TEST 4(Differentiation and Integration)

Question 1

Find $\frac{d}{dx}\sqrt{1+x^3}$ and hence evaluate $\int_0^2 \frac{x^2}{\sqrt{1+x^3}}$ dx. [4]

$$\frac{d}{dx}(1+x^3)^{\frac{1}{2}} = \frac{1}{2}(1+x^3)^{-1/2} \cdot 3x^2$$

$$= \frac{3}{2}(x^2)(1+x^3)^{-1/2} \quad — 1$$

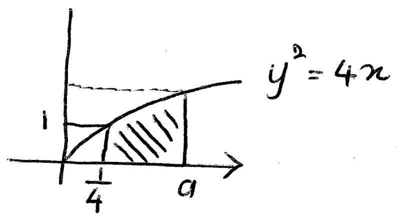
$$\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx = \frac{2}{3}\int_0^2 \frac{\frac{3}{2}x^2}{\sqrt{1+x^3}}\,dx \quad — 1$$

$$= \frac{2}{3}\left[(1+x^3)^{1/2}\right]_0^2 \quad — 1$$

$$= \frac{2}{3}(3-1) = \frac{4}{3} \quad — 1$$

Question 2

A cup is obtained by rotating the parabola $y^2 = 4x$ about the x-axis. If it holds exactly 32π cm^3, find the diameter of the rim of the cup, given that the radius of the base is 1 cm. [5]



$$\pi\int_{1/4}^{a} y^2\,dx = 32\pi \quad — 1$$

$$= \pi\int_{1/4}^{a} 4x\,dx$$

$$\pi\left[2x^2\right]_{1/4}^{a} = 32\pi \quad — 1$$

$$a = \frac{\sqrt{257}}{4} \quad — 1$$

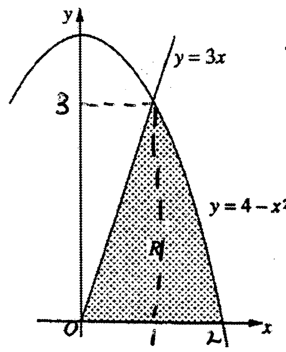
when $x = a = \frac{\sqrt{257}}{4}$

$$y^2 = 4\left(\frac{\sqrt{257}}{4}\right)$$

$$y = \sqrt[4]{257} \quad — 1$$

diameter of rim $= \sqrt[4]{257} \times 2 \approx 8$ cm — 1

Question 3



$x^2 = \left(\frac{y}{3}\right)^2$

$x^2 = 4 - y$

The region R shown in the diagram is bounded by the line y = 3x, the curve y = 4 – x² and the x-axis.

(a) Find the area of R. [3]

(b) Find the volume generated when R is rotated through 2π about the y-axis. [4]
(in exact form)

(a) Intersection

$3x = 4 - x^2$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$x = 1$ — 1

$$\text{Area} = \int_0^1 3x\,dx + \int_1^2 4 - x^2\,dx$$

$$= \left[\tfrac{3}{2}x^2\right]_0^1 + \left[4x - \tfrac{1}{3}x^3\right]_1^2 \quad — 1$$

$$= \tfrac{3}{2} + \left[\left(8 - \tfrac{8}{3}\right) - \left(4 - \tfrac{1}{3}\right)\right]$$

$$= 3\tfrac{1}{6} \,/\, \tfrac{19}{6} \quad — 1$$

(b)

$$V = \pi\int_0^3 (4 - y)\,dy - \pi\int_0^3 \frac{y^2}{9}\,dy \quad — 1$$

$$= \pi\left[4y - \tfrac{1}{2}y^2\right]_0^3 - \pi\left[\tfrac{1}{27}y^3\right]_0^3 \quad — 1$$

$$= \pi\left[12 - \tfrac{9}{2} - 1\right] \quad — 1$$

$$= 6\tfrac{1}{2}\pi \,/\, \tfrac{13}{2}\pi \quad — 1$$

Question 4

(a) The gradient of a curve at any point is given by $\frac{dy}{dx} = 2 - \frac{x^3}{8}$. The curve intersects the x-axis at the point P. Given that the gradient of the curve at P is 1, find the equation of the curve. [5]

(b) Two variables, x and y, are related by the equation $y = \frac{3}{4}\left(\frac{x}{12} - 1\right)^6$. Given that both x and y vary with time, find the value of y when the rate of change of y is 12 times the rate of change of x. [7]

