OCTOBER / NOVEMBER 2013

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1. C: $r = 2e^{\theta}$, $\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$

i) The area of the region from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$

is $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4e^{2\theta}}{2}\, d\theta$

$$= \left[e^{2\theta} \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= e^{\pi} - e^{\frac{\pi}{3}}$$

ii) The arc length of C is

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4e^{2\theta} + 4e^{2\theta}}\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{8e^{2\theta}}\, d\theta$$

$$= 2\sqrt{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} e^{\theta}\, d\theta$$

$$= 2\sqrt{2} \left[e^{\theta} \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 2\sqrt{2}\left(e^{\frac{\pi}{2}} - e^{\frac{\pi}{6}}\right)$$

2. $x^3 - px - q = 0$

α, β, γ are the roots.

$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \alpha\gamma + \beta\gamma = -p \quad \alpha\beta\gamma = q$

i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 0^2 - 2(-p)$

$= 2p$

ii) If $S_n = \alpha^n + \beta^n + \gamma^n$

$S_{3+r} - pS_{1+r} - qS_r = 0$

$S_0 = \alpha^0 + \beta^0 + \gamma^0 = 1 + 1 + 1 = 3$

$S_1 = \alpha^1 + \beta^1 + \gamma^1 = \alpha + \beta + \gamma = 0$

$r = 0: S_3 - pS_1 - qS_0 = 0$

$\therefore S_3 = 3q$

iii) $r = 2: S_5 - pS_3 - qS_2 = 0$

$S_5 - p(3q) - q(2p) = 0$

$S_5 = 5pq$

$6S_5 = 6(5pq) = 30pq = 5(6pq)$

$= 5(2p)(3q)$

$= 5S_3S_2$

$\therefore 6(\alpha^5 + \beta^5 + \gamma^5) = 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^2 + \beta^2 + \gamma^2)$

3. $S_n = \sum_{r=1}^{n} u_r = 2n^2 + n$

$S_1 = 2 + 1 = 3$

$S_2 = 8 + 2 = 10$

$S_3 = 18 + 3 = 21$

$S_4 = 32 + 4 = 36$

$S_n = \sum_{r=1}^{n} u_r = u_1 + u_2 + \cdots + u_{n-1} + u_n$

$S_{n-1} = \sum_{r=1}^{n-1} u_r = u_1 + u_2 + \cdots + u_{n-1}$

$\therefore u_n = S_n - S_{n-1}$

$= 2n^2 + n - (2(n-1)^2 + n - 1)$

$= 2n^2 + n - 2(n-1)^2 - n + 1$

$= 2n^2 + n - 2(n^2 - 2n + 1) - n + 1$

$= 2n^2 + n - 2n^2 + 4n - 2 - n + 1$

$= 4n - 1$

$\therefore u_r = 4r - 1$

$\sum_{r=n+1}^{2n} u_r = \sum_{r=1}^{2n} u_r - \sum_{r=1}^{n} u_r$

$= S_{2n} - S_n$

$= 2(2n)^2 + 2n - (2n^2 + n)$

$= 8n^2 + 2n - 2n^2 - n$

$= 6n^2 + n$

4. $I_n = \int_0^1 \frac{x^n}{\sqrt{1+2x}}\,dx$

$u = x^n \qquad dv = (1+2x)^{-\frac{1}{2}}dx$

$du = nx^{n-1}dx \qquad v = (1+2x)^{\frac{1}{2}}$

$= \left[x^n(1+2x)^{\frac{1}{2}}\right]_0^1 - \int_0^1 nx^{n-1}(1+2x)^{\frac{1}{2}}dx$

$= \sqrt{3} - 0 - n\int_0^1 x^{n-1}(1+2x)^{\frac{1}{2}}dx$

$= \sqrt{3} - n\int_0^1 \frac{x^{n-1}(1+2x)}{\sqrt{1+2x}}dx$

$= \sqrt{3} - n\int_0^1 \frac{x^{n-1}}{\sqrt{1+2x}} + \frac{2x^n}{\sqrt{1+2x}}dx$

