9 Use a vector method to calculate the angles between the following pairs of vectors, giving your answers in degrees to one place of decimals, where appropriate.

(a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ 　　　(b) $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$ 　　　(c) $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 9 \end{pmatrix}$

(d) $\begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ 　　　(e) $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ 　　　(f) $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$

10 Let $\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Calculate $|\mathbf{r}_2 - \mathbf{r}_1|$ and interpret your result geometrically.

11 Find the angle between the line joining $(1,2)$ and $(3,-5)$ and the line joining $(2,-3)$ to $(1,4)$.

12 Find the angle between the line joining $(1,3,-2)$ and $(2,5,-1)$ and the line joining $(-1,4,3)$ to $(3,2,1)$.

13 Find the angle between the diagonals of a cube.

14 *ABCD* is the base of a square pyramid of side 2 units, and *V* is the vertex. The pyramid is symmetrical, and of height 4 units. Calculate the acute angle between *AV* and *BC*, giving your answer in degrees correct to 1 decimal place.

15 Two aeroplanes are flying in directions given by the vectors $300\mathbf{i} + 400\mathbf{j} + 2\mathbf{k}$ and $-100\mathbf{i} + 500\mathbf{j} - \mathbf{k}$. A person from the flight control centre is plotting their paths on a map. Find the acute angle between their paths on the map.

16 The roof of a house has a rectangular base of side 4 metres by 8 metres. The ridge line of the roof is 6 metres long, and centred 1 metre above the base of the roof. Calculate the acute angle between two opposite slanting edges of the roof.

Miscellaneous exercise 13

1 Find which pairs of the following vectors are perpendicular to each other.

$\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ 　　　$\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ 　　　$\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ 　　　$\mathbf{d} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

2 The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ respectively. The vector $\overrightarrow{AD}$ is the sum of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Determine the acute angle, in degrees correct to one decimal place, between the diagonals of the parallelogram defined by the points A, B, C and D. 　　　(OCR)

3 The vectors **AB** and **AC** are $\begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix}$ respectively.

(a) Determine the lengths of the vectors.

(b) Find the scalar product **AB.AC**.

(c) Use your result from part (b) to calculate the acute angle between the vectors. Give the angle in degrees correct to one decimal place. 　　　(OCR)

4 The points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$

respectively, with respect to a fixed origin. The point D is such that $ABCD$, in that order, is a parallelogram.

  (a) Find the position vector of D.

  (b) Find the position vector of the point at which the diagonals of the parallelogram intersect.

  (c) Calculate the angle BAC, giving your answer to the nearest tenth of a degree.

                                                                 (OCR)

5 A vertical aerial is supported by three straight cables, each attached to the aerial at a point P, 30 metres up the aerial. The cables are attached to the horizontal ground at points A, B and C, each x metres from the foot O of the aerial, and situated symmetrically around it (see the diagrams).

Suppose that $\mathbf{i}$ is the unit vector in the direction $\overrightarrow{OA}$, $\mathbf{j}$ is the unit vector perpendicular to $\mathbf{i}$ in the plane of the ground, as shown in the Plan view, and $\mathbf{k}$ is the unit vector in the direction $\overrightarrow{OP}$.
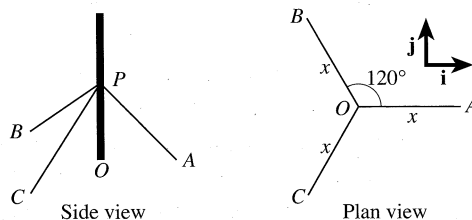
  (a) Write down expressions for the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ in terms of x, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

  (b) (i) Write down an expression for the vector $\overrightarrow{AP}$ in terms of vectors $\overrightarrow{OA}$ and $\overrightarrow{OP}$.

      (ii) Hence find expressions for the vectors $\overrightarrow{AP}$ and $\overrightarrow{BP}$ in terms of x, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

  (c) Given that $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are perpendicular to each other, find the value of x.   (OCR)

6 The position vectors of three points A, B and C with respect to a fixed origin O are $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ respectively. Find unit vectors in the directions of $\overrightarrow{CA}$ and $\overrightarrow{CB}$. Calculate angle ACB in degrees, correct to 1 decimal place.   (OCR)

7 (a) Find the angle between the vectors $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.

