Integration

P1/9/1: Integration as the reverse process of differentiation; The constant of integration

P1/9/2: Problem involving the evaluation of a constant of integration

P1/9/3: Definite integrals including simple cases of improper integrals

P1/9/4: Find the area bounded by the curve and the axes

P1/9/5: Volumes of revolution

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P1/9/1:

(a) Integration as the reverse process of differentiation;

(b) The constant of integration

P1/9/2:

Problem involving the evaluation of a constant of integration

Learning Outcome

Students should be able to:

- **Relate integration as the reverse process of differentiation and integrate $(ax + b)^n$ (for any rational n except $n = -1$), together with constant multiples, sums and differences.**
- **Find the constant of integration and should be able to solve problems involving the evaluation of the constant of integration.**

The reverse of differentiation

If $y = f(x)$, we can find the gradient function $\dfrac{dy}{dx}$.

If we are given the gradient function $\dfrac{dy}{dx}$,

can we obtain the equation of the curve?

This reverse process is called **integration**.

We can now state the rule for integrating ax^n :

$$\text{If } \quad y = \frac{ax^{n+1}}{n+1} + c \text{ then } \frac{dy}{dx} = ax^n \text{ provided } n \neq -1$$

The process of finding all ***y***'s (antiderivatives) of a function is called **integration**.

$$y = \int ax^n dx = \frac{ax^{n+1}}{n+1} + c \text{ for } n \neq -1$$

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c \text{ for } n \neq -1$$

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1)(a)} + C, \qquad n \neq -1$$

Basic Rules

$$\boxed{\int k\,dx = kx + C}$$

$$\boxed{\int kf(x)\,dx = k\int f(x)\,dx}$$

$$\boxed{\int (f \pm g)\,dx = \int f\,dx \pm \int g\,dx}$$

Example 1:

Find y in term of x of the following cases.

$$(i)\quad \frac{dy}{dx} = 5\pi$$

$$(ii)\quad \frac{dy}{dx} = \frac{5}{\sqrt[3]{x}}$$

$$(iii)\quad \frac{dy}{dx} = \frac{1}{\sqrt{2x-1}}$$

Example 2:

A tree is growing so that, after t years, its height is increasing at a rate of $\dfrac{30}{\sqrt[3]{t}}$ cm per year. Assume that, when $t = 0$, the height is 5 cm.

(a) Find the height of the tree after 4 years.

(b) After how many years will the height be 4.1 metres?

Practice Exercise

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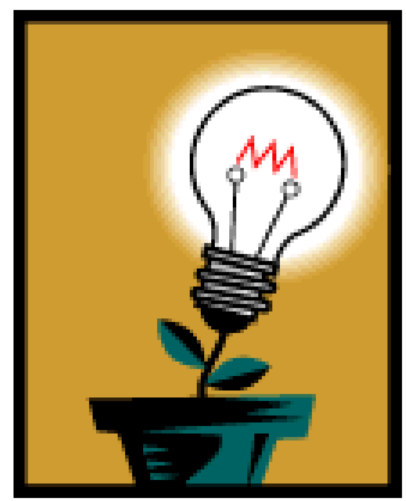
Exercise 16A (Page 238)
Q15

Exercise 16D (Page 253)
Q1

Exercise 16A (Page 239)
Q14, Q17



P1/9/3:

Definite integrals including simple cases of improper integrals

Learning Outcome

Students should be able to:

- **Evaluate definite integrals and solve problems involving the techniques of integration.**

Properties of the Definite Integral

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ (c is a constant)

4. $\int_a^b \left(f(x) \pm g(x) \right) dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
$(a < c < b)$

Example 3:

Evaluate the following definite integrals.

