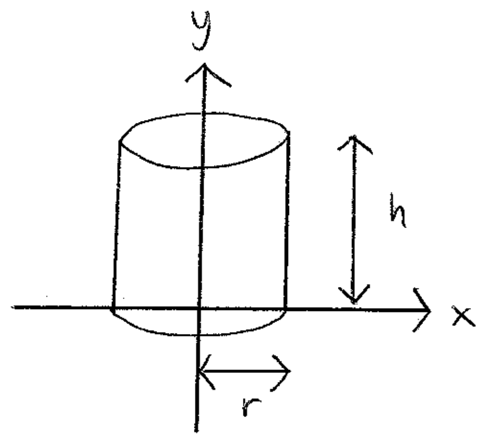
1. Rotate line $x=r$ about the y-axis,

curved surface area

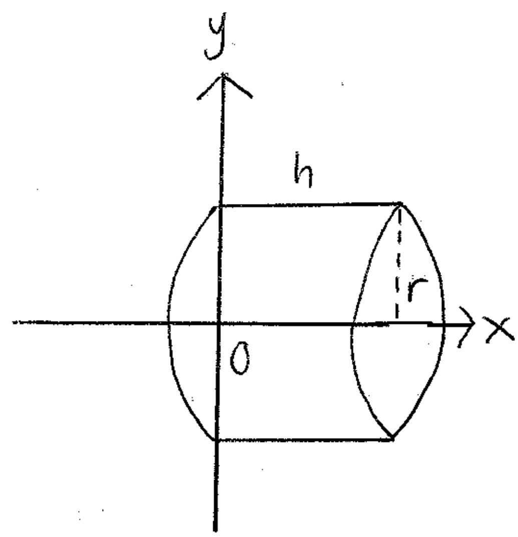
$$= \int_0^h 2\pi r\sqrt{1+0^2}\, dy$$

$$= 2\pi r\left[y\right]_0^h$$

$$= 2\pi rh$$



or



$$y = r$$

$$\frac{dy}{dx} = 0$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1}$$

$$= 1$$

surface area of revolution about the x-axis

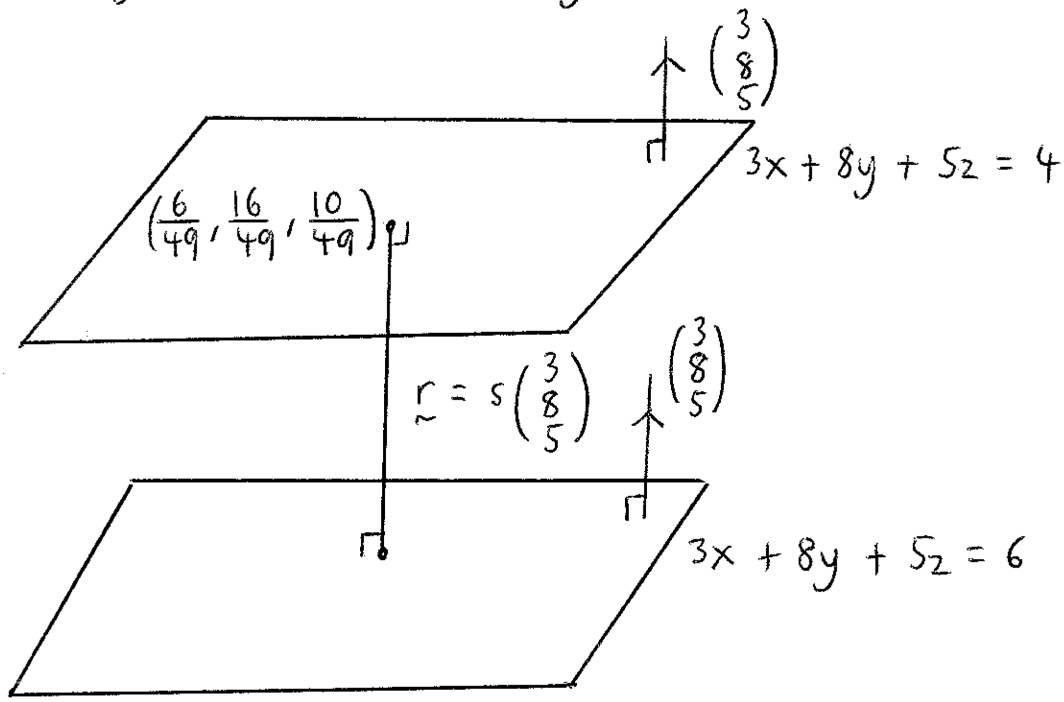
$$= \int_0^h 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^h 2\pi r\, dx$$

