Discrete random variables

S1/4/1: Probability distribution table

Learning Outcome

Students should be able to:

- Define discrete random variable and describe it's characteristics
- Construct a probability distribution table relating to a given situation involving a discrete random variable, X.

S1/4/2 and S1/4/3: Expectation and variance of X Miscellaneous questions on DRV

Learning Outcome

Students should be able to:

- Calculate the expectation of X, E(X) and the variance of X, Var(X)
- Solve problems that involves the expectation of X, E(X), and the variance of X, Var(X).

S1/4/5: Binomial distribution

Learning Outcome

Students should be able to:

- Use formulae for the expectation and variance of the binomial distribution

Discrete & Random Variables

Discrete Random Variable:

A quantitative random variable that can assume a countable number of values

Note: **Usually associated with counting**

Probability Distribution & Function

Probability Distribution:

A distribution of the probabilities associated with each of the values of a random variable.

Probability Function:

A rule that assigns probabilities to the values of the random variable

Reminder!

Every probability function must satisfy the two basic properties of probability:

1. The probability assigned to each value of the random variable must be between 0 and 1, inclusive:

$$0 \le P(x) \le 1$$

2. The sum of the probabilities assigned to all the values of the random variable must equal 1:

$$\sum_{\text{all } x} P(x) = 1$$

Example 1:

The number of people staying in a randomly selected room at a local hotel is a random variable ranging in value from 0 to 4. The probability distribution is known and is given in various forms below:

x	0	1	2	3	4
$P(x)$	2/15	4/15	d	3/15	1/15

Notes:

This chart implies the only values x takes on are 0, 1, 2, 3, and 4

$$\sum_{\text{all } x} P(W = w) = 1$$

Find d.

$$2/15 + 4/15 + d + 3/15 + 1/15 = 1 \qquad d = 5/15$$

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x	0	1	2	3	4
$P(x)$	2/15	4/15	d	3/15	1/15

Find the probability that four pupils staying in a room.

$$P(\text{the random variable } x \text{ equals } 4) = P(4) = \frac{1}{15}$$

Example 1:

The number of people staying in a randomly selected room at a local hotel is a random variable ranging in value from 0 to 4. The probability distribution is known and is given in various forms below:

x	0	1	2	3	4
$P(x)$	2/15	4/15	d	3/15	1/15

Find P($0 < x \le 2$).

$$P(0 < x \le 2) \quad = P(1) + P(2) \quad = 4/15 + 5/15 = 9/15$$

Example 1:

The number of people staying in a randomly selected room at a local hotel is a random variable ranging in value from 0 to 4. The probability distribution is known and is given in various forms below:

x	0	1	2	3	4
$P(x)$	2/15	4/15	d	3/15	1/15

Find the mode.

The value of x with the highest probability is 2, so the mode is 2.

Example 2

Construct the probability distribution of sixes when three fair dice are thrown.

X = the number of sixes

$$\mathrm{P}(\mathrm{X}=0) = \left(\frac{5}{6}\right)^3 = \frac{125}{216} \qquad \mathrm{P}(\mathrm{X}=1) = \left(\frac{5}{6}\right)^2 \times \left(\frac{1}{6}\right) \times \frac{3!}{2!} = \frac{75}{216}$$

$$\mathrm{P}(\mathrm{X}=2) = 3 \times \left(\frac{5}{6}\right) \times \left(\frac{1}{6}\right)^2 = \frac{15}{216} \qquad \mathrm{P}(\mathrm{X}=3) = \left(\frac{1}{6}\right)^3 = \frac{1}{216}$$

x	0	1	2	3
P(X = x)	125/216	75/216	15/216	1/216

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<u>**Exercise 6A (Page 104)**</u>
Q3, Q6, Q7

