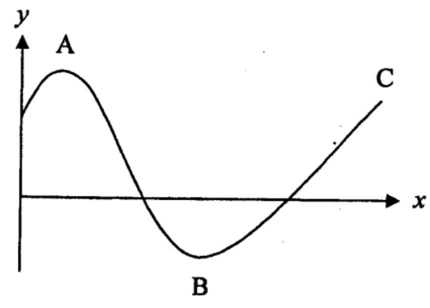
Pure Mathematics 1

Tutorial 8 Differentiation

1	Given that $f(x) = \dfrac{4}{x-4} - \dfrac{1}{x-1}$. (i) Write down the expressions for $f'(x)$ and $f''(x)$. You need not simplify your answers. [2] (ii) Find the coordinates of the two stationary points on the graph of f and identify each as a maximum or minimum. [4] $\left[-4(x-4)^{-2} + (x-1)^{-2}\,; 8(x-4)^{-3} - 2(x-1)^{-3}\,; (2\,,-3)\,max\,; \left(-2\,,-\frac{1}{3}\right) min\right]$
2	The radius of a solid cylinder is always equal its height. Given that the volume of the cylinder is increasing at a constant rate of $0.05\ m^3s^{-1}$, calculate the rate of change of the total surface area of the cylinder when its radius is 0.5 m. [6] [4/15]
3	Air is being pumped into a spherical balloon at a constant rate of $5cm^3$ per second. Find the rate of increase, at the instant when radius is 10cm, of i) the radius, [3] ii) the surface area of the balloon. [3] $\left[\dfrac{1}{80\pi};1\right]$
4	Given that $x + 2y = 4$. Let $V = x^2 + xy - y^2$, find the minimum value of V. [8] [-20]
5	 The diagram above is a graph of the function $y = f(x)$ where $0 \le x \le c$. The x coordinates of the points A,B and C are a, b, c respectively. The function is a cubic polynomial and A and B are the maximum and minimum points respectively. i) State the range of x, in terms of a, b and c, for which $f(x)$ is increasing. [2] ii) State the values of x for which the graph has stationary points. [1] iii) Copy the graph and mark clearly on the graph, two other points D and E for which $f(x) = 0$. [2] iv) At point A, is $f''(x)$ positive or negative ? Give a reason for your answer. [2]

$[0 \le x \le a, b \le x \le c; x = a, x = b; f''(x) < 0, A \text{ is max. point}]$

6

A curve has an equation of the form $y = x^3 + bx^2 - 9x$ where b is a constant.

a) Given that the curve has a stationary point at $x = -1$, find the value of the constant b. [3]

b) Determine whether the stationary point at $x = -1$ is a maximum or a minimum point. [3]

c) Find the set of values of x for which y increases as x increases. [3]

[-3; max; x < -1 and x > 3]

7

a) Find the range of values of x for which the function $1 - x + 2x^2 - x^3$ is increasing. [4]

b) The normal to the curve $y = 2x - \dfrac{1}{1-x}$ where $x = 2$, meets the curve again at point P. Find the equation of this normal and the coordinates of P. [6]

[1/3 < x < 1; y = -x +7; P(4/3, 17/3)]

8

A closed cylindrical can with radius R cm is made from 300cm^2 of thin sheet metal. Show that its height, H cm , is given by $H = \dfrac{150 - \pi R^2}{\pi R}$. Find R and H so that the can will contain the maximum possible volume and find this maximum volume. [11]

$$\left[\sqrt{\frac{50}{\pi}}; \frac{100}{\sqrt{50\pi}}; \frac{5000}{\sqrt{50\pi}}\right]$$

9

(a) Two parallel sides of a rectangle respectively lengthen at a rate of 2 cm per second, while the other two parallel sides shorten such that the area of the rectangle is always 50 cm^2. If, at time t, the length of each lengthening side is x, the length of each shortening side is y, and the perimeter of the rectangle is p, show that $\dfrac{dp}{dx} = 2 - \dfrac{100}{x^2}$ and $\dfrac{dp}{dt} = 4\left(1 - \dfrac{50}{x^2}\right)$. [5]

