ALGEBRA

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(i) Expand $\frac{1}{\sqrt{(1-4x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [3]

(ii) Hence find the coefficient of x^2 in the expansion of $\frac{1+2x}{\sqrt{(4-16x)}}$. [2]

(i) $1+2x+6x^2$
(ii) 5

MAY/JUNE2012 9709/31

The polynomial $p(x)$ is defined by

$$p(x) = x^3 - 3ax + 4a,$$

where a is a constant.

(i) Given that $(x-2)$ is a factor of $p(x)$, find the value of a. [2]

(ii) When a has this value,

(a) factorise $p(x)$ completely, [3]

(b) find all the roots of the equation $p(x^2) = 0$. [2]

(i) 4
(i) (a) $(x-2)^2(x+4)$ (b) $\pm\sqrt{2}, \pm 2i$

MAY/JUNE2012 9709/32

Expand $\sqrt{\left(\dfrac{1-x}{1+x}\right)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [5]

$1 - x + \frac{1}{2}x^2$

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Expand $\dfrac{1}{\sqrt{(4+3x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

$\frac{1}{2} - \frac{3}{16}x + \frac{27}{256}x^2$

OCTOBER/NOVEMBER 2011 9709/31

The polynomial $x^4 + 3x^3 + ax + 3$ is denoted by $\text{p}(x)$. It is given that $\text{p}(x)$ is divisible by $x^2 - x + 1$.

(i) Find the value of a. [4]

(ii) When a has this value, find the real roots of the equation $\text{p}(x) = 0$. [2]

(i) 1
(ii) -3, -1

OCTOBER/NOVEMBER 2011 9709/33

Expand $\dfrac{16}{(2+x)^2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

$-4x + 3x^2$

OCTOBER/NOVEMBER 2011 9709/33

The polynomial $p(x)$ is defined by

$$p(x) = ax^3 - x^2 + 4x - a,$$

where a is a constant. It is given that $(2x - 1)$ is a factor of $p(x)$.

(i) Find the value of a and hence factorise $p(x)$. [4]

(ii) When a has the value found in part **(i)**, express $\dfrac{8x - 13}{p(x)}$ in partial fractions. [5]

(i) $a = 2, (2x-1)(x^2+2)$

(ii) $-\frac{4}{2x-1} + \frac{2x+5}{x^2+2}$

MAY/JUNE2011 9709/31

Expand $\sqrt[3]{(1 - 6x)}$ in ascending powers of x up to and including the term in x^3, simplifying the coefficients. [4]

$1 - 2x - 4x^2 - \dfrac{40}{3}x^3$

MAY/JUNE2011 9709/32

Solve the inequality $|x| < |5 + 2x|$. [3]

$x < -5, x > -\frac{5}{3}$

MAY/JUNE2011 9709/32

(i) Express $\dfrac{5x - x^2}{(1 + x)(2 + x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{5x - x^2}{(1 + x)(2 + x^2)}$ in ascending powers of x, up to and including the term in x^3. [5]

(i) $-\frac{2}{1+x} + \frac{x+4}{2+x^2}$

(ii) $\frac{5}{2}x - 3x^2 + \frac{7}{4}x^3$

MAY/JUNE2011 9709/33

The polynomial $ax^3 + bx^2 + 5x - 2$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x - 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x - 2)$ the remainder is 12.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the quadratic factor of $p(x)$. [2]

(i) $a = 2, b = -3$
(ii) $x^2 - x + 2$

OCTOBER/NOVEMBER 2010 9709/31

Solve the inequality $2|x - 3| > |3x + 1|$. [4]

$-7 < x < 1$

OCTOBER/NOVEMBER 2010 9709/31

Let $f(x) = \dfrac{3x}{(1+x)(1+2x^2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [5]

(i) $-\frac{1}{1+x}+\frac{2x+!}{1+2x^2}$

(ii) $3x-3x^2-3x^3$

OCTOBER/NOVEMBER 2010 9709/33

Expand $(1+2x)^{-3}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [3]

$1-6x+24x^2$

MAY/JUNE2010 9709/31

Solve the inequality $|x + 3a| > 2|x - 2a|$, where a is a positive constant. [4]

$\frac{1}{3}a < x < 7a$

MAY/JUNE2010 9709/32

The polynomial $2x^3 + 5x^2 + ax + b$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x + 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x + 2)$ the remainder is 9.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $p(x)$ completely. [3]

(i) $a = -4, b = -3$
(ii) $(2x + 1)(x + 3)(x - 1)$

MAY/JUNE2010 9709/33

Solve the inequality $|x - 3| > 2|x + 1|$. [4]

$-5 < x < \frac{1}{3}$

MAY/JUNE2010 9709/33

(i) Express $\dfrac{4 + 5x - x^2}{(1 - 2x)(2 + x)^2}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{4 + 5x - x^2}{(1 - 2x)(2 + x)^2}$ in ascending powers of x, up to and including the term in x^2. [5]

(i) $\frac{1}{1-2x} + \frac{1}{2+x} - \frac{2}{(2+x)^2}$

(ii) $1 + \frac{9}{4}x + \frac{15}{4}x^2$

OCTOBER/NOVEMBER 2009 9709/31

Solve the inequality $2 - 3x < |x - 3|$. [4]

$x > -\frac{1}{2}$

OCTOBER/NOVEMBER 2009 9709/31

(i) Express $\dfrac{5x+3}{(x+1)^2(3x+2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{5x+3}{(x+1)^2(3x+2)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [5]

(i) $\frac{1}{x+1} + \frac{2}{(x+1)^2} - \frac{3}{3x+2}$

(ii) $\frac{3}{2} - \frac{11}{4}x + \frac{29}{8}x^2$

OCTOBER/NOVEMBER 2009 9709/32

The polynomial $2x^3 + ax^2 + bx - 4$, where a and b are constants, is denoted by $\mathrm{p}(x)$. The result of differentiating $\mathrm{p}(x)$ with respect to x is denoted by $\mathrm{p}'(x)$. It is given that $(x+2)$ is a factor of $\mathrm{p}(x)$ and of $\mathrm{p}'(x)$.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $\mathrm{p}(x)$ completely. [3]

(i) $a = 7, b = 4$
(ii) $(x+2)^2(2x-1)$

OCTOBER/NOVEMBER 2009 9709/32

(i) Express $\dfrac{1+x}{(1-x)(2+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{1+x}{(1-x)(2+x^2)}$ in ascending powers of x, up to and including the term in x^2. [5]

(i) $\frac{2}{3(1-x)} + \frac{2x+1}{3(2+x^2)}$
(ii) $\frac{1}{2} + x + \frac{3}{4}x^2$

MAY/JUNE2009

When $(1+2x)(1+ax)^{\frac{2}{3}}$, where a is a constant, is expanded in ascending powers of x, the coefficient of the term in x is zero.

(i) Find the value of a. [3]

(ii) When a has this value, find the term in x^3 in the expansion of $(1+2x)(1+ax)^{\frac{2}{3}}$, simplifying the coefficient. [4]

(i) -3

(ii) $-\frac{10}{3}x^3$

OCTOBER/NOVEMBER 2008

Expand $(1+x)\sqrt{(1-2x)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

$1-\frac{3}{2}x^2$

OCTOBER/NOVEMBER 2008

The polynomial $4x^3 - 4x^2 + 3x + a$, where a is a constant, is denoted by $p(x)$. It is given that $p(x)$ is divisible by $2x^2 - 3x + 3$.

(i) Find the value of a. [3]

(ii) When a has this value, solve the inequality $p(x) < 0$, justifying your answer. [3]

(i) 3

(ii) $x < -\frac{1}{2}$

MAY/JUNE2008

Solve the inequality $|x - 2| > 3|2x + 1|$. [4]

$-1 < x < -\dfrac{1}{7}$

OCTOBER/NOVEMBER 2007

The polynomial $x^4 + 3x^2 + a$, where a is a constant, is denoted by $\text{p}(x)$. It is given that $x^2 + x + 2$ is a factor of $\text{p}(x)$. Find the value of a and the other quadratic factor of $\text{p}(x)$. [4]

$a = 4, x^2 - x + 2$

OCTOBER/NOVEMBER 2007

(i) Express $\dfrac{2 - x + 8x^2}{(1 - x)(1 + 2x)(2 + x)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{2 - x + 8x^2}{(1 - x)(1 + 2x)(2 + x)}$ in ascending powers of x, up to and including the term in x^2. [5]

(i) $\frac{1}{1-x} + \frac{2}{1+3x} - \frac{4}{2-z}$
(ii) $1 - 2x + \frac{17}{2}x^2$

MAY/JUNE2007

Expand $(2 + 3x)^{-2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. **[4]**

$\frac{1}{4} - \frac{3}{4}x + \frac{27}{16}x^2$

MAY/JUNE2007

The polynomial $x^3 - 2x + a$, where a is a constant, is denoted by $\mathrm{p}(x)$. It is given that $(x + 2)$ is a factor of $\mathrm{p}(x)$.

(i) Find the value of a. [2]

(ii) When a has this value, find the quadratic factor of $\mathrm{p}(x)$. [2]

(i) 4
(ii) $x^2 - 2x + 2$

OCTOBER/NOVEMBER 2006

(i) Simplify $\left(\sqrt{(1+x)}+\sqrt{(1-x)}\right)\left(\sqrt{(1+x)}-\sqrt{(1-x)}\right)$, showing your working, and deduce that

$$\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}}=\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{2x}.$$ [2]

(ii) Using this result, or otherwise, obtain the expansion of

$$\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}}$$

in ascending powers of x, up to and including the term in x^2. [4]

(ii) $\frac{1}{2}+\frac{1}{16}x^2$

MAY/JUNE2006

Solve the inequality $2x > |x-1|$. [4]

$x > \frac{1}{3}$

MAY/JUNE2006

(i) Express $\dfrac{10}{(2-x)(1+x^2)}$ in partial fractions. [5]

(ii) Hence, given that $|x| < 1$, obtain the expansion of $\dfrac{10}{(2-x)(1+x^2)}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [5]

(i) $\frac{2}{2-x}+\frac{2x+4}{1+x^2}$

(ii) $5+\frac{5}{2}x-\frac{15}{4}x^2-\frac{15}{8}x^3$

OCTOBER/NOVEMBER 2005

Given that a is a positive constant, solve the inequality

$$|x-3a| > |x-a|.$$ [4]

$x < 2a$

OCTOBER/NOVEMBER 2005

(i) Express $\dfrac{3x^2 + x}{(x+2)(x^2+1)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{3x^2 + x}{(x+2)(x^2+1)}$ in ascending powers of x, up to and including the term in x^3. [5]

(i) $\frac{2}{x+2} + \frac{x-1}{x^2+1}$

(ii) $\frac{1}{2}x + \frac{5}{4}x^2 - \frac{9}{8}x^3$

MAY/JUNE2005

Expand $(1 + 4x)^{-\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

$1 - 2x + 6x^2 - 20x^3$

MAY/JUNE2005

The polynomial $x^4 + 5x + a$ is denoted by $p(x)$. It is given that $x^2 - x + 3$ is a factor of $p(x)$.

(i) Find the value of a and factorise $p(x)$ completely. [6]

(ii) Hence state the number of real roots of the equation $p(x) = 0$, justifying your answer. [2]

(i) $a = -6, (x^2 - x + 3)(x + 2)(x - 1)$

(ii) two real roots. $x^2 - x + 3 = 0$ has no real root.

OCTOBER/NOVEMBER 2004

Expand $\dfrac{1}{(2+x)^3}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

$\dfrac{1}{8} - \dfrac{3}{16}x + \dfrac{3}{16}x^2$

OCTOBER/NOVEMBER 2004

The polynomial $2x^3 + ax^2 - 4$ is denoted by $p(x)$. It is given that $(x - 2)$ is a factor of $p(x)$.

(i) Find the value of a. [2]

When a has this value,

(ii) factorise $p(x)$, [2]

(iii) solve the inequality $p(x) > 0$, justifying your answer. [2]

(i) -3

(ii) $(x-2)(2x^2+x+2)$

(iii) $x > 2$

MAY/JUNE2004

Solve the inequality $|2x + 1| < |x|$. [4]

$-1 < x < -\frac{1}{3}$

MAY/JUNE2004

Let $f(x) = \dfrac{x^2 + 7x - 6}{(x-1)(x-2)(x+1)}$.

(i) Express $f(x)$ in partial fractions. [4]

(ii) Show that, when x is sufficiently small for x^4 and higher powers to be neglected,

$$f(x) = -3 + 2x - \tfrac{3}{2}x^2 + \tfrac{11}{4}x^3. \qquad [5]$$

(i) $-\frac{1}{x-1} + \frac{4}{x-2} - \frac{2}{x+1}$

OCTOBER/NOVEMBER 2003

Expand $(2 + x^2)^{-2}$ in ascending powers of x, up to and including the term in x^4, simplifying the coefficients. [4]

$\frac{1}{4} - \frac{1}{4}x^2 + \frac{3}{16}x^4$

MAY/JUNE2003

Solve the inequality $|x-2| < 3-2x$. [4]

$x < 1$

MAY/JUNE2003

The polynomial $x^4 - 2x^3 - 2x^2 + a$ is denoted by $f(x)$. It is given that $f(x)$ is divisible by $x^2 - 4x + 4$.

(i) Find the value of a. [3]

(ii) When a has this value, show that $f(x)$ is never negative. [4]

(i) a=8

MAY/JUNE2003

Let $f(x) = \dfrac{9x^2 + 4}{(2x + 1)(x - 2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that, when x is sufficiently small for x^3 and higher powers to be neglected,

$$f(x) = 1 - x + 5x^2.$$ [4]

(i) $\frac{1}{2x+1} + \frac{4}{x-2} + \frac{8}{(x-2)^2}$

OCTOBER/NOVEMBER 2002

Solve the inequality $|9 - 2x| < 1$. [3]

$4 < x < 5$

OCTOBER/NOVEMBER 2002

Let $f(x) = \dfrac{6+7x}{(2-x)(1+x^2)}$.

(i) Express $f(x)$ in partial fractions. [4]

(ii) Show that, when x is sufficiently small for x^4 and higher powers to be neglected,

$$f(x) = 3 + 5x - \tfrac{1}{2}x^2 - \tfrac{15}{4}x^3. \qquad [5]$$

(i) $\frac{4}{2-x} + \frac{4x+1}{x^2+1}$

MAY/JUNE2002

Expand $(1-3x)^{-\frac{1}{3}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

$1 + x + 2x^2 + \dfrac{14}{3}x^3$

MAY/JUNE2002

The polynomial $x^4 + 4x^2 + x + a$ is denoted by $p(x)$. It is given that $(x^2 + x + 2)$ is a factor of $p(x)$.

Find the value of a and the other quadratic factor of $p(x)$. [4]

$a = 6, x^2 - x + 3$

LOGARITHMIC & EXPONENTIAL FUNCTIONS

MAY/JUNE2012 9709/31

Solve the equation $|4 - 2^x| = 10$, giving your answer correct to 3 significant figures. [3]

3.81

MAY/JUNE2012 9709/32

Solve the equation

$$\ln(3x + 4) = 2\ln(x + 1),$$

giving your answer correct to 3 significant figures. [4]

2.30

MAY/JUNE2012 9709/33

Solve the equation $\ln(2x + 3) = 2\ln x + \ln 3$, giving your answer correct to 3 significant figures. [4]

1.39

OCTOBER/NOVEMBER 2011 9709/31

Using the substitution $u = e^x$, or otherwise, solve the equation

$$e^x = 1 + 6e^{-x},$$

giving your answer correct to 3 significant figures. [4]

1.10

MAY/JUNE2011 9709/31

The polynomial $f(x)$ is defined by

$$f(x) = 12x^3 + 25x^2 - 4x - 12.$$

(i) Show that $f(-2) = 0$ and factorise $f(x)$ completely. [4]

(ii) Given that

$$12 \times 27^y + 25 \times 9^y - 4 \times 3^y - 12 = 0,$$

state the value of 3^y and hence find y correct to 3 significant figures. [3]

(i) $(x+2)(4x+3)(3x-2)$
(ii) -0.369

MAY/JUNE2011 9709/32

(i) Show that the equation

$$\log_2(x+5) = 5 - \log_2 x$$

can be written as a quadratic equation in x. [3]

(ii) Hence solve the equation

$$\log_2(x+5) = 5 - \log_2 x.$$ [2]

(ii) 3.68

MAY/JUNE2011 9709/33

Use logarithms to solve the equation $5^{2x-1} = 2(3^x)$, giving your answer correct to 3 significant figures. [4]

1.09

OCTOBER/NOVEMBER 2010 9709/31

Solve the equation

$$\ln(1 + x^2) = 1 + 2\ln x,$$

giving your answer correct to 3 significant figures. [4]

0.763

MAY/JUNE2010 9709/31

The variables x and y satisfy the equation $x^n y = C$, where n and C are constants. When $x = 1.10$, $y = 5.20$, and when $x = 3.20$, $y = 1.05$.

(i) Find the values of n and C. [5]

(ii) Explain why the graph of $\ln y$ against $\ln x$ is a straight line. [1]

(i) $n = 1.50, C = 6.00$

MAY/JUNE2010 9709/32

Solve the equation

$$\frac{2^x + 1}{2^x - 1} = 5,$$

giving your answer correct to 3 significant figures. [4]

0.585

MAY/JUNE2010 9709/33

The variables x and y satisfy the equation $y^3 = Ae^{2x}$, where A is a constant. The graph of $\ln y$ against x is a straight line.

