MAY/ JUNE 2006

1. $u_n = \dfrac{1}{4n^2 - 1}$

Expressing u_n as partial fractions,

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$= \frac{A(2n+1) + B(2n-1)}{(2n+1)(2n-1)}$$

$$\therefore 1 = A(2n+1) + B(2n-1)$$

$$= 2(A+B)n + A - B$$

$$\therefore 2(A+B) = 0 \qquad A - B = 1$$

$$A + B = 0$$

$$2A = 1$$

$$A = \frac{1}{2}$$

$$B = -\frac{1}{2}$$

$$\therefore u_n = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

$$= \frac{1}{2 \cdot 1} - \frac{1}{2 \cdot 3}$$

$$+ \frac{1}{2 \cdot 3} - \frac{1}{2 \cdot 5}$$

$$+\frac{1}{2\cdot 5}-\frac{1}{2\cdot 7}$$

$$+\frac{1}{2(2(N-1)-1)}-\frac{1}{2(2(N-1)+1)}$$

$$+\frac{1}{2(2N-1)}-\frac{1}{2(2N+1)}$$

$$=\frac{1}{2}-\frac{1}{2(2N+1)}$$

$$u_1+u_2+u_3+\cdots=\sum_{n=1}^{\infty}u_n$$

$$=\lim_{N\to\infty}\sum_{n=1}^{N}u_n$$

Since $\displaystyle\sum_{n=1}^{N}u_n=\frac{1}{2}-\frac{1}{2(2N+1)}$

$$\lim_{N\to\infty}\sum_{n=1}^{N}u_n=\lim_{N\to\infty}\left(\frac{1}{2}-\frac{1}{2(2N+1)}\right)$$

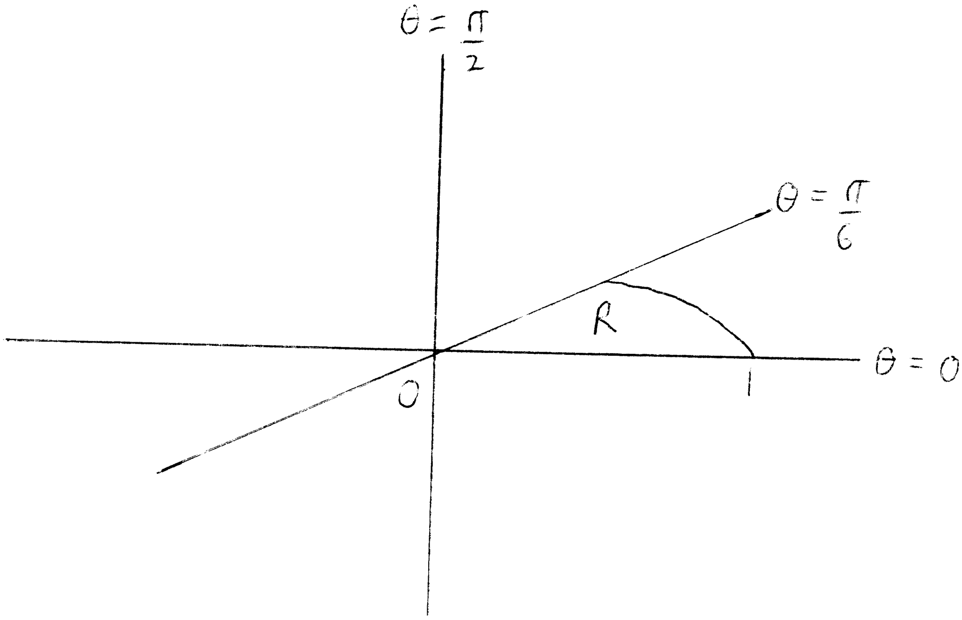
$$=\frac{1}{2}-0$$

$$=\frac{1}{2}$$

$\therefore$ The infinite series $u_1+u_2+u_3+\cdots$ converges to $\frac{1}{2}$.

2. $r = \cos 2\theta \qquad \theta = 0,\ \theta = \frac{\pi}{6}$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$
r	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$



The area of the region R is

$$\int_0^{\frac{\pi}{6}} \frac{r^2}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos^2 2\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}\left(\frac{\cos 4\theta + 1}{2}\right) d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos 4\theta}{4} + \frac{1}{4}\, d\theta$$

$$= \left[\frac{\sin 4\theta}{4(4)} + \frac{\theta}{4} \right]_0^{\frac{\pi}{6}}$$

$$= \left[\frac{\sin 4\theta}{16} + \frac{\theta}{4} \right]_0^{\frac{\pi}{6}}$$

$$= \frac{\sin \frac{4\pi}{6}}{16} + \frac{\pi}{6(4)} - \left(\frac{\sin 0}{16} + 0 \right)$$

$$= \frac{\sin \frac{2\pi}{3}}{16} + \frac{\pi}{24} - 0$$

$$= \frac{\sin\left(\pi - \frac{2\pi}{3}\right)}{16} + \frac{\pi}{24}$$

$$= \frac{\sin \frac{\pi}{3}}{6} + \frac{\pi}{24}$$

$$= \frac{\sqrt{3}}{2(16)} + \frac{\pi}{24}$$

$$= \frac{\sqrt{3}}{32} + \frac{\pi}{24}$$

3. Let $f(n) = 23^{2n} + 31^{2n} + 46$

when $n = 0$.

