Shifting the Origin

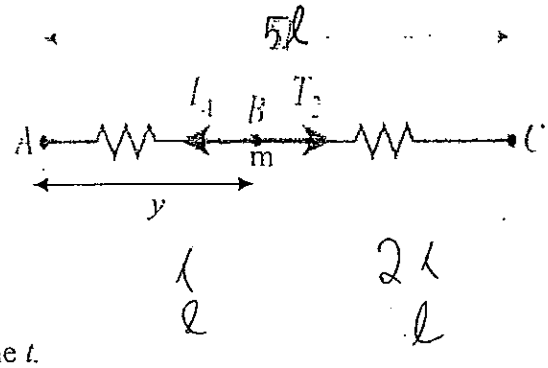
1. An elastic string of natural length 50 cm and modulus of elasticity 0.9 newtons hangs from a fixed point. While the string is unstretched, a ring of mass 50 grams is attached to the free end and then released from rest. Show that, if the extension of the string is x metres after t seconds, then x satisfies the differential equation $\ddot{x} = 9.8 - 36x$. Hence find the period of oscillation of the ring, and the extension of the string when the ring is in equilibrium. [$\frac{1}{3}\pi$ s, 27.2 cm]

2. A weather balloon has mass 25 kg. Since the density of air decreases with height, the force of buoyancy supporting the balloon at a height x metres above the ground is modelled by the formula $(400 - 0.04x)$ newtons. The balloon oscillates in a vertical line with amplitude 200 metres. Show that the motion is simple harmonic, and find the period of the oscillation.
The cloud base is at a height of 4000 metres. Find for how long the balloon disappears from view during each oscillation. [157 s, 44.8 s]

3. A particle of mass m is attached to one end of a light elastic string of modulus $3mg$ and natural length $3l$. The other end of the string is attached to a fixed point A. The particle can move freely under gravity in the vertical line containing A. Let y be the displacement below A of the particle at time t.
(a) Show that $y = 4l$ when the particle hangs in equilibrium.
(b) The particle is pulled down a distance $\frac{1}{2}l$ from the equilibrium position and released from rest. Write down the equation of motion of the mass at time t. Show that the motion is simple harmonic. Find the period and amplitude and explain why the string does not become slack.
(c) The particle is now pulled down a distance $2l$ from the equilibrium position and released from rest when the time t is zero. Show that the string first becomes slack when $t = \frac{2}{3}\pi\sqrt{\frac{l}{g}}$.

[(b) $\ddot{y} = -\frac{g}{l}(y - 4l)$, $2\pi\sqrt{\frac{l}{g}}$, $\frac{1}{2}l$; $4l - \frac{1}{2}l > 3l$]

4. AB and BC are two light springs each of natural length l but with modulus of elasticity λ and 2λ respectively, as shown in the diagram. The ends A and C are attached to fixed points A and C a distance $5l$ apart. The other end of each spring is attached to a mass m and oscillations can take place along the line ABC on a smooth horizontal table. The variable y gives the distance of the mass m from A at time t.



The mass m is held at rest at the mid-point of AC, when $y = \frac{5}{2}l$, and is then released.

(a) Find the tensions T_1 and T_2 in the two springs AB and BC respectively, in terms of y, at any later time.

(b) Write down the equation of motion of the mass at time t. Show that the motion is simple harmonic and that the centre of the motion is at the point where $y = 3l$.

(c) Write down the period and amplitude.

(d) Whilst the mass is once again held at rest when $y = \frac{5}{2}l$, the end C is moved a further distance $\frac{5}{2}l$ away from A in the same line. The distance AC is now $\frac{15}{2}l$. The mass is then released from rest. Show that the period of the resulting oscillation is unaltered but that the amplitude is increased by $\frac{5}{3}l$.

[(a) $\frac{\lambda}{l}(y-l)$, $\frac{2\lambda}{l}(4l-y)$ (b) $\ddot{y} = -\frac{3\lambda}{ml}(y-3l)$ (c) $2\pi\sqrt{\frac{ml}{3\lambda}}$, $\frac{1}{2}l$]

Simple Pendulum

1. If $g = 9.8 \text{ m s}^{-2}$, find the length of a pendulum whose period is 2.4 seconds. [1.43 m]
2. A pendulum of length 2236 mm has a period of 3 seconds. Calculate, to three decimal places, the value of g. [9.808 m s^{-2}]
3. The length of a pendulum is reduced by 30%. Find the percentage change in the period. [16.3 % reduction]
4. The period of a pendulum is T seconds. If the length of the pendulum is doubled, write the new period in terms of T. [$\sqrt{2}T$]
5. Calculate, to the nearest mm, the length required for a seconds pendulum at a place where $g = 9.802 \text{ ms}^{-2}$. [993 mm]
6. A grandfather clock keeps perfect time when the length of its pendulum is 994 mm Determine, to three significant figures, the value of g. [9.81 m s^{-2}]
7. A seconds pendulum beats exact seconds at a location where $g = 9.80 \text{ms}^{-2}$. If it is taken to a place where $g = 9.81 \text{ ms}^{-2}$, find:

   a by how many seconds per day it will be wrong [Gain 44s]

   b by how much the length should be altered to correct it. [increase by 1.01 mm]

Using approximations.

