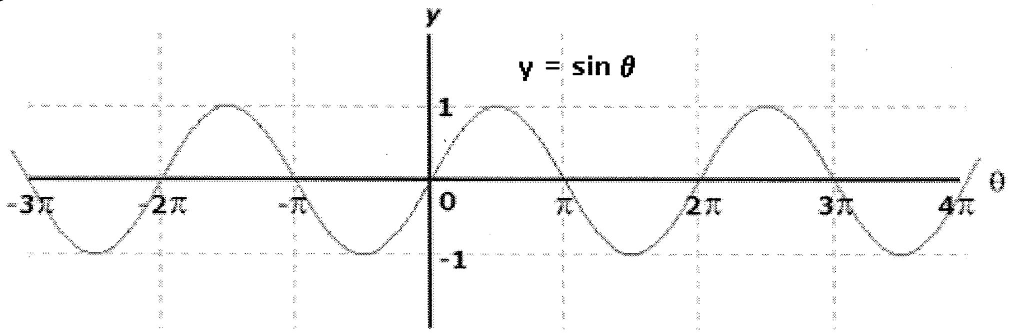
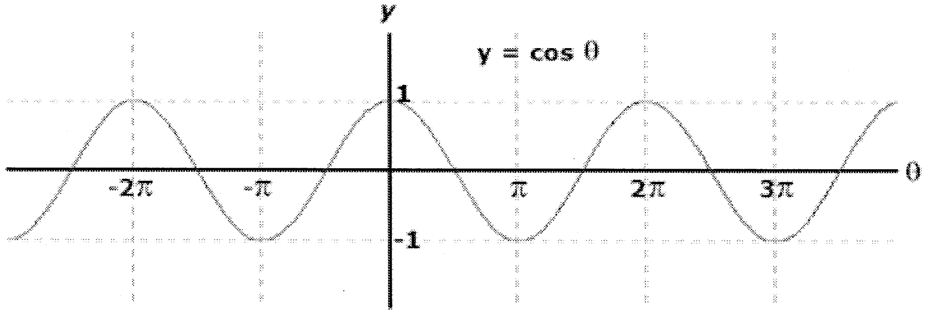
Topic 3: Trigonometry

Trigonometric Graphs ➔ AS

$y = \sin\theta$



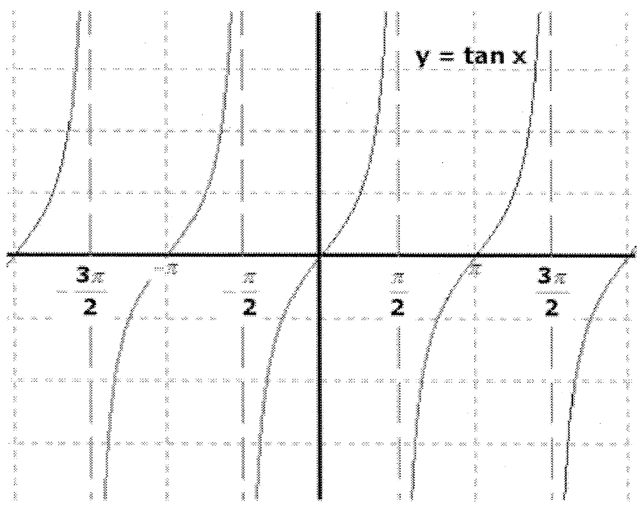
$y = \cos\theta$



Properties of the Sine and Cosine Curve

1. The period is 360°.
2. The amplitude is 1, meaning the range is between -1 and 1 inclusive.
3. The curve is periodic and continuous

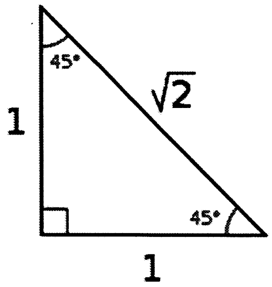
y = tangent x



Properties of Tangent Curve

1. The period is 180°.
2. The range is all real numbers.
3. The graph is symmetrical about the origin.
4. The curve is not continuous. There are vertical asymptotes at x = 90°, 270°,...

