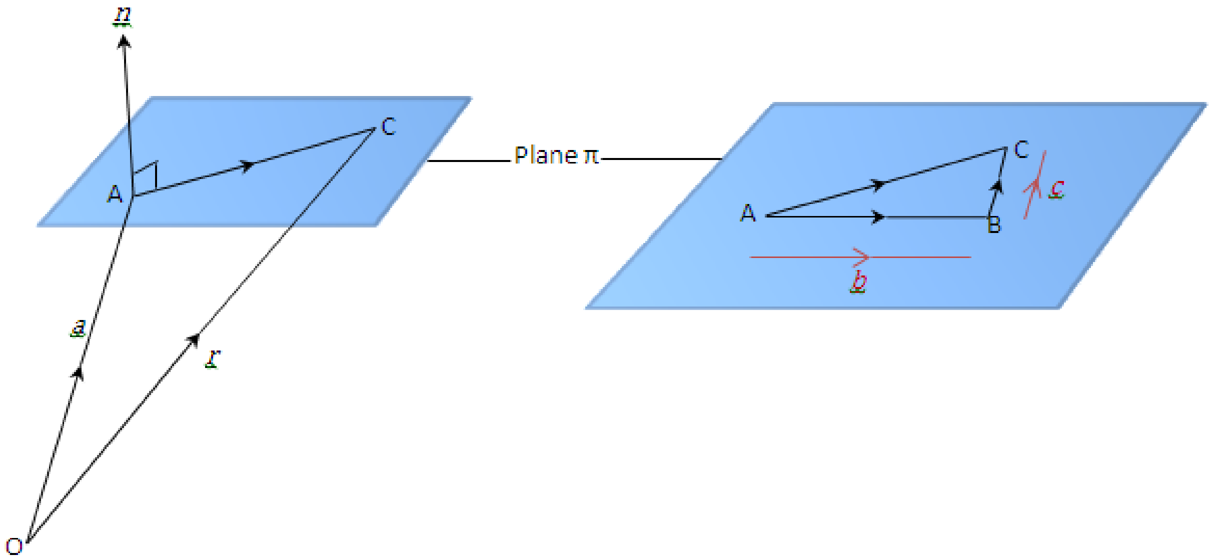
PLANES
Equations of a Plane



Vector Equation - Parametric	Vector Equation -Scalar Product	Cartesian Equation
$\underline{\boldsymbol{r}} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{BC}$	$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$ $\underline{r} \cdot \underline{n} = d$	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d$ $ax + by + cz = d$

O is the point of origin. *A, B* and *C* are three points on a plane and $\underline{n}$ is a normal to the plane π.

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$
$$= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$$

Since AB is parallel to $\underline{b}$ and BC is parallel to $\underline{c}$.

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$$
$$\underline{r} = \underline{a} + \lambda\underline{b} + \mu\underline{c}$$

Since $\overrightarrow{AC}$is perpendicular to $\underline{n}$, and $\overrightarrow{AC} = \underline{r} - \underline{a}$.

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0$$
$$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$$

$\underline{a} \cdot \underline{n} = d$, where d is a constant.Let $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Since $\underline{n}$ is the common perpendicular. We can use cross product to find the direction of $\underline{n}$.

$$\underline{\boldsymbol{n}} = \overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{BC}}$$

Examples

1. Find a vector equation for the plane containing the three points A, B, and C whose position vectors are $2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $3\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $5\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ respectively.

2. Given the points A(1, 2, -1), B(4, 0,3) and C(2, -1, 5) in a plane. Find the equation of the plane.

3. Find the vector equation of the plane which passes through the point with position vector $2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and is perpendicular to the vector $3\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}$.

4. Given the normal vector, $3\mathbf{i} + 1\mathbf{j} - 2\mathbf{k}$ to the plane containing the point (2, 3, -1), Write the equation of the plane in
   i) scalar product form.
   ii) cartesian form.

Question 5

The line l and m have vector equations $\mathbf{r} = 13\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + s(\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 14\mathbf{i} - 8\mathbf{j} - 13\mathbf{k} + t(-3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k})$ respectively. The two lines l and m intersect at point A.

(i) Find the position vector of A. [4]

(ii) Find the vector equation of the plane containing l and m, giving your answer in the form $ax + by + cz = d$. [5]

Question 6 - June 05

10 With respect to the origin O, the points A and B have position vectors given by

$$\overrightarrow{OA} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation $\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

(i) Prove that the line l does not intersect the line through A and B. [5]

(ii) Find the equation of the plane containing l and the point A, giving your answer in the form $ax + by + cz = d$. [6]

Question 7 - June 06

10 The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}.$$

The line l passes through A and is parallel to OB. The point N is the foot of the perpendicular from B to l.