Find the rate of change in the perimeter when

(i) $x = 5$ cm. [1]

(ii) $y = 5$ cm. [2]

(b) Given that the gradient of the curve $y = 3x - \dfrac{8}{x^2}$ at the point (a, b) is 5. Find the values of a and b. [4]

[-4; 2; a = 2 and b = 4]

10	. The function $y = ax^3 + bx^2 - 12x + 13$ passes through the point (1,0) and has a stationary point where $x = -1$. Find a) the value of a and b. [5] b) the coordinate of the other stationary point and determine its nature. [4] [2, -3; (2, -7) minimum]
11	Given that $y = x^2 + \frac{8}{(1-x)}$ ($x \in R$, $x \neq 1$), a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ [2] b) Verify that there is a turning point on the graph of y when $x = -1$ and find whether it is a maximum or a minimum point. [2] c) Show that there are no other turning points. [6] $\left[2x + \frac{8}{(1-x)^2}\ ; 2 + \frac{16}{(1-x)^3}\ ; minimum\right]$
12	Find all turning points on the graph $y = \sqrt{x-1} + \frac{1}{\sqrt{x-1}}$, $x > 1$ and determine whether they are maximum or minimum points. [8] $[(2\,,2) minimum]$
13	**a) The tangent to the curve $y = 2x^2 + ax + b$ at the point (−2, 11) is perpendicular to the line $2y = x + 7$. Find the value of a and b.** [4] **b) Find the radius and height of a cylindrical can if it is to have a volume of 128π cm^3 with minimum possible surface area?** [7] $[a = 6\,, b = 15\,; r = 4\,; h = 8]$

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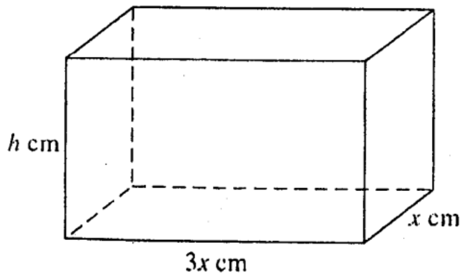
I.



The diagram shows a closed cardboard box in the shape of a cuboid with base $3x$ cm by x cm and height h cm. The surface area of the cardboard (i.e. the outside of the box) is 1152 cm^2.

(i) By first expressing h in terms of x, show that the volume of the box, V cm^3, is given by

$$V = 432x - \frac{9}{4}x^3 .$$ [4]

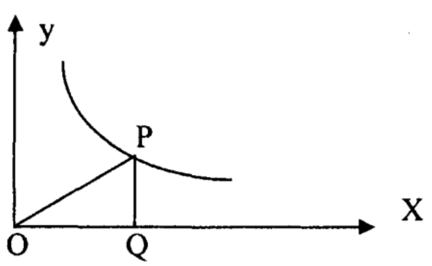
(ii) Given that x can vary, find the maximum value of V, explaining why it is a maximum and not a minimum. [6]

$$\left[2304; \frac{d^2V}{dx^2} = -108 < 0 \text{ max}\right]$$

15

I. Point P moves on the graph of $y = f(x)$ where $f(x) = \frac{4}{x}$ for $x > 0$

O is the origin and point Q is on the x-axis. Let OQ be x units long and OP be L units long.



(i) Show that $L^2 = x^2 + \frac{16}{x^2}$ [1 mark]

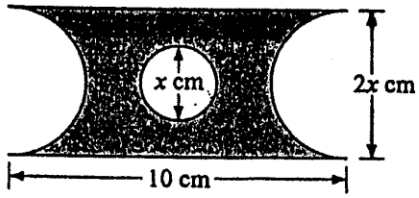
(ii) Find the position of P at which L^2 is minimized [4 marks]

(iii) Show that when L^2 is minimized, OP is normal to $y = f(x)$ [5 marks]

[(2, 2)]

16

(a) The diagram below shows a metal strip 10 cm long and $2x$ cm wide, with two semi-circular ends. A circular piece, x cm in diameter, has been removed from the strip.



