Paper 3 - Topic 9: Complex Numbers

- understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, modulus, argument, conjugate, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal;
- carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in cartesian form $x + \mathrm{i}y$;
- use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs;
- represent complex numbers geometrically by means of an Argand diagram;
- carry out operations of multiplication and division of two complex numbers expressed in polar form $r(\cos\theta + \mathrm{i}\sin\theta) \equiv r\mathrm{e}^{\mathrm{i}\theta}$;
- find the two square roots of a complex number;
- understand in simple terms the geometrical effects of conjugating a complex number and of adding, subtracting, multiplying and dividing two complex numbers;
- illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram, e.g. $|z-a| < k$, $|z-a| = |z-b|$, $\arg(z-a) = \alpha$.

Introduction to Complex Numbers

To solve the quadratic equation $x^2 + 4x + 5 = 0$.
Using quadratic formula, the solutions would be:

$$x = \frac{-4 \pm \sqrt{4^2 - 4\times1\times5}}{2} = \frac{-4 \pm \sqrt{-4}}{2}.$$

We notice a problem however since $\sqrt{-4}$ is not a real number. So the equation $x^2 + 4x + 5 = 0$ does not have any real roots.

However, suppose we introduced the symbol *i* to represent $\sqrt{-1}$. We could then find expressions for the solutions of the quadratic:

$$x = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm \sqrt{4\times -1}}{2} = \frac{-4 \pm 2i}{2}.$$

So the equation has two solutions: $x = -2 + i$ or $x = -2 - i$.
These two solutions are called **complex roots**.

Definitions

Suppose that z is a complex number. Let $z = a + ib$.

The **real part** of z, written Re(z), is a.

The **imaginary part** of z, written Im(z), is b.

The **complex conjugate** of z, written z^* or $\bar{z}$, is $z^* = a - ib$.

Example : Let $z = 5 - 2i$, $w = -2 + i$ and $u = 7i$.
Then:

Re(z) = 5	Im(z) = -2	z^* = 5 + 2i
Re(w) = -2	Im(w) = 1	w* = -2 – i.
Re(u) = 0	Im(u) = 7	u^* = -7i.

Solving Quadratic Equations

Example: Solve the quadratic equation $2x^2 + 4x + 5 = 0$, giving your answers as complex numbers in surd form.

Solution: Using the quadratic formula:

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 2 \times 5}}{4} = \frac{-4 \pm \sqrt{-24}}{4}$$

$$= \frac{-4 \pm i\sqrt{24}}{4} = \frac{-4 \pm 2\sqrt{6}\,i}{4}$$

So the solutions are $x = -1 + \frac{\sqrt{6}}{2}i$ or $x = -1 - \frac{\sqrt{6}}{2}i$.

Notice that the two solutions are **complex conjugates** of each other. The solutions form a **conjugate pair**. This leads to this very important result:

Consider the equation $ax^2 + bx + c = 0$ where a, b and c are **real numbers**. If the equation has complex roots, then the two roots are always **conjugates** of each other.
Note: If a quadratic equation has any complex coefficients then this result doesn't apply.

Example: If $z = 1 - i$ is one solution of the quadratic equation $z^2 - 2z + 2 = 0$, then the second solution must be the **complex conjugate** (as the quadratic has real coefficients). So the second solution is $z = 1 + i$.

Calculating with Complex Numbers

Adding and subtracting

Two complex numbers are added or subtracted by collecting together their real and imaginary parts. So

$$(x + iy) + (u + iv) = (x + u) + i(y + v)$$

and

$$(x + iy) - (u + iv) = (x - u) + i(y - v)$$

We can also easily multiply a complex number by a real number:

$$k(x + iy) = kx + iky$$

Example: If $z = 4 + 2i$ and $w = 3 - i$, then

$z + w = (4 + 2i) + (3 – i) = 7 + i$

$z – w = (4 + 2i) – (3 – i) = 1 + 3i$ (being careful with the negative signs!)

$3z + 2w = 3(4 + 2i) + 2(3 – i) = (12 + 6i) + (6 – 2i) = 18 + 4i$

$2w – z^* = 2(3 – i) – (4 – 2i) = (6 – 2i) – (4 – 2i) = 2$

Multiplying

Complex numbers can be multiplied using the general method for expanding brackets.

Examples:

$$\begin{aligned}(2 + 5i)(4 – 3i) &= 8 – 6i + 20i – 15i^2\\ &= 8 – 6i + 20i – 15(-1)\\ &= 23 + 14i\end{aligned}$$

Remember: $i^2 = -1$

$$\begin{aligned}(3 + 2i)^2 = (3 + 2i)(3 + 2i) &= 9 + 6i + 6i + 4i^2\\ &= 9 + 12i + 4(-1)\\ &= 5 + 12i\end{aligned}$$

Dividing

To divide complex conjugates, you multiply through by the complex conjugate of the denominator:

Example: If $z = 3 – i$ and $w = 1 – 2i$, then

$$\begin{aligned}z \div w &= \frac{3-i}{1-2i}\\ &= \frac{3-i}{1-2i}\times\frac{1+2i}{1+2i}\\ &= \frac{3+6i-i-2i^2}{1+2i-2i-4i^2} = \frac{3+5i-2(-1)}{1-4(-1)}\\ &= \frac{5+5i}{5}\end{aligned}$$

(as the complex conjugate of w is $w^* = 1 + 2i$)

(as $i^2 = -1$)

Therefore:

$$z \div w = 1 + i$$

Note: When a complex number is multiplied by its complex conjugate the answer is always purely a real number.

