Exercise 1: Reciprocal Trigonometric Functions

1 Solve the following equations for $0° \leqslant x \leqslant 360°$.

(i) $\operatorname{cosec} x = 1$ **(ii)** $\sec x = 2$ **(iii)** $\cot x = 4$

(iv) $\sec x = -3$ **(v)** $\cot x = -1$ **(vi)** $\operatorname{cosec} x = -2$

2 Find the following giving your answers as fractions or in surd form.
You should not need your calculator.

(i) $\cot 135°$ **(ii)** $\sec 150°$ **(iii)** $\operatorname{cosec} 240°$

(iv) $\sec 210°$ **(v)** $\cot 270°$ **(vi)** $\operatorname{cosec} 225°$

3 In triangle ABC, angle $A = 90°$ and $\sec B = 2$.

(i) Find the angles B and C.
(ii) Find $\tan B$.
(iii) Show that $1 + \tan^2 B = \sec^2 B$.

4 In triangle LMN, angle $M = 90°$ and $\cot N = 1$.

(i) Find the angles L and N.
(ii) Find $\sec L$, $\operatorname{cosec} L$, and $\tan L$.
(iii) Show that $1 + \tan^2 L = \sec^2 L$.

5 Malini is 1.5 m tall.
At 8 pm one evening her shadow is 6 m long.
Given that the angle of elevation of the sun at that moment is α

(i) show that $\cot \alpha = 4$
(ii) find α.

6 **(i)** For what values of α, where $0° \leqslant \alpha \leqslant 360°$, are $\sec\alpha$, $\operatorname{cosec}\alpha$ and $\cot\alpha$ all positive?
(ii) Are there any values of α for which $\sec\alpha$, $\operatorname{cosec}\alpha$ and $\cot\alpha$ are all negative? Explain your answer.
(iii) Are there any values of α for which $\sec\alpha$, $\operatorname{cosec}\alpha$ and $\cot\alpha$ are all equal? Explain your answer.

7 Solve the following equations for $0° \leqslant x \leqslant 360°$.

(i) $\cos x = \sec x$ **(ii)** $\operatorname{cosec} x = \sec x$

(iii) $2 \sin x = 3 \cot x$ **(iv)** $\operatorname{cosec}^2 x + \cot^2 x = 2$

(v) $3 \sec^2 x - 10 \tan x = 0$ **(vi)** $1 + \cot^2 x = 2 \tan^2 x$

Exercise 2: The Pythagorean Identities

Simplify the following expressions.

1 $\dfrac{1 - \sec^2 A}{1 - \operatorname{cosec}^2 A}$

2 $\dfrac{\sqrt{(1 + \tan^2 \theta)}}{\sqrt{(1 - \sin^2 \theta)}}$

3 $\dfrac{1}{\cos \theta \sqrt{(1 + \cot^2 \theta)}}$

4 $\dfrac{\sin \theta}{1 + \cot^2 \theta}$

Eliminate θ from the following pairs of equations.

5 $x = 4 \sec \theta$
$y = 4 \tan \theta$

6 $x = a \operatorname{cosec} \theta$
$y = b \cot \theta$

7 $x = a \sec \theta$
$y = b \sin \theta$

Prove the following identities.

8 $\cot \theta + \tan \theta \equiv \sec \theta \operatorname{cosec} \theta$

9 $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} \equiv \sin A + \cos A$

10 $\tan^2 \theta + \cot^2 \theta \equiv \sec^2 \theta + \operatorname{cosec}^2 \theta - 2$

Solve the equations for angles in the range $0 \leqslant \theta \leqslant 360°$

11 $\cot^2 \theta = \operatorname{cosec} \theta$

12 $\sec^2 \theta + \tan^2 \theta = 6$

13 $\tan \theta + \cot \theta = 2 \sec \theta$

14 $\tan \theta + 3 \cot \theta = 5 \sec \theta$

15 $4 \sec^2 \theta - 3 \tan \theta = 5$

16 $4 \cot^2 \theta + 12 \operatorname{cosec} \theta + 1 = 0$

Exercise 3: Compound Angle Formulae

1 Use the compound-angle formulae to write the following as surds.

(i) $\sin 75° = \sin(45° + 30°)$ **(ii)** $\cos 135° = \cos(90° + 45°)$
(iii) $\tan 15° = \tan(45° - 30°)$ **(iv)** $\tan 75° = \tan(45° + 30°)$

2 Expand each of the following expressions.

(i) $\sin(\theta + 45°)$ **(ii)** $\cos(\theta - 30°)$ **(iii)** $\sin(60° - \theta)$
(iv) $\cos(2\theta + 45°)$ **(v)** $\tan(\theta + 45°)$ **(vi)** $\tan(\theta - 45°)$

