Topic 1: Quadratics

- carry out the process of completing the square for a quadratic polynomial ax^2+bx+c, and use this form, e.g. to locate the vertex of the graph of $y=ax^2+bx+c$ or to sketch the graph;
- find the discriminant of a quadratic polynomial ax^2+bx+c and use the discriminant, e.g. to determine the number of real roots of the equation $ax^2+bx+c=0$;
- solve quadratic equations, and linear and quadratic inequalities, in one unknown;
- solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic;
- recognise and solve equations in x which are quadratic in some function of x, e.g. $x^4-5x^2+4=0$.

An example of a **Quadratic Equation**:

this makes it Quadratic

$$5x^2 - 3x + 3 = 0$$

The name **Quadratic** comes from "quad" meaning square, because the variable gets squared

(like x^2). It is also called an "Equation of degree 2" (because of the "2" on the x).

The **Standard Form** of a Quadratic Equation looks like this:

$$ax^2 + bx + c = 0$$

- The letters **a**, **b** and **c** are coefficient (you know those values).
  They can have any value, except that **a** can't be 0.
- The letter "***x***" is the variable or unknown (you don't know it yet).

Here are some more examples:

$2x^2+5x+3=0$ In this one **a=2**, **b=5** and **c=3**

$x^2-3x=0$ This one is a little more tricky:

- Where is **a**? In fact **a=1**, as we don't usually write "$1x^2$"
- **b = -3**
- And where is **c**? Well, **c=0**, so is not shown.

$5x-3=0$ **Oops!** This one is **not** a quadratic equation, because it is missing x^2 (in other words **a=0**, and that means it can't be quadratic)

But sometimes a quadratic equation doesn't look like that! For example:

In disguise	→	In Standard Form	a, b and c
$x^2 = 3x - 1$	Move all terms to left hand side	$x^2 - 3x + 1 = 0$	a=1, b=-3, c=1
$2(w^2 - 2w) = 5$	Expand (undo the brackets), and move 5 to left	$2w^2 - 4w - 5 = 0$	a=2, b=-4, c=-5
$z(z-1) = 3$	Expand, and move 3 to left	$z^2 - z - 3 = 0$	a=1, b=-1, c=-3
$5 + 1/x - 1/x^2 = 0$	Multiply by x^2	$5x^2 + x - 1 = 0$	a=5, b=1, c=-1

How to Solve a Quadratic Equation? (The solutions are called "roots").

1. You can **Complete the Square**,
2. You can **Factor the Quadratic** (find what to multiply to make the Quadratic Equation)
3. You can use the special **Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Just plug in the values of a, b and c, and do the calculations.

Complete the Square

Completing the Square" is where you ...

... take a Quadratic Equation like this: $ax^2 + bx + c = 0$

→

and turn it into this: $a(x+d)^2 + e = 0$

For those of you in a hurry, I can tell you that: $d = \frac{b}{2a}$ and: $e = c - \frac{b^2}{4a}$

Example 1:

Start with: $$x^2 + 6x + 7$$

("b" is 6 in this case)

Complete the Square: $$x^2 + 6x + \underbrace{\quad}_{+\left(\frac{6}{2}\right)^2} + 7 \underbrace{\quad}_{-\left(\frac{6}{2}\right)^2}$$

Also **subtract** the new term

Simplify it and we are done.

$$\underbrace{x^2 + 6x + \left(\frac{6}{2}\right)^2}_{\left(x+\frac{6}{2}\right)^2} + \underbrace{7 - \left(\frac{6}{2}\right)^2}_{7-9} = (x+3)^2 - 2$$

The result:

$$x^2 + 6x + 7 = (x+3)^2 - 2$$

Example 2:

Solve $x^2 + 6x + 7 = 0$

Example 3:

i) Express $4x^2 + x = 3$ in the form $a(x+d)^2 + e = 0$.
ii) Solve $4x^2 + x = 3$.

Example 4:

i) Express $3 – 4x – 2x^2$ in completed square form.
ii) Solve $3 – 4x – 2x^2 = 0$.

