1. $u_n = e^{nx} - e^{(n+1)x}$

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} e^{nx} - e^{(n+1)x}$$

$$\begin{aligned} &= e^{x} - e^{2x} \\ &\quad + e^{2x} - e^{3x} \\ &\quad + e^{3x} - e^{4x} \\ &\quad \vdots \\ &\quad + e^{(N-2)x} - e^{(N-1)x} \\ &\quad + e^{(N-1)x} - e^{Nx} \\ &\quad + e^{Nx} - e^{(N+1)x} \\ &= e^{x} - e^{(N+1)x} \end{aligned}$$

$$\sum_{n=1}^{\infty} u_n = \lim_{N\to\infty} \sum_{n=1}^{N} u_n$$

$$= \lim_{N\to\infty} e^{x} - e^{(N+1)x}$$

If $x = 0$: $\displaystyle\sum_{n=1}^{\infty} u_n = \lim_{N\to\infty} 1 - 1$

$$= \lim_{N\to\infty} 0$$

$$= 0$$

If $x < 0$: $\displaystyle\sum_{n=1}^{\infty} u_n = \lim_{N\to\infty} e^{x} - e^{(N+1)x}$

$$= e^{x} - 0$$

$$= e^{x}.$$

2. $x^4 + x^3 + Ax^2 + 4x - 2 = 0$

$\alpha, \beta, \gamma, \delta$ are the roots

$\alpha + \beta + \gamma + \delta = -1$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = A$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -4$

$\alpha\beta\gamma\delta = -2$

$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$

Let $u = \frac{1}{\alpha}$

$\alpha = \frac{1}{u}$

α is a root

$\therefore \alpha^4 + \alpha^3 + A\alpha^2 + 4\alpha - 2 = 0$

$\left(\frac{1}{u}\right)^4 + \left(\frac{1}{u}\right)^3 + A\left(\frac{1}{u}\right)^2 + 4\left(\frac{1}{u}\right) - 2 = 0$

$\frac{1}{u^4} + \frac{1}{u^3} + \frac{A}{u^2} + \frac{4}{u} - 2 = 0$

$1 + u + Au^2 + 4u^3 - 2u^4 = 0$

$2u^4 - 4u^3 - Au^2 - u - 1 = 0$

$\therefore$ The equation $2u^4 - 4u^3 - Au^2 - u - 1 = 0$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$.

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = 2$

$\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\alpha\delta} + \frac{1}{\beta\gamma} + \frac{1}{\beta\delta} + \frac{1}{\gamma\delta} = \frac{-A}{2}$

$\frac{1}{\alpha\beta\gamma} + \frac{1}{\alpha\beta\delta} + \frac{1}{\alpha\gamma\delta} + \frac{1}{\beta\gamma\delta} = \frac{1}{2}$

$\frac{1}{\alpha\beta\gamma\delta} = \frac{-1}{2}$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$= (-1)^2 - 2A$

$= 1 - 2A$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right)^2 - 2\left(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\alpha\delta} + \frac{1}{\beta\gamma} + \frac{1}{\beta\delta} + \frac{1}{\gamma\delta}\right)$$

$= 2^2 - 2\left(\frac{-A}{2}\right)$

$= 4 + A$

If $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$,

$1 - 2A = 4 + A$

$3A = -3$

$A = -1$

CONFIDENTIAL

3. $a_n = 17^{2n} + 3(9)^n + 20, \quad n = 0, 1, 2, 3, \ldots$

$$
\begin{aligned}
a_{n+1} - a_n &= 17^{2(n+1)} + 3(9)^{n+1} + 20 \\
&\quad -(17^{2n} + 3(9)^n + 20) \\
&= 17^{2n+2} + 3(9^n)9 + 20 \\
&\quad -17^{2n} - 3(9^n) - 20 \\
&= 17^{2n}17^2 + 27(9^n) - 17^{2n} - 3(9^n) \\
&= 17^{2n}289 - 17^{n} + 24(9^n) \\
&= 17^{2n}288 + 24(9^n) \\
&= 24(17^{2n}12 + 9^n)
\end{aligned}
$$

When $n = 0$:
$$
\begin{aligned}
a_0 &= 17^0 + 3(9^0) + 20 \\
&= 1 + 3 + 20 \\
&= 24 \\
&= 24(1)
\end{aligned}
$$

$\therefore 24 \mid a_0$

Assume the statement is true when $n = k$.

$n = k$: $24 \mid a_k$

$a_k = 24s$, s is an integer.

when $n = k+1$: $a_{k+1} - a_k = 24(17^{2k}12 + 9^k)$

$$
\begin{aligned}
a_{k+1} &= a_k + 24(17^{2k}12 + 9^k) \\
&= 24s + 24(17^{2k}12 + 9^k) \\
&= 24(s + 17^{2k}12 + 9^k)
\end{aligned}
$$

Since s is an integer and k is an integer, $s + 17^{2k}12 + 9^k$ is an integer.

