NUMERICAL METHODS

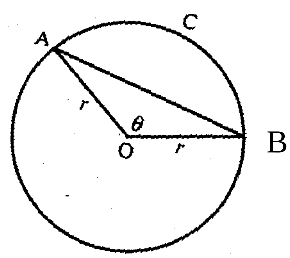
Introduction to numerical methods

- A significant part of pure mathematics is concerned with precision - with equations that have exact solutions.
- Most equations do not have solutions that can be expressed exactly and most functions cannot be integrated precisely.
  - Example 1. $\frac{1}{x} - \ln x = 2$ and

    2. $\int_1^3 e^{-x^2}\,dx$ cannot be solved exactly.
- In 'real world' practical problems, numerical solutions are required depending on the situation.
- Numerical methods can be used to solve problems to whatever degree of accuracy is required.

Example :

OACB is a sector of a circle, centre O, radius r, where angle AOB = θ. Find θ if the area of segment ACB equals the area of triangle OAB.



Area of segment ACB

$= \text{Area of sector OACB} - \text{Area of } \triangle\text{OAB}$

$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$= \frac{1}{2}r^2(\theta - \sin\theta)$

But Area of segment ACB = Area of $\triangle$OAB

$\therefore \quad \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{2}r^2\sin\theta$

$\theta - \sin\theta = \sin\theta$

$\theta - 2\sin\theta = 0$

Does this equation have a solution?

Can this equation be solved algebraically?

How can this equation be solved?

Three approaches will be used:

Graphical Method

– locate the root(s) by sketching.

Location of roots in an interval

– find an interval for the root(s) using the method sign change.
– narrow down the interval by using decimal search

Iteration

– find an approximated value for the root(s)

GRAPHICAL METHOD

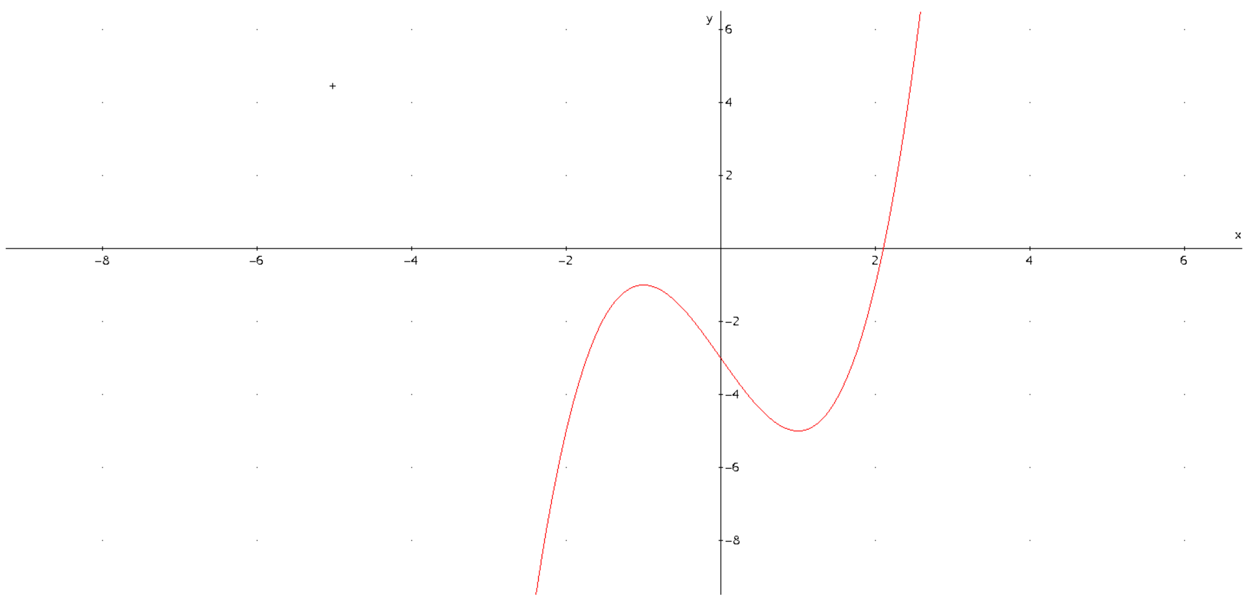
This section looks at solving equations $f(x) = 0$ and $f(x) = g(x)$ graphically.

