Exercise 5.1 – Finding Probabilities(Standard Normal Distribution Table)

In all the following questions $Z \sim N(0, 1)$.

Always **draw sketches** to illustrate the probabilities and also check that your answer is sensible. For example, if you get a negative answer for a probability it is definitely wrong. It is also wrong if you get a probability greater than 1.

1 Find

    (i) $P(Z < 0.874)$ (ii) $P(Z > -0.874)$

    (iii) $P(Z > 0.874)$ (iv) $P(Z < -0.874)$

2 Find

    (i) $P(Z > 1.8)$ (ii) $P(Z < -0.65)$

    (iii) $P(Z \geqslant -2.46)$ (iv) $P(Z \leqslant 1.36)$

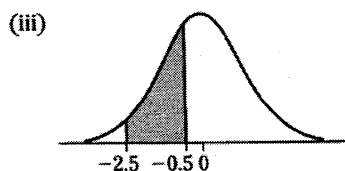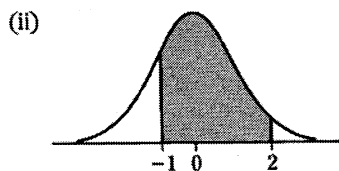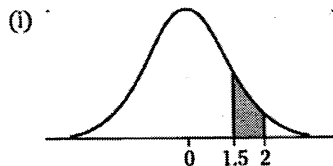   (v) $P(Z > 2.58)$ (vi) $P(Z > -2.37)$

    (vii) $P(Z < 1.86)$ (viii) $P(Z \leqslant -0.725)$

    (ix) $P(Z > 1.863)$ (x) $P(Z < 1.63)$
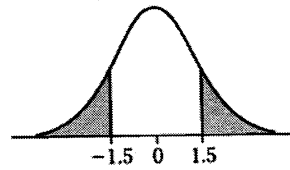
    (xi) $P(Z > -2.061)$ (xii) $P(Z < -2.875)$

3 Find

    (i) $P(Z \geqslant 1.645)$ (ii) $P(Z < -1.645)$

    (iii) $P(Z > 1.282)$ (iv) $P(Z > 1.96)$

    (v) $P(Z > -2.575)$ (vi) $P(Z \geqslant 2.326)$

    (vii) $P(Z \leqslant -2.808)$ (viii) $P(Z < 1.96)$

4 Find the probabilities represented by the shaded areas in the diagrams.

    (i)

    (ii)

    (iii)
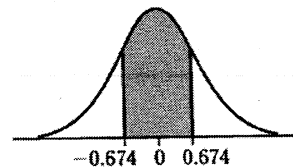

(iv)

5 Find

    (i) $P(0.829 < Z < 1.834)$

    (ii) $P(-2.56 < Z < 0.134)$ ·

    (iii) $P(-1.762 \leqslant Z \leqslant -0.246)$

    (iv) $P(0 < Z < 1.73)$

    (v) $P(-2.05 < Z < 0)$

    (vi) $P(-2.08 < Z < 2.08)$

    (vii) $P(1.764 < Z < 2.567)$

    (viii) $P(-1.65 \leqslant Z < 1.725)$

    (ix) $P(-0.98 < Z < -0.16)$

    (x) $P(Z < -1.97 \text{ or } Z > 2.5)$

6 Find

    (i) $P(-1.78 < Z < 1.78)$

    (ii) $P(-1.645 < Z < 1.645)$

7 Complete this statement:

The central ......% of the distribution lies between -0.674 and 0.674.

8 It is given that $P(Z < a) = 0.3$ and $P(a < Z < b) = 0.6$.
    Find   (i) $P(Z < b)$ (ii) $P(Z > a)$

9 It is given that $P(Z < a) = 0.7$ and $P(Z > b) = 0.45$.
    Find   (i) $\Phi(b)$ (ii) $P(b < Z < a)$

10 It is given that $P(-a < Z < a) = 0.8$.
    Find   (i) $P(Z < a)$ (ii) $P(Z > a)$

Exercise 5.2 – Standardising Normal Distribution

Always draw sketches and check that your answer is sensible.

