OCTOBER / NOVEMBER 2012

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1. $$\sum_{r=n+1}^{2n} r^2 = \frac{n(2n+1)(7n+1)}{6}$$

Since $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$,

$$\sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2$$

$$= \frac{2n(2n+1)(2(2n)+1)}{6} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(2n+1)}{6}\left(2(4n+1)-(n+1)\right)$$

$$= \frac{n(2n+1)}{6}(8n+2-n-1)$$

$$= \frac{n(2n+1)(7n+1)}{6}$$

2. $$\left.\begin{aligned} ax + y + 2z &= 0 \\ 3x - 2y &= 4 \\ 3x - 4y - 6az &= 14 \end{aligned}\right\}$$

$-3\times①+a\times②$: $ax + y + 2z = 0$

$-3\times①+a\times③$: $(-3-2a)y - 6z = 0$

$(-3-4a)y + (-6-6a^2)z = 0$

$(3+4a)\times② - (3+2a)\times③$:

$$\left.\begin{aligned} ax + y + 2z &= 0 \\ (-3-2a)y - 6z &= 0 \\ [-6(3+4a)-(3+2a)(-6-6a^2)]z &= 0 \end{aligned}\right\}$$

when $-6(3+4a)-(3+2a)(-6-6a^2)=0$,

$-6(3+4a)+(3+2a)(6a^2+6)=0$

$-3-4a+(3+2a)(a^2+1)=0$

$-3-4a+3a^2+3+2a^3+2a=0$

$2a^3+3a^2-2a=0$

$a(2a^2+3a-2)=0$

$a(2a-1)(a+2)=0$

$a=0,\frac{1}{2},-2$

$\therefore$ If the system has a unique solution,

$a\neq 0,\frac{1}{2},-2$.

3. $S_N = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{N}{(N+1)!}$

$S_N = 1 - \frac{1}{(N+1)!}$

when $N=1$: $\frac{1}{(1+1)!} = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2\cdot 1} = 1 - \frac{1}{2!} = 1 - \frac{1}{(1+1)!}$

Assume the statement is true when $N=k$.

$N=k$: $S_k = 1 - \frac{1}{(k+1)!} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!}$

when $N=k+1$: $S_{k+1} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= 1 - \frac{1}{(k+2)!}$

(what needs to be proved)

$S_{k+1} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= 1 - \frac{1}{(k+1)!} + \frac{(k+1)}{(k+2)!}$

$= 1 - \frac{(k+2)}{(k+2)(k+1)!} + \frac{(k+1)}{(k+2)!}$

$= 1 - \frac{(k+2)}{(k+2)!} + \frac{(k+1)}{(k+2)!}$

$= 1 + \frac{(k+1) - (k+2)}{(k+2)!}$

$= 1 - \frac{1}{(k+2)!}$

$\therefore S_N = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{N}{(N+1)!}$

$= 1 - \frac{1}{(N+1)!}$ for every positive integer N.

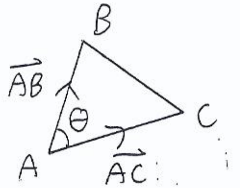
4. $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$ $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

i)



$$\text{Area} = \frac{AB \cdot AC \sin\theta}{2}$$

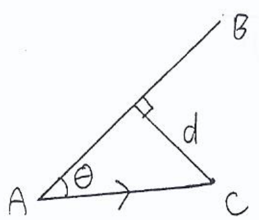
$$= \frac{|\overrightarrow{AB}||\overrightarrow{AC}|\sin\theta}{2}$$

$$= \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2}$$

$$= \frac{1}{2}\left|\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right|$$

$$= \frac{\sqrt{3}}{2}$$

ii)



The perpendicular distance from C to AB is

$$d = AC \sin\theta$$

$$= |\overrightarrow{AC}| \sin\theta$$

$$= \frac{|\overrightarrow{AB}||\overrightarrow{AC}|\sin\theta}{|\overrightarrow{AB}|}$$

$$= \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{|\overrightarrow{AB}|}$$

$$= \frac{\left|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|}{\left|\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right|}$$

