Permutations & Combinations

S1/2/1: Permutations
S1/2/2: Combinations
S1/2/3: Repetition of objects in arrangements
S1/2/4: Arrangement of objects with restrictions
S1/2/5: Discussion of exam type and miscellaneous questions

S1/2/1:
Permutations

Learning Outcome

Students should be able to:

- **Understand the term permutation**
- **Solve simple problems involving permutations.**

Permutations

Suppose that you have the three letters, **A**, **B** and **C**, and you are going to arrange them in a line to form 'words'.

How many three-letter words are there?

You have 3 choices for the first letter:

$1st$
A or B or C

Having chosen the first letter, you then have just 2 choices for the second letter:

$1st$	$2nd$
A	B or C
B	A or C
C	A or B

Having chosen the first two letters, you have only 1 choice for the third letter:

$1st$	$2nd$	$3rd$
A	B	only C
B	C	only A
C	A	only B

There are $3 \times 2 \times 1 = 6$ possible words that you can make with three letters.

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ABC *BAC* *CAB* *ACB* *BCA* *CBA*

Using a similar argument you can find the number of words which make from 4 letters **A**, **B, C and D.**

There are $4 \times 3 \times 2 \times 1 = 24$ possible words that you can make with three letters.

The different arrangements of the objects are called **permutations**.

The number of permutations of *n* distinct objects is *n*!, *n* factorial where

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1.$$

- Use calculator: Example: Find 4!.

4	SHIFT	*x*!	=

The expression $n\times(n-1)\times(n-2)\times\ldots\times2\times1$ is called n factorial and written as $n!$.

The number of permutations of *n* distinct objects is *n*!, where

$$n!=n\times(n-1)\times(n-2)\times\ldots\times2\times1.$$

Use calculator: Example: Find 4!.

4	SHIFT	$x!$	=

$4! = 24$

Example 1:

Seven different cars are to be loaded on to a transporter truck. In how many different ways can the cars be arranged?

Number of ways of arrangements

$$7! = 5040$$

Example 2:

A witness reported that a car seen speeding away from the scene of the crime had a number plate that began with V or W, the digits were 4, 7 and 8 and the end letters were A, C, E. He could not however remember the order of the digits or the end letters. How many cars would need to be checked to be sure of including the suspect car?

There are 2 choices for the initial letter. 2!

3! Ways of arranging the digits 4, 7, 8.

3! Ways of arranging the letters A, C, E.

The total number of different plates $= 2! \times 3! \times 3! = 72$

72 cars would need to be checked.

Permutations when the objects are not distinct

If instead of the letter A, B, C, D you have the letters

A, A, A, **D** then the arrangements are as follow:

AAAD AADA ADAA DAAA

So the number of ways of arranging the 4 objects, of which 3 are alike

$$= \frac{4!}{3!} = \frac{4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 4$$

The number of ways of arranging in a line *n* objects of which *p* of one type are alike, *q* of a second type are alike, *r* of a third type are alike, and so on, is

$$\frac{n!}{p!q!r!\ldots}$$

Example 3:

The letters of the word MATHEMATICS are written, one on each of 11 separate cards.
The cards are laid out in a line.
Calculate the number of different arrangements of these letters.

Number of different arrangements

$$\frac{11!}{2!\times 2!\times 2!} = 4989600$$

2Ms **2A**s **2T**s

Permutations of *r* objects from *n* objects

Consider the number of ways of placing 3 of the letters **A**, **B**, **C**, **D**, **E**, **F**, **G** in 3 empty spaces.

The first space can be filled in 7 ways.
The second space can be filled in 6 ways.
The third space can be filled in 5 ways.

Therefore there are $7 \times 6 \times 5$ ways of arranging 3 letters taken from 7 letters.

This is the number of permutations of 3 objects taken from 7 and written as ${}^{7}P_{3}$.

Use calculator:

$${}^{7}P_{3} = 7 \times 6 \times 5 = 210$$

7	SHIFT	${}^{n}P_{r}$	3	=

The number of permutations, or ordered arrangements, of r objects taken from n distinct objects is written ${}^{n}P_{r}$

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!} $$

Note:

0! is defined to be 1 $\rightarrow$ $0! = 1$

Example 4:

Eight runners are hoping to take part in a race, but the track has only six lanes. In how many ways can six of the eight runners be assigned to lanes?