$= \sqrt{3} - n\int_0^1 \frac{x^{n-1}}{\sqrt{1+2x}}dx - 2n\int_0^1 \frac{x^n}{\sqrt{1+2x}}dx$

$= \sqrt{3} - nI_{n-1} - 2nI_n$

$(2n+1)I_n = \sqrt{3} - nI_{n-1}$

$n=3: \ 7I_3 = \sqrt{3} - 3I_2$

$n=2: \ 5I_2 = \sqrt{3} - 2I_1$

$n=1: \ 3I_1 = \sqrt{3} - I_0$

$I_0 = \int_0^1 \frac{1}{\sqrt{1+2x}}dx = \left[(1+2x)^{\frac{1}{2}}\right]_0^1$

$= \sqrt{3} - 1$

$$3I_1 = \sqrt{3} - (\sqrt{3} - 1)$$
$$= 1$$
$$I_1 = \frac{1}{3}$$
$$5I_2 = \sqrt{3} - 2\left(\frac{1}{3}\right)$$
$$= \sqrt{3} - \frac{2}{3}$$
$$I_2 = \frac{\sqrt{3}}{5} - \frac{2}{15}$$
$$7I_3 = \sqrt{3} - 3\left(\frac{\sqrt{3}}{5} - \frac{2}{15}\right)$$
$$= \frac{2\sqrt{3}}{5} + \frac{2}{5}$$
$$I_3 = \frac{2\sqrt{3}}{35} + \frac{2}{35}$$
$$= \frac{2}{35}(\sqrt{3} + 1)$$
$$\vdots$$

5. $y = (1+x)^2 \ln(1+x)$

$$\frac{dy}{dx} = \frac{(1+x)^2}{1+x} + 2(1+x)\ln(1+x) = 1+x+2(1+x)\ln(1+x)$$

$$\frac{d^2y}{dx^2} = 1 + 2\ln(1+x) + \frac{2(1+x)}{1+x} = 3 + 2\ln(1+x)$$

$$\frac{d^3y}{dx^3} = \frac{2}{1+x}$$

$$\frac{d^ny}{dx^n} = (-1)^{n-1}\frac{2(n-3)!}{(1+x)^{n-2}}, \quad n \geq 3$$

when $n=3$: $\frac{d^3y}{dx^3} = (-1)^{3-1}\frac{2(3-3)!}{(1+x)^{3-2}}$

$$= \frac{(-1)^2 2(0!)}{(1+x)^1}$$

$$= \frac{1 \cdot 2 \cdot 1}{1+x} = \frac{2}{1+x}$$

Assume the statement is true when $n=k$.

$n=k$: $\frac{d^ky}{dx^k} = (-1)^{k-1}\frac{2(k-3)!}{(1+x)^{k-2}}$.

when $n=k+1$: $\frac{d^{k+1}y}{dx^{k+1}} = (-1)^k\frac{2(k-2)!}{(1+x)^{k-1}}$

(what needs to be proved)

$$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left(\frac{(-1)^{k-1}2(k-3)!}{(1+x)^{k-2}}\right)$$

$$= (-1)^{k-1}2(k-3)!\,\frac{d}{dx}\left[(1+x)^{-(k-2)}\right]$$

$$= (-1)^{k-1}2(k-3)!\left[-(k-2)(1+x)^{-k+2-1}\right]$$

$$= (-1)^{k-1}(-1)2(k-3)!\,(1+x)^{-k+1}(k-2)$$

$$= (-1)^{k}2(k-2)(k-3)!\,(1+x)^{-(k-1)}$$

$$= (-1)^{k}\frac{2(k-2)!}{(1+x)^{k-1}}$$

$$\therefore \frac{d^n y}{dx^n} = (-1)^{n-1}\frac{2(n-3)!}{(1+x)^{n-2}}$$ for every integer $n \geqslant 3$.

