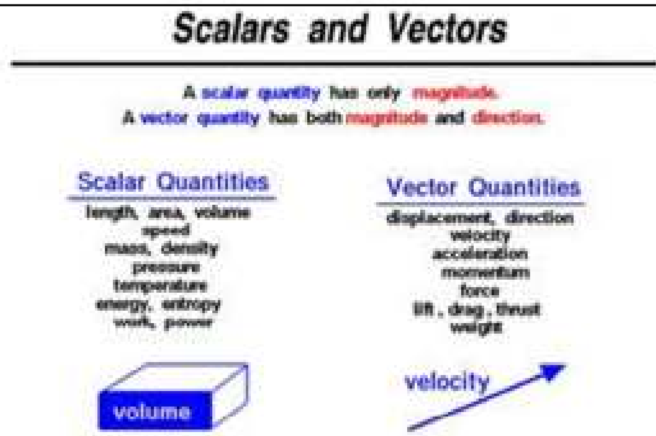
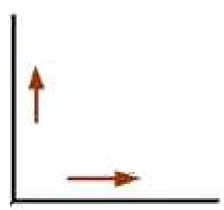
TOPIC 6 - VECTORS

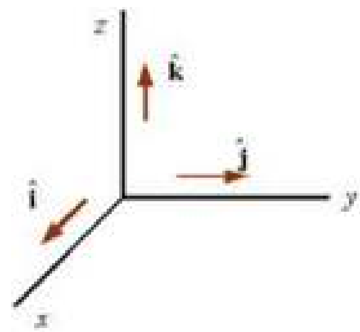
- use standard notations for vectors, i.e. $\begin{pmatrix} x \\ y \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\overrightarrow{AB}$, $\mathbf{a}$;
- carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometrical terms;
- use unit vectors, displacement vectors and position vectors;
- calculate the magnitude of a vector and the scalar product of two vectors;
- use the scalar product to determine the angle between two directions and to solve problems concerning perpendicularity of vectors.



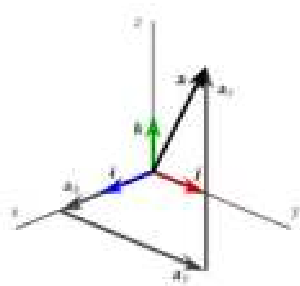
Two Dimension Vectors



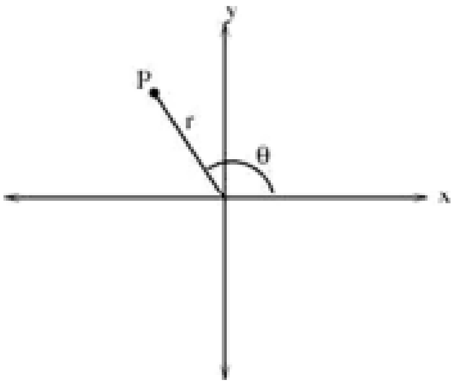
Three Dimension Vectors



Example of a 3-D vector



Magnitude(modulus) and Direction of a vector



Let P(x, y)

$r = |\overrightarrow{OP}| = \sqrt{x^2 + y^2}$

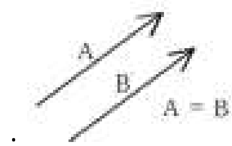
θ= the direction of $\overrightarrow{OP}$.

For 3-Dimension

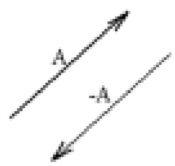
Let $\underline{a} = x\underline{i} + y\underline{j} + z\underline{k}$