$\therefore 24 \mid a_{k+1}$

$\therefore a_n$ is divisible by 24 for every integer $n \geq 0$.

4. $I_n = \int_0^1 x^n e^{-x^2}\,dx\,, \quad n \geq 0.$

i) $I_1 = \int_0^1 x e^{-x^2}\,dx$

$= \left[\dfrac{-e^{-x^2}}{2}\right]_0^1$

$= \dfrac{-e^{-1}}{2} - \left(\dfrac{-1}{2}\right)$

$= \dfrac{1-e^{-1}}{2}$

ii) $I_{n+2} = \int_0^1 x^{n+2} e^{-x^2}\,dx$

$= \int_0^1 x^{n+1}\, x e^{-x^2}\,dx$

$u = x^{n+1} \qquad dv = x e^{-x^2}\,dx$

$du = (n+1)x^n\,dx \qquad v = \dfrac{e^{-x^2}}{-2}$

$= \left[\dfrac{x^{n+1}e^{-x^2}}{-2}\right]_0^1 - \int_0^1 \dfrac{(n+1)x^n e^{-x^2}}{-2}\,dx$

$= \dfrac{1^{n+1}e^{-1}}{-2} - 0 + \dfrac{(n+1)}{2}\int_0^1 x^n e^{-x^2}\,dx$

$= \dfrac{e^{-1}}{-2} + \dfrac{(n+1)}{2} I_n$

$= \dfrac{(n+1)I_n}{2} - \dfrac{1}{2e}$

iii) $n=3: \; I_5 = 2I_3 - \dfrac{1}{2e}$

$I_3 = I_1 - \dfrac{1}{2e}$

$= \dfrac{1}{2} - \dfrac{1}{2e} - \dfrac{1}{2e}$

$= \dfrac{1}{2} - \dfrac{1}{e}$

$I_5 = 1 - \dfrac{2}{e} - \dfrac{1}{2e}$

$= 1 - \dfrac{5}{2e}$

CONFIDENTIAL

5. $C: r\theta = 1$, $0 < \theta \leq 2\pi$.

i) $r = \dfrac{1}{\theta}$

$$\frac{y}{\sin\theta} = \frac{1}{\theta}$$

$$y = \frac{\sin\theta}{\theta}$$

$$\lim_{\theta \to 0} y = \lim_{\theta \to 0} \frac{\sin\theta}{\theta}$$

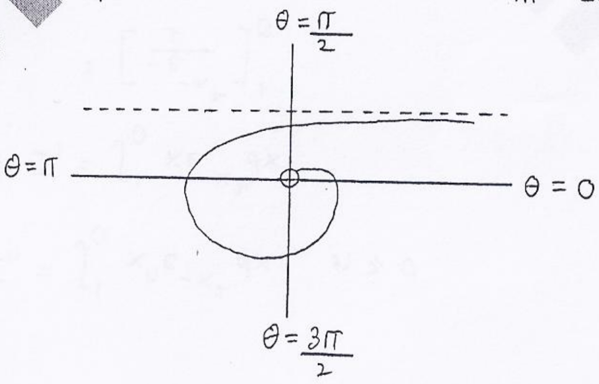
Since $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$,

$$\lim_{\theta \to 0} y = 1$$

$\therefore$ $y = 1$ is an asymptote to C.

ii)

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	$\frac{4}{\pi}$	$\frac{2}{\pi}$	$\frac{4\pi}{3}$	$\frac{1}{\pi}$	$\frac{4}{5\pi}$	$\frac{2}{3\pi}$	$\frac{4}{7\pi}$	$\frac{1}{2\pi}$



$P\left(\frac{6}{\pi}, \frac{\pi}{6}\right)$ $Q\left(\frac{3}{\pi}, \frac{\pi}{3}\right)$

iii) The area of OPQ is

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{r^2}{2}\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2\theta^2}\, d\theta$$

$$= \left[\frac{-1}{2\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{-3}{2\pi} - \left(\frac{-6}{2\pi}\right)$$

$$= \frac{3}{2\pi}$$

iv) The arc length from P to Q is

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{\left(\frac{1}{\theta}\right)^2 + \left(\frac{-1}{\theta^2}\right)^2}\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{1+\theta^2}}{\theta^2}\, d\theta$$

6. $x^3 + xy^2 - y^3 = 0$

i) $\frac{d}{dx}(x^3 + xy^2 - y^3) = 0$

$$3x^2 + y^2 + 2xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$(3y^2 - 2xy)\frac{dy}{dx} = 3x^2 + y^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{3y^2 - 2xy}$$