$$= \left[2\pi rx\right]_0^h$$

$$= 2\pi rh$$

2. $3x + 8y + 5z = 4 \quad 3x + 8y + 5z = 6$



$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}3\\8\\5\end{pmatrix} = 4 \qquad \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}3\\8\\5\end{pmatrix} = 6$$

Since $\begin{pmatrix}3\\8\\5\end{pmatrix}$ is normal to both planes,

the line ℓ passing through the origin and perpendicular to both planes has equation $\underset{\sim}{r} = s\begin{pmatrix}3\\8\\5\end{pmatrix}$.

When ℓ meets the plane $3x + 8y + 5z = 4$,

$$3(3s) + 8(8s) + 5(5s) = 4$$
$$9s + 64s + 25s = 4$$
$$98s = 4$$
$$s = \frac{4}{98}$$

$\therefore$ The line meets the plane $3x + 8y + 5z = 4$ at the point $\left(\frac{6}{49}, \frac{16}{49}, \frac{10}{49}\right)$.

When ℓ meets the plane $3x + 8y + 5z = 6$,

$$3(3s) + 8(8s) + 5(5s) = 6$$

$$9s + 64s + 25s = 6$$

$$98s = 6$$

$$s = \frac{3}{49}$$

The line meets the plane $3x + 8y + 5z = 6$ at the point $\left(\frac{9}{49}, \frac{24}{49}, \frac{15}{49}\right)$.

Since ℓ is perpendicular to both planes, the minimum distance between the two planes is

$$\sqrt{\left(\frac{9}{49} - \frac{6}{49}\right)^2 + \left(\frac{24}{49} - \frac{16}{49}\right)^2 + \left(\frac{15}{49} - \frac{10}{49}\right)^2}$$

$$= \sqrt{\left(\frac{3}{49}\right)^2 + \left(\frac{8}{49}\right)^2 + \left(\frac{5}{49}\right)^2}$$

$$= \frac{\sqrt{3^2 + 8^2 + 5^2}}{49}$$

$$= \frac{\sqrt{9 + 64 + 25}}{49}$$

$$= \frac{\sqrt{98}}{49}$$

$$= \frac{7\sqrt{2}}{49}$$

$$= \frac{\sqrt{2}}{7}$$

3. $\left(\frac{3iz+5}{2iz+7}\right)^4 = 1$

$$= \cos 0 + i\sin 0$$

$$= \cos(0 + 2k\pi) + i\sin(0 + 2k\pi),\ k \in \mathbb{Z}$$

$$= \cos 2k\pi + i\sin 2k\pi$$

$$\frac{3iz+5}{2iz+7} = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{4}}$$

$$= \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2},\ k = 0,1,2,3$$

$$= -1, 1, -i, i$$

$$3iz + 5 = -2iz - 7, \qquad 3iz + 5 = 2iz + 7$$

$$5iz = -12 \qquad iz = 2$$

$$z = \frac{-12}{5i} \qquad z = \frac{2}{i}$$

$$= \frac{12i}{5} \qquad = -2i$$

$$3iz + 5 = i(2iz + 7) \qquad 3iz + 5 = -i(2iz + 7)$$

$$= -2z + 7i \qquad = 2z - 7i$$

$$(3i + 2)z = -5 + 7i \qquad (3i - 2)z = -5 - 7i$$

$$z = \frac{-5+7i}{2+3i} \qquad z = \frac{5+7i}{2-3i}$$

$$= \frac{(-5+7i)(2-3i)}{(2+3i)(2-3i)} \qquad = \frac{(5+7i)(2+3i)}{4+9}$$

$$= \frac{11+29i}{13} \qquad = \frac{-11+29i}{13}$$

$$z = \frac{12i}{5}, -2i, \frac{11+29i}{13}, \frac{-11+29i}{13}$$

4. i) $a_r = \frac{e-1}{e^r} = e\left[\frac{1}{e^r} - \frac{1}{e^{r+1}}\right]$

$$\sum_{r=1}^{n} a_r = e\left[\frac{1}{e} - \frac{1}{e^{n+1}}\right] = 1 - e^{-n}$$