(i) Find the gradient of this line. [2]

(ii) Given that the line intersects the axis of $\ln y$ at the point where $\ln y = 0.5$, find the value of A correct to 2 decimal places. [2]

(i) $\frac{2}{3}$

(ii) 4.48

OCTOBER/NOVEMBER 2009 9709/31

Solve the equation $3^{x+2} = 3^x + 3^2$, giving your answer correct to 3 significant figures. [4]

0.107

OCTOBER/NOVEMBER 2009 9709/32

Solve the equation

$$\ln(5 - x) = \ln 5 - \ln x,$$

giving your answers correct to 3 significant figures. [4]

1.38, 3.62

MAY/JUNE2009

Solve the equation $\ln(2 + e^{-x}) = 2$, giving your answer correct to 2 decimal places. [4]

-1.68

OCTOBER/NOVEMBER 2008

Solve the equation

$$\ln(x + 2) = 2 + \ln x,$$

giving your answer correct to 3 decimal places. [3]

0.313

MAY/JUNE2008

Solve, correct to 3 significant figures, the equation

$$e^x + e^{2x} = e^{3x}.$$ [5]

0.481

MAY/JUNE2007

Using the substitution $u = 3^x$, or otherwise, solve, correct to 3 significant figures, the equation

$$3^x = 2 + 3^{-x}.$$ **[6]**

0.802

OCTOBER/NOVEMBER 2006

Find the set of values of x satisfying the inequality $|3^x - 8| < 0.5$, giving 3 significant figures in your answer. [4]

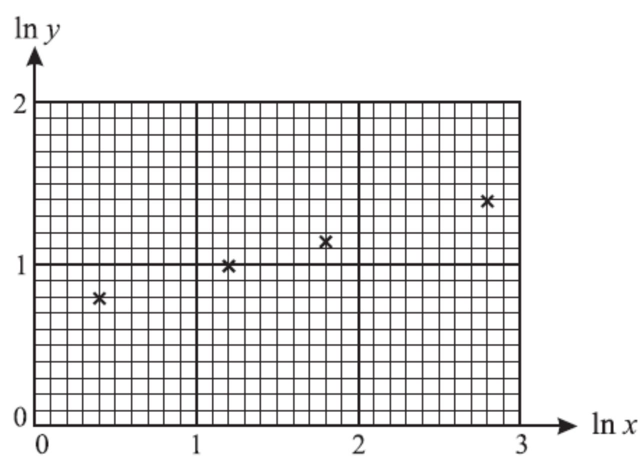
$1.83 < x < 1.95$

MAY/JUNE2006

Given that $x = 4(3^{-y})$, express y in terms of x. [3]

$y = \dfrac{ln4 - lnx}{ln3}$

OCTOBER/NOVEMBER 2005



Two variable quantities x and y are related by the equation $y = Ax^n$, where A and n are constants. The diagram shows the result of plotting $\ln y$ against $\ln x$ for four pairs of values of x and y. Use the diagram to estimate the values of A and n. [5]

$A = 1.97\ to\ 2.03, n = 0{,}25$

OCTOBER/NOVEMBER 2004

Solve the equation

$$\ln(1 + x) = 1 + \ln x,$$

giving your answer correct to 2 significant figures. [4]

0.58

MAY/JUNE2004

(i) Show that if $y = 2^x$, then the equation

$$2^x - 2^{-x} = 1$$

can be written as a quadratic equation in y. [2]

(ii) Hence solve the equation

$$2^x - 2^{-x} = 1.$$ [4]

(i) $y^2 - y - 1 = 0$

(ii) 0.694

OCTOBER/NOVEMBER 2003

Solve the inequality $|2^x - 8| < 5$. [4]

$1.58 < x < 3.70$

OCTOBER/NOVEMBER 2002

(i) Show that the equation

$$\log_{10}(x+5) = 2 - \log_{10} x$$

may be written as a quadratic equation in x. [3]

(ii) Hence find the value of x satisfying the equation

$$\log_{10}(x+5) = 2 - \log_{10} x.$$ [2]

(i) $x^2 + 5x = 100$

(ii) 7.81

TRIGONOMETRY

MAY/JUNE2012 9709/32

Solve the equation

$$\operatorname{cosec} 2\theta = \sec\theta + \cot\theta,$$

giving all solutions in the interval $0^\circ < \theta < 360^\circ$. [6]

201.5°, 338.5°

MAY/JUNE2012 9709/33

It is given that $\tan 3x = k\tan x$, where k is a constant and $\tan x \neq 0$.

(i) By first expanding $\tan(2x + x)$, show that

$$(3k - 1)\tan^2 x = k - 3.$$ [4]

(ii) Hence solve the equation $\tan 3x = k\tan x$ when $k = 4$, giving all solutions in the interval $0^\circ < x < 180^\circ$. [3]

(iii) Show that the equation $\tan 3x = k\tan x$ has no root in the interval $0^\circ < x < 180^\circ$ when $k = 2$. [1]

(ii) 16.8°, 163.2°

OCTOBER/NOVEMBER 2011 9709/31

(i) Express $\cos x + 3\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $\cos 2\theta + 3\sin 2\theta = 2$, for $0^\circ < \theta < 90^\circ$. [5]

(i) $R = \sqrt{10}, \alpha = 71.57°$
(ii) 61.2°, 10.4°

OCTOBER/NOVEMBER 2011 9709/33

(i) Express $8\cos\theta + 15\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $8\cos\theta + 15\sin\theta = 12$, giving all solutions in the interval $0^\circ < \theta < 360^\circ$. [4]

(i) $R = 17, \alpha = 61.93°$
(ii) 107.0°, 16.8°

MAY/JUNE2011 9709/31

(i) Prove the identity $\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3$. [4]

(ii) Hence

(a) solve the equation $\cos 4\theta + 4\cos 2\theta = 1$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, [3]

(b) find the exact value of $\int_0^{\frac{1}{4}\pi} \cos^4\theta \, d\theta$. [3]

(ii) (a) 0.572, -0.572

(b) $\frac{3}{32}\pi + \frac{1}{4}$

MAY/JUNE2011 9709/32

Solve the equation

$$\cos\theta + 4\cos 2\theta = 3,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 180°$. [5]

$29.0°, 180°$

MAY/JUNE2011 9709/33

(i) Show that the equation

$$\tan(60^\circ + \theta) + \tan(60^\circ - \theta) = k$$

can be written in the form

$$(2\sqrt{3})(1 + \tan^2\theta) = k(1 - 3\tan^2\theta).$$ [4]

(ii) Hence solve the equation

$$\tan(60^\circ + \theta) + \tan(60^\circ - \theta) = 3\sqrt{3},$$

giving all solutions in the interval $0^\circ \leqslant \theta \leqslant 180^\circ$. [3]

16.8°, 163.2°

OCTOBER/NOVEMBER 2010 9709/31

Solve the equation

$$\cos(\theta + 60^\circ) = 2\sin\theta,$$

giving all solutions in the interval $0^\circ \leqslant \theta \leqslant 360^\circ$. [5]

9.9°, 189.9°

OCTOBER/NOVEMBER 2010 9709/33

(i) Express $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence, in each of the following cases, find the smallest positive angle θ which satisfies the equation

(a) $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta = -4$, [2]

(b) $(\sqrt{6})\cos\frac{1}{2}\theta + (\sqrt{10})\sin\frac{1}{2}\theta = 3$. [4]

(i) $R = 4, \propto = 52.24$

(ii) (a) 232.2 (b) 21.7

MAY/JUNE2010 9709/31

Solve the equation

$$\sin\theta = 2\cos 2\theta + 1,$$

giving all solutions in the interval $0^\circ \leqslant \theta \leqslant 360^\circ$. [6]

48.6°, 131.4°, 270°

MAY/JUNE2010 9709/32

It is given that $\cos a = \frac{3}{5}$, where $0^\circ < a < 90^\circ$. Showing your working and without using a calculator to evaluate a,

(i) find the exact value of $\sin(a - 30^\circ)$, [3]

(ii) find the exact value of $\tan 2a$, and hence find the exact value of $\tan 3a$. [4]

(i) $\frac{1}{10}(4\sqrt{3} - 3)$

(ii) $-\frac{24}{7}, -\frac{44}{117}$

MAY/JUNE2010 9709/33

Solve the equation

$$\tan(45^\circ - x) = 2\tan x,$$

giving all solutions in the interval $0^\circ < x < 180^\circ$. [5]

15.7°, 119.3°

OCTOBER/NOVEMBER 2009 9709/32

The angles α and β lie in the interval $0° < x < 180°$, and are such that

$$\tan\alpha = 2\tan\beta \quad \text{and} \quad \tan(\alpha + \beta) = 3.$$

Find the possible values of α and β. [6]

$\alpha = 45°, \beta = 26.6°$ or $\alpha = 116.6°, \beta = 135°$

MAY/JUNE2009

(i) Prove the identity $\operatorname{cosec} 2\theta + \cot 2\theta \equiv \cot\theta$. [3]

(ii) Hence solve the equation $\operatorname{cosec} 2\theta + \cot 2\theta = 2$, for $0° \leqslant \theta \leqslant 360°$. [2]

$26.6°$, $206.6°$

OCTOBER/NOVEMBER 2008

(i) Express $5\sin x + 12\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$5\sin 2\theta + 12\cos 2\theta = 11,$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

(i) $R = 13, \alpha = 67.38°$
(ii) 27.4°, 175.2°

MAY/JUNE2008

(i) Show that the equation $\tan(30° + \theta) = 2\tan(60° - \theta)$ can be written in the form

$$\tan^2\theta + (6\sqrt{3})\tan\theta - 5 = 0.$$ [4]

(ii) Hence, or otherwise, solve the equation

$$\tan(30° + \theta) = 2\tan(60° - \theta),$$

for $0° \leqslant \theta \leqslant 180°$. [3]

24.7°, 95.3°

OCTOBER/NOVEMBER 2007

(i) Show that the equation

$$\tan(45^\circ + x) - \tan x = 2$$

can be written in the form

$$\tan^2 x + 2\tan x - 1 = 0.$$ [3]

(ii) Hence solve the equation

$$\tan(45^\circ + x) - \tan x = 2,$$

giving all solutions in the interval $0^\circ \leqslant x \leqslant 180^\circ$. [4]

(ii) 22.5°, 112.5°

OCTOBER/NOVEMBER 2006

Solve the equation

$$\tan x \tan 2x = 1,$$

giving all solutions in the interval $0^\circ < x < 180^\circ$. [4]

30°, 150°

MAY/JUNE2006

(i) Express $7\cos\theta + 24\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$7\cos\theta + 24\sin\theta = 15,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [4]

(i) $R = 25, \alpha = 73.74°$
(ii) $126.9°, 20{,}6°$

OCTOBER/NOVEMBER 2005

By expressing $8\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, solve the equation

$$8\sin\theta - 6\cos\theta = 7,$$

for $0° \leqslant \theta \leqslant 360°$. [7]

$R = 10, \alpha = 36.9°$
$81.3°, 172.4°$

MAY/JUNE2005

(i) Prove the identity

$$\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3. \qquad [4]$$

(ii) Hence solve the equation

$$\cos 4\theta + 4\cos 2\theta = 2,$$

for $0^\circ \leqslant \theta \leqslant 360^\circ$. [4]

(ii) 27.2° (or 27.3°), 332.8°, 152.8°, 207.2°

OCTOBER/NOVEMBER 2004

(i) Show that the equation

$$\tan(45^\circ + x) = 2\tan(45^\circ - x)$$

can be written in the form

$$\tan^2 x - 6\tan x + 1 = 0. \qquad [4]$$

(ii) Hence solve the equation $\tan(45^\circ + x) = 2\tan(45^\circ - x)$, for $0^\circ < x < 90^\circ$. [3]

(ii) 80.3°, 9.7°

MAY/JUNE2004

Sketch the graph of $y = \sec x$, for $0 \leqslant x \leqslant 2\pi$. [3]

OCTOBER/NOVEMBER 2003

Solve the equation

$$\cos\theta + 3\cos 2\theta = 2,$$

giving all solutions in the interval $0^\circ \leqslant \theta \leqslant 180^\circ$. [5]

33.6°, 180°

MAY/JUNE2003

(i) Show that the equation

$$\sin(x - 60^\circ) - \cos(30^\circ - x) = 1$$

can be written in the form $\cos x = k$, where k is a constant. [2]

(ii) Hence solve the equation, for $0^\circ < x < 180^\circ$. [2]

(ii) 125.3°

OCTOBER/NOVEMBER 2002

(i) Express $4\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, stating the value of α correct to 2 decimal places. [3]

Hence

(ii) solve the equation

$$4\sin\theta - 3\cos\theta = 2,$$

giving all values of θ such that $0° < \theta < 360°$, [4]

(iii) write down the greatest value of $\dfrac{1}{4\sin\theta - 3\cos\theta + 6}$. [1]

(i) $R = 5, \alpha = 36.87°$
(ii) $60.4°\ (or\ 60.5°), 193.3°$
(iii) 1

MAY/JUNE2002

Prove the identity

$$\cot\theta - \tan\theta \equiv 2\cot 2\theta.$$

[3]

DIFFERENTIATION

MAY/JUNE2012 9709/31

The equation of a curve is $3x^2 - 4xy + y^2 = 45$.

(i) Find the gradient of the curve at the point $(2, -3)$. [4]

(ii) Show that there are no points on the curve at which the gradient is 1. [3]

(i) $\frac{2}{7}$

MAY/JUNE2012 9709/32

The equation of a curve is $y = 3\sin x + 4\cos^3 x$.

(i) Find the x-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. [6]

(ii) Determine the nature of the stationary point in this interval for which x is least. [2]

(i) $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{2}$

(ii) maximum at $\frac{\pi}{12}$.

MAY/JUNE2012 9709/33

The parametric equations of a curve are

$$x = \sin 2\theta - \theta, \quad y = \cos 2\theta + 2\sin\theta.$$

Show that $\dfrac{dy}{dx} = \dfrac{2\cos\theta}{1 + 2\sin\theta}$. [5]

MAY/JUNE2012 9709/33

The curve with equation $y = \dfrac{e^{2x}}{x^3}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

(i) $\frac{3}{2}$
(ii) minimum

OCTOBER/NOVEMBER 2011 9709/31

The parametric equations of a curve are

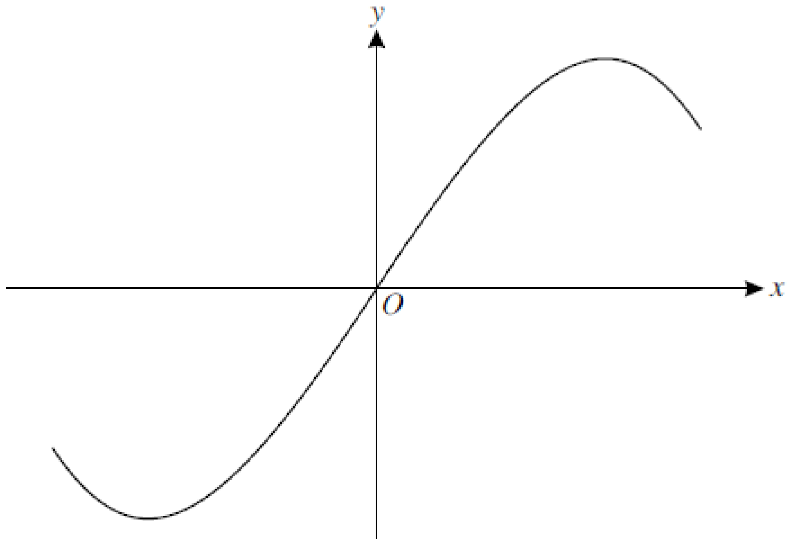
$$x = 3(1 + \sin^2 t), \quad y = 2\cos^3 t.$$

Find $\frac{dy}{dx}$ in terms of t, simplifying your answer as far as possible. [5]

$-cost$

OCTOBER/NOVEMBER 2011 9709/33

The equation of a curve is $y = \frac{e^{2x}}{1 + e^{2x}}$. Show that the gradient of the curve at the point for which $x = \ln 3$ is $\frac{9}{50}$. [4]



The diagram shows the curve with parametric equations

$$x = \sin t + \cos t, \quad y = \sin^3 t + \cos^3 t,$$

for $\frac{1}{4}\pi < t < \frac{5}{4}\pi$.

(i) Show that $\frac{dy}{dx} = -3\sin t \cos t$. [3]

(ii) Find the gradient of the curve at the origin. [2]

(iii) Find the values of t for which the gradient of the curve is 1, giving your answers correct to 2 significant figures. [4]

(ii) $\frac{3}{2}$
(iii) 1.9, 2.8

MAY/JUNE2011 9709/31

Find $\frac{dy}{dx}$ in each of the following cases:

(i) $y = \ln(1 + \sin 2x)$, [2]

(ii) $y = \frac{\tan x}{x}$. [2]

(i) $\frac{2cos2x}{1+sin2x}$

(ii) $\frac{xsec^2x-tanx}{x^2}$

MAY/JUNE2011 9709/31

The curve with equation

$$6e^{2x} + ke^{y} + e^{2y} = c,$$

where k and c are constants, passes through the point P with coordinates $(\ln 3, \ln 2)$.

(i) Show that $58 + 2k = c$. [2]

(ii) Given also that the gradient of the curve at P is -6, find the values of k and c. [5]

(ii) $k = 5, c = 68$

MAY/JUNE2011 9709/32

The parametric equations of a curve are

$$x = \ln(\tan t), \quad y = \sin^2 t,$$

where $0 < t < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of t. [4]

(ii) Find the equation of the tangent to the curve at the point where $x = 0$. [3]

(i) $2sin^2tcos^2t$

(ii) $y = \frac{1}{2}x + \frac{1}{2}$

MAY/JUNE2011 9709/33

The curve $y = \frac{\ln x}{x^3}$ has one stationary point. Find the x-coordinate of this point. [4]

1.40

OCTOBER/NOVEMBER 2010 9709/33

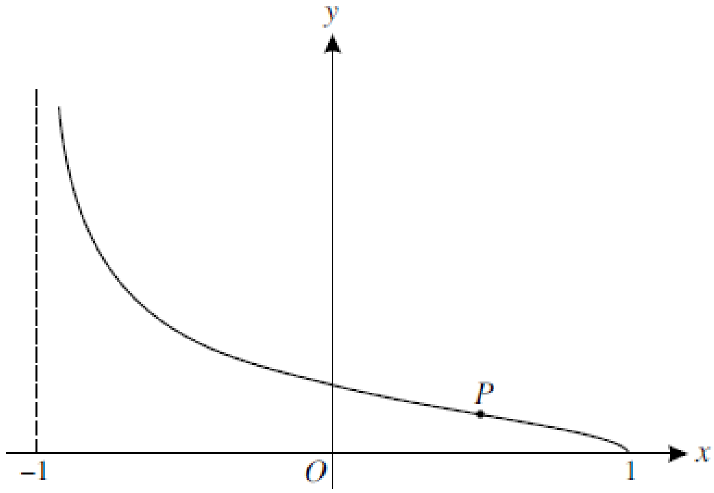
The parametric equations of a curve are

$$x = \frac{t}{2t+3}, \qquad y = e^{-2t}.$$

Find the gradient of the curve at the point for which $t = 0$. [5]

-6

MAY/JUNE2010 9709/31



The diagram shows the curve $y = \sqrt{\left(\frac{1-x}{1+x}\right)}$.