<u>**Exercise 6B (Page 106)**</u>
Q7, Q8

Mean and Variance of a Discrete Probability Distribution

Describe the center and spread of a population

μ, σ, σ^2 : population parameters

μ is the *mean of the population*

σ^2 is the *variance of the population*

σ is the *standard deviation of the population*

Theoretical approach

The expectation of X (expected value or mean),

$$E(X) = \sum_{\text{all } x} x P(X = x)$$

$$E(X) = \sum x_i \, p_i \qquad i = 1, 2, \ldots, n$$

$$\mu = E(X)$$

Discrete Random Variables

<u>**Variance of a Discrete Random Variable**</u>**:**

$$\sigma^2 = \sum\left[x^2 P(x)\right] - \left\{\sum x P(x)\right\}^2$$

$$Var(X) = E(X^2) - \mu^2$$

$$Var(X) = E(X^2) - E^2(X)$$

<u>**Standard Deviation of a Discrete Random Variable**</u>**:**
The positive square root of the variance:

$$\sigma = \sqrt{\sigma^2}$$

Example 3

The number of standby passengers who get seats on a daily commuter flight from Boston to New York is a random variable, x, with probability distribution given below (in an extensions table). Find the mean, variance, and standard deviation:

x	0	1	2	3	4	5
$p(x)$	0.30	0.25	0.20	0.15	0.05	0.05

Example 3

x	$P(x)$	$xP(x)$	x^2	$x^2 P(x)$
0	0.30	0.00	0	0.00
1	0.25	0.25	1	0.25
2	0.20	0.40	4	0.80
3	0.15	0.45	9	1.35
4	0.05	0.20	16	0.80
5	0.05	0.25	25	1.25
Totals	1.00	1.55		4.45
	$\sum P(x)$ (check)	$E(X) = \sum[xP(x)]$		$E(X^2) = \sum[x^2 P(x)]$

$$\mu = E(X) = \sum\left[x P(x)\right] = 1.55$$

Example 3

x	$P(x)$	$xP(x)$	x^2	$x^2P(x)$
0	0.30	0.00	0	0.00
1	0.25	0.25	1	0.25
2	0.20	0.40	4	0.80
3	0.15	0.45	9	1.35
4	0.05	0.20	16	0.80
5	0.05	0.25	25	1.25
Totals	1.00	1.55		4.45
	$\sum P(x)$ (check)	$E(X) = \sum[xP(x)]$		$E(X^2) = \sum[x^2P(x)]$

$$\sigma^2 = E(X^2) - E^2(X) = \sum\left[x^2 P(x)\right] - \left\{\sum\left[x P(x)\right]\right\}^2$$

$$= 4.45 - 1.55^2 \quad = 2.0475 \quad \sigma = \sqrt{2.0745} \approx 1.43$$

Example 4

The probability distribution for a random variable x is given by the probability function:

$$P(x) = \frac{8-x}{15} \quad \text{for} \quad x = 3, 4, 5, 6, 7$$

Find the mean, variance, and standard deviation

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Example 4

Find the mean, variance, and standard deviation

$$P(x) = \frac{8-x}{15} \quad \text{for} \quad x = 3, 4, 5, 6, 7$$

$$P(3) = \frac{8-3}{15} = \frac{5}{15} \quad P(4) = \frac{8-4}{15} = \frac{4}{15} \quad P(5) = \frac{8-5}{15} = \frac{3}{15}$$

$$P(6) = \frac{8-6}{15} = \frac{2}{15} \quad P(7) = \frac{8-7}{15} = \frac{1}{15}$$

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Finding the Population Parameters

Use an extensions table to find the population parameters:

x	$P(x)$	$xP(x)$	x^2	$x^2P(x)$
3	5/15	15/15	9	45/15
4	4/15	16/15	16	64/15
5	3/15	15/15	25	75/15
6	2/15	12/15	36	72/15
7	1/15	7/15	49	49/15
Totals	15/15	65/15		305/15
		$E(X) = \sum[xP(x)]$		$E(X^2) = \sum[x^2P(x)]$

$$\mu = E(X) = \sum\left[x P(x)\right] = \frac{65}{15} \approx 4.33$$

$$\sigma^2 = E(X^2) - E^2(X) = \sum\left[x^2 P(x)\right] - \left\{\sum\left[x P(x)\right]\right\}^2 = \frac{305}{15} - \left(\frac{65}{15}\right)^2 \approx 1.56$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{1.56} \approx 1.25$$

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Example 5

In each round of a particular game, Albert draws 3 marbles without replacement from a box containing 2 red marbles and 8 blue marbles. The number of red marbles drawn in each round is denoted R. Find the probability distribution for R and show that the mean of R is 3/5. [5]

