Probability

S1/3/1: Introduction to probability
S1/3/2: Calculation of probabilities using permutations or combinations
S1/3/3: Mutually exclusive and independent events
S1/3/4: Conditional probability
S1/3/5: Tree diagram
S1/3/6: Past years questions and solutions

S1/3/1:
Introduction to probability

Learning Outcome

Students should be able to:

- **Define and calculate the probability of an event where the possible outcomes are equally likely**

Sample Space & Event

Sample Space:

The set of all possible outcomes of an experiment.

The sample space is typically called *S*.

The individual outcomes in a sample space are called *sample points*.

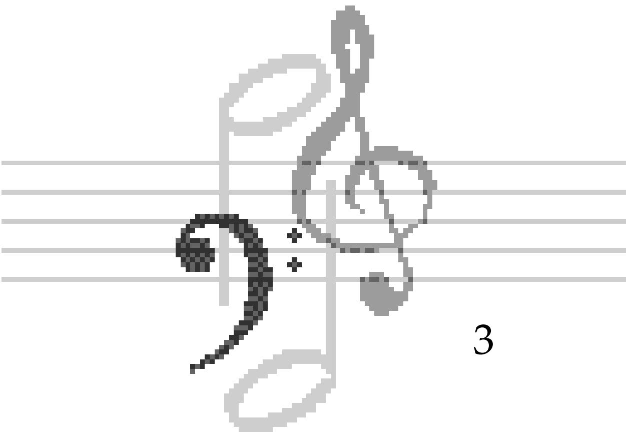
n(S) is the number of sample points in the sample space.

Consider tossing a fair coin.

In a single toss of the coin, there are two possible outcomes.

$$S = \{h, t\}$$

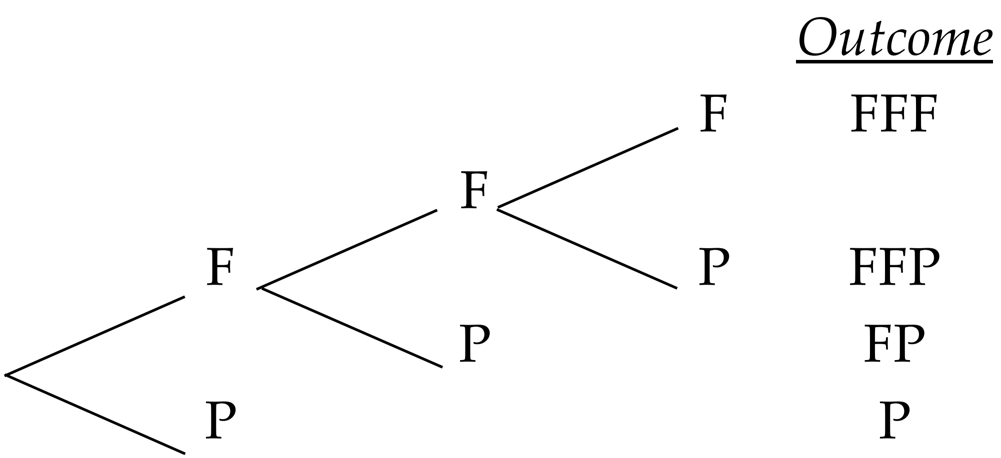
$$n(S) = 2$$



Examples 1:

An experiment consists of selecting electronic parts from an assembly line and testing each to see if it passes inspection (P) or fails (F). The experiment terminates as soon as one acceptable part is found or after three parts are tested.

Construct the sample space and find $n(S)$:



$S = \{ \text{FFF, FFP, FP, P} \}$

$n(S) = 4$

Sample Space & Event

Event:

If A is an event, then n(A) is the number of sample points that belong to A.

Consider tossing a fair coin.

In a single toss of the coin, there are two possible outcomes.

$S = \{h, t\}$ $n(S) = 2$

Define the event H as the occurrence of a head.

$H = \{h\}$ $n(H) = 1$

Equally Likely Events

In a sample space,
suppose all sample points are equally likely to occur.

The probability of an event H is the ratio of the number of sample points in H to the number of sample points in S.

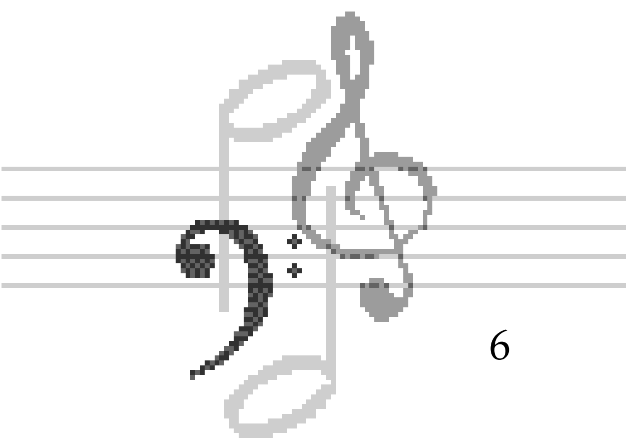
Consider tossing a fair coin.

$S = \{h, t\}$ $\qquad\qquad$ $n(S) = 2$

H = the occurrence of a head.

$H = \{h\}$ $\qquad\qquad$ $n(H) = 1$

$$P(H) = \frac{n(H)}{n(S)} = \frac{1}{2}$$



Notes:

1. The outcomes in a sample space can never overlap (*mutually exclusive*)

2. All possible outcomes must be represented (*all inclusive*)

3. These two characteristics are called *mutually exclusive* and *all inclusive*

Properties:

- The probability of any event A is between 0 and 1.

$$0 \le P(A) \le 1$$

- The sum of the probabilities of all outcomes in the sample space is 1.

$$\sum_{\text{all outcomes}} P(A) = 1$$

Note:

1. The probability is zero if the event is impossible.
2. The probability is one if the event is certain to happen.

Example 2

A fair coin is tossed 5 times, and a head (H) or a tail (T) is recorded each time. What is the probability of:

A = {exactly one head in 5 tosses},

$2^5 = 32$

There are 2 outcomes in each toss. Tossed 5 times.

There are 32 possible outcomes,

A = {HTTTT, THTTT, TTHTT, TTTHT, TTTTH}

$$P(\mathrm{A}) = \frac{n(\mathrm{A})}{n(S)} = \frac{5}{32}$$

Example 3:

When two dice is thrown, find the probability for the event "the sum of the two scores is 6".

There are 36 possible outcomes. $6^2 = 36$

$$A = \text{event "the sum of the two scores is 6"}$$

$$A = \{(5,1),(4,2),(3,3),(2,4),(1,5)\}$$

$$P(A) = \frac{5}{36}$$

Complement of An Event

Complement of an Event:

The set of all sample points in the sample space that do not belong to event A. The complement of event A is denoted by A' or $\bar{A}$ (read “A complement”).

1. The complement of the event "success" is "failure"
2. The complement of the event "rain" is "no rain"
3. The complement of the event "at least 3 patients recover" out of 5 patients is "2 or fewer recover"

Notes:

- $P(\mathrm{A}) + P(\overline{\mathrm{A}}) = 1$ for any event A
- $P(\overline{\mathrm{A}}) = 1 - P(\mathrm{A})$
- Complementary probabilities are very useful when the question asks for the probability of "at least one."

