1. $ax^3 + bx^2 + cx + d = 0$

α, β, γ are the roots

$\alpha + \beta + \gamma = \frac{-b}{a}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ $\alpha\beta\gamma = \frac{-d}{a}$

$\alpha + \beta = \gamma$

$\therefore \gamma + \gamma = \frac{-b}{a}$ $\alpha\beta + (\alpha + \beta)\gamma = \frac{c}{a}$ $\alpha\beta\gamma = \frac{-d}{a}$

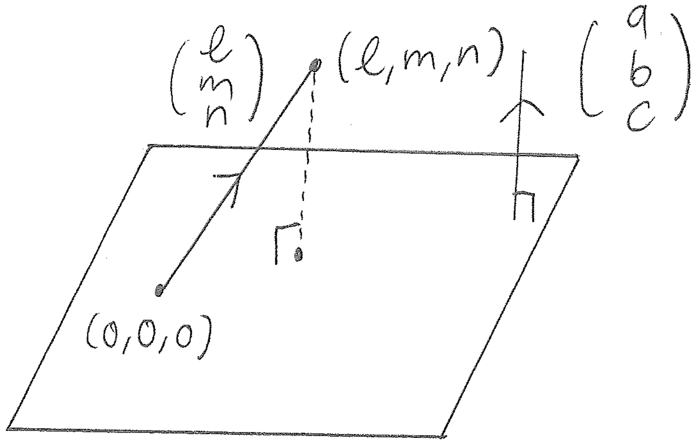
$2\gamma = \frac{-b}{a}$ $\alpha\beta + \gamma^2 = \frac{c}{a}$ $\frac{-\alpha\beta b}{2a} = \frac{-d}{a}$

$\gamma = \frac{-b}{2a}$ $\alpha\beta + \frac{b^2}{4a^2} = \frac{c}{a}$ $\alpha\beta = \frac{2d}{b}$

$\frac{2d}{b} + \frac{b^2}{4a^2} = \frac{c}{a}$

$8a^2d + b^3 = 4abc$

2. $ax+by+cz=0 \quad (l,m,n)$



Since the line perpendicular to the plane through (l,m,n) is $\underset{\sim}{r} = s\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

and $(0,0,0)$ is on the plane,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \left[\begin{pmatrix} l \\ m \\ n \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right] = \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| \left|\begin{pmatrix} l \\ m \\ n \end{pmatrix}\right| \cos\theta$$

$$al + bm + cn = \sqrt{a^2+b^2+c^2}\left|\begin{pmatrix} l \\ m \\ n \end{pmatrix}\right| \cos\theta$$

$$\therefore \left|\begin{pmatrix} l \\ m \\ n \end{pmatrix}\right| \cos\theta = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}}$$

$x = 5+10s \quad y = 7-9s \quad z = 8-2s$

$3x + 2y + 6z = 0$

$15 + 30s + 14 - 18s + 48 - 12s = 0$

$0s = -77$

no solution

$\therefore$ The line and plane are parallel

$$\underset{\sim}{r} = \begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix} + s\begin{pmatrix} 10 \\ -9 \\ -2 \end{pmatrix} \quad 3x + 2y + 6z = 0$$

Since $(5,7,8)$ is a point on the plane,

the perpendicular distance from

$\underset{\sim}{r} = \begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix}$ to $3x + 2y + 6z = 0$

is $$\frac{3(5) + 2(7) + 6(8)}{\sqrt{3^2 + 2^2 + 6^2}}$$

$$= \frac{15 + 14 + 48}{7}$$

$$= \frac{77}{7}$$

$$= 11$$

3. $(2r+1)^4 - (2r-1)^4 = 64r^3 + 16r$

$$\sum_{r=1}^{n} 64r^3 + 16r = \sum_{r=1}^{n} (2r+1)^4 - (2r-1)^4$$

$$64\sum_{r=1}^{n} r^3 + 16\sum_{r=1}^{n} r = (2n+1)^4 - (2n-1)^4$$
$$+ (2n-1)^4 - (2n-3)^4$$
$$\vdots$$
$$+ 5^4 - 3^4$$
$$+ 3^4 - 1^4$$

$$64\sum_{r=1}^{n} r^3 + \frac{16n(n+1)}{2} = (2n+1)^4 - 1$$

$$64\sum_{r=1}^{n} r^3 + 8n^2 + 8n = 16n^4 + 32n^3 + 24n^2 + 8n + 1 - 1$$
$$= 16n^4 + 32n^3 + 24n^2 + 8n$$

$$64\sum_{r=1}^{n} r^3 = 16n^4 + 32n^3 + 16n^2$$
$$= 16n^2(n^2 + 2n + 1)$$
$$= 16n^2(n+1)^2$$

