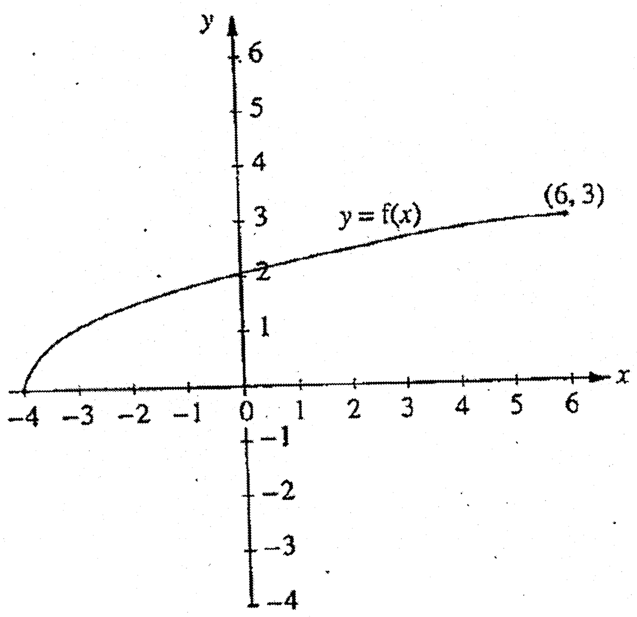
Tutorial (Discussion 2) – Topic 2 (Functions)

Question 1

(a)



The diagram shows the graph of $y = f(x)$, where f is a function with domain $-4 \leqslant x \leqslant 6$ and range $0 \leqslant f(x) \leqslant 3$. Make a sketch of the diagram shown and add a sketch of the graph of $y = f^{-1}(x)$. State the range of f^{-1}. $-4 \leq f^{-1}(x) \leq 6$ [3]

(b) The functions f and g are defined for real values of x by

$$f: x \mapsto 2x - 1,$$

$$g: x \mapsto \frac{1}{x-1}, \text{ where } x \neq 1.$$

Express (i) $fg(x)$, (ii) $g^{-1}(x)$, in the form of a single fraction in terms of x, stating in each case the value of x for which the function is not defined. [5]

$fg = \frac{3-x}{x-1}$, $x = 1$

$g^{-1} = \frac{x+1}{x}$, $x = 0$

(c) The functions f and g are defined for real values of x by

$$f: x \mapsto x^2,$$

$$g: x \mapsto x - 3.$$

Express each of the following in terms of f and g only.

(i) $x \mapsto x - 6$, (ii) $x \mapsto x^2 - 6$; (iii) $x \mapsto (x+3)^2$. [4]

g^2 g^2f fg^{-1}

Question 2

(a) The functions f and g are defined for all real values of x by

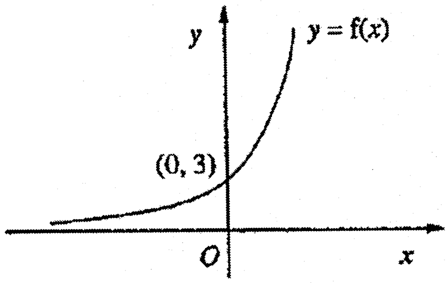
$f(x) = 2x - 1,$
$g(x) = x^2 + 3.$

(i) Find $f^{-1}(x)$ and $gf(x)$. $\frac{x+1}{2}$, $(2x-1)^2 + 3$

(ii) Explain why g has no inverse function. not one-one.

[5]

(b)



The diagram shows the graph of $y = f(x), x \in \mathbb{R}$.
The same scale is used for each axis.
Copy the diagram.
On the same axes, sketch the graph of $y = f^{-1}(x)$, marking in any relevant point on the axes and any other line you may have used.

[3]

(c) The functions f and g are defined, for all real values of x, by

$f : x \mapsto x^3,$
$g : x \mapsto x + 2.$

Express in terms of f and g only

(i) $x \mapsto x^3 + 2,$ gf

(ii) $x \mapsto x^9 + 2,$ gf^2

(iii) $x \mapsto (x-2)^3.$ fg^{-1}

[4]

Question 3

The function f is such that $f(x) = 2 + \frac{3}{x}, \quad x > 0.$

Find

(i) an expression for $f^{-1}(x)$, $\frac{3}{x-2}$

(ii) the domain of f^{-1}, $x > 2$

(iii) an expression for $f^2(x)$. $2 + \frac{3x}{2x+3}$

Solve the equation $f^{-1}(x) = f^2(x)$.

$x = 3, \quad x \neq -1$

Question 4

(a) Functions f and g are defined for real values of x by

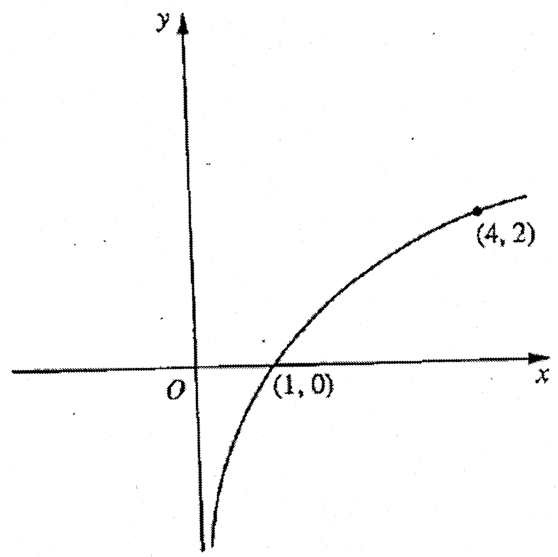
$$f(x) = 2x - 3, \quad g(x) = x^2 + 1.$$