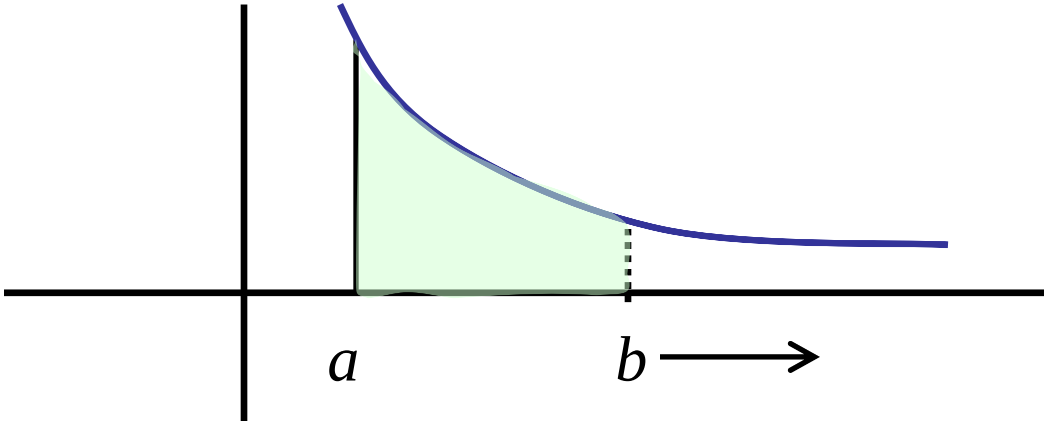
$$\int_{1}^{3} \frac{1}{(x+2)^2}\,dx$$

Improper Integrals

Integrals with unbounded intervals of integration are called ***improper integrals***.

Let f be continuous on $[a,\infty)$.

Then $\int_a^\infty f(x)dx = \lim\limits_{b\to\infty}\int_a^b f(x)dx$ if the limit exists.



Example 4:

Find the values of the improper integral.

$$(i) \quad \int_1^{\infty} \frac{1}{x^3}\, dx$$

$$(ii) \quad \int_0^{16} \frac{1}{\sqrt[4]{x}}\, dx$$

Practice Exercise

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Exercise 16B (Page 244)
Q15(d)(f)

Exercise 16C (Page 250)
Q3(b), Q4(b)

Exercise 16D (Page 253)
Q2(b)

P1/9/4:

Find the area bounded by the curve and the axes

Learning Outcome

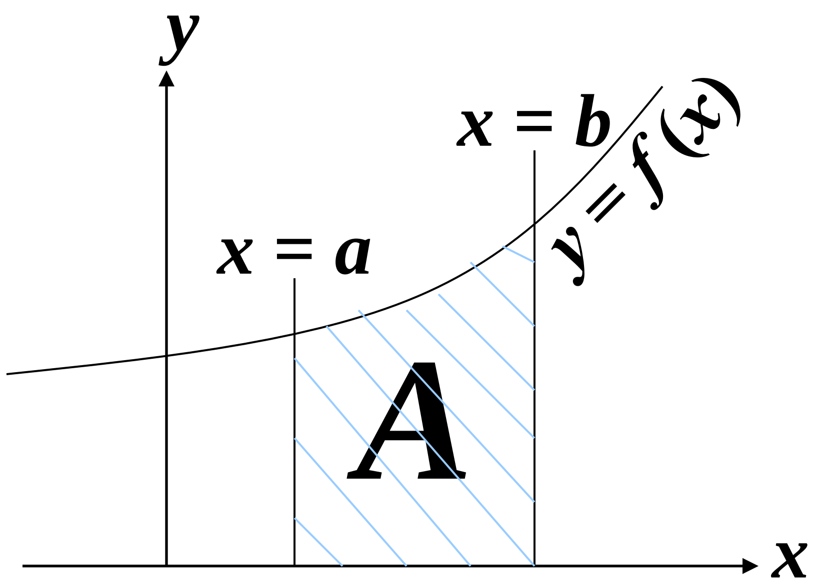
Students should be able to:

- **Locate the bounded region under the curve and calculate the area of that region.**

The area under a curve

Area Between a curve and the x-axis

Suppose A is the area bounded by the curve $y = f(x)$, the x-axis and the line $x = a$ and $x = b$.