(i) State a vector equation for the line l. **[1]**

(ii) Find the position vector of N and show that $BN = 3$. **[6]**

(iii) Find the equation of the plane containing A, B and N, giving your answer in the form $ax + by + cz = d$. **[5]**

Distance from a Point to a Plane

Examples

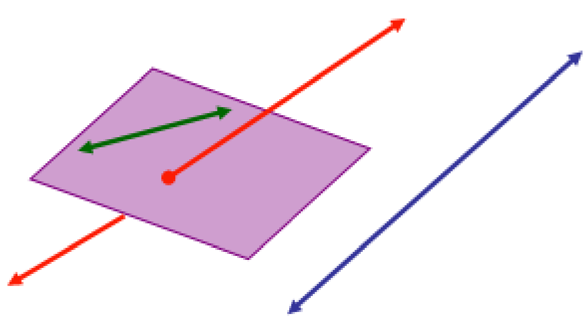
1. Find the foot of the perpendicular from point (2, -1, 6) to the plane $x + 2y - 3z = 10$ and hence find the shortest distance from point to the plane.

2. A plane contains a point A, position vector $3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$ and is perpendicular to the vector $\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$. Find

a) a vector equation of the plane.
b) the foot of the perpendicular from the point (1, 1, 1).
c) the perpendicular distance of the plane from the point (1, 1, 1).
d) the perpendicular distance of the plane from the origin.

Geometrical relationship between a line and a plane

There are three possibilities:



- Line passes through plane – intersection is a point**(RED LINE)**
- Line lies on the plane - intersection is a line**(GREEN LINE)**
- Line is parallel to the plane - no common point. **(BLUE LINE)**

Example

(Line lies on a plane)

1. Show that the line $\underline{r} = (3\underline{i} + 3\underline{j} - 2\underline{k}) + \lambda(\underline{i} + \underline{j} - \underline{k})$ lies in the plane $\underline{r} \cdot (3\underline{i} - 2\underline{j} + \underline{k}) = 1$.

(Line is parallel to the plane)

2. Find whether the line $\underline{r} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ lies on the plane $\underline{r} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 1$

(Line intersects plane at one point)

3. Find the position vector of the point of intersection of the line $\underline{r} = (2\underline{i} - \underline{k}) + \lambda(\underline{i} + 3\underline{j})$ and the plane $\underline{r} \cdot (5\underline{i} - \underline{j} + 7\underline{k}) = 9$.

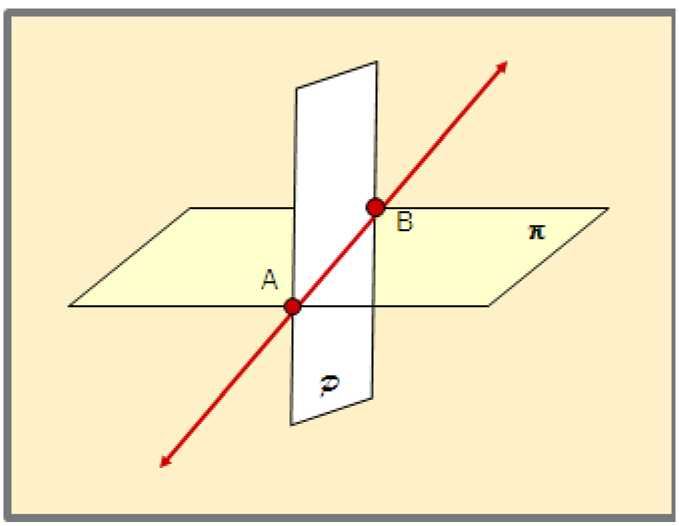
Question 4 - Nov 06

7 The line l has equation $\mathbf{r} = \mathbf{j} + \mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. The plane p has equation $x + 2y + 3z = 5$.

(i) Show that the line l lies in the plane p. [3]

(ii) A second plane is perpendicular to the plane p, parallel to the line l and contains the point with position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. Find the equation of this plane, giving your answer in the form $ax + by + cz = d$. [6]

Intersection between 2 planes



Intersection of Plane π and Plane P.

Two non-parallel planes will always meet in a straight line, $\overrightarrow{AB}$.

