Differential Equations

A2 Trial Examination

August 2009

7 The rate, in cm^3s^{-1}, at which oil is leaking from an engine sump at any time t seconds is proportional to the volume of oil, V cm^3, in the sump at that instant.

At time $t = 0$, $V = A$.

(i) By performing and integrating a differential equation, show that

$$V = Ae^{-kt},$$ [4]

Where k is a positive constant.

(ii) Sketch a graph to show the relation between V and t. [2]

Given further that $V = \frac{1}{2}A$ at $t = T$,

(iii) Show that $kT = \ln 2$. [2]

[745]

March 2009

7 The organizer of a sale, which lasted for 3 hours and raised a total of £ 1000, attempted to create a model to represent the relationship between s and t and £s is the amount which has been raised at time t hours after the start of the sale. In the model, s and t were taken to be continuous variables .The organizer assumed that the rate of raising money varied directly as the time remaining and inversely as the amount already raised .Show that for this model,

$$\frac{ds}{dt} = k\left(\frac{3-t}{s}\right) \quad \text{where } k \text{ is a constant}$$

[2]

Solve the differential equation and show that it can be written in the form

$$\frac{s^2}{1000^2} + \frac{(3-t)^2}{3^2} = 1.$$

[6]

Hence, find the amount raised during the first hour of the sale. [1]

[745]

August 2008

8 The surface area of a pond is 10000 m^2 and a part of it is covered by algae. At any instant, the area of algae is increasing at a rate which is directly proportional to its area at that instant.

(i) Obtain a differential equation that relates the area of algae x m^2, with time t (in days). [2]

Given that when $t = 0$, $x = 100$ and when $t = 7$, $x = 1000$,

(ii) Show that $\ln\left(\dfrac{x}{100}\right) = \dfrac{t}{7}\ln(10)$ [5]

By giving your answer correct to 3 significant figures,

(iii) Find the area of the pond not covered by algae when $t = 10.5$ days. [2]

(iv) Find value of t when the algae covers half the surface of the pond. [2]

$$\left[\frac{dx}{dt} = kx; 6837.3m^2; 12 \text{ days}\right]$$

March 2008

11. In an irrigation system, water is stored in a rectangular tank with a square horizontal base of edge 200 cm and with a vertical height of 400 cm. A tap in the base is opened and water flows out. After t seconds, when the depth of water in the tank is x cm, the rate of flow is $100\sqrt{x}\ cm^3s^{-1}$.

i) Show that $400\dfrac{dx}{dt} + \sqrt{x} = 0$. [3]

The tank is initially full.

ii) Find an expression for t in terms of x. [4]

iii) When the tap is opened, find to the nearest half hour how long the tank takes to empty. [2]

iv) Find the depth of water in the tank after the tap has been open for 2 hours. [2]

[$t = 16000 - 800\sqrt{x}$; 4.5 hours; 121cm]

August 2007

8. (i) Express $\dfrac{1}{y^2 - y}$ in partial fractions. [3]

(ii) Given the differential equation $\dfrac{1}{2}\left(\dfrac{dy}{dx}\right) + xy = xy^2$ and that when $x = 0, y = 2$, find y in terms of x. [5]

$$\left[\frac{1}{y-1} - \frac{1}{y}; y = \frac{2}{2 - e^{x^2}}\right]$$

March 2007

10. In places where very heavy rainfall occurs, 'storm drains' are used to collect rainwater and prevent flooding. A storm drain can be modelled as a rectangular tank with vertical sides, so that the volume of water in the tank is proportional to the depth of the water. During a storm, rainwater flows into the tank at a constant rate. Water drains away from the bottom of the tank at a rate which is proportional to the square root of the depth of water in the tank. At time t hours the depth of water in the tank is h metres. Explain how the information given above leads to the differential equation

$$\frac{dh}{dt} = A - B\sqrt{h} \quad ,$$

where A and B are positive constants. (3)

When rainfall stops and no more water is flowing into the tank, the value of A is zero. For this case, find the value of B, given that it takes half an hour for the depth of water in the tank to fall from 1 metre to zero. (4)

In another case, $A = 2$ and $B = 1$, so that the differential equation is $\frac{dh}{dt} = 2 - \sqrt{h}$.

