

## TOPIC 4 – Differentiation

### Differentiating Exponential Functions

The derivative of  $e^x$  is quite remarkable. The expression for the derivative is the same as the expression that we started with; that is,  $e^x$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$$

Example: i)  $\frac{d}{dx} e^{2x} =$       ii)  $\frac{d}{dx} e^{x+\sin x} =$

#### Exercise 1

In questions 1 to 7 differentiate the given functions with respect to  $x$ .

- |                                      |                            |  |
|--------------------------------------|----------------------------|--|
| 1. (a) $e^{2x}$                      | (b) $5e^{-x}$              | (c) $e^{3x+5}$                           |
| 2. (a) $e^{2x^3}$                    | (b) $e^{\sqrt{x}}$         | (c) $e^{-1/x}$                           |
| 3. (a) $e^{\sin x}$                  | (b) $e^{\cos 2x}$          | (c) $e^{4 \tan x}$                       |
| 4. (a) $xe^{x^2}$                    | (b) $x^2 e^{5x}$           | (c) $e^{x \cos x}$                       |
| 5. (a) $e^{2x} \cos 3x$              | (b) $e^{-x^2} \sin x$      | (c) $(x+1)^3 e^{x/2}$                    |
| 6. (a) $\sqrt{1-e^{4x}}$             | (b) $e^{(e^x)}$            | (c) $\sin(e^x)$                          |
| 7. (a) $\frac{1}{2}(e^x - e^{-x})^2$ | (b) $\frac{e^x}{1+e^{-x}}$ | (c) $\frac{e^{\sin^2 x}}{e^{-\cos^2 x}}$ |

#### Question 8

If  $y = e^x \sin x$ , show that  $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$ .

#### Question 9

Find the equation of the tangent to the curve  $y = e^{3x-5}$  at the point where  $x = 2$ .

#### Question 10

Find the values of  $x$  for which the function  $(x^2 - 2x - 1)e^{2x}$  has maximum or minimum values, distinguish between them.

#### Question 11

Find the values of  $x$  between 0 and  $2\pi$  for which the function  $e^x \cos x$  has maximum or minimum values, distinguish between them.

### Question 12

The curve  $C$  has the equation  $y = \sqrt{x} + e^{1-4x}$ ,  $x \geq 0$ .

- (i) Find an equation for the normal to the curve at the point  $(\frac{1}{4}, \frac{3}{2})$ .

### Question 13

The curve  $y = e^x + 4e^{-2x}$  has one stationary point.

- (i) Find the  $x$ -coordinate of this point.

[4]

- (ii) Determine whether the stationary point is a maximum or a minimum point.

[2]

Answers:

1. (a)  $2e^{2x}$ , (b)  $-5e^{-x}$ , (c)  $3e^{3x+5}$ .
2. (a)  $6x^2e^{2x^3}$ , (b)  $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ , (c)  $\frac{1}{x^2}e^{-1/x}$ .
3. (a)  $\cos x e^{\sin x}$ , (b)  $-2 \sin 2x e^{\cos 2x}$ , (c)  $4 \sec^2 x e^{4 \tan x}$ .
4. (a)  $(2x^2 + 1)e^{x^2}$ , (b)  $(5x + 2)xe^{5x}$ , (c)  $(\cos x - x \sin x)e^{x \cos x}$ .
5. (a)  $e^{2x}(2 \cos 3x - 3 \sin 3x)$ , (b)  $e^{-x^2}(\cos x - 2x \sin x)$ ,  
(c)  $\frac{1}{2}(x+1)^2(x+7)e^{x/2}$ .
6. (a)  $-2e^{4x}(1 - e^{4x})^{-\frac{1}{2}}$ , (b)  $e^{(x+e^x)}$ , (c)  $e^x \cos(e^x)$ .
7. (a)  $e^{2x} - e^{-2x}$ , (b)  $(2 + e^x)/(1 + e^{-x})^2$ , (c) 0.
9.  $y = e(3x - 5)$ . 10.  $x = -1$ , max;  $x = 2$ , min. 11.  $x = \pi/4$ , max;  $x = 5\pi/4$ , min.
12.  $4x - 12y + 17 = 0$  13. i)  $x = \ln 2$ , ii) minimum

### Differentiating Logarithmic Functions

The derivative of the logarithmic function  $y = \ln x$  is given by:

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

If  $y = \ln f(x)$ , then the derivative of  $y$  is given by:

$$\frac{d}{dx} \ln[f(x)] = \frac{f'(x)}{f(x)}$$

Example : Find the derivative of

- i)  $y = 2 \ln(3x^2 - 1)$  ii)  $y = \ln(1 - 2x)^3$  iii)  $y = \ln(\cos x^2)$  iv)  $y = [\ln(x^2 + 3)]^4$

## Exercise 2

In questions 1 to 7 differentiate the given functions with respect to  $x$ .

