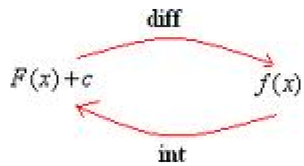
Integration

The expression to be integrated is the derivative of some function eg f(x) called the **integrand**.



When this expression is integrated the original function is restored plus a constant (C) called the constant of integration.

$$F(x) = \int f(x)dx + C$$

This is called the **indefinite integral** when the integration is not between two limiting values of x.

$$\int_a^b f(x)dx = \left[F(x) \right]_a^b = F(b) - F(a)$$

However when the integration is between two limiting values of x then the integral is called the **definite integral** and the constant of integration is not involved.

Most of the following basic formulas directly follow the differentiation rules.

1. $\int kf(x)\,dx = k\int f(x)\,dx$
2. $\int [f(x) \pm g(x)]dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\int kdx = kx + C$
4. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

The Substitution method(or 'changing the variable')

Example:

$\int (2x+5)^3 dx$ $\qquad$ $\int_1^2 \sqrt{2-x^2}\,dx$

Integration of Trigonometric Function

5. $\int \sin x \, dx = -\cos x + C$
6. $\int \cos x \, dx = \sin x + C$
7. $\int \sec^2 x \, dx = \tan x + C$
8. $\int \csc^2 x \, dx = -\cot x + C$
9. $\int \sec x \tan x \, dx = \sec x + C$
10. $\int \csc x \cot x \, dx = -\csc x + C$

To integrate **tan** ***x*** and **cot** ***x*** ➔Use substitution method or the short cut ⇨ recognition method

Integration of Exponential Functions

11. $\int e^x \, dx = e^x + C$

Integration using Logarthms

12. $\int \frac{dx}{x} = \ln|x| + C$

Integration of tan *x*

13. $\int \tan x \, dx = -\ln|\cos x| + C$ **why** 

Integration of cot *x*

14. $\int \cot x \, dx = \ln|\sin x| + C$

What about the following integrals? Is recognition formula possible???

$$\int \sec x \, dx =$$

Strategy: The strategy is not obvious. Multiply and divide by (sec x + tan x); use Substitution.

$$\int \sec x \, dx = \int \sec x \; \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

set

$u = \sec x + \tan x$

then we find

$du = (\sec x \tan x + \sec^2 x)\, dx$

substitute $du = (\sec x \tan x + \sec^2 x)\, dx$, $u = \sec x + \tan x$

$$\int \sec x \; \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{(\sec^2 x + \sec x \tan x)\, dx}{\sec x + \tan x}$$

$$= \int \frac{du}{u}$$

solve integral

$= \ln |u| + C$

substitute back u=sec x + tan x

$= \ln |\sec x + \tan x| + C$

$$\int \csc x \, dx =$$

Strategy: The strategy is not obvious. Multiply and divide by (csc x + cot x); use Substitution.

$$\int \csc x \, dx = \int \csc x \, \frac{\csc x + \cot x}{\csc x + \cot x} \, dx$$

set

$u = \csc x + \cot x$

then we find

$du = (-\csc x \cot x - \csc^2 x) \, dx$

substitute $du = (-\csc x \cot x - \csc^2 x)\, dx$, $u = \csc x + \cot x$

$$\int \csc x \, \frac{\csc x + \cot x}{\csc x + \cot x} \, dx = -\int \frac{(-\csc^2 x - \csc x \cot x)\, dx}{\csc x + \cot x}$$

$$= -\int \frac{du}{u}$$

solve integral

$= -\ln |u| + C$

substitute back $u=\csc x + \cot x$

$= -\ln |\csc x + \cot x| + C$

Integration of the form ($ax + b$)

1. $\int (ax+b)^n dx =$ where n ≠-1

2. $\int \frac{1}{ax+b} dx =$

3. $\int e^{ax+b} \, dx =$

4. $\int \sin(ax+b) dx =$

5. $\int \cos(ax+b) dx =$

6. $\int \sec^2(ax+b) dx =$

The Recognition method

1. $\int \frac{kf'(x)}{f(x)} dx = k \ln|f(x)| + c$

2. $\int k[f(x)]^n \; f'(x) \, dx = \frac{k}{n+1}[f(x)]^{n+1} + c$, $\quad n \neq -1$