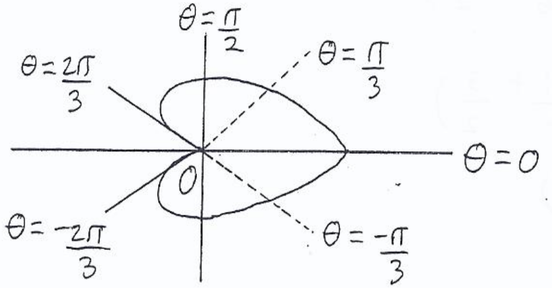
$$= \frac{\sqrt{3}}{\sqrt{14}}$$

$$= \sqrt{\frac{3}{14}}$$

5. $C: r = 1 + 2\cos\theta \quad , \quad -\frac{2\pi}{3} \le \theta < \frac{2\pi}{3}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
r	3	$1+\sqrt{3}$	$1+\sqrt{2}$	$1+1$	1	0

$1 + 2\cos(-\theta) = 1 + 2\cos\theta$



The area bounded by C, $\theta = -\frac{\pi}{3}$ and $\theta = \frac{\pi}{3}$ is

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{r^2}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{(1+2\cos\theta)^2}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1 + 4\cos\theta + 4\cos^2\theta}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1 + 4\cos\theta + 2\cos 2\theta + 2}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos 2\theta + 2\cos\theta + \frac{3}{2}\, d\theta$$

$$= \left[\frac{\sin 2\theta}{2} + 2\sin\theta + \frac{3\theta}{2}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\sin\frac{2\pi}{3}}{2} + 2\sin\frac{\pi}{3} + \frac{\pi}{2} - \left(\frac{\sin\left(-\frac{2\pi}{3}\right)}{2} + 2\sin\left(-\frac{\pi}{3}\right) - \frac{\pi}{2}\right)$$

$$= \frac{\sin\frac{2\pi}{3}}{2} + 2\sin\frac{\pi}{3} + \frac{\pi}{2} + \frac{\sin\frac{2\pi}{3}}{2} + 2\sin\frac{\pi}{3} + \frac{\pi}{2}$$

$$= \sin\frac{2\pi}{3} + 4\sin\frac{\pi}{3} + \pi$$

$$= \frac{\sqrt{3}}{2} + 4\left(\frac{\sqrt{3}}{2}\right) + \pi$$

$$= \frac{5\sqrt{3}}{2} + \pi$$

6. $C: x = t^2,\ y = \frac{t^4}{4} - \ln t,\ 1 \le t \le 2.$

$$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = t^3 - \frac{1}{t}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + \left(t^3 - \frac{1}{t}\right)^2$$

$$= 4t^2 + t^6 - 2t^2 + \frac{1}{t^2}$$

$$= t^6 + 2t^2 + \frac{1}{t^2}$$

$$= \left(t^3 + \frac{1}{t}\right)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t^3 + \frac{1}{t}$$

The surface area of revolution from $t=1$ to $t=2$ about the y-axis is

$$\int_1^2 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^2 2\pi t^2 \left(t^3 + \frac{1}{t}\right) dt$$

$$= 2\pi \int_1^2 t^5 + t \, dt$$

$$= 2\pi \left[\frac{t^6}{6} + \frac{t^2}{2}\right]_1^2$$

$$= 2\pi \left(\frac{64}{6} + 2 - \frac{1}{6} - \frac{1}{2}\right)$$

$$= 2\pi \left(\frac{63}{6} + \frac{3}{2}\right)$$

$$= 2\pi \left(\frac{21}{2} + \frac{3}{2}\right)$$

$$= 2\pi (12)$$

$$= 24\pi$$

7. $ax^3+bx^2+cx+d=0$, $a\neq 0$.

α, β, γ are the roots

$\alpha+\beta+\gamma = -\frac{b}{a}$ $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$, $\alpha\beta\gamma=-\frac{d}{a}$

If $\alpha+\beta+\gamma = 4$

$\alpha^2+\beta^2+\gamma^2=14$

$\alpha^3+\beta^3+\gamma^3=34$,

Since $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$,

$4^2 = 14+2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$\therefore \alpha\beta+\alpha\gamma+\beta\gamma = 1$

α,β,γ are the roots,

$(x-\alpha)(x-\beta)(x-\gamma)=0$

$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma=0$

$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0$

$\therefore -\frac{b}{a}=\alpha+\beta+\gamma=4$, $\frac{c}{a}=\alpha\beta+\alpha\gamma+\beta\gamma=1$, $\frac{d}{a}=-\alpha\beta\gamma$

$\frac{b}{a}=-4$.