To show this, suppose $z = x + iy$.

Then $zz^* = (x + iy)(x – iy) = x^2 – 2ixy + 2ixy – i^2y^2$
$= x^2 + y^2$

Exercise 1- Calculating Complex Numbers

Question 1

The complex numbers z and w are given by $z = 1 + 2i$ and $w = 2 + i$. Find $z + w$, $z – w$, zw, $\frac{z}{w}$ and $z + \frac{1}{w}$.

Question 2

Given that $\left(\frac{3 - i}{2+i}\right)z = \frac{6+2i}{1+2i}$, find z in the form $a + ib$.

Question 3

$z_1 = -3 + 4i$ $\qquad$ $z_2 = 1 + 2i$

Express $z_1 z_2$ and $\frac{z_1}{z_2}$ each in the form $a + ib$, where a, b $\in \Re$.

Question 4

The complex numbers z and w are such that

$z = -2 + 5i$ $\qquad$ $zw = 14 + 23i$

Find w in the form $p + qi$, where p and q are real.

Question 5

Given that $z = \frac{1+i}{1-2i}$, find z in the form $a + ib$.

Question 6

Given that $z = -2 + 2\sqrt{3}i$, show that $z^2 + 4z$ is real.

Exercise 2 – Solving Quadratic Equation

Question 7

(a) Show that $(3 - i)^2 = 8 - 6i$.

(b) The quadratic equation

$$az^2 + bz + 10i = 0,$$

where a and b are real, has a root 3 – i.

(i) Show that a = 3 and find the value of b.

(ii) Determine the other root of the equation, giving your answer in the form $p + iq$.

(Hint for (ii): Use the fact that the sum of the roots in any quadratic is –b/a).

Equivalence of two complex numbers

Two complex numbers are equal to each other if and only if their real parts are equal and their imaginary parts are equal, i.e

If $z = x + iy$ and $w = u + iv$, then $z = w$ if and only if $x = u$ and $y = v$.

Example:

1. Given that $(x + iy)^2 = i$, find the possible values of the real numbers x and y in the exact form.

2. Solve the equation $(-2 + 3i)^2 + \lambda(-2 + 3i) + \mu = 0$ where μ and λ are real numbers.

Finding the square root of a complex number

The square root of a complex number which could be determined as follows

$\sqrt{x + iy} = a + ib$ where a and b are real numbers.

$x + iy = (a + ib)^2$

$x + iy = a^2 + 2abi + i^2b^2$

Equate Re and Im parts

Example:

Find the square roots of the following complex numbers:

a) $\sqrt{3 + i4}$ $\qquad$ b) $\sqrt{8 + i6}$ $\qquad$ c) $\sqrt{-2i}$

Note: All questions about finding the square root of a complex number can be solved using the same method.

Solving Polynomial Equations with all Real Coefficients

Example:

1. Find the real root of the equation $3z^3 - 10z^2 + 7z + 10 = 0$ given that one root is 2 – i.

- the coefficients of the cubic equation are real, (a,b,c,and d ∈ ℝ)
- the roots always appear as **complex conjugate**. The second root is 2 + i.

[The third root is a real root. Can be found using long division or equating coefficients.]

2. Two of the roots of a cubic equation, in which all the coefficients are real, are 2 and $1+3i$.

   a) State the third root of the equation. [1-3i]

   b) Find the equation, giving it in the form $z^3 + az^2 + bz + c = 0$. [$z^3 - 4z^2 + 14z - 20$]

3. Prove that 1 – i is a root of the equation $z^3 - 4z^2 + 6z - 4 = 0$. Find all the other roots.

4. Show that $(1+i)^4 = -4$. Hence find all the roots of the equation $z^4 + 4 = 0$.

Note:

✓ When solving polynomial equations with **real coefficients**, complex roots occur in **conjugate pairs**.

Solving linear equations with complex coefficients

Linear equations can be solved by substituting $z = x + iy$.

Example:

1. Solve 4z – 2 + 5i = 6 – 7i

Solution: $z = 2 - 3i$.

2. Find z when $2z - 5z^* = 9 + 14i$.

Solution: $z = -3 + 2i$.

3. Solve (4 + 2i)z + (3 – 2i) = 9 – 4i

Solution: $z = 1 - i$.

4. It is given that $z = x + iy$ and that z^* is the complex conjugate of z.
   a) Express $2z - 3z^*$ in the form $p + qi$.
   b) Find the value of z for which $2z - 3z^* = -5 + 15i$

Solution:
   a) $-x + 5iy$.
   b) $x = 5$ and $y = 3$.

Solving Polynomial Equations with Complex Coefficients

✓ When solving polynomial equations with at **least one non-real coefficient**, complex **roots does not occur in conjugate pairs**.

Example:

1. Solve the following quadratic equation $z^2 + z + (1 - i) = 0$.

2. Find the roots of the equation $z^2 - 21 + 20i = 0$.

3. Express in the form $a + ib$,

(i) $(3 + i)^2$

(ii) $(2 + 4i)(3 + i)$.

