FURTHER MATHEMATICS

Paper 9231/11

Paper 11

General comments

The scripts for this paper were of a generally good quality. There were a number of outstanding scripts and once again there were very few poor scripts. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was sound. This year there was a marked improvement at vector work, which was good to see.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

The Examiners felt candidates had a sound knowledge of most topics on the syllabus.

As well as the vector work, already mentioned, candidates performed particularly well on matrices, roots of equations and whichever alternative of the final question that they chose, each of which was answered by a large number of candidates.

Comments on specific questions

Question 1

Many candidates scored full marks on this question. Only the weakest candidates found difficulty dealing with cube roots and differentiating, either implicitly or explicitly after rearranging the equation. Those differentiating implicitly found difficulty with the third term, or left 29 on the right-hand side. Those differentiating explicitly sometimes failed to employ the chain rule correctly.

Answers: 3, $\frac{4}{27}$.

Question 2

In part **(i)**, most sketches scored at least 1 mark for being of approximately the correct shape and location for $0 \leqslant \theta \leqslant 2\pi$. Marks were lost for not showing the initial line to be a tangent at the pole, or for not showing asymptotic approach to the circle $r = a$.

In part **(ii)**, those who expanded $(1-e^{-\theta})^2$ invariably went on to correctly obtain the printed answer.

Question 3

Almost all candidates correctly obtained the length of C, since the derivatives of the parametric equations were given and the expression for $\frac{ds}{dt}$ was a basic integral. A substantial number of candidates were unable to determine y and were thus unable to make further progress. Many of those who found y, successfully completed the question.

Answer: $\frac{128\sqrt{2}\pi}{15}$.

Question 4

The initial mark in this question was a method mark for applying the method of differences to $\sum_{n=1}^{N}\left\{\left(n+\frac{1}{2}\right)^6-\left(n-\frac{1}{2}\right)^6\right\}$. Those not realising this were unable to score any of the first 4 marks. Candidates were not required to simplify their expression for S_N, although some did, which wasted valuable time. There were a considerable number of completely correct answers, although some who obtained S_N correctly wrote at considerable length in part **(i)** to no avail. Those who wrote S_N in a form such as that below quickly realised that part **(i)** required $\frac{1}{6}\times 1$ and part **(ii)** $\frac{1}{6}\times 0$.

Answers: $\frac{1}{6}\left(N+\frac{1}{2}\right)^6-\frac{5}{24}N^2(N+1)^2-\frac{1}{32}N(N+1)-\frac{1}{384}$; **(i)** $\frac{1}{6}$; **(ii)** 0.

Question 5

Many candidates proved the reduction formula correctly, using integration by parts.

There were a disappointing number of candidates who tried, in vain, to integrate $(\ln x)^n$. A considerable number of candidates had difficulty with the notation in the proof by induction. Consequently they were unable to formulate a correct inductive hypothesis and make progress. Some candidates did not realise that they needed to show that $I_1=\frac{1}{4}e^2+\frac{1}{4}$ for the base case, thus producing a good deal of unnecessary work. There were, pleasingly, a substantial number of correct proofs, however.

Question 6

There were many completely correct answers to this question. Only the weaker candidates had difficulty in obtaining the initial result and both values in part **(i).**

Some candidates did not obtain the correct values in part **(ii)** because they relied on mis-remembered formulae. Some candidates, realising that they had gone wrong in finding S_6, began again with a different approach and obtained S_8 correctly. Mostly, candidates employed the method in the mark scheme.

Answers: **(i)** –2; **(ii)** 1, –6.

Question 7

The responses to this question were most encouraging. Nearly all candidates were able to show in part **(i)** that the lines l_1 and l_2 intersected, with very few forgetting to check the values obtained for λ and μ in the third equation. In part **(ii)** most candidates were able to find the normal vector to the plane containing l_1 and l_2 using the relevant vector product. The usual method, then, was to find the equation of the plane containing l_1 and l_2, before using the distance of a point from a plane formula. Others considered the distance between parallel planes having the normal vector that they had obtained. Some, unnecessarily long, solutions found the point where the perpendicular from *P* met the plane containing l_1 and l_2, before using Pythagoras to obtain the result. In part **(iii)** few used the method of finding the magnitude of the cross product between a vector joining *P* to any point of l_1 and the unit vector in the direction of l_1. The preferred approach was to call *Q* the foot of the perpendicular from *P* to l_1 and obtain the value, $\left(-\frac{29}{21}\right)$, of the parameter at *Q*, then use Pythagoras to determine the required distance. Where candidates employed methods that required calculating the value of a parameter, the rather awkward fractions involved inevitably led to arithmetical errors for some.

Answers: **(ii)** $\frac{7}{\sqrt{38}}$ or 1.14; **(iii)** $\sqrt{\frac{125}{21}}$ or 2.44.

Question 8

Almost all candidates were able to use the result **Ae** = λ**e** to obtain the required eigenvalue and the required eigenvector when the eigenvalue was 4. Many could then write down the required matrices **P** and **D** and state, correctly, that **Q** = **P**$^{-1}$. Errors rather frequently occurred in calculating **P**$^{-1}$, whether by row operations or co-factors and determinant. These methods occurred with approximately equal frequency.