Also $(\alpha+\beta+\gamma)^3 = \alpha^3+3\alpha^2(\beta+\gamma)+3\alpha(\beta+\gamma)^2+(\beta+\gamma)^3$,

$= \alpha^3+3\alpha^2\beta+3\alpha^2\gamma+3\alpha(\beta^2+2\beta\gamma+\gamma^2)$

$+\beta^3+3\beta^2\gamma+3\beta\gamma^2+\gamma^3$

$=\alpha^3+\beta^3+\gamma^3+3\alpha^2\beta+3\alpha^2\gamma$

$+3\alpha\beta^2+3\alpha\gamma^2+3\beta^2\gamma+3\beta\gamma^2+6\alpha\beta\gamma$

Since $(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)$

$=\alpha^2\beta+\alpha^2\gamma+\alpha\beta\gamma$

$+\alpha\beta^2+\alpha\beta\gamma+\beta^2\gamma$

$+\alpha\beta\gamma+\alpha\gamma^2+\beta\gamma^2$

$=\alpha^2\beta+\alpha^2\gamma+\alpha\beta^2+\alpha\gamma^2+\beta^2\gamma+\beta\gamma^2+3\alpha\beta\gamma$,

$(\alpha+\beta+\gamma)^3 = \alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)-3\alpha\beta\gamma$

$4^3 = 34+3(4)1-3\alpha\beta\gamma$

$64 = 34+12-3\alpha\beta\gamma$

$3\alpha\beta\gamma = -18$

$\alpha\beta\gamma = -6$

$\therefore \frac{d}{a}=6$

$\therefore$ The cubic equation is $x^3-4x^2+x+6=0$

-1 is a root since $(-1)^3-4(-1)^2-1+6=0$

$$\begin{array}{r|l} & x^2-5x+6 \\ \hline x+1 & x^3-4x^2+x+6 \\ & \underline{x^3+x^2} \\ & -5x^2+x \\ & \underline{-5x^2-5x} \\ & 6x+6 \\ & \underline{6x+6} \\ & 0 \end{array}$$

$\therefore (x+1)(x^2-5x+6)=0$

$(x+1)(x-2)(x-3)=0$

$\therefore x=-1,2,3$

8. $z = \cos\theta + i\sin\theta$

$= e^{i\theta}$

$1+z = 1+\cos\theta + i\sin\theta$

$= 1 + 2\cos^2\frac{\theta}{2} - 1 + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

$(1+z)^n = \sum_{r=0}^{n} \binom{n}{r} z^r$

$= \sum_{r=0}^{n} \binom{n}{r} (\cos\theta + i\sin\theta)^r$

$= \sum_{r=0}^{n} \binom{n}{r} (\cos r\theta + i\sin r\theta)$

$= \sum_{r=0}^{n} \binom{n}{r} \cos r\theta + i\sum_{r=0}^{n} \binom{n}{r} \sin r\theta$

since $1+z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

$(1+z)^n = \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^n$

$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^n$

$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right)$

$$\therefore \sum_{r=0}^{n} \binom{n}{r} \cos r\theta + i \sum_{r=0}^{n} \binom{n}{r} \sin r\theta$$

$$= 2^n \cos^n \frac{\theta}{2} \left(\cos \frac{n\theta}{2} + i \sin \frac{n\theta}{2} \right)$$

$$= 2^n \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2} + 2i \cos^n \frac{\theta}{2} \sin \frac{n\theta}{2}$$

$$\therefore \sum_{r=0}^{n} \binom{n}{r} \sin r\theta = 2 \cos^n \frac{\theta}{2} \sin \frac{n\theta}{2}.$$