<u>Exact values for common angles</u>

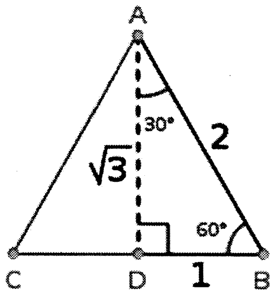


Isosceles right-angled triangle

sin 45° =

cos 45° =

tan 45° =



Equilateral triangle

sin 30° =

cos 30° =

tan 30° =

sin 60° =

cos 60° =

tan 60° =

<u>Trigonometric Formulas/Identitiy</u>

$$\mathbf{\sin^2 x + \cos^2 x = 1} \quad \text{and} \quad \mathbf{\tan x = \frac{\sin x}{\cos x}}$$

<u>Positive and Negative Angles</u>

$\sin(-\theta) = -\sin\theta$
$\tan(-\theta) = -\tan\theta$
$\cos(-\theta) = \cos\theta$

<u>Additional Formulas:</u>

$\sin(\pi - \theta) = \sin\theta$
$\cos(\pi - \theta) = -\cos\theta$
$\tan(\pi - \theta) = -\tan\theta$

$\sin(\pi/2 - \theta) = \cos\theta$
$\cos(\pi/2 - \theta) = \sin\theta$

Trigonometry → A2

Objectives:

- understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude;
- use trigonometrical identities for the simplification and exact evaluation of expressions and in the course of solving equations, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of

  $\sec^2\theta \equiv 1+\tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1+\cot^2\theta$,

  the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$,

  the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$,

  the expressions of $a\sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$.

Trigonometric Graphs

-sine, cosine, tangent, secant, cotangent and cosecant

$$y = \frac{1}{\sin x} = \operatorname{cosec} x$$

Properties of the Cosecant Curve

1. The period is 360° or 2 π.
2. The graph is symmetrical about the origin.
3. There are vertical asymptotes at x = -2 π, - π, 0, π, 2 π, etc.
4. There are turning points at (π /2, 1), (-π /2, -1), etc
5. The curve is not defined for the range -1 < y < 1.
6. The curve is not continuous.

$$y = \frac{1}{\cos x} = \sec x$$

Properties of the Secant Curve

1. The period is 360° or 2π.
2. The graph is symmetrical about the y-axis.
3. There are vertical asymptotes at x = - π /2, π /2, 3 π /2, etc
4. There are turning points at (-π,-1), (0, 1), (π, 1), etc
5. The curve is not defined for the range -1 < y < 1.
6. The curve is not continuous.

$$y = \frac{\cos x}{\sin x} = \frac{1}{\tan x} = \cot x$$

Properties of Cotangent Curve

1. The period is 180° or π.
2. The range is all real numbers.
3. The graph is symmetrical about the origin.
4. The curve is not continuous. There are vertical asymptotes at x = 0°, 180°,...

Exercise 1 - Solving Simple Equations

1. Solve for $-180° \leqslant \theta \leqslant 180°$

a $\operatorname{cosec}\theta = 2$ **b** $\sec 2\theta = 3$ **c** $\cot\theta = 1$

d $\sec(\theta - 30°) = 2$ **e** $\operatorname{cosec}\frac{1}{2}\theta = \sqrt{2}$ **f** $\operatorname{cosec} 2\theta = -1$

g $\cot(2\theta + 10°) = \sqrt{3}$ **h** $2\cot(30° - \theta) + 1 = 0$ **i** $\sec(\theta - 150°) = 4$

j $\operatorname{cosec} 3\theta = -5$ **k** $\sec(17° - 2\theta) = -2$ **l** $\cot(\theta + 40°) = -3$

2. Solve for $-\pi < \theta < \pi$

a $\sec\theta = 2\cos\theta$ **b** $\cot\theta = 5\cos\theta$ **c** $\tan\theta = 4\cot\theta + 3$

d $5\sin\theta + 6\operatorname{cosec}\theta = 17$ **e** $4\sin\theta + \operatorname{cosec}\theta = 4$ **f** $10\cos\theta + 1 = 2\sec\theta$

