Exercise 1: Long Division

Question 1 - 12

Find the quotient when $(x^3 + x^2 - 6x)$ is divided by $(x - 2)$.

Divide $(2x^3 - x^2 - 5x + 10)$ by $(x + 2)$.

Find the quotient when $(x^4 + x^2 - 2)$ is divided by $(x - 1)$.

Divide $(2x^3 - 10x^2 + 3x - 15)$ by $(x - 5)$.

Find the quotient when $(x^4 + 5x^3 + 6x^2 + 5x + 15)$ is divided by $(x + 3)$.

Divide $(2x^4 + 5x^3 + 4x^2 + x)$ by $(2x + 1)$.

Find the quotient when $(4x^4 + 4x^3 - x^2 + 7x - 4)$ is divided by $(2x - 1)$.

Divide $(2x^4 + 2x^3 + 5x^2 + 2x + 3)$ by $(x^2 + 1)$.

Find the quotient when $(x^4 + 3x^3 - 8x^2 - 27x - 9)$ is divided by $(x^2 - 9)$.

Divide $(x^4 + x^3 + 4x^2 + 4x)$ by $(x^2 + x)$.

Find the quotient when $(2x^4 - 5x^3 - 16x^2 - 6x)$ is divided by $(2x^2 + 3x)$.

Divide $(x^4 + 3x^3 + x^2 - 2)$ by $(x^2 + x + 1)$.

Exercise 2: Factor and Remainder Theorem

1 Given that $f(x) = x^3 + 2x^2 - 9x - 18$:

(i) find $f(-3)$, $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$ and $f(3)$

(ii) factorise $f(x)$

(iii) solve the equation $f(x) = 0$

(iv) sketch the curve with the equation $y = f(x)$.

2 The polynomial $p(x)$ is given by $p(x) = x^3 - 4x$.

(i) Find the values of $p(-3)$, $p(-2)$, $p(-1)$, $p(0)$, $p(1)$, $p(2)$ and $p(3)$.

(ii) Factorise $p(x)$.

(iii) Solve the equation $p(x) = 0$.

(iv) Sketch the curve with the equation $y = p(x)$.

3 You are given that $f(x) = x^3 - 19x + 30$.

(i) Calculate $f(0)$ and $f(3)$. Hence write down a factor of $f(x)$.

(ii) Find p and q such that $f(x) \equiv (x - 2)(x^2 + px + q)$.

(iii) Solve the equation $x^3 - 19x + 30 = 0$.

(iv) Without further calculation draw a sketch of $y = f(x)$.

4 **(i)** Show that $x-3$ is a factor of $x^3-5x^2-2x+24$.

(ii) Solve the equation $x^3-5x^2-2x+24=0$.

(iii) Sketch the curve with the equation $y=x^3-5x^2-2x+24$.

5 **(i)** Show that $x=2$ is a root of the equation $x^4-5x^2+2x=0$ and write down another integer root.

(ii) Find the other two roots of the equation $x^4-5x^2+2x=0$.

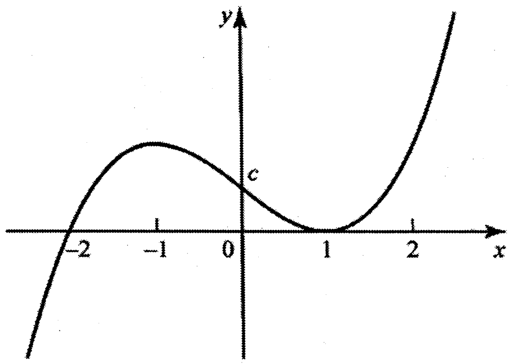
(iii) Sketch the curve with the equation $y=x^4-5x^2+2x$.

6 **(i)** The polynomial $p(x)=x^3-6x^2+9x+k$ has a factor $x-4$. Find the value of k.

(ii) Find the other factors of the polynomial.

(iii) Sketch the curve with the equation $y=p(x)$.

7 The diagram shows the curve with the equation $y=(x+a)(x-b)^2$ where a and b are positive integers.



(i) Write down the values of a and b, and also of c, given that the curve crosses the y axis at $(0, c)$.

(ii) Solve the equation $(x+a)(x-b)^2=c$ using the values of a, b and c you found in part **(i)**.

