1. $x = \cos y$

$$\frac{dx}{dy} = -\sin y \quad \Rightarrow \quad \frac{dy}{dx} = -\csc y$$

$$\frac{d^2y}{dx^2} = \csc y \cot y \frac{dy}{dx}$$

$$= \frac{\cos y}{\sin^2 y} \frac{dy}{dx}$$

$$(1 - \cos^2 y)\frac{d^2y}{dx^2} = \cos y \frac{dy}{dx}$$

$$(1 - x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx}$$

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

2. $f(n) = 7^{2n-1} + 3^{2n}$

$n=1$: $f(1) = 7 + 9 = 16 = 8(2)$

$f(1)$ is divisible by 8.

$n=k$: Assume $f(k) = 7^{2k-1} + 3^{2k}$ is divisible by 8.

$n=k+1$: $f(k+1) = 7^{2k+1} + 3^{2k+2}$

$$= 49(7^{2k-1}) + 9(3^{2k})$$

$$= 40(7^{2k-1}) + 9(7^{2k-1} + 3^{2k})$$

$$= 40(7^{2k-1}) + 9f(k)$$

If $f(k)$ is divisible by 8 then $f(k+1)$ is also divisible by 8.

Alternatively: $f(k+1) - f(k) = 7^{2k+1} + 3^{2k+2} - 7^{2k-1} - 3^{2k}$

$$= 7^{2k-1}(7^2 - 1) + 3^{2k}(3^2 - 1)$$

$$= 48(7^{2k-1}) + 8(3^{2k})$$

$$= 40(7^{2k-1}) + 8(7^{2k-1} + 3^{2k})$$

If $f(k)$ is divisible by 8 then $f(k+1)$ is also divisible by 8.

$\therefore$ By induction $f(n)$ is divisible by 8 for all positive integers n.

3. $x^3 + px^2 + qx + r = 0$

roots: $\alpha, \beta, \alpha + \beta$

$\alpha + \beta + \alpha + \beta = -p$

$\alpha + \beta = \frac{-p}{2}$ — ①

one roots is $-\frac{p}{2}$ (shown)

$\alpha\beta + \alpha^2 + \alpha\beta + \alpha\beta + \beta^2 = q$

$(\alpha + \beta)^2 + \alpha\beta = q$

$\alpha\beta(\alpha + \beta) = -r$

① in ③: $\alpha\beta = \frac{2r}{p}$

① & ④ in ②: $\frac{p^2}{4} + \frac{2r}{p} = q$

$p^3 + 8r = 4pq$

$x^3 + 4x^2 + 7x + 6 = 0$

One root $= \frac{-4}{2} = -2$

$\therefore x^3 + 4x^2 + 7x + 6 = 0$

$(x + 2)(x^2 + 2x + 3) = 0$

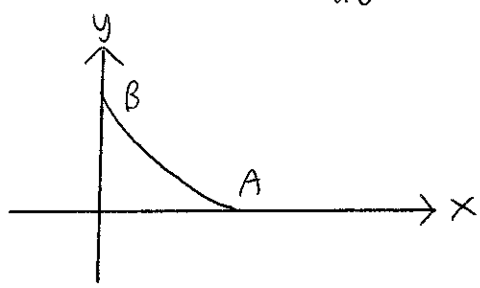
$x = -2$ @ $x = \frac{-2 \pm \sqrt{4 - 12}}{2}$

$= \frac{-2 \pm 2\sqrt{2}i}{2}$

$= -1 \pm \sqrt{2}i$

4. $x = a\cos^3\theta \Rightarrow \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

$y = a\sin^3\theta \Rightarrow \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$



At A, $y = 0 \Rightarrow \theta = 0$

At B, $x = 0 \Rightarrow \theta = \frac{\pi}{2}$

$$\text{circumference} = 4\int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} (3a\cos\theta\sin\theta)\sqrt{\sin^2\theta + \cos^2\theta}\ d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{3a}{2}\sin 2\theta\ d\theta$$

$$= 3a\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 3a(1-(-1))$$

$$= 6a \text{ unit}^2$$

5. $\left(\begin{array}{ccc|c} 1 & 2 & p & 1 \\ 2 & -1 & 5 & 11 \\ 3 & 1 & 8 & q \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & p & 1 \\ 0 & 5 & 2p-5 & -9 \\ 0 & 0 & 3-p & q-12 \end{array}\right)$