(i) By first differentiating $\frac{1-x}{1+x}$, obtain an expression for $\frac{dy}{dx}$ in terms of x. Hence show that the gradient of the normal to the curve at the point (x, y) is $(1+x)\sqrt{(1-x^2)}$. [5]

(ii) The gradient of the normal to the curve has its maximum value at the point P shown in the diagram. Find, by differentiation, the x-coordinate of P. [4]

(ii) $x = \frac{1}{3}$

MAY/JUNE2010 9709/32

The equation of a curve is

$$x \ln y = 2x + 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{y}{x^2}$. [4]

(ii) Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

(ii) $y + 4x + 1 = 0$

OCTOBER/NOVEMBER 2009 9709/31

A curve has equation $y = e^{-3x} \tan x$. Find the x-coordinates of the stationary points on the curve in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. Give your answers correct to 3 decimal places. [6]

0.365, 1.206

OCTOBER/NOVEMBER 2009 9709/32

The equation of a curve is $x^3 - x^2y - y^3 = 3$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [4]

(ii) Find the equation of the tangent to the curve at the point $(2, 1)$, giving your answer in the form $ax + by + c = 0$. [2]

(i) $\frac{3x^2-2xy}{x^2+3y^2}$

(ii) $8x - 7y - 9 = 0$

MAY/JUNE2009

The parametric equations of a curve are

$$x = a\cos^3 t, \quad y = a\sin^3 t,$$

where a is a positive constant and $0 < t < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x\sin t + y\cos t = a\sin t\cos t. \quad [3]$$

(iii) Hence show that, if this tangent meets the x-axis at X and the y-axis at Y, then the length of XY is always equal to a. [2]

(i) $\frac{dx}{dt} = -3acos^2tsint. \frac{dy}{dt} = 3asin^2tcost$

OCTOBER/NOVEMBER 2008

The curve $y = \dfrac{e^x}{\cos x}$, for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, has one stationary point. Find the x-coordinate of this point. [5]

$-\dfrac{\pi}{4}$

OCTOBER/NOVEMBER 2008

The parametric equations of a curve are

$$x = a(2\theta - \sin 2\theta), \qquad y = a(1 - \cos 2\theta).$$

Show that $\dfrac{dy}{dx} = \cot\theta$. [5]

MAY/JUNE2008

The equation of a curve is $xy(x + y) = 2a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [8]

$x = a, y = -2a$

OCTOBER/NOVEMBER 2007

The curve with equation $y = e^{-x} \sin x$ has one stationary point for which $0 \leqslant x \leqslant \pi$.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

(i) $\frac{\pi}{4}$

(ii) maximum point

MAY/JUNE2007

The equation of a curve is $y = x\sin 2x$, where x is in radians. Find the equation of the tangent to the curve at the point where $x = \frac{1}{4}\pi$. **[4]**

$y = x$

OCTOBER/NOVEMBER 2006

The curve with equation $y = 6e^x - e^{3x}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

(i) $\frac{1}{2}ln2$
(ii) maximum point

OCTOBER/NOVEMBER 2006

The equation of a curve is $x^3 + 2y^3 = 3xy$.

(i) Show that $\frac{dy}{dx} = \frac{y - x^2}{2y^2 - x}$. [4]

(ii) Find the coordinates of the point, other than the origin, where the curve has a tangent which is parallel to the x-axis. [5]

(ii) $x = 1. y = 1$

MAY/JUNE2006

The parametric equations of a curve are

$$x = 2\theta + \sin 2\theta, \quad y = 1 - \cos 2\theta.$$

Show that $\frac{dy}{dx} = \tan\theta$. [5]

OCTOBER/NOVEMBER 2005

The equation of a curve is $y = x + \cos 2x$. Find the x-coordinates of the stationary points of the curve for which $0 \leqslant x \leqslant \pi$, and determine the nature of each of these stationary points. [7]

$x = \frac{\pi}{12}, maximum. x = \frac{5}{12}\pi, minimum$

MAY/JUNE2004

Find the gradient of the curve with equation

$$2x^2 - 4xy + 3y^2 = 3,$$

at the point $(2, 1)$. [4]

OCTOBER/NOVEMBER 2003

The equation of a curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a},$$

where a is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of x and y. [3]

(ii) The straight line with equation $y = x$ intersects the curve at the point P. Find the equation of the tangent to the curve at P. [3]

(i) $-\sqrt{\frac{y}{x}}$

(ii) $x + y = \frac{1}{2}a$

OCTOBER/NOVEMBER 2002

The curve $y = e^x + 4e^{-2x}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether the stationary point is a maximum or a minimum point. [2]

(i) $ln2$

(ii) minimum

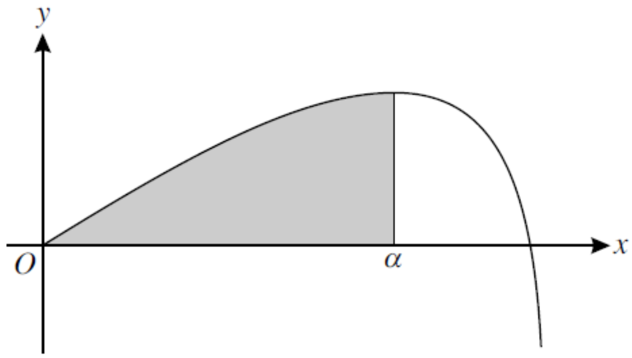
MAY/JUNE2002

The equation of a curve is $y = 2\cos x + \sin 2x$. Find the x-coordinates of the stationary points on the curve for which $0 < x < \pi$, and determine the nature of each of these stationary points. [7]

$x = \frac{\pi}{6}, maximum. x = \frac{5}{6}\pi, minimum$

INTEGRATION

MAY/JUNE2012 9709/31



The diagram shows the curve

$$y = 8\sin\tfrac{1}{2}x - \tan\tfrac{1}{2}x$$

for $0 \leqslant x < \pi$. The x-coordinate of the maximum point is α and the shaded region is enclosed by the curve and the lines $x = \alpha$ and $y = 0$.

(i) Show that $\alpha = \frac{2}{3}\pi$. [3]

(ii) Find the exact value of the area of the shaded region. [4]

(ii) $8 + 2ln\frac{1}{2}$

MAY/JUNE2012 9709/31

By first expressing $\frac{4x^2 + 5x + 3}{2x^2 + 5x + 2}$ in partial fractions, show that

$$\int_0^4 \frac{4x^2 + 5x + 3}{2x^2 + 5x + 2}\,dx = 8 - \ln 9.$$ [10]

MAY/JUNE2012 9709/32

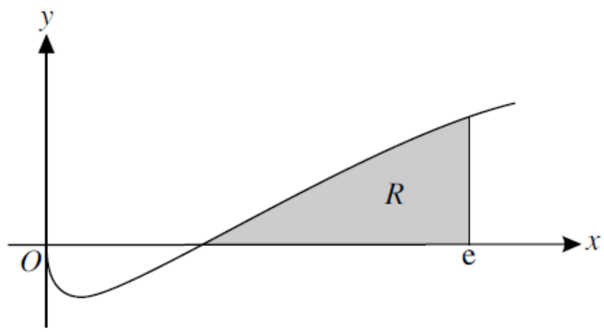
Let $I = \int_2^5 \frac{5}{x + \sqrt{(6 - x)}}\,dx$.

(i) Using the substitution $u = \sqrt{(6 - x)}$, show that

$$I = \int_1^2 \frac{10u}{(3 - u)(2 + u)}\,du.$$ [4]

(ii) Hence show that $I = 2\ln\left(\frac{9}{2}\right)$. [6]

MAY/JUNE2012 9709/32



The diagram shows the curve $y = x^{\frac{1}{2}} \ln x$. The shaded region between the curve, the x-axis and the line $x = \mathrm{e}$ is denoted by R.

(i) Find the equation of the tangent to the curve at the point where $x = 1$, giving your answer in the form $y = mx + c$. [4]

(ii) Find by integration the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π and e. [7]

(i) $y = x - 1$
(ii) $\frac{1}{4}\pi(e^2 - 1)$

MAY/JUNE2012 9709/33

Let $\mathrm{f}(x) = \dfrac{4x^2 - 7x - 1}{(x+1)(2x-3)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

(ii) Show that $\displaystyle\int_2^6 \mathrm{f}(x)\,\mathrm{d}x = 8 - \ln\left(\tfrac{49}{3}\right)$. [5]

(i) $2 - \frac{2}{x+1} - \frac{1}{2x-3}$

OCTOBER/NOVEMBER 2011 9709/31

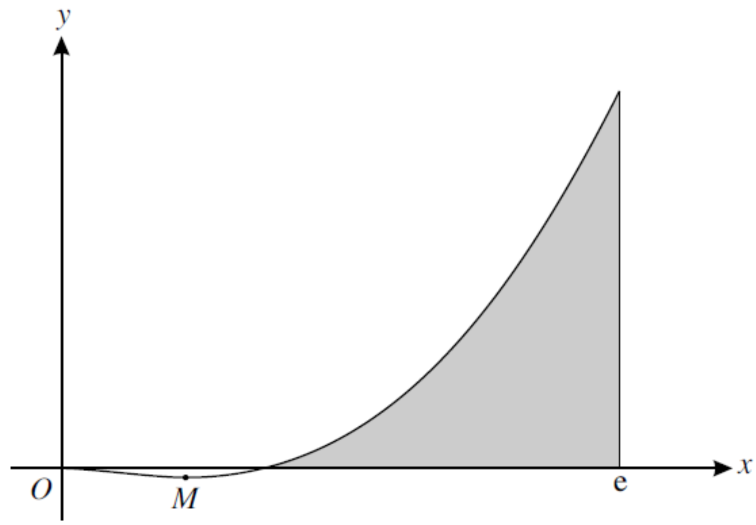
Let $\mathrm{f}(x) = \dfrac{12 + 8x - x^2}{(2-x)(4+x^2)}$.

(i) Express $\mathrm{f}(x)$ in the form $\dfrac{A}{2-x} + \dfrac{Bx+C}{4+x^2}$. [4]

(ii) Show that $\displaystyle\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = \ln\left(\tfrac{25}{2}\right)$. [5]

(i) $A = 3, B = 4. C = 0$

OCTOBER/NOVEMBER 2011 9709/31



The diagram shows the curve $y = x^2 \ln x$ and its minimum point M.

(i) Find the exact values of the coordinates of M. [5]

(ii) Find the exact value of the area of the shaded region bounded by the curve, the x-axis and the line $x = \mathrm{e}$. [5]

(i) $x = e^{-\frac{1}{2}}, y = \frac{1}{2}e^{-1}$

(ii) $\frac{1}{9}(2e^3 + 1)$

OCTOBER/NOVEMBER 2011 9709/33

(i) Use the substitution $u = \tan x$ to show that, for $n \neq -1$,

$$\int_0^{\frac{1}{4}\pi} (\tan^{n+2} x + \tan^n x)\, dx = \frac{1}{n+1}.$$ [4]

(ii) Hence find the exact value of

(a) $\int_0^{\frac{1}{4}\pi} (\sec^4 x - \sec^2 x)\, dx$, [3]

(b) $\int_0^{\frac{1}{4}\pi} (\tan^9 x + 5\tan^7 x + 5\tan^5 x + \tan^3 x)\, dx$. [3]

(ii) $\frac{1}{3}$

(iii) $\frac{25}{24}$

MAY/JUNE2011 9709/31

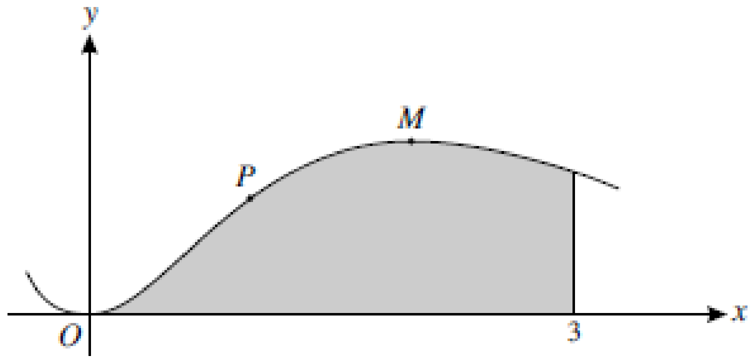
The integral I is defined by $I = \int_0^2 4t^3 \ln(t^2+1)\, dt$.

(i) Use the substitution $x = t^2 + 1$ to show that $I = \int_1^5 (2x-2)\ln x\, dx$. [3]

(ii) Hence find the exact value of I. [5]

(ii) $15 ln5 - 4$

MAY/JUNE2011 9709/32



The diagram shows the curve $y = x^2 e^{-x}$.

(i) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 3$ is equal to $2 - \frac{17}{e^3}$. [5]

(ii) Find the x-coordinate of the maximum point M on the curve. [4]

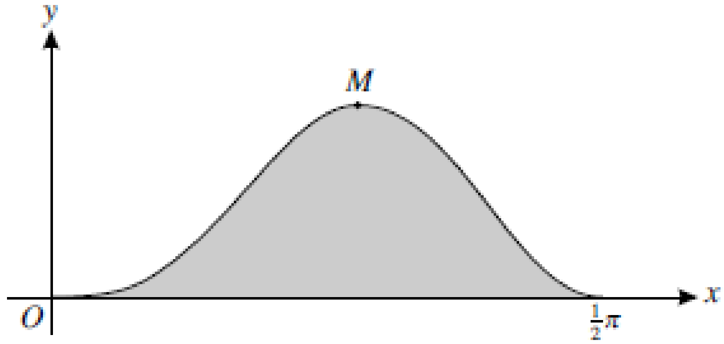
(iii) Find the x-coordinate of the point P at which the tangent to the curve passes through the origin. [2]

(ii) 2
(iii) 1

MAY/JUNE2011 9709/33

Show that $\int_0^1 (1-x)e^{-\frac{1}{2}x}\,dx = 4e^{-\frac{1}{2}} - 2.$ [5]

MAY/JUNE2011 9709/33



The diagram shows the curve $y = 5\sin^3 x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. [5]

(ii) Using the substitution $u = \cos x$, find by integration the area of the shaded region bounded by the curve and the x-axis. [5]

(i) 0.886
(ii) $\frac{2}{3}$

OCTOBER/NOVEMBER 2010 9709/31

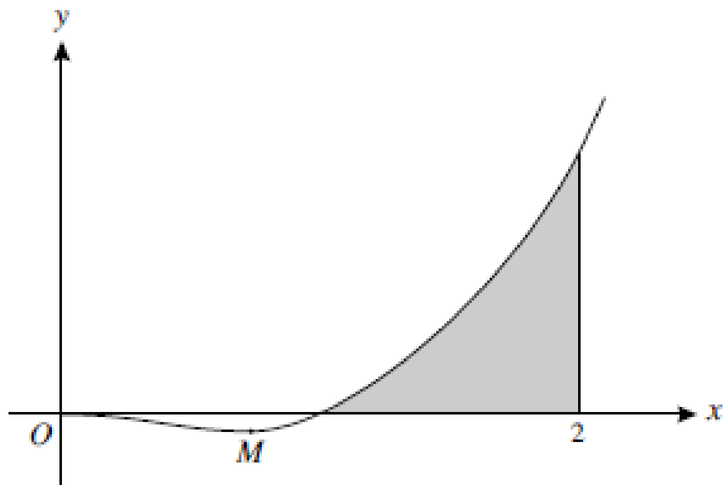
Let $I = \int_0^1 \frac{x^2}{\sqrt{(4 - x^2)}}\, dx$.

(i) Using the substitution $x = 2\sin\theta$, show that

$$I = \int_0^{\frac{1}{6}\pi} 4\sin^2\theta \, d\theta. \qquad [3]$$

(ii) Hence find the exact value of I. [4]

(ii) $\frac{1}{3}\pi - \frac{\sqrt{3}}{2}$



The diagram shows the curve $y = x^3 \ln x$ and its minimum point M.

(i) Find the exact coordinates of M. [5]

(ii) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 2$. [5]

(i) $x = \exp\left(-\frac{1}{3}\right), y = -\frac{1}{3e}$

(ii) $4ln2 - \frac{15}{16}$

OCTOBER/NOVEMBER 2010 9709/33

It is given that $f(x) = 4\cos^2 3x$.

(i) Find the exact value of $f'(\frac{1}{9}\pi)$. [3]

(ii) Find $\int f(x)\,dx$. [3]

(i) $-6\sqrt{3}$

(ii) $2z + \frac{1}{3} sin6x$

OCTOBER/NOVEMBER 2010 9709/33

Show that $\int_0^7 \frac{2x+7}{(2x+1)(x+2)}\,dx = \ln 50$. [7]

MAY/JUNE2010 9709/31

(i) Using the expansions of $\cos(3x - x)$ and $\cos(3x + x)$, prove that

$$\tfrac{1}{2}(\cos 2x - \cos 4x) \equiv \sin 3x \sin x. \qquad [3]$$

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin 3x \sin x \, dx = \tfrac{1}{8}\sqrt{3}. \qquad [3]$$

MAY/JUNE2010 9709/31

(i) Express $\dfrac{2}{(x+1)(x+3)}$ in partial fractions. [2]

(ii) Using your answer to part **(i)**, show that

$$\left(\frac{2}{(x+1)(x+3)}\right)^2 \equiv \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+3} + \frac{1}{(x+3)^2}. \qquad [2]$$

(iii) Hence show that $\displaystyle\int_0^1 \frac{4}{(x+1)^2(x+3)^2}\, dx = \tfrac{7}{12} - \ln\tfrac{3}{2}$. [5]

(i) $\frac{1}{x+1} - \frac{1}{x+3}$

MAY/JUNE2010 9709/32

Show that $\int_0^{\pi} x^2 \sin x \, dx = \pi^2 - 4$. [5]

MAY/JUNE2010 9709/32

(i) Find the values of the constants A, B, C and D such that

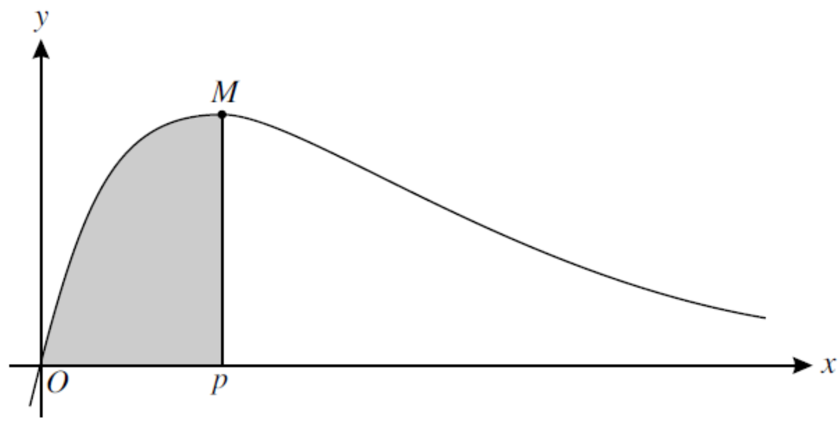
$$\frac{2x^3 - 1}{x^2(2x-1)} \equiv A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{2x-1}.$$ [5]

(ii) Hence show that

$$\int_1^2 \frac{2x^3 - 1}{x^2(2x-1)} \, dx = \tfrac{3}{2} + \tfrac{1}{2}\ln\left(\tfrac{16}{27}\right).$$ [5]

(i) $A = 1, B = 2, C = 1, D = -3$

MAY/JUNE2010 9709/33



The diagram shows the curve $y = e^{-x} - e^{-2x}$ and its maximum point M. The x-coordinate of M is denoted by p.

(i) Find the exact value of p. [4]

(ii) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = p$ is equal to $\frac{1}{8}$. [4]

(i) $ln2$

MAY/JUNE2010 9709/33

(i) Prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [4]

(ii) Using this result, find the exact value of

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \cos^3\theta \, d\theta.$$ [4]

OCTOBER/NOVEMBER 2009 9709/31

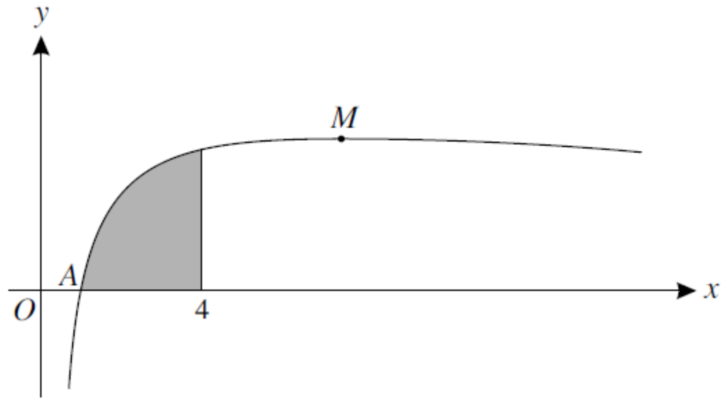
(i) Prove the identity $\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$. [4]

(ii) Using this result find, in simplified form, the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin^4\theta \, d\theta.$$ [4]

(ii) $\frac{1}{32}(2\pi - \sqrt{3})$

OCTOBER/NOVEMBER 2009 9709/31



The diagram shows the curve $y = \frac{\ln x}{\sqrt{x}}$ and its maximum point M. The curve cuts the x-axis at the point A.

