Vectors

P1/6/1: Standard notations for vectors and basic concepts

P1/6/2: Unit vectors, displacement vectors and position vectors

P1/6/3: The scalar product

P1/6/4: The scalar product

P1/6/5: Revision

Prepared by

Tan Bee Hong

P1/6/1:
Standard notations for vectors and basic concepts

Learning Outcome

Students should be able to:

- know how to use the standard notations for vectors and carry out addition, subtraction and multiplication of a vector by a scalar.
- interpret these operations in geometrical terms.

P1/6/2:

Unit vectors, displacement vectors and position vectors

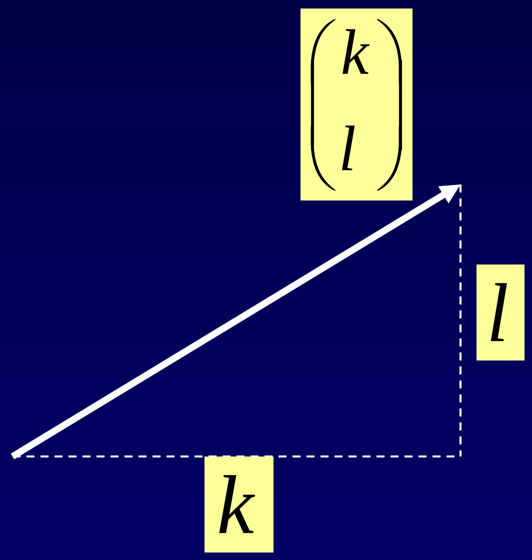
Learning Outcome

Students should be able to:

- Define unit vectors, displacement vectors and position vectors
- Use of standard symbols for the vectors
- Locate points or coordinates in a 2 or 3-D diagram on the Cartesian plane.

Vector Algebra

A general translation of k units across and l units up the grid is called a vector and is written $\begin{pmatrix} k \\ l \end{pmatrix}$.



Vector Algebra

In handwriting vectors are indicated as follow:

$$\underset{\sim}{p} = \begin{pmatrix} k \\ l \end{pmatrix}$$

Vector

Components of $\underset{\sim}{p}$ in the *x*-direction

Components of $\underset{\sim}{p}$ in the *y*-direction

In print, bold type is used to distinguish vectors from numbers.

$$\mathbf{p} = \begin{pmatrix} k \\ l \end{pmatrix}$$

Basic Unit Vectors

A unit vector is a vector of length one unit in a given direction.

$$\mathbf{p} = \begin{pmatrix} k \\ l \end{pmatrix} = k\begin{pmatrix} 1 \\ 0 \end{pmatrix} + l\begin{pmatrix} 0 \\ 1 \end{pmatrix} = k\mathbf{i} + l\mathbf{j}$$

