1. $\sin y = x$

$\cos y = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y} = \sec y$

$\frac{d^2y}{dx^2} = \sec y \tan y \frac{dy}{dx}$

$= \frac{1}{\cos^2 y} \sin y \frac{dy}{dx}$

$\cos^2 y \frac{d^2y}{dx^2} = \sin y \frac{dy}{dx}$

$(1 - x^2) \frac{d^2y}{dx^2} = x \frac{dy}{dx}$

2. $\frac{2}{x(x+1)(x+2)} = \frac{1}{x} - \frac{2}{x+1} + \frac{1}{x+2}$

$\frac{1}{2} \sum_{n=1}^{N} \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$

$= \frac{1}{2} \left\{ \left(\frac{1}{1} - \frac{1}{2} \right) + \dots + \left(\frac{-1}{N+1} + \frac{1}{N+2} \right) \right\}$

$= \frac{1}{4} - \frac{1}{2(N+1)(N+2)}$

when $N \to \infty$, $\frac{1}{N+1} \to 0$, $\frac{1}{N+2} \to 0$

$S_\infty = \frac{1}{4}$

3. Proposition : $2^n > 2n$, $n \in \mathbb{Z}^+$ where $n > 2$

Basis case : when $n = 3$, $2^3 > 6$ True for $n = 3$.

Induction step: Assume true for $n = k$.

$\therefore 2^k > 2k$

when $n = k+1$, $2^{k+1} > 4k$ & $4k > 2k+2$ for $k \geq 3$

$\therefore 2^{k+1} > 2k+2$

$2^{k+1} > 2(k+1)$

$\therefore$ True for $n = k+1$

Completion: $2^n > 2n$, $n \in \mathbb{Z}^+$ where $n > 2$

4 $$I_n = \int_0^{\frac{\pi}{3}} \sec^n x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^{n-2} x \sec^2 x \, dx$$

$u = \sec^{n-2} x$ $\qquad\qquad$ $\frac{dv}{dx} = \sec^2 x$

$\frac{du}{dx} = (n-2)\sec^{n-2} x \sec x \tan x$ $\qquad\qquad$ $v = \tan x$

$$= \left[\sec^{n-2} x \tan x\right]_0^{\frac{\pi}{3}} - (n-2)\int_0^{\frac{\pi}{3}} \sec^{n-2} x \tan^2 x \, dx$$

$$= \left[\sec^{n-2} x \tan x\right]_0^{\frac{\pi}{3}} - (n-2)\int_0^{\frac{\pi}{3}} \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sqrt{3}\left(\frac{1}{\frac{1}{2}}\right)^{n-2} - (n-2)\left[I_n - I_{n-2}\right]$$

$$(n-1) I_n = 2^{n-2}\sqrt{3} + (n-2) I_{n-2}$$

$$I_n = \frac{2^{n-2}\sqrt{3}}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$I_7 = \frac{2^5\sqrt{3}}{6} + \frac{5}{6}\left[\frac{2^3\sqrt{3}}{4} + \frac{3}{4}\left(\frac{2\sqrt{3}}{2} + \frac{1}{2} I_1\right)\right]$$

$$= \frac{16\sqrt{3}}{3} + \frac{5\sqrt{3}}{3} + \frac{5\sqrt{3}}{8} + \frac{5}{16}\ln(2+\sqrt{3})$$

$$= \frac{61\sqrt{3}}{8} + \frac{5}{16}\ln(2+\sqrt{3})$$

where $$I_1 = \int_0^{\frac{\pi}{3}} \sec x \, dx$$

$$= \left[\ln|\sec x + \tan x|\right]_0^{\frac{\pi}{3}}$$

$$= \ln(2+\sqrt{3})$$

5. i) $\begin{pmatrix} 3+2\lambda \\ 5+6\lambda \\ 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 7$

$6 + 4\lambda - 15 - 18\lambda + 1 - \lambda = 7$

$\lambda = -1$

Point $(1, -1, 2)$

ii) Distance $= \dfrac{7}{\sqrt{4+9+1}} = \dfrac{7}{\sqrt{14}} = \dfrac{\sqrt{14}}{2}$

iii) Line M : $\underset{\sim}{r} = \begin{pmatrix} 4 \\ 7 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 8 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 8 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

6. $A = \begin{pmatrix} 3 & 0 & -2 & 3 \\ -4 & 2 & 0 & -8 \\ 1 & -2 & 2 & 5 \\ 3 & -3 & 2 & 9 \end{pmatrix}$

i) $\begin{pmatrix} 3 & 0 & -2 & 3 \\ 0 & 6 & -8 & -12 \\ 0 & -6 & 8 & 12 \\ 0 & -3 & 4 & 6 \end{pmatrix}$ $\begin{array}{l} R_2 \to 3R_2 + 4R_1 \\ R_3 \to 3R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}$

