Discrete random variables

S1/4/1: Probability distribution table
S1/4/2: Expectation and variance of X
S1/4/3: Miscellaneous questions on DRV
S1/4/4: Binomial distribution
S1/4/5: Binomial distribution
S1/4/6: Past years questions and solutions

S1/4/1:
Probability distribution table

Learning Outcome

Students should be able to:

- Define discrete random variable and describe it's characteristics
- Construct a probability distribution table relating to a given situation involving a discrete random variable, X.

S1/4/2 and S1/4/3:
Expectation and variance of X
Miscellaneous questions on DRV

Learning Outcome

Students should be able to:

- Calculate the expectation of X, E(X) and the variance of X, Var(X)
- Solve problems that involves the expectation of X, E(X), and the variance of X, Var(X).

S1/4/5:
Binomial distribution

Learning Outcome

Students should be able to:

- Use formulae for the expectation and variance of the binomial distribution

Discrete & Random Variables

Discrete Random Variable:

A quantitative random variable that can assume a countable number of values

Note: **Usually associated with counting**

Probability Distribution & Function

Probability Distribution:

A distribution of the probabilities associated with each of the values of a random variable.

Probability Function:

A rule that assigns probabilities to the values of the random variable

Reminder!

Every probability function must satisfy the two basic properties of probability:

1. The probability assigned to each value of the random variable must be between 0 and 1, inclusive:

$$0 \leq P(x) \leq 1$$

2. The sum of the probabilities assigned to all the values of the random variable must equal 1:

$$\sum_{\text{all } x} P(x) = 1$$

Example 1:

The number of people staying in a randomly selected room at a local hotel is a random variable ranging in value from 0 to 4. The probability distribution is known and is given in various forms below:

x	0	1	2	3	4
$P(x)$	2/15	4/15	d	3/15	1/15

Notes:

This chart implies the only values *x* takes on are 0, 1, 2, 3, and 4

$$\sum_{\text{all } x} P(W = w) = 1$$

Find *d*.

$$2/15 + 4/15 + d + 3/15 + 1/15 = 1$$

$$d = 5/15$$

Example 1:

The number of people staying in a randomly selected room at a local hotel is a random variable ranging in value from 0 to 4. The probability distribution is known and is given in various forms below:

x	0	1	2	3	4
$P(x)$	2/15	4/15	d	3/15	1/15

Find the probability that four pupils staying in a room.

$$P(\text{the random variable } x \text{ equals } 4) = P(4) = \frac{1}{15}$$

Example 1:

The number of people staying in a randomly selected room at a local hotel is a random variable ranging in value from 0 to 4. The probability distribution is known and is given in various forms below:

x	0	1	2	3	4
$P(x)$	2/15	4/15	d	3/15	1/15

Find P(0 < x ≤ 2).

$$P(0 < x \le 2) \quad = P(1) + P(2) \quad = 4/15 + 5/15 = 9/15$$

Example 1:

The number of people staying in a randomly selected room at a local hotel is a random variable ranging in value from 0 to 4. The probability distribution is known and is given in various forms below:

x	0	1	2	3	4
$P(x)$	2/15	4/15	d	3/15	1/15

Find the mode.

The value of x with the highest probability is 2, so the mode is 2.

Example 2

Construct the probability distribution of sixes when three fair dice are thrown.

X = the number of sixes

$$P(X=0)=\left(\frac{5}{6}\right)^3=\frac{125}{216}$$

$$P(X=1)=\left(\frac{5}{6}\right)^2\times\left(\frac{1}{6}\right)\times\frac{3!}{2!}=\frac{75}{216}$$

$$P(X=2)=3\times\left(\frac{5}{6}\right)\times\left(\frac{1}{6}\right)^2=\frac{15}{216}$$

$$P(X=3)=\left(\frac{1}{6}\right)^3=\frac{1}{216}$$

x	0	1	2	3
$P(X = x)$	125/216	75/216	15/216	1/216

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<u>Exercise 6A (Page 104)</u>

