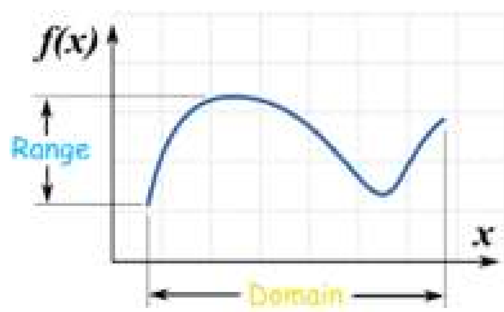
Topic 2: Functions

- understand the terms function, domain, range, one-one function, inverse function and composition of functions;
- identify the range of a given function in simple cases, and find the composition of two given functions;
- determine whether or not a given function is one-one, and find the inverse of a one-one function in simple cases;
- illustrate in graphical terms the relation between a one-one function and its inverse.

Domain, Codomain and Range



There are special name for **what can go into**, and **what can come out** of a function:

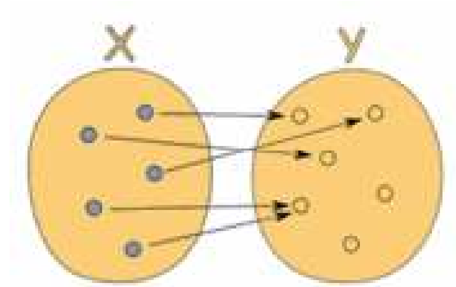
What can go **into** a function is called the **Domain**

What **may possibly come out** of a function is called the **Codomain**

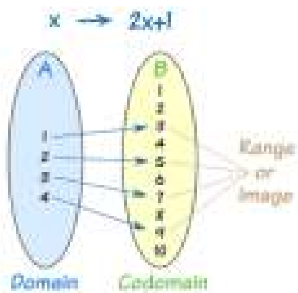
What **actually comes out** of a function is called the **Range**

Definition of a Function

A function relates each element of a set with exactly one element of another set (possibly the same set).



Let us look at a simple example:



In this illustration:

- the set "A" is the Domain,
- the set "B" is the Codomain,
- and the set of elements that get pointed to in B (the actual values produced by the function) are the Range, also called the Image.

In that example:

- Domain: {1, 2, 3, 4}
- Codomain: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Domain

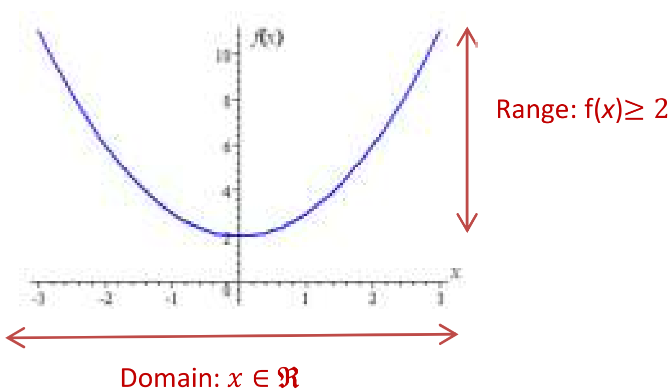
The **domain** of a function is the complete set of possible values of the independent variable in the function.

Range

The **range** of a function is the complete set of all possible **resulting values** of the dependent variable (*y,* usually) of a function, after we have substituted the domain values.

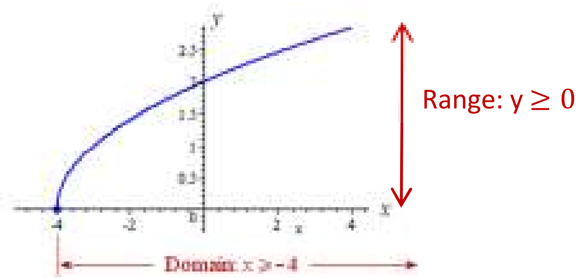
Example 1:

Find the domain and range for the function $f(x) = x^2 + 2$.



Example 2:

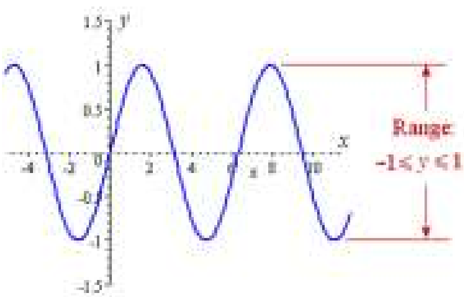
The function $y = \sqrt{x+4}$ has the following graph.



