Section 1 of Unit 05 (Mechanics 2)

Motion of a projectile, uniform motion in a circle and linear motion under a variable force

Recommended Prior Knowledge
Students must have studied M1.

Context
Section 1 follows from M1 and it is recommended that it is studied before Section 2. Liaison between the teachers of M2 and P3 will be necessary since the topic on motion under a variable force, topic 5, cannot be studied until the work on differential equations in P3 has been completed.

Outline
The motion of a particle is considered in three separate situations:
- moving freely under gravity as a projectile
- moving in a horizontal circle with constant speed
- moving under the action of a variable force.

Topic	Learning Outcomes	Suggested Teaching Activities	Resources	On-Line Resources
1	**Motion of a projectile** • To model the motion of a projectile as a particle moving with constant acceleration and understand the limitations of the model.	Revise motion in a straight line with uniform speed and also with uniform acceleration, especially motion vertically under gravity. Discuss generally the limitations of the model, in particular how air resistance, the shape and dimensions of the projectile and changes in gravity can affect the motion.	There are professional packages available illustrating most of the ideas in this section.	www.Mathsrevision.net/alevel/mechanics/projectiles (revision notes)

	• To use the horizontal and vertical equations of motion to solve problems on the motion of projectiles, including finding the magnitude and direction of the velocity at a given time or position, the range on a horizontal plane and the greatest height reached.	Discuss the problem of what will happen to a particle being projected with a velocity having components 30m/s horizontally and 40 m/s vertically. Students should realise that the horizontal component remains constant (there is no force that can affect it) and the vertical component will change because of the effect of gravity. The independence of the vertical motion can be demonstrated (with a little practice) by a simple experiment. A coin or marble is projected (or rolled) along a horizontal table. A second coin (or marble) is released from rest at the edge of the table at the same instant that the first coin leaves the table. Both should hit the floor at the same time. It is then worthwhile providing a 2cm, 2mm piece of graph paper and requiring students to plot the calculated positions at times 1s, 2s, 3s etc. The students should realise that the horizontal displacements at these times are 30m, 60m, 90m etc. and should be able to show (using $s = ut + \frac{1}{2}at^2$ with $a = -g$) that the vertical displacements are 35m, 60m, 75m etc. The students should be able to plot the accurate trajectory for the motion. It is also worth introducing the ideas of "time of flight", "range" and "maximum height". Students should be able to find these from their graph. Discussion can then follow on how to find the components of speed at each of the times, 1s, 2s, 3s etc. They should realise that after 1s, the horizontal speed is still 30m/s and that the vertical speed (using $v = u + at$ with $a = -g$) is also 30m/s. They should be encouraged to deduce that the velocity after 1s is √1800 m/s at 45° to the horizontal.

Discussion should follow on how the time of flight, maximum height and range can be calculated without drawing a graph. Students should practice on other situations where the original components of velocity are given. | Practice illustrating this experiment.

2mm Graph paper.

OHP showing this trajectory

OHP showing this trajectory. | |

		The drawing of the graph should then be repeated, if time allows, for a given initial speed (say 40m/s) and direction of motion (say 30° to the horizontal). The students should be encouraged to realise that this is equivalent to the first graph with a horizontal component of 40cos30 and a vertical component of 40sin30.	OHP showing these proofs.	
		A bright group of students should be able to prove the formulae $H = \dfrac{V^2 \sin^2 \theta}{2g}$, $T = \dfrac{2V \sin \theta}{g}$ and $R = \dfrac{V^2 \sin 2\theta}{g}$, though these are not required for this syllabus.		
	• To derive and use the Cartesian equations of the trajectory of a particle, including problems in which the initial speed and/or angle of projection may be unknown.	Discussion should follow on how the equation of the curve (the equation of the trajectory) can be found for the graphs drawn above. Students should realise that the coordinates of the projectile at time t seconds after projection for the first example are $x = 30t$ and $y = 40t - 5t^2$. They should be able to derive the equation of this trajectory as $y = \dfrac{4x}{3} - \dfrac{x^2}{180}$ and $y = \dfrac{x}{\sqrt{3}} - \dfrac{x^2}{4800}$ for the second example. Students should be able to prove the general form of the trajectory as $y = x \tan \vartheta - \dfrac{gx^2}{2V^2 \cos^2 \vartheta}$. They should realise that this is a quadratic, either in x for given V, θ and y or in $\tan\theta$ for given V, x and y. Students should be presented with a variety of problems using all the ideas discussed in this section.	OHP showing this proof.	

| 3 | **Uniform motion in a circle**

• To understand the concept of angular speed for a particle moving in a circle, and use the relation *v=rω*.

• To understand that the acceleration of a particle moving in a circle with constant speed is directed towards the centre of the circle, and use the formulae *rω²* $and\ \dfrac{v^2}{r}$. | Discuss the concept of angular speed. Look at the example of 5 recruits *A, B, C, D* and *E* standing in a line, 1m apart. They march so that each turns in a circle about *A* and with such speeds that they are always in a straight line. They are also moving so that the angular speed is constant. Students should appreciate that the last recruit *E* will be moving with the greatest speed. If the angular speed is 0.4 radians/s, the students should appreciate that the speeds of *B, C, D* and *E* are 0.4, 0.8, 1.2 and 1.6 m/s respectively. This should enable them to develop the formula *v=rω*. Text books have a number of examples illustrating this formula and giving examples in converting such units as revolutions/minute (common in engineering and in the record industry) to radians/s or to degrees per s etc.