(i) State the coordinates of A. [1]

(ii) Find the exact value of the x-coordinate of M. [4]

(iii) Using integration by parts, show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 4$ is equal to $8 \ln 2 - 4$. [5]

(i) (1, 0)
(ii) $x = e^2$

OCTOBER/NOVEMBER 2009 9709/32

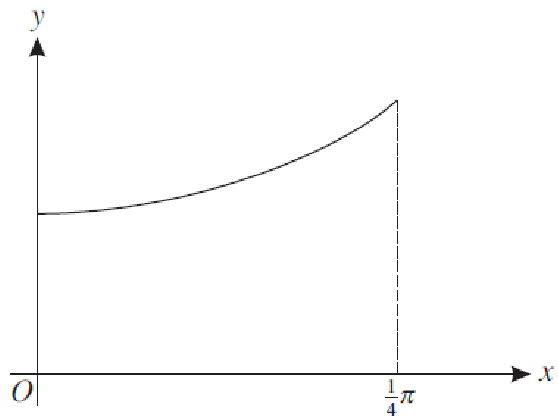
(i) Use the substitution $x = 2\tan\theta$ to show that

$$\int_0^2 \frac{8}{(4+x^2)^2}\,dx = \int_0^{\frac{1}{4}\pi} \cos^2\theta\,d\theta. \qquad [4]$$

(ii) Hence find the exact value of

$$\int_0^2 \frac{8}{(4+x^2)^2}\,dx. \qquad [4]$$

$$\frac{1}{8}(\pi + 2)$$



The diagram shows the curve $y = \sqrt{(1 + 2\tan^2 x)}$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$.

(i) Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{4}\pi} \sqrt{(1 + 2\tan^2 x)}\, dx,$$

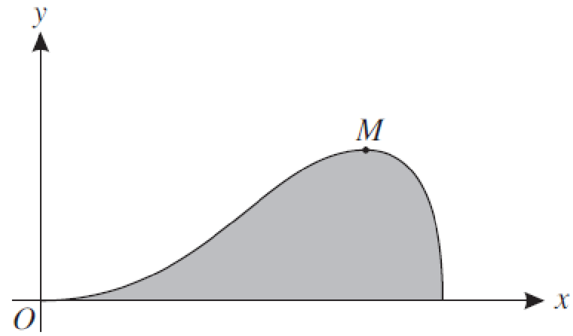
giving your answer correct to 2 decimal places. [3]

(ii) The estimate found in part **(i)** is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with six intervals would be greater than E or less than E. [1]

(i) 0.98

(ii) second estimate would be less than E.

MAY/JUNE2009



The diagram shows the curve $y = x^2\sqrt{(1-x^2)}$ for $x \geqslant 0$ and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Show, by means of the substitution $x = \sin\theta$, that the area A of the shaded region between the curve and the x-axis is given by

$$A = \frac{1}{4}\int_0^{\frac{1}{2}\pi} \sin^2 2\theta \, d\theta. \qquad [3]$$

(iii) Hence obtain the exact value of A. [4]

(i) $\sqrt{\frac{2}{3}}$

(iii) $\frac{\pi}{16}$

MAY/JUNE2008

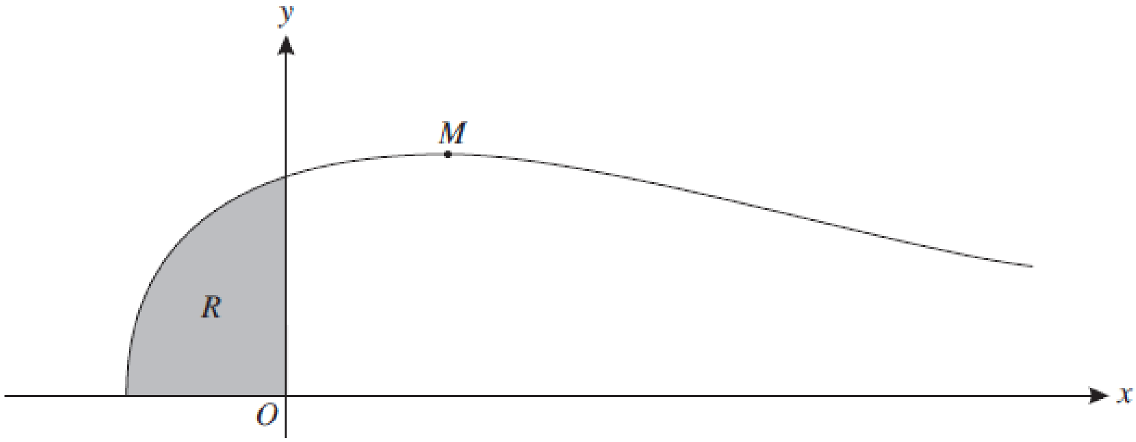
Let $\mathrm{f}(x) \equiv \dfrac{x^2 + 3x + 3}{(x+1)(x+3)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

(ii) Hence show that $\displaystyle\int_0^3 \mathrm{f}(x)\,\mathrm{d}x = 3 - \tfrac{1}{2}\ln 2$. [4]

(i) $\mathbf{1} + \dfrac{\mathbf{1}}{\mathbf{2}(x+\mathbf{1})} - \dfrac{\mathbf{3}}{\mathbf{2}(x+\mathbf{3})}$

MAY/JUNE2008



The diagram shows the curve $y = e^{-\frac{1}{2}x}\sqrt{(1+2x)}$ and its maximum point M. The shaded region between the curve and the axes is denoted by R.

(i) Find the x-coordinate of M. [4]

(ii) Find by integration the volume of the solid obtained when R is rotated completely about the x-axis. Give your answer in terms of π and e. [6]

(i) $\frac{1}{2}$

(ii) $\pi(2\sqrt{e} - 3)$

OCTOBER/NOVEMBER 2007

Find the exact value of the constant k for which $\int_1^k \frac{1}{2x-1}\,dx = 1.$ **[4]**

$\frac{1}{2}(e^2+1)$

OCTOBER/NOVEMBER 2007

Use integration by parts to show that

$$\int_2^4 \ln x\,dx = 6\ln 2 - 2.$$ **[4]**

MAY/JUNE2007

(i) Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. **[3]**

(ii) Hence show that $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{1}{\left(\cos\theta + (\sqrt{3})\sin\theta\right)^2}\,d\theta = \frac{1}{\sqrt{3}}$. **[4]**

(i) $R = 2, \alpha = \frac{\pi}{6}$

MAY/JUNE2007

Let $I = \displaystyle\int_1^4 \frac{1}{x(4-\sqrt{x})}\,dx$.

(i) Use the substitution $u = \sqrt{x}$ to show that $I = \displaystyle\int_1^2 \frac{2}{u(4-u)}\,du$. **[3]**

(ii) Hence show that $I = \frac{1}{2}\ln 3$. **[6]**

OCTOBER/NOVEMBER 2006

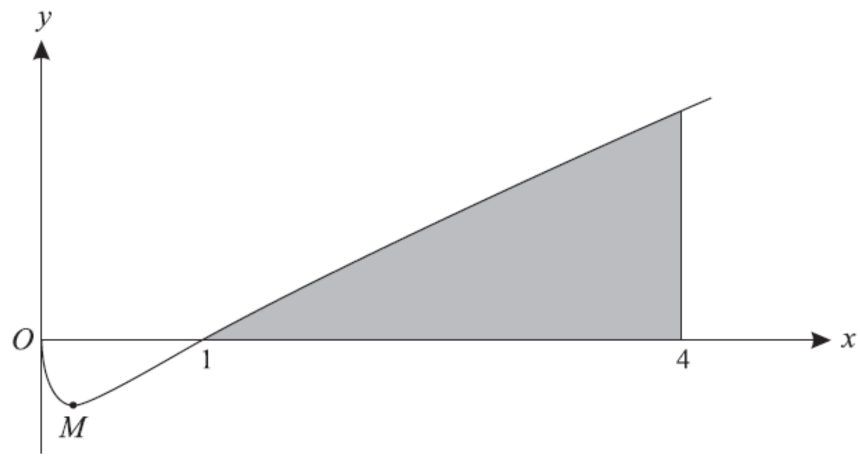
Let $f(x) = \dfrac{7x+4}{(2x+1)(x+1)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^2 f(x)\,dx = 2 + \ln\frac{5}{3}$. [5]

(i) $\frac{2}{2x+1} - \frac{1}{x+1} + \frac{3}{(x+1)^2}$

MAY/JUNE2006



The diagram shows a sketch of the curve $y = x^{\frac{1}{2}} \ln x$ and its minimum point M. The curve cuts the x-axis at the point $(1, 0)$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Use integration by parts to find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 4$. Give your answer correct to 2 decimal places. [5]

(i) e^{-2}
(ii) 4.28

OCTOBER/NOVEMBER 2005

(i) Use the substitution $x = \sin^2\theta$ to show that

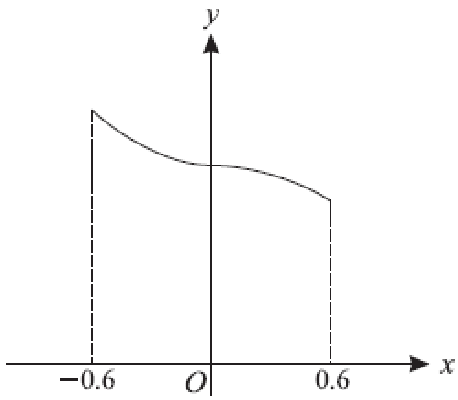
$$\int \sqrt{\left(\frac{x}{1-x}\right)}\,dx = \int 2\sin^2\theta\,d\theta. \qquad [4]$$

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx. \qquad [4]$$

(ii) $\frac{\pi}{6} - \frac{1}{4}\sqrt{3}$

MAY/JUNE2005



The diagram shows a sketch of the curve $y = \dfrac{1}{1+x^3}$ for values of x from -0.6 to 0.6.

(i) Use the trapezium rule, with two intervals, to estimate the value of

$$\int_{-0.6}^{0.6} \frac{1}{1+x^3}\,dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Explain, with reference to the diagram, why the trapezium rule may be expected to give a good approximation to the true value of the integral in this case. [1]

(i) 1.23
(ii) one trapezium over-estimates area and the other under-estimates.

MAY/JUNE2005

(i) Use the substitution $x = \tan\theta$ to show that

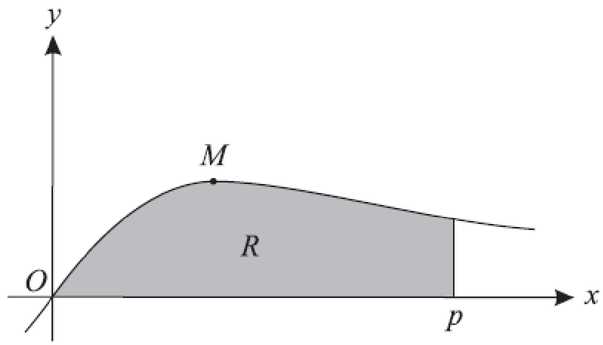
$$\int \frac{1 - x^2}{(1 + x^2)^2}\, dx = \int \cos 2\theta \, d\theta. \qquad [4]$$

(ii) Hence find the value of

$$\int_0^1 \frac{1 - x^2}{(1 + x^2)^2}\, dx. \qquad [3]$$

(ii) 0.5

MAY/JUNE2005



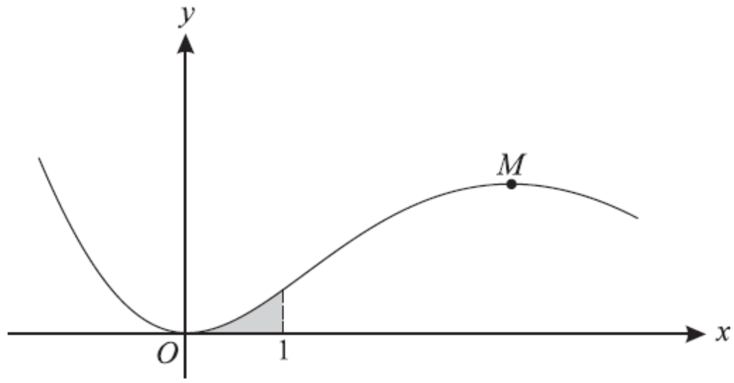
The diagram shows part of the curve $y = \dfrac{x}{x^2+1}$ and its maximum point M. The shaded region R is bounded by the curve and by the lines $y = 0$ and $x = p$.

(i) Calculate the x-coordinate of M. [4]

(ii) Find the area of R in terms of p. [3]

(iii) Hence calculate the value of p for which the area of R is 1, giving your answer correct to 3 significant figures. [2]

(i) 1
(ii) $\frac{1}{2}\ln(p^2+1)$
(iii) 2.53



The diagram shows the curve $y = x^2 e^{-\frac{1}{2}x}$.

(i) Find the x-coordinate of M, the maximum point of the curve. [4]

(ii) Find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 1$, giving your answer in terms of e. [5]

(i) 4

(ii) $16 - 26e^{-\frac{1}{2}}$

OCTOBER/NOVEMBER 2004

An appropriate form for expressing $\dfrac{3x}{(x+1)(x-2)}$ in partial fractions is

$$\frac{A}{x+1} + \frac{B}{x-2},$$

where A and B are constants.

(a) Without evaluating any constants, state appropriate forms for expressing the following in partial fractions:

(i) $\dfrac{4x}{(x+4)(x^2+3)}$, [1]

(ii) $\dfrac{2x+1}{(x-2)(x+2)^2}$. [2]

(b) Show that $\displaystyle\int_3^4 \frac{3x}{(x+1)(x-2)}\,dx = \ln 5$. [6]

(a) (i) $\frac{A}{x+4} + \frac{Bx+C}{x^2+3}$

(ii) $\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

MAY/JUNE2004

(i) Prove the identity

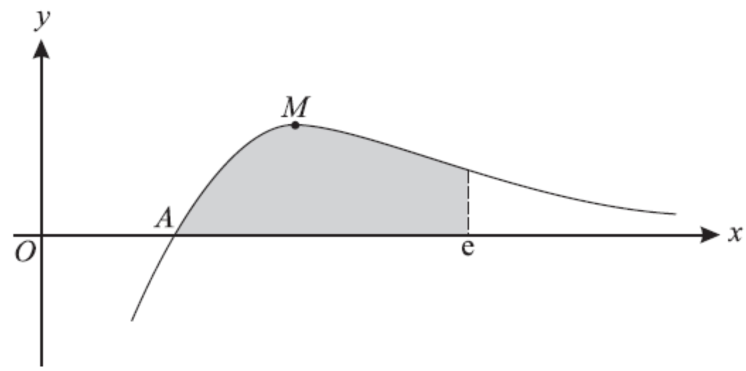
$$\sin^2\theta\cos^2\theta \equiv \tfrac{1}{8}(1-\cos 4\theta). \qquad [3]$$

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} \sin^2\theta\cos^2\theta\,\mathrm{d}\theta. \qquad [3]$$

(ii) $\frac{1}{8}(\frac{1}{3}\pi+\frac{\sqrt{3}}{8})$

MAY/JUNE2004



The diagram shows the curve $y = \frac{\ln x}{x^2}$ and its maximum point M. The curve cuts the x-axis at A.

(i) Write down the x-coordinate of A. [1]

(ii) Find the exact coordinates of M. [5]

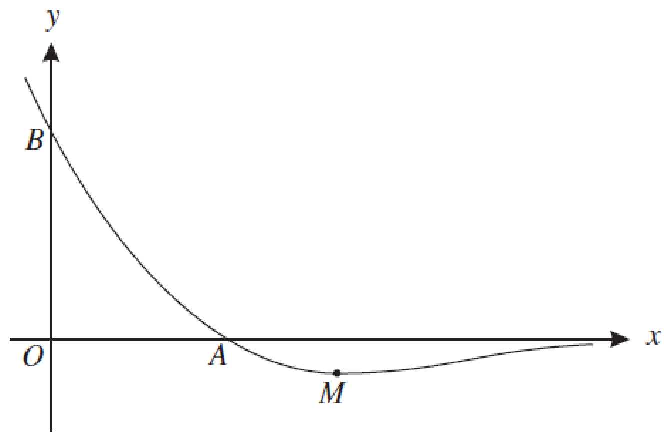
(iii) Use integration by parts to find the exact area of the shaded region enclosed by the curve, the x-axis and the line $x = \mathrm{e}$. [5]

(i) 1

(ii) $x = e^{\frac{1}{2}}, y = \frac{1}{2e}$

(iii) $1 - \frac{2}{e}$

OCTOBER/NOVEMBER 2003



The diagram shows the curve $y = (3 - x)e^{-2x}$ and its minimum point M. The curve intersects the x-axis at A and the y-axis at B.

(i) Calculate the x-coordinate of M. [4]

(ii) Find the area of the region bounded by OA, OB and the curve, giving your answer in terms of e. [5]

(i) $\frac{7}{2}$

(ii) $\frac{1}{4}(5 + e^{-6})$

OCTOBER/NOVEMBER 2003

Let $f(x) = \dfrac{x^3 - x - 2}{(x-1)(x^2+1)}$.

