Forces and Equilibrium

Unit of Force
The unit of force is Newton (N).

Types of Forces
Certain types of forces have particular names. Forces occur in mechanics in various ways. Some of the most common are described below.

Weight *W*
The weight of a body is the force with which the earth attracts the body. It acts at the body's centre of gravity and is always vertically downwards.

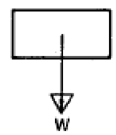


Figure 1

A light body is considered to be weightless.

Push and pull *P*
Pushes and pulls are forces which act on a body at the point(s) where they are applied.

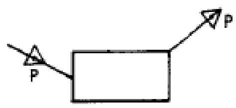


Figure 2

Normal reaction *R*
A normal reaction is a force which acts on a body in contact with a surface. It acts in a direction at right angles to the surfaces in contact.

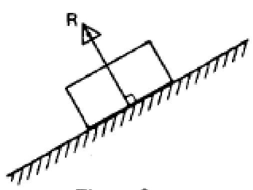


Figure 3

Friction *F*
Friction is a force which acts on a body in contact with a rough surface. It acts tangentially between the two surfaces and in a direction to resist the motion of the body.

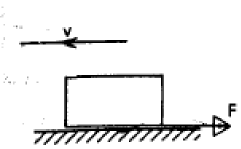


Figure 4

Smooth surfaces are considered to be frictionless. Air resistance is ignored unless stated otherwise.

Tension *T*
The tension in a string is a force which acts on a body to which the string is attached. Tensions can also come from springs, rods, etc.

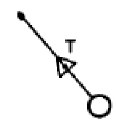


Figure 5

Forces acting on a particle / body

(a) Forces acting on a block on a smooth horizontal plane:
W – weight (vertically down)
R – normal reaction (at right angles to the surfaces in contact)

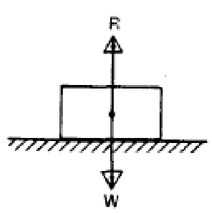


Figure 7

(b) Forces acting on a block at rest on a rough inclined plane:
W – weight
R – normal reaction
F – friction (acting to resist motion)

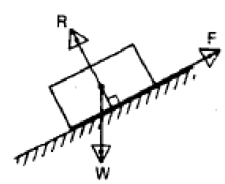


Figure 8

(c) Forces acting on a block being pulled (by a string) along a rough horizontal plane:
W – weight
R – normal reaction
F – friction
T – tension in string (acting away from body)

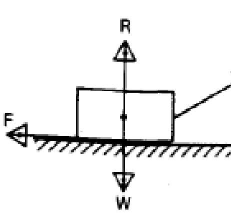
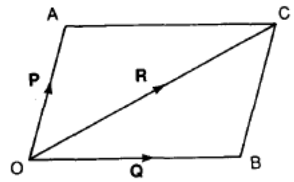


Figure 9

Composition of two forces
Forces as vectors can be combined to produce a *single resultant force*.

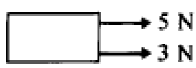


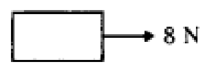
If two forces P and Q are represented by the adjacent sides of a parallelogram, then the resultant force is R.

Resultant of two forces

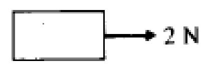
The resultant R of two forces P and Q is that single force which could completely take the place of the two forces. The resultant R must have the same effect as the two forces P and Q.

When only parallel forces are involved, it is easy to find the resultant. For example:

the forces 

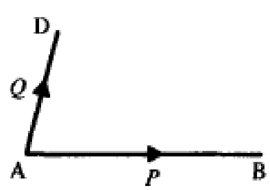
could be replaced by 

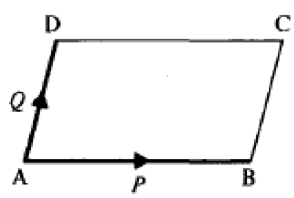
the forces 

could be replaced by 

Parallelogram of forces

Two forces P and Q are represented by the line segments AB and AD.





