MAY / JUNE 2012

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1\. a) $x^3 - 7x^2 + 2x - 3 = 0$

α, β, r are the roots

$\alpha + \beta + r = 7 \quad \alpha\beta + \alpha r + \beta r = 2 \quad \alpha\beta r = 3$

i) $\alpha^2 + \beta^2 + r^2 = (\alpha + \beta + r)^2 - 2(\alpha\beta + \alpha r + \beta r)$

$= 7^2 - 2(2)$

$= 49 - 4$

$= 45$

ii) $(\alpha + \beta + r)^3 = (\alpha + \beta)^3 + 3(\alpha + \beta)^2 r + 3(\alpha + \beta) r^2 + r^3$

$= \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 + 3\alpha^2 r + 6\alpha\beta r + 3\beta^2 r + 3\alpha r^2 + 3\beta r^2 + r^3$

$= \alpha^3 + \beta^3 + r^3 + 6\alpha\beta r + 3(\alpha^2\beta + \alpha\beta^2 + \alpha^2 r + \beta^2 r + \alpha r^2 + \beta r^2)$

$= \alpha^3 + \beta^3 + r^3 + 6\alpha\beta r + 3[(\alpha + \beta + r)(\alpha\beta + \alpha r + \beta r) - 3\alpha\beta r]$

$= \alpha^3 + \beta^3 + r^3 - 3\alpha\beta r + 3(\alpha + \beta + r)(\alpha\beta + \alpha r + \beta r)$

$\therefore \alpha^3 + \beta^3 + r^3 = (\alpha + \beta + r)^3 + 3\alpha\beta r - 3(\alpha + \beta + r)(\alpha\beta + \alpha r + \beta r)$

$= 7^3 + 3(3) - 3(7)2$

$= 343 + 9 - 42$

$= 310$

2. $4^n > 3^n + 2^n$, $n \geq 2$

When $n=2$: $4^2 = 16 > 13 = 9 + 4 = 3^2 + 2^2$

Assume the statement is true when $n=k$.

$n=k$: $4^k > 3^k + 2^k$.

when $n=k+1$: $4^{k+1} > 3^{k+1} + 2^{k+1}$

(what needs to be proved)

$4^k > 3^k + 2^k$

$4^k 4 > 3^k 4 + 2^k 4$

$4^{k+1} > 3^k 4 + 2^k 4$

$3^k 4 + 2^k 4 > 3^k 3 + 2^k 2 = 3^{k+1} + 2^{k+1}$

Since $4^{k+1} > 3^k 4 + 2^k 4$ and $3^k 4 + 2^k 4 > 3^{k+1} + 2^{k+1}$,

$4^{k+1} > 3^{k+1} + 2^{k+1}$.

$\therefore 4^n > 3^n + 2^n$ for every positive integer $n \geq 2$.

3. $$f(r) = \frac{1}{(r+1)(r+2)}$$

$$f(r-1) - f(r) = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

$$= \frac{1}{r+1}\left(\frac{1}{r} - \frac{1}{r+2}\right)$$

$$= \frac{1}{r+1}\left(\frac{r+2-r}{r(r+2)}\right)$$

$$= \frac{2}{r(r+1)(r+2)}$$

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \sum_{r=1}^{n} f(r-1) - f(r)$$

$$2\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = f(n-1) - f(n)$$
$$+ f(n-2) - f(n-1)$$
$$+ f(n-3) - f(n-2)$$
$$\vdots$$
$$+ f(2) - f(3)$$
$$+ f(1) - f(2)$$
$$+ f(0) - f(1)$$
$$= f(0) - f(n)$$
$$= \frac{1}{2} - \frac{1}{(n+1)(n+2)}$$

$$\therefore \sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

$$\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+2)} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$$