1 The masses of packages from a particular machine are normally distributed with mean 200 g and standard deviation 2 g. Find the probability that a randomly selected package from the production line of the machine weighs

   (i) less than 197 g,

   (ii) more than 200.5 g,

   (iii) between 198.5 g and 199.5 g.

2 The heights of boys at a particular age follow a normal distribution with mean 150.3 cm and standard deviation 5 cm. Find the probability that the height of a boy picked at random from this age group is

   (i) less than 153.2 cm,

   (ii) more than 158 cm,

   (iii) between 150 cm and 158 cm,
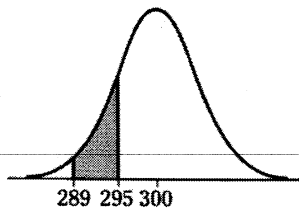
   (iv) more than 10 cm from the mean height.

3 $X \sim N(300, 25)$

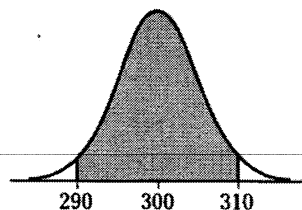Find the probabilities represented by the shaded areas in the diagrams:
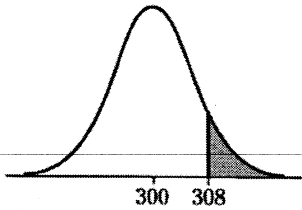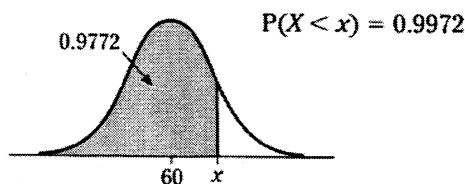

4 The random variable X is distributed normally with mean 50 and variance 20. Find

   (i) $P(X > 60.3)$,

   (ii) $P(X < 59.8)$.

5 $X \sim N(-8, 12)$. Find

   (i) $P(X < -9.8)$,

   (ii) $P(X > -8.2)$,

   (iii) $P(-7 \leqslant X \leqslant 0.5)$.

6 The masses of a certain type of cabbage are normally distributed with mean 1000 g and standard deviation 150 g.

   (a) Find the proportion of cabbages with a mass

      (i) greater than 850 g,

      (ii) between 750 g and 1290 g,

   (b) Estimate the number of cabbages in a batch of 800 with a mass less than 900 g or greater than 1375 g.

7 The number of hours of life of a certain type of torch battery is normally distributed with mean 150 and standard deviation 12. In a quality control test two batteries are chosen at random from a batch. If both batteries have a life less than 120 hours, the batch is rejected. Find the probability that the batch is rejected.

8 Cartons of milk from a particular supermarket are advertised as containing 1 litre of milk, but in fact the volume of the milk in a carton is normally distributed with mean 1012 ml and standard deviation 5 ml.

   (i) Find the probability that a randomly chosen carton contains more than 1010 ml.

   (ii) Find the probability that exactly 3 cartons in a sample of 10 cartons contain more than 1012 ml.

   (iii) Estimate how many cartons in a batch of 1000 cartons contain less than the advertised volume of milk.

9 The lifetime, in hours, of a certain make of electric light bulb is known to be normally distributed with mean 2000 and standard deviation 120.
Estimate the probability that the lifetime of a bulb of this make will be

   (i) greater than 2150 hours,

   (ii) greater than 1910 hours,

   (iii) between 1850 hours and 2090 hours.

10 The weights of vegetable marrows supplied to retailers by a wholesaler have a normal distribution with mean 1.5 kg and standard deviation 0.02 kg. The wholesaler supplies three sizes of marrow:

      Size 1   under 1.48 kg

      Size 2   from 1.48 kg to 1.53 kg

      Size 3   over 1.53 kg

Find, to three decimal places, the proportions of marrows in the three sizes.

11 A normal distribution has mean μ and standard deviation σ. 1000 observations are taken from the distribution. How many would you expect to be between $\mu - 1.5\sigma$ and $\mu + 1.5\sigma$?