Answers: 3, $\begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 \\ -4 & -2 & -4 \\ 1 & -1 & -4 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 243 & 0 \\ 0 & 0 & 1024 \end{pmatrix}$.

Question 9

Once again, candidates demonstrated rather limited knowledge of complex numbers. Part **(i)** was successfully done by many, possibly more from memory than reasoning.

The chief problem in part **(ii)** was being able to express z^5 as $32\mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}}$. Often the argument was incorrect. Those who accomplished this step mostly obtained the fifth roots correctly. Thus, many only obtained 1 mark for their Argand diagram as the roots were not in the correct place. Many, of the relatively few who attempted part **(iii),** made the fatal mistake of summing only 4 terms of the GP, rather than 5. Those who summed 5 usually realised the connection with part **(ii)** and were able to obtain the printed result. Part **(iv)** was seldom and not very successfully attempted. It might be helpful for candidates to think of $|2-w|$ as 'the distance of *w* from 2 in the Argand diagram'.

Answers: **(i)** $\mathrm{e}^{-\frac{2k\pi \mathrm{i}}{5}}$, k = 0, 1, 2, 3, 4; **(ii)** $2\mathrm{e}^{\left(-\frac{2\pi \mathrm{i}}{15}+\frac{2k\pi \mathrm{i}}{5}\right)}$, k = 0, 1, 2; **(iv)** $2\mathrm{e}^{-\frac{2\pi \mathrm{i}}{15}}$.

Question 10

This question was somewhat different to recent linear algebra questions. There were few complete answers and a good deal of muddled thinking. Those attempting the first part by row operations seldom gave a complete set of values of *a* for which the system of equations had a unique solution. This was surprising, as intelligent reading of the next two parts of the question would give a good clue as to what was expected. Formal set notation was not required here. Those who said that the system of equations has no unique solution when the appropriate determinant was zero fared much better. A sizeable number of candidates thought that because the appropriate determinant was zero, when *a* took the value 18, that the system would have no solution. The distinction between 'no solution' and 'no unique solution' was clearly not appreciated. Those who used row operations usually found a contradiction such as $0z = -5$ and deduced that no solution was possible. In the case when *a* took the value 18, only a small number obtained the two equations $x+4y=2$ and $z=\frac{1}{4}$ and hardly any of those gave the parametric solution, although some said there were infinite solutions. Some, who ignored any reference to infinite solutions, did manage to find the particular solution for which $x+y+z=1$.

Answers: all values of *a* except *a* = 8 and *a* = 18; $x=\frac{1}{3}$, $y=\frac{5}{12}$, $z=\frac{1}{4}$.

Question 11 EITHER

The initial request in this question was much more easily dealt with by expressing $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ as $3x^2\frac{\mathrm{d}z}{\mathrm{d}x}$ and $6z\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2+3z^2\frac{\mathrm{d}^2z}{\mathrm{d}x^2}$ respectively, rather than trying to express $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2z}{\mathrm{d}x^2}$ in terms of *x* and *y*. The majority realised this and had little difficulty in establishing the first result. The remainder of the question was well done by a large number of those doing this alternative. The only significant difficulty was in finding the coefficient of $\sin 2x$ in the particular solution. Most knew how to do it, but only a few could do it accurately. This did not prevent success on the final request, where nearly all said that $\mathrm{e}^{-x}\to 0$ as $x\to\infty$ and obtained the printed result.

Answer: $z=\left\{\mathrm{e}^{-x}(\cos 2x-\sin 2x)+x\right\}^{\frac{1}{3}}$.

Question 11 OR

Nearly all candidates attempting this alternative were able to find the equation of the horizontal and vertical asymptote. Likewise, almost all were able to find the only point of intersection of *C* with the asymptotes. Most candidates were able to accurately find the only stationary point, although some slipped up in calculating the *y*-coordinate. Finding the set of values for which the gradient of *C* was negative caused problems. This was largely due to candidates looking at the equation they had solved for the stationary points, rather than at the expression for the gradient function itself. This led them to give $\left\{x:-\frac{1}{3}<x<1\right\}$ (set notation was not required). Nevertheless, they invariably drew the correct graph.

Answers: **(i)** $x=1$, $y=1$; **(ii)** ($\frac{1}{3}$, 1); **(iii)(a)** $\left(-\frac{1}{3},-\frac{1}{8}\right)$, **(b)** $x<-\frac{1}{3}$ or $x > 1$.

FURTHER MATHEMATICS

Paper 9231/12

Paper 12

General comments

The scripts for this paper were of a generally good quality. There were a number of outstanding scripts and once again there were very few poor scripts. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was sound. This year there was a marked improvement at vector work, which was good to see.

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The initial request in this question was much more easily dealt with by expressing $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as $3x^2\frac{dz}{dx}$ and $6z\left(\frac{dz}{dx}\right)^2+3z^2\frac{d^2z}{dx^2}$ respectively, rather than trying to express $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$ in terms of x and y. The majority realised this and had little difficulty in establishing the first result. The remainder of the question was well done by a large number of those doing this alternative. The only significant difficulty was in finding the coefficient of $\sin 2x$ in the particular solution. Most knew how to do it, but only a few could do it accurately. This did not prevent success on the final request, where nearly all said that $e^{-x}\to 0$ as $x\to\infty$ and obtained the printed result.