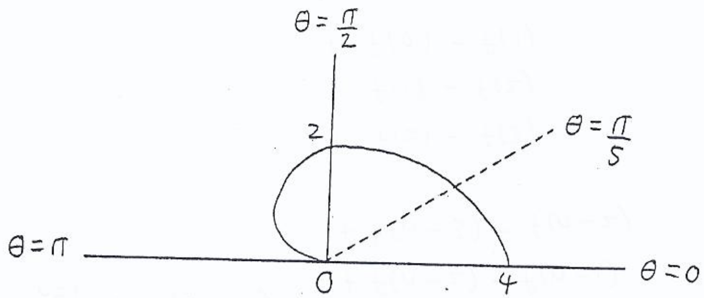
$$= \lim_{n\to\infty} \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

$$= \frac{1}{4} - 0$$

$$= \frac{1}{4}$$

4. $C: r = 2 + 2\cos\theta,\ 0 \le \theta \le \pi.$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	$2+\sqrt{3}$	$2+\sqrt{2}$	3	2	1	$2-\sqrt{2}$	$2-\sqrt{3}$	0



The area enclosed by C and the initial line is $\int_0^{\pi} \frac{r^2}{2}\, d\theta$

$$= \int_0^{\pi} \frac{(2+2\cos\theta)^2}{2}\, d\theta$$

$$= 2\int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta \ d\theta$$

$$= \int_0^{\pi} 2 + 4\cos\theta + 1 + \cos 2\theta \ d\theta$$

$$= \int_0^{\pi} 3 + 4\cos\theta + \cos 2\theta \ d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= 3\pi$$

If $\theta = \frac{\pi}{5}$ divides R into two parts,

$$\int_0^{\frac{\pi}{5}} \frac{r^2}{2}\, d\theta = \int_0^{\frac{\pi}{5}} \frac{(2+2\cos\theta)^2}{2}\, d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{5}}$$

$$= \frac{3\pi}{5} + 4\sin\frac{\pi}{5} + \frac{\sin\frac{2\pi}{5}}{2} - 0$$

$$= 4.712$$

$$\int_{\frac{\pi}{5}}^{\pi} \frac{r^2}{2}\, d\theta = \int_{\frac{\pi}{5}}^{\pi} \frac{(2+2\cos\theta)^2}{2}\, d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{5}}^{\pi}$$

$$= 3\pi - \left(\frac{3\pi}{5} + 4\sin\frac{\pi}{5} + \frac{\sin\frac{2\pi}{5}}{2}\right)$$

$$= 4.713$$

$\therefore$ The area of each part is 4.712 and 4.713

5. If A has eigenvalues $-1, 1$, and 2 with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$,

let $P = \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -3 \\ -2 & 3 & 5 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\therefore A = PDP^{-1}$

$$= \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -3 \\ -2 & 3 & 5 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -3 \\ -2 & 3 & 5 \end{pmatrix}^{-1}$$

$$P^{-1} = \frac{1}{|P|}\left(\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \quad \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \right)^T$$

$$|P| = 0(-5+9) - (-1)(5-6) + 2(3-2)$$
$$= 0 - 1 + 2$$
$$= 1$$

$$\therefore P^{-1} = \begin{pmatrix} 4 & 1 & 1 \\ 11 & 4 & 2 \\ 5 & 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} 0 & -1 & 2 \\ -1 & -3 & -3 \\ -2 & 3 & 5 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 4 \\ -1 & -1 & -6 \\ 2 & 3 & 10 \end{pmatrix} \begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & 2 \\ -11 & -27 & -13 \\ 21 & 54 & 26 \end{pmatrix}$$

