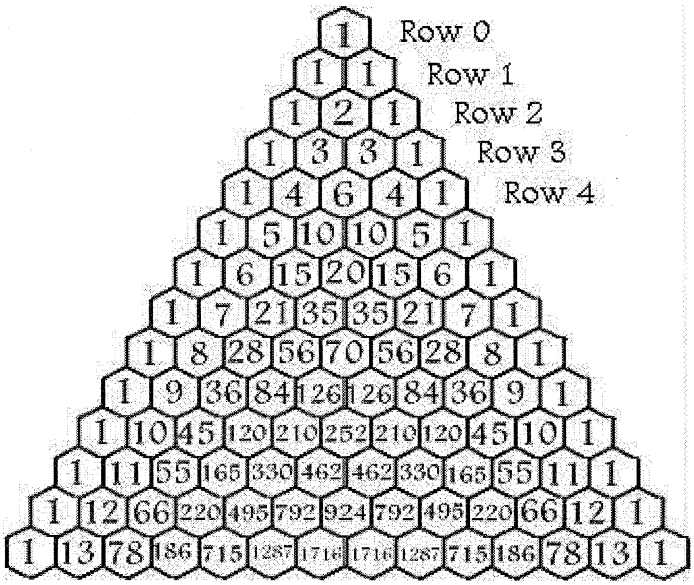
TOPIC 1: Algebra

Binomial Expansion /Binomial Series
Objective:

- use the expansion of $(1+x)^n$, where n is a rational number and $|x|<1$ (finding a general term is not included, but adapting the standard series to expand e.g. $(2-\frac{1}{2}x)^{-1}$ is included).

BINOMIAL EXPANSION OF $(a+b)^n$
n is a positive integer ➔ AS component

a) The coefficients of a binomial expansion can be obtained from Pascal Triangle



b) General formula

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + b^n \text{, where } n \text{ is a positive integer}$$

$$\text{and } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

n is negative , a fraction or a positive integer ➔ A2 Component
The binomial expansion gives an *infinite series* which converges for certain values of the variable.

➔It would be easier if the binomial expression is in the form $(1+x)^n$

The General Formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \cdots \text{, where } n \text{ is rational and } |x|<1$$

Note:
Make sure the first term is always a 1. Eg: $(1+x)^n$

Example:

1. Find the expansion of $(1-2x)^{-2}$ in ascending powers of x up to and including the term in x^3.
2. Find the binomial expansion of $(4-3x^2)^{\frac{1}{2}}$ up to and including the term in x^4.

$[2-\frac{3}{4}x^2-\frac{9}{64}x^4]$

3. Expand $\frac{5+x}{2-x+x^2}$ in ascending powers of x up to the term x^3.

$[\frac{5}{2}+\frac{7}{4}x-\frac{3}{8}x^2-\frac{17}{16}x^3+\cdots]$

4. Obtain the first three terms in the expansion, in ascending powers of x, of $(4+x)^{\frac{1}{2}}$. State the set of values of x for which the expansion is valid.

$[2+\frac{1}{4}x-\frac{1}{64}x^2, |x|<4]$

Note:
For an expansion $(1+x)^n$ to be valid:
1. Make sure the variable x only takes values: $\mathbf{-1 < x < 1} \Leftrightarrow |x|<1$
2. Remember to check the values of x for <u>which the expansion is valid.</u>

Exercise 1 – Binomial Series
Question 1

Find the first four terms in the expansion of each of the following in ascending powers of x. State the interval of values of x for which each expansion is valid.

(a) $\sqrt{1-6x}$ (b) $\frac{1}{1+5x}$ (c) $\frac{1}{\sqrt[3]{1+9x}}$ (d) $\frac{1}{(1-2x)^4}$

(e) $\sqrt{1+2x^2}$ (f) $\sqrt[3]{8-16x}$ (g) $\frac{10}{\left(1+\frac{1}{5}x\right)^2}$ (h) $\frac{2}{2-x}$

(i) $\frac{1}{(2+x)^3}$ (j) $\frac{4x}{\sqrt{4+x^3}}$ (k) $\sqrt[4]{1+8x}$ (l) $\frac{12}{\left(\sqrt{3}-x\right)^4}$

(a) $1-3x-\frac{9}{2}x^2-\frac{27}{2}x^3, |x|<\frac{1}{6}$
(b) $1-5x+25x^2-125x^3, |x|<\frac{1}{5}$
(c) $1-3x+18x^2-126x^3, |x|<\frac{1}{9}$
(d) $1+8x+40x^2+160x^3, |x|<\frac{1}{2}$
(e) $1+x^2-\frac{1}{2}x^4+\frac{1}{2}x^4, |x|<\frac{1}{2}\sqrt{2}$
(f) $2-\frac{4}{3}x-\frac{8}{9}x^2-\frac{80}{81}x^3, |x|<\frac{1}{2}$
(g) $10-4x+\frac{6}{5}x^2-\frac{8}{25}x^3, |x|<5$
(h) $1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3, |x|<2$
(i) $\frac{1}{8}-\frac{3}{16}x+\frac{3}{16}x^2-\frac{5}{32}x^3, |x|<2$
(j) $2x-\frac{1}{4}x^4+\frac{3}{64}x^7-\frac{5}{512}x^{10}, |x|<\sqrt[3]{4}$
(k) $1+2x-6x^2+28x^3, |x|<\frac{1}{8}$
(l) $\frac{4}{3}-\frac{16}{9}\sqrt{3}x+\frac{40}{9}x^2+\frac{80}{27}\sqrt{3}x^3, |x|<\sqrt{3}$

Question 2

Expand $\frac{1}{(2+x)^3}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

[1/8 - 3x/16 + 3x²/16]

Question 3

Find the expansion of $\frac{1+x^2}{\sqrt{(1+4x)}}$ in ascending powers of x up to and including the term in x^2. [3]

[1-2x+7x²]

Question 4

Obtain the first three terms in the expansion, in ascending powers of x, of $(8+3x)^{\frac{2}{3}}$ stating the set of values of x for which the expansion is valid.

[$4+x-\frac{1}{16}x^2, |x|<\frac{8}{3}$ *]*

Question 5

Expand $\sqrt{1+8x}$ in ascending powers of x up to and including the term in x^3. By giving a suitable value to x, find an approximation for $\sqrt{1.08}$. Deduce approximations for

(a) $\sqrt{108}$, (b) $\sqrt{3}$.

$1+4x-8x^2+32x^3, 1.039\,232$

[(a) 10.392 32 (b) 1.732 05 *]*

Question 6

Expand $(1-x)^{-2}$ as a series of ascending powers of x, given that $|x|<1$. Hence express $\frac{1+x}{(1-x)^2}$ in the form $1+3x+ax^2+bx^3+\ldots$, where the values of a and b are to be stated.

[$1+2x+3x^2+4x^3, a=5, b=7$ *]*

Question 7

Expand $\frac{2+\left(1+\frac{1}{2}x\right)^6}{2+3x}$ in ascending powers of x up to and including the term in x^2.

[$\frac{3}{2}-\frac{3}{4}x+3x^2$ *]*