Basic unit vectors in *x*- direction

Basic unit vectors in *y*- direction

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Vectors in three dimensions

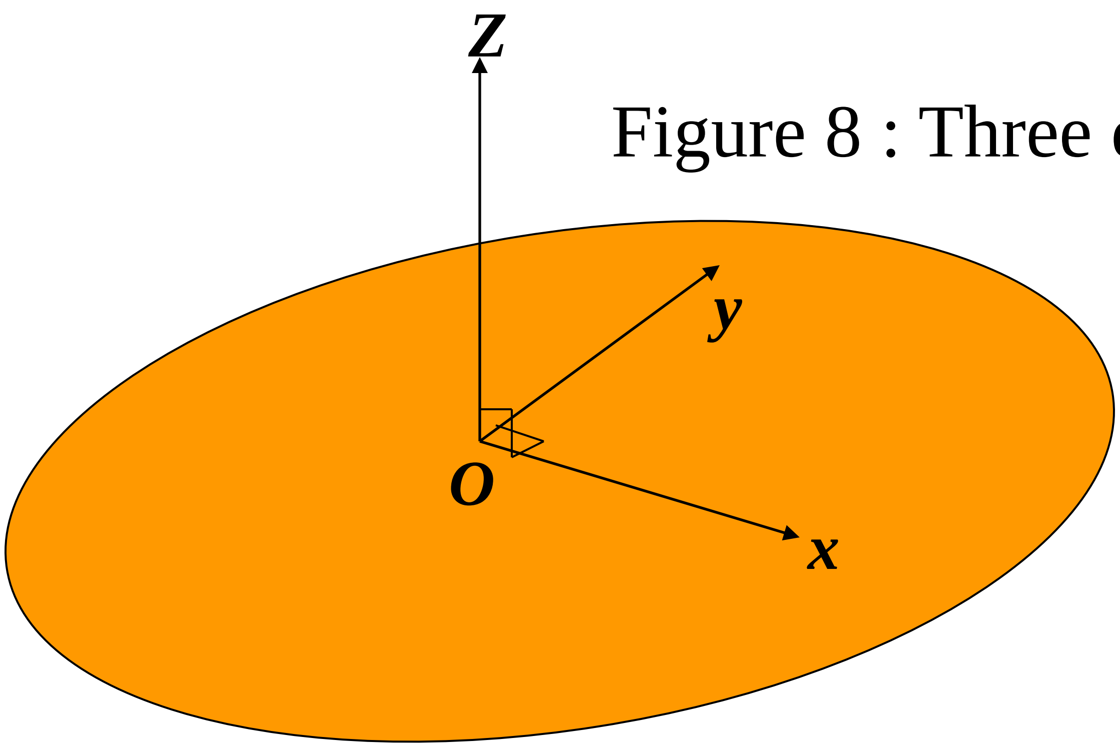


Figure 8 : Three dimensions

The position of a point is given by its three coordinates (x, y, z).

A vector **p** in three dimensions is a translation of the whole of space relative to a fixed coordinate framework.

It is written as $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$.

Translation of l, m and n units in the x-, y- and z-directions.

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$$

where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

are basic unit vectors in the x-, y- and z-directions.

Multiplication of a vector by a scalar

If s is a scalar and $\mathbf{p} = \begin{pmatrix} k \\ l \\ m \end{pmatrix}$, then $s\mathbf{p} = \begin{pmatrix} sk \\ sl \\ sm \end{pmatrix}$

If $s > 0$, the vector $s\mathbf{p}$ is a translation in the same direction as $\mathbf{p}$ but s times as large.

If $s < 0$, it is in the opposite direction $|s|$ times as large.

(2) Addition of vector

Vector are added by performing one translation after another.

If $\mathbf{p} = \begin{pmatrix} k \\ l \\ m \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $\mathbf{p} + \mathbf{q} = \begin{pmatrix} k + x \\ l + y \\ m + z \end{pmatrix}$

Position Vectors

The position vector of the point A with coordinates (*u*, *v*) is

$$\overrightarrow{OA} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

Parallelogram rule of addition for position vectors

The arrows from O to A and from O to B are not related in the way needed for addition.

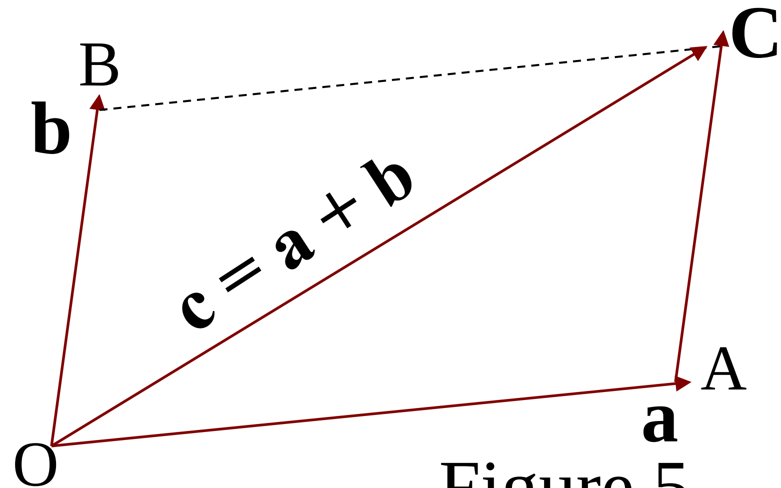


Figure 5

It is therefore necessary to complete the parallelogram OACB.