ii) $b_r = \frac{-1}{r(r+1)} = \frac{1}{r+1} - \frac{1}{r}$

$$\sum_{r=1}^{n} b_r = \frac{1}{n+1} - \frac{1}{1} = \frac{1}{n+1} - 1$$

iii) $S_n = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r$

$$= \frac{1}{n+1} - \frac{1}{e^n}$$

As $n \to \infty$, $\frac{1}{n+1} \to 0$ and $\frac{1}{e^n} \to 0$

$S_n \to 0$

5. $1^2 - 3^2 + 5^2 - 7^2 + \cdots - [2 \times 2n - 1]^2 = -8n^2$

$n=1$: LHS $= 1^2 - 3^2 = -8$ RHS $= -8(1^2) = -8$

LHS $=$ RHS

$n=k$: Assume statement is true for $n=k$ i.e.

$$1^2 - 3^2 + 5^2 - 7^2 + \cdots - [2 \times 2k - 1]^2 = -8k^2$$

$n=k+1$: Need to show that

$$1^2 - 3^2 + 5^2 - 7^2 + \cdots - [2 \times 2k - 1]^2$$
$$+ [2 \times (2k+1) - 1]^2 - [2 \times (2k+2) - 1]^2$$
$$= -8(k+1)^2$$

$$1^2 - 3^2 + 5^2 - 7^2 + \cdots - [2 \times 2k - 1]^2$$
$$+ [2(2k+1) - 1]^2 - [2 \times (2k+2) - 1]^2$$
$$= -8k^2 + (4k+1)^2 - (4k+3)^2$$
$$= -8k^2 + [(4k+1) - (4k+3)][(4k+1) + (4k+3)]$$
$$= -8k^2 + (-2)(8k+4)$$
$$= -8(k^2 + 2k + 1)$$
$$= -8(k+1)^2$$

If statement is true for $n=k$ then it's true for $n=k+1$.

By induction statement is true.

6. $x=1 \Rightarrow \quad y = e^{y(1-1)} = 1$

$$y = e^{y(x-1)}$$

$$\ln y = y(x-1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}(x-1) + y \qquad ①$$

$$\text{At } (1,1): \quad \frac{1}{1}\frac{dy}{dx} = \frac{dy}{dx}(1-1) + 1$$

$$\therefore \frac{dy}{dx} = 1$$

From ①: $$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = \frac{d^2y}{dx^2}(x-1) + \frac{dy}{dx} + \frac{dy}{dx}$$

At $(1,1)$: $$\frac{1}{1}\frac{d^2y}{dx^2} - \frac{1}{1}(1^2) = \frac{d^2y}{dx^2}(1-1) + 1 + 1$$

$\frac{dy}{dx} = 1$

$$\frac{d^2y}{dx^2} - 1 = 2$$

$$\frac{d^2y}{dx^2} = 3$$

7. $x^3 + ax + b = 0$

α, β, γ are the roots.

$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \alpha\gamma + \beta\gamma = a \quad \alpha\beta\gamma = -b$

$$\frac{\alpha\beta}{\alpha^2+\beta^2}, \frac{\alpha\gamma}{\alpha^2+\gamma^2}, \frac{\beta\gamma}{\beta^2+\gamma^2}$$