A. Intersection points on the x-axis

The roots of the equation $f(x) = 0$ are the values of x where the curve $y = f(x)$ cuts the x-axis.

Example:

The equation $x^3 - 3x - 3 = 0$ has one root real root, α. Show, by sketching a graph, that the equation $x^3 - 3x - 3 = 0$, has only one positive root.

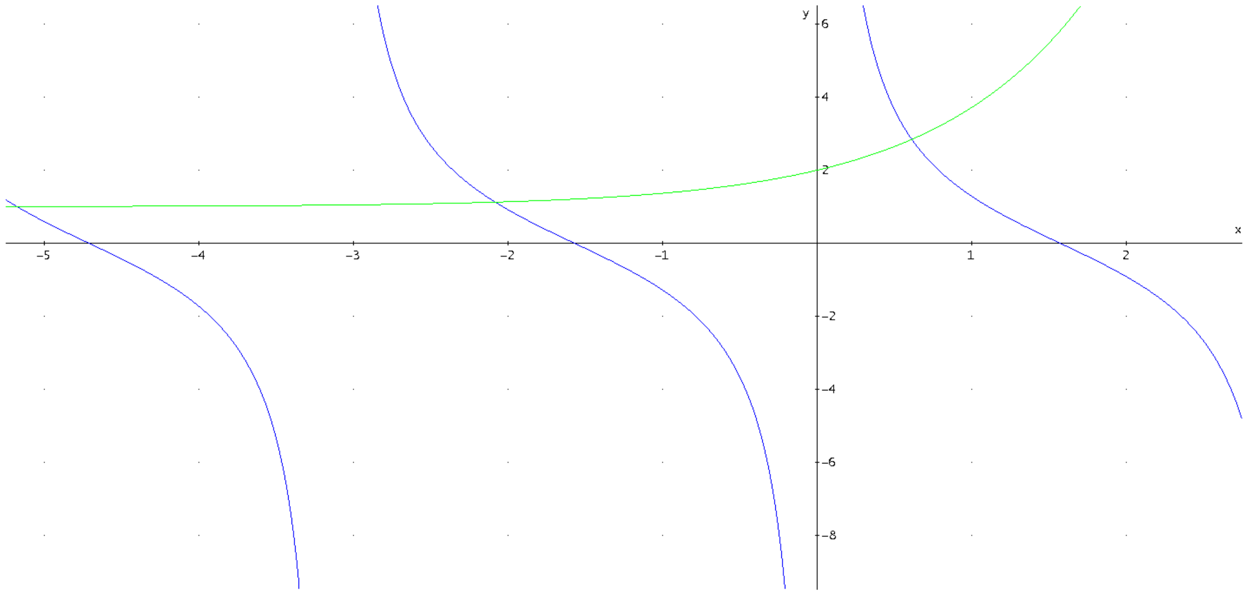


B. Intersection points of two curves

An equation $f(x) = g(x)$ can be solved by finding the x-coordinate(s) where the graphs of curve $y = f(x)$ and curve $y = g(x)$ intersect.

Example:

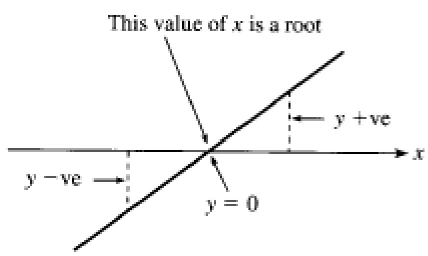
By sketching a suitable pair of graphs, show that the equation $2\cot x = 1 + e^x$, where x is in radians, has only one positive real root.

