Exercise 1: First Order Differential Equations with Separable Variables

Solve each differential equation.

1 $y\dfrac{dy}{dx} = \sin x$

2 $x^2\dfrac{dy}{dx} = y^2$

3 $\dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{1}{y^2 - 2}$

4 $\tan y\dfrac{dy}{dx} = \dfrac{1}{x}$

5 $\dfrac{dy}{dx} = y^2$

6 $\dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{1}{1 - x^2}$

7 $(x - 3)\dfrac{dy}{dx} = y$

8 $\tan y\dfrac{dx}{dy} = 4$

9 $u\dfrac{du}{dv} = v + 2$

10 $e^x\dfrac{dy}{dx} = \dfrac{x}{y}$

11 $\dfrac{dv}{du} = \dfrac{v + 1}{u + 2}$

12 $y(x + 1) = (x^2 + 2x)\dfrac{dy}{dx}$

13 $v^2\dfrac{dv}{dt} = (2 + t)^3$

14 $x\dfrac{dy}{dx} = \dfrac{1}{y} + y$

15 $y\sin^3 x\dfrac{dy}{dx} = \cos x$

16 $\dfrac{uv}{u - 1} = \dfrac{du}{dv}$

17 $e^x\dfrac{dy}{dx} = e^{y-1}$

Exercise 2: Calculation of Constant of Integration

Solve each of the following differential equations.

1 $y^2\dfrac{dy}{dx} = x^2 + 1$ and $y = 1$ when $x = 2$

2 $e^t\dfrac{ds}{dt} = \sqrt{s}$ and $s = 4$ when $t = 0$

3 $\dfrac{y}{x}\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}$ and $y = 3$ when $x = 2$

4 A curve passes through the origin and its gradient function is $2x - 1$. Find the equation of the curve and sketch it.

5 A curve for which $e^{-x}\dfrac{dy}{dx} = 1$ passes through the point $(0, -1)$. Find the equation of the curve.

6 A curve passes through the points $(1, 2)$ and $\left(\frac{1}{5}, -10\right)$ and its gradient is inversely proportional to x^2. Find the equation of the curve.

7 Given that $y = 2$ when $x = 1$, find the coordinates of the point where the curve represented by $\dfrac{2y}{3}\dfrac{dy}{dx} = e^{-3x}$ crosses the y-axis.

8 Find the equation of the curve whose gradient function is $\dfrac{y + 1}{x^2 - 1}$ and that passes through the point $(-3, 1)$.

9 The gradient function of a curve is proportional to $x + 3$. The curve passes through the origin and the point $(2, 8)$. Find its equation.

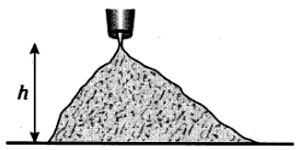
10 Solve the differential equation $(1 + x^2)\dfrac{dy}{dx} - y(y + 1)x = 0$, given that $y = 1$ when $x = 0$

11 Solve the differential equation $\dfrac{dy}{dx} = 3x^2y^2$ given that $y = 1$ when $x = 0$

12 $\dfrac{dy}{dx} = x(y^2 + 1)$ and $y = 0$ when $x = 2$. Solve the differential equation.

Exercise 3: Rates of Change

1 Grain is pouring from a container on to a barn floor where it forms a conical pile whose height h is increasing at a rate that is inversely proportional to h^3. The initial height of the pile is h_0 and the height doubles after a time T. Find, in terms of T, the time after which its height has grown to $3h_0$



2 The gradient of any point of a curve is proportional to the square root of the x-coordinate. Given that the curve passes through the point (1, 2) and at that point the gradient is 0.6, form and solve the differential equation representing the given relationship. Show that the curve passes through the point (4, 4.8) and find the gradient at this point.

3 The number of bacteria in a liquid is growing at a rate proportional to the number of bacteria present at any time. Initially there are 100 bacteria.

(a) Form a differential equation that models the growth in the number of bacteria.