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + \overrightarrow{OB} \\ &= \mathbf{a} + \mathbf{b}\end{aligned}$$

Subtraction for position vectors

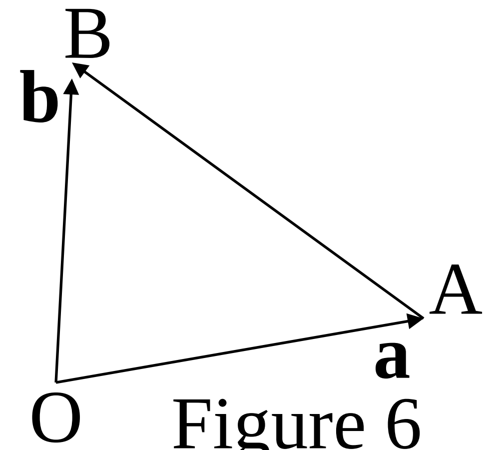


Figure 6

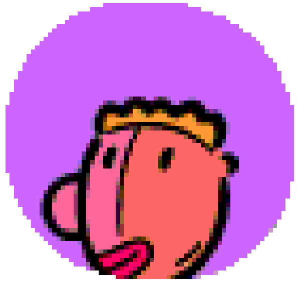
To interpret $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ as a position vector:

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \mathbf{b} - \mathbf{a}$$

Example 1:



Two points *A* and *B* have position vectors $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$.

C is the point on the line segment *AB* such that $\dfrac{AC}{CB} = 2$.

Find

(a) The vector $\overrightarrow{AB}$,

(b) The vector $\overrightarrow{AC}$,

(c) The position vector of *C*.

Parallel vectors

If **a** and **b** are two parallel vectors they can be represented by

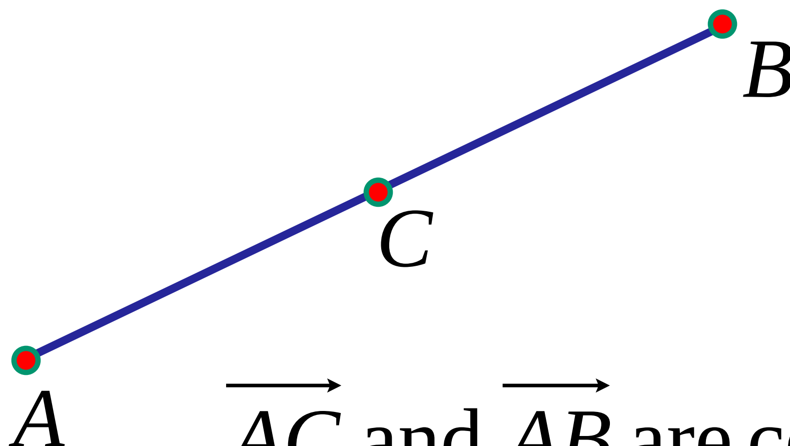
$$\mathbf{a} = k\mathbf{b}$$

where k is constant. k can also be negative.

Collinear vectors

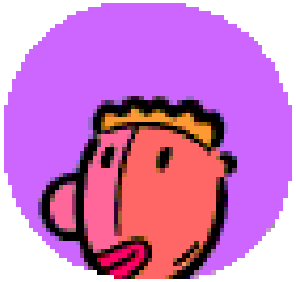
In collinear vectors, all the vectors are on one straight line as shown in the below diagram. If point C lies on the line passing through A and B,

$$\overrightarrow{AC} = k\,\overrightarrow{AB}$$



$\overrightarrow{AC}$ and $\overrightarrow{AB}$ are collinear vectors.

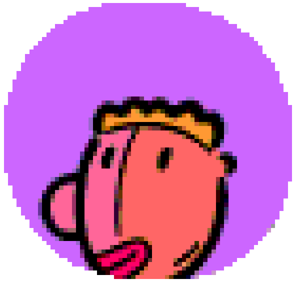
Example 2:



If C is the point AB such that $\overrightarrow{AC} = t\,\overrightarrow{AB}$,

prove that $\mathbf{c} = t\mathbf{b} + (1-t)\mathbf{a}$.

