Topic 2: Logarithmic and Exponential Functions

Objectives

- understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base);
- understand the definition and properties of e^x and $\ln x$, including their relationship as inverse functions and their graphs;
- use logarithms to solve equations of the form $a^x = b$, and similar inequalities;
- use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/or intercept.

Logarithmic Functions

General form: $y = log_a x$ where a > 0.

Definitions

log x means $\log_{10} x$. All $\log_a$ rules apply for log. When a logarithm is written without a base it means common logarithm.

Rules:

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

$$\log_a a = 1$$

Change of base:

$$\log_b a = \frac{\log_c a}{\log_c b}$$

Additional rules:

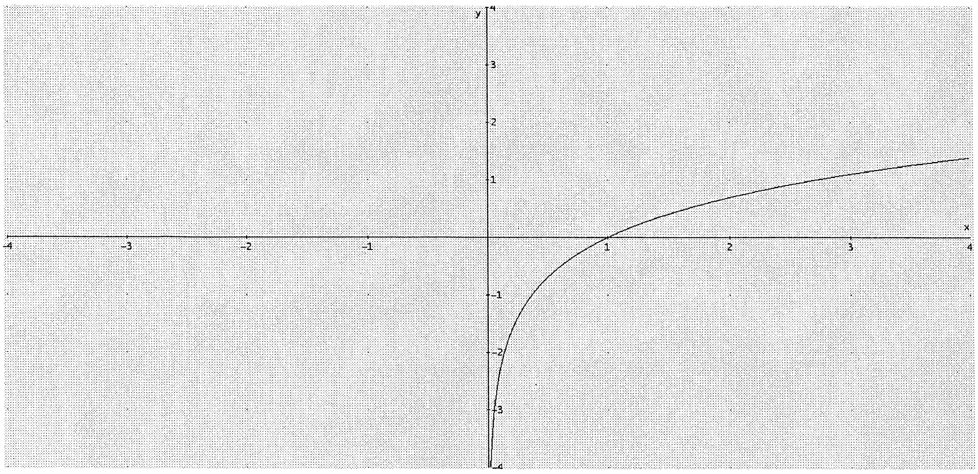
$$\log_a 1 = 0 \qquad \log_a\left(\tfrac{1}{a}\right) = -1$$

Careful!!!

1. $\log(x+y) \neq$

2. $\log(x-y) \neq$

Graph of $y = log_a x$ where a > 0



1. For any base, the *x*-intercept is 1. Why?
2. Which numbers are those that have negative logarithms?
3. The function is defined only for positive values of *x*.
   The **domain** is *x > 0.*
4. The **range** of the function is all real numbers, f(*x*) ε R.

Exercise 1: Logarithmic Functions

Question 1

Express each of the following in the form $\log_a[f(x)]$

(a) $2\log_a x$
(b) $\log_a x + \log_a(x+3)$
(c) $\log_a(x+1) - \log_a 2$
(d) $\log_a(x^2-1) - \log_a(x+1)$
(e) $2\log_a x - \log_a x(x+1)$
(f) $3\log_a x + \log_a(x+1)$
(g) $4\log_a x - \log_a(x^2+x^3)$
(h) $\frac{5}{2}\log_a x + \log_a(x+1) - \log_a\sqrt{x}$

2. Solve $log_b(x^2) = log_b(2x - 1)$.

3. Solve $2log_b(x) = log_b(4) + log_b(x - 1)$.

4. Solve the equation

   a) lg $(2x+5) = 1 +$ lg x
   b) $\log_4(3x + 10) - \log_4(x-1) = 1$

Answers -Exercise 1
Question 1

(a) $\log_a x^2$ (b) $\log_a [x(x+3)]$ (c) $\log_a \left(\frac{x+1}{2}\right)$ (d) $\log_a (x-1)$

(e) $\log_a \left(\frac{x}{x+1}\right)$ (f) $\log_a [x^3(x+1)]$ (g) $\log_a \left(\frac{x^2}{1+x}\right)$ (h) $\log_a [x^2(x+1)]$

