TOPIC 7 – SERIES
(GEOMETRIC PROGRESSION)

- use the expansion of $(a+b)^n$, where n is a positive integer (knowledge of the greatest term and properties of the coefficients are not required, but the notations $\binom{n}{r}$ and $n!$ should be known);
- recognise arithmetic and geometric progressions;
- use the formulae for the nth term and for the sum of the first n terms to solve problems involving arithmetic or geometric progressions;
- use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression.

<u>Geometric Progression</u>

In a **Geometric Sequence** each term is found by **multiplying** the previous term by a **constant**.

In General you could write a Geometric Sequence like this:

$$a, ar, ar^2, ar^3, \ldots$$

where:

- **a** is the first term, and
- **r** is the factor between the terms (called the **"common ratio"**)

But be careful, **r** should not be 0:

When **r=0**, you get the sequence {a,0,0,...} which is not geometric

The Rule

You can also calculate **any term** using the Rule:

$$a_n = ar^{(n-1)}$$

(We use "n-1" because ar^0 is for the 1st term)

Summing a Geometric Series

When you need to sum a Geometric Sequence, there is a handy formula.

To sum:

$$a + ar + ar^2 + ... + ar^{(n-1)}$$

Each term is ar^n, where n starts at 0 and goes up to n-1

Use this formula:

For r > 1, $S_n = \frac{a(r^n-1)}{r-1}$ or For r < 1, $S_n = \frac{a(1-r^n)}{1-r}$

Infinite Geometric Series

So what happens when *n* goes to infinity?

Well ... when r is less than 1, then r^n goes to **zero** and we get:

$$S_\infty = \frac{a}{1-r}$$

NOTE: this does not work when r is 1 or more (or less than -1):

r must be between (but not including) **-1 and 1**

and **r should not be 0** because you get the sequence {a,0,0,...} which is not geometric.

Examples:

Question 1

For each of the following GPs, find the common ratio, r.

(a) $\frac{1}{2}, -2, 8, \ldots$ **(b)** $x, 2x^2, 4x^3, \ldots$

Question 2

Find the sixth term of the GP 1, 2, 4, . . . , 256. How many terms are there?

Question 3

In a GP, each term is positive, the third term is 18 and the fifth term is 162. Find the eighth term.

Question 4

In a GP, the first term exceeds the third term by 72 and the sum of the second and third terms is 36. Find the first term.

Question 5

The first three terms of a geometric progression are x, $x + 2$, $x + 3$. Calculate

(a) the value of x,

(b) the fifth term.

Question 6

Find the sum of the first 10 terms of the GP 1, 2, 4, . . .

Question 7

Find the least number of terms of the GP 4, 8, 16, . . . which must be taken for their sum to exceed 500.

Question 8

A child lives 200 metres from school. He walks 60 metres in the first minute, and in each subsequent minute he walks 75% of the distance he walked in the previous minute. Show that he takes between 6 and 7 minutes to get to school.

Question 9

The second term of a GP is 2 while its sum to infinity is 8. Find the sum of the first 8 terms.

Question 10

The first term of an arithmetic progression is -2 and its last term is 85. The sum of the whole series is 1 245. Calculate

(a) the number of terms,

(b) the common difference,

(c) the sum of all the terms from the 10^{th} to the 25^{th} terms (both inclusive).

Question 11

An infinite geometric series, whose terms are positive, has a finite sum. The ratio of the sum of the first 4 terms to the sum of the first 2 terms is 13 : 9. Calculate the value of the common ratio.

If the third term is $2\frac{2}{3}$, calculate the value of the sum to infinity. Find also the least value of n for which the sum to n terms differs from the sum to infinity by less than 0.09.

Exercise 1 – Geometric Progression

4. In a GP the fourth term is 18 and the seventh term is 486. Find the second term.

5. The terms of a GP are distinct. If the second term is 4 and the sum of its third and fourth terms is 8, find the sixth term.

6. In a GP whose terms are all positive, the third term exceeds the first term by 32 while the fifth term exceeds the first term by 320. Calculate the first term.

7. Find the geometric means of the following:
   (a) 3, 12 (b) $\frac{1}{3}, \frac{4}{27}$ (c) $\sqrt{24}, \sqrt{6}$

8. Find the indicated sums of these GPs.
   (a) 5, 10, 20, . . . ; S_5 and S_7 (b) 2, −4, 8, . . . ; S_6 and S_9

9. Given x, $x + 4$, $2x + 2$ are the first three positive terms of a GP, calculate the sum of the first 6 terms.

10. The sum of the first 4 terms of a GP is 40. Its third term exceeds the first term by 8. If the first term is greater than 1, calculate the sum of the first 6 terms.

11. The sum of the first 3 terms of a GP is 117 and the sum of the next 3 terms is $4\frac{1}{3}$. Calculate the third term.

12. Find the least number of terms of the GP 2, 4, 8, . . . which must be taken for their sum to exceed 2 500.

13. The second term of a GP is 1 less than the first term and the sum of the first 3 terms is $6\frac{1}{3}$. If the GP contains positive and negative terms, find the seventh term.