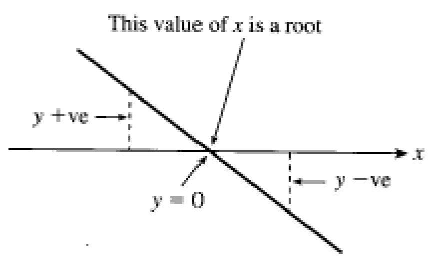


LOCATION OF ROOTS IN AN INTERVAL

Sign-Change Method

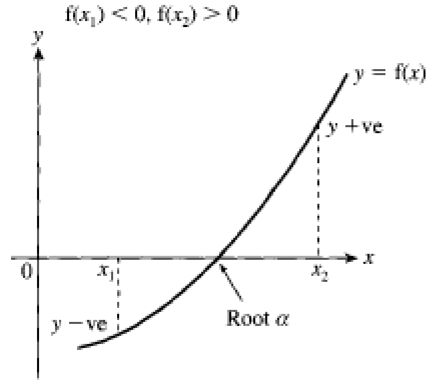
The diagrams show that, when the graph of a continuous function, $y = f(x)$, crosses the x-axis, there is a change of sign.

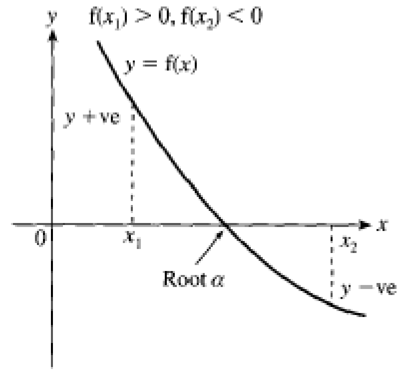




- The function will be negative on one side of the crossing point and positive on the other.
- The crossing point corresponds to root of the equation y = 0.

Example:





If two values x_1, x_2 can be found such that $x_1 < x_2$ and $f(x_1)$ and $f(x_2)$ have *different* signs, then $f(x) = 0$ has at least one root in (x_1, x_2), provided $f(x)$ is continuous in the interval (x_1, x_2).

These two diagrams show a change in sign for y in the interval (x_1, x_2), so y = 0 has at least one root, α, in this interval $(x_1 < \alpha < x_2)$

Example 1

The equation $x^3 - 3x - 3 = 0$ has one root real root, α. Show that α lies between 2 and 3.

Example 2

The equation $x^3 + x + 1 = 0$ has one real root. Show by calculation that this root lies between -1 and 0. [2]

Example 3

(i) By sketching a suitable pair of graphs, show that the equation

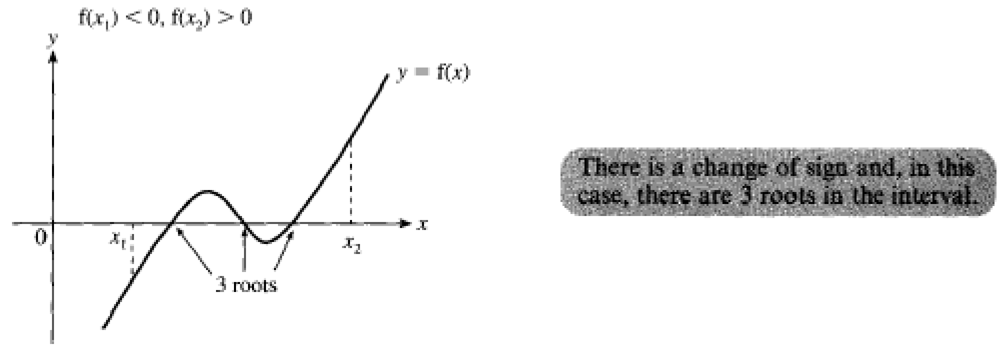
$$\operatorname{cosec} x = \tfrac{1}{2}x + 1,$$

where x is in radians, has a root in the interval $0 < x < \frac{1}{2}\pi$.