The parallelogram ABCD is completed by drawing BC and DC.

To find the resultant of the forces P and Q, we have to consider:

$$\overrightarrow{AB} + \overrightarrow{AD}$$

But $\overrightarrow{AD} = \overrightarrow{BC}$, as these are equivalent vectors.

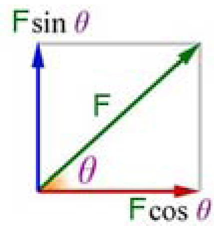
$$\therefore \quad \overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC}$$
$$= \overrightarrow{AC}$$

Hence the resultant of the two forces P and Q, which are represented by the line segments AB and AD, is fully represented by the line segment AC. This is the diagonal AC of the parallelogram ABCD, which is therefore referred to as a parallelogram of forces.

Resolving Forces

A force **F** can be replaced by two vectors that are at right angles to each other, passing through the point of application. Hence if the angle between one component vector and the original vector is θ, then the two components are F_y = **F**sinθ and

F_x = **F**cosθ .



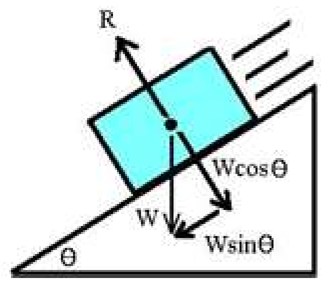
Problems are solved by resolving all the vectors into their horizontal and vertical components. The components are then resolved vertically and then horizontally to obtain two equations. These can be solved as simultaneous equations.

Example 1

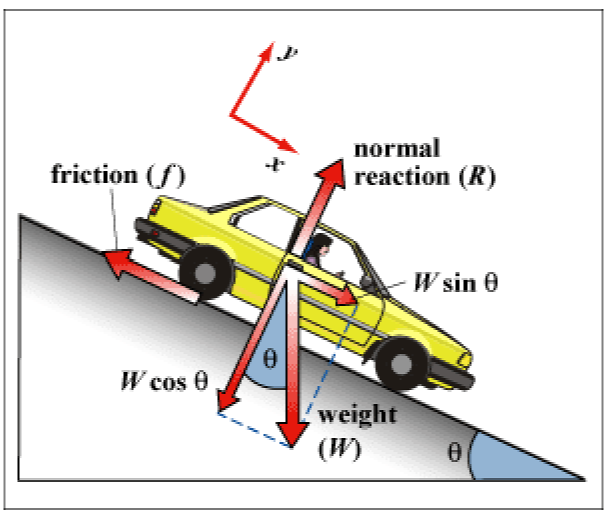
A particle is placed on a inclined plane at an angle θ.

W_y = **Wcos**θ and

W_x = **W**sinθ .



Example 2

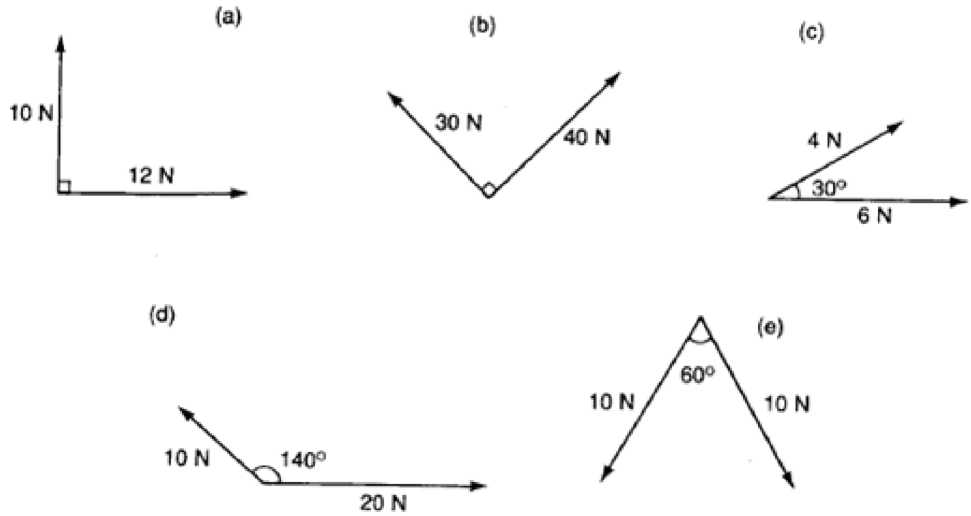


Note:

