

MAY/ JUNE 2006

$$1. \quad u_n = \frac{1}{4n^2 - 1}$$

Expressing u_n as partial fractions,

$$\begin{aligned} \frac{1}{4n^2 - 1} &= \frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} \\ &= \frac{A(2n+1) + B(2n-1)}{(2n+1)(2n-1)} \end{aligned}$$

$$\begin{aligned} \therefore 1 &= A(2n+1) + B(2n-1) \\ &= 2(A+B)n + A - B \end{aligned}$$

$$\begin{aligned} \therefore 2(A+B) &= 0 \quad A - B = 1 \\ A + B &= 0 \end{aligned}$$

$$2A = 1$$

$$A = \frac{1}{2}$$

$$B = -\frac{1}{2}$$

$$\therefore u_n = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

$$\begin{aligned} \sum_{n=1}^N u_n &= \sum_{n=1}^N \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)} \\ &= \frac{1}{2 \cdot 1} - \frac{1}{2 \cdot 3} \end{aligned}$$

$$+ \frac{1}{2 \cdot 3} - \frac{1}{2 \cdot 5}$$

$$+ \frac{1}{2 \cdot 5} - \frac{1}{2 \cdot 7}$$

$$+ \frac{1}{2(2(N-1)-1)} - \frac{1}{2(2(N-1)+1)}$$

$$+ \frac{1}{2(2N-1)} - \frac{1}{2(2N+1)}$$

$$= \frac{1}{2} - \frac{1}{2(2N+1)}$$

$$u_1 + u_2 + u_3 + \dots = \sum_{n=1}^{\infty} u_n$$

$$= \lim_{N \rightarrow \infty} \sum_{n=1}^N u_n$$

Since $\sum_{n=1}^N u_n = \frac{1}{2} - \frac{1}{2(2N+1)}$

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N u_n = \lim_{N \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{2(2N+1)} \right)$$

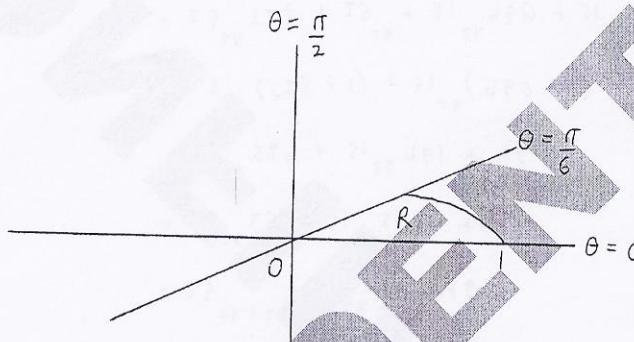
$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

The infinite series $u_1 + u_2 + u_3 + \dots$ converges to $\frac{1}{2}$.

2. $r = \cos 2\theta$ $\theta = 0, \theta = \frac{\pi}{6}$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$
r	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$



The area of the region R is

$$\int_0^{\pi/6} \frac{r^2}{2} d\theta$$

$$= \int_0^{\pi/6} \frac{\cos^2 2\theta}{2} d\theta$$

$$= \int_0^{\pi/6} \frac{1}{2} \left(\frac{\cos 4\theta + 1}{2} \right) d\theta$$

$$= \int_0^{\pi/6} \frac{\cos 4\theta}{4} + \frac{1}{4} d\theta$$

$$= \left[\frac{\sin 4\theta}{4(4)} + \frac{\theta}{4} \right]_0^{\frac{\pi}{6}}$$

$$= \left[\frac{\sin 4\theta}{16} + \frac{\theta}{4} \right]_0^{\frac{\pi}{6}}$$

$$= \frac{\sin \frac{4\pi}{6}}{16} + \frac{\pi}{6(4)} - \left(\frac{\sin 0}{16} + 0 \right)$$

$$= \frac{\sin \frac{2\pi}{3}}{16} + \frac{\pi}{24} - 0$$

$$= \frac{\sin \left(\pi - \frac{2\pi}{3}\right)}{16} + \frac{\pi}{24}$$

$$= \frac{\sin \frac{\pi}{3}}{6} + \frac{\pi}{24}$$

$$= \frac{\sqrt{3}}{2(16)} + \frac{\pi}{24}$$

$$= \frac{\sqrt{3}}{32} + \frac{\pi}{24}$$

3. Let $f(n) = 23^{2n} + 31^{2n} + 46$

when $n = 0$:

$$f(0) = 23^{2(0)} + 31^{2(0)} + 46 = 23^0 + 31^0 + 46 = 1 + 1 + 46 = 48 \text{ (1)}$$

$\therefore f(0)$ is divisible by 48.

when $n = k$:

Assume that $f(k)$ is divisible by 48

$$f(k) = 23^{2k} + 31^{2k} + 46$$

$$48 | f(k)$$

$$\therefore f(k) = 48s, s \in \mathbb{Z}^+$$

$$23^{2k} + 31^{2k} + 46 = 48s$$

when $n = k+1$:

$$f(k+1) = 23^{2(k+1)} + 31^{2(k+1)} + 46$$

$$= 23^{2k+2} + 31^{2k+2} + 46$$

$$= 23^{2k} \cdot 23^2 + 31^{2k} \cdot 31^2 + 46$$

$$= 23^{2k} \cdot 529 + 31^{2k} \cdot 961 + 46$$

$$= 23^{2k} (528 + 1) + 31^{2k} (960 + 1) + 46$$

$$= 23^{2k} \cdot 528 + 23^{2k} + 31^{2k} \cdot 960 + 31^{2k} + 46$$

$$= 23^{2k} \cdot 528 + 31^{2k} \cdot 960 + 23^{2k} + 31^{2k} + 46$$

$$= 23^{2k} \cdot 11 \cdot 48 + 31^{2k} \cdot 20 \cdot 48 + 48s$$

$$= 48(23^{2k} \cdot 11 + 31^{2k} \cdot 20 + s)$$

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since k is a positive integer and s is a positive integer, $23^{2k}11 + 31^{2k}20 + s$ is a positive integer.

$$\therefore 48 \mid f(k+1)$$

Since $f(k+1)$ is divisible by 48 if $f(k)$ is divisible by 48 and $f(0)$ is divisible by 48, $f(n)$ is divisible by 48 for all integers $n \geq 0$.
 $\therefore 23^{2n} + 31^{2n} + 46$ is divisible by 48 for all $n \geq 0$.

$$4. A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -3 & 4 & 5 \\ 5 & -6 & 10 & 14 \\ 4 & -5 & 8 & 11 \end{pmatrix}$$

$$\xrightarrow{-2r_1 + r_2} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{-r_2 + r_3} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The dimension of the range space of T is 2.

$$\text{If } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4,$$

$$\text{since } \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -3 & 4 & 5 \\ 5 & -6 & 10 & 14 \\ 4 & -5 & 8 & 11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 - x_2 + 2x_3 + 3x_4 \\ 2x_1 - 3x_2 + 4x_3 + 5x_4 \\ 5x_1 - 6x_2 + 10x_3 + 14x_4 \\ 4x_1 - 5x_2 + 8x_3 + 11x_4 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ 2x_1 \\ 5x_1 \\ 4x_1 \end{pmatrix} + \begin{pmatrix} -x_2 \\ -3x_2 \\ -6x_2 \\ -5x_2 \end{pmatrix} + \begin{pmatrix} 2x_3 \\ 4x_3 \\ 10x_3 \\ 8x_3 \end{pmatrix} + \begin{pmatrix} 3x_4 \\ 5x_4 \\ 14x_4 \\ 11x_4 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ 2 \\ 5 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -3 \\ -6 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 4 \\ 10 \\ 8 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 5 \\ 14 \\ 11 \end{pmatrix}$$

$$= (x_1 + 2x_3 + 4x_4) \begin{pmatrix} 1 \\ 2 \\ 5 \\ 4 \end{pmatrix} + (x_2 + x_4) \begin{pmatrix} -1 \\ -3 \\ -6 \\ -5 \end{pmatrix}$$

$\therefore \left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -6 \\ -5 \end{pmatrix} \right\}$ is a basis for the range space of T .

If $S = \{M\mathbf{A}\mathbf{x} : \mathbf{x} \in \mathbb{R}^4\}$ and $\{\underline{b}_1, \underline{b}_2\}$ is a basis of the range space of T , $\lambda \underline{b}_1 + M\underline{b}_2 = \underline{0}$ if and only if $\lambda = M = 0$, since \underline{b}_1 and \underline{b}_2 are linearly independent. If M is a non-singular 4×4 matrix and $\lambda(M\underline{b}_1) + M(M\underline{b}_2) = \underline{0}$,

$$M^{-1}(\lambda(M\underline{b}_1) + M(M\underline{b}_2)) = M^{-1}\underline{0}$$

$$M^{-1}(\lambda(M\underline{b}_1)) + M^{-1}(M(M\underline{b}_2)) = \underline{0}$$

$$\lambda(M^{-1}M\underline{b}_1) + M(M^{-1}M\underline{b}_2) = \underline{0}$$

$$\lambda\underline{b}_1 + M\underline{b}_2 = \underline{0}$$

Since $\lambda = M = 0$, $M\underline{b}_1$ and $M\underline{b}_2$ are linearly independent and $M\mathbf{A}\mathbf{x} = M(\lambda\underline{b}_1 + M\underline{b}_2)$

$$= \lambda(M\underline{b}_1) + M(M\underline{b}_2)$$

Since S consists of vectors of the form $M\mathbf{A}\mathbf{x}$, the dimension of S is 2.

5. C : $y = 2x + \frac{3(x-1)}{x+1}$

i)

$$x+1 \begin{array}{c} 1 \\ \hline x-1 \\ \hline x+1 \\ \hline -2 \end{array}$$

$$\therefore y = 2x + 3\left(1 - \frac{2}{x+1}\right)$$

$$= 2x + 3 - \frac{6}{x+1}$$

$$x \rightarrow \pm\infty \quad y \rightarrow 2x + 3$$

$\therefore y = 2x + 3$ is an asymptote of C.

$$x \rightarrow -1 \quad y \rightarrow \pm\infty$$

$\therefore x = -1$ is an asymptote of C.