A is the area 'under' the curve from $x = a$ to $x = b$.

$$\Rightarrow A = \int_a^b y \, dx$$

Area Between a curve and the *y*-axis

Suppose that we wish to find the area between some curve $y = f(x)$ and the y-axis, from $y = a$ to $y = b$.

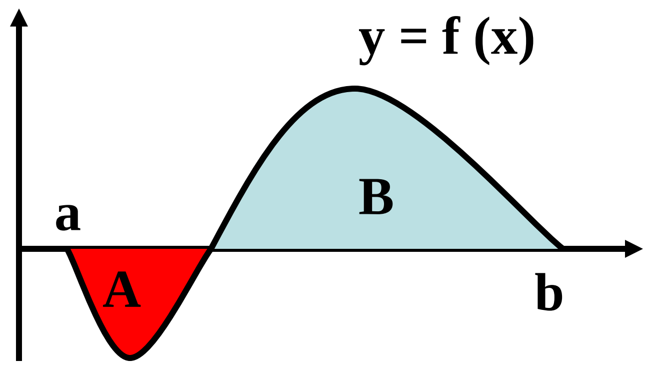
$$A = \int_a^b x\,dy$$

Negative Values

If $f(x)$ is positive for some values of x on $[a,b]$ and negative for others, then the definite integral symbol

$$\int_a^b f(x)\,dx$$

Represents the cumulative sum of the signed areas between the graph of $f(x)$ and the x axis where areas above are positive and areas below are counted negatively.



$$\int_a^b f(x)\,dx = |A| + B$$

Remember area measure is never negative!

Example 5:

Find the area of the region enclosed by the curve $y = 2\sqrt{x}$, the x-axis and the lines $x = 1$ and $x = 4$.

Example 6:

Find the area enclosed between the curve $y^2 = 9 - x$ and the y-axis.

Example 7:

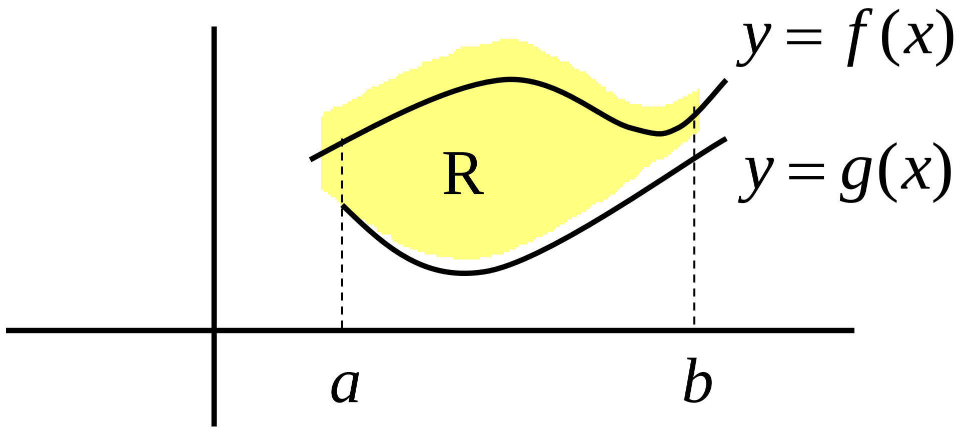
Find the area between the curve $y = x(4 - x)$ and the x-axis from $x = 0$ to $x = 5$.

Area Between Two Curves

Let f and g be continuous functions, the area bounded above by $y = f(x)$ and below by $y = g(x)$ on $[a, b]$ is

$$\boxed{\int_a^b \left[f(x) - g(x) \right] dx}$$

provided that $f(x) \geq g(x)$ on $\left[a, b\right]$.



Example 8:

Find the area bounded by the curve

$$f(x) = x^2 - 1 \text{ and } g(x) = x - 3$$

and the vertical lines $x = -1$ and $x = 2$.