Example 4:

A fair coin is tossed 5 times, and a head (H) or a tail (T) is recorded each time. What is the probability of

1) A = {at least one head in 5 tosses}

2) B = {at most 3 heads in 5 tosses}

$$n(s) = 2^5 = 32 \qquad x = \text{number of heads}$$

$$P(\text{A}) = P(x \geq 1) = 1 - P(x = 0) = 1 - P(\overline{A})$$

$$= 1 - P(0 \text{ heads in 5 tosses})$$

$$= 1 - \frac{1}{32} = \frac{31}{32}$$

Example 4:

A fair coin is tossed 5 times, and a head (H) or a tail (T) is recorded each time. What is the probability of

1) A = {at least one head in 5 tosses}

2) B = {at most 3 heads in 5 tosses}

$$n(s) = 2^5 = 32 \qquad x = \text{number of heads}$$

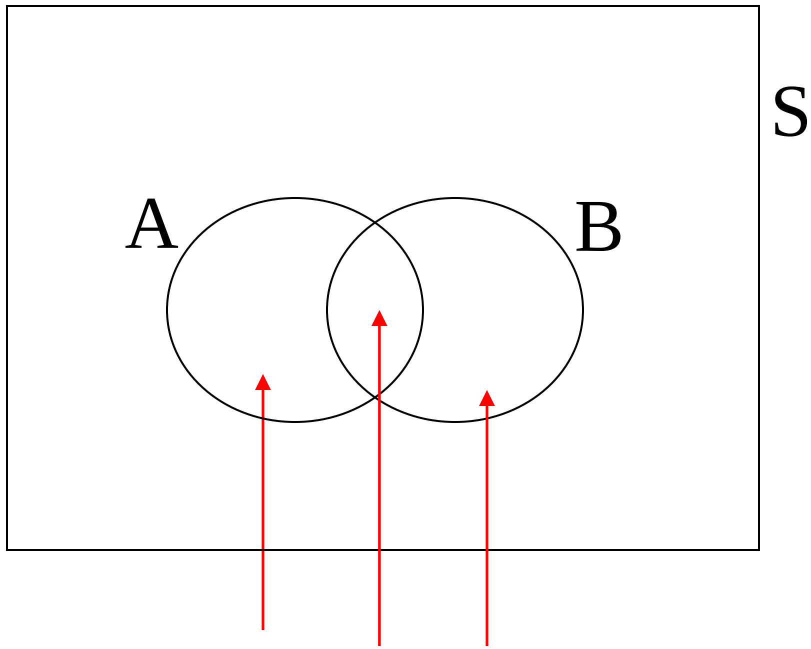
$$P(\mathrm{B}) = 1 - P(\overline{\mathrm{B}}) \; = 1 - P(4 \text{ or } 5 \text{ heads})$$

$$= 1 - (P(4 \text{ heads}) + P(5 \text{ heads}))$$

$$= 1 - \left(\frac{5}{32} + \frac{1}{32}\right) = 1 - \frac{6}{32} = \frac{26}{32} = \frac{13}{16}$$

Illustrating 2 or more events using Venn Diagrams

$(a)\quad A \cup B$

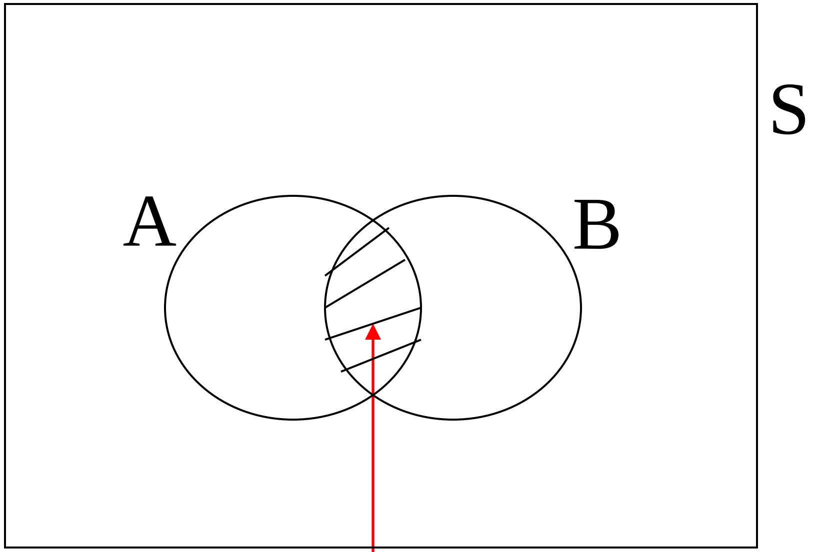


A or B or both = the union of A and B

Illustrating 2 or more events using Venn Diagrams

$(b)\quad A \cap B$

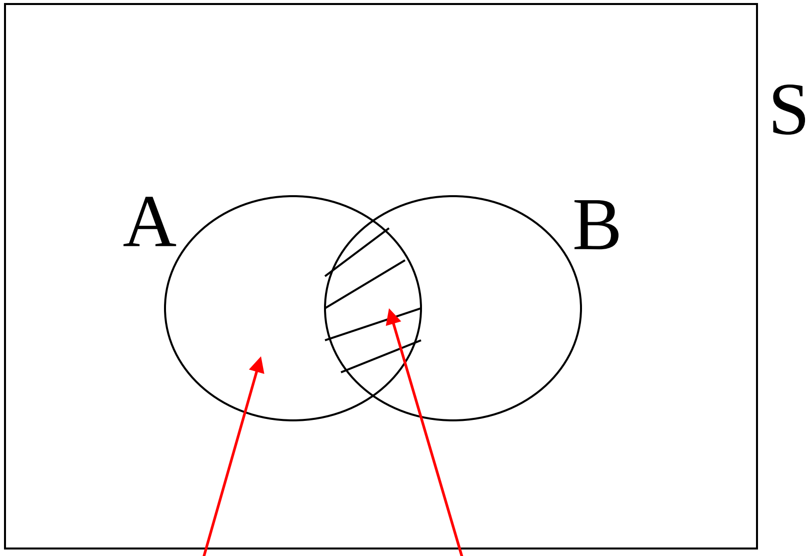
$$P(A \cap B) = P(B \cap A)$$



Both A and B = the intersection of A and B

Some useful results relating two events A and B

$$(c)\quad \mathrm{P}(A) = \mathrm{P}(A\cap B) + \mathrm{P}(A\cap B')$$
$$= \mathrm{P}(A \text{ and } B) + \mathrm{P}(A \text{ but not } B)$$

