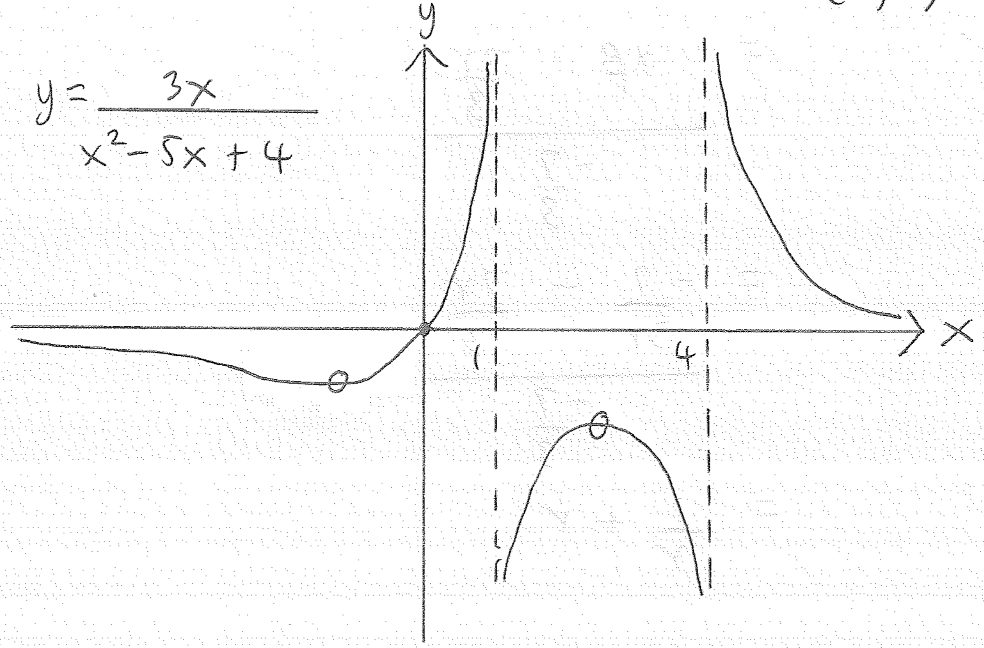
At the stationary points, $\dfrac{dy}{dx} = 0$

$\Rightarrow (x-4)^2 = 4(x-1)^2 \Rightarrow x = \pm 2$

$x = 2 \Rightarrow y = -3 \Rightarrow \dfrac{d^2y}{dx^2} = -1 - 2 < 0 \Rightarrow$ Maximum turning point is $(2, -3)$

$x = -2 \Rightarrow y = \dfrac{-1}{3} \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-1}{27} + \dfrac{2}{27} > 0 \Rightarrow$ Minimum turning point is $(-2, \frac{-1}{3})$.

c. The curve crosses the axes at $(0, 0)$



6. i) $3x^3 + 5x^2 + 7 = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = -\frac{5}{3}$ $\quad \alpha\beta + \alpha\gamma + \beta\gamma = 0$ $\quad \alpha\beta\gamma = -\frac{7}{3}$

$\frac{1}{\alpha} + \frac{1}{\beta} + \gamma, \ \frac{1}{\alpha} + \frac{1}{\gamma} + \beta, \ \frac{1}{\beta} + \frac{1}{\gamma} + \alpha$

Let $u = \frac{1}{\alpha} + \frac{1}{\beta} + \gamma$

$$= \frac{\alpha + \beta}{\alpha\beta} + \gamma$$

$$= \frac{\alpha + \beta + \alpha\beta\gamma}{\alpha\beta}$$

$$= \frac{\alpha + \beta - \frac{7}{3}}{\alpha\beta}$$

$$= \frac{\alpha\gamma + \beta\gamma - \frac{7}{3}}{\alpha\beta\gamma}$$

$$= \frac{-\alpha\beta - \frac{7\gamma}{3}}{-\frac{7}{3}}$$

$$= \frac{-\frac{7}{3\gamma} - \frac{7\gamma}{3}}{-\frac{7}{3}}$$

$$= \frac{1}{\gamma} + \gamma$$

$$r^2+1=ru$$
$$r^2-ru+1=0$$
$$r=\frac{u\pm\sqrt{u^2+4}}{2}$$

r is a root

$$\therefore\ 3r^3+5r^2+7=0$$

$$3\left(\frac{u\pm\sqrt{u^2+4}}{2}\right)^3+5\left(\frac{u\pm\sqrt{u^2+4}}{2}\right)^2+7=0$$

$$\frac{3}{8}\left(u^3\pm 3u^2\sqrt{u^2+4}+3u(u^2+4)\pm(u^2+4)\sqrt{u^2+4}\right)$$
$$+\frac{5}{4}\left(u^2\pm 2u\sqrt{u^2+4}+u^2+4\right)+7=0$$

$$3u^3\pm 9u^2\sqrt{u^2+4}+9u(u^2+4)\pm 3(u^2+4)\sqrt{u^2+4}$$
$$+10u^2\pm 20u\sqrt{u^2+4}+10u^2+40+56=0$$

$$\pm\sqrt{u^2+4}\,(9u^2+3u^2+12+20u)=-3u^3-9u^3-36u-20u^2-96$$

$$(u^2+4)(12u^2+20u+12)^2=(-12u^3-20u^2-36u-96)^2$$

$$(u^2+4)(144u^4+240u^3+144u^2+240u^3+400u^2$$
$$+240u+144u^2+240u+144)$$

$$=144u^6+240u^5+432u^4+1152u^3$$
$$+240u^5+400u^4+720u^3+1920u^2$$
$$+432u^4+720u^3+1296u^2+3456u$$
$$+1152u^3+1920u^2+3456u+9216$$

