Solutions
2004 Senior Division

1. (Also I1)

$$\frac{2004+6}{100}=\frac{2010}{100}=20.1,$$

hence (D).

2. (Also J5 & I3)

$\frac{4}{5}$ is closer to 1 than any other integer,

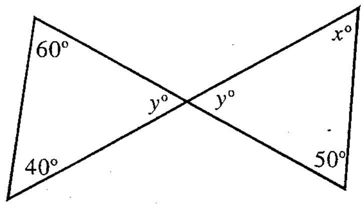
hence (B).

3. $z=2-3y=2-3\times 3x=2-9x$,

hence (E).

4. (Also I7)

The angles marked $y°$ are vertically opposite and are equal.
So, from the angle sums of the two triangles, $60 + 40 = x + 50$ and $x = 50$,



hence (A).

5.

$$\begin{aligned} 2x+3&>9\\ 2x&>6\\ x&>3, \end{aligned}$$

hence (A).

6. $2^{15}=4\times 2^n=2^2\times 2^n=2^{n+2}$, so $n+2=15$ and $n=13$,

hence (B).

7. (Also I11)

Let the rectangle have length 25 units, then its breadth is 1 unit. The area of the rectangle is $25\times 1=25$ sq units and its perimeter is 25 + 1 + 25 + 1 = 52 units. The square with area 25 sq units has side 5 units and perimeter 20 units.

The ratio of the perimeter of the rectangle to that of the square is then

$$52:20=13:5,$$

hence (A).

8. (Also I10)
The new wage is 120% of the old wage, so the old wage was

$$\$360\times\frac{100}{120}=\$300,$$

hence (B).

9. The minimum number of sheets needed is 4, and $4\times 90-300=60$ cm. Since there are 3 equal overlaps, each mush have width 20 cm,

hence (C).

---

10. For $x>2$, the order of magnitude is

$$\frac{1}{\sqrt{x}}<\sqrt{x}<2x<x^2<x^3,$$

hence (E).

11. *Alternative 1*
Before: Uses machine 3 times per week $=3\times 120=360$ L.
After: Uses machine $2\frac{1}{3}$ times per week $=2\frac{1}{3}\times 120=\frac{8}{3}\times 120=280$ L.
So the average amount saved per week is $360-280=80$ L

hence (C).

Alternative 2
Consider a period of days divisible by both 7 and 3, such as 21. So, in 21 days, the use is
Before: $3\times 3\times 120=1080$ L.
After: $7\times 120=840$ L.
The amount of water saved in 3 weeks is $1080-840=240$ L, so the average amount saved per week is 240/3 = 80L,

hence (C).

12. (Also I14)
Let the smallest of the four integers be x. Then the sum of the four consecutive integers is $x+x+1+x+2+x+3=4x+6$. So

$$4x+6=2000 \text{ or } 2001 \text{ or } 2002 \text{ or } 2003 \text{ or } 2004$$
$$4x=1994 \text{ or } 1995 \text{ or } 1996 \text{ or } 1997 \text{ or } 1998.$$

The only one of these with an integer solution is $4x = 1996$, $x = 499$, and $4x + 6 = 2002$,

hence (C).

13. If P is the population at the start of the 4 year period, the population after 4 years is

$$P\times 1.2\times 1.2\times 0.8\times 0.8=0.9216P\approx 92\%P$$

So there has been then an 8% decrease,

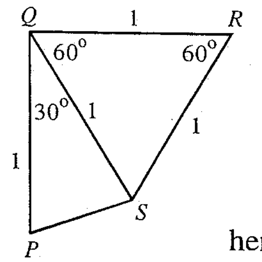
hence (A).

14. (Also I23)

Drawing the diagonal QS as shown, we obtain an equilateral triangle QRS and an isosceles triangle PQS as shown.

$\angle QPS = \angle QSP = 75°$, so

$\angle RSP = x = 75 + 60 = 135°$,



hence (C).

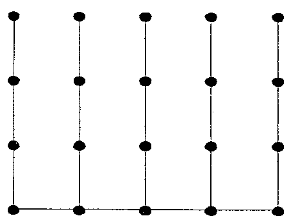
15. Drawing up the table of outcomes for the difference when tossing two dice,

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

we get $P(0) = \frac{6}{36}$, $P(1) = \frac{10}{36}$, $P(2) = \frac{8}{36}$, $P(3) = \frac{6}{36}$, $P(4) = \frac{4}{36}$ and $P(5) = \frac{2}{36}$,

hence (B).