12 An intelligence test used in a particular country has scores which are normally distributed with mean 100 and standard deviation 15. In a randomly selected group of 500 people sitting the test, estimate how many have a score

   (i) higher than 140,

   (ii) below 120,

   (iii) between 100 and 110,

   (iv) between 85 and 90.
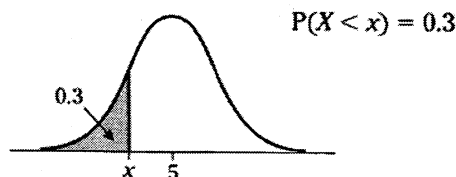
Exercise 5.3 – Using the Standard Normal table in reverse
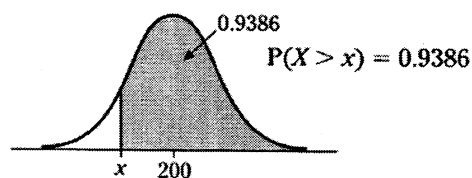
1 Find x in each of the following.
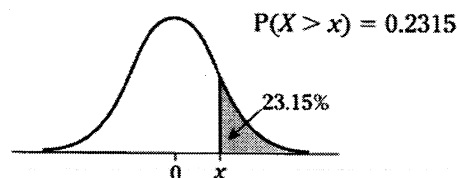
(i) $X \sim N(60, 5^2)$

P($X < x$) = 0.9972

0.9772

60   x

(ii) $X \sim N\left(5, \frac{4}{9}\right)$

P($X < x$) = 0.3

0.3

x 5

(iii) $X \sim N(200, 6^2)$

0.9386

P($X > x$) = 0.9386

x 200

(iv) $X \sim N(0, 4)$

P($X > x$) = 0.2315

23.15%

0   x

2 The heights of female students at a particular college are normally distributed with mean 169 cm and standard deviation 9 cm.

(i) Given that 80% of these female students have a height less than h cm, find the value of h.

(ii) Given that 60% of these female students have a height greater than s cm, find the value of s.

3 The masses of lettuces sold at a market stall are normally distributed with mean 600 g and standard deviation 20 g.

(i) Find the mass exceeded by 10% of the lettuces.

(ii) 5% of the lettuces have a mass less than M g. Find M.

4 $X \sim N(400, 64)$.

(i) Find the limits within which the central 95% of the distribution lies.

(ii) Find the interquartile range of the distribution.

5 The lengths of metal rods produced by a machine are normally distributed with mean 120 cm and standard deviation 10 cm.

(i) Find the probability that a rod selected at random has a mass within 5 cm of the mean.

Rods shorter than L cm are rejected.

(ii) Estimate the value of L if 1% of all rods are rejected.

6 The lifetime, in hours, of a certain make of batteries follows a normal distribution with mean 160 and standard deviation 30. Calculate the interval, symmetrical about the mean, within which 75% of the battery lifetimes lie.

7 The times for a certain car journey have a normal distribution with mean 100 minutes and standard deviation 7 minutes. Journey times are classified as follows:

'short'   (the shortest 33% of times)

'long'   (the longest 33% of times)

'standard'   (the remaining 34% of times)

(i) Find the probability that a randomly chosen car journey takes between 85 and 100 minutes.

(ii) Find the least and greatest times for the 'standard' journey.

Cambridge Paper 61 Q3 N09

Exercise 5.4 – Finding μ or σ or both

You are advised to draw sketches and check that your answer is reasonable.

1 The random variable X is normally distributed with standard deviation 25.
If $P(X < 27.5) = 0.3085$, find the mean.

2 The random variable X is normally distributed with mean 45 and standard deviation σ. Given that $P(X > 51) = 0.288$, find the value of σ.

3 The volume of orange juice in cartons sold by a particular company is normally distributed with mean 333 ml. It is known that 20% of the cans contain more than 340 ml.

(i) Find the standard deviation of the volume of orange juice in a can.

(ii) Find the percentage of cans that contain less than 330 ml.

4 The random variable X is distributed as $N(\mu, 12)$. It is given that $P(X > 32) = 0.8438$.

(i) Find the value of μ.