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Answers: **(i)** $x=1$, $y=1$; **(ii)** $(\frac{1}{3}, 1)$; **(iii)(a)** $\left(-\frac{1}{3},-\frac{1}{8}\right)$, **(b)** $x<-\frac{1}{3}$ or $x > 1$.

FURTHER MATHEMATICS

Paper 9231/13

Paper 13

General comments

There were a pleasing number of scripts of outstanding quality, many of a good quality and very few of a poor quality. As in previous May/June sessions, work was set out in a very good fashion by the vast majority of candidates, with solutions set out in a clear and logical manner. Arithmetical accuracy and algebraic manipulation continued to be of a high standard.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all twelve questions. Once again there were few misreads and few rubric infringements.

The Examiners felt candidates had a sound knowledge of most topics on the syllabus.

The Examiners noted a pleasing improvement in vector work. Indeed, the vectors alternative in **Question 12** was more popular than the linear spaces alternative. Although there was some good work done on proof by mathematical induction, it remains a difficult area of the syllabus for some candidates. As will be noticed in the comments that follow, the later stages of some questions caused difficulty for some candidates this year.

Comments on specific questions

Question 1

The vast majority of candidates were able to produce a complete solution to this question in an economical fashion. There were a small number of candidates who wasted time by unnecessarily solving characteristic equations and finding corresponding eigenvectors. The number of marks in the question should have been an indication that something less demanding was required.

Answers: $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$; 30, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Question 2

Despite the identity being stated in the question, a small number of candidates thought that they needed to derive it, which again wasted considerable time. An even smaller group decided to ignore the identity and try to obtain the displayed result using de Moivre's theorem. The vast majority, however, used the identity and the method of differences to obtain the first 4 marks. Only the very best candidates were able to earn the 5th mark. One hurdle was noticing that $\alpha = \frac{\pi}{3}$ and the second hurdle was realising that $\cos[(2N+1)\alpha]$ oscillated between 0.5 and –1.

Question 3

This question on proof by mathematical induction caused a number of problems for a good many candidates. Some candidates could not state a proper induction hypothesis, since they mistook the recurrence relation for the thing that they were trying to prove. Others were confused over the base case, which was that: $x_1 = 3 > 2 \Rightarrow \mathrm{H}_1$ is true. In attempting to prove that $\mathrm{H}_k \Rightarrow \mathrm{H}_{k+1}$, weaker candidates used the fallacious argument $a > 14$, $b > 7 \Rightarrow \frac{a}{b} > 2$, which was a surprising misconception at this level. A sizeable minority produced a full proof, while a few spoilt an otherwise full proof by not stating a conclusion. 'Hence true for all positive integers *n*' would have been adequate.

Question 4

This question was answered well by around half the candidates. All but the weakest candidates found correct expressions for $\dot{x}$ and $\dot{y}$ and applied them to the integral formula for the surface area. Weaker candidates made errors in the subsequent integration by parts, but stronger candidates successfully obtained the correct result.

Answer: $6\pi - \frac{\pi^3}{2}$.

Question 5

This year showed a pleasing improvement in complex number work. The majority of candidates correctly established the identity for $\sin 5\theta$ and substituted $x = \sin\theta$ in order to deduce that $\sin 5\theta = -\frac{1}{2}$. Only the best candidates gave a full set of five distinct roots of the equation, the problem being that many candidates did not identify duplicate roots.

Answer: $\sin\left(\frac{7}{30}\pi\right)$, $\sin\left(\frac{11}{30}\pi\right)$, $\sin\left(\frac{31}{30}\pi\right)$, $\sin\left(\frac{35}{30}\pi\right)$, $\sin\left(\frac{43}{30}\pi\right)$, or an equivalent set.

Question 6

Weaker candidates occasionally found difficulty with division in order to obtain the equation of the oblique asymptote. Some also found difficulty in getting $\frac{\mathrm{d}y}{\mathrm{d}x}$ into an appropriate form in order to demonstrate that it was always positive. In drawing the graph, a small number of candidates, even some stronger ones, did not take sufficient care to ensure that the curve approached the asymptotes at the extremes. There were, however, many candidates who obtained full marks.

Answers: **(i)** $x = -1$, $y = x - 4$.

Question 7

Most candidates were able to find correct expressions for $\dot{x}$ and $\dot{y}$ and hence $\frac{\mathrm{d}y}{\mathrm{d}x}$.

A good many candidates attempted to use $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right).\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Weaker candidates made errors in applying the quotient rule and/or omitted the multiplier $\frac{\mathrm{d}t}{\mathrm{d}x}$.

Answer: **(ii)** $\frac{(-1+2t^2-2t^4)\mathrm{e}^{t^2}}{4t^3(1-t^2)^3}$.

Question 8

The majority of candidates were successful in obtaining the complementary function and also in choosing the correct form for the particular integral. Lack of care in solving equations led to incorrect particular integrals. Only the best candidates successfully completed the last part of the question. Substantial numbers failed to justify why the exponential terms vanished as x became large. Others had negative values for R, and very few could correctly identify the quadrant into which ϕ fell.