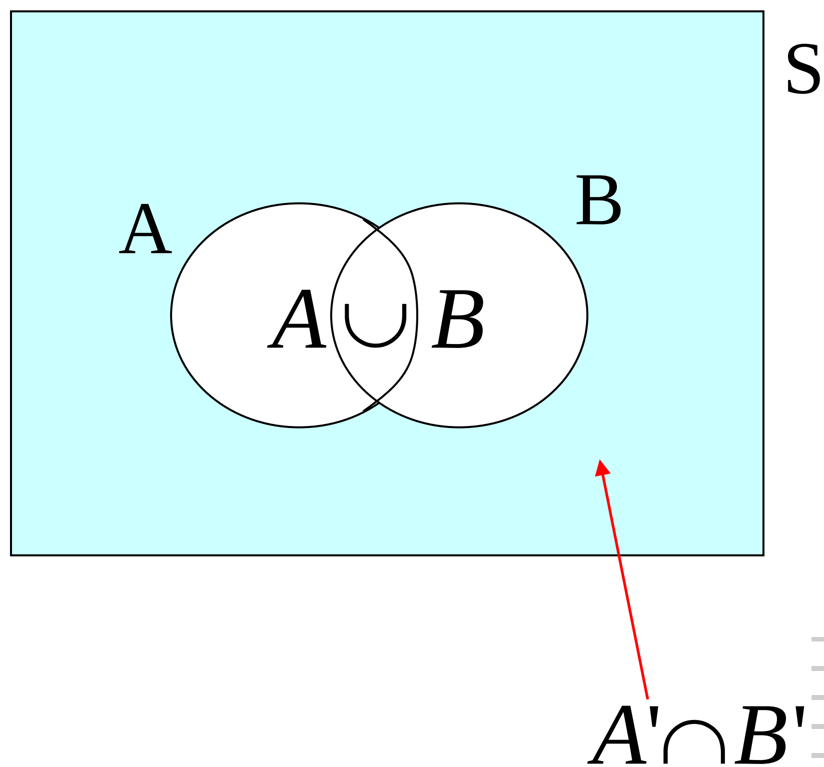


$A\cap B'$ $\quad (A\cap B)$

Some useful results relating two events A and B

$$(d) \quad \mathrm{P}(\text{neither } A \text{ nor } B) = 1 - \mathrm{P}(A \text{ or } B)$$

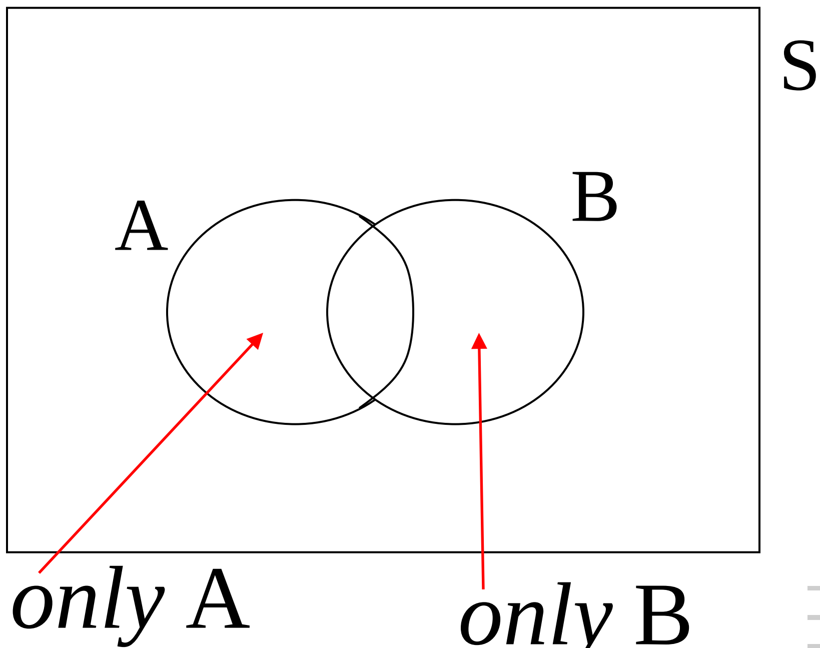
$$\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B)$$



Some useful results relating two events A and B

$$(e)\quad \mathrm{P}(\text{only } A \text{ or } B) = \mathrm{P}(A \text{ or } B) - \mathrm{P}(A \text{ and } B)$$

$$\mathrm{P}(\text{only } A \text{ or } B) = \mathrm{P}(A \cup B) - \mathrm{P}(A \cap B)$$



Probability Rule for combined events

$$\mathrm{P}(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$\boxed{\mathrm{P}(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Example 5:

In a class of 20 children, 4 of the 9 boys and 3 of the 11 girls are in the athletics team. A person from the class is chosen to be in the 'egg and spoon' race on Sports Day. Find the probability that the person chosen is

A female or in the athletics team.

$n(s) = 20 \text{ people}$ $\quad A = \text{athletics team.}$ $\quad F = \text{female}$

$$P(A) = \frac{7}{20} = 0.35 \qquad P(F) = \frac{11}{20} = 0.55$$

$$P(A \text{ and } F) = \frac{3}{20} = 0.15$$

$$P(A \text{ or } F) = P(A) + P(F) - P(A \text{ and } F)$$
$$= 0.75$$

Example 6:

In a survey, 15% of the participants said that they had never bought lottery tickets and a premium bonds, 73% had bought lottery tickets and 49% had bought premium bonds. Find the probability that a person chosen at random from those taking part in the survey

(a) had bought lottery tickets or premium bonds,

$$\text{L : person has bought lottery tickets; P(L)} = 0.73$$

$$\text{B : person has bought premium bonds; P(B)} = 0.49$$

$$\text{P(neither L nor B)} = P(L' \cap B') = 0.15$$

$$(a)\ \ \mathrm{P}(L \text{ or } B) = 1 - P(\text{neither } L \text{ nor } B)$$

$$= 1 - 0.15 = 0.85$$

Example 6:

In a survey, 15% of the participants said that they had never bought lottery tickets or a premium bonds, 73% had bought lottery tickets and 49% had bought premium bonds. Find the probability that a person chosen at random from those taking part in the survey

(b) had bought lottery tickets and premium bonds,

$$\text{L : person has bought lottery tickets; P(L)} = 0.73$$

$$\text{B : person has bought premium bonds; P(B)} = 0.49$$

$$(b)\ \ \mathrm{P}(L \text{ or } B) = P(L) + P(B) - P(L \text{ and } B)$$

$$0.85 = 0.73 + 0.49 - P(L \text{ and } B)$$

$$P(L \text{ and } B) = 0.73 + 0.49 - 0.85 = 0.37$$

Example 6:

In a survey, 15% of the participants said that they had never bought lottery tickets or a premium bonds, 73% had bought lottery tickets and 49% had bought premium bonds. Find the probability that a person chosen at random from those taking part in the survey

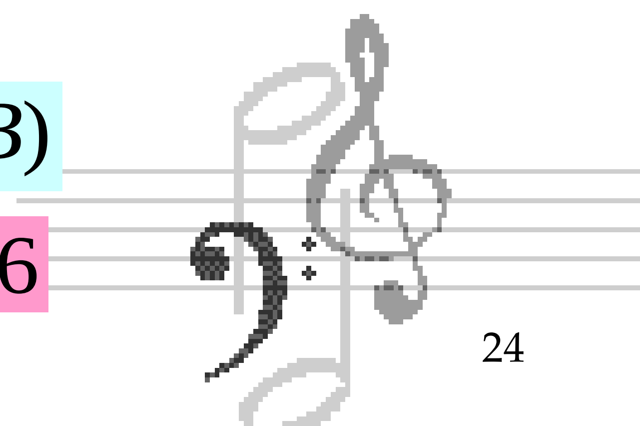
(c) had bought lottery tickets only.

$$\text{L : person has bought lottery tickets; } \mathrm{P(L)} = 0.73$$

$$\text{B : person has bought premium bonds; } \mathrm{P(B)} = 0.49$$

$$(c)\ \ \mathrm{P}(L \text{ only}) = P(L) - P(L \text{ and } B)$$

$$= 0.73 - 0.37 = 0.36$$



Example 7:

Events A and B are such that $P(A) = 0.3$, $P(B) = 0.4$, $P(A \cap B) = 0.1$.

