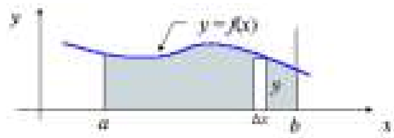
Application of Integration - Area Under a Curve

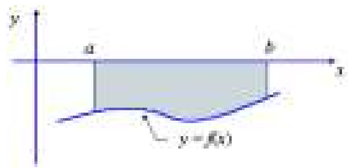
Case 1: Curves which are entirely above the *x*-axis.



In this case, we find the area by simply finding the integral: $\text{Area} = \int_a^b f(x)dx$

Case 2: Curves which are entirely below the *x*-axis

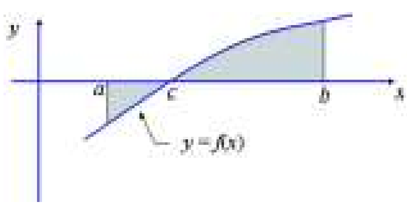
(for the range of *x* values being considered):



In this case, the integral gives a **negative number.** We need to take the **absolute value** of this to find our area:

$$\text{Area} = \left| \int_a^b f(x)dx \right|$$

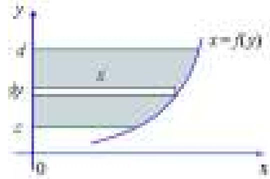
Case 3: Part of the curve is below the *x*-axis and part of the curve is above the *x*-axis.



In this case, we have to sum the individual parts, taking the absolute value for the section where the curve is below the *x*-axis (from $x = a$ to $x = c$).

$$A = \left| \int_a^c f(x)dx \right| + \int_c^b f(x)dx$$

<u>Case 4: Certain curves are much easier to sum vertically</u> (or only *possible* to sum vertically).

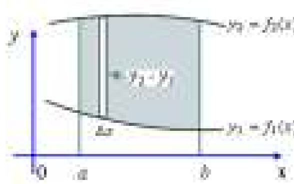


In this case, we find the area is the sum of the rectangles, heights $x = f(y)$ and width dy.

If we are given $y = f(x)$, then we need to re-express this as $x = f(y)$ and we need to **sum from bottom to top.**

So, in case 4 we have: $A = \int_c^d f(y)dy$

<u>Case 5: Area Between 2 Curves</u>



We are trying to find the area between 2 curves, $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and the lines $x = a$ and $x = b$.

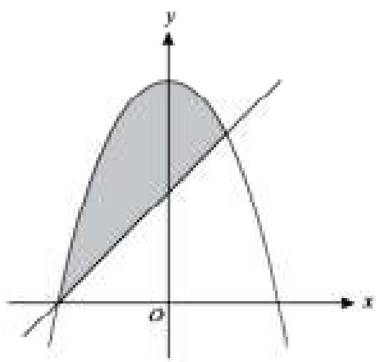
We see that if we subtract the area under lower curve $y_1 = f_1(x)$ from the area under the upper curve

$y_2 = f_2(x)$, then we will find the required area. This can be achieved in one step:

$$A = \int_a^b (y_2 - y_1)dx$$

Exercise – Area Under a Curve

Question 1

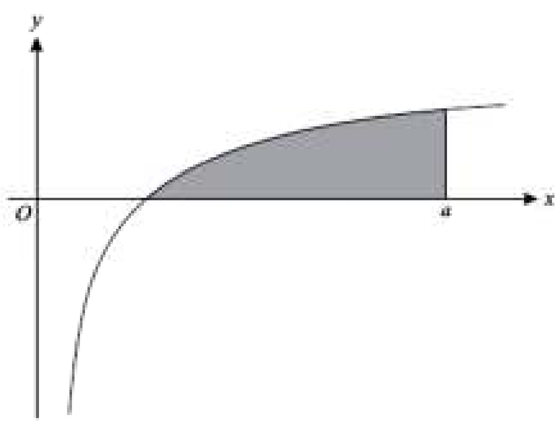


The diagram shows the curve $y = 4 - x^2$ and the line $y = x + 2$.

(i) Find the x-coordinates of the points of intersection of the curve and the line. [2]

(ii) Use integration to find the area of the shaded region bounded by the line and the curve. [6]

Question 2



The diagram shows the graph of $y = 1 - 3x^{-\frac{1}{2}}$.