6. $T: R^4 \to R^4$

$$M = \begin{pmatrix} 1 & -3 & -1 & 2 \\ 4 & -10 & 0 & 2 \\ 1 & -1 & 3 & -4 \\ 5 & -12 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[-5r_1 + r_4]{\substack{-4r_1 + r_2 \\ -r_1 + r_3}} \begin{pmatrix} 1 & -3 & -1 & 2 \\ 0 & 2 & 4 & -6 \\ 0 & 2 & 4 & -6 \\ 0 & 3 & 6 & -9 \end{pmatrix}$$

$$\xrightarrow{\frac{r_2}{2},\ \frac{r_3}{2},\ \frac{r_4}{3}} \begin{pmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & 2 & -3 \end{pmatrix}$$

$$\xrightarrow{\substack{-r_2 + r_3 \\ -r_3 + r_4}} \begin{pmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank $M = 2$

If $\begin{pmatrix} 1 & -3 & -1 & 2 \\ 4 & -10 & 0 & 2 \\ 1 & -1 & 3 & -4 \\ 5 & -12 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & -3 & -1 & 2 & 0 \\ 4 & -10 & 0 & 2 & 0 \\ 1 & -1 & 3 & -4 & 0 \\ 5 & -12 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -3 & -1 & 2 & 0 \\ 0 & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s, s \in \mathbb{R}$ and $w = t, t \in \mathbb{R}$

$$y = 3t - 2s$$

$$x - 3y - z + 2w = 0$$

$$x - 3(3t - 2s) - s + 2t = 0$$

$$x - 9t + 6s - s + 2t = 0$$

$$x = 7t - 5s$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 7t - 5s \\ 3t - 2s \\ s \\ t \end{pmatrix} = t \begin{pmatrix} 7 \\ 3 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -5 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ A basis for the null space K of T is $\left\{ \begin{pmatrix} 7 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$$M \begin{pmatrix} 1 \\ -2 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 & -3 & -1 & 2 \\ 4 & -10 & 0 & 2 \\ 1 & -1 & 3 & -4 \\ 5 & -12 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 16 \\ 10 \\ 22 \end{pmatrix}.$$

If $M\underset{\sim}{x} = \begin{pmatrix} 2 \\ 16 \\ 10 \\ 22 \end{pmatrix}$

since $\begin{pmatrix} 1 \\ -2 \\ -3 \\ -4 \end{pmatrix}$ is a solution of $M\underset{\sim}{x} = \begin{pmatrix} 2 \\ 16 \\ 10 \\ 22 \end{pmatrix}$

and $\left\{ \begin{pmatrix} 7 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis of the null space of T

$\underset{\sim}{x} = \begin{pmatrix} 1 \\ -2 \\ -3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ is the general solution of $M\underset{\sim}{x} = \begin{pmatrix} 2 \\ 16 \\ 10 \\ 22 \end{pmatrix}$.

7. $A\underset{\sim}{e} = \lambda \underset{\sim}{e}$

$A^2\underset{\sim}{e} = A(A\underset{\sim}{e}) = A(\lambda\underset{\sim}{e}) = \lambda(A\underset{\sim}{e}) = \lambda(\lambda\underset{\sim}{e}) = \lambda^2\underset{\sim}{e}$

$\therefore \underset{\sim}{e}$ is an eigenvector of A^2 with eigenvalue λ^2.

$$B = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

$$B - \lambda I = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 3 & 0 \\ 2 & -\lambda & 2 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$|B - \lambda I| = (1-\lambda)[-\lambda(2-\lambda) - 2] - 3(4 - 2\lambda - 2) + 0$$
$$= (1-\lambda)\lambda(\lambda-2) - 2(1-\lambda) - 3(2-2\lambda)$$
$$= \lambda(\lambda - 2 - \lambda^2 + 2\lambda) - 2 + 2\lambda - 6 + 6\lambda$$
$$= \lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 - 8 + 8\lambda$$
$$= -\lambda^3 + 6\lambda^2 + 3\lambda - 8$$
$$= (1-\lambda)(\lambda^2 - 2\lambda) - 2(1-\lambda) - 6(1-\lambda)$$
$$= (1-\lambda)(\lambda^2 - 2\lambda - 2 - 6)$$
$$= (1-\lambda)(\lambda^2 - 2\lambda - 8)$$
$$= (1-\lambda)(\lambda - 4)(\lambda + 2)$$