(i) Express $f(x)$ in the form

$$A + \frac{B}{x-1} + \frac{Cx+D}{x^2+1},$$

where A, B, C and D are constants. [5]

(ii) Hence show that $\int_2^3 f(x)\,dx = 1$. [4]

(i) $A = 1, B = -1, C = 2, D = 0$

MAY/JUNE2003

Find the exact value of $\int_0^1 xe^{2x}\,dx$. [4]

$\frac{1}{4}(e^2 + 1)$

MAY/JUNE2003

(i) Prove the identity

$$\cot x - \cot 2x \equiv \operatorname{cosec} 2x. \qquad [3]$$

(ii) Show that $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x \, dx = \frac{1}{2} \ln 2$. [3]

(iii) Find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \operatorname{cosec} 2x \, dx$, giving your answer in the form $a \ln b$. [4]

(iii) $\frac{1}{4} ln3$

OCTOBER/NOVEMBER 2002

Find the exact value of $\int_1^2 x \ln x \, dx$. [4]

$2ln2 - \frac{3}{4}$

MAY/JUNE2002

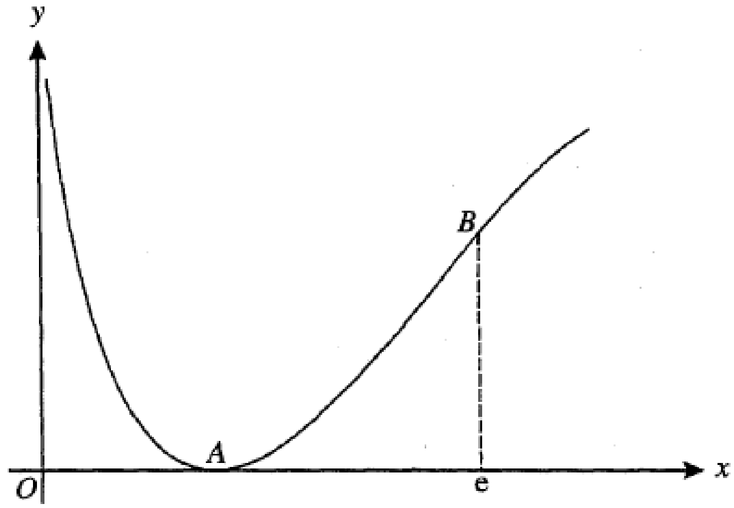
Let $f(x) = \dfrac{4x}{(3x+1)(x+1)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^1 f(x)\,dx = 1 - \ln 2$. [5]

(i) $-\frac{3}{3x+1} + \frac{2}{(x+1)^2} + \frac{1}{x+1}$

MAY/JUNE2002



The function f is defined by $f(x) = (\ln x)^2$ for $x > 0$. The diagram shows a sketch of the graph of $y = f(x)$. The minimum point of the graph is A. The point B has x-coordinate e.

(i) State the x-coordinate of A. [1]

(ii) Show that $f''(x) = 0$ at B. [4]

(iii) Use the substitution $x = e^u$ to show that the area of the region bounded by the x-axis, the line $x = e$, and the part of the curve between A and B is given by

$$\int_0^1 u^2 e^u \, du.$$ [3]

(iv) Hence, or otherwise, find the exact value of this area. [3]

(i) 1

(iv) e-2

DIFFERENTIAL EQUATION

MAY/JUNE2012 9709/31

The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{6xe^{3x}}{y^2}.$$

It is given that $y = 2$ when $x = 0$. Solve the differential equation and hence find the value of y when $x = 0.5$, giving your answer correct to 2 decimal places. [8]

2.44

MAY/JUNE2012 9709/32

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{2x+y},$$

and $y = 0$ when $x = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [6]

$y = \ln(\frac{2}{3 - e^{2x}})$

MAY/JUNE2012 9709/33

In a certain chemical process a substance A reacts with another substance B. The masses in grams of A and B present at time t seconds after the start of the process are x and y respectively. It is given that $\frac{dy}{dt} = -0.6xy$ and $x = 5e^{-3t}$. When $t = 0$, $y = 70$.

(i) Form a differential equation in y and t. Solve this differential equation and obtain an expression for y in terms of t. [6]

(ii) The percentage of the initial mass of B remaining at time t is denoted by p. Find the exact value approached by p as t becomes large. [2]

(i) $y = \exp(e^{-3t} - 1)$
(ii) $\frac{100}{e}$

OCTOBER/NOVEMBER 2011 9709/31

The variables x and θ are related by the differential equation

$$\sin 2\theta \frac{dx}{d\theta} = (x+1)\cos 2\theta,$$

where $0 < \theta < \frac{1}{2}\pi$. When $\theta = \frac{1}{12}\pi$, $x = 0$. Solve the differential equation, obtaining an expression for x in terms of θ, and simplifying your answer as far as possible. [7]

$x = \sqrt{(2sin2\theta)} - 1$

OCTOBER/NOVEMBER 2011 9709/33

During an experiment, the number of organisms present at time t days is denoted by N, where N is treated as a continuous variable. It is given that

$$\frac{dN}{dt} = 1.2e^{-0.02t}N^{0.5}.$$

When $t = 0$, the number of organisms present is 100.

(i) Find an expression for N in terms of t. [6]

(ii) State what happens to the number of organisms present after a long time. [1]

(i) $N = (40 - 30e^{-0.002t})^2$
(ii) Number approaches 1600.

MAY/JUNE2011 9709/31

The number of birds of a certain species in a forested region is recorded over several years. At time t years, the number of birds is N, where N is treated as a continuous variable. The variation in the number of birds is modelled by

$$\frac{dN}{dt} = \frac{N(1800 - N)}{3600}.$$

It is given that $N = 300$ when $t = 0$.

(i) Find an expression for N in terms of t. [9]

(ii) According to the model, how many birds will there be after a long time? [1]

(i) $N = \frac{1800e^{\frac{1}{2}t}}{5+e^{\frac{1}{2}t}}$

(ii) N approaches 1800

MAY/JUNE2011 9709/32

A certain curve is such that its gradient at a point (x, y) is proportional to xy. At the point $(1, 2)$ the gradient is 4.

(i) By setting up and solving a differential equation, show that the equation of the curve is $y = 2e^{x^2-1}$. [7]

(ii) State the gradient of the curve at the point $(-1, 2)$ and sketch the curve. [2]

(ii) -4

MAY/JUNE2011 9709/33

In a chemical reaction, a compound X is formed from two compounds Y and Z. The masses in grams of X, Y and Z present at time t seconds after the start of the reaction are x, $10 - x$ and $20 - x$ respectively. At any time the rate of formation of X is proportional to the product of the masses of Y and Z present at the time. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 2$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.01(10 - x)(20 - x).$$ [1]

(ii) Solve this differential equation and obtain an expression for x in terms of t. [9]

(iii) State what happens to the value of x when t becomes large. [1]

(ii) $x = 20(\exp(0.1t) - 1)/(2\exp(0.1t) - 1)$
(iii) x approaches 10.

OCTOBER/NOVEMBER 2010 9709/31

A certain substance is formed in a chemical reaction. The mass of substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance is proportional to $(20 - x)$. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.05(20 - x).$$ [2]

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find x when $t = 10$, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of x as t becomes very large. [1]

(ii) $ln20 - \ln(20 - x) = \frac{1}{20}t$

(iii) 7.9

(iv) x approaches 20.

OCTOBER/NOVEMBER 2010 9709/33

A biologist is investigating the spread of a weed in a particular region. At time t weeks after the start of the investigation, the area covered by the weed is A m^2. The biologist claims that the rate of increase of A is proportional to $\sqrt{(2A - 5)}$.

(i) Write down a differential equation representing the biologist's claim. [1]

(ii) At the start of the investigation, the area covered by the weed was 7 m^2 and, 10 weeks later, the area covered was 27 m^2. Assuming that the biologist's claim is correct, find the area covered 20 weeks after the start of the investigation. [9]

(i) $k\sqrt{2A-5}$

(ii) 63

MAY/JUNE2010 9709/31

Given that $y = 0$ when $x = 1$, solve the differential equation

$$xy\frac{dy}{dx} = y^2 + 4,$$

obtaining an expression for y^2 in terms of x. [6]

$y^2 = 4(x^2 - 1)$

MAY/JUNE2010 9709/32

The variables x and t are related by the differential equation

$$e^{2t}\frac{dx}{dt} = \cos^2 x,$$

where $t \geqslant 0$. When $t = 0$, $x = 0$.

(i) Solve the differential equation, obtaining an expression for x in terms of t. [6]

(ii) State what happens to the value of x when t becomes very large. [1]

(iii) Explain why x increases as t increases. [1]

(i) $x = tan^{-1}(\frac{1}{2} - \frac{1}{2}e^{-2t})$

(ii) x approaches $tan^{-1}(\frac{1}{2})$

(iii) $e^{-2t}cos^2 x$ is positive.

MAY/JUNE2010 9709/33

Given that $x = 1$ when $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{x} - \frac{x}{4},$$

obtaining an expression for x^2 in terms of t. [7]

$x^2 = 4 - 3\exp\left(-\frac{1}{2}t\right)$

OCTOBER/NOVEMBER 2009 9709/31

In a model of the expansion of a sphere of radius r cm, it is assumed that, at time t seconds after the start, the rate of increase of the surface area of the sphere is proportional to its volume. When $t = 0$, $r = 5$ and $\frac{dr}{dt} = 2$.

(i) Show that r satisfies the differential equation

$$\frac{dr}{dt} = 0.08r^2. \qquad [4]$$

[The surface area A and volume V of a sphere of radius r are given by the formulae $A = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$.]

(ii) Solve this differential equation, obtaining an expression for r in terms of t. [5]

(iii) Deduce from your answer to part **(ii)** the set of values that t can take, according to this model. [1]

(ii) $r = \frac{5}{(1-0.4t)}$

(iii) $0 \leq t < 2.5$

OCTOBER/NOVEMBER 2009 9709/32

The temperature of a quantity of liquid at time t is θ. The liquid is cooling in an atmosphere whose temperature is constant and equal to A. The rate of decrease of θ is proportional to the temperature difference $(\theta - A)$. Thus θ and t satisfy the differential equation

$$\frac{d\theta}{dt} = -k(\theta - A),$$

where k is a positive constant.

(i) Find, in any form, the solution of this differential equation, given that $\theta = 4A$ when $t = 0$. [5]

(ii) Given also that $\theta = 3A$ when $t = 1$, show that $k = \ln \frac{3}{2}$. [1]

(iii) Find θ in terms of A when $t = 2$, expressing your answer in its simplest form. [3]

(i) $\ln(\theta - A) = -kt + ln3A$

(iii) $\theta = \frac{7}{3}A$

MAY/JUNE2009

(i) Express $\dfrac{100}{x^2(10-x)}$ in partial fractions. [4]

(ii) Given that $x = 1$ when $t = 0$, solve the differential equation

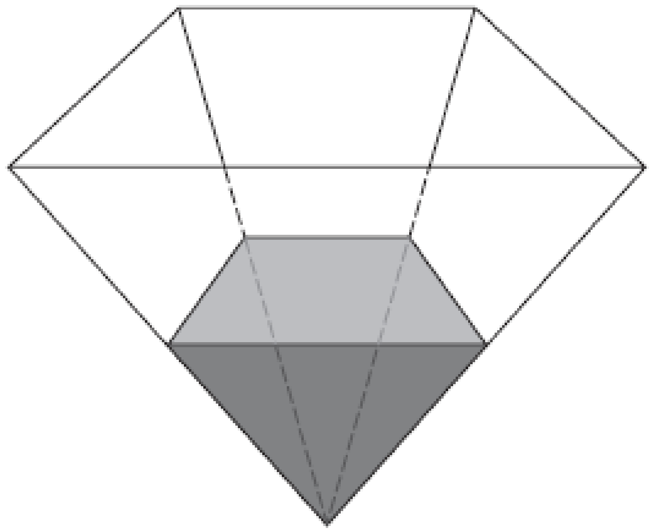
$$\frac{dx}{dt} = \frac{1}{100}x^2(10-x),$$

obtaining an expression for t in terms of x. [6]

(i) $\frac{1}{x} + \frac{10}{x^2} + \frac{1}{10-x}$

(ii) $t = \ln\left(\frac{9x}{10-x}\right) - \frac{10}{x} + 10$

OCTOBER/NOVEMBER 2008



An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time t hours after filling begins, the volume of liquid is V m^3 and the depth of liquid is h m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of 20 m^3 per hour, but owing to leakage, liquid is lost at a rate proportional to h^2. When $h = 1$, $\frac{dh}{dt} = 4.95$.

(i) Show that h satisfies the differential equation

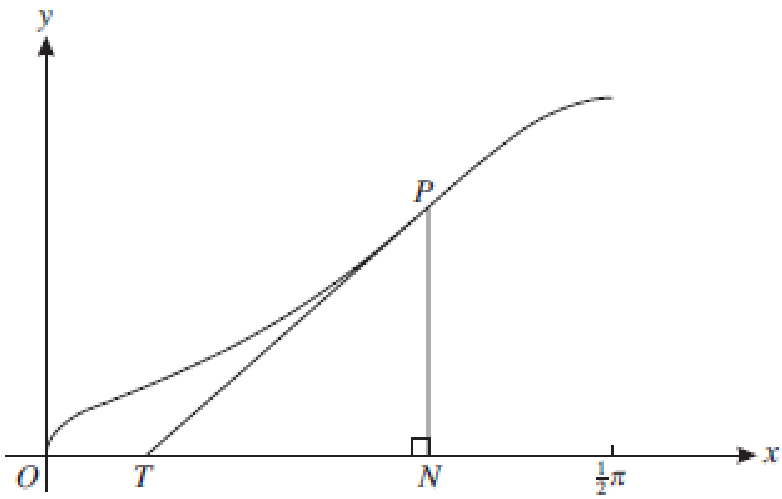
$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}.$$ [4]

(ii) Verify that $\frac{20h^2}{100 - h^2} \equiv -20 + \frac{2000}{(10-h)(10+h)}$. [1]

(iii) Hence solve the differential equation in part (i), obtaining an expression for t in terms of h. [5]

(iii) $t = 100\ln(\frac{10+h}{10-h})$

MAY/JUNE2008



In the diagram the tangent to a curve at a general point P with coordinates (x, y) meets the x-axis at T. The point N on the x-axis is such that PN is perpendicular to the x-axis. The curve is such that, for all values of x in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle PTN is equal to $\tan x$, where x is in radians.

(i) Using the fact that the gradient of the curve at P is $\frac{PN}{TN}$, show that

$$\frac{dy}{dx} = \frac{1}{2}y^2 \cot x. \qquad [3]$$

(ii) Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

(ii) $y = 2/(1 - \ln(2sinx))$

OCTOBER/NOVEMBER 2007

The number of insects in a population t days after the start of observations is denoted by N. The variation in the number of insects is modelled by a differential equation of the form

$$\frac{dN}{dt} = kN\cos(0.02t),$$

where k is a constant and N is taken to be a continuous variable. It is given that $N = 125$ when $t = 0$.

(i) Solve the differential equation, obtaining a relation between N, k and t. **[5]**

(ii) Given also that $N = 166$ when $t = 30$, find the value of k. **[2]**

(iii) Obtain an expression for N in terms of t, and find the least value of N predicted by this model. **[3]**

(i) $\ln N = 50k\sin(0.02t) + \ln 125$

(ii) 0.0100479… (accept 0.01)

(iii) $N = 125\exp(0.502\sin(0.02t))$, 75.6

MAY/JUNE2007

A model for the height, h metres, of a certain type of tree at time t years after being planted assumes that, while the tree is growing, the rate of increase in height is proportional to $(9-h)^{\frac{1}{3}}$. It is given that, when $t=0$, $h=1$ and $\frac{\mathrm{d}h}{\mathrm{d}t}=0.2$.

(i) Show that h and t satisfy the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t}=0.1(9-h)^{\frac{1}{3}}.$$ **[2]**

(ii) Solve this differential equation, and obtain an expression for h in terms of t. **[7]**

(iii) Find the maximum height of the tree and the time taken to reach this height after planting. **[2]**

(iv) Calculate the time taken to reach half the maximum height. **[1]**

(ii) $h=9-(4-\frac{1}{15}t)^{\frac{3}{2}}$

(iii) 9, 60 years

(iv) 19.1

OCTOBER/NOVEMBER 2006

Given that $y = 2$ when $x = 0$, solve the differential equation

$$y\frac{dy}{dx} = 1 + y^2,$$

obtaining an expression for y^2 in terms of x. [6]

$y^2 = 5e^{2x} - 1$

MAY/JUNE2006

In a certain industrial process, a substance is being produced in a container. The mass of the substance in the container t minutes after the start of the process is x grams. At any time, the rate of formation of the substance is proportional to its mass. Also, throughout the process, the substance is removed from the container at a constant rate of 25 grams per minute. When $t = 0$, $x = 1000$ and $\frac{dx}{dt} = 75$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.1(x - 250).$$ [2]

(ii) Solve this differential equation, obtaining an expression for x in terms of t. [6]

(ii) $x = 250(3e^{0.1t} + 1)$

OCTOBER/NOVEMBER 2005

In a certain chemical reaction the amount, x grams, of a substance present is decreasing. The rate of decrease of x is proportional to the product of x and the time, t seconds, since the start of the reaction. Thus x and t satisfy the differential equation

$$\frac{dx}{dt} = -kxt,$$

where k is a positive constant. At the start of the reaction, when $t = 0$, $x = 100$.

(i) Solve this differential equation, obtaining a relation between x, k and t. [5]

(ii) 20 seconds after the start of the reaction the amount of substance present is 90 grams. Find the time after the start of the reaction at which the amount of substance present is 50 grams. [3]

(i) $lnx = -\frac{1}{2}kt^2 + ln100$

(ii) 51.3

MAY/JUNE2005

(i) Using partial fractions, find

$$\int \frac{1}{y(4-y)}\,dy. \qquad [4]$$

(ii) Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = y(4-y),$$

obtaining an expression for y in terms of x. [4]

(iii) State what happens to the value of y if x becomes very large and positive. [1]

(i) $\frac{1}{4}lny - \frac{1}{4}\ln(4-y)$

(ii) $y = 4/(3e^{-4x}+1)$

(iii) y approaches 4 as x becomes very large

OCTOBER/NOVEMBER 2004

A rectangular reservoir has a horizontal base of area $1000\,\text{m}^2$. At time $t = 0$, it is empty and water begins to flow into it at a constant rate of $30\,\text{m}^3\,\text{s}^{-1}$. At the same time, water begins to flow out at a rate proportional to $\sqrt{h}$, where h m is the depth of the water at time t s. When $h = 1$, $\dfrac{dh}{dt} = 0.02$.

(i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = 0.01(3 - \sqrt{h}).$$ [3]

It is given that, after making the substitution $x = 3 - \sqrt{h}$, the equation in part **(i)** becomes

$$(x - 3)\frac{dx}{dt} = 0.005x.$$

(ii) Using the fact that $x = 3$ when $t = 0$, solve this differential equation, obtaining an expression for t in terms of x. [5]

(iii) Find the time at which the depth of water reaches 4 m. [2]

(ii) $t = 200(x - 3 - 3\ln x + 3\ln 3)$
(iii) 259

MAY/JUNE2004

Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = \frac{y^3 + 1}{y^2},$$

obtaining an expression for y in terms of x. [6]

$y = (2e^{3x} - 1)^{\frac{1}{3}}$

OCTOBER/NOVEMBER 2003

Compressed air is escaping from a container. The pressure of the air in the container at time t is P, and the constant atmospheric pressure of the air outside the container is A. The rate of decrease of P is proportional to the square root of the pressure difference $(P - A)$. Thus the differential equation connecting P and t is

$$\frac{dP}{dt} = -k\sqrt{(P - A)},$$

where k is a positive constant.

(i) Find, in any form, the general solution of this differential equation. [3]

(ii) Given that $P = 5A$ when $t = 0$, and that $P = 2A$ when $t = 2$, show that $k = \sqrt{A}$. [4]

(iii) Find the value of t when $P = A$. [2]

(iv) Obtain an expression for P in terms of A and t. [2]

(i) $2\sqrt{P - A} = -kt$

(iii) 4

(iv) $P = \frac{1}{4}A(4 + (4 - t)^2)$

MAY/JUNE2003

In a chemical reaction a compound X is formed from a compound Y. The masses in grams of X and Y present at time t seconds after the start of the reaction are x and y respectively. The sum of the two masses is equal to 100 grams throughout the reaction. At any time, the rate of formation of X is proportional to the mass of Y at that time. When $t = 0$, $x = 5$ and $\frac{dx}{dt} = 1.9$.

