Differentiation

P1/8/1: Introduction to differentiation
P1/8/2: Sum and difference of functions, Chain Rule
P1/8/3: Applications of differentiation to gradients, tangents and normals
P1/8/4: Increasing and decreasing functions, Stationary points and curve sketching
P1/8/5: Rates of change
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P1/8/1:

Introduction to differentiation

P1/8/2:

Sum and differences of functions, chain Rule.

P1/8/3:

Applications of differentiation to gradients, tangents and normals

Learning Outcome

Students should be able to:

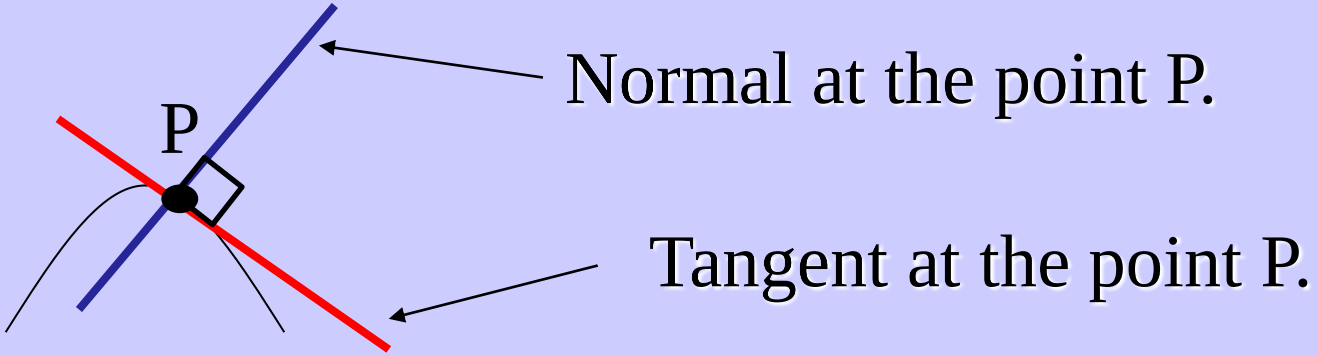
- Differentiate a constant and functions of the form $y = ax^{n}$ where "a" is a constant and n ε **R**.
- Use the rules to find first derivatives.
- Find gradients, equations of tangent and equations of normal at a given point.

The gradient of a curve
First Principles

- The gradient of a straight line is the same at all points on the **line**.

But the gradient of a curve will depend upon where we are on the **curve**.

- The gradient at point P on a curve is defined as the gradient of the *tangent* drawn to the curve at the point P.



The Gradient of a Curve

Gradient of a curve $y = f(x)$ at point $P(x_1, y_1)$

$= $ Gradient of the tangent to a curve at point $P(x_1, y_1)$

$= f'(x_1)$

The Derivative of a Constant

Constant function rule

Theorem 1

Let $y = f(x) = C$ be a constant function, then

$$f'(x) = \frac{dy}{dx} = \frac{d(C)}{dx} = 0$$

Power Rule

Theorem 2

Let $y = f(x) = x^n$ be a power function, then

$$f'(x) = \frac{dy}{dx} = \frac{d(x^n)}{dx} = nx^{n-1}$$

Constant Multiple Property

Constant times a function rule.

Theorem 3

Let $y = f(x) = k \,.\, u(x)$ be a constant k times a differential function $u(x)$. Then

$$f'(x) = \frac{dy}{dx} = \frac{d(k.u(x))}{dx} = k.u'(x)$$

Sum and difference properties

Theorem 4

If $y = f(x) = u(x) \pm v(x)$, then

$$f'(x) = \frac{dy}{dx} = u'(x) \pm v'(x)$$

Example 1:

Differentiate the following functions:

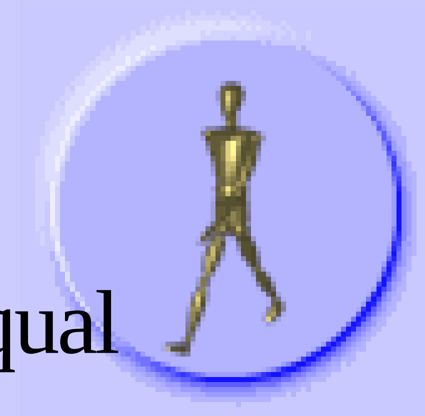
$$(a) \quad y = 10\pi$$

$$(b) \quad y = 3x^5$$

$$(c) \quad y = \sqrt[3]{x}$$

$$(d) \quad y = 3x^2 - 6x + \frac{2}{x^2}$$

Example 2:



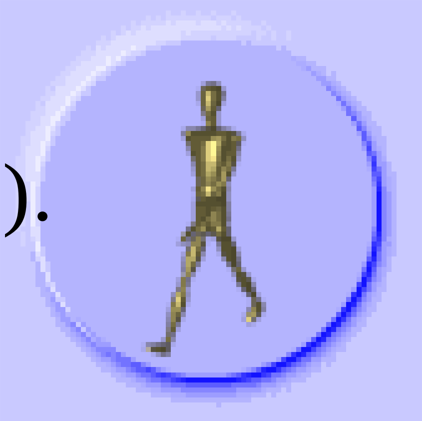
For the following function $f(x)$ find the value(s) of x such that $f'(x)$ is equal to the given number.

$$f(x) = 3x - 3x^2 + x^3 \qquad 108$$

Example 3:

Differentiate the following function $f(x)$.

Give our answer $f'(x)$ in a similar form, without negative or fractional indices.



$$f(x) = \frac{1+x}{\sqrt[4]{x}}$$

Practice Exercise

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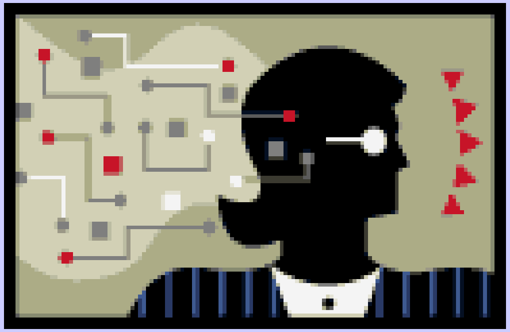
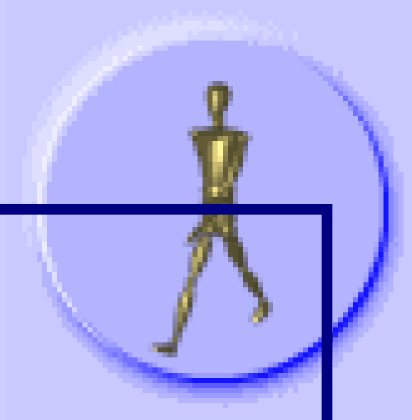
Exercise 6D (Page 86)

Q3(a), Q4(a), Q14(c)(d)(h)(i)(j)



Chain Rule: Power Rule



If $u(x)$ is a differential function, n is any real number, and

$$y = f(x) = [u(x)]^n$$

then

$$y' = f'(x) = n[u(x)]^{n-1}u'(x)$$

or

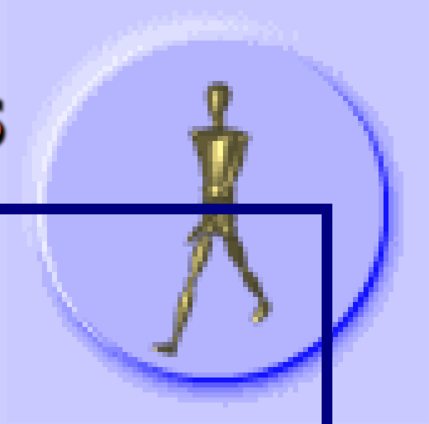
$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$$

Example 4:

Find $\dfrac{dy}{dx}$ for the following function.

$$y = \frac{4}{(4x-1)^4}$$

Product and quotient rules



$$y = f(x)g(x)$$

then

$$y' = f'(x)g(x) + f(x)g'(x)$$

$$y = \frac{f(x)}{g(x)}$$

then

$$y' = \frac{g(x)f'(x) - g'(x)f(x)}{\left[g(x)\right]^2}$$

Example 5:

Find $\dfrac{dy}{dx}$.