3. $\int kf'(x) e^{f(x)} \; dx = ke^{f(x)} + c$

Example:

1. $\int x(x^2+3)^{10} \, dx$ 2. $\int \frac{1}{(7x+3)^3} \, dx$

3. Find

a) $\int 7x^2\left(x^3+1\right)^5 dx$

b) $\int (3x+4)^4\, dx$

c) $\int_0^1 \frac{x}{\sqrt{(x^2+1)}}\, dx$

d) $\int 6x^3\left(3x^4+4\right)^4 dx$

e) $\int (x+1)\, e^{x^2+2x}\, dx$

Exercise 1

Find

a) $\int \frac{2x-3}{3x^2-9x+4}\, dx$ $\quad \left\{\frac{1}{3}\ln\left|3x^2-9x+4\right|+c\right\}$

b) $\int \frac{e^x}{e^x+2}\, dx$ $\quad \left\{\ln\left(2+e^x\right)+c\right\}$

c) $\int_0^{10} \frac{x}{x^2+1}\, dx$ $\quad \left\{\frac{1}{2}\ln 101\right\}$

Integration of Algebraic Fractions – **Integration using Partial Fractions**

Look at the following integration:

$\int \frac{3}{5x+2}\, dx \qquad \int \frac{2x-2}{2x^2-2x-5}\, dx \qquad \int \frac{2x+2}{(x+1)^2}\, dx \qquad \int \frac{5+x}{(1-x)(5+x^2)}\, dx$

Exercise 2

1. Find a) $\int_3^4 \frac{3x}{(x+1)(x-2)}\,dx$ b) $\int \frac{1}{x^2-9}\,dx$

   $\{\ln 5\}$ $\left\{\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|+c\right\}$

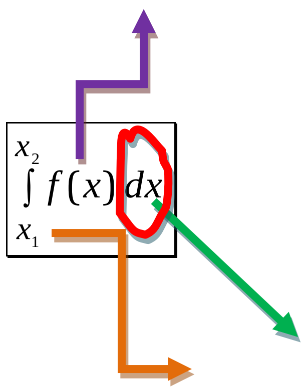
2. If $\frac{2x-1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$, find A and B. Hence find $\int \frac{2x-1}{(x+1)^2}\,dx$.

   $\{A=2, B=-3\}\left\{2\ln|x+1| + \frac{3}{x+1} + c\right\}$

3. Show that $\int_2^3 \frac{5+x}{(1-x)(5+x^2)}\,dx = \frac{1}{2}\ln\frac{7}{18}$.

Integration using a given Substitution

The 3 step method:



Exercise 3

1. Use the substitution $u = \ln x$, find $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx.$ $\{2\}$

2. Find $\int \frac{x(x-8)}{(x-4)^2}\,dx$ by substituting $u = x-4$. $\left\{(x-4) + \frac{16}{(x-4)} + c\right\}$

3. By substituting $x = \tan\theta$, find the exact value of $\int_0^\infty \frac{x^2}{(1+x^2)^2}\,dx.$ $\left\{\frac{\pi}{4}\right\}$

4. By substituting $\theta = \sin x$, show that $\int_{\pi/6}^{\pi/4} \cot x\,dx = \frac{1}{2}\ln 2.$

5. Find the exact value of $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{5+3\sin\theta}}\,d\theta.$ Using the substitution $u = 5 + 3\sin\theta.$

Integration of $\sin^n x$ and $\cos^n x$

The only 2 squared functions that can be integrated at this level is $\sec^2$x → tan x and cosec^2x → -cot x. The rest must use other fomulas (e.g. Double Angle or Trig Identities)

Double Angle Formula (When there is sine and cosine)	$\sin 2A = 2\sin A\cos A$ $\sin A\cos A = \frac{1}{2}\sin 2A$
Double Angle Formula (when there is $\sin^2 x$)	$\cos 2A = 1 - 2\sin^2 A$ $\cos 2A - 1 = -2\sin^2 A$ $2\sin^2 A = 1 - \cos 2A$ $\therefore \sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$ OR $\sin^2 x = 1 - \cos^2 x$
Double Angle Formula (when there is $\cos^2 x$)	$\cos 2A = 2\cos^2 A - 1$ $\cos 2A + 1 = 2\cos^2 A$ $\therefore \cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$ OR $\cos^2 x = 1 - \sin^2 x$
Trigonometry Identity (when there is $\tan^2 x$	$\tan^2 x + 1 = \sec^2 x$ $\tan^2 x = \sec^2 x - 1$

