Exercise 1: Equation of a Straight Line

1 Write down a vector that is parallel to each of these lines.

(a) $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 5\mathbf{k})$

(b) $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + s(3\mathbf{j} - 5\mathbf{k})$

(c) $\mathbf{r} = (1 - 2s)\mathbf{i} + (4s - 3)\mathbf{j} + (1 + s)\mathbf{k}$

(d) $\mathbf{r} = t\mathbf{i} + 3\mathbf{j} - (1 - t)\mathbf{k}$

2 Write down equations in vector form for the line through a point A with position vector $\mathbf{a}$ and in the direction of vector $\mathbf{b}$ where

(a) $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ $\quad \mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

(b) $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ $\quad \mathbf{b} = 3\mathbf{j} - \mathbf{k}$

(c) A is the origin $\quad \mathbf{b} = \mathbf{i} - \mathbf{j} - \mathbf{k}$

3 State whether or not the following pairs of lines are parallel.

(a) $\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$
$\mathbf{r} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k})$

(b) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + s(\mathbf{i} + \mathbf{j} - \mathbf{k})$
$\mathbf{r} = (3 + t)\mathbf{i} + (t - 1)\mathbf{j} + (5 - t)\mathbf{k}$

(c) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$
$\mathbf{r} = \mu(2\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$

(d) $\mathbf{r} = \lambda(3\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$
$\mathbf{r} = 4\mathbf{j} + \lambda(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

4 The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of a parallelogram $ABCD$.

(a) Find vector equations for the sides AB, BC and AD.

(b) Find the perpendicular distance of the point A from BC.

5 Write down a vector equation for the line through A and B if

(a) $\overrightarrow{AB}$ is $3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $\overrightarrow{OB}$ is $\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}$

(b) A and B have coordinates $(1, 1, 7)$ and $(3, 4, 1)$.

Find, in each case, the coordinates of the points where the line crosses the xy-plane, the yz-plane and the zx-plane.

6 Find the distance of the point with position vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ from the line $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

Exercise 2: Pairs of Line

1 Find whether the following pairs of lines are parallel, intersecting or skew. When the lines intersect, state the position vector of the common point, and find the angle between the lines.

(a) $\mathbf{r} = \mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$,
$\mathbf{r} = \mu(-9\mathbf{i} + 12\mathbf{j} - 3\mathbf{k})$

(b) $\mathbf{r} = (4 - t)\mathbf{i} + (8 - 2t)\mathbf{j} + (3 - t)\mathbf{k}$,
$\mathbf{r} = (7 + 6s)\mathbf{i} + (6 + 4s)\mathbf{j} + (5 + 5s)\mathbf{k}$

(c) $\mathbf{r} = \mathbf{i} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k})$,
$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j})$

2 Two lines which intersect have equations

$$\mathbf{r} = 2\mathbf{i} + 9\mathbf{j} + 13\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

and $$\mathbf{r} = a\mathbf{i} + 7\mathbf{j} - 2\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

Find the value of a, the position vector of the point of intersection, and the angle between the lines.

3 Show that the lines

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

and $$\mathbf{r} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$$

are skew.

Exercise 3: Equation of a Plane

1 Write down the Cartesian equations of these planes.

(a) $\mathbf{r}.(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 2$

(b) $\mathbf{r}.(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = 1$

2 Write down the vector equations of these planes.

(a) $3x - 2y + z = 5$

(b) $5x - 3y - 4z = 7$

3 Write down a vector that is perpendicular to each plane given in questions **1** and **2**.

4 Find the distance from the origin of each plane given in questions **1** and **2**.

5 Two planes P_1 and P_2 have vector equations $\mathbf{r}.(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3$ and $\mathbf{r}.(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 9$

Explain why P_1 and P_2 are parallel and hence find the distance between them.

6 Find the vector equation of the line through the origin that is perpendicular to the plane $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 3$

7 Find the vector equation of the line through the point (2, 1, 1) that is perpendicular to the plane $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = 6$

8 A plane goes through the three points whose position vectors are **a**, **b** and **c** where

(a) $= \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ (b) $= 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

(c) $= -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

Find the vector equation of this plane in scalar product form and hence find the distance of the plane from the origin.

