TOPIC 7 – SERIES
(BINOMIAL EXPANSION)

- use the expansion of $(a+b)^n$, where n is a positive integer (knowledge of the greatest term and properties of the coefficients are not required, but the notations $\binom{n}{r}$ and $n!$ should be known);
- recognise arithmetic and geometric progressions;
- use the formulae for the nth term and for the sum of the first n terms to solve problems involving arithmetic or geometric progressions;
- use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression.

Binomial

A binomial is a polynomial with two terms.

Multiplying

The Binomial Theorem shows what happens when you multiply a binomial by itself (as many times as you want).

It works because there is a pattern ... let us see if we can discover it.

$a + b)^0 = 1$

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$(a + b)^6 =$

$(a + b)^7 =$

The Pattern - Binomial Expansion $(a + b)^n$

- There are $n + 1$ terms.
- The first term is a^n and the final term is b^n.
- Progressing from the first term to the last, the exponent of a decreases by 1 from term to term while the exponent of b increases by 1. In addition, the sum of the exponents of a and b in each term is n.
- If the coefficient of each term is multiplied by the exponent of a in that term, and the product is divided by the number of that term, we obtain the coefficient of the next term.

The Coefficient - Pascal's Triangle

We note that the coefficients (the numbers in front of each term) follow a pattern.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

You can use this pattern to form the coefficients, rather than multiply everything out as we did above.

The Binomial Theorem:

Examples:

Question 1

Expand $(a + b)^3$ and hence find the expansion of $(2 + 3x)^3$.

Question 2

Find, in descending powers of x, the first 4 terms in the binomial expansion of

(a) $(2x - 3)^5$, **(b)** $\left(x + \frac{1}{x^2}\right)^6$.

Question 3

Find the first 4 terms in the expansion of $(1 + x^2)^8$. Use your result to estimate the value of $(1.01)^8$.

Question 4

Given that $\left(p - \frac{1}{2}x\right)^6 = r - 96x + sx^2 + \ldots$, find p, r and s.

Question 5

Write down and simplify the expansion of $(1 - p)^5$. Use this result to find the expansion of $(1 - x + x^2)^5$ in ascending powers of x as far as the term in x^3. Find the value of x which would enable you to estimate $(1.11)^5$ from this expansion.

Question 6

Write down, in ascending powers of x, the first three terms in the expansion of $(2 + ax)^5$.
Given that the first three terms in the expansion of $(b + 2x)(2 + ax)^5$ are $96 - 176x + cx^2$, find the values of a, b and c.

A specific term for $(a+b)^n$ is $\binom{n}{r} a^r b^{n-r}$

Question 7

Find the terms in x^2 and x^5 in the expansion of $\left(1 - \frac{x}{2}\right)^{12}$. Hence find the coefficient of x^5 in the expansion of $(3 + 2x^3)\left(1 - \frac{x}{2}\right)^{12}$.

Exercise 1 – Binomial Expansion

1. Expand the following:
   (a) $(3-2x)^4$ (b) $(2+x^2)^5$ (c) $\left(2-\frac{1}{2x}\right)^6$

2. Find the first 4 terms in the expansion of
   (a) $(2+3x)^6$, (b) $\left(4-\frac{1}{2x}\right)^5$, (c) $\left(2x^2-\frac{1}{2x}\right)^8$.

3. Obtain, in ascending powers of x, the first 4 terms in the expansion of $\left(2-\frac{x}{2}\right)^7$.
   Hence find the value of $(1.995)^7$ correct to 4 decimal places.

4. Find the values of a and b for which $(2x-a)^3 = 8x^3 - bx^2 + \frac{3}{2}bx - a^3$.

5. Find the first 3 terms, in ascending powers of x, in the expansion of $(1-2x)^9$ and of $(2+x)^5$. Hence expand $(1-2x)^9(2+x)^5$ up to the terms in x^2.

6. Find the indicated term in each of the following expansions.
   (a) $(2+x)^{10}$, 7th term (b) $(3x-2)^9$, 4th term
   (c) $(y-2x)^{10}$, 5th term (d) $\left(x+\frac{1}{2x^2}\right)^{12}$, middle term

7. In the expansion of $\left(x^3-\frac{2}{x^2}\right)^{10}$, find
   (a) the term in x^{10}, (b) the coefficient of $\frac{1}{x^5}$, (c) the constant term.

8. Expand $\left(\frac{1}{2}-2x\right)^5$ up to the term in x^3. If the coefficient of x^2 in the expansion of $(1+ax+3x^2)\left(\frac{1}{2}-2x\right)^5$ is $\frac{13}{2}$, find the coefficient of x^3.

9. In the expansion of $(1+x)(a-bx)^{12}$, the coefficient of x^8 is zero. Find in its simplest form the value of the ratio $\frac{a}{b}$.