$= \begin{pmatrix} 3 & 0 & -2 & 3 \\ 0 & 6 & -8 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{array}{l} R_3 \to R_3 + R_2 \\ R_4 \to 2R_4 + R_2 \end{array}$

Rank of $A = 2$

ii) $\left(\begin{array}{cccc|c} 3 & 0 & -2 & 3 & 0 \\ 0 & 6 & -8 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Let $x_4 = \lambda$

$x_3 = \beta$

$\therefore x_2 = \dfrac{4\beta + 6\lambda}{3}$

$\therefore x_1 = \dfrac{2\beta - 3\lambda}{3}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \frac{\beta}{3} \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix}$$

Basis $= \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} \right\}$

iii) $\underset{\sim}{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

7. $\sum \alpha = 0 \quad \sum \alpha\beta = p \quad \alpha\beta\gamma = -q$

i) $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

$= -2p$

$\sum \alpha^4 + p\sum \alpha^2 + q\sum \alpha = 0$

$\sum \alpha^4 = 2p^2$

ii) $\sum \alpha^5 + p\sum \alpha^3 + q\sum \alpha^2 = 0$

$\sum \alpha^5 = 5pq$

$\sum \alpha^7 + p\sum \alpha^5 + q\sum \alpha^4 = 0$

$\sum \alpha^7 = -7p^2q$

$2\sum \alpha^7 = -14p^2q$

$7\alpha\beta\gamma(\sum \alpha^4) = 7(-q)(2p^2) = -14p^2q$

8. $x = t - \sin t \qquad y = 1 - \cos t$

$$\frac{dx}{dt} = 1 - \cos t \qquad \frac{dy}{dt} = \sin t \qquad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2(1 - \cos t)$$

i) Length $= \displaystyle\int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \sqrt{2} \int_0^{\pi} \sqrt{1 - \cos t}\, dt$$

$$= \sqrt{2} \int_0^{\pi} \sqrt{2} \sin \frac{t}{2}\, dt$$

$$= 2 \left[\frac{-\cos \frac{t}{2}}{\frac{1}{2}} \right]_0^{\pi}$$

$$= 4$$

ii) surface area $= \displaystyle 2\pi \int_0^{\pi} y \sqrt{2(1 - \cos t)}\, dt$

$$= 2\sqrt{2}\pi \int_0^{\pi} \left[2 \sin^2 \frac{t}{2} \right]^{\frac{3}{2}} dt$$

$$= 2\pi \int_0^{\pi} \sin^3 \frac{t}{2}\, dt$$

$$= 2\pi \int_0^{\pi} 3 \sin \frac{t}{2} - \sin \frac{3t}{2}\, dt$$

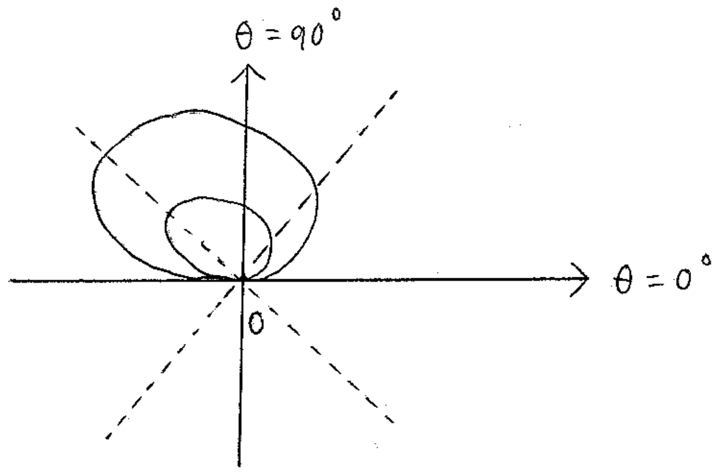
$$= 2\pi \left[-6\cos \frac{t}{2} + \frac{2}{3} \cos \frac{3t}{2} \right]_0^{\pi}$$

$$= 2\pi \left[6 - \frac{2}{3} \right]$$

$$= \frac{32\pi}{3}$$

9. $r = \theta \sin\theta \qquad 0 \leq \theta \leqslant 2\pi$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	0·56	1·57	1·67	0	−2·78	−4·71	−3·89	0



$$\text{Area} = \frac{1}{2}\int \theta^2 \sin^2\theta \; d\theta$$

$$= \frac{1}{2}\int \frac{\theta^2(1-\cos 2\theta)}{2} \; d\theta$$

$$= \frac{1}{4}\left[\frac{\theta^3}{3} - \frac{\theta^2 \sin 2\theta}{2} + \int \theta \sin 2\theta \; d\theta\right]$$

