8A. Differentiation

8A.1 Revision - Differentiation of Standard Functions

$\frac{d}{dx}(c) = 0\,, c = \text{constant}$	
$\frac{d}{dx}(x^n) = nx^{n-1}\,, n \in \mathbb{R}$	$\frac{d}{dx}[f^n(x)] = n[f^{n-1}(x)].f'(x)$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)}.f'(x)$
$\frac{d}{dx}(\sin mx) = m\cos x$	$\frac{d}{dx}(\sin^n x) = n\sin^{n-1} x\,.\cos x$
$\frac{d}{dx}(\cos mx) = -m\sin x$	$\frac{d}{dx}(\cos^n x) = -n\cos^{n-1} x\,.\sin x$
$\frac{d}{dx}(\tan mx) = \frac{m}{\cos^2 x} = m\sec^2 x$	$\frac{d}{dx}(\tan^n x) = n\tan^{n-1} x\,.\sec^2 x$
$\frac{d}{dx}(\sec mx) = m\sec x\tan x$	
$\frac{d}{dx}(\csc mx) = -m\csc x\cot x$	
$\frac{d}{dx}(\cot mx) = -m\csc^2 x$	

Example 8A.1:

Find the derivatives of the following functions.

a) $y = x^3$

b) $y = \frac{1}{\sqrt{x}}$

c) $y = x^{-\frac{3}{4}}$

d) $y = \ln 4x^3$

e) $y = \ln(4 - x^3)$

f) $y = e^{\tan(1-x^2)}$

g) $y = e^{3x^4}$

h) $y = \tan 5x$

i) $y = \cos^4 x$

j) $y = \sec^5 x$

Solution:

a) $y = x^3$

$\frac{dy}{dx} = 3x^{3-1} = 3x^2$

b) $y = \frac{1}{\sqrt{x}}$

$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$

c) $y = x^{-\frac{3}{4}}$

$\frac{dy}{dx} = -\frac{3}{4}x^{-\frac{3}{4}-1} = -\frac{3}{4}x^{-\frac{7}{4}}$

d) $y = \ln 4x^3$

$\frac{dy}{dx} = \frac{1}{4x^3}(12x^2)$

$= \frac{3}{x}$

e) $y = \ln(4 - x^3)$

$\frac{dy}{dx} = \frac{1}{4 - x^3}(-3x^2)$

$= \frac{3x^2}{x^3 - 4}$

f) Using the rule $\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)}.f'(x)$

$\frac{dy}{dx} = e^{\tan(1-x^2)}.\frac{d}{dx}[\tan(1 - x^2)]$

$= e^{\tan(1-x^2)}.\sec^2(1 - x^2).(-2x)$

$= -2xe^{\tan(1-x^2)}\sec^2(1 - x^2)$

g) $y = e^{3x^4}$

$\frac{dy}{dx} = e^{3x^4}(12x^3)$

$= 12x^3e^{3x^4}$

h) $y = \tan 5x$

$\frac{dy}{dx} = (\sec^2 5x)(5)$

$= 5\sec^2 5x$

i) $y = \cos^4 x$

$\frac{dy}{dx} = 4(\cos^{4-1} x)(-\sin x)(1)$

$= -4\cos^3 x \sin x$

j) $y = \sec^5 x$

$\frac{dy}{dx} = 5(\sec^{5-1} x)(\sec x \tan x)(1)$

$= 5\sec^5 x \tan x$

8A.2 Revision: Rules for Differentiation

$\frac{d}{dx}\{k[f(x)]\} = k\frac{d}{dx}[f(x)]$, $k =$ constant (constant multiple rule)

$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$, u and v are functions of x (product rule)

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, u and v are functions of x (quotient rule)

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$; $y = f(u), u = g(x)$ (chain rule)

Example 8A.2:

Differentiate each of the following with respect to x.

a) $3e^x - 4\sin x$

b) $(x^2+1)\left(\frac{1}{x} - x^3\right)$

c) $x^5 e^x$

d) $\cos x \ln x$

e) $\dfrac{x^2}{x+3}$

f) $\dfrac{\ln x}{\sin x}$

g) $(3x-2)^5$

h) $\ln(4-x^3)$

i) 5^{2x}

j) $\tan 5x$

k) $\cos^4 x$

l) $\sec^5 x$

Solution:

(a) Let $y = 3e^x - 4\sin x$

$$\frac{dy}{dx} = 3\frac{d}{dx}(e^x) - 4\frac{d}{dx}(\sin x)$$
$$= 3e^x - 4\cos x$$

(b) Let $y = (x^2+1)\left(\frac{1}{x} - x^3\right)$

If $\quad u = x^2 + 1 \quad$ and $\quad v = \frac{1}{x} - x^3 = x^{-1} - x^3$

then $\quad \frac{du}{dx} = 2x \quad$ and $\quad \frac{dv}{dx} = -x^{-2} - 3x^2 = -\frac{1}{x^2} - 3x^2$

Using the product rule,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= (x^2+1)\left(-\frac{1}{x^2} - 3x^2\right) + \left(\frac{1}{x} - x^3\right)(2x)$$
$$= -1 - 3x^4 - \frac{1}{x^2} - 3x^2 + 2 - 2x^4$$
$$= 1 - 3x^2 - 5x^4 - \frac{1}{x^2}$$

(c) Let $y = x^5 e^x$

If $u = x^5$ and $v = e^x$

then $\frac{du}{dx} = 5x^4$ and $\frac{dv}{dx} = e^x$

Using the product rule,

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x^5 e^x + e^x(5x^4) \\ &= x^4 e^x(x+5)\end{aligned}$$