Let $u = \dfrac{\alpha\beta}{\alpha^2+\beta^2}$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2 - 2\alpha\beta}$$

$$= \frac{\alpha\beta}{(-\gamma)^2 - 2\alpha\beta}, \text{ since } \alpha+\beta+\gamma = 0$$

$$= \frac{\alpha\beta}{\gamma^2 - 2\alpha\beta}$$

$$= \frac{\alpha\beta\gamma}{\gamma^3 - 2\alpha\beta\gamma}, \text{ since } \alpha\beta\gamma = -b$$

$$= \frac{-b}{\gamma^3 - 2(-b)}$$

$$= \frac{-b}{\gamma^3 + 2b}$$

$$\gamma^3 + 2b = \frac{-b}{u}$$

$$\gamma^3 = \frac{-b}{u} - 2b$$

$$= \frac{-b - 2bu}{u}$$

$$= -\frac{(1+2u)b}{u}$$

$$r = \frac{-(1+2u)^{\frac{1}{3}} b^{\frac{1}{3}}}{u^{\frac{1}{3}}}$$

r is a root

$$r^3 + ar + b = 0$$

$$\left(\frac{-(1+2u)^{\frac{1}{3}} b^{\frac{1}{3}}}{u^{\frac{1}{3}}}\right)^3 + a\left(\frac{-(1+2u)^{\frac{1}{3}} b^{\frac{1}{3}}}{u^{\frac{1}{3}}}\right) + b = 0$$

$$-\frac{(1+2u)b}{u} - \frac{a(1+2u)^{\frac{1}{3}} b^{\frac{1}{3}}}{u^{\frac{1}{3}}} + b = 0$$

$$\frac{a(1+2u)^{\frac{1}{3}} b^{\frac{1}{3}}}{u^{\frac{1}{3}}} = -\frac{b}{u} - 2b + b$$

$$= -\frac{b}{u} - b$$

$$= -\left(\frac{1}{u} + 1\right)b$$

$$= -\frac{(1+u)b}{u}$$

$$a(1+2u)^{\frac{1}{3}} u^{\frac{2}{3}} = -(1+u)b^{\frac{2}{3}}$$

$$\left(a(1+2u)^{\frac{1}{3}} u^{\frac{2}{3}}\right)^3 = \left(-(1+u)b^{\frac{2}{3}}\right)^3$$

$$a^3(1+2u)u^2 = -(1+u)^3 b^2$$

$$a^3(u^2+2u^3) = -(1+3u+3u^2+u^3)b^2$$

$$a^3u^2 + 2a^3u^3 = -b^2 - 3b^2u - 3b^2u^2 - b^2u^3$$

$$a^3u^2 + 2a^3u^3 + b^2 + 3b^2u + 3b^2u^2 + b^2u^3 = 0$$

$$(2a^3 + b^2)u^3 + (a^3 + 3b^2)u^2 + 3b^2u + b^2 = 0$$

$\therefore$ The equation having roots

$\dfrac{\alpha\beta}{\alpha^2+\beta^2}, \dfrac{\alpha r}{\alpha^2+r^2}, \dfrac{\beta r}{\beta^2+r^2}$ is

$$(2a^3 + b^2)u^3 + (a^3 + 3b^2)u^2 + 3b^2u + b^2 = 0$$

8. $A = \begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & 4 \\ 6 & 5-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (3-\lambda)(5-\lambda) - 24$$

$$= 15 - 8\lambda + \lambda^2 - 24$$

$$= \lambda^2 - 8\lambda - 9$$

$$= (\lambda - 9)(\lambda + 1)$$

When $|A - \lambda I| = 0$:

$$(\lambda - 9)(\lambda + 1) = 0$$

$$\lambda = 9, -1.$$

When $\lambda = 9$: $\begin{pmatrix} -6 & 4 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} -6 & 4 & 0 \\ 6 & -4 & 0 \end{array}\right)$$

Let $y = 3s$, $s \in R$

$x = 2s$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2s \\ 3s \end{pmatrix}$$

$$= s\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

When $\lambda = -1$: $\begin{pmatrix} 4 & 4 \\ 6 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 4 & 4 & 0 \\ 6 & 6 & 0 \end{array}\right)$$

Let $y = s, s \in R$

$x = -s$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -s \\ s \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are $9, -1$ with corresponding eigenvectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