Q2. $x = 1$ Q3. $x = 2$ Q4. a) $x = 5/8$, b) $x = 14$

<u>Logarithm base *e* – Natural logarithm</u>

$$\mathbf{lg}\, x = \mathbf{log}\, x$$

means $\log_{10}$ - common logarithm

$$\mathbf{ln}\, x = \mathbf{log}_e\, x$$

means $\log_e$ – natural logarithm

The value of $e \approx 2.718281828...$

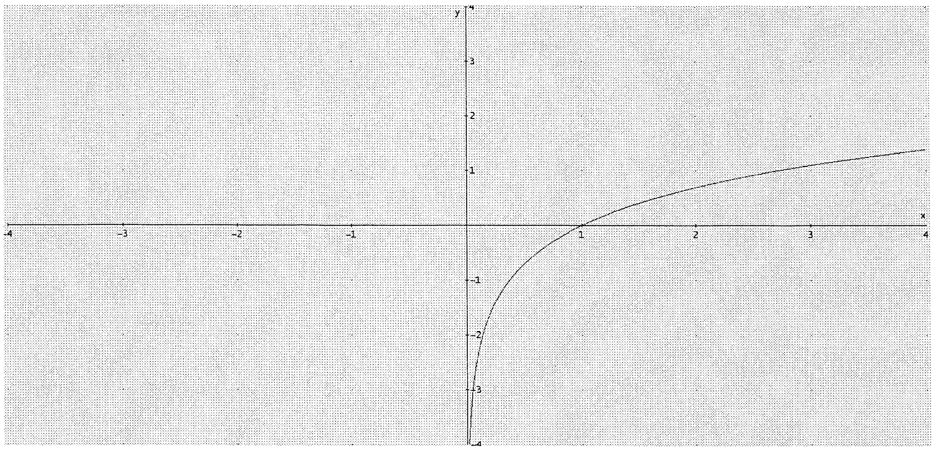
Rules:

1. $\ln xy =$
2. $\ln x/y =$
3. $\ln x^m =$

Additional rules:

$\ln 1 =$ $\ln e =$

Graph of $y = log_e x$ = $\ln x$



<u>Exercise 2: Natural Logarithm</u>

Question 1

Sketch the graphs of the folowing functions giving in each case the coordinates of the points where it crosses the x-axis.

(a) $y = \ln(x+1)$ (b) $y = 1 + \ln x$

(c) $y = |\ln x|$ (d) $y = 1 - \ln x$

(e) $y = |1 - \ln x|$ (f) $y = 1 - \ln(x+2)$

Question 2

Given that $\ln 5 = 1{\cdot}6$ and $\ln 11 = 2{\cdot}4$ find the values of:

(a) $\ln 625$ (b) $\ln 55$ (c) $\ln(2{\cdot}2)$ (d) $\ln 275$

(e) $\ln(0{\cdot}2)$ (f) $\ln(0{\cdot}04)$ (g) $\log_5 11$

3. It is given that $\ln a = x$ and $\ln b = y$. Express $ln\left(\dfrac{a^2 b}{e}\right)$ in terms of x and y.

4. Write $ln\, x^3 + ln\, xy - ln\, y^2$ as a single term.

5. Solve the following simultaneous equations.

$$\ln 6 + \ln(x - 3) = 2\ln y \quad \text{and} \quad 2y - x = 3$$

Answers – Exercise 2

Question 2

(a) 6·4 (b) 4 (c) 0·8 (d) 5·6 (e) − 1·6 (f) − 3·2 (g) 1·5

Q3. $2x + y - 1$ Q4. ln (x^4/y) Q5. $x = 9$, $y = 6$

Exponential Functions

Previously, you have dealt with such functions as

$$f(x) = x^2, \rightarrow$$

where the variable x was the base and the number 2 was the power.

In the case of exponentials, however, you will be dealing with functions such as

$$g(x) = 2^x, \rightarrow$$

where the base is the fixed number, and the power is the variable.

Rules:

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$$

$$(a^m)^n = a^{mn}$$

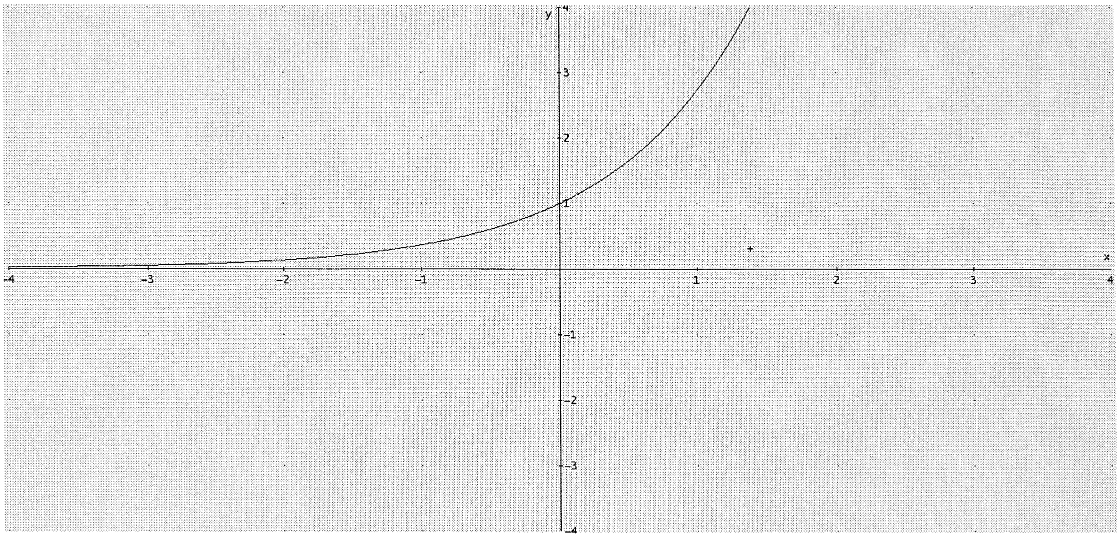
$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^0 = 1 \quad (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$$

Graph of $y = a^x$, $a > 0$



1. For $a > 0$, the y-intercept is 1. Why?
2. The function is defined for real numbers of x.
   The **domain** is $x \varepsilon R$.
3. The **range** of the function is defined only for positive values of f(x), f(x) > 0.

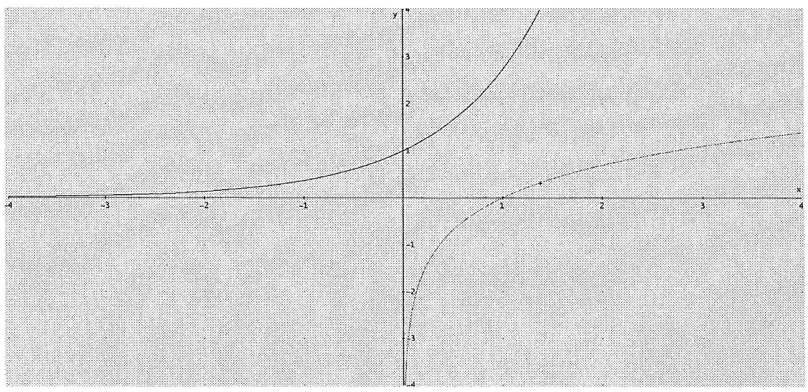
<u>The Relationship between $y = a^x$ and $y = \log_a x$</u>

log

$$a^b = c \Leftrightarrow \log_a c = b$$

base

Note:
The graphs below are f(x) = ln x and f^{-1}(x) = e^x. What is the relationship between this two functions?



Exercise 3: Exponential Functions

Question 1

Sketch the graphs of the following functions giving in each case the coordinates of the point where it crosses the y-axis.

(a) $y = e^x - 3$

(b) $y = e^{-x} + 1$

(c) $y = 2 - e^x$

(d) $y = -e^{-x}$

(e) $y = 3e^x$

(f) $y = e^{-x} - 1$

(g) $y = |(e^x) - 1|$

(h) $y = |1 - e^{-x}|$

(i) $y = |e^{-x} - 3|$

(j) $y = e^{|x|}$

2. Solve the following :
   a) $5^{x+1} = 625$
   b) $2^{x-5} = 32$
   c) $\log_5(2x + 3) = 3$
   d) $\ln(5x - 1) = \ln(2x + 8)$.
   e) $\ln(1 + x) - \ln(1 - x) = 1$.

3. Solve for x. The solution may be expressed as a logarithm.

$$10^{3x-1} = 2^{2x+1}$$

4. Solve the equation $ln(5 - x) = ln5 - lnx$ giving your answers to 3 significant figures.

5. Solve for x the equation $e^{2x} - e^x = 12$.

6. Given the simultaneous equations $2^x = 3^y, x + y = 1$. Show that $x = \dfrac{ln3}{ln6}$.

7. Given that $f(x) = 3x^3 - 4x^2 - 5x + 2$, show that $(x - 2)$ is a factor of $f(x)$. Express $f(x)$ as a product of three linear factors. Hence, solve for x, the equation $f(x) = 3e^{3x} - 4e^{2x} - 5e^x + 2 = 0$.