(ii) Find the probability that a random observation of X lies between 34.5 and 35.3.

5 The heights, in metres, of 500 people are normally distributed with standard deviation 0.080. Given that the heights of 129 of these people are greater than the mean height, but less than 1.806 m, estimate the mean height.

Hint: Draw a sketch.

6 The masses of boxes of apples are normally distributed such that 20% of the boxes are heavier than 5.08 kg and 15% of the boxes are heavier than 5.62 kg.

Estimate the mean and standard deviation of the masses.

7 The random variable X has a normal distribution with mean μ and standard deviation σ. Given that $P(X > 80) = 0.0113$ and $P(X < 30) = 0.0287$, find μ and σ.

8 Metal rods produced by a machine have lengths that are normally distributed. It is known that 2% of the rods are rejected as being too short and 5% are rejected as being too long.

    (i) Given that the least and greatest acceptable lengths of the rods are 6.32 cm and 7.52 cm, calculate the mean and variance of the lengths of the rods.

    (ii) Ten rods are chosen at random from a batch produced by the machine. Find the probability that exactly three of them are rejected for being too long.

9 The random variable X is distributed as $N(\mu, \sigma^2)$.
$P(X < 35) = 0.2$ and $P(35 < X < 45) = 0.65$.
Find μ and σ.

10   (i) The height of sunflowers follows a normal distribution with mean 112 cm and standard deviation 17.2 cm. Find the probability that the height of a randomly chosen sunflower is greater than 120 cm.

    (ii) When a new fertiliser is used, the height of sunflowers follows a normal distribution with mean 115 cm. Given that 80% of the heights are now greater than 103 cm, find the standard deviation.

Cambridge Paper 6 Q3 J03

11 A farmer cuts hazel twigs to make into bean poles to sell at the market. He says that the sticks are each 240 cm long but in fact the lengths of the sticks are normally distributed such that 55% of the sticks are longer than 240 cm and 10% are longer than 250 cm.

Find the probability that a randomly selected stick is shorter than 235 cm.

12 The diameters of bolts produced by a particular machine follow a normal distribution with mean 1.34 cm and standard deviation 0.04 cm. A bolt is rejected if its diameter is less than 1.24 cm or more than 1.40 cm.

    (i) Find the percentage of bolts which are accepted.

The setting of the machine is altered so that the mean diameter changes but the standard deviation remains the same. With the new setting, 3% of the bolts are rejected because they are too large in diameter.

    (ii) Find the new mean diameter of bolts produced by the machine.

    (iii) Find the percentage of bolts that are now rejected because they are too small in diameter.

13 Tea is sold in packages marked 750 g. The masses of the packages are normally distributed with mean 760 g. The probability that a package weighs more than 750 g is 0.975. Find the standard deviation of the distribution.

14 In a normal distribution, 4% of the distribution is less than 53 and 97% of the distribution is less than 65.

    (i) Find the mean and standard deviation of the distribution.

    (ii) Find the interquartile range of the distribution.

15 A certain make of car tyre can be safely used for 25 000 km on average before it is replaced. The manufacturer guarantees to pay compensation to anyone whose tyre does not last for 20 000 km. It is thought that 5% of all tyres sold will qualify for compensation. The distance travelled, X km, before a tyre is replaced has a normal distribution. Find the standard deviation of X.

16 A machine dispenses raisins into bags so that the mass of raisins in a bag is normally distributed.

    (i) The standard deviation is 1.25 g. It is found that 2.5% of bags contain less than 826 g of raisins. Find the mean mass of raisins in a bag.

    (ii) After the machine is serviced, the mean mass of raisins in a bag is 828.1 g and 0.1% of bags contain more than 830 g. Find the standard deviation of the mass of raisins in a bag after the service.

17 A machine is used to fill cans of soup with a nominal volume of 0.5 litres. The quantity of soup actually delivered is normally distributed with mean μ litres and standard deviation σ litres. It is required that no more than 1% of cans should contain less than the nominal volume.

   (i) Find the least value of μ which will comply with the requirement when $\sigma = 0.003$.

