MARK SCHEME for the October/November 2015 series

9231 FURTHER MATHEMATICS

9231/23   Paper 2, maximum raw mark 100

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Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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Question Number	Mark Scheme Details		Part Mark	Total
1	Find 3 independent equations for T, R_A, R_B :			
	Resolve horizontally:	$R_B = T \cos \alpha$ **M1 A1**		
	Resolve vertically:	$R_A = W + T \sin \alpha$ **M1 A1**		
	Take moments about A: (*a* may be omitted from moment eqns)	$R_B \, 3a \sin \theta = W(3a/2) \cos \theta$ $+ \, T \, a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ *or* $+ \, T \, a \sin(\alpha + \theta)$ *or* $+ \, T \, 3a \cos \theta \sin \alpha$ **M1 A1**		
	Take moments about B:	$R_A 3a \cos \theta = W(3a/2) \cos \theta$ $+ \, T \, 2a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ *or* $+ \, T \, 2a \sin(\alpha + \theta)$ *or* $+ \, T \, 3a \sin \theta \cos \alpha$ **(M1 A1)**		
	Take moments about C:	$R_A \, a \cos \theta + W(a/2) \cos \theta$ $= R_B \, 2a \sin \theta$ **(M1 A1)**		
	Take moments about D:	$R_A \, 3a \cos \theta - W(3a/2) \cos \theta$ $= R_B \, 3a \sin \theta$ **(M1 A1)**		
	Solve for T, R_A, R_B (AEF in W and α):	$T = W / 2 \sin \alpha$ *or* ½W cosec α **B1** $R_A = 3W / 2$ **B1** $R_B = W / 2 \tan \alpha$ *or* ½W cot α **B1**	9	**9**
2	For A & B use conservation of momentum,	e.g.: $2mv_A + mv_B = 2mu$ (allow $2v_A + v_B = 2u$) **M1**		
	Use Newton's law of restitution (consistent signs):	$v_B - v_A = eu$ **M1**		
	Combine to find v_A and v_B :	$v_A = (2 - e)\,u/3$, $v_B = 2(1 + e)\,u/3$ **A1, A1**	4	
	Find e from $v_A = \lvert v_B{}' \rvert$ with $v_B{}' = [-] \, 0{\cdot}4 \, v_B$:	$(2 - e) = 0{\cdot}8(1 + e)$, $e = 2/3$ **M1 A1**	2	
	EITHER: Equate times in terms of reqd. distance x: [speeds need not be found: Use $v_A = v_B{}' = 0{\cdot}4 \, v_B$ to solve for x:	$(d - x)/v_A = d/v_B + x/v_B{}'$ (AEF) **M1 A1** $v_A = v_B{}' = 4u/9$, $v_B = 10u/9$] $d - x = 0{\cdot}4 \, d + x$, $x = 0{\cdot}3 \, d$ **M1 A1**		
	OR: Find dist. moved by A when B reaches wall: Find reqd. distance x:	$d_A = (d/v_B)\,v_A = 0{\cdot}4 \, d$ **(M1 A1)** $x = ½(d - d_A) = 0{\cdot}3 \, d$ **(M1 A1)**	4	**10**