(b) The number of bacteria increases by 50% in 10 hours. Find the number of hours for the bacteria to double from the initial number of 100.

4 In a chemical reaction, a substance is transformed into a compound. The mass of the substance after time t is m. The substance is being transformed at a rate that is proportional to the mass of the substance at that time. Given that the original mass is 50 g and that 20 g is transformed after 200 seconds

(a) form and solve the differential equation relating m and t

(b) find the mass of the substance transformed in 300 seconds.

5 The rate of decrease of the temperature of a liquid is proportional to the amount by which this temperature is greater than the temperature of its surroundings. (This is known as Newton's Law of Cooling.) Take θ as the difference in temperature at any time t, and 80° as the initial difference.

(a) Show that $\theta = 80e^{-kt}$

A pan of water at 65 °C is standing in a kitchen whose temperature is 15 °C.

(b) Show that, after cooling for t minutes, the water temperature, ϕ, can be modelled by the equation

$\phi = 15 + 50e^{-kt}$ where k is a constant.

(c) Given that after 10 minutes the temperature of the water has fallen to 50 °C, find the value of k.

(d) Find the temperature after 15 minutes.

6 A virus has infected the population of rabbits on an island. The growth in the number of rabbits infected is proportional to the number already infected. Initially 20 rabbits were infected.

(a) Form a differential equation that models the growth in the number infected.

(b) Thirty days after the initial number of infections, 60 rabbits were infected. After how many further days does the model predict that 200 rabbits will be infected?

Topic 6 – Differential Equations

Answers

Exercise 1

1. $y^2 = A - 2\cos x$
2. $\frac{1}{y} - \frac{1}{x} = A$
3. $2y^3 = 3(x^2 + 4y + A)$
4. $x = A \sec y$
5. $(A - x)y = 1$
6. $y = \ln \frac{A}{\sqrt{1 - x^2}}$
7. $y = A(x - 3)$
8. $x + A = 4 \ln |\sin y|$
9. $u^2 = v^2 + 4v + A$
10. $y^2 + 2(x + 1)e^{-x} = A$
11. $u + 2 = A(v + 1)$
12. $y^2 = Ax(x + 2)$
13. $4v^3 = 3(2 + t)^4 + A$
14. $1 + y^2 = Ax^2$
15. $y^2 = A - \operatorname{cosec}^2 x$
16. $v^2 + A = 2u - 2 \ln |u|$
17. $e^{-x} = e^{1-y} + A$

Exercise 2

1. $y^3 = x^3 + 3x - 13$
2. $e^t(5 - 2\sqrt{s}) = 1$
3. $3(y^2 - 1) = 8(x^2 - 1)$
4. $y = x^2 - x$
5. $y = e^x - 2$
6. $y = 5 - \frac{3}{x}$
7. $\left(0, \sqrt{3 + e^{-3}}\right), \left(0, -\sqrt{3 + e^{-3}}\right)$
8. $(y + 1)^2(x + 1) = 2(x - 1)$
9. $2y = x^2 + 6x$
10. $4y^2 = (y + 1)^2(x^2 + 1)$
11. $x^3 y = y - 1$
12. $y = \tan\left\{\frac{1}{2}(x^2 - 4)\right\}$

Exercise 3

1. $t = \frac{16T}{3}$
2. $\frac{dy}{dx} = k\sqrt{x}$; $y = 0.4x^{\frac{3}{2}} + 1.6$; 1.2
3. (a) $\frac{dn}{dt} = kn$
   (b) $t = \frac{10 \ln 2}{\ln 1.5} = 1.71$ (hours)
4. (a) $-\frac{dm}{dt} = km$; $m = 50e^{-kt}$ where $k = 0.002554\ldots$
   (b) 26.8 g (3 s.f.)
5. (c) $k = 0.0357$ (3 s.f.)
   (d) 44° (nearest degree)
6. (a) $\frac{dN}{dt} = kN$ (b) 33