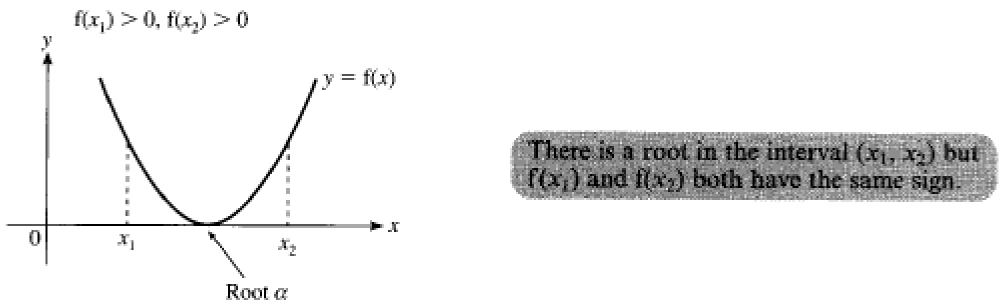
(ii) Verify, by calculation, that this root lies between 0.5 and 1.

Possible problems with the sign-change method

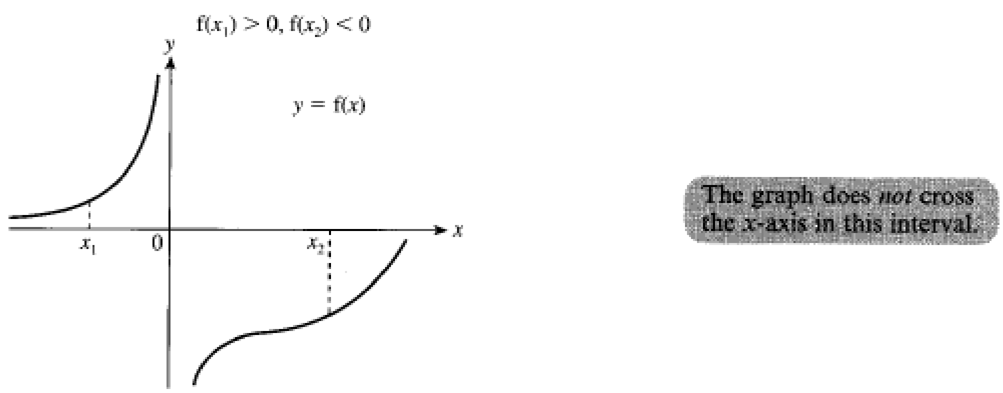
There could be *more than one root* between x_1 and x_2.



There could be a root *without* a change of sign.



There could be a change of sign but *no* root if $f(x)$ is discontinuous.



- When locating roots by looking for a change of sign, using graphical method as a guideline may help to avoid possible problems.

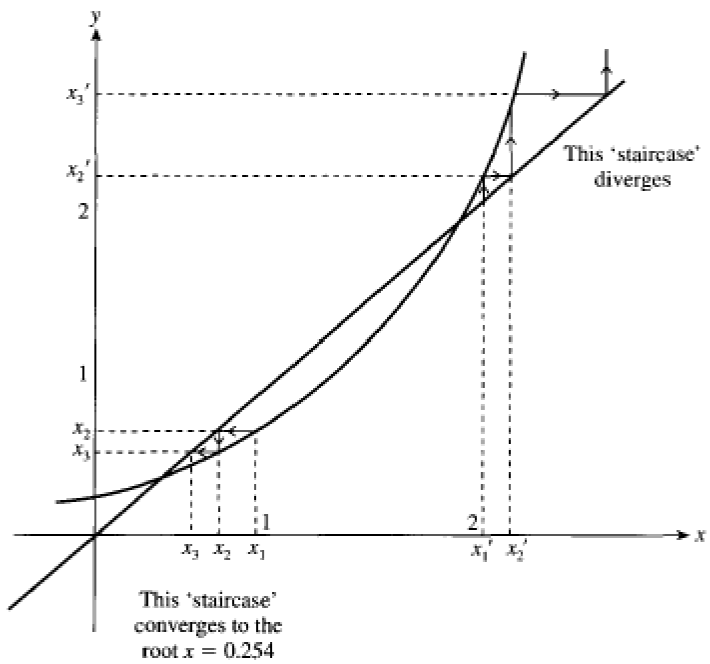
ITERATION

- A sequence of approximation $x_0, x_1, x_2, x_3, \ldots$ is found, each one closer to the root α of $f(x) = 0$.
- Each approximation is found from the one before it using the intersection method..
- The process is continued until the required accuracy is reached.