If $P = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix}$,

$$A^{10} = PD^{10}P^{-1}$$

$$= \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix}^{10} \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 9^{10} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 9^{10} & 9^{10} \\ -3 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 2(9^{10})+3 & 2(9^{10})-2 \\ 3(9^{10})-3 & 3(9^{10})+2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 6973568805 & 6973568800 \\ 10460353200 & 10460353205 \end{pmatrix}$$

$$= \begin{pmatrix} 1394713761 & 1394713760 \\ 2092070640 & 2092070641 \end{pmatrix}$$

9. $x = t^{\frac{1}{3}} \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{3}t^{-\frac{2}{3}}$

i) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\frac{dy}{dt}}{\frac{1}{3}t^{-\frac{2}{3}}} = 3t^{\frac{2}{3}}\dfrac{dy}{dt}$

ii) $$\frac{d^2y}{dt^2} = \frac{\frac{d}{dt}\left(3t^{\frac{2}{3}}\frac{dy}{dt}\right)}{\frac{dx}{dt}}$$

$$= \frac{3t^{\frac{2}{3}}\frac{d^2y}{dt^2} + 2t^{-\frac{1}{3}}\frac{dy}{dt}}{\frac{1}{3}t^{-\frac{2}{3}}}$$

$$= 9t^{\frac{4}{3}}\frac{d^2y}{dt^2} + 6t^{\frac{1}{3}}\frac{dy}{dt}$$

From, $\dfrac{1}{x}\dfrac{d^2y}{dx^2} - \dfrac{2}{x^2}\dfrac{dy}{dx} + 36yx^3 = 0$

$$t^{-\frac{1}{3}}\left[9t^{\frac{4}{3}}\frac{d^2y}{dt^2} + 6t^{\frac{1}{3}}\frac{dy}{dt}\right] - 2t^{-\frac{2}{3}}\left[3t^{\frac{2}{3}}\frac{dy}{dt}\right] + 36yt = 0$$

$$9t\frac{d^2y}{dt^2} + \frac{6dy}{dt} - \frac{6dy}{dt} + 36yt = 0$$

$$9t\frac{d^2y}{dt^2} + 36yt = 0$$

$$\frac{d^2y}{dt^2} + 4y = 0$$

AE./CE.: $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

$y(t) = A\cos 2t + B\sin 2t$

$y(x) = A\cos 2x^3 + B\sin 2x^3$

10.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{pmatrix}$$

$$\xrightarrow[-bcr_1 + r_3]{-ar_1 + r_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & ac-bc & ab-bc \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & (a-b)c & (a-c)b \end{pmatrix}$$

$$\xrightarrow{cr_2 + r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & ab-bc-ac+c^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$$

If $a = b = c$: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

rank $(A) = 1$

If $a = b \neq c$: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$

$$\xrightarrow{\frac{r_2}{c-a},\ \frac{r_3}{(c-a)(c-b)}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{-r_2 + r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

rank $(A) = 2$

If $a = c \neq b$: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\xrightarrow{\frac{r_2}{b-a}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 2$

If $a \neq b = c$: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & 0 \end{pmatrix}$

$$\xrightarrow{\frac{r_2}{b-a}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{c-a}{b-a} \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 2$

If $a \neq b,\ a \neq c,\ b \neq c$:

$$\xrightarrow{\frac{r_2}{b-a},\ \frac{r_3}{(c-a)(c-b)}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{c-a}{b-a} \\ 0 & 0 & 1 \end{pmatrix}$$

$\text{rank}(A) = 3$

If $a = b = c$: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Let $z = s,\ s \in R$

and $y = t,\ t \in R$

$x = -s - t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s-t \\ t \\ s \end{pmatrix}$$

$$= s\begin{pmatrix}-1\\0\\1\end{pmatrix} + t\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

If $a = b \neq c$: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$z = 0$

Let $y = s$, $s \in \mathbb{R}$

$x = -s$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-s\\s\\0\end{pmatrix}$$

$$= s\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

If $a = c \neq b$: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$y = 0$

Let $z = s$, $s \in \mathbb{R}$

$x = -s$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-s\\0\\s\end{pmatrix}$$

$$= s\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

If $a \neq b = c$: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{c-a}{b-a} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Let $z = (b-a)s$, $s \in \mathbb{R}$