Answers: $y = Ae^{-x} + Be^{-4x} - 0.2\cos 3x - 1.4\sin 3x$, $R = 1.41$, $\phi = 3.28$.

Question 9

Most candidates realised that integration by parts, or a reversal of differentiating a product, would be necessary to obtain the reduction formula. A good many were able to complete the process. Many obtained a correct integral representation for $\bar{y}$, but very few indeed were able to use a substitution $u = mx$ to establish that $\bar{y} = \dfrac{I_8}{2I_4}$.

Most assumed that this was the case, and many used the reduction formula correctly to obtain the correct value of $\bar{y}$.

Answer: $\bar{y} = \dfrac{35}{96}$ or 0.365.

Question 10

The majority of candidates did well on the first three parts of this question. If they made errors in part **(ii)**, a follow through mark was awarded if they followed the correct procedure in part **(iii)**. Part **(iv)** proved to be too demanding for all but the very best candidates. Quite a number did manage to show that $S = 0$, but were then unable to make the deduction that the roots were equal to ± 1 and consequently move to a contradiction when assuming that all the roots were equal.

Answers: **(i)** $2y^4 - 4y^3 - cy^2 - y - 1 = 0$; **(ii)** $1 - 2c$, $4 + c$; **(iii)** $-c - 3$.

Question 11

This question on polar graphs caused a surprisingly high number of problems. In part **(i),** those who adopted a differentiation approach were largely successful. The alternative invariably lacked rigour. It was necessary to say that a was both constant and positive and that $1+\theta$ was positive and increasing, before one could deduce the required result. A considerable number of candidates successfully maximised $y = r\sin\theta = \dfrac{a\sin\theta}{1+\theta}$ to obtain the displayed result in part **(ii)**. There were some fallacial arguments and arithmetical errors in demonstrating that the root of the equation lay in the required interval. Sketches of the graph were reasonable, although some did not have a maximum point in approximately the correct position and some sketches lacked any indication of scale. The requisite formula for area was well known, but some integration was incorrect and final answers inaccurate.

Answer: **(iv)** $\dfrac{\pi a^2}{2(\pi + 2)}$.

Question 12 EITHER

As remarked earlier, the work on this question showed a pleasing improvement on vector work last year. The direct approach to this question saw candidates using the triple scalar product in part **(i)**, then using $\overrightarrow{PQ} = k(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ in part **(ii)**. Weaker candidates, however, incorrectly assumed $k = 1$. A sizeable minority combined both of these parts by forming a two parameter expression for $\overrightarrow{PQ}$ and obtaining the values of the parameters from a pair of simultaneous equations arising from orthogonality conditions. The position vector for Q and its distance from the plane followed immediately. In part **(iii)** the result could be obtained from a further triple scalar product, but many candidates found the normal to the plane and hence its cartesian equation. The formula for the distance of a point from a plane was then employed. There was a pleasing number of candidates scoring full marks on this question.

Answers: **(i)** 3; **(ii)** $-\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Question 12 OR

Most candidates successfully used row reduction of **M** to find the dimension of R. They then went on to state a correct basis for R, although a sizeable minority lost time and marks by finding a basis for the null space. A variety of methods were used in part **(ii)**, although the best quickly saw that $\begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 9 \\ 7 \\ 6 \end{pmatrix}$ having shown $\left\{\begin{pmatrix} 1 \\ -15 \\ -17 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ 7 \\ 6 \end{pmatrix}\right\}$ as their basis for R. It was not uncommon for a mark to be lost here for not checking for consistency on all components. Part **(iii)** proved difficult for the majority of candidates who attempted this alternative. Even the minority, who knew what to do, sometimes lost a mark for not showing sufficient working when the answer was displayed. Nevertheless, even those who did not do part **(iii)**, or had an incorrect solution, often picked up both marks in part **(iv)** for obtaining a correct vector.

Answers: **(i)(b)** $\left\{\begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ 7 \\ 6 \end{pmatrix}\right\}$; **(iv)** $\begin{pmatrix} 37 \\ 0 \\ -3 \\ -3 \end{pmatrix}$ (for example).

FURTHER MATHEMATICS

Paper 9231/21

Paper 21

General comments

As in previous years, the paper worked well in discriminating between candidates, producing a wide range of marks. There appeared to be no undue time pressure since almost all candidates attempted the required number of questions. Candidates from some Centres performed better at Statistics than Mechanics, but the opposite was also true. In the single question which offered a choice, the Statistics alternative was probably more popular overall. Good attempts were seen at all the questions. A few questions appeared to be found more challenging than others, most notably **Questions 7** and **11** and to a lesser extent **Question 6**.

Comments on specific questions

Question 1

Almost all candidates made use of the period to find the SHM parameter ω correctly, and also appreciated that the required force is the product of mass and acceleration. Those who found the largest magnitude of the acceleration by using $-\omega^2 x$ often took the maximum possible value of x as 0.3 though some wrongly used 0.6, but some of those who started instead from an SHM equation $x = 0.3 \sin \omega t$ or $0.3 \cos \omega t$ substituted an inappropriate value for t in the second derivative instead of simply replacing the sine or cosine factor by unity.

Answer: 9.47.

