Topic 8 – Complex Numbers

Exercise 1: Operations on Complex Numbers

1 Simplify: i^7, i^{-3}, i^9, i^{-5}, i^{4n}, i^{4n+1}.

2 Add the following pairs of complex numbers:

(a) $3 + 5i$ and $7 - i$ (b) $4 - i$ and $3 + 3i$

(c) $2 + 7i$ and $4 - 9i$ (d) $a - bi$ and $c + di$

3 Subtract the second number from the first in each part in question **2**.

4 Simplify:

(a) $(2 + i)(3 - 4i)$ (b) $(5 + 4i)(7 - i)$

(c) $(3 - i)(4 - i)$ (d) $(3 + 4i)(3 - 4i)$

(e) $(2 - i)^2$ (f) $(1 + i)^3$

(g) $i(3 + 4i)$ (h) $(x + iy)(x - iy)$

(i) $i(1 + i)(2 + i)$ (j) $(a + bi)^2$

5 Express each of the following fractions in the form $a + bi$.

(a) $\dfrac{2}{1 - i}$ (b) $\dfrac{3 + i}{4 - 3i}$

(c) $\dfrac{4i}{4 + i}$ (d) $\dfrac{1 + i}{1 - i}$

(e) $\dfrac{7 - i}{1 + 7i}$ (f) $\dfrac{x + iy}{x - iy}$

(g) $\dfrac{3 + i}{i}$ (h) $\dfrac{-2 + 3i}{-i}$

6 Solve the following equations for x and y.

(a) $x + iy = (3 + i)(2 - 3i)$

(b) $\dfrac{2 + 5i}{1 - i} = x + iy$

(c) $3 + 4i = (x + iy)(1 + i)$

(d) $x + iy = 2$

(e) $x + iy = (3 + 2i)(3 - 2i)$

(f) $x + iy = (4 + i)^2$

(g) $\dfrac{x + iy}{2 + i} = 5 - i$

(h) $(x + iy)^2 = 3 + 4i$

7 Find the real and imaginary parts of:

(a) $(2 - i)(3 + i)$ (b) $(1 + i)^3$

(c) $\dfrac{3 + 2i}{4 - i}$ (d) $\dfrac{2}{3 + i} + \dfrac{3}{2 + i}$

(e) $\dfrac{1}{x + iy} - \dfrac{1}{x - iy}$

(f) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3$

(g) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^2$

Exercise 2: Square Roots of a Complex Number

Find the square roots of each complex number.

1 $3 - 4i$

2 $2i$

3 $3 + 4i$

4 $-1 + 2\sqrt{6}i$

5 $4 - 4i$

Exercise 3: Complex Roots of an Equation

1 Solve the following equations.

(a) $x^2 + x + 1 = 0$

(b) $2x^2 + 7x + 1 = 0$

(c) $x^2 + 9 = 0$

(d) $x^2 + x + 3 = 0$

(e) $x^4 - 1 = 0$

2 Form the equation whose roots are

(a) $i, -i$ (b) $2 + i, 2 - i$

(c) $1 - 3i, 1 + 3i$ (d) $1 + i, 1 - i, 2$

3 The complex number $2 - i$ is one root of the equation $x^3 - 5x^2 + ax - 5$

Find the value of a.

Exercise 4:The Polar Coordinate Form

1 Represent the following complex numbers by lines on Argand diagrams.

Find the modulus and argument of each complex number.

(a) $3 - 2i$ (b) $-4 + i$

(c) $-3 - 4i$ (d) $5 + 12i$

(e) $1 - i$ (f) $-1 + i$

(g) 4 (h) $-2i$

(i) $a + bi$ (j) $1 + i$

(k) $i(1 + i)$ (l) $i^2(1 + i)$

(m) $i^3(1 + i)$ (n) $(3 + i)(4 + i)$

(o) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(p) $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

(q) $3\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]$

2 $z_1 = 3 - i$, $z_2 = 1 + 4i$, $z_3 = -4 + i$, $z_4 = -2 - 5i$. Represent the following by lines on Argand diagrams, showing the direction of each line by an arrow.