Answers – Exercise 1

Question 1

a $\theta = 30°,\ 150°$

b $\theta = \pm 35.3°,\ \pm 144.7°$

(c) -135, 45 (d) -30, 90 (e) 90 (f) -45, 135

g $\theta = -170°,\ -80°,\ 10°,\ 100°$

h $\theta = -86.6°,\ 93.4°$

i $\theta = -134.5°,\ 74.5°$

j $\theta = -176.2°,\ -123.8°,\ -56.2°,\ -3.8°,\ 63.8°,\ 116.2°$

k $\theta = -111\frac{1}{2}°,\ -51\frac{1}{2}°,\ 68\frac{1}{2}°,\ 128\frac{1}{2}°$

l $\theta = -58.4°,\ 121.6°$

Question 2

a $\theta = \pm\frac{\pi}{4},\ \pm\frac{3\pi}{4}$

b $\theta = 0.20,\ 2.94$

c $\theta = -\frac{\pi}{4},\ \frac{3\pi}{4};\ \theta = -1.82,\ 1.32$

d $\theta = 0.41,\ 2.73$

e $\theta = \frac{\pi}{6},\ \frac{5\pi}{6}$

f $\theta = \pm\frac{2\pi}{3},\ \pm 1.16$

Trigonometric Identities/Pythagorean Identities

This important identity can be derived as a direct result of Pythagoras's theorem, when applied to angles in trigonometry:

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

Dividing by $\cos^2\theta$:
$$\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$\therefore \boxed{1+\tan^2\theta = \sec^2\theta.}$$

Dividing by $\sin^2\theta$:
$$\frac{\cos^2\theta}{\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

$$\therefore \boxed{\cot^2\theta+1 = \operatorname{cosec}^2\theta.}$$

Exercise 2 - Trigonometric Identities

1. If $\cot x = 7/24$ and $90° < x < 360°$, find $\sin x$, $\cos x$ and $\operatorname{cosec} x$.

$-24/25,\ -7/25,\ -25/24.$

2. Prove the following identities

(a) $\sec\theta-\cos\theta = \sin\theta\tan\theta$,

(b) $\sec^2\theta+\operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$,

(c) $\dfrac{1-\sin\theta}{1+\sin\theta} = (\sec\theta-\tan\theta)^2$,

(d) $\dfrac{\cot^2\theta-1}{\cot^2\theta+1} = 1-2\sin^2\theta$.

3. Solve the given equations for $0° \leqslant \theta \leqslant 360°$.

a) $4\cot^2\theta - 3\operatorname{cosec}^2\theta = 2\cot\theta$ b) $\sec\theta\tan\theta = 2$ c) $\sec^2\theta + \tan\theta - 1 = 0$

a) 18·4°, 135°, 198·4°, 315° b) 51·3°, 128·7° c) 0°, 135°, 180°, 315° and 360°

4. Find the range of the function $f: x \to \sec^2 x - 2\tan x + 1$.

$y \geqslant 1$

Compound Angles

sin(A + B) = sinAcosB + cosAsinB

cos(A + B) = cosAcosB – sinAsinB

tan(A + B) = $\dfrac{\textbf{tanA + tanB}}{\textbf{1 – tanAtanB}}$

sin(A - B) = sinAcosB – cosAsinB

cos(A - B) = cosAcosB + sinAsinB

tan(A - B) = $\dfrac{\textbf{tanA - tanB}}{\textbf{1 + tanAtanB}}$

Exercise 3 - Compound Angles

1. Find without using the table, expressions in surd form for
   a) $\sin 15°$ b) $\tan 105°$

2. Solve the equation $\cos(\theta - 30°) = 2\sin\theta$ for values of $0° \le \theta \le 360°$.

3. Find, without using tables, the values of
   a) $\frac{1}{\sqrt{2}}(\cos 75° - \sin 75°)$ b) $\frac{1+\tan 15°}{1-\tan 15°}$

4. The angle x, measured in degrees, satisfies the equation $\sin(x+30°) = 2\cos(x+60°)$. By expanding each side, show that this equation may be simplified to $\cos x = 3\sqrt{3}\sin x$.

5. If $\cos A = \frac{5}{7}$ and $\sin B = \frac{1}{5}$, where A is acute and B is obtuse, find $\sin(A-B)$ *and* $\cos(A-B)$.

6. Solve the equation $\cos 3x \cos 2x - \sin 3x \sin 2x = 0.5$ *for* $0° \le x \le 180°$.

Answers:

1.	a) $\frac{1}{4}(\sqrt{6}-\sqrt{2})$ b) $-2-\sqrt{3}$.
2.	$\theta = 30°$ or $210°$
3.	a) -1/2 b) $\sqrt{3}$
5.	$-29/35$, $-8\sqrt{6}/35$.
6.	12°, 60°, 84°, 132°, 156°.