(i) Verify that the curve intersects the x-axis at $(9, 0)$. [1]

(ii) The shaded region is enclosed by the curve, the x-axis and the line $x = a$ (where $a > 9$). Given that the area of the shaded region is 4 square units, find the value of a. [9]

Question 3

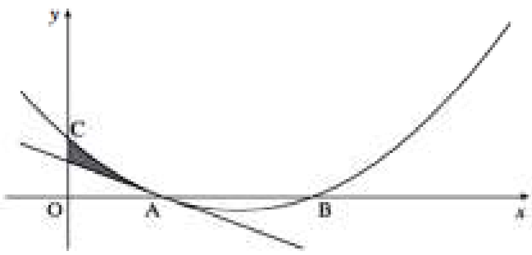


Fig. 9

The curve $y = x^2 - 7x + 10$ cuts the x-axis at A and B and the y-axis at C as shown in Fig. 9.

(i) Write down the coordinates of C.

Show that A is (2, 0) and find the coordinates of B.

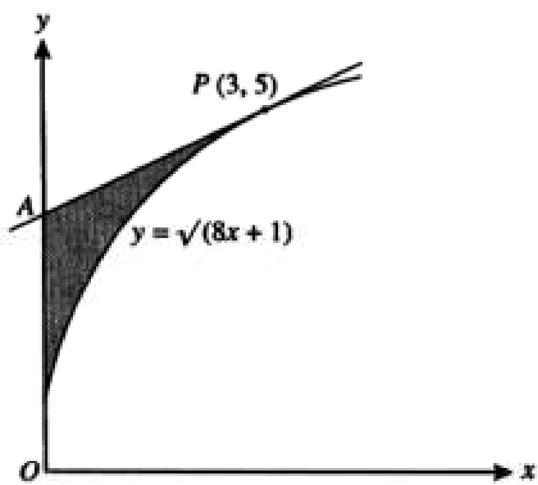
Hence or otherwise find the coordinates of the turning point of the curve. [5]

(ii) Solve the inequality $x^2 - 7x + 10 > 0$. [2]

(iii) Show that the equation of the tangent to the curve at A is $y = -3x + 6$. [3]

(iv) Calculate the area of the region bounded by the portion AC of the curve, the tangent at A and the y-axis. This region is shown shaded in Fig. 9. [5]

Question 4



The diagram shows the curve $y = \sqrt{(8x+1)}$ and the tangent at the point $P\,(3, 5)$ on the curve. This tangent meets the y-axis at A. Find

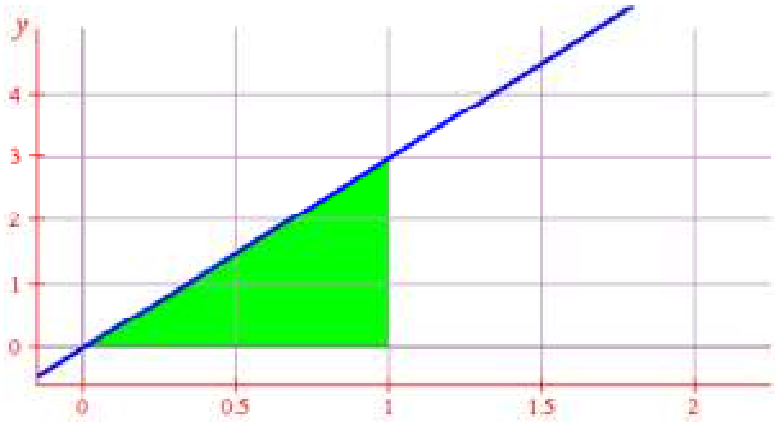
(i) the equation of the tangent at P, [4]

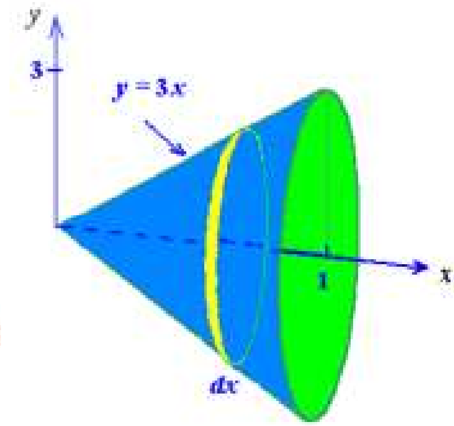
(ii) the coordinates of A, [1]