6. $$1^{\frac{1}{5}} = (\cos 0 + i\sin 0)^{\frac{1}{5}}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{5}}, \quad k \in \mathbb{Z}$$

$$= \cos \frac{2k\pi}{5} + i\sin \frac{2k\pi}{5}, \quad k = 0,1,2,3,4$$

$$= \cos\theta + i\sin\theta, \quad \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}.$$

$$(z+1)^5 = z^5$$

$$\frac{(z+1)^5}{z^5} = 1$$

$$\left(\frac{z+1}{z}\right)^5 = 1$$

$$\frac{z+1}{z} = 1^{\frac{1}{5}}$$

$$1 + \frac{1}{z} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}, \quad k = 0,1,2,3,4$$

$$= 1 - 2\sin^2\frac{k\pi}{5} + 2i\sin\frac{k\pi}{5}\cos\frac{k\pi}{5}$$

$$\frac{1}{z} = 2i\sin\frac{k\pi}{5}\cos\frac{k\pi}{5} - 2\sin^2\frac{k\pi}{5}$$

$$= -2\sin\frac{k\pi}{5}\left(\sin\frac{k\pi}{5} - i\cos\frac{k\pi}{5}\right)$$

$$z = \frac{1}{-2\sin\frac{k\pi}{5}\left(\sin\frac{k\pi}{5} - i\cos\frac{k\pi}{5}\right)}$$

$$= \frac{\sin\frac{k\pi}{5} + i\cos\frac{k\pi}{5}}{-2\sin\frac{k\pi}{5}\left(\sin\frac{k\pi}{5} - i\cos\frac{k\pi}{5}\right)\left(\sin\frac{k\pi}{5} + i\cos\frac{k\pi}{5}\right)}$$

$$= \frac{\sin\frac{k\pi}{5} + i\cos\frac{k\pi}{5}}{-2\sin\frac{k\pi}{5}}$$

$$= -\frac{1}{2}\left(1 + i\cot\frac{k\pi}{5}\right), \quad k = 1,2,3,4.$$

$$(z+1)^5 = z^5$$

$$z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1 = z^5$$

$$5z^4 + 10z^3 + 10z^2 + 5z + 1 = 0$$

Since $5z^4 + 10z^3 + 10z^2 + 5z + 1 = 0$ is a quartic equation, $z = -\frac{1}{2}\left(1 + i\cot\frac{k\pi}{5}\right)$, $k = 1,2,3,4$.

7. $T_1 : R^4 \to R^4 \quad T_2 : R^4 \to R^4$

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 1 & 4 & 11 \\ 3 & 4 & 1 & 9 \\ 4 & -3 & 18 & 37 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 3 & 0 & 1 \\ 3 & 4 & 1 & 0 \\ 4 & 5 & 2 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 0 \\ 2 & 1 & 4 & 11 & 0 \\ 3 & 4 & 1 & 9 & 0 \\ 4 & -3 & 18 & 37 & 0 \end{array}\right)$$

$$\xrightarrow[-4r_1+r_4]{\substack{-2r_1+r_2 \\ -3r_1+r_3}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 0 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & -7 & 14 & 21 & 0 \end{array}\right)$$

$$\xrightarrow[-7r_2+r_3]{r_2+r_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 0 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

If $\begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 1 & 4 & 11 \\ 3 & 4 & 1 & 9 \\ 4 & -3 & 18 & 37 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Let $w = s,\ s \in R$ and $z = t,\ t \in R$

$y = 2t + 3s$

$x + 2t + 3s + t + 4s = 0$

$x = -3t - 7s$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3t - 7s \\ 2t + 3s \\ t \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

If $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 3 & 0 & 1 \\ 3 & 4 & 1 & 0 \\ 4 & 5 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 4 & 5 & 2 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-4r_1+r_4]{\substack{-2r_1+r_2 \\ -3r_1+r_3}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 1 & -2 & 4 & 0 \end{array}\right)$$

$$\xrightarrow[-r_2+r_4]{-r_2+r_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_3 \leftrightarrow r_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$w = 0$

Let $z = t,\ t \in R$

$y = 2t$

$x = -2t - t = -3t$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3t \\ 2t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ The dimension of K_1 is 2 and the dimension of K_2 is 1.

A basis of K_1 is $\left\{ \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$.

If $a \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} = b \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$,

$b = 0$ and $c = a$.

$\therefore K_2 \subset K_1$.

