Differentiation – Implicit Functions

Many equations involving 2 variables x and y can be solved for y in terms of x:

➔ Example: $= 2x^2 + 4$, $y = sin^2x - 3$, $y = 3e^{2x} - 5e^x + 1$ (explicit function)

For great variety of equations, it is very difficult if not impossible to do so:

➔ Example: $x^2 + 2y^2 - 3xy + 4x - 5y = 0$, $xy + cosy - x^2 = 1$ (implicit function)

Remember The chain rule for differentiating composite functions: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Consider, for example, this implicit function:

$$x^2 + y^2 = 25 \qquad ①$$

The gradient of the curve is given by $\frac{dy}{dx}$.

Differentiating both sides of ① with respect to x

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25) \qquad ②$$

To differentiate y^2 with respect to x, the chain rule must be used:

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 2y\frac{dy}{dx}$$

Finally $\frac{dy}{dx} =$

Example 1

Find $\frac{dy}{dx}$ for each of these implicit functions.

a $2x^3 + 3y^2 = 7$ **b** $\frac{x^2}{x - y^2} = 5$ **c** $2x^2 + y^3 = 5xy$ **d** $x \ln y + y^2 = 10$

Example 2

Find $\frac{dy}{dx}$, in terms of x and y, for the curve $\tan(x + y) = y^2$.

Example 3

Find the gradient of the curve $x^2 - 3xy + y^2 = 31$ at the point $(2, -3)$.

Example 4

Find the equations of the tangent and normal to the curve $y^2e^x + x^2 = 9$ at the point $(0, 3)$.

Exercise 1

1 Differentiate with respect to x

a y^2 **b** y^3 **c** $3y^4$ **d** xy

e x^2y **f** xy^2 **g** $\ln y$ **h** $\ln y^5$

i $\ln x^2y^3$ **j** $\sin y$ **k** $x\cos y$ **l** x^2e^{2y}

m e^xy^2 **n** $\dfrac{1}{y}$ **o** $\dfrac{x}{y^3}$ **p** $\sin(x+y)$

2 Find $\dfrac{dy}{dx}$ for each of these implicit functions.

a $x^2+y^2=8$ **b** $2x^3+3y^4=10$ **c** $x^2+3xy=2y^2+4$

d $x^3-2xy^2+7x=0$ **e** $4x^2+6y^2=3x^2y^2$ **f** $3x^3+2x^2y+5xy^2+4y^3=8$

g $\dfrac{1}{x}+\dfrac{1}{y}=2$ **h** $\dfrac{x^2}{2x+5y^2}=2$

3 For each of these implicit functions, find $\dfrac{dy}{dx}$ in terms of x and y.

a $4e^xy-3xe^y=10$ **b** $x\tan y=10$ **c** $x\sin y+y\sin x=1$

d $3x\ln y=2y^2+8$ **e** $2\sin 2x\cos 3y=1$ **f** $e^x\ln y=y$

g $4xy-x\ln y^3=8$ **h** $2x\sin^2 y=3(x+y)^2$

4 Find the gradient of each for these curves at the points specified.

a $xy^2=20$ at $(5, 2)$ **b** $x^2+3xy+2y^2=15$ at $(1, 2)$

c $(x-1)^2+(y+2)^2=2$ at $(2, -3)$ **d** $\sec y=x+y$ at $(1, 0)$

e $e^xy+x^2y=2$ at $(0, 2)$ **f** $\dfrac{\sin x}{\sin y}=2$ at $\left(\dfrac{\pi}{2}, \dfrac{\pi}{6}\right)$

g $x\ln y^3=6$ at $(2, e)$ **h** $\dfrac{8x^2}{4x^2-3y^3}=3y$ at $(3, 2)$

Question 5

Find the equations of the tangent and normal to the curve $\sin x\sin y=\dfrac{\sqrt{3}}{4}$ at the point $\left(\dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$.

Question 6

Show that one of the points of intersection of the line $3y=2x+1$ and the curve $2x^2-3xy+y^2=5$ is the point $(4, 3)$. Find the equation of the tangent to the curve at this point. What is the area of the triangle bounded by the tangent and the axes?