Question 2

P is most easily found by taking moments for the rod about A, and the friction F and normal reaction R at A by vertical and horizontal resolutions respectively. However a variety of other moment and resolution equations were seen, and errors in omitting forces or using the wrong angles were common. More credit might be gained for seemingly incorrect equations if candidates indicated what they were trying to do, for example by stating that they were resolving forces on the rod in a direction normal to it. Most candidates appreciated that $F < \mu R$, though incorrect multiples of W for F and R often precluded the set of possible values of the coefficient of friction μ being determined correctly.

Answers: $\frac{\sqrt{3}}{2}W$; $\mu > \frac{1}{\sqrt{3}}$.

Question 3

Use of the restitution equation and conservation of momentum, and sometimes conservation of kinetic energy, enabled most candidates to find the speeds of A and B immediately after they first collide, though quite a few candidates took the coefficient of restitution between the spheres to be either $\frac{1}{2}$ or an unknown value. Almost all realised that the speed of B would be halved during collision with the barrier. Thereafter a variety of different approaches were used, usually equating either the times for both A and B to their second collision, or their distances from the barrier, and inevitably errors were often made in the process. A few candidates needlessly tried to find the speeds of A and B after their second collision.

Answers: $\frac{1}{2}u$, $\frac{3}{2}u$; $\frac{2}{5}a$.

Question 4

The angular speed of the smaller disc when t = 2 was frequently taken to equal that of the larger disc, namely 1 rad s^{-1}, whereas it follows from a realisation that the linear speeds of the two discs must be the same at their point of contact. A few even less plausible attempts were seen, involving moments of inertia for example. The second part was also poorly answered, with many candidates finding only the radial or the tangential acceleration, whereas both must be found and combined. Angular and linear acceleration were sometimes confused, or mistakenly combined together.

Answers: **(i)** $\frac{5}{3}$ rad s^{-1}; **(ii)** $\frac{\sqrt{5}}{4}$ ms^{-2}, 0.464 rad.

Question 5

While most candidates performed a vertical resolution satisfactorily in order to obtain the tension T in the form $\frac{mg}{2\cos\beta}$ and then attempted to find an alternative expression $\frac{mg}{2\sin\beta}$ for T using Hooke's Law in order to show that $\beta = \frac{1}{4}\pi$, problems in finding the relevant extension frequently occurred. Some candidates became bogged down in trigonometry while others effectively subtracted a from the correct extension $\frac{a}{\sin\beta}$. The derivation of the given SHM equation was not done well in general, with some candidates effectively taking the additional extension of the elastic band as x, while others obtained an expression for the extension which involved powers of x, and mistakenly ignored higher-order powers. Many wrongly assumed that the angle made by the band with the vertical would be unaltered when the particle moves down. However many went on to find the amplitude $A = \frac{a}{\sqrt{2}}$ correctly and to compare this with $\frac{a}{2}$, and made some attempt at finding the required time t. The most direct way is to use the standard SHM equation $x = A \sin \omega t$ with here $x = -\frac{a}{2}$ and $\omega = \sqrt{\frac{2g}{a}}$. This is preferable to the cosine form in this case since the motion commences from the equilibrium point rather than a point of maximum amplitude. Some candidates wrongly assumed t to be some particular multiple of the period $\frac{2\pi}{\omega}$ such as $\frac{3}{4}$.

Answers: $\frac{mg}{2\sin\beta}$; $\frac{5\pi}{4}\sqrt{\frac{a}{2g}}$.

Question 6

In general candidates seemed to find this question challenging, particularly the first part. The median requires only the evaluation of $10^{1.5}$, while the required probability is $1-\Phi\left(\frac{\log 50 - 1.5}{0.2}\right)$ Premature approximation of log 50 is unwise since the final answer is not then correct to the required 3 significant figures.

Answers: 31.6; 0.160.

Question 7

Both parts of this question were also found difficult by many candidates. The key to answering the first part is to express P(X < 2 < 4X) as P($\frac{1}{2}$ < X < 2) and hence F(2) – F($\frac{1}{2}$) which can then be readily evaluated, but many incorrect variations on this were seen. A similar variety of incorrect approaches were used for the second part too, frequently involving extensive integration. The simplest solution is probably to realise that the required width is 3E(X) and to state or prove that E(X) = 2.

Answers: 0.411; 6.

Question 8

Most candidates made a reasonable attempt at finding the difference in the sample means (the sign of which is undefined by the question), apart from the occasional premature approximation, and at estimating the common population variance. The confidence interval then follows from the usual formula, with the required tabular t-value 2.101 corresponding to 18 degrees of freedom. Instead of finding the confidence interval, however, a number of candidates performed a test of equality of the means, no doubt prompted by the final part of the question even though this explicitly refers to using the confidence interval. The appropriate reason for observing that the means are not equal is that the confidence interval does not contain zero, though some candidates mistakenly referred instead to the size of the interval. As is often the case in such questions, the required assumptions that the two populations from which the samples are drawn should have Normal distributions and the same variance, were wrongly or insufficiently precisely stated.

Answer: [–7.36, –1.31] or [1.31, 7.36].

