UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/11**

Paper 1 **October/November 2011**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

JC11 11_9231_11/2R

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1 The equation $x^3 + px + q = 0$ has a repeated root. Prove that $4p^3 + 27q^2 = 0$. [5]

2 The position vectors of points A, B, C, relative to the origin O, are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, where

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \qquad \mathbf{b} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{c} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Find $\mathbf{a} \times \mathbf{b}$ and deduce the area of the triangle OAB. [3]

Hence find the volume of the tetrahedron $OABC$, given that the volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height. [2]

3 Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \sin x) = 2^{\frac{1}{2}n}\mathrm{e}^x \sin\left(x + \tfrac{1}{4}n\pi\right).$$ [7]

4 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 3 & 4 & 2 & 5 \\ 6 & 7 & 5 & 8 \\ 9 & 9 & 9 & 9 \\ 15 & 16 & 14 & 17 \end{pmatrix}.$$

Find

(i) the rank of $\mathbf{M}$ and a basis for the range space of T, [4]

(ii) a basis for the null space of T. [3]

5 The point $P\,(2,\ 1)$ lies on the curve with equation

$$x^3 - 2y^3 = 3xy.$$

Find

(i) the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at P, [3]

(ii) the value of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at P. [4]

6 Let $I_n = \displaystyle\int_0^1 x^n(1-x)^{\frac{1}{2}}\,\mathrm{d}x$, for $n \geqslant 0$. Show that, for $n \geqslant 1$,

$$(3 + 2n)I_n = 2nI_{n-1}.$$ [5]

Hence find the exact value of I_3. [3]

7 The curve C has equation $y = \dfrac{x^2 + px + 1}{x - 2}$, where p is a constant. Given that C has two asymptotes, find the equation of each asymptote. [3]

Find the set of values of p for which C has two distinct turning points. [5]

Sketch C in the case $p = -1$. Your sketch should indicate the coordinates of any intersections with the axes, but need not show the coordinates of any turning points. [3]

8 The vector $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{A}$, with corresponding eigenvalue λ, and is also an eigenvector of the matrix $\mathbf{B}$, with corresponding eigenvalue μ. Show that $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{AB}$ with corresponding eigenvalue $\lambda\mu$. [2]

State the eigenvalues of the matrix $\mathbf{C}$, where

$$\mathbf{C} = \begin{pmatrix} -1 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix},$$

and find corresponding eigenvectors. [4]

Show that $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{D}$, where

$$\mathbf{D} = \begin{pmatrix} 1 & -1 & 1 \\ -6 & -3 & 4 \\ -9 & -3 & 7 \end{pmatrix},$$

and state the corresponding eigenvalue. [3]

Hence state an eigenvector of the matrix $\mathbf{CD}$ and give the corresponding eigenvalue. [2]

9 The curve C has equation $y = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)$ for $0 \leqslant x \leqslant \ln 5$. Find

(i) the mean value of y with respect to x over the interval $0 \leqslant x \leqslant \ln 5$, [4]

(ii) the arc length of C, [4]

(iii) the surface area generated when C is rotated through 2π radians about the x-axis. [4]

10 The curve C has polar equation $r = 3 + 2\cos\theta$, for $-\pi < \theta \leqslant \pi$. The straight line l has polar equation $r\cos\theta = 2$. Sketch both C and l on a single diagram. [3]

Find the polar coordinates of the points of intersection of C and l. [4]

The region R is enclosed by C and l, and contains the pole. Find the area of R. [6]

[Question 11 is printed on the next page.]

11 Answer only **one** of the following two alternatives.

EITHER

Let $\omega = \cos \frac{1}{5}\pi + \mathrm{i} \sin \frac{1}{5}\pi$. Show that $\omega^5 + 1 = 0$ and deduce that

$$\omega^4 - \omega^3 + \omega^2 - \omega = -1.$$ [2]

Show further that

$$\omega - \omega^4 = 2\cos \tfrac{1}{5}\pi \quad \text{and} \quad \omega^3 - \omega^2 = 2\cos \tfrac{3}{5}\pi.$$ [4]

Hence find the values of

$$\cos \tfrac{1}{5}\pi + \cos \tfrac{3}{5}\pi \quad \text{and} \quad \cos \tfrac{1}{5}\pi \cos \tfrac{3}{5}\pi.$$ [4]

Find a quadratic equation having roots $\cos \frac{1}{5}\pi$ and $\cos \frac{3}{5}\pi$ and deduce the exact value of $\cos \frac{1}{5}\pi$. [4]

OR

Given that

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4x(1 + x)\frac{\mathrm{d}y}{\mathrm{d}x} + 2(1 + 4x + 2x^2)y = 8x^2$$

and that $x^2 y = z$, show that

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + 4\frac{\mathrm{d}z}{\mathrm{d}x} + 4z = 8x^2.$$ [4]

Find the general solution for y in terms of x. [8]

Describe the behaviour of y as $x \to \infty$. [2]