$y = (a-c)s$

$x = (c-b)s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (c-b)s \\ (a-c)s \\ (b-a)s \end{pmatrix}$$

$$= s \begin{pmatrix} c-b \\ a-c \\ b-a \end{pmatrix}$$

If $a \neq b,\ a \neq c,\ b \neq c$: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{c-a}{b-a} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$z = 0$

$y = 0$

$x = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

11. $I_n = \int \csc^n x \, dx$

$= \int \csc^{n-2} x \csc^2 x \, dx$

$= \int \csc^{n-2} x (1 + \cot^2 x) \, dx$

$= \int \csc^{n-2} x + \csc^{n-2} x \cot^2 x \, dx$

$= \int \csc^{n-2} x \, dx + \int \csc^{n-2} x \cot^2 x \, dx$

$= I_{n-2} + \int \csc^{n-3} x \csc x \cot x \cot x \, dx$

$u = \cot x \qquad dv = \csc^{n-3} x \csc x \cot x \, dx$

$du = -\csc^2 x \, dx \qquad v = \int \csc^{n-3} x \csc x \cot x \, dx$

$w = \csc x$

$dw = -\csc x \cot x \, dx$

$= -\int w^{n-3} \, dw$

$= \dfrac{-w^{n-2}}{n-2}$

$= -\dfrac{\csc^{n-2} x}{n-2}$

$= I_{n-2} - \dfrac{\cot x \csc^{n-2} x}{n-2} - \int \dfrac{\csc^{n-2} x \csc^2 x \, dx}{n-2}$

$= I_{n-2} - \dfrac{\cot x \csc^{n-2} x}{n-2} - \dfrac{1}{n-2} \int \csc^n x \, dx$

$= I_{n-2} - \dfrac{\cot x \csc^{n-2} x}{n-2} - \dfrac{1}{n-2} I_n$

$$\left(1+\frac{1}{n-2}\right)I_n = I_{n-2} - \frac{\cot x \csc^{n-2}x}{n-2}$$

$$\left(\frac{n-1}{n-2}\right)I_n = I_{n-2} - \frac{\cot x \csc^{n-2}x}{n-2}$$

$$(n-1)I_n = (n-2)I_{n-2} - \cot x \csc^{n-2}x$$

i) $n=3:\ 2I_3 = I_1 - \cot x \csc x$

$$I_1 = \int \csc x \, dx$$

$$= \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x}\,dx$$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx$$

$$= -\int \frac{-\csc^2 x - \csc x \cot x}{\csc x + \cot x}\,dx$$

$$= -\ln|\csc x + \cot x|$$

$$2I_3 = -\ln|\csc x + \cot x| - \cot x \csc x$$

$$I_3 = -\frac{\cot x \csc x}{2} - \frac{\ln|\csc x + \cot x|}{2} + C$$

$n=4:\ 3I_4 = 2I_2 - \cot x \csc^2 x$

$$I_2 = \int \csc^2 x \, dx$$

$$= -\cot x$$

$$3I_4 = -2\cot x - \cot x \csc^2 x$$

$$I_4 = -\frac{\cot x \csc^2 x}{3} - \frac{2\cot x}{3} + C$$

ii) $x = 2t^2 \qquad y = t^4$

$\frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 4t^3$

$$S = \int_0^1 \sqrt{16t^2 + 16t^6}\, dt = \int_0^1 4t\sqrt{1+t^4}\, dt$$

let $t^2 = \cot x$

$2t\,dt = -\csc^2 x\, dx$

$t\,dt = -\frac{\csc^2 x}{2}\, dx$

$\sqrt{1+t^4} = \sqrt{1+\cot^2 x} = \csc x$

when $t = 1$, $x = \cot^{-1} 1 = \frac{\pi}{4}$

when $t = 0$, $x = \cot^{-1} 0 = \frac{\pi}{2}$

$$S = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\left(\frac{-1}{2}\csc^2 x\right)(\csc x)\, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\csc^3 x\, dx$$

$$= \Big[-\ln|\csc x + \cot x| - \cot x \csc x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \big[-\ln(1+0) - 0\times 1\big] - \big[-\ln(\sqrt{2}+1) - \sqrt{2}\times 1\big]$$