(i) $y = (7x^2+6)^3(x-6)^{-5}$

(ii) $y = \dfrac{x+3}{(5x+1)^2}$.

Practice Exercise

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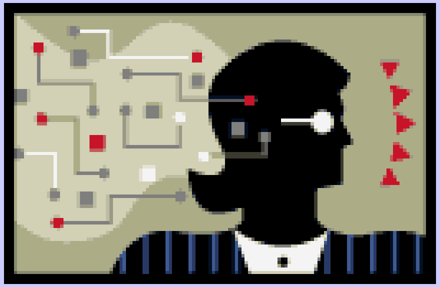
Exercise 6D (Page 86)

Q4, Q14

Exercise 12B (Page 176)

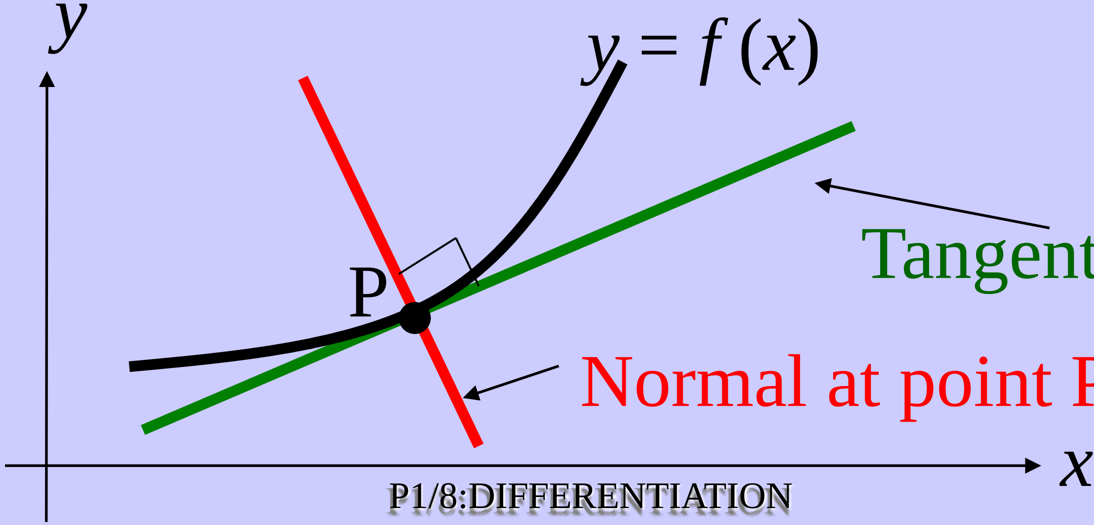
Q2, Q3, Q4, Q5



Tangents and Normals

Suppose that some point P lies on a curve $y = f(x)$

The line passing through P, perpendicular to the tangent to the curve at P, is said to be the ***normal*** to the curve at P.



The relationship of the gradient of tangent and the gradient of the normal

If m_T = gradient of the tangent = $f'(x)$

If m_N = gradient of the normal

Normal is *perpendicular* to **tangent**

Then $$\boxed{m_T \times m_N = -1}$$

$$m_N = -\frac{1}{m_T}$$

Equation of the tangent to a curve $y = f(x)$ at point $P(x_1, y_1)$

$$m_T = \text{gradient of the tangent} = f'(x)$$

$$\boxed{y - y_1 = m_T(x - x_1)}$$

Equation of the normal to a curve $y = f(x)$ *at point* $P(x_1, y_1)$

$$m_T = \text{gradient of the tangent} = f'(x)$$

$$m_N = \text{gradient of the normal} = -\frac{1}{m_T}$$

$$\boxed{y - y_1 = m_N(x - x_1)}$$

Example 6:

Find the equation of the tangent to the graph $y = x^2 - 2$ at the point whose y-coordinate is -2.

Example 7:

Find the equation of the normal to the curve $y=(x-1)^2$ which is parallel to the y-axis.

