TOPIC 4 – Differentiation

Objectives **(A2 syllabus)**:

- use the derivatives of e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, together with constant multiples, sums, differences and composites;
- differentiate products and quotients;
- find and use the first derivative of a function which is defined parametrically or implicitly.

<u>Rules of Differentiation</u>

1. $\frac{d}{dx}(k) = 0$ where k is a constant
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
4. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\frac{d}{dx}[f(x)]$
5. $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ ---→ The Chain Rule

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.}$$

The Product Rule and Quotient Rule

Product Rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1

Differentiate the following with respect to *x*.

i) $x^2(x+3)^3$ ii) $x^2\sqrt{(1+x^2)}$.

Example 2

Differentiate the following with respect to *x*.

i) $\frac{x-2}{x+2}$ ii) $\frac{x^2}{2x+1}$

Exercise 1 - Product Rule and Quotient Rule

Question 1

Find $\frac{d^2y}{dx^2}$ given that

(a) $y = \frac{1}{x^2+1}$, (b) $y = \frac{x}{x-1}$, (c) $y = \sqrt{(x^2+1)}$.

(a) $\frac{2(3x^2-1)}{(x^2+1)^3}$, (b) $\frac{2}{(x-1)^3}$, (c) $\frac{1}{(x^2+1)^{3/2}}$

Question 2

Differentiate the following with respect to x

(a) $\frac{(x^2+1)(x-1)^2}{2x-1}$, (b) $\frac{x\sqrt{(x-1)}}{x+1}$, (c) $\sqrt{\left\{\frac{x^3+1}{(x+1)^3}\right\}}$.

(a) $\frac{2x(x-1)(3x^2-3x+2)}{(2x-1)^2}$, (b) $\frac{x^2+3x-2}{2(x+1)^2\sqrt{(x-1)}}$, (c) $\frac{3(x-1)}{2\sqrt{\{(x+1)^3(x^3+1)\}}}$.

Question 3

Given that $y = \frac{1-x}{1+x}$, show that

$$\frac{d^2y}{dx^2} = \frac{4}{(1+x)^3}$$

Question 4

$f(x) \equiv \frac{x}{x^2+2}$, $x \in \mathbb{R}$

Find the set of values of x for which $f'(x) < 0$.

$x < -\sqrt{2},\ x > \sqrt{2}$

Question 5

Use differentiation to find the x-coordinates of the points at which the graph of $y = \frac{x^2}{2x-1}$ has either a maximum or a minimum and distinguish between them.

x = 0, 1

Question 6

The curve y= $\frac{2x+1}{2x-1}$ crosses the *x*-axis at A and the *y*-axis at B. Find the point of intersection of the tangents to the curve at A and B.

(-1/6, -1/3)

Differentiating Trigonometric Functions

The function $\sin x$ is differentiated from first principles as follows:

$$y = \sin x \quad \text{(where } x \text{ is in radians)}$$
$$\Rightarrow y + \delta y = \sin(x + \delta x)$$
$$\Rightarrow \quad \delta y = \sin(x+\delta x) - \sin x = 2\cos(x + \tfrac{1}{2}\delta x)\sin\tfrac{1}{2}\delta x$$
$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{2\cos(x+\frac{1}{2}\delta x)\sin\frac{1}{2}\delta x}{\delta x} = \cos(x+\tfrac{1}{2}\delta x).\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}.$$

As $\delta x \to 0$, $\cos(x + \frac{1}{2}\delta x) \to \cos x$ and $\dfrac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x} \to 1$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \cos x \,.\, 1.$$

Hence $\boxed{\dfrac{d}{dx}(\sin x) = \cos x.}$

Similarly it can be shown that $\boxed{\dfrac{d}{dx}(\cos x) = -\sin x}$

Example 3

i) Find $\frac{d}{dx}(tan\ x)$ ------------------>> differentiate by using quotient rule.

ii) Find $\frac{d}{dx}(sec\ x)$

iii) Find $\frac{d}{dx}(cot\ x)$

iv) Find $\frac{d}{dx}(cosec\ x)$

Example 4

Differentiate (a) $\sin^3 x$, (b) $\cos(4-3x)$, (c) $\sin x°$

Exercise 2

Differentiate the following with respect to x, simplifying your answers where possible.

1. (a) $\sin 2x$, (b) $3\cos 4x$, (c) $4\tan\frac{1}{2}x$.
2. (a) $\cos^4 x$, (b) $\sqrt{(\sin x)}$, (c) $\sin^2 5x$.
3. (a) $(x+\sin x)^3$, (b) $\sin(x^2)$, (c) $\tan(6x°)$.
4. (a) $x^4\cos x$, (b) $(4x^2+1)\tan x$, (c) $x^2\sin 3x$.
5. (a) $\operatorname{cosec}(x+1)$, (b) $2\cot(1-2x)$, (c) $\sec(3x-4)$.
6. (a) $\tan(\cos x)$, (b) $\sec(1+\sqrt{x})$, (c) $\cot(1/x)$.
7. (a) $\sin x\cos 2x$, (b) $\cos^3 x\sin 3x$, (c) $\cos x\tan x$.

Answers:

1. (a) $2\cos 2x$, (b) $-12\sin 4x$, (c) $2\sec^2 \frac{1}{2}x$.
2. (a) $-4\cos^3 x \sin x$, (b) $\cos x/2\sqrt{(\sin x)}$, (c) $5\sin 10x$.
3. (a) $3(x+\sin x)^2(1+\cos x)$, (b) $2x\cos(x^2)$, (c) $\frac{\pi}{30}\sec^2(6x°)$.
4. (a) $x^3(4\cos x - x\sin x)$, (b) $8x\tan x + (4x^2+1)\sec^2 x$, (c) $x(2\sin 3x + 3x\cos 3x)$.
5. (a) $-\operatorname{cosec}(x+1)\cot(x+1)$, (b) $4\operatorname{cosec}^2(1-2x)$, (c) $3\sec(3x-4)\tan(3x-4)$.
6. (a) $-\sin x\sec^2(\cos x)$, (b) $\frac{1}{2\sqrt{x}}\sec(1+\sqrt{x})\tan(1+\sqrt{x})$, (c) $\frac{1}{x^2}\operatorname{cosec}^2\left(\frac{1}{x}\right)$.
7. (a) $\cos x\cos 2x - 2\sin x\sin 2x$, (b) $3\cos^2 x\cos 4x$, (c) $\cos x$.

Question 8

Differentiate with respect to x

(a) $\sin^2(2x-5)$, (b) $x^4\tan 4x$,

(c) $\dfrac{\sec x + \tan x}{\sec x - \tan x}$, (d) $\dfrac{\sin x}{\sqrt{(\cos 2x)}}$.

(a) $2\sin(4x-10)$, (b) $4x^3(\tan 4x + x\sec^2 4x)$, (c) $\dfrac{2\sec x}{(\sec x - \tan x)^2}$, (d) $\dfrac{\cos x}{(\cos 2x)^{3/2}}$.

Question 9

The equation of a curve is $y = 2\cos x + \sin 2x$. Find the x-coordinates of the stationary points on the curve for which $0 < x < \pi$, and determine the nature of each of these stationary points. [7]

$\frac{\pi}{6} - max, \quad \frac{5\pi}{6} - min$

Question 10

Find the equation of normal to the curve $y = tan^2x$ when $x = \frac{\pi}{4}$.

$4y + x = 4 + \frac{\pi}{4}$