$f(0) = 23^{2(0)} + 31^{2(0)} + 46 = 23^{0} + 31^{0} + 46 = 1 + 1 + 46 = 48 = 48(1)$

$\therefore f(0)$ is divisible by 48.

When $n = k$:

Assume that $f(k)$ is divisible by 48

$f(k) = 23^{2k} + 31^{2k} + 46$

$48 \mid f(k)$

$\therefore f(k) = 48s, \quad s \in \mathbb{Z}^{+}$

$23^{2k} + 31^{2k} + 46 = 48s$

When $n = k+1$

$f(k+1) = 23^{2(k+1)} + 31^{2(k+1)} + 46$

$= 23^{2k+2} + 31^{2k+2} + 46$

$= 23^{2k} 23^{2} + 31^{2k} 31^{2} + 46$

$= 23^{2k} 529 + 31^{2k} 961 + 46$

$= 23^{2k} (528 + 1) + 31^{2k} (960 + 1) + 46$

$= 23^{2k} 528 + 23^{2k} + 31^{2k} 960 + 31^{2k} + 46$

$= 23^{2k} 528 + 31^{2k} 960 + 23^{2k} + 31^{2k} + 46$

$= 23^{2k} 11 \cdot 48 + 31^{2k} 20 \cdot 48 + 48s$

$= 48(23^{2k} 11 + 31^{2k} 20 + s)$

Since k is a positive integer and s is a positive integer, $23^{2k} 11 + 31^{2k} 20 + s$ is a positive integer.

$\therefore 48 \mid f(k+1)$

Since $f(k+1)$ is divisible by 48 if $f(k)$ is divisible by 48 and $f(0)$ is divisible by 48, $f(n)$ is divisible by 48 for all integers $n \geqslant 0$.

$23^{2n} + 31^{2n} + 46$ is divisible by 48 for all $n \geqslant 0$.

4. $A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -3 & 4 & 5 \\ 5 & -6 & 10 & 14 \\ 4 & -5 & 8 & 11 \end{pmatrix}$

$$\xrightarrow[-4r_1 + r_4]{\substack{-2r_1 + r_2 \\ -5r_1 + r_3}} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow[-r_2 + r_4]{-r_2 + r_3} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The dimension of the range space of T is 2.

If $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in R^4$,

since $\begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -3 & 4 & 5 \\ 5 & -6 & 10 & 14 \\ 4 & -5 & 8 & 11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$$= \begin{pmatrix} x_1 - x_2 + 2x_3 + 3x_4 \\ 2x_1 - 3x_2 + 4x_3 + 5x_4 \\ 5x_1 - 6x_2 + 10x_3 + 14x_4 \\ 4x_1 - 5x_2 + 8x_3 + 11x_4 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ 2x_1 \\ 5x_1 \\ 4x_1 \end{pmatrix} + \begin{pmatrix} -x_2 \\ -3x_2 \\ -6x_2 \\ -5x_2 \end{pmatrix} + \begin{pmatrix} 2x_3 \\ 4x_3 \\ 10x_3 \\ 8x_3 \end{pmatrix} + \begin{pmatrix} 3x_4 \\ 5x_4 \\ 14x_4 \\ 11x_4 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ 2 \\ 5 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -3 \\ -6 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 4 \\ 10 \\ 8 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 5 \\ 14 \\ 11 \end{pmatrix}$$

$$= (x_1 + 2x_3 + 4x_4)\begin{pmatrix}1\\2\\5\\4\end{pmatrix} + (x_2 + x_4)\begin{pmatrix}-1\\-3\\-6\\-5\end{pmatrix}$$

$\therefore \left\{ \begin{pmatrix}1\\2\\5\\4\end{pmatrix}, \begin{pmatrix}-1\\-3\\-6\\-5\end{pmatrix} \right\}$ is a basis for the range space of T.

If $S = \{ M A \underset{\sim}{x} : \underset{\sim}{x} \in R^4 \}$ and $\{ \underset{\sim}{b}_1, \underset{\sim}{b}_2 \}$ is a basis of the range space of T, $\lambda \underset{\sim}{b}_1 + \mu \underset{\sim}{b}_2 = \underset{\sim}{0}$ if and only if $\lambda = \mu = 0$, since $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ are linearly independent. If M is a non-singular 4×4 matrix and $\lambda (M \underset{\sim}{b}_1) + \mu (M \underset{\sim}{b}_2) = \underset{\sim}{0}$,

$$M^{-1}(\lambda (M \underset{\sim}{b}_1) + \mu (M \underset{\sim}{b}_2)) = M^{-1} \underset{\sim}{0}$$

$$M^{-1}(\lambda (M \underset{\sim}{b}_1)) + M^{-1}(\mu (M \underset{\sim}{b}_2)) = \underset{\sim}{0}$$

$$\lambda (M^{-1} M \underset{\sim}{b}_1) + \mu (M^{-1} M \underset{\sim}{b}_2) = \underset{\sim}{0}$$

$$\lambda \underset{\sim}{b}_1 + \mu \underset{\sim}{b}_2 = \underset{\sim}{0}$$

Since $\lambda = \mu = 0$, $M \underset{\sim}{b}_1$ and $M \underset{\sim}{b}_2$ are linearly independent and

$$M A \underset{\sim}{x} = M(\lambda \underset{\sim}{b}_1 + \mu \underset{\sim}{b}_2)$$
$$= \lambda (M \underset{\sim}{b}_1) + \mu (M \underset{\sim}{b}_2)$$

Since S consists of vectors of the form $M A \underset{\sim}{x}$, the dimension of S is 2.