Exercise 1:

1. Express the following functions in completed square form.

a) $f(x) = x^2 + 2x + 2$

b) $f(x) = 5 - 6x + x^2$

c) $f(x) = 7 - 8x - 4x^2$

d) $f(x) = 3x^2 - 12x + 3$

Miscellaneous Exercise 4 – Pure Mathematics 1 (page 62) Question 2, 5, 8 and 10

Factoring Quadratics

Example 1:

The factors of $x^2 + 3x - 4$ are:

$$(x+4) \text{ and } (x-1)$$

Why? Well, let us multiply them to see:

$$\begin{aligned}(x+4)(x-1) &= x(x-1) + 4(x-1)\\ &= x^2 - x + 4x - 4\\ &= x^2 + 3x - 4\end{aligned}$$

To solve $x^2 + 3x - 4 = 0$

$$(x + 4)(x - 1) = 0$$

And we can figure out that

$(x + 4)$ would be zero when $x = -4$

and

$(x - 1)$ would be zero when $x = 1$

So the roots/solution of $x^2 + 3x - 4 = 0$ are:

$$-4 \text{ and } 1$$

Example 2:

Factorise $6x^2 - 47x + 77$. Solve $6x^2 - 47x + 77 = 0$.

Example 3:

Solve $2 - \frac{1}{x} = \frac{3}{x+2}$.

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Plus/Minus

First of all what is that plus/minus thing that looks like ± ?

The ± means there are TWO answers:

$$x = (-b + \sqrt{b^2 - 4ac})/2a$$
$$x = (-b - \sqrt{b^2 - 4ac})/2a$$

But sometimes you don't get two real answers, and the "Discriminant" shows why ...

Discriminant

Do you see **b^2 - 4ac** in the formula above? It is called the **Discriminant**, because it can "discriminate" between the possible types of answer:

- when **b^2 - 4ac** is positive, you get two real solutions.
- when it is zero you get just ONE real solution (both answers are the same).
- when it is negative you get two complex/imaginary solutions.

Using the Quadratic Formula

Example 1: Solve $5x^2 + 6x + 1 = 0$

Coefficients are: a = 5, b = 6, c = 1

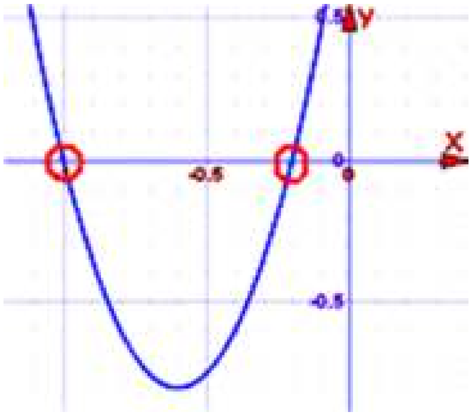
Quadratic Formula: $x = [-b \pm \sqrt{(b^2-4ac)}] / 2a$

Put in a, b and c: $x = [-6 \pm \sqrt{(6^2-4\times5\times1)}] / (2\times5)$

Solve: $x = [-6 \pm \sqrt{(36-20)}]/10$

$x = [-6 \pm \sqrt{(16)}]/10$

$x = (-6 \pm 4)/10$

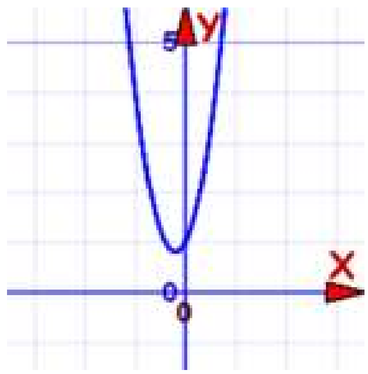


x = -0.2 **or** -1 (2 real roots)

Example 2: Solve $5x^2 + 2x + 1 = 0$

Coefficients are: a = 5, b = 2, c = 1

Note that The **Discriminant** is negative: $b^2 - 4ac =$ **-16**



Exercise 2

1. State the number of distinct real roots of the given equations:
   (a) $3x^2 - 2x - 5 = 0$ b) $x^2 - 2x - 5 = 0$ c) $x^2 + 4 = 0$

2. Show that the $x^2 + 4x + 8$ is always positive.

3. Find the values of *k* for which $2(k-1)x^2 + 2kx + k - 1 = 0$ has equal roots.

4. The quadratic equation $(k+1)x^2 + 12x + k - 4 = 0$ has real roots.
   (a) Show that $k^2 - 3k - 40 \leq 0$.

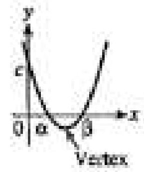
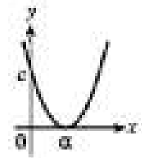
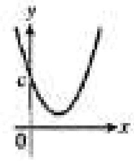
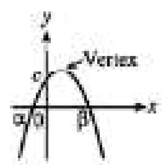
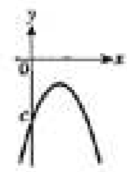
   (b) Hence find the possible values of k.

