Perspectives - Summer School 2014

Mathematics

Countability - Taming the Infinites

The Oxford University Malaysia Club is proudly sponsored by Shell Malaysia

Introduction

This document consists mainly of results together with their proofs and a few small exercises. You are advised to attempt, or at least give some thought to these exercises. More importantly, you should generate your own questions and ask it during the supervision session. The first two sections on sets and functions are meant to set the stage for our exploration of countability, so they do not contain anything deep at all. Rather, they are mainly here to set the language and terminology right. The third section on countability, however, is quite dense with many ideas, so you are advised to spend more time on this part.

The flavour of this document may or may not be familiar to you as it is certainly not the same as secondary school maths, or indeed, pre-university maths. For one, it is *a lot* more wordy than is customary in secondary school. Nonetheless, just as a word of assurance: the majority of the ideas herein are, when you think about it, not very complicated at all. So if you are willing to persist a little bit more to cut through the language, then be sure that you *will* gain some nice and cool insights on infinity.

Countability - Taming the Infinities

Throughout human history, the infinite has entranced, amused, bewildered and sparked the curiosity of man. There is something inherently awe-inspiring about the concept of infinity, something mysterious and elusive. However, mathematics has advanced by leaps and strides over the centuries and we have a suitable way to classify different types of infinities. So now, we shall set out to tame it with the idea of countability. But before we can gain some ground on our quest, let us acquaint ourselves with two concepts, namely that of *sets* and *functions*.

1 Sets

Due to the nature of this short, introductory course, we will not go very deep into the investigation of sets. Hence, we will treat here mainly on the terminology and some basic ideas.

A set is a collection of **distinct** objects. The word 'distinct' was emphasised to remind the reader that there is no multiple membership. For example, the set $\{1, 3, 3, 1\}$ is really just the set $\{1, 3\}$. Also, the order does not matter, ie. $\{1, 3\} = \{3, 1\}$.

Terminology and Notations

• A **set** is denoted by curly braces. For instance, a set S consisting of the elements *car, bikes, and trains* is written as $S = \{cars, bikes, trains\}$.
• The **union**, $\cup$ of two sets is a set consisting of the elements of both sets. For example, let $A = \{2, 3, 5, 7\}$ and $B = \{2, 4, 6\}$. Then $C = A \cup B = \{2, 3, 4, 5, 6, 7\}$.
• The **intersection**, $\cap$ of two sets is a set consisting of the common elements of the two sets. For example, using A and B as above, $D = A \cap B = \{2\}$.
• If x is an **element** of S, then we write $x \in S$.
• A **subset** is a set contained within a set. For example, let $E = \{3, 5, 7\}$. Then we write $E \subset A$.
• $\emptyset$ denotes the **null set**, ie. the set that contains nothing.
• The **product set**, $A \times B$ of two sets A and B is the set of all points of the form (a, b) where $a \in A$ and $b \in B$. For instance, the familiar Cartesian plane is $\mathbb{R} \times \mathbb{R}$.
• $\forall$ is a shorthand for the phrase **for all**.
• $\exists$ is a shorthand for the phrase **there exists**.
• $\mathbb{N}$ means the **natural numbers**, ie. $\mathbb{N} = \{1, 2, 3, \ldots\}$. Note that the naturals does *not* include 0.
• $\mathbb{Z}$ means the **integers**, ie. $\mathbb{Z} = \{1, 2, 3, \ldots\} \cup \{0\} \cup \{-1, -2, -3, \ldots\}$.
• $\setminus$ means **exclusion**. For example, using the above set A, $A\setminus\{2, 5\} = \{3, 7\}$.
• $\mathbb{Q}$ denotes the **rational numbers**, ie. $\mathbb{Q}=$ {the set of numbers that can be written in the form $\frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ }. Numbers that cannot be written in this form, such as $\sqrt{2}$ (if you're interested, try proving it by contradiction, that is, assume it *can* be written as such, and try to reach a contradiction), are called **irrational numbers**.
• $\mathbb{R}$ denotes the **real numbers**, where $\mathbb{R} = \mathbb{Q} \cup \{\textit{irrational numbers}\}$

2 Functions

For our purposes, a function can be considered a mapping from one set (the domain set) to another (the codomain set) where *each* element in the domain has a *unique image* in the codomain. As usual, some examples will make things much clearer. Consider the function $f(x) = 2x + 1$ on the set $S = \{1, 2, 3, 4, 5\}$ to the set $T = \{1, 3, 5, 7, 9, 11, 13\}$. Then, we have the following mapping:

$$1 \to 3$$

$$2 \to 5$$

$$\vdots$$

$$5 \to 11$$

Note that not every element in the codomain set is hit by the function. The set of elements that *are* hit is called the image set.

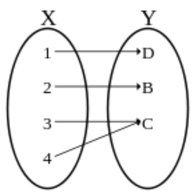
We note that the mapping $g(x) = x^{\frac{1}{2}}$ on the set of positive integers is **not** a function as every element in the domain has two images. Recall that for a mapping to be a function, it has to assign a unique image to an element in the domain. But, for example, $4^{\frac{1}{2}} = \pm 2$, so 4 has two images.

