TOPIC 1: Algebra
Factor and Remainder Theorem

Objectives:

- divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero);
- use the factor theorem and the remainder theorem, e.g. to find factors, solve polynomial equations or evaluate unknown coefficients;

<u>Definition of a polynomial</u>

A **polynomial function** of degree n is of the form:

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ where $a_0 \neq 0$ and n is a positive integer, called the **degree** of the polynomial.

$f(x) = x^3 - 2x^2 - 5x + 6$ is a polynomial function of degree 3.
$f(x) = x^4 - x^3 - 19x^2 - 11x + 31$ is a polynomial function of degree 4.
$f(x) = 7x^5 + 4x^3 - 2x^2 - 8x + 1$ is a polynomial function of degree 5

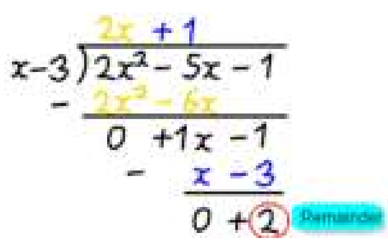
<u>Division of Polynomials</u>

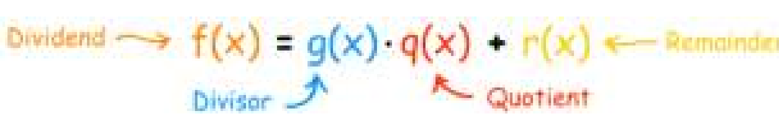
Consider what happens when we divide numbers. Example: $\frac{37}{4}$

Division of polynomials (long division) is something like our number example.

Example 1: Divide a) $f(x) = 2x^2 - 5x - 1$ by $(x - 3)$.

Using **long division**





Example 2

Divide

b) $f(x) = x^5 + 4x^4 - 6x^2 + 2$ by $(x + 2)$

c) $f(x) = 6x^4 + 4x^3 - 6x^2 + 2$ by $(2x^2 - 1)$

➔ ***What is Synthetic Division???***

What is the setback of Synthetic Division???

Divide $x^2 + 5x + 6$ by $x - 1$

$$
\begin{array}{r}
x + 6 \\
x-1\overline{\big)\, x^2 + 5x + 6} \\
\underline{x^2 - x} \\
6x + 6 \\
\underline{6x - 6} \\
12
\end{array}
\qquad\qquad
\begin{array}{r|rrr}
1 & 1 & 5 & 6 \\
 & & 1 & 6 \\
\hline
 & 1 & 6 & 12
\end{array}
$$

<u>Exercise 1</u>

1. Find values of p, q, r, s such that

$$(2x+3)(px^3+qx^2+rx+s) = 2x^4+13x^3+7x^2+18$$

for all values of x.

2. Find the quotient and the remainder when x^3+2x^2+x-3 is divided by $(x-4)$.

Question 3

Find the quotient and the remainder when

(a) $x^4-2x^3-7x^2+7x+5$ is divided by x^2+2x-1,

(b) x^4-x^3+7x+2 is divided by x^2+x-1,

(c) $2x^4-4x^3+3x^2+6x+5$ is divided by x^3+x^2+1,

(d) $6x^4+x^3+13x+10$ is divided by $2x^2-x+4$.

3 (a) $x^2-4x+2, -x+7$

(b) $x^2-2x+3, 2x+5$

(c) $2x-6, 9x^2+4x+11$

(d) $3x^2+2x-5, 30$

Factor and Remainder Theorem

Factor Theorem
$(x - a)$ is a factor of a polynomial $f(x)$ if $f(a) = 0$.

Remainder Theorem
The remainder when a polynomial $f(x)$ is divided by $(x - a)$ is $f(a)$.

Extended version of the factor theorem:
$(ax + b)$ is a factor of a polynomial $f(x)$ if $f\left(\frac{-b}{a}\right) = 0$.

Example 3
Find the remainder when $2x^3 - 5x^2 + 2x + 7$ is divided by $(x + 2)$.