Find (a) $P(A \cap B')$, (b) $P(A' \cap B')$

(a) $P(A) = P(A \cap B) + P(A \cap B')$

$0.3 = 0.1 + P(A \cap B')$

$P(A \cap B') = 0.2$

(b) $P(A' \cap B') = 1 - P(A \cup B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.4 - 0.1 = 0.6$

$P(A' \cap B') = 1 - P(A \cup B) = 1 - 0.6 = 0.4$

Statistics 1

Steve Dobbs and Jane Miller

Exercise 4A (Page 71)

Q2, Q3, Q4, Q5

S1/3/3 & S1/3/4 & S1/3/5:

Mutually exclusive and independent events & Conditional Probability & Tree Diagram

Learning Outcome

Students should be able to:

- **Use the addition of probabilities applicable to mutually exclusive events.**
- **Use the multiplication of probabilities applicable to independent events.**
- **Calculate and use conditional probabilities in simple cases.**
- Calculate and use conditional probabilities in situations that can be represented by means of a tree diagram.

Conditional Probability

Conditional Probability:

The symbol $P(\text{A} \mid \text{B})$ represents the probability that A will occur given B has occurred. This is called conditional probability.

$$P(\text{A}|\text{B}) = \frac{P(\text{A and B})}{P(\text{B})}$$

- Given B has occurred, the relevant sample space is no longer S, but B (reduced sample space)

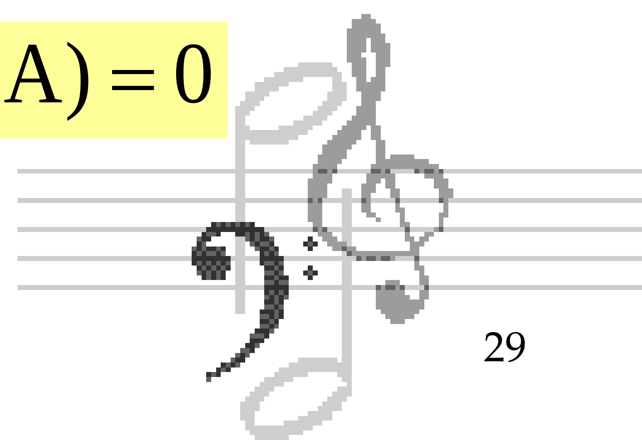
Mutually Exclusive Events

- **Mutually Exclusive Events:**
- If events A and B are mutually exclusive:

$$P(\text{A and B}) = P(\text{A} \cap \text{B}) = 0$$

(i) $\mathrm{P}(A \cup B) = P(A) + P(B)$

(ii) $P(\text{A} \mid \text{B}) = 0 \qquad P(B \mid \text{A}) = 0$



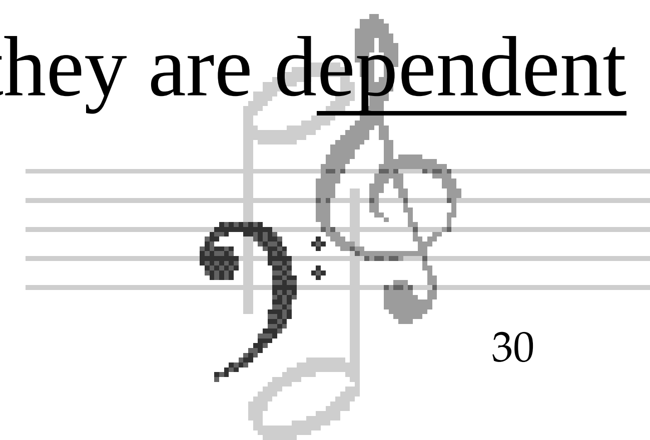
Independence, the Multiplication Rule, & Conditional Probability

Independent Events:

Two events A and B are independent events if the occurrence (or nonoccurrence) of one does not affect the probability assigned to the occurrence of the other.

Note:

If two events are *not* independent, they are dependent



Independence, the Multiplication Rule, & Conditional Probability

Independent Events:

$$P(\text{A and B}) = P(\text{A}) \times \text{P(B)}$$

$$\text{(i)} \quad \text{P}(A \cup B) = P(A) + P(B) - P(A) \times P(B)$$

$$\text{(ii)} \quad P(\text{A} \mid \text{B}) = P(\text{A}) \quad \text{or} \quad P(\text{B} \mid \text{A}) = P(\text{B})$$

Notes: If A and B are independent, the occurrence of B does not affect the occurrence of A

Example 8:

Consider the experiment in which a single fair die is rolled: $S = \{1, 2, 3, 4, 5, 6\}$. Define the following events:

A = "a 1 occurs," B = "an odd number occurs," and

C = "an even number occurs"

Find : $(i)\ P(\text{A}\,|\,\text{B})$

$$P(\text{A}\,|\,\text{B}) = \frac{P(\text{A and B})}{P(\text{B})} = \frac{P(\text{A} \cap \text{B})}{P(\text{B})} = \frac{1/6}{3/6} = \frac{1}{3}$$

Example 8:

Consider the experiment in which a single fair die is rolled: $S = \{1, 2, 3, 4, 5, 6\}$. Define the following events:

A = "a 1 occurs," B = "an odd number occurs," and

C = "an even number occurs"

Find : $(ii)\ P(\text{A}|\text{C})$

$$P(\text{A}|\text{C}) = \frac{P(\text{A and C})}{P(C)}$$

$$= \frac{P(\text{A} \cap \text{C})}{P(\text{C})} = \frac{0}{3/6} = 0$$

Example 8:

Consider the experiment in which a single fair die is rolled: $S = \{1, 2, 3, 4, 5, 6\}$. Define the following events:

A = "a 1 occurs," B = "an odd number occurs," and

C = "an even number occurs"

Find : $(iii)\ P(\mathrm{B} \mid \mathrm{A})$

$$P(B \mid A) = \frac{P(B \text{ and } \mathrm{A})}{P(A)}$$

$$= \frac{P(\mathrm{B} \cap \mathrm{A})}{P(A)} = \frac{1/6}{1/6} = 1$$

Example 8:

Consider the experiment in which a single fair die is rolled: $S = \{1, 2, 3, 4, 5, 6\}$. Define the following events:

A = “a 1 occurs,” B = “an odd number occurs,” and

C = “an even number occurs”

(iv) Are A and B independent events and explain why?