16. The diagram



shows that 12 roads can be closed while all cities are still connected. Also, we need for 19 roads in order to get a connected system of roads. To see this, start with the 20 cities as isolated points and add the roads of the connected system one by one. Then the number of isolated parts decreases by at most one for any new road, that is at least 19 roads are necessary for a connected system.

hence (B).

17. (Also I19)

In 1L of the fruit juice, there is 800 mL of water. If 75% of the water is removed, there will be 200 mL of water left in the remaining 400 mL, so the concentrated juice contains 50% water,

hence (C).

18. (Also J24 & I24)

Writing T for telling the truth and L for lying, we get the table:

	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Sun	...
John	L	T	T	T	T	L	L	L	...
Deiter	T	T	L	L	L	T	T	T	...

Consider the statement "Yesterday I lied". This can only be true if the speaker lied one day and told the truth the next (L T), or, told the truth one day and lied the next (T L).

So, we are looking in the table for each to have a sequence LT or TL together, and this occurs once on the Friday,

hence (D).

19. If $a+\cfrac{1}{b+\cfrac{1}{c}}=\dfrac{37}{16}=2+\dfrac{5}{16}$,

then $a=2$ and $b+\dfrac{1}{c}=\dfrac{16}{5}=3+\dfrac{1}{5}$.

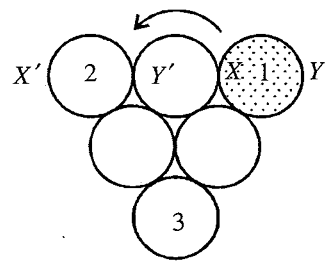
Then, $b=3$ and $c=5$ giving $a+b+c=10$,

hence (A).

Comment: This structure is called a *continued fraction*.

20. (Also I21 & J22)

In rolling coin 1 around the coin to its left from position 1 to position 2, the point Y on coin 1 rolls around the circle to go to the point Y' in position 2, and X goes to the position X' in 2. So coin 1 rotates through an angle of $360°$ when it rolls from position 1 to position 2. Similarly, it will rotate another $360°$ in rolling from position 2 to position 3, and another $360°$ in rolling from position 3 to position 1.



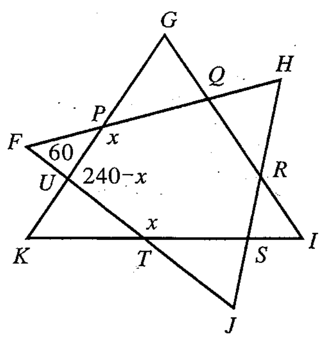
This means that it rolls through an angle of $3\times 360° = 1080°$ to get back to its original position (and it will also have the same orientation),

hence (E).

21. Produce the sides PQ, RS and UT respectively to meet at H, J and F respectively, then ΔFHJ is equilateral. Similarly we obtain the equilateral triangle ΔIKG from the intersection of the sides QR, ST and UP.

Let $\angle UPQ = x°$. Then $\angle FPU = 180-x$ and the exterior $\angle PUT = 60+180-x=240-x$.

Similarly, $\angle UTS = x$, $\angle TSR = 240-x$, $\angle SRQ = x$ and $\angle RQP = 240-x$. The angles of the hexagon $PQRSTU$ are either $x°$ or $(240-x)°$, so there are at most two angle sizes,



hence (B).

22. Let the ages of Ann, Ben and Cathy be a, b,and c respectively. Then $a+b+c=23$ and $abc=113+(a-1)(b-1)(c-1)$

The second equation gives

$$abc=113+abc-(ab+bc+ca)+a+b+c-1$$

$\therefore \quad ab+bc+ca=113+23-1=135$

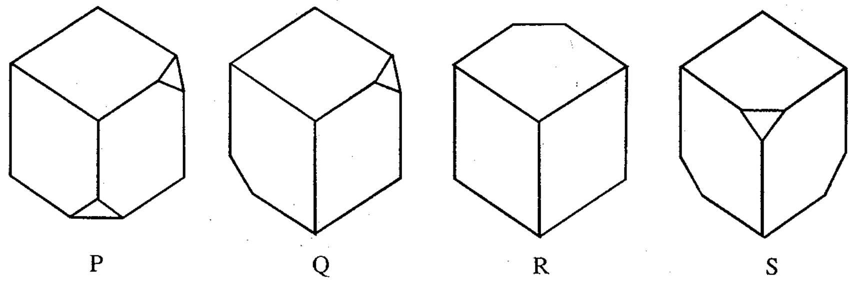
Now $\quad a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)$