8 The function $f(x)$ is given by $f(x)=x^4-3x^2-4$ for real values of x.

(i) By treating $f(x)$ as a quadratic in x^2, factorise it in the form $(x^2+\ldots)(x^2+\ldots)$.

(ii) Complete the factorisation as far as possible.

(iii) How many real roots has the equation $f(x)=0$? What are they?

9 **(i)** Show that $x-2$ is not a factor of $2x^3+5x^2-7x-3$.

(ii) Find the quotient and the remainder when $2x^3+5x^2-7x-3$ is divided by $x-2$.

10 The equation $f(x)=x^3-4x^2+x+6=0$ has three integer roots.

(i) List the eight values of a for which it is sensible to check whether $f(a)=0$ and check each of them.

(ii) Solve $f(x)=0$.

11 Factorise, as far as possible, the following expressions.

(i) $x^3 - x^2 - 4x + 4$ given that $(x - 1)$ is a factor.
(ii) $x^3 + 1$ given that $(x + 1)$ is a factor.
(iii) $x^3 + x - 10$ given that $(x - 2)$ is a factor.
(iv) $x^3 + x^2 + x + 6$ given that $(x + 2)$ is a factor.

12 (i) Show that neither $x = 1$ nor $x = -1$ is a root of $x^4 - 2x^3 + 3x^2 - 8 = 0$.
(ii) Find the quotient and the remainder when $x^4 - 2x^3 + 3x^2 - 8$ is divided by
(a) $(x - 1)$ **(b)** $(x + 1)$ **(c)** $(x^2 - 1)$.

13 When $2x^3 + 3x^2 + kx - 6$ is divided by $x + 1$ the remainder is 7.
Find the value of k.

14 When $x^3 + px^2 + p^2x - 36$ is divided by $x - 3$ the remainder is 21.
Find a possible value of p.

15 When $x^3 + ax^2 + bx + 8$ is divided by $x - 3$ the remainder is 2 and when it is divided by $x + 1$ the remainder is -2.
Find a and b and hence obtain the remainder on dividing by $x - 2$.

16 When $f(x) = 2x^3 + ax^2 + bx + 6$ is divided by $x - 1$ there is no remainder and when $f(x)$ is divided by $x + 1$ the remainder is 10.
Find a and b and hence solve the equation $f(x) = 0$.

Exercise 3: The Modulus Function

1 Solve the following equations.

(i) $|x + 4| = 5$
(ii) $|x - 3| = 4$
(iii) $|3 - x| = 4$
(iv) $|4x - 1| = 7$
(v) $|2x + 1| = 5$
(vi) $|8 - 2x| = 6$
(vii) $|2x + 1| = |x + 5|$
(vii) $|4x - 1| = |9 - x|$
(ix) $|3x - 2| = |4 - x|$

2 Solve the following inequalities.

(i) $|x + 3| < 5$
(ii) $|x - 2| \leqslant 2$
(iii) $|x - 5| > 6$
(iv) $|x + 1| \geqslant 2$
(v) $|2x - 3| < 7$
(vi) $|3x - 2| \leqslant 4$

3 Express each of the following inequalities in the form $|x-a|<b$, where a and b are to be found.

(i) $-1<x<3$
(ii) $2<x<8$
(iii) $-2<x<4$
(iv) $-1<x<6$
(v) $9.9<x<10.1$
(vi) $0.5<x<7.5$

4 Sketch each of the following graphs on a separate set of axes.

(i) $y=|x+2|$
(ii) $y=|2x-3|$
(iii) $y=|x+2|-2$
(iv) $y=|x|+1$
(v) $y=|2x+5|-4$
(vi) $y=3+|x-2|$

5 Solve the following inequalities.

(i) $|x+3|<|x-4|$
(ii) $|x-5|>|x-2|$
(iii) $|2x-1|\leqslant|2x+3|$
(iv) $|2x|\leqslant|x+3|$
(v) $|2x|>|x+3|$
(vi) $|2x+5|\geqslant|x-1|$

Exercise 4A: Partial Fractions

Express the following fractions in partial fractions.