9. $C: \; y = \dfrac{x^2 - 3x + 3}{x - 2}$

$$\begin{array}{r} x - 1 \\ x-2 \,\overline{)\, x^2 - 3x + 3} \\ \underline{x^2 - 2x} \\ -x + 3 \\ \underline{-x + 2} \\ 1 \end{array}$$

$$= x - 1 + \frac{1}{x - 2}$$

As $x \to \pm\infty$ $\quad y \to x - 1$

As $x \to 2$ $\quad y \to \pm\infty$

$\therefore$ The asymptotes of C are $y = x - 1$ and $x = 2$.

$$(x - 2)y = x^2 - 3x + 3$$

$$xy - 2y = x^2 - 3x + 3$$

$$x^2 - (y+3)x + 2y + 3 = 0$$

$A = 1$, $B = -(y+3)$, $C = 2y + 3$

$$B^2 - 4AC = (y+3)^2 - 4(2y+3)$$

$$= y^2 + 6y + 9 - 8y - 12$$

$$= y^2 - 2y - 3$$

$$= (y-3)(y+1)$$

when $B^2 - 4AC < 0$, $(y-3)(y+1) < 0$

$$\therefore -1 < y < 3$$

$\therefore$ There are no points on C for $-1 < y < 3$.

$$\frac{dy}{dx} = 1 - \frac{1}{(x-2)^2}$$

when $\frac{dy}{dx} = 0$: $1 - \frac{1}{(x-2)^2} = 0$

$$(x-2)^2 = 1$$
$$x - 2 = \pm 1$$
$$x = 1, 3$$
$$y = -1, 3$$

$$\frac{d^2y}{dx^2} = \frac{2}{(x-2)^3}$$

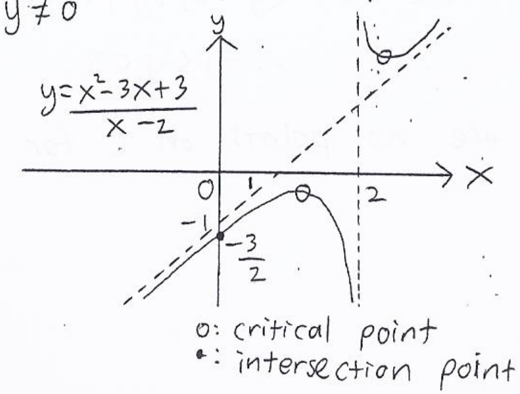
when $x = 1$: $\frac{d^2y}{dx^2} = -2 < 0$

when $x = 3$: $\frac{d^2y}{dx^2} = 2 > 0$

$\therefore$ $(1,-1)$ is a maximum point and $(3,3)$ is a minimum point.

when $x = 0$: $y = -\frac{3}{2}$

$y \neq 0$



10. $C: x^3 + y^3 = 3xy,\ x > 0, y > 0.$

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3xy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

when $\frac{dy}{dx} = 0$: $x^2 = y$

$$\frac{d}{dx}\left(x^2 + y^2\frac{dy}{dx}\right) = \frac{d}{dx}\left(x\frac{dy}{dx} + y\right)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}\left(y^2\frac{dy}{dx}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{d}{dx}(y)$$

$$2x + y^2\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2 = x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx}$$

$$2x + y^2\frac{d^2y}{dx^2} = x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}$$

when $\frac{dy}{dx} = 0$: $2x + y^2\frac{d^2y}{dx^2} = x\frac{d^2y}{dx^2}$

Since $y = x^2$, $x^3 + (x^2)^3 = 3x(x^2)$

$$x^3 + x^6 = 3x^3$$
$$x^6 - 2x^3 = 0$$
$$x^3(x^3 - 2) = 0$$
$$x = 0,\ 2^{\frac{1}{3}}$$
$$y = 0,\ 2^{\frac{2}{3}}.$$

since $x \neq 0$, the point is $\left(2^{\frac{1}{3}}, 2^{\frac{2}{3}}\right)$.