Having covered those things, we have only one more group of concepts before we are ready to put a leash on the infinities! We shall now familiarise ourselves with the (intuitive) ideas of *injection*, *surjection*, and *bijection*. In what are to follow, we let A be the domain and B the codomain. As another piece of notation to remind the reader of the domain and codomain sets, we write the function f on A to B as $f : A \longrightarrow B$.

Definition 1 *A function f is said to be* ***injective*** *if the following condition is satisfied:*

$$\forall x, y \in A,\ f(x) = f(y) \Rightarrow x = y$$

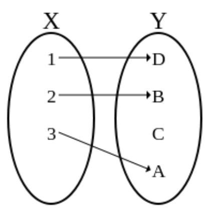
In laymen terms, an injective function is one-to-one, two different elements remain different after the mapping. Here is an example of a non-injective function:



Definition 2 *A function f is said to be* ***surjective*** *if the following condition is satisfied:*

$$\forall y \in B,\ \exists x \in A \text{ such that } f(x) = y$$

In other words, everything in the codomain is hit. Here is an example of a non-surjective function:



Definition 3 *A function f is said to be* ***bijective*** *if it is both injective and surjective.*

So if you think about it, if there is a bijection between A and B, then there is a pairing between the elements of the sets and that, roughly speaking, the sets have the same 'sizes'.

Exercises

Identify which of the following functions are injective, surjective, or bijective. *(Hint: Try to draw the arrow diagrams as above to better see what's going on)*
i) The identity function, $\imath : \mathbb{N} \longrightarrow \mathbb{Z}$, where $\forall x \in \mathbb{N} : \imath(x) = x$.
ii) The function f, $f : \mathbb{N} \longrightarrow \mathbb{N}$, where $f(1) = 1$ and $\forall x \in \mathbb{N}\backslash\{1\} : f(x) = x - 1$.
iii) The function g, $g : \mathbb{Z} \longrightarrow \mathbb{Z}$, where $g(x) = x + 1$.

3 Countability

Now with the preliminary ideas set, we are prepared to deal with the objective of this course: countability. Essentially, the motivation of this section is to study the different sizes of infinity. So naturally, the first thing that we would want is a method to measure the 'size' of a set. Obviously, if the set is finite, then the 'size' of the set is just the number of elements it contains. But for sets with an infinite number of elements, such as $\mathbb{Z}$ and $\mathbb{R}$, measuring 'sizes' is not so straightforward. With this in mind, let us introduce the concept of countability.

Definition 4 *A set, X, is said to be* ***countable*** *if there exists a bijection between X and $\mathbb{N}$, or if X is a finite set.*

Intuitively speaking, a set is countable if it's elements can be listed, ie. the elements can written as $a_1, a_2, a_3, \ldots$. Using that definition, let us prove a very useful result.

Lemma 1 *A set X is countable if and only if there exists an injection from X to $\mathbb{N}$.*

Proof. *(This is not essential, so by all means, skip the proof if you want to. The second part of the proof is a bit long-winded, so it might appear daunting at first. But in actual fact, the idea is extremely simple.)*
Suppose X countable. Then there is a bijection between X and $\mathbb{N}$. But a bijection is an injection. So there exists an injection from X to $\mathbb{N}$.
Suppose there exists an injection from X to $\mathbb{N}$. If X is finite, then we are done. So suppose X is an infinite set. Call this injection f, ie. $f : X \longrightarrow \mathbb{N}$ is an injection. Consider the set $f(X)$, that is, the set of all images of X under the function f. Clearly, there is a bijection between X and $f(X)$ as f is an injection and by definition, f surjects X to $f(X)$ *(by definition because $f(X) = \{f(x) : x \in X\}$, the set of images of all the elements of X).*
Since $f(X) \subset \mathbb{N}$, we can find the minimal element of $f(X)$ and call it a_1. We can then find the minimal element of $f(X)\backslash\{a_1\}$ and call it a_2. Going on in this way, we see that $f(X) = \{a_1, a_2, \ldots\}$. But we can biject this with $\mathbb{N}$ by $a_k \longrightarrow k$. So we can biject $f(X)$ with $\mathbb{N}$. Since there is a bijection between X and $f(X)$ and there is a bijection between $f(X)$ and $\mathbb{N}$, there is a bijection between X and $\mathbb{N}$. So X countable. $\square$

Corollary 1 *A subset of a countable set is countable.*

Proof. Let A be a subset of a countable set, S. We are done if A is finite. So now suppose A infinite. We know by Lemma 1 that there exists an injection f from S to $\mathbb{N}$. Then $f : A \longrightarrow \mathbb{N}$ also an injection. So A also countable. $\square$

With Lemma 1 in our hands, let us prove a famous proposition which will be the first hint of weirder things to come. Another point of interest for the following result is its proof, which embodies a technique both useful and elegant. We say useful because the idea of the proof can be used for proving a host of other things.

Proposition 1 $\mathbb{N} \times \mathbb{N}$ *is countable, ie.* $\mathbb{N} \times \mathbb{N}$ *bijects with* $\mathbb{N}$.