(v) Are A and C mutually exclusive events and explain why?

$$(iv) \quad \text{No. Because} \quad P(\mathrm{A} \mid \mathrm{B}) = \frac{1}{3} \neq P(A)$$

$$(v) \quad \text{Yes. Because} \; P(\mathrm{A} \mid \mathrm{C}) = 0$$

Notes

1. Independence and mutually exclusive are two very different concepts
   a. Mutually exclusive says the two events cannot occur together, that is, they have no intersection
   b. Independence says each event does not affect the other event's probability

2. P(A and B) = P(A) P(B) when A and B are independent
   a. Since P(A) and P(B) are not zero, P(A and B) is nonzero
   b. Thus, independent events have an intersection

Notes

3. Events cannot be both mutually exclusive and independent
   a. If two events are independent, then they are not mutually exclusive
   b. If two events are mutually exclusive, then they are not independent

Statistics 1

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Exercise 4B (Page 78)

Q2, Q3, Q4, Q7, Q8, Q10

General Multiplication Rule

General Multiplication Rule:

Let A and B be two events defined in sample space S. Then:

$$P(\mathrm{B} \mid \mathrm{A}) = \frac{P(B \cap A)}{P(A)} \qquad P(A \mid \mathrm{B}) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \quad P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}) \cdot P(\mathrm{B} \mid \mathrm{A})$$

or

$$\Rightarrow \quad P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{B}) \cdot P(\mathrm{A} \mid \mathrm{B})$$

Example 9:

A certain company uses three overnight delivery services: A, B, and C. The probability of selecting service A is 1/2, of selecting B is 3/10, and of selecting C is 1/5. Suppose the event T is "on time delivery."

$P(T|A) = 9/10$, $P(T|B) = 7/10$, and $P(T|C) = 4/5$.

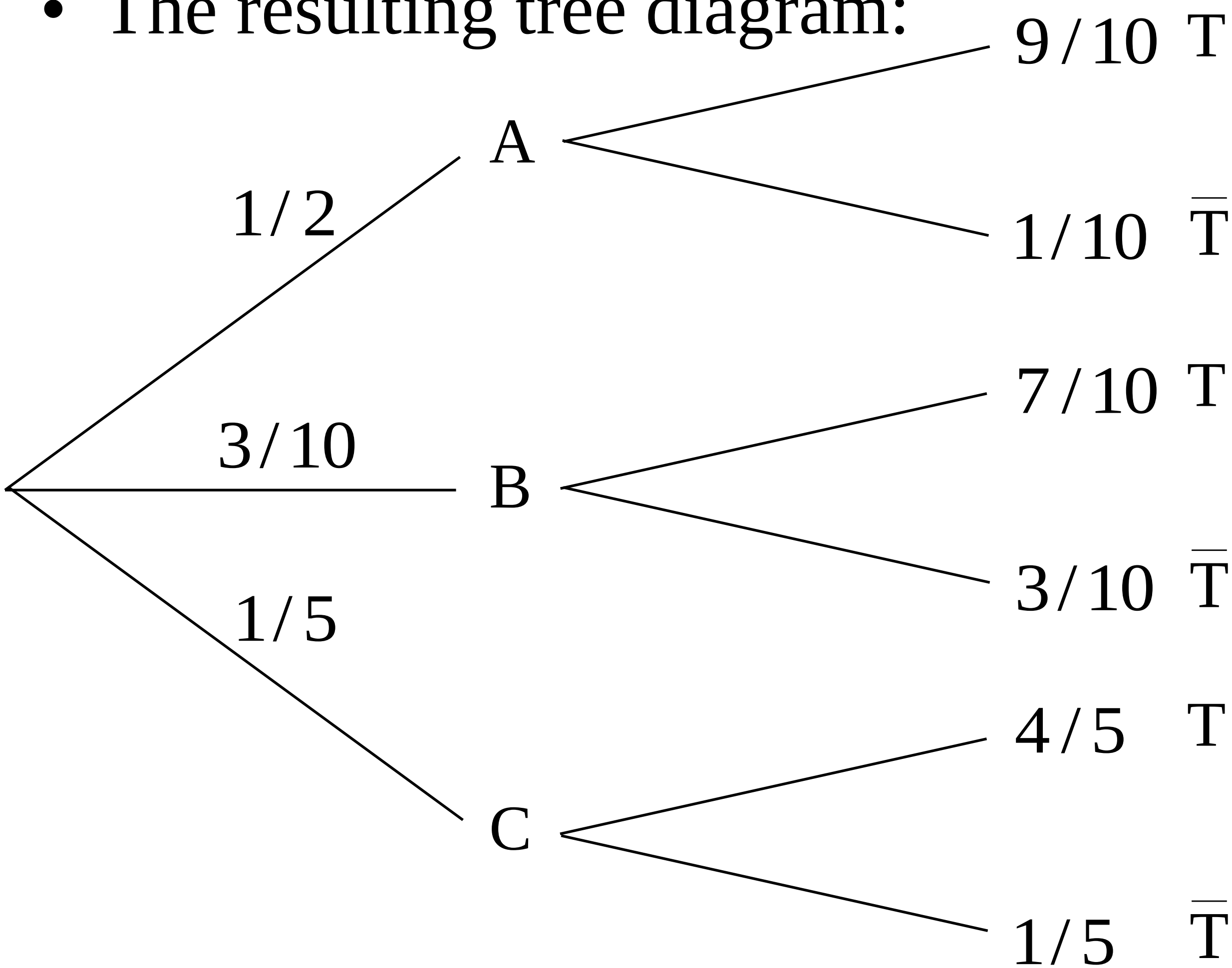
A service is randomly selected to deliver a package overnight.

1. The probability of selecting service A and having the package delivered on time:
2. The probability of having the package delivered on time:

Example 9:

Solution

- The resulting tree diagram:



P(A) = 1/2,

P(B) = 3/10,

P(C) = 1/5.

T = “on time delivery.”