(a) $z_1 + z_2$ (b) $z_2 - z_3$

(c) $z_1 - z_3$ (d) $z_2 + z_4$

(e) $z_4 - z_1$ (f) $z_3 - z_4$

(g) z_1 (h) z_4

(i) $z_2 - z_1$ (j) $z_1 + z_3$

3 Express in the form $r(\cos\theta + i\sin\theta)$:

(a) $1 + i$ (b) $\sqrt{3} - i$

(c) $-3 - 4i$ (d) $-5 + 12i$

(e) $2 - i$ (f) 6

(g) -3 (h) $4i$

(i) $-3 - i\sqrt{3}$ (j) $24 + 7i$

4 Express in the form $x + iy$ the complex number:

(a) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(b) $3\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

(c) $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

(d) $\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$

(e) 3

(f) $2(\cos\pi + i\sin\pi)$

(g) $4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

(h) $\cos\pi + i\sin\pi$

(i) $3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

(j) $\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$

5 By using $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ show on an Argand diagram the position of the point representing:

(a) $\frac{1}{2}(z_1 + z_2)$ (b) $\frac{1}{3}(2z_1 + z_2)$

Exercise 5: The Exponential Form

1 Find the modulus and argument of:

(a) $2(1 + i)$ (b) $(3 - i\sqrt{3})(1 - i)$

(c) $\dfrac{-2 - i\sqrt{3}}{i\sqrt{3} - 2}$

2 Given that $z = re^{i\theta}$ show that $z^2 = r^2e^{i2\theta}$

Hence find the square roots of $2\sqrt{3} - 2i$.

Let $z^2 = 2\sqrt{3} - 2i$

3 Given that $z = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, find the modulus and argument of z^3.

4 Convert $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z_2 = 3\frac{\sqrt{3}}{2} + \frac{3}{2}i$ into polar form.

Hence or otherwise illustrate z_1z_2 and $\dfrac{z_1}{z_2}$ on an Argand diagram.

Exercise 6: Loci

Sketch the locus on an Argand diagram of points for which

1 $|z| = 1$

2 $|z - 1| = 3$

3 $|z - 2i| = 3$

4 $|z + 2| = 2$

5 $|z - 1 + i| = 4$

6 $\arg z = \frac{\pi}{3}$

7 $|z - 2 - 3i| = |z + 4 - 5i|$

8 $|z| = |z + 4i|$

9 Shade on an Argand diagram the areas represented by the following inequalities.

(a) $|z - 1| < 4$ (b) $|z + 3i| > 2$

(c) $|z + 1 - i| < 1$ (d) $\frac{\pi}{3} < \arg z < \frac{2\pi}{3}$

10 Shade on an Argand diagram the region occupied by the set of points $P(x, y)$ for which $|z| < 5$ *and* $-\frac{\pi}{6} < \arg z < \frac{\pi}{6}$, where $z = x + iy$

11 Show, on an Argand diagram, the set of points for which

(a) $|z| = 4$ and $\arg z = \frac{\pi}{4}$

(b) $|z + 2 + i| = 5$ and $\mathrm{Re}(z - 1) = 0$

12 Find the complex number represented by the points of intersection of the loci on an Argand diagram defined by

$\arg z = -\frac{\pi}{4}$ and $|z| = 2$

13 Indicate on an Argand diagram the set of points $P(x, y)$ for which

(a) $0 \leqslant \arg(z + 1) \leqslant \frac{\pi}{3}$ and $|z + i| = 3$

(b) $|z + 3 - 2i| < 4$ and $\arg(z + 1) = \frac{5\pi}{6}$

(c) $|z| > 1, |z| < 4$ and $\arg z = -\frac{3\pi}{4}$

Exercise 7: Mixed Exercise

1 If $z_1 = \frac{2 - i}{2 + i}$, $z_2 = \frac{2i - 1}{1 - i}$, express z_1 and z_2 in the form $a + ib$.

Sketch an Argand diagram showing points P and Q representing the complex numbers $5z_1 + 2z_2$ and $5z_1 - 2z_2$ respectively.