14. A GP has first term 5 and last term 2 560. If the sum of all its terms is 5 115, how many terms are there?

15. The r^{th} term of a GP is $\frac{6 \times 5^{r-1}}{2^r}$. Find the common ratio and an expression for the sum of the first $2n$ terms.

16. Find three numbers in a GP such that their sum is 42 and their product is 512.

17. The sum of the first $(n + 10)$ terms of the geometric series $4 + 8 + 16 + \ldots$ is twice the sum of the first n terms of the geometric series $6 + 24 + 96 + \ldots$. Calculate the value of n.

18. In the GP $3, 6, 12, \ldots$, the r^{th} term is the largest term in the progression which has a value less than 1 000. Find r.

19. The sum of the first 4 terms of a GP is 5 times the sum of its first 2 terms. If the fifth term is 128, find the possible values of the eighth term.

20. The sum of the first n terms of a GP is $k(3^{n-1}) - 9$. If the third term is 162, find k and the common ratio.

21. The lengths of the sides of a quadrilateral are in geometric progression and the longest side is 81 cm. Given that the perimeter is 120 cm, find the lengths of the other 3 sides.

Answers:

5. 64	6. 4	7. (a) 6 (b) $\frac{2}{9}$	(c) 2
8. (a) 155, 635	(b) −42, 342	9. $166\frac{1}{4}$	10. 168
11. 9	12. 11	13. $29\,127\frac{1}{9}$	14. 10
15. $\frac{5}{2}$, $2\left[\left(\frac{5}{2}\right)^{2n} - 1\right]$	16. 32, 8, 2	17. 10	18. 9
19. $\pm 1\,024$	20. 27, 3	21. 3 cm, 9 cm, 27 cm	

Exercise 2 – Geometric Progression

3. The first term of a GP exceeds the second term by 1 while its sum to infinity is 16. Find the third term.

4. The second term of a GP is 24 and its sum to infinity is 100. Find the two possible values of the common ratio.

5. The sum of the first and second terms of a GP is 10 while its sum to infinity is 18. If this GP contains both positive and negative terms, find the fourth term. Show also that the sum to eight terms differs from the sum to infinity by less than 0.71.

6. 4, $x - 4$ and $5 - 2x$ are the first three terms of an infinite GP whose sum is finite. Calculate the value of this finite sum.

9. Find the least value of n for which the sum to n terms of the geometric series $1 + 0.98 + (0.98)^2 + \ldots$ is greater than half the sum to infinity.

10. The length of the first swing of a pendulum is 40 cm. Each succeeding swing is $\frac{4}{5}$ the length of the preceding one. Find the length of the fourth swing and the total length of all swings.

11. An infinite GP is such that the sum of all the terms after the n^{th} term is thrice the n^{th} term. Show that the sum to infinity of the whole progression is 4 times the first term.

12. In a certain GP, the second term is 32 and the fifth term is 4. The sum of the first 4 terms is $k\%$ of the sum to infinity. Find the value of k.

Answers:

3. $2\frac{1}{4}$ 4. $\frac{2}{5}, \frac{3}{5}$ 5. $-8\frac{8}{9}$ 6. $2\frac{2}{3}$

9. 35 10. 20.48 cm; 200 cm 12. 93.75

Exercise 3 – Arithmetic and Geometric Progression

1. The sum of the first n terms of a sequence is given by $S_n = 2n^3 - 4n$. Find the sum of the first 6 terms and the value of the seventh term.

2. The n^{th} term of a sequence is $T_n = pn + q$. If the eighth term is 13 and the sum of the first 3 terms is 3, find the values of p and q.

3. The sum to n terms of a certain progression is given by $S_n = kn^2(n + 1)^2$. Given that the fifth term is 125, find the value of k and the first four terms. Hence, deduce the sum $1^3 + 2^3 + 3^3 + \ldots + 15^3$.

*4. The sum of the first $(n + 1)$ terms of a progression is $4n^2 + 5n + 1$. Find, in terms of n, the sum of the first n terms and the n^{th} term.

5. The 29^{th} term of an AP is 10 and the sum of the first 33 terms is 231. Find the sum of the first 29 terms.

6. In an AP, the third term is four times the first term and the fifth term is 14. Find the tenth term and the sum of the first 8 terms. How many terms are there if the sum of the whole progression is 1 855?

7. An AP is such that the sum of the first 8 terms is half the sum of the next 8 terms. Find the ratio of the 9^{th} term to the 4^{th} term.

8. For the AP 50, 49.2, 48.4, . . . , find the last positive term. How many terms must be taken for their sum to be negative?

9. An AP has 14 terms. The sum of the odd-numbered terms (1^{st}, 3^{rd}, 5^{th}, etc.) is 140 and the sum of the even-numbered terms (2^{nd}, 4^{th}, 6^{th}, etc.) is 161. Find the common difference of the progression and the 14^{th} term.

10. 20 m of hose are wound onto a reel of circumference 2 m. Taking into account the thickness of the hose, each turn is 0.05 m longer than the previous one. How many complete turns are needed?