If we need to integrate $\cos^3$ x , $\sin^3$ x and $\tan^3$ x **?**

There are two methods

METHOD 1 **METHOD 2**

Method 2

Example 1

(i) Prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [4]

(ii) Using this result, find the exact value of

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \cos^3\theta \, d\theta. \qquad [4]$$

Example 2

Show that $\frac{d}{d\theta}(\tan^3\theta) = 3\tan^4\theta + 3\sec^2\theta - 3$. Hence, find $\int_0^{\frac{\pi}{4}} \tan^4\theta \; d\theta$. [4]

Integration by Parts

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}v dx$$

Example:

1. $\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}v dx$

$\int \ln x dx$ =

2. $\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}vdx$

$$\int_0^{\infty} xe^{-x}dx =$$

$u = x \qquad \Rightarrow \frac{du}{dx} =$

$\frac{dv}{dx} = e^{-x} \qquad \Rightarrow v =$

Now deal with the limits:

3. $\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}vdx$

$$\int x^2 e^{2x}dx =$$

$u = x^2 \qquad \Rightarrow \frac{du}{dx} =$

$\frac{dv}{dx} = e^{2x} \qquad \Rightarrow v =$

We now need to focus on finding $\int xe^{2x}dx$

$$\int xe^{2x}dx =$$

$u = x \qquad \Rightarrow \frac{du}{dx} =$

$\frac{dv}{dx} = e^{2x} \qquad \Rightarrow v =$

Therefore $\int x^2 e^{2x}dx =$

4. $\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx$

$$\int_0^{\pi} e^x \cos x \, dx$$

Exercise 4

1. Find a) $\int \ln x \, dx$ $\{x \ln x - x + c\}$

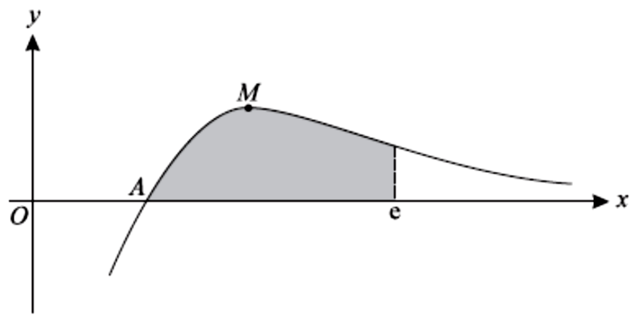
   b) $\int x(\ln x)^2 dx$

   $\left\{\frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + c\right\}$

2. Find $\int_1^2 x^3 \ln(4x) dx$. $\left\{\frac{23}{2}\ln 2 - \frac{15}{16}\right\}$

3. Find $\int \frac{x}{2e^x} dx$. $\left\{\frac{-1}{2}xe^{-x} - \frac{1}{2}e^{-x} + c\right\}$

Question 4



The diagram shows the curve $y = \frac{\ln x}{x^2}$ and its maximum point M. The curve cuts the x-axis at A.

(i) Write down the x-coordinate of A. [1]

(ii) Find the exact coordinates of M. [5]

(iii) Use integration by parts to find the exact area of the shaded region enclosed by the curve, the x-axis and the line $x = \mathrm{e}$. [5]

Exercise 5 – Past Year Questions

June 2002

Let $\mathrm{f}(x) = \frac{4x}{(3x+1)(x+1)^2}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = 1 - \ln 2$. [5]

June 2003

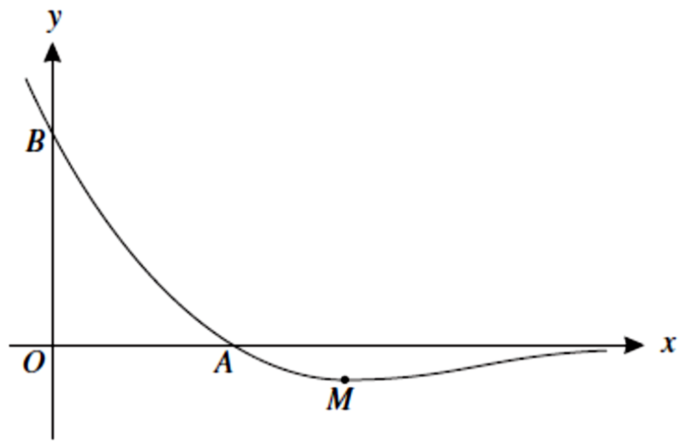
(i) Prove the identity

$$\cot x - \cot 2x \equiv \operatorname{cosec} 2x. \qquad [3]$$

(ii) Show that $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x\,\mathrm{d}x = \frac{1}{2}\ln 2$. [3]