8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10e^{-2x}$, $y = 5$ $\frac{dy}{dx} = 1$ $x = 0$.

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

$m^2 + 2m + 5 = 0$

$(m+1)^2 + 4 = 0$

$(m+1)^2 = -4$

$m + 1 = \pm 2i$

$m = -1 \pm 2i$

$\therefore$ The complementary function, y_c, is

$y_c = e^{-x}(A\cos 2x + B\sin 2x)$.

The particular integral, y_p, is given by

$y_p = Ce^{-2x}$

$\frac{dy_p}{dx} = -2Ce^{-2x}$

$\frac{d^2y_p}{dx^2} = 4Ce^{-2x}$

$\therefore \frac{d^2y_p}{dx^2} + 2\frac{dy_p}{dx} + 5y_p = 4Ce^{-2x} - 4Ce^{-2x} + 5Ce^{-2x}$

$= 5Ce^{-2x}$

$= 10e^{-2x}$

$5C = 10$

$C = 2$.

$\therefore y_p = 2e^{-2x}$

$y = y_c + y_p$

$= e^{-x}(A\cos 2x + B\sin 2x) + 2e^{-2x}$

$\frac{dy}{dx} = -e^{-x}(A\cos 2x + B\sin 2x)$

$+ e^{-x}(-2A\sin 2x + 2B\cos 2x) - 4e^{-2x}$

$x = 0$ $y = 5$: $5 = A + 2$

$A = 3$

$x = 0$ $\frac{dy}{dx} = 1$: $1 = -A + 2B - 4$

$B = 4$

$y = e^{-x}(3\cos 2x + 4\sin 2x) + 2e^{-2x}$.

9. $C: \quad y = \dfrac{2x^2 + 2x + 3}{x^2+2}$

$$(x^2+2)y = 2x^2 + 2x + 3$$
$$x^2y + 2y = 2x^2 + 2x + 3$$
$$(y-2)x^2 - 2x + 2y - 3 = 0$$
$$a = y-2 \quad b = -2 \quad c = 2y-3$$
$$\begin{aligned} b^2 - 4ac &= 4 - 4(y-2)(2y-3) \\ &= 4 - 4(2y^2 - 7y + 6) \\ &= 4 - 8y^2 + 28y - 24 \\ &= -8y^2 + 28y - 20 \\ &= -4(2y^2 - 7y + 5) \\ &= -4(y-1)(2y-5) \end{aligned}$$

If $b^2 - 4ac \geq 0$

$$-4(y-1)(2y-5) \geq 0$$
$$(y-1)(2y-5) \leq 0$$
$$\therefore 1 \leq y \leq \frac{5}{2}.$$

$$\begin{array}{r|l} & 2 \\ \hline x^2+2 & 2x^2 + 2x + 3 \\ & 2x^2 + 4 \\ \hline & 2x - 1 \end{array}$$

$$y = 2 + \frac{2x-1}{x^2+2}$$

$$\frac{dy}{dx} = \frac{2}{x^2+2} - \frac{2x(2x-1)}{(x^2+2)^2}$$

when $\dfrac{dy}{dx} = 0$: $\dfrac{2}{x^2+2} - \dfrac{2x(2x-1)}{(x^2+2)^2} = 0$

$$2(x^2+2) = 2x(2x-1)$$
$$x^2 + 2 = 2x^2 - x$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, -1$$
$$y = \frac{5}{2}, 1$$

$\therefore$ The critical points of C are $(2, \frac{5}{2})$ and $(-1, 1)$.

$$\frac{d^2y}{dx^2} = \frac{-4x}{(x^2+2)^2} - \frac{2(2x-1)}{(x^2+2)^2} - \frac{4x(2x-1)}{(x^2+2)^2} + \frac{8x^2(2x-1)}{(x^2+2)^3}.$$

when $x = 2$: $\dfrac{d^2y}{dx^2} < 0$

when $x = -1$: $\dfrac{d^2y}{dx^2} > 0$.

