Vector Equation of a Plane

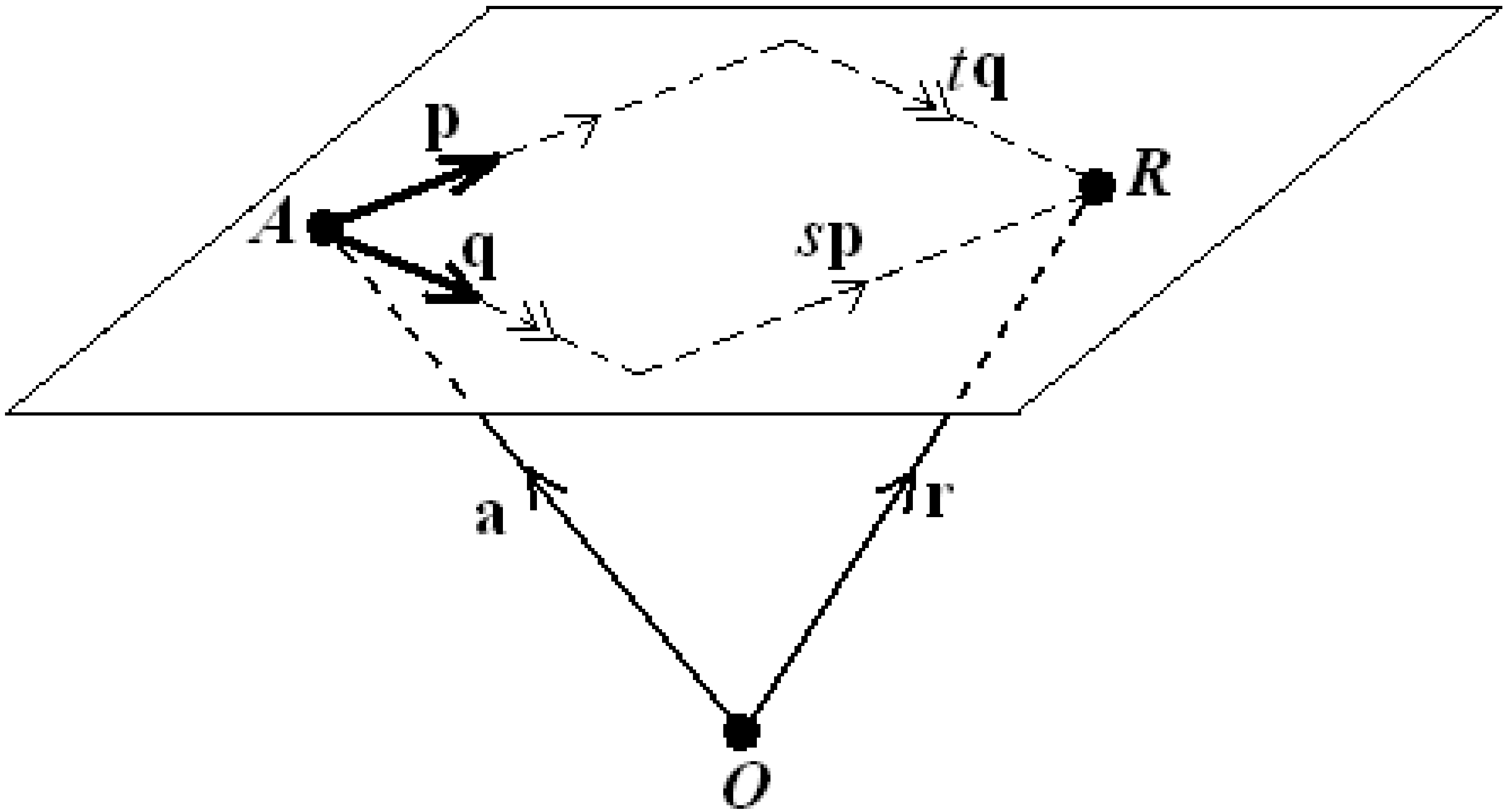
If a plane has the following properties:

(a) direction defined by vectors $\mathbf{p}, \mathbf{q} \notin \mathbf{0}$ which are not parallel, and

(b) containing a known point A with position vector $\mathbf{a}$,

then its <u>vector equation</u> is given by

$\mathbf{r} = \mathbf{a} + s\mathbf{p} + t\mathbf{q}$ where $s, t \in \mathrm{R}$.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Example:

Find the vector equation of the plane which is parallel to vectors $\mathbf{p}$ and $\mathbf{q}$, and passes through point A.

(a) $\mathbf{p} = -2\mathbf{i} - 2\mathbf{k}$, $\mathbf{q} = -\mathbf{i} - 3\mathbf{k}$, $A:(1, 1, 1)$.