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

4. $y = e^{ax}\cos bx$

$$\frac{d^n y}{dx^n} = (a^2+b^2)^{\frac{n}{2}} e^{ax}\cos(bx+n\phi),\ \phi=\tan^{-1}\frac{b}{a}$$

when $n=1$: $\frac{d^1 y}{dx^1} = \frac{dy}{dx} = ae^{ax}\cos bx - be^{ax}\sin bx$

$$= e^{ax}(a\cos bx - b\sin bx)$$

$\tan\phi = \frac{b}{a}$

$$= e^{ax}(\sqrt{a^2+b^2}\cos bx\cos\phi - \sqrt{a^2+b^2}\sin bx\sin\phi)$$

$\therefore \sin\phi = \frac{b}{\sqrt{a^2+b^2}}$

$\cos\phi = \frac{a}{\sqrt{a^2+b^2}}$

$$= \sqrt{a^2+b^2}\,e^{ax}(\cos bx\cos\phi - \sin bx\sin\phi)$$

$$= \sqrt{a^2+b^2}\,e^{ax}\cos(bx+\phi)$$

$$= (a^2+b^2)^{\frac{1}{2}} e^{ax}\cos(bx+1\phi),$$

$$\phi = \tan^{-1}\frac{b}{a}$$

Assume the statement is true when $n=k$.

$n=k$: $\frac{d^k y}{dx^k} = (a^2+b^2)^{\frac{k}{2}} e^{ax}\cos(bx+k\phi),\ \phi=\tan^{-1}\frac{b}{a}$

when $n=k+1$: $\frac{d^{k+1} y}{dx^{k+1}} = (a^2+b^2)^{\frac{k+1}{2}} e^{ax}\cos(bx+(k+1)\phi),$

$$\phi = \tan^{-1}\frac{b}{a}$$

(what needs to be proved)

$$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left(a^2+b^2\right)^{\frac{k}{2}} e^{ax} \cos\left(bx + k\phi\right)$$

$$= \left(a^2+b^2\right)^{\frac{k}{2}} \frac{d}{dx}\left(e^{ax}\cos\left(bx+k\phi\right)\right)$$

$$= \left(a^2+b^2\right)^{\frac{k}{2}}\left(ae^{ax}\cos\left(bx+k\phi\right) - be^{ax}\sin\left(bx+k\phi\right)\right)$$

$$= \left(a^2+b^2\right)^{\frac{k}{2}} e^{ax}\left(a\cos\left(bx+k\phi\right) - b\sin\left(bx+k\phi\right)\right)$$

$$= \left(a^2+b^2\right)^{\frac{k}{2}} e^{ax}\left(\sqrt{a^2+b^2}\cos\left(bx+k\phi\right)\cos\phi - \sqrt{a^2+b^2}\sin\left(bx+k\phi\right)\sin\phi\right)$$

$$= \left(a^2+b^2\right)^{\frac{k+1}{2}} e^{ax}\left(\cos\left(bx+k\phi\right)\cos\phi - \sin\left(bx+k\phi\right)\sin\phi\right)$$

$$= \left(a^2+b^2\right)^{\frac{k+1}{2}} e^{ax}\left(\cos\left(bx+(k+1)\phi\right)\right), \ \phi = \tan^{-1}\frac{b}{a}$$

$$\therefore \frac{d^n y}{dx^n} = \left(a^2+b^2\right)^{\frac{n}{2}} e^{ax}\cos\left(bx+n\phi\right), \ \phi = \tan^{-1}\frac{b}{a}$$

for every positive integer n.