4. The quadratic equation

$$z^2 - (2+4i)z + 8i - 6 = 0$$

has roots z_1 and z_2.

i) Verify that $z_1 = 3 + i$ is a root of the equation.

ii) By considering the coefficients of the quadratic, write down the sum of the roots;

iii) Explain why z_1^*, the complex conjugate of z_1, is **not** a root of the quadratic;

iv) Find the other root, z_2, in the form $a + ib$.

Exercise 3

1. Express $\frac{-1+2i}{1+3i}$ in the form a + ib. $[\frac{1}{2} + \frac{1}{2}i]$

2. The complex numbers z and w are such that $w = 1 + ia$ and $z = -b - i$ where a and b are real and positive. Given that $wz = 3 - 4i$, find the exact values of a and b.

$\left[a = \frac{3+\sqrt{21}}{2}\ ,\ b = \frac{-3+\sqrt{21}}{2}\right]$

3. Given that z is a complex number such that $z + 3z^* = 12 + 8i$, find z, giving your answer in the form $z = x + iy$. $[z = 3 - 4i]$

4. Find the square roots of the complex number 5-12i, giving your answers in the form a + ib.

[3 – 2i, -3 + 2i]

5. Two of the roots of a cubic equation, in which all the coefficients are real, are 2 and $1+3i$.

c) State the third root of the equation. [1-3i]

d) Find the equation, giving it in the form $z^3 + az^2 + bz + c = 0$.

$[z^3 - 4z^2 + 14z - 20]$

6. It is given that $z = x + iy$ and that z^* is the complex conjugate of z.

a) Express $z + 2z^*$ in the form $p + qi$.

b) Find the value of z for which $z + 2z^* = 9 + 2i$.

7. It is given that $z = x + iy$, where x and y are real numbers.

a) Write down, in terms of x and y, an expression for z^*, the complex conjugate of z.

b) Find, in terms of x and y, the real and imaginary parts of

$$2z - iz^*$$

c) Find the complex number z such that

$$2z - iz^* = 3i.$$

8.(a) (i) Calculate $(2 + i\sqrt{5})(\sqrt{5} - i)$.

(ii) Hence verify that $\sqrt{5} - i$ is a root of the equation $(2 + i\sqrt{5})z = 3z^*$ where z^* is the complex conjugate of z.

(b) The quadratic equation $x^2 + px + q = 0$ in which the coefficients p and q are real, has a complex root $\sqrt{5} - i$.

(i) Write down the other root of the equation.

(ii) Find the sum and the product of the two roots of the equation.

9. Solve the simultaneous equations

$$iz + 2\overset{w}{\cancel{x}} = 1$$
$$z - (1 + i)w = i$$

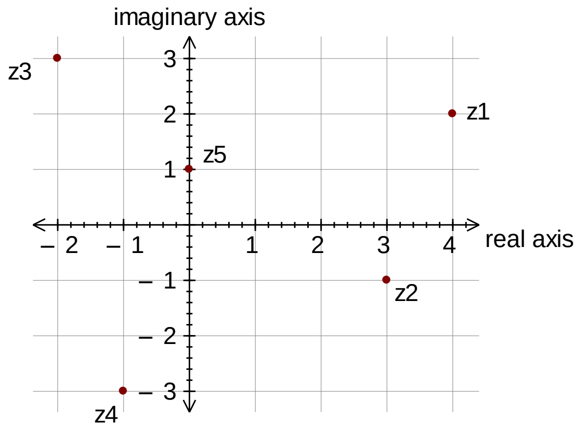
giving your answers for z and w in the form $a + ib$.

<u>Argand Diagrams</u>

Representing complex numbers on Argand diagrams

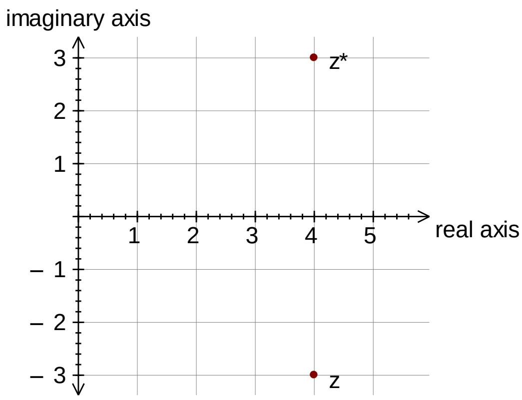
Complex numbers can be shown on an Argand diagram. The horizontal axis represents the real part of the complex number whilst the vertical axis represents the imaginary part.

Example 1:
Plot the complex numbers $z_1 = 4 + 2i$, $z_2 = 3 - i$, $z_3 = -2 + 3i$, $z_4 = -1 - 3i$ and $z_5 = i$ on an Argand diagram.

