# DIFFERENTIAL EQUATIONS

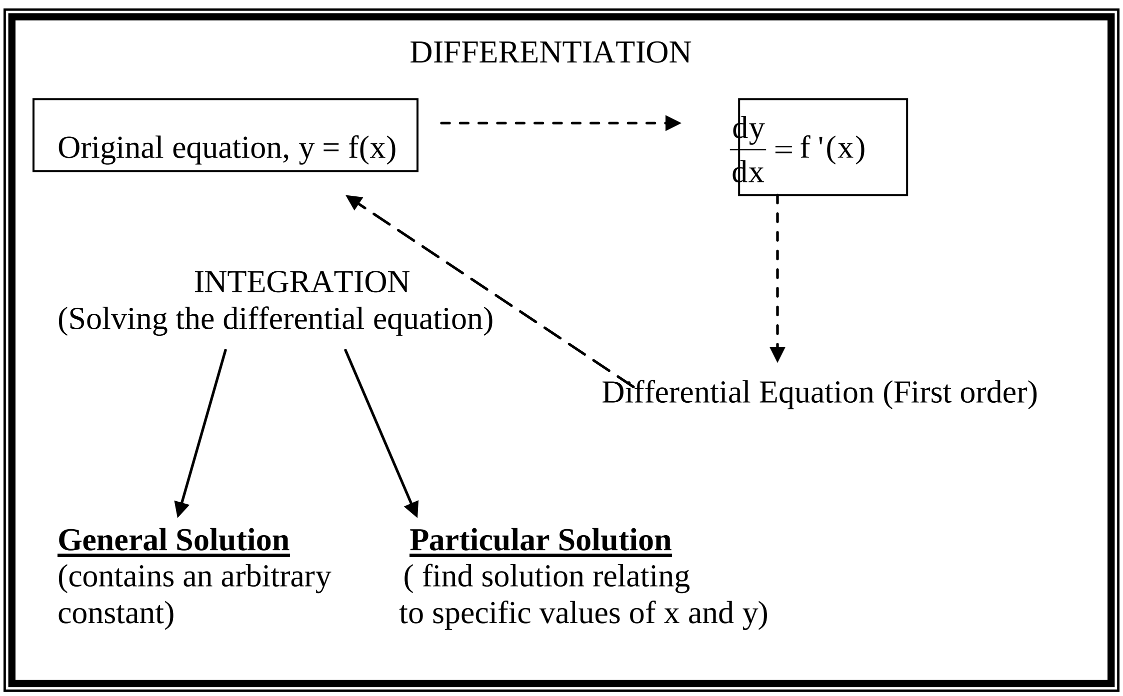
A differential equation in *x* and *y* is an equation containing at least one of the derivates

$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3} \ldots$

First order differentiation

Examples of first order differentiation:

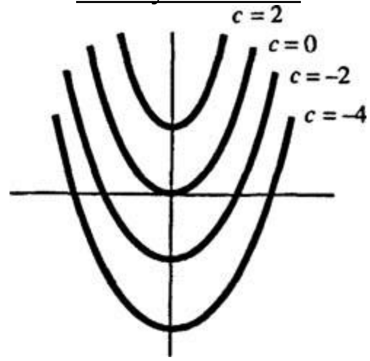
1. $\frac{dy}{dx} = 5x^2$
2. $\frac{dy}{dx} - xy = 0$
3. $e^y \frac{dy}{dx} = \sin x$



Example:

Given $\dfrac{dy}{dx} = 2x$ ⇨ $\boldsymbol{y = x^2 + c}$

<u>Family of Curves</u>




**1. First order differential equations of the form** $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{f}(\mathrm{x})$

- this is the simplest form of differential equation
- to solve this form, rearrange the terms and
- integrate both side with respect to x.

$$\text{if} \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{f}(\mathrm{x}), \quad \text{then } \mathrm{y} = \int \mathrm{f}(\mathrm{x})\,\mathrm{dx}$$

Example 1: Solve the differential equation $\dfrac{dy}{dx} = 6x^2 - 5$.

Example 2: Solve the differential equation $\cos^2 x \dfrac{dy}{dx} = \sin x$.

Example 3: Solve the differential equation $e^x \dfrac{dy}{dx} + x = 0$, given that $y = 2$ when $x = 0$.