(b) $\mathbf{p} = \mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{q} = -2\mathbf{i} - \mathbf{j} - \mathbf{k}$, $A:(0, 1, 2)$.

Example:

Find the vector equation of the plane which passes through points A, B and C.

(a) $A:(1,1,1)$, $B:(1,-3,2)$, $C:(1,0,1)$

(b) $A:(4,-1,2)$, $B:(0,0,3)$, $C:(-1,2,0)$.

Cartesian Equation of a Plane

The Cartesian equation of a plane, i.e. $P(x, y, z) = d$ can be obtained from the vector equation by first substituting $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and then eliminating all parameters.

Example:

Obtain the Cartesian equation from the vector equation.

(a) $\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + \lambda(-2\mathbf{i} - 2\mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{k})$.

(b) $\mathbf{r} = (s - 2t)\mathbf{i} + (1 - 3s - t)\mathbf{j} + (2 - 2s - t)\mathbf{k}$.

From the examples, the Cartesian equation of a plane parallel to vectors $\mathbf{p} = \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}$ and passing through a point with position vector $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$ is $ax + by + cx = d$.

$\underline{\textit{The aim of this activity is to investigate the relationship}}$

$\underline{\textit{between } a, b, c \,\&\, d \text{ with } \mathbf{p}, \mathbf{q} \,\&\, \mathbf{a}}$. *Follow the instruction* :

(a) Write down the vector equation of the plane.

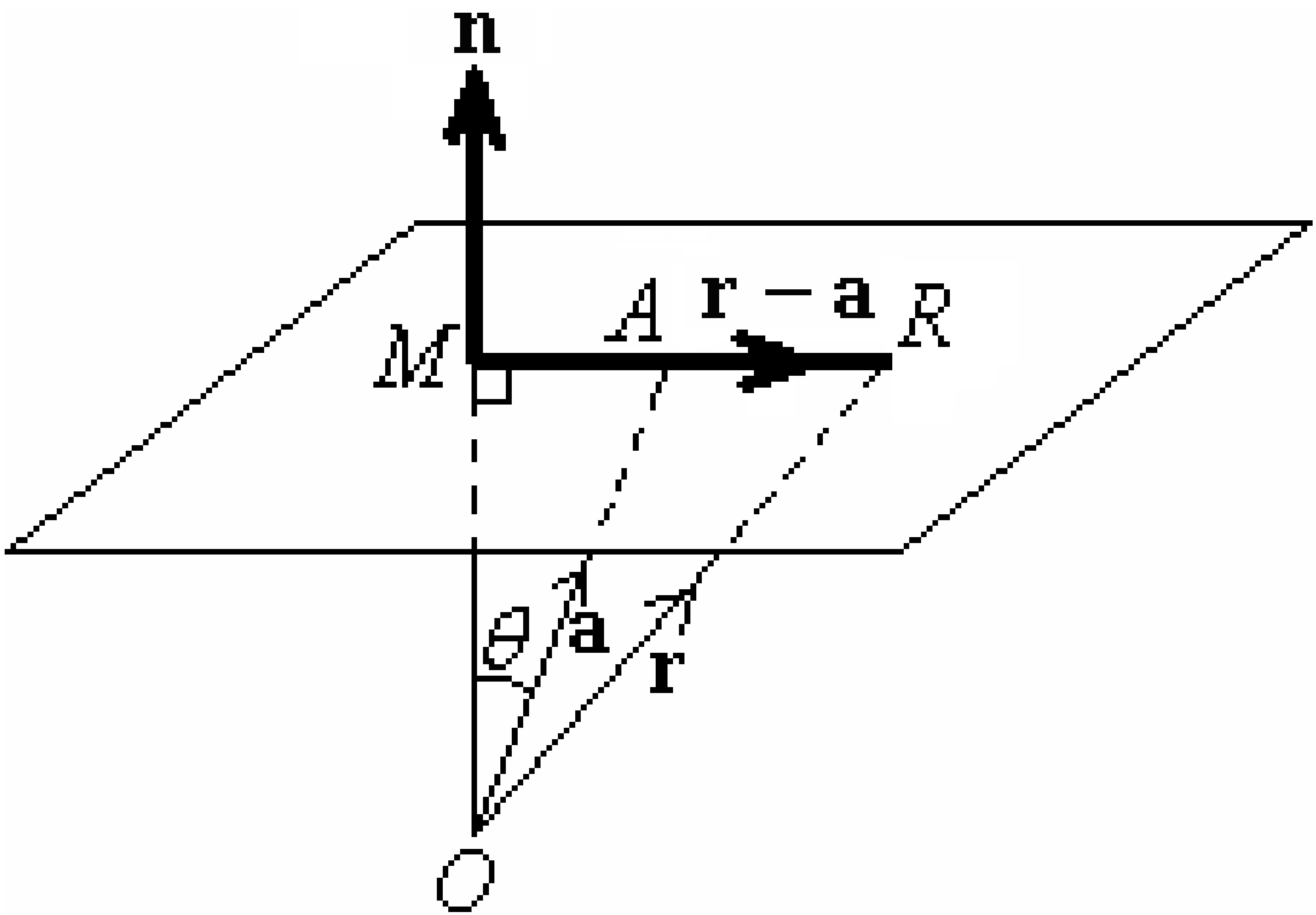
(b) From (a), obtain the Cartesian equation of the plane in the form $ax + by + cz = d$

(c) From (c), show that $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$ where $\mathbf{n}$ is a vector to determine. What is $\mathbf{n}$?