$\therefore (2, \frac{5}{2})$ is a maximum point and $(-1, 1)$ is a minimum point.

As $x \to \pm\infty$ $y \to 2$.

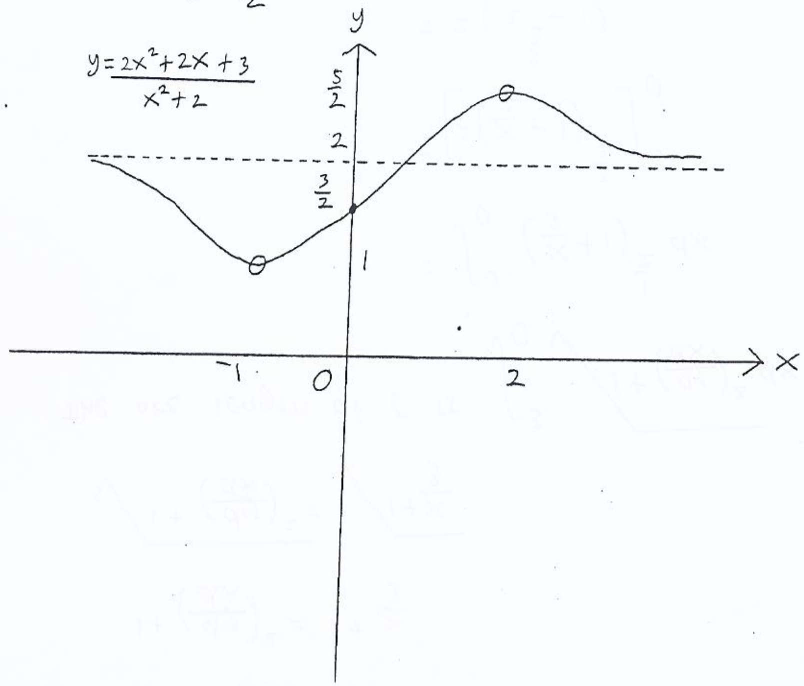
$\therefore y = 2$ is the asymptote of C.

when $x = 0$: $y = \frac{3}{2}$ $y \neq 0$.

when $y = 2$:

$2 = 2 + \frac{2x-1}{x^2+2}$

$\therefore \frac{2x-1}{x^2+2} = 0$ $\therefore x = \frac{1}{2}$



o: Critical point

•: Intersection point.

10. C: $y = 2\left(\frac{x}{3}\right)^{\frac{3}{2}}$, $0 \le x \le 3$.

$$\frac{dy}{dx} = 3\left(\frac{x}{3}\right)^{\frac{1}{2}} \frac{1}{3}$$

$$= \left(\frac{x}{3}\right)^{\frac{1}{2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x}{3}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x}{3}}$$

The arc length of C is $\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^3 \left(\frac{x}{3} + 1\right)^{\frac{1}{2}} dx$$

$$= \left[2\left(\frac{x}{3} + 1\right)^{\frac{3}{2}}\right]_0^3$$

$$= 2\left(2^{\frac{3}{2}} - 1\right)$$

$$= 2(2\sqrt{2} - 1).$$

The area of the region enclosed by C, the x-axis and the line $x = 3$, A, is

$$\int_0^3 y\, dx = \int_0^3 2\left(\frac{x}{3}\right)^{\frac{3}{2}} dx$$

$$= \left[\frac{12}{5}\left(\frac{x}{3}\right)^{\frac{5}{2}}\right]_0^3$$

$$= \frac{12}{5}.$$

If $(\bar{x}, \bar{y})$ are the coordinates of the centroid of the region enclosed by C, the x-axis and the line $x = 3$,

$$\bar{x} = \frac{\int_0^3 xy\, dx}{A}$$

$$= \frac{\int_0^3 x\left(2\left(\frac{x}{3}\right)^{\frac{3}{2}}\right) dx}{A}$$