Discuss the nature of circular motion – of why a ball-bearing on a string can perform a horizontal circle on a smooth table. Students should realise that the ball-bearing has an acceleration towards the centre of the circle because the tension in the string has a horizontal component. Discuss what happens when the string breaks (there is now no horizontal force acting on the ball-bearing so it moves at constant speed along the tangent to its original path). Remind students that, even though the speed is constant, the direction is changing and this means that the particle is accelerating.. Discuss with students why it is possible to maintain a circular path, without the ball-bearing actually moving towards the centre of the circle. State, (or prove to brighter students), that the magnitude of the acceleration is $r\omega^2$ or $\dfrac{v^2}{r}$ and that there must be a force (the horizontal component of the tension in the string) equal in magnitude to $mr\omega^2$ or $m\dfrac{v^2}{r}$ (from Newton's second law) acting towards the centre of the circle. | OHP illustrating the example of 5 recruits marching so that they are always in a straight line. | www.mathsrevision.net/alevel/mechanics/motioninacircle revision notes. |

	• To solve problems which can be modelled by the motion of a particle moving in a horizontal circle with constant speed.	Look at different situations - a particle turning on a record turntable, a car turning a bend, a plane banking, a conical pendulum, the wall of death and other fairground rides. Although not specifically on the syllabus the example of a satellite orbiting the earth is probably the best example to give a student understanding of "circular motion". Students should realise that if a particle is moving in a horizontal circle, two equations can be found connecting the forces acting on the particle. • Firstly, since the particle stays at the same horizontal level, forces must balance in the vertical direction. • Secondly, since the particle is travelling in a circle, there must be a resultant force towards the centre of the circle to provide the "F" in the equation "$F=ma$". In this case the acceleration can be used in the form $r\omega^2$ or $\dfrac{v^2}{r}$ depending on the requirements of the question. Students should look specifically at the conical pendulum and be encouraged to write down the two equations connecting the forces for themselves. Students should have plenty of practice with as many different types of examples as possible.	OHP showing the forces acting in all of these situations	
5	**Linear Motion under a variable force**	Discuss generally with the students problems posed by such questions as "why does a parachutist reach a steady speed?" or "why does a cyclist moving on a horizontal road have a maximum speed?" or "why is there no danger from a bullet fired vertically upwards when it returns to the ground?" or "why does a feather and an apple released from the same point at the same time reach the ground at different times?". Look at the velocity-time graphs for these situations.	Have available on OHP the velocity-time graph for a free-fall parachutist.	

• To use the formulae $v = \dfrac{dx}{dt}$ and $a = \dfrac{dv}{dt} = v\dfrac{dv}{dx}$ as appropriate.	Reinforce the statement that acceleration is rate of change of velocity and velocity is rate of change of distance, both with respect to time. Show the students why acceleration can be expressed both as $\dfrac{dv}{dt}$ and as $v\dfrac{dv}{dx}$. Point out that using acceleration as $\dfrac{dv}{dt}$ with forces in terms of v and/or t in Newton's second law, leads to a differential equation in the variables *v* and *t*. Similarly using acceleration as $v\dfrac{dv}{dx}$ with forces in terms of *v* and/or *x* in Newton's second law, leads to a differential equation in the variables *v* and *x*.		
• Solve problems which can be modelled as the linear motion of a particle under the action of a variable force, by setting up and solving an appropriate differential equation (restricted to equations in which the variables are separable).	Point out to the students that questions on this topic generally come as two types: (1) The student is told that the acceleration (or retardation) is given by a particular formula. (2) The question involves a force which is variable. Use of *F=ma* reduces the question to that of type (1) above. Discuss in general terms situations where a force is variable – mentioning in particular (A) resistance proportional to speed, (*kv* or *kv²*), (B) tension proportional to extension (Hooke's law), (C) the variable force when an object moves under a constant power source (Power =*Fv).* Discuss with students, the solution of the differential equations obtained from the examples A,B and C above. (revise the section on differential equations from Syllabus P3).		

| | | Solve a variety of problems including that of a particle moving vertically upwards under gravity and subject to a resistance proportional to speed. Discuss with students why the differential equation for the upwards motion is different to that of the downwards motion. | | |
| | | Show the students that if air resistance is given by *kmv,* then the equation for a particle moving downwards is given by either $mg - kmv = m\dfrac{\mathrm{d}v}{\mathrm{d}t}$ or $mg - kmv = mv\dfrac{\mathrm{d}v}{\mathrm{d}x}$. In the first case, cancellation of *m* reduces the equation to $g - kv = \dfrac{\mathrm{d}v}{\mathrm{d}t}$. Point out to students that separation of the variables can only be achieved by dividing the equation by $(g - kv)$. | Have available on OHP the solution to a typical A Level question on motion under gravity with a resistance proportional to *v* or v^2. | |