Example 8:

Find the equation of the normal to the curve $y = 2x^2 + 3x + 4$ which is perpendicular to the line $y = 7x - 5.$

Practice Exercise

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Exercise 6B (Page 79)

Q6, Q7, Q8

Exercise 6C (Page 82)

Q9, Q10, Q11, Q12

Exercise 6D (Page 86)

Q6, Q11, Q13, Q16

P1/8/4:

Increasing and decreasing functions, Stationary points and curve sketching

Learning Outcome

Students should be able to:

- Distinguish between increasing and decreasing functions.
- Locate the stationary points
- ** Higher Derivatives

Increasing and Decreasing Functions

Theorem 1: Increasing and decreasing functions.

For the interval (a, b)	
$f\,'(x)$	$f\,(x)$
+	**Increasing**
-	**Decreasing**

Example 9:

Find the intervals where $y = x^2 + 6x + 7$ is increasing or decreasing.

Maximum and Minimum points

Theorem:

If (q, $f(q)$) is a minimum or maximum point of the graph of $y = f(x)$, then either $f'(q) = 0$ or $f'(q)$ does not exist (is not defined).

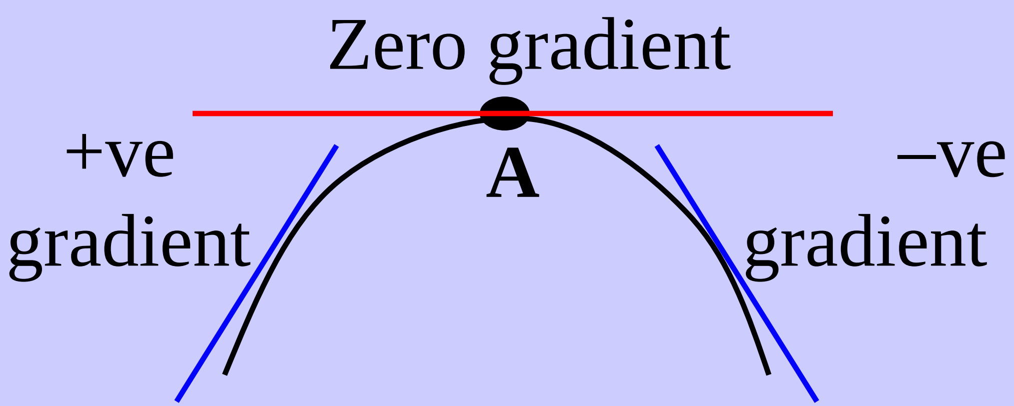
A point of a graph where the gradient is 0 is called a **stationary point**.

A stationary point may be a minimum or maximum point, but may be neither.

Maximum and Minimum points

When the graph of a function changes from increasing to decreasing,

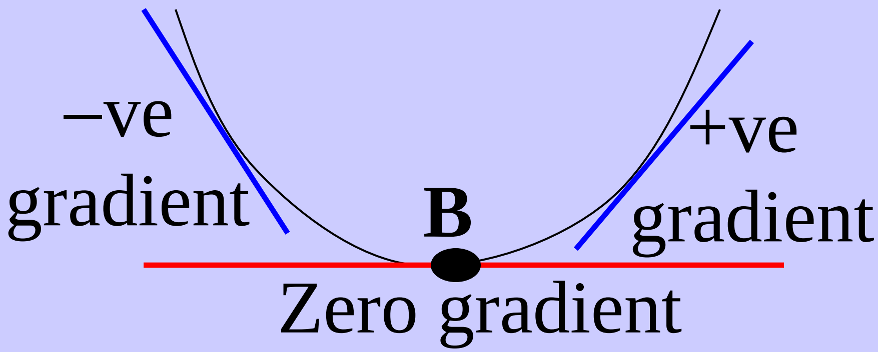
a high point, or a maximum occurs.



f '(*x*)	*f* (*x*)
+	**Increasing**
-	**Decreasing**

When the graph of a function changes from decreasing to increasing,

a low point, of a minimum occurs.



$f'(x)$	$f(x)$
+	**Increasing**
-	**Decreasing**

Minimum and maximum points are sometimes called **turning points**.

Procedures to find the minimum and maximum points on the graph of $y = f(x)$

Step 1: Decide the domain in which you are interested.

Step 2: Find the gradient function $f'(x)$.

Step 3: Find the values of x in the domain for which $f'(x) = 0$ or undefined.