Answers

1 **a** $2y\frac{dy}{dx}$ **b** $3y^2\frac{dy}{dx}$

c $12y^3\frac{dy}{dx}$ **d** $x\frac{dy}{dx}+y$

e $x^2\frac{dy}{dx}+2xy$ **f** $2xy\frac{dy}{dx}+y^2$

g $\frac{1}{y}\frac{dy}{dx}$ **h** $\frac{5}{y}\frac{dy}{dx}$

i $\frac{2}{x}+\frac{3}{y}\frac{dy}{dx}$ **j** $\cos y\frac{dy}{dx}$

k $\cos y - x\sin y\frac{dy}{dx}$

l $2xe^{2y}\left(x\frac{dy}{dx}+1\right)$

m $e^x y\left(2\frac{dy}{dx}+y\right)$

n $-\frac{1}{y^2}\frac{dy}{dx}$

o $\dfrac{y-3x\frac{dy}{dx}}{y^4}$

p $\cos(x+y)\left[1+\frac{dy}{dx}\right]$

2 **a** $-\frac{x}{y}$ **b** $-\frac{x^2}{2y^3}$

c $\frac{2x+3y}{4y-3x}$ **d** $\frac{3x^2-2y^2+7}{4xy}$

e $\frac{x(4-3y^2)}{3y(x^2-2)}$

f $-\frac{9x^2+4xy+5y^2}{2x^2+10xy+12y^2}$

g $-\frac{y^2}{x^2}$ **h** $\frac{x-2}{10y}$

3 **a** $\frac{3e^y-4e^x y}{4e^x-3xe^y}$

b $-\frac{1}{x}\sin y\cos y$

c $-\frac{\sin y+y\cos x}{x\cos y+\sin x}$

d $\frac{3y\ln y}{4y^2-3x}$

e $\frac{2}{3}\cot 2x\cot 3y$

f $\frac{e^x y\ln y}{y-e^x}$

g $\frac{y(4y-3\ln y)}{3x-4xy}$

h $\frac{3(x+y)-\sin^2 y}{x\sin 2y-3(x+y)}$

4 **a** $-\frac{1}{5}$ **b** $-\frac{8}{11}$ **c** 1

d -1 **e** -2 **f** 0

g $-\frac{e}{2}$ **h** $\frac{24}{45}$

Question 5

$x+3y-\frac{5\pi}{6}=0$, $y=3x-\frac{5\pi}{6}$

Question 6

$7x-6y-10=0$,

Area $=\frac{25}{21}$ square units

Exercise 2 – Implicit Differentiation

Question 1

The equation of a curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a},$$

where a is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of x and y. **[3]**

(ii) The straight line with equation $y = x$ intersects the curve at the point P. Find the equation of the tangent to the curve at P. **[3]**

Question 2

The equation of a curve is $x^3 + 2y^3 = 3xy$.

(i) Show that $\frac{dy}{dx} = \frac{y - x^2}{2y^2 - x}$. **[4]**

(ii) Find the coordinates of the point, other than the origin, where the curve has a tangent which is parallel to the x-axis. **[5]**

Question 3

The equation of a curve is $xy(x + y) = 2a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. **[8]**

Question 4

The equation of a curve is

$$x \ln y = 2x + 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{y}{x^2}$. [4]

(ii) Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

Question 5

The equation of a curve is $x^3 - x^2y - y^3 = 3$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [4]

(ii) Find the equation of the tangent to the curve at the point (2, 1), giving your answer in the form $ax + by + c = 0$. [2]

Question 6

A curve C has the equation $x^3 + xy + 2y^3 = k$, where k is a constant.

Determine $\frac{dy}{dx}$ in terms of x and y.

Given that C has a tangent parallel to the *y-axis*, verify that the y-coordinate of the point of contact of the tangent with C must satisfy $216y^6 + 4y^3 + k = 0$.

Hence verify that $k \le \frac{1}{54}$.

<u>Answers</u>

1. i) $-\sqrt{\frac{y}{x}}$ ii) $x + y = \frac{1}{2}a$
2. ii) (1,1)
3.
4. ii) $4x + y + 1 = 0$
5. i) $\frac{3x^2-2xy}{x^2+3y^2}$ ii) $8x - 7y - 9 = 0$
6. $\frac{dy}{dx} = -\frac{3x^2+y}{x+6y^2}$