Iteration of the form $x_{n+1} = g(x_n)$

- Rewrite the equation $f(x) = 0$ in the form $x = g(x)$
- Rewrite $x = g(x)$ as an iterative formula $x_{n+1} = g(x_n)$
- to use the formula chose an initial value, x_0.

ILLUSTRATING ITERATION GRAPHICALLY



The intersection of the line y = x and of the curve y = g(x) will give the roots of the x = g(x).

Example 1: May/June 2002

The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right),$$

with initial value $x_1 = 1$, converges to α.

(i) **Use this formula to find α correct to 2 decimal places, showing the result of each iteration.** [3]

(ii) **State an equation satisfied by α, and hence find the exact value of α.** [2]

Example 2: Oct/Nov 2003

(i) By sketching suitable graphs, show that the equation

$$\sec x = 3 - x^2$$

has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

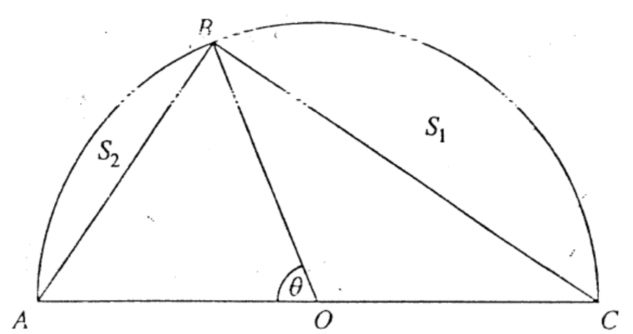
(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \cos^{-1}\left(\frac{1}{3 - x_n^2}\right)$$

converges, then it converges to a root of the equation given in part **(i)**. [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to determine the root in the interval $0 < x < \frac{1}{2}\pi$ correct to 2 decimal places, showing the result of each iteration. [3]

Example 3



The diagram shows a semicircle ABC on AC as diameter. The mid-point of AC is O, and angle $AOB = \theta$ radians, where $0 < \theta < \frac{1}{2}\pi$. The area of the segment S_1 bounded by the chord BC is twice the area of the segment S_2 bounded by the chord AB. Show that

$$3\theta = \pi + \sin\theta.$$ [3]

Use the iterative formula

$$\theta_{n+1} = \frac{1}{3}(\pi + \sin\theta_n),$$

together with a suitable starting value, to find θ correct to 3 significant figures. You should show the value of each approximation that you calculate. [3]

Iteration which go wrong

There is more than one way to arrange f(x) = 0 as x = g(x).

For example:

The solution for $x^3 - 3x - 5 = 0$ can be found using iteration:

The equation can be split into $x_{n+1} = g(x_n)$.

a) $x = \sqrt[3]{3x+5}$ or b) $x = \frac{1}{3}\left(x^3 - 5\right)$

Perform the iteration for the above formula with the initial value of $x_0 = 2$.

Example: Oct/Nov 05

4 The equation $x^3 - x - 3 = 0$ has one real root, α.

(i) Show that α lies between 1 and 2. [2]

Two iterative formulae derived from this equation are as follows:

$$x_{n+1} = x_n^3 - 3, \qquad (A)$$

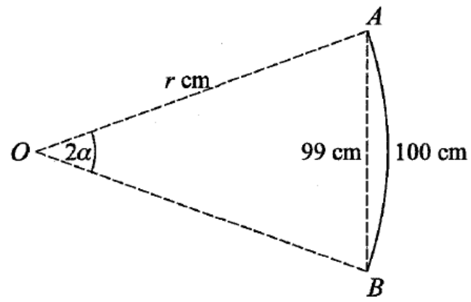
$$x_{n+1} = (x_n + 3)^{\frac{1}{3}}. \qquad (B)$$

Each formula is used with initial value $x_1 = 1.5$.

