TOPIC 8 – DIFFERENTIATION

Learning Outcome:

- understand the idea of the gradient of a curve, and use the notations f'(x), f''(x), $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (the technique of differentiation from first principles is not required);
- use the derivative of x^n (for any rational n), together with constant multiples, sums, differences of functions, and of composite functions using the chain rule;
- apply differentiation to gradients, tangents and normals, increasing and decreasing functions and rates of change (including connected rates of change);
- locate stationary points, and use information about stationary points in sketching graphs (the ability to distinguish between maximum points and minimum points is required, but identification of points of inflexion is not included).

FORMULA:

1. $\frac{d}{dx}(k) = 0$ where k is a constant
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
4. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\frac{d}{dx}[f(x)]$
5. $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ $\dashrightarrow$ The Chain Rule

Exercise 6D (page 86)

1 Differentiate the following functions.

(a) x^3+2x^2 (b) $1-2x^3+3x^2$ (c) $x^3-6x^2+11x-6$

(d) $2x^3-3x^2+x$ (e) $2x^2(1-3x^2)$ (f) $(1-x)(1+x+x^2)$

2 Find $f'(-2)$ for each of the following functions $f(x)$.

(a) $2x-x^3$ (b) $2x-x^2$ (c) $1-2x-3x^2+4x^3$

(d) $2-x$ (e) $x^2(1+x)$ (f) $(1+x)(1-x+x^2)$

3 For each of the following functions $f(x)$ find the value(s) of x such that $f'(x)$ is equal to the given number.

(a) x^3 12 (b) x^3-x^2 8 (c) $3x-3x^2+x^3$ 108

(d) x^3-3x^2+2x -1 (e) $x(1+x)^2$ 0 (f) $x(1-x)(1+x)$ 2

4 Differentiate the following functions.

(a) $2\sqrt{x}$ (b) $\left(1+\sqrt{x}\right)^2$ (c) $y=x-\frac{1}{2}\sqrt{x}$ (d) $x\left(1-\frac{1}{\sqrt{x}}\right)^2$

(e) $x-\frac{1}{x}$ (f) $\frac{x^3+x^2+1}{x}$ (g) $\frac{(x+1)(x+2)}{x}$ (h) $\left(\frac{\sqrt{x}+x}{\sqrt{x}}\right)^2$

14 Differentiate each of these functions $f(x)$. Give your answers $f'(x)$ in a similar form, without negative or fractional indices.

(a) $\frac{1}{4x}$ (b) $\frac{3}{x^2}$ (c) x^0 (d) $\sqrt[4]{x^3}$

(e) $6\sqrt[3]{x}$ (f) $\frac{4}{\sqrt{x}}$ (g) $\frac{3}{x}+\frac{1}{3x^3}$ (h) $\sqrt{16x^5}$

(i) $x\sqrt{x}$ (j) $\frac{1}{\sqrt[3]{8x}}$ (k) $\frac{x-2}{x^2}$ (l) $\frac{1+x}{\sqrt[4]{x}}$

Exercise 12C (page 179)

4 Use the substitution $u = 2x^2 + 3$ to differentiate the following with respect to x.

(a) $y = (2x^2 + 3)^6$ (b) $y = \dfrac{1}{2x^2 + 3}$ (c) $y = \dfrac{1}{\sqrt{2x^2 + 3}}$

5 Differentiate $y = (3x^4 + 2)^2$ with respect to x by using the chain rule. Confirm your answer by expanding $(3x^4 + 2)^2$ and then differentiating.

6 Differentiate $y = (2x^3 + 1)^3$ with respect to x

(a) by using the binomial theorem to expand $y = (2x^3 + 1)^3$ and then differentiating term by term,

(b) by using the chain rule.

Check that your answers are the same.

16 Use the substitutions $u = x^2 - 1$ and $v = \sqrt{u} + 1$ with the chain rule in the form $\dfrac{dy}{dx} = \dfrac{dy}{dv} \times \dfrac{dv}{du} \times \dfrac{du}{dx}$ to differentiate $y = \left(\sqrt{x^2 - 1} + 1\right)^6$.