3 Simplify each of the following expressions.

(i) $\sin 2\theta \cos\phi - \cos 2\theta \sin\theta$
(ii) $\cos\phi \cos 7\phi - \sin\phi \sin 7\phi$
(iii) $\sin 120° \cos 60° + \cos 120° \sin 60°$
(iv) $\cos\theta \cos\theta - \sin\theta \sin\theta$

4 Solve the following equations for values of θ in the range $0° \leqslant \theta \leqslant 180°$.

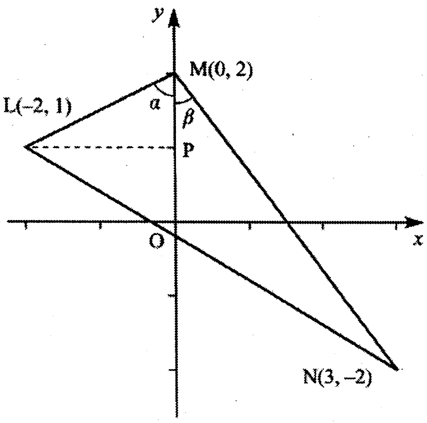
(i) $\cos(60° + \theta) = \sin\theta$
(ii) $\sin(45° - \theta) = \cos\theta$
(iii) $\tan(45° + \theta) = \tan(45° - \theta)$
(iv) $2\sin\theta = 3\cos(\theta - 60°)$
(v) $\sin\theta = \cos(\theta + 120°)$

5 Solve the following equations for values of θ in the range $0 \leqslant \theta \leqslant \pi$.
(When the range is given in radians, the solutions should be in radians, using multiples of π where appropriate.)

(i) $\sin\left(\theta + \frac{\pi}{4}\right) = \cos\theta$

(ii) $2\cos\left(\theta - \frac{\pi}{3}\right) = \cos\left(\theta + \frac{\pi}{2}\right)$

6 Calculators are not to be used in this question.
The diagram shows three points L(−2, 1), M(0, 2) and N(3, −2) joined to form a triangle. The angles α and β and the point P are shown in the diagram.



(i) Show that $\sin\alpha = \frac{2}{\sqrt{5}}$ and write down the value of $\cos\alpha$.

(ii) Find the values of $\sin\beta$ and $\cos\beta$.

(iii) Show that $\sin\angle\text{LMN} = \frac{11}{5\sqrt{5}}$.

(iv) Show that $\tan\angle\text{LNM} = \frac{11}{27}$.

Exercise 4: Double Angle Formulae

1 Solve the following equations for $0° \leqslant \theta \leqslant 360°$.

(i) $2\sin 2\theta = \cos\theta$

(ii) $\tan 2\theta = 4\tan\theta$

(iii) $\cos 2\theta + \sin\theta = 0$

(iv) $\tan\theta\tan 2\theta = 1$

(v) $2\cos 2\theta = 1 + \cos\theta$

2 Solve the following equations for $-\pi \leqslant \theta \leqslant \pi$.

(i) $\sin 2\theta = 2\sin\theta$

(ii) $\tan 2\theta = 2\tan\theta$

(iii) $\cos 2\theta - \cos\theta = 0$

(iv) $1 + \cos 2\theta = 2\sin^2\theta$

(v) $\sin 4\theta = \cos 2\theta$

Hint: Write the expression in part **(v)** as an equation in 2θ.

3 By first writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, express $\sin 3\theta$ in terms of $\sin\theta$.
Hence solve the equation $\sin 3\theta = \sin\theta$ for $0 \leqslant \theta \leqslant 2\pi$.

4 Solve $\cos 3\theta = 1 - 3\cos\theta$ for $0° \leqslant \theta \leqslant 360°$.

5 Simplify $\dfrac{1+\cos 2\theta}{\sin 2\theta}$.

6 Express $\tan 3\theta$ in terms of $\tan\theta$.

7 Show that $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$.

8 **(i)** Show that $\tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left(\dfrac{\pi}{4}-\theta\right) = 1$.

(ii) Given that $\tan 26.6° = 0.5$, solve $\tan\theta = 2$ without using your calculator. Give θ to 1 decimal place, where $0° < \theta < 90°$.

9 **(i)** Sketch on the same axes the graphs of

$$y = \cos 2x \quad \text{and} \quad y = 3\sin x - 1 \quad \text{for} \quad 0 \leqslant x \leqslant 2\pi.$$

(ii) Show that these curves meet at points whose x co-ordinates are solutions of the equation $2\sin^2 x + 3\sin x - 2 = 0$.

(iii) Solve this equation to find the values of x in terms of π for $0 \leqslant x \leqslant 2\pi$.