8. Find x and y given that $e^x + 3e^y = 3$ and $e^{2x} - 9e^{2y} = 6$ expressing each answer as a logarithm to base e.

9. By means of the substitution $y = 8^x$, or otherwise, find the exact values of x which satisfy the equation $64^x - 5(8^x) + 4 = 0$.

10. Solve the equation $\ln(2 + e^{-x}) = 2$, giving your answer correct to 2 decimal places.

11. Solve, correct to 3 significant figures, the equation $e^x + e^{2x} = e^{3x}$.

12. Solve the equation $e^{\ln x} + \ln e^x = 8$.

13. If $5^x.25^{2y} = 1$ and $3^{5x}.9^y = \frac{1}{9}$, calculate the value of x and y.

14. Use the substitution $y = 2^x$ to solve for x the equation $2^{2x+1} - 2^{x+1} + 1 = 2^x$.

15. Find the value of x for which $2^{3x+1} = 3^{x+2}$, giving three significant figures in your answer.

Question 16

Using the substitution $u = 3^x$, or otherwise, solve, correct to 3 significant figures, the equation

$$3^x = 2 + 3^{-x}.$$

Answers – Exercise 3

2.	a) $x = 3$ b) $x = 10$ c) $x = 61$ d) $x = 3$ e) $x = (e - 1) / (e + 1)$
3.	$x = \frac{1 + \log 2}{3 - 2 \log 2}$
4.	1.38 and 3.62
5.	1.39
7.	$(x-2)(3x-1)(x+1)$, ln2, -ln3
8.	
9.	$x = 0$ or $2/3$
10.	−1.68
11.	0.481
12.	4
13.	- 4/9 , 1/9
14.	-1 , 0
15.	1.53
16.	0.804

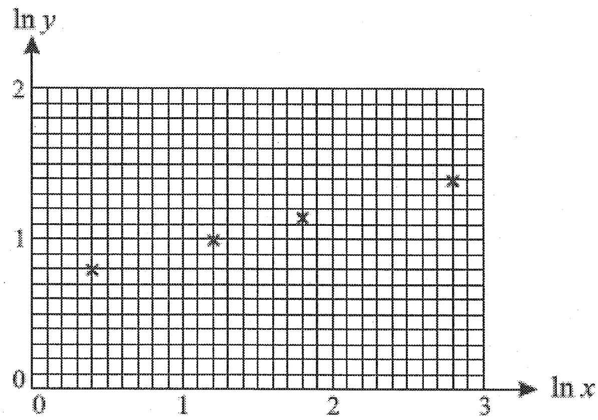
Exponential Functions: Conversion to Linear Functions

$y = ax^b$ ⟹ Linear equation

Exercise 4 - Conversion to Linear Functions

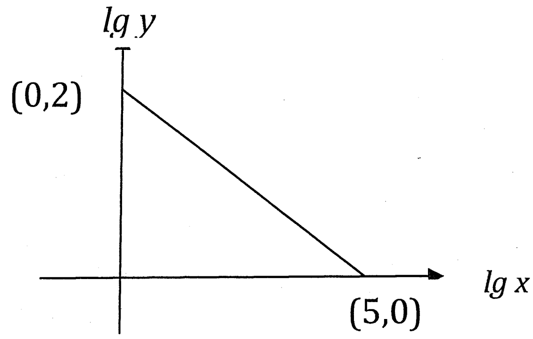
1. Two variable quantities x and y are related by the equation , $y = ax^b$ where a and b are constants. When a graph is plotted showing values of $\ln y$ on the vertical axis and values of $\ln x$ on the horizontal axis, the points lie on a straight line having gradient 1.8 and crossing the vertical axis at the point (0,4.1). Find the values of a and b.

2



Two variable quantities x and y are related by the equation $y = Ax^n$, where A and n are constants. The diagram shows the result of plotting $\ln y$ against $\ln x$ for four pairs of values of x and y. Use the diagram to estimate the values of A and n. [5]

3. The figure below shows the graph of $lg\, y$ against $lg\, x$, where $y = ax^b$. Find the values of a and b.



Question 4

The variables x and y satisfy the equation $y^3 = Ae^{2x}$, where A is a constant. The graph of $\ln y$ against x is a straight line.

(i) Find the gradient of this line. [2]

(ii) Given that the line intersects the axis of $\ln y$ at the point where $\ln y = 0.5$, find the value of A correct to 2 decimal places. [2]