i) One common point if $3 - p \neq 0 \Rightarrow p \neq 3$

$\therefore p_1 = 3$

ii) No common point if $3 - p = 0$ and $q - 12 \neq 0$

$p = 3$ $\quad q \neq 12$

$\therefore p_2 = 3$ $\quad q_2 = 12$

iii) $p_2 = 3$, $q_2 = 12$

$\left(\begin{array}{ccc|c} 1 & 2 & p & 1 \\ 0 & 5 & 2p-5 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 5 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Row 1 : $x = 1 - 3z - 2y$ $\quad$ Row 2 : $5y = -9 - z$

let, $z = 5t$

Row 2 : $y = \frac{-9}{5} - t$ $\quad$ Row 1 : $x = \frac{23}{5} - 13t$

$$\therefore \underset{\sim}{r} = \begin{pmatrix} \frac{23}{5} \\ -\frac{9}{5} \\ 0 \end{pmatrix} + \begin{pmatrix} -13 \\ -1 \\ 5 \end{pmatrix} t$$

or $$\underset{\sim}{r} = \begin{pmatrix} 28 \\ 0 \\ -9 \end{pmatrix} + \begin{pmatrix} 13 \\ 1 \\ -5 \end{pmatrix} t$$

6: i) $$\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1} = \frac{n^2+n+1-(n^2-n+1)}{n^4+n^3+n^2-n^3-n^2-n+n^2+n+1}$$

$$= \frac{2n}{n^4+n^2+1}$$

ii) $$n^2+n+1 = \left(n+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$n^2-n+1 = \left(n-\frac{1}{2}\right)^2+\frac{3}{4}$$

iii) $$S_n = \sum_{n=1}^{N} \frac{n}{n^4+n^2+1} = \frac{1}{2}\sum_{n=1}^{2N} \frac{2n}{n^4+n^2+1}$$

$$= \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{\left(n-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\left(n+\frac{1}{2}\right)^2+\frac{3}{4}}\right)$$

$$= \frac{1}{2}\left[\frac{1}{\left(1-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\left(N+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{\left(N+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

Since $\left(N+\frac{1}{2}\right)^2+\frac{3}{4} > 0 \quad \forall N$

$$\therefore 1 - \frac{1}{\left(N+\frac{1}{2}\right)^2+\frac{3}{4}} < 1$$

$$\therefore S_N < \frac{1}{2}(1) = \frac{1}{2}$$

7. i) $\begin{pmatrix} 1 & 2 & 4 & 7 \\ -2 & -3 & 5 & 0 \\ 3 & 2 & 6 & 11 \\ 8 & 3 & 1 & 12 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Dimension of range space of $K = 3$

ii) Basis for $K = \left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \\ 1 \end{pmatrix} \right\}$

iii) Let

$$a \begin{pmatrix} 1 \\ -2 \\ 3 \\ 8 \end{pmatrix} + b \begin{pmatrix} 2 \\ -3 \\ 2 \\ 3 \end{pmatrix} + c \begin{pmatrix} 4 \\ 5 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 45 \\ 9 \\ -15 \end{pmatrix}$$

Using Rows 1 − 3 we get

$a = 1, \ b = -9, \ c = 4$

Check in Row 4: LHS $= 1(8) - 9(3) + 4(1)$
$= -15 = $ RHS

$\therefore \begin{pmatrix} -1 \\ 45 \\ 9 \\ -15 \end{pmatrix}$ belongs to K.

8. $$\frac{d^2y}{dx^2} + 4y = \sin 2x \quad - (1)$$

Given $y_p = (a\cos 2x + b\sin 2x)x \quad - (2)$

$y_p' = (-2a\sin 2x + 2b\cos 2x)x$

$\quad + a\cos 2x + b\sin 2x \quad - (3)$

$y_p'' = (-4a\cos 2x - 4b\sin 2x)x$

$\quad + -2a\sin 2x + 2b\cos 2x + -2a\sin 2x + 2b\cos 2x$

$= -4y_p - 4a\sin 2x + 4b\cos 2x$

$y_p'' + 4y_p = -4a\sin 2x + 4b\cos 2x$

(4) in (1) : $a = \frac{-1}{4}$, $b = 0$

$\therefore y_p = -\frac{1}{4}x\cos 2x$

CE / AE for (1) : $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

C.F. for (1) $y_c = A\cos 2x + B\sin 2x$

$\therefore y = (A - \frac{1}{4}x)\cos 2x + B\sin 2x$

given $y(0) = \pi$: $\pi = (A - 0) + 0 \Rightarrow A = \pi$

$y' = -2(A - \frac{1}{4}x)\sin 2x - \frac{1}{4}\cos 2x + 2B\cos 2x$

given $y'(0) = -\frac{1}{4}$: $-\frac{1}{4} = 0 - \frac{1}{4} + 2B \Rightarrow B = 0$