2 If $(1 + 3i)z_1 = 5(1 + i)$, express z_1 and z_1^2 in the form $x + iy$, where x and y are real.

Sketch in an Argand diagram the circle $|z - z_1| = |z_1|$, giving the coordinates of its centre.

3 (a) If $z = 4 - 3i$ express $z + \frac{1}{z}$ in the form $a + ib$.

(b) Find the two square roots of $4i$ in the form $a + ib$.

(c) If $z_1 = 5 - 5i$ and $z_2 = -1 + 7i$ prove that:

$$|z_1 + z_2| < |z_1 - z_2| < |z_1| + |z_2|$$

4 Express the complex number $\frac{5 + 12i}{3 + 4i}$ in the form $a + ib$ and in the form $r(\cos\theta + i\sin\theta)$, giving the values of a, b, r, $\cos\theta$, $\sin\theta$.

5 The complex numbers $z_1 = \frac{a}{1 + i}$, $z_2 = \frac{b}{1 + 2i}$ where a and b are real, are such that $z_1 + z_2 = 1$. Find a and b.

With these values of a and b, find the distance between the points that represent z_1 and z_2 in the Argand diagram.

6 Find the modulus and argument of $z_1 = \sqrt{3} + i$. If $z_2 = \sqrt{3} - i$ express $q = \frac{z_1}{z_2}$ in the form $a + ib$ where a and b are real.

7 (a) The complex number z and its conjugate z^* satisfy the equation

$$zz^* + 2iz = 12 + 6i$$

Find the values of z.

(b) Mark on an Argand diagram the points representing $4 + 3i$, $4 - 3i$ and $\frac{4 + 3i}{4 - 3i}$

8 (a) If $z = 3 + 4\mathrm{i}$, express $z + \frac{25}{z}$ in its simplest form.

(b) If $z = x + \mathrm{i}y$, find the real part and the imaginary part of $z + \frac{1}{z}$.

Find the locus of points in the Argand diagram for which the imaginary part of $z + \frac{1}{z}$ is zero.

9 (a) Find the square roots of $(5 + 12\mathrm{i})$.

(b) Find the modulus and argument of each of the numbers

(i) $(1 - \mathrm{i})$

(ii) $(4 + 3\mathrm{i})$

(iii) $(1 - \mathrm{i})(4 + 3\mathrm{i})$.

If these numbers are represented in an Argand diagram by the points A, B, C, calculate the area of the triangle ABC.

(c) Find the ratio of the greatest value of $|z + 1|$ to its least value when $|z - \mathrm{i}| = 1$

10 (a) Find the modulus and one value for the argument of $\frac{(\mathrm{i} + 1)^2}{(\mathrm{i} - 1)^4}$

(b) Find the two square roots of $5 - 12\mathrm{i}$ in the form $a + \mathrm{i}b$ where a and b are real. Show the points P and Q representing the square roots in an Argand diagram. Find the complex numbers represented by points R_1, R_2 such that the triangles PQR_1, PQR_2 are equilateral.

11 Prove that the modulus of $2 + \cos\theta + \mathrm{i}\sin\theta$ is $(5 + 4\cos\theta)^{\frac{1}{2}}$

Hence show that the modulus of

$$\frac{2 + \cos\theta + \mathrm{i}\sin\theta}{2 + \cos\theta - \mathrm{i}\sin\theta}$$

is unity.