Example 4
When $2x^3 - x^2 - 13x + k$ is divided by $x - 2$ the remainder is -20. Show that k = -6.

Example 5
When $2x^3 - x^2 + ax + b$ is divided by $x - 2$ the remainder is 25. When divided by $x + 1$ the remainder is -5. Find the values of a and b.

Example 6
Show that $(x - 2)$ is a factor of $P(x)$, where $P(x) = x^3 - 3x^2 - 10x + 24$, and find the other two factors.

Example 7
Given $g(x) = x^3 + 3x^2 - 13x - 15$. Show that $g(-5) = 0$ and $g(3) = 0$.Hence factorise $g(x)$.

Example 8
a) Show that $(x - 3)$ is a factor of $x^3 + x^2 - 8x - 12$ and find the other two factors.
b) Sketch the graph of $y = x^3 + x^2 - 8x - 12$.
c) Solve the inequality $x^3 + x^2 - 8x - 12 > 0$.

Exercise 2

Question 1
The polynomial $x^4 + 4x^2 + x + a$ is denoted by $p(x)$. It is given that $(x^2 + x + 2)$ is a factor of $p(x)$.

Find the value of a and the other quadratic factor of $p(x)$. **[4]**

[6, $x^2 - x + 3$]

Question 2

The polynomial $x^4 - 2x^3 - 2x^2 + a$ is denoted by $f(x)$. It is given that $f(x)$ is divisible by $x^2 - 4x + 4$.

(i) Find the value of a. **[3]**

(ii) When a has this value, show that $f(x)$ is never negative. **[4]**

[i) a = 8]

Question 3

The polynomial $2x^3 + ax^2 - 4$ is denoted by $p(x)$. It is given that $(x - 2)$ is a factor of $p(x)$.

(i) Find the value of a. [2]

When a has this value,

(ii) factorise $p(x)$. [2]

(iii) solve the inequality $p(x) > 0$, justifying your answer. [2]

[i) -3, ii) $(x-2)(2x^2+x+2)$]

Question 4

The polynomial $x^4 + 5x + a$ is denoted by $p(x)$. It is given that $x^2 - x + 3$ is a factor of $p(x)$.

(i) Find the value of a and factorise $p(x)$ completely. [6]

(ii) Hence state the number of real roots of the equation $p(x) = 0$, justifying your answer. [2]

[i) -6, $(x+2)(x-1)$, *ii) 2]*

Additional Example:
Show that (x – 3) is a factor of $x^3 - 2x^2 - 5x + 6$ and find the other two factors.
Sketch the graph of $y = x^3 - 2x^2 - 5x + 6$, showing the points where it cuts the x and y axes.
Solve the inequality $x^3 - 2x^2 - 5x + 6 > 0$.

Solution:
Let $f(x) = x^3 - 2x^2 - 5x + 6$.
$(x - 3)$ is a factor if $f(3) = 0$.

$$f(3) = 3^3 - 2\times 3^2 - 5\times 3 + 6 = 27 - 18 - 15 + 6 = 0.$$

So $(x - 3)$ is a factor.

To find the other factors we divide $x^3 - 2x^2 - 5x + 6$ by $(x - 3)$:

$$\begin{array}{r|l} & x^2 + x - 2 \\ \hline x - 3 & x^3 - 2x^2 - 5x + 6 \\ & \underline{x^3 - 3x^2} \\ & \qquad x^2 - 5x \\ & \qquad \underline{x^2 - 3x} \\ & \qquad\quad -2x + 6 \\ & \qquad\quad \underline{-2x + 6} \\ & \qquad\qquad\quad 0 \end{array}$$

To find the other factors we have to factorise $x^2 + x - 2 = (x-1)(x+2)$
So $x^3 - 2x^2 - 5x + 6 = (x - 3)(x - 1)(x + 2)$.

From the graph we see that $x^3 - 2x^2 - 5x + 6 > 0$ if:

$x > 3$ or $-2 < x < 1$