$r = |\underline{a}| = \sqrt{x^2 + y^2 + z^2}$

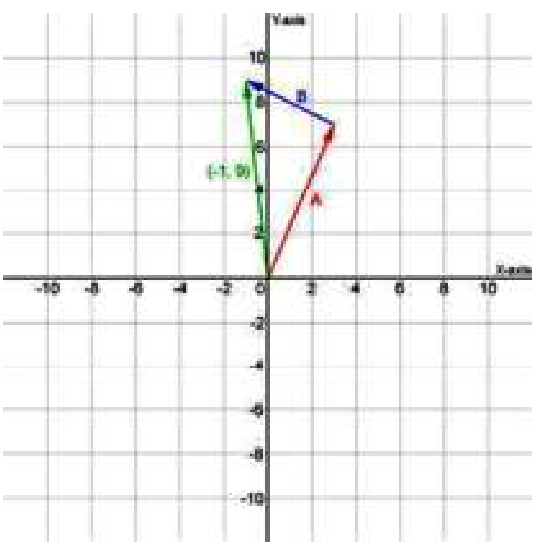
- Two vectors, **A** and **B** are equal if they have the same magnitude and direction, regardless of whether they have the same initial points.



- A vector having the same magnitude as **A** but in the opposite direction to **A** is denoted by **-A**.



Position Vector and Displacement Vector



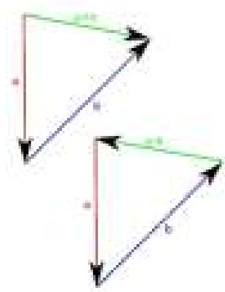
$\overrightarrow{OA} = position\ vector, \quad vector\ \boldsymbol{B}\ is\ a\ displacement\ vector.$

Scalar Multiplication of a Vector

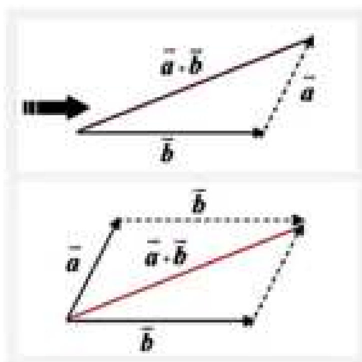
- $\underline{b} = k\underline{a}$ and k > 0 ⇔ $\underline{b}$ and $\underline{a}$ are in the same direction and $|\underline{b}| = k|\underline{a}|$.
- $\underline{b} = k\underline{a}$ and k < 0 ⇔ $\underline{b}$ and $\underline{a}$ are in the same direction and $|\underline{b}| = k|\underline{a}|$.

Addition of Vectors

- Triangle Law of Addition



- Parallelogram Law



Example 1

Fig. 23.11 shows a regular hexagon with vertices $ABCDEF$ and centre O. Express each of the following sums as a single vector.

(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$ (b) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AF}$

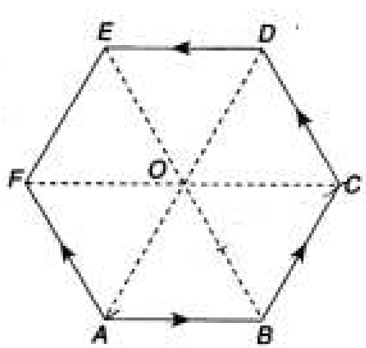


Fig. 23.11

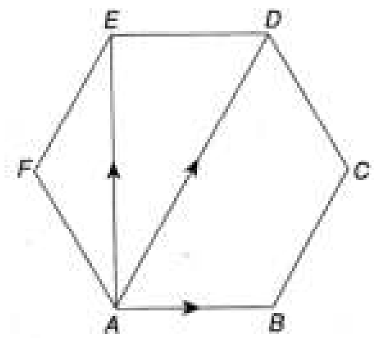
Example 2

ABCDEF is a regular hexagon. Express the sum of the following vectors in terms of a single vector.

(a) $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$

(b) $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AE}$, $\overrightarrow{AF}$

(c) $\overrightarrow{AB}$, $\overrightarrow{FE}$, $\overrightarrow{CD}$



Example 3

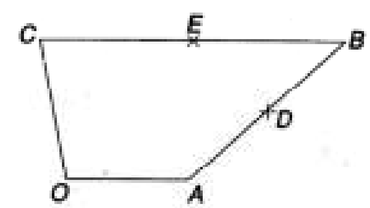
PQRS is a quadrilateral. The points *A*, *B*, *C* and *D* are midpoints of *PQ*, *QR*, *RS* and *SP* respectively. Show that *ABCD* is a parallelogram.