   (ii) Find the greatest value of σ which will comply with the requirement when $\mu = 0.506$.

18 In a large consignment of packets of sugar, it is found that 5% have a mass less than 515 g and 2% have a mass less than 510 g. The masses of packets of sugar are normally distributed. Estimate the mean and the standard deviation of the mass of sugar in a packet.

19 The speeds of cars passing a checkpoint on a certain motorway follow a normal distribution. Observations show that 95% of the cars passing the checkpoint are travelling at less than 136 km/h and 10% are travelling at less than 88 km/h.

   (i) Find the average speed of cars passing the checkpoint.

   (ii) Find the proportion of cars travelling at more than 112 km/h.

20 The random variable X has a normal distribution with mean μ and standard deviation σ.

   (i) If $2\mu = 1.5\sigma$, find $P(X < 3\mu)$.

   (ii) If $P(X > \frac{1}{3}\mu) = 0.7257$, express μ in terms of σ.

Exercise 5.5 – Normal Approximation to Binomial

1 A fair cubical die is thrown 120 times. Using a suitable approximation, find the probability of obtaining at least 24 sixes.

2 The random variable X follows a binomial distribution with $n = 25$ and $p = 0.38$.

  (i) Find the mean and variance of X.

  (ii) Verify that the distribution can be approximated by a normal distribution.

  (iii) Use the normal approximation to find
    (a) $P(10 \leqslant X \leqslant 15)$ (b) $P(X = 12)$
    (c) $P(X < 7)$ (d) $P(X \geqslant 9)$

3 On average 1 in 10 of the chocolates produced in a factory are mis-shapes. In a random sample of 1000 chocolates, find the probability that

  (i) fewer than 80 are mis-shapes,

  (ii) between 90 and 115 (inclusive) are mis-shapes,

  (iii) 120 or more are mis-shapes.

4 When Alex tries to send a fax, the probability that he can successfully send it is 0.85 and each attempt is independent of all other attempts. He tries to send 50 faxes. Use a suitable approximation to find the probability that he can successfully send at least 46 faxes.

5 At a particular hospital, records show that each day, on average, 80% of people keep their appointment at the outpatients' clinic.

  (i) Find the probability that in a random sample of 10 patients, more than 7 keep their appointment.

  (ii) Using a suitable approximation, find the probability that on a day when 200 appointments have been booked
    (a) more than 170 patients keep their appointments,
    (b) at least 155 patients keep their appointments.

6 The random variable X is distributed as $B(200, 0.7)$. Use the normal approximation to the binomial distribution to find

  (i) $P(X \geqslant 130)$ (ii) $P(136 \leqslant X < 148)$

  (iii) $P(X < 142)$ (iv) $P(X = 152)$

7 One-fifth of a certain population has a minor eye defect. Use the normal distribution as an approximation to the binomial distribution to estimate the probability that the number of people with this eye defect is

  (i) more than 20 in a random sample of 100 people,

  (ii) exactly 20 in a random sample of 100 people,

  (iii) more than 200 in a random sample of 1000 people.

8 A certain variety of flower seed is sold in packets containing a large number of seeds. It is claimed on the packet that 40% will bloom white and 60% will bloom red and this may be assumed to be accurate.

  (i) Five seeds are planted. Find the probability that
    (a) exactly three will bloom white,
    (b) at least one will bloom white.

  (ii) One hundred seeds are planted. Use a suitable approximation to estimate the probability that at least 30 but at most 45 will bloom white.

9 A certain tribe is distinguished by the fact that 45% of the males have six toes on their right foot. Find the probability that, in a group of 200 males from the tribe, more than 97 have six toes on their right foot.

10 The random variable X follows a binomial distribution with $n = 12$ and $p = 0.42$.

  (i) Use the binomial distribution to calculate, to 4 significant figures, the probability that $X \leqslant 3$.

  (ii) Use the normal distribution as an approximation to the binomial distribution to calculate, to 4 significant figures, the probability that $X \leqslant 3$.

  (iii) Calculate the percentage error when using the normal approximation.

