11

$$y = \frac{-x^2 + x + 5}{x^2 - x - 2}$$

$$x^2 - x - 2 \overline{\smash{\big)}\, -x^2 + x + 5}$$ quotient -1:

$$\begin{array}{r} -1 \\ x^2 - x - 2 \,\big)\, -x^2 + x + 5 \\ -x^2 + x + 2 \\ \hline 3 \end{array}$$

$$= -1 + \frac{3}{x^2 - x - 2}$$

$$\frac{3}{x^2 - x - 2} = \frac{A}{x - 2} + \frac{B}{x + 1}$$

$$= \frac{A(x + 1) + B(x - 2)}{(x - 2)(x + 1)}$$

$$3 = A(x + 1) + B(x - 2)$$

$$= (A + B)x + A - 2B$$

$A + B = 0$

$A - 2B = 3$

$3B = -3$

$B = -1$

$A = 1$

$$\frac{3}{x^2 - x - 2} = \frac{1}{x - 2} - \frac{1}{x + 1}$$

$$y = -1 + \frac{1}{x - 2} - \frac{1}{x + 1}$$

$$y(x^2 - x - 2) = -x^2 + x + 5$$

$$x^2y - xy - 2y = -x^2 + x + 5$$

$$(y+1)x^2 - (y+1)x - 2y - 5 = 0$$

$a = y + 1$ $\quad b = -(y+1)$ $\quad c = -2y - 5$

$$\begin{aligned} b^2 - 4ac &= (y+1)^2 - 4(y+1)(-2y-5) \\ &= y^2 + 2y + 1 + 4(y+1)(2y+5) \\ &= y^2 + 2y + 1 + 4(2y^2 + 7y + 5) \\ &= y^2 + 2y + 1 + 8y^2 + 28y + 20 \\ &= 9y^2 + 30y + 21 \\ &= 3(3y^2 + 10y + 7) \\ &= 3(y+7)(3y+1) \end{aligned}$$

When $b^2 - 4ac \geqslant 0$

$$3(y+7)(3y+1) \geqslant 0$$

$$(y+7)(3y+1) \geqslant 0$$

$$y \leqslant -7 \text{ or } y \geqslant \frac{-1}{3}$$

$\therefore$ y cannot take values in the interval

$$-7 \leqslant y \leqslant \frac{-1}{3}$$

Asymptotes:

As $x \to \pm\infty$ $y \to -1$

As $x \to 2$ $y \to \pm\infty$

As $x \to -1$ $y \to \pm\infty$

$y = -1$

$x = 2$

$x = -1$

Intersection points:

when $x = 0$: $y = -\frac{5}{2}$

when $y = 0$: $\frac{-x^2 + x + 5}{x^2 - x - 2} = 0$

$$-x^2 + x + 5 = 0$$

$$x^2 - x - 5 = 0$$

$$(x - \tfrac{1}{2})^2 = \frac{21}{4}$$

$$x - \frac{1}{2} = \frac{\pm\sqrt{21}}{2}$$

$$x = \frac{1 \pm \sqrt{21}}{2}$$

Critical points:

$$\frac{dy}{dx} = \frac{-1}{(x-2)^2} + \frac{1}{(x+1)^2}$$

when $\dfrac{dy}{dx} = 0$

$$\frac{-1}{(x-2)^2} + \frac{1}{(x+1)^2} = 0$$

$$\frac{1}{(x-2)^2} = \frac{1}{(x+1)^2}$$

$$(x-2)^2 = (x+1)^2$$

$$x^2 - 4x + 4 = x^2 + 2x + 1$$

$$6x = 3$$

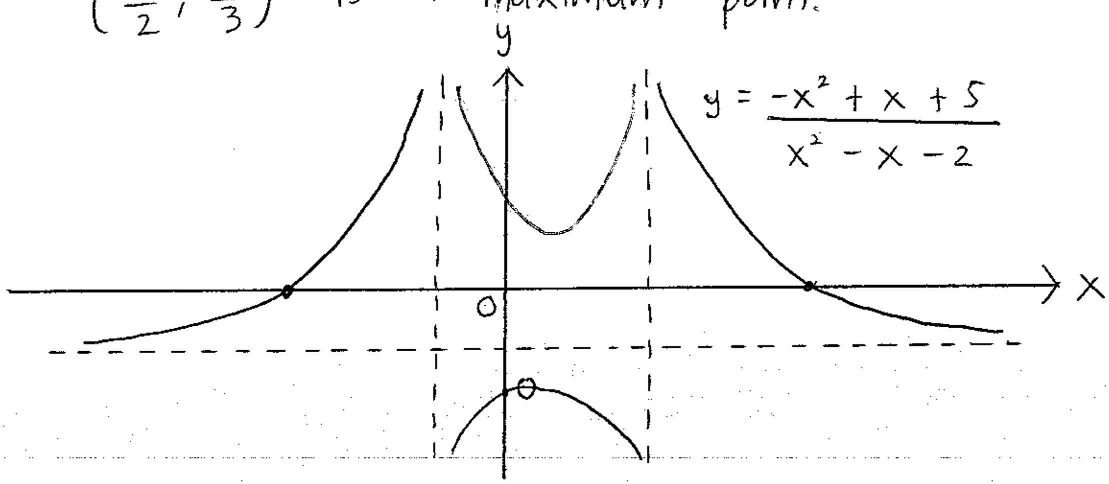
$$x = \frac{1}{2}$$

$$y = \frac{-7}{3}$$

$$\frac{d^2y}{dx^2} = \frac{2}{(x-2)^3} - \frac{2}{(x+1)^3}$$

when $x = \frac{1}{2}$: $\dfrac{d^2y}{dx^2} = \dfrac{-16}{27} - \dfrac{16}{27} = \dfrac{-32}{27} < 0$

$\left(\frac{1}{2}, \frac{7}{3}\right)$ is a maximum point.



o : Critical points

• : Intersection points.