Resultant Force , $R^2 = (R_x)^2 + (R_y)^2$ and $\tan\theta = R_y/R_x$

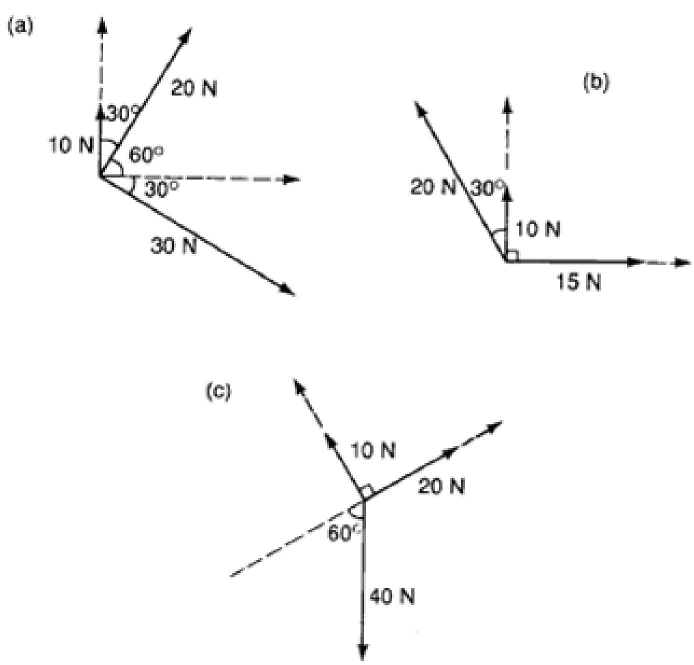
Example: Resultant Force

1. Find the resultant of the following forces, in magnitude and direction.



(a) 15.6 N at 39.8° to 12 N (b) 50 N at 36.9° to 40 N (c) 9.7 N at 11.9° to 6 N
(d) 13.9 N at 27.5° to 20 N (e) 17.3 N bisecting angle

2. Find the sum of the components of the forces acting as shown in the following figures in the direction of the dotted lines and hence find their resultant.

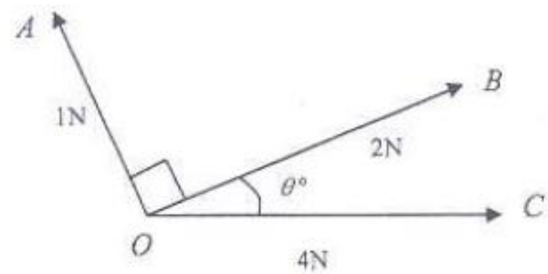


(a) 36, 12.3 N; 38 N at 18.9° to horizontal
(b) 5, 27.3 N; 27.8 N at 79.6° to horizontal
(c) 0, –24.6 N; 24.6 N opposite 10 N force

Exercise 1: Resultant Force - Magnitude and Direction

1. Two forces, P and Q, are such that the sum of their magnitudes is 45 N. The resultant of P and Q is perpendicular to P and has a magnitude of 15 N. Calculate
   i) the magnitude of P and of Q, [4]
   ii) the angle between P and Q. [2]