Since $3x^2 + y^2 > 0$, $\frac{dy}{dx} \neq 0$ at every point of the curve.

ii) $\frac{d}{dx}\left(3x^2 + y^2 + 2xy\frac{dy}{dx} - 3y^2\frac{dy}{dx}\right) = 0$

$$6x + 2y\frac{dy}{dx} + 2xy\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(2xy) - 3y^2\frac{d}{dx}\left(\frac{dy}{dx}\right) - \frac{dy}{dx}\frac{d}{dx}(3y^2) = 0$$

$$6x + 2y\frac{dy}{dx} + 2xy\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(2x\frac{dy}{dx} + 2y\right) - 3y^2\frac{d^2y}{dx^2} - 6y\left(\frac{dy}{dx}\right)^2 = 0$$

$$6x + 2y\frac{dy}{dx} + 2xy\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} - 3y^2\frac{d^2y}{dx^2} - 6y\left(\frac{dy}{dx}\right)^2 = 0$$

$$6x + 4y\frac{dy}{dx} + 2xy\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 - 3y^2\frac{d^2y}{dx^2} - 6y\left(\frac{dy}{dx}\right)^2 = 0$$

At $(1, -1)$: $\frac{dy}{dx} = \frac{4}{5}$

$$6 - 4\left(\frac{4}{5}\right) - 2\frac{d^2y}{dx^2} + 2\left(\frac{16}{25}\right) - 3\frac{d^2y}{dx^2} + 6\left(\frac{16}{25}\right) = 0$$

$$6 - \frac{16}{5} + \frac{32}{25} + \frac{96}{25} - \frac{5d^2y}{dx^2} = 0$$

$$\frac{5d^2y}{dx^2} = \frac{198}{25}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{198}{125}$$

CONFIDENTIAL

7. $z = \cos\theta + i\sin\theta$

i) $z^{-1} = (\cos\theta + i\sin\theta)^{-1}$

$\frac{1}{z} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

$z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

ii) $z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$= 2\cos\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$

$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$(2i\sin\theta)^6 = \left(z - \frac{1}{z}\right)^6$

$-64\sin^6\theta = z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$

$= z^6 + \frac{1}{z^6} - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$

$= 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$

$\therefore \sin^6\theta = \frac{10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta}{32}$

$(2\cos\theta)^6 = \left(z + \frac{1}{z}\right)^6$

$64\cos^6\theta = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$

$= z^6 + \frac{1}{z^6} + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$

$\therefore \cos^6\theta = \frac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{32}$

$\cos^6\theta - \sin^6\theta = \frac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{32}$

$\frac{-(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)}{32}$

$= \frac{\cos 6\theta}{16} + \frac{15\cos 2\theta}{16}$

CONFIDENTIAL

8. $\dfrac{d^2y}{dt^2} + \dfrac{5dy}{dt} + 6y = 15\cos 3t - 3\sin 3t$, $y = 3$, $\dfrac{dy}{dt} = -2$

$t = 0$

$$\frac{d^2y}{dt^2} + \frac{5dy}{dt} + 6y = 0$$

$m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$m = -2, -3$

$\therefore$ The complementary function, y_c, is

$y_c = Ae^{-2t} + Be^{-3t}$

The particular integral, y_p, is given by

$y_p = C\cos 3t + D\sin 3t$

$\dfrac{dy_p}{dt} = -3C\sin 3t + 3D\cos 3t$

$\dfrac{d^2y_p}{dt^2} = -9C\cos 3t - 9D\sin 3t$

$$\begin{aligned}\frac{d^2y_p}{dt^2} + \frac{5dy_p}{dt} + 6y_p &= -9C\cos 3t - 9D\sin 3t \\ &\quad + 5(-3C\sin 3t + 3D\cos 3t) \\ &\quad + 6(C\cos 3t + D\sin 3t) \\ &= (-3C + 15D)\cos 3t \\ &\quad + (-15C - 3D)\sin 3t \\ &= 15\cos 3t - 3\sin 3t\end{aligned}$$

$-3C + 15D = 15$

$C - 5D = -5$

$C = 5D - 5$.

$-15C - 3D = -3$

$5C + D = 1$

$5(5D - 5) + D = 1$

$25D - 25 + D = 1$

$26D = 26$

$D = 1$

$C = 0$

$\therefore y_p = \sin 3t$

$y = y_c + y_p$

$= Ae^{-2t} + Be^{-3t} + \sin 3t$

$\dfrac{dy}{dt} = -2Ae^{-2t} - 3Be^{-3t} + 3\cos 3t$

$t = 0$ $y = 3$: $3 = A + B$

$t = 0$ $\dfrac{dy}{dt} = -2$: $-2 = -2A - 3B + 3$

$A + B = 3$

$-2A - 3B = -5$

$-B = 1$

$B = -1$

$A = 4$

$\therefore y = 4e^{-2t} - e^{-3t} + \sin 3t$

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As $t \to \infty$, since $e^{-2t} \to 0$ and $e^{-3t} \to 0$,

$y \to \sin 3t$.