Question 9

Part **(i)** explicitly requires an indication of the appearance of a scatter diagram, such as the points lying close to a straight line with negative gradient, and not (as many candidates gave) a comment on the degree of correlation or on the regression lines. Alternatively an illustration of a possible scatter diagram would have been acceptable. Most candidates knew how to find the equation of the regression line, with some using the given value of the product moment correlation coefficient r to find the gradient b directly from $r\sqrt{\frac{S_{yy}}{S_{xx}}}$ and others first determining S_{xy} from $r\sqrt{(S_{xx}\,S_{yy})}$ and then b from $\frac{S_{xy}}{S_{xx}}$. The value of b' may be found by using either $\frac{S_{xy}}{S_{yy}}$. or $r^2 = bb'$. A common error was to take a positive value rather than a negative one for b and/or b'.

Answers: **(ii)** $y = 77.0 - 6.50x$; **(iii)** –0.151.

Question 10

Although the general approach to conducting a χ^2-test was widely known, many candidates did not base their test on the appropriate sets of observed and expected values. The contingency table of observed data should contain the numbers of volunteers who caught flu after receiving each of the three vaccines, namely 30, 29 and 16, and also the corresponding numbers who did not catch flu, namely 148, 120 and 157. Two variations on this were often seen, however, with some candidates using a table with only the first three values and others replacing the number of volunteers not catching flu by the total number of volunteers receiving each of the three vaccines, namely 178, 149 and 173. The null hypothesis, that catching flu is independent of the vaccine given, was not always clearly or unambiguously stated. Indeed in extreme cases the hypothesis was the single word ‘independence’. Most candidates knew how to calculate the table of expected values and then the value of χ^2, here 7.30. Comparison with the tabular value 5.991 then leads to the conclusion that the probability of catching flu does depend on which vaccine is administered. Despite the admonition in the question that a statistical test should not be used to decide which vaccine appears to be most effective, many candidates referred to the numbers calculated during their χ^2-test. Instead a simple comparison of the proportions of volunteers receiving each vaccine who went on to contract flu (or equivalently to escape it) suggests that C appears to be more effective than the other two.

Question 11 EITHER

This alternative was attempted by rather fewer candidates than the Statistics alternative, but some good answers were seen to the first and second part, with the final part seemingly more challenging. The given moment of inertia of the wheel is probably obtained most quickly by adding together the moments of inertia of the component parts about O, and then applying the parallel axes theorem once to find the combined moment of inertia about A. The alternative of finding the component moments of inertia about A before adding them is equally valid, but requires more applications of the parallel axes theorem with a consequent greater risk of error. Equating the rotational and potential energies then yields the required angular speed, and while the principle was usually understood there were occasional minor errors such as taking the wheel's mass as m rather than $8m$ or the distance of the centre of mass from A as a rather than $2a$. A second energy equation completes the final part of the question, but here the moment of inertia has decreased to $24ma^2$ and the mass to $4m$.

Answers: $\dfrac{1.95}{\sqrt{a}}$; 34.8°.

Question 11 OR

This alternative, although popular, was not well done by many candidates. The distribution function F(t) is of course found by integration, but many candidates did not apply the appropriate limits 2 and t or introduce a constant of integration, so that their result omitted the unit constant. This inevitably produced an incorrect value when evaluating p = P(T > 5) from 1 – F(5), and also affected the remainder of the question. Since the distribution in part **(ii)** is a geometric one with parameter p, the required value P(N > E(N)) is found from $(1 - p)^{1/p}$. In the final part many candidates simply noted that $T = 1 + \dfrac{1}{Y}$ and therefore replaced t in their expression for F(t) by $1 + \dfrac{1}{y}$, taking the result to be the cumulative distribution G(y) of Y without any apparent understanding, and of course obtaining an incorrect result for G(y). The correct approach is to show that G(y) = P($T > 1 + \dfrac{1}{y}$), which in turn equals 1 – F($1 + \dfrac{1}{y}$), though a number of candidates who did this then made an error when replacing t by $1 + \dfrac{1}{y}$ in their earlier expression for F(t) and simplifying. Whether or not they had found G(y) correctly, most candidates knew that the probability density function of Y is obtained by differentiation, though the intervals of validity were rarely stated correctly.

Answers: **(i)** $1 - \dfrac{1}{(t-1)^2}$ (t > 2), $\dfrac{1}{16}$; **(ii)** 0.356; **(iii)** $2y$ for $0 < y < 1$, 0 otherwise.

FURTHER MATHEMATICS

Paper 9231/22

Paper 22

General comments

As in previous years, the paper worked well in discriminating between candidates, producing a wide range of marks. There appeared to be no undue time pressure since almost all candidates attempted the required number of questions. Candidates from some Centres performed better at Statistics than Mechanics, but the opposite was also true. In the single question which offered a choice, the Statistics alternative was probably more popular overall. Good attempts were seen at all the questions. A few questions appeared to be found more challenging than others, most notably **Questions 7** and **11** and to a lesser extent **Question 6**.

Comments on specific questions

Question 1

Almost all candidates made use of the period to find the SHM parameter ω correctly, and also appreciated that the required force is the product of mass and acceleration. Those who found the largest magnitude of the acceleration by using $-\omega^2 x$ often took the maximum possible value of x as 0.3 though some wrongly used 0.6, but some of those who started instead from an SHM equation $x = 0.3 \sin \omega t$ or $0.3 \cos \omega t$ substituted an inappropriate value for t in the second derivative instead of simply replacing the sine or cosine factor by unity.