**2. First order differential equations of the form** $\dfrac{1}{\mathrm{g}(\mathrm{y})}\dfrac{\mathrm{dy}}{\mathrm{dx}} = 1$

- integrate both side with respect to x.

$$\text{if} \quad \frac{1}{\mathrm{g}(\mathrm{y})}\frac{\mathrm{dy}}{\mathrm{dx}} = 1, \quad \text{then } \int \frac{1}{\mathrm{g}(\mathrm{y})}\frac{\mathrm{dy}}{\mathrm{dx}}\mathrm{dx} = \int \mathrm{dx}$$

$$\text{finally} \quad \int \frac{1}{\mathrm{g}(\mathrm{y})}\mathrm{dy} = \int \mathrm{dx}$$

Example 4: Solve the differential equation $\dfrac{dy}{dx} = y^2$.

Example 5: Solve the differential equation $\dfrac{dy}{dx} = \cos^2 y$.

Example 6: Solve the differential equation $2\dfrac{dy}{dx} - y^2 + 1 = 0$, given that $y = 2$, when $x = 0$.

**3. First order differential equations of the form** $g(y)\frac{dy}{dx}=f(x)$

- separate the variables, in terms of x and y.
- the terms in y is taken to the left-hand side, next to $\frac{dy}{dx}$.
- integrate both sides with respect to x.

$$\text{if } \; g(y)\frac{dy}{dx}=f(x), \quad \text{then} \int g(y)\frac{dy}{dx}dx=\int f(x)\,dx$$

$$\text{finally} \quad \int g(y)\,dy=\int f(x)\,dx$$

Example 7: Solve the differential equation $2y\frac{dy}{dx}=\cos x$.

Example 8: Solve the differential equation $\frac{dy}{dx}=2xy^2-y^2$.

Example 9: Express y in terms of x given that $\frac{dy}{dx}=xy-x$ and that $y=4$ when $x=0$.

Example 10: Find the general solution of the equation $2x\frac{dy}{dx}-(2y+1)(x+1)=0$.

**Exercise 1**

Find the general solutions to the following differential equations, giving $y$ in terms of $x$, in each case.

**2.** $\frac{dy}{dx}=x+2$ **3.** $2y\frac{dy}{dx}=3x^2$ **4.** $x^5\frac{dy}{dx}=4y^2$

**5.** $\frac{dy}{dx}-e^xy^2=0$ **6.** $\frac{dy}{dx}=\frac{\cos x}{y}$ **7.** $\frac{dy}{dx}=\frac{1}{1-x}$

**8.** $\frac{dy}{dx}+2x\operatorname{cosec} y=0$ **9.** $\frac{dy}{dx}=2y$ **10.** $\frac{dy}{dx}=3x^2(3+y)$

**11.** $x\frac{dy}{dx}=\sec y$ **13.** $\frac{dy}{dx}=4xe^{-y}$

**14.** $\frac{dy}{dx}=\frac{y}{x+1}$ **16.** $2\frac{dy}{dx}=3x^2(y^2-1)$

Solve the following differential equations, giving $y$ in terms of $x$, in each case.

**17.** $\frac{dy}{dx}+4=12x$; when $x=-2$, $y=30$.

**18.** $\frac{dy}{dx}=y^2$; when $x=3$, $y=-1$.

**19.** $3y^2\frac{dy}{dx}=2x+1$; when $x=2$, $y=2$.

**20.** $(\cos y)\frac{dy}{dx}=x^2\operatorname{cosec}^2 y$; when $x=\frac{1}{2}$, $y=\frac{\pi}{2}$.

Solve the following differential equations, giving $y$ in terms of $x$, in each case.

**21.** $x\dfrac{dy}{dx} = 2$; $x > 0$ and when $x = 1$, $y = -3$.

**22.** $x\dfrac{dy}{dx} = 2 + \dfrac{dy}{dx}$; $x > 1$ and when $x = 2$, $y = 1$.

**23.** $\dfrac{dy}{dx} = 4xy$; when $x = 0$, $y = 4$.

**24.** $\dfrac{dy}{dx} = 6x(y + 1)$; when $x = 0$, $y = 3$.

**25.** $\dfrac{dy}{dx} = (y - 3)(4x + 3)$; when $x = -1$, $y = 3(e^{-1} + 1)$.

**26.** $e^{y}\dfrac{dy}{dx} + \sin x = 0$; when $x = \dfrac{\pi}{2}$, $y = 1$.

**27.** $x\dfrac{dy}{dx} + y^2 = 0$; $x > 0$ and when $x = 1$, $y = \frac{1}{2}$.

**28.** $x\dfrac{dy}{dx} = y + 2$; when $x = 3$, $y = 7$.

**29.** $x\dfrac{dy}{dx} + 3 = y - 4\dfrac{dy}{dx}$; when $x = 1$, $y = 13$.

<u>Exam Questions</u>

June 2004

**6** Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = \frac{y^3 + 1}{y^2},$$

obtaining an expression for $y$ in terms of $x$. [6]