5. $C: \quad y = 2x + \dfrac{3(x-1)}{x+1}$

i)

$$\begin{array}{r|l} & \;1 \\ \hline x+1 & x-1 \\ & x+1 \\ \hline & \;\;-2 \end{array}$$

$$\therefore y = 2x + 3\left(1 - \frac{2}{x+1}\right)$$

$$= 2x + 3 - \frac{6}{x+1}$$

$x \to \pm\infty \quad y \to 2x+3$

$\therefore$ $y = 2x+3$ is an asymptote of C.

$x \to -1 \quad y \to \pm\infty$

$x = -1$ is an asymptote of C.

$\therefore$ The asymptotes of C are $y = 2x+3$ and $x = -1$

ii) The set of values of x for which C is above it's oblique asymptote is given by

$$y > 2x + 3$$

$$2x + \frac{3(x-1)}{x+1} > 2x + 3$$

$$\frac{3(x-1)}{x+1} > 3$$

$$\frac{x-1}{x+1} > 1$$

$$\frac{x-1}{x+1} - 1 > 0$$

$$\frac{x-1-x-1}{x+1} > 0$$

$$\frac{-2}{x+1} > 0$$

$$\frac{1}{x+1} < 0$$

$$x < -1$$

$\therefore$ C is above it's oblique asymptote if $x < -1$

The set of values of x for which C is below it's oblique asymptote is given by

$$y < 2x + 3$$

$$2x + \frac{3(x-1)}{x+1} < 2x + 3$$

$$\frac{3(x-1)}{x+1} < 3$$

$$\frac{x-1}{x+1} < 1$$

$$\frac{x-1}{x+1} - 1 < 0$$

$$\frac{x-1-x-1}{x+1} < 0$$

$$\frac{-2}{x+1} < 0$$

$$\frac{1}{x+1} > 0$$

$x > -1$

$\therefore$ C is below it's oblique asymptote if $x > -1$

iii) $y = 2x + \frac{3(x-1)}{x+1}$

$= \frac{2x(x+1) + 3(x-1)}{x+1}$

$= \frac{2x^2 + 2x + 3x - 3}{x+1}$

$= \frac{2x^2 + 5x - 3}{x+1}$

$= \frac{(2x-1)(x+3)}{x+1}$

when $x = 0$: $y = -3$

when $y = 0$: $\frac{(2x-1)(x+3)}{x+1} = 0$

$(2x-1)(x+3) = 0$

$x = \frac{1}{2}, -3$

$\therefore$ The intersection points of C are $(0,-3)$, $(\frac{1}{2}, 0)$

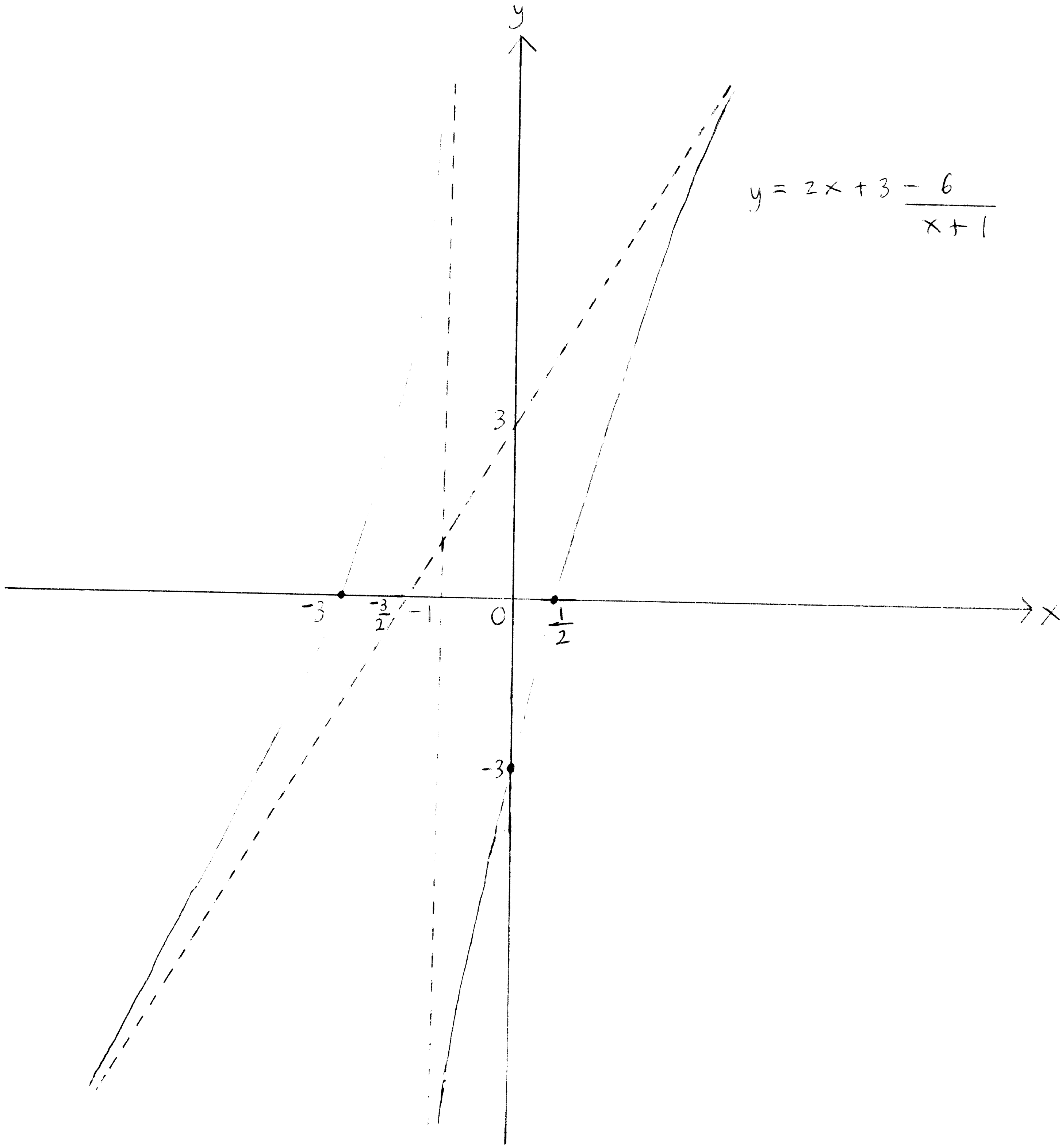
and $(-3, 0)$

$y = 2x + 3 - \frac{6}{x+1}$

$\frac{dy}{dx} = 2 + \frac{6}{(x+1)^2}$

> 0 since $\frac{1}{(x+1)^2} > 0$

Since $\frac{dy}{dx} \neq 0$, there are no critical points.



6. a) $y = \frac{2\sqrt{3}}{3} x^{\frac{3}{2}}$

$$\frac{dy}{dx} = \sqrt{3} x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = 3x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 3x$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 3x}$$

The length of the arc from the origin to the point where $x = 1$ is $\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^1 \sqrt{1 + 3x}\, dx$$