The asymptotes of C are $y = 2x + 3$ and $x = -1$.

ii) The set of values of x for which C is above its oblique asymptote is given by

$$y > 2x + 3$$

$$2x + \frac{3(x-1)}{x+1} > 2x + 3$$

$$\frac{3(x-1)}{x+1} > 3$$

$$\frac{x-1}{x+1} > 1$$

$$\frac{x-1}{x+1} - 1 > 0$$

$$\frac{x-1 - x-1}{x+1} > 0$$

$$\frac{-2}{x+1} > 0$$

$$\frac{1}{x+1} < 0$$

$$x < -1$$

$\therefore C$ is above it's oblique asymptote if $x < -1$

The set of values of x for which C is below its oblique asymptote is given by

$$y < 2x + 3$$

$$2x + \frac{3(x-1)}{x+1} < 2x + 3$$

$$\frac{3(x-1)}{x+1} < 3$$

$$\frac{x-1}{x+1} < 1$$

$$\frac{x-1}{x+1} - 1 < 0$$

$$\frac{x-1 - x-1}{x+1} < 0$$

$$\frac{-2}{x+1} < 0$$

$$\frac{1}{x+1} > 0$$

$$x > -1$$

$\therefore C$ is below it's oblique asymptote if $x > -1$

$$\text{iii) } y = 2x + \frac{3(x-1)}{x+1}$$

$$= \frac{2x(x+1) + 3(x-1)}{x+1}$$

$$= \frac{2x^2 + 2x + 3x - 3}{x+1}$$

$$= \frac{2x^2 + 5x - 3}{x+1}$$

$$= \frac{(2x-1)(x+3)}{x+1}$$

$$\text{when } x = 0 : y = -3$$

$$\text{when } y = 0 : \frac{(2x-1)(x+3)}{x+1} = 0$$

$$(2x-1)(x+3) = 0$$

$$x = \frac{1}{2}, -3$$

\therefore The intersection points of C are $(0, -3), (\frac{1}{2}, 0)$

and $(-3, 0)$

$$y = 2x + 3 - \frac{6}{x+1}$$

$$\frac{dy}{dx} = 2 + \frac{6}{(x+1)^2}$$

$$> 0 \quad \text{since } \frac{1}{(x+1)^2} > 0$$

Since $\frac{dy}{dx} \neq 0$, there are no critical points.

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b. a) $y = \frac{2\sqrt{3}}{3} x^{\frac{3}{2}}$

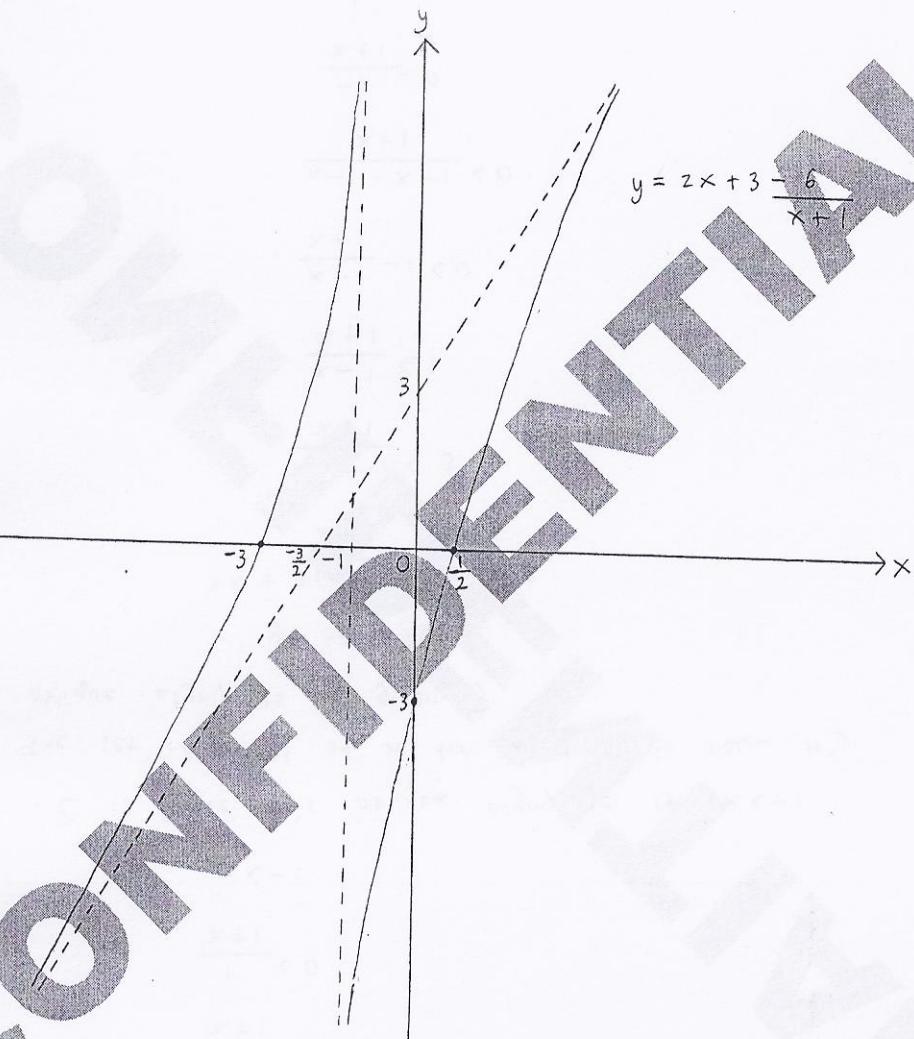
$$\frac{dy}{dx} = \sqrt{3} x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = 3x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 3x$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 3x}$$

