

TOPIC 4 – Differentiation

Objectives (A2 syllabus):

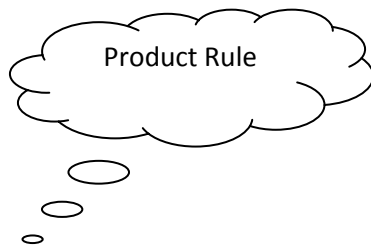
- use the derivatives of e^x , $\ln x$, $\sin x$, $\cos x$, $\tan x$, together with constant multiples, sums, differences and composites;
- differentiate products and quotients;
- find and use the first derivative of a function which is defined parametrically or implicitly.

Rules of Differentiation

1. $\frac{d}{dx}(k) = 0$ where k is a constant
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
4. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \frac{d}{dx}[f(x)]$
5. $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \text{---} \rightarrow \text{The Chain Rule}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The Product Rule and Quotient Rule

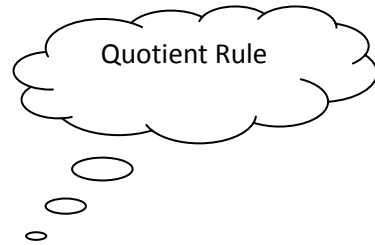


$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Example 1

Differentiate the following with respect to x .

i) $x^2(x+3)^3$ ii) $x^2\sqrt{1+x^2}$.



$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Example 2

Differentiate the following with respect to x .

i) $\frac{x-2}{x+2}$ ii) $\frac{x^2}{2x+1}$

Exercise 1 - Product Rule and Quotient Rule

Question 1

Find $\frac{d^2y}{dx^2}$ given that

(a) $y = \frac{1}{x^2 + 1}$, (b) $y = \frac{x}{x-1}$, (c) $y = \sqrt{x^2 + 1}$.

(a) $\frac{2(3x^2-1)}{(x^2+1)^3}$, (b) $\frac{2}{(x-1)^3}$, (c) $\frac{1}{(x^2+1)^{3/2}}$

Question 2

Differentiate the following with respect to x

(a) $\frac{(x^2+1)(x-1)^2}{2x-1}$, (b) $\frac{x\sqrt{x-1}}{x+1}$, (c) $\sqrt{\left\{\frac{x^3+1}{(x+1)^3}\right\}}$.
(a) $\frac{2x(x-1)(3x^2-3x+2)}{(2x-1)^2}$, (b) $\frac{x^2+3x-2}{2(x+1)^2\sqrt{x-1}}$,
(c) $\frac{3(x-1)}{2\sqrt{\{(x+1)^3(x^3+1)\}}}$.

Question 3

Given that $y = \frac{1-x}{1+x}$, show that

$$\frac{d^2y}{dx^2} = \frac{4}{(1+x)^3}$$

Question 4

$$f(x) = \frac{x}{x^2+2}, \quad x \in \mathbb{R}$$

Find the set of values of x for which $f'(x) < 0$.

$$x < -\sqrt{2}, \quad x > \sqrt{2}$$

Question 5

Use differentiation to find the x -coordinates

of the points at which the graph of $y = \frac{x^2}{2x-1}$

has either a maximum or a minimum and distinguish between them.

$$x = 0, 1$$

Question 6

The curve $y = \frac{2x+1}{2x-1}$ crosses the x -axis at A and the y -axis at B. Find the point of intersection of the tangents to the curve at A and B.

$$(-1/6, -1/3)$$

Differentiating Trigonometric Functions

The function $\sin x$ is differentiated from first principles as follows:

$$\begin{aligned}y &= \sin x \quad (\text{where } x \text{ is in radians}) \\ \Rightarrow y + \delta y &= \sin(x + \delta x) \\ \Rightarrow \delta y &= \sin(x + \delta x) - \sin x = 2 \cos(x + \tfrac{1}{2}\delta x) \sin \tfrac{1}{2}\delta x \\ \Rightarrow \frac{\delta y}{\delta x} &= \frac{2 \cos(x + \tfrac{1}{2}\delta x) \sin \tfrac{1}{2}\delta x}{\delta x} = \cos(x + \tfrac{1}{2}\delta x) \cdot \frac{\sin \tfrac{1}{2}\delta x}{\tfrac{1}{2}\delta x}.\end{aligned}$$

$$\text{As } \delta x \rightarrow 0, \quad \cos(x + \tfrac{1}{2}\delta x) \rightarrow \cos x \quad \text{and} \quad \frac{\sin \tfrac{1}{2}\delta x}{\tfrac{1}{2}\delta x} \rightarrow 1$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \cos x \cdot 1.$$