$$= \left[\frac{2(1 + 3x)^{\frac{3}{2}}}{3(3)}\right]_0^1$$

$$= \left[\frac{2}{9}(1 + 3x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{9}(1 + 3(1))^{\frac{3}{2}} - \frac{2}{9}(1 + 3(0))^{\frac{3}{2}}$$

$$= \frac{2}{9}(4^{\frac{3}{2}}) - \frac{2}{9}(1)^{\frac{3}{2}}$$

$$= \frac{2}{9}(8) - \frac{2}{9}(1)$$

$$= \frac{16}{9} - \frac{2}{9}$$

$$= \frac{14}{9}$$

b) $y^3 + \left(\frac{dy}{dx}\right)^3 = x^4 + 6$ $\quad y = -1$ when $x = 1$

$x = 1 \quad y = -1.$

$$(-1)^3 + \left(\frac{dy}{dx}\right)^3 = 1^4 + 6$$

$$-1 + \left(\frac{dy}{dx}\right)^3 = 1 + 6$$

$$\left(\frac{dy}{dx}\right)^3 = 8$$

$$\frac{dy}{dx} = 2$$

$\therefore$ when $x = 1$, $\frac{dy}{dx} = 2.$

$$\frac{d}{dx}(y^3) + \frac{d}{dx}\left(\frac{dy}{dx}\right)^3 = \frac{d}{dx}(x^4 + 6)$$

$$\frac{dy}{dx}\frac{d}{dy}(y^3) + 3\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}\left(\frac{dy}{dx}\right) = 4x^3$$

$$3y^2\frac{dy}{dx} + 3\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} = 4x^3$$

when $x = 1$, since $y = -1$ and $\frac{dy}{dx} = 2$.

$$3(-1)^2(2) + 3(2)^2\frac{d^2y}{dx^2} = 4(1^3)$$

$$6 + 12\frac{d^2y}{dx^2} = 4$$

$$\frac{d^2y}{dx^2} = \frac{-1}{6}$$

7. $$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \quad n \geq 0$$

$$I_{n+2} = \int_0^{\frac{\pi}{2}} \sin^{n+2} x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x \sin^2 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x (1 - \cos^2 x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x - \sin^n x \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x \, dx - \int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x \, dx$$

$$= I_n - \int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x \, dx$$

$$= I_n - \int_0^{\frac{\pi}{2}} \sin^n x \cos x \cos x \, dx$$

$$u = \cos x \qquad dv = \sin^n x \cos x \, dx$$

$$du = -\sin x \, dx \qquad v = \frac{\sin^{n+1} x}{n+1}$$

$$= I_n - \left(\left[\frac{\sin^{n+1} x \cos x}{n+1} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-\sin^{n+1} x \sin x \, dx}{n+1} \right)$$