5. For what value(s) of k does the quadratic equation

   (a) $x^2 + 2(k - 4)x + 2k = 0$ have equal roots?

   (b) $kx^2 - 2x - 7 = 0$ has two real roots?

Summary:

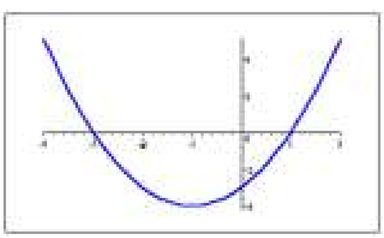
	Discriminant		
	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Number of roots	Two distinct real roots	One repeated real root (2 equal roots)	No real roots (2 complex roots)
Intersection with the x-axis	Two distinct points	Curve touches x-axis (or curve meets x-axis in two coincident points)	Curve and x-axis do not meet
Sketch for $a > 0$			
Sketch for $a < 0$			

The Graph of the Quadratic Function

In general, the graph of a quadratic equation $y = ax^2 + bx + c$ is a **parabola**.

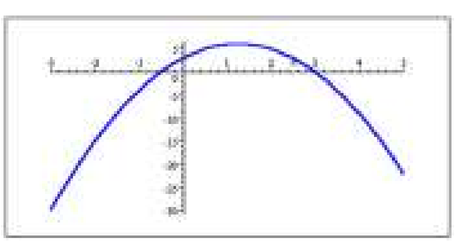
If $a > 0$, then the parabola has a **MINIMUM POINT** and it opens upwards (U-shaped)

eg. $y = x^2 + 2x - 3$



If a < 0, then the parabola has a **MAXIMUM POINT** and it opens downwards (n-shaped)

eg. $y = -2x^2 + 5x + 3$

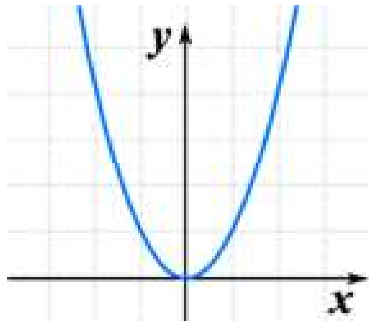


The Simplest Quadratic

The simplest Quadratic Equation is:

$$f(x) = x^2$$

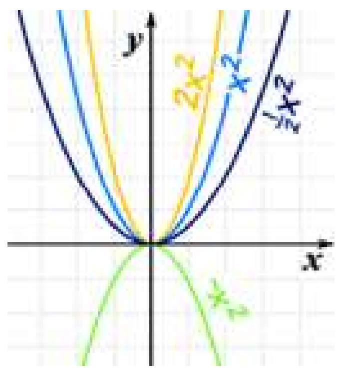
And its graph is simple too:



This is the curve f(x) = x^2
It is a parabola

Now let us see what happens when we introduce the "a" value:

$$f(x) = ax^2$$



- Larger values of **a** squash the curve
- Smaller values of **a** expand it
- And negative values of **a** flip it upside down

The "General" Quadratic

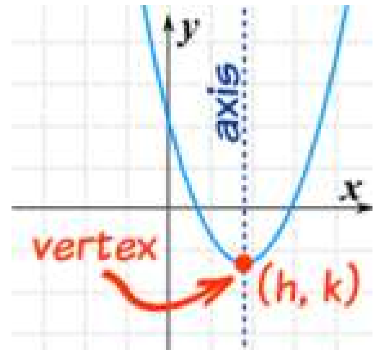
Before graphing we **rearrange** the equation, from

f(x) = ax^2 + bx + c	➡	$f(x) = a(x-h)^2 + k$
(general form)		**(Completed square form)**

First of all ... Why?

Well, the wonderful thing about completed square form is that h and k show you the very lowest (or very highest) point, called the **vertex**:

And also the curve is symmetrical (mirror image) about the **axis** that passes through ***x* = h**, making it easy to graph



So ...