If the value of the assets, $\$(10^8 y + 10^9)$, is less than $\$9.5 \times 10^8$ as $t \to \infty$,

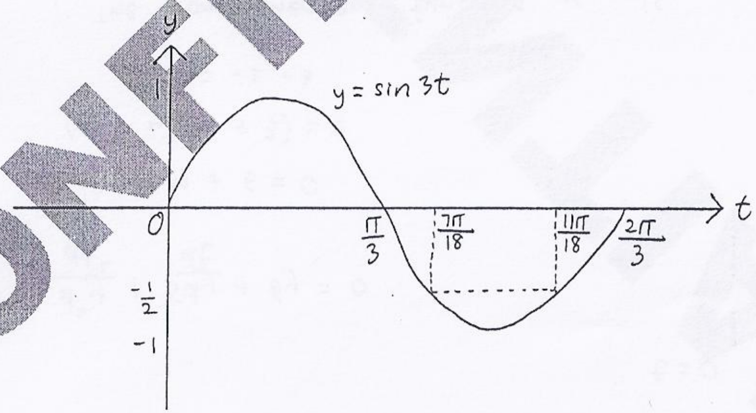
$$10^8 y + 10^9 < 9.5 \times 10^8$$

$$y + 10 < 9.5$$

$$y < -0.5$$

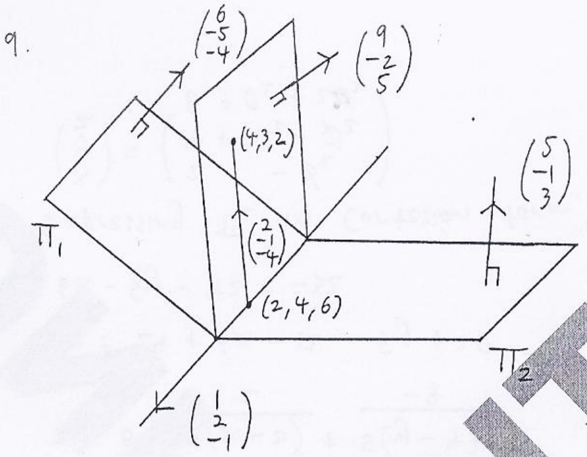
$$\sin 3t < -\frac{1}{2}$$

$$\frac{7\pi}{18} < t < \frac{11\pi}{18}$$



$\therefore$ The value of the assets is less than $\$9.5 \times 10^8$ for about $\dfrac{\frac{11\pi}{18} - \frac{7\pi}{18}}{\frac{2\pi}{3} - 0} = \dfrac{1}{3}$ of the time.

9.



$\Pi_1 : \underset{\sim}{r} = 2\underset{\sim}{i} + 4\underset{\sim}{j} + 6\underset{\sim}{k} + \theta_1(2\underset{\sim}{i} + 3\underset{\sim}{k}) + \phi_1(-4\underset{\sim}{j} + 5\underset{\sim}{k})$

$\Pi_2 : \underset{\sim}{r} = 2\underset{\sim}{i} + 4\underset{\sim}{j} + 6\underset{\sim}{k} + \theta_2(3\underset{\sim}{j} + \underset{\sim}{k}) + \phi_2(-\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k})$

Since $\begin{pmatrix}2\\0\\3\end{pmatrix}$ and $\begin{pmatrix}0\\-4\\5\end{pmatrix}$ are parallel to Π_1,

the normal of Π_1 is parallel to $\begin{pmatrix}2\\0\\3\end{pmatrix} \times \begin{pmatrix}0\\-4\\5\end{pmatrix}$.

$$\begin{pmatrix}2\\0\\3\end{pmatrix} \times \begin{pmatrix}0\\-4\\5\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 0 & 3 \\ 0 & -4 & 5 \end{vmatrix} = \begin{pmatrix}12\\-10\\-8\end{pmatrix} = 2\begin{pmatrix}6\\-5\\-4\end{pmatrix}$$

Since $\begin{pmatrix}0\\3\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\2\end{pmatrix}$ are parallel to Π_2,

the normal of Π_2 is parallel to $\begin{pmatrix}0\\3\\1\end{pmatrix} \times \begin{pmatrix}-1\\1\\2\end{pmatrix}$.

$$\begin{pmatrix}0\\3\\1\end{pmatrix} \times \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 3 & 1 \\ -1 & 1 & 2 \end{vmatrix} = \begin{pmatrix}5\\-1\\3\end{pmatrix}$$

