Test 1 (Algebra, Exponential & Logarithmic, Trigonometry) (52 marks)

1 The quadratic expression $f(x)$ is such that

- when $f(x)$ is divided by $x-1$, the remainder is 3;
- when $f(x)$ is divided by $x+1$, the remainder is 7;
- $f(0)=1$.

Find $f(x)$. [7]

2 Find the values of x for which

$$x^2-x+4<|4x-2|.$$ [7]

3 In the expansion of

$$\frac{1}{\sqrt{1+ax}}-\frac{1}{1+2x}$$

in ascending powers of x, the first non-zero term is the term in x^2. Find the value of the constant a and hence find the terms in x^2 and x^3. [7]

4 (a) The variables p and q are related by the law

$q=ap^{\,b}$, where a and b are constants.

Given that $\ln p=1.32$ when $lnq=1.73$,

and $\ln p=0.44$ when $lnq=1.95$ find the values of b and lna. [5]

(b) Given that $y=log_2x$ and that

$$log_2x-log_x8+log_22^k+klog_x4=0,$$

prove that

$$y^2+ky+(2k-3)=0.$$ [4]

(i) Hence deduce the set of values of k for which y is real. [4]

(ii) Find the values of x when $k=1.5$. [3]

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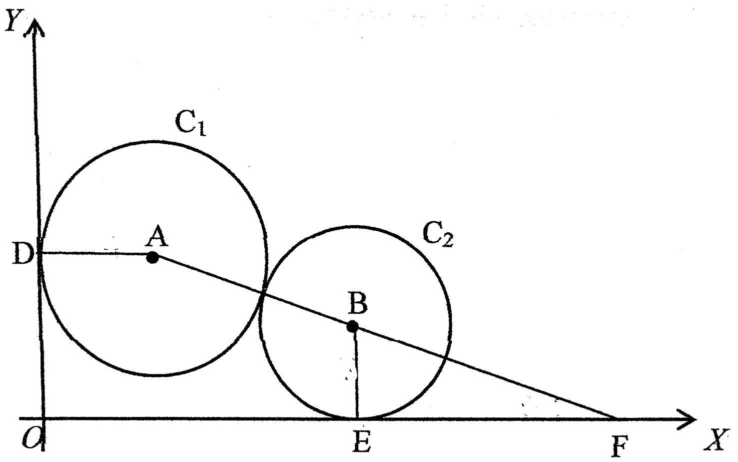


The figure shows two perpendicular axes *OX* and *OY*. Also shown are two circles C_1 and C_2 which touch each other and which both lie in the first quadrant (i.e. above *OX* and to the right of *OY*).

C_1 has radius 4 and touches *OY* at D; C_2 has radius 3 and touches *OX* at E. The line AB, joining the centres of C_1 and C_2, meets *OX* at F and $B\hat{F}O = \theta$.

(a) Find expression for OD and OE in terms θ and show that

$$DE^2 = a + p\sin\theta + q\cos\theta$$

where a, p and q are constants.
Hence express DE^2 in the form $a + r\cos(\theta - \alpha)$ where the values of r and α are to be found. [6]

(b) By considering the extreme positions in which
(i) Both circles touch *OX*, and
(ii) Both circles touch *OY*,

Show that, correct to 1 decimal place,

$$8.2 \leq \theta \leq 98.2.$$ [4]

(c) Find the greatest and least possible lengths of DE and state the corresponding values of θ. [5]

Test 1

1. $f(1)=3$

$f(-1)=7$

$f(0)=1$

$$\frac{f(x)}{x-1} = (ax+b)+\frac{3}{x-1}$$

$$\begin{array}{r} ax+b \\ x-1\overline{)\,f(x)} \\ \vdots \\ \hline 3 \end{array}$$

$\rightarrow f(0)=1$

$$\frac{1}{0-1} = [a(0)+b] + \frac{3}{0-1}$$

$b = 2$

$\rightarrow f(-1)=7$

$$\frac{7}{-1-1} = [a(-1)+2] + \frac{3}{-1-1}$$

$a = 4$

$\therefore f(x) = (4x+2)(x-1)+3$

$f(x) = 4x^2-2x+1$

2. $x^2 - x + 4 < |4x-2|$

$(x^2-x+4)^2 < (4x-2)^2$

$x^4 - x^3 + 4x^2 - x^3 + x^2 - 4x + 4x^2 - 4x + 16$

$< 16x^2 - 16x + 4$

$\Rightarrow x^4 - 2x^3 - 7x^2 + 8x + 12 < 0$

$f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

TSE $f(-1)=0 \Rightarrow (x+1)$ is a factor

$f(2)=0 \Rightarrow (x-2)$ is a factor

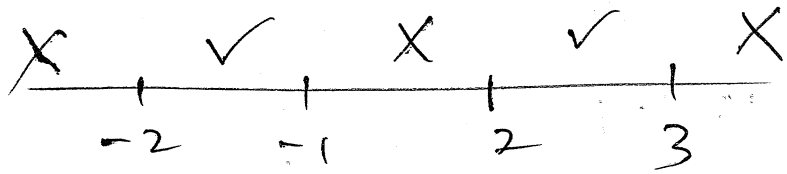
$$\begin{array}{l} x^2-x-2\overline{)\,f(x)} \longrightarrow x^2-x-6 \\ \vdots \end{array}$$

