PAPER 4 : Mechanics 1

Kinematics - Motion in a straight line

Questions set will be mainly numerical, and will aim to test mechanical principles without involving difficult algebra or trigonometry. However, candidates should be familiar in particular with the following trigonometrical results: $\sin(90° - \theta) \equiv \cos\theta$, $\cos(90° - \theta) \equiv \sin\theta$, $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$, $\sin^2\theta + \cos^2\theta \equiv 1$.

2. Kinematics of motion in a straight line	• understand the concepts of distance and speed as scalar quantities, and of displacement, velocity and acceleration as vector quantities (in one dimension only); • sketch and interpret displacement-time graphs and velocity-time graphs, and in particular appreciate that the area under a velocity-time graph represents displacement, the gradient of a displacement-time graph represents velocity, the gradient of a velocity-time graph represents acceleration; • use differentiation and integration with respect to time to solve simple problems concerning displacement, velocity and acceleration (restricted to calculus within the scope of unit P1); • use appropriate formulae for motion with constant acceleration in a straight line.

Kinematics is the study of **displacement**, **velocity** and **acceleration**.
Displacement is the position of a point relative to an origin, O. It is a **vector**.
SI unit is metre (m). Other metric units are centimetre (cm), kilometre (km).
Distance is the magnitude of the displacement. It is a **scalar**.

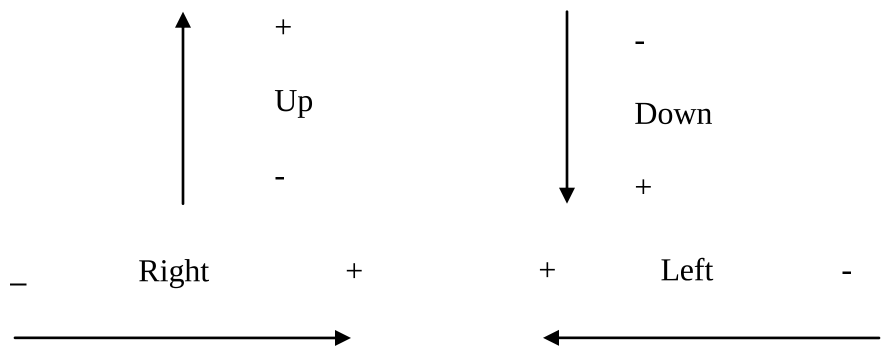
Velocity is the rate of change of displacement with respect to time. It is a **vector**.
SI unit is metre per second (ms^{-1}). Other metric units are cms^{-1} (cm/s) , kmh^{-1} (km/h).
Speed is the magnitude of the velocity. It is a **scalar**.
Uniform velocity is constant speed in a fixed direction.

Average velocity $= \dfrac{\text{change in displacement}}{\text{time taken}}$

Average speed $= \dfrac{\text{total distance travelled}}{\text{time taken}}$

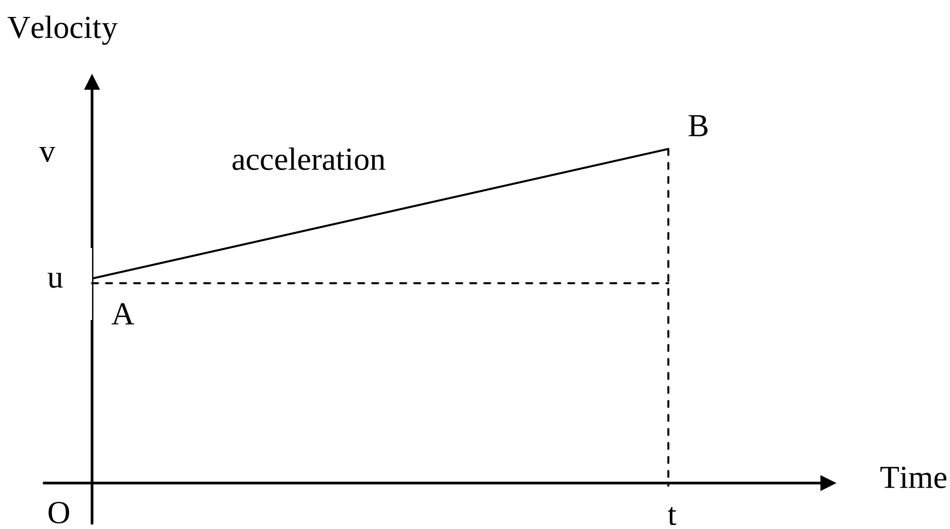
Acceleration is the rate of change of velocity with respect to time. It is a **vector**.
SI unit is metre per second squared (ms^{-2}). Other metric units are cm s^{-2} or cm/s^2, kmh^{-2} or km/h^2.
Negative acceleration is called **deceleration** or **retardation**.
Uniform acceleration is constant acceleration in a fixed direction.