Q3, Q6, Q7

<u>Exercise 6B (Page 106)</u>

Q7, Q8

Mean and Variance of a Discrete Probability Distribution

Describe the center and spread of a population

μ, σ, σ^2 : population parameters

μ is the *mean of the population*

σ^2 is the *variance of the population*

σ is the *standard deviation of the population*

Theoretical approach

The expectation of X(expected value or mean),

$$E(X) = \sum_{\text{all } x} x\, P(X = x)$$

$$E(X) = \sum x_i \; p_i \qquad\qquad i = 1,2,\ldots, n$$

$$\mu = E(X)$$

Discrete Random Variables

Variance of a Discrete Random Variable:

$$\sigma^2 = \sum\left[x^2 P(x)\right] - \left\{\sum xP(x)\right\}^2$$

$$Var(X) = E(X^2) - \mu^2$$

$$Var(X) = E(X^2) - E^2(X)$$

Standard Deviation of a Discrete Random Variable:

The positive square root of the variance:

$$\sigma = \sqrt{\sigma^2}$$

Example 3

The number of standby passengers who get seats on a daily commuter flight from Boston to New York is a random variable, x, with probability distribution given below (in an extensions table). Find the mean, variance, and standard deviation:

x	0	1	2	3	4	5
$p(x)$	0.30	0.25	0.20	0.15	0.05	0.05

Example 3

	x	$P(x)$	$xP(x)$	x^2	$x^2P(x)$
	0	0.30	0.00	0	0.00
	1	0.25	0.25	1	0.25
	2	0.20	0.40	4	0.80
	3	0.15	0.45	9	1.35
	4	0.05	0.20	16	0.80
	5	0.05	0.25	25	1.25
Totals		1.00	1.55		4.45
		$\sum P(x)$ (check)	$E(X) = \sum[xP(x)]$		$E(X^2) = \sum[x^2P(x)]$

$$\mu = E(X) = \sum\left[x\,P(x)\right] = 1.55$$

Example 3

x	$P(x)$	$xP(x)$	x^2	$x^2 P(x)$
0	0.30	0.00	0	0.00
1	0.25	0.25	1	0.25
2	0.20	0.40	4	0.80
3	0.15	0.45	9	1.35
4	0.05	0.20	16	0.80
5	0.05	0.25	25	1.25
Totals	1.00	1.55		4.45
	$\sum P(x)$ (check)	$E(X) = \sum[xP(x)]$		$E(X^2) = \sum[x^2 P(x)]$

$$\sigma^2 = E(X^2) - E^2(X) = \sum\left[x^2\ P(x)\right] - \left\{\sum\left[x\,P(x)\right]\right\}^2$$

$$= 4.45 - 1.55^2 \qquad = 2.0475 \qquad \sigma = \sqrt{2.0745} \approx 1.43$$

Example 4

The probability distribution for a random variable x is given by the probability function:

$$P(x) = \frac{8 - x}{15} \quad \text{for} \quad x = 3,\ 4,\ 5,\ 6,\ 7$$

Find the mean, variance, and standard deviation

Example 4

Find the mean, variance, and standard deviation

$$P(x) = \frac{8-x}{15} \quad \text{for} \quad x = 3,\ 4,\ 5,\ 6,\ 7$$

$$P(3) = \frac{8-3}{15} = \frac{5}{15} \qquad P(4) = \frac{8-4}{15} = \frac{4}{15} \qquad P(5) = \frac{8-5}{15} = \frac{3}{15}$$

$$P(6) = \frac{8-6}{15} = \frac{2}{15} \qquad P(7) = \frac{8-7}{15} = \frac{1}{15}$$

Finding the Population Parameters

Use an extensions table to find the population parameters:

	x	$P(x)$	$xP(x)$	x^2	$x^2P(x)$
	3	5/15	15/15	9	45/15
	4	4/15	16/15	16	64/15
	5	3/15	15/15	25	75/15
	6	2/15	12/15	36	72/15
	7	1/15	7/15	49	49/15
Totals		15/15	65/15		305/15
			$E(X)=\sum[xP(x)]$		$E(X^2)=\sum[x^2P(x)]$

$$\mu = E(X) = \sum\left[x\,P(x)\right] = \frac{65}{15} \approx 4.33$$

$$\sigma^2 = E(X^2) - E^2(X) = \sum\left[x^2\,P(x)\right] - \left\{\sum\left[x\,P(x)\right]\right\}^2 = \frac{305}{15} - \left(\frac{65}{15}\right)^2 \approx 1.56$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{1.56} \approx 1.25$$