1. (a)  $\ln 3x$ , (b)  $\ln(x + 3)$ , (c)  $\ln(2x - 1)$ .
2. (a)  $\ln(x^3 + 4)$ , (b)  $\ln(\sin 2x)$ , (c)  $\ln(\sec x)$ .
3. (a)  $x^2 \ln x$ , (b)  $(\ln x)/x^2$ , (c)  $\ln(\ln x)$ .
4. (a)  $\ln(x^3)$ , (b)  $\ln\sqrt{4x + 5}$ , (c)  $x \ln(1/x)$ .
5. (a)  $\ln(\sec x + \tan x)$ , (b)  $\ln(x^2 + 4)^2$ , (c)  $\log_{10} x$ .
6. (a)  $\ln\left(\frac{x^2}{3x - 2}\right)$ , (b)  $\ln\sqrt{\left(\frac{x}{1 + x}\right)}$ , (c)  $\ln\frac{\cos x}{\sqrt{1 - x^2}}$ .
7. (a)  $e^{2 \ln x}$ , (b)  $\ln(e^{\tan x})$ , (c)  $\ln(3x^2 e^{-x})$ .

8. Find the equation of the tangent to the curve  $y = \ln(3x - 5)$  at the point where  $x = 2$ .

### Question 9

If  $y = \ln(x^2 - 5)$ , show that  $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 2e^{-y}$ .

### Question 10

If  $y = \sin 2x \ln(\tan x)$ , show that  $\frac{d^2 y}{dx^2} + 4y = 4 \cot 2x$ .

### Question 11

The equation of a curve is  $y = \ln x + \frac{2}{x}$ , where  $x > 0$ .

- (i) Find the coordinates of the stationary point of the curve and determine whether it is a maximum or a minimum point. [5]

### Question 12

A curve has the equation  $x = y^2 - 3 \ln 2y$ .

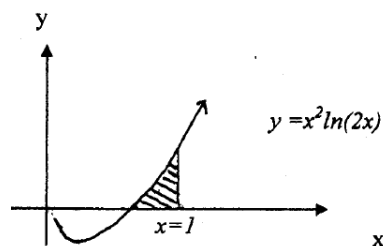
- (i) Show that

$$\frac{dy}{dx} = \frac{y}{2y^2 - 3}. \quad [3]$$

- (ii) Find an equation for the tangent to the curve at the point where  $y = \frac{1}{2}$ .  
Give your answer in the form  $ax + by + c = 0$  where  $a$ ,  $b$  and  $c$  are integers. [3]

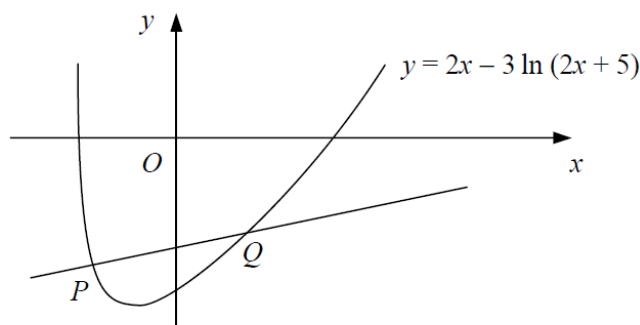
Question 13

The sketch below shows the curve with equation  $y = x^2 \ln(2x)$



- (i) Find the exact coordinates of the stationary point.
- (ii) Find the  $x$ -coordinate of the point where the curve cuts the  $x$ -axis.

Question 14



The diagram shows the curve with equation  $y = 2x - 3 \ln(2x + 5)$  and the normal to the curve at the point  $P(-2, -4)$ .

- (i) Find an equation for the normal to the curve at  $P$ . [4]

Question 15

The curve  $C$  has the equation  $y = 2e^x - 6 \ln x$  and passes through the point  $P$  with  $x$ -coordinate 1.

- (i) Find an equation for the tangent to  $C$  at  $P$ . [4]

The tangent to  $C$  at  $P$  meets the coordinate axes at the points  $Q$  and  $R$ .

- (ii) Show that the area of triangle  $OQR$ , where  $O$  is the origin, is  $\frac{9}{3-e}$ . [4]

Answers:

1. (a)  $\frac{1}{x}$ , (b)  $\frac{1}{x+3}$ , (c)  $\frac{2}{2x-1}$ .

2. (a)  $\frac{3x^2}{x^3+4}$ , (b)  $2 \cot 2x$ , (c)  $\tan x$ .

3. (a)  $x(2 \ln x + 1)$ , (b)  $(1 - 2 \ln x)/x^3$ , (c)  $1/x \ln x$ .

4. (a)  $\frac{3}{x}$ , (b)  $\frac{2}{4x+5}$ , (c)  $-1 - \ln x$ .

5. (a)  $\sec x$ , (b)  $\frac{4x}{x^2+4}$ , (c)  $(\log_{10} e)/x$ .

6. (a)  $\frac{3x-4}{x(3x-2)}$ , (b)  $\frac{1}{2x(x+1)}$ , (c)  $\frac{x}{1-x^2} - \tan x$ .

7. (a)  $2x$ , (b)  $\sec^2 x$ , (c)  $\frac{2}{x} - 1$ . 8.  $y = 3x - 6$ .

11.  $(2, \ln 2 + 1)$

12. ii)  $4x + 20y - 11 = 0$

13. i)  $\left( \frac{1}{2\sqrt{e}}, -\frac{1}{8e} \right)$ , ii)  $\frac{1}{2}$

14. i)  $y = \frac{1}{4}x - \frac{7}{2}$

15. i)  $y = (2e - 6)x + 6$