17

$$y = \frac{x - 3}{(x-2)(x+1)}$$

$$(x-2)(x+1)y = x - 3$$

$$(x^2 - x - 2)y = x - 3$$

$$x^2y - xy - 2y = x - 3$$

$$x^2y - (y+1)x + 3 - 2y = 0$$

$a = y \quad b = -(y+1) \quad c = 3 - 2y$

$$\begin{aligned} b^2 - 4ac &= (y+1)^2 - 4y(3-2y) \\ &= y^2 + 2y + 1 - 12y + 8y^2 \\ &= 9y^2 - 10y + 1 \\ &= (y-1)(9y-1) \end{aligned}$$

when $b^2 - 4ac \geqslant 0$

$(y-1)(9y-1) \geqslant 0$

$y \geqslant 1$ or $y \leqslant \frac{1}{9}$

. The value of y cannot lie between

$\frac{1}{9} \leqslant y \leqslant 1$

$$\frac{x-3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

$$x - 3 = A(x+1) + B(x-2)$$

$$= (A+B)x + A - 2B$$

$$A + B = 1$$

$$A - 2B = -3$$

$$3B = 4$$

$$B = \frac{4}{3}$$

$$A = \frac{-1}{3}$$

$$\therefore\ y = \frac{-1}{3(x-2)} + \frac{4}{3(x+1)}$$

Asymptotes

As $x \to 2$ $y \to \pm\infty$

As $x \to -1$ $y \to \pm\infty$

As $x \to \pm\infty$ $y \to 0$

$x = 2$

$x = -1$

$y = 0$

when $y = \frac{1}{9}$

$$\frac{1}{9} = \frac{x - 3}{(x - 2)(x + 1)}$$

$$(x - 2)(x + 1) = 9(x - 3)$$

$$x^2 - x - 2 = 9x - 27$$

$$x^2 - 10x + 25 = 0$$

$$(x - 5)^2 = 0$$

$$x = 5$$

when $y = 1$

$$1 = \frac{x - 3}{(x - 2)(x + 1)}$$

$$(x - 2)(x + 1) = x - 3$$

$$x^2 - x - 2 = x - 3$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0$$

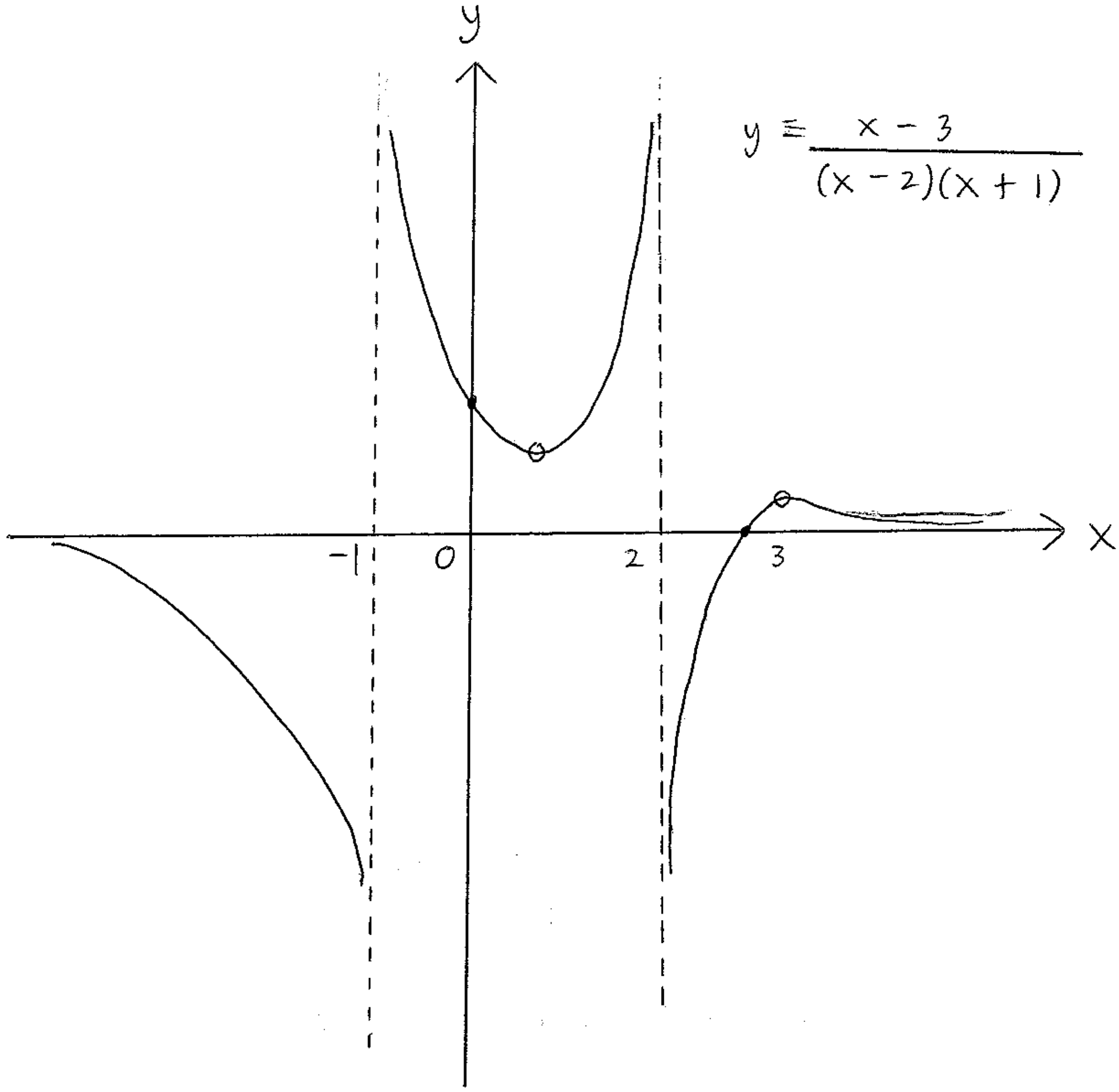
$$x = 1$$

Intersection points:

when $x = 0$ $\quad y = \frac{3}{2}$

when $y = 0$: $\frac{x - 3}{(x - 2)(x + 1)} = 0$

$x = 3.$



o : Critical points

• : Intersection points.