Answers

1. (a) $81 - 216x + 216x^2 - 96x^3 + 16x^4$ (b) $32 + 80x^2 + 80x^4 + 40x^6 + 10x^8 + x^{10}$
   (c) $64 - \frac{96}{x} + \frac{60}{x^2} - \frac{20}{x^3} + \frac{15}{4x^4} - \frac{3}{8x^5} + \frac{1}{64x^6}$
2. (a) $64 + 576x + 2\,160x^2 + 4\,320x^3 + \cdots$ (b) $1\,024 - \frac{640}{x} + \frac{160}{x^2} - \frac{20}{x^3} + \cdots$
   (c) $256x^{16} - 512x^{13} + 448x^{10} - 224x^7 + \cdots$
3. $128 - 224x + 168x^2 - 70x^3 + \cdots$; $125.776\,7$ 4. $a = 3$, $b = 36$
5. $1 - 18x + 144x^2 + \cdots$; $32 + 80x + 80x^2 + \cdots$; $32 - 496x + 3\,248x^2 + \cdots$
6. (a) $3\,360x^6$ (b) $-489\,888x^6$ (c) $3\,360x^4y^6$ (d) $14\frac{7}{16}x^{-6}$
7. (a) $3\,360x^{10}$ (b) $-15\,360$ (c) $13\,440$
8. $\frac{1}{32} - \frac{5}{8}x + 5x^2 - 20x^3 + \cdots$; $-33\frac{1}{8}$ 9. $\frac{5}{8}$

Exercise 2 – Arithmetic Progression (Textbook-Mis. Exercise 9 – page 135)

1 Expand $(3+4x)^3$.

2 Find the first three terms in the expansions, in ascending powers of x, of

(a) $(1+4x)^{10}$, (b) $(1-2x)^{16}$.

3 Find the coefficient of a^3b^5 in the expansions of

(a) $(3a-2b)^8$, (b) $\left(5a+\frac{1}{2}b\right)^8$.

4 Expand $(3+5x)^7$ in ascending powers of x up to and including the term in x^2. By putting $x = 0.01$, find an approximation, correct to the nearest whole number, to 3.05^7.

5 Obtain the first four terms in the expansion of $\left(2+\frac{1}{4}x\right)^8$ in ascending powers of x. By substituting an appropriate value of x into this expansion, find the value of 2.0025^8 correct to three decimal places. (OCR)

6 Find, in ascending powers of x, the first three terms in the expansion of $(2-3x)^8$. Use the expansion to find the value of 1.997^8 to the nearest whole number. (OCR)

7 Expand $\left(x^2+\frac{1}{x}\right)^3$, simplifying each of the terms.

8 Expand $\left(2x-\frac{3}{x^2}\right)^4$.

9 Expand and simplify $\left(x+\frac{1}{2x}\right)^6+\left(x-\frac{1}{2x}\right)^6$. (OCR)

10 Find the coefficient of x^2 in the expansion of $\left(x^4+\frac{4}{x}\right)^3$.

11 Find the term independent of x in the expansion of $\left(2x+\frac{5}{x}\right)^6$.

12 Find the coefficient of y^4 in the expansion of $(1+y)^{12}$. Deduce the coefficient of

(a) y^4 in the expansion of $(1+3y)^{12}$,

(b) y^8 in the expansion of $\left(1-2y^2\right)^{12}$,

(c) x^8y^4 in the expansion of $\left(x+\frac{1}{2}y\right)^{12}$.

13 Determine the coefficient of p^4q^7 in the expansion of $(2p-q)(p+q)^{10}$.

14 Find the first three terms in the expansion of $(1+2x)^{20}$. By substitution of a suitable value of x in each case, find approximations to

(a) 1.002^{20}, (b) 0.996^{20}.

15 Write down the first three terms in the binomial expansion of $\left(2-\frac{1}{2x^2}\right)^{10}$ in ascending powers of x. Hence find the value of 1.995^{10} correct to three significant figures. (OCR)

16 Two of the following expansions are correct and two are incorrect. Find the two expansions which are incorrect.

A: $(3+4x)^5 = 243+1620x+4320x^2+5760x^3+3840x^4+1024x^5$

B: $\left(1-2x+3x^2\right)^3 = 1+6x-3x^2+28x^3-9x^4+54x^5-27x^6$

C: $(1-x)(1+4x)^4 = 1+15x+80x^2+160x^3-256x^5$

D: $(2x+y)^2(3x+y)^3 = 108x^5+216x^4y+171x^3y^2+67x^2y^3+13xy^4+y^6$

17 Find and simplify the term independent of x in the expansion of $\left(\frac{1}{2x}+x^3\right)^8$. (OCR)

18 Find the term independent of x in the expansion of $\left(2x+\frac{1}{x^2}\right)^9$.

19 Evaluate the term which is independent of x in the expansion of $\left(x^2-\frac{1}{2x^2}\right)^{16}$. (OCR)

20 Find the coefficient of x^{-12} in the expansion of $\left(x^3-\frac{1}{x}\right)^{24}$. (OCR)

21 Expand $\left(1+3x+4x^2\right)^4$ in ascending powers of x as far as the term in x^2. By substituting a suitable of x, find an approximation to 1.0304^4.

22 Expand and simplify $(3x+5)^3-(3x-5)^3$.
Hence solve the equation $(3x+5)^3-(3x-5)^3=730$.

23 Solve the equation $(7-6x)^3+(7+6x)^3=1736$.