Answers

Exercise 1

1 $-\mathrm{i}, \mathrm{i}, \mathrm{i}, -\mathrm{i}, 1, \mathrm{i}$

2 (a) $10 + 4\mathrm{i}$ (b) $7 + 2\mathrm{i}$
(c) $6 - 2\mathrm{i}$ (d) $(a + c) + (d - b)\mathrm{i}$

3 (a) $-4 + 6\mathrm{i}$ (b) $1 - 4\mathrm{i}$
(c) $-2 + 16\mathrm{i}$ (d) $(a - c) - (b + d)\mathrm{i}$

4 (a) $10 - 5\mathrm{i}$ (b) $39 + 23\mathrm{i}$
(c) $11 - 7\mathrm{i}$ (d) 25
(e) $3 - 4\mathrm{i}$ (f) $-2 + 2\mathrm{i}$
(g) $-4 + 3\mathrm{i}$ (h) $x^2 + y^2$
(i) $-3 + \mathrm{i}$ (j) $(a^2 - b^2) + 2ab\mathrm{i}$

5 (a) $1 + \mathrm{i}$ (b) $\frac{9}{25} + \frac{13}{25}\mathrm{i}$
(c) $\frac{4}{17} + \frac{16}{17}\mathrm{i}$ (d) i
(e) $-\mathrm{i}$ (f) $\frac{x^2 - y^2}{x^2 + y^2} + \frac{2xy}{x^2 + y^2}\mathrm{i}$
(g) $1 - 3\mathrm{i}$ (h) $-3 - 2\mathrm{i}$

6 (a) $x = 9, y = -7$
(b) $x = -\frac{3}{2}, y = \frac{7}{2}$
(c) $x = \frac{7}{2}, y = \frac{1}{2}$
(d) $x = 2, y = 0$
(e) $x = 13, y = 0$
(f) $x = 15, y = 8$
(g) $x = 11, y = 3$
(h) $x = 2, y = 1$ or $x = -2, y = -1$

7 (a) $7, -1$ (b) $-2, 2$
(c) $\frac{10}{17}, \frac{11}{17}$ (d) $\frac{9}{5}, -\frac{4}{5}$
(e) $0, \frac{-2y}{x^2 + y^2}$ (f) $-1, 0$
(g) $-\frac{1}{2}, \frac{1}{2}\sqrt{3}$

Exercise 2

1 $2 - \mathrm{i}$ and $-2 + \mathrm{i}$

2 $1 + \mathrm{i}$ and $-(1 + \mathrm{i})$

3 $2 + \mathrm{i}$ and $-2 - \mathrm{i}$

4 $\sqrt{2} + \mathrm{i}\sqrt{3}$ and $-(\sqrt{2} + \mathrm{i}\sqrt{3})$

5 $2.20 - 0.910\mathrm{i}$ and $-2.20 + 0.910\mathrm{i}$

Exercise 3

1 (a) $-\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\mathrm{i}$ (b) $-\frac{7}{4} \pm \frac{1}{4}\sqrt{41}$
(c) $\pm 3\mathrm{i}$ (d) $-\frac{1}{2} \pm \frac{1}{2}\sqrt{11}\mathrm{i}$
(e) $\pm 1, \pm \mathrm{i}$

2 (a) $x^2 + 1 = 0$
(b) $x^2 - 4x + 5 = 0$
(c) $x^2 - 2x + 10 = 0$
(d) $x^3 - 4x^2 + 6x - 4 = 0$

3 $a = 9$

Exercise 4

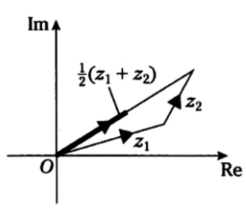
1 (a) $\sqrt{13}, -0.59$ (b) $\sqrt{17}, -0.24$
(c) $5, -2.21$ (d) $13, 1.18$
(e) $\sqrt{2}, -\frac{1}{4}\pi$ (f) $\sqrt{2}, \frac{3}{4}\pi$
(g) $4, 0$ (h) $2, -\frac{1}{2}\pi$
(i) $\sqrt{a^2+b^2}, \tan^{-1}\dfrac{b}{a}$
(j) $\sqrt{2}, \frac{1}{4}\pi$ (k) $\sqrt{2}, \frac{3}{4}\pi$
(l) $\sqrt{2}, -\frac{3}{4}\pi$ (m) $\sqrt{2}, -\frac{1}{4}\pi$
(n) $\sqrt{170}, 0.57$ (o) $2, \frac{1}{3}\pi$
(p) $1, \frac{3}{4}\pi$ (q) $3, -\frac{5}{6}\pi$