The number of permutation is

$$^{8}P_{6} = \frac{8!}{(8-6)!} = \frac{8!}{2!} = 20160$$

Practice Exercise

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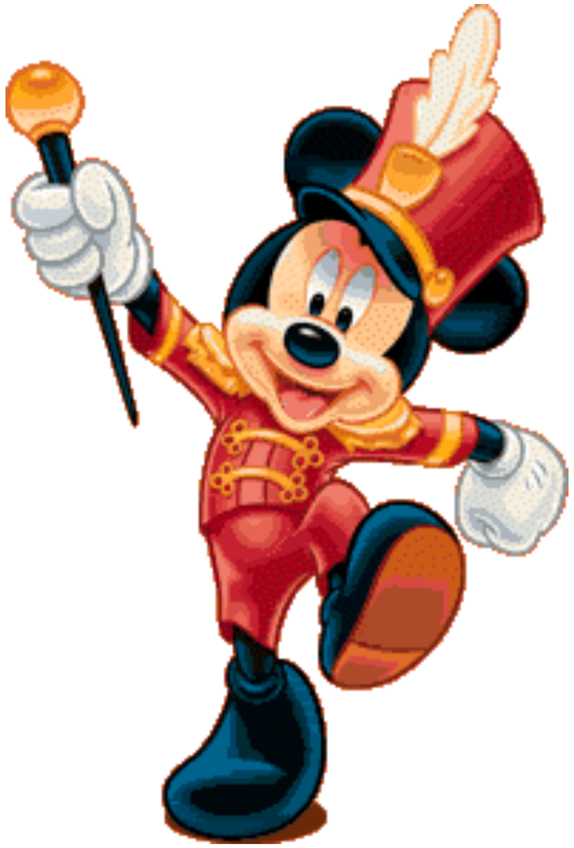
Exercise 5A (Page 89)
Q4, Q6, Q7, Q8, Q9



S1/2/2:
Combinations

Learning Outcome

Students should be able to:

- **Understand the term combination**
- **Solve simple problems involving combinations.**

Combinations

When a selection is made from a set of objects and the order of selection is unimportant it is called **combination**.

When considering the number of combinations of r objects from n objects, the order in which they are placed is not important.

Consider what happen when you select 3 cards from the 4 cards *A*, *B*, *C* and *D*.

Start with permutation of 3 letters from *A*, *B*, *C* and *D*.

ABC	*ABD*	*ACD*	*BCD*
ACB	*ADB*	*ADC*	*BDC*
BAC	*BAD*	*CAD*	*CBD*
BCA	*BDA*	*CDA*	*CDB*
CAB	*DAB*	*DAC*	*DBC*
CBA	*DBA*	*DCA*	*DCB*

The permutations are all different, but they all give rise to the same combination for each column.

To count the combinations, it is sufficient to count the columns.

ABC	*ABD*	*ACD*	*BCD*
ACB	*ADB*	*ADC*	*BDC*
BAC	*BAD*	*CAD*	*CBD*
BCA	*BDA*	*CDA*	*CDB*
CAB	*DAB*	*DAC*	*DBC*
CBA	*DBA*	*DCA*	*DCB*

There are 4 combinations of three cards from the four cards *A*, *B*, *C* and *D*.

Use calculator: Find ${}^{4}C_{3}$.

4	**SHIFT**	${}^{n}C_{r}$	**3**	=

$$\binom{4}{3} = {}^{4}C_{3} = 4$$

A combination is a selection in which the order of the objects selected is unimportant.

The number of different combinations of r objects selected from n distinct objects is $\binom{n}{r}$ where

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{(n-r)! \times r!}.$$

Example 5:

The manager of a football team has a squad of 16 players. He needs to choose 11 to play in a match. How many possible teams can be chosen?

The number of teams is

$$^{16}C_{11} = \binom{16}{11} = \frac{16!}{(16-11)!11!} = 4368$$

Example 6:

A team of 5 people, which must contain 3 men and 2 women, is chosen from 8 men and 7 women. How many different teams can be selected?

The number of different teams of 3 men which can be selected from 8 is $\binom{8}{3}$

The number of different teams of 3 women which can be selected from 7 is $\binom{7}{2}$

The number of possible teams is $\binom{8}{3} \times \binom{7}{2} = 1176$

Example 7:

Five cards are dealt without replacement from a standard pack of 52 cards. Find the probability that exactly 3 of 5 cards are hearts.

The number of choices is $\begin{pmatrix} 52 \\ 5 \end{pmatrix}$ because the order of selection is irrelevant.

In a pack of 52 cards, 13 cards are hearts, therefore 39 cards 'non-hearts'.

There are $\begin{pmatrix} 13 \\ 3 \end{pmatrix}$ combinations of 3 hearts.

There are $\begin{pmatrix} 39 \\ 2 \end{pmatrix}$ combinations of 'non - hearts'.

Example 7:

Therefore the number of sets of 5 cards with 3 hearts is

$$\binom{13}{3} \times \binom{39}{2}$$

The probability that exactly 3 of 5 cards are hearts

$$\frac{\binom{13}{3} \times \binom{39}{2}}{\binom{52}{5}} = 0.0815$$

Practice Exercise

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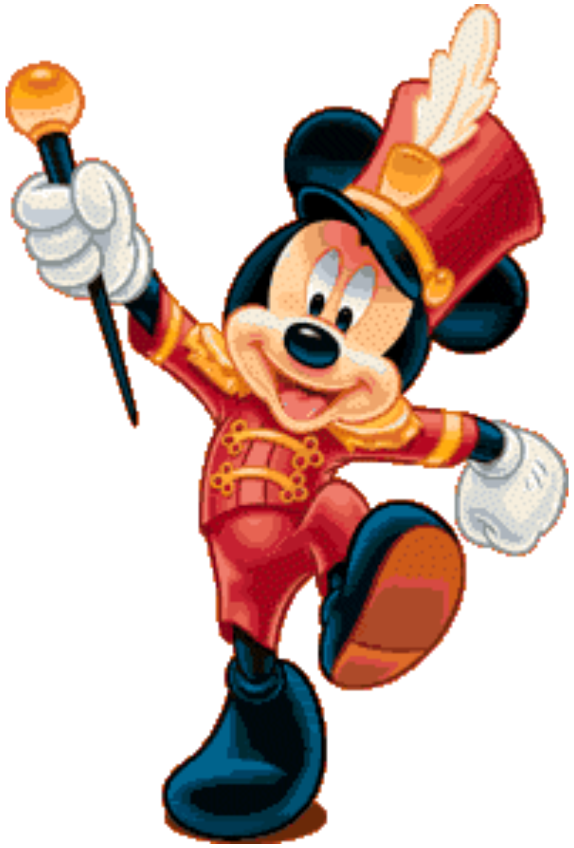
Exercise 5B (Page 93)
Q2, Q3, Q4, Q5, Q7



S1/2/3:
Repetition of objects in arrangements

S1/2/4:
Arrangement of objects with restrictions

Learning Outcome

Students should be able to:

- Solve problems about arrangements of objects in a line, including those involving repetition.
- Solve problems involving restrictions.

Example 8:

If the letters of the word MINIMUM are arranged in a line at random, what is the probability that the arrangement begins with MMM?

The number of arrangements $= \dfrac{7!}{3!\,2!} = 420$

The letters must be arranged in the order MMMxxxx.

The remaining 4 letters can be arranged in $\dfrac{4!}{2!} = 12$ ways.

$$P(\text{MMMxxxx}) = \frac{12}{420} = \frac{1}{35}$$

Example 9:

If a four-digit number is formed from the digits 1, 2, 3 and 5 and repetition are not allowed, find the probability that the number is divisible by 5.

The number of arrangements $= 4! = 24$

The number is divisible by 5 must end with the digit 5.

Number of ways of arranging the digits 1, 2, 3 $= 3!$

$$P(\text{the number is divisible by 5}) = \frac{3!}{24} = \frac{1}{4}$$

Example 10:

The six letters of the word LONDON are each written on a card and the six cards are then shuffled and placed in a line.

(a) Calculate the number of different arrangements.

Number of different arrangements of LONDON

$$= \frac{6!}{2! \times 2!} = 180$$

Example 10:

The six letters of the word LONDON are each written on a card and the six cards are then shuffled and placed in a line.

(b) Find the probability that the middle two cards both have the letter N on them.

$$\text{The number of arrangements} = \frac{4!}{2!} = 12$$

Number of different arrangements of LONDON =180

$$P(\text{middle two letters are NN}) = \frac{12}{180} = \frac{1}{15}$$

Example 10:

The six letters of the word LONDON are each written on a card and the six cards are then shuffled and placed in a line.

(c) Find the probability that the two cards with letter O are adjacent and the two cards with letter N are also adjacent.

L, OO, NN, D

The number of arrangements $= 4! = 24$

$$P\,(\text{two Os, two Ns are adjacent}) = \frac{24}{180} = \frac{2}{15}$$

Example 10:

The six letters of the word LONDON are each written on a card and the six cards are then shuffled and placed in a line.

The cards are shuffled again and placed in a line, face down. The first two cards in the line are turned over and reveal the letter L and O.

(d) Find the probability that when the other four cards are turned over the letters will spell LONDON.

$$\text{The number of arrangements} = \frac{4!}{2!} = 12$$

$$P\text{ (L, O and four remaining letters spell LONDON)} = \frac{1}{12}$$

Example 11:

Eight people A, B, …, H are arranged randomly in a line. What is the probability that

(a) A and B are next to each other,

Treating the A and B as one item,

A and B can be arranged in two ways AB or BA

There are 2 x 7! ways of arranging the eight people in line with A and B next to each other.

If there are no restrictions, there are 8! ways of arranging the eight people in line.

$$P(\text{A and B are next to each other}) = \frac{2 \times 7!}{8!} = \frac{1}{4}$$

Example 11:

Eight people A, B, …, H are arranged randomly in a line. What is the probability that

(b) A and B are not next to each other?

P(A and B are not together) = 1 – P(A and B are together)

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

Example 12:

Ten pupils are placed at random in a line.
What is the probability that the two youngest pupils are separated?

The number of arrangements $= 10!$

Treating the two youngest pupils as one item, there are 9 items to arrange.

9 items can be arranged in 9! ways.

The two youngest can be arranged in two ways

$Y_1 Y_2$ or $Y_2 Y_1$

$$P(\text{two youngest are together}) = \frac{2 \times 9!}{10!} = 0.2$$

$$P(\text{two youngest are separated}) = 1 - 0.2 = 0.8$$

Example 13:

Find the number of permutations of four letters from the letters of the word MATHEMATICS.

Practice Exercise

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Exercise 5C (Page 96)
Q1, Q4, Q5, Q6, Q7