- **h** shows you how far left (or right) the curve has been shifted from $x = 0$
- **k** shows you how far up (or down) the curve has been shifted from $y = 0$

Lets see an example of how to do this:

Example 1: Plot $f(x) = 2x^2 - 12x + 16$

First, let's note down:

- **a = 2, b = -12,** and **c = 16**

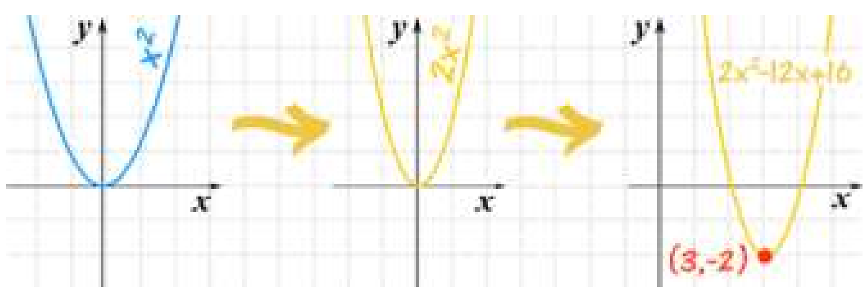
Now, what do we know?

- a is positive, so it is an "upwards" graph ("U" shaped)
- a is 2, so it is a little "squashed" compared to the $\mathbf{x^2}$ graph

Next, let's express $2x^2 - 12x + 16$ in completed square form

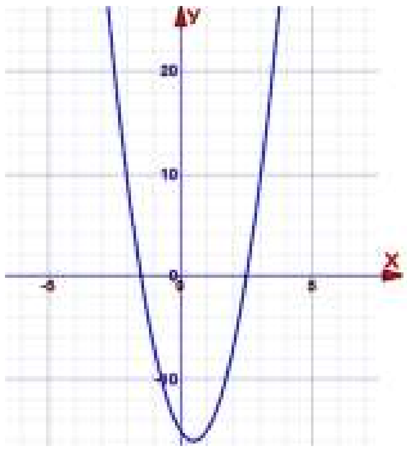
$$f(x) = 2(x - 3)^2 - 2$$

So now we can plot the graph (with real understanding!):



We also know: the **vertex** is (3,-2), and the **axis of symmetry** is $x = 3$

Example 2:



The parabola shown above has its minimum point at (0.5, -16). What is its equation?

Example 3: What is the vertex and axis of symmetry of the parabola f(x) = $-3x^2 + 12x + 1$?

Example 4:

Use the method of completing the square to

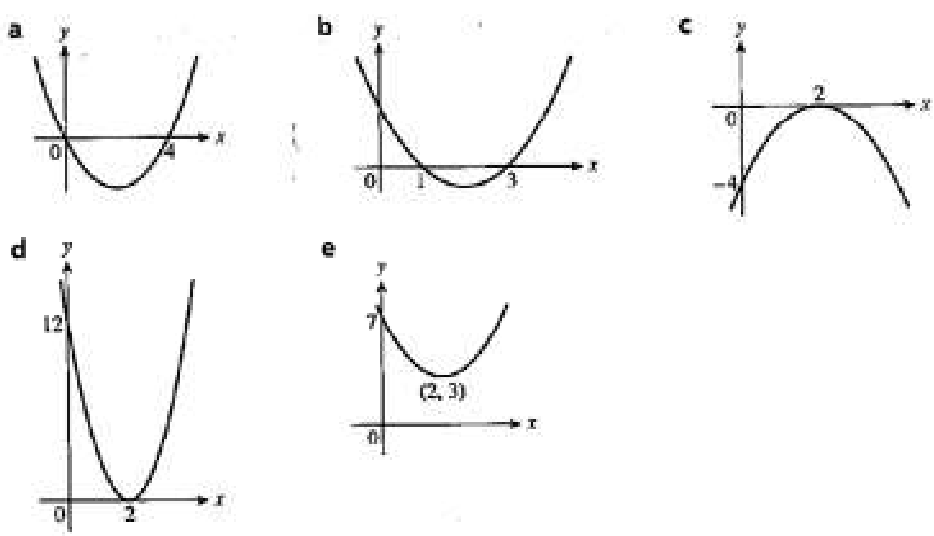
(i) solve the equation $x^2 + 8x - 1 = 0$.
(ii) Find the minimum value of $x^2 + 8x - 1$ and the value of x for which this occurs
(iii) Sketch the curve $y = x^2 + 8x - 1$, showing where the curve cuts the axes.

Example 6: Use the method of completing the square to sketch the curve $y = 7 - 6x - x^2$.

Example 7:A parabola has its maximum point at (-2, 11) and y-intercept 5. What is the equation of the parabola?

Example 8:

The graphs of five quadratic functions are shown. Find an equation for each graph.