If $|B - \lambda I| = 0$: $(1-\lambda)(\lambda-4)(\lambda+2) = 0$

$\lambda = 4, 1, -2.$

If $B\underset{\sim}{e} = \lambda\underset{\sim}{e} \therefore B^2\underset{\sim}{e} = \lambda^2\underset{\sim}{e}$

$$(B^4 + 2B^2 + 3I)\underset{\sim}{e} = B^4\underset{\sim}{e} + 2B^2\underset{\sim}{e} + 3I\underset{\sim}{e}$$

$$= B^2(B^2\underset{\sim}{e}) + 2(B^2\underset{\sim}{e}) + 3\underset{\sim}{e}$$

$$= B^2(\lambda^2\underset{\sim}{e}) + 2(\lambda^2\underset{\sim}{e}) + 3\underset{\sim}{e}$$

$$= \lambda^2(B^2\underset{\sim}{e}) + 2\lambda^2\underset{\sim}{e} + 3\underset{\sim}{e}$$

$$= \lambda^2(\lambda^2\underset{\sim}{e}) + 2\lambda^2\underset{\sim}{e} + 3\underset{\sim}{e}$$

$$= \lambda^4\underset{\sim}{e} + 2\lambda^2\underset{\sim}{e} + 3\underset{\sim}{e}$$

$$= (\lambda^4 + 2\lambda^2 + 3)\underset{\sim}{e}$$

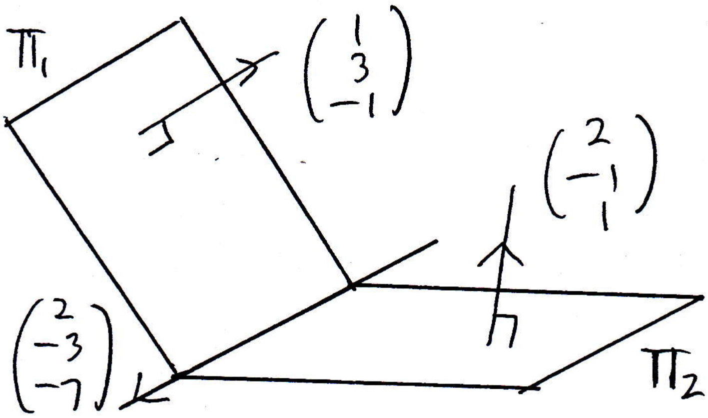
Since the eigenvalues of B are $4, 1, -2$,

the eigenvalues of $B^4 + 2B^2 + 3I$ are

$4^4 + 2(4^2) + 3$, $1^4 + 2(1^2) + 3$, $(-2)^4 + 2(-2)^2 + 3$

$291, 6, 27$.

8. $\Pi_1 : \underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$



Since $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ are parallel to Π_1,

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ is perpendicular to Π_1.

Since $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ is normal to Π_1 and $(2,3,-1)$ is a point on Π_1, if $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is any point on Π_1, $\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = 2 + 9 + 1$$

$$x + 3y - z = 12.$$

$\therefore$ The Cartesian equation of Π_1 is $x + 3y - z = 12$.

$\Pi_2 : 2x - y + z = 10$

$$\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \left|\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right| \cos\theta$$

$$2 - 3 - 1 = \sqrt{11}\sqrt{6} \cos\theta$$

$$-2 = \sqrt{11}\sqrt{6} \cos\theta$$

$\therefore \cos\theta = \dfrac{-2}{\sqrt{6}\sqrt{11}}$

$\theta = \cos^{-1} \dfrac{-2}{\sqrt{6}\sqrt{11}} \approx 104.3^{\circ}$

$\therefore$ The acute angle between Π_1 and Π_2 is 75.7°

$x + 3y - z = 12 \quad -①$

$2x - y + z = 10 \quad -②$

$① + ② : 3x + 2y = 22$

Let $x = 2s, s \in \mathbb{R}$

$3(2s) + 2y = 22$

$3s + y = 11$

$y = 11 - 3s$

$2s + 3(11 - 3s) - z = 12$.