(i) Find an expression for $fg(x)$. $2x^2 - 1$ [2]

(ii) Find an expression for $f^2(x)$. $4x - 9$ [1]

(iii) Sketch the graph of $y = |f(x)|$. [2]

(b)



The diagram, in which the same scale is used for each axis, shows the graph of $y = h(x)$, passing through the points (1, 0) and (4, 2).
Copy the diagram and, on the same set of axes, sketch the graph of $y = h^{-1}(x)$. Show the coordinates of two points on your graph of $y = h^{-1}(x)$. [3]

Question 5

Functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto 3x + 1,$$
$$g : x \mapsto 2x^2 - 1, \quad x \geqslant 0.$$

Find

(i) fg in terms of x, $6x^2 - 2$ [1]

(ii) g^{-1} in terms of x, $\left(\frac{x+1}{2}\right)^{\frac{1}{2}}$ [2]

(iii) the domain and range of g^{-1}. [2]

$x \geqslant -1$, $g^{-1}(x) \geqslant 0$

Question 6

$f(x) = 1 + \sqrt{x}$, $x \in \mathbb{R}$, $x \geq 0$.

(a) On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.
State the number of roots of the equation $f(x) = f^{-1}(x)$. one.

(b) Solve the equation $f(x) = f^{-1}(x)$.

$\frac{1}{2}(3 + \sqrt{5})$

Question 7

The function f is given by

$$f: x \mapsto x^2 - 6\lambda x, \quad x \in \mathbb{R},$$

where λ is a positive constant. Find, in terms of λ,

(i) $ff(\lambda)$, $25\lambda^4 + 30\lambda^3$

(ii) the range of f. $f(x) \geqslant -9\lambda^2$

Give a reason why f does not have an inverse. not one-one

The function f has an inverse if its domain is restricted to $x \geq k$ and also has an inverse if its domain is restricted to $x \leq k$. Find k in terms of λ, and find an expression for $f^{-1}(x)$ corresponding to each of these domains for f. $k = 3\lambda$

$f^{-1}(x) = 3\lambda \pm \sqrt{9\lambda^2 + x} \quad x \geqslant -9\lambda^2$

Question 8

$f(x)$ is defined by $f(x) = (x-2)^2 + 1$, $x \in \mathbb{R}$, $x \geq 2$.

(a) On a single diagram, sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = ff^{-1}(x)$.

$g(x)$ is defined by $g(x) = \sqrt{x-1}$, $x \in \mathbb{R}$.

(b) Find $gf(x)$ for $x \geq 2$. $x - 2$

$h(x)$ is defined by $h(x) = px + q$, $x \in \mathbb{R}$, p, q constant.

(c) If $fh(x) = 25x^2 + 20x + 5$, $x \in \mathbb{R}$, $x \geq r$, find the values of p, q and r.

5, 4, $-\frac{2}{5}$

Question 9

Functions f and g are defined by

$$f: x \mapsto ax + 2, \ a \neq 0$$

$$g: x \mapsto \frac{bx}{x+1}, \ x \neq -1, \ b > 0$$

Find in similar form, g^{-1} and fg^{-1}.

Given that $fg^{-1}(-1) = g\left(\frac{1}{b}\right)$, find a in terms of b. If $fg^{-1}(2) = 12$, find the value of a and of b.

$a = 5$, $b = 3$.

Question 10

Functions f and g are defined by $f : x \mapsto \frac{2x}{x-1}$, $x \neq 1$, and $g : x \mapsto \frac{x+a}{x}$, $x \neq 0$.

Find f^{-1} in similar form. Given that $gf^{-1}(3) = 4$, calculate the value of a.

$f^{-1} : x \to \frac{x}{x-2}$ $x \neq 2$, $a = 9$

Question 11

Functions f and g are defined on the set of real numbers by $f : x \mapsto \frac{3}{x+2}$, $x \neq k$, and

$g : x \mapsto 2x + 1$.

(a) State the value of k. -2

(b) Express fg in similar form and state the value of x for which fg is not defined. $\frac{3}{2x+3}$ $x \neq \cancel{-\frac{3}{3}}\ -\frac{3}{2}$

(c) Find the value of p for which $f^{-1}(p) = g(9)$. $\frac{1}{7}$

Question 12 (Oct/Nov 2008 Examination)

The function f is defined by

$$f : x \mapsto 3x - 2 \quad \text{for } x \in \mathbb{R}.$$

(i) Sketch, in a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the two graphs. [2]

The function g is defined by

$$g : x \mapsto 6x - x^2 \quad \text{for } x \in \mathbb{R}.$$

(ii) Express $gf(x)$ in terms of x, and hence show that the maximum value of $gf(x)$ is 9. [5]

$-9x^2 + 30x - 16$

The function h is defined by

$$h : x \mapsto 6x - x^2 \quad \text{for } x \geqslant 3.$$

(iii) Express $6x - x^2$ in the form $a - (x - b)^2$, where a and b are positive constants. [2]

$9 - (x-3)^2$

(iv) Express $h^{-1}(x)$ in terms of x. [3]

$3 + \sqrt{9 - x}$