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Answers: $\frac{1.95}{\sqrt{a}}$; 34.8°.

Question 11 OR

This alternative, although popular, was not well done by many candidates. The distribution function F(t) is of course found by integration, but many candidates did not apply the appropriate limits 2 and t or introduce a constant of integration, so that their result omitted the unit constant. This inevitably produced an incorrect value when evaluating p = P(T > 5) from 1 – F(5), and also affected the remainder of the question. Since the distribution in part **(ii)** is a geometric one with parameter p, the required value P(N > E(N)) is found from $(1 - p)^{1/p}$. In the final part many candidates simply noted that $T = 1 + \frac{1}{Y}$ and therefore replaced t in their expression for F(t) by $1 + \frac{1}{y}$, taking the result to be the cumulative distribution G(y) of Y without any apparent understanding, and of course obtaining an incorrect result for G(y). The correct approach is to show that $G(y) = P(T > 1 + \frac{1}{y})$, which in turn equals $1 - F(1 + \frac{1}{y})$, though a number of candidates who did this then made an error when replacing t by $1 + \frac{1}{y}$ in their earlier expression for F(t) and simplifying. Whether or not they had found G(y) correctly, most candidates knew that the probability density function of Y is obtained by differentiation, though the intervals of validity were rarely stated correctly.

Answers: **(i)** $1 - \frac{1}{(t-1)^2}$ (t > 2), $\frac{1}{16}$; **(ii)** 0.356; **(iii)** $2y$ for $0 < y < 1$, 0 otherwise.

FURTHER MATHEMATICS

Paper 9231/23

Paper 23

General comments

As in previous years, the paper worked well in discriminating between candidates, producing a range of performance varying from low to very high marks. There appeared to be no undue time pressure since almost all candidates attempted the required number of questions. Candidates from some Centres performed better at Statistics than Mechanics, but the opposite was also true. In the single question which offered a choice, the Statistics alternative was probably more popular overall. Good attempts were seen at all the questions, although a few appeared to be found more challenging than others, most notably **Questions 8** and **11**.

Comments on specific questions

Question 1

The best approach is probably to formulate a separate equation of motion for the particle, which is subject to the opposing forces of the tension T and weight $2mg$, and for the disc under the action of the same T. Substitution for the moment of inertia of the disc and elimination of T then yields the required acceleration. A common fault was to consider the disc only, in the mistaken belief that $T = 2mg$, sometimes augmenting the moment of inertia of the disc as if the particle was attached directly to its circumference rather than by means of the string. Confusion was often apparent between linear and angular acceleration, and using the same symbol for the acceleration as for the disc's radius was a further source of confusion for some. The alternative approach based on energy was rarely seen, and is probably less straightforward here.

Answer: $\frac{4}{5}g$.

Question 2

Two independent resolutions are needed, though they may be chosen from six reasonable alternatives. The most obvious ones are probably horizontally and vertically, from which R can be eliminated to find T, and hence R. As an alternative, resolutions along and then normal to BV yield T and R more directly. Some candidates concocted a wholly invalid resolution, seemingly expressly designed to yield the given expression for T. In all cases it would be helpful if candidates stated the direction of their resolution equation. The given inequality follows from the requirement that $R > 0$, though some candidates wrongly argued from $T > 0$, ignoring the inconvenient minus sign which results.

Answer: $\frac{m}{\sqrt{3}}\left(g - \frac{2u^2}{a}\right)$.

Question 3

Most candidates easily showed that $\lambda = 25$ by equating the Hooke's Law expression for the tension to the downward gravitational force at the equilibrium point where the extension is 0.01. Derivation of the given SHM equation was less well done in general, however. It requires the application of Newton's Law at the general point where the tension is found from a total extension $0.01 + x$, assuming it is chosen to take the downward direction as positive. The most direct way to find the required time t is to use the standard SHM equation $x = A \cos \omega t$ with here $x = -0.01$, $A = 0.02$ and $\omega = \sqrt{1000}$. This is preferable to the sine form in this case since the motion commences from a point of maximum amplitude rather than the equilibrium point. Some candidates wrongly assumed t to be some multiple of the period $\frac{2\pi}{\omega}$ other than the correct $\frac{1}{3}$.

Answer: 0.0662 s.

Question 4

Finding the impulse requires the addition of the magnitudes of Q's speeds immediately before and after striking the plane, and not subtracting them as some candidates did. There are then two broad approaches to determining the height at which the balls collide. One is to consider the distances moved by P and Q relative to the plane during the total time from the balls being dropped until their collision, while the other considers only the time from Q striking the plane, making use of the obvious fact that P will then be 5 m above it with the same downward speed with which Q struck the plane. What is not required is to follow the example of a minority of candidates in needlessly attempting to find the speeds of the balls after they collide.

Answers: 0.6 Ns; 1.11 m.