Also, $2x + y^2 \frac{d^2y}{dx^2} = x\frac{d^2y}{dx^2}$

$$2x + (x^2)^2 \frac{d^2y}{dx^2} = x\frac{d^2y}{dx^2}$$

$$2x + x^4 \frac{d^2y}{dx^2} = x\frac{d^2y}{dx^2}$$

$$2 + x^3 \frac{d^2y}{dx^2} = \frac{d^2y}{dx^2}$$

$$(1-x^3)\frac{d^2y}{dx^2} = 2$$

$$\frac{d^2y}{dx^2} = \frac{2}{1-x^3}$$

At $\left(2^{\frac{1}{3}}, 2^{\frac{2}{3}}\right)$, $\frac{d^2y}{dx^2} = \frac{2}{1-\left(2^{\frac{1}{3}}\right)^3}$

$$= \frac{2}{1-2}$$

$$= -2 < 0$$

$\therefore \left(2^{\frac{1}{3}}, 2^{\frac{2}{3}}\right)$ is a maximum point

11. $\int x(1-x^2)^{\frac{1}{2}}\,dx = \frac{(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}(-2)} = \frac{-(1-x^2)^{\frac{3}{2}}}{3} + c$

$$I_n = \int_0^1 x^n(1-x^2)^{\frac{1}{2}}\,dx$$

$$= \int_0^1 x^{n-1}x(1-x^2)^{\frac{1}{2}}\,dx$$

$u = x^{n-1}$ $\qquad dv = (1-x^2)^{\frac{1}{2}}dx$

$du = (n-1)x^{n-2}dx$ $\qquad v = \frac{-(1-x^2)^{\frac{3}{2}}}{3}$

$$= \left[\frac{-x^{n-1}(1-x^2)^{\frac{3}{2}}}{3}\right]_0^1 - \int_0^1 \frac{-(n-1)x^{n-2}(1-x^2)^{\frac{3}{2}}}{3}dx$$

$$= 0 + \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)^{\frac{3}{2}}dx$$

$$= \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)(1-x^2)^{\frac{1}{2}}dx$$

$$= \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)^{\frac{1}{2}} - x^n(1-x^2)^{\frac{1}{2}}dx$$

$$= \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)^{\frac{1}{2}}dx - \frac{(n-1)}{3}\int_0^1 x^n(1-x^2)^{\frac{1}{2}}dx$$

$$= \frac{(n-1)}{3}I_{n-2} - \frac{(n-1)}{3}I_n$$

$$3I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$(3+n-1)I_n = (n-1)I_{n-2}$$

$$\therefore (n+2)I_n = (n-1)I_{n-2}$$

$$\int_0^1 (1-x^2)^{\frac{1}{2}} dx$$

$x = \sin u$
$dx = \cos u \, du$
$x=0 \quad u=0$
$x=1 \quad u=\frac{\pi}{2}$

$$\therefore \int_0^1 (1-x^2)^{\frac{1}{2}} dx = \int_0^{\frac{\pi}{2}} (\cos^2 u)^{\frac{1}{2}} \cos u \, du$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 u \, du$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2u}{2} du$$

$$= \left[\frac{u}{2} + \frac{\sin 2u}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

$n=4: \; 6I_4 = 3I_2$
$n=2: \; 4I_2 = I_0$

$$I_0 = \int_0^1 x^0(1-x^2)^{\frac{1}{2}} dx$$

$$= \int_0^1 (1-x^2)^{\frac{1}{2}} dx$$

$$= \frac{\pi}{4}$$

$$4I_2 = \frac{\pi}{4}$$

$$I_2 = \frac{\pi}{16}$$

$$2I_4 = \frac{\pi}{16}$$

$$I_4 = \frac{\pi}{32}$$

12. EITHER

$A\underset{\sim}{e} = \lambda \underset{\sim}{e}$, $B\underset{\sim}{e} = \mu \underset{\sim}{e}$

$(AB)\underset{\sim}{e} = A(B\underset{\sim}{e}) = A(\mu\underset{\sim}{e}) = \mu(A\underset{\sim}{e}) = \mu(\lambda\underset{\sim}{e}) = (\lambda\mu)\underset{\sim}{e}$

$$A = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}$$

If $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are the eigenvectors,

$$A\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\therefore$ The corresponding eigenvalues are 0 and 1