Since the line of intersection of Π_1 and Π_2, ℓ is perpendicular to $\begin{pmatrix}6\\-5\\-4\end{pmatrix}$ and $\begin{pmatrix}5\\-1\\3\end{pmatrix}$, it is parallel to $\begin{pmatrix}6\\-5\\-4\end{pmatrix} \times \begin{pmatrix}5\\-1\\3\end{pmatrix}$.

$$\begin{pmatrix}6\\-5\\-4\end{pmatrix} \times \begin{pmatrix}5\\-1\\3\end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 6 & -5 & -4 \\ 5 & -1 & 3 \end{vmatrix} = \begin{pmatrix}-19\\-38\\19\end{pmatrix} = -19\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

Since ℓ has direction $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and $(2,4,6)$ is a point on Π_1 and Π_2, a vector equation of ℓ is $\underset{\sim}{r} = \begin{pmatrix}2\\4\\6\end{pmatrix} + s\begin{pmatrix}1\\2\\-1\end{pmatrix}$.

If Π_3 contains ℓ and $(4,3,2)$, since $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\-4\end{pmatrix}$ are parallel to Π_3, a vector equation of Π_3 is $\underset{\sim}{r} = \begin{pmatrix}2\\4\\6\end{pmatrix} + s\begin{pmatrix}1\\2\\-1\end{pmatrix} + t\begin{pmatrix}2\\-1\\-4\end{pmatrix}$.

Expressing Π_1 in Cartesian form,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2+2\theta_1\\4-4\phi_1\\6+3\theta_1+5\phi_1\end{pmatrix}$$

$$\left.\begin{aligned} x &= 2+2\theta_1 \\ y &= 4-4\phi_1 \\ z &= 6+3\theta_1+5\phi_1 \end{aligned}\right\}$$

①: $\theta_1 = \dfrac{x-2}{2}$ ②: $\phi_1 = \dfrac{y-4}{-4}$

$$z = 6 + \frac{3(x-2)}{2} + \frac{5(y-4)}{-4}$$

$$4z = 24 + 6x - 12 - 5y + 20$$

$$6x - 5y - 4z = -32$$

Expressing Π_2 in Cartesian form,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2-\phi_2\\4+3\theta_2+\phi_2\\6+\theta_2+2\phi_2\end{pmatrix}$$

$$\left.\begin{aligned} x &= 2-\phi_2 \\ y &= 4+3\theta_2+\phi_2 \\ z &= 6+\theta_2+2\phi_2 \end{aligned}\right\}$$

①: $\phi_2 = 2-x$

$y = 4+3\theta_2+2-x$

$\theta_2 = \dfrac{x+y-6}{3}$

$z = 6+\theta_2+2(2-x)$

$ = 6+\theta_2+4-2x$

$\theta_2 = 2x+z-10$

$$\therefore \frac{x+y-6}{3} = 2x+z-10$$

$$x+y-6 = 6x+3z-30$$

$$5x - y + 3z = 24$$

Expressing Π_3 in Cartesian form,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2+s+2t\\4+2s-t\\6-s-4t\end{pmatrix}$$

$$\left.\begin{aligned} x &= 2+s+2t \\ y &= 4+2s-t \\ z &= 6-s-4t \end{aligned}\right\}$$

$$\left.\begin{aligned} s+2t &= x-2 \\ 2s-t &= y-4 \\ -s-4t &= z-6 \end{aligned}\right\}$$

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$$\left.\begin{aligned} -2\times ① + ②:\quad & s + 2t = x - 2 \\ ① + ③:\quad & -5t = -2x + y \\ & -2t = x + z - 8 \end{aligned}\right\}$$

$$5\times ③:\quad \left.\begin{aligned} s + 2t &= x - 2 \\ -5t &= -2x + y. \\ -10t &= 5x + 5z - 40 \end{aligned}\right\}$$

$$-2\times ② + ③:\quad \left.\begin{aligned} s + 2t &= x - 2 \\ -5t &= -2x + y \\ 0 &= 9x - 2y + 5z - 40 \end{aligned}\right\}$$

$$9x - 2y + 5z = 40$$

$$\left.\begin{aligned} 6x - 5y - 4z &= -32 \\ 5x - y + 3z &= 24 \\ 9x - 2y + 5z &= 40 \end{aligned}\right\}$$

Since Π_1, Π_2, and Π_3 meet at ℓ, the system $\left.\begin{aligned} 6x - 5y - 4z &= -32 \\ 5x - y + 3z &= 24 \\ 9x - 2y + 5z &= 40 \end{aligned}\right\}$ has an infinite number of solutions.