(a) $\frac{dy}{dx} = 2 - \frac{x^3}{8} = 1$ — 1

$\frac{x^3}{8} = 1$

$x = 2$ at point P (2,0) — 1

$\int \frac{dy}{dx} = \int 2 - \frac{x^3}{8} \, dx$ — 1

$y = 2x - \frac{1}{32}x^4 + c$, at P

$0 = 4 - \frac{1}{32}(16) + c$, $c = -3\frac{1}{2}$ — 1

$y = 2x - \frac{x^4}{32} - \frac{7}{2}$ — 1

(b) $y = \frac{3}{4}\left(\frac{x}{12} - 1\right)^6$, $\frac{dy}{dt} = 12\frac{dx}{dt}$ — 1

$\frac{dy}{dx} = \frac{3}{4}(6)\left(\frac{x}{12} - 1\right)^5\left(\frac{1}{12}\right)$

$= \frac{3}{8}\left(\frac{x}{12} - 1\right)^5$ — 1

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$12\frac{dx}{dt} = \frac{3}{8}\left(\frac{x}{12} - 1\right)^5 \frac{dx}{dt}$ — 2

$32 = \left(\frac{x}{12} - 1\right)^5$ — 1

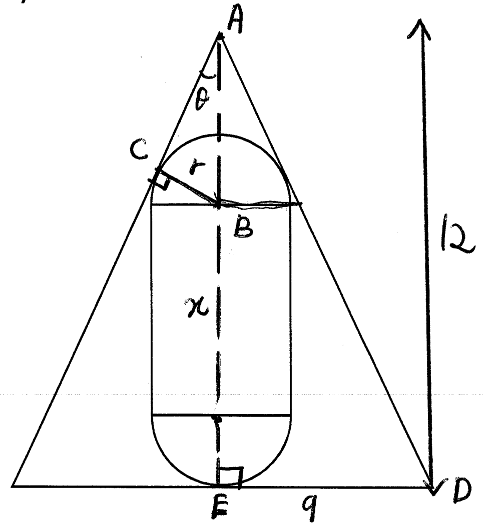
$x = 36$ — 1

$y = \frac{3}{4}\left(\frac{36}{12} - 1\right)^6$

$= 48$ — 1

Question 5

A capsule made of two similar hemispheres of radius r cm, one at each end of a cylinder, is inscribed into a fixed cone of height 12 cm and base radius 9 cm. The two figures have a common axis of symmetry and the cross-section is shown below.



(i) Show that the height of the cylinder of the capsule, x cm, is related to the radius of the hemisphere through the equation, $x = 12 - \frac{8}{3}r$. [4]

(ii) Given that, when $r = 2$ cm, x is decreasing at a rate of 0.1 cms^{-1}, find the rate of change at this instant of the volume of the capsule. [8]

[Given: Volume of sphere = $\frac{4}{3}\pi r^3$]

(i) $\sin\theta = \frac{9}{15} = \frac{r}{AB}$ — ① $AD = \sqrt{12^2 + 9^2} = 15$ — ①

$AB = \frac{5r}{3}$

$x = 12 - r - \frac{5r}{3}$ — ①

$= 12 - \frac{8r}{3}$ — ①

or

compare ABC and AED

$\frac{AB}{AD} = \frac{BC}{ED}$

$\frac{12-x-r}{15} = \frac{r}{9}$, $x = 12 - \frac{8}{3}r$

(ii) $\frac{dx}{dt} = -0.1$

$V = \frac{4}{3}\pi r^3 + \pi r^2 x$ — ①

$= \frac{4}{3}\pi r^3 + \pi r^2\left(12 - \frac{8}{3}r\right)$ — ①

$= \frac{4}{3}\pi r^3 + 12\pi r^2 - \frac{8}{3}\pi r^3$

$V = 12\pi r^2 - \frac{4}{3}\pi r^3$ — ①

$\frac{dv}{dr} = 24\pi r - 4\pi r^2$ — ①

$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dx} \times \frac{dx}{dt}$

where $\frac{dx}{dr} = -\frac{8}{3}$ — ①

$= (24\pi r - 4\pi r^2)\left(\frac{-3}{8}\right)(-0.1)$ — ②

sub. $r = 2$

$\frac{dv}{dt} = -3.77$ cm^{3}s^{-1} — ①

$= \frac{6}{5}\pi$