Example 4

OABC is a trapezium with $\overrightarrow{CB} = 3\overrightarrow{OA}$. The points *D* and *E* are midpoints of *AB* and *BC*.

(a) Express $\overrightarrow{OD}$, $\overrightarrow{OE}$ and $\overrightarrow{DE}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(b) Express $\overrightarrow{OD}$, $\overrightarrow{OE}$ and $\overrightarrow{DE}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

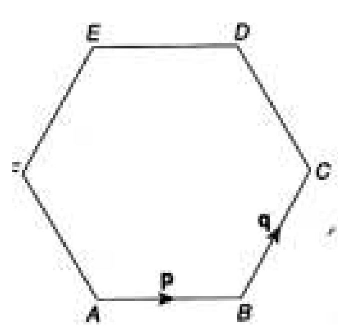


Example 5

In Fig. 23.13, *ABCDEF* is a regular hexagon and $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$.

(a) Express $\overrightarrow{FD}$, $\overrightarrow{CD}$ and $\overrightarrow{BE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

(b) Given that $\mathbf{p}$ is a unit vector, evaluate $|\overrightarrow{AC}|$.



Unit Vector

A vector with a magnitude of 1, $|\underline{a}| = 1$.

$$\underline{\hat{a}} = \frac{\underline{a}}{|\underline{a}|}$$

Example 6

Fig. 23.6 shows a grid with parallel lines. $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{MN} = \mathbf{b}$.

(a) Express **a** and **b** in terms of **u**.

(b) If **u** is a unit vector, find the magnitude of **a** and of **b**.

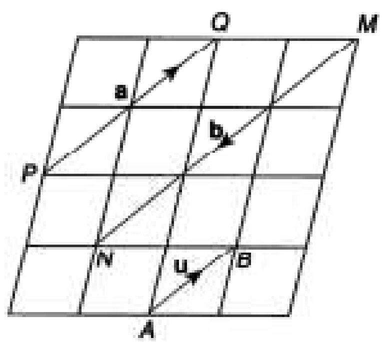


Fig. 23.6

Scalar Product

There are two ways of 'multiplying' vectors:

- Scalar product (dot product) – gives a scalar result.
- Vector product (cross product) – the product gives a vector result.

The dot product(scalar product) of vectors $\underline{a}$ and $\underline{b}$ is defined as

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$$

where θ is the angle between two vectors.

Example 7

Find the value of the following scalar products:

(a) $(3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (4\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

(b) $(3\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})$

(c) $(5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - 7\mathbf{k})$

(d) $(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j})$

(e) $(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) \cdot (\mathbf{j} - 3\mathbf{k})$

(f) $(\mathbf{i} + 4\mathbf{k}) \cdot (\mathbf{j} - \mathbf{k})$

Example 8

Find the angle between the vectors:

(a) $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

(b) $\mathbf{p} = 3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$ and $\mathbf{q} = -\mathbf{j} + \mathbf{k}$

(c) $\mathbf{u} = \begin{pmatrix} a \\ -2a \\ -5a \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -2a \\ 7a \\ 5a \end{pmatrix}$, where a is a scalar

Example 9

Find the angle between the line joining $(1,3,-2)$ and $(2,5,-1)$ and the line joining $(-1,4,3)$ to $(3,2,1)$.

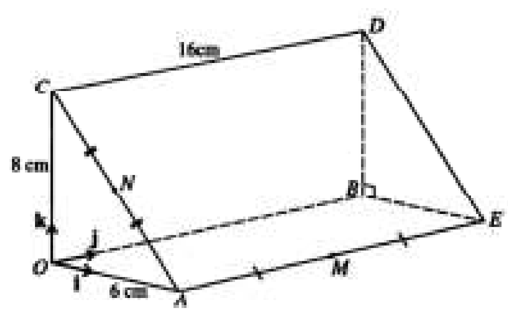
Example 10

The points A, B, C and D have position vectors $3\mathbf{i} + 2\mathbf{k}$, $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $2\mathbf{j} + 7\mathbf{k}$ and $-2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}$ respectively.