(i) By means of the substitution $x = 2 - \sqrt{h}$, or otherwise, find

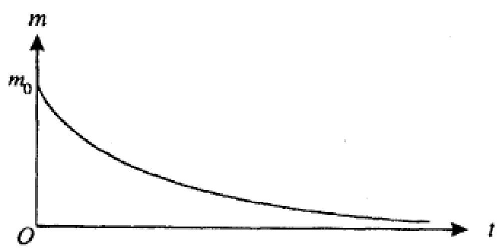
$$\int \frac{1}{2-\sqrt{h}}\, dh .$$ (5)

(ii) Given that the tank is initially empty, find the time taken for the depth of water in the tank to reach 1 metre. (3)

[B = 4; $2\left(2-\sqrt{h}-2\ln(2-\sqrt{h})\right)+c$; 0.773 hour]

August 2006

5. During a chemical reaction, the mass m of one of the chemicals involved decreases at a rate which is proportional to m. Express this information as a differential equation involving m and the time t.
When $t = 0$ the mass of the chemical is m_0. Show by integration that the solution of the differential equation is $m = m_0e^{-kt}$, where k is a positive constant. [4]



A sketch of the graph of $m = m_0e^{-kt}$ is shown in the diagram. The mass M of another chemical involved in the reaction varies in such a way that $m + M = A$, where A is a constant.

(i) Describe what happens to the value of M as t becomes large. [1]

(ii) Show that $\frac{dM}{dt} = k(A - M)$. [2]

$$\left[\frac{dm}{dt} = -km; \text{ when } t \to \infty, m \to A\right]$$

March 2006

6. The rate of increase in sales S (in thousands of units) of a product is proportional to the current level of sales and inversely proportional to the square of the time t. This is described by the differential equation $\frac{dS}{dt} = \frac{kS}{t^2}$, where t is the time in years. The saturation point for the market is 50,000 units. That is, the limit of S as $t \to \infty$ is 50. After 1 year, 10,000 units have been sold. Show that k = ln 5. [6]

August 2005

7. The area of a circle of radius r metres is $A\ m^2$

(i) Find $\frac{dr}{dA}$ in terms of r . [1]

(ii) The area increases with time t seconds in such a way that

$\frac{dA}{dt} = \frac{2}{(t+1)^3}$. Find an expression, in terms of r and t, for $\frac{dr}{dt}$. [2]

(iii) Solve the differential equation $\frac{dA}{dt} = \frac{2}{(t+1)^3}$ to obtain A in terms of t,

given that $A = 0$ when $t = 0$. [3]

(iv) Show that, when t = 1, $\frac{dr}{dt} = 0.081$, correct to 2 significant figures. [3]

$$\left[\frac{1}{2\pi r}; \frac{1}{\pi r(t+1)^3}; A = 1 - \frac{1}{(t+1)^2}\right]$$

March 2005

7 The rate of change per minute in the temperature (T °C) of an object is proportional to the difference between T and T_m where T_m is the room temperature.

(i) Show that $\frac{dT}{dt} = k(T - T_m)$ where k is a constant called the Cooling Coefficient. [1]

Copper has a Cooling Coefficient of -0.19 and a copper rod which was out in the sun cools from $65\,^0$C to room temperature of $15\,^\circ$C.

(ii) Solve the differential equation, expressing T in terms of t. [3]

(iii) At what rate is the rod's temperature decreasing when ,

(a) its temperature is $40\,^\circ$C, [1]

(b) it has cooled for 2 minutes . [2]

(iv) How long would it take for the temperature to reach $20\,^\circ$C. ? [2]

$\left[T = 50e^{-0.19t} + 15;\ 4.75^o C \text{ per minute};\ 6.50^o C \text{ per minute};\ 12.1 \text{minutes}\right]$