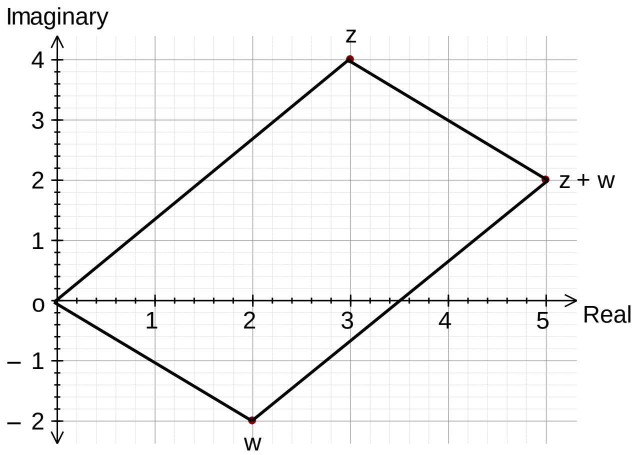


4-3i

Example 2: Let z = ~~3 – 4~~i. Show z and z^* on an Argand diagram.
Solution: z* = 3 + 4i.



Note: In an Argand diagram, z* is a reflection of z in the real axis.
Example 3: Let z = 3 + 4i and w = 2 – 2i. Show z, w and z + w on an Argand diagram.
Solution: z + w = 5 + 2i.



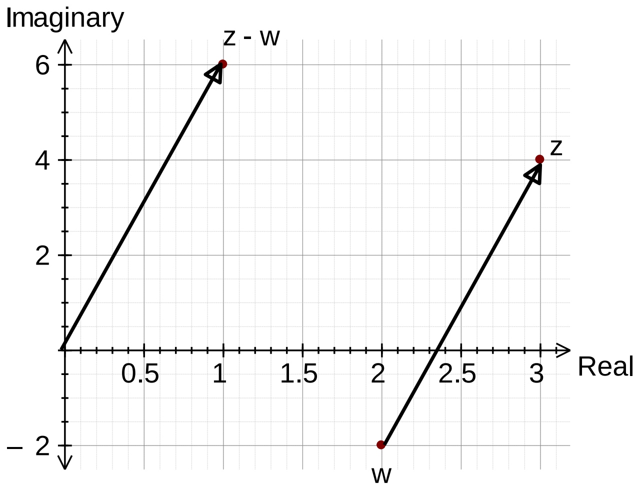
Notice that the points representing the origin, z, w and z + w form a parallelogram.

General result: If complex numbers z, w and z + w are represented by points Z, W and S in an Argand diagram, then OZSW is a parallelogram.

Example 4:

Let z = 3 + 4i and w = 2 – 2i. Show z, w and z - w on an Argand diagram.

Solution: z – w = 1 + 6i.



Notice this general result: If the complex numbers z and w are represented in an Argand diagram by points Z and W, then the translation which takes W to Z represents the complex number z – w.

Question:

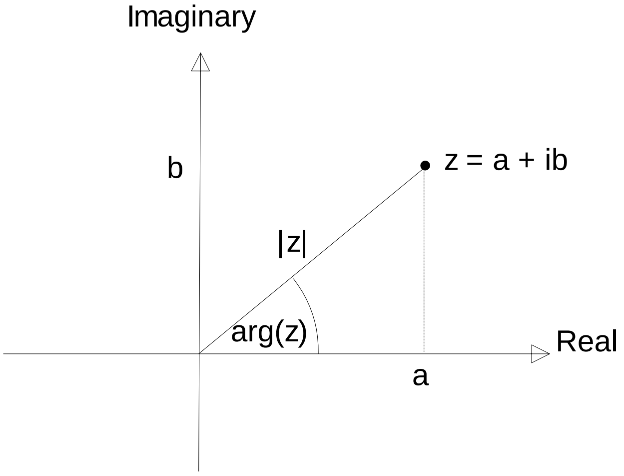
Given that $z_1 = 1 + 2i$ and $z_2 = \frac{3}{5} + \frac{4}{5}i$, write $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $p + iq$, where p, q $\in \Re$.

Plot the points representing $z_1, z_2, z_1 z_2$ and $\frac{z_1}{z_2}$ on an Argand diagram.

In the diagram, the origin and the points representing $z_1 z_2, \frac{z_1}{z_2}, z_3$ are the vertices of a rhombus. Find z_3 and sketch the rhombus on this Argand diagram.

Modulus and Argument of a Complex Number

Consider the complex number $z = a + ib$:



The **modulus** of z, written |z|, is the distance of z from the origin.

Therefore: r = $|z| = \sqrt{a^2 + b^2}$.

The **argument** of a complex number z , arg z is the angle that the line joining O to z makes with the positive real axis. Anticlockwise rotation is positive and clockwise rotation is negative. The argument is usually measured in radians and is chosen so that $-\pi < \arg(z) \leq \pi$.