18

$$y = x + \frac{4}{x^2}$$

As $x \to \pm\infty$ $y \to x$

As $x \to 0$ $y \to \pm\infty$

$y = x$

$x = 0$

$\therefore$ The asymptotes of y are $y = x$ and $x = 0$.

Intersection points

When $y = 0$

$$x + \frac{4}{x^2} = 0$$

$$x = \frac{-4}{x^2}$$

$$x^3 = -4$$

$$x = -4^{\frac{1}{3}}$$

Critical points:

$$\frac{dy}{dx} = 1 - \frac{8}{x^3}$$

when $\frac{dy}{dx} = 0$

$$1 - \frac{8}{x^3} = 0$$

$$\frac{8}{x^3} = 1$$

$$x^3 = 8$$

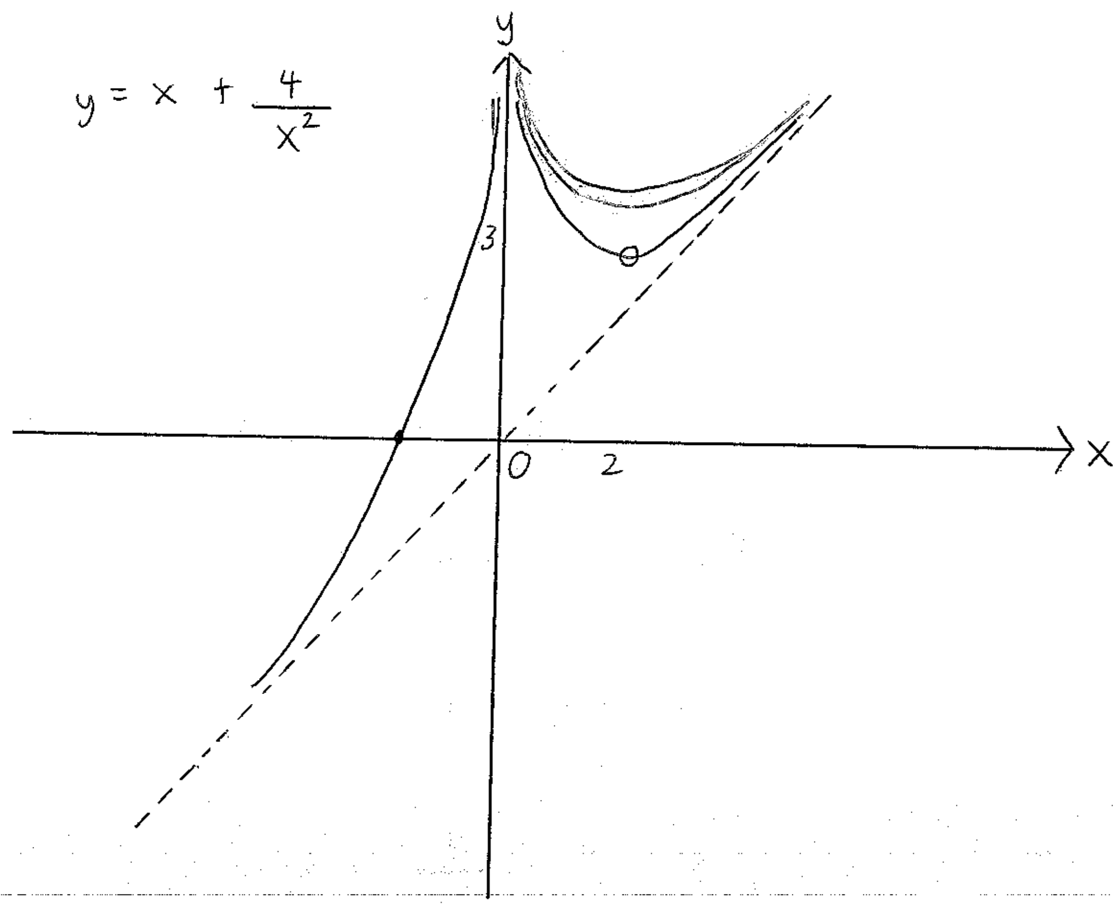
$$x = 2$$

$$y = 3$$

$$\frac{d^2y}{dx^2} = \frac{24}{x^4}$$

when $x = 2$: $\dfrac{d^2y}{dx^2} = \dfrac{3}{2} > 0$

$\therefore (2, 3)$ is a minimum point.



o : Critical point

• : Intersection point.

19.

$$y = \frac{x^2}{x+1}$$

$$\begin{array}{r|l} & x - 1 \\ \cline{2-2} x + 1 & x^2 \\ & x^2 + x \\ \cline{2-2} & \quad -x \\ & \quad -x - 1 \\ \cline{2-2} & \qquad\quad 1 \end{array}$$

$$= x - 1 + \frac{1}{x+1}$$

As $x \to \pm\infty$ $y \to x - 1$

As $x \to -1$ $y \to \pm\infty$

$y = x - 1$

$x = -1$

$\therefore$ The asymptotes of y are the lines $y = x - 1$ and $x = -1$

Intersection points :

when $x = 0$: $y = 0$

Critical points :