The length of the arc from the origin to the point where $x=1$ is



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$$\begin{aligned}& \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\&= \int_0^1 \sqrt{1 + 3x} dx \\&= \left[\frac{2(1+3x)^{\frac{3}{2}}}{3(3)} \right]_0^1 \\&= \left[\frac{2}{9} (1+3x)^{\frac{3}{2}} \right]_0^1 \\&= \frac{2}{9} (1+3(1))^{\frac{3}{2}} - \frac{2}{9} (1+3(0))^{\frac{3}{2}} \\&= \frac{2}{9} (4^{\frac{3}{2}}) - \frac{2}{9} (1)^{\frac{3}{2}} \\&= \frac{2}{9} (8) - \frac{2}{9} (1) \\&= \frac{16}{9} - \frac{2}{9} \\&= \frac{14}{9}\end{aligned}$$

b) $y^3 + \left(\frac{dy}{dx}\right)^3 = x^4 + 6$ $y = -1$ when $x = 1$

$x = 1$ $y = -1$:

$$(-1)^3 + \left(\frac{dy}{dx}\right)^3 = 1^4 + 6$$

$$-1 + \left(\frac{dy}{dx}\right)^3 = 1 + 6$$

$$\left(\frac{dy}{dx}\right)^3 = 8$$

$$\frac{dy}{dx} = 2$$

∴ when $x = 1$, $\frac{dy}{dx} = 2$.

$$\frac{d(y^3)}{dx} + \frac{d}{dx} \left(\frac{dy}{dx}\right)^3 = \frac{d}{dx}(x^4 + 6)$$

$$\frac{dy}{dx} \frac{d(y^3)}{dy} + 3\left(\frac{dy}{dx}\right)^2 \frac{d}{dx} \left(\frac{dy}{dx}\right) = 4x^3$$

$$3y^2 \frac{dy}{dx} + 3\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} = 4x^3$$

when $x = 1$, since $y = -1$ and $\frac{dy}{dx} = 2$:

$$3(-1)^2(2) + 3(2)^2 \frac{d^2y}{dx^2} = 4(1^3)$$

$$6 + 12 \frac{d^2y}{dx^2} = 4$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{6}$$

7. $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, n \geq 0$

$$I_{n+2} = \int_0^{\frac{\pi}{2}} \sin^{n+2} x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x \sin^2 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x (1 - \cos^2 x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x - \sin^n x \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x \, dx - \int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x \, dx$$

$$= I_n - \int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x \, dx$$

$$= I_n - \int_0^{\frac{\pi}{2}} \sin^n x \cos x \cos x \, dx$$

$$u = \cos x \quad dv = \sin^n x \cos x \, dx$$

$$du = -\sin x \, dx \quad v = \frac{\sin^{n+1} x}{n+1}$$

$$= I_n - \left(\left[\frac{\sin^{n+1} x \cos x}{n+1} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-\sin^{n+1} x \sin x \, dx}{n+1} \right)$$

$$= I_n - \left(\frac{\sin^{n+1} \frac{\pi}{2} \cos \frac{\pi}{2}}{n+1} - \frac{\sin^{n+1} 0 \cos 0}{n+1} \right) - \int_0^{\frac{\pi}{2}} \frac{\sin^{n+2} x}{n+1} \, dx$$

$$= I_n - 0 - \frac{1}{n+1} \int_0^{\pi} \sin^{n+1} x dx$$

$$= I_n - \frac{1}{n+1} I_{n+2}$$

$$I_{n+2} + \frac{1}{n+1} I_{n+2} = I_n$$

$$\left(1 + \frac{1}{n+1}\right) I_{n+2} = I_n$$

$$\left(\frac{n+1+1}{n+1}\right) I_{n+2} = I_n$$

$$\left(\frac{n+2}{n+1}\right) I_{n+2} = I_n$$

$$I_{n+2} = \left(\frac{n+1}{n+2}\right) I_n$$

The area of the region bounded by the x -axis and the arc of the curve $y = \sin^4 x$ from $x=0$ to $x=\pi$, R is

$$\int_0^{\pi} y dx = \int_0^{\pi} \sin^4 x dx$$

$$\text{If } I_n = \int_0^{\pi} \sin^n x dx,$$

$$I_{n+2} = \int_0^{\pi} \sin^{n+2} x dx$$

$$= \int_0^{\pi} \sin^n x \sin^2 x dx$$

$$= \int_0^{\pi} \sin^n x (1 - \cos^2 x) dx$$

$$= \int_0^{\pi} \sin^n x - \sin^n x \cos^2 x dx$$

$$= \int_0^{\pi} \sin^n x dx - \int_0^{\pi} \sin^n x \cos^2 x dx$$

$$= I_n - \int_0^{\pi} \sin^n x \cos x \cos x dx$$

$$u = \cos x \quad dv = \sin^n x \cos x dx \\ du = -\sin x dx \quad v = \frac{\sin^{n+1} x}{n+1}$$