$$= I_n - \left(\frac{\sin^{n+1} \frac{\pi}{2} \cos \frac{\pi}{2}}{n+1} - \frac{\sin^{n+1} 0 \cos 0}{n+1} \right) - \int_0^{\frac{\pi}{2}} \frac{\sin^{n+2}}{n+1} x \, dx$$

$$= I_n - 0 - \frac{1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{n+1} x \, dx$$

$$= I_n - \frac{1}{n+1} I_{n+2}$$

$$I_{n+2} + \frac{1}{n+1} I_{n+2} = I_n$$

$$\left(1 + \frac{1}{n+1}\right) I_{n+2} = I_n$$

$$\left(\frac{n+1+1}{n+1}\right) I_{n+2} = I_n$$

$$\left(\frac{n+2}{n+1}\right) I_{n+2} = I_n$$

$$I_{n+2} = \left(\frac{n+1}{n+2}\right) I_n$$

The area of the region bounded by the x-axis and the arc of the curve $y = \sin^4 x$ from $x = 0$ to $x = \pi$, R is

$$\int_0^{\pi} y \, dx = \int_0^{\pi} \sin^4 x \, dx$$

If $I_n = \int_0^{\pi} \sin^n x \, dx$,

$$I_{n+2} = \int_0^{\pi} \sin^{n+2} x \, dx$$

$$= \int_0^{\pi} \sin^n x \, in^2 x \, dx$$

$$= \int_0^{\pi} \sin^n x (1 - \cos^2 x) \, dx$$

$$= \int_0^{\pi} \sin^n x - \sin^n x \cos^2 x \, dx$$

$$= \int_0^{\pi} \sin^n x \, dx - \int_0^{\pi} \sin^n x \cos^2 x \, dx$$

$$= I_n - \int_0^{\pi} \sin^n x \cos x \cos x \, dx$$

$u = \cos x$ $\qquad$ $dv = \sin^n x \cos x \, dx$

$du = -\sin x \, dx$ $\qquad$ $v = \dfrac{\sin^{n+1} x}{n+1}$

$$= I_n - \left(\left[\frac{\sin^{n+1} x \cos x}{n+1} \right]_0^{\pi} - \int_0^{\pi} \frac{-\sin^{n+1} x \sin x}{n+1} \, dx \right)$$

$$= I_n - \left(\frac{\sin^{n+1} \pi \cos \pi}{n+1} - \frac{\sin^{n+1} 0 \cos 0}{n+1} \right) - \int_0^{\pi} \frac{\sin^{n+2} x}{n+1} \, dx$$

$$= I_n - 0 - \frac{1}{n+1} \int_0^{\pi} \sin^{n+2} x \, dx$$

$$= I_n - \frac{1}{n+1} I_{n+2}$$

$$I_{n+2} + \frac{1}{n+1} I_{n+2} = I_n$$

$$\left(1 + \frac{1}{n+1} \right) I_{n+2} = I_n$$

$$\left(\frac{n+1+1}{n+1} \right) I_{n+2} = I_n$$

$$\left(\frac{n+2}{n+1} \right) I_{n+2} = I_n$$

$$I_{n+2} = \left(\frac{n+1}{n+2} \right) I_n$$

$\therefore$ The area of R is $I_4 = \int_0^{\pi} \sin^4 x \, dx$

$$= \frac{3}{4} I_2$$

$$= \frac{3}{4} \cdot \frac{1}{2} I_0$$

$$= \frac{3}{8} I_0$$

$$= \frac{3\pi}{8}$$

since $I_0 = \int_0^{\pi} \sin^0 x \, dx$

$$= \int_0^{\pi} 1 \, dx$$

$$= [x]_0^{\pi}$$

$$= \pi - 0$$

$$= \pi$$

The y-coordinate of the centroid of R is

$$\frac{\int_0^{\pi} \frac{y^2}{2} \, dx}{\frac{3\pi}{8}}$$

$$= \frac{\int_0^{\pi} \frac{(\sin^4 x)^2}{2} \, dx}{\frac{3\pi}{8}}$$

$$= \frac{\int_0^{\pi} \frac{\sin^8 x}{2} \, dx}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2}\int_0^{\pi} \sin^8 x \, dx}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} I_8}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} I_6}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} \cdot \frac{5}{6} I_4}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} I_2}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \pi}{\frac{3\pi}{8}}$$

$$= \frac{\frac{105\pi}{768}}{\frac{3\pi}{8}}$$

$$= \frac{35}{96}$$

8 $$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = 80e^{-3t}$$

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = 0$$

$$m^2 + 6m + 25 = 0$$

$$(m+3)^2 + 16 = 0$$

$$m = -3 \pm 4i$$

$$\therefore y_c = e^{-3t}(A\cos 4t + B\sin 4t)$$

Let $y_p = Ce^{-3t}$

$$\frac{dy_p}{dt} = -3Ce^{-3t}$$

$$\frac{d^2y_p}{dt^2} = 9Ce^{-3t}$$

$$\therefore \frac{d^2y_p}{dt^2} + 6\frac{dy_p}{dt} + 25y_p = 9Ce^{-3t} - 18Ce^{-3t} + 25Ce^{-3t}$$