(iii) the area of the shaded region. [6]

Integration: Volume of Solid of Revolution

Consider the area bounded by the straight line $y = 3x$ the x-axis, and $x = 1$





Find the volume of the cone using $V = \frac{\pi r^2 h}{3} = \frac{\pi(3)^2(1)}{3} = \frac{9\pi}{3} = 3\pi \text{ unit}^3$

Rotation around the x-axis

When the shaded area is rotated 360° about the x-axis, the volume that is generated:

$\mathrm{V} = \pi\int_a^b y^2\,dx$ which means $\mathrm{V} = \pi\int_a^b [f(x)]^2\,dx$ where:

$y = f(x)$ is the equation of the curve whose area is being rotated

a and b are the limits of the area being rotated

dx shows that the area is being rotated about the x-axis

Applying the formula $\mathrm{V} = \pi\int_a^b y^2\,dx$ to the earlier example, we have:

$$\begin{aligned} V &= \pi\int_a^b y^2\,dx \\ &= \pi\int_0^1 (3x)^2\,dx \\ &= \pi\int_0^1 9x^2\,dx \\ &= \pi[3x^3]_0^1 \\ &= \pi[3] - \pi[0] \\ &= 3\pi \text{ unit}^3 \end{aligned}$$

Example 1:

Find the volume if the area bounded by the curve $y = x^3 + 1$, the x-axis and the limits of $x = 0$ and $x = 3$ is rotated around the x-axis.

Rotation around the y-axis

When the shaded area is rotated 360° about the y-axis, the volume that is generated:

$$V = \pi\int_c^d x^2 dy$$ which means $$V = \pi\int_c^d [f(y)]^2 dy$$

where:

$x = f(y)$ is the equation of the curve **expressed in terms of *y***

c and d are the upper and lower y limits of the area being rotated

dy shows that the area is being rotated about the y-axis

Example 2:

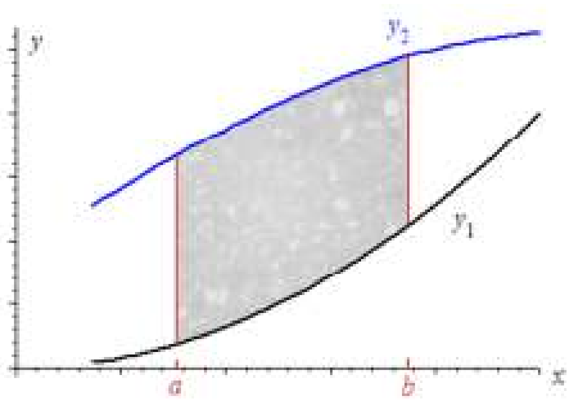
Find the volume of the solid of revolution generated by rotating the curve $y = x^3$ between $y = 0$ and $y = 4$ about the y-axis.

Volume by Rotating the Area Enclosed Between 2 Curves

If we have 2 curves y_2 and y_1 that enclose some area and we rotate that area around the x-axis, then the volume of the solid formed is given by:

$$\boxed{\text{Volume} = \pi\int_a^b [(y_2)^2 - (y_1)^2]dx}$$

The lower and upper limits for the region to be rotated are indicated in dark red: $x = a$ to $x = b$.

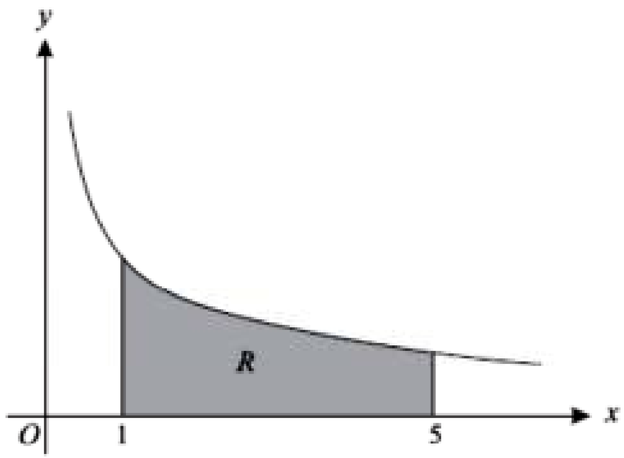


Example 3:

A cup is made by rotating the area between $y = 2x^2$ and $y = x + 1$ with $x \geq 0$ around the x-axis. Find the volume of the material needed to make the cup. Units are in cm.