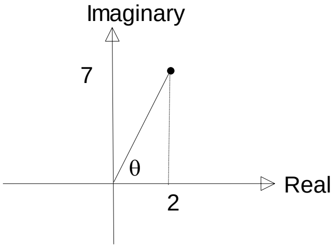
Example: Find the modulus and argument of these complex numbers:

a) 2 + 7i
b) 5 – 2i
c) -4 + 3i
d) -2 – 3i
e) -5
f) 3i

Solution: It helps to sketch an Argand diagram in each case:

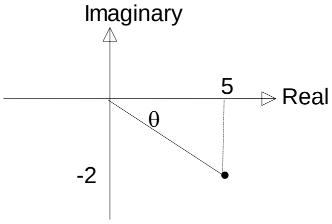
a) | 2 + 7i | =

arg(2 + 7i) is angle θ in the diagram:

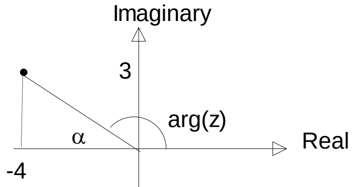


b) | 5 - 2i | =

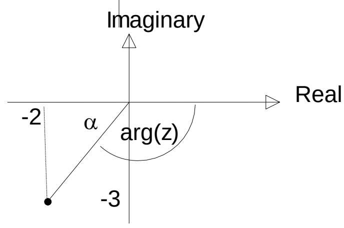
arg(5 - 2i) is shown by angle θ in the diagram.



c) | - 4 + 3i | =

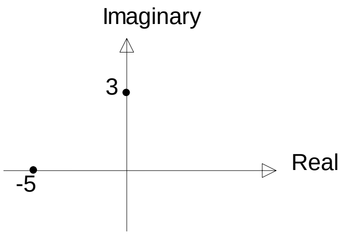


d) |-2 – 3i| =

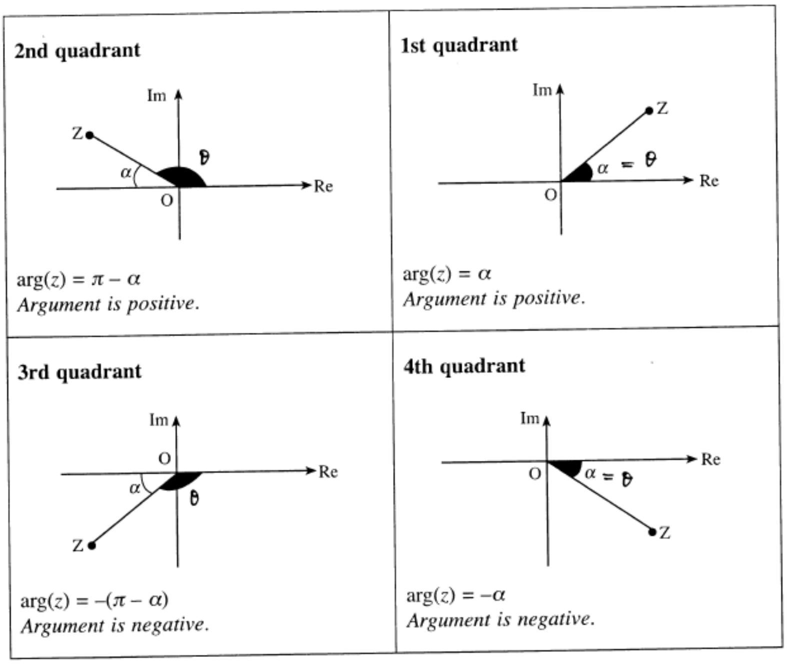


e) |-5| = 5
arg(-5) = π rads

f) |3i| = 3
arg(3i) = ½ π rads



Conclusion: (Based on principal argument)

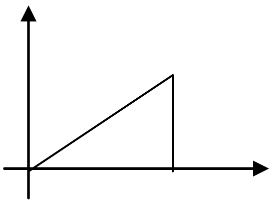


The Modulus – Argument Form (Polar Form) of a Complex Number

Let z = x + iy

$$\Rightarrow z = r\cos\theta + ir\sin\theta$$

$$\Rightarrow z = r(\cos\theta + i\sin\theta)$$



Where $r = |z| = \sqrt{(x^2 + y^2)}$

and $-\pi < \theta \leq \pi$ --- principal argument

r = modulus of z

θ = the argument of z

Products and Quotients

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ (multiplication)

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ (division)

Example:

1. The complex numbers z and w are such that $|z| = 2,\ \arg(z) = -\frac{2}{3}\pi$,

   $|w| = 5,\ \arg(w) = \frac{3}{4}\pi$. Find the exact values of

   a) the real parts of z and the imaginary parts of z.

   b) the modulus and argument of $\frac{w}{z^2}$. [a) -1 , $-\sqrt{3}$, b) $\frac{5}{4}$, $\frac{\pi}{12}$]

2. Given that $(5 + 12i)z = 63 + 16i$, *find* $|z|$ and $\arg(z)$, giving this answer in radians correct to 3s.f.

   Given also that $w = 3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$, find

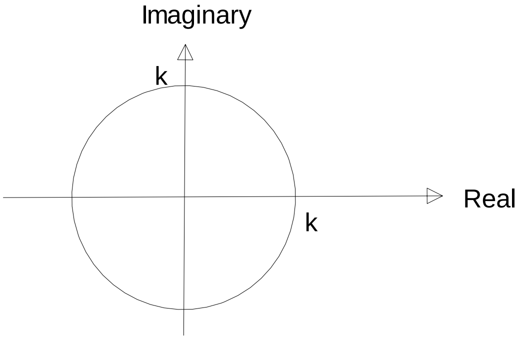
   a) $\left|\frac{z}{w}\right|$, b) $\arg(zw)$. [a) $\frac{5}{3}$, b) 0.120]

The Exponential Form z = re$^{i\theta}$ of a Complex Number

Euler's formula states that $e^{i\theta} = \cos\theta + i\sin\theta$,
Therefore the polar form of a complex number, $z = r(\cos\theta + i\sin\theta)$,
can also be written as $z = re^{i\theta}$.