The **domain** of the function is *x* ≥ −4, since *x* cannot take values less than −4.

Example 3:

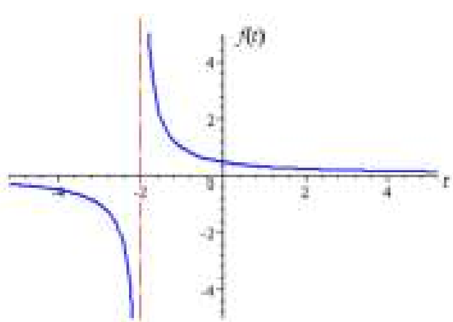
Consider the following curve, *y* = sin *x.*



The **domain** of the function *y* = sin *x* is "all values of *x*", since there are no restrictions on the values for *x*.

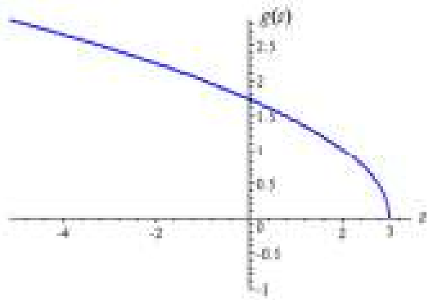
Example 4:

Find the domain and range function $f(t) = \frac{1}{t+2}$.



Example 5:

Find the domain and range for the function $f(x) = \sqrt{3-x}$.

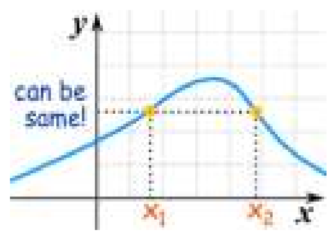


Exercise 3A – Pure Mathematics 1(page 35)   Question 7, 8, 9a),9b),9e) and 13

Type of Functions

One-to-One (Injective)

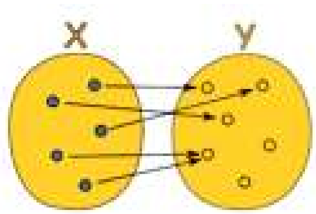
Many-to-One



A Function has **special rules**:

- It must work for **every** possible input value
- And you can only have **one relationship** for each input value

This can be said in one definition:

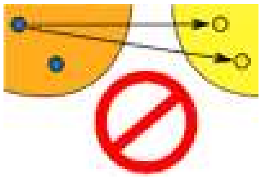


A function relates **each element** of a set with **exactly one** element of another set.

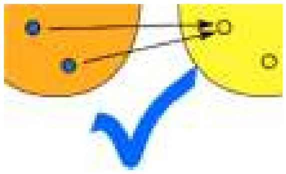
The Two Important Things!

1. "...each element..." means that every element in **X** is related to some element in **Y**.
2. "...exactly one..." means that a function is ***single/unique valued***. It will not give back 2 or more results for the same input.

If a relationship does not follow those two rules then it is **not a function** ... it would still be a relationship, just not a function.



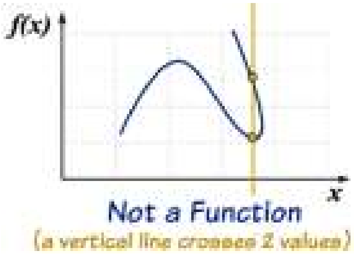
(one-to-many)
This is **NOT** OK in a function



(many-to-one)
But this **is** OK in a function

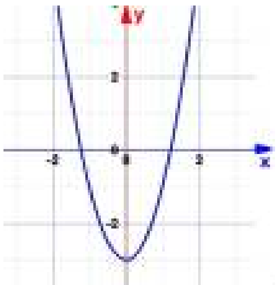
Vertical Line Test

On a graph, the idea of **single/unique valued** means that no vertical line would ever cross more than one value. If it **crosses more than once** it is still a valid curve, but it would **not be a function**.

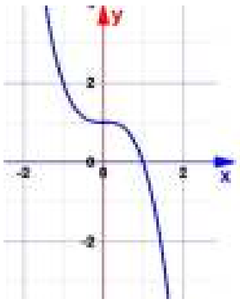


Example 6: Which one of the following is not a function?