(iii) Find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \operatorname{cosec} 2x\,\mathrm{d}x$, giving your answer in the form $a \ln b$. [4]

Nov 2003



The diagram shows the curve $y = (3 - x)e^{-2x}$ and its minimum point M. The curve intersects the x-axis at A and the y-axis at B.

(i) Calculate the x-coordinate of M. [4]

(ii) Find the area of the region bounded by OA, OB and the curve, giving your answer in terms of e. [5]

June 2004

(i) Prove the identity

$$\sin^2\theta\cos^2\theta \equiv \tfrac{1}{8}(1 - \cos 4\theta).$$ [3]

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} \sin^2\theta\cos^2\theta\,d\theta.$$ [3]

Additional Questions

Question 1

6 **(i)** Use the substitution $x = \sin^2\theta$ to show that

$$\int \sqrt{\left(\frac{x}{1-x}\right)}\,dx = \int 2\sin^2\theta\,d\theta. \qquad [4]$$

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx. \qquad [4]$$

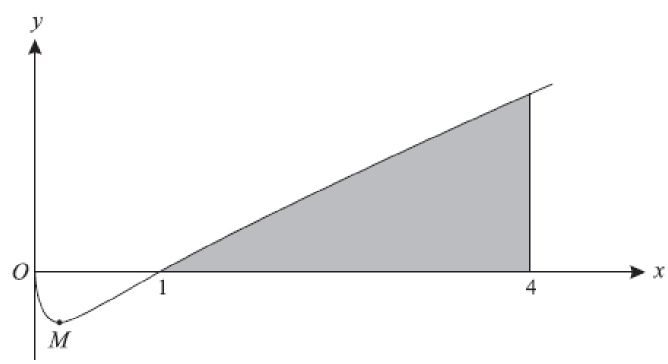
Question 2

5 **(i)** Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(ii) Hence show that $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{1}{(\cos\theta + (\sqrt{3})\sin\theta)^2}\,d\theta = \frac{1}{\sqrt{3}}$. [4]

Question 3

8



The diagram shows a sketch of the curve $y = x^{\frac{1}{2}}\ln x$ and its minimum point M. The curve cuts the x-axis at the point $(1, 0)$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Use integration by parts to find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 4$. Give your answer correct to 2 decimal places. [5]

Question 4

7 Let $I = \int_1^4 \frac{1}{x(4-\sqrt{x})}\,dx$.

(i) Use the substitution $u = \sqrt{x}$ to show that $I = \int_1^2 \frac{2}{u(4-u)}\,du$. [3]

(ii) Hence show that $I = \frac{1}{2}\ln 3$. [6]

Question 5

(a) Show that $\int_2^4 4x \ln x \, dx = 56 \ln 2 - 12$. [5]

(b) Use the substitution $u = \sin 4x$ to find the exact value of $\int_0^{\frac{1}{24}\pi} \cos^3 4x \, dx$. [5]

Question 6

(i) Express $4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the value of α correct to 4 decimal places. [3]

(ii) Hence

(a) solve the equation $4\cos\theta + 3\sin\theta = 2$ for $0 < \theta < 2\pi$, [4]

(b) find $\int \frac{50}{(4\cos\theta + 3\sin\theta)^2}\,d\theta$. [3]

Question 7

(i) By differentiating $\frac{1}{\cos x}$, show that the derivative of $\sec x$ is $\sec x \tan x$. Hence show that if $y = \ln(\sec x + \tan x)$ then $\frac{dy}{dx} = \sec x$. [4]

(ii) Using the substitution $x = (\sqrt{3})\tan\theta$, find the exact value of

$$\int_1^3 \frac{1}{\sqrt{(3+x^2)}}\,dx,$$

expressing your answer as a single logarithm. [4]