5. $C: y = \dfrac{1}{ax+b} + \dfrac{1}{cx+d}$, $a, b, c, d > 0$

i) As $x \to \pm\infty$ $y \to 0$

As $x \to -\dfrac{b}{a}$ $y \to \pm\infty$

As $x \to -\dfrac{d}{c}$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y=0$, $x=-\dfrac{b}{a}$ and $x=-\dfrac{d}{c}$

ii) When $x=0$: $y = \dfrac{1}{b} + \dfrac{1}{d}$

When $y=0$: $\dfrac{1}{ax+b} + \dfrac{1}{cx+d} = 0$

$$\frac{ax+b+cx+d}{(ax+b)(cx+d)} = 0$$

$$(a+c)x + b + d = 0$$

$$x = \frac{-b-d}{a+c}$$

$\therefore$ The intersection points of C are $\left(0, \dfrac{1}{b} + \dfrac{1}{d}\right)$ and $\left(\dfrac{-b-d}{a+c}, 0\right)$.

iii) $\dfrac{dy}{dx} = \dfrac{-a}{(ax+b)^2} - \dfrac{c}{(cx+d)^2} = -\left(\dfrac{a}{(ax+b)^2} + \dfrac{c}{(cx+d)^2}\right)$

$\therefore \dfrac{dy}{dx} < 0$ at every point of C.

6. $\left(1+\frac{2}{z}\right)^n = 1$

$$1 + \frac{2}{z} = 1^{\frac{1}{n}}$$

$$= (\cos 0 + i\sin 0)^{\frac{1}{n}}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{n}}, \quad k \in \mathbb{Z}$$

$$= \cos \frac{2k\pi}{n} + i\sin \frac{2k\pi}{n}, \quad k = 0, 1, 2, \cdots, n-1.$$

$$= 1 - 2\sin^2 \frac{k\pi}{n} + 2i\sin \frac{k\pi}{n} \cos \frac{k\pi}{n}$$

$$\frac{2}{z} = 2i\sin \frac{k\pi}{n} \cos \frac{k\pi}{n} - 2\sin^2 \frac{k\pi}{n}$$

$$\frac{1}{z} = i\sin \frac{k\pi}{n} \cos \frac{k\pi}{n} - \sin^2 \frac{k\pi}{n}$$

$$= i\sin \frac{k\pi}{n} \left(\cos \frac{k\pi}{n} + i\sin \frac{k\pi}{n} \right)$$

$$z = \frac{1}{i\sin \frac{k\pi}{n} \left(\cos \frac{k\pi}{n} + i\sin \frac{k\pi}{n} \right)}$$

$$= \frac{\cos \frac{k\pi}{n} - i\sin \frac{k\pi}{n}}{i\sin \frac{k\pi}{n} \left(\cos \frac{k\pi}{n} + i\sin \frac{k\pi}{n} \right)\left(\cos \frac{k\pi}{n} - i\sin \frac{k\pi}{n} \right)}$$

$$= \frac{\cos \frac{k\pi}{n} - i\sin \frac{k\pi}{n}}{i\sin \frac{k\pi}{n} \left(\cos^2 \frac{k\pi}{n} + \sin^2 \frac{k\pi}{n} \right)} = \frac{\cos \frac{k\pi}{n} - i\sin \frac{k\pi}{n}}{i\sin \frac{k\pi}{n}}$$

$$= -1 - i\cot \frac{k\pi}{n}$$

$$|z| = \sqrt{1 + \cot^2 \frac{k\pi}{n}} = \sqrt{\csc^2 \frac{k\pi}{n}} = \csc \frac{k\pi}{n}, \quad k = 1, 2, \cdots, n-1$$

7. $I_n = \int \tan^n x \, dx$

$= \int \tan^{n-2} x \tan^2 x \, dx$

$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int \tan^{n-2} x \sec^2 x - \tan^{n-2} x \, dx$