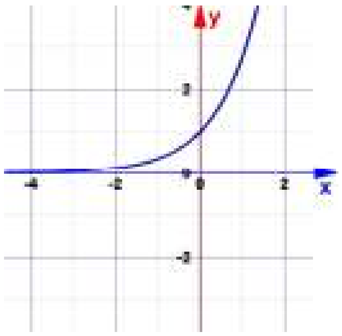
A



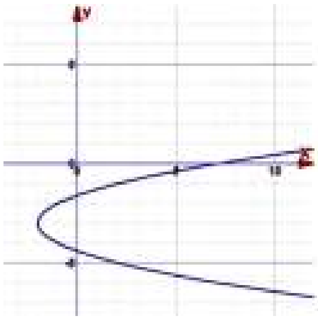
B



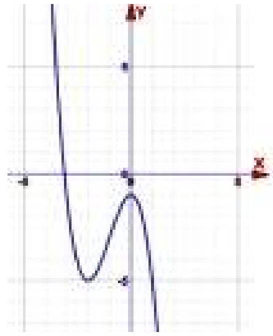
C



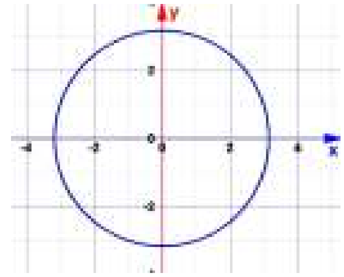
D



E



F



Composition of Functions

"Function Composition" is applying one function to the results of another:

The result of f( ) is sent through g( )

It is written: $(g \circ f)(x)$

Which means: $g(f(x))$

Symbol

The symbol for composition is a small circle: $(g \circ f)(x)$

Note:

1. $f(x)$ can be written as f.
2. $g[f(x)] = gf(x) = gf$ and usually $fg(x) \neq gf(x)$.
3. $f[f(x)]$ can be written as ff or f^2.
4. $f^4 = ffff = fff^2 = ff^3 = f^2f^2$.

Exercise 11A – Pure Mathematics 1(page 162) Question 10, 12, 15, 19 and 20.

Inverse Function

An inverse function goes in the opposite direction!

Symbol

The symbol for inverse is $f^{-1}(x)$.

The cool thing about the inverse is that it should give you back the original value:

If the function f turns the apple into a banana,
Then the **inverse** function f^{-1} turns the banana back to the apple.

So applying a function f and then its inverse f^{-1} gives us the original value back again:

$$f^{-1}(\, f(x)\,) = x$$

We could also have put the functions in the other order and it still works:

$$f(\, f^{-1}(x)\,) = x$$

Example 7:

The function: **f(*x*)** = **2*x*+3**

Solution (put "f^{-1}(y)" for "x") : $\mathbf{f^{-1}(y) = (y-3)/2}$

Example 8:

Find the inverse of each of these functions.

a) $f(x) = \dfrac{2x+5}{3}$

b) $f(x) = \dfrac{2x}{3} + 5$

c) $f(x) = 2\left(\dfrac{x}{3} + 5\right)$

d) $f(x) = \sqrt{x+4}, x \ge -4$

e) $f(x) = 3x^2 + 2, x > 0$

f) $f(x) = \dfrac{1}{x},\ x \ge 0$

g) $f(x) = \dfrac{1}{1-\dfrac{1}{x}}, x \ne 0, -1$

h) $f(x) = x^2 + 6x, x > -3$

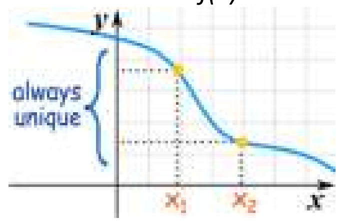
i) $f(x) = 2 - 2x - x^2, x \ge -1$

Self-Inverse Functions

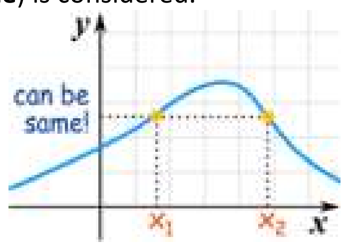
When $f(x) = f^{-1}(x)$ the function, $f(x)$ is said to be **self-inverse**.