$144u^6 + 240u^5 + 144u^4 + 240u^5 + 400u^4 + 240u^3$
$+ 144u^4 + 240u^3 + 144u^2 + 576u^4 + 960u^3 + 576u^2$
$+ 960u^3 + 1600u^2 + 960u + 576u^2 + 960u + 576$

$= 144u^6 + 240u^5 + 432u^4 + 1152u^3$
$+ 240u^5 + 400u^4 + 720u^3 + 1920u^2$
$+ 432u^4 + 720u^3 + 1296u^2 + 3456u$
$+ 1152u^3 + 1920u^2 + 3456u + 9216$

$1344u^3 + 2240u^2 + 4992u + 8640 = 0$
$21u^3 + 35u^2 + 78u + 135 = 0$

$\therefore$ The equation having roots
$\frac{1}{\alpha} + \frac{1}{\beta} + \gamma, \frac{1}{\alpha} + \frac{1}{\gamma} + \beta, \frac{1}{\beta} + \frac{1}{\gamma} + \alpha$ is
$21u^3 + 35u^2 + 78u + 135 = 0$

ii) $ax^4 + bx^3 + cx^2 + dx + e = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$$

$$\alpha\beta\gamma\delta = \frac{e}{a}$$

$$\beta + \gamma + \delta = \alpha^{k+\beta\gamma\delta}\beta^{\beta\gamma\delta}\gamma^{\beta\gamma\delta}\delta^{\beta\gamma\delta}$$

$$\alpha + \gamma + \delta = \alpha^{\alpha\gamma\delta}\beta^{k+\alpha\gamma\delta}\gamma^{\alpha\gamma\delta}\delta^{\alpha\gamma\delta}$$

$$\alpha + \beta + \delta = \alpha^{\alpha\beta\delta}\beta^{\alpha\beta\delta}\gamma^{k+\alpha\beta\delta}\delta^{\alpha\beta\delta}$$

$$\alpha + \beta + \gamma = \alpha^{\alpha\beta\gamma}\beta^{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\delta^{k+\alpha\beta\gamma}$$

$$(\beta+\gamma+\delta)(\alpha+\gamma+\delta)(\alpha+\beta+\delta)(\alpha+\beta+\gamma)$$

$$= \alpha^{k+\beta\gamma\delta}\beta^{\beta\gamma\delta}\gamma^{\beta\gamma\delta}\delta^{\beta\gamma\delta}\alpha^{\alpha\gamma\delta}\beta^{k+\alpha\gamma\delta}\gamma^{\alpha\gamma\delta}\delta^{\alpha\gamma\delta}$$
$$\alpha^{\alpha\beta\delta}\beta^{\alpha\beta\delta}\gamma^{k+\alpha\beta\delta}\delta^{\alpha\beta\delta}\alpha^{\alpha\beta\gamma}\beta^{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\delta^{k+\alpha\beta\gamma}$$

$$\left(-\frac{b}{a} - \alpha\right)\left(-\frac{b}{a} - \beta\right)\left(-\frac{b}{a} - \gamma\right)\left(-\frac{b}{a} - \delta\right)$$

$$= \alpha^{k+\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}\beta^{k+\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}$$
$$\gamma^{k+\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}\delta^{k+\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}$$

$$\left(\frac{b}{a}+\alpha\right)\left(\frac{b}{a}+\beta\right)\left(\frac{b}{a}+\gamma\right)\left(\frac{b}{a}+\alpha\right)$$

$$=(\alpha\beta\gamma\delta)^{k+\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}$$

$$\left(\frac{b}{a}\right)^4+(\alpha+\beta+\gamma+\delta)\left(\frac{b}{a}\right)^3$$

$$+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)\left(\frac{b}{a}\right)^2$$

$$+(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)\frac{b}{a}+\alpha\beta\gamma\delta=\left(\frac{e}{a}\right)^{k-\frac{d}{a}}$$

$$\frac{b^4}{a^4}-\frac{b}{a}\left(\frac{b^3}{a^3}\right)+\frac{c}{a}\left(\frac{b^2}{a^2}\right)-\frac{d}{a}\left(\frac{b}{a}\right)+\frac{e}{a}=\left(\frac{e}{a}\right)^{k-\frac{d}{a}}$$

$$\left(\frac{e}{a}\right)^{k-\frac{d}{a}}=\frac{b^2c-abd+a^2e}{a^3}$$

$$\left(k-\frac{d}{a}\right)\lg\frac{e}{a}=\lg\left(\frac{b^2c-abd+a^2e}{a^3}\right)$$

$$k-\frac{d}{a}=\lg_{\frac{e}{a}}\left(\frac{b^2c-abd+a^2e}{a^3}\right)$$

$$k=\frac{d}{a}+\lg_{\frac{e}{a}}\left(\frac{b^2c-abd+a^2e}{a^3}\right)$$