10. $T: R^4 \to R^4$

$$H = \begin{pmatrix} 1 & 2 & -3 & -5 \\ -1 & 4 & 5 & 1 \\ 2 & 3 & 0 & -3 \\ -3 & 5 & 7 & 2 \end{pmatrix}$$

i) $\xrightarrow[3r_1 + r_4]{r_1 + r_2,\ -2r_1 + r_3} \begin{pmatrix} 1 & 2 & -3 & -5 \\ 0 & 6 & 2 & -4 \\ 0 & -1 & 6 & 7 \\ 0 & 11 & -2 & -13 \end{pmatrix}$

$\xrightarrow{r_2 \Leftrightarrow r_3} \begin{pmatrix} 1 & 2 & -3 & -5 \\ 0 & -1 & 6 & 7 \\ 0 & 6 & 2 & -4 \\ 0 & 11 & -2 & -13 \end{pmatrix}$

$\xrightarrow[11r_2 + r_4]{6r_2 + r_3} \begin{pmatrix} 1 & 2 & -3 & -5 \\ 0 & -1 & 6 & 7 \\ 0 & 0 & 38 & 38 \\ 0 & 0 & 64 & 64 \end{pmatrix}$

$\xrightarrow{\frac{r_3}{38},\ \frac{r_4}{64}} \begin{pmatrix} 1 & 2 & -3 & -5 \\ 0 & -1 & 6 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$\xrightarrow{-r_3 + r_4} \begin{pmatrix} 1 & 2 & -3 & -5 \\ 0 & -1 & 6 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ The dimension of the range space of T is 3.

ii)
$$\begin{pmatrix} 1 & 2 & -3 & -5 \\ -1 & 4 & 5 & 1 \\ 2 & 3 & 0 & -3 \\ 3 & 5 & 7 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & -5 & 0 \\ -1 & 4 & 5 & 1 & 0 \\ 2 & 3 & 0 & -3 & 0 \\ 3 & 5 & 7 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & -5 & 0 \\ 0 & -1 & 6 & 7 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w = s$, $s \in R$

$z = -s$

$-y + 6z + 7w = 0$

$-y - 6s + 7s = 0$

$y = s$

$x + 2y - 3z - 5w = 0$

$x + 2s + 3s - 5s = 0$

$x = 0$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ -s \\ s \end{pmatrix} = s\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ A basis for the null space of T is $\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}\right\}$.

iii) $H\underset{\sim}{x} = \begin{pmatrix} 2 \\ -10 \\ -1 \\ -15 \end{pmatrix}$

$$H\begin{pmatrix} 1 \\ -3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -1 \\ -15 \end{pmatrix}$$

If $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, since $H\begin{pmatrix} 1 \\ -3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -1 \\ -15 \end{pmatrix}$

and $\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}\right\}$ is a basis for the null space of T,

$$\underset{\sim}{x} = \begin{pmatrix} 1 \\ -3 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ s-3 \\ 1-s \\ s-2 \end{pmatrix}$$

$$|\underset{\sim}{x}|^2 = 1^2 + (s-3)^2 + (1-s)^2 + (s-2)^2$$

$$= 1 + s^2 - 6s + 9 + 1 - 2s + s^2 + s^2 - 4s + 4$$

$$= 3s^2 - 12s + 15$$

$$= 3(s-1)^2 + 3$$

$$|\underset{\sim}{x}| = \sqrt{3(s-1)^2 + 3}$$

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$$|\underset{\sim}{x}|_{min} = \sqrt{3}$$

$\therefore$ The least possible value of $|\underset{\sim}{x}|$ is $\sqrt{3}$.

11. EITHER

If the matrix G has eigenvalue λ with corresponding eigenvector $\underset{\sim}{e}$, $G\underset{\sim}{e} = \lambda\underset{\sim}{e}$.

i) $(G + kI)\underset{\sim}{e} = G\underset{\sim}{e} + kI\underset{\sim}{e}$

$= \lambda\underset{\sim}{e} + k\underset{\sim}{e}$

$= (\lambda + k)\underset{\sim}{e}$

$\therefore$ The matrix $G + kI$ has eigenvalue $(\lambda + k)$ with corresponding eigenvector $\underset{\sim}{e}$.

ii) $G^2\underset{\sim}{e} = G(G\underset{\sim}{e})$

$= G(\lambda\underset{\sim}{e})$

$= \lambda(G\underset{\sim}{e})$

$= \lambda(\lambda\underset{\sim}{e})$

$= \lambda^2\underset{\sim}{e}$.