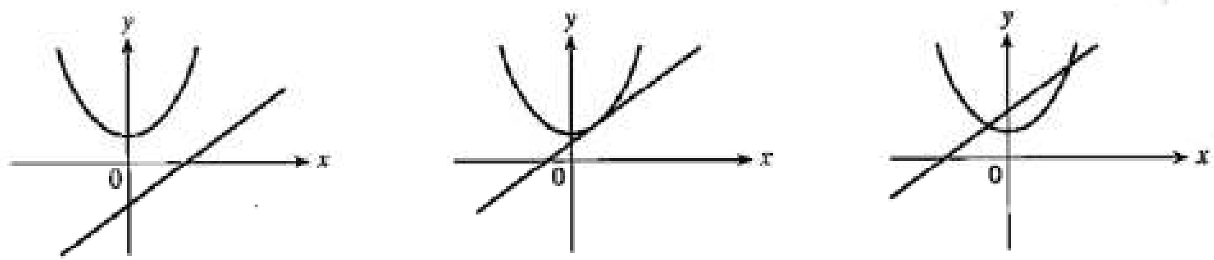
Example 9:

By expressing the function f(x)=(2x + 3)(x - 4) in completed square form, find the range of the function f(x).

Exercise 3C – Pure Mathematics 1(page 41)   Question 8, 11 and 12

Solving Simultaneous Equation

Consider the possible points of intersection of a parabola and a general straight line:



No point of intersection
The line misses the curve altogether.

One point of intersection (or two coincident points)
The line is a tangent to the curve.

Two points of intersection
The line cuts the curve in two distinct points.

Discriminant can be used to test for the number of points of intersection of a linear and quadratic curve.

Example 1:

Find the values of k for which $kx + y = 4$ is a tangent to the curve $y = x^2 + 8$.

Example 2:

Find the range of values of k for which $y = 2x + k$ meets $2x^2 + y^2 = 3$ in two distinct points.

Exercise 4C – Pure Mathematics 1 (page 61)   Question 1, 2 and 3

Equations which are Reduced to Quadratic Equation

Sometimes there are equations which are not quadratic, but which can be changed into quadratic equations, usually by making the right **SUBSTITUTION**.

Example: Solve the equations

1. $t^4 - 13t^2 + 36 = 0$ **2.** $\sqrt{x} = 6 - x$

Exercise 4C – Pure Mathematics 1 (page 61)   Question 4, 5 and 6

Miscellaneous Exercise 4 – Pure Mathematics 1 (page 62)   Question 6

Properties of Inequalities

1. Operations on Inequalities
   - You can add or subtract a number on both sides of an inequality.
   - You can multiply or divide an inequality by a positive number.
   - You can multiply or divide an inequality by a negative number, but you must change the direction of the inequality.
2. Solution of an Inequality
   - The solution of an inequality consists of all the values of the variable that make the inequality a true statement.
   - **Conditional inequalities** are those which are true for some, but not all, values of the variable.

     Example

     The inequality $x + 1 > 0$ is true for all values of x greater than -1.

     Hence the solution of the inequality is written as $x > -1$ and so this is a conditional inequality.
   - **Absolute inequalities** are those which are true for all values of the variable.

     Example

     The inequality $x^2 + 1 > 0$ is true for all values of x and hence is an absolute inequality.
   - A solution of an inequality consists of only real numbers as the terms "less than or greater than" are not defined for complex numbers.

Solving Linear Inequalities

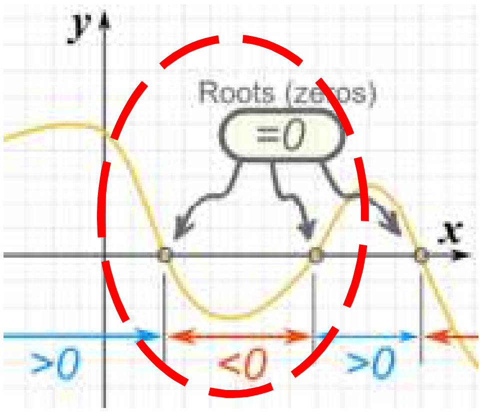
The procedure for solving linear inequalities in one variable is similar to solving basic equations.

Example:

1. Solve the inequality $\frac{3}{2}(1 - x) > \frac{1}{4} - x$.
2. Solve $-1 < 2x + 3 < 6$.
3. Solve $2x < x - 4 \leq 3x + 8$.
4. Solve $\frac{x}{5} - 2 \geq \frac{2}{3}(x + 3)$.
5. Solve $x - 1 < 2x + 2 < 3x + 1$.