$2s + 33 - 9s - z = 12$

$z = 21 - 7s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ 11 - 3s \\ 21 - 7s \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 11 \\ 21 \end{pmatrix} + s \begin{pmatrix} 2 \\ -3 \\ -7 \end{pmatrix}$$

$\therefore$ The line of intersection of Π_1 and Π_2 is

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 11 \\ 21 \end{pmatrix} + s \begin{pmatrix} 2 \\ -3 \\ -7 \end{pmatrix}.$$

9. $C: x = t^2 \quad y = t - \frac{t^3}{3}, \quad 0 \le t \le 1.$

$$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 1 - t^2$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (2t)^2 + (1-t^2)^2$$

$$= 4t^2 + 1 - 2t^2 + t^4$$

$$= t^4 + 2t^2 + 1$$

$$= (t^2+1)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t^2 + 1$$

Surface area of revolution of 2π radians about the x-axis is

$$\int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 2\pi \left(t - \frac{t^3}{3}\right)(t^2+1)\, dt$$

$$= \int_0^1 2\pi \left(t^3 - \frac{t^5}{3} + t - \frac{t^3}{3}\right) dt$$

$$= \int_0^1 2\pi \left(\frac{2t^3}{3} + t - \frac{t^5}{3}\right) dt$$

$$= 2\pi \left[\frac{t^4}{6} + \frac{t^2}{2} - \frac{t^6}{18}\right]_0^1$$

$$= 2\pi \left(\frac{11}{18} - 0\right)$$

$$= \frac{11\pi}{9} \text{ unit}^3.$$

The area, A, of the region bounded by C, the x-axis and the line $x=1$ is $\quad x = 1 = t^2$

$\therefore\ t = 1$

$$A = \int_0^1 y\,dx = \int_0^1 y\frac{dx}{dt}\,dt = \int_0^1 \left(t - \frac{t^3}{3}\right)2t\,dt$$

$$= \int_0^1 2t^2 - \frac{2t^4}{3}\,dt = \left[\frac{2t^3}{3} - \frac{2t^5}{15}\right]_0^1 = \frac{8}{15} - 0 = \frac{8}{15}$$

If $(\bar{x}, \bar{y})$ are the coordinates of the centroid of the region bounded by C, the x-axis and the line $x=1$,

$$A\bar{x} = \int_0^1 xy\,dx = \int_0^1 xy\frac{dx}{dt}\,dt = \int_0^1 t^2\left(t - \frac{t^3}{3}\right)2t\,dt$$

$$= \int_0^1 2t^4 - \frac{2t^6}{3}\,dt = \left[\frac{2t^5}{5} - \frac{2t^7}{21}\right]_0^1 = \frac{32}{105} - 0 = \frac{32}{105}$$

$$A\bar{y} = \int_0^1 \frac{y^2}{2}\,dx = \int_0^1 \frac{y^2}{2}\frac{dx}{dt}\,dt = \int_0^1 \frac{1}{2}\left(t - \frac{t^3}{3}\right)^2 2t\,dt$$

$$= \int_0^1 t\left(t^2 - \frac{2t^4}{3} + \frac{t^6}{9}\right)dt = \int_0^1 t^3 - \frac{2t^5}{3} + \frac{t^7}{9}\,dt$$

$$= \left[\frac{t^4}{4} - \frac{t^6}{9} + \frac{t^8}{72}\right]_0^1 = \frac{11}{72} - 0 = \frac{11}{72}$$

$A\bar{x} = \frac{32}{105} \qquad A\bar{y} = \frac{11}{72}$

$\bar{x} = \frac{4}{7} \qquad \bar{y} = \frac{55}{192}$

$$\therefore (\bar{x}, \bar{y}) = \left(\frac{4}{7}, \frac{55}{192}\right)$$

10. $C: y = \frac{px^2 + 4x + 1}{x+1}, \quad p > 0, p \neq 3.$

$$\begin{array}{r} px + 4 - p \\ x+1 \overline{)\ px^2 + 4x + 1} \\ \underline{px^2 + px} \\ (4-p)x + 1 \\ \underline{(4-p)x + 4 - p} \\ p - 3 \end{array}$$