Example 3:

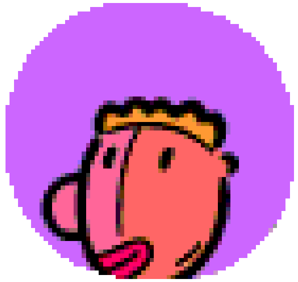


Four points *A*, *B*, *C* and *D* have coordinates (0,1,-2), (1,3,2), (4,3,4), and (5,-1,-2) respectively. Find the position vectors of

(a) The mid-point *E* of *AC*,

(b) The point *F* on *BD* such that $\dfrac{BF}{FD} = \dfrac{1}{3}$.

Example 4:



ABC is a triangle. *D* is the mid-point of *BC*, *E* is the mid-point of *AD* and *F* is the point of trisection of *AC* closer to *A*. *G* is the point on *FB* such that $\vec{FG} = \frac{1}{4}\vec{FB}$.

Express **d**, **e**, **f** and **g** in terms of **a**, **b** and **c**, and deduce that *G* is the same point as *E*. Draw a figure to illustrate this result.

Practice Exercise

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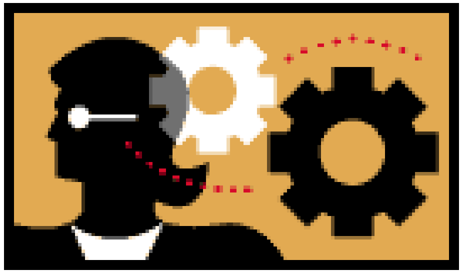
Exercise 13B (Page 197)

Q2, Q4

Exercise 13C (Page 200)

Q2, Q3, Q5



P1/6/3 P1/6/4:

The scalar product

Learning Outcome

Students should be able to:

- understand and the use of scalar product
- Calculate the magnitude of a vector and
  the scalar product of two vectors
- Determine the angle between two directions
- Solving problems concerning perpendicularity of vectors.

The Magnitude of a vector

The **magnitude** of a vector $\mathbf{p} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$, is $|\,\mathbf{p}\,|$.

$$|\mathbf{p}| = \sqrt{l^2 + m^2 + n^2}$$

The Magnitude of a vector

A vector of magnitude 1 is called a **unit vector**.

The **unit vector** in the same direction of $\mathbf{p} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$:

$$\text{Unit Vector} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

Example 5:

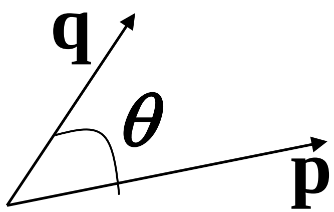
(a) Find the unit vector in the same direction as $\mathbf{p} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

(b) Hence, find the vector which is in the same direction as $\mathbf{p}$ and has magnitude 30.

Angle between two directions

The **scalar product**, or **dot product**, of vectors **p** and **q**

$$\mathbf{p}.\mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\theta,$$



$$\Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{p}.\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}\right)$$

The angle θ may be acute or obtuse, and it is the angle between **p** and **q**.

Properties of the scalar products

$\Rightarrow$ if $\mathbf{p}$ and $\mathbf{q}$ are parallel,

$$\mathbf{p}.\mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos 0^o \Rightarrow \mathbf{p}.\mathbf{q} = |\mathbf{p}||\mathbf{q}|$$

$\Rightarrow$ If $\mathbf{p}$ and $\mathbf{q}$ are in perpendicular direction.

$$\mathbf{p}.\mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos 90^o \Rightarrow \mathbf{p}.\mathbf{q} = \mathbf{0}$$

