TAYLOR'S COLLEGE

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CAMBRIDGE A LEVEL PROGRAMME

A2 TRIAL EXAMINATION AUGUST 2011

(June 2010 Intake)

Wednesday **24 August 2011** **12.30 pm – 3.30 pm**

FURTHER MATHEMATICS 9231/12

PAPER 1

3 hours

Additional materials: Answer Booklet/Paper
List of formulae (MF 10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **4** printed pages.

[Turn over

1 Given that $f(n)=(2n+1)7^n-1$, where n is a natural number.

(i) Show that $f(k+1)-f(k)=(ak+b)7^k$, stating the values of the constants a and b. [3]

(ii) Use induction to prove that $f(n)$ is divisible by 4. [4]

2 By expressing $\dfrac{r^2+3r+1}{(r+2)!}$ in the form $\dfrac{1}{a!}-\dfrac{1}{b!}$, find $\displaystyle\sum_{r=1}^{n}\frac{r^2+3r+1}{(r+2)!}$. [5]

Hence find $\dfrac{5}{6}+\dfrac{11}{24}+\dfrac{19}{120}+\cdots$. [2]

3 The vectors **a** and **b** are given by **a** = **i** + (sin 2α)**j** + (cos 2α)**k** and **b** = **i** + (sin 2β)**j** + (cos 2β)**k**, where $0 \le \alpha \le \beta \le \pi$. Find an expression for $|\mathbf{a}\times\mathbf{b}|$ in terms of θ, where $\theta = \alpha - \beta$ and deduce that the angle ϕ between **a** and **b** is given by $\sin\phi=\sin\theta\sqrt{\cos^2\theta+1}$. [8]

4 The linear transformation $\mathrm{T}:\mathfrak{R}^4\to\mathfrak{R}^4$ is represented by the matrix $\mathbf{M}$, where

$$\mathbf{M}=\begin{pmatrix}1 & 2 & -3 & -4\\ 2 & 5 & -4 & -5\\ 3 & a^2+5 & 2a-7 & 3a-9\\ 6 & a^2+12 & 2a-14 & 3a-18\end{pmatrix}$$

and a is a constant.

Show that the dimension of the range space of T is 3 provided $a\ne a_1$ and $a \ne a_2$, where a_1 and a_2 are integers that need to be found and that $a_1<0$ and $a_2>0$. [5]

In the case where $a=2$, find a basis for the null space of T. [3]

5 The matrix **A** has eigenvalue λ with corresponding eigenvector **x**. If **M** is a non-singular matrix and $(\mathbf{M}^{-1})^n\mathbf{A}\mathbf{M}^n$ has an eigenvector $(\mathbf{M}^{-1})^n\mathbf{x}$, find the corresponding eigenvalue. [3]

If the matrix **A** has eigenvalues a, b, c with corresponding eigenvectors $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix},\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$ and $\mathbf{M}^{-1}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}$, find the eigenvalues and corresponding eigenvectors of $(\mathbf{M}^{-1})^n\mathbf{A}\mathbf{M}^n$. [3]

If $\mathbf{P}^{-1}[(\mathbf{M}^{-1})^n\mathbf{A}\mathbf{M}^n]^k\mathbf{P}=\mathbf{D}^k$, where **P** is a square matrix and **D** is a diagonal matrix, find **P** and **D**. [2]

6 The curve C is given by the equation $y = ax + \dfrac{b}{cx+d}, a, b, c > 0$.

(i) Find the equations of the asymptotes of C. [2]

(ii) Find the maximum and minimum values of y. [3]

(iii) Find the range of x if C has no stationary points. [3]

7 **(i)** Given $x = \sin t + \cos t$ and $y = \sin t - \cos t$, find $\dfrac{d^2 y}{dx^2}$. [4]

(ii) If $x = \sin t + \sin t \cos t + \cos t$ and $y = \sin t - \sin t \cos t + \cos t$, express the curve in the form $(ax+b)^2 + (cy+d)^2 = exy + f$. [5]

8 **(i)** Find the general solution of the differential equation

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 2y = 3e^{-x}.$$ [5]

(ii) Find the particular solution given that $y = 1$ and $\dfrac{dy}{dx} = 1$ when $t = 0$. [4]

9 The roots of the equation $x^4 - 2x - 1 = 0$ are α, β, γ and δ. Show that the equation $ay^4 - by^3 + cy^2 - dy + e = 0$, where a, b, c, d and e are constants that need to be determined, has roots $\dfrac{1}{1+\alpha}, \dfrac{1}{1+\beta}, \dfrac{1}{1+\gamma}$ and $\dfrac{1}{1+\delta}$. [3]

Let $S_n = \dfrac{1}{(1+\alpha)^n} + \dfrac{1}{(1+\beta)^n} + \dfrac{1}{(1+\gamma)^n} + \dfrac{1}{(1+\delta)^n}$. Show that $S_3 = 6$. [7]

10 If $I_n = \int \sec^2 y \sec^n(\tan y) \csc^n(\tan y)\, \mathrm{d}y$, by using the substitution $x = \tan y$, show that $I_n = \int \sec^n x \csc^n x\, \mathrm{d}x$ and find a reduction formula for I_n, in terms of x. [7]

Hence find I_2, I_3, and I_4. [5]

11 Answer only **one** of the following two alternatives.

EITHER

(i) Find $\int e^{ax} \cos bx \, \mathrm{d}x$. [3]

(ii) If C: $x = e^t \cos t$ and $y = e^t \sin t$, $t = 0, 1$ find the arc length of C from $t = 0$ to $t = 1$. [4]

(iii) Find the surface area of revolution of C from $t = 0$ to $t = 1$ about the y-axis. [3]

(iv) If $y = e^{2x} \sin 3x$, $x = 4, 5$ and the y coordinate of the centroid of the region bounded by the x-axis, the curve $y = e^{2x} \sin 3x$ and the lines $x = 4$ and $x = 5$ with area A is denoted c, find Ac. [4]

OR

The curve C has polar equation $r = 3a \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. The curve D has polar equation $r = a(1 + \cos \theta)$, $-\pi \leq \theta < \pi$. Given that a is a positive constant,

(i) sketch, on the same diagram, the graphs of C and D, indicating where each curve cuts the initial line. [4]

The graphs of C intersect at the pole O and at the points P and Q.

(ii) Find the polar coordinates of P and Q. [3]

(iii) Use integration to find the exact value of the area enclosed by the curve D and the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$. [7]