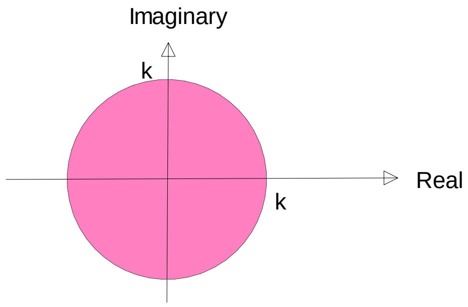
Loci

Situation 1: $|z| = k$
Since |z| denotes the distance in an Argand diagram that a complex number is from the origin, the loci of points satisfying the relationship $|z| = k$ is a circle centre the origin, radius k.



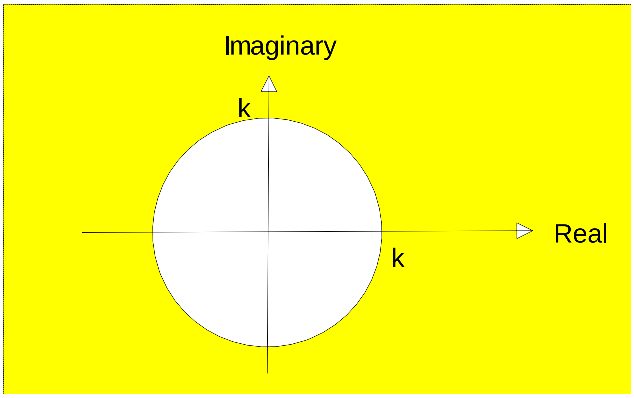
Related loci are:

*** $|z| < k$** the inside of a circle radius k, centre O (not including the circle circumference);



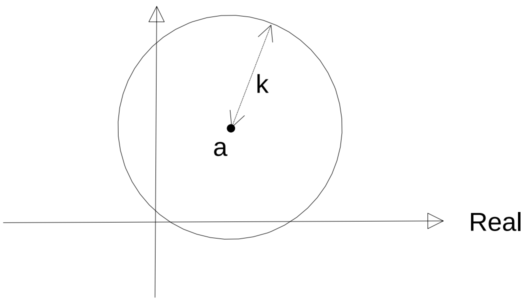
(the boundary is shown with a dotted line as it does not form part of the locus.)

*** $|z| \geq k$**   the outside of a circle radius k, centre O (including the circle circumference).



Situation 2: $|z - a| = k$

Since $|z - w|$ represents the distance between the two complex numbers z and w on an Argand diagram, the loci of points satisfying the relation $|z - a| = k$ is a circle centre the complex number *a* and radius *k*.



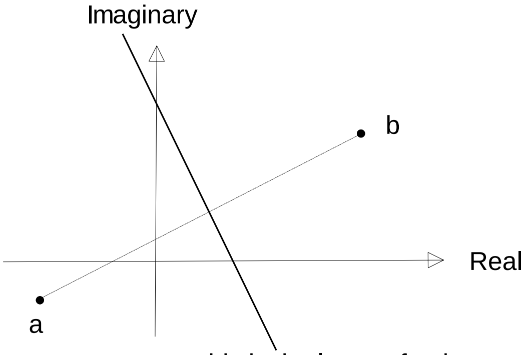
Related loci are:

*** $|z - a| < k$**   the inside of a circle radius *k*, centre *a* (not including the circle circumference);

* $|z - a| \geq k$ the outside of a circle radius *k*, centre *a* (including the circumference).

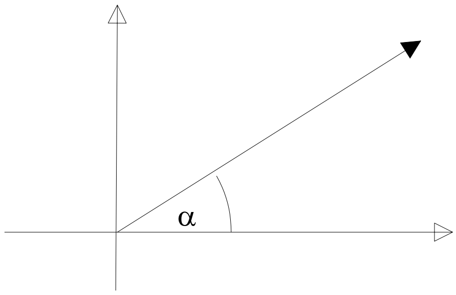
Situation 3: $|z - a| = |z - b|$

The points satisfying the relation $|z - a| = |z - b|$ will be the complex numbers that are equidistant from the complex numbers *a* and *b*. So the locus consists of the perpendicular bisector of the line segment joining complex numbers *a* and b.

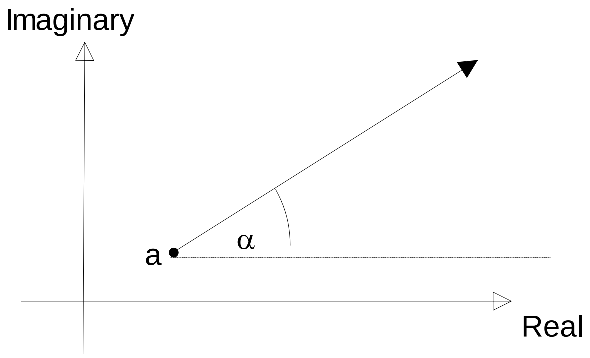


Likewise the locus of points satisfying the relationship $|z - a| < |z - b|$ consists of points closer to *a* than to *b*.