$\therefore y = (\pi - \frac{1}{4}x)\cos 2x$

9. $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$$= \cos^5\theta + 5\cos^4\theta\sin\theta\, i - 10\cos^3\theta\sin^2\theta$$
$$- (10\cos^2\theta\sin^3\theta)i + 5\cos\theta\sin^4\theta + \sin^5\theta\, i$$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$$\therefore \tan 5\theta = \frac{\sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} \div \frac{\cos^5\theta}{\cos^5\theta}$$

$$= \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

let $t = \tan\theta \Rightarrow \tan 5\theta = \dfrac{t^5 - 10t^3 + 5t}{1 - 10t^2 + 5t^4}$

let $\theta = \dfrac{\pi}{20} \Rightarrow \tan 5\left(\dfrac{\pi}{20}\right) = 1$ and $t = \tan\dfrac{\pi}{20}$

$$\therefore 1 = \frac{t^5 - 10t^3 + 5t}{1 - 10t^2 + 5t^4}$$

$$t^5 - 5t^4 - 10t^3 + 10t^2 + 5t - 1 = 0$$

$$(t-1)(t^4 - 4t^3 - 14t^2 - 4t + 1) = 0$$

since $t \neq 1 \Rightarrow t^4 - 4t^3 - 14t^2 - 4t + 1 = 0$

$$\tan 5\theta = 1 \Rightarrow 5\theta = \frac{\pi}{4} + n\pi = \frac{(n+1)\pi}{4}$$

$$\theta = \frac{(n+1)\pi}{20}$$

$n = 0 \Rightarrow \theta = \frac{\pi}{20}$

$n = 1 \Rightarrow \theta = \frac{\pi}{4}$

$n = 2 \Rightarrow \theta = \frac{9\pi}{20}$

$n = 3 \Rightarrow \theta = \frac{13\pi}{20}$

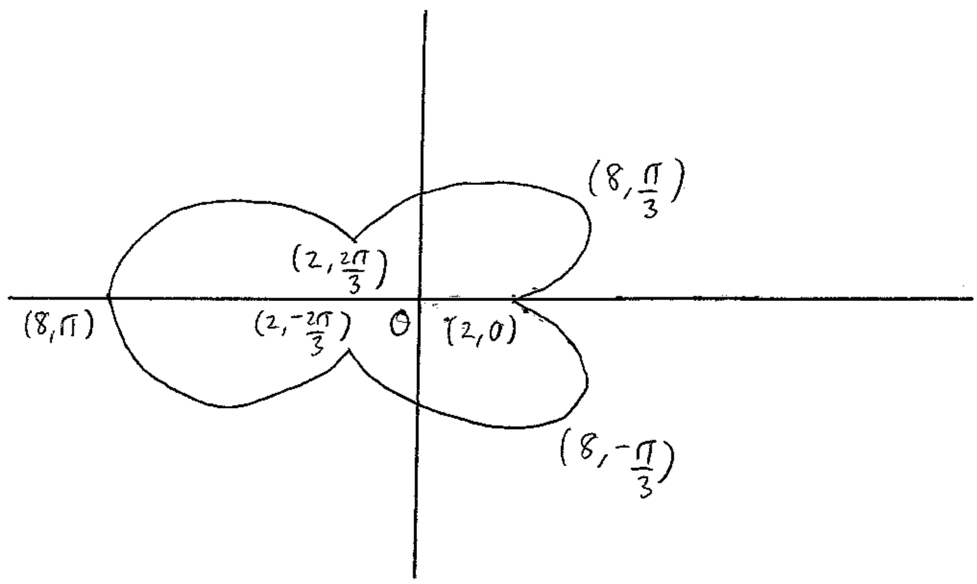
$n = 4 \Rightarrow \theta = \frac{17\pi}{20}$

$\therefore$ The other roots are : $\frac{9\pi}{20}, \frac{13\pi}{20}, \frac{17\pi}{20}$

10 i) $r = 5 - 3\cos 3\theta$

$r_{max} = 8$ when $\cos 3\theta = -1 \Rightarrow \theta = \pm\frac{\pi}{3}, \pi$

$r_{min} = 2$ when $\cos 3\theta = 1 \Rightarrow \theta = 0, \pm\frac{2\pi}{3}$



ii) $A >$ quarter circle with radius $r_{min} = 2$

$A <$ a quarter circle with radius $r_{max} = 8$

$\frac{1}{2}(2)^2\left(\frac{\pi}{2}\right) < A < \frac{1}{2}(8)^2\left(\frac{\pi}{2}\right) \Rightarrow \pi < A < 16\pi$

iii) $A = \int_0^{\frac{\pi}{2}} \frac{1}{2}(5 - 3\cos 3\theta)^2 \, d\theta$

$= \int_0^{\frac{\pi}{2}} \left(\frac{25}{2} - 15\cos 3\theta + \frac{9}{2}\cos^2 3\theta\right) d\theta$