***11.** 21 books are arranged from left to right in increasing order of their prices. The price of each book differs from that of each adjacent book by $1.40. For the price of the book at the extreme right plus an extra $1, a customer can buy the middle book and an adjacent one. Find how much it would cost to buy all the books.

12. In a certain AP whose terms are all positive, the tenth term is 83 and the sum of the first 15 terms is 1 335. Find the sum of the first 10 terms and the maximum possible number of terms in this progression.

13. The sum of the first n terms of a certain sequence is given by $S_n = 3n^2 - 4n$. Calculate the sixth term of this sequence. Show that the n^{th} term is $T_n = 6n - 7$ and that this sequence is an AP.

14. The sum of the first 6 terms of an AP is p and the sum of the first 12 terms is q. Find, in terms of p and q, the sum of the first 30 terms.

Answers:

1. 408, 250 **2.** $p = 2$, $q = -3$ **3.** $k = \frac{1}{4}$; 1, 8, 27, 64; 14 400

4. $4n^2 - 3n$; $8n - 7$ **5.** $188\frac{1}{2}$ **6.** 29, 100; 35 **7.** 5 : 3

8. 0.4; 127 **9.** 3, 41 **10.** 9 **11.** $344.40 **12.** 965, 37

13. 29 **14.** $10q - 15p$

17. The three positive numbers $2x - 3$, x, $x - 2$ are successive terms of a GP. Given that x is the third term of the progression, calculate the value of the sum to infinity.

18. The terms of a certain GP are all positive. If the sum of the second and third terms is 60 and the sum of the third and fourth terms is 180, find the values of the common ratio and the first term.

19. The positive quantities $\frac{36}{25}$, x, y, z, $\frac{100}{9}$ are in geometric progression. Calculate the values of x, y and z.

20. An infinite geometric series has a finite sum. The sum of the first three terms is 112 and the sum of the next three terms is 14. Calculate the sum to infinity and the least number of terms that must be taken for their sum to exceed 125.

21. A GP containing 20 terms has 48 as its fifth term. The ratio of the sum of its first 6 terms to that of its first 3 terms is 9 : 1. Find the sum of its last 5 terms.

22. The series $(x + 1) + 2(x + 1)^2 + 4(x + 1)^3 + \ldots$ has a finite sum. What is the range of values of x? If its sum is 4, find x.

23. A nail 3 cm long is driven into wood by blows of a hammer. The first blow drives it in 1.25 cm and each successive blow drives it in by $\frac{3}{5}$ of the previous distance except the last for which the distance is less. Find how many blows must be used.

24. The sum to infinity of a geometric series is 162 .The sum of the first 3 terms is 114. Find the least value of n for which the sum to n terms differs from the sum to infinity by less than 0.02.

25. The n^{th} term of a certain progression is $T_n = 2(4)^n$. Show that the progression is geometric. Hence, find the sum of the first $2n$ terms in terms of n.

Answers:

17. $40\frac{1}{2}$

18. 3, 5

19. $x = \frac{12}{5}$, $y = 4$, $z = \frac{20}{3}$

20. 128, 6

21. 3 047 424

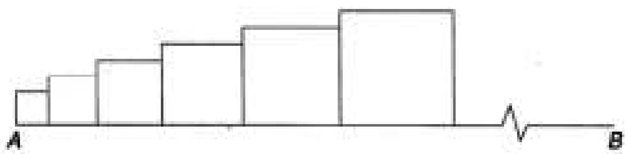
22. $-\frac{3}{2} < x < -\frac{1}{2}$; $-\frac{5}{9}$

23. 7

24. 23

25. $\frac{8}{3}(4^{2n} - 1)$

***26.**



The diagram shows a series of squares lined up side by side along a line AB. The squares are labelled 1^{st}, 2^{nd}, 3^{rd}, . . . from the left and their areas form a GP. The area of the 2^{nd} square is 3.6 cm^2 and the total area of the 3^{rd} and 4^{th} squares is 9.504 cm^2. Calculate the total area of the first seven squares, correct to 4 significant figures. If the line segment AB has length 60 cm, what is the maximum number of complete squares that can be lined up in the same manner along AB?

27. The second, sixth and eighth terms of an AP are three distinct consecutive terms of a GP. Find the common ratio of the GP. If the sum of the first 5 terms of the GP is 31, find its first term.

28. If $\frac{1}{b+a}$, $\frac{1}{2b}$ and $\frac{1}{b+c}$ are in AP, prove that a, b and c are in GP.

29. An AP has first term 7 and common difference 4. A GP has first term 6 and common ratio 2. A new sequence is formed by adding together corresponding terms of these progressions (1^{st} to 1^{st}, 2^{nd} to 2^{nd}, etc.). Show that the n^{th} term of this new sequence is given by $T_n = 3(2^n) + 4n + 3$.
Hence, for this new sequence, find the fifth term and the sum of the first 10 terms.

30. Three distinct successive terms of an AP are also the first, second and fourth terms of a certain GP. Given that the sum to infinity of the GP is $3 + \sqrt{5}$, find its first term.

Answers:

26. 38.75 cm^2; 16 **27.** $\frac{1}{2}$; 16 **29.** 119; 6 388 **30.** 2