Proof. Consider the function $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that $\forall (x, y) \in \mathbb{N} \times \mathbb{N} : f(x, y) = 2^x \cdot 3^y$.
We can show that it is an injection as follows:
Suppose $f(x, y) = f(p, q)$ for some $x, y, p, q \in \mathbb{N}$. So
$2^x \cdot 3^y = 2^p \cdot 3^q$
$\Rightarrow 2^{(x-p)} \cdot 3^{(y-q)} = 1$
$\Rightarrow x - p = y - q = 0$
$\Rightarrow x = p$ and $y = q$
So $(x, y) = (p, q)$.
So we have an injection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. By Lemma 1, $\mathbb{N} \times \mathbb{N}$ is countable. $\square$

Theorem 1 *A countable union of countable sets is countable, ie. if* $A_1, A_2, \ldots$ *are countable sets, then* $\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \ldots$ *is countable.*

Proof. The proof idea here is similar to the one in Proposition 1.
Suppose we have a countable collection of countable sets $A_1, A_2, \ldots$. Since it is countable, we can list their elements in the following way:

$$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$$
$$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots\}$$
$$\vdots$$

Hence, all the elements of $\bigcup_{i=1}^{\infty} A_i$ have the form a_{mn} for some m and n. Suppose an element is repeated in this representation, that is, $a_{mn} = a_{kl}$ for some m, n, k, and l. Then, we choose to represent in this form by taking the minimum between m and n, that is, by choosing the one with minimum first subscript.

Our goal is to find an injection from $\bigcup_{i=1}^{\infty} A_i$ to $\mathbb{N}$ and use Lemma 1 to conclude that $\bigcup_{i=1}^{\infty} A_i$ is countable. We see that there are two numbers that specify an element in the subscripts. So the proof for Proposition 1 should spring to mind and we shall use that.

Consider the function $f : \bigcup_{i=1}^{\infty} A_i \longrightarrow \mathbb{N}$ such that $f(a_{mn}) = 2^m \cdot 3^n$. As shown in Proposition 1, this form of function is injective. So there exists an injection from $\bigcup_{i=1}^{\infty} A_i$ to $\mathbb{N}$. So by Lemma 1, $\bigcup_{i=1}^{\infty} A_i$ is countable. $\square$

Exercises

1. Show that $\mathbb{Z}$ is countable.
2. Show that $\mathbb{Q}$ is countable.

The two exercises are interesting because they challenge our intuitive ideas of 'sizes'. Although $\mathbb{N}$ is contained within $\mathbb{Z}$, for example, we can still biject $\mathbb{N}$ with $\mathbb{Z}$. So that means that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ have the same 'sizes'.

Everything that we've met so far is countable. If every infinite set is countable, then the whole concept of countability would be a futile one. However, in the 19th century, Georg Cantor, who pioneered the study of countability, found a counterexample by employing his brilliant brainchild, the now very well-known *diagonal argument*, to prove that $\mathbb{R}$ is an uncountable set, ie. that there is no bijection between $\mathbb{R}$ and $\mathbb{N}$. So without wanting to beat around the bush, let us prove that $\mathbb{R}$ is uncountable.

Proposition 2 *The set of real numbers $\mathbb{R}$ is uncountable.*

Proof (This is probably the most beautiful proof in the first year mathematics course in university, so you are advised to read, understand and appreciate the ingenuity of the proof.) We prove by contradiction.
Suppose $\mathbb{R}$ is countable. Consider the subset $A = \{a \in \mathbb{R} : 0 < a < 1\}$, that is, all the numbers between 0 and 1. We write all the elements in this set in decimal expansion form, $a = 0.a_1a_2a_3\ldots$.
We know by Corollary 1 that any infinite subset of $\mathbb{R}$ will be countable. So A countable, hence we can list down the elements of A as follows:

$$a_1 = 0.\underline{a_{11}}a_{12}a_{13}\ldots$$

$$a_2 = 0.a_{21}\underline{a_{22}}a23\ldots$$

$$a_3 = 0.a_{31}a_{32}\underline{a_{33}}\ldots$$

$$\vdots$$

Now here comes the clever bit. Let us construct an element of A, r, such that $r_n \neq a_{nn}$, ie. the n^{th} term of r is different from the n^{th} term of a_n. So r differs from a_n at the n^{th} term in the decimal expansion, for any n.
Therefore, $r \neq a_n$ for any n. So r is not in the listing. This is a contradiction, because our initial supposition was that A is countable, so every number between 0 and 1 should be in the lisiting. Hence, A not countable, so $\mathbb{R}$ not countable. $\square$

3.1 Extra Exercises

These exercises are much harder than the previous ones and requires a bit of extra mathematical knowledge. They are not compulsory.

1. Let $A_1, A_2, \ldots$ be sets such that $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$ for all n. Can $A_1 \cap A_2 \cap \ldots$ be true?
2. Find an injection from $\mathbb{R}^2$ to $\mathbb{R}$.
3. Is there an injection from the set of all real sequences to $\mathbb{R}$?
4. Show that there does not exist an uncountable family of pairwise disjoint discs on the 2D plane.
5. Show that the 2D plane is not a countable collection of lines.