Question 2 – Trial Exam

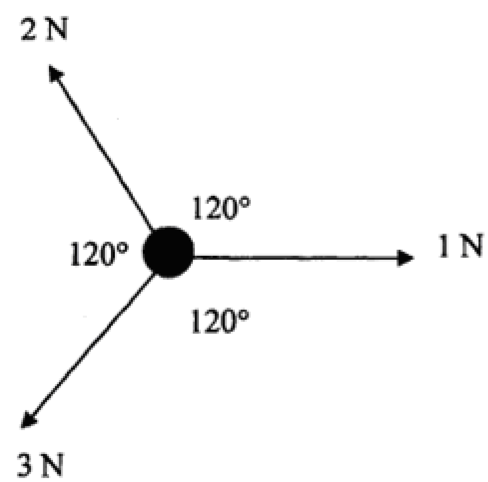


Three coplanar forces of magnitudes 1N, 2N and 4N act at a point O in the directions shown in the diagram. The angle between OA and OB is 90° and the angle between OB and OC is θ°. The resultant of the three forces acts in the direction OB.

(i) Find θ. [2]

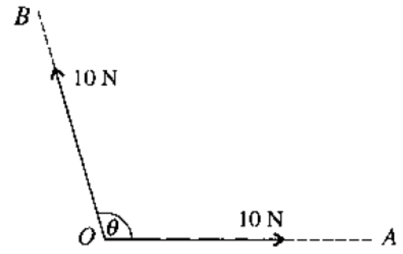
(ii) Calculate the magnitude of the resultant force. [2]

Question 3 – Trial Exam



Coplanar forces of magnitudes 1 N, 2 N, 3 N act on a particle, as shown in the diagram; the angle between the directions of each pair of the forces is 120°. Find the magnitude of the resultant of the three forces. [4]

Question 4 – June 2002

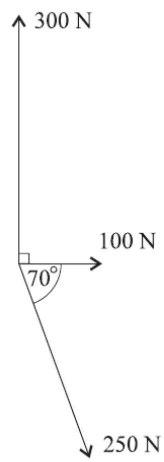


Two forces, each of magnitude 10 N, act at a point O in the directions of OA and OB, as shown in the diagram. The angle between the forces is θ. The resultant of these two forces has magnitude 12 N.

(i) Find θ. [3]

(ii) Find the component of the resultant force in the direction of OA. [2]

Question 5- June 2004



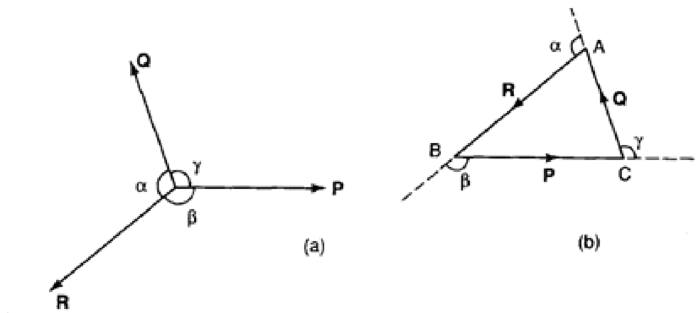
Coplanar forces of magnitudes 250 N, 100 N and 300 N act at a point in the directions shown in the diagram. The resultant of the three forces has magnitude R N, and acts at an angle $\alpha°$ anticlockwise from the force of magnitude 100 N. Find R and α. [6]

<u>A particle in equilibrium</u>

- Forces on a particle are said to be in equilibrium if their **resultant is zero**.
- We saw earlier that the resultant of a set of forces can be found by collecting components in each of two perpendicular directions.
- If the resultant is zero, the collected components in each direction must individually be zero, Horizontal component, $F_x = 0$ and Vertical component, $F_y = 0$.

Lami's Theorem

If three forces are in equilibrium, **Lami's theorem** relates them to the angles between their directions. (It is in fact a version of the sine rule.)