$$= \frac{5}{12}\int_0^3 \frac{2}{3\sqrt{3}} x^{\frac{5}{2}}\, dx$$

$$= \frac{5}{18\sqrt{3}}\left[\frac{2x^{\frac{7}{2}}}{7}\right]_0^3$$

$$= \frac{5}{18\sqrt{3}} \, \frac{2}{7} \left(3^{\frac{7}{2}} - 0\right)$$

$$= \frac{5}{18\sqrt{3}} \, \frac{2}{7} \left(27\sqrt{3}\right)$$

$$= \frac{15}{7}$$

$$\bar{y} = \frac{\int_0^3 \frac{y^2}{2} \, dx}{A}$$

$$= \frac{\frac{1}{2}\int_0^3 \frac{4x^3}{27} \, dx}{A}$$

$$= \frac{5}{12} \int_0^3 \frac{2x^3}{27} \, dx$$

$$= \frac{5}{12} \left[\frac{2x^4}{27(4)}\right]_0^3$$

$$= \frac{5}{12} \left[\frac{2(81-0)}{27(4)}\right]$$

$$= \frac{5}{12} \left(\frac{3}{2}\right)$$

$$= \frac{5}{8}$$

12. EITHER

$$\int_0^{\pi} e^x \sin x \, dx$$

$u = e^x$ $\qquad dv = \sin x \, dx$

$du = e^x dx$ $\qquad v = -\cos x$

$$\int_0^{\pi} e^x \sin x \, dx = \left[-e^x \cos x\right]_0^{\pi} - \int_0^{\pi} -e^x \cos x \, dx$$

$$= e^{\pi} + 1 + \int_0^{\pi} e^x \cos x \, dx$$

$u = e^x$ $\qquad dv = \cos x \, dx$

$du = e^x dx$ $\qquad v = \sin x$

$$= e^{\pi} + 1 + \left[e^x \sin x\right]_0^{\pi} - \int_0^{\pi} e^x \sin x \, dx$$

$$= e^{\pi} + 1 - \int_0^{\pi} e^x \sin x \, dx$$

$$2\int_0^{\pi} e^x \sin x \, dx = e^{\pi} + 1$$

$$\int_0^{\pi} e^x \sin x \, dx = \frac{e^{\pi} + 1}{2}.$$

$$I_n = \int_0^{\pi} e^x \sin^n x \, dx$$

$u = \sin^n x$ $\qquad dv = e^x dx$

$du = n \sin^{n-1} x \cos x \, dx$ $\qquad v = e^x$

$$= \left[e^x \sin^n x\right]_0^{\pi} - \int_0^{\pi} n e^x \sin^{n-1} x \cos x \, dx$$

$$= 0 - n \int_0^{\pi} e^x \sin^{n-1} x \cos x \, dx$$

$u = \sin^{n-1} x \cos x$ $\qquad dv = e^x dx$

$du = \left[(n-1) \sin^{n-2} x \cos^2 x - \sin^{n-1} x \sin x\right] dx$ $\qquad v = e^x$

$$= -n \left(\left[e^x \sin^{n-1} x \cos x\right]_0^{\pi} - \int_0^{\pi} e^x (n-1) \sin^{n-2} x \cos^2 x - \sin^n x \, dx \right)$$

$$= -n \left(0 - \int_0^{\pi} e^x (n-1) \sin^{n-2} x \cos^2 x \, dx + \int_0^{\pi} \sin^n x \, dx\right)$$

$$= n(n-1) \int_0^{\pi} e^x \sin^{n-2} x \cos^2 x \, dx - n \int_0^{\pi} \sin^n x \, dx$$

$$= n(n-1) \int_0^{\pi} e^x \sin^{n-2} x \cos^2 x \, dx - n I_n$$

$$= n(n-1) \int_0^{\pi} e^x \sin^{n-2} x (1 - \sin^2 x) \, dx - n I_n$$

$$= n(n-1) \int_0^{\pi} e^x \sin^{n-2} x - e^x \sin^n x \, dx - n I_n$$

$$= n(n-1)\int_0^{\pi} e^x \sin^{n-2} x \, dx - n(n-1)\int_0^{\pi} e^x \sin^n x \, dx - nI_n$$