2 (a) $4+3\mathrm{i}$ (b) $5+3\mathrm{i}$
(c) $7-2\mathrm{i}$ (d) $-1-\mathrm{i}$
(e) $-5-4\mathrm{i}$ (f) $-2+6\mathrm{i}$
(i) $-2+5\mathrm{i}$ (j) -1

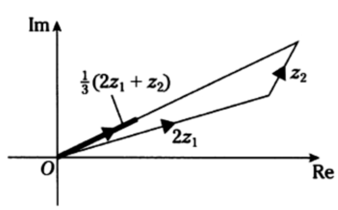
3 (a) $\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$
(b) $2\left\{\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right\}$
(c) $5\{\cos(-2.214^c) + \mathrm{i}\sin(-2.214^c)\}$
(d) $13(\cos 1.966^c + \mathrm{i}\sin 1.966^c)$
(e) $\sqrt{5}\{\cos(-0.464^c) + \mathrm{i}\sin(-0.464^c)\}$
(f) $6(\cos 0 + \mathrm{i}\sin 0)$
(g) $3(\cos\pi + \mathrm{i}\sin\pi)$
(h) $4\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$
(i) $2\sqrt{3}\left\{\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right\}$
(j) $25(\cos 0.284^c + \mathrm{i}\sin 0.284^c)$

4 (a) $\sqrt{3}+\mathrm{i}$
(b) $\frac{3}{2}\sqrt{2} - \frac{3}{2}\sqrt{2}\mathrm{i}$
(c) $-\frac{1}{2} + \frac{1}{2}\sqrt{3}\mathrm{i}$
(d) $-\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\mathrm{i}$
(e) 3
(f) -2
(g) $2\sqrt{3} - 2\mathrm{i}$
(h) -1
(i) $-3\mathrm{i}$
(j) $-\dfrac{1}{2} - \dfrac{\mathrm{i}\sqrt{3}}{2}$

5 (a)



(b)



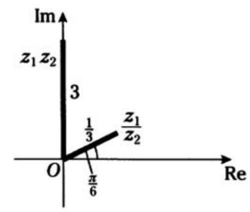
Exercise 5

1 (a) $2\sqrt{2}, \frac{1}{4}\pi$
(b) $2\sqrt{6}, -\frac{5}{12}\pi$
(c) $1, 2\tan^{-1}\sqrt{\frac{3}{2}}$

2 $\pm 2\left\{\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right\}$

3 $9, \dfrac{3\pi}{4}$

4 $z_1 = \mathrm{e}^{\frac{\mathrm{i}\pi}{3}}, z_2 = 3\mathrm{e}^{\frac{\mathrm{i}\pi}{6}},$



Exercise 6

1 Circle, centre O, radius 1
2 Circle, centre $(1, 0)$, radius 3
3 Circle, centre $(0, 2)$, radius 3
4 Circle, centre $(0, -2)$, radius 2
5 Circle, centre $(1, -1)$, radius 4
6 Straight line through O at $\frac{\pi}{3}$ to Ox
7 The perpendicular bisector of the line joining $(2, 3)$ to $(-4, 5)$
8 The perpendicular bisector of the line joining O to $(0, -4)$
9 (a) Inside the circle centre $(1, 0)$, radius 4
(b) Outside the circle centre $(0, -3)$, radius 2
(c) Inside the circle centre $(-1, 1)$, radius 1
(d) Between the lines through O inclined at $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