$$= I_n - \left(\left[\frac{\sin^{n+1} x \cos x}{n+1} \right]_0^{\pi} - \int_0^{\pi} -\frac{\sin^{n+1} x \sin x}{n+1} dx \right)$$

$$= I_n - \left(\frac{\sin^{n+1} \pi \cos \pi}{n+1} - \frac{\sin^{n+1} 0 \cos 0}{n+1} \right) - \int_0^{\pi} \frac{\sin^{n+2} x}{n+1} dx$$

$$= I_n - 0 - \frac{1}{n+1} \int_0^{\pi} \sin^{n+2} x dx$$

$$= I_n - \frac{1}{n+1} I_{n+2}$$

$$I_{n+2} + \frac{1}{n+1} I_{n+2} = I_n$$

$$\left(1 + \frac{1}{n+1}\right) I_{n+2} = I_n$$

$$\left(\frac{n+1+1}{n+1}\right) I_{n+2} = I_n$$

$$\left(\frac{n+2}{n+1}\right) I_{n+2} = I_n$$

$$I_{n+2} = \left(\frac{n+1}{n+2}\right) I_n$$

\therefore The area of R is $I_4 = \int_0^{\pi} \sin^4 x dx$

$$= \frac{3}{4} I_2$$

$$= \frac{3}{4} \cdot \frac{1}{2} I_0$$

$$= \frac{3}{8} I_0$$

$$= \frac{3\pi}{8}$$

$$\text{since } I_0 = \int_0^\pi \sin^0 x \, dx$$

$$= \int_0^\pi 1 \, dx$$

$$= [x]_0^\pi$$

$$= \pi - 0$$

$$= \pi$$

The y -coordinate of the centroid of R is

$$\frac{\int_0^\pi \frac{y^2}{2} dx}{\frac{3\pi}{8}}$$

$$= \frac{\int_0^\pi (\sin^4 x)^2 dx}{\frac{3\pi}{8}}$$

$$= \frac{\int_0^\pi \frac{\sin^8 x}{2} dx}{\frac{3\pi}{8}}$$

$$= \frac{3\pi}{8}$$

$$= \frac{\frac{1}{2} \int_0^\pi \sin^8 x \, dx}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} I_8}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} I_6}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} \cdot \frac{5}{6} I_4}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} I_2}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0}{\frac{3\pi}{8}}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \pi}{\frac{3\pi}{8}}$$

$$= \frac{\frac{105\pi}{768}}{\frac{3\pi}{8}}$$

$$= \frac{35}{96}$$

$$8. \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = 80e^{-3t}$$

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = 0$$

$$m^2 + 6m + 25 = 0$$

$$(m+3)^2 + 16 = 0$$

$$m = -3 \pm 4i$$

$$\therefore y_c = e^{-3t}(A\cos 4t + B\sin 4t)$$

$$\text{Let } y_p = Ce^{-3t}$$

$$\frac{dy_p}{dt} = -3Ce^{-3t}$$

$$\frac{d^2y_p}{dt^2} = 9Ce^{-3t}$$

$$\begin{aligned} \therefore \frac{d^2y_p}{dt^2} + 6\frac{dy_p}{dt} + 25y_p &= 9Ce^{-3t} - 18Ce^{-3t} + 25Ce^{-3t} \\ &= 16Ce^{-3t} \\ &= 80e^{-3t} \end{aligned}$$

$$\therefore 16C = 80$$

$$C = 5$$

$$\therefore y_p = 5e^{-3t}$$

$$y = y_c + y_p$$

$$= e^{-3t}(A\cos 4t + B\sin 4t) + 5e^{-3t}$$

$$\frac{dy}{dx} = -3e^{-3t}(A\cos 4t + B\sin 4t) + e^{-3t}(-4A\sin 4t + 4B\cos 4t)$$

$$-15e^{-3t}$$

when $t=0$ $y=8 : 8 = A+5$

when $t=0$ $\frac{dy}{dt} = -8 : -8 = -3A + 4B - 15$

$$A = 3$$

$$-3A + 4B = 7$$

$$-3(3) + 4B = 7$$

$$4B = 16$$

$$B = 4$$

$$y = e^{-3t}(3\cos 4t + 4\sin 4t) + 5e^{-3t}$$

$$= e^{-3t}(3\cos 4t + 4\sin 4t + 5)$$

$$\therefore ye^{-3t} = 3\cos 4t + 4\sin 4t + 5$$

Expressing $3\cos 4t + 4\sin 4t$ as $R\cos(4t - \alpha)$:

$$3\cos 4t + 4\sin 4t = R\cos(4t - \alpha)$$

$$= R(\cos 4t \cos \alpha + \sin 4t \sin \alpha)$$

$$= R\cos 4t \cos \alpha + R\sin 4t \sin \alpha$$

$$R\cos \alpha = 3 \quad R\sin \alpha = 4$$

$$(R\cos \alpha)^2 + (R\sin \alpha)^2 = 3^2 + 4^2$$

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 9 + 16$$

$$R^2 (\cos^2 \alpha + \sin^2 \alpha) = 25$$

$$R^2 = 25$$

$$R = 5$$

$$\frac{R\sin \alpha}{R\cos \alpha} = \frac{4}{3}$$

$$\tan \alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1} \frac{4}{3}$$

$$\therefore ye^{-3t} = 5\cos(4t - \alpha) + 5$$

since $-1 \leq \cos(4t - \alpha) \leq 1$,

$$-5 \leq 5\cos(4t - \alpha) \leq 5$$

$$0 \leq 5\cos(4t - \alpha) + 5 \leq 10$$

$$0 \leq ye^{-3t} \leq 10$$

If $y=8$ and $\frac{dy}{dt} = -8$ when $t=0$, $0 \leq ye^{-3t} \leq 10$ for all t .

$$9. z = e^{i\theta}$$

$$= \cos \theta + i \sin \theta$$

$$z^n = (\cos \theta + i \sin \theta)^n$$

$$= \cos n\theta + i \sin n\theta$$

$$\text{and } z^{-n} = (\cos \theta + i \sin \theta)^{-n}$$

$$= \cos(-n\theta) + i \sin(-n\theta)$$

$= \cos n\theta - i \sin n\theta$, from De Moivre's Theorem.

$$\therefore z^n + z^{-n} = \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta$$

$$= 2 \cos n\theta$$

and

$$z^n - z^{-n} = \cos n\theta + i \sin n\theta - (\cos n\theta - i \sin n\theta)$$

$$= \cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta$$

$$= 2i \sin n\theta.$$

Since

$$z + \frac{1}{z} = 2 \cos \theta$$

$$(2 \cos \theta)^7 = \left(z + \frac{1}{z}\right)^7$$

$$128 \cos^7 \theta = z^7 + 7z^5 + 21z^3 + 35z + \frac{35}{z} + \frac{21}{z^3} + \frac{7}{z^5} + \frac{1}{z^7}$$

$$= z^7 + \frac{1}{z^7} + 7\left(z^5 + \frac{1}{z^5}\right) + 21\left(z^3 + \frac{1}{z^3}\right) + 35\left(z + \frac{1}{z}\right)$$

$$= 2 \cos 7\theta + 7(2 \cos 5\theta) + 21(2 \cos 3\theta) + 35(2 \cos \theta)$$

$$\therefore \cos^7 \theta = \frac{1}{64} (\cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta)$$

$$= \frac{\cos 7\theta}{64} + \frac{7 \cos 5\theta}{64} + \frac{21 \cos 3\theta}{64} + \frac{35 \cos \theta}{64}$$

The mean value of $\cos^7 2\theta$ over the interval $0 \leq \theta \leq \frac{\pi}{4}$ is

$$\frac{1}{\frac{\pi}{4} - 0} \int_0^{\frac{\pi}{4}} \cos^7 2\theta \, d\theta$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \cos^7 2\theta \, d\theta$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{\cos 7(2\theta)}{64} + \frac{7 \cos 5(2\theta)}{64} + \frac{21 \cos 3(2\theta)}{64} + \frac{35 \cos 2\theta}{64} \, d\theta$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{\cos 14\theta}{64} + \frac{7 \cos 10\theta}{64} + \frac{21 \cos 6\theta}{64} + \frac{35 \cos 2\theta}{64} \, d\theta$$

$$= \frac{4}{\pi} \left[\frac{\sin 14\theta}{64(14)} + \frac{7 \sin 10\theta}{64(10)} + \frac{21 \sin 6\theta}{64(6)} + \frac{35 \sin 2\theta}{64(2)} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{4}{\pi} \left(\frac{\sin \frac{7\pi}{2}}{896} + \frac{7 \sin \frac{5\pi}{2}}{640} + \frac{21 \sin \frac{3\pi}{2}}{384} + \frac{35 \sin \frac{\pi}{2}}{128} \right)$$

$$- \left(\frac{\sin 0}{896} + \frac{7 \sin 0}{640} + \frac{21 \sin 0}{384} + \frac{35 \sin 0}{128} \right)$$

$$= \frac{4}{\pi} \left(\frac{-1}{896} + \frac{7}{640} - \frac{21}{384} + \frac{35}{128} - 0 \right)$$

$$= \frac{4}{\pi} \left(\frac{8}{35} \right)$$

$$= \frac{32}{35\pi}$$

$$10. \Pi: 2x + 3y + 4z = 48$$

Let $y = -2\lambda$ and $z = -M$, $\lambda, M \in \mathbb{R}$

$$2x + 3(-2\lambda) + 4(-M) = 48$$

$$2x - 6\lambda - 4M = 48$$

$$x - 3\lambda - 2M = 24$$

$$x = 24 + 3\lambda + 2M$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24 + 3\lambda + 2M \\ -2\lambda \\ -M \end{pmatrix}$$

$$= \begin{pmatrix} 24 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3\lambda \\ -2\lambda \\ 0 \end{pmatrix} + \begin{pmatrix} 2M \\ 0 \\ -M \end{pmatrix}$$

$$= \begin{pmatrix} 24 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + M \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