Step 4: Consider the sign of $f'(x)$ on either side of these points.

Step 5: If these signs are – and +, the graph has minimum point. If they are + and – it has a maximum point. If the signs are the same, it has neither.

Step 6: For each value of x which gives a minimum or maximum, calculate $f(x)$.

Maximum and minimum

If $f'(q) = 0$ and $f''(q) > 0$, then $f(x)$ has a minimum at $x = q$

If $f'(q) = 0$ and $f''(q) < 0$, then $f(x)$ has a maximum at $x = q$

Procedures to find the minimum and maximum points on the graph of $y = f(x)$

Step 1: Decide the domain in which you are interested.

Step 2: Find the gradient function $f'(x)$.

Step 3: Find the values of x in the domain for which $f'(x) = 0$.
(If there are values where $f'(x)$ is undefined, use the old procedure)

Procedures to find the minimum and maximum points on the graph of $y = f(x)$

Step 4: Find $f''(x)$.

Step 5: For each value of x in Step 3, find the sign of $f''(x)$.
If the sign is +, the graph has minimum point.
If –, a maximum.
(If the value of $f''(x)$ is 0, follow the old procedure.)

Step 6: For each value of x which gives a minimum or maximum, calculate $f(x)$.

Example 10:

Find the minimum or maximum point of $f\,(x)$ and sketch the graph.

$$f\,(x) = x^3 - 12x + 2$$

Practice Exercise

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Exercise 7B (Page 103)

Q5(d)(e), Q6(f), Q7(l)

Exercise 15B (Page 232)

Q2(e)(f)

P1/8/5:

Rates of change

Learning Outcome

Students should be able to:

- **Apply differentiation in determining the rates of change including related rate of change.**

Derivatives as rates of change

$$\boxed{y} = f(\boxed{x})$$

Dependent variable   Independent variable

It is convenient to use letters for these quantities :

$t = \text{time},\ V = \text{volume}, C = \text{cost}, P = \text{Population},$

$A = \text{Area}, \ldots$

Derivatives as rates of change

If $t = \text{time}$, $A = \text{area}$, $r = \text{radius}$, and $\theta = \text{temperature}$

$$\frac{dA}{dt} = \text{rate of change of area,}$$

$$\frac{dr}{dt} = \text{rate of change of radius,}$$

$$\frac{d\theta}{dt} = \text{rate of change of temperature with respect}$$

t o the time of the day.

Derivatives as rates of change

If x and y are the independent and dependent variables respectively in a functional relationship, $y = f(x)$,

then the derivative, $\dfrac{dy}{dx} = f'(x)$,

measures the **rate of change of y with respect to x.**

Example 11:

The product of two positive real numbers *x* and *y* is 20. Find the minimum possible value of their sum.

Example 12:

A loop of string of length 1 metre is formed into a rectangle with one pair of opposite sides each x cm. Calculate the value of x which will maximise the area enclosed by the string.

Example 13:

An open rectangular box is to be made with a square base, and its capacity is to be 4000 cm^3. Find the length of the side of the base when the amount of material used to make the box is as small as possible.

Example 14:

A circular stain is spreading so that its radius is increasing at a constant rate of 3 mm s^{-1}.

Find the rate at which the area is increasing when the radius is 50 mm.

Example 15:

Air is being lost from a spherical balloon at a constant rate of 0.6 $m^3 s^{-1}$.

Find the rate at which the radius is decreasing at the instant when the radius is 2.5 m.

Sphere

$$\text{volume} = \frac{4}{3}\pi r^3$$

$$\text{surface area} = 4\pi r^2$$

Cylinder

$$\text{volume} = \pi r^2 h$$

$$\text{surface area} = 2\pi r h + 2\pi r^2$$

Cone

$$\text{volume} = \frac{1}{3}\pi r^2 h$$

$$\text{surface area} = \pi r l + \pi r^2$$

Pyramid

$$\text{volume} = \frac{1}{3} A h$$

$$A = \text{area of the base}$$

Practice Exercise

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Exercise 7C (Page 109)

Q7, Q8, Q15

Exercise 12D (Page 182)

Q4, Q6, Q8, Q9