1. A particle P of mass 0.25 kg moves in a straight line AOB on a horizontal surface. At time t seconds, the displacement of P from O towards B is x metres. The only horizontal forces acting on P are:

   a force of constant magnitude 1 N in the direction $\overrightarrow{AB}$;

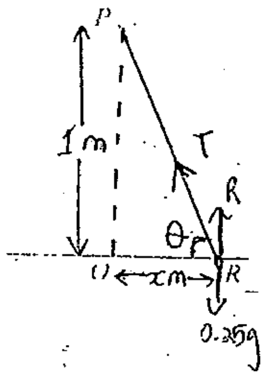
   a force of variable magnitude $\sqrt{1+x}$ in the direction $\overrightarrow{BA}$.

   Obtain an expression for $\frac{d^2x}{dt^2}$ in terms of x. It is given that, in the motion, x remains small.

   (a) Write down the first two terms of the binomial expansion of $\sqrt{1+x}$.
   (b) Hence obtain and simplify an equation for the approximate acceleration of P.
   (c) For the approximate motion of P, state its type, and find the period.
   Explain why the force of magnitude 1 N cannot be a frictional force.

   [$4(1-\sqrt{1+x})$ (a) $1+\frac{1}{2}x$ (b) $\ddot{x}=-2x$ (c) simple harmonic, $\pi\sqrt{2}$,]
   [A friction force would change sign when the velocity changes direction]

2. A ring R of mass 0.25 kg is threaded on a smooth horizontal wire. P is a fixed point in the same vertical plane as the wire, and O is the point on the wire such that OP is perpendicular to the wire. The distance OP is 1 m. The ring is connected to P by a light elastic string of natural length 0.75 m and modulus of elasticity 27 N. The ring is projected from O along the wire. After t seconds the displacement of R from O is x metres (see diagram). In a simple model for the subsequent motion, air resistance is ignored. Show that



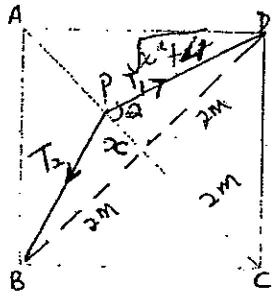
$$\frac{d^2x}{dt^2}=-108x\left(\frac{4}{3}-\frac{1}{\sqrt{1+x^2}}\right).$$

   It is given that x remains small, and that in the binomial expansion of $(1+x^2)^{-\frac{1}{2}}$ terms involving x^2 and higher powers of x can be ignored. Show that R moves with approximate simple harmonic motion of period $\frac{1}{3}\pi$ seconds.

   Given that the amplitude of the motion is 0.05 m, find the approximate position of the ring 1.9 s after it is projected from O.

   [0.0460 m in the direction opposite to the direction of projection]

3. A particle of mass 1 kg rests on a smooth horizontal table. It is attached by means of two elastic strings to each of points B and D, of a square $ABCD$ whose diagonals are of length 4 m. The natural length of each string is 1 m and each has modulus of elasticity 40 N. The particle is initially at rest at the centre of the square. It is then displaced slightly in the direction of A and is, released from rest when AP = 1 m.



   (i) Find the period of oscillation of the subsequent motion. [0.993 s]
   (ii) Find the time taken to travel from O to Q where AQ = 1.75 m. [0.0400s]

4. An acrobat of mass 50 kg hangs from two points A and B at the same level, 24 metres apart, by two elastic cords which have natural length 3.2 metres and modulus of elasticity 208 newtons. Find an equation for the depth, x metres, below AB at which she is in equilibrium. Show that this equation is satisfied by $x = 5$.
She now oscillates vertically about this equilibrium position. Using a binomial approximation, find the period of her oscillation. [$13x\left(1-\dfrac{32}{\sqrt{144+x^2}}\right) = 49$; 4.38 s]

[Question 5 is optional]

5. A particle of mass m hangs by two elastic strings from pegs which are at the same horizontal level at a distance $2a$ apart. Each string has natural length l and modulus of elasticity λ. In equilibrium the particle is a distance b below the level of the pegs.
Show that $\dfrac{1}{l} - \dfrac{1}{c} = \dfrac{mg}{2\lambda b}$, where $c = \sqrt{a^2 + b^2}$.
The particle is pulled down a small distance and then released. Show that its oscillation is described approximately by the simple harmonic motion equation
$$\ddot{x} = -\left(\frac{g}{b} + \frac{2\lambda b^2}{mc^3}\right)x.$$