Exercise 5A – Pure Mathematics 1 (page 68) Question 5 and 6

Solving Quadratic Inequalities



When we solve **inequalities** we try to find **interval(s)**

Example 1:

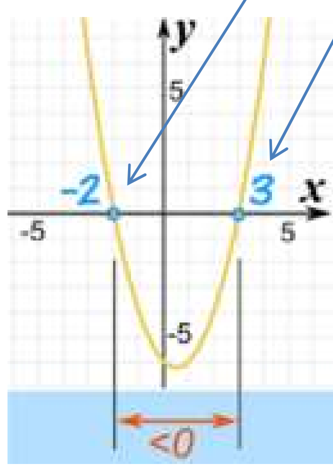
Solve the inequality $x^2 - x - 6 < 0$

$x^2 - x - 6$ has these simple factors $(x + 2)(x - 3) < 0$

Firstly, let us find where it is **equal to** zero:

$$(x + 2)(x - 3) = 0$$

It will be equal to zero when $x = -2$ or $x = +3$



Note: $x^2 - x - 6 < 0$ in the interval (-2, 3)

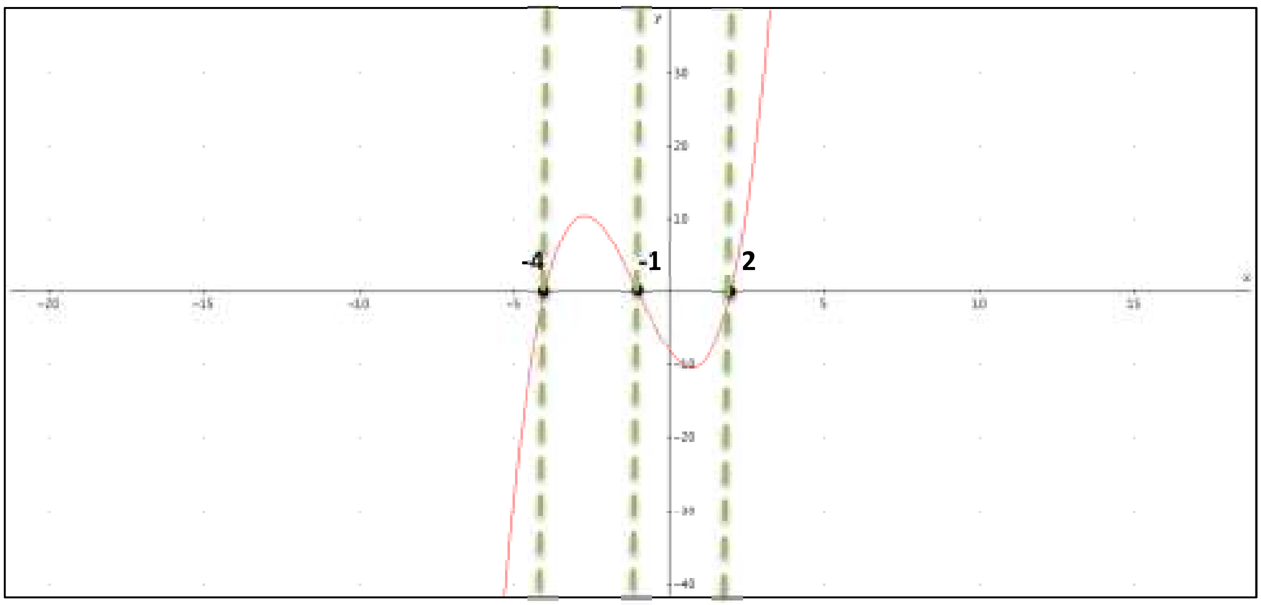
Exercise 5B – Pure Mathematics 1 (page 71)   Question 4

Higher Than Quadratic

The same ideas can help you solve more complicated inequalities:

Example: Solve the inequality $(x - 2)(x + 1)(x + 4) \geq 0$

This is a **cubic equation** (the highest exponent is a cube, i.e. x^3), let us graph it instead:



The interval that we want is **≥ 0**. The set of solution should be $-4 \leq x \leq -1$ or $x \geq 2$.

This is where the graph is above the *x*-axis.

Miscellaneous Exercise 5 – Pure Mathematics 1 (page 72)   Question 2,4,5,6,7,8,9,10