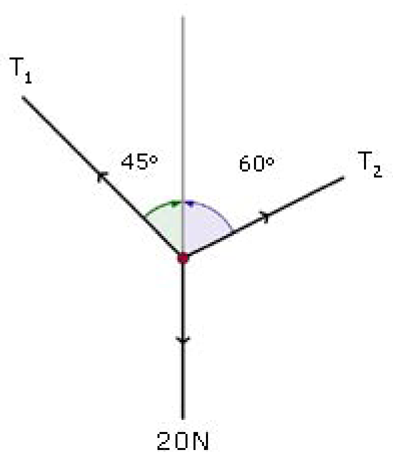
By the sine rule,

$$\frac{BC}{\sin(180^\circ - \alpha)} = \frac{AC}{\sin(180^\circ - \beta)} = \frac{AB}{\sin(180^\circ - \gamma)}$$

i.e. $\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$ (where $P = |\mathbf{P}|$, $Q = |\mathbf{Q}|$ and $R = |\mathbf{R}|$)

Example

A 2 kg mass is suspended by two light inextensible strings inclined at 60° and 45° to the vertical. What are the tensions in the strings?(to 2 d.p.) (assume g=10 ms^{-2})



resolving horizontally $T_1 \sin 60^\circ = T_2 \sin 45^\circ$

$$T_1 = T_2 \frac{\sin 45^\circ}{\sin 60^\circ} = \frac{0.7071}{0.8660} T_2 = 0.8165 T_2$$

$$T_1 = 0.8165 T_2 \qquad \text{(i}$$

resolving vertically

$$T_1 \cos 60^\circ + T_2 \cos 45^\circ = 20 \qquad \text{(ii}$$

substituting for T_1 in (ii

$0.8165 T_2 (0.5) + 0.7071 T_2 = 20$

$0.4083 T + 0.7071 T_2 = 20$

$1.1154 T_2 = 20$

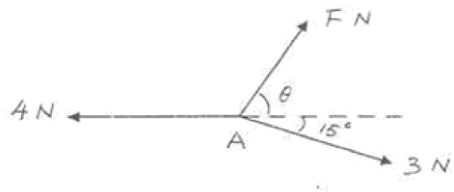
$$T_2 = \frac{20}{1.1154} = 17.9308$$

from (i $T_1 = 0.8165 T_2$

$T_1 = 0.8165 \times 17.9308 = 14.6405$

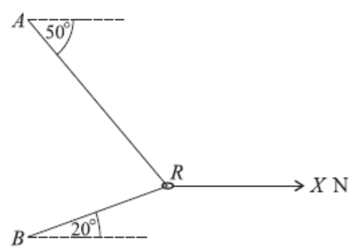
Exercise 2: Forces and Equilibrium

Question 1 – Trial Exam Question



Three forces with magnitudes and directions as shown in the diagram, act in a horizontal plane at the point A. If they are in equilibrium, find the value of F and θ. [6]

Question 2 – Nov 2002

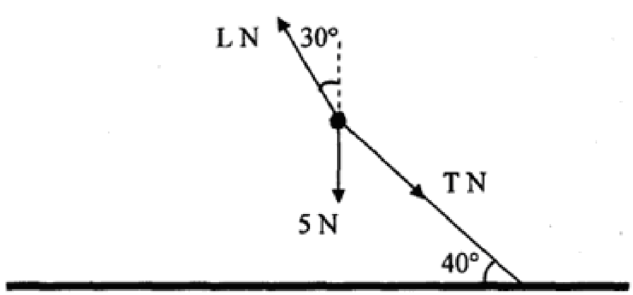


A light inextensible string has its ends attached to two fixed points A and B, with A vertically above B. A smooth ring R, of mass 0.8 kg, is threaded on the string and is pulled by a horizontal force of magnitude X newtons. The sections AR and BR of the string make angles of 50° and 20° respectively with the horizontal, as shown in the diagram. The ring rests in equilibrium with the string taut. Find

(i) the tension in the string, [3]

(ii) the value of X. [3]

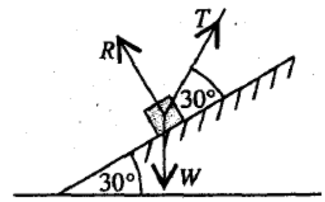
Question 3 – Trial Exam Question



The diagram shows the forces acting on a kite of weight 5 newtons. The kite is stationary and is held in position by a straight inextensible light string inclined at 40° to the horizontal and a lifting force of magnitude L netwons making an angle of 30° with the vertical. The tension in the string is T newtons. Calculate T and L. [6]

Question 4 – Trial Exam

A small block of weight W rests on a smooth plane inclined at 30° to the horizontal and is held in equilibrium by a light string inclined at 30° to the plane. Find, in terms of W, the tension in the string.