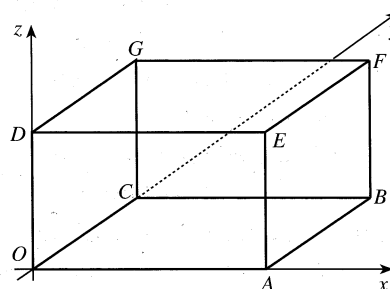 (b) The vectors $\mathbf{a}$ and $\mathbf{b}$ are non-zero.

    (i) Given that $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{a} - \mathbf{b}$, prove that $|\mathbf{a}| = |\mathbf{b}|$.

    (ii) Given instead that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, prove that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

                                                                 (OCR)

8 $OABCDEFG$, shown in the figure, is a cuboid. The position vectors of A, C and D are $4\mathbf{i}$, $2\mathbf{j}$ and $3\mathbf{k}$ respectively. Calculate

  (a) $|AG|$,

  (b) the angle between AG and OB.

                        (OCR)

9 The three-dimensional vector $\mathbf{r}$, which has positive components, has magnitude 1 and makes angles of 60° with each of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

  (a) Write $\mathbf{r}$ as a column vector.

  (b) State the angle between $\mathbf{r}$ and the unit vector $\mathbf{k}$.

10 The points A, B and C have position vectors given respectively by $\mathbf{a} = 7\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{c} = 6\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$.

  (a) Find the angle BAC.   (b) Find the area of the triangle ABC.   (OCR)

11 The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively relative to the origin O. P is the point on BC such that $\overrightarrow{PC} = \frac{1}{10}\overrightarrow{BC}$.

  (a) Show that the position vector of P is $\frac{1}{10}(9\mathbf{c} + \mathbf{b})$.

  (b) Given that the line AP is perpendicular to the line BC, show that
    $(9\mathbf{c} + \mathbf{b}).(\mathbf{c} - \mathbf{b}) = 10\mathbf{a}.(\mathbf{c} - \mathbf{b})$.

  (c) Given also that OA, OB and OC are mutually perpendicular, prove that $OC = \frac{1}{3}OB$.
                                        (OCR)

12 A mathematical market trader packages fruit in three sizes. An Individual bag holds 1 apple and 2 bananas; a Jumbo bag holds 4 apples and 3 bananas; and a King-size bag holds 8 apples and 7 bananas. She draws two vector arrows $\mathbf{a}$ and $\mathbf{b}$ to represent an apple and a banana respectively, and then represents the three sizes of bag by vectors $\mathbf{I} = \mathbf{a} + 2\mathbf{b}$, $\mathbf{J} = 4\mathbf{a} + 3\mathbf{b}$ and $\mathbf{K} = 8\mathbf{a} + 7\mathbf{b}$. Find numbers s and t such that $\mathbf{K} = s\mathbf{I} + t\mathbf{J}$.

  By midday she has sold all her King-size bags, but she has plenty of Individual and Jumbo bags left. She decides to make up some more King-size bags by using the contents of the other bags. How can she do this so that she has no loose fruit left over?

13 $ABCD$ is a parallelogram. The coordinates of A, B, D are $(4,2,3)$, $(18,4,8)$ and $(-1,12,13)$ respectively. The origin of coordinates is O.

  (a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Find the coordinates of C.

  (b) Show that $\overrightarrow{OA}$ can be expressed in the form $\lambda\,\overrightarrow{AB} + \mu\,\overrightarrow{AD}$, stating the values of λ and μ. What does this tell you about the plane $ABCD$?   (MEI)

14 A balloon flying over flat land reports its position at 7.40 a.m. as $(7.8, 5.4, 1.2)$, the coordinates being given in kilometres relative to a checkpoint on the ground. By 7.50 a.m. its position has changed to $(9.3, 4.4, 0.7)$. Assuming that it continues to descend at the same speed along the same line, find the coordinates of the point where it would be expected to land, and the time when this would occur.

15 Prove that, if $(\mathbf{c} - \mathbf{b}).\mathbf{a} = 0$ and $(\mathbf{c} - \mathbf{a}).\mathbf{b} = 0$, then $(\mathbf{b} - \mathbf{a}).\mathbf{c} = 0$. Show that this can be used to prove the following geometrical results.

  (a) The lines through the vertices of a triangle ABC perpendicular to the opposite sides meet in a point.

  (b) If the tetrahedron $OABC$ has two pairs of perpendicular opposite edges, the third pair of edges is perpendicular.

  Prove also that, in both cases, $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$.