Topic 7 – Numerical Solutions of Equations

Exercise 1: Locating the Roots of an Equation

1 Use sketch graphs to determine the number of real roots of each equation. (Some may have an infinite set of roots.)

(a) $\sin x = \frac{1}{x}$ (b) $\cos x = x^2 - 1$

(c) $x^2 = \tan x$ (d) $\sin x = x$

(e) $(x^2 - 4) = \frac{1}{x}$ (f) $\sin x = x^2$

(g) $x \ln x = 1$ (h) $xe^x = 1$

(i) $\ln x + e^x = 0$

2 For each equation in question **1** with a finite number of roots, locate the root, or the larger root where there is more than one, within an interval of half a unit.

3 Find the turning points on the curve whose equation is $y = x^3 - 3x^2 + 1$. Hence sketch the curve and use your sketch to find the number of real roots of the equation $x^3 - 3x^2 + 1 = 0$

4 Using a method similar to that given in question **3**, or otherwise, determine the number of real roots of each equation.

(a) $x^4 - 3x^3 + 1 = 0$

(b) $x^3 - 24x + 1 = 0$

(c) $x^5 - 5x^2 + 4 = 0$

5 Show that the equation $e^{-x} = x^2 + 2$ has just one root and find this root to the nearest integer.

6 Find the successive integers between which the smallest root of the equation $e^x = \frac{1}{2}(x + 3)$ lies.

Exercise 2: Iteration

Show by calculation that each of the following equations has a root between 0 and 1.

1 $x^3 - x^2 + 10x - 2 = 0$

2 $3x^3 - 2x^2 - 9x + 2 = 0$

3 $2x^3 + x^2 + 6x - 1 = 0$

4 $x^2 + 8x - 8 = 0$

5 The sequence of values given by the iteration formula $x_n = \sqrt[3]{3x_n + 3}$ converges to α.

Starting with $x_1 = 2$, find x_2, x_3 and x_4 giving your answers to 4 decimal places.

6 (a) For the equation in question **1**, explain why $x_{n+1} = \frac{1}{10}(2 + x_n^2 - x_n^3)$ is an iteration formula that gives a sequence that converges to the root between 0 and 1.

(b) Use the formula with $x_1 = 0.5$ to find this root correct to 2 decimal places.

7 (a) Show that equation $x \ln x = 1$ has a root between 1.5 and 2.

(b) Show that the sequence given by the iteration formula $x_{n+1} = \frac{1}{\ln x_n}$ with $x_1 = 1.5$ does not converge.

8 The sequence of values given by the iteration formula $x_{n+1} = \frac{4}{3} + \frac{4}{3x_n}$ converges to α when $x_1 = 1.8$

(a) Find the value of α correct to 1 decimal place.

(b) Write down the equation for which $x = \alpha$ is a solution. Hence find the exact value of α.

9 Starting with $x_1 = 3$, the sequence of values given by the iteration formula $x_{n+1} = \ln x_n + 2$ converges to α.

(a) Find the value of α correct to 2 decimal places.

(b) Write down the equation for which $x = \alpha$ is a solution.

P3**10** Use an iteration formula to find the root of the equation given in question **3** correct to 2 decimal places.

P3**11** Find iteration formulae for questions **1**, **2** and **4**. Determine in each case whether the iteration converges or fails.

Answers

Exercise 1

1 (a) infinite (b) 2 (c) 2
(d) 1 (e) 3 (f) 2
(g) 1 (h) 1 (i) 1

2 (b) $1 < x < 1.5$ (c) 0 (exact)
(d) 0 (exact) (e) $2 < x < 2.5$
(f) $0.5 < x < 1$ (g) $1.5 < x < 2$
(h) $0.5 < x < 1$ (i) $0 < x < 0.5$

3 (0, 1) max, (2, −3) min, 3

4 (a) 2 (b) 3 (c) 3

5 −1

6 $-3 < x < -2$

Exercise 2

5 2.0801, 2.0984, 2.1026

6 0.20

8 (a) 2.0 (b) $3x^2 = 4x + 4$, 2

9 (a) 3.15 (b) $x = \ln x + 2$

10 0.16

11 any rearrangement of the equations in the form $x = f(x)$