$= px + 4 - p + \frac{p-3}{x+1}$

i) As $x \to \pm\infty$ $y \to px + 4 - p$

As $x \to -1$ $y \to \pm\infty$

$\therefore$ The asymptotes of C are $x = -1$ and $y = px + 4 - p$.

ii) when C intersects the x-axis, $y = 0$

$y = 0: \frac{px^2 + 4x + 1}{x+1} = 0$

$px^2 + 4x + 1 = 0$

If the x-axis is a tangent, C intersects the axis once $\therefore b^2 = 4ac$

$16 = 4p$

$p = 4.$

$p = 1: y = x + 3 - \frac{2}{x+1}$

$\frac{dy}{dx} = 1 + \frac{2}{(x+1)^2}$

≥ 1

> 0

$\therefore$ C has no turning points when $p = 1$.

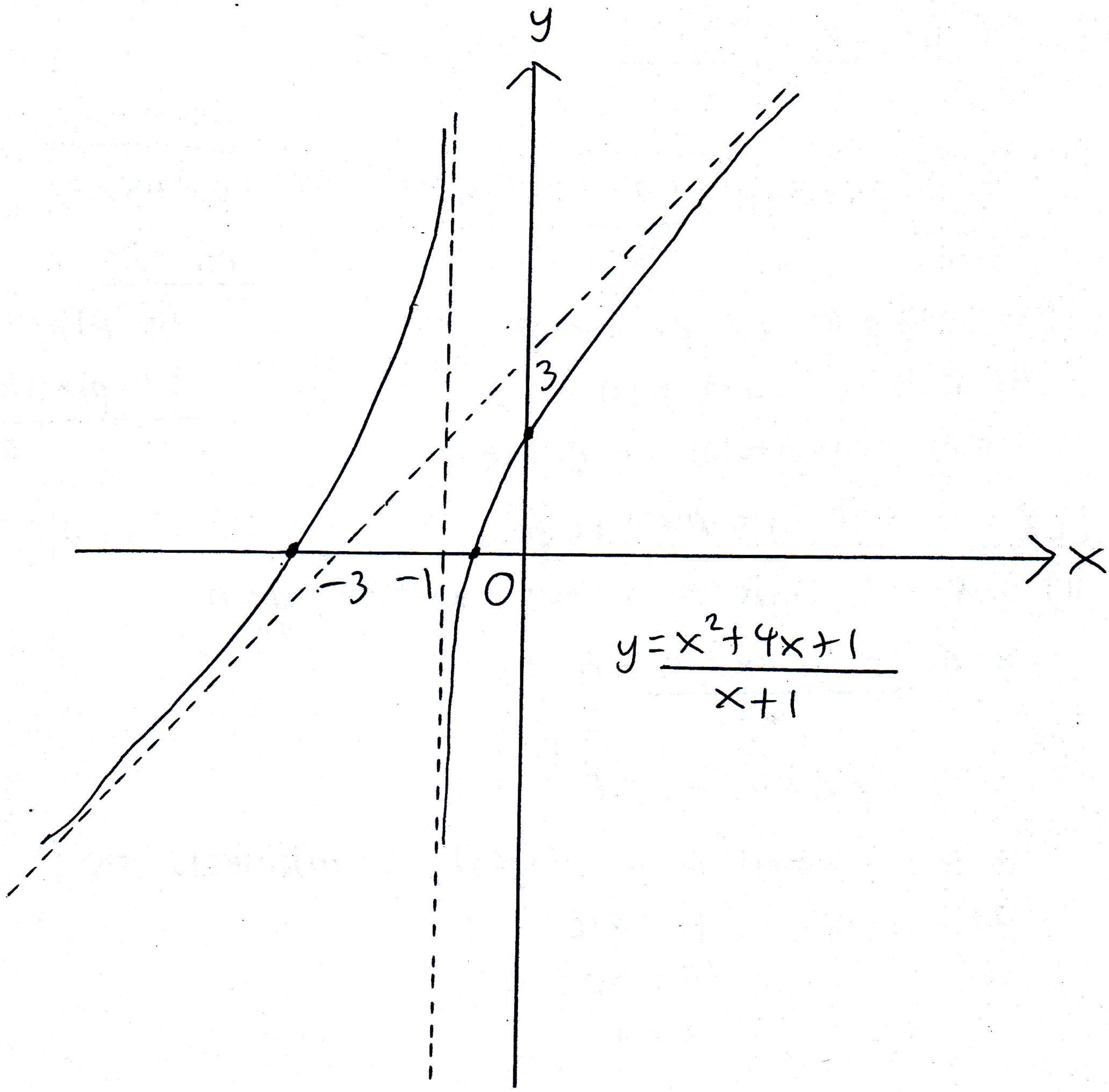
when $x = 0$: $y = 1$.