If Albert scores 4 points for each red marble drawn but loses 1 point for every blue marble drawn, calculate the number of points Albert expects to have at the end of 5 rounds. [5]

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Example 5

In each round of a particular game, Albert draws 3 marbles without replacement from a box containing 2 red marbles and 8 blue marbles. The number of red marbles drawn in each round is denoted R. Find the probability distribution for R and show that the mean of R is 3/5. [5]

2R , 8B R = number of red marbles drawn in 3 draws.

$$P(R=2) = \left(\frac{2}{10}\right)\left(\frac{1}{9}\right)\left(\frac{8}{8}\right) + \left(\frac{2}{10}\right)\left(\frac{8}{9}\right)\left(\frac{1}{8}\right) + \left(\frac{8}{10}\right)\left(\frac{2}{9}\right)\left(\frac{1}{8}\right) = \frac{1}{15}$$

$$P(R=1) = \left(\frac{2}{10}\right)\left(\frac{8}{9}\right)\left(\frac{7}{8}\right) + \left(\frac{8}{10}\right)\left(\frac{2}{9}\right)\left(\frac{7}{8}\right) + \left(\frac{8}{10}\right)\left(\frac{7}{9}\right)\left(\frac{2}{8}\right) = \frac{7}{15}$$

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Example 5

In each round of a particular game, Albert draws 3 marbles without replacement from a box containing 2 red marbles and 8 blue marbles. The number of red marbles drawn in each round is denoted R. Find the probability distribution for R and show that the mean of R is 3/5. [5]

r	0	1	2
$P(R = r)$	7/15	7/15	1/15

$$\mu = E(X) = \sum [x\,P(x)] = (0)\left(\frac{7}{15}\right) + (1)\left(\frac{7}{15}\right) + (2)\left(\frac{1}{15}\right) = \frac{3}{5}$$

Example 5

If Albert scores 4 points for each red marble drawn but loses 1 point for every blue marble drawn, calculate the number of points Albert expects to have at the end of 5 rounds. [5]

W = number of points scored per game

w	-3	2	7
$P(R = r)$	7/15	7/15	1/15

$$E(W) = (-3)\left(\frac{7}{15}\right) + (2)\left(\frac{7}{15}\right) + (7)\left(\frac{1}{15}\right) = 0$$

r	0	1	2
$P(R = r)$	7/15	7/15	1/15

Expected points in 5 rounds = 5 x 0 = 0

Expectation and Variance of the Binomial Distribution

If $X \sim B(n, p)$

$$E(X) = \mu = np$$

$$Var(X) = \sigma^2 = npq \quad \text{where } q = 1 - p$$

Example 6

Nails are sold in packets of 100. Occasionally a nail is faulty. The number of faulty nails in a randomly chosen packets is denoted by *X*. Assuming that faulty nails occur independently and at random, calculate the mean and standard deviation of *X*, given that the probability of any nail being faulty is 0.04.

Solutions: $n = 100$ and $p = 0.04$

$$E(X) = \mu = np = 100 \times 0.04 = 4$$

$$Var(X) = \sigma^2 = np(1-p) = 100 \times 0.04 \times 0.96 = 3.84$$

$$\sigma = \sqrt{3.84} = 1.96$$

Example 7

Given that Y is B(n , p), E(Y) = 24 and Var(Y) = 8, find the values of n and p.

Solutions:

$$E(Y) = np = 24 \quad \text{——— (1)}$$

$$Var(Y) = npq = np(1-p) = 8 \quad \text{——— (2)}$$

(1) into (2)

$$24(1-p) = 8$$

$$(1-p) = \frac{1}{3} \Rightarrow p = \frac{2}{3}$$

$$np = 24$$

$$n\left(\frac{2}{3}\right) = 24 \Rightarrow n = 36$$

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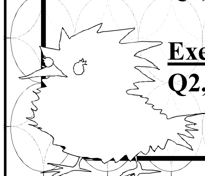
<u>Exercise 8A (Page 127)</u>
Q4, Q8, Q9, Q10

<u>Exercise 8B (Page 130)</u>
Q2, Q5



S1/4/4:
Binomial distribution

Learning Outcome

Students should be able to:

- Use formulae for probabilities for the binomial distribution

The Binomial Probability Distribution

Based on a series of repeated trials whose outcomes can be classified in one of two categories: ***success*** or ***failure***

Distribution based on a ***binomial probability experiment***

Binomial Probability Experiment

Binomial Probability Experiment:

An experiment that is made up of repeated trials that possess the following properties:

1. There are n repeated independent trials
2. Each ***trial*** has two possible outcomes (success, failure)
3. $P(\text{success}) = p$, $P(\text{failure}) = q$, and $p + q = 1$
4. The ***binomial random variable*** x is the count of the number of successful trials that occur; $x = 0, 1, 2, 3, \ldots, n$

Binomial Probability Function

For a binomial experiment, let p represent the probability of a "success" and q represent the probability of a "failure" on a single trial; $x \sim B(n, p)$

Then $P(x)$, the probability that there will be exactly x successes on n trials is:

$$P(x) = \binom{n}{x}(p^x)(q^{n-x}), \text{ for } x = 0, 1, 2, \ldots, \text{ or } n$$

Both x and p must be associated with "success"

Example 8

According to a recent study, 65% of all homes in a certain county have high levels of radon gas leaking into their basements. Four homes are selected at random and tested for radon. The random variable x is the number of homes with high levels of radon (out of the four).

Find the probability distribution.

Example 8

x = number of homes with high levels of radon, $x = 0, 1, 2, 3, 4$

$x \sim B(4, 0.65)$ $x = 0, 1, 2, 3, 4$

$$P(x) = \binom{4}{x}(0.65)^x(0.35)^{4-x}, \text{ for } x = 0, 1, 2, 3, 4$$

$$P(0) = \binom{4}{0}(0.65)^0(0.35)^4 = 0.0150 \quad P(1) = \binom{4}{1}(0.65)^1(0.35)^3 = 0.1115$$

$$P(2) = \binom{4}{2}(0.65)^2(0.35)^2 = 0.3105 \quad P(3) = \binom{4}{3}(0.65)^3(0.35)^1 = 0.3845$$

$$P(4) = \binom{4}{4}(0.65)^4(0.35)^0 = 0.1785$$

Example 9

In a certain automobile dealership, 70% of all customers purchase an extended warranty with their new car.

For 15 customers selected at random:

1) Find the probability that exactly 12 will purchase an extended warranty

$$P(x)=\binom{15}{x}(0.7)^x(0.3)^{15-x},\quad \text{for } x=0,1,2,\ldots,15$$

$$P(12)=\binom{15}{12}(0.7)^{12}(0.3)^{3}$$ $$=0.1700$$

Example 9

In a certain automobile dealership, 70% of all customers purchase an extended warranty with their new car.

For 15 customers selected at random:

2) Find the probability at least 13 will purchase an extended warranty

$$P(x)=\binom{15}{x}(0.7)^x(0.3)^{15-x},\quad \text{for } x=0,1,2,\ldots,15$$

$$P(x\geq 13)=P(13)+P(14)+P(15)$$ $$=P(13)+P(14)+P(15)$$

$$=\binom{15}{13}(0.7)^{13}(0.3)^{2}+\binom{15}{14}(0.7)^{14}(0.3)^{1}+\binom{15}{15}(0.7)^{15}(0.3)^{0}$$

$$=0.1268$$

Example 9

In a certain automobile dealership, 70% of all customers purchase an extended warranty with their new car.

For 15 customers selected at random:

3) Find the probability at most 13 will purchase an extended warranty

$$P(x)=\binom{15}{x}(0.7)^x(0.3)^{15-x},\quad \text{for } x=0,1,2,\ldots,15$$

$$P(x\leq 13)=P(0)+P(1)+\ldots+P(13)$$

$$=1-\left[P(14)+P(15)\right]$$ $$=1-\left[\binom{15}{14}(0.7)^{14}(0.3)^{1}+\binom{15}{15}(0.7)^{15}(0.3)^{0}\right]$$

$$=0.9648$$

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<u>Exercise 7A (Page 114)</u>
Q3, Q5
<u>Exercise 7B (Page 118)</u>
Q4
<u>Miscellaneous exercise 7 (Page 119)</u>
Q1, Q7