NOTE:
When a particle moves in one dimension, along a straight line, it has only two possible directions in which to move. Positive and negative signs are used to identify the two directions.



Equation of Motion - Constant Acceleration

The notation used:
s = displacement, t = time, u = initial velocity, v = velocity at time t, a = acceleration



1. Acceleration = Gradient of AB

$$a = \frac{v-u}{t}, \qquad at = v - u, \qquad \boxed{v = u + at}$$

2. The area under the graph represents the displacement $\boxed{s = \frac{1}{2}(u+v)t}$

3. Substituting for v = u +at in equation (2)

$$s = \frac{1}{2}(u+u+at)t \qquad \boxed{s = ut + \frac{1}{2}at^2}$$

4. Substituting $t = \frac{v-u}{a}$ into (2)

$$s = \frac{1}{2}(u+v)(\frac{v-u}{a})$$

$$\boxed{v^2 = u^2 + 2as}$$

NOTE:

- These four equations are applicable *only* to a body traveling in a straight line with *constant acceleration*.
- These equations do not apply to acceleration which is not uniform. *Non-uniform* acceleration must be solved by *calculus*.

Constant Velocity (a = 0)

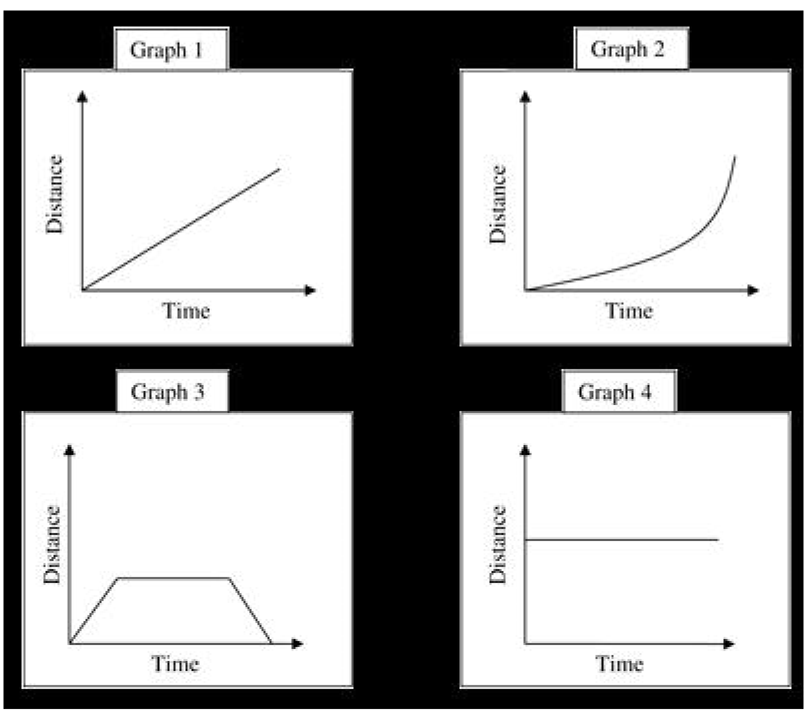
An object moving with constant velocity u in a particular direction for a time t makes a displacement s in that direction, where

$$\boxed{s = ut}$$

Graphs in Kinematics

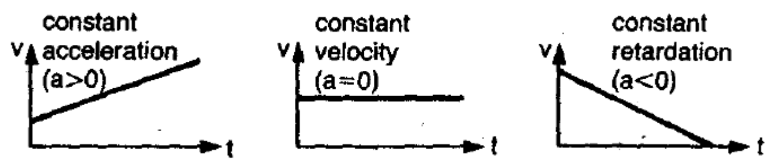
1. Displacement –time graph (s – t graph)

    - The **velocity** v of a body at the time t is given by the **gradient** of the s-t graph, since v = $\frac{ds}{dt}$.
    - The s-t graph for a body moving with **constant velocity** is a **straight line**.
    - The s-t graph for a body moving with **variable velocity** is a **curve**.