Question 5

The given relation between $\tan\theta$ and $\tan\phi$ follows from the speed component along the edge being unchanged, while that normal to the edge changing by a factor $\frac{3}{4}$. Most candidates also noted that LX equals $a \cot\theta + a \cot\phi$, substituted for $\cot\phi$ and finally subtracted the result from the length $2a$ of LM to yield XM. Denoting the angle MXN by ψ, XM can also be expressed as $a \cot\psi$, and using an extension $\tan\psi = \left(\frac{3}{4}\right)^2 \tan\theta$ of the relationship found initially and equating these two expressions for XM yields the required value of θ. Perhaps the easiest way to find the speed with which the ball reaches N is to combine the components of its speed along and normal to XM, expressed in terms of θ. Finding both the value of θ and this speed defeated a large number of candidates.

Answers: 64.1°; $0.669u$.

Question 6

Almost all candidates knew how to calculate the confidence interval, and most of them used the correct tabular t-value of 1.711.

Answer: [108.0, 112.8].

Question 7

This question was also generally well done, with most candidates calculating a χ^2-value of 4.23 and concluding from a comparison with the tabular value 9.236 that the null hypothesis should not be rejected. Formulating an appropriate null hypothesis, in this case that the data conforms to Benford's Law, was the principal weakness, with some candidates either giving a vague, generalised statement or else saying that the data should conform to the Poisson distribution, for example.

Question 8

Many candidates found this question difficult, and a variety of invalid approaches were seen. One common error was to use 2.5 as the parameter λ of the exponential distribution throughout, rather than the reciprocal 0.4 of the given mean 2.5. Since the probability p_3 of one component operating after 3 months is $e^{-3\lambda}$, the probability of all 5 doing so is found from p_3^5, while the probability that exactly two components fail within one month is ${}^5C_2\, p_1^2\, (1 - p_1)^3$, where $p_1 = e^{-\lambda}$. The final part makes straightforward use of $(e^{-c\lambda})^n = e^{-nc\lambda}$, and this holds of course irrespective of what value is used for λ.

Answers: **(a)(i)** 0.00248; **(ii)** 0.327.

Question 9

Part **(a)** explicitly requires an indication of the appearance of a scatter diagram in each of the three cases, such as the points lying close to a straight line with positive gradient in part **(iii)**, and not (as many candidates gave) a comment on the degree of correlation or on the regression lines. Alternatively a possible illustration of each scatter diagram would be acceptable. Other faults sometimes seen were an assumption that the lines in parts **(i)** and **(iii)** would have gradients –1 and 1 respectively, or that the lines would pass through the origin. Most candidates were able to use the standard formula to calculate the product moment correlation coefficient r, though a surprisingly common fault was to use a value of 10 000 for n rather than 7. The final test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho < 0$, after which comparison of the magnitude of the previously calculated value of r with the tabular value 0.669 leads to a conclusion of no evidence of negative correlation.

Answer: **(b)(i)** –0.636.

Question 10

The appropriate test here is a paired sample one and not the two-sample one used by some candidates. The differences between the post-treatment and pre-treatment grips (or the reverse) are first calculated, followed by the sample mean $\bar{x} = \frac{17.7}{8}$ and the estimated unbiased population variance $s^2 = 3.03$ (or 2.65 if the biased estimate is found, though $n - 1 = 7$ must then replace $n = 8$ in the usual formula for t). Comparison of the calculated value 3.60 of t with the tabular value 2.998 suggests that the mean grip increases after treatment. The hypotheses should be stated in the test, as should the assumption that the population of differences has a normal population. A common fault here, as often occurs in similar questions, was not to state the assumption sufficiently precisely, using ‘sample’ instead of ‘population’ for example. The required inequality for w follows from $\dfrac{\bar{x} - w}{\frac{s}{\sqrt{8}}} > 1.415$.

Answer: $w < 1.34$.

Question 11 EITHER

The required friction F_B at B is found most quickly from two moment equations involving only F_B and a second unknown such as the reaction R_B at B, and these can be obtained by taking moments about A for BA and about C for the system of rods. Other equations involving the corresponding forces F_C and R_C at C will suffice when used with $R_B = R_C$ and $F_B + F_C = 3W$, while a few candidates needlessly introduced forces between the rods at A, though this did not necessarily prevent their answering the question correctly. As in **Question 2**, more credit might be gained for seemingly incorrect equations if candidates indicated what they were trying to do, for example by stating that they were taking moments about A for BA. Having determined the friction and reaction at both B and C, the lower limit on μ is the larger of the two ratios $\frac{F_B}{R_B}$ and $\frac{F_C}{R_C}$, though not many candidates appreciated this, with some only considering limiting friction at the point B, or at C without justifying this.

Answer: $\mu > \dfrac{7}{3\tan\beta}$.

Question 11 OR

Although a somewhat more popular alternative than the Mechanics one, this question caused difficulties for many candidates. The key to both the first and second parts is to use the appropriate expected time of $1.5n$ and variance of 0.4^2n for packing n items, but the variance was often incorrect. Use of the normal approximation then yields the required probability in the first part, with $n = 50$, while in the second $\frac{70 - 1.5n}{0.4\sqrt{n}}$ must be no less than the tabular t-value 1.282, from which the greatest integral value of n may be determined. The final probability comes from evaluating $\frac{\Phi(10)}{\sqrt{20.5}}$, where the mean 10 and variance 20.5 in this expression come from combining the corresponding values for the packing of 50 items by Aram and by Huldu.

Answers: 0.0385; 44; 0.986.