2 is an eigenvalue

If $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\left.\begin{aligned} 3x + 2y + 2z &= 2x \\ -2x - 2y - 2z &= 2y \\ x + 2y + 2z &= 2z \end{aligned}\right\}$$

$$\left.\begin{aligned} x + 2y + 2z &= 0 \\ x + 2y + 2z &= 0 \\ x + 2y \quad &= 0 \end{aligned}\right\}$$

$\therefore z = 0$

Let $y = s$, $s \in \mathbb{R}$

$\therefore x = -2s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s \\ s \\ 0 \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ A corresponding eigenvector is $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

$B = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$ has the same eigenvectors as A.

$$\therefore B\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$B\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$B\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = -2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ B has eigenvalues $0, -3, -2$

If $C = AB$

$\therefore$ C has eigenvalues $0, -3, -4$ with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

If $P^{-1}C^2P = D$, $P = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{pmatrix}$

OR

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 13x = 75\cos 2t.$$

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 13x = 0$$

$m^2 + 6m + 13 = 0$

$(m+3)^2 + 4 = 0$

$(m+3)^2 = -4$

$m + 3 = \pm 2i$

$m = -3 \pm 2i$

$\therefore$ The complementary function, x_c is

$x_c = e^{-3t}(A\cos 2t + B\sin 2t).$

The particular integral, x_p, is given by

$x_p = C\cos 2t + D\sin 2t$

$$\frac{dx_p}{dt} = -2C\sin 2t + 2D\cos 2t$$

$$\frac{d^2x_p}{dt^2} = -4C\cos 2t - 4D\sin 2t$$

$$\frac{d^2x_p}{dt^2} + 6\frac{dx_p}{dt} + 13x_p = -4C\cos 2t - 4D\sin 2t$$
$$-12C\sin 2t + 12D\cos 2t$$
$$+13C\cos 2t + 13D\sin 2t$$
$$= (9C + 12D)\cos 2t + (9D - 12C)\sin 2t$$
$$= 75\cos 2t$$

$9C + 12D = 75 \quad 9D - 12 = 0$

$3C + 4D = 25 \qquad 3D = 4C$

$D = \frac{4C}{3}$

$3C + \frac{16C}{3} = 25$

$9C + 16C = 75$

$25C = 75$

$C = 3$

$D = 4$

$\therefore x_p = 3\cos 2t + 4\sin 2t$

$x = x_c + x_p$

$= e^{-3t}(A\cos 2t + B\sin 2t) + 3\cos 2t + 4\sin 2t$

$$\frac{dx}{dt} = -3e^{-3t}(A\cos 2t + B\sin 2t) + e^{-3t}(-2A\sin 2t + 2B\cos 2t)$$
$$-6\sin 2t + 8\cos 2t$$

$t = 0 \quad x = 5 : 5 = A + 3$

$t = 0 \quad \frac{dx}{dt} = 0 : 0 = -3A + 2B + 8$

$A = 2 \quad B = -1$

$\therefore x = e^{-3t}(2\cos 2t - \sin 2t) + 3\cos 2t + 4\sin 2t.$

If $3\cos 2t + 4\sin 2t = R\cos(2t - \phi)$

$$= R\cos 2t \cos\phi + R\sin 2t \sin\phi,$$

$R\cos\phi = 3 \quad R\sin\phi = 4$

$$R^2(\cos^2\phi + \sin^2\phi) = 3^2 + 4^2$$

$$R^2 = 25$$

$$R = 5$$

$$\frac{R\sin\phi}{R\cos\phi} = \frac{4}{3}$$

$$\tan\phi = \frac{4}{3}$$

$$\phi = \tan^{-1}\frac{4}{3}$$

$\therefore 3\cos 2t + 4\sin 2t = 5\cos(2t - \phi), \phi = \tan^{-1}\frac{4}{3}$.

$\therefore x = e^{-3t}(A\cos 2t + B\sin 2t) + 5\cos(2t - \phi), \phi = \tan^{-1}\frac{4}{3}$.

As $t \to \infty$, since $e^{-3t} \to 0$,

$$e^{-3t}(A\cos 2t + B\sin 2t) \to 0$$

$\therefore x \approx 5\cos(2t - \phi), \phi = \tan^{-1}\frac{4}{3}$