Hence

$$\boxed{\frac{d}{dx}(\sin x) = \cos x.}$$

Similarly it can be shown that $\boxed{\frac{d}{dx}(\cos x) = -\sin x}$

Example 3

- i) Find $\frac{d}{dx}(\tan x)$ ----->> differentiate by using quotient rule.
- ii) Find $\frac{d}{dx}(\sec x)$
- iii) Find $\frac{d}{dx}(\cot x)$
- iv) Find $\frac{d}{dx}(\operatorname{cosec} x)$

Example 4

Differentiate (a) $\sin^3 x$, (b) $\cos(4 - 3x)$, (c) $\sin x^\circ$

Exercise 2

Differentiate the following with respect to x , simplifying your answers where possible.

1. (a) $\sin 2x$, (b) $3 \cos 4x$, (c) $4 \tan \tfrac{1}{2}x$.
2. (a) $\cos^4 x$, (b) $\sqrt{(\sin x)}$, (c) $\sin^2 5x$.
3. (a) $(x + \sin x)^3$, (b) $\sin(x^2)$, (c) $\tan(6x^\circ)$.
4. (a) $x^4 \cos x$, (b) $(4x^2 + 1) \tan x$, (c) $x^2 \sin 3x$.
5. (a) $\operatorname{cosec}(x + 1)$, (b) $2 \cot(1 - 2x)$, (c) $\sec(3x - 4)$.
6. (a) $\tan(\cos x)$, (b) $\sec(1 + \sqrt{x})$, (c) $\cot(1/x)$.
7. (a) $\sin x \cos 2x$, (b) $\cos^3 x \sin 3x$, (c) $\cos x \tan x$.

Answers:

- (a) $2 \cos 2x$, (b) $-12 \sin 4x$, (c) $2 \sec^2 \frac{1}{2}x$.
- (a) $-4 \cos^3 x \sin x$, (b) $\cos x / 2\sqrt{\sin x}$, (c) $5 \sin 10x$.
- (a) $3(x + \sin x)^2(1 + \cos x)$, (b) $2x \cos(x^2)$, (c) $\frac{\pi}{30} \sec^2(6x^\circ)$.
- (a) $x^3(4 \cos x - x \sin x)$, (b) $8x \tan x + (4x^2 + 1) \sec^2 x$,
(c) $x(2 \sin 3x + 3x \cos 3x)$.
- (a) $-\operatorname{cosec}(x+1) \cot(x+1)$, (b) $4 \operatorname{cosec}^2(1-2x)$,
(c) $3 \sec(3x-4) \tan(3x-4)$.
- (a) $-\sin x \sec^2(\cos x)$, (b) $\frac{1}{2\sqrt{x}} \sec(1+\sqrt{x}) \tan(1+\sqrt{x})$,
(c) $\frac{1}{x^2} \operatorname{cosec}^2\left(\frac{1}{x}\right)$.
- (a) $\cos x \cos 2x - 2 \sin x \sin 2x$, (b) $3 \cos^2 x \cos 4x$, (c) $\cos x$.

Question 8

Differentiate with respect to x

- (a) $\sin^2(2x-5)$, (b) $x^4 \tan 4x$,
(c) $\frac{\sec x + \tan x}{\sec x - \tan x}$, (d) $\frac{\sin x}{\sqrt{(\cos 2x)}}$, (a) $2 \sin(4x-10)$, (b) $4x^3(\tan 4x + x \sec^2 4x)$,
(c) $\frac{2 \sec x}{(\sec x - \tan x)^2}$, (d) $\frac{\cos x}{(\cos 2x)^{3/2}}$.

Question 9

The equation of a curve is $y = 2 \cos x + \sin 2x$. Find the x -coordinates of the stationary points on the curve for which $0 < x < \pi$, and determine the nature of each of these stationary points. [7]

$$\frac{\pi}{6} - \max, \quad \frac{5\pi}{6} - \min$$

Question 10

Find the equation of normal to the curve $y = \tan^2 x$ when $x = \frac{\pi}{4}$.

$$4y + x = 4 + \frac{\pi}{4}$$