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3	Find k by equating equilibrium tensions: (vertical motion can earn M1 only)	$mg\,(a/2)/a = 2mg\,(3a/2 - ka)\,/\,ka$ **M1 A1** $\frac{1}{2} = 3/k - 2,\quad k = 6/5\ \text{or}\ 1{\cdot}2$ **A1**	3	
	Apply Newton's law at general point, e.g.: (lose A1 for each incorrect term)	$m\,\mathrm{d}^2x/\mathrm{d}t^2 = -mg\,(a/2 + x)/a$ $+ 2mg\,(3a/2 - ka - x)\,/\,ka$ $or\quad m\,\mathrm{d}^2y/\mathrm{d}t^2 = +mg\,(a/2 - y)/a$ $- 2mg\,(3a/2 - ka + y)\,/\,ka$ **M1 A2**		
	Simplify to give standard SHM eqn, e.g.: **S.R.**: B1 if no derivation (max 2/5)	$\mathrm{d}^2x/\mathrm{d}t^2 = -(1 + 2/k)gx\,/\,a$ $= -8gx/3a$ **A1**		
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$: ($\sqrt{}$ on ω)	$T = 2\pi\sqrt{(3a/8g)}\ or\ \pi\sqrt{(3a/2g)}$ $or\ 3{\cdot}85\sqrt{(a/g)}\ or\ 1{\cdot}22\sqrt{a}\ [\text{s}]$ **B1**✓	5	
	Substitute values in $v^2 = \omega^2\,(x_0^2 - x^2)$:	$0{\cdot}7^2 = (8g/3a)\{(0{\cdot}2a)^2 - (0{\cdot}05a)^2\}$ **M1 A1**		
	Solve to find numerical value of a:	$0{\cdot}49 = (8g/3) \times 0{\cdot}0375a,\ a = 0{\cdot}49$ **A1**	3	**11**

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4	*EITHER:* Find tension at top from $F = ma$ vertically: $$T = mu^2/a - mg \qquad \textbf{B1}$$			
	OR: Use energy at e.g. θ to upward vertical: $\frac{1}{2} mv^2 = \frac{1}{2} mu^2 + mga (1 - \cos \theta)$			
	Find tension T by using $F = ma$ radially: $T' = mv^2/a - mg \cos \theta$			
	Eliminate v^2: $= mu^2/a + mg (2 - 3 \cos \theta)$			
	Find T at top by taking $\theta = 0$: $T = mu^2/a - mg$ **(B1)**			
	Find u_{min} by requiring $T \geqslant 0$ at top [or $T > 0$]: $u^2/a - g \geqslant 0$ so $u_{min} = \sqrt{ag}$ **A.G.** **B1**		2	
	Find v at bottom from conservation of energy: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mg \times 2a$ **M1** $v^2 = ag + 4ag, \; v = \sqrt{5ag}$ **A1**			
	Find new speed V from conservation of momentum: $$m'V = mv \text{ with } m' = m + \frac{1}{4}m \qquad \textbf{M1}$$ $V = 4v/5 = 4\sqrt{(ag/5)}$ $or \quad (4/5) \sqrt{(5ag)}$ AEF **A1**		4	
	Find w^2 at angle θ from conservation of energy: (condone m instead of m' here since cancels out) $\frac{1}{2} m'w^2 = \frac{1}{2} m'V^2$ $- m'ga (1 + \cos \theta)$ **M1 A1** $[\; w^2 = ag (6/5 - 2 \cos \theta)]$			
	S.R. Invalid energy method (max 2/5): [gives $T' = (5mg/4)(2 - 3 \cos \theta)$] $\frac{1}{2} m'w^2 = \frac{1}{2} mu^2$ $+ mga (1 - \cos \theta)$ $- \frac{1}{4}mga (1 + \cos \theta)$ **(B1)**			
	Find tension T' there by using $F = ma$ radially: $T' = m'w^2/a - m'g \cos \theta$ **B1**			
	Eliminate w^2: $= m'V^2/a - m'g (2 + 3 \cos \theta)$ **A1**			
	Substitute for m' and V: $= (5mg/4)(6/5 - 3 \cos \theta)$ AEF $or \quad 3mg/2 - (15/4) mg \cos \theta$ **A1**		5	
	Find $\cos \theta$ when string becomes slack from $T' = 0$: $\cos \theta = \frac{1}{3} \times 6/5 = 2/5 \; or \; 0.4$ **M1 A1** **S.R.** Allow if found from $T' = mg (6/5 - 3 \cos \theta)$		2	**13**
5	Find or use sample mean $\bar{x} = 222.8 / 10 = 22.28$ and estimate population variance: $s^2 = 4.12 / 9$ (allow biased here: 0.412 *or* 0.642^2) $= 0.458$ *or* $103/225$ *or* 0.677^2 **M1**			
	Find confidence interval (allow z in place of t) e.g.: $22.28 \pm t \sqrt{(0.458 / 10)}$ **M1 A1**			
	Use of correct tabular value: $t_{9, \, 0.975} = 2.26[2]$ **A1**			
	Evaluate C.I. correct to 3 s.f.: $22.3 \pm 0.48[4]$ *or* $[21.8, 22.8]$ **A1**		5	**5**