(i) Show that x satisfies the differential equation

$$\frac{dx}{dt} = 0.02(100 - x).$$ [2]

(ii) Solve this differential equation, obtaining an expression for x in terms of t. [6]

(iii) State what happens to the value of x as t becomes very large. [1]

(ii) $x = 100 - 95\exp(-0{,}02t)$

(iii) x tends to 100 as t becomes very large.

OCTOBER/NOVEMBER 2002

In an experiment to study the spread of a soil disease, an area of $10\,\text{m}^2$ of soil was exposed to infection. In a simple model, it is assumed that the infected area grows at a rate which is proportional to the product of the infected area and the uninfected area. Initially, $5\,\text{m}^2$ was infected and the rate of growth of the infected area was $0.1\,\text{m}^2$ per day. At time t days after the start of the experiment, an area $a\,\text{m}^2$ is infected and an area $(10 - a)\,\text{m}^2$ is uninfected.

(i) Show that $\dfrac{\text{d}a}{\text{d}t} = 0.004a(10 - a)$. [2]

(ii) By first expressing $\dfrac{1}{a(10 - a)}$ in partial fractions, solve this differential equation, obtaining an expression for t in terms of a. [6]

(iii) Find the time taken for 90% of the soil area to become infected, according to this model. [2]

(ii) $t = 25\ln(\frac{a}{10-a})$

(iii) 54.9 or 55

MAY/JUNE2002

In a certain chemical process a substance is being formed, and t minutes after the start of the process there are m grams of the substance present. In the process the rate of increase of m is proportional to $(50 - m)^2$. When $t = 0$, $m = 0$ and $\frac{dm}{dt} = 5$.

(i) Show that m satisfies the differential equation

$$\frac{dm}{dt} = 0.002(50 - m)^2. \qquad [2]$$

(ii) Solve the differential equation, and show that the solution can be expressed in the form

$$m = 50 - \frac{500}{t + 10}. \qquad [5]$$

(iii) Calculate the mass of the substance when $t = 10$, and find the time taken for the mass to increase from 0 to 45 grams. [2]

(iv) State what happens to the mass of the substance as t becomes very large. [1]

(iii) $m = 25, t = 10$; $t = 90, m = 45$

(iv) m approaches 50

NUMERICAL METHODS

MAY/JUNE2012 9709/31

(i) It is given that $2\tan 2x + 5\tan^2 x = 0$. Denoting $\tan x$ by t, form an equation in t and hence show that either $t = 0$ or $t = \sqrt[3]{(t + 0.8)}$. [4]

(ii) It is given that there is exactly one real value of t satisfying the equation $t = \sqrt[3]{(t + 0.8)}$. Verify by calculation that this value lies between 1.2 and 1.3. [2]

(iii) Use the iterative formula $t_{n+1} = \sqrt[3]{(t_n + 0.8)}$ to find the value of t correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

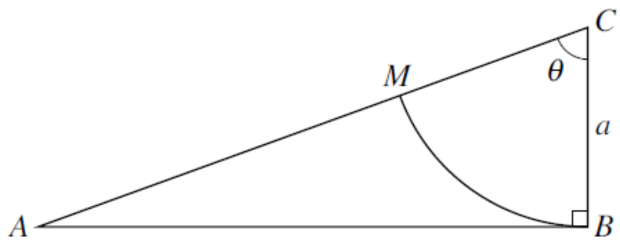
(iv) Using the values of t found in previous parts of the question, solve the equation

$$2\tan 2x + 5\tan^2 x = 0$$

for $-\pi \leqslant x \leqslant \pi$. [3]

(iii) 1.276

(iv) $-2.24, 0.906, -\pi, 0, \pi$



In the diagram, ABC is a triangle in which angle ABC is a right angle and $BC = a$. A circular arc, with centre C and radius a, joins B and the point M on AC. The angle ACB is θ radians. The area of the sector CMB is equal to one third of the area of the triangle ABC.

(i) Show that θ satisfies the equation

$$\tan\theta = 3\theta.$$ [2]

(ii) This equation has one root in the interval $0 < \theta < \frac{1}{2}\pi$. Use the iterative formula

$$\theta_{n+1} = \tan^{-1}(3\theta_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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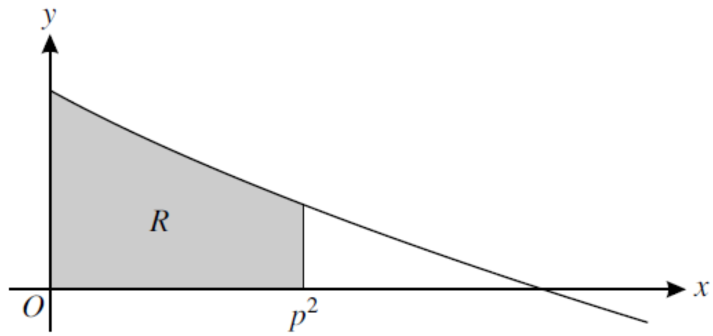


The diagram shows part of the curve $y = \cos(\sqrt{x})$ for $x \geqslant 0$, where x is in radians. The shaded region between the curve, the axes and the line $x = p^2$, where $p > 0$, is denoted by R. The area of R is equal to 1.

(i) Use the substitution $x = u^2$ to find $\int_0^{p^2} \cos(\sqrt{x})\,dx$. Hence show that $\sin p = \dfrac{3 - 2\cos p}{2p}$. [6]

(ii) Use the iterative formula $p_{n+1} = \sin^{-1}\left(\dfrac{3 - 2\cos p_n}{2p_n}\right)$, with initial value $p_1 = 1$, to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(i) $2u\ sinu + 2cosu$

(ii) 1.25

OCTOBER/NOVEMBER 2011 9709/31

(i) By sketching a suitable pair of graphs, show that the equation

$$\sec x = 3 - \tfrac{1}{2}x^2,$$

where x is in radians, has a root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 1 and 1.4. [2]

(iii) Show that this root also satisfies the equation

$$x = \cos^{-1}\left(\frac{2}{6 - x^2}\right). \quad [1]$$

(iv) Use an iterative formula based on the equation in part **(iii)** to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

1.13

OCTOBER/NOVEMBER 2011 9709/33

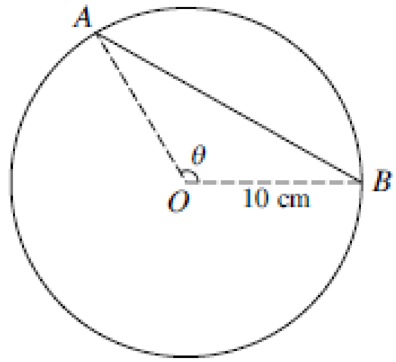
It is given that $\int_1^a x \ln x \, dx = 22$, where a is a constant greater than 1.

(i) Show that $a = \sqrt{\left(\dfrac{87}{2\ln a - 1}\right)}$. [5]

(ii) Use an iterative formula based on the equation in part **(i)** to find the value of a correct to 2 decimal places. Use an initial value of 6 and give the result of each iteration to 4 decimal places. [3]

(ii) 5.86

MAY/JUNE2011 9709/31



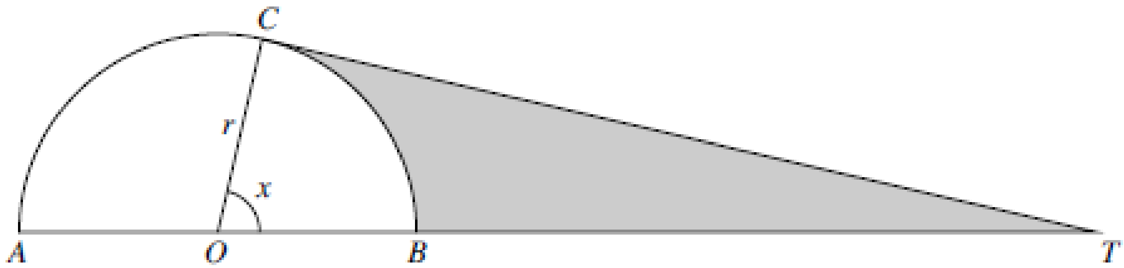
The diagram shows a circle with centre O and radius 10 cm. The chord AB divides the circle into two regions whose areas are in the ratio 1 : 4 and it is required to find the length of AB. The angle AOB is θ radians.

(i) Show that $\theta = \frac{2}{5}\pi + \sin\theta$. [3]

(ii) Showing all your working, use an iterative formula, based on the equation in part (i), with an initial value of 2.1, to find θ correct to 2 decimal places. Hence find the length of AB in centimetres correct to 1 decimal place. [5]

(ii) $\theta = 2.11, 17.4$

MAY/JUNE2011 9709/32



The diagram shows a semicircle ACB with centre O and radius r. The tangent at C meets AB produced at T. The angle BOC is x radians. The area of the shaded region is equal to the area of the semicircle.

(i) Show that x satisfies the equation

$$\tan x = x + \pi. \qquad [3]$$

(ii) Use the iterative formula $x_{n+1} = \tan^{-1}(x_n + \pi)$ to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(ii) 1.35

MAY/JUNE2011 9709/33

(i) By sketching a suitable pair of graphs, show that the equation

$$\cot x = 1 + x^2,$$

where x is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.5 and 0.8. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{1}{1 + x_n^2}\right)$$

to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) 0.62

OCTOBER/NOVEMBER 2010 9709/31

(i) By sketching suitable graphs, show that the equation

$$4x^2 - 1 = \cot x$$

has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.6 and 1. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tfrac{1}{2}\sqrt{(1 + \cot x_n)}$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) 0.73

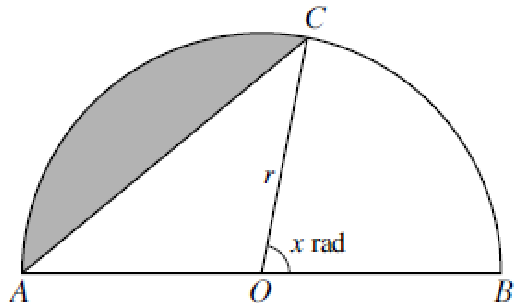
OCTOBER/NOVEMBER 2010 9709/33

(i) Given that $\displaystyle\int_1^a \frac{\ln x}{x^2}\,dx = \frac{2}{5}$, show that $a = \frac{5}{3}(1 + \ln a)$. [5]

(ii) Use an iteration formula based on the equation $a = \frac{5}{3}(1 + \ln a)$ to find the value of a correct to 2 decimal places. Use an initial value of 4 and give the result of each iteration to 4 decimal places. [3]

(ii) 3.96

MAY/JUNE2010 9709/31



The diagram shows a semicircle ACB with centre O and radius r. The angle BOC is x radians. The area of the shaded segment is a quarter of the area of the semicircle.

(i) Show that x satisfies the equation

$$x = \tfrac{3}{4}\pi - \sin x.$$ [3]

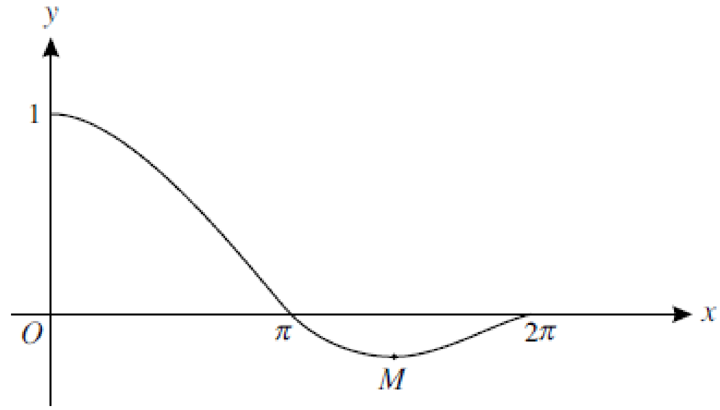
(ii) This equation has one root. Verify by calculation that the root lies between 1.3 and 1.5. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tfrac{3}{4}\pi - \sin x_n$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) 1.38



The diagram shows the curve $y = \frac{\sin x}{x}$ for $0 < x \leqslant 2\pi$, and its minimum point M.

(i) Show that the x-coordinate of M satisfies the equation

$$x = \tan x.$$ [4]

(ii) The iterative formula

$$x_{n+1} = \tan^{-1}(x_n) + \pi$$

can be used to determine the x-coordinate of M. Use this formula to determine the x-coordinate of M correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

4.49

MAY/JUNE2010 9709/33

The curve $y = \dfrac{\ln x}{x+1}$ has one stationary point.

(i) Show that the x-coordinate of this point satisfies the equation

$$x = \frac{x+1}{\ln x},$$

and that this x-coordinate lies between 3 and 4. [5]

(ii) Use the iterative formula

$$x_{n+1} = \frac{x_n + 1}{\ln x_n}$$

to determine the x-coordinate correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(ii) 3.59

OCTOBER/NOVEMBER 2009 9709/31

The sequence of values given by the iterative formula

$$x_{n+1} = \frac{3x_n}{4} + \frac{15}{x_n^3},$$

with initial value $x_1 = 3$, converges to α.

(i) Use this iterative formula to find α correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

(ii) State an equation satisfied by α and hence find the exact value of α. [2]

(i) 2.78

(ii) $x = \frac{3}{4}x + \frac{15}{x^3}$, $\sqrt[4]{60}$

OCTOBER/NOVEMBER 2009 9709/32

The equation $x^3 - 8x - 13 = 0$ has one real root.

(i) Find the two consecutive integers between which this root lies. [2]

(ii) Use the iterative formula

$$x_{n+1} = (8x_n + 13)^{\frac{1}{3}}$$

to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(i) 3 and 4
(ii)3.43

MAY/JUNE2009

The equation $x^3 - 2x - 2 = 0$ has one real root.

(i) Show by calculation that this root lies between $x = 1$ and $x = 2$. [2]

(ii) Prove that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 2}$$

converges, then it converges to this root. [2]

(iii) Use this iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) 1.77

OCTOBER/NOVEMBER 2008

The constant a is such that $\int_0^a x e^{\frac{1}{2}x}\,dx = 6$.

(i) Show that a satisfies the equation

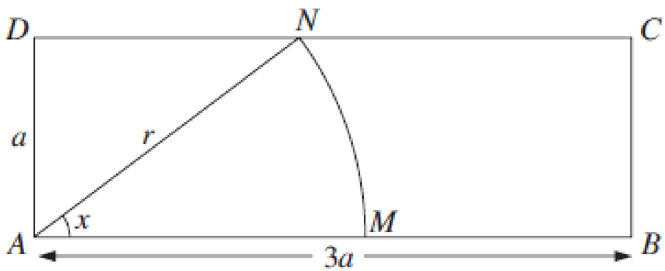
$$x = 2 + e^{-\frac{1}{2}x}. \qquad [5]$$

(ii) By sketching a suitable pair of graphs, show that this equation has only one root. [2]

(iii) Verify by calculation that this root lies between 2 and 2.5. [2]

(iv) Use an iterative formula based on the equation in part **(i)** to calculate the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iv) 2.31



In the diagram, $ABCD$ is a rectangle with $AB = 3a$ and $AD = a$. A circular arc, with centre A and radius r, joins points M and N on AB and CD respectively. The angle MAN is x radians. The perimeter of the sector AMN is equal to half the perimeter of the rectangle.

(i) Show that x satisfies the equation

$$\sin x = \tfrac{1}{4}(2 + x).$$ [3]

(ii) This equation has only one root in the interval $0 < x < \frac{1}{2}\pi$. Use the iterative formula

$$x_{n+1} = \sin^{-1}\left(\frac{2 + x_n}{4}\right),$$

with initial value $x_1 = 0.8$, to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

0.76

OCTOBER/NOVEMBER 2007

(i) By sketching a suitable pair of graphs, show that the equation

$$2 - x = \ln x$$

has only one root. [2]

(ii) Verify by calculation that this root lies between 1.4 and 1.7. [2]

(iii) Show that this root also satisfies the equation

$$x = \tfrac{1}{3}(4 + x - 2\ln x).$$ [1]

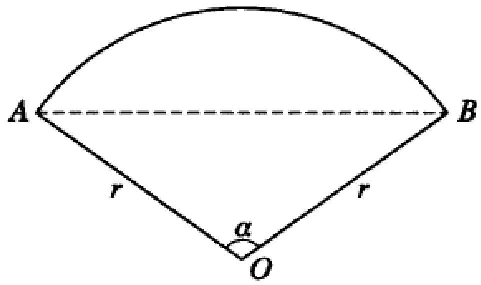
(iv) Use the iterative formula

$$x_{n+1} = \tfrac{1}{3}(4 + x_n - 2\ln x_n),$$

with initial value $x_1 = 1.5$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iv) 1.56

MAY/JUNE2007



The diagram shows a sector AOB of a circle with centre O and radius r. The angle AOB is α radians, where $0 < \alpha < \pi$. The area of triangle AOB is half the area of the sector.

(i) Show that α satisfies the equation

$$x = 2\sin x. \qquad [2]$$

(ii) Verify by calculation that α lies between $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$. [2]

(iii) Show that, if a sequence of values given by the iterative formula

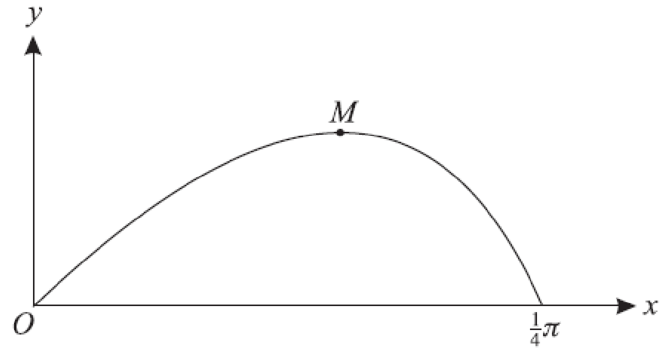
$$x_{n+1} = \tfrac{1}{3}(x_n + 4\sin x_n)$$

converges, then it converges to a root of the equation in part **(i)**. [2]

(iv) Use this iterative formula, with initial value $x_1 = 1.8$, to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iv) 1.90

OCTOBER/NOVEMBER 2006



The diagram shows the curve $y = x\cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The point M is a maximum point.

(i) Show that the x-coordinate of M satisfies the equation $1 = 2x\tan 2x$. [3]

(ii) The equation in part **(i)** can be rearranged in the form $x = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x}\right)$. Use the iterative formula

$$x_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_n}\right),$$

with initial value $x_1 = 0.4$, to calculate the x-coordinate of M correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) Use integration by parts to find the exact area of the region enclosed between the curve and the x-axis from 0 to $\frac{1}{4}\pi$. [5]

(ii) 0.43

(iii) $\frac{1}{8}\pi - \frac{1}{4}$

MAY/JUNE2006

(i) By sketching a suitable pair of graphs, show that the equation

$$2\cot x = 1 + e^x,$$

where x is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.5 and 1.0. [2]

(iii) Show that this root also satisfies the equation

$$x = \tan^{-1}\left(\frac{2}{1 + e^x}\right).$$ [1]

(iv) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{2}{1 + e^{x_n}}\right),$$

with initial value $x_1 = 0.7$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iv) 0.61

OCTOBER/NOVEMBER 2005

The equation $x^3 - x - 3 = 0$ has one real root, α.