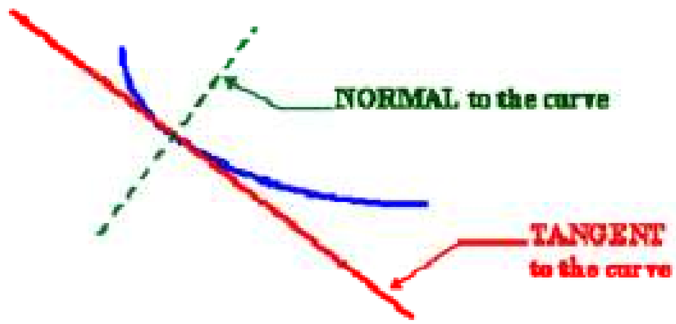
17 Use two substitutions to find $\dfrac{d}{dx}\left(\sqrt{1 + \sqrt{4x + 3}}\right)$.

Application of Differentiation

1. Slope, Tangent and Normal

A **tangent** to a curve is a line that touches the curve at one point and has the same **slope** as the curve at that point.

A **normal** to a curve is a line perpendicular to a tangent to the curve.



Exercise - Slope, Tangent and Normal

Question 1

Find the gradient of the curve $y = 2x + \frac{1}{x}$ at the point (1, 3).

Question 2

Given that the curve $y = ax^2 + \frac{b}{x}$ has gradient 4 at the point (1, 5), calculate the value of a and of b.

Question 3

Find the equations of the tangent and the normal to the curve

(a) $y = 2x^2 - 3x + 1$ at the point (2, 3),

(b) $y = x^3 + 3x^2$ at the point where $x = -1$,

(c) $y = x + \frac{2}{x}$ at the point where $x = 1$.

Question 4

A curve has equation $y = \frac{4}{\sqrt{x}}$.

(i) The normal to the curve at the point (4, 2) meets the x-axis at P and the y-axis at Q. Find the length of PQ, correct to 3 significant figures. [6]

<u>Exercise 6D (page 86)</u>

6 One of the tangents to the curve with equation $y = 4x - x^3$ is the line with equation $y = x - 2$. Find the equation of the other tangent parallel to $y = x - 2$.

10 The graphs of $y = x^2 - 2x$ and $y = x^3 - 3x^2 - 2x$ both pass through the origin. Show that they share the same tangent at the origin.

15 Find the equations of the tangent and the normal to $y = \sqrt[3]{x^2}$ at the point $(8,4)$.

16 The tangent to the curve with equation $y = \frac{1}{x^2}$ at the point $\left(\frac{1}{2},4\right)$ meets the axes at P and Q. Find the coordinates of P and Q.

<u>Exercise 12C (page 179)</u>

13 Find the equation of the tangent to the curve $y = \left(x^2 - 5\right)^3$ at the point $(2,-1)$.

14 Find the equation of the tangent to the curve $y = \frac{1}{\sqrt{x-1}}$ at the point $(4,1)$.

15 Find the equation of the normal to the curve $y = \frac{8}{1-x^3}$ at the point $(-1,4)$.

18 A curve has equation $y = \left(x^2+1\right)^4 + 2\left(x^2+1\right)^3$. Show that $\frac{dy}{dx} = 4x\left(x^2+1\right)^2\left(2x^2+5\right)$ and hence show that the curve has just one stationary point. State the coordinates of the stationary point and, by considering the gradient of the curve on either side of the stationary point, determine its nature.

2. Increasing and Decreasing Functions

- A function, f(x) is increasing when both f(x) and x is increasing and $\frac{dy}{dx} > 0$ for a set of values.
- A function, f(x) is decreasing when x is increasing and f(x) is decreasing and $\frac{dy}{dx} < 0$ for a set of values.

Exercise 7D (page 103)

4 For each of the following functions $f(x)$, find $f'(x)$ and any intervals in which $f(x)$ is decreasing. In part (i), n is an integer.