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6	Find prob. p of head from mean = 2 × variance: $\quad 1/p = 2 \times (1 - p)/p^2, \quad p = \frac{2}{3}$ **A.G.** $\quad$ **M1 A1**		2	
(i)	Find P($X = 4$) (denoting $1 - p$ by q [$= \frac{1}{3}$]): $\qquad$ P($X = 4$) $= q^3 \times p$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 2/81 \;\; or \;\; 0.0247 \qquad$ **B1**		1	
(ii)	Find or state P($X > 4$): $\qquad$ P($X > 4$) $[= 1 - (1 + q + q^2 + q^3) \times p$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 1 - (1 - q^4)\,] \; = \; q^4$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 1/81 \;\; or \;\; 0.0123 \qquad$ **M1 A1**		2	
(iii)	Formulate condition for N: $\qquad\qquad 1 - q^N > 0.999, \;\; [\,(\frac{1}{3})^N < 0.001\,] \qquad$ **M1** Take logs (any base) to give bound for N: $\quad N > \log 0.001 \,/\, \log \frac{1}{3} \qquad$ **M1** Find N_{min}: $\qquad\qquad\qquad\qquad\qquad\qquad N > 6.29, \;\; N_{min} = 7 \qquad$ **A1** ($N < 6.29$ *or* $N = 6.29$ earns M2 A0)		3	**8**
7	Find F(x) for $1 \leqslant x \leqslant 4$: $\qquad\qquad$ F(x) $= (x^3 - 1)/63 \qquad$ **B1**			
	Find G(y) from $Y = X^2$ for $1 \leqslant x \leqslant 4$: $\quad$ G(y) $=$ P($Y < y$) $=$ P($X^2 < y$) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = $ P($X < y^{1/2}$) $=$ F($y^{1/2}$) (result may be stated) $\qquad\qquad\qquad\qquad = (y^{3/2} - 1)/63 \qquad$ **M1 A1**			
	Find g(y) for corresponding range of y: $\qquad$ g(y) $= y^{1/2}/42$ **A.G.** $\qquad$ **A1**			
	Find or state corresponding range of y: $\qquad 1 \leqslant y \leqslant 16$ **A.G.** $\qquad$ **B1**		5	
(i)	Find median value m of Y: $\qquad\qquad (m^{3/2} - 1)/63 = \frac{1}{2}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad m = 32.5^{2/3} = 10.2 \qquad$ **M1 A1**		2	
(ii)	Find E(Y) [or equivalently E(X^2)]: $\qquad$ E(Y) $= \int y\,$g(y) d$y = \int y^{3/2}$ d$y\,/42$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad = [y^{5/2}]_1^{16} /105 = 1023/105$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 341/35 \;\; or \;\; 9.74 \qquad$ **M1 A1**		2	**9**

