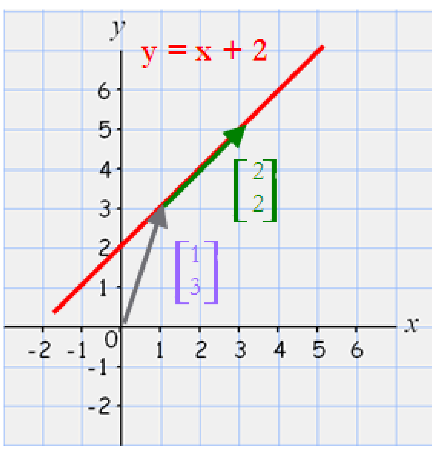
VECTORS IN SPACE

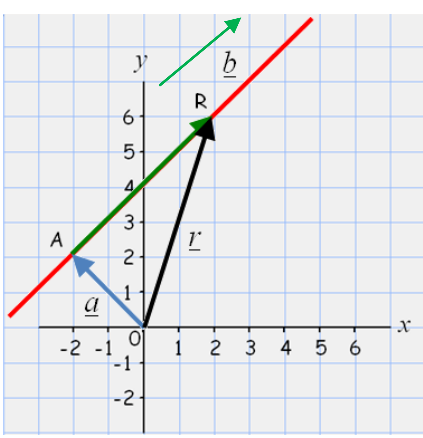
7. Vectors	• understand the significance of all the symbols used when the equation of a straight line is expressed in the form **r** = **a** + *t***b**; • determine whether two lines are parallel, intersect or are skew; • find the angle between two lines, and the point of intersection of two lines when it exists; • understand the significance of all the symbols used when the equation of a plane is expressed in either of the forms $ax + by + cz = d$ or $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$; • use equations of lines and planes to solve problems concerning distances, angles and intersections, and in particular find the equation of a line or a plane, given sufficient information, determine whether a line lies in a plane, is parallel to a plane, or intersects a plane, and find the point of intersection of a line and a plane when it exists, find the line of intersection of two non-parallel planes, find the perpendicular distance from a point to a plane, and from a point to a line, find the angle between two planes, and the angle between a line and a plane.

Vector Equation of a Line



$$y = c + xm$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



$\overrightarrow{AR} = t\underline{b}$ where $\underline{b}$ =

$\underline{r} = \overrightarrow{OR}$

$= \overrightarrow{OA} + \overrightarrow{AR}$

$\underline{r} = \underline{a} + t\underline{b}$ where $\underline{a}$ is known as

Examples

1. Find the vector equation of the line which passes through the point with position vector $2\underline{i} - \underline{j} + 3\underline{k}$ and which is parallel to the vector $2\underline{j} - \underline{k}$.

2. Find the vector equation of the straight line which passes through the points A(1, 0, -2) and B(2, 3, -1).

3. Find a vector equation of the straight line
   i) passing through the point *A*(3, 2, 10) and parallel to the vector $\underline{i} + 2j - 4\underline{k}$. Hence find the point on the line when λ= 4.

   ii) passing through the points *A*(1,-3,-4) and *B*(6,2,4).

4. A line has the vector equation of $\underline{r} = (\underline{i} + 2\underline{j} - 4\underline{k}) + \lambda(3\underline{i} + 2\underline{j} + 10\underline{k})$. Show that the point with position vector $13\underline{i} + 10\underline{j} + 36\underline{k}$ is on the line.

5. Determine whether the points with coordinates (5, 1, -6) and (-7, 5, 9) lie on the line joining A(1, 2, -1) to B(-3, 3, 4).

Different ways of writing the vector equation of a line:

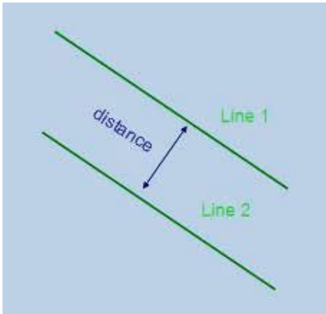
Vector Equation	**Parametric Equation**	**Cartesian Equation**
$\underline{r} = \underline{a} + \lambda\underline{b}$	$x = a_1 + \lambda b_1$ $y = a_2 + \lambda b_2$ $z = a_3 + \lambda b_3$	$\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$

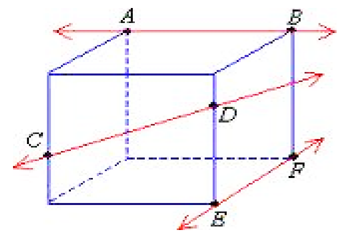
Example

Write the following Cartesian equation $\dfrac{x-1}{3} = \dfrac{y-2}{1} = \dfrac{z+3}{4}$ in the

i) Vector equation
ii) Parametric equation

Geometrical Interpretation between Two Lines





Parallel lines — equal directional vectors

Intersecting lines — share a common point

Skew lines — the lines do not meet & not parallel

Example

1. For each of these pairs of lines determine whether the lines are parallel, intersecting or skew, giving a reason for your answer.