A vector equation of Π in the form

$$\underline{r} = \underline{a} + \lambda \underline{b} + M \underline{c},$$

where $\underline{a} = p\underline{i}$, $\underline{b} = q\underline{i} + r\underline{j}$ and $\underline{c} = s\underline{i} + t\underline{k}$,

and where, p, q, r, s, t are integers is

$$\underline{r} = 24\underline{i} + \lambda(3\underline{i} - 2\underline{j}) + M(2\underline{i} - \underline{k})$$

$$\ell: \underline{r} = 29\underline{i} - 2\underline{j} - \underline{k} + \theta(5\underline{i} - 6\underline{j} + 2\underline{k})$$

$$= (29 + 5\theta)\underline{i} + (-2 - 6\theta)\underline{j} + (-1 + 2\theta)\underline{k}$$

If ℓ intersects Π ,

$$2(29 + 5\theta) + 3(-2 - 6\theta) + 4(-1 + 2\theta) = 48$$

$$58 + 10\theta - 6 - 18\theta - 4 + 8\theta = 48$$

$$48 = 48$$

Since $x = 29 + 5\theta$, $y = -2 - 6\theta$ and $z = -1 + 2\theta$

satisfy the equation $2x + 3y + 4z = 48$ for all values of θ , ℓ lies in Π .

The plane which contains ℓ and is perpendicular to Π has a normal vector in the direction

$$\begin{pmatrix} 5 \\ -6 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

since the normal of Π , $2\underline{i} + 3\underline{j} + 4\underline{k}$ is also perpendicular to the normal of the plane.

$$\begin{pmatrix} 5 \\ -6 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & -6 & 2 \\ 2 & 3 & 4 \end{vmatrix} = -30\underline{i} - 16\underline{j} + 27\underline{k}$$

Since the plane contains ℓ , $(29, -2, -1)$ is a point in the plane, and since $-30\underline{i} - 16\underline{j} + 27\underline{k}$ is a normal to the plane, the equation of the plane can be expressed as

$$\underline{r} \cdot \begin{pmatrix} -30 \\ -16 \\ 27 \end{pmatrix} = \begin{pmatrix} 29 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -30 \\ -16 \\ 27 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -30 \\ -16 \\ 27 \end{pmatrix} = \begin{pmatrix} 29 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -30 \\ -16 \\ 27 \end{pmatrix}$$

$$-30x - 16y + 27z = -870 + 32 - 27$$

$$= -865$$

$$30x + 16y + 27z = 865$$

∴ The equation of the plane containing ℓ and perpendicular to Π is $30x + 16y + 27z = 865$.

$$\text{II. } x^4 + 3x^3 + 5x^2 + 12x + 4 = 0$$

If α, β, r, s are the roots of the equation

$$\alpha + \beta + r + s = -3$$

$$\alpha\beta + \alpha r + \alpha s + \beta r + \beta s + rs = 5$$

$$\alpha\beta r + \alpha\beta s + \alpha rs + \beta rs = -12$$

$$\alpha\beta rs = 4$$

The sum of the squares of the roots of the equation,

$$\begin{aligned} \alpha^2 + \beta^2 + r^2 + s^2 &= (\alpha + \beta + r + s)^2 \\ &\quad - 2(\alpha\beta + \alpha r + \alpha s + \beta r + \beta s + rs) \\ &= (-3)^2 - 2(5) \\ &= 9 - 10 \\ &= -1 \end{aligned}$$

If $\alpha, \beta, r, s \in \mathbb{R}$, $\alpha^2 > 0, \beta^2 > 0, r^2 > 0, s^2 > 0$ and $\alpha^2 + \beta^2 + r^2 + s^2 > 0$.

Since $\alpha^2 + \beta^2 + r^2 + s^2 = -1$, the equation does not have all real roots and since the coefficients of the equation are real, the complex roots occur in conjugate pairs.

\therefore The equation does not have more than 2 real roots.

$$\text{Let } f(x) = x^4 + 3x^3 + 5x^2 + 12x + 4$$

$$\begin{aligned} f(-3) &= (-3)^4 + 3(-3)^3 + 5(-3)^2 + 12(-3) + 4 \\ &= 13 \end{aligned}$$

$$\begin{aligned} f(-1) &= (-1)^4 + 3(-1)^3 + 5(-1)^2 + 12(-1) + 4 \\ &= -5 \end{aligned}$$

\therefore The equation has a root in the interval $-3 < x < -1$

$$\begin{aligned} f(0) &= 0^4 + 3(0^3) + 5(0^2) + 12(0) + 4 \\ &= 4 \end{aligned}$$

\therefore The equation has a root in the interval $-1 < x < 0$

\therefore The equation has exactly 2 real roots in the interval $-3 < x < 0$.