$$= \frac{1}{4}\left[\frac{\theta^3}{3} - \frac{\theta^2 \sin 2\theta}{2} - \frac{\theta\cos 2\theta}{2} + \frac{1}{4}\sin 2\theta\right]$$

$$\text{Smaller loop} = \frac{1}{4}\left[\frac{\theta^3}{3} - \frac{\theta^2 \sin 2\theta}{2} - \frac{\theta\cos 2\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{\pi}$$

$$= \frac{1}{4}\left(\frac{\pi^3}{3} - \frac{\pi}{2}\right)$$

$$\text{Larger loop} = \frac{1}{4}\left[\frac{\theta^3}{3} - \frac{\theta^2 \sin 2\theta}{2} - \frac{\theta\cos 2\theta}{2} + \frac{1}{4}\sin 2\theta\right]_{\pi}^{2\pi}$$

$$= \frac{1}{4}\left(\frac{8\pi^3}{3} - \pi - \frac{\pi^3}{3} + \frac{\pi}{2}\right)$$

$$= \frac{1}{4}\left(\frac{7\pi^3}{3} - \frac{\pi}{2}\right)$$

$$\text{Area} = \frac{1}{4}\left(\frac{7\pi^3}{3} - \frac{\pi}{2}\right) - \frac{1}{4}\left(\frac{\pi^3}{3} - \frac{\pi}{2}\right) = \frac{\pi^3}{2}$$

10. Let $z = \cos\theta + i\sin\theta$

$$z^n = \cos n\theta + i\sin n\theta$$

$$z^5 = (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$$

$$- 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

$$\cos 5\theta = \mathrm{Re}(z^5)$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^3\theta$$

$$+ 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$$

$$= \frac{16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta} \div \frac{\cos^5\theta}{\cos^5\theta}$$

$$= \frac{16\tan\theta - \frac{20\tan^3\theta}{\cos^2\theta} + \frac{5\tan\theta}{\cos^4\theta}}{16 - \frac{20}{\cos^2\theta} + \frac{5}{\cos^4\theta}} \qquad \sec^2\theta = 1 + \tan^2\theta$$

$$= \frac{16\tan^5\theta - 20\tan^3\theta(1 + \tan^2\theta) + 5\tan\theta(1 + \tan^2\theta)^2}{16 - 20(1 + \tan^2\theta) + 5(1 + \tan^2\theta)^2}$$

$$= \frac{16\tan^5\theta - 20\tan^3\theta - 20\tan^5\theta + 5\tan\theta(1 + 2\tan^2\theta + \tan^4\theta)}{16 - 20 - 20\tan^2\theta + 5\tan^4\theta + 10\tan^2\theta + 5}$$

$$= \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 + 5\tan^4\theta - 10\tan^2\theta} \quad \text{where } t = \tan\theta$$

$$T_n = \cos^n\theta + \sin^n\theta$$

$$16T_5 - 20T_3 + 5T_1 = 16[\cos^5\theta + \sin^5\theta] - 20[\cos^3\theta + \sin^3\theta]$$

$$+ 5[\cos\theta + \sin\theta]$$

$$= \cos 5\left(\frac{\pi}{15}\right) + \sin 5\left(\frac{\pi}{15}\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}$$

11. EITHER

i) $|A - \lambda I| = 0$

$$\downarrow$$

$$12\lambda^3 - 4\lambda^2 - 3\lambda + 1 = 0$$

$$(2\lambda - 1)(2\lambda + 1)(3\lambda - 1) = 0$$

$$\lambda = \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}$$

For $\lambda = \frac{1}{2}$, $\left(\begin{array}{ccc|c} -2 & 4 & \frac{11}{3} & 0 \\ 1 & -1 & -2 & 0 \\ -1 & 2 & \frac{11}{6} & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -2 & 4 & \frac{11}{3} & 0 \\ 0 & 2 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Let $x_3 = \beta$, $x_2 = \dfrac{\beta}{6}$, $x_1 = \dfrac{13}{6}\beta$

$$\therefore \underset{\sim}{e}_1 = \begin{pmatrix} 13 \\ 1 \\ 6 \end{pmatrix}$$

$\lambda = -\frac{1}{2}$, $\underset{\sim}{e}_2 = \begin{pmatrix} 24 \\ -5 \\ 12 \end{pmatrix}$ $\quad \lambda = \frac{1}{3}$, $\underset{\sim}{e}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

ii) $M^n = QD^nQ^{-1}$

$$= Q \begin{pmatrix} (\frac{1}{2})^n & 0 & 0 \\ 0 & (-\frac{1}{2})^n & 0 \\ 0 & 0 & (\frac{1}{3})^n \end{pmatrix} Q^{-1}$$