$$= n(n-1) I_{n-2} - n(n-1) I_n - n I_n$$

$$= n(n-1) I_{n-2} + (n - n^2 - n) I_n$$

$$= n(n-1) I_{n-2} - n^2 I_n$$

$$(n^2+1) I_n = n(n-1) I_{n-2}$$

$$y = e^x \sin^5 x$$

The mean value of y over $0 \leq x \leq \pi$ is

$$\frac{1}{\pi - 0}\int_0^{\pi} y \, dx = \frac{1}{\pi}\int_0^{\pi} e^x \sin^5 x \, dx$$

$$= \frac{1}{\pi} I_5$$

$$= \frac{1}{\pi}\left(\frac{5 \cdot 4}{26} I_3\right)$$

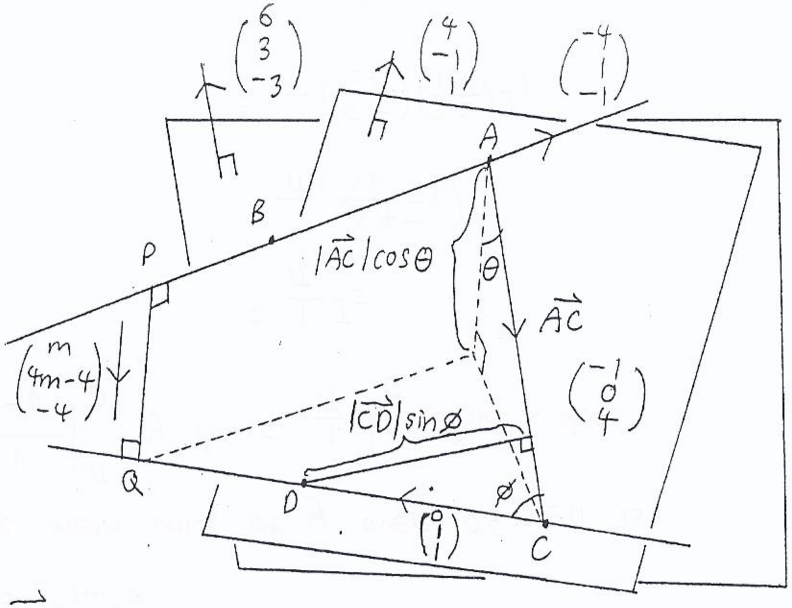
$$= \frac{1}{\pi}\left(\frac{5 \cdot 4}{26}\right)\left(\frac{3 \cdot 2}{10}\right) I_1$$

$$= \frac{1}{\pi} \frac{6}{13} I_1$$

$$= \frac{1}{\pi} \frac{6}{13}\left(\frac{e^{\pi}+1}{2}\right) = \frac{3(e^{\pi}+1)}{13\pi}$$

OR

$\vec{OA} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$ $\vec{OB} = \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix}$ $\vec{OC} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ $\vec{OD} = \begin{pmatrix} 1 \\ 5 \\ m \end{pmatrix}$



$\vec{AB} = \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$

$\underset{\sim}{r} = \vec{OA} + s\vec{AB}$

$= \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + s\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$

$\vec{CD} = \begin{pmatrix} 1 \\ 5 \\ m \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ m-1 \end{pmatrix}$

$\underset{\sim}{r} = \vec{OC} + t\vec{CD}$

$= \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ m-1 \end{pmatrix}$

$\therefore$ The line through A and B has equation $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + s\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$ and the line through C and D has equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ m-1 \end{pmatrix}$.

The common perpendicular to both lines, PQ, has direction $\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ m-1 \end{pmatrix} = \begin{pmatrix} m \\ 4m-4 \\ -4 \end{pmatrix}$.