$$= 16Ce^{-3t}$$

$$= 80e^{-3t}$$

$$\therefore 16C = 80$$

$$C = 5$$

$$\therefore y_p = 5e^{-3t}$$

$$y = y_c + y_p$$

$$= e^{-3t}(A\cos 4t + B\sin 4t) + 5e^{-3t}$$

$$\frac{dy}{dx} = -3e^{-3t}(A\cos 4t + B\sin 4t) + e^{-3t}(-4A\sin 4t + 4B\cos 4t)$$

$$-15e^{-3t}$$

when $t=0$ $y=8$: $8 = A+5$

when $t=0$ $\frac{dy}{dt} = -8$: $-8 : -3A + 4B - 15$

$$A = 3$$

$$-3A + 4B = 7$$

$$-3(3) + 4B = 7$$

$$4B = 16$$

$$B = 4$$

$$y = e^{-3t}(3\cos 4t + 4\sin 4t) + 5e^{-3t}$$

$$= e^{-3t}(3\cos 4t + 4\sin 4t + 5)$$

$$\therefore ye^{3t} = 3\cos 4t + 4\sin 4t + 5$$

Expressing $3\cos 4t + 4\sin 4t$ as $R\cos(4t-\alpha)$.

$$3\cos 4t + 4\sin 4t = R\cos(4t-\alpha)$$

$$= R(\cos 4t\cos\alpha + \sin 4t\sin\alpha)$$

$$= R\cos 4t\cos\alpha + R\sin 4t\sin\alpha$$

$R\cos\alpha = 3$ $\quad$ $R\sin\alpha = 4$

$$(R\cos\alpha)^2 + (R\sin\alpha)^2 = 3^2 + 4^2$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 9 + 16$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 25$$

$$R^2 = 25$$

$$R = 5$$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{3}$$

$$\tan\alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\frac{4}{3}$$

$$\therefore \; ye^{3t} = 5\cos(4t-\alpha) + 5$$

since $-1 \leq \cos(4t-\alpha) \leq 1$,

$$-5 \leq 5\cos(4t-\alpha) \leq 5$$

$$0 \leq 5\cos(4t-\alpha) + 5 \leq 10$$

$$0 \leq ye^{3t} \leq 10$$

If $y=8$ and $\frac{dy}{dt} = -8$ when $t=0$, $0 \leq ye^{3t} \leq 10$ for all t.

9 $z = e^{i\theta}$

$= \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$

and $z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$, from De Moiv're's Theorem.

$\therefore z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

and

$z^n - z^{-n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= 2i\sin n\theta.$

Since $z + \frac{1}{z} = 2\cos\theta$

$(2\cos\theta)^7 = \left(z + \frac{1}{z}\right)^7$

$128\cos^7\theta = z^7 + 7z^5 + 21z^3 + 35z + \frac{35}{z} + \frac{21}{z^3} + \frac{7}{z^5} + \frac{1}{z^7}$

$= z^7 + \frac{1}{z^7} + 7\left(z^5 + \frac{1}{z^5}\right) + 21\left(z^3 + \frac{1}{z^3}\right) + 35\left(z + \frac{1}{z}\right)$

$= 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos\theta)$

$\therefore \cos^7\theta = \frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$

$= \frac{\cos 7\theta}{64} + \frac{7\cos 5\theta}{64} + \frac{21\cos 3\theta}{64} + \frac{35\cos\theta}{64}$

The mean value of $\cos^7 2\theta$ over the interval $0 \leq \theta \leq \frac{\pi}{4}$ is

$$\frac{1}{\frac{\pi}{4} - 0} \int_0^{\frac{\pi}{4}} \cos^7 2\theta \, d\theta$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \cos^7 2\theta \, d\theta$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{\cos 7(2\theta)}{64} + \frac{7\cos 5(2\theta)}{64} + \frac{21\cos 3(2\theta)}{64} + \frac{35\cos 2\theta}{64} \, d\theta$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{\cos 14\theta}{64} + \frac{7\cos 10\theta}{64} + \frac{21\cos 6\theta}{64} + \frac{35\cos 2\theta}{64} \, d\theta$$

$$= \frac{4}{\pi} \left[\frac{\sin 14\theta}{64(14)} + \frac{7\sin 10\theta}{64(10)} + \frac{21\sin 6\theta}{64(6)} + \frac{35\sin 2\theta}{64(2)} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{4}{\pi} \left(\frac{\sin \frac{7\pi}{2}}{896} + \frac{7\sin \frac{5\pi}{2}}{640} + \frac{21\sin \frac{3\pi}{2}}{384} + \frac{35\sin \frac{\pi}{2}}{128} \right.$$

$$\left. - \frac{\sin 0}{896} - \frac{7\sin 0}{640} - \frac{21\sin 0}{384} - \frac{35\sin 0}{128} \right)$$