Situation 4: arg(z) = α
The points satisfying this relationship consist of a half-line emanating from the origin making an angle α with the positive real axis.



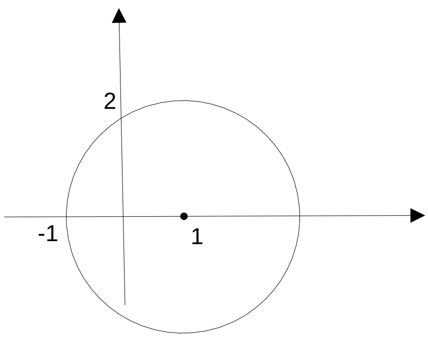
Situation 5: arg(z – a) = α
The required locus here is a half-line this time emanating from a, making an angle α with the (positive) horizontal direction:



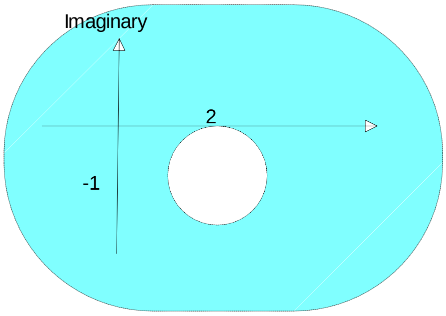
Example:
Sketch the loci satisfying these equations:

a) $|z-1| = 2$
b) $|z-2+i| > 1$
c) $|z-3| = |z-1-i|$
d) $|z| = |z+2i|$
e) $\arg(z-2) = \pi/3$
f) $\arg(z-2-i) = -\pi/4$.

Solution:
a) The required locus is a circle centre 1, radius 2.



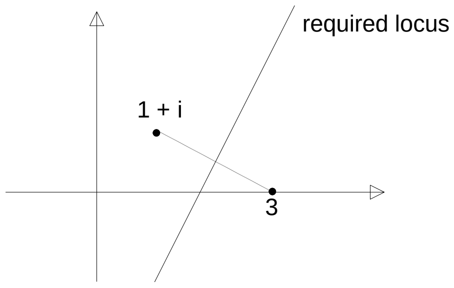
b) First we re-write the locus into the form $|z-a| > 1$:
We get: $|z-(2-i)| > 1$.
This is the locus of points outside a circle centre $2-i$, radius 1.



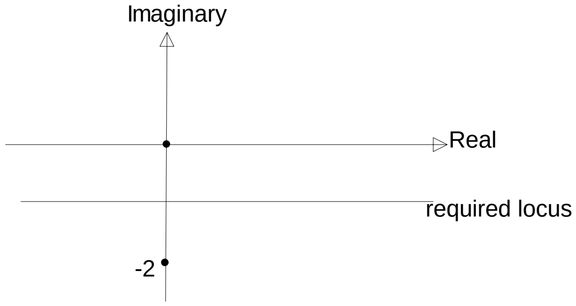
c) The equation $|z - 3| = |z - 1 - i|$ can be rewritten as

$|z - 3| = |z - (1 + i)|$.

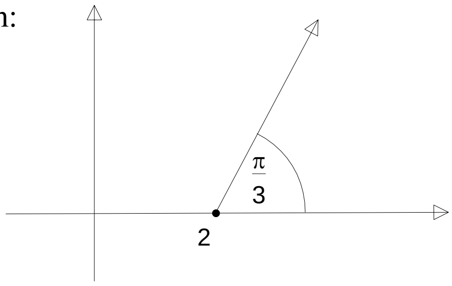
So the required locus is the perpendicular bisector of the line segment joining the complex numbers 3 and 1 + i:



d) The equation $|z| = |z + 2i|$ represents the locus of points that are equidistant from the origin O and the complex number – 2i:

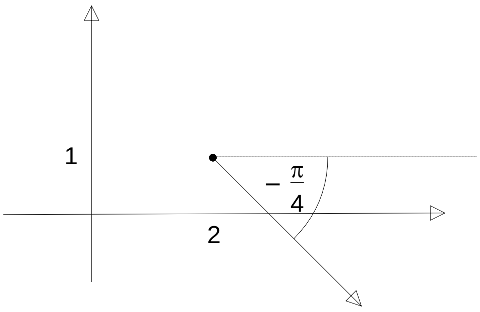


e) $\arg(z - 2) = \pi/3$ represents the locus shown in the diagram:



f) The equation $\arg(z - 2 - i) = -\pi/4$ can be rewritten as $\arg(z - (2 + i)) = -\pi/4$.

This locus is shown on the diagram:



Examination Questions:

1.a) On the same Argand diagram, sketch the loci of points satisfying:

(i) $|z + 3 + i| = 5$;

(ii) $\arg(z + 3) = -\frac{3}{4}\pi$.

b) (i) From your sketch, explain why there is only one complex number satisfying both equations.

(ii) Verify that this complex number is -7 – 4i.

[Hint: To verify the last part, you need to check that the number -7 – 4i satisfies both equations].

2.a) Sketch on one Argand diagram:

(i) the locus of points satisfying $|z - i| = |z - 2|$;

(ii) the locus of points satisfying $\arg(z - i) = \frac{1}{4}\pi$.