$\therefore$ The matrix G^2 has eigenvalue λ^2 with corresponding eigenvector $\underset{\sim}{e}$.

$$A = \begin{pmatrix} 3 & -3 & 0 \\ 1 & 0 & 1 \\ -1 & 3 & 2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 3 & -3 & 0 \\ 1 & 0 & 1 \\ -1 & 3 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & -3 & 0 \\ 1 & -\lambda & 1 \\ -1 & 3 & 2-\lambda \end{pmatrix}$$

$$|A-\lambda I| = (3-\lambda)\begin{vmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 1 & 1 \\ -1 & 2-\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & -\lambda \\ -1 & 3 \end{vmatrix}$$

$$= (3-\lambda)\left[-2\lambda+\lambda^2-3\right] + 3(2-\lambda+1)$$

$$= (3-\lambda)(\lambda^2-2\lambda-3) + 3(3-\lambda)$$

$$= (3-\lambda)(\lambda^2-2\lambda-3+3)$$

$$= (3-\lambda)(\lambda^2-2\lambda)$$

$$= \lambda(3-\lambda)(\lambda-2)$$

$|A-\lambda I| = 0$:

$\lambda(3-\lambda)(\lambda-2) = 0$

$\lambda = 0, 2, 3$

$\lambda = 0$: $\begin{pmatrix} 3 & -3 & 0 \\ 1 & 0 & 1 \\ -1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & -3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Longleftrightarrow r_2}\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 3 & -3 & 0 & 0 \\ -1 & 3 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[r_1+r_3]{-3r_1+r_2}\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{r_2+r_3}\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{-3}}\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s$, $s \in R$

$y = -s$

$x = -s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ -s \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 2$: $\begin{pmatrix} 1 & -3 & 0 \\ 1 & -2 & 1 \\ -1 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1+r_3]{-r_1+r_2}\left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s$, $s \in R$

$y = -s$

$x = -3s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3s \\ -s \\ s \end{pmatrix} = s\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda=3:$ $\begin{pmatrix} 0 & -3 & 0 \\ 1 & -3 & 1 \\ -1 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -1 & 3 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \Leftrightarrow r_3} \left(\begin{array}{ccc|c} -1 & 3 & -1 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{r_3}{-3}]{r_1 + r_2,} \left(\begin{array}{ccc|c} -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$y = 0$

Let $z = s$, $s \in R$

$x = -s$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are $0, 2, 3$ with corresponding eigenvectors $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

$$B = \begin{pmatrix} -5 & -3 & 0 \\ 1 & -8 & 1 \\ -1 & 3 & -6 \end{pmatrix}$$

$B = A - 8I$

$\therefore$ If λ is an eigenvalue of A with corresponding eigenvector $\underset{\sim}{x}$, $\lambda - 8$ is an eigenvalue of B with corresponding eigenvector $\underset{\sim}{x}$ and $(\lambda-8)^2$ is an eigenvalue of B^2 with corresponding eigenvector $\underset{\sim}{x}$.

$\therefore$ The eigenvalues of B are $64, 36, 25$ with corresponding eigenvectors $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

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OR

C: $y = \dfrac{(x-a)(x-b)}{x-c}$, $0<a<b<c$.

i) $y = \dfrac{x^2-(a+b)x+ab}{x-c}$

$$\begin{array}{r|l} & x + c - a - b \\ \hline x-c & x^2-(a+b)x+ab \\ & x^2 - cx \\ \hline & (c-a-b)x+ab \\ & (c-a-b)x - c^2+ac+bc \\ \hline & c^2-ac-bc+ab \end{array}$$

$= x+c-a-b+\dfrac{c^2-ac-bc+ab}{x-c}$

$= x+c-a-b+\dfrac{(c-a)(c-b)}{x-c}$

$= x+P+\dfrac{Q}{x-c}$, $P = c-a-b$, $Q=(c-a)(c-b)$

ii) As $x \to \pm\infty$ $y \to x+c-a-b$

As $x \to c$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = x+c-a-b$ and $x = c$.

iii) $\dfrac{dy}{dx} = 1 - \dfrac{(c-a)(c-b)}{(x-c)^2}$

when $\dfrac{dy}{dx}=0$: $1-\dfrac{(c-a)(c-b)}{(x-c)^2}=0$

$\dfrac{(c-a)(c-b)}{(x-c)^2}=1$

$(x-c)^2 = (c-a)(c-b)$

$x-c = \pm\sqrt{(c-a)(c-b)}$

$x = c \pm \sqrt{(c-a)(c-b)}$, since $a<c$ and $b<c$.