(i) Use a scalar product to show that BA and BC are perpendicular. [4]

(ii) Show that BC and AD are parallel and find the ratio of the length of BC to the length of AD. [4]

Example 11



The diagram shows a prism with cross-section in the shape of a right-angled triangle OAC where $OA = 6$ cm and $OC = 8$ cm. The cross-section through E is the triangle BED. The length of the prism is 16 cm. M is the mid-point of AE and N is the mid-point of AC.

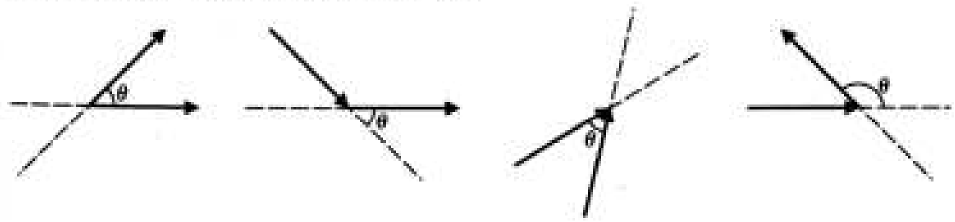
Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OB and OC respectively as shown.

(i) Express each of the vectors $\overrightarrow{MN}$ and $\overrightarrow{MD}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [4]

(ii) Evaluate $\overrightarrow{MN}.\overrightarrow{MD}$ and hence find the value of angle NMD, giving your answer to the nearest degree. [5]

Note:

The phrase "the angle between two vectors" always refers to the angle between the directions of the vectors when these directions are either both towards their point of intersection or both away from their point of intersection. Thus, in each of the following diagrams, θ is the angle between the two vectors.



Properties of the scalar product

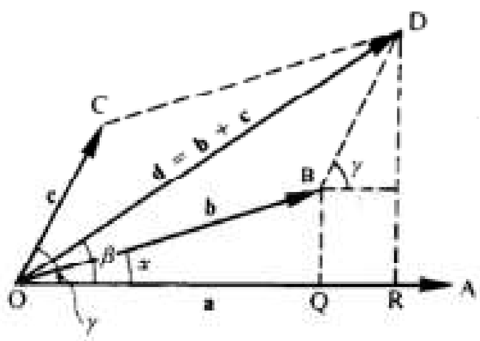
1. From the definition $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$, it follows that two perpendicular vectors will have a scalar product of zero.

2. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}| \cos 0$
   $\quad = |\mathbf{a}||\mathbf{a}|$
   This is usually written $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, or just $\mathbf{a} \cdot \mathbf{a} = a^2$

3. The scalar product is commutative i.e. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
   Proof: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$
   $\quad = |\mathbf{b}||\mathbf{a}| \cos \theta = \mathbf{b} \cdot \mathbf{a}$

4. The scalar product is distributive over addition i.e. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

   Proof: Consider vectors **a**, **b**, **c** and **d** with $\mathbf{d} = \mathbf{b} + \mathbf{c}$
   It is clear from the diagram that $OR = OQ + QR$
   but $OR = |\mathbf{d}| \cos \beta$, $OQ = |\mathbf{b}| \cos \alpha$ and $QR = |\mathbf{c}| \cos \gamma$
   $\therefore \quad |\mathbf{d}| \cos \beta = |\mathbf{b}| \cos \alpha + |\mathbf{c}| \cos \gamma$
   $\therefore \quad |\mathbf{a}||\mathbf{d}| \cos \beta = |\mathbf{a}||\mathbf{b}| \cos \alpha + |\mathbf{a}||\mathbf{c}| \cos \gamma$
   So $\quad \mathbf{a} \cdot \mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
   giving $\quad \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ as required.



5. The properties for multiplication by a scalar λ are that:
   $$\lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda |\mathbf{a}||\mathbf{b}| \cos \theta$$