Example 5

In each round of a particular game, Albert draws 3 marbles without replacement from a box containing 2 red marbles and 8 blue marbles. The number of red marbles drawn in each round is denoted R. Find the probability distribution for R and show that the mean of R is 3/5. [5]

If Albert scores 4 points for each red marble drawn but loses 1 point for every blue marble drawn, calculate the number of points Albert expects to have at the end of 5 rounds. [5]

Example 5

In each round of a particular game, Albert draws 3 marbles without replacement from a box containing 2 red marbles and 8 blue marbles. The number of red marbles drawn in each round is denoted R. Find the probability distribution for R and show that the mean of R is 3/5. [5]

2R , 8B R = number of red marbles drawn in 3 draws.

$$P(\,P(R=2)=\left(\frac{2}{10}\right)\left(\frac{1}{9}\right)\left(\frac{8}{8}\right)+\left(\frac{2}{10}\right)\left(\frac{8}{9}\right)\left(\frac{1}{8}\right)+\left(\frac{8}{10}\right)\left(\frac{2}{9}\right)\left(\frac{1}{8}\right)=\frac{1}{15}$$

$$P(R=1)=\left(\frac{2}{10}\right)\left(\frac{8}{9}\right)\left(\frac{7}{8}\right)+\left(\frac{8}{10}\right)\left(\frac{2}{9}\right)\left(\frac{7}{8}\right)+\left(\frac{8}{10}\right)\left(\frac{7}{9}\right)\left(\frac{2}{8}\right)=\frac{7}{15}$$

Example 5

In each round of a particular game, Albert draws 3 marbles without replacement from a box containing 2 red marbles and 8 blue marbles. The number of red marbles drawn in each round is denoted R. Find the probability distribution for R and show that the mean of R is 3/5. [5]

r	0	1	2
$P(R = r)$	7/15	7/15	1/15

$$\mu = E(X) = \sum\left[x\,P(x)\right] = (0)\left(\frac{7}{15}\right) + (1)\left(\frac{7}{15}\right) + (2)\left(\frac{1}{15}\right) = \frac{3}{5}$$

Example 5

If Albert scores 4 points for each red marble drawn but loses 1 point for every blue marble drawn, calculate the number of points Albert expects to have at the end of 5 rounds. [5]

W = number of points scored per game

w	-3	2	7
$P(R = r)$	7/15	7/15	1/15

$$E(W) = (-3)\left(\frac{7}{15}\right) + (2)\left(\frac{7}{15}\right) + (7)\left(\frac{1}{15}\right) = 0$$

r	0	1	2
$P(R = r)$	7/15	7/15	1/15

Expected points in 5 rounds = 5 x 0 = 0

Expectation and Variance of the Binomial Distribution

If $X \sim B(n, p)$

$$E(X) = \mu = np$$

$$Var(X) = \sigma^2 = npq \quad \text{where } q = 1 - p$$

Example 6

Nails are sold in packets of 100. Occasionally a nail is faulty. The number of faulty nails in a randomly chosen packets is denoted by *X*. Assuming that faulty nails occur independently and at random, calculate the mean and standard deviation of *X*, given that the probability of any nail being faulty is 0.04.

Solutions: $n = 100 \text{ and } p = 0.04$

$$E(X) = \mu = np = 100 \times 0.04 = 4$$

$$Var(X) = \sigma^2 = np(1-p) = 100 \times 0.04 \times 0.96 = 3.84$$

$$\sigma = \sqrt{3.84} = 1.96$$

Example 7

Given that Y is B(n , p), E(Y) = 24 and Var(Y) = 8, find the values of n and p.

Solutions:

$$E(Y) = np = 24 \qquad (1)$$

$$Var(Y) = npq = np(1-p) = 8 \qquad (2)$$

1 into 2

$$24(1-p) = 8$$

$$(1-p) = \frac{1}{3}$$

$$\Rightarrow p = \frac{2}{3}$$

$$np = 24$$

$$n\left(\frac{2}{3}\right) = 24 \Rightarrow n = 36$$

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<u>Exercise 8A (Page 127)</u>

Q4, Q8, Q9, Q10

<u>Exercise 8B (Page 130)</u>

Q2, Q5

S1/4/4:
Binomial distribution

Learning Outcome

Students should be able to:

- Use formulae for probabilities for the binomial distribution

The Binomial Probability Distribution

Based on a series of repeated trials whose outcomes can be classified in one of two categories: ***success*** or ***failure***

Distribution based on a ***binomial probability experiment***

Binomial Probability Experiment

Binomial Probability Experiment:

An experiment that is made up of repeated trials that possess the following properties:

1. There are n repeated independent trials
2. Each ***trial*** has two possible outcomes (success, failure)
3. P(success) $= p$, P(failure) $= q$, and $p + q = 1$
4. The ***binomial random variable*** x is the count of the number of successful trials that occur; $x = 0, 1, 2, 3, \ldots, n$

Binomial Probability Function

For a binomial experiment, let p represent the probability of a "success" and q represent the probability of a "failure" on a single trial; $x \sim B(n, p)$

Then $P(x)$, the probability that there will be exactly x successes on n trials is:

$$P(x) = \binom{n}{x}(p^x)(q^{n-x}), \quad \text{for } x = 0,\ 1,\ 2,\ \ldots\ ,\ \text{or } n$$

Both x and p must be associated with "success"

Example 8

According to a recent study, 65% of all homes in a certain county have high levels of radon gas leaking into their basements. Four homes are selected at random and tested for radon. The random variable x is the number of homes with high levels of radon (out of the four).

Find the probability distribution.

Example 8

x = number of homes with high levels of radon,
$x = 0, 1, 2, 3, 4$

$x \sim B(4, 0.65)$ $\quad$ $x = 0, 1, 2, 3, 4$

$$P(x) = \binom{4}{x}(0.65)^x(0.35)^{4-x}, \text{ for } x = 0, 1, 2, 3, 4$$

$$P(0) = \binom{4}{0}(0.65)^0(0.35)^4 = 0.0150$$

$$P(1) = \binom{4}{1}(0.65)^1(0.35)^3 = 0.1115$$

$$P(2) = \binom{4}{2}(0.65)^2(0.35)^2 = 0.3105$$

$$P(3) = \binom{4}{3}(0.65)^3(0.35)^1 = 0.3845$$

$$P(4) = \binom{4}{4}(0.65)^4(0.35)^0 = 0.1785$$

Example 9

In a certain automobile dealership, 70% of all customers purchase an extended warranty with their new car.

For 15 customers selected at random:

1) Find the probability that exactly 12 will purchase an extended warranty

$$P(x) = \binom{15}{x}(0.7)^{x}(0.3)^{15-x}, \qquad \text{for } x = 0, 1, 2, \ldots, 15$$

$$P(12) = \binom{15}{12}(0.7)^{12}(0.3)^{3} \quad = 0.1700$$

Example 9

In a certain automobile dealership, 70% of all customers purchase an extended warranty with their new car.

For 15 customers selected at random:

2) Find the probability at least 13 will purchase an extended warranty

$$P(x) = \binom{15}{x}(0.7)^x(0.3)^{15-x}, \qquad \text{for } x = 0, 1, 2, \dots, 15$$

$$P(x \geq 13) = P(13) + P(14) + P(15)$$

$$= P(13) + P(14) + P(15)$$

$$= \binom{15}{13}(0.7)^{13}(0.3)^2 + \binom{15}{14}(0.7)^{14}(0.3)^1 + \binom{15}{15}(0.7)^{15}(0.3)^0$$

$$= 0.1268$$

Example 9

In a certain automobile dealership, 70% of all customers purchase an extended warranty with their new car.

For 15 customers selected at random:

3) Find the probability at most 13 will purchase an extended warranty

$$P(x) = \binom{15}{x}(0.7)^{x}(0.3)^{15-x}, \qquad \text{for } x = 0, 1, 2, \ldots, 15$$

$$P(x \le 13) = P(0) + P(1) + \ldots + P(13)$$

$$= 1 - \left[P(14) + P(15)\right]$$

$$= 1 - \left[\binom{15}{14}(0.7)^{14}(0.3)^{1} + \binom{15}{15}(0.7)^{15}(0.3)^{0}\right]$$

$$= 0.9648$$

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<u>Exercise 7A (Page 114)</u>

Q3, Q5

<u>Exercise 7B (Page 118)</u>

Q4

<u>Miscellaneous exercise 7 (Page 119)</u>

Q1, Q7