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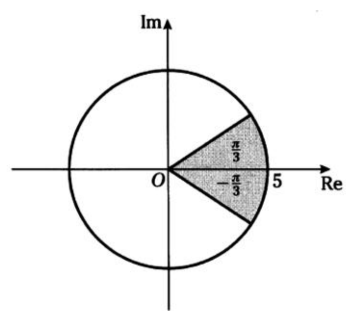
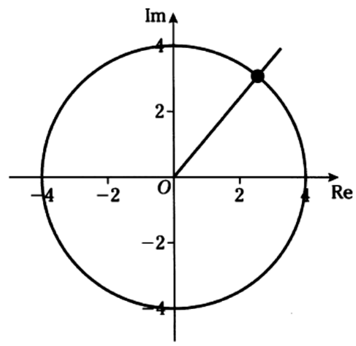
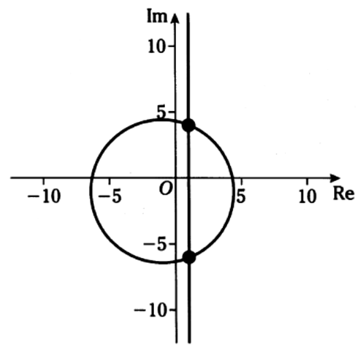


Exercise 6

11 (a)

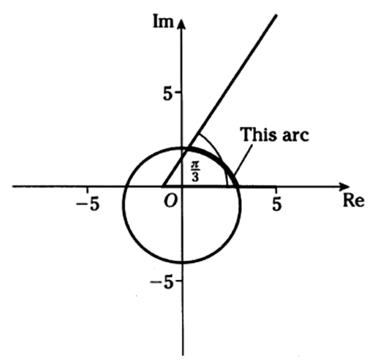


(b)



12 $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

13 (a)



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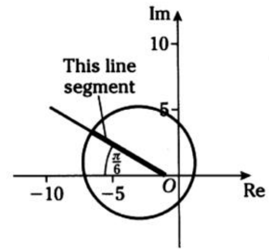

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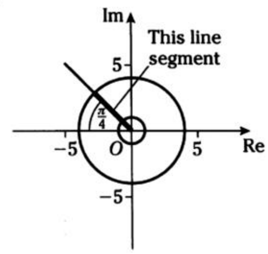


Exercise 7

1 $\frac{1}{5}(3-4i), \frac{1}{2}(-3+i)$
Diagram showing the points $-3i$ and $6-5i$

2 $2-i, 3-4i$, a circle centre $(2-i)$, radius $\sqrt{5}$

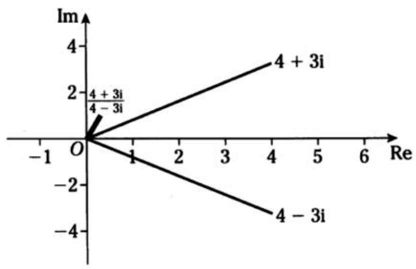
3 (a) $\frac{104}{25} - \frac{72}{25}i$
(b) $\pm\sqrt{2}(1+i)$

4 $a = \frac{63}{25},\ b = \frac{16}{25},\ r = \frac{13}{5},\ \cos\theta = \frac{63}{65},\ \sin\theta = \frac{16}{65}$

5 4, −5; 5

6 2, $\frac{1}{6}\pi$, $\frac{1}{2}(1 + i\sqrt{3})$

7 (a) $3-i, 3+3i$
(b)



8 (a) 6
(b) $\frac{x(x^2+y^2+1)}{x^2+y^2}, \frac{y(x^2+y^2-1)}{x^2+y^2}$;
The points $(x+iy)$ where $y(x^2+y^2-1) = 0$

9 (a) $\pm(3+2i)$
(b) (i) $\sqrt{2}, -\frac{1}{4}\pi$
(ii) 5, 0.643^c
(iii) $5\sqrt{2}, -0.142^c$; 12
(c) $\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)}$

10 (a) $\frac{1}{2}, -\frac{1}{2}\pi$
(b) $3-2i, -3+2i$;
Diagram showing $\pm(2+3i)\sqrt{3}$