9 Find the vector equation of the plane that contains the points $A(0, 1, 1)$, $B(-1, 2, 1)$ and $C(2, 0, 2)$.

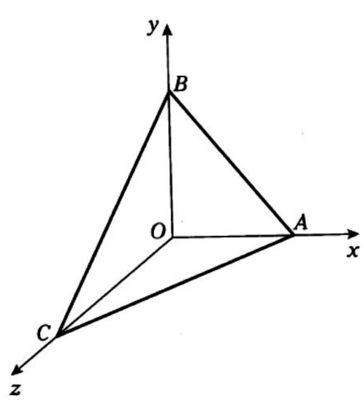
10 Find the vector equation of the plane that contains the lines

$\mathbf{r} = -3\mathbf{i} - 2\mathbf{j} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ and

$\mathbf{r} = \mathbf{i} - 11\mathbf{j} + 4\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

11 Find the point of intersection of the line $\mathbf{r} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$ and the plane $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 2$

12



OABC is a tetrahedron. The coordinates of *A*, *B* and *C* are (2, 0, 0), (0, 3, 0) and (0, 0, 4) respectively. Find

(a) a Cartesian equation for the plane *ABC*

(b) the distance of the plane *ABC* from *O*

(c) the angle between the edge *OB* and the plane *ABC*.

13 A line passes through the point with position vector $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

The line is parallel to the vector $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

(a) Write down the equation of this line.

The equation of a plane is $2x - 5y + z = 8$

(b) Find the position vector of the point where the line intersects the plane.

(c) Find the acute angle between the line and the plane.

14 (a) Find the equation of the line of intersection of the planes $x + y - z = 5$ and $2x - y + z = 1$

(b) Find the angle between the planes in part (a).

15 Repeat question **14** for the planes in question **1**.

16 The position vectors of three points are $\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Find the equation of the plane containing these three points.

Answers

Exercise 1

1 (a) $2\mathbf{i} - \mathbf{j} - 5\mathbf{k}$ (b) $3\mathbf{j} - 5\mathbf{k}$
(c) $-2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ (d) $\mathbf{i} + \mathbf{k}$

2 (a) $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + t(5\mathbf{i} + 4\mathbf{j} - \mathbf{k})$
(b) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + t(3\mathbf{j} - \mathbf{k})$
(c) $\mathbf{r} = t(\mathbf{i} - \mathbf{j} - \mathbf{k})$

3 (a) no (b) yes (c) yes (d) yes

4 (a) $\mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + 10\mathbf{k} + s(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$
$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + s(\mathbf{i} + \mathbf{j} + 5\mathbf{k})$
$\mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + 10\mathbf{k} + s(\mathbf{i} + \mathbf{j} + 5\mathbf{k})$
(b) 1.09

5 (a) $\mathbf{r} = \mathbf{i} + 7\mathbf{j} + 8\mathbf{k} + s(3\mathbf{i} + \mathbf{j} - 4\mathbf{k})$
$(7, 9, 0), \left(0, \frac{20}{3}, \frac{28}{3}\right), (-20, 0, 36)$
(b) $\mathbf{r} = \mathbf{i} + \mathbf{j} + 7\mathbf{k} + s(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$
$\left(\frac{10}{3}, \frac{9}{2}, 0\right), \left(0, -\frac{1}{2}, 10\right), \left(\frac{1}{3}, 0, 9\right)$

6 1.67

Exercise 2

1 (a) parallel (b) intersect at $\mathbf{i} + 2\mathbf{j}$; 27.9°
(c) skew

2 -3, $-\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, 64.6°

Exercise 3

1 (a) $x + y - z = 2$ (b) $2x + 3y - 4z = 1$

2 (a) $\mathbf{r}.(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 5$ (b) $\mathbf{r}.(5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}) = 7$

3 1. (a) $\mathbf{i} + \mathbf{j} - \mathbf{k}$ (b) $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$
2. (a) $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ (b) $5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$

4 1. (a) $\frac{2}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{29}}$
2. (a) $\frac{5}{\sqrt{14}}$ (b) $\frac{7}{\sqrt{50}}$

5 The normals are parallel, 2 units

6 $\mathbf{r} = s(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

7 $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$

8 $\mathbf{r}.(7\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = 3, \frac{3}{\sqrt{62}}$

9 $\mathbf{r}.(\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = -4$

10 $\mathbf{r}.(-\mathbf{i} + \mathbf{k}) = 3$

11 $\left(\frac{5}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

12 (a) $6x + 4y + 3z = 12$
(b) $\frac{12}{\sqrt{61}}$ (c) 30.8°

13 (a) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$
(b) $\frac{1}{10}(33\mathbf{i} - 3\mathbf{j} - \mathbf{k})$
(c) 33.4°

14 (a) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + t(3\mathbf{j} + 3\mathbf{k})$
(b) 90°

15 (a) $\mathbf{r} = 5\mathbf{j} + 3\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$
(b) 15.2°

16 $2x - y - 2z = 9$