To find the equation of the line of intersection of the two planes

Method 1 - Using two coordinates.
Method 2 - Using one coordinates and directional vector**(Cross Product).**

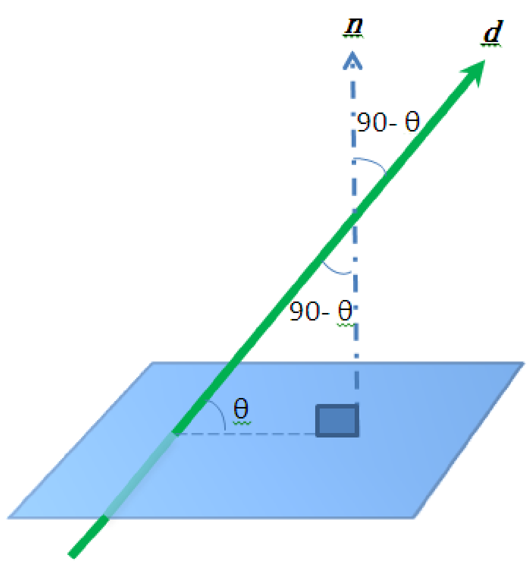
Example

1. Find the vector equation of the line of intersection of the planes $\underline{r} \cdot (2\underline{i} - \underline{j} - 3\underline{k}) = 10$ and $\underline{r} \cdot (\underline{i} + 3\underline{j} - 2\underline{k}) = 16$.

2. Find the line formed at the intersection of the planes:

$$\pi_1 : x - 3y + 2z = 3$$
$$\pi_2 : 2x + 2y + z = 5$$

<u>Angle between a plane and a line</u>



Using dot product,

$$\underline{n} \cdot \underline{d} = |\underline{n}||\underline{d}| \cos(90 - \theta)$$

Since $\cos(90 - \theta) = sin\,\theta$

$$\underline{n} \cdot \underline{d} = |\underline{n}||\underline{d}| \sin \theta$$

Example

1. Find the angle between the line $\underline{r} = 3\underline{k} + \lambda(7\underline{i} - \underline{j} + 4\underline{k})$ and the plane $\underline{r} \cdot (2\underline{i} - 5\underline{j} - 2\underline{k}) = 8$.

Question 2 - Nov 05

10 The straight line l passes through the points A and B with position vectors

$$2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

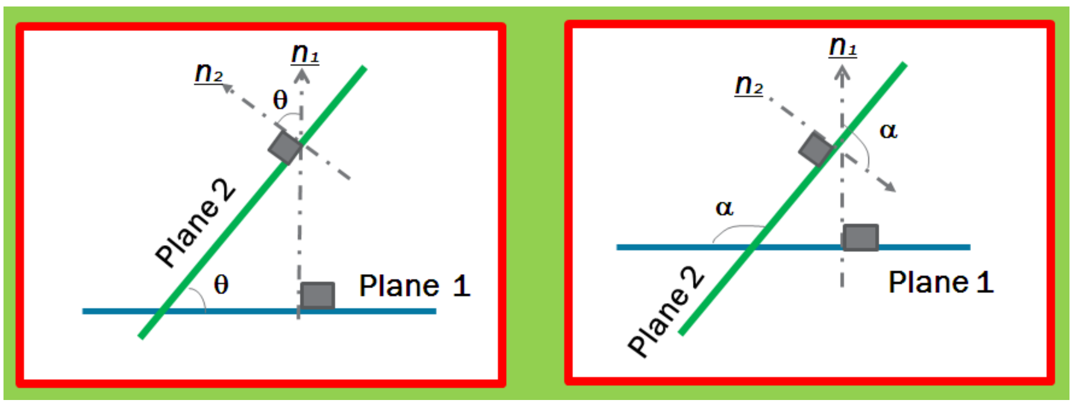
respectively. This line intersects the plane p with equation $x - 2y + 2z = 6$ at the point C.

(i) Find the position vector of C. [4]

(ii) Find the acute angle between l and p. [4]

(iii) Show that the perpendicular distance from A to p is equal to 2. [3]

<u>Angle between two planes</u>



In most cases the question will ask you to find the acute angle between two planes as there is always the acute and obtuse angle between them.

Using dot product,

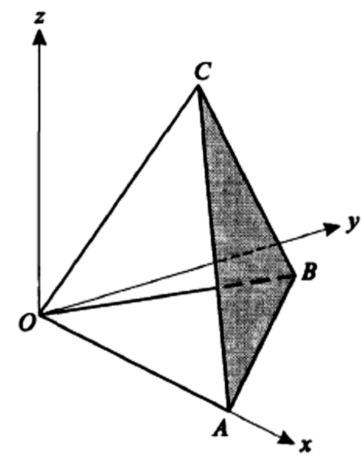
$$n_1 \cdot n_2 = |n_1||n_2|\cos\theta$$

Examples

1. Find the acute angle between $\underline{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = -4$ and $\underline{r} \cdot \begin{pmatrix} 1 \\ -1 \\ -8 \end{pmatrix} = 3$

Question 2

9



The diagram shows a set of rectangular axes Ox, Oy and Oz, and three points A, B and C with position vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

(i) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. **[6]**

(ii) Calculate the acute angle between the planes ABC and OAB. **[4]**