Logarithmic Functions

A logarithmic function is defined by $y = \log_a x, a \in \mathrm{R}^+$.

$y = \log_a x$ is only valid for $x \in \mathrm{R}^+$.

Example:

Find the domain of the function

$$f : x \to \ln\left[x\left(x-1\right)\right]$$

Example:

Given that $f : x \to \ln x$ and $g : x \to x^3, x \in \mathrm{R}$.

Determine whether fg is defined.

By stating the largest possible domain, define fg.

Logarithm Vs Exponential

Theorem:

Let $a, x \in \mathrm{R}^{+}$ and $a \neq 1$. $f(x) = \log_a x$ and $g(x) = a^x$ are inverses of one another i.e. $f = g^{-1}$ and $g = f^{-1}$.

$\therefore$ Hence, $y = f(x)$ and $y = g(x)$ are mirror image of one another in the line $y = x$.

Range of Natural Logarithm

$$\begin{cases} \ln x > 0 \text{ when } x > 1 \\ \ln x = 0 \text{ when } x = 1 \\ \ln x < 0 \text{ when } 0 < x < 1 \end{cases}$$

Derivative of Logarithmic Functions

Theorem :

Let $a, x \in \mathrm{R}^+$

$$(\mathrm{a})\ \frac{d}{dx}\left(\log_a x\right) = \frac{1}{x}\log_a e,$$

$$(\mathrm{b})\ \frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$

Theorem :

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{f'(x)}{f(x)},\ f(x)\in \mathrm{R}^{+}$$

Example: Find

$$\text{(i)}\ \frac{d}{dx}\ln\left(x^{2}\right)= \qquad \text{(ii)}\ \frac{d}{dx}\ln\left(6x^{4}\right)=$$

$$(\text{iii})\ \frac{d}{dx}\ln\left(\frac{5}{x^2}\right)=$$

$$(\text{iv})\ \frac{d}{dx}\ln\left(3+5x^2\right)=$$

$$(\text{v})\frac{d}{dx}\ln\left(x^2+2x+5\right)= \quad (\text{vi})\frac{d}{dx}\ln\sqrt{1-x^2}=$$

Example:

Find the gradient of the tangent to the curve

$y = \ln\left(\dfrac{x}{x^2+1}\right)$ when $x = 1$.

Example:

Find the equation of the tangent to the curve

$y=\ln\sqrt{\dfrac{x+1}{x-1}}$ when $x=2$.

Example:

Find the $x-$coordinates of the stationary points of the curve $y = \ln\left(x\sqrt{x^2 - 1}\right)$.

Example:

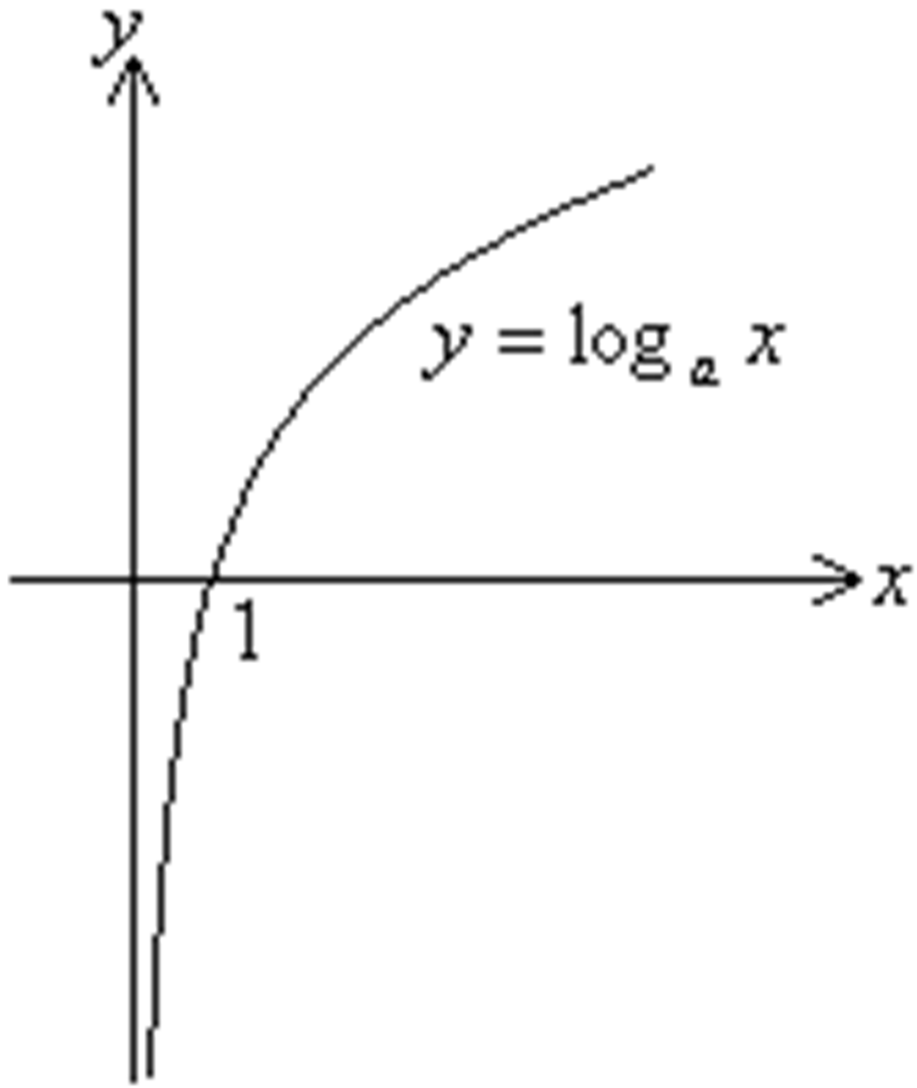
Find the domain of the function $f(x)=\ln\left(\dfrac{x^2}{\sqrt{x-1}}\right)$.

State the domain in which f is increasing.

Example: Find

$$(\mathrm{a})\ \frac{d}{dx}\ln\left(\ln x^{2}\right)= \qquad (\mathrm{b})\ \frac{d}{dx}(\ln x)^{4} =$$

Graph of Logarithmic Functions



$$\text{The curve } y = \ln x \text{ is}$$

$$(\text{a})\,\text{increasing because } \frac{d}{dx}(\ln x) > 0.$$

$$(\text{b})\,\text{one - to - one}$$

Homework

Please attempt all the questions in the following slides.

Questions are to be discussed on the next day of the instruction.

Example:

Find $\dfrac{d}{dx}\ln\left(x^3 + 4x\right)$.

Example:

Prove that the tangent at $x = e$ to the curve $y = \ln x$ passes through the origin.

Example:

Find the equation of the normal at $x = 2$ to the curve $y = \ln(2x - 3)$.

Example:

Using differentiation, find the equation of the tangent to the curve $y = 4 + \ln(x+1)$ at the point where $x = 0$.

Example:

Find the coordinates of the stationary point of the curve $y = \ln\left(x^2 - 6x + 10\right)$ and show that this stationary point is a minimum.

Example: (Napier's inequality)

(a) Find the stationary value of $y = \ln x - x$, and deduce that $\ln x \le x - 1$ for $x > 0$ with equality only when $x = 1$.

(b) Find the stationary value of $\ln x + \dfrac{1}{x}$, and deduce that $\dfrac{x-1}{x} \le \ln x$ for $x > 0$ with equality only when $x = 1$.

(c) By putting $x = \dfrac{z}{y}$ where $0 < y < z$, deduce that

$$\frac{1}{z} < \frac{\ln z - \ln y}{z - y} < \frac{1}{y}.$$