$$=529-2\times 135$$
$$=259,$$

hence (E).

23. (Also J27 & I26)

Given the cubes as shown,



P and Q are not the same, as P does not have a pair of corners cut off at the ends of a long (or internal) diagonal. Similarly, R does not have a pair cut at the ends of a long diagonal, so Q and R are not the same.

P and R are not the same, since if R had two corners cut off, they would have to be on the same edge.

R and S are not the same S has three corners cut off and R has at most two.

If Q and S were the same, then the hidden corner of Q would have to be cut off. But since Q has a pair of corners cut off at the ends of a long diagonal, the hidden fourth corner of S would also be cut off. Hence Q and S are not the same.

So, the only possibility is that P and S are the same, and with the hidden corner of S cut off, they are the same.

hence (D).

24. (Also J28 & I28)

If the product that remains is even, we must remove all multiple of 5, otherwise the last digit would be 0.

The 80 numbers which remain all end in 1, 2, 3, 4, 6, 7, 8 or 9 with ten of each type. A direct calculation shows $1\times2\times3\times4\times6\times7\times8\times9$ end with as 6. Similarly, if we take $11\times12\times13\times14\times16\times17\times18\times19$ this ends with a 6.

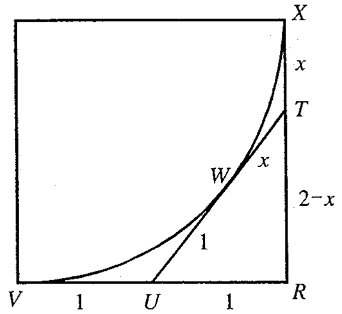
Hence the product of the whole 80 numbers has a last digit which is the last digit of 6^{10} which is a six.

Next if we remove the number 3, the last digit of the remaining 79 numbers is a 2.

Thus the minimum number to be removed is 21,

hence (B).

25. Let the side of the square be 4 units and consider the bottom right quadrant of the circle as shown in the diagram. Then $WU = UV = 1$. Let $TX = x$, then $TW = x$ and $RT = 2 - x$.



Hence $(1+x^2) = (2-x)^2 + 1$ and so $1 + 2x + x^2 = 4 - 4x + x^2 + 1$. Thus $6x = 4$ and $x = \frac{2}{3}$.

So the length of RT is $2 - \frac{2}{3} = \frac{4}{3}$, which is $\frac{1}{3}$ of the side of the square,

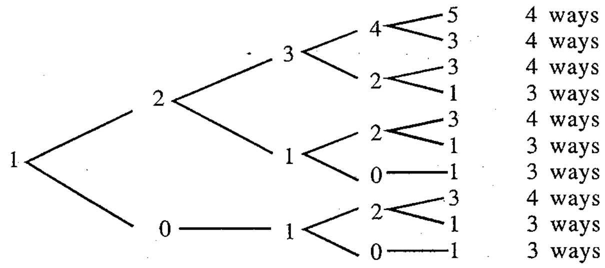
hence (A).

26. *Alternative 1*

If the sum of the first 5 elements of the sequence is 2 or more, there are 4 ways of completing the sequence as either $+1$ or -1 can be used in each position 6 and 7.

If the sum of the first 5 elements of sequence is 1, then there are 3 (valid) ways of completing the sequence, $+1,\ +1;\ +1,\ -1$ and $-1,\ +1$.

Now, from the following tree diagram listing the possible sequence of the first 5 terms:



We get 4 lots of 5 ways and 5 lots of 3 ways giving $5 \times 4 + 5 \times 3 = 35$ ways,

hence (A).

Alternative 2

The first digit must be 1.

There are $2^6 = 64$ sequences of