$= \int_0^{\frac{\pi}{2}} \left[\frac{25}{2} - 15\cos 3\theta + \frac{9}{4} + \frac{9}{4}\cos 6\theta\right] d\theta$

$= \left[\frac{59\theta}{4} - 5\sin 3\theta + \frac{3}{8}\sin 6\theta\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{59\pi}{8} + 5 + 0\right) - (0)$

$= \frac{59\pi}{8} + 5$

11. EITHER

$$y = \frac{x^2 - a}{(x-a)^2}, \quad a > 1$$

i) At the x-axis, $y = 0 \Rightarrow x = \pm\sqrt{a}$

$(\sqrt{a}, 0)$, $(-\sqrt{a}, 0)$

At the y-axis, $x = 0 \Rightarrow y = \frac{-a}{a^2} = \frac{-1}{a}$ $\left(0, -\frac{1}{a}\right)$

ii) $$y = \frac{x^2 - a}{(x-a)^2} = 1 + \frac{2a}{x-a} + \frac{a^2 - a}{(x-a)^2}$$

Asymptotes: $y = 1$, $x = a$

iii) $$\frac{dy}{dx} = 0 \Rightarrow \frac{-2a}{(x-a)^2} - \frac{2(a^2-a)}{(x-a)^3} = 0$$

$$2a(x-a) = 2a(1-a)$$

$$x = 1$$

$$y(1) = \frac{1-a}{(1-a)^2} = \frac{1}{1-a}$$

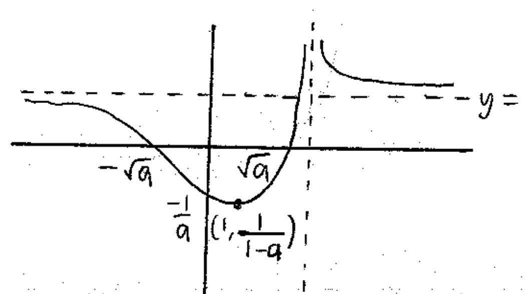
$$\frac{d^2y}{dx^2} = \frac{4a}{(x-a)^3} + \frac{6(a^2-a)}{(x-a)^4}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{4a}{(1-a)^3} + \frac{6(a^2-a)}{(1-a)^4} = \frac{2a(a-1)}{(1-a)^4} > 0$$

$\therefore \left(1, \frac{1}{1-a}\right)$ is a minimum point.

iv) Since $a > 1$

$$\frac{1}{1-a} < 0$$



11. OR

i) $$\begin{pmatrix} \frac{2}{3} & -\frac{1}{2} & -\frac{1}{6} \\ \frac{1}{6} & 0 & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{4} & \frac{5}{12} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

eigenvalue $= \frac{1}{2}$

ii) $$\begin{vmatrix} \frac{2}{3} - \lambda & -\frac{1}{2} & -\frac{1}{6} \\ \frac{1}{6} & -\lambda & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{4} & \frac{5}{12} - \lambda \end{vmatrix} = 0$$

$$\frac{-1}{24}(24\lambda^3 - 26\lambda^2 + 9\lambda - 1) = 0$$

$$(2\lambda - 1)(12\lambda^2 - 7\lambda + 1) = 0$$

$$(2\lambda - 1)(3\lambda - 1)(4\lambda - 1) = 0$$

$$\lambda = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$$

$\lambda = \frac{1}{3}$

$$\begin{pmatrix} \frac{2}{3} - \frac{1}{3} & -\frac{1}{2} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{4} & \frac{5}{12} - \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x = -z$ and $y = -z$

let $z = -t \Rightarrow x = t = y$

eigenvector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\lambda = \frac{1}{4}$

$$\begin{pmatrix} \frac{2}{3} - \frac{1}{4} & -\frac{1}{2} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{4} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{4} & \frac{5}{12} - \frac{1}{4} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

let $z = -t \Rightarrow x = 2t = y$

eigenvector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

Matrix $Q = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{pmatrix}$ and matrix $D = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$

iii) $A^n = QD^nQ^{-1}$

$$= Q \begin{pmatrix} \frac{1}{2^n} & 0 & 0 \\ 0 & \frac{1}{3^n} & 0 \\ 0 & 0 & \frac{1}{4^n} \end{pmatrix} Q^{-1}$$

as $n \rightarrow \infty$, $\frac{1}{2^n} > 0$, $\frac{1}{3^n} > 0$, $\frac{1}{4^n} > 0$.

$\therefore$ All elements of A^n tend to zero.