$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$

$= \int \tan^{n-2} x \sec^2 x \, dx - I_{n-2}$

$w = \tan x$

$dw = \sec^2 x \, dx$

$= \int w^{n-2} \, dw - I_{n-2}$

$= \dfrac{w^{n-1}}{n-1} - I_{n-2}$

$= \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$

$n=5$: $I_5 = \dfrac{\tan^4 x}{4} - I_3$

$I_3 = \dfrac{\tan^2 x}{2} - I_1$

$I_1 = \int \tan x \, dx$

$= \int \dfrac{\sin x}{\cos x} \, dx$

$= -\ln|\cos x| + c$

$= \ln|\sec x| + c$

$\therefore I_3 = \frac{\tan^2 x}{2} - \ln|\sec x| + c$

$I_5 = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + c$

$n=4: \quad I_4 = \frac{\tan^3 x}{3} - I_2$

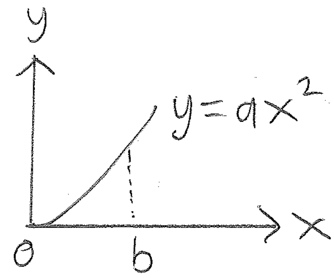
$I_2 = \int \tan^2 x \, dx$

$= \int \sec^2 x - 1 \, dx$

$= \tan x - x + c$

$\therefore I_4 = \frac{\tan^3 x}{3} - \tan x + x + c$

8.i) $y = ax^2$, $x = 0$ $x = b$



$$\text{Volume, } V = \int_0^b \pi y^2 \, dx$$

$$= \int_0^b \pi a^2 x^4 \, dx$$

$$= \pi a^2 \left[\frac{x^5}{5} \right]_0^b$$

$$= \frac{\pi a^2 b^5}{5}$$

$$V\bar{x} = \int_0^b \pi x y^2 \, dx$$

$$= \int_0^b \pi x \, a^2 x^4 \, dx$$

$$= \int_0^b \pi a^2 x^5 \, dx$$

$$= \pi a^2 \left[\frac{x^6}{6} \right]_0^b$$

$$= \frac{\pi a^2 b^6}{6}$$

$$\therefore \bar{x} = \frac{5b}{6}$$

ii) $x = e^t \cos t \quad y = e^t \sin t, \quad t=0 \quad t=\pi$

$$\frac{dx}{dt} = e^t \cos t - e^t \sin t \quad \frac{dy}{dt} = e^t \sin t + e^t \cos t$$

a) $$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t$$
$$+ \sin^2 t + 2\sin t \cos t + \cos^2 t)$$

$$= 2e^{2t}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2}e^t$$

The arc length from $t=0$ to $t=\pi$ is

$$\int_0^\pi \sqrt{2}e^t \, dt$$

$$= \left[\sqrt{2}e^t\right]_0^\pi$$

$$= \sqrt{2}(e^\pi - 1)$$

b) $$2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2\pi e^t \cos t\,(\sqrt{2}e^t)$$

$$= 2\sqrt{2}\pi e^{2t} \cos t$$

The surface area of revolution from $t=0$ to $t=\pi$ about the y-axis is

$$\int_0^\pi 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

$$= \int_0^{\pi} 2\sqrt{2}\pi e^{2t} \cos t \, dt$$

$$\int e^{2t} \cos t \, dt = e^{2t} \sin t - \int 2e^{2t} \sin t \, dt$$

$$= e^{2t} \sin t - 2\int e^{2t} \sin t \, dt$$

$$= e^{2t} \sin t - 2\left(-e^{2t} \cos t - \int -2e^{2t} \cos t \, dt \right)$$

$$= e^{2t} \sin t + 2e^{2t} \cos t - 4\int e^{2t} \cos t \, dt$$

$$\therefore \int e^{2t} \cos t \, dt = \frac{e^{2t}}{5} (\sin t + 2\cos t)$$

$$\int 2\sqrt{2}\pi e^{2t} \cos t \, dt = \frac{2\sqrt{2}\pi e^{2t}}{5} (\sin t + 2\cos t)$$

$$= \int_0^{\frac{\pi}{2}} 2\sqrt{2}\pi e^{2t} \cos t \, dt + \int_{\frac{\pi}{2}}^{\pi} 2\sqrt{2}\pi e^{2t} \cos t \, dt$$