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8	Find mean of sample data [for use in Poisson distn.]: $\lambda = 220/100 = 2\cdot2$	**B1**		
	State (at least) null hypothesis (AEF): H_0: Poisson distn. fits data or $\lambda = 2\cdot2$	**B1**		
	Find expected values $100\lambda^r e^{-\lambda}/r!$ (to 1 d.p.): $11\cdot080\ 24\cdot377\ 26\cdot814\ 19\cdot664$ (ignore incorrect final value here for M1) $10\cdot8151\ 4\cdot759\ 2\cdot491$	**M1 A1**		
	Combine last two cells so that exp. value $\geqslant 5$: O_i: 3 E_i: $7\cdot25$	**M1***		
	Calculate value of χ^2 (to 2 d.p.; A1 dep M1*): $\chi^2 = 0\cdot076 + 2\cdot879 + 0\cdot653 + 1\cdot448$ $+ 0\cdot441 + 2\cdot491$ (allow 7·95 if 1 d.p. exp.values used) $= 7\cdot99$	**M1 A1**		
	State or use consistent tabular value (to 3 s.f.): 5 cells: $\chi_{3,\,0.95}^2 = 7\cdot815$ 6 cells: $\chi_{4,\,0.95}^2 = 9\cdot488$ (correct) 7 cells: $\chi_{5,\,0.95}^2 = 11\cdot07$	**B1**		
	State or imply valid method for conclusion e.g.: Accept H_0 if $\chi^2 <$ tabular value	**M1**		
	Conclusion (AEF, requires both values correct): Distn fits or $\lambda = 2\cdot2$ Not combining cells [so $\chi^2 = 8\cdot64$] can earn B1 B1 M1 A1 M0 M1 B1 M1 (max 7)	**A1**	10	**10**
9	Calculate gradient b_1 in $y - \bar{y} = b_1(x - \bar{x})$: $S_{xy} = 24\,879 - 472 \times 400/8$ $= 1\,279$ $S_{xx} = 29\,950 - 472^2/8 = 2\,102$ $b_1 = S_{xy} / S_{xx} = 0\cdot6085$ (3 s.f.)	**M1 A1**		
	Find regression line of y on x: $y = 400/8 + b_1 (x - 472/8)$ $= 50 + 0\cdot6085 (x - 59)$ $= 0\cdot6085x + 14\cdot1$	**M1 A1**		
	Find y when $x = 72$: $= 57\cdot9$ or 58 Allow use of regression line of x on y (since neither variable clearly independent): $S_{yy} = 21\,226 - 400^2/8 = 1\,226$ $b_2 = S_{xy} / S_{yy} = 1\cdot043$ $x = 472/8 + b_2 (y - 400/8)$ $= 1\cdot043\,y + 6\cdot85$ $Y = 62\cdot5$ or 62	**(M1 A1)** **(M1 A1)** **(A1)**	5	

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	Find product moment correlation coefficient r:	$r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}$ or $\sqrt{(0.6085 \times 1.043)} = 0.797$ **M1 A1***		
	State both hypotheses (B0 for r …):	$H_0: \rho = 0$, $H_1: \rho \neq 0$ **B1**		
	State or use correct tabular two-tail r-value:	$r_{8,\,5\%} = 0.707$ **B1***		
	State or imply valid method for conclusion e.g.:	Reject H_0 if $|r| >$ tab. value (AEF) **M1**		
	Correct conclusion (AEF, dep A1*, B1*):	There is non-zero correlation **A1**	6	**11**
10A	Find MI of lamina about Q:	$I_{\text{lamina}} = \tfrac{1}{3}m\{(3a)^2 + (3a/2)^2\}$ $\quad + m(9a/2)^2$ **M1 A1** $[= (15/4 + 81/4)\,ma^2 = 24\,ma^2]$		
	State or find MI of rod about Q:	$I_{\text{rod}} = (\tfrac{1}{3} + 1)\,M\,(3a/2)^2$ $[= 3Ma^2]$ **B1**		
	Sum to find MI of object about Q:	$I_1 = 24\,ma^2 + 3\,Ma^2$ $= 3\,(8m + M)\,a^2$ **A.G.** **A1**	4	
	Find MI of object about mid-point of PQ:	$I_2 = (15/4 + 3^2)\,ma^2 + \tfrac{1}{3}\,M\,(3a/2)^2$ $= (51/4)\,ma^2 + \tfrac{3}{4}\,Ma^2$ $= \tfrac{3}{4}\,(17m + M)\,a^2$ **A.G.** **M1 A1**	2	
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_1:	$[-]I_1\,d^2\theta/dt^2 = mg \times (9a/2)\sin\theta$ $\quad + Mg \times (3a/2)\sin\theta$ **M1 A1** $[= (9m/2 + 3M/2)\,ga\sin\theta]$		
	[Approximate $\sin\theta$ by θ and] find ω_1^2 in SHM eqn:	$\omega_1^2 = (3m + M)g / 2(8m + M)\,a$ **M1**		
	Find period T_1 for axis l_1 from $2\pi/\omega_1$: (AEF)	$T_1 = 2\pi\sqrt{\{2(8m + M)\,a / (3m + M)g\}}$ **A1**		
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_2:	$[-]I_2\,d^2\theta/dt^2 = mg \times 3a\sin\theta$ **M1**		
	[Approximate $\sin\theta$ by θ and] find ω_2^2 in SHM eqn:	$\omega_2^2 = 4mg / (17m + M)\,a$ **M1**		
	Find period T_2 for axis l_2 from $2\pi/\omega_2$: (AEF)	$T_2 = 2\pi\sqrt{\{(17m + M)\,a / 4mg\}}$ **A1**		
	Verify that $T_1 = T_2$ when $m = M$: (AEF) [Taking $m = M$ throughout 2nd part can earn: M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)]	$T_1 = 2\pi\sqrt{(18\,a / 4g)} = T_2$ **B1**	8	**14**