Exercise – Volume of Solid of Revolution

Question 1

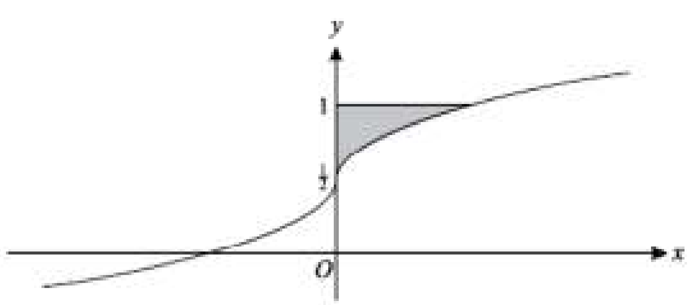


The diagram shows the curve $y = \dfrac{2}{\sqrt{x}}$. The region R, shaded in the diagram, is bounded by the curve and by the lines $x = 1$, $x = 5$ and $y = 0$. The region R is rotated completely about the x-axis. Find the exact volume of the solid formed. [4]

Question 2

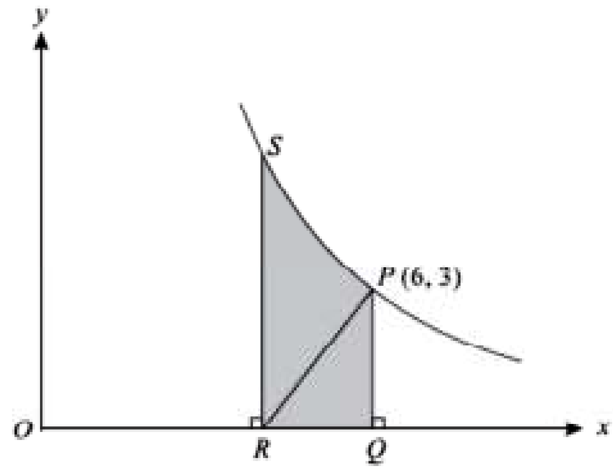
(i) Given that $y = \frac{1}{4}(2 + \sqrt[5]{x})$, show that x may be expressed in the form $(ay + b)^5$, where the values of the constants a and b are to be stated. [2]

(ii)



The diagram shows a sketch of the curve $y = \frac{1}{4}(2 + \sqrt[5]{x})$. The shaded region is bounded by part of the curve and the lines $x = 0$ and $y = 1$. The shaded region is rotated through four right angles about the y-axis. Find the exact volume of the solid produced. [4]

Question 3



The diagram shows part of the graph of $y = \frac{18}{x}$ and the normal to the curve at P (6, 3). This normal meets the x-axis at R. The point Q on the x-axis and the point S on the curve are such that PQ and SR are parallel to the y-axis.

(i) Find the equation of the normal at P and show that R is the point $(4\frac{1}{2}, 0)$. [5]

(ii) Show that the volume of the solid obtained when the shaded region $PQRS$ is rotated through 360° about the x-axis is 18π. [4]

Question 4

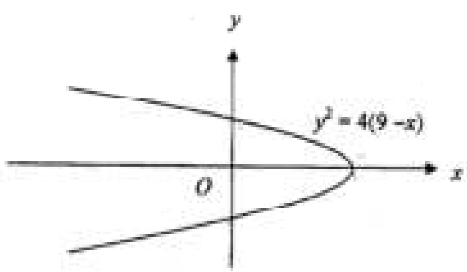


Diagram shows a region bounded by the parabola $y^2 = 4(9 - x)$ and the y-axis.

Find

(i) the points of intersections between the parabola and the two axes; [3]

(ii) the enclosed area, and [3]

(iii) the volume generated by rotating the enclosed region about the x-axis. [3]