Topic 2 – Logarithm and Exponential Functions

Exercise 1: Logarithm Functions

Solve the equations in questions **1** to **11**.

1 $3^x = 9$

2 $3^x = \frac{1}{9}$

3 $9^x = 27$

4 $3^x = 6$

5 $2^{2x} = 5$

6 $5^x = 4$

7 $3^{x-1} = 7$

8 $4^{2x+1} = 8$

9 $\log_2 x = \log_2 (2x - 1)$

10 $\log_4 x = 2$

11 $\log x = 2 \log (x - 2)$

Solve the inequalities in questions **12** to **16**.

12 $3^x > 27$

13 $2^x < 32$

14 $5^{x-2} > 125$

15 $3^x > 10$

16 $2^{2x} < 3 \times 2^{3x}$

17 Express $\log_x 5 - 2 \log_x 3$ as a single log term.
Hence find the value of x when
$\log_x 5 - 2 \log_x 3 = 2$

18 Express $\log_3 y - 2 \log_3 x$ as a single logarithm.
Hence express y in terms of x when
$\log_3 y - 2 \log_3 x = 1$

19 Given that $y = 2^x$, express 2^{2x} in terms of y.
By substituting y for 2^x, solve the equation
$2^{2x} - 2^x - 2 = 0$

P3 20 Use the substitution $y = a^x$ or otherwise to solve the equation $a^{3x} - 2a^{2x} + a^x = 0$

P3 21 Solve the equation $5^x = 5^{x+2} - 5^3$

Question 22

Solve these inequalities.

(i) $2^x < 128$

(ii) $3^x + 5 \geqslant 32$

(iii) $4^x + 6 \geqslant 70$

(iv) $0.6^x < 0.8$

(v) $0.4^x - 0.1 \geqslant 0.3$

(vi) $0.5^x + 0.2 \leqslant 1$

(vii) $2 \leqslant 5^x < 8$

(viii) $1 \leqslant 7^x < 5$

(ix) $|2^x - 4| < 2$

(x) $|5^x - 7| < 4$

Question 23

Express the following as a single logarithm.

$$2\log_{10} x - \log_{10} 7$$

Hence solve

$$2\log_{10} x - \log_{10} 7 = \log_{10} 63.$$

Question 24

Find how many terms there are in these geometric sequences.

(i) $-1, 2, -4, 8, \ldots, -16\,777\,216$

(ii) $0.1, 0.3, 0.9, 2.7, \ldots, 4\,304\,672.1$

Exercise 2: Exponential Functions

1 Find the value of

(a) e^2 (b) e^{-1}

(c) $e^{1.5}$ (d) $e^{-0.3}$

(e) e^3 (f) $e^{1.8}$

(g) e^{-2} (h) $e^{0.05}$

2 Sketch the given curve.

(a) $y = 1 - e^x$ (b) $y = e^x + 1$

(c) $y = 1 - e^{-x}$ (d) $y = 1 + e^{-x}$

Question 3

A colony of humans settles on a previously uninhabited planet. After t years, their population, P, is given by $P = 100e^{0.05t}$.

(i) Sketch the graph of P against t.

(ii) How many settlers land on the planet initially?

(iii) What is the population after 50 years?

(iv) How long does it take the population to reach 1 million?

Exercise 3: Natural Logarithm

1 Convert each equation to logarithmic form.

(a) $e^x = 4$ (b) $e^2 = y$

(c) $e^a = b$

2 Convert each equation to index form.

(a) $\ln x = 4$ (b) $\ln 0.5 = a$

(c) $\ln a = b$

5 Express as a sum or difference of logarithms or as a product

(a) $\ln 5x$ (b) $\ln 5x^2$

(c) $\ln 3(x + 1)$ (d) $\ln \frac{x+1}{x}$

(e) $\ln \frac{2x-1}{x}$ (f) $\ln xy^2$

(g) $\ln \sqrt{x+1}$ (h) $\ln x(x + 4)$

(i) $\ln (x^2 - 1)$ (j) $\ln x^2(x + y)$

(k) $\ln ex$ (l) $\ln e^2x(x - e)$

(m) $\ln \frac{x^2}{x+1}$ (n) $\ln (a^2 - b^2)$

(o) $\ln \tan x$

7 Solve the following equations for x.

(a) $e^x = 8.2$

(b) $e^{2x} + e^x - 2 = 0$ (Hint: use $e^{2x} = (e^x)^2$)

(c) $e^{2x-1} = 3$ (d) $e^{4x} + e^x = 0$

(d) $\ln 2 + 2 \ln x = \ln (x + 3)$

(e) $\ln 4 - 2 \ln (x + 1) = \ln e$

3 Use a calculator to evaluate, correct to 3 significant figures

(a) $\ln 3$ (b) $\ln 2.4$

(c) $\ln 0.201$ (d) $\ln 17.3$

4 Evaluate

(a) $\ln e$ (b) $\ln e^2$ (c) $\ln 1$

6 Express as a single logarithm

(a) $\ln 2 + \ln x$

(b) $\ln 3 - \ln x$

(c) $2 \ln x - \ln 4$

(d) $\ln x - 2 \ln (1 - x)$

(e) $1 - \ln x$

(f) $2 + \ln x$

(g) $2 \ln x - \frac{1}{2} \ln (x - 1)$

(h) $\ln \cos x - \ln \sin x$

(i) $1 + \ln x$

(j) $\frac{2}{3} \ln (x - 1)$

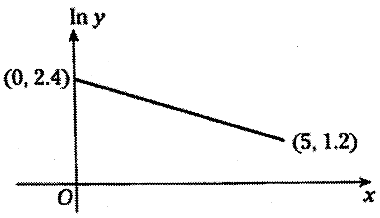
P3 **8** Solve the equation $\ln (e^x - 1) = 1$