The Double Angle Identities

sin(A + B) = sinAcosB + cosAsinB

Replacing B by A in the above formula becomes:
sin(2A) = sinAcosA + cosAsinA
so **sin2A = 2sinAcosA**

similarly, **cos2A = $\cos^2$A - $\sin^2$A**
Replacing $\cos^2$A by 1 - $\sin^2$A (see Pythagorean identities) in the above formula gives:

cos2A = 1 - 2$\sin^2$A
Replacing $\sin^2$A by 1 - $\cos^2$A gives:

cos2A = 2$\cos^2$A – 1

It can also be shown that:

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Exercise 4 - Double Angles

1. Express $\sin 4\theta$ in terms of $\sin 2\theta$ and $\cos 2\theta$ and hence express $\frac{\sin 4\theta}{\sin \theta}$ in terms of $\cos\theta$ only.

2. Express sin 3A in terms of sin A.

3. Given that A is an acute angle such that $\tan A = \frac{1}{2}$, find sin 2A, tan 2A and tan 3A.

4. Prove the identity $\tan(\theta+45°)+\tan(\theta-45°)\equiv 2\tan 2\theta$.
   Hence find the acute angle θ such that $\tan(\theta+45°)+\tan(\theta-45°)=2$.

5. Prove the identity $\cos 2\theta+\tan\theta\sin 2\theta\equiv 1$. By letting $\theta=15°$, find the exact value of $\tan 15°$.

6. Prove the given identities

   a) $\tan A+\cot A\equiv\dfrac{2}{\sin 2A}$

   b) $\operatorname{cosec} x-\cot x=\tan\dfrac{1}{2}x$

7. Prove the given identity.

   a) $\cos^4 A-\sin^4 A=\cos 2A.$

   b) $\cot\theta-\tan\theta=2\cot 2\theta.$

   c) $\dfrac{\sin 3A}{\sin A}+\dfrac{\cos 3A}{\cos A}=4\cos 2A.$

   d) $\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\dfrac{\cos 2\theta}{1+\sin 2\theta}$

   e) $\dfrac{\sec 2x-1}{\sec 2x+1}=\sec^2 x-1.$

Answers:

2.	$3\sin A-4\sin^3 A.$
3.	$4/5,\ 4/3,\ 5\frac{1}{2}.$
4.	22.5
5.	$2-\sqrt{3}$

Exercise 5 - Miscellaneous Exercise

Question 1

Prove the identity

$$\sin^2\theta\cos^2\theta\equiv\tfrac{1}{8}(1-\cos 4\theta).$$

Question 2

Given that $\sec A=\dfrac{25}{7}$ where $-180°< A < 0°$, find the exact value of cosec 2A. [3]

Question 3

(i) Prove the identity

$$\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3. \qquad [4]$$

(ii) Hence solve the equation

$$\cos 4\theta + 4\cos 2\theta = 2,$$

for $0° \leqslant \theta \leqslant 360°$. [4]

[27.2°,152.8° and 207.2°]

Question 4

(i) Show that the equation

$$\tan(45° + x) = 2\tan(45° - x)$$

can be written in the form

$$\tan^2 x - 6\tan x + 1 = 0. \qquad [4]$$

(ii) Hence solve the equation $\tan(45° + x) = 2\tan(45° - x)$, for $0° < x < 90°$. [3]

[80.3, 9.7°]

Question 5

The angles α and β lie in the interval $0° < x < 180°$, and are such that

$$\tan\alpha = 2\tan\beta \quad \text{and} \quad \tan(\alpha + \beta) = 3.$$

Find the possible values of α and β. [6]

[$\alpha = 45°$, $\beta = 26.6°$, $\alpha = 116.6°$, $\beta = 135°$]

Question 6

i) Prove that $1 + \sin\theta - \cos\theta = 2\sin\left(\frac{\theta}{2}\right)\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]$.

ii) Hence, solve the equation $1 + \sin\theta - \cos\theta = 0, \quad 0 \le \theta \le 2\pi$.

$\theta = 0, \frac{3\pi}{2}, 2\pi$

TRIGONOMETRY LAWS AND IDENTITIES

TANGENT IDENTITIES

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

RECIPROCAL IDENTITIES

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

PYTHAGOREAN IDENTITIES

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

EVEN/ODD IDENTITIES

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta$$

$$\sec(-\theta) = \sec\theta$$

$$\cot(-\theta) = -\cot\theta$$

DOUBLE ANGLE IDENTITIES

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

SUM/DIFFERENCES IDENTITIES

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$