(i) Show that α lies between 1 and 2. [2]

Two iterative formulae derived from this equation are as follows:

$$x_{n+1} = x_n^3 - 3, \qquad (A)$$

$$x_{n+1} = (x_n + 3)^{\frac{1}{3}}. \qquad (B)$$

Each formula is used with initial value $x_1 = 1.5$.

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [5]

(ii) (A) fails to converge. (B) converges to 1.67.

MAY/JUNE2005

(i) By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec} x = \tfrac{1}{2}x + 1,$$

where x is in radians, has a root in the interval $0 < x < \tfrac{1}{2}\pi$. [2]

(ii) Verify, by calculation, that this root lies between 0.5 and 1. [2]

(iii) Show that this root also satisfies the equation

$$x = \sin^{-1}\left(\frac{2}{x+2}\right).$$ [1]

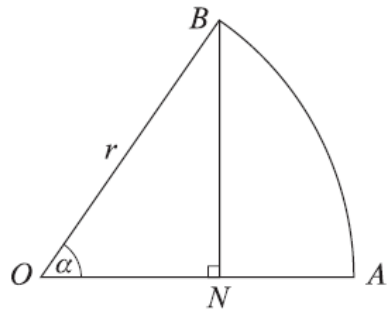
(iv) Use the iterative formula

$$x_{n+1} = \sin^{-1}\left(\frac{2}{x_n+2}\right),$$

with initial value $x_1 = 0.75$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iv) 0.80

OCTOBER/NOVEMBER 2004



The diagram shows a sector OAB of a circle with centre O and radius r. The angle AOB is α radians, where $0 < \alpha < \frac{1}{2}\pi$. The point N on OA is such that BN is perpendicular to OA. The area of the triangle ONB is half the area of the sector OAB.

(i) Show that α satisfies the equation $\sin 2x = x$. [3]

(ii) By sketching a suitable pair of graphs, show that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(iii) Use the iterative formula

$$x_{n+1} = \sin(2x_n),$$

with initial value $x_1 = 1$, to find α correct to 2 decimal places, showing the result of each iteration. [3]

(iii) 0.95

MAY/JUNE2004

(i) The equation $x^3 + x + 1 = 0$ has one real root. Show by calculation that this root lies between -1 and 0. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 + 1}$$

converges, then it converges to the root of the equation given in part **(i)**. [2]

(iii) Use this iterative formula, with initial value $x_1 = -0.5$, to determine the root correct to 2 decimal places, showing the result of each iteration. [3]

(iii) -0.68

OCTOBER/NOVEMBER 2003

(i) By sketching suitable graphs, show that the equation

$$\sec x = 3 - x^2$$

has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \cos^{-1}\left(\frac{1}{3 - x_n^2}\right)$$

converges, then it converges to a root of the equation given in part **(i)**. [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to determine the root in the interval $0 < x < \frac{1}{2}\pi$ correct to 2 decimal places, showing the result of each iteration. [3]

(iii) 1.03

MAY/JUNE2003

The equation of a curve is $y = \ln x + \dfrac{2}{x}$, where $x > 0$.

(i) Find the coordinates of the stationary point of the curve and determine whether it is a maximum or a minimum point. [5]

(ii) The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2}{3 - \ln x_n},$$

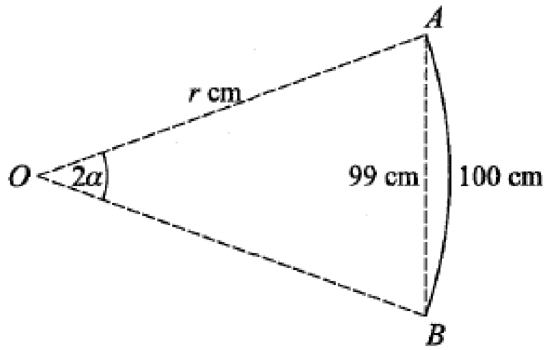
with initial value $x_1 = 1$, converges to α. State an equation satisfied by α, and hence show that α is the x-coordinate of a point on the curve where $y = 3$. [2]

(iii) Use this iterative formula to find α correct to 2 decimal places, showing the result of each iteration. [3]

(i) $(2, ln2 + 1)$, minimum

(ii) $3 = ln \propto + \frac{2}{\propto}$

(iv) 0.56



The diagram shows a curved rod AB of length 100 cm which forms an arc of a circle. The end points A and B of the rod are 99 cm apart. The circle has radius r cm and the arc AB subtends an angle of 2α radians at O, the centre of the circle.

(i) Show that α satisfies the equation $\frac{99}{100}x = \sin x$. [3]

(ii) Given that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$, verify by calculation that this root lies between 0.1 and 0.5. [2]

(iii) Show that if the sequence of values given by the iterative formula

$$x_{n+1} = 50 \sin x_n - 48.5x_n$$

converges, then it converges to a root of the equation in part **(i)**. [2]

(iv) Use this iterative formula, with initial value $x_1 = 0.25$, to find α correct to 3 decimal places, showing the result of each iteration. [2]

(iv) 0.245

MAY/JUNE2002

The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right),$$

with initial value $x_1 = 1$, converges to α.

(i) Use this formula to find α correct to 2 decimal places, showing the result of each iteration. [3]

(ii) State an equation satisfied by α, and hence find the exact value of α. [2]

(i) 1.26

(ii) $x = \frac{2}{3}\left(x + \frac{1}{x}\right)$, $\sqrt[3]{2}$

COMPLEX NUMBERS

MAY/JUNE2012 9709/31

The complex number u is defined by $u = \dfrac{(1+2\mathrm{i})^2}{2+\mathrm{i}}$.

(i) Without using a calculator and showing your working, express u in the form $x + \mathrm{i}y$, where x and y are real. [4]

(ii) Sketch an Argand diagram showing the locus of the complex number z such that $|z - u| = |u|$. [3]

(i) $-\frac{2}{5} + \frac{11}{5}i$

(ii) CIRCLE

MAY/JUNE2012 9709/32

Throughout this question the use of a calculator is not permitted.

The complex number u is defined by

$$u = \frac{1+2\mathrm{i}}{1-3\mathrm{i}}.$$

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Show on a sketch of an Argand diagram the points A, B and C representing the complex numbers u, $1 + 2\mathrm{i}$ and $1 - 3\mathrm{i}$ respectively. [2]

(iii) By considering the arguments of $1 + 2\mathrm{i}$ and $1 - 3\mathrm{i}$, show that

$$\tan^{-1} 2 + \tan^{-1} 3 = \tfrac{3}{4}\pi.$$ [3]

(i) $-\frac{1}{2} + \frac{1}{2}i$

MAY/JUNE2012 9709/33

(a) The complex numbers u and w satisfy the equations

$$u - w = 4\text{i} \quad \text{and} \quad uw = 5.$$

Solve the equations for u and w, giving all answers in the form $x + \text{i}y$, where x and y are real. [5]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 2 + 2\text{i}| \leqslant 2$, $\arg z \leqslant -\frac{1}{4}\pi$ and $\operatorname{Re} z \geqslant 1$, where $\operatorname{Re} z$ denotes the real part of z. [5]

(ii) Calculate the greatest possible value of $\operatorname{Re} z$ for points lying in the shaded region. [1]

(a) $u = 1 + 2i, w = 1 - 2i$
(b) (ii) $2 + \sqrt{2}$

OCTOBER/NOVEMBER 2011 9709/31

(a) Showing your working, find the two square roots of the complex number $1 - (2\sqrt{6})\text{i}$. Give your answers in the form $x + \text{i}y$, where x and y are exact. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy the inequality $|z - 3\text{i}| \leqslant 2$. Find the greatest value of $\arg z$ for points in this region. [5]

(a) $\pm(\sqrt{3} - i\sqrt{2})$
(b) 2.30 rad

OCTOBER/NOVEMBER 2011 9709/33

The complex number w is defined by $w = -1 + \mathrm{i}$.

(i) Find the modulus and argument of w^2 and w^3, showing your working. [4]

(ii) The points in an Argand diagram representing w and w^2 are the ends of a diameter of a circle. Find the equation of the circle, giving your answer in the form $|z - (a + b\mathrm{i})| = k$. [4]

(i) $|w^2| = 2, |w^3| = 2\sqrt{2}, \arg(w^2) = -\frac{1}{2}\pi, \arg(w^3) = \frac{\pi}{4}$

(ii) $\left|z + \frac{1}{2} + \frac{1}{2}i\right| = \frac{1}{2}\sqrt{10}$

MAY/JUNE2011 9709/31

The complex number u is defined by $u = \dfrac{6 - 3\mathrm{i}}{1 + 2\mathrm{i}}$.

(i) Showing all your working, find the modulus of u and show that the argument of u is $-\frac{1}{2}\pi$. [4]

(ii) For complex numbers z satisfying $\arg(z - u) = \frac{1}{4}\pi$, find the least possible value of $|z|$. [3]

(iii) For complex numbers z satisfying $|z - (1 + \mathrm{i})u| = 1$, find the greatest possible value of $|z|$. [3]

(i) 3

(ii) $\frac{3}{2}\sqrt{2}$

(iii) $3\sqrt{2} + 1$

MAY/JUNE2011 9709/32

(a) The complex number u is defined by $u = \dfrac{5}{a+2\mathrm{i}}$, where the constant a is real.

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Find the value of a for which $\arg(u^*) = \frac{3}{4}\pi$, where u^* denotes the complex conjugate of u. [3]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z| < 2$ and $|z| < |z - 2 - 2\mathrm{i}|$. [4]

(a) (i) $\frac{5a}{a^2+4} - \frac{10}{a^2+4}i$

(ii) -2

MAY/JUNE2011 9709/33

(i) Find the roots of the equation

$$z^2 + (2\sqrt{3})z + 4 = 0,$$

giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) State the modulus and argument of each root. [3]

(iii) Showing all your working, verify that each root also satisfies the equation

$$z^6 = -64.$$ [3]

(i) $-\sqrt{3} \pm i$

(ii) 2, 2, $\frac{5}{6}\pi, -\frac{5}{6}\pi$

OCTOBER/NOVEMBER 2010 9709/31

The complex number z is given by

$$z = (\sqrt{3}) + \mathrm{i}.$$

(i) Find the modulus and argument of z. [2]

(ii) The complex conjugate of z is denoted by z^*. Showing your working, express in the form $x + \mathrm{i}y$, where x and y are real,

(a) $2z + z^*$,

(b) $\dfrac{\mathrm{i}z^*}{z}$.

[4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z and $\mathrm{i}z^*$ respectively. Prove that angle $AOB = \frac{1}{6}\pi$. [3]

(i) $2, \frac{\pi}{6}$

(ii) (a) $3\sqrt{3} + i$ (b) $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

OCTOBER/NOVEMBER 2010 9709/33

The complex number w is defined by $w = 2 + \mathrm{i}$.

(i) Showing your working, express w^2 in the form $x + \mathrm{i}y$, where x and y are real. Find the modulus of w^2. [3]

(ii) Shade on an Argand diagram the region whose points represent the complex numbers z which satisfy

$$|z - w^2| \leqslant |w^2|.$$ [3]

(i) $3 + 4i, 5$

OCTOBER/NOVEMBER 2010 9709/33

The polynomial $p(z)$ is defined by

$$p(z) = z^3 + mz^2 + 24z + 32,$$

where m is a constant. It is given that $(z + 2)$ is a factor of $p(z)$.

(i) Find the value of m. [2]

(ii) Hence, showing all your working, find

(a) the three roots of the equation $p(z) = 0$, [5]

(b) the six roots of the equation $p(z^2) = 0$. [6]

(i) 6

(ii) (a) $-2, -2 \pm 2\sqrt{3}i$ (b) $\pm\sqrt{2}i, \pm(1 + \sqrt{3}i), \pm(1 - \sqrt{3}i)$

MAY/JUNE2010 9709/31

The complex number $2 + 2\text{i}$ is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Sketch an Argand diagram showing the points representing the complex numbers 1, i and u. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1| \leqslant |z - \text{i}|$ and $|z - u| \leqslant 1$. [4]

(iii) Using your diagram, calculate the value of $|z|$ for the point in this region for which $\arg z$ is least. [3]

(i) $\sqrt{8}. \frac{\pi}{4}$
(iii) $\sqrt{7}$

MAY/JUNE2010 9709/32

The variable complex number z is given by

$$z = 1 + \cos 2\theta + \mathrm{i}\sin 2\theta,$$

where θ takes all values in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(i) Show that the modulus of z is $2\cos\theta$ and the argument of z is θ. [6]

(ii) Prove that the real part of $\dfrac{1}{z}$ is constant. [3]

MAY/JUNE2010 9709/33

(a) The equation $2x^3 - x^2 + 2x + 12 = 0$ has one real root and two complex roots. Showing your working, verify that $1 + \mathrm{i}\sqrt{3}$ is one of the complex roots. State the other complex root. [4]

(b) On a sketch of an Argand diagram, show the point representing the complex number $1 + \mathrm{i}\sqrt{3}$. On the same diagram, shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1 - \mathrm{i}\sqrt{3}| \leqslant 1$ and $\arg z \leqslant \frac{1}{3}\pi$. [5]

(i) $1 - i\sqrt{3}$

OCTOBER/NOVEMBER 2009 9709/31

The complex number $-2 + \mathrm{i}$ is denoted by u.

(i) Given that u is a root of the equation $x^3 - 11x - k = 0$, where k is real, find the value of k. [3]

(ii) Write down the other complex root of this equation. [1]

(iii) Find the modulus and argument of u. [2]

(iv) Sketch an Argand diagram showing the point representing u. Shade the region whose points represent the complex numbers z satisfying both the inequalities

$$|z| < |z - 2| \quad \text{and} \quad 0 < \arg(z - u) < \tfrac{1}{4}\pi. \qquad [4]$$

(i) 20
(ii) $-2 - i$
(iii) $\sqrt{5}, 2.68rad$

OCTOBER/NOVEMBER 2009 9709/32

The complex numbers $-2+\mathrm{i}$ and $3+\mathrm{i}$ are denoted by u and v respectively.

(i) Find, in the form $x+\mathrm{i}y$, the complex numbers

(a) $u+v$, [1]

(b) $\frac{u}{v}$, showing all your working. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, v and $u+v$ respectively.

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

(iv) State fully the geometrical relationship between the line segments OA and BC. [2]

(i) (a) $1+2i$

(b) $-\frac{1}{2}+\frac{1}{2}i$

(ii) $\frac{3}{4}\pi$

(iv) OA=BC, OA is parallel to BC.

MAY/JUNE2009

(i) Solve the equation $z^2 + (2\sqrt{3})iz - 4 = 0$, giving your answers in the form $x + iy$, where x and y are real. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) Find the modulus and argument of each root. [3]

(iv) Show that the origin and the points representing the roots are the vertices of an equilateral triangle. [1]

(i) $1 - \sqrt{3}i, -1 - \sqrt{3}i$

(iii) For $1 - \sqrt{3}i$, modulus=2, argument =$-\frac{\pi}{3}$

For $-1 - \sqrt{3}i$, modulus =2, argument = $-\frac{2}{3}\pi$

OCTOBER/NOVEMBER 2008

The complex number w is given by $w = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$.

(i) Find the modulus and argument of w. [2]

(ii) The complex number z has modulus R and argument θ, where $-\frac{1}{3}\pi < \theta < \frac{1}{3}\pi$. State the modulus and argument of wz and the modulus and argument of $\frac{z}{w}$. [4]

(iii) Hence explain why, in an Argand diagram, the points representing z, wz and $\frac{z}{w}$ are the vertices of an equilateral triangle. [2]

(iv) In an Argand diagram, the vertices of an equilateral triangle lie on a circle with centre at the origin. One of the vertices represents the complex number $4 + 2\mathrm{i}$. Find the complex numbers represented by the other two vertices. Give your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [4]

(i) $1, \frac{2}{3}\pi$

(ii) For wz, modulus = R, argument = $\theta + \frac{2}{3}\pi$

For z/w, modulus = R, argument = $\theta - \frac{2}{3}\pi$

(iv) $-(2+\sqrt{3}) + (2\sqrt{3}-1)i, -(2-\sqrt{3}) - (2\sqrt{3}+1)i,$

MAY/JUNE2008

The variable complex number z is given by

$$z = 2\cos\theta + \mathrm{i}(1 - 2\sin\theta),$$

where θ takes all values in the interval $-\pi < \theta \leqslant \pi$.

(i) Show that $|z - \mathrm{i}| = 2$, for all values of θ. Hence sketch, in an Argand diagram, the locus of the point representing z. [3]

(ii) Prove that the real part of $\dfrac{1}{z + 2 - \mathrm{i}}$ is constant for $-\pi < \theta < \pi$. [4]

OCTOBER/NOVEMBER 2007

(a) **The complex number z is given by $z = \dfrac{4 - 3\mathrm{i}}{1 - 2\mathrm{i}}$.**

(i) **Express z in the form $x + \mathrm{i}y$, where x and y are real.** **[2]**

(ii) **Find the modulus and argument of z.** **[2]**

(b) **Find the two square roots of the complex number $5 - 12\mathrm{i}$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real.** **[6]**

(a) (i) $2 + i$

(ii) $\sqrt{5}, 0.464rad$

(b) $3 - 2i, -3 + 2i$

MAY/JUNE2007

The complex number $\frac{2}{-1+\mathrm{i}}$ is denoted by u.

(i) Find the modulus and argument of u and u^2. [6]

(ii) Sketch an Argand diagram showing the points representing the complex numbers u and u^2. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z| < 2$ and $|z - u^2| < |z - u|$. [4]

(i) For u, modulus = $\sqrt{2}$, argument = $-\frac{3}{4}\pi$

For u^2, modulus = 2, argument = $\frac{1}{2}\pi$

OCTOBER/NOVEMBER 2006

The complex number u is given by

$$u = \frac{3 + i}{2 - i}.$$

(i) Express u in the form $x + iy$, where x and y are real. [3]

(ii) Find the modulus and argument of u. [2]

(iii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the point representing the complex number z such that $|z - u| = 1$. [3]

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. [2]

(i) $1 + i$
(ii) $\sqrt{2}, \frac{\pi}{4}$
(iv) $\sqrt{2} - 1$

MAY/JUNE2006

The complex number $2 + \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) Show, on a sketch of an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u + u^*$ respectively. Describe in geometrical terms the relationship between the four points O, A, B and C. [4]

(ii) Express $\dfrac{u}{u^*}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\dfrac{u}{u^*}$, or otherwise, prove that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = 2\tan^{-1}\left(\tfrac{1}{2}\right).$$ [2]

(i) OACB is a parallelogram, OACB is a rhombus

(ii) $\frac{3}{5} + \frac{4}{5}i$

OCTOBER/NOVEMBER 2005

The equation $2x^3 + x^2 + 25 = 0$ has one real root and two complex roots.