$$= \frac{4}{\pi} \left(\frac{-1}{896} + \frac{7}{640} - \frac{21}{384} + \frac{35}{128} - 0 \right)$$

$$= \frac{4}{\pi} \left(\frac{8}{35} \right)$$

$$= \frac{32}{35\pi}$$

10. $\Pi : 2x + 3y + 4z = 48$

Let $y = -2\lambda$ and $z = -\mu$, $\lambda, \mu \in \mathbb{R}$

$\therefore 2x + 3(-2\lambda) + 4(-\mu) = 48$

$2x - 6\lambda - 4\mu = 48$

$x - 3\lambda - 2\mu = 24$

$x = 24 + 3\lambda + 2\mu$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24 + 3\lambda + 2\mu \\ -2\lambda \\ -\mu \end{pmatrix}$$

$$= \begin{pmatrix} 24 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3\lambda \\ -2\lambda \\ 0 \end{pmatrix} + \begin{pmatrix} 2\mu \\ 0 \\ -\mu \end{pmatrix}$$

$$= \begin{pmatrix} 24 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

A vector equation of Π in the form

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$,

where $\underset{\sim}{a} = p\underset{\sim}{i}$, $\underset{\sim}{b} = q\underset{\sim}{i} + r\underset{\sim}{j}$ and $\underset{\sim}{c} = s\underset{\sim}{i} + t\underset{\sim}{k}$,

and where p, q, r, s, t are integers is

$\underset{\sim}{r} = 24\underset{\sim}{i} + \lambda(3\underset{\sim}{i} - 2\underset{\sim}{j}) + \mu(2\underset{\sim}{i} - \underset{\sim}{k})$

$l : \underset{\sim}{r} = 29\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k} + \theta(5\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k})$

$= (29 + 5\theta)\underset{\sim}{i} + (-2 - 6\theta)\underset{\sim}{j} + (-1 + 2\theta)\underset{\sim}{k}$

If l intersects Π,

$2(29 + 5\theta) + 3(-2 - 6\theta) + 4(-1 + 2\theta) = 48$

$58 + 10\theta - 6 - 18\theta - 4 + 8\theta = 48$

$48 = 48$

Since $x = 29 + 5\theta$, $y = -2 - 6\theta$ and $z = -1 + 2\theta$ satisfy the equation $2x + 3y + 4z = 48$ for all values of θ, l lies in Π.

The plane which contains l and is perpendicular to Π has a normal vector in the direction $\begin{pmatrix} 5 \\ -6 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

since the normal of Π, $2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$ is also perpendicular to the normal of the plane.

$$\begin{pmatrix} 5 \\ -6 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 5 & -6 & 2 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= -30\underset{\sim}{i} - 16\underset{\sim}{j} + 27\underset{\sim}{k}$$

Since the plane contains l, $(29, -2, -1)$ is a point in the plane, and since $-30\underset{\sim}{i} - 16\underset{\sim}{j} + 27\underset{\sim}{k}$ is a normal to the plane, the equation of the plane can be expressed as

$$\underset{\sim}{r} \cdot \begin{pmatrix} -30 \\ -16 \\ 27 \end{pmatrix} = \begin{pmatrix} 29 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -30 \\ -16 \\ 27 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -30 \\ -16 \\ 27 \end{pmatrix} = \begin{pmatrix} 29 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -30 \\ -16 \\ 27 \end{pmatrix}$$

$$-30x - 16y + 27z = -870 + 32 - 27$$

$$= -865$$

$$30x + 16y + 27z = 865$$

$\therefore$ The equation of the plane containing l and perpendicular to Π is $30x + 16y + 27z = 865$.

11. $x^4 + 3x^3 + 5x^2 + 12x + 4 = 0$

If $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$\alpha + \beta + \gamma + \delta = -3$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 5$$

$$\alpha\beta\gamma + \alpha\beta\gamma + \alpha\gamma\delta + \beta\gamma\delta = -12$$

$$\alpha\beta\gamma\delta = 4$$

The sum of the squares of the roots of the equation,

$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 + \delta^2 &= (\alpha + \beta + \gamma + \delta)^2 \\ &\quad -2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) \\ &= (-3)^2 - 2(5) \\ &= 9 - 10 \\ &= -1\end{aligned}$$

If $\alpha, \beta, \gamma, \delta \in R$, $\alpha^2 > 0$, $\beta^2 > 0$, $\gamma^2 > 0$, $\delta^2 > 0$ and

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 > 0$$

Since $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -1$, the equation does not have all real roots and since the coefficients of the equation are real, the complex roots occur in conjugate pairs.

$\therefore$ The equation does not have more than 2 real roots.

Let $f(x) = x^4 + 3x^3 + 5x^2 + 12x + 4$

$f(-3) = (-3)^4 + 3(-3)^3 + 5(-3)^2 + 12(-3) + 4$

$= 13$

$f(-1) = (-1)^4 + 3(-1)^3 + 5(-1)^2 + 12(-1) + 4$

$= -5$

$\therefore$ The equation has a root in the interval

$-3 < x < -1$

$f(0) = 0^4 + 3(0^3) + 5(0^2) + 12(0) + 4$

$= 4$

$\therefore$ The equation has a root in the interval

$-1 < x < 0$

The equation has exactly 2 real roots in the interval $-3 < x < 0$.