June 2005

**8** **(i)** Using partial fractions, find

$$\int \frac{1}{y(4 - y)}\, dy. \qquad [4]$$

**(ii)** Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = y(4 - y),$$

obtaining an expression for $y$ in terms of $x$. [4]

**(iii)** State what happens to the value of $y$ if $x$ becomes very large and positive. [1]

Nov 2006

4 Given that $y = 2$ when $x = 0$, solve the differential equation

$$y\frac{dy}{dx} = 1 + y^2,$$

obtaining an expression for $y^2$ in terms of $x$. [6]

May/June 2010

The equation of a curve is

$$x \ln y = 2x + 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{y}{x^2}$. [4]

(ii) Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

## **FORMULATION OF DIFFERENTIAL EQUATIONS**

Differential equations often occur when a mathematical model is used to describe a physical situation.

Example 11:
A circular inkblot, with radius r cm, is enlarging such that the rate of increase of the radius at the time t seconds is given by

$$\frac{dr}{dt} = \frac{0.1}{r}$$

Initially the radius of the inkblot is 0.4 cm. Find the area of the inkblot after 2 seconds.

Example 12:
The tangent at any point P on a curve in the first quadrant cuts the x-axis at Q. Given that OP = PQ, where O is the origin, and that the point (1,4) lies on the curve, find the equation of the curve.

## Exercise 2

May/June 2002

**7** In a certain chemical process a substance is being formed, and $t$ minutes after the start of the process there are $m$ grams of the substance present. In the process the rate of increase of $m$ is proportional to $(50 - m)^2$. When $t = 0$, $m = 0$ and $\frac{dm}{dt} = 5$.

**(i)** Show that $m$ satisfies the differential equation

$$\frac{dm}{dt} = 0.002(50 - m)^2.$$ [2]

**(ii)** Solve the differential equation, and show that the solution can be expressed in the form

$$m = 50 - \frac{500}{t + 10}.$$ [5]

**(iii)** Calculate the mass of the substance when $t = 10$, and find the time taken for the mass to increase from 0 to 45 grams. [2]

**(iv)** State what happens to the mass of the substance as $t$ becomes very large. [1]

Oct/Nov 2003

**9** Compressed air is escaping from a container. The pressure of the air in the container at time $t$ is $P$, and the constant atmospheric pressure of the air outside the container is $A$. The rate of decrease of $P$ is proportional to the square root of the pressure difference $(P - A)$. Thus the differential equation connecting $P$ and $t$ is

$$\frac{dP}{dt} = -k\sqrt{(P - A)},$$

where $k$ is a positive constant.

**(i)** Find, in any form, the general solution of this differential equation. [3]

**(ii)** Given that $P = 5A$ when $t = 0$, and that $P = 2A$ when $t = 2$, show that $k = \sqrt{A}$. [4]

**(iii)** Find the value of $t$ when $P = A$. [2]

**(iv)** Obtain an expression for $P$ in terms of $A$ and $t$. [2]

Oct/Nov 2005

**8** In a certain chemical reaction the amount, $x$ grams, of a substance present is decreasing. The rate of decrease of $x$ is proportional to the product of $x$ and the time, $t$ seconds, since the start of the reaction. Thus $x$ and $t$ satisfy the differential equation

$$\frac{dx}{dt} = -kxt,$$

where $k$ is a positive constant. At the start of the reaction, when $t = 0$, $x = 100$.

**(i)** Solve this differential equation, obtaining a relation between $x$, $k$ and $t$. [5]

**(ii)** 20 seconds after the start of the reaction the amount of substance present is 90 grams. Find the time after the start of the reaction at which the amount of substance present is 50 grams. [3]

**Miscellaneous Exercise**
**Question 1**

An infectious disease is spreading in an isolated village of 400 people. The number of people with the disease at time $t$ days is $N$.

**(a)** In one model of the spread of the disease it is assumed that the rate of increase of the number of people with the disease is proportional to the number with the disease at the time. When $t = 0$, 40 people have the disease and it is spreading at a rate of 20 people per day. Taking $N$ to be a continuous variable, show that it satisfies the differential equation

$$\frac{dN}{dt} = \tfrac{1}{2}N. \qquad [2]$$

**(i)** Solve the differential equation, obtaining an expression for $N$ in terms of $t$, and sketch the solution curve of $N$ against $t$. [4]

**(ii)** Show that the time predicted for half of the people in the village to have the disease is 3.2 days to 2 significant figures. [2]