$$\frac{dy}{dx} = 1 - \frac{1}{(x+1)^2}$$

when $\frac{dy}{dx} = 0$

$$1 - \frac{1}{(x+1)^2} = 0$$

$$\frac{1}{(x+1)^2} = 1$$

$$(x+1)^2 = 1$$

$$x + 1 = \pm 1$$

$$x = -2, 0$$

$$y = -4, 0$$

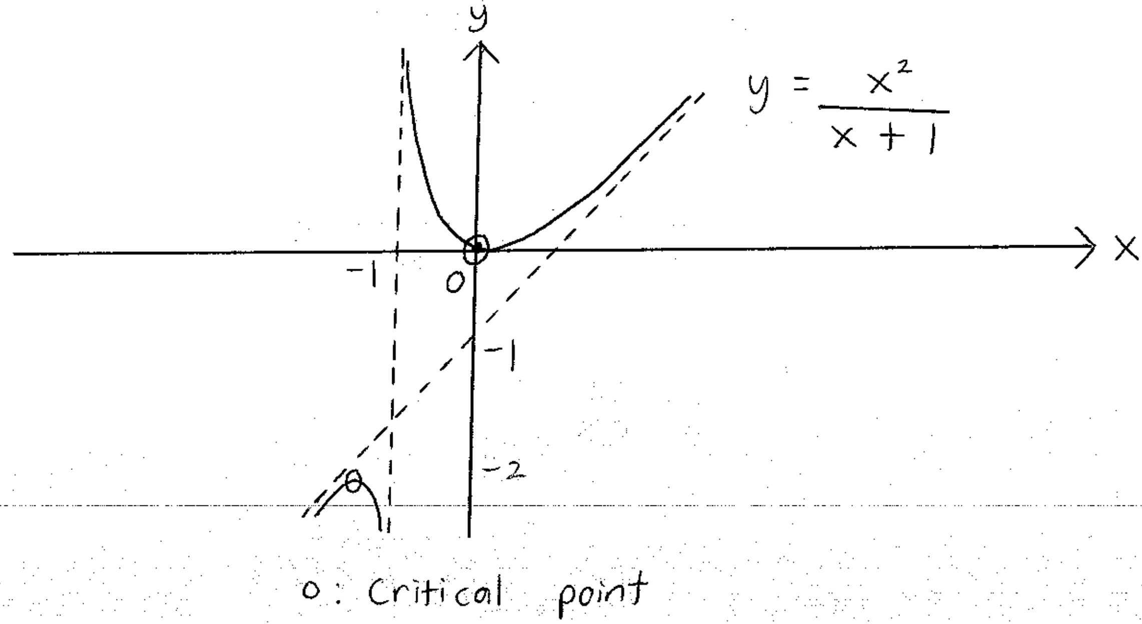
$$\frac{d^2y}{dx^2} = \frac{2}{(x+1)^3}$$

when $x = -2$: $\frac{d^2y}{dx^2} = -2 < 0$

when $x = 0$: $\frac{d^2y}{dx^2} = 2 > 0$

$(-2, -4)$ is a maximum point and

$(0, 0)$ is a minimum point



o : Critical point

• : Intersection point.

Q :

$$y = \frac{2x^2 + 3x + 1}{x - 1}$$

i)

$$\begin{array}{r l}
 & 2x + 5 \\
x - 1 \,\big) & 2x^2 + 3x + 1 \\
 & \underline{2x^2 - 2x} \\
 & \quad\quad 5x + 1 \\
 & \quad\quad \underline{5x - 5} \\
 & \quad\quad\quad\quad 6
\end{array}$$

$$y = 2x + 5 + \frac{6}{x - 1}$$

As $x \to \pm\infty$ $y \to 2x + 5$

As $x \to 1$ $y \to \pm\infty$

$\therefore$ The asymptotes of y are $y = 2x + 5$ and $x = 1$

ii) $$\frac{dy}{dx} = 2 - \frac{6}{(x-1)^2}$$

when $\frac{dy}{dx} = 0$

$$2 - \frac{6}{(x-1)^2} = 0$$

$$\frac{6}{(x-1)^2} = 2$$

$$(x-1)^2 = 3$$

$$x - 1 = \pm\sqrt{3}$$

$$x = 1 \pm \sqrt{3}$$

$$y = 7 \pm 4\sqrt{3}$$

$$\frac{d^2y}{dx^2} = \frac{12}{(x-1)^3}$$

when $x = 1 + \sqrt{3}$: $\dfrac{d^2y}{dx^2} = \dfrac{4}{\sqrt{3}} > 0$

when $x = 1 - \sqrt{3}$: $\dfrac{d^2y}{dx^2} = \dfrac{-4}{\sqrt{3}} < 0$

$\therefore$ $(1 + \sqrt{3}, 7 + 4\sqrt{3})$ is a minimum point

and $(1 - \sqrt{3}, 7 - 4\sqrt{3})$ is a maximum point

iii) Intersection points:

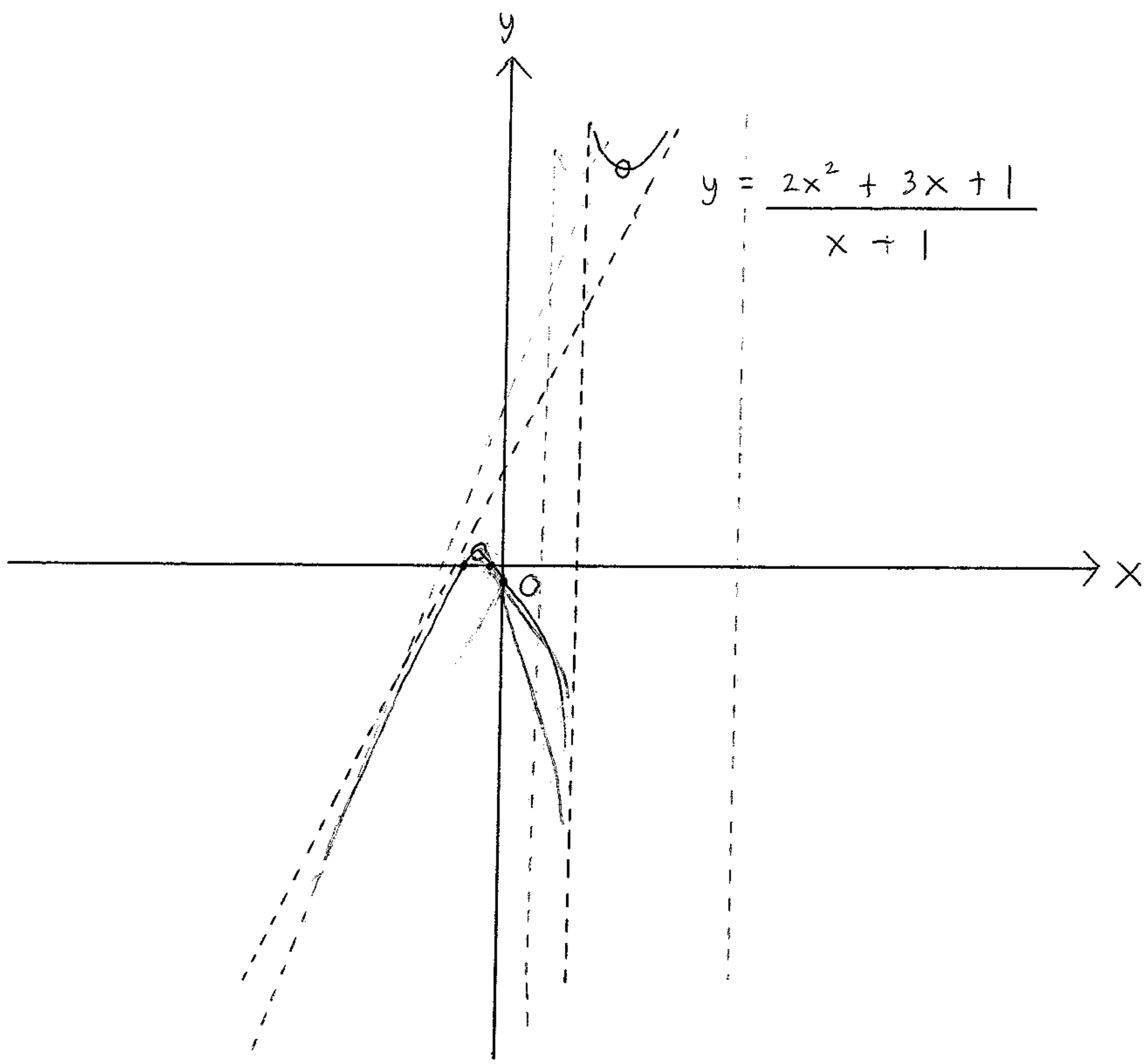
when $x = 0$: $y = -1$

when $y = 0$: $\dfrac{2x^2 + 3x + 1}{x - 1} = 0$

$$2x^2 + 3x + 1 = 0$$

$$(2x+1)(x+1) = 0$$

$$x = \frac{-1}{2}, -1$$



o : Critical point

• : Intersection point.

22

C :

$$y = 2x + 1 - \frac{5}{2x+1}$$

i) As $x \to \pm \infty$ $y \to 2x + 1$

As $x \to \frac{-1}{2}$ $y \to \pm \infty$

$\therefore$ The asymptotes of y are $y = 2x + 1$

and $x = \frac{-1}{2}$

ii) $$\frac{dy}{dx} = 2 + \frac{5}{(2x+1)^2}$$

since $$\frac{5}{(2x+1)^2} > 0$$

$$2 + \frac{5}{(2x+1)^2} > 2 > 0$$

$$\therefore \frac{dy}{dx} > 0.$$

no critical points.