***P*(T|A) = 9/10,**

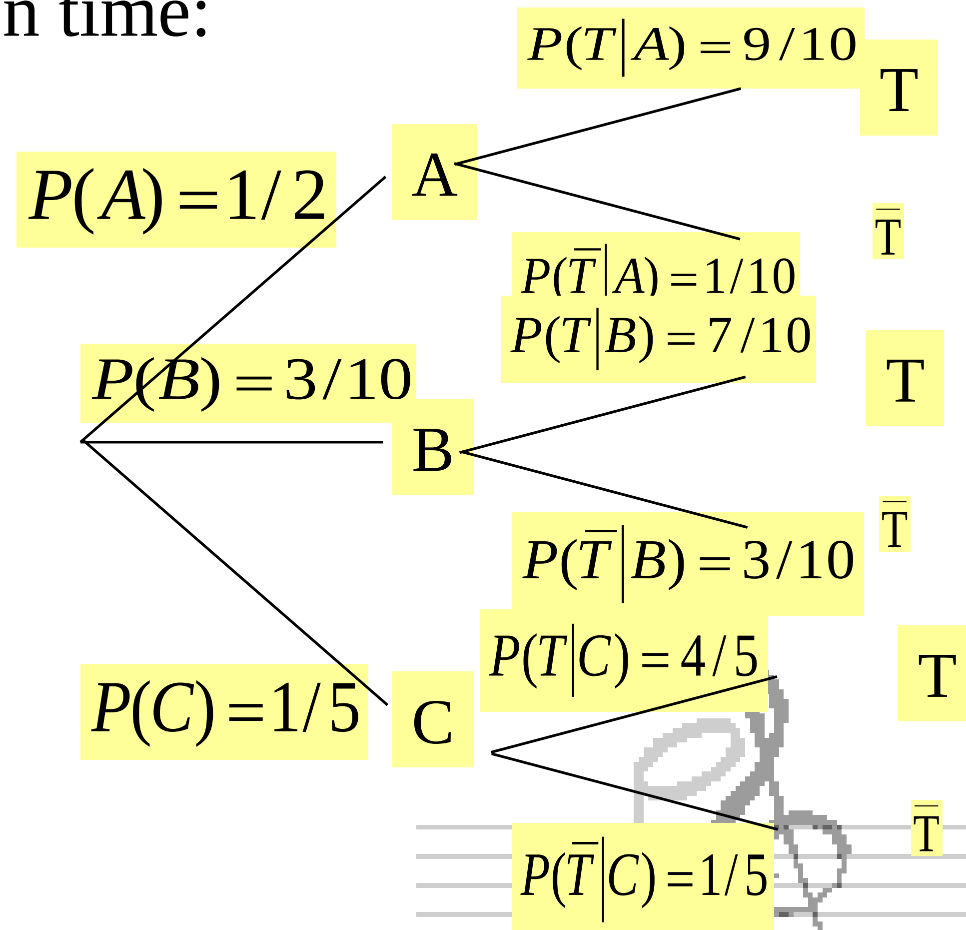
***P*(T|B) = 7/10, and**

***P*(T|C) = 4/5.**

Example 9:

1. The probability of selecting service A and having the package delivered on time:

$$P(\mathrm{T}\,|\,\mathrm{A}) = \frac{P(\mathrm{A\ and\ T})}{P(\mathrm{A})}$$



$$P(\mathrm{A\ and\ T}) = P(\mathrm{A})\cdot P(\mathrm{T}\,|\,\mathrm{A})$$

$$= \frac{1}{2}\cdot\frac{9}{10} = \frac{9}{20}$$

Example 9:

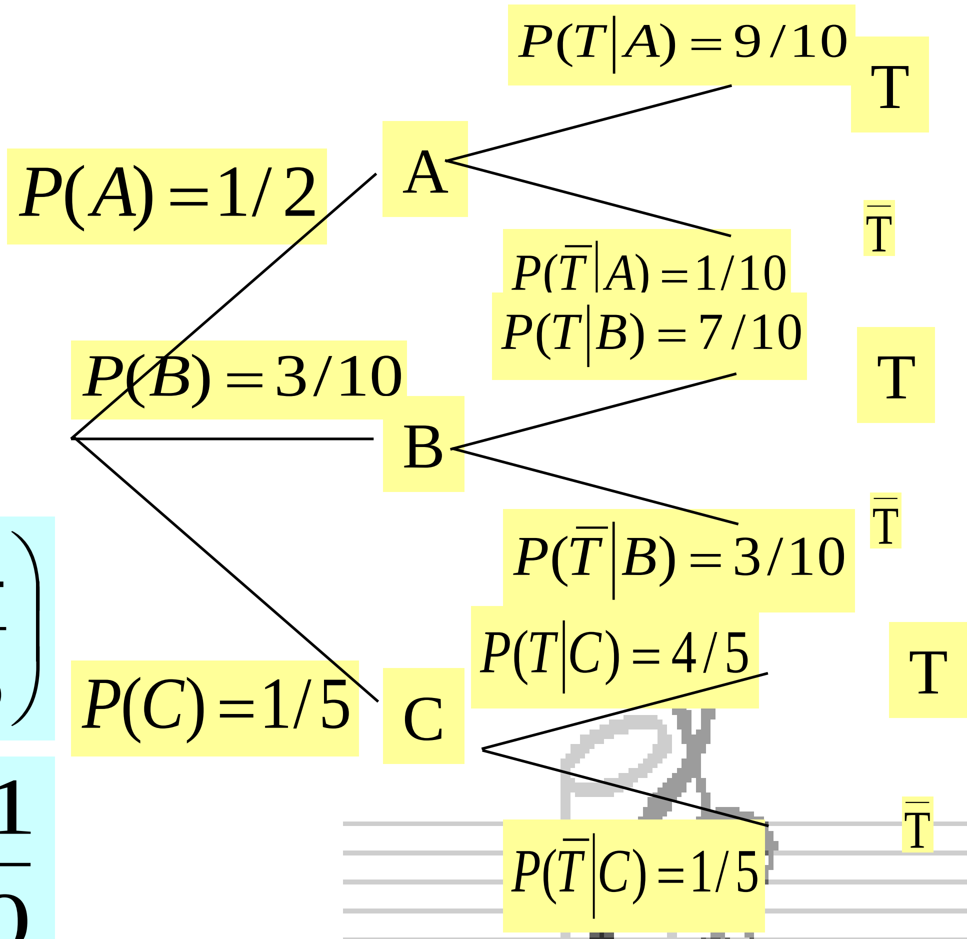
2. The probability of having the package delivered on time:

$$P(\mathrm{T}) = P(\mathrm{A\ and\ T}) + P(\mathrm{B\ and\ T}) + P(\mathrm{C\ and\ T})$$

$$= P(\mathrm{A})\cdot P(\mathrm{T}\,|\,\mathrm{A}) + P(\mathrm{B})\cdot P(\mathrm{T}\,|\,\mathrm{B}) + P(\mathrm{C})\cdot P(\mathrm{T}\,|\,\mathrm{C})$$