$\left[\frac{W}{\sqrt{3}} \right]$

Question 5 – Trial Exam

Two forces X and Y act at a point O and are at right angle to each other. X has magnitude 12 N and acts along a bearing of 090°. Y has magnitude 15 N and acts along a bearing of 000°.

(i) Calculate the magnitude and bearing of the resultant of X and Y. [3]

(ii) A third force E is now applied at O. The three forces X, Y and E are in equilibrium. State the magnitude of E, and give the bearing along which it acts. [2]

Question 6 – Nov 2004

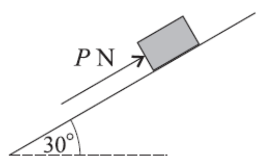


Fig. 1

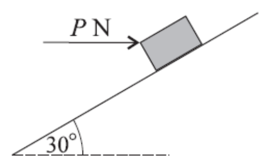


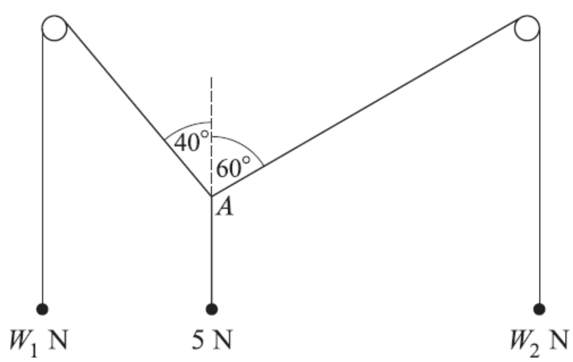
Fig. 2

A small block of weight 18 N is held at rest on a smooth plane inclined at 30° to the horizontal, by a force of magnitude P N. Find

(i) the value of P when the force is parallel to the plane, as in Fig. 1, [2]

(ii) the value of P when the force is horizontal, as in Fig. 2. [3]

Question 7 – Nov 2005



Each of three light strings has a particle attached to one of its ends. The other ends of the strings are tied together at a point A. The strings are in equilibrium with two of them passing over fixed smooth horizontal pegs, and with the particles hanging freely. The weights of the particles, and the angles between the sloping parts of the strings and the vertical, are as shown in the diagram. Find the values of W_1 and W_2. [6]

Friction

- When the surfaces of two objects are in rough contact, and have a tendency to move relative to each other, equal and opposite frictional forces act, one on each of the potential movement.
- Until it reaches its limiting value, the magnitude of the frictional force F is just sufficient to prevent motion.
- When the limiting value is reached, $F = \mu R$ where R is the normal reaction between the surfaces and μ is the coefficient of friction for those two surfaces.
- For all rough contacts $0 < F \le \mu R$.
- If a contact is smooth $\mu = 0$.
- The frictional force cannot exceed a certain magnitude, called the **limiting friction.** If the object is at rest and equilibrium is possible with a frictional force less than this limiting friction, the object will remain in equilibrium.

If the object is at rest and the forces are in equilibrium with the limiting friction, the object is said to be in **limiting equilibrium**, and to be 'on the point of moving'.

Example 1

A particle of mass 1 kg rests on a horizontal floor. The coefficient of friction between the particle and the floor is $\frac{1}{2}$. What force is required just to make the particle move when

(a) pulling horizontally,

(b) pulling at an angle of 30° to the horizontal?

Example 2

A particle of mass 1 kg is placed on a rough plane inclined at an angle 30° to the horizontal. The coefficient of friction is $\frac{2}{5}$. Find the least force parallel to the plane that is required

(a) to hold the particle at rest,

(b) to make the particle slide up the plane.