(d) Let $y = \cos x \ln x$

If $u = \cos x$ and $v = \ln x$

then $\frac{du}{dx} = -\sin x$ and $\frac{dv}{dx} = \frac{1}{x}$

Using the product rule,

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= (\cos x)\left(\frac{1}{x}\right) + (\ln x)(-\sin x) \\ &= \frac{1}{x}\cos x - (\ln x)\sin x\end{aligned}$$

(e) Let $y = \frac{x^2}{x+3}$

If $u = x^2$ and $v = x + 3$

then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 1$

Using the quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(x+3)(2x) - x^2(1)}{(x+3)^2} \\ &= \frac{2x^2 + 6x - x^2}{(x+3)^2} \\ &= \frac{x(x+6)}{(x+3)^2}\end{aligned}$$

(f) Let $y = \frac{\ln x}{\sin x}$

If $u = \ln x$ and $v = \sin x$

then $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = \cos x$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(\sin x)\left(\frac{1}{x}\right) - (\ln x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin x - x\ln x \cos x}{x\sin^2 x}$$

(g) Let $y = (3x-2)^5$

If $u = 3x - 2$, then $y = u^5$

$\frac{du}{dx} = 3$ and $\frac{dy}{du} = 5u^4$

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 5u^4 \times 3$$

$$= 15u^4$$

$$= 15(3x-2)^4$$

(h) Let $y = \ln(4 - x^3)$

If $u = 4 - x^3$, then $y = \ln u$

$\frac{du}{dx} = -3x^2$ $\frac{dy}{du} = \frac{1}{u}$

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u}(-3x^2)$$

$$= \frac{-3x^2}{4 - x^3}$$

$$= \frac{3x^2}{x^3 - 4}$$

(i) Let $y = 5^{2x}$, then $\ln y = 2x \ln 5$

Differentiate both sides with respect to x

$$\frac{d}{dx}\ln y = \frac{d}{dx}(2x \ln 5)$$

Apply chain rule on the left term,

$$\frac{d}{dy}(\ln y).\frac{dy}{dx} = 2 \ln 5$$

$$\frac{1}{y}\frac{dy}{dx} = 2 \ln 5$$

$$\frac{dy}{dx} = 2y \ln 5$$

(j) Let $y = \tan 5x$

If $u = 5x$, then $y = \tan u$

$\frac{du}{dx} = 5$ $\qquad$ $\frac{dy}{du} = \sec^2 u$

Using chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 5 \sec^2 u \\ &= 5 \sec^2 5x\end{aligned}$$

(k) Let $y = \cos^4 x$

If $u = \cos x$, $y = u^4$

$\frac{du}{dx} = -\sin x$ $\qquad$ $\frac{dy}{du} = 4u^3$

Using chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 4u^3(-\sin x) \\ &= -4 \cos^3 x \sin x\end{aligned}$$

(l) Let $y = \sec^5 x$

If $u = \sec x$, then $y = u^5$

$\frac{du}{dx} = \sec x \tan x$ $\qquad$ $\frac{dy}{du} = 5u^4$

Using chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 5u^4 \times \sec x \tan x \\ &= 5 \sec^4 x (\sec x \tan x) \\ &= 5 \sec^5 x \tan x\end{aligned}$$

8A.3 Differentiation of Parametric Equations

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \ ; \ y = f(t), x = g(t)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} \ ; \ y = f(t), x = g(t)$$

Example 8A.3:

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following parametric equations.

a) $x = 2t \, , y = t^2$

b) $x = \frac{1}{t} \, , y = 4t$

c) $x = \sin\theta \, , y = \cos\theta$

d) $x = \sin^2\theta \, , y = 2\sin\theta$

e) $x = t - \frac{1}{t} \, , y = t + \frac{1}{t}$

f) $x = e^t \, , y = \sin t$

8A.4 Differentiation of Implicit Functions

So far, the differentiation has been done involving functions of the form $y = f(x)$.

Now consider the equation $x^2 + x^3y^2 = y^3$. In this equation, the functions are defined implicitly since it cannot be put easily in the form $y = f(x)$.

Take for example the term x^3y^2. To differentiate this term,

$$\frac{d}{dx}(x^3y^2) = x^3\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x^3)$$

$$= x^3\frac{d}{dy}(y^2)\frac{dy}{dx} + y^2(3x^2)$$

$$= x^3(2y)\frac{dy}{dx} + 3x^2y^2$$

$$= 2x^3y\frac{dy}{dx} + 3x^2y^2$$

To differentiate implicit functions,

(1) Differentiate both sides of the equation with respect to x
(2) Place all the terms with $f'(x)$ on one side
(3) Factorize and solve for $f'(x)$

Second Derivatives

Consider the function $y = f(x)$.

By differentiating y with respect to x, the first derivative of f, that is, $\frac{dy}{dx} = f'(x)$ is obtained.

If $\frac{dy}{dx}$ is differentiated with respect to x again, the second derivative of f is obtained, that is,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

Example 8A.4:

(i) Find $\frac{dy}{dx}$ of the following functions.

(a) $x^3 + y^3 = 3x$

(b) $y\sqrt{x} - y^2 = 6$

(c) $x \sin y + \ln y = x^2$

(d) $\sqrt{xy} = x + y$

(ii) (a) If $y = x^4$, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 1$.

(b) If $y = ax^2 + \frac{b}{x}$, prove that $x^2 \frac{d^2y}{dx^2} = 2y$.

(c) Find $\frac{d^2y}{dx^2}$ of the implicit functions $xy = 1$ and $y^2 - xy = 8$.