If γ and δ are the complex roots,

let $\gamma = c + di$ and $\delta = c - di$

$\gamma\delta = (c + di)(c - di) = c^2 - cdi + cdi + d^2$

$= c^2 + d^2$

$= |\gamma|^2$

$= |\delta|^2$

Since $\alpha\beta\gamma\delta = 4$,

$$|\alpha\beta\gamma\delta| = |4|$$

$$|\alpha\beta||\gamma||\delta| = 4$$

$$\alpha\beta|\gamma||\delta| = 4$$

$$|\gamma||\delta| = \frac{4}{\alpha\beta}$$

$$|\gamma|^2 = |\delta|^2 = \frac{4}{\alpha\beta}$$

$$|\gamma| = |\delta| = \frac{2}{\sqrt{\alpha\beta}}$$

$$A\underset{\sim}{x} = \lambda\underset{\sim}{x}$$

$$B = MAM^{-1}$$

$$\therefore B(M\underset{\sim}{x}) = MAM^{-1}(M\underset{\sim}{x})$$

$$= MAM^{-1}M\underset{\sim}{x}$$

$$= MAI\underset{\sim}{x}$$

$$= MA\underset{\sim}{x}$$

$$= M(\lambda\underset{\sim}{x})$$

$$= \lambda(M\underset{\sim}{x})$$

$\therefore$ $M\underset{\sim}{x}$ is an eigenvector of the matrix B, where $B = MAM^{-1}$, and that λ is the corresponding eigenvalue.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & -3 & 0 \\ b & c & -5 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i) $$A - \lambda I = \begin{pmatrix} 1 & 0 & 0 \\ a & -3 & 0 \\ b & c & -5 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 0 & 0 \\ a & -3-\lambda & 0 \\ b & c & -5-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} -3-\lambda & 0 \\ c & -5-\lambda \end{vmatrix} - 0\begin{vmatrix} a & 0 \\ b & -5-\lambda \end{vmatrix}$$

$$+ 0\begin{vmatrix} a & -3-\lambda \\ b & c \end{vmatrix}$$

$$= (1-\lambda)[(-3-\lambda)(-5-\lambda) - 0] - 0 + 0$$

$$= (1-\lambda)(\lambda+3)(\lambda+5)$$

$$|A - \lambda I| = 0$$

$$(1-\lambda)(\lambda+3)(\lambda+5) = 0$$

$$\lambda = 1, -3, -5$$

when $\lambda = 1$: $\begin{pmatrix} 0 & 0 & 0 \\ a & -4 & 0 \\ b & c & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ a & -4 & 0 & 0 \\ b & c & -6 & 0 \end{array}\right)$$

Let $y = 6as$, $s \in R$

$$ax - 4(6as) = 0$$

$$x = 24s$$

$$b(24s) + c(6as) - 6z = 0$$

$$z = 4bs + acs$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24s \\ 6as \\ 4bs + acs \end{pmatrix}$$

$$= s\begin{pmatrix} 24 \\ 6a \\ 4b + ac \end{pmatrix}$$

when $\lambda = -3$: $\begin{pmatrix} 4 & 0 & 0 \\ a & 0 & 0 \\ b & c & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & c & -2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{4}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & c & -2 & 0 \end{array}\right)$$

$$\xrightarrow[-br_1 + r_3]{-ar_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c & - & 0 \end{array}\right)$$

Let $z = cs, \ s \in R$

$$cy - 2cs = 0$$

$$y = 2s$$

$$x = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2s \\ cs \end{pmatrix}$$

$$= s \begin{pmatrix} 0 \\ 2 \\ c \end{pmatrix}$$

when $\lambda = -5$:
$$\begin{pmatrix} 6 & 0 & 0 \\ a & 2 & 0 \\ b & c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ a & 2 & 0 & 0 \\ b & c & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{6}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ a & 2 & 0 & 0 \\ b & c & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{-ar_1 + r_2 \\ -br_1 + r_3}]{} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & c & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_2}{2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-cr_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = s, s \in R$

$y = 0$

$x = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvalues of A are $1, -3$, and -5 and the corresponding eigenvectors are $\begin{pmatrix} 24 \\ 6a \\ 4b + ac \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ c \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

ii) The eigenvalues of B are $1, -3$ and -5 and the corresponding eigenvectors are $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 24 \\ 6a \\ 4b+ac \end{pmatrix}$,

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ c \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ since $B = MAM^{-1}$

$\therefore$ The eigenvalues of B are $1, -3$ and -5 and the corresponding eigenvectors are $\begin{pmatrix} 24+4b+ac \\ 6a \\ 4b+ac \end{pmatrix}$, $\begin{pmatrix} c \\ 2 \\ c \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

iii) If $B^n = QDQ^{-1}$,

$$\therefore D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-3)^n & 0 \\ 0 & 0 & (-5)^n \end{pmatrix}$$ is a diagonal matrix

and $$Q = \begin{pmatrix} 24+4b+ac & c & 1 \\ 6a & 2 & 0 \\ 4b+ac & c & 1 \end{pmatrix}.$$