$$= \frac{2\sqrt{2}\pi}{5} \Big[e^{2t} (\sin t + 2\cos t) \Big]_0^{\frac{\pi}{2}} - \frac{2\sqrt{2}\pi}{5} \Big[e^{2t} (\sin t + 2\cos t) \Big]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2\sqrt{2}\pi}{5} \left(e^{\pi} - 2 \right) - \frac{2\sqrt{2}\pi}{5} \left(e^{2\pi}(-2) - e^{\pi} \right)$$

$$= \frac{2\sqrt{2}\pi}{5} \left(e^{\pi} - 2 + 2e^{2\pi} + e^{\pi} \right)$$

$$= \frac{2\sqrt{2}\pi}{5} \left(2e^{2\pi} + 2e^{\pi} - 2 \right)$$

$$= \frac{4\sqrt{2}\pi}{5} \left(e^{2\pi} + e^{\pi} - 1 \right)$$

9. Eigenvalues 3, 5, 7 with corresponding eigenvectors

$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$

If $P = \begin{pmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix}$ and $D = \begin{pmatrix}3 & 0 & 0\\0 & 5 & 0\\0 & 0 & 7\end{pmatrix}$,

$A = PDP^{-1}$

$|P| = -2$

$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}$ $\begin{pmatrix}1\\0\\1\end{pmatrix} \times \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$ $\begin{pmatrix}0\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$

$\begin{pmatrix}-1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & -1\end{pmatrix}^T = \begin{pmatrix}-1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & -1\end{pmatrix}$

$\therefore P^{-1} = \frac{1}{|P|}\begin{pmatrix}-1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & -1\\1 & -1 & 1\\-1 & 1 & 1\end{pmatrix}$

$\therefore A = PDP^{-1}$

$= \begin{pmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}3 & 0 & 0\\0 & 5 & 0\\0 & 0 & 7\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1 & -1\\1 & -1 & 1\\-1 & 1 & 1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}3 & 5 & 0\\3 & 0 & 7\\0 & 5 & 7\end{pmatrix}\begin{pmatrix}1 & 1 & -1\\1 & -1 & 1\\-1 & 1 & 1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}8 & -2 & 2\\-4 & 10 & 4\\-2 & 2 & 12\end{pmatrix}$

$= \begin{pmatrix}4 & -1 & 1\\-2 & 5 & 2\\-1 & 1 & 6\end{pmatrix}$

10. $C: (x+y)(x^2+y^2) = xy$

i) $x = r\cos\theta \quad y = r\sin\theta$

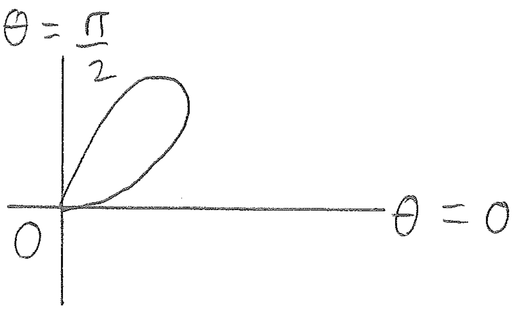
$(r\cos\theta + r\sin\theta)(r^2\cos^2\theta + r^2\sin^2\theta) = r^2\sin\theta\cos\theta$

$r(\cos\theta + \sin\theta)(\cos^2\theta + \sin^2\theta) = \sin\theta\cos\theta$

$$\therefore r = \frac{\sin\theta\cos\theta}{\sin\theta + \cos\theta}$$

ii)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	$\frac{\sqrt{3}}{2+\sqrt{3}}$	$\frac{1}{2\sqrt{2}}$	$\frac{\sqrt{3}}{2+\sqrt{3}}$	0



iii) $\dfrac{\sin^2 2\theta}{1+\sin 2\theta} = \sin 2\theta - 1 + \dfrac{1}{1+\sin 2\theta}$