Practice Exercise

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Exercise 16B (Page 245)

Q13(e)(f), Q22

Exercise 16C (Page 250)

Q2(b), Q10

Exercise 16D (Page 253)

Q5(b), Q7

P1/9/5:

Volumes of revolution

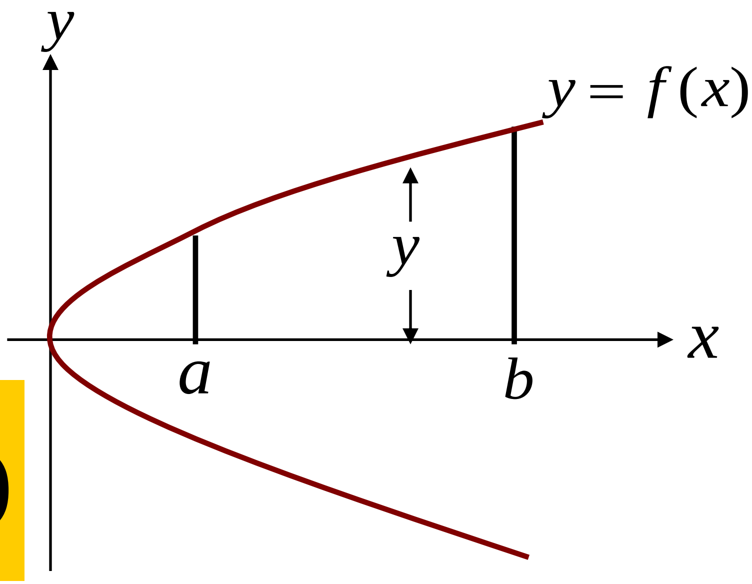
Learning Outcome

Students should be able to:

- **Locate the bounded region under the curve and find the volume of revolution.**

When the region under the graph of $y = f(x)$ between $x = a$ and $x = b$ (where $a < b$) is **rotated about the *x*-axis**,

The volume of the solid of revolution formed is

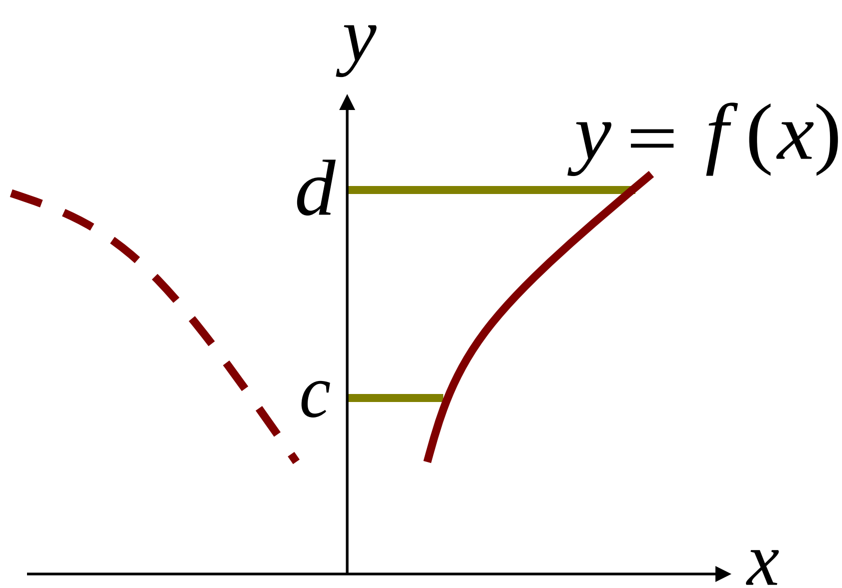


$$V = \int_a^b \pi y^2 dx \quad \text{where} \quad y = f(x)$$

Rotation about the *y*-axis

The volume of the solid of revolution formed by revolution an area through one revolution about the y-axis can be found in a similar way.

