TOPIC 9 – INTEGRATION

Learning Outcome

- understand integration as the reverse process of differentiation, and integrate $(ax+b)^n$ (for any rational n except -1), together with constant multiples, sums and differences;

- solve problems involving the evaluation of a constant of integration, e.g. to find the equation of the curve through $(1,-2)$ for which $\frac{dy}{dx}=2x+1$;

- evaluate definite integrals (including simple cases of 'improper' integrals, such as $\int_0^1 x^{-\frac{1}{2}}\,dx$ and $\int_1^\infty x^{-2}\,dx$);

- use definite integration to find

  the area of a region bounded by a curve and lines parallel to the axes, or between two curves,

  a volume of revolution about one of the axes.

<u>Indefinite Integrals - Simple Algebraic Functions</u>

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$$ where c is an arbitrary constant

Example 1

Integrate with respect to x.

(a) $x^2 + \frac{1}{x^2}$ **(b)** $\frac{x^2+1}{2x^2}$ **(c)** $3-\sqrt{x}$ **(d)** $\sqrt{x}(\sqrt{x}+3)$

Example 2

Find an expression for y if $\frac{dy}{dx}$ is each of the following:

(a) $6x + 3$ **(b)** 4 **(c)** $3x(x + 2)$

(d) $(x - 1)(x + 2)$ **(e)** $x\left(2 + \frac{1}{x}\right)$ **(f)** $\frac{2x^2 + 3}{x^2}$

Example 3

Find the equation of the curve which passes through the point (2, 3) and for which $\frac{dy}{dx} = 3x^2 + x$.

Example 4

Integrate $(2x + 1)^3$ with respect to x.

Example 5

Find $\int \frac{1}{(2x - 1)^2}\, dx$.

Definite Integrals

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Note:

a) $\int_a^a f(x)dx = 0$ b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ c) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Example 6

Evaluate the following definite integrals.

(a) $\int_{-1}^{1} (8x - 4)\, dx$ **(b)** $\int_{-1}^{0} (3x^2 - 2x + 5)\, dx$

(c) $\int_{1}^{4} (6x - 3\sqrt{x})\, dx$ **(d)** $\int_{1}^{4} \left(\sqrt{x} - \frac{2}{\sqrt{x}}\right) dx$

(e) $\int_{1}^{2} \left(x^2 - \frac{4}{x^2}\right) dx$ **(f)** $\int_{1}^{2} \left(8x^3 - 2 + \frac{1}{2x^2}\right) dx$

Example 7

Evaluate the following definite integrals.

(a) $\int_1^4 \frac{x^2+1}{x^2}\,dx$ **(b)** $\int_1^2 \frac{1-2x^3}{x^2}\,dx$

(c) $\int_1^4 \frac{2x-1}{\sqrt{x}}\,dx$ **(d)** $\int_1^9 \frac{3-2\sqrt{x}}{x^2}\,dx$

(e) $\int_1^3 \frac{1-4x+x^3}{2x^3}\,dx$ **(f)** $\int_1^2 \frac{(x+3)(x-3)}{x^2}\,dx$

Example 8

Find the value of k if $\int_1^2 (4x+k)\,dx = 1$.

Example 9

Find the values of the infinite integrals

(a) $\int_2^\infty \frac{6}{x^4}\,dx$, (b) $\int_4^\infty \frac{6}{x\sqrt{x}}\,dx$, (c) $\int_1^\infty x^{-1.01}\,dx$.

Example 10

A tree is growing so that, after t years, its height is increasing at a rate of $\frac{30}{\sqrt[3]{t}}$ cm per year. Assume that, when $t=0$, the height is 5 cm.

(a) Find the height of the tree after 4 years.

(b) After how many years will the height be 4.1 metres?

Example 11

A pond, with surface area 48 square metres, is being invaded by a weed. At a time t months after the weed first appeared, the area of the weed on the surface is increasing at a rate of $\frac{1}{3}t$ square metres per month. How long will it be before the weed covers the whole surface of the pond?

Example 12

The function $f(x)$ is such that $f'(x) = 9x^2 + 4x + c$, where c is a particular constant. Given that $f(2) = 14$ and $f(3) = 74$, find the value of $f(4)$.