<u>Exercise 5</u>: The Form r cos($\theta \pm \alpha$), r sin ($\theta \pm \alpha$)

1 Express each of the following in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $0° < \alpha < 90°$.

(i) $\cos\theta + \sin\theta$ **(ii)** $20\cos\theta + 21\sin\theta$

(iii) $\cos\theta + \sqrt{3}\sin\theta$ **(iv)** $\sqrt{5}\cos\theta + 2\sin\theta$

2 Express each of the following in the form $r\cos(\theta + \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(i) $\cos\theta - \sin\theta$ **(ii)** $\sqrt{3}\cos\theta - \sin\theta$

3 Express each of the following in the form $r\sin(\theta + \alpha)$, where $r > 0$ and $0° < \alpha < 90°$.

(i) $\sin\theta + 2\cos\theta$ **(ii)** $2\sin\theta + \sqrt{5}\cos\theta$

4 Express each of the following in the form $r\sin(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(i) $\sin\theta - \cos\theta$ **(ii)** $\sqrt{7}\sin\theta - \sqrt{2}\cos\theta$

5 Express each of the following in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $-180° < \alpha < 180°$.

(i) $\cos\theta - \sqrt{3}\sin\theta$ **(ii)** $2\sqrt{2}\cos\theta - 2\sqrt{2}\sin\theta$

(iii) $\sin\theta + \sqrt{3}\cos\theta$ **(iv)** $5\sin\theta + 12\cos\theta$

(v) $\sin\theta - \sqrt{3}\cos\theta$ **(vi)** $\sqrt{2}\sin\theta - \sqrt{2}\cos\theta$

6 **(i)** Express $5\cos\theta - 12\sin\theta$ in the form $r\cos(\theta + \alpha)$, where $r > 0$ and $0° < \alpha < 90°$.

(ii) State the maximum and minimum values of $5\cos\theta - 12\sin\theta$.

(iii) Sketch the graph of $y = 5\cos\theta - 12\sin\theta$ for $0° \leqslant \theta \leqslant 360°$.

(iv) Solve the equation $5\cos\theta - 12\sin\theta = 4$ for $0° \leqslant \theta \leqslant 360°$.

7 **(i)** Express $3\sin\theta - \sqrt{3}\cos\theta$ in the form $r\sin(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(ii) State the maximum and minimum values of $3\sin\theta - \sqrt{3}\cos\theta$ and the smallest positive values of θ for which they occur.

(iii) Sketch the graph of $y = 3\sin\theta - \sqrt{3}\cos\theta$ for $0 \leqslant \theta \leqslant 2\pi$.

(iv) Solve the equation $3\sin\theta - \sqrt{3}\cos\theta = \sqrt{3}$ for $0 \leqslant \theta \leqslant 2\pi$.

8 **(i)** Express $2\sin 2\theta + 3\cos 2\theta$ in the form $r\sin(2\theta + \alpha)$, where $r > 0$ and $0° < \alpha < 90°$.

(ii) State the maximum and minimum values of $2\sin 2\theta + 3\cos 2\theta$ and the smallest positive values of θ for which they occur.

(iii) Sketch the graph of $y = 2\sin 2\theta + 3\cos 2\theta$ for $0° \leqslant \theta \leqslant 360°$.

(iv) Solve the equation $2\sin 2\theta + 3\cos 2\theta = 1$ for $0° \leqslant \theta \leqslant 360°$.

9 **(i)** Express $\cos\theta + \sqrt{2}\sin\theta$ in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $0° < \alpha < 90°$.

(ii) State the maximum and minimum values of $\cos\theta + \sqrt{2}\sin\theta$ and the smallest positive values of θ for which they occur.

(iii) Sketch the graph of $y = \cos\theta + \sqrt{2}\sin\theta$ for $0° \leqslant \theta \leqslant 360°$.

(iv) State the maximum and minimum values of

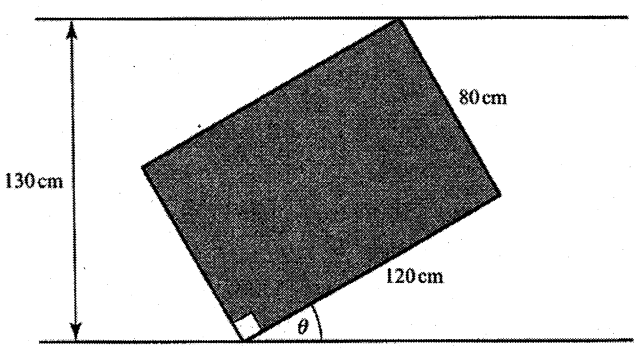
$$\frac{1}{3 + \cos\theta + \sqrt{2}\sin\theta}$$

and the smallest positive values of θ for which they occur.

10 The diagram shows a table jammed in a corridor. The table is 120 cm long and 80 cm wide, and the width of the corridor is 130 cm.

(i) Show that $12\sin\theta + 8\cos\theta = 13$.