$$= \sqrt{2} + \ln(1+\sqrt{2})$$

12. EITHER

i) $x = 0, \ x = -6, \ y = 0$

ii) $y = \dfrac{3(x-2)}{x^2 + 6x} \Rightarrow x^2 y + 6xy = 3x - 6$

$$x^2 y + (6y - 3)x + 6 = 0$$

Dis $\geq 0 \Rightarrow 9(2y-1)^2 - 4(y)(6) \geq 0$

$$12y^2 - 20y + 3 \geq 0$$

$$y \leq \frac{1}{6} \quad @ \quad y \geq \frac{3}{2}$$

Curve does not exist for $\frac{1}{6} < y < \frac{3}{2}$

Max point at $y = \frac{1}{6}$ Min. point at $y = \frac{3}{2}$

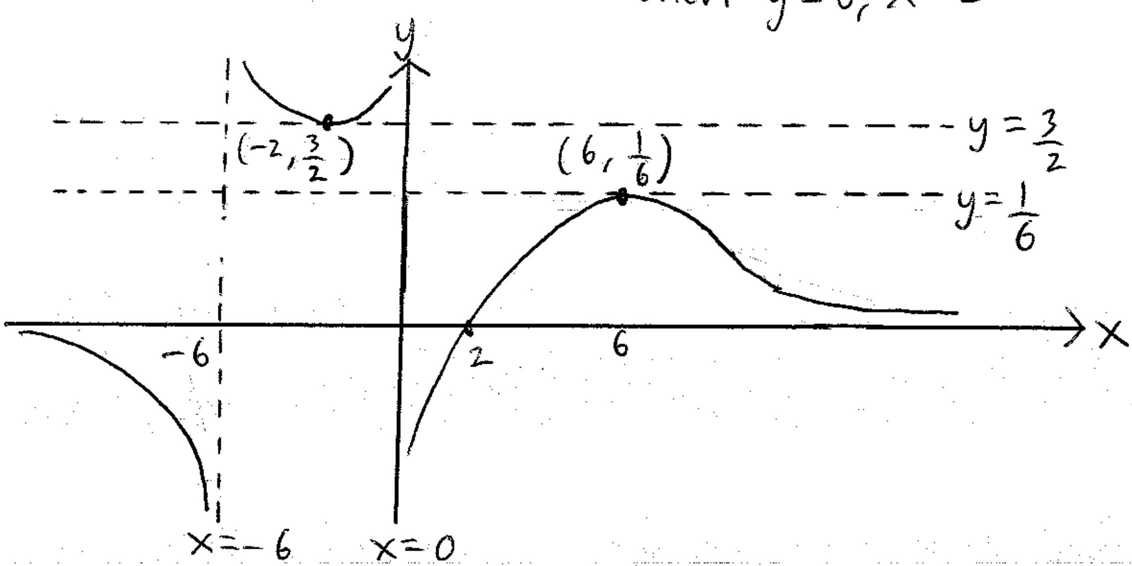
$\frac{1}{6}x^2 + (1-3)x + 6 = 0$ $\frac{3x^2}{2} + (9-3)x + 6 = 0$

$(x-6)^2 = 0$ $(x+2)^2 = 0$

$x = 6$ $x = -2$

Max. point $\left(6, \frac{1}{6}\right)$ Min. point at $\left(-2, \frac{3}{2}\right)$