1 $\dfrac{x-2}{(x+1)(x-1)}$

2 $\dfrac{2x-1}{(x-1)(x-7)}$

3 $\dfrac{4}{(x+3)(x-2)}$

4 $\dfrac{7x}{(2x-1)(x+4)}$

5 $\dfrac{2}{x(x-2)}$

6 $\dfrac{2x-1}{x^2-3x+2}$

7 $\dfrac{3}{x^2-9}$

8 $\dfrac{6x+7}{3x(x+1)}$

9 $\dfrac{9}{2x^2+x}$

10 $\dfrac{x+1}{3x^2-x-2}$

11 $\dfrac{2}{(x+1)(x-1)}$

12 $\dfrac{3}{(x-2)(x+1)}$

13 $\dfrac{1}{x(x-3)}$

14 $\dfrac{4}{(x-1)(x+3)}$

15 $\dfrac{1}{(x^2-1)}$

16 $\dfrac{2}{(2x+1)(2x-1)}$

17 $\dfrac{3x^2+4x-1}{(x-1)(x+1)(x+2)}$

18 $\dfrac{x}{(x-2)(x+2)(x-1)}$

Exercise 4B: Partial Fractions

Express in partial fractions

1 $\dfrac{2}{(x-1)(x+1)^2}$

2 $\dfrac{x^2+3}{x(x^2+2)}$

3 $\dfrac{2x^2+x+1}{(x-3)(x+1)^2}$

4 $\dfrac{x^2+1}{x(2x^2+1)}$

5 $\dfrac{x}{(x-1)(x-2)^2}$

6 $\dfrac{(x^2-1)}{x^2(2x+1)}$

7 $\dfrac{x^2-2}{(x+3)(x-1)^2}$

8 $\dfrac{(x-1)}{(x+1)(x+2)^2}$

9 Express $\dfrac{x}{(x^2-4)(x-1)}$ in partial fractions

(a) by first treating (x^2-4) as a quadratic factor

(b) by first factorising (x^2-4).

State which method you think is better and explain why.

Exercise 4C: Partial Fractions

Express in partial fractions.

1 $\dfrac{x^2}{(x+1)(x-1)}$

2 $\dfrac{x^2+3}{(x-1)(x+1)}$

3 $\dfrac{x^2-2}{(x+3)(x-1)}$

4 $\dfrac{x^3}{(x+2)(x^2+1)}$

Exercise 4D: Partial Fractions

In questions **3** to **8** express the given fraction in partial fractions.

3 $\dfrac{3x-1}{x^2(x-3)}$

4 $\dfrac{1-4x}{(x^2+1)(x+4)}$

5 $\dfrac{8}{(x+3)(x-1)^2}$

6 $\dfrac{x^2}{(x+1)^2(x-1)}$

7 $\dfrac{x}{(x-1)(x^2+5)}$

8 $\dfrac{3-x}{(x^2+2)(x+2)}$

9 Express as the sum of a constant and partial fractions.

(a) $\dfrac{x^2}{(x+1)(x+2)}$ (b) $\dfrac{x^3+3}{x^2(x+1)}$

10 Express y in partial fractions and hence find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

(a) $y=\dfrac{2x}{(x-1)(x-2)}$

(b) $y=\dfrac{x}{(x+3)(x-2)}$

(c) $y=\dfrac{x^2+x+3}{(x+1)(x+2)(x+3)}$

Exercise 5: Binomial Expansion

Expand the following functions as series of ascending powers of x up to and including the term in x^3. In each case give the range of values of x for which the expansion is valid.

1 $(1-2x)^{\frac{1}{2}}$

2 $(1+5x)^{-2}$

3 $\left(1-\frac{1}{2}x\right)^{-3}$

4 $(1+x)^{\frac{3}{2}}$

5 $(3+x)^{-1}$

6 $\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}$

7 $\dfrac{1}{(1-x)^2}$

8 $\sqrt{\dfrac{1}{1+x}}$

9 $(1+x)\sqrt{1-x}$

10 $\dfrac{x+2}{x-1}$

11 $\dfrac{2-x}{\sqrt{1-3x}}$

12 $\dfrac{1}{(2-x)(1+2x)}$

13 $\sqrt{\dfrac{1+x}{1-x}}$

14 $\left(1+\frac{x^2}{9}\right)^{-1}$

15 $\dfrac{x}{(1+x)(1-2x)}$

16 $\left(1+\frac{1}{x}\right)^{-1}$ $\left[\text{Hint: } \left(1+\frac{1}{x}\right)^{-1} \equiv \left(\frac{x+1}{x}\right)^{-1} \equiv \frac{x}{1+x}\right]$

17 Expand $\left(1+\frac{1}{p}\right)^{-3}$ as a series of descending powers of p, as far as and including the term containing p^{-4}. State the range of values of p for which the expansion is valid.