iii) Intersection points

when $x = 0$: $y = -4$

when $y = 0$: $2x + 1 - \frac{5}{2x+1} = 0$

$$\frac{(2x+1)^2-5}{2x+1}=0$$

$$(2x+1)^2-5=0$$

$$4x^2+4x+1-5=0$$

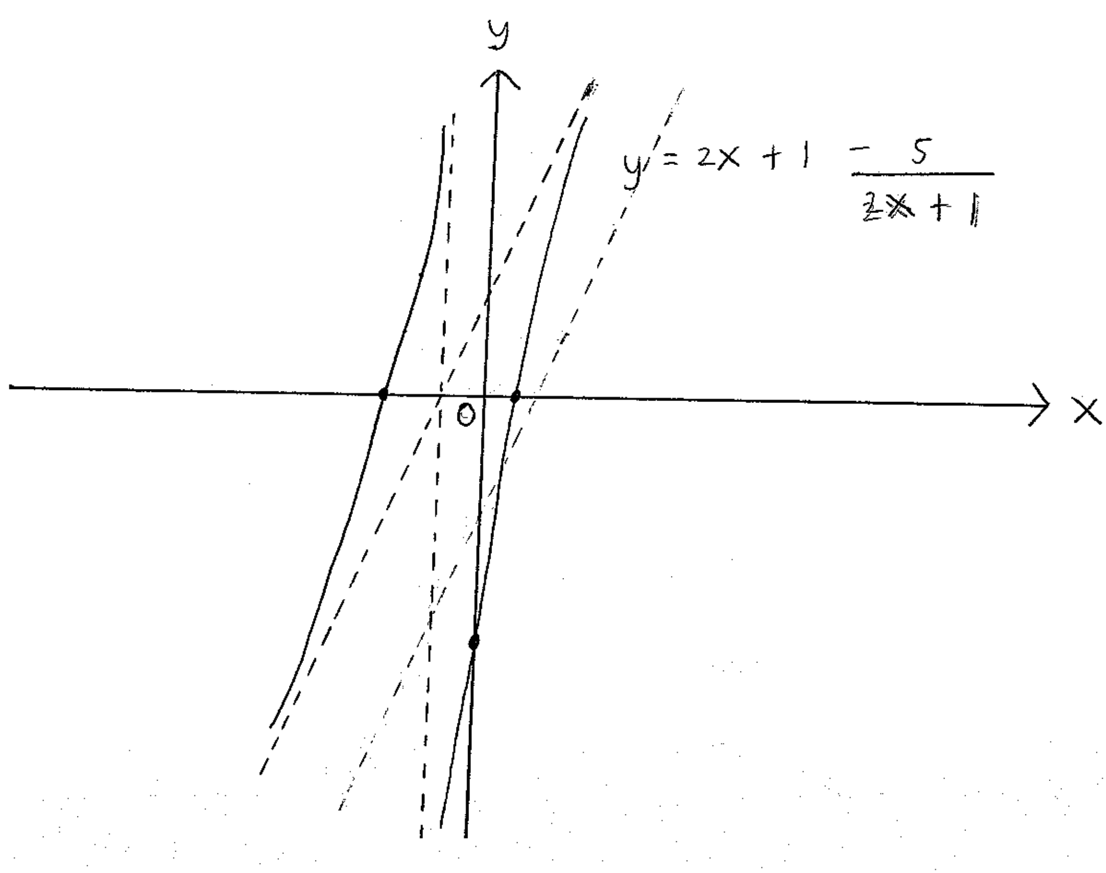
$$4x^2+4x-4=0$$

$$x^2+x-1=0$$

$$\left(x+\frac{1}{2}\right)^2=\frac{5}{4}$$

$$x+\frac{1}{2}=\frac{\pm\sqrt{5}}{2}$$

$$x=\frac{-1\pm\sqrt{5}}{2}$$



• : Intersection point

24.

$$y = \frac{4 - ax^2}{b + x}$$

$$\begin{array}{r|l} & -ax + ab \\ \hline x + b & -ax^2 + 0x + 4 \\ & -ax^2 - abx \\ \hline & \quad abx + 4 \\ & \quad abx + ab^2 \\ \hline & \qquad 4 - ab^2 \end{array}$$

$$= ab - ax + \frac{4 - ab^2}{x + b}$$

As $x \longrightarrow \pm\infty$ $y \longrightarrow ab - ax$

As $x \longrightarrow -b$ $y \longrightarrow \pm\infty$

$\therefore$ The asymptotes are $y = ab - ax$ and $x = -b$. Since $y = 1 - x$ and $x = -1$ are the asymptotes, $a = 1$ $b = 1$

$$y = \frac{4 - x^2}{1 + x}$$

Intersection points:

$x = 0$: $y = 4$

$y = 0$: $\dfrac{4 - x^2}{1 + x} = 0$

$$4 - x^2 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

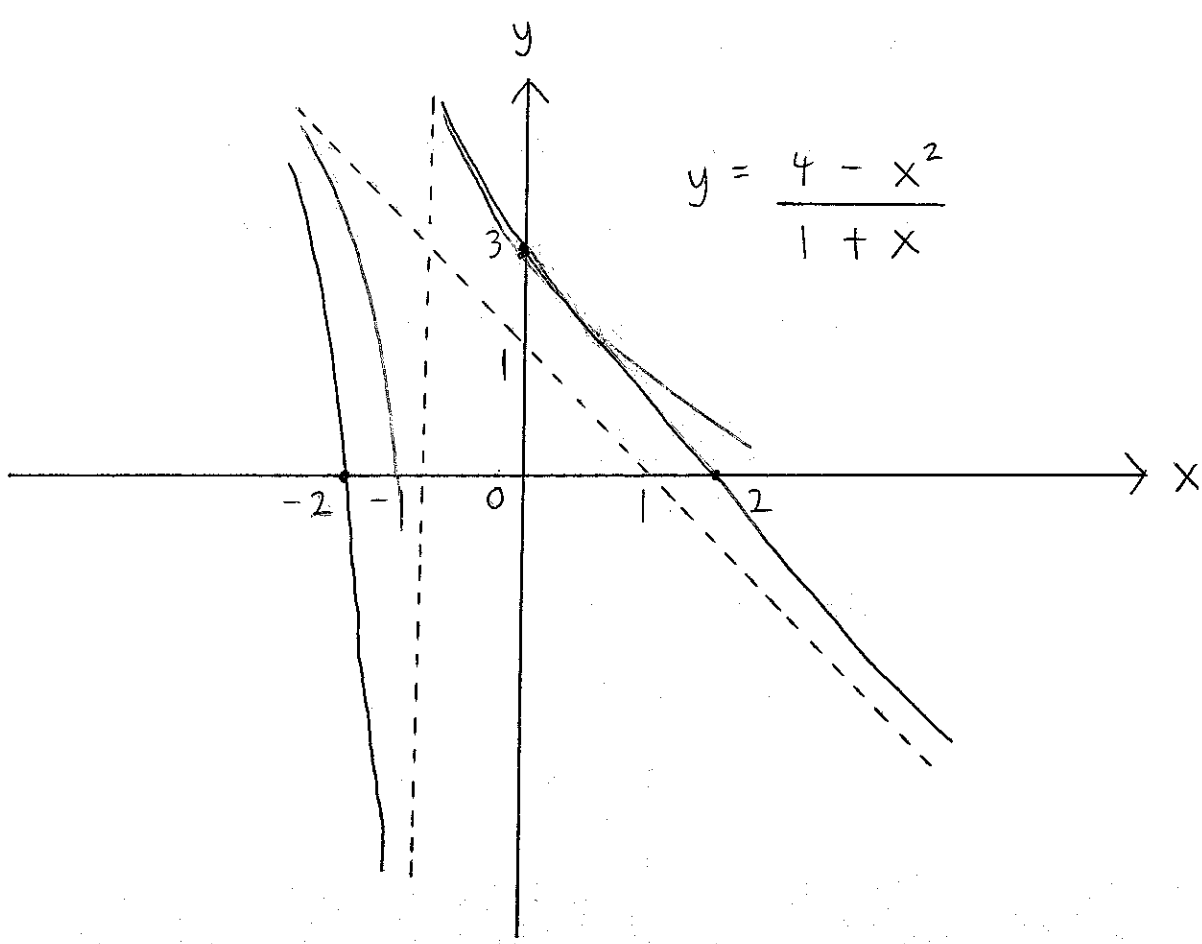
Critical points:

$$y = 1 - x + \frac{3}{x+1}$$

$$\frac{dy}{dx} = -1 - \frac{3}{(x+1)^2}$$

$$= -\left(1 + \frac{3}{(x+1)^2}\right)$$

$$< 0, \text{ since } 1 + \frac{3}{(x+1)^2} > 0$$



•: Intersection point

25. $C: \quad y = \dfrac{(x-1)^2}{x+1}$

i)

$$\begin{array}{r|l} & x - 3 \\ \hline x+1 & x^2 - 2x + 1 \\ & \underline{x^2 + x} \\ & \quad -3x + 1 \\ & \quad \underline{-3x - 3} \\ & \qquad\quad 4 \end{array}$$

$$= x - 3 + \frac{4}{x+1}$$

As $x \to \pm\infty \quad y \to x - 3$

As $x \to -1 \quad y \to \pm\infty$

The asymptotes of C are $y = x - 3$ and $x = -1$.

ii) $\dfrac{dy}{dx} = 1 - \dfrac{4}{(x+1)^2}$

when $\dfrac{dy}{dx} = 0$:

$$1 - \frac{4}{(x+1)^2} = 0$$

$$\frac{4}{(x+1)^2} = 1$$

$$(x+1)^2 = 4$$

$$x + 1 = \pm 2$$

$$x = -3, 1$$

$$y = -8, 0$$

$$\frac{d^2y}{dx^2} = \frac{8}{(x+1)^3}$$

When $x = -3$: $\dfrac{d^2y}{dx^2} = -1 < 0$

When $x = 1$: $\dfrac{d^2y}{dx^2} = 1 > 0$

$\therefore$ $(-3, 8)$ is a maximum point and

$(1, 0)$ is a minimum point.

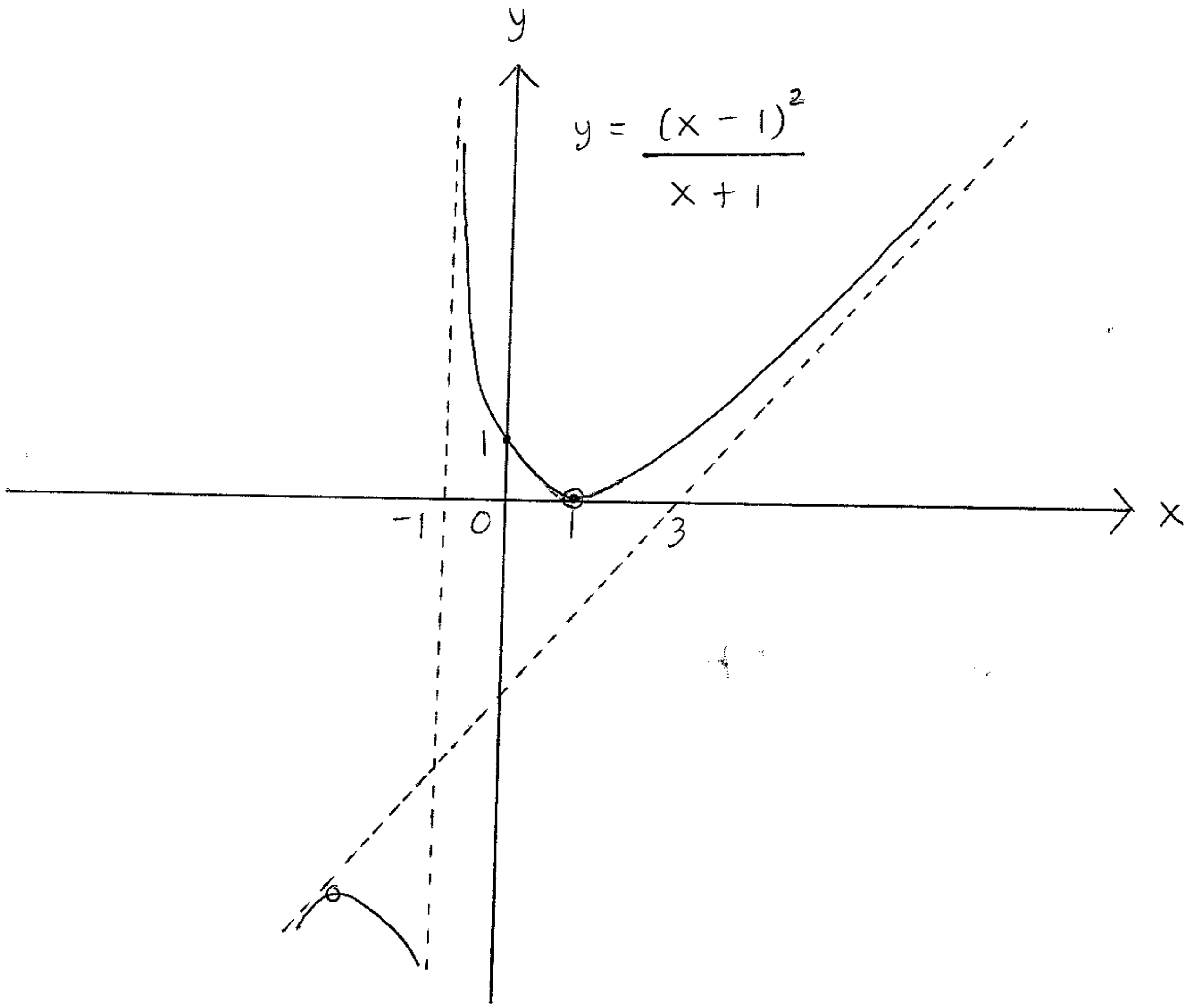
iii) Intersection points:

when $x = 0$: $y = 1$

when $y = 0$: $\dfrac{(x-1)^2}{x+1} = 0$

$$(x-1)^2 = 0$$

$$x = 1$$



o : Critical points

• : Intersection points

27. $C: \quad y = \dfrac{x^2 + 3}{x + 1}$

i)

$$\begin{array}{r|l} & x \quad -1 \\ \hline x+1 & x^2 + 0x + 3 \\ & \underline{x^2 + x} \\ & \quad -x + 3 \\ & \quad \underline{-x - 1} \\ & \qquad\quad 4 \end{array}$$

$$y = x - 1 + \frac{4}{x + 1}$$

As $x \to \pm\infty \quad y \to x - 1$

As $x \to -1 \quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = x - 1$ and $x = -1$

ii) $\dfrac{dy}{dx} = 1 - \dfrac{4}{(x+1)^2}$

when $\dfrac{dy}{dx} = 0$

$$1 - \frac{4}{(x+1)^2} = 0$$

$$\frac{4}{(x+1)^2} = 1$$

$(x+1)^2 = 4$

$x + 1 = \pm 2$

$x = -3, 1$

$y = -6, 2$

$$\frac{d^2y}{dx^2} = \frac{8}{(x+1)^3}$$

when $x = -3$: $\frac{d^2y}{dx^2} = -1 < 0$

when $x = 1$: $\frac{d^2y}{dx^2} = \frac{18}{25} > 0$

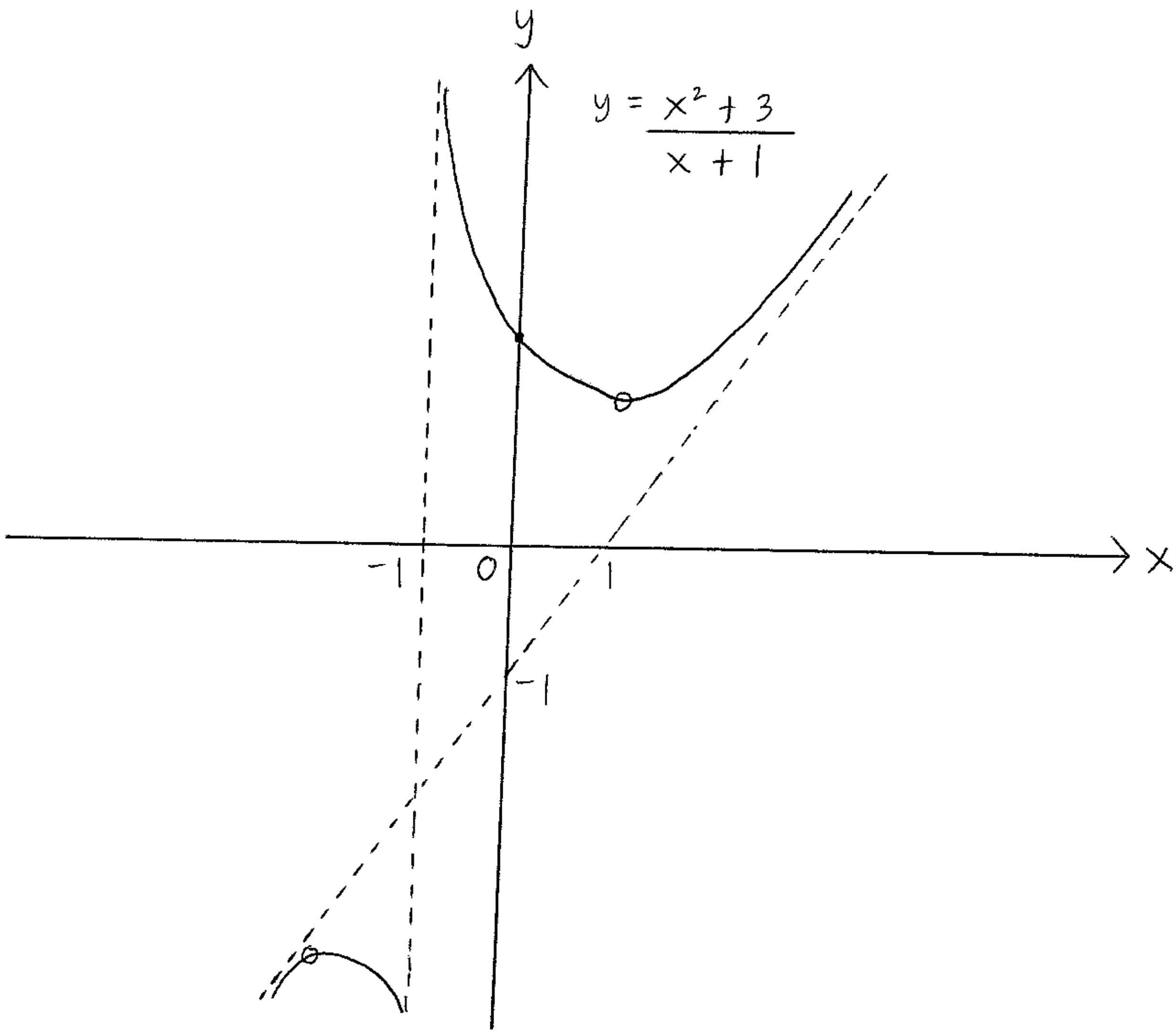
$(-3, -6)$ is a maximum point and

$(1, 2)$ is a minimum point.

iii) Intersection points

when $x = 0$: $y = 3$.

when $y =$



o: Critical points

•: Intersection points