P3 **9** Solve the equation $-3 + \ln x = \ln (x - 3)$

Exercise 4: Reduction of a Relationship to a Linear Form

Throughout this exercise, a and b are constants.

1 The variables x and y satisfy the relation $ay = b^x$. The graph of $\ln y$ against x is shown in the diagram.



Find the values of a and b.

2 The variables s and t satisfy the relation $s = ab^{-t}$. The graph of $\ln s$ plotted against t gives a straight line that crosses the s-axis at (0, 5) and crosses the t-axis at (10, 0). Find the values of a and b.

3 The variables x and y satisfy the relation $y = a(x + 1)^b$

(a) Take logarithms to show that plotting values of $\ln y$ against $\ln (x + 1)$ gives a straight line.

(b) Given that $y = 20$ when $x = 2$ and that $y = 40$ when $x = 5$, find the gradient of the line.

4 The variables x and y satisfy the relation $3^x = 5^{y-2}$. Take natural logarithms to show that plotting values of y against x gives a straight line. Find the value of the intercept of this line on the vertical axis.

5 The variables x and y satisfy the relationship $3y + 5 = 7^{(x-1)}$. Explain why plotting $\ln (3y + 5)$ against x gives a straight line. Find the coordinates of the point where the line crosses the x-axis.

P3 6 The variables x and y satisfy the relationship $y - 2000 = ab^{-x}$

It is given that $y = 8800$ when $x = 2$ and $y = 6100$ when $x = 6$

(a) Find the values of a and b.

(b) Explain why the graph of $\ln (y - 200)$ against x is a straight line.

P3 7 The variables p and q satisfy the relationship $p = aq^2$

It is given that $p = 100$ when $q = 50$

(a) Find the value of a.

(b) Explain why the graph of $\ln p$ against q is a straight line.

P3 8 Two variables s and t are related by a law of the form $s = ke^{-nt}$ where k and n are constants. The values in the table were obtained from an experiment.

t	1	1.5	2	2.5	3
s	1230	590	260	140	60

(a) Show how the relationship between s and t can be reduced to a linear form.

(b) Draw a graph and use it to find the values of k and n.

Question 9

In a spectacular experiment on cell growth the following data were obtained, where N is the number of cells at a time t minutes after the start of the growth.

t	1.5	2.7	3.4	8.1	10
N	9	19	32	820	3100

At $t = 10$ a chemical was introduced which killed off the culture.

The relationship between N and t was thought to be modelled by $N = ab^t$, where a and b are constants.

(i) Show that the relationship is equivalent to $\log N = t\log b + \log a$.

(ii) Plot the values of $\log N$ against t and say how they confirm the supposition that the relationship is of the form $N = ab^t$.

(iii) Find the values of a and b.

(iv) If the growth had not been stopped at $t = 10$ and had continued according to your model, how many cells would there have been after 20 minutes?

Question 10

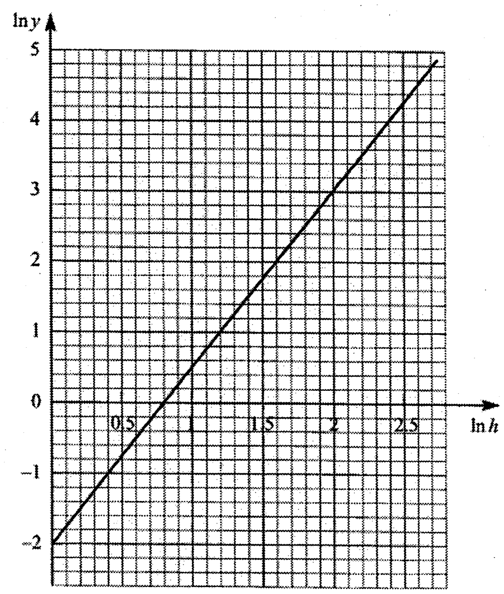
The height h metres of a species of pine tree t years after planting is modelled by the equation $h = 20 - 19 \times 0.9^t$.

(i) What is the height of the trees when they are planted?

(ii) Calculate the height of the trees after 2 years, and the time taken for the height to reach 10 metres.

The relationship between the market value $\$y$ of the timber from the tree and the height h metres of the tree is modelled by the equation $y = ah^b$, where a and b are constants.

The diagram shows the graph of $\ln y$ plotted against $\ln h$.



(iii) Use the graph to calculate the values of a and b.

(iv) Calculate how long it takes to grow trees worth $100.