$\therefore f(x) = (x+1)(x-2)(x+2)(x-3)$

$\underline{\underline{-2 < x < -1}}$ or $\underline{\underline{2 < x < 3}}$

2. $\Rightarrow (x+1)(x+2)(x-2)(x-3) = f(x)$

check interval



$\therefore$ $-2 < X < -1$ or

$2 < X < 3$

3. $\dfrac{1}{\sqrt{1+ax}} - \dfrac{1}{1+2x}$

$= (1+ax)^{-\frac{1}{2}} - (1+2x)^{-1}$

$= \left[1 - \frac{1}{2}ax + \frac{3}{8}a^2x^2 - \frac{5}{16}a^3x^3\right] - \left[1 - 2x + 4x^2 - 8x^3\right]$

$= (2 - \frac{1}{2}a)x + (\frac{3}{8}a^2 - 4)x^2 + (8 - \frac{5}{16}a^3)x^3$

$2 - \frac{1}{2}a = 0$ $\qquad \downarrow \qquad\qquad \downarrow$

$a = 4$ $\qquad 2x^2 \qquad\qquad -12x^3$

4. (a) $q = ap^b$

$\ln q = b\ln p + \ln a$

$\ln p = 1.32, \ln q = 1.73$

$1.73 = b(1.32) + \ln a$ — ①

$\ln p = 0.44, \ln q = 1.95$

$1.95 = b(0.44) + \ln a$ — ②

① − ②

$-0.22 = 0.88b \Rightarrow \boxed{b = -0.25}$

$\ln a = 2.06$

(b) $y = \log_2 x$

$2^y = x$

$\log_x 2^y = \log_x x$

$\log_x 2 = \frac{1}{y}$

$\Rightarrow \log_2 x - \log_x 8 + \log_2 2^k + k\log_x 4 = 0$

$y - \log_x 2^3 + k + k\log_x 2^2 = 0$

$y - 3\left(\frac{1}{y}\right) + k + k(2)\left(\frac{1}{y}\right) = 0$

$y - \frac{3}{y} + k + \frac{2k}{y} = 0$

$y^2 - 3 + ky + 2k = 0$

$y^2 + ky + (2k-3) = 0$

☆

4b) i) $b^2 - 4ac \geq 0$

$(k^2 - 4(1)(2k-3) \geq 0$

$(k-6)(k-2) \geq 0$

$k \leq 2$ or $k \geq 6$

(ii) $y^2 + \frac{3}{2}y = 0$

$y(y + \frac{3}{2}) = 0$

$y = 0, \quad y = -\frac{3}{2}$

$x = 2^y$

$x = 2^0 = 1$

$x = 2^{-3/2}$

$x = \frac{1}{2\sqrt{2}}$

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4(b) (i) $b^2-4ac \geq 0$

$k^2-4(1)(2k-3) \geq 0$

$k^2-8k+12 \geq 0$ $\quad$ $k-6$ $\quad$ $k-2$

$(k-6)(k-2) \geq 0$

$k \leq 2$ or $k \geq 6$

(ii) $y^2+\frac{3}{2}y=0$

$y(y+\frac{3}{2})=0$

$y=0$, $y=-\frac{3}{2}$

$x=2^y$

$x=2^0=1$

$x=2^y$

$x=2^{-3/2}$

$x=\frac{1}{2\sqrt{2}}$

5/

$OD = 7\sin\theta + 3 \Rightarrow OD^2 = 49\sin^2\theta + 42\sin\theta + 9$

(a) $OE = 7\cos\theta + 4 \Rightarrow OE^2 = 49\cos^2\theta + 56\cos\theta + 16$

$\Rightarrow DE^2 = OD^2 + OE^2$

$DE^2 = 74 + 42\sin\theta + 56\cos\theta$

$42\sin\theta + 56\cos\theta = R\cos(\theta - \alpha)$

$R = \sqrt{42^2 + 56^2} = 70$

$\alpha = \tan^{-1} 42/56 = 36.87^\circ$

(b) (i)

C_1 touches OX, $OD = 4$

$OD = 7\sin\theta + 3$

$4 = 7\sin\theta + 3 \Rightarrow \theta = 8.2^\circ$

C_2 toches OY, $OE = 3$

$OE = 7\cos\theta + 4$

$3 = 7\cos\theta + 4 \Rightarrow \theta = 98.2$

$\therefore 8.2^\circ \leq \theta \leq 98.2^\circ$

(c) $DE^2 = 74 + 70\cos(\theta - 36.87)^\circ$ $-28.7^\circ \leq \theta - 36.87 \leq 61$

max $DE = \sqrt{74 + 70(1)} = 12$

when $\cos(\theta - 36.87) = 1 \Rightarrow \theta = 36.87^\circ$

min $DE = \sqrt{74 + 70\cos(98.2 - 36.87)^\circ} = 10.37$

when $\theta = 98.2$