(ii) Hence find the angle θ. (There are two answers.)



11 **(i)** Use a trigonometrical formula to expand $\cos(x+\alpha)$.

(ii) Express $y = 2\cos x - 5\sin x$ in the form $r\cos(x+\alpha)$, giving the positive value of r and the smallest positive value of α.

(iii) State the maximum and minimum values of y and the corresponding values of x for $0° \leqslant x \leqslant 360°$.

(iv) Solve the equation

$$2\cos x - 5\sin x = 3, \quad \text{for } 0° \leqslant x \leqslant 360°.$$

[MEI]

12 **(i)** Find the value of the acute angle α for which

$$5\cos x - 3\sin x = \sqrt{34}\cos(x+\alpha)$$

for all x.

Giving your answers correct to 1 decimal place,

(ii) solve the equation $5\cos x - 3\sin x = 4$ for $0° \leqslant x \leqslant 360°$

(iii) solve the equation $5\cos 2x - 3\sin 2x = 4$ for $0° \leqslant x \leqslant 360°$.

[MEI]

13 **(i)** Find the positive value of R and the acute angle α for which

$$6\cos x + 8\sin x = R\cos(x-\alpha).$$

(ii) Sketch the curve with equation

$$y = 6\cos x + 8\sin x, \quad \text{for } 0° \leqslant x \leqslant 360°.$$

Mark your axes carefully and indicate the angle α on the x axis.

(iii) Solve the equation

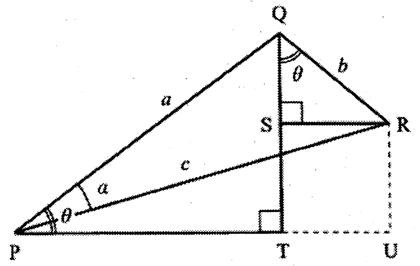
$$6\cos x + 8\sin x = 4, \quad \text{for } 0° \leqslant x \leqslant 360°.$$

(iv) Solve the equation

$$8\cos\theta + 6\sin\theta = 4, \quad \text{for } 0° \leqslant \theta \leqslant 360°.$$

[MEI]

14 In the diagram below, angle QPT = angle SQR = θ, angle QPR = α, PQ = a, QR = b, PR = c, angle QSR = angle QTP = 90°, SR = TU.



(i) Show that angle PQR = 90°, and write down the length of c in terms of a and b.

(ii) Show that PU may be written as $a\cos\theta + b\sin\theta$ and as $c\cos(\theta - \alpha)$. Write down the value of $\tan\alpha$ in terms of a and b.

(iii) In the case when $a = 4$, $b = 3$, find the acute angle α.

(iv) Solve the equation

$$4\cos\theta + 3\sin\theta = 2 \quad \text{for} \quad 0° \leqslant \theta \leqslant 360°.$$

Exercise 6: Miscellaneous

1 Eliminate θ from the equations
$x = \sin\theta$ and $y = \cos 2\theta$

2 Prove the identity $\dfrac{\sin 2\theta}{1+\cos 2\theta} \equiv \tan\theta$

3 Prove that $\tan\left(\theta + \frac{1}{4}\pi\right)\tan\left(\frac{1}{4}\pi - \theta\right) \equiv 1$

4 If $\cos A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$ find the possible values of $\cos(A+B)$

5 Eliminate θ from the equations
$x = \cos 2\theta$ and $y = \cos^2\theta$

6 Prove the identity
$$\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta} \equiv 2\sec\theta$$

7 Express $4\sin\theta - 3\cos\theta$ in the form $r\sin(\theta - \alpha)$. Hence find the maximum and minimum values of $4\sin\theta - 3\cos\theta + 2$

8 Express $\sin 2\theta - \cos 2\theta$ in the form $r\sin(2\theta - \alpha)$. Hence find the smallest positive value of θ for which $\sin 2\theta - \cos 2\theta$ has a maximum value.

9 Solve the equation $\cos^2\theta - \sin^2\theta = 1$ for values of θ in the range $-\pi \leqslant \theta \leqslant \pi$

P3**10** Prove the identity $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$

P3**11** Simplify the expression $\dfrac{1+\cos 2x}{1-\cos 2x}$

P3**12** Find the values of A between 0 and 360° for which $\sin(60° - A) + \sin(120° - A) = 0$

P3**13** (a) Express $2\sin^2\theta + 1$ in terms of $\cos 2\theta$.

(b) Express $4\cos^2 2A$ in terms of $\cos 4A$

(Hint: use $2A = x$)

P3**14** Find all the values of x between 0 and 180° for which $\cos x - 2\sin x = 1$

P3**15** Solve the equation $3\cos x - 2\sin x = 1$ for values of x in the range $0 \leqslant x \leqslant 180°$