Normal Equation of a Plane

A plane can also be defined by a normal AND a known point.

If the plane contains a normal parallel to vector $\mathbf{n}$ and a point with position vector $\mathbf{a}$, then the vector equation of the plane is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.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Example:

Find the normal equation and the Cartesian equation of the plane through the point $(-2,4,6)$ with normal parallel to $6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

Example:

Find the normal equation and the Cartesian equation of the plane through the point $(0,0,0)$ with normal parallel to $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$. Is point $(1,1,-1)$ on the plane?

Example:

Find the normal equation and Cartesian equation of the plane through the point $(1,2,3)$ with normal parallel to $4\mathbf{i}+5\mathbf{j}+6\mathbf{k}$. Explain why $(-2,1,1)$ is not in the plane.

Example:

State a vector of normal of the following plane, and rewrite the equation in normal form.

(a) $x - 2y + z = -1$ (b) $x + z = 2$

(c) $-3x + y = 0$ (d) $3x + 2y + z = 1$

Example.

A plane is parallel to vectors $\mathbf{p}$ and $\mathbf{q}$ and contains point A. Find a vector perpendicular to the plane. Then, find the equation of the plane in vector, normal and cartesian forms.

(a) $\mathbf{p} = -\mathbf{i} - \mathbf{j}\,,\ \mathbf{q} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k},\ A : (0\,,0\,,3).$

(b) $\mathbf{p} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\,,\ \mathbf{q} = 3\mathbf{i} - \mathbf{j} + \mathbf{k},\ A : (2\,,0\,,-3).$

Example:

Find, in vector, normal and Cartesian forms, the equation of the plane that contains the line $\mathbf{r} = \mathbf{i} - \mathbf{k} + \lambda\left(-\mathbf{i} + \mathbf{j} + \mathbf{k}\right)$ and parallel to the vector $4\mathbf{j} + 6\mathbf{k}$

Example.

A plane contains points A, B and C. Find a vector perpendicular to the plane. Then, find the equation of the plane in vector, normal and Cartesian forms.

(i) $A:(0,1,2)$, $B:(1,-2,0)$ and $C:(-2,0,1)$.

(ii) $A:(1,-1,0)$, $B:(0,1,-1)$ and $C:(-1,0,1)$.

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example.

A plane is parallel to vectors $-\mathbf{i}+\mathbf{j}$ and $\mathbf{j}-\mathbf{k}$ and contains point $(1\,,-1\,,-3)$. Find a vector perpendicular to the plane. Then, find the equation of the plane in vector, normal and cartesian forms.

Example:

A plane contains line $\mathbf{r} = \begin{pmatrix} 3+2\lambda & -1-3\lambda & 2-\lambda \end{pmatrix}$ and is parallel to the vector $\begin{pmatrix} 4 & -3 & 2 \end{pmatrix}$. Find the equation of the plane in vector, normal and Cartesian forms.

Example.

A plane contains points A, B and C. Find a vector perpendicular to the plane. Then, find the equation of the plane in vector, normal and Cartesian forms.

(i) $A:(2,1,0)$, $B:(1,-3,0)$ and $C:(4,3,3)$.

(ii) $A:(1,0,0)$, $B:(0,0,0)$ and $C:(0,1,0)$.

Example:

Show that the equation of the plane containing the points with position vectors $a\mathbf{i}, b\mathbf{j}$ and $c\mathbf{k}$ is given by $\mathbf{r}\cdot\left(\frac{1}{a}\mathbf{i}+\frac{1}{b}\mathbf{j}+\frac{1}{c}\mathbf{k}\right)=1.$

Example:

l_1 : passing through point $P(2,1,-1)$ and direction $\mathbf{i}-\mathbf{j}$.

l_2 : passing through point $Q(5,-2,-1)$ and direction $\mathbf{j}+2\mathbf{k}$.

(a) Write down equations for l_1 and l_2 in $\mathbf{r}=\mathbf{a}+\mu\mathbf{b}$.

(b) Show that Q lies on l_1.

(c) Find the acute angle between l_1 and l_2.

(d) Find the cartesian equation for the plane containing l_1 and l_2.