when $y = 0$: $x^2 + 4x + 1 = 0$

$x^2 + 4x + 4 = 3$

$(x+2)^2 = 3$

$x = -2 \pm \sqrt{3}$



y
3
x
-3
-1
0
$y = \frac{x^2+4x+1}{x+1}$

11. EITHER

$$1^{\frac{1}{5}} = (\cos 0 + i\sin 0)^{\frac{1}{5}}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{5}}, \quad k \in \mathbb{Z}$$

$$= \cos \frac{2k\pi}{5} + i\sin \frac{2k\pi}{5}, \quad k = 0, 1, 2, 3, 4.$$

$$= \cos 0 + i\sin 0, \ \cos \frac{2\pi}{5} + i\sin \frac{2\pi}{5}, \ \cos \frac{4\pi}{5} + i\sin \frac{4\pi}{5},$$

$$\cos \frac{6\pi}{5} + i\sin \frac{6\pi}{5}, \ \cos \frac{8\pi}{5} + i\sin \frac{8\pi}{5}$$

$$= \cos 0 + i\sin 0, \ \cos \frac{2\pi}{5} + i\sin \frac{2\pi}{5}, \ \cos \frac{4\pi}{5} + i\sin \frac{4\pi}{5},$$

$$\cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right), \ \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$$

$$\left(x - \left[\cos \frac{2\pi}{5} + i\sin \frac{2\pi}{5}\right]\right)\left(x - \left[\cos \frac{2\pi}{5} - i\sin \frac{2\pi}{5}\right]\right)$$

$$= \left(x - \left[\cos \frac{2\pi}{5} + i\sin \frac{2\pi}{5}\right]\right)\left(x - \left[\cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)\right]\right)$$

$$= \left(x - e^{\frac{2\pi i}{5}}\right)\left(x - e^{\frac{-2\pi i}{5}}\right)$$

$$= x^2 - \left(e^{\frac{2\pi i}{5}} + e^{\frac{-2\pi i}{5}}\right)x + 1$$

$$= x^2 - 2\left(\cos \frac{2\pi}{5}\right)x + 1$$

$\therefore \quad x^5-1=(x-1)(x^2-2x\cos\frac{2\pi}{5}+1)(x^2-2x\cos\frac{4\pi}{5}+1)$.

$x^6-x^3+1=0$

$a=1 \quad b=-1 \quad c=1$

$x^3=\frac{1\pm\sqrt{-3}}{2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$x^3=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

$=\cos\left(\frac{\pi}{3}+2k\pi\right)+i\sin\left(\frac{\pi}{3}+2k\pi\right), k\in\mathbb{Z}$

$x=\cos\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right), k=0,1,2$

$x=\cos\frac{\pi}{9}+i\sin\frac{\pi}{9},\ \cos\frac{7\pi}{9}+i\sin\frac{7\pi}{9},\ \cos\frac{13\pi}{9}+i\sin\frac{13\pi}{9}$

$x^3=\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$

$=\cos\left(\frac{5\pi}{3}+2k\pi\right)+i\sin\left(\frac{5\pi}{3}+2k\pi\right), k\in\mathbb{Z}$

$=\cos\left(\frac{5\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{5\pi}{9}+\frac{2k\pi}{3}\right), k=0,1,2$