$$V = \int_{y=c}^{y=d} \pi x^2 \, dy$$



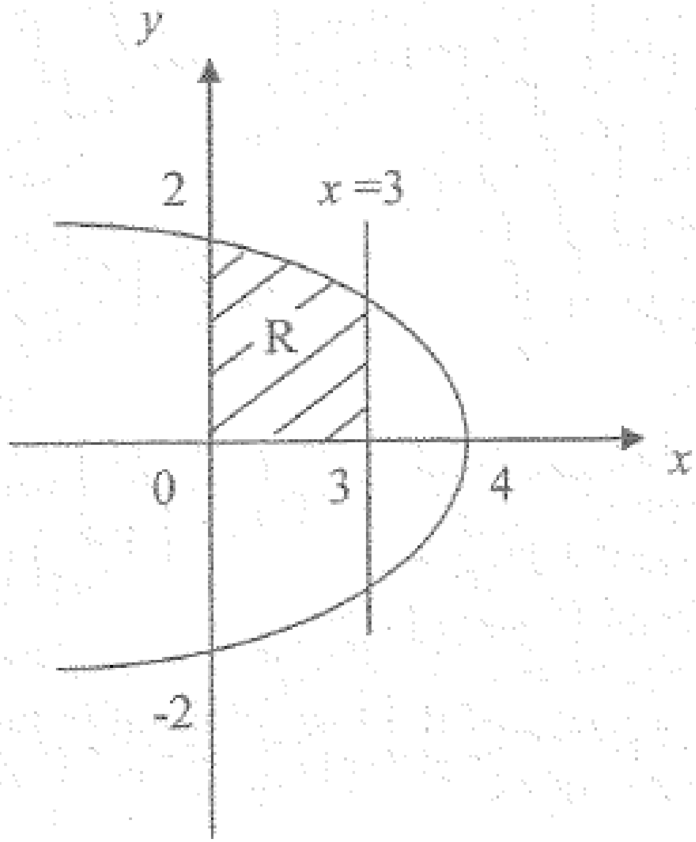
Example 9:

Find the volume of the solid of revolution formed by rotating the area enclosed by the $x = 2$ and $x = 3$ through one revolution curve $y = x + x^2$, the x-axis and the ordinates.

Example 10:

Find the volume of the solid of revolution formed by rotating the area enclosed by the $y = 9 - x^2$ and the x-axis about the y-axis through 360°.

Example 11:



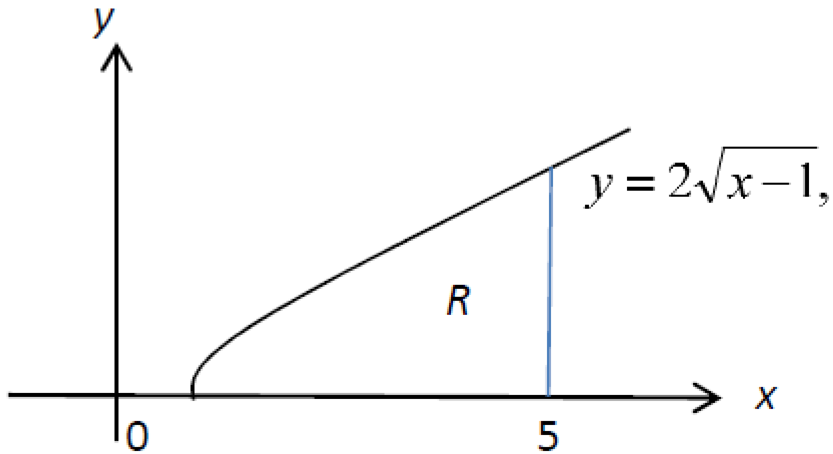
The region R in the first quadrant is bounded by the y-axis, the x-axis, the line $x = 3$ and the curve $y^2 = 4 - x$.

(i) Calculate the area of the region R. [4]

(ii) Calculate the volume of the solid formed when R is rotated about the y-axis through one revolution. (Leave your answer in terms of π) [5]

Example 12:

The diagram shows the curve with equation $y = 2\sqrt{x-1}$, the line $x = 5$, and the x-axis.



(i) Find the exact area of the region R. [3]

(ii) Find the exact volume of the solid formed when the region R is rotated through 360^0 about the y-axis. [6]

Practice Exercise

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Exercise 17 (Page 261)

Q2(c), Q3(e), Q5, Q9