2. Velocity – time graph (v – t graph)

- The **acceleration** a of a body at time t is given by the **gradient** of the v-t graph, since a = $\frac{dv}{dt}$.
- The **displacement** s is in a time interval is given by the **area** under the v-t graph for that time interval.
- The v-t graph for a body moving with **uniform acceleration** is a **straight line**.



- The v-t graph for a body moving with variable acceleration is a curve.

Example 1

A cyclist reaches the top of a slope with a speed of 1.5 m s^{-1}, and accelerates at 2 m s^{-2}. The slope is 22 m long. How long does she take to reach the bottom of the slope, and how fast is she moving then?

Example 2

A police car accelerates from 15 m s^{-1} to 35 m s^{-1} in 5 seconds. The acceleration is constant. Illustrate this with a velocity–time graph. Use the equation $v = u + at$ to calculate the acceleration. Find also the distance travelled by the car in that time.

Example 3

A balloon at a height of 300 m is descending at 10 m s^{-1} and decelerating at a rate of 0.4 m s^{-2}. How long will it take for the balloon to stop descending, and what will its height be then?

Example 4

An ocean liner leaves the harbour entrance travelling at 3 m s^{-1}, and accelerates at 0.04 m s^{-2} until it reaches its cruising speed of 15 m s^{-1}.

(a) How far does it travel in accelerating to its cruising speed?

(b) How long does it take to travel 2 km from the harbour entrance?

Example 5

A downhill skier crosses the finishing line at a speed of 30 m s^{-1} and immediately starts to decelerate at 10 m s^{-2}. There is a barrier 50 metres beyond the finishing line.

(a) Find an expression for the skier's speed when she is s metres beyond the finishing line.

(b) How fast is she travelling when she is 40 metres beyond the finishing line?

(c) How far short of the barrier does she come to a stop?

(d) Display an (s, v) graph to illustrate the motion.

Example 6

A car travelling at 10 m s^{-1} is 25 metres from a pedestrian crossing when the traffic light changes from green to amber. The light remains at amber for 2 seconds before it changes to red. The driver has two choices: to accelerate so as to reach the crossing before the light changes to red, or to try to stop at the light. What is the least acceleration which would be necessary in the first case, and the least deceleration which would be necessary in the second?

Example 7

A cheetah is pursuing an impala. The impala is running in a straight line at a constant speed of 16 m s^{-1}. The cheetah is 10 m behind the impala, running at 20 m s^{-1} but tiring, so that it is decelerating at 1 m s^{-2}. Find an expression for the gap between the cheetah and the impala t seconds later. Will the impala get away?

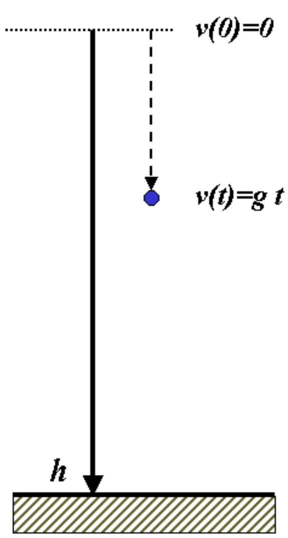
Example 8

A cyclist travels from A to B, a distance of 240 metres. He passes A at 12 m s^{-1}, maintains this speed for as long as he can, and then brakes so that he comes to a stop at B. If the maximum deceleration he can achieve when braking is 3 m s^{-2}, what is the least time in which he can get from A to B?

<u>Vertical Motion Under Gravity</u>

A free falling object is an object that is falling under the sole influence of gravity. Any object that is being acted upon only by the force of gravity is said to be in a state of free fall. There are two important motion characteristics that are true of free-falling objects:

- Free-falling objects do not encounter air resistance.
- All free-falling objects (on Earth) accelerate downwards approximated as g = 10 m/s.



1. displacement & velocity are positive(+) upwards & negative (-) downwards.

2. acceleration(g) <u>always</u> acts downwards and is therefore positive(+) downwards & negative (-) upwards.

3. acceleration due to gravity(g) is a constant

Example 1

A stone is thrown vertically upwards at 15 ms^{-1}.

(i) what is the maximum height attained?
(ii) how long is the stone in the air before hitting the ground?

Example 2

A boy throws a stone vertically down a well at 12 ms^{-1}. If he hears the stone hit the water 3 secs. later,

(i) how deep is the well?
(ii) what is the speed of the stone when it hits the water?

Motion with Variable Acceleration

- It is very important to appreciate that the standard relationships linking velocity, time, displacement and acceleration can be used only for *motion with constant acceleration.*
- There are many different types of motion in which the acceleration is not constant (**variable acceleration**).
- Calculus is used to find expressions for acceleration, velocity and displacement as a function of time.

