Partial Fractions

recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than

$(ax+b)(cx+d)(ex+f)$,

$(ax+b)(cx+d)^2$,

$(ax+b)(x^2+c^2)$,

and where the degree of the numerator does not exceed that of the denominator;

Now we will consider the reverse process by which a rational function is expressed as a sum of two or simpler fractions called *partial fractions*.

Note:

Proper fractions ➢ fractions in which the degree of the numerator is less than the degree of the denominator.

Example:

Improper fractions ➢ fractions in which the degree of the numerator is more than or the same as the degree of the denominator.

Example:

Expressing Rational Functions In Partial Fractions:

1. For a linear factor $(x-a)$ in the denominator.

Example 1:

a) Express $\dfrac{2x-5}{(x+2)(x-1)}$ in partial fractions.

b) Express $\dfrac{x+1}{2x^3-5x^2+2x}$ in partial fractions.

2. For a quadratic factor (ax^2+bx+c) in the denominator.

Example 2

Express $\dfrac{4x^2}{(x-3)(x^2+3)}$ in partial fractions.

3. For a repeated factor $(x-a)^n$ and $(ax^2+bx+c)^n$ in the denominator.

Example 3

Express $\dfrac{x-5}{(x+1)^2(x-1)}$ in partial fractions.

Expressing Improper Fractions In Partial Fractions:

☑ Change the improper fraction to a proper fraction first ➔ long division.

☑ Write out the partial forms.

Example 4

a) Express $\dfrac{x^2+2}{(2x+1)(x-1)}$ in partial fractions.

b) Express $\dfrac{x^3-3x^2+1}{x^2-x-2}$ in partial fractions.

Exercise 1 – Binomial Expansion

1 Express $\dfrac{4}{(x-3)(x+1)}$ in partial fractions.

2 Express $\dfrac{2}{x(x-1)(x+1)}$ in partial fractions.

3 Express $\dfrac{2x^2+1}{x(x-1)^2}$ in partial fractions.

4 Express $\dfrac{x^2-11}{(x+2)^2(3x-1)}$ in partial fractions.

1 $\dfrac{1}{x-3}-\dfrac{1}{x+1}$

2 $-\dfrac{2}{x}+\dfrac{1}{x-1}+\dfrac{1}{x+1}$

3 $\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{3}{(x-1)^2}$

4 $\dfrac{1}{x+2}+\dfrac{1}{(x+2)^2}-\dfrac{2}{3x-1}$

Question 5

Express $\dfrac{1}{x^2(x-1)}$ in the form $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-1}$, where A, B and C are constants.

$$-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x-1}$$

Question 6

Express in partial fractions

(a) $\dfrac{x^2-x+3}{x(2x^2+3)}$, (b) $\dfrac{7x^2-2x+5}{(x-1)(3x^2+2)}$

Question 7

Find the values of A, B and C for which $\dfrac{x^2-2}{(x-2)^2} \equiv A+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$.

$$A=1, B=4, C=2$$

(a) $\dfrac{1}{x}-\dfrac{x+1}{2x^2+3}$ (b) $\dfrac{2}{x-1}+\dfrac{x-1}{3x^2+2}$

Binomial Expansion and Partial Fractions

Example 5

Let $f(x) = \dfrac{x^2+7x-6}{(x-1)(x-2)(x+1)}$.

(i) Express $f(x)$ in partial fractions. [4]

(ii) Show that, when x is sufficiently small for x^4 and higher powers to be neglected,

$$f(x) = -3 + 2x - \tfrac{3}{2}x^2 + \tfrac{11}{4}x^3. \quad [5]$$

Example 6

Expand $\dfrac{6x^3+21x+7}{(2x+1)(x^2+4)}$ in ascending powers of x up to and including the term in x^2.

$$\frac{7}{4}+\frac{7}{4}x-\frac{63}{16}x^2$$

Example 7

Let $f(x) = \dfrac{9x^2+4}{(2x+1)(x-2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that, when x is sufficiently small for x^3 and higher powers to be neglected,

$$f(x) = 1 - x + 5x^2. \qquad [4]$$

Example 8

Given that $f(x) = \dfrac{x}{(x-1)(x+2)}$

(i) Express f(x) in partial fractions. [4]

(ii) Show that the coefficient of x^3 in the expansion of f(x) in ascending powers of x is $-\dfrac{3}{8}$. [5]

(iii) State the set of values of x for which this expansion is valid. [2]

<u>Exercise 2 – Binomial Expansion</u>

Question 1

It is given that $g(x) = (2x-1)(x+2)(x-3)$.

(a) Express $g(x)$ in the form $Ax^3 + Bx^2 + Cx + D$, giving the values of the constants A, B, C and D.

(b) Find the value of the constant a, given that $x+3$ is a factor of $g(x)+ax$.

(c) Express $\dfrac{x-3}{g(x)}$ in partial fractions. (OCR)

(a) $2x^3 - 3x^2 - 11x + 6$ (b) -14

(c) $\dfrac{2}{5(2x-1)} - \dfrac{1}{5(x+2)}$

Question 2

Let $y = \dfrac{4+7x}{(2-x)(1+x)^2}$. Express y in the form $\dfrac{A}{2-x} + \dfrac{B}{1+x} + \dfrac{C}{(1+x)^2}$, where the numerical values of A, B and C are to be found. Hence, or otherwise, expand y in a series of ascending powers of x up to and including the term in x^3, simplifying the coefficients. Use your result to find the value of $\dfrac{dy}{dx}$ when $x = 0$.

$A=2, B=2, C=-1;\ 2+\frac{1}{2}x-\frac{3}{4}x^2+\frac{17}{8}x^3;\ \frac{1}{2}$