If r and s are the complex roots,

$$\text{let } r = c + di \text{ and } s = c - di$$

$$\begin{aligned} rs &= (c + di)(c - di) = c^2 - cdi + cdi + d^2 \\ &= c^2 + d^2 \\ &= |r|^2 \\ &= |s|^2 \end{aligned}$$

since $\alpha\beta r s = 4$,

$$|\alpha\beta r s| = |4|$$

$$|\alpha\beta||r||s| = 4$$

$$\alpha\beta|r||s| = 4$$

$$|r||s| = \frac{4}{\alpha\beta}$$

$$|v|^2 = |s|^2 = \frac{4}{\alpha\beta}$$

$$\therefore |v| = |s| = \frac{2}{\sqrt{\alpha\beta}}$$

CONFIDENTIAL

$$Ax = \lambda x$$

$$B = MAM^{-1}$$

$$\therefore B(Mx) = MAM^{-1}(Mx)$$

$$= MAM^{-1}Mx$$

$$= MAIx$$

$$= MAX$$

$$= M(\lambda x)$$

$$= \lambda(Mx)$$

$\therefore Mx$ is an eigenvector of the matrix B , where $B = MAM^{-1}$, and that λ is the corresponding eigenvalue.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & -3 & 0 \\ b & c & -5 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{i)} A - \lambda I = \begin{pmatrix} 1 & 0 & 0 \\ a & -3 & 0 \\ b & c & -5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 0 & 0 \\ a & -3-\lambda & 0 \\ b & c & -5-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda) \begin{vmatrix} -3-\lambda & 0 \\ c & -5-\lambda \end{vmatrix} - 0 \begin{vmatrix} a & 0 \\ b & -5-\lambda \end{vmatrix}$$

$$+ 0 \begin{vmatrix} a & -3-\lambda \\ b & c \end{vmatrix}$$

$$\begin{aligned}
 &= (1-\lambda)[(-3-\lambda)(-5-\lambda) - 0] - 0 + 0 \\
 &= (1-\lambda)(\lambda+3)(\lambda+5)
 \end{aligned}$$

$$|A - \lambda I| = 0$$

$$(1-\lambda)(\lambda+3)(\lambda+5) = 0$$

$$\lambda = 1, -3, -5$$

when $\lambda = 1$:

$$\begin{pmatrix} 0 & 0 & 0 \\ a & -4 & 0 \\ b & c & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ a & -4 & 0 \\ b & c & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Let } y = 6as, s \in \mathbb{R}$$

$$ax - 4(6as) = 0$$

$$x = 24s$$

$$b(24s) + c(6as) - 6z = 0$$

$$z = 4bs + acs$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24s \\ 6as \\ 4bs + acs \end{pmatrix}$$

$$= s \begin{pmatrix} 24 \\ 6a \\ 4b + ac \end{pmatrix}$$

when $\lambda = -3$:

$$\begin{pmatrix} 4 & 0 & 0 \\ a & 0 & 0 \\ b & c & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & c & -2 & 0 \end{array} \right)$$

$$\xrightarrow{\frac{r_1}{4}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & c & -2 & 0 \end{array} \right)$$

$$\xrightarrow{-ar_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c & -2 & 0 \end{array} \right)$$

$$\text{Let } z = cs, s \in \mathbb{R}$$

$$cy - 2cs = 0$$

$$y = 2s$$

$$x = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2s \\ cs \end{pmatrix}$$

$$= s \begin{pmatrix} 0 \\ 2 \\ c \end{pmatrix}$$

when $\lambda = -5$:

$$\begin{pmatrix} 6 & 0 & 0 \\ a & 2 & 0 \\ b & c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ a & 2 & 0 & 0 \\ b & c & 0 & 0 \end{array} \right)$$

$$\xrightarrow{\frac{r_1}{6}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ a & 2 & 0 & 0 \\ b & c & 0 & 0 \end{array} \right)$$

$$\begin{array}{l} -ar_1 + r_2 \\ -br_1 + r_3 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & c & 0 & 0 \end{array} \right)$$

$$\xrightarrow{\frac{r_2}{2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 0 & 0 \end{array} \right)$$

$$\xrightarrow{-cr_2 + r_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Let $z = s, s \in \mathbb{R}$

$$y = 0$$

$$x = 0$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvalues of A are 1, -3, and -5 and the corresponding eigenvectors are

$$\begin{pmatrix} 24 \\ 6a \\ 4b+ac \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ c \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

ii) The eigenvalues of B are 1, -3 and -5 and the corresponding eigenvectors are

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ 6a \\ 4b+ac \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ c \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ since } B = MAM^{-1}$$

∴ The eigenvalues of B are 1, -3 and -5 and the corresponding eigenvectors are

$$\begin{pmatrix} 24+4b+ac \\ 6a \\ 4b+ac \end{pmatrix}, \begin{pmatrix} c \\ 2 \\ c \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

iii) If $B^n = QDQ^{-1}$,

$$\therefore D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-3)^n & 0 \\ 0 & 0 & (-5)^n \end{pmatrix} \text{ is a diagonal matrix}$$

$$\text{and } Q = \begin{pmatrix} 24+4b+ac & c & 1 \\ 6a & 2 & 0 \\ 4b+ac & c & 1 \end{pmatrix}.$$