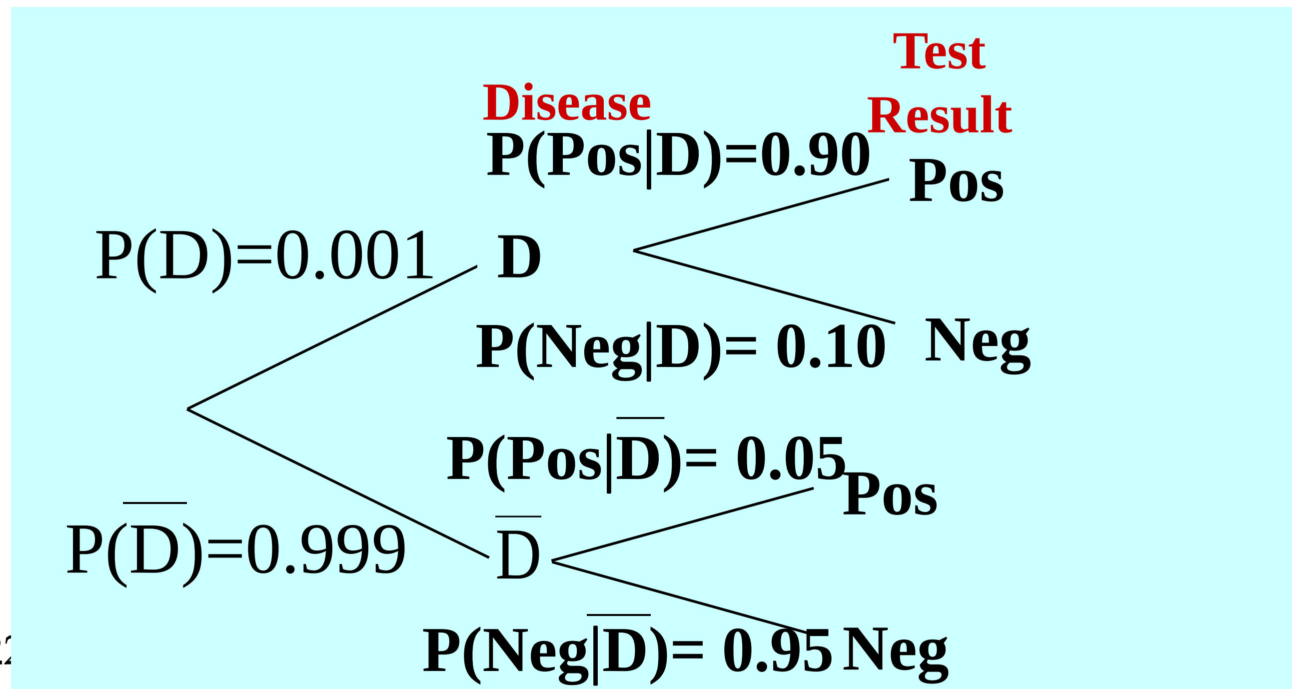
$$= \left(\frac{1}{2}\right)\left(\frac{9}{10}\right) + \left(\frac{3}{10}\right)\left(\frac{7}{10}\right) + \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)$$

$$= \frac{9}{20} + \frac{21}{100} + \frac{4}{25} = \frac{41}{50}$$



Example 10:

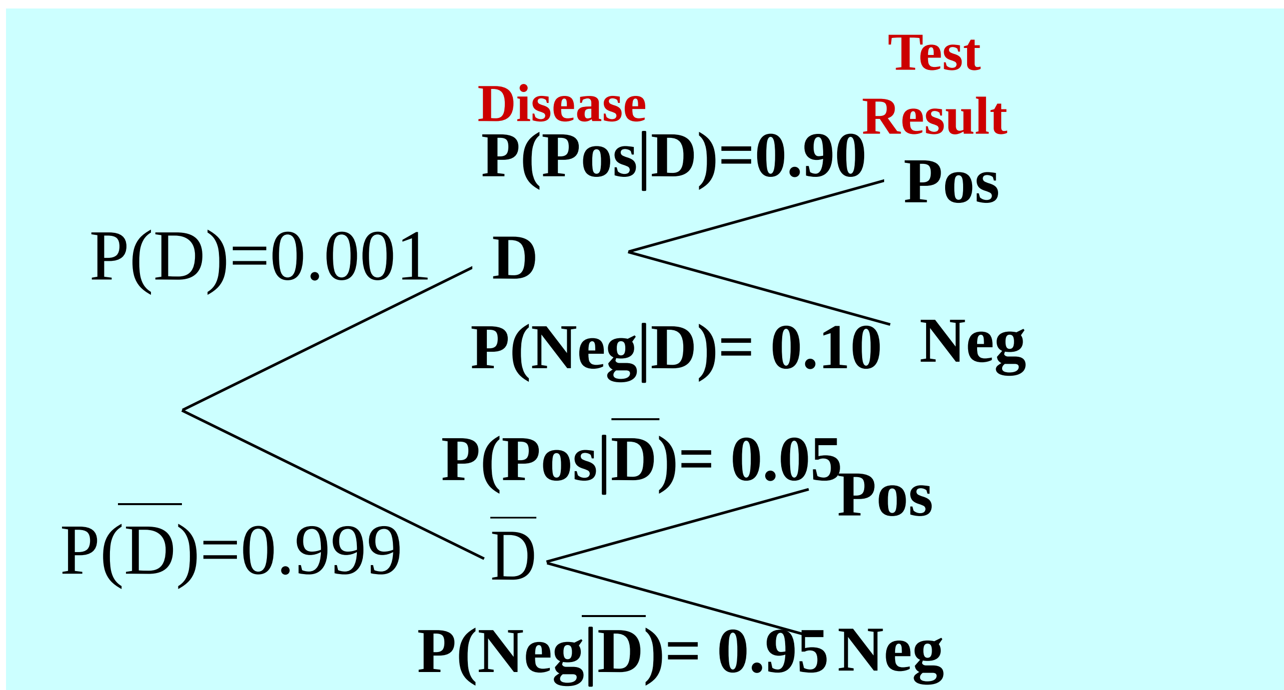
This problem involves testing individuals for the presence of a disease. Suppose the probability of having the disease (D) is 0.001. If a person has the disease, the probability of a positive test result (Pos) is 0.90. If a person does not have the disease, the probability of a negative test result (Neg) is 0.95. For a person selected at random:



Example 10:

1) Find the probability of a negative test result given the person has the disease

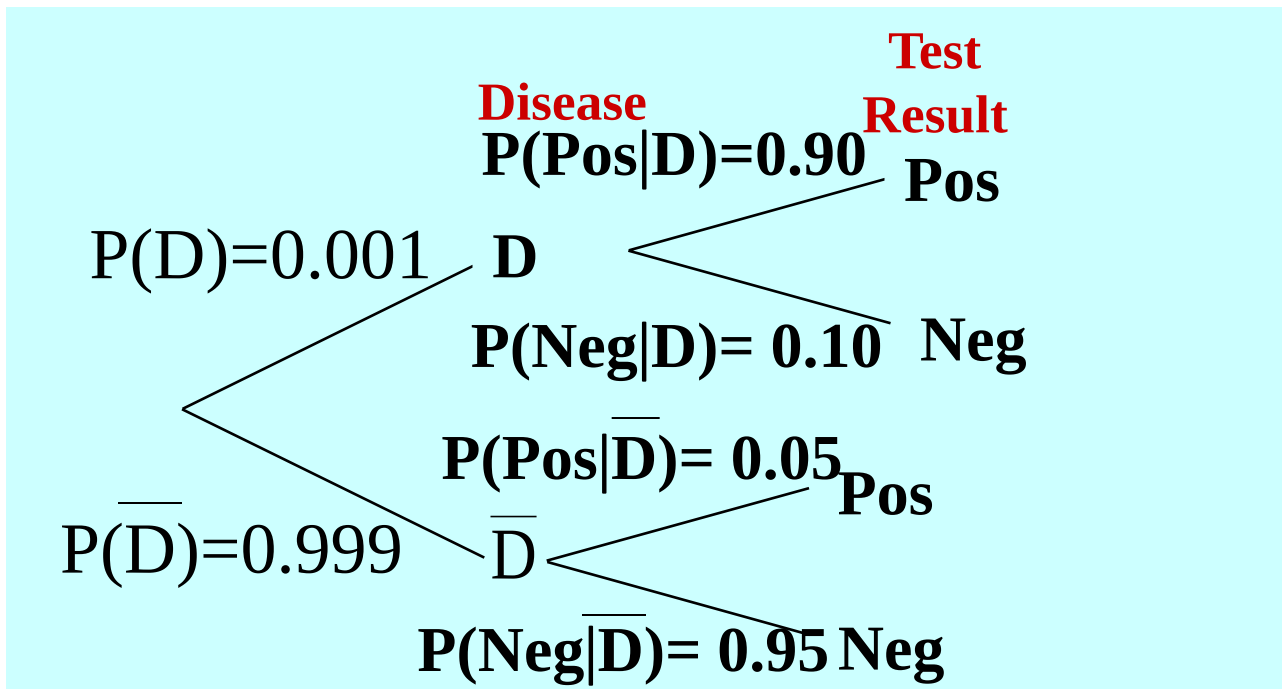
$$P(\mathrm{Neg}|\mathrm{D}) = 1 - P(\mathrm{Pos}|\mathrm{D})$$
$$= 1 - 0.90 = 0.10$$