Exercise 3: Friction

Question 1 – June 2001

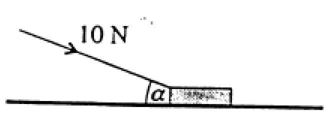


Fig. 1

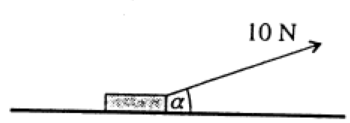


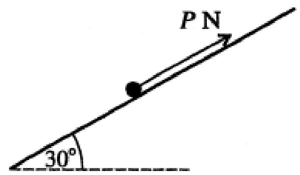
Fig. 2

A block of mass 1 kg is at rest on a horizontal floor. The coefficient of friction between the block and the floor is μ.

(i) When a force of magnitude 10 N is applied to the block, acting *downwards* at an angle α to the horizontal, as shown in Fig. 1, the block remains at rest. Given that $\tan\alpha = \frac{7}{24}$, show that $\mu \geqslant \frac{3}{4}$. [4]

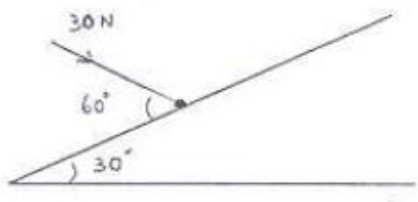
(ii) When a force of magnitude 10 N is applied to the block, acting *upwards* at the same angle α to the horizontal, as shown in Fig. 2, the block slides along the floor. Show that $\mu < \frac{4}{3}$. [4]

Question 2 - Nov 2001



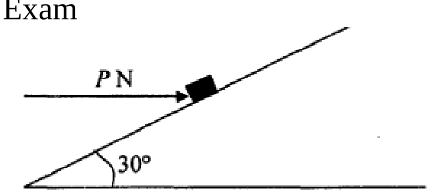
The diagram shows a particle of mass 0.5 kg resting on a rough plane inclined at 30° to the horizontal. The coefficient of friction between the particle and the plane is 0.4. A force of magnitude *P* N, acting directly up the plane, is just sufficient to prevent the particle sliding down the plane. Find the value of *P*. **[6]**

Question 3 – Trial Exam



A rough plane is inclined at 30° to the horizontal. A particle of mass 1 kg is held on the plane by a force of 30 N. This force makes an angle of 60° with a line of greatest slope. Given that the particle is on the point of moving up the plane, calculate the value of μ, the coefficient of friction between the particle and the plane. If the 30 N force is removed, determine whether or not the particle will slide down the plane. [8]

Question 4 - Trial Exam



A box of mass 6 kg lies on a rough plane inclined at 30° to the horizontal. The box is held in equilibrium by means of a horizontal force of magnitude *P* newtons, as shown in the diagram. The coefficient of friction between the box and plane is 0.4. The box is modeled as a particle.

Given that the box is in limiting equilibrium and on the point of moving up the plane, find,

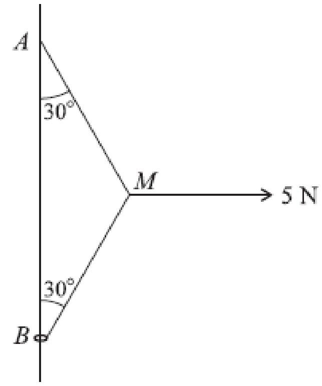
(a) the normal reaction exerted on the box by the plane. [3]

(b) the value of *P*. [3]

The horizontal force is now removed.

(c) Determine whether the box will still remain in equilibrium. Give a reason to you answer. [2]

Question 5 – Nov 2003



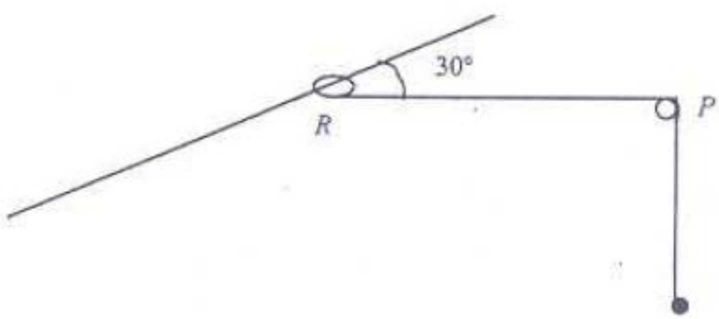
One end of a light inextensible string is attached to a fixed point A of a fixed vertical wire. The other end of the string is attached to a small ring B, of mass 0.2 kg, through which the wire passes. A horizontal force of magnitude 5 N is applied to the mid-point M of the string. The system is in equilibrium with the string taut, with B below A, and with angles ABM and BAM equal to 30° (see diagram).