   a) $\underline{r} = 13\underline{i} + 5\underline{j} - 3\underline{k} + \lambda(2\underline{i} - 4\underline{j} + 7\underline{k})$ and $\underline{r} = 4\underline{i} - 5\underline{j} - 7\underline{k} + \eta(-6\underline{i} + 12\underline{j} - 21\underline{k})$

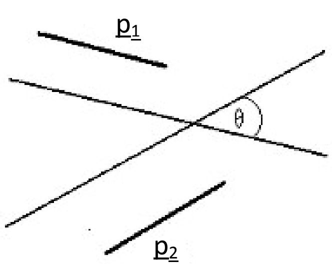
   b) $\underline{r} = \underline{i} + \underline{j} - 3\underline{k} + \delta(-4\underline{i} + \underline{j})$ and $\underline{r} = 9\underline{i} + 2\underline{j} + \underline{k} + \alpha(2\underline{i} + \underline{j} + 2\underline{k})$

   c) $\underline{r} = -2\underline{i} + \underline{j} + 9\underline{k} + s(2\underline{i} + 5\underline{j} + 4\underline{k})$ and $\underline{r} = 11\underline{i} + 8\underline{j} + 3\underline{k} + t(3\underline{i} - \underline{j} + 5\underline{k})$

2. Show that the lines with vector equations $\underline{r} = 2\lambda\underline{i} - 3\underline{j} + (\lambda - 2)\underline{k}$ and $\underline{r} = (\mu + 1)\underline{i} + (2 - \mu)\underline{j} + (2\mu - 5)\underline{k}$ do not intersect.

3. Show that the lines with vector equations $\underline{r} = \underline{k} + \lambda(\underline{i} - \underline{j} - 3\underline{k})$ and $\underline{r} = 2\underline{i} + \underline{j} + \mu(3\underline{j} + 5\underline{k})$, intersect, and find the position vector of their point of intersection.

Angle between two lines



l_1: $\underline{r} = \underline{a_1} + \lambda\,\underline{p_1}$

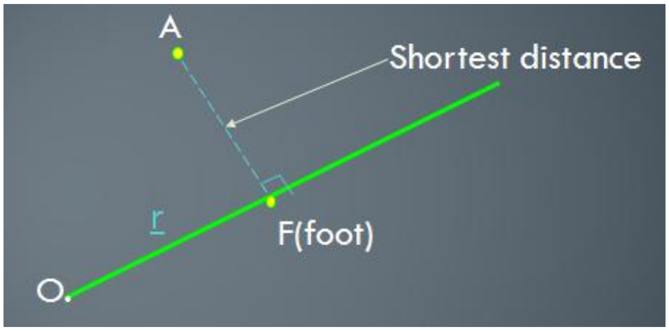
l_2: $\underline{r} = \underline{a_2} + \mu\,\underline{p_2}$

To find the angle between the two lines use scalar product $\cos\theta = \dfrac{\underline{p_1} \bullet \underline{p_2}}{|\underline{p_1}||\underline{p_2}|}$

Example

Find the angle between the lines $\underline{r} = \underline{k} + \lambda(\underline{i} - \underline{j} - 3\underline{k})$ and $\underline{r} = 2\underline{i} + \underline{j} + \mu(3\underline{j} + 5\underline{k})$.

<u>Perpendicular (Shortest) distance from a point to a line</u>



Example

1. Find the perpendicular distance from the point A, position vector $\begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}$ to the line L, vector equation

$$\underline{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

2. Find the distance of the point Q with coordinates (1, 2, 3) from the straight line with equation $\underline{r} = 3\underline{i} + 4\underline{j} - 2\underline{k} + \eta(\underline{i} - 2\underline{j} + 2\underline{k})$.

3. The lines l_1 and l_2 have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} a \\ 2 \\ \frac{3}{2} \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ b \end{pmatrix}$ where $\lambda, \mu \in \Re$ and $a, b \in \Re$.

   (i) Given that l_1 and l_2 are parallel, write down the values of a and b.
   (ii) Find the shortest distance between l_1 and l_2.

4. Relative to a fixed origin O, the points A, B and C have position vectors given respectively by

$$\mathbf{a} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \qquad \mathbf{b} = 2\mathbf{j} + 5\mathbf{k}, \qquad \mathbf{c} = -5\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

   (i) Find the vector equation of the line l that passes through point A and is parallel to the vector $\mathbf{a}$
   (ii) Find the point D, the foot of perpendicular from the point B to l.