Displacement and Velocity

Velocity is the rate of change of displacement with respect to time.

Velocity and Acceleration

Acceleration is the rate of change of velocity with respect to time.

For an object moving in a straight line, if x denotes the displacement from a fixed point O of the line at time t, v denotes the velocity and a the acceleration, Using symbols:

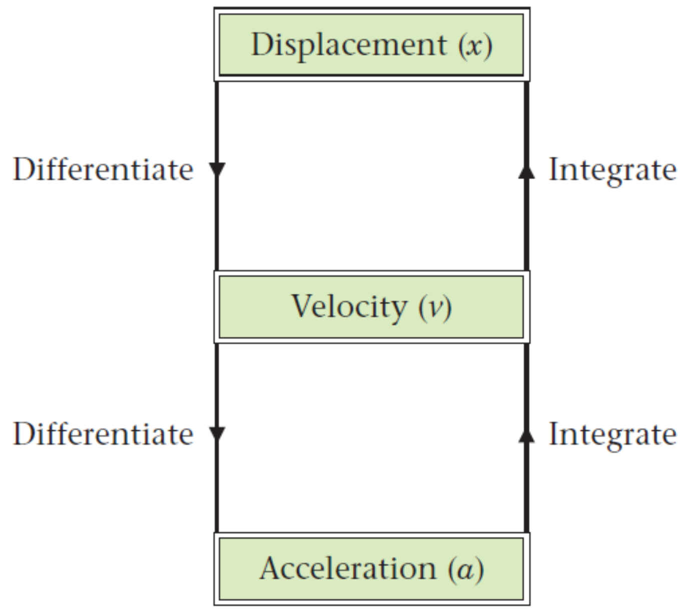
$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = \int a\,dt$$

$$x = \int v\,dt$$

This particle is moving in a straight line with acceleration a, displacement x and velocity v. The relationship between these three variables is shown in the diagram below.



- **Velocity is the rate of change of displacement with time. To find the velocity from the displacement, you differentiate with respect to time.**
- **Acceleration is the rate of change of velocity with time. To find the acceleration from the velocity you differentiate with respect to time.**
- **To obtain the velocity from the acceleration, you integrate with respect to time.**
- **To obtain the displacement from the velocity, you integrate with respect to time.**

When you integrate, it is important that you remember to include a constant of integration. The constant of integration often represents the initial displacement or initial velocity of a particle. Many questions include information which enables you to find the value of this constant.

For an object moving in a straight line, with the velocity v given as a function of the time t, the displacement between times t_1 and t_2 is given by

$$\boxed{\int_{t_2}^{t_1} v\,dt}$$

This displacement is represented by the area under the v-t graph for the interval $t_1 \le t \le t_2$.

Example 1

A particle P is moving on the x-axis. At time t seconds, the displacement x metres from O is given by $x = t^4 - 32t + 12$. Find

a the speed of P when $t = 3$,

b the value of t for which P is instantaneously at rest,

c the magnitude of the acceleration of P when $t = 1.5$.

a) 76 b) 2 c) 27

Example 2

A particle is moving on the x-axis. At time $t = 0$, the particle is at the point where $x = 5$. The velocity of the particle at time t seconds (where $t \geqslant 0$) is $(6t - t^2)\,\mathrm{m\,s^{-1}}$. Find

a the acceleration of the particle when $t = 2$,

b the acceleration of the particle when $t = 4$,

c an expression for the displacement of the particle from O at time t seconds,

d the distance of the particle from its starting point when $t = 6$.

a) 2 b) 2 c) $3t^2-(1/3)t^3 + 5$ d) 36

Example 3

A particle P is moving along a straight line. At time $t = 0$, the particle is at a point A and is moving with velocity $8\,\mathrm{m\,s^{-1}}$ towards a point B on the line, where $AB = 30\,\mathrm{m}$. At time t seconds (where $t \geqslant 0$), the acceleration of P is $(2 - 2t)\,\mathrm{m\,s^{-2}}$ in the direction $\overrightarrow{AB}$.

a Find an expression, in terms of t, for the displacement of P from A at time t seconds.

b Show that P does not reach B.

c Find the value of t when P returns to A, giving your answer to 3 significant figures.

d Find the total distance travelled by P in the interval between the two instants when it passes through A.

a) $8t + t^2-(1/3)t^3$ c) 6.62 d) 53 1/3