(i) Show that the tension in BM is 5 N. [3]

(ii) The ring is on the point of sliding up the wire. Find the coefficient of friction between the ring and the wire. [5]

(iii) A particle of mass m kg is attached to the ring. The ring is now on the point of sliding down the wire. Given that the coefficient of friction between the ring and the wire is unchanged, find the value of m. [2]

Question 6 – Trial Exam



A fixed straight wire is inclined at an angle of 30° to the horizontal. A small ring R, of mass 0.5 kg, is threaded onto the wire. One end of a light inextensible string is attached to R, and the other end is attached to a particle of mass 0.2 kg. The string passes over a fixed smooth peg P, so that the string and the wire are in the same vertical plane, with the particle hanging freely. The system is in equilibrium with RP horizontal. Given that the system is in limiting equilibrium, find the coefficient of friction between the wire and the ring. [6]

Question 7- Nov 2002

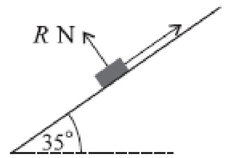


Fig. 1

A force, whose direction is upwards parallel to a line of greatest slope of a plane inclined at 35° to the horizontal, acts on a box of mass 15 kg which is at rest on the plane. The normal component of the contact force on the box has magnitude R newtons (see Fig. 1).

(i) Show that $R = 123$, correct to 3 significant figures. [1]

about to move down

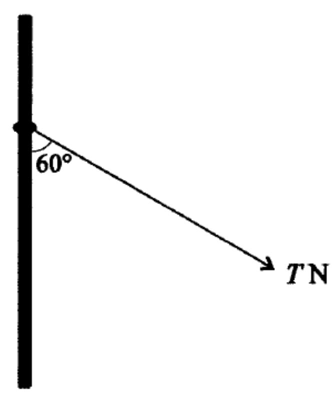
about to move up

Fig. 2

When the force parallel to the plane acting on the box has magnitude X newtons the box is about to move *down* the plane, and when this force has magnitude $5X$ newtons the box is about to move *up* the plane (see Fig. 2).

(ii) Find the value of X and the coefficient of friction between the box and the plane. [7]

Question 8 – Nov 2007



A ring of mass 4 kg is threaded on a fixed rough vertical rod. A light string is attached to the ring, and is pulled with a force of magnitude T N acting at an angle of 60° to the downward vertical (see diagram). The ring is in equilibrium.

(i) The normal and frictional components of the contact force exerted on the ring by the rod are R N and F N respectively. Find R and F in terms of T. [4]

(ii) The coefficient of friction between the rod and the ring is 0.7. Find the value of T for which the ring is about to slip. [3]

Exercise 1

1. i) 20, 25 ii) 143.1°
2. i) 14.5° ii) 5.87
3. $\sqrt{3}$
4. i) 106.3° ii) 7.2
5. 197, 19.3°

Exercise 2

1. 35.2°, 1.35
2. i) 18.9 ii)29.9
3. 11.2, 7.3
4. $\frac{w}{\sqrt{3}}$
5. i) 19.2, 038.7° ii) 19.2, 219°
6. i) 9 ii) 10.4
7. 4.40, 3.26

Exercise 3

2. 0.768
3. 0.289, will slide down
4.
5. ii) 0.932 iii) 0.466
6. 0.144
7. ii) 28.7, 0.467
8. i) Tsin 60°, 40 + T cos 60° ii) 337