11 Kamal has 30 hens. The probability that any hen lays an egg on any day is 0.7. Hens do not lay more than one egg per day, and the days on which a hen lays an egg are independent.

   (i) Calculate the probability that, on any particular day, Kamal's hens lay exactly 24 eggs.

   (ii) Use a suitable approximation to calculate the probability that Kamal's hens lay fewer than 20 eggs on any particular day.

<div align="right">Cambridge Paper 6 Q4 J03</div>

12 In a certain city 37% of all shops advertise in the local newspaper.

   (i) A random sample of 12 shops is taken. Find the probability that more than 9 advertise in the local newspaper.

   (ii) A random sample of 60 shops is taken. Estimate, using a normal approximation, the probability that

     (a) at least 30 advertise in the local newspaper,

     (b) fewer than 39 do **not** advertise in the local newspaper.

13 A manufacturer makes two sizes of elastic bands: large and small. 40% of the bands produced are large bands and 60% are small bands. Assuming that each pack of these elastic bands contains a random selection, calculate the probability that, in a pack of 20 bands, there are

   (i) equal numbers of large and small bands,

   (ii) more than 17 small bands.

An office pack contains 150 elastic bands.

   (iii) Using a suitable approximation, calculate the probability that the number of small bands in the office pack is between 88 and 97 inclusive.

<div align="right">Cambridge Paper 6 Q7 N06</div>

14 On a certain road 20% of the vehicles are trucks, 16% are buses and the remainder are cars.

A random sample of 125 vehicles is taken. Using a suitable approximation, find the probability that more than 73 are cars.

<div align="right">Cambridge Paper 6 Q3(ii) J09</div>

15 A box contains 3 red balloons and 5 green balloons. There are 150 similar boxes in a storeroom. A balloon is taken at random from each box. Using a suitable approximation, find the probability that no more than 100 green balloons are taken.

Exercise 5.6 – Miscellaneous Exercise

1 The lengths of certain items follow a normal distribution with mean μ mm and standard deviation 6.0 mm. It is known that 4.78% of the items are longer than 82.0 mm.

   (i) Find the value of μ.

   (ii) Find the probability that the length of a randomly selected item is between 63.7 mm and 71.3 mm.

2 The breaking strength of a particular type of paving slab is normally distributed with mean 50 units and standard deviation 4 units.

   (i) Find the probability that a paving slab from a batch of slabs of this type has a breaking strength greater than 59.5 units.

   (ii) 95% of the slabs have a breaking strength between the limits $(50 - b)$ units and $(50 + b)$ units. Find the value of b and hence find these limits.

3 (i) The daily minimum temperature in degrees Celsius (°C) in January in Ottawa is a random variable with distribution $N(-15.1, 62.0)$. Find the probability that a randomly chosen day in January in Ottawa has a minimum temperature above 0 °C.

   (ii) In another city the daily minimum temperature in °C in January is a random variable with distribution $N(\mu, 40.0)$. In this city the probability that a randomly chosen day in January has a minimum temperature above 0 °C is 0.8888. Find the value of μ.

   Cambridge Paper 6 Q3 N08

4 The time spent by customers in a particular supermarket is normally distributed with mean 16.3 minutes and standard deviation 4.2 minutes.

   (i) Find the probability that a customer spends less than 5 minutes in the supermarket.

   (ii) Find the probability that in a random sample of six customers at least one spends more than 25 minutes in the supermarket.

5 In a normal distribution with mean μ and standard deviation σ, $P(X < 16.2) = 0.5$ and $P(X > 18.3) = 0.1049$.

   (i) Write down the value of μ and calculate the value of σ.

   (ii) Find $P(14.5 < X < 15.9)$.

6 Melons are sold in three sizes: small, medium and large. The weights follow a normal distribution with mean 450 grams and standard deviation 120 grams. Melons weighing less than 350 grams are classified as small.

   (i) Find the proportion of melons which are classified as small.

   (ii) The rest of the melons are divided in equal proportions between medium and large. Find the weight above which melons are classified as large.

   Cambridge Paper 6 Q4 J04

7 In a certain country the time taken for a common infection to clear up is normally distributed with mean μ days and standard deviation 2.6 days. 25% of these infections clear up in less than 7 days.