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10B	State hypotheses (B0 for $\bar{x}$ …), e.g.:	H_0: $\mu_X = \mu_Y$, H_1: $\mu_X \neq \mu_Y$ **B1**		
	State assumption (AEF):	Distributions have equal variances **B1**		
	Find sample means <u>and</u> estimate popln. variances: (allow biased here: 0.36 or 0.6^2) (allow biased here: 0.4183 or 0.6468^2)	$\bar{x} = 4.2$, $\bar{y} = 4.8$ $s_X^2 = (180 - 42^2/10) / 9$ $= 0.4$ or 0.6325^2 $s_Y^2 = (281.5 - 57.6^2/12) / 11$ $= 0.4564$ or $251/550$ or 0.6755^2 **M1**		
	Estimate (pooled) common variance: (note s_X^2 and s_Y^2 not needed explicitly)	$s^2 = (9\,s_X^2 + 11\,s_Y^2) / 20$ (AEF) or $(180 - 42^2/10 + 281.5 - 57.6^2/12) / 20$ $= 0.431$ or 0.6565^2 **M1 A1**		
	Calculate value of t (to 3 s.f.):	$[-]\ t = (\bar{y} - \bar{x}) / s \sqrt{(1/10 + 1/12)}$ $= 2.13$ **M1 A1**		
	State or use correct tabular t value: (or can compare $\bar{y} - \bar{x} = 0.6$ with 0.586)	$t_{20, 0.975} = 2.086$ [allow 2.09] **B1***		
	Correct conclusion (AEF, √ on t, dep *B1): **S.R.** Implicitly taking s_X^2, s_Y^2 as popln. variances: (may also earn first B1 B1 M1) Comparison with $z_{0.975}$ and conclusion: (can earn at most 5/9)	$t > 2.09$ so mean masses not same **B1√** $z = (\bar{y} - \bar{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)}$ $= 0.6 / \sqrt{(0.078} = 2.15$ $2.15 > 1.96$ so mean masses not same **(B1)**	9	
	State hypotheses (B0 for $\bar{x}$ …), e.g.:	H_0: $\mu_X = 3.8$, H_1: $\mu_X > 3.8$ or H_0: $\mu_X = \mu_Z$, H_1: $\mu_X > \mu_Z$ **B1**		
	Calculate value of t using s_X from above:	$t = (4.2 - 3.8) / (s_X/\sqrt{10}) = 2.0$ **M1 A1**		
	State or use correct tabular t value: (or can compare 0.4 with 0.367)	$t_{9, 0.95} = 1.833$ [allow 1.83] **B1***		
	Correct conclusion (A.E.F., √ on t, dep *B1):	$t > 1.833$, so claim is justified *or* mean mass of Royals > mean mass of Crowns **B1√**	5	**14**