Show that the area of the shaded region (in cm^2) is given by

$$A = 20x - \frac{5\pi}{4}x^2.$$ [4]

Hence, find the value of x for which A is a maximum. [3]

(b) A funnel is shaped in the form of an inverted cone of radius 5 cm and height 10 cm.

If the funnel is filled with water, the volume of water contained in the cone may be expressed as $\frac{1}{12}\pi h^3$ where h is the depth of the cone.

A small hole is made at the side of the funnel such that water flows out at a constant rate of 8π cm^3/s. Calculate the rate of change of the depth of water when the depth of water is is 4 cm. (Volume of a cone = $\frac{1}{3}\pi r^2 h$) [5]

$$\left[\frac{40}{5\pi};-2\right]$$

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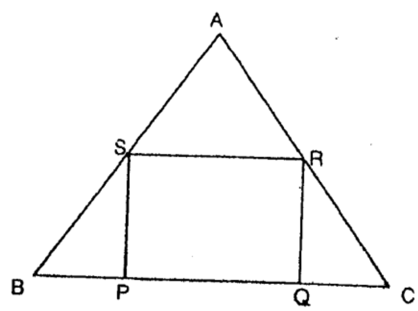
ABC is an isosceles triangle (as shown in the diagram below) with AB = AC = 20cm and BC = 24cm. A rectangle PQRS is drawn inside the triangle with PQ on BC, S and R on AB and AC respectively.

a) If $PQ = 2x$ cm, show that the area M cm^2 of the rectangle is given by

$M = \dfrac{8x(12-x)}{3}$. [5]

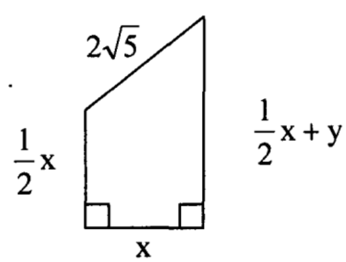
b) Hence, find the maximum value of M. [6]



[96]

18

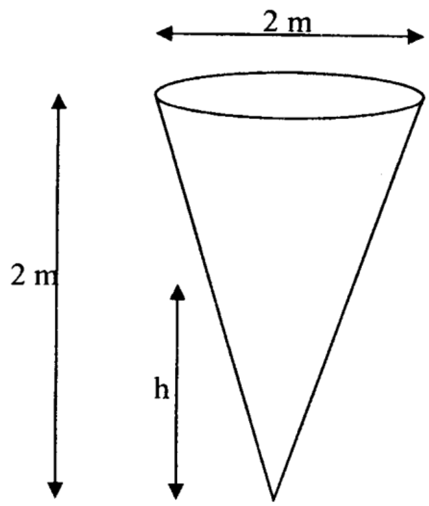
The diagram below represents (in meter) the dimensions of a small garden.



(i) Show that $y = \sqrt{20 - x^2}$. [2]

(ii) Write an expression, in terms of x, for the perimeter, P (in meters) of the garden and find a value of x for which $\dfrac{dP}{dx} = 0$. [5]

$\left[2 - \frac{x}{\sqrt{2-x^2}}; 4\right]$

19	 A container is in the shape of a right circular cone with both height and diameter 2 metres (as shown) It is being filled with water at a rate of π m^3 per minute. Find the rate of change of height h of water when the container is one - eighth full (by volume). (Volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$). [7] $[4m/min]$
20	**(a)** Gas is being pumped into a spherical balloon at the rate of 9 m^3 per minute. Find the rate at which the radius of the sphere is increasing at the instant when the radius is 2 m. [4] **(b)** A closed box with a square base is to have a volume of 2000 m^3. The material for the top and the bottom of the box is to cost \$3 per square metre, and the material for the sides is to cost \$1.50 per square metre. What are the dimensions of the box if the cost for the materials is to be minimum ? [7] $[0.179\ ;\ 10, 20]$