(a) $x^3 - 27x$ for $x \geqslant 0$
(b) $x^4 + 4x^2 - 5$ for $x \geqslant 0$
(c) $x^3 - 3x^2 + 3x - 1$
(d) $12x - 2x^3$
(e) $2x^3 + 3x^2 - 36x - 7$
(f) $3x^4 - 20x^3 + 12$
(g) $36x^2 - 2x^4$
(h) $x^5 - 5x$
(i) $x^n - nx \quad (n > 1)$

5 For each of the following functions $f(x)$, find $f'(x)$, the intervals in which $f(x)$ is decreasing, and the intervals in which $f(x)$ is increasing.

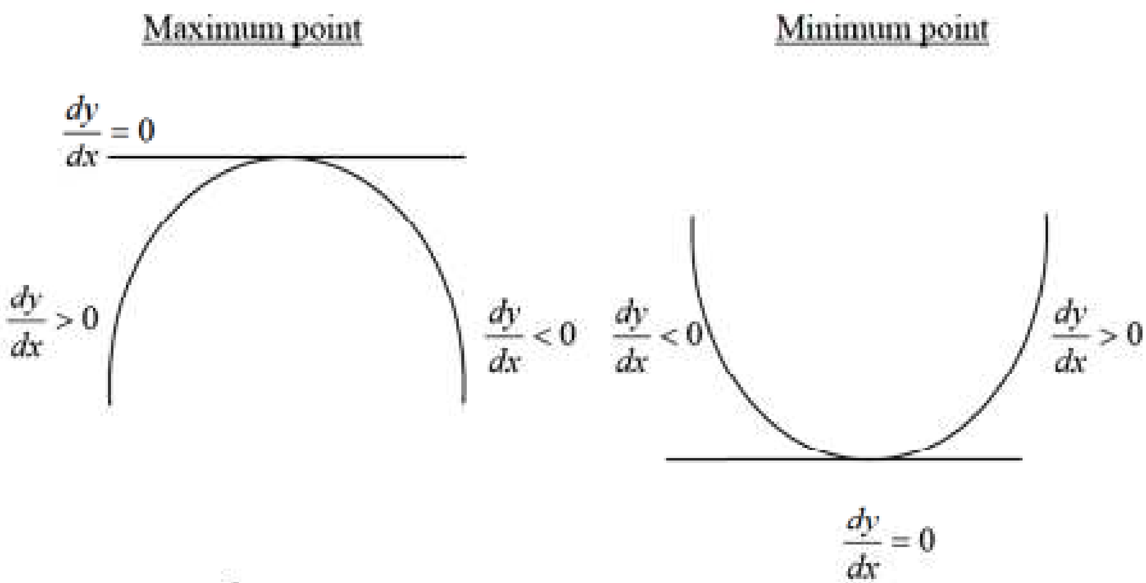
(a) $x^{\frac{3}{2}}(x-1)$, for $x > 0$
(b) $x^{\frac{3}{4}} - 2x^{\frac{7}{4}}$, for $x > 0$
(c) $x^{\frac{3}{5}}(x+2)$
(d) $x^{\frac{1}{3}}\left(x^2 - 13\right)$
(e) $x + \frac{3}{x}$, for $x \neq 0$
(f) $\sqrt{x} + \frac{1}{\sqrt{x}}$, for $x > 0$

3. Stationary Points

At stationary point on a curve, the gradient is zero.

To find the nature of the stationary point(s)/ to distinguish between the point(s).

Method 1 – work out the sign of the gradient on either side of the stationary point(s).



Method 2 – Find $\frac{d^2y}{dx^2}$ and substitute the stationary point(s):

- $\frac{d^2y}{dx^2} > 0$ $\longrightarrow$ Minimum point
- $\frac{d^2y}{dx^2} < 0$ $\longrightarrow$ Maximum point
- $\frac{d^2y}{dx^2} = 0$ method 2 does not work, use method 1.

<u>Exercise – Stationary Points</u>

1. Use differentiation to find the coordinates of the stationary points on the curve $y = x + \frac{4}{x}$ and determine whether each stationary point is a maximum point or a minimum point. Find the set of value of *x* for which *y* increases as *x* increases.