$\therefore$ C has two stationary points

$x = c\pm\sqrt{(c-a)(c-b)}$

$y = \dfrac{\left(c-a\pm\sqrt{(c-a)(c-b)}\right)\left(c-b\pm\sqrt{(c-a)(c-b)}\right)}{\pm\sqrt{(c-a)(c-b)}}$

$\dfrac{d^2y}{dx^2} = \dfrac{2(c-a)(c-b)}{(x-c)^3}$

$x = c+\sqrt{(c-a)(c-b)}$:

$\dfrac{d^2y}{dx^2} = \dfrac{2(c-a)(c-b)}{(c-a)(c-b)\sqrt{(c-a)(c-b)}} = \dfrac{2}{\sqrt{(c-a)(c-b)}} > 0$

$x = c - \sqrt{(c-a)(c-b)}$:

$$\frac{d^2y}{dx^2} = \frac{2(c-a)(c-b)}{-(c-a)(c-b)\sqrt{(c-a)(c-b)}} = \frac{-2}{\sqrt{(c-a)(c-b)}} < 0$$

$$\therefore \left(c + \sqrt{(c-a)(c-b)},\ \frac{(c-a+\sqrt{(c-a)(c-b)})(c-b+\sqrt{(c-a)(c-b)})}{\sqrt{(c-a)(c-b)}}\right)$$

is a minimum point and

$$\left(c - \sqrt{(c-a)(c-b)},\ \frac{(c-a-\sqrt{(c-a)(c-b)})(c-b-\sqrt{(c-a)(c-b)})}{-\sqrt{(c-a)(c-b)}}\right)$$

is a maximum point

iv) $a + b > c$

when $x = 0$: $y = \frac{ab}{-c}$

when $y = 0$: $\frac{(x-a)(x-b)}{(x-c)} = 0$

$x = a, b$.

Since $a < b$,

$c - b < c - a$

$(c-a)(c-b) < (c-a)^2$

$\sqrt{(c-a)(c-b)} < c - a$

$a < c - \sqrt{(c-a)(c-b)}$

$a < b$

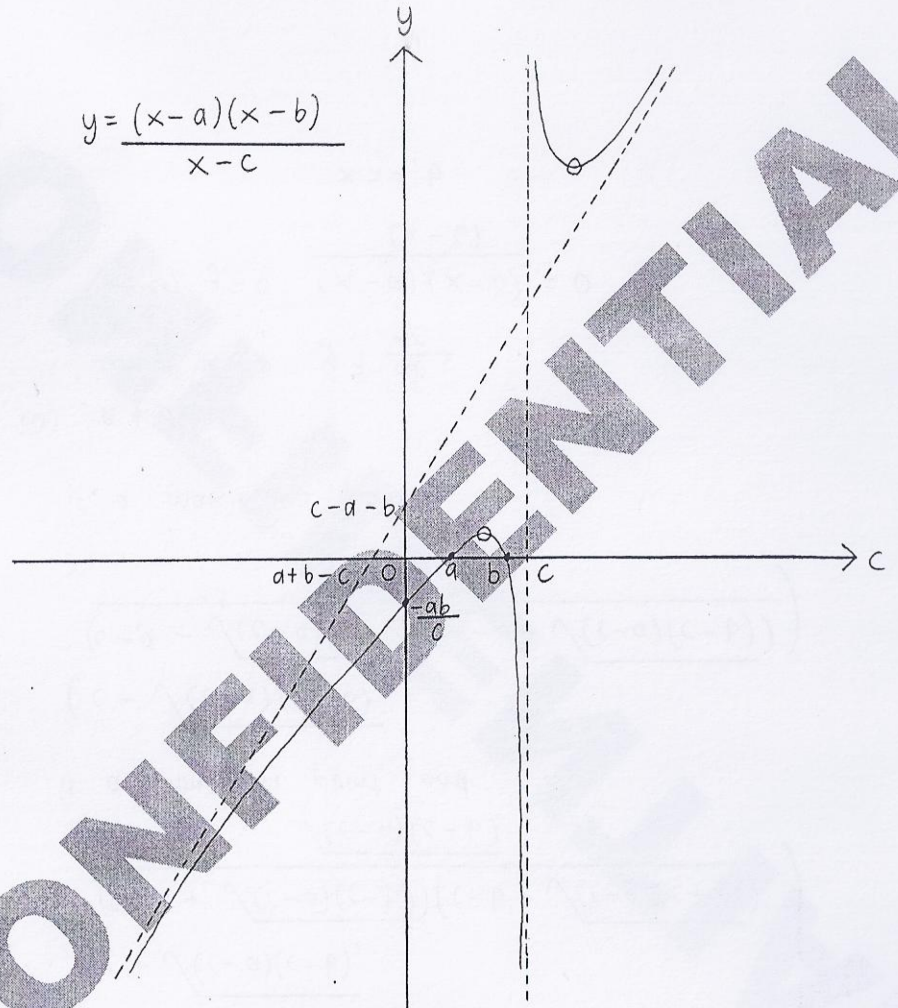
$c - b < c - a$

$(c-b)^2 < (c-a)(c-b)$

$c - b < \sqrt{(c-a)(c-b)}$

$c - \sqrt{(c-a)(c-b)} < b$

$\therefore a < c - \sqrt{(c-a)(c-b)} < b$



o : Critical point

• : Intersection point.