$=\cos\frac{5\pi}{9}+i\sin\frac{5\pi}{9},\ \cos\frac{11\pi}{9}+i\sin\frac{11\pi}{9},\ \cos\frac{17\pi}{9}+i\sin\frac{17\pi}{9}$

$=\cos\frac{13\pi}{9}-i\sin\frac{3\pi}{9},\ \cos\frac{7\pi}{9}-i\sin\frac{7\pi}{9},\ \cos\frac{\pi}{9}-i\sin\frac{\pi}{9}$

$\therefore$ The roots of $x^6 - x^3 + 1 = 0$ are

$\cos\frac{\pi}{9} \pm i\sin\frac{\pi}{9}$, $\cos\frac{7\pi}{9} \pm i\sin\frac{7\pi}{9}$, $\cos\frac{13\pi}{9} \pm i\sin\frac{13\pi}{9}$

$\therefore$ The factors of $x^6 - x^3 + 1$ are

$$x^6 - x^3 + 1$$

$$= \left(x - \left[\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right]\right)\left(x - \left[\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right]\right)$$

$$\left(x - \left[\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right]\right)\left(x - \left[\cos\frac{7\pi}{9} - i\sin\frac{7\pi}{9}\right]\right)$$

$$\left(x - \left[\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right]\right)\left(x - \left[\cos\frac{13\pi}{9} - i\sin\frac{13\pi}{9}\right]\right).$$

$$= \left(x^2 - 2x\cos\frac{\pi}{9} + 1\right)\left(x^2 - 2x\cos\frac{7\pi}{9} + 1\right)\left(x^2 - 2x\cos\frac{13\pi}{9} + 1\right).$$

11. OR

$$y^2\frac{d^2y}{dx^2} - 6y^2\frac{dy}{dx} + 2y\left(\frac{dy}{dx}\right)^2 + 3y^3 = 25e^{-2x}$$

$v = y^3$

$\frac{dv}{dy} = 3y^2$

$\frac{dv}{dx} = 3y^2\frac{dy}{dx}$

$\frac{d^2v}{dx^2} = 3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2$

$$3y^2\frac{d^2y}{dx^2} - 18y^2\frac{dy}{dx} + 6y\left(\frac{dy}{dx}\right)^2 + 9y^3 = 75e^{-2x}$$

$$3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 - 6\left(3y^2\frac{dy}{dx}\right) + 9y^3 = 75e^{-2x}$$

$\therefore \frac{d^2v}{dx^2} - 6\frac{dv}{dx} + 9v = 75e^{-2x}$

$m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

$m = 3$

$\therefore$ The complementary function, v_c, is

$v_c = (Ax+B)e^{3x}$.

The particular integral, v_p, is

$v_p = ce^{-2x}$

$$\frac{dv_p}{dx} = -2Ce^{-2x}$$

$$\frac{d^2v_p}{dx^2} = 4Ce^{-2x}$$

$$\therefore \frac{d^2v_p}{dx^2} - 6\frac{dv_p}{dx} + 9v_p = 4Ce^{-2x} + 12Ce^{-2x} + 9Ce^{-2x}$$

$$= 25Ce^{-2x}$$

$$= 75e^{-2x}$$

$$25C = 75$$

$$C = 3$$

$$\therefore v_p = 3e^{-2x}$$

$$\therefore v = v_c + v_p$$

$$= (Ax+B)e^{3x} + 3e^{-2x}$$

$$y^3 = (Ax+B)e^{3x} + 3e^{-2x}$$

$x=0$ $y=2$: $8 = B + 3$

$$B = 5$$

$$3y^2\frac{dy}{dx} = Ae^{3x} + 3(Ax+B)e^{3x} - 6e^{-2x}$$

$x=0$ $y=2$ $\frac{dy}{dx}=1$: $12 = A + 3B - 6$

$$A = 3$$

$$\therefore y^3 = (3x+5)e^{3x} + 3e^{-2x}$$

$$\therefore y = \left(3xe^{3x} + 5e^{3x} + 3e^{-2x}\right)^{\frac{1}{3}}.$$