$\therefore \vec{PQ} = k\begin{pmatrix} m \\ 4m-4 \\ -4 \end{pmatrix}, \quad k \neq 0.$

$\vec{AC} \cdot \vec{PQ} = |\vec{AC}||\vec{PQ}|\cos\theta$

$\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \cdot k\begin{pmatrix} m \\ 4m-4 \\ -4 \end{pmatrix} = |\vec{AC}|\left|k\begin{pmatrix} m \\ 4m-4 \\ -4 \end{pmatrix}\right|\cos\theta$

$= k|\vec{AC}|\left|\begin{pmatrix} m \\ 4m-4 \\ -4 \end{pmatrix}\right|\cos\theta$

$\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} m \\ 4m-4 \\ -4 \end{pmatrix} = |\vec{AC}|\left|\begin{pmatrix} m \\ 4m-4 \\ -4 \end{pmatrix}\right|\cos\theta$

$-m - 16 = \sqrt{m^2 + 16m^2 - 32m + 16 + 16}\ |\vec{AC}|\cos\theta$

$= \sqrt{17m^2 - 32m + 32}\ |\vec{AC}|\cos\theta$

$\therefore |\vec{AC}|\cos\theta = \dfrac{-(m+16)}{\sqrt{17m^2 - 32m + 32}}$

Since $PQ = 3$,

$$\frac{-(m+16)}{\sqrt{17m^2 - 32m + 32}} = 3$$

$$(m+16)^2 = 9(17m^2 - 32m + 32)$$

$$m^2 + 32m + 256 = 153m^2 - 288m + 288$$

$$152m^2 - 320m + 32 = 0$$

$$76m^2 - 160m + 16 = 0$$

$$38m^2 - 80m + 8 = 0$$

$$19m^2 - 40m + 4 = 0$$

$$(m-2)(19m-2) = 0$$

$m \neq \frac{2}{19}$, $\therefore m = 2$.

$$|\vec{CD} \times \vec{AC}| = |\vec{CD}||\vec{AC}|\sin\phi.$$

$$\therefore |\vec{CD}|\sin\phi = \frac{|\vec{CD} \times \vec{AC}|}{|\vec{AC}|}$$

$$= \frac{\left|\begin{pmatrix}0\\1\\1\end{pmatrix} \times \begin{pmatrix}-1\\0\\4\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\0\\4\end{pmatrix}\right|}$$

$$= \frac{\left|\begin{pmatrix}4\\-1\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\0\\4\end{pmatrix}\right|}$$

$$= \frac{\sqrt{18}}{\sqrt{17}}$$

$$\simeq 1.03$$

$\vec{CD} \times \vec{AC} = \begin{pmatrix}4\\-1\\1\end{pmatrix}$ $\vec{CD} \times \vec{BC} = \begin{pmatrix}0\\1\\1\end{pmatrix} \times \begin{pmatrix}3\\-1\\5\end{pmatrix} = \begin{pmatrix}6\\3\\-3\end{pmatrix}$

Since $\begin{pmatrix}4\\-1\\1\end{pmatrix}$ is normal to the plane ACD and $\begin{pmatrix}6\\3\\-3\end{pmatrix}$ is normal to the plane BCD, if Ω is the angle between the planes,

$$\begin{pmatrix}4\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}6\\3\\-3\end{pmatrix} = \left|\begin{pmatrix}4\\-1\\1\end{pmatrix}\right|\left|\begin{pmatrix}6\\3\\-3\end{pmatrix}\right| \cos\Omega$$

$$24 - 3 - 3 = \sqrt{18}\sqrt{54}\cos\Omega$$

$$18 = \sqrt{18}\sqrt{54}\cos\Omega$$

$$\cos\Omega = \sqrt{\frac{18}{54}} = \frac{1}{\sqrt{3}}$$

$$\therefore \Omega = \cos^{-1}\frac{1}{\sqrt{3}}.$$

$\therefore$ The acute angle between the planes is $\cos^{-1}\frac{1}{\sqrt{3}}$.