when $y = 0, x = 2$



OR

i) $(x^2+y^2)^5 = (x^4+y^4)^2$

$(r^2)^5 = ((r\cos\theta)^4 + (r\sin\theta)^4)^2$

$r^{10} = (r^4\cos^4\theta + r^4\sin^4\theta)^2$

$r^5 = r^4\cos^4\theta + r^4\sin^4\theta$

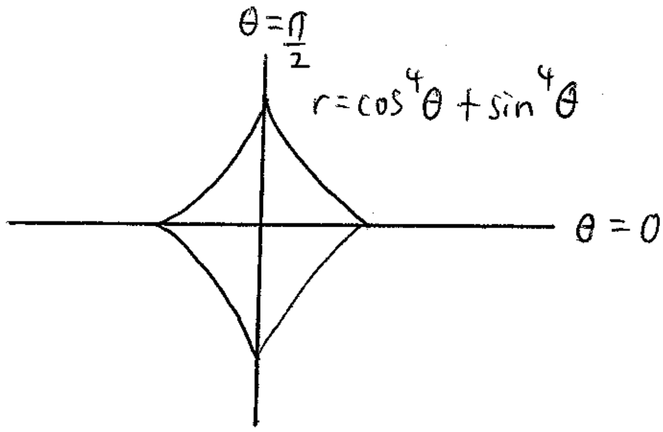
$r = \cos^4\theta + \sin^4\theta$

$\therefore$ C has the polar form $r = \cos^4\theta + \sin^4\theta$

ii)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	1	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	1	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	1

$\cos^4(-\theta) + \sin^4(-\theta) = \cos^4\theta + \sin^4\theta$



iii) Area $= \int_0^{\pi} \frac{r^2}{2}\,d\theta$

$= \int_0^{\pi} \frac{(\cos^4\theta + \sin^4\theta)^2}{2}\,d\theta$

$= \int_0^{\pi} \frac{[(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta]^2}{2}\,d\theta$

$$= \int_0^{\pi} \frac{(1 - 2\sin^2\theta\cos^2\theta)^2}{2} \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \left(1 - \frac{\sin^2 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \left(1 - \frac{1}{2}\left(\frac{1 - \cos 4\theta}{2}\right)\right)^2 d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \left(1 - \frac{1}{4} + \frac{\cos 4\theta}{4}\right)^2 d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \left(\frac{3}{4} + \frac{\cos 4\theta}{4}\right)^2 d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \frac{9 + 6\cos 4\theta + \cos^2 4\theta}{16} \, d\theta$$

$$= \frac{1}{32} \int_0^{\pi} 9 + 6\cos 4\theta + \frac{1 + \cos 8\theta}{2} \, d\theta$$

$$= \frac{1}{32} \int_0^{\pi} \frac{19 + 12\cos 4\theta + \cos 8\theta}{2} \, d\theta$$

$$= \frac{1}{64} \left[19\theta + 3\sin 4\theta + \frac{\sin 8\theta}{8} \right]_0^{\pi}$$

$$= \frac{1}{64}(19\pi - 0)$$

$$= \frac{19\pi}{64}$$

$\therefore$ Area enclosed by C for $0 \leq \theta < 2\pi$ is

$$2 \times \frac{19\pi}{64} = \frac{19\pi}{32}$$

iv) $r = \cos^4\theta + \sin^4\theta$

$$\frac{dr}{d\theta} = 4\cos^3\theta(-\sin\theta) + 4\sin^3\theta\cos\theta$$

$$= -4\cos^3\theta\sin\theta + 4\sin^3\theta\cos\theta$$

$$= 4\sin\theta\cos\theta(\sin^2\theta - \cos^2\theta)$$

when $\frac{dr}{d\theta} = 0$:

$$4\sin\theta\cos\theta(\sin^2\theta - \cos^2\theta) = 0$$

$\sin\theta = 0$, $\cos\theta = 0$, $\sin^2\theta - \cos^2\theta = 0$

$\sin^2\theta = \cos^2\theta$

$\tan^2\theta = 1$

$\tan\theta = \pm 1$

$$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

$$r = 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}.$$

$$\frac{d^2r}{d\theta^2} = 12\sin^2\theta\cos^2\theta - 4\sin^4\theta + 12\cos^2\theta\sin^2\theta - 4\cos^4\theta$$

$$= 24\sin^2\theta\cos^2\theta - 4\sin^4\theta - 4\cos^4\theta$$

when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$:

$$\frac{d^2r}{d\theta^2} = 6 - 1 - 1 = 4 > 0$$

when $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$:

$$\frac{d^2r}{d\theta^2} = -4 < 0$$

The points with minimum distance from the pole are $\left(\frac{1}{2}, \frac{\pi}{4}\right)$, $\left(\frac{1}{2}, \frac{3\pi}{4}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{4}\right)$ and $\left(\frac{1}{2}, \frac{7\pi}{4}\right)$.