Example 10:

2) Find the probability of having the disease and a positive test result

$$P(\text{D and Pos}) = P(\text{D}) \cdot P(\text{Pos}|D)$$
$$= (0.001)(0.90) = 0.0009$$



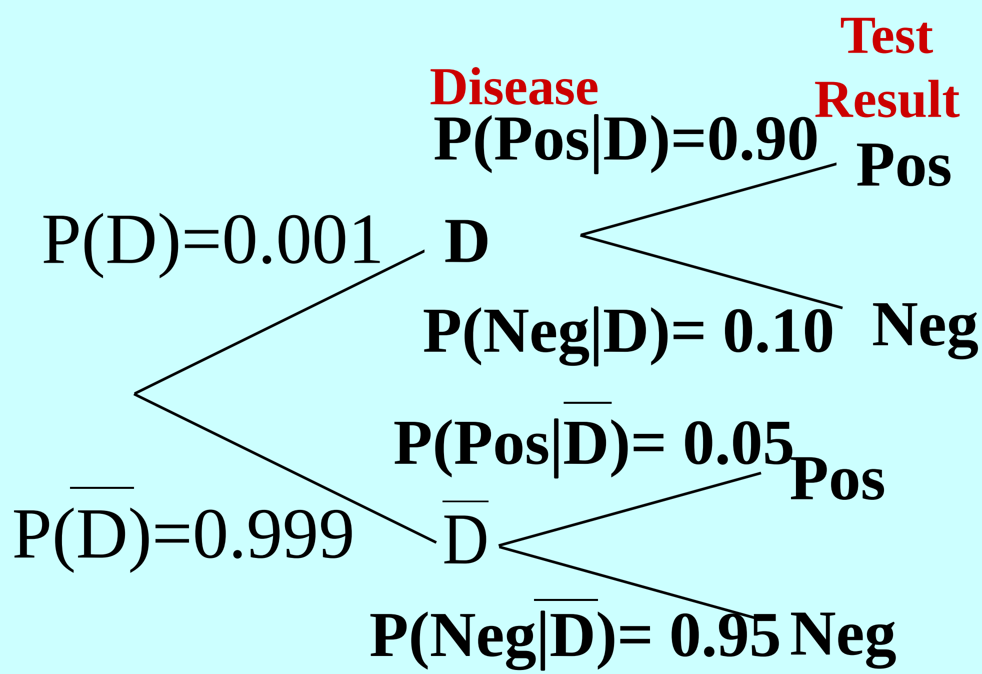
Example 10:

3) Find the probability of a positive test result

$$P(\text{Pos}) = P(\text{Pos and D}) + P(\text{Pos and } \overline{\text{D}})$$

$$= P(\text{D}) \cdot P(\text{Pos} | \text{D}) + P(\overline{\text{D}}) \cdot P(\text{Pos} | \overline{\text{D}})$$

$$= (0.001)(0.90) + (0.999)(0.05) = 0.05085$$

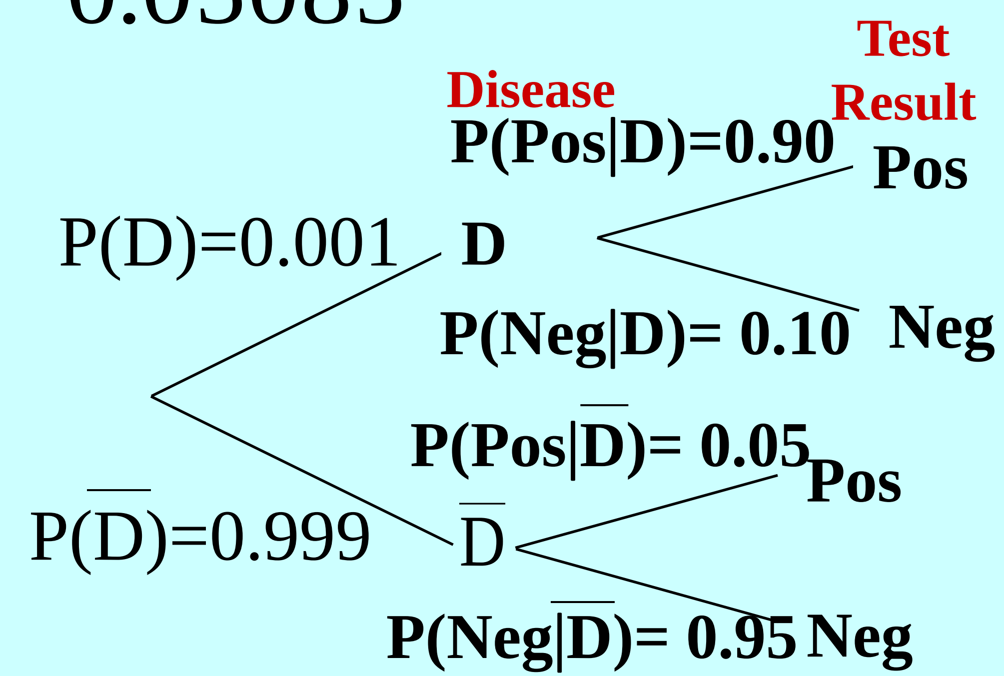


Example 10:

4) Find the probability of $P\left(D \middle| Pos\right)$

$$P\left(D \middle| Pos\right) = \frac{P(\mathrm{D}) \cdot P\left(\mathrm{Pos} \middle| D\right)}{P(\mathrm{Pos})}$$

$$= \frac{0.0009}{0.05085} = 0.0177$$



Statistics 1

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Exercise 4B (Page 78)

Q1, Q5, Q6, Q12