2. Find the coordinates of maximum and minimum points on the graph of

   i) $y = x^3 - 6x^2 + 9x + 6$ ii) $y = 16x + \frac{1}{x^2}$ iii) $y = \sqrt{x} + \sqrt{30 - 5x}$

Question 3

The equation of a curve is $y = (2x - 3)^3 - 6x$.

(i) Express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x. **[3]**

(ii) Find the x-coordinates of the two stationary points and determine the nature of each stationary point. **[5]**

<u>Exercise - Maxima and Minima</u>

1. If the perimeter of a rectangle is to be 80 m, calculate the maximum area of the rectangle. *[400m^2]*

2. Find the minimum value of $x^2 + 2y^2$ if *x* and *y* are connected by the relation $x + 2y = 1$. *[1/3]*

Question 3

A piece of wire of length *8 cm* is cut into two pieces, one of length *x cm*, the other of length *(8-x) cm*. the piece of length *x cm* is bent to form a circle with circumference *x cm*. The other piece is bent to form a square with perimeter *(8-x) cm*. As x varies, find the minimum total area enclosed by these two pieces of wire.

Question 4

A company that manufactures dog food wishes to pack the food in closed cylindrical tins. What should be the dimensions of each tin if each is to have a volume of 128π cm^3 and the minimum possible surface area?

[h = 8, r = 4, S = 96π]

Question 5

A hollow circular cylinder, open at one end, is constructed of thin sheet metal. The total external surface area of the cylinder is 192π cm^2. The cylinder has a radius of r cm and a height of h cm.

(i) Express h in terms of r and show that the volume, V cm^3, of the cylinder is given by

$$V = \tfrac{1}{2}\pi(192r - r^3).$$ [4]

Given that r can vary,

(ii) find the value of r for which V has a stationary value, [3]

(iii) find this stationary value and determine whether it is a maximum or a minimum. [3]

[i) r = 8, ii) V = 1610, max.]

4. Sketching Curves

- Find stationary points
- Determine the nature of the stationary points
- Determine the y-intercept(s) and if possible the x-intercept(s).

Exercise- Curve Sketching

1. Find the coordinates of the stationary points on the graph $y = x^3 - 12x - 12$ and sketch the graph.

2. Find the coordinates of the stationary points on the graph $y = x^3 + x^2$. Sketch the graph and hence write down the set of values of the constant *k* for which the equation $x^3 + x^2 = k$ has three distinct real roots.

5. Rates of change

The rate at which one variable is changing with respect to some other variable.
Use chain rule to find possible solution.

Exercise- Rates of Change

Question 1

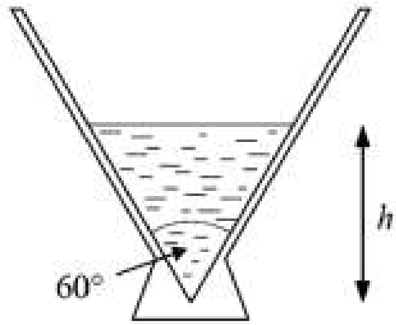
1. A closed right circular cylinder has base radius r cm and height 3r cm. If r is increased at a rate of 1 mm/s, find expression in terms of r for the rate of increase of

   a) the total external surface area and

   b) the volume of the cylinder *[a) $8/5\ \pi r$ b) $9/10\ \pi r^2$]*

2. A funnel has a circular top of diameter 20 cm and a height of 30 cm. When the depth of liquid in the funnel is 12 cm, the liquid is dripping from the funnel at the rate of 0.2 cm^3s^{-1}. At what rate is the depth of the liquid in the funnel decreasing at? *[$0.004 cms^{-1}$]*

Question 3



The diagram shows a vertical cross-section through a vase.

The inside of the vase is in the shape of a right-circular cone with the angle between the sides in the cross-section being 60°. When the depth of water in the vase is h cm, the volume of water in the vase is V cm^3.

(a) Show that $V = \frac{1}{9}\pi h^3$. [2]

The vase is initially empty and water is poured in at a constant rate of 120 cm^3 s^{-1}.

(b) Find, to 2 decimal places, the rate at which h is increasing

(i) when $h = 6$, [5]

(ii) after water has been poured in for 8 seconds. [2]