(i) Verify that $1 + 2i$ is one of the complex roots. [3]

(ii) Write down the other complex root of the equation. [1]

(iii) Sketch an Argand diagram showing the point representing the complex number $1 + 2i$. Show on the same diagram the set of points representing the complex numbers z which satisfy

$$|z| = |z - 1 - 2i|.$$ [4]

(ii) $1 - 2i$

MAY/JUNE2005

(i) Solve the equation $z^2 - 2iz - 5 = 0$, giving your answers in the form $x + iy$ where x and y are real. [3]

(ii) Find the modulus and argument of each root. [3]

(iii) Sketch an Argand diagram showing the points representing the roots. [1]

(i) $2 + i, -2 + i$

(ii) For $2 + i$, modulus = $\sqrt{5}$, argument = 0.464rad

For $-2 + i$, modulus = $\sqrt{5}$, argument = 2.68rad

OCTOBER/NOVEMBER 2004

The complex numbers $1 + 3i$ and $4 + 2i$ are denoted by u and v respectively.

(i) Find, in the form $x + iy$, where x and y are real, the complex numbers $u - v$ and $\frac{u}{v}$. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram, with origin O, the points A, B and C represent the numbers u, v and $u - v$ respectively.

(iii) State fully the geometrical relationship between OC and BA. [2]

(iv) Prove that angle $AOB = \frac{1}{4}\pi$ radians. [2]

(i) $-3 + i, \frac{1}{2} + \frac{1}{2}i$

(ii) $\frac{\pi}{4}$

(iii) length of OC and BA are equal. OC and BA are parallel.

MAY/JUNE2004

(i) Find the roots of the equation $z^2 - z + 1 = 0$, giving your answers in the form $x + iy$, where x and y are real. [2]

(ii) Obtain the modulus and argument of each root. [3]

(iii) Show that each root also satisfies the equation $z^3 = -1$. [2]

(i) $\frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}$

(ii) For $\frac{1}{2} + i\frac{\sqrt{3}}{2}$, modulus = 1, argument = $\frac{\pi}{3}$

For $\frac{1}{2} + i\frac{\sqrt{3}}{2}$, modulus = 1, argument = $\frac{5}{3}\pi$

OCTOBER/NOVEMBER 2003

The complex number u is given by $u = \dfrac{7 + 4i}{3 - 2i}$.

(i) Express u in the form $x + iy$, where x and y are real. [3]

(ii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the complex number z such that $|z - u| = 2$. [3]

(iii) Find the greatest value of $\arg z$ for points on this locus. [3]

(i) $1 + 2i$

(iii) 2.21rad

MAY/JUNE2003

The complex number 2i is denoted by u. The complex number with modulus 1 and argument $\frac{2}{3}\pi$ is denoted by w.

(i) Find in the form $x + \mathrm{i}y$, where x and y are real, the complex numbers w, uw and $\frac{u}{w}$. [4]

(ii) Sketch an Argand diagram showing the points U, A and B representing the complex numbers u, uw and $\frac{u}{w}$ respectively. [2]

(iii) Prove that triangle UAB is equilateral. [2]

(i) $w = cos\frac{2}{3}\pi + isin\frac{2}{3}\pi,\ uw = -\sqrt{3} - i, \frac{u}{w} = \sqrt{3} - i$

OCTOBER/NOVEMBER 2002

(a) Find the two square roots of the complex number $-3 + 4\mathrm{i}$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) The complex number z is given by

$$z = \frac{-1 + 3\mathrm{i}}{2 + \mathrm{i}}.$$

(i) Express z in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Show on a sketch of an Argand diagram, with origin O, the points A, B and C representing the complex numbers $-1 + 3\mathrm{i}$, $2 + \mathrm{i}$ and z respectively. [1]

(iii) State an equation relating the lengths OA, OB and OC. [1]

(a) $1 + 2i, -1 - 2i$

(b) (i) $\frac{1}{5} + \frac{7}{5}i$

(iii) $OC = \frac{OA}{OB}$

MAY/JUNE2002

The complex number $1 + i\sqrt{3}$ is denoted by u.

(i) Express u in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Hence, or otherwise, find the modulus and argument of u^2 and u^3. [5]

(ii) Show that u is a root of the equation $z^2 - 2z + 4 = 0$, and state the other root of this equation. [2]

(iii) Sketch an Argand diagram showing the points representing the complex numbers i and u. Shade the region whose points represent every complex number z satisfying both the inequalities

$$|z - i| \leqslant 1 \quad \text{and} \quad \arg z \geqslant \arg u.$$ [4]

(i) $2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$

For u^2, modulus = 4, argument = $\frac{2}{3}\pi$

For u^3, modulus = 8, argument = π

(ii) $1 - i\sqrt{3}$

VECTORS

MAY/JUNE2012 9709/31

The point P has coordinates $(-1, 4, 11)$ and the line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

(i) Find the perpendicular distance from P to l. [4]

(ii) Find the equation of the plane which contains P and l, giving your answer in the form $ax + by + cz = d$, where a, b, c and d are integers. [5]

(i) $\sqrt{104}$
(ii) $3x - 9y + z = -28$

MAY/JUNE2012 9709/32

Two planes, m and n, have equations $x + 2y - 2z = 1$ and $2x - 2y + z = 7$ respectively. The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

(i) Show that l is parallel to m. [3]

(ii) Find the position vector of the point of intersection of l and n. [3]

(iii) A point P lying on l is such that its perpendicular distances from m and n are equal. Find the position vectors of the two possible positions for P and calculate the distance between them. [6]

[The perpendicular distance of a point with position vector $x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ from the plane $ax + by + cz = d$ is $\dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{(a^2 + b^2 + c^2)}}$.]

(ii) $5i + 3j + 3k$
(iii) 6

MAY/JUNE2012 9709/33

The lines l and m have equations $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} - \mathbf{k})$ respectively, where a and b are constants.

(i) Given that l and m intersect, show that

$$2a - b = 4.$$ [4]

(ii) Given also that l and m are perpendicular, find the values of a and b. [4]

(iii) When a and b have these values, find the position vector of the point of intersection of l and m. [2]

(ii) a=3, b=2
(iii) $i + 2j + 3k$

OCTOBER/NOVEMBER 2011 9709/31

With respect to the origin O, the position vectors of two points A and B are given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line through A and B, and $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

(i) Show that $\overrightarrow{OP} = (1 + 2\lambda)\mathbf{i} + (2 + 2\lambda)\mathbf{j} + (2 - 2\lambda)\mathbf{k}$. [2]

(ii) By equating expressions for $\cos AOP$ and $\cos BOP$ in terms of λ, find the value of λ for which OP bisects the angle AOB. [5]

(iii) When λ has this value, verify that $AP : PB = OA : OB$. [1]

(ii) $\frac{3}{8}$

OCTOBER/NOVEMBER 2011 9709/33

The line l has equation $\mathbf{r} = \begin{pmatrix} a \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$, where a is a constant. The plane p has equation $2x - 2y + z = 10$.

(i) Given that l does not lie in p, show that l is parallel to p. [2]

(ii) Find the value of a for which l lies in p. [2]

(iii) It is now given that the distance between l and p is 6. Find the possible values of a. [5]

(ii) 4
(iii) -5, 13

MAY/JUNE2011 9709/31

Points A and B have coordinates $(-1, 2, 5)$ and $(2, -2, 11)$ respectively. The plane p passes through B and is perpendicular to AB.

(i) Find an equation of p, giving your answer in the form $ax + by + cz = d$. [3]

(ii) Find the acute angle between p and the y-axis. [4]

(i) $3x - 4y + 6z = 80$
(ii) 30.8°

MAY/JUNE2011 9709/32

Two planes have equations $x + 2y - 2z = 7$ and $2x + y + 3z = 5$.

(i) Calculate the acute angle between the planes. [4]

(ii) Find a vector equation for the line of intersection of the planes. [6]

(i) 79.7°

(ii) $r = i + 3j + \lambda(8i - 7j - 3k)$

MAY/JUNE2011 9709/33

With respect to the origin O, the lines l and m have vector equations $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ respectively.

(i) Prove that l and m do not intersect. [4]

(ii) Calculate the acute angle between the directions of l and m. [3]

(iii) Find the equation of the plane which is parallel to l and contains m, giving your answer in the form $ax + by + cz = d$. [5]

(ii) 47.1°

(iii) $-2x + 4y + 3z = 26$

OCTOBER/NOVEMBER 2010 9709/31

With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line AB and OP is perpendicular to AB.

(i) Find a vector equation for the line AB. [1]

(ii) Find the position vector of P. [4]

(iii) Find the equation of the plane which contains AB and which is perpendicular to the plane OAB, giving your answer in the form $ax + by + cz = d$. [4]

(i) $r = i + 2j + 2k + \lambda(2i + 2j - 2k)$
(ii) $\frac{2}{3}i + \frac{5}{3}j + \frac{7}{3}k$
(iii) $2x + 5y + 7z = 26$

OCTOBER/NOVEMBER 2010 9709/33

The straight line l passes through the points with coordinates $(-5, 3, 6)$ and $(5, 8, 1)$. The plane p has equation $2x - y + 4z = 9$.

(i) Find the coordinates of the point of intersection of l and p. [4]

(ii) Find the acute angle between l and p. [4]

(i) (-1, 5, 4)
(ii) 5.1°

MAY/JUNE2010 9709/31

The lines l and m have vector equations

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 4\mathbf{i} + 6\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m intersect. [4]

(ii) Calculate the acute angle between the lines. [3]

(iii) Find the equation of the plane containing l and m, giving your answer in the form $ax + by + cz = d$. [5]

(ii) 74.2°
(iii) $5x - 3y - 4z = -2$

MAY/JUNE2010 9709/32

The plane p has equation $3x + 2y + 4z = 13$. A second plane q is perpendicular to p and has equation $ax + y + z = 4$, where a is a constant.

(i) Find the value of a. [3]

(ii) The line with equation $\mathbf{r} = \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ meets the plane p at the point A and the plane q at the point B. Find the length of AB. [6]

(i) a=-2
(ii) 3

MAY/JUNE2010 9709/33

The straight line l has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. The plane p has equation $3x - y + 2z = 9$. The line l intersects the plane p at the point A.

(i) Find the position vector of A. [3]

(ii) Find the acute angle between l and p. [4]

(iii) Find an equation for the plane which contains l and is perpendicular to p, giving your answer in the form $ax + by + cz = d$. [5]

(i) $4i + 3j$

(ii) 26.5°

(iii) $6x + 4y - 7z = 36$

OCTOBER/NOVEMBER 2009 9709/31

With respect to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} - \mathbf{k}, \quad \overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.$$

The mid-point of AB is M. The point N lies on AC between A and C and is such that $AN = 2NC$.

(i) Find a vector equation of the line MN. [4]

(ii) It is given that MN intersects BC at the point P. Find the position vector of P. [4]

(i) $r = 2i + j - 2k + \lambda(i - 3j + 3k)$
(ii) $5i - 8j + 7k$

OCTOBER/NOVEMBER 2009 9709/32

The plane p has equation $2x - 3y + 6z = 16$. The plane q is parallel to p and contains the point with position vector $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

(i) Find the equation of q, giving your answer in the form $ax + by + cz = d$. [2]

(ii) Calculate the perpendicular distance between p and q. [3]

(iii) The line l is parallel to the plane p and also parallel to the plane with equation $x - 2y + 2z = 5$. Given that l passes through the origin, find a vector equation for l. [5]

(i) $2x - 3y + 6z = 2$
(ii)2
(iii) $r = \lambda(-6i - 2j + k)$

MAY/JUNE2009

The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$. It is given that l lies in the plane with equation $2x + by + cz = 1$, where b and c are constants.

(i) Find the values of b and c. [6]

(ii) The point P has position vector $2\mathbf{j} + 4\mathbf{k}$. Show that the perpendicular distance from P to l is $\sqrt{5}$. [5]

(i) -2, 3

OCTOBER/NOVEMBER 2008

Two planes have equations $2x - y - 3z = 7$ and $x + 2y + 2z = 0$.

(i) Find the acute angle between the planes. [4]

(ii) Find a vector equation for their line of intersection. [6]

(i) 57.7°

(ii) $r = 2i - k + \lambda(4i - 7j + 5k)$

MAY/JUNE2008

The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation

$$\mathbf{r} = (1 - 2t)\mathbf{i} + (5 + t)\mathbf{j} + (2 - t)\mathbf{k}.$$

(i) Show that l does not intersect the line passing through A and B. [4]

(ii) The point P lies on l and is such that angle PAB is equal to $60°$. Given that the position vector of P is $(1 - 2t)\mathbf{i} + (5 + t)\mathbf{j} + (2 - t)\mathbf{k}$, show that $3t^2 + 7t + 2 = 0$. Hence find the only possible position vector of P. [6]

(ii) $5i + 3j + 4k$

OCTOBER/NOVEMBER 2007

The straight line l has equation $\mathbf{r} = \mathbf{i} + 6\mathbf{j} - 3\mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$. The plane p has equation $(\mathbf{r} - 3\mathbf{i}).(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = 0$. The line l intersects the plane p at the point A.

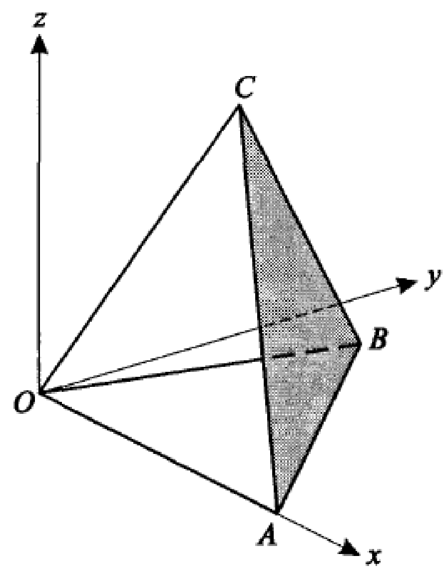
(i) Find the position vector of A. **[3]**

(ii) Find the acute angle between l and p. **[4]**

(iii) Find a vector equation for the line which lies in p, passes through A and is perpendicular to l. **[5]**

(i) $3i + 2j + k$
(ii) 72.2°
(iii) $r = 3i + 2 + k + \lambda(6i + 2j - k)$

MAY/JUNE2007



The diagram shows a set of rectangular axes Ox, Oy and Oz, and three points A, B and C with position vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

(i) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [6]

(ii) Calculate the acute angle between the planes ABC and OAB. [4]

(i) $4x + 2y + z = 8$
(ii) 77.4°

OCTOBER/NOVEMBER 2006

The line l has equation $\mathbf{r} = \mathbf{j} + \mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. The plane p has equation $x + 2y + 3z = 5$.

(i) Show that the line l lies in the plane p. [3]

(ii) A second plane is perpendicular to the plane p, parallel to the line l and contains the point with position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. Find the equation of this plane, giving your answer in the form $ax + by + cz = d$. [6]

(ii) $4x + y - 2z = 1$

MAY/JUNE2006

The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}.$$

The line l passes through A and is parallel to OB. The point N is the foot of the perpendicular from B to l.

(i) State a vector equation for the line l. [1]

(ii) Find the position vector of N and show that $BN = 3$. [6]

(iii) Find the equation of the plane containing A, B and N, giving your answer in the form $ax + by + cz = d$. [5]

(i) $r = -i + 3j + 5k + \lambda(3i - j - 4k)$
(ii) $5i + j - 3k$
(iii) $7x - 11y + 8z = 0$

OCTOBER/NOVEMBER 2005

The straight line l passes through the points A and B with position vectors

$$2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

respectively. This line intersects the plane p with equation $x - 2y + 2z = 6$ at the point C.

(i) Find the position vector of C. [4]

(ii) Find the acute angle between l and p. [4]

(iii) Show that the perpendicular distance from A to p is equal to 2. [3]

(i) $4i - 2j - k$
(ii) 24.1°

MAY/JUNE2005

With respect to the origin O, the points A and B have position vectors given by

$$\overrightarrow{OA} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \overrightarrow{OB} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation $\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

(i) Prove that the line l does not intersect the line through A and B. [5]

(ii) Find the equation of the plane containing l and the point A, giving your answer in the form $ax + by + cz = d$. [6]

(ii) $6x + y - 8z = 6$

OCTOBER/NOVEMBER 2004

The lines l and m have vector equations

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + s(\mathbf{i} + \mathbf{j} - \mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m do not intersect. [4]

The point P lies on l and the point Q has position vector $2\mathbf{i} - \mathbf{k}$.

(ii) Given that the line PQ is perpendicular to l, find the position vector of P. [4]

(iii) Verify that Q lies on m and that PQ is perpendicular to m. [2]

(ii) $4i + j + 2k$

MAY/JUNE2004

With respect to the origin O, the points P, Q, R, S have position vectors given by

$$\overrightarrow{OP} = \mathbf{i} - \mathbf{k}, \quad \overrightarrow{OQ} = -2\mathbf{i} + 4\mathbf{j}, \quad \overrightarrow{OR} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OS} = 3\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}.$$

(i) Find the equation of the plane containing P, Q and R, giving your answer in the form $ax + by + cz = d$. [6]

(ii) The point N is the foot of the perpendicular from S to this plane. Find the position vector of N and show that the length of SN is 7. [6]

(i) $2x + 3y - 6z = 8$
(ii) $i + 2j$

OCTOBER/NOVEMBER 2003

The lines l and m have vector equations

$$\mathbf{r} = \mathbf{i} - 2\mathbf{k} + s(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 6\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m intersect, and find the position vector of their point of intersection. [5]

(ii) Find the equation of the plane containing l and m, giving your answer in the form $ax + by + cz = d$. [6]

(i) $3i + j + k$
(ii) $7x + y - 5z = 17$

MAY/JUNE2003

Two planes have equations $x + 2y - 2z = 2$ and $2x - 3y + 6z = 3$. The planes intersect in the straight line l.

(i) Calculate the acute angle between the two planes. [4]

(ii) Find a vector equation for the line l. [6]

(i) $40.4°$ (or $40.3°$ or $139.6°$)
(ii) $r = 3j + 2k + \lambda(6i - 10j - 7k)$

OCTOBER/NOVEMBER 2002

With respect to the origin O, the points A, B, C, D have position vectors given by

$$\overrightarrow{OA} = 4\mathbf{i} + \mathbf{k}, \quad \overrightarrow{OB} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}, \quad \overrightarrow{OC} = \mathbf{i} + \mathbf{j}, \quad \overrightarrow{OD} = -\mathbf{i} - 4\mathbf{k}.$$

(i) Calculate the acute angle between the lines AB and CD. [4]

(ii) Prove that the lines AB and CD intersect. [4]

(iii) The point P has position vector $\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$. Show that the perpendicular distance from P to the line AB is equal to $\sqrt{3}$. [4]

(i) 45.6°

MAY/JUNE2002

The straight line l passes through the points A and B whose position vectors are $\mathbf{i} + \mathbf{k}$ and $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively. The plane p has equation $x + 3y - 2z = 3$.

(i) Given that l intersects p, find the position vector of the point of intersection. [4]

(ii) Find the equation of the plane which contains l and is perpendicular to p, giving your answer in the form $ax + by + cz = 1$. [6]

(i) $-2i + j - k$
(ii) $-4x + 8y + 10z = 6$