Topic 3 - Trigonometry

Trigonometric Functions - The form $a\sin x + b\cos x$

$a\sin x + b\cos x \equiv R\sin(x+\alpha)$
$a\sin x - b\cos x \equiv R\sin(x-\alpha)$
$a\cos x + b\sin x \equiv R\cos(x-\alpha)$
$a\cos x - b\sin x \equiv R\cos(x+\alpha)$

where $R > 0$ and α is an acute angle.

Example 1

Find the value of R and $\tan\alpha$ in these identities.

a $4\cos\theta + 3\sin\theta \equiv R\cos(\theta - \alpha)$ **b** $2\cos\theta - 3\sin\theta \equiv R\cos(\theta + \alpha)$

c $7\cos\theta + \sin\theta \equiv R\sin(\theta + \alpha)$ **d** $3\sin\theta - 4\cos\theta \equiv R\sin(\theta - \alpha)$

a $5, \frac{3}{4}$ **b** $\sqrt{13}, \frac{3}{2}$
c $5\sqrt{2}, 7$ **d** $5, \frac{4}{3}$

Example 2

Solve these equations for $0° \leqslant \theta \leqslant 360°$.

a $\sqrt{3}\cos\theta + \sin\theta = 1$ **b** $5\sin\theta - 12\cos\theta = 6$

c $\sin\theta + \cos\theta = \frac{1}{2}$ **d** $2\cos 2\theta + \sin 2\theta = 1$

a $\theta = 90°, 330°$
b $\theta = 94.9°, 219.9°$
c $\theta = 114.3°, 335.7°$
d $\theta = 45°, 225°; 161.6°, 341.6°$

Example 3

a Show that $3\cos\theta + 4\sin\theta$ may be written in the form $5\cos(\theta - \alpha)$ where $\tan\alpha = \frac{4}{3}$.

b Hence find the values of these functions at their local maxima and minima, giving the smallest positive values of θ, in degrees, for which they occur.

i $3\cos\theta + 4\sin\theta$ **ii** $6 + 3\cos\theta + 4\sin\theta$ **iii** $10 - 3\cos\theta - 4\sin\theta$

iv $\dfrac{1}{3\cos\theta + 4\sin\theta}$ **v** $\dfrac{1}{2 + 3\cos\theta + 4\sin\theta}$ **vi** $\dfrac{7}{12 + 3\cos\theta + 4\sin\theta}$

vii $(3\cos\theta + 4\sin\theta)^2$ **viii** $\dfrac{1}{1 + (3\cos\theta + 4\sin\theta)^2}$

b **i** 5, 53.1°; −5, 233.1°
ii 11, 53.1°; 1, 233.1°
iii 15, 233.1°; 5, 53.1°
iv $-\frac{1}{5}$, 233.1°; $\frac{1}{5}$, 53.1°
v $-\frac{1}{3}$, 233.1°; $\frac{1}{7}$, 53.1°
vi 1, 233.1°; $\frac{7}{17}$, 53.1°
vii 25, 53.1°; 0, 143.1°
viii 1, 143.1°; $\frac{1}{26}$, 53.1°

Exercise - The form *asinx + bcosx.*

Question 1

By expressing $8\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, solve the equation

$$8\sin\theta - 6\cos\theta = 7,$$

for $0° \leqslant \theta \leqslant 360°$. **[7]**

10sin (θ- 36.9°), 81.3°, 172.4°

Question 2

(i) Express $3\cos x° + \sin x°$ in the form $R\cos(x - \alpha)°$ where $R > 0$ and $0 < \alpha < 90$.

(ii) Using your answer to part *(a)*, or otherwise, solve the equation

$$6\cos^2 x° + \sin 2x° = 0,$$

for x in the interval $0 \le x \le 360$, giving your answers to 1 decimal place where appropriate.

i)$\sqrt{10}$ cos (x – 18.4)°, ii) x = 90, 108.4 (1dp), 270, 288.4 (1dp)

Question 3

(i) **Express $7\cos\theta + 24\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places.** **[3]**

(ii) **Hence solve the equation**

$$7\cos\theta + 24\sin\theta = 15,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. **[4]**

i)25, 73.74 ii) 20.6, 126.9

Question 4

Given that $f(x) \equiv 4\cos x - b\sin x \equiv \sqrt{20}\cos(x+\alpha)$, $b > 0$, $0 < \alpha < \frac{\pi}{2}$,

(i) find the value of b and the value of $\tan\alpha$.

(ii) Solve the equation $f(x) = \sqrt{5}$, for $0 < x < 2\pi$, giving your answer to 3 significant figures.

(iii) Find the maximum value of $\dfrac{1}{f(x)+5}$, $x \in \Re$.

i) 2, ½ ii) 0.584, 4.77 iii) 1.89

Question 5

(i) Express $4\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, stating the value of α correct to 2 decimal places. [3]

Hence

(ii) solve the equation

$$4\sin\theta - 3\cos\theta = 2,$$

giving all values of θ such that $0° < \theta < 360°$, [4]

(iii) write down the greatest value of $\dfrac{1}{4\sin\theta - 3\cos\theta + 6}$. [1]

i)5,36.87 ii) 60.4,193.3 iii) 1

Question 6

It is given that $f(x) = 10\cos^2 x - 8\sin x\cos x + 4\sin^2 x$.

Express $f(x)$ in the form $a\cos 2x + b\sin 2x + c$, where a, b and c are constants. [3]

Hence, by rewriting $f(x)$ in the form $R\cos(2x+\alpha) + c$, or otherwise, show that the greatest and least values of $f(x)$ are 12 and 2 respectively. [3]

$3\cos 2x - 4\sin 2x + 7$, $R = 5$ $c = 7$

Question 7

Show that the equation $2\sec\theta° - \tan\theta° = 3$ can be expressed in the form $R\cos(\theta - \alpha)° = 2$ where the values of R and α with $(0 < \alpha < 90)$are to be stated. Hence solve the equation $2\sec\theta° - \tan\theta° = 3$, giving all values of θ such that $0 < \theta < 360$.

$\sqrt{10}$, 18.4; 69.2, 327.7

Question 8

. (i) Show that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec}2\theta$. [2]

(ii) Find all solutions of the equation $2\operatorname{cosec}2\theta = 3\tan\theta + 1$ for $-\pi \leq \theta \leq \pi$.

[6]

(iii) Express $3\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$ where $R > 0$ and $0^0 < \alpha < 90^0$. Hence or otherwise, solve the equation $3\cos\theta - \sin\theta = -1$ for $-180^0 \leq \theta \leq 180^0$. [4]

ii) 0.464, -0.786, -2.68, 2.36 iii) 90, -126.9