**(b)** In an alternative model it is assumed that the rate of increase of $N$ varies as the product of the number with the disease at the time and the number of those not yet infected. You are given that with the initial conditions stated in part **(a)**, $N$ satisfies the differential equation

$$\frac{dN}{dt} = \frac{N(400 - N)}{720}.$$

Solve this differential equation and show that the time predicted for half of the people in the village to have the disease is 4.0 days to 2 significant figures. [6]

## Question 2

In places where very heavy rainfall occurs, 'storm drains' are used to collect rainwater and prevent flooding. A storm drain can be modelled as a rectangular tank with vertical sides, so that the volume of water in the tank is proportional to the depth of the water. During a storm, rainwater flows into the tank at a constant rate. Water drains away from the bottom of the tank at a rate which is proportional to the square root of the depth of water in the tank. At time $t$ hours the depth of water in the tank is $h$ metres. Explain how the information given above leads to the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = A - B\sqrt{h},$$

where $A$ and $B$ are positive constants. [3]

When rainfall stops and no more water is flowing into the tank, the value of $A$ is zero. For this case, find the value of $B$, given that it takes half an hour for the depth of water in the tank to fall from 1 metre to zero. [4]

In another case, $A = 2$ and $B = 1$, so that the differential equation is $\frac{\mathrm{d}h}{\mathrm{d}t} = 2 - \sqrt{h}$.

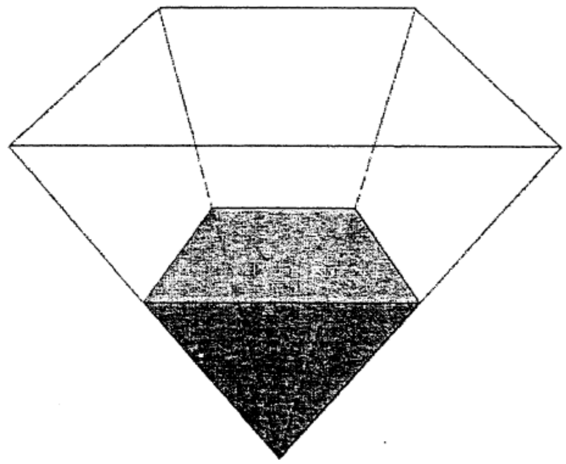
(i) By means of the substitution $x = 2 - \sqrt{h}$, or otherwise, find

$$\int \frac{1}{2 - \sqrt{h}}\, \mathrm{d}h.$$

[5]

(ii) Given that the tank is initially empty, find the time taken for the depth of water in the tank to reach 1 metre. [3]

**Question 3**



An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time $t$ hours after filling begins, the volume of liquid is $V$ m$^3$ and the depth of liquid is $h$ m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of 20 m$^3$ per hour, but owing to leakage, liquid is lost at a rate proportional to $h^2$. When $h = 1$, $\frac{dh}{dt} = 4.95$.

(i) Show that $h$ satisfies the differential equation

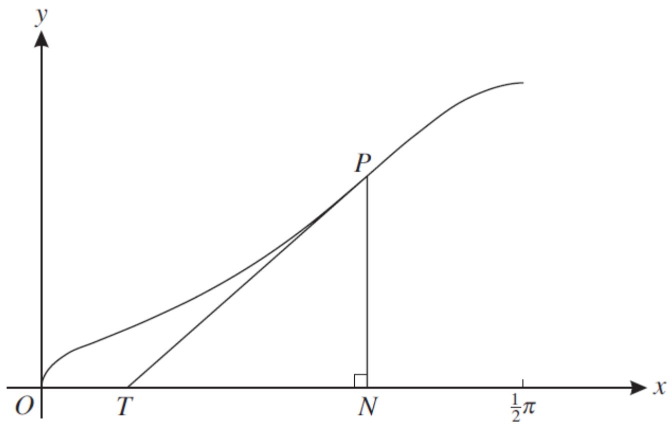
$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}.$$ [4]

(ii) Verify that $\frac{20h^2}{100 - h^2} \equiv -20 + \frac{2000}{(10 - h)(10 + h)}$. [1]

(iii) Hence solve the differential equation in part (i), obtaining an expression for $t$ in terms of $h$. [5]

**Question 4** (May/June 2008)

**8**




In the diagram the tangent to a curve at a general point $P$ with coordinates $(x, y)$ meets the $x$-axis at $T$. The point $N$ on the $x$-axis is such that $PN$ is perpendicular to the $x$-axis. The curve is such that, for all values of $x$ in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle $PTN$ is equal to $\tan x$, where $x$ is in radians.

**(i)** Using the fact that the gradient of the curve at $P$ is $\frac{PN}{TN}$, show that

$$\frac{dy}{dx} = \frac{1}{2}y^2 \cot x.$$ [3]

**(ii)** Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing $y$ in terms of $x$. [6]