Differentiation -Parametric Functions

Some curves are defined by expressing *x* and *y* in terms of a third variable, called a parameter.

For example:

$x = t^2, \quad y = 3t + 1$ ⇨ t is a parameter.

To find $\frac{dy}{dx}$ ➔

Example 1

Find the equation of the tangent to the curve $x = 4 + \cos\theta$, $y = \sin^2\theta$ at the point $(4\frac{1}{2}, \frac{3}{4})$.

Exercise 1

1 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t for the following curves.

(a) $x = t^3, y = 2t$

(b) $x = \sin t, y = \cos t$

(c) $x = 2\cos t, y = 3\sin t$

(d) $x = t^3 + t, y = t^2 - t$

2 Find the gradients of the tangents to the following curves, at the specified values of t.

(a) $x = 3t^2, y = 6t$ when $t = 0.5$

(b) $x = t^3, y = t^2$ when $t = 2$

(c) $x = 1 - \frac{1}{t}, y = 1 + \frac{1}{t}$ when $t = 2$

(d) $x = t^2, y = \frac{1}{t}$ when $t = 3$

3 Find the gradients of the normals to the following curves, at the specified values of t.

(a) $x = 5t^2, y = 10t$ when $t = 3$

(b) $x = \cos^2 t, y = \sin^2 t$ when $t = \frac{1}{3}\pi$

(c) $x = \cos^3 t, y = \sin^3 t$ when $t = \frac{1}{6}\pi$

(d) $x = t^2 + 2, y = t - 2$ when $t = 4$

4 Show that the equation of the tangent to the curve $x = 3\cos t, y = 2\sin t$ when $t = \frac{3}{4}\pi$ is $3y = 2x + 6\sqrt{2}$.

5 (a) Find the gradient of the curve $x = t^3, y = t^2 - t$ at the point $(1,0)$.

(b) Hence find the equation of the tangent to the curve at this point.

6 A curve has parametric equations $x = t - \cos t, y = \sin t$. Find the equation of the tangent to the curve when $t = \pi$.

7 Find the equations of the tangents to these curves at the specified values.

(a) $x = t^2, y = 2t$ when $t = 3$

(b) $x = 5\cos t, y = 3\sin t$ when $t = \frac{11}{6}\pi$

8 Find the equations of the normals to these curves at the specified values.

(a) $x = 5t^2$, $y = 10t$ when $t = 3$ (b) $x = \cos t$, $y = \sin t$ when $t = \frac{2}{3}\pi$

9 (a) Find the equation of the normal to the hyperbola $x = 4t$, $y = \frac{4}{t}$ at the point $(8,2)$.

(b) Find the coordinates of the point where this normal crosses the curve again.

10 (a) Find the equation of the normal to the parabola $x = 3t^2$, $y = 6t$ at the point where $t = -2$.

(b) Find the coordinates of the point where this normal crosses the curve again.

Answers – Exercise 1

1 (a) $\dfrac{2}{3t^2}$ (b) $-\tan t$

(c) $-\frac{3}{2}\cot t$ (d) $\dfrac{2t-1}{3t^2+1}$

2 (a) 2 (b) $\frac{1}{3}$ (c) -1 (d) $-\frac{1}{54}$

3 (a) -3 (b) 1 (c) $\sqrt{3}$ (d) -8

5 (a) $\frac{1}{3}$ (b) $3y = x - 1$

6 $x + y = 1 + \pi$

7 (a) $3y = x + 9$ (b) $5y = 3\sqrt{3}x - 30$

8 (a) $3x + y = 165$ (b) $y = -\sqrt{3}x$

9 (a) $y = 4x - 30$ (b) $\left(-\frac{1}{2}, -32\right)$

10 (a) $y = 2x - 36$ (b) $(27,18)$

Exercise 2 – Exam Based Questions

Question 1

The parametric equations of a curve are

$$x = a\cos^3 t, \quad y = a\sin^3 t,$$

where a is a positive constant and $0 < t < \frac{1}{2}\pi$.

(i) Express $\dfrac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x\sin t + y\cos t = a\sin t\cos t.$$ [3]

(iii) Hence show that, if this tangent meets the x-axis at X and the y-axis at Y, then the length of XY is always equal to a. [2]

[i) -tan t]

Question 2

The parametric equations of a curve are

$$x = a(2\theta - \sin 2\theta), \qquad y = a(1 - \cos 2\theta).$$

Show that $\dfrac{dy}{dx} = \cot\theta$. [5]

Question 3

A curve is given parametrically by the equations

$$x = t^2, \qquad y = \frac{1}{t}.$$

(i) Find $\frac{dy}{dx}$ in terms of t, giving your answer in its simplest form. [3]

(ii) Show that the equation of the tangent at the point $P\left(4, -\frac{1}{2}\right)$ is

$$x - 16y = 12.$$ [3]

(iii) Find the value of the parameter at the point where the tangent at P meets the curve again. [4]

Question 4

The parametric equations of a curve are $x = \ln(\cos\theta)$, $y = \ln(\sin\theta)$, $0 < \theta < \frac{\pi}{2}$. Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$, leaving your answer in the form of $y = ax + b$ where *a* and *b* are exact values to be found.
Explain, using an algebraic method, why the tangent will not meet the curve again.

[$y = -x - \ln 2$ *]*

Question 5

The parametric equations of a curve *C* are

$$x = -1 - t^2 \quad \text{and} \quad y = \ln(2 - t), \quad t < 2.$$

(i) Find the equations of the normal to the curve at $(-1, \ln 2)$ and the tangent to the curve at $(-5, 2\ln 2)$.

(ii) Find exactly the coordinates of the point of intersection of the tangent and the normal.

(iii) Find, in radians, the acute angle between the tangent and the normal.

i) $y = \ln 2$, $y = -\frac{x}{16} - \frac{5}{16} + 2\ln 2$ ii) $P = (16\ln 2 - 5,\ \ln 2)$ iii) $\tan\theta = \frac{1}{16} \Rightarrow \theta = 0.0624$