<u>Telescoping Series</u>

Definition :

<u>**Telescoping series**</u> is a series whose partial sums cancel off, leaving a fixed number of terms.

Example :

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}.$$

Example:

Determine whether they are telescoping series.

(a)
$$
\begin{aligned}
&\left(0^5-2^5\right)\\
+&\left(1^5-3^5\right)\\
+&\left(2^5-4^5\right)\\
+&\quad\vdots\\
+&\left[(n-1)^5-(n+1)^5\right]
\end{aligned}
$$

(b)
$$
\begin{aligned}
&(1+2)\\
+&(2+3)\\
+&(3+4)\\
+&\quad\vdots\\
+&\left[r+(r+1)\right]
\end{aligned}
$$

Example :

Explain whether they are telescoping series. Hence, evaluate the summations.

$$\text{(a)} \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1} \right) \qquad \text{(b)} \sum_{r=1}^{n} \left(\frac{1}{2r-1} - \frac{1}{2r+1} \right)$$

Method of Differences

Definition :

Method of differences form telescoping series by using *difference* between two *composite* functions.

Example:

Evaluate $\sum_{r=1}^{n} g(r)$, for $r \in \mathrm{N}$, if

(a) $g(r) = f(r) - f(r+1)$.

(b) $g(r) = f(r-2) - f(r)$.

Example:

Express $g(r)=\dfrac{1}{(r+1)(r+2)}$ as partial fractions.

Hence, or otherwise, evaluate $\sum_{r=1}^{n} g(r)$.

Example:

Express $g(r) = \dfrac{1}{(3r-2)(3r+1)}$ as partial fractions.

Hence, or otherwise, evaluate $\sum_{r=1}^{n} g(r)$.

Example :

Express $\dfrac{1}{r(r+2)}$ as partial fraction, and use your result to find $S_n = \dfrac{1}{1\times 3} + \dfrac{1}{2\times 4} + \ldots + \dfrac{1}{n(n+2)}$. Hence, find $\lim_{n\to\infty} S_n$.

Example :

Evaluate $S_n = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \ldots + \frac{1}{n(n+1)(\mathrm{n}+2)}$.

Hence, show that $\lim_{n\to\infty} S_n = \frac{1}{4}$.

Example:

Show that $(r+1)^4-(r-1)^4=8r^3+8r$. Hence,

Show that $\sum_{r=1}^{n} r^3=\frac{1}{4}n^2(n+1)^2$.

Example :

Show that $(r+1)! - r! \equiv r \cdot r!$.

Hence, sum the series $1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n!$

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example :

Let $S = \dfrac{2}{1\times 3} + \dfrac{2}{3\times 5} + \dfrac{2}{5\times 7} + \ldots$

(a) Write down the r – th term of S, and express it in partial fractions.

(b) Find S.

Example :

Given that $T_r = \dfrac{1}{r^3 - r}$.

(a) Express T_r in partial fractions, and hence find $\sum_{r=2}^{n} T_r$.

(b) Explain why the series $\dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{60} + \ldots + \dfrac{1}{r^3 - r} + \ldots$ is convergent.

Example :

Show that $(2r+1)^5-(2r-1)^5=2+40r^2+80r^4$. Hence,

Show that $\sum_{r=1}^{n} r^4=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

Example :

Simplify $\frac{1}{4}r(r+1)(r+2)(r+3)-\frac{1}{4}(r-1)r(r+1)(r+2)$.

Hence, find $\sum_{r=1}^{n} r(r+1)(r+2)$.

Example :

Express $\dfrac{r}{(r-1)(r+1)}$ in partial fractions, and use your

results to sum the series $\dfrac{2}{1\times 3}-\dfrac{4}{3\times 5}+\ldots+\dfrac{(-1)^{n+1}2n}{(2n-1)(2n+1)}$.

Find also $\lim_{n\to\infty} S_n$.