Scalar products in component form

In component form, the scalar product is

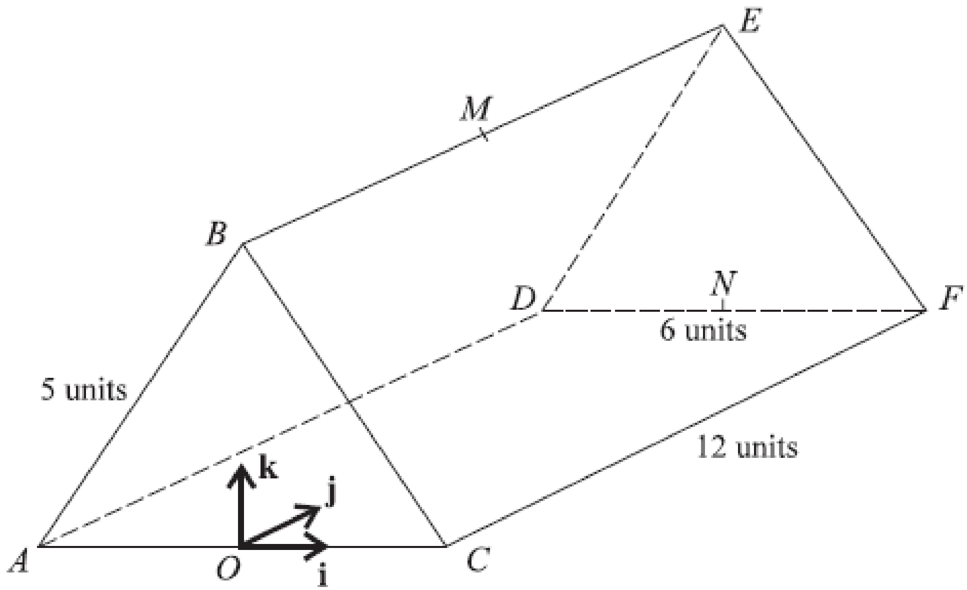
$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = (l\mathbf{i} + m\mathbf{j} + n\mathbf{k}) \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$$

$$= lu + mv + nw$$

Example 6:

Find the angle between the line joining (1,2) and (3,-5) and the line joining (2,-3) to (1,4).

Example 7:



The diagram shows a triangular prism with a horizontal rectangular base $ADFC$, where $CF = 12$ units and $DF = 6$ units. The vertical ends ABC and DEF are isosceles triangles with $AB = BC = 5$ units. The mid-points of BE and DF are M and N respectively. The origin O is at the mid-point of AC.

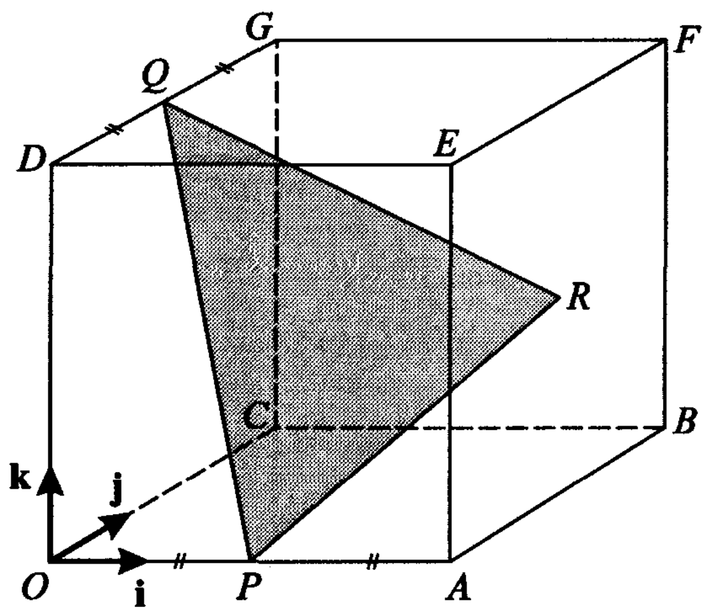
Unit vectors **i**, **j** and **k** are parallel to OC, ON and OB respectively.

(i) Find the length of OB. [1]

(ii) Express each of the vectors $\overrightarrow{MC}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**. [3]

(iii) Evaluate $\overrightarrow{MC}.\overrightarrow{MN}$ and hence find angle CMN, giving your answer correct to the nearest degree. [4]

Example 8:



The diagram shows a cube $OABCDEFG$ in which the length of each side is 4 units. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively. The mid-points of OA and DG are P and Q respectively and R is the centre of the square face $ABFE$.

(i) Express each of the vectors $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

(ii) Use a scalar product to find angle QPR. [4]

(iii) Find the perimeter of triangle PQR, giving your answer correct to 1 decimal place. [3]

Example 9:

Find the angle between the diagonals of a cube.

Practice Exercise

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Exercise 13D (Page 206)

Q7, Q12, Q15