As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$

$\left(\frac{-1}{2}\right)^n \to 0$

$\left(\frac{1}{3}\right)^n \to 0$

$$\approx Q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}$$

$\therefore$ All elements of $M^n \to 0$

iii) Given $M\underset{\sim}{e} = \lambda \underset{\sim}{e}$

$$
\begin{aligned}
(kI + qM)\underset{\sim}{e} &= kI\underset{\sim}{e} + qM\underset{\sim}{e} \\
&= k\underset{\sim}{e} + q\lambda\underset{\sim}{e} \\
&= (k + q\lambda)\underset{\sim}{e}
\end{aligned}
$$

N has eigenvector $\underset{\sim}{e}$ with corresponding eigenvalue $(k + q\lambda)$.

$\underset{\sim}{e}_1 = \begin{pmatrix} 13 \\ 1 \\ 6 \end{pmatrix}$ with eigenvalue $k + \frac{q}{2}$

$\underset{\sim}{e}_2 = \begin{pmatrix} 24 \\ -5 \\ 12 \end{pmatrix}$ with eigenvalue $k - \frac{q}{2}$

$\underset{\sim}{e}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ with eigenvalue $k + \frac{q}{3}$

11 OR

$$y = x^{-\frac{1}{2}} z$$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} \frac{dz}{dx} + \left(-\frac{1}{2} x^{-\frac{3}{2}}\right) z$$

$$\frac{d^2y}{dx^2} = \frac{3}{4} x^{-\frac{5}{2}} z - \frac{1}{2} x^{-\frac{3}{2}} \frac{dz}{dx} - \frac{1}{2} x^{-\frac{3}{2}} \frac{dz}{dx} + x^{-\frac{1}{2}} \frac{d^2z}{dx^2}$$

Substitute into D.E.

$$x^2 \left[\frac{3}{4} x^{-\frac{5}{2}} z - x^{-\frac{3}{2}} \frac{dz}{dx} + x^{-\frac{1}{2}} \frac{d^2z}{dx^2} \right]$$

$$+ (2x^2 + x)\left[x^{-\frac{1}{2}} \frac{dz}{dx} + z\left(-\frac{1}{2} x^{-\frac{3}{2}}\right) \right]$$

$$+ (2x^2 + x - 0.25) x^{-\frac{1}{2}} z = x^{\frac{5}{2}}$$

$$\downarrow$$

$$\frac{d^2z}{dx^2} + \frac{2dz}{dx} + 2z = x$$

where $a = 2$, $b = 2$ & $c = 1$

AE: $m^2 + 2m + 2 = 0$

$m = -1 \pm i$

CF: $z = e^{-x}(A\cos x + B\sin x)$

Final solution : Let $z = \lambda x + \beta$

$z' = \lambda \quad z'' = 0$

Substitute into D.E.

$0 + 2\lambda + 2\lambda x + \beta = x$

$\lambda = \frac{1}{2}, \ \beta = -\frac{1}{2}$

PI: $z = \frac{x}{2} - \frac{1}{2}$

General solution: $z = e^{-x}(A\cos x + B\sin x) + \frac{x}{2} - \frac{1}{2}$

when $x = \pi, y = 0, z = 0$ $\quad \frac{dy}{dx} = 0 \Rightarrow \frac{dz}{dx} = 0$

when $x = \pi, z = 0 \Rightarrow 0 = e^{-\pi}(-A) + \frac{\pi}{2} - \frac{1}{2}$

$$A = \frac{(\pi - 1)e^{\pi}}{2}$$

$x = \pi, \frac{dz}{dx} = 0 \Rightarrow z' = e^{-x}(-A\sin x + B\cos x)$

$$+ e^{-x}(-1)(A\cos x + B\sin x) + \frac{1}{2}$$

$$0 = e^{-\pi}(-B) - e^{-\pi}(-A) + \frac{1}{2}$$

$$\therefore B = \frac{\pi e^{\pi}}{2}$$

GS: $y\sqrt{x} = e^{-x}\left[\frac{(\pi-1)e^{\pi}\cos x}{2} + \frac{\pi e^{\pi}\sin x}{2}\right] + \frac{x}{2} - \frac{1}{2}$

$$y = \frac{e^{-x+\pi}}{2\sqrt{x}}\left\{[(\pi-1)\cos x + \pi\sin x] + \frac{x}{2} - \frac{1}{2}\right\}$$