$\left(\text{Hint: Replace } x \text{ by } \frac{1}{p} \text{ in } (1+x)^{-3}\right)$

18 Expand $\sqrt{\dfrac{1+2x}{1-2x}}$ as a series of ascending powers of x up to and including the term in x^2.

19 If x is so small that x^2 and higher powers of x may be neglected, show that

$$\frac{1}{(x-1)(x+2)} \approx -\frac{1}{2}-\frac{1}{4}x$$

20 By neglecting x^3 and higher powers of x, find a quadratic function that approximates to the function $\dfrac{1-2x}{\sqrt{1+2x}}$ for small values of x.

21 Find a quadratic function that approximates to

$$f(x)=\frac{1}{\sqrt[3]{(1-3x)^2}}$$

for values of x that are small enough for x^3 and higher powers to be neglected.

22 Use partial fractions and the binomial series to show that

$$\frac{3}{(1-2x)(2-x)} \approx \frac{3}{2}+\frac{15}{4}x$$

23 If terms containing x^4 and higher powers of x can be neglected, show that

$$\frac{2}{(x+1)(x^2+1)} \approx 2(1-x)$$

24 Show that

$$\frac{12}{(3+x)(1-x)^2} \approx 4+\frac{20}{3}x+\frac{88}{9}x^2$$

provided that x is small enough to neglect powers higher than 2.

25 If x is very small, show that

$$\frac{1}{(3-x)^3} \approx \frac{1}{729}(27+27x+18x^2+10x^3)$$

Exercise 6: Partial Fractions and Binomial Expansion

1 Find the first three terms in ascending powers of x in the binomial expansion of the following fractions.

(i) $\dfrac{4}{(1-3x)(1-x)^2}$ **(ii)** $\dfrac{4+2x}{(2x-1)(x^2+1)}$

(iii) $\dfrac{5-2x}{(x-1)^2(x+2)}$ **(iv)** $\dfrac{2x+1}{(x-2)(x^2+4)}$

2 **(i)** Express $\dfrac{7-4x}{(2x-1)(x+2)}$ in partial fractions as $\dfrac{A}{(2x-1)}+\dfrac{B}{(x+2)}$ where A and B are to be found.

(ii) Find the expansion of $\dfrac{1}{(1-2x)}$ in the form $a+bx+cx^2+\ldots$ where a, b and c are to be found.
Give the range of values of x for which this expansion is valid.

(iii) Find the expansion of $\dfrac{1}{(2+x)}$ as far as the term containing x^2.
Give the range of values of x for which this expansion is valid.

(iv) Hence find a quadratic approximation for $\dfrac{7-4x}{(2x-1)(x+2)}$ when $|x|$ is small.
Find the percentage error in this approximation when $x = 0.1$.

[MEI]

3 **(i)** Expand $(2-x)(1+x)$.
Hence express $\dfrac{3x}{2+x-x^2}$ in partial fractions.

(ii) Use the binomial expansion of the partial fractions in part **(i)** to show that

$$\frac{3x}{2+x-x^2}=\frac{3}{2}x-\frac{3}{4}x^2+\ldots.$$

State the range of values of x for which this result is valid.

[MEI, *part*]

4 **(i)** Given that $f(x)=\dfrac{8x-6}{(1-x)(3-x)}$, express $f(x)$ in partial fractions.

Hence show that

$$f'(x)=(1-x)^{-2}-\left(1-\frac{x}{3}\right)^{-2}.$$

(ii) Using the results in part **(i)**, or otherwise, find the x co-ordinates of the stationary points on the graph of $y=f(x)$.

(iii) Use the binomial expansion, together with the result in part **(i)**, to expand $f'(x)$ in powers of x up to and including the term in x^2.

(iv) Show that, when $f'(x)$ is expanded in powers of x, the coefficients of all the powers of x are positive.

[MEI]