Restricted Domain and One-to-One Functions

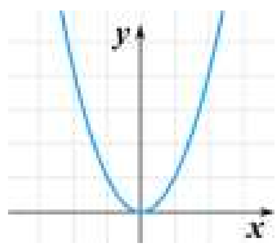
1. For an inverse function, $f^{-1}(x)$ to exist the function $f(x)$ must be one-to-one.



2. For a many-to-one function, an inverse can be defined by **restricting the domain** so that, a portion of the function(**one-to-one**) is considered.



Example 9:



The graph f(x) = x^2, is a many-to-one function. You are aware that a many to one function does not have an inverse function.

a) Determine a restricted domain so an inverse function exists.
b) Find the inverse function $f^{-1}(x) = x^2$.
c) Sketch y = f(x) and y = $f^{-1}(x)$ on one diagram.

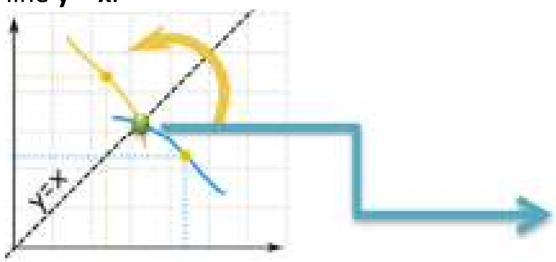
Example 10:

The function $f: x \to 6+2x+x^2$ has domain $x \in \Re$ and $x \le k$. Given that f is one to one, determine the greatest possible value of k. When k has this value,

a) Determine the range of f.
b) Find the inverse function $f^{-1}(x)$ and state its domain and range.

Graphing Inverse Functions

- If f is a one-one function, the graphs of **y = f(x)** and **y = f(x)$^{-1}$** are reflections of each other in the line **y = x**.



ff $^{-1}(x)$ = f $^{-1}$f(x) = x.

Or

f(x) = f(x)$^{-1}$ or f(x) = x or f(x)$^{-1}$ = x

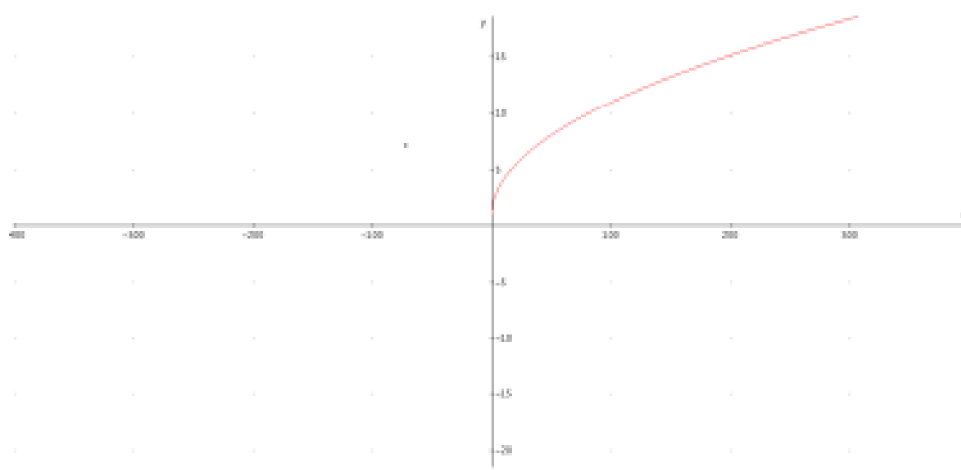
- graph of **f(x)** and **f^{-1}(x)** are symmetric across the line **y = x.**
  (**f(x)** and **f^{-1}(x)** are like mirror images)

Domain and Range of y = f(x) and y = f(x)$^{-1}$

- **Domain of *f* (*x*)** is the **Range of *f(x)$^{-1}$***
- **Range of *f* (*x*)** is the **Domain of *f(x)$^{-1}$***

Example 11:

The diagram shows the graph of the function *f* defined for $x \geq 0$ by f : x → 1 + $\sqrt{x}$.



a) Copy the sketch, and show on the same diagram the graph of f^{-1}(x), making clear the relationship between the two graphs.
b) Give an expression in terms of x for f^{-1}(x), and state the domain of f^{-1}(x).
c) Find the point of intersection between f(x) and f^{-1}(x).

Exercise 11B – Pure Mathematics 1(page 169) Question 7, 12, 13 and 14(Transformation of graph).