(b) Shade on your diagram the region in which

$$|z - i| \leq |z - 2| \text{ and } -\tfrac{\pi}{2} \leq \arg(z - i) \leq \tfrac{\pi}{4}$$

3.Shade on an Argand diagram the region in which

$$|z - 2i| \leq 1.$$

4.The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + \sqrt{3}i \text{ and } z_2 = iz.$$

a) (i) Express z_2 in the form $a + ib$.

(ii) Find the modulus and argument of z_2.

b) Label the points representing z_1 and z_2 on an Argand diagram.

c) On the same Argand diagram sketch the locus of points z satisfying

(i) $|z - z_1| = |z - z_2|$;

(ii) $\arg(z - z_1) = \arg(z_2)$.

5. Two loci, L_1 and L_2, in the Argand diagram are defined by the following equations:

$$L_1 : \quad |z + 2 - 3i| = 1;$$

$$L_2 : \quad \arg(z - 4) = \tfrac{1}{2}\pi.$$

a) Sketch the two loci on one Argand diagram;

b) Find the smallest possible value of $|z_1 - z_2|$, where the points z_1 and z_2 lie on the loci L_1 and L_2 respectively.

Question 6

8 The complex number $\dfrac{2}{-1 + i}$ is denoted by u.

(i) Find the modulus and argument of u and u^2. [6]

(ii) Sketch an Argand diagram showing the points representing the complex numbers u and u^2. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z| < 2$ and $|z - u^2| < |z - u|$. [4]

Question 7

9 The complex number u is given by

$$u = \frac{3 + i}{2 - i}.$$

(i) Express u in the form $x + iy$, where x and y are real. [3]

(ii) Find the modulus and argument of u. [2]

(iii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the point representing the complex number z such that $|z - u| = 1$. [3]

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. [2]

Question 8

7 The equation $2x^3 + x^2 + 25 = 0$ has one real root and two complex roots.

(i) Verify that $1 + 2i$ is one of the complex roots. [3]

(ii) Write down the other complex root of the equation. [1]

(iii) Sketch an Argand diagram showing the point representing the complex number $1 + 2i$. Show on the same diagram the set of points representing the complex numbers z which satisfy

$$|z| = |z - 1 - 2i|.$$ [4]

Question 9

The complex number $-2 + \mathrm{i}$ is denoted by u.

(i) Given that u is a root of the equation $x^3 - 11x - k = 0$, where k is real, find the value of k. [3]

(ii) Write down the other complex root of this equation. [1]

(iii) Find the modulus and argument of u. [2]

(iv) Sketch an Argand diagram showing the point representing u. Shade the region whose points represent the complex numbers z satisfying both the inequalities

$$|z| < |z - 2| \quad \text{and} \quad 0 < \arg(z - u) < \tfrac{1}{4}\pi.$$ [4]

Answers for questions 6 - 9:

6. (i) u: $\sqrt{2}, \ -\frac{3}{4}\pi$ $\quad u^2$: $2, \ \frac{\pi}{2}$

7. (i) $1 + i$, (ii) $\sqrt{2}, \ \frac{\pi}{4}$, (iv) 0.414

8. (ii) $1 - 2i$

9. (i) 20, (ii) $-2 - i$, (iii) $\sqrt{5}, \ 153.4°$

Worked examination style question:

The complex number 2 + 5i is one root of the equation $x^2 + px + q = 0$, where p and q are real.

a) Write down the second root of this equation.
b) Find the values of p and q.

Solution:

a) As the quadratic equation has real coefficients, the second root must be the complex conjugate of the first, i.e. 2 – 5i.

b) We know that the sum of the roots is given by $-b/a = -p$.
But the sum of the roots is (2 + 5i) + (2 – 5i) = 4.
Therefore $-p = 4$ i.e. $p = -4$.

The product of the roots is $c/a = q$.
But the product of the roots is (2 + 5i)(2 – 5i) = 4 – 10i + 10i + 25 = 29.
So $q = 29$.

Examination question

(i) The complex number z is such that $z^2 = 1 + i\sqrt{3}$. Find the two possible values of z in the form $a + ib$, where a and b are exact real numbers.
(ii) With the value of z from part (i) such that the real part of z is positive, show on an Argand diagram the points A and B representing z and z^2 respectively.

Examination question

The complex numbers z_1 and z_2 are such that $z_1 = 1 - i$ and $z_2 = -\sqrt{3} + i$.

(i) Find $\frac{z_1}{z_2}$ in the form $x + iy$, where x and y are exact real numbers.

(i) Find the exact modulus and argument (in terms of π) of z_1 and z_2.

Self-review:

Syllabus	Self-review ☺ 😐 ☹
Understand the idea of a complex number, recall the meaning of the terms *real part*, *imaginary part*, *modulus*, *argument*, *conjugate*, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal;	
Carry out operations of *addition*, *subtraction*, *multiplication* and *division* of two complex numbers;	
Be able to use the result that, for a polynomial equation with real coefficients, any non-real roots occur in *conjugate pairs*;	
Represent complex numbers geometrically by means of an *Argand diagram*, and understand the geometrical effects of conjugating a complex number and of adding and subtracting two complex numbers;	
Find the two square roots of a complex number;	
Illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram, e.g. $\|z-a\| < k$, $\|z-a\| = \|z-b\|$, $\arg(z-a) = \alpha$.	