$= \sin 2\theta - 1 + \dfrac{1-\sin 2\theta}{(1+\sin 2\theta)(1-\sin 2\theta)}$

$= \sin 2\theta - 1 + \dfrac{1-\sin 2\theta}{\cos^2\theta}$

$= \sin 2\theta - 1 + \sec^2 2\theta - \tan 2\theta \sec 2\theta$

$$\text{Area} = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{(\sin\theta+\cos\theta)^2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{1+2\sin\theta\cos\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4(1+\sin 2\theta)}\, d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{1+\sin 2\theta}\, d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} \sin 2\theta - 1 + \sec^2 2\theta - \tan 2\theta \sec 2\theta \, d\theta$$

$$= \frac{1}{8}\left[-\frac{\cos 2\theta}{2} - \theta + \frac{\tan 2\theta}{2} - \frac{\sec 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}(0+1+1-\pi-(0-1-1-0))$$

$$= \frac{4-\pi}{16}$$

11. EITHER

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ b+c+d & a+c+d & a+b+d & a+b+c \end{pmatrix}$$

i) $$\xrightarrow[-(b+c+d)r_1 + r_3]{-ar_1 + r_2,} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & a-b & a-c & a-d \end{pmatrix}$$

$$\xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $a > b > c > d$,

$\therefore \text{rank}(M) = 2$

ii) A basis for the range space of T is

$$\left\{ \begin{pmatrix} 1 \\ a \\ b+c+d \end{pmatrix}, \begin{pmatrix} 1 \\ b \\ a+c+d \end{pmatrix} \right\}.$$

iii) If $$\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ b+c+d & a+c+d & a+b+d & a+b+c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ a & b & c & d & 0 \\ b+c+d & a+c+d & a+b+d & a+b+c & 0 \end{array} \right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & b-a & c-a & d-a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

If $a=b=c=d$: $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

let $y=r,\ z=s,\ w=t,\ r,s,t \in \mathbb{R}$

$\therefore x=-r-s-t$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -r-s-t \\ r \\ s \\ t \end{pmatrix}$$

$$= r\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ The nullspace is $\underset{\sim}{r}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

iv) Since $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution of $M\underset{\sim}{x} = \begin{pmatrix} 4 \\ a+b+c+d \\ 3(a+b+c+d) \end{pmatrix}$

and $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is the basis of the

nullspace of T, $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + r\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

is a general solution of $M\underset{\sim}{x} = \begin{pmatrix} 4 \\ a+b+c+d \\ 3(a+b+c+d) \end{pmatrix}$.

OR

$$\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 5y = 10e^{-2x}, \quad y = 5, \ \frac{dy}{dx} = 1 \text{ when } x = 0.$$

$$\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 5y = 0$$

$$m^2 + 2m + 5 = 0$$

$$(m+1)^2 + 4 = 0$$

$$(m+1)^2 = -4$$

$$m + 1 = \pm 2i$$

$$m = -1 \pm 2i$$

$\therefore$ The complementary function, y_c, is

$$y_c = e^{-x}(A\cos 2x + B\sin 2x).$$

The particular integral, y_p, is given by

$$y_p = Ce^{-2x}$$

$$\frac{dy_p}{dx} = -2Ce^{-2x}$$

$$\frac{d^2y_p}{dx^2} = 4Ce^{-2x}$$

$$\frac{d^2y_p}{dx^2} + \frac{2dy_p}{dx} + 5y_p = 4Ce^{-2x} - 4Ce^{-2x} + 5Ce^{-2x}$$

$$= 5Ce^{-2x}$$

$$= 10e^{-2x}$$

$$5C = 10$$
$$C = 2$$
$$\therefore \; y_p = 2e^{-2x}$$

$$\therefore \; y = y_c + y_p$$
$$= e^{-x}(A\cos 2x + B\sin 2x) + 2e^{-2x}$$

$$\frac{dy}{dx} = -e^{-x}(A\cos 2x + B\sin 2x)$$
$$+ e^{-x}(-2A\sin 2x + 2B\cos 2x) - 4e^{-2x}$$

$x=0 \; y=5$: $5 = A + 2$

$x=0 \; \frac{dy}{dx}=1$: $1 = 2B - A - 4$

$$A = 3$$
$$B = 4$$

$$\therefore \; y = e^{-x}(3\cos 2x + 4\sin 2x) + 2e^{-2x}$$