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [5]

Past Year Questions – Numerical Methods

Oct/Nov 2002



The diagram shows a curved rod AB of length 100 cm which forms an arc of a circle. The end points A and B of the rod are 99 cm apart. The circle has radius r cm and the arc AB subtends an angle of 2α radians at O, the centre of the circle.

(i) Show that α satisfies the equation $\frac{99}{100}x = \sin x$. [3]

(ii) Given that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$, verify by calculation that this root lies between 0.1 and 0.5. [2]

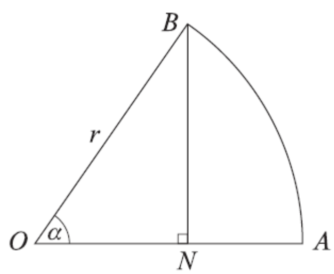
(iii) Show that if the sequence of values given by the iterative formula

$$x_{n+1} = 50\sin x_n - 48.5x_n$$

converges, then it converges to a root of the equation in part **(i)**. [2]

(iv) Use this iterative formula, with initial value $x_1 = 0.25$, to find α correct to 3 decimal places, showing the result of each iteration. [2]

Oct/Nov 2004



The diagram shows a sector OAB of a circle with centre O and radius r. The angle AOB is α radians, where $0 < \alpha < \frac{1}{2}\pi$. The point N on OA is such that BN is perpendicular to OA. The area of the triangle ONB is half the area of the sector OAB.

(i) Show that α satisfies the equation $\sin 2x = x$. [3]

(ii) By sketching a suitable pair of graphs, show that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(iii) Use the iterative formula

$$x_{n+1} = \sin(2x_n),$$

with initial value $x_1 = 1$, to find α correct to 2 decimal places, showing the result of each iteration. [3]

<u>May/June 05</u>

7 **(i)** By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec} x = \tfrac{1}{2}x + 1,$$

where x is in radians, has a root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify, by calculation, that this root lies between 0.5 and 1. [2]

(iii) Show that this root also satisfies the equation

$$x = \sin^{-1}\left(\frac{2}{x+2}\right). \quad [1]$$

(iv) Use the iterative formula

$$x_{n+1} = \sin^{-1}\left(\frac{2}{x_n+2}\right),$$

with initial value $x_1 = 0.75$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

<u>May/June 06</u>

6 **(i)** By sketching a suitable pair of graphs, show that the equation

$$2\cot x = 1 + e^x,$$

where x is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.5 and 1.0. [2]

(iii) Show that this root also satisfies the equation

$$x = \tan^{-1}\left(\frac{2}{1+e^x}\right). \quad [1]$$

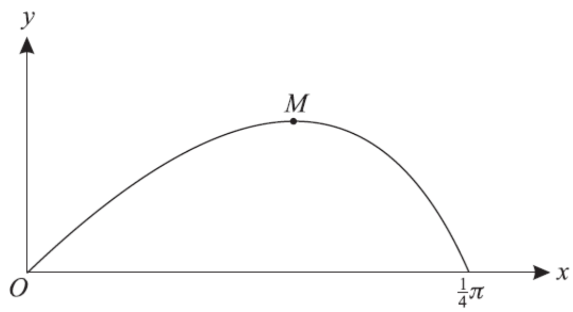
(iv) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{2}{1+e^{x_n}}\right),$$

with initial value $x_1 = 0.7$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Oct/Nov 2006

10



The diagram shows the curve $y = x\cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The point M is a maximum point.

(i) Show that the x-coordinate of M satisfies the equation $1 = 2x\tan 2x$. [3]

(ii) The equation in part **(i)** can be rearranged in the form $x = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x}\right)$. Use the iterative formula

$$x_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_n}\right),$$

with initial value $x_1 = 0.4$, to calculate the x-coordinate of M correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) Use integration by parts to find the exact area of the region enclosed between the curve and the x-axis from 0 to $\frac{1}{4}\pi$. [5]