   (i) Find the value of μ.

   In another country the standard deviation of the time taken for the infection to clear up is the same as in part (i), but the mean is 6.5 days. The time taken is normally distributed.

   (ii) Find the probability that, in a randomly chosen case from this country, the infection takes longer than 6.2 days to clear up.

   Cambridge Paper 6 Q4 J08

8 The lengths of fish of a certain type have a normal distribution with mean 38 cm. It is found that 5% of the fish are longer than 50 cm.

   (i) Find the standard deviation.

   (ii) When fish are chosen for sale, those shorter than 30 cm are rejected. Find the proportion of fish rejected.

   (iii) Nine fish are chosen at random. Find the probability that at least one of them is longer than 50 cm.

   Cambridge Paper 6 Q3 J06

9 On any occasion when a particular gymnast performs a certain routine, the probability that she will perform it correctly is 0.65, independently of all other occasions.

(i) Find the probability that she will perform the routine correctly on exactly 5 occasions out of 7.

(ii) On one day she performs the routine 50 times. Use a suitable approximation to estimate the probability that she will perform the routine correctly on fewer than 29 occasions.

(iii) On another day she performs the routine n times. Find the smallest value of n for which the expected number of correct performances is at least 8.

Cambridge Paper 6 Q6 N07

10 The times spent by people visiting a certain dentist are independent and normally distributed with a mean of 8.2 minutes. 79% of people who visit the dentist have visits lasting less than 10 minutes.

(i) Find the standard deviation of the times spent visiting the dentist.

(ii) Find the probability that the time spent visiting this dentist by a randomly chosen person deviates from the mean by more than 1 minute.

(iii) Find the probability that, of 6 randomly chosen people, more than 2 have visits lasting longer than 10 minutes.

(iv) Find the probability that, of 35 randomly chosen people, fewer than 16 have visits lasting less than 8.2 minutes.

Cambridge Paper 63 Q7 N10

11 The length of time a person undergoing a routine operation stays in hospital can be modelled by a normal distribution with mean 7.8 days and standard deviation 2.8 days.

(i) Calculate the proportion of people who spend between 7.8 days and 11.0 days in hospital.

(ii) Calculate the probability that, of 3 people selected at random, exactly 2 spend longer than 11.0 days.

(iii) A health worker plotted a box-and-whisker plot of the times that 100 patients, chosen randomly, stayed in hospital. The results are shown below.
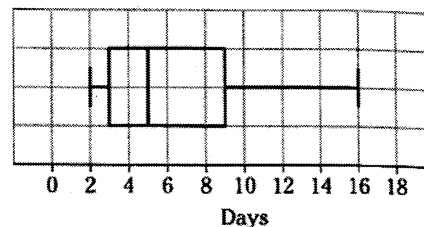
State with a reason whether or not this agrees with the model used in parts (i) and (ii).

Cambridge Paper 6 Q7 J03

12 A shop sells old video tapes, of which 1 in 5 on average are known to be damaged.

(i) A random sample of 15 tapes is taken. Find the probability that at most 2 are damaged.

(ii) Find the smallest value of n if there is a probability of at least 0.85 that a random sample of n tapes contains at least one damaged tape.

(iii) A random sample of 1600 tapes is taken. Use a suitable approximation to find the probability that there are at least 290 damaged tapes.

Cambridge Paper 6 Q7 J04

13 (i) In a spot check of the speeds, $x \, \text{km h}^{-1}$, of 30 cars on a motorway, the data were summarised by $\sum(x - 110) = -47.2$ and $\sum(x - 110)^2 = 5460$. Calculate the mean and standard deviation of these speeds.

(ii) On another day the mean speed of cars on the motorway was found to be $107.6 \, \text{km h}^{-1}$ and the standard deviation was $13.8 \, \text{km h}^{-1}$. Assuming these speeds follow a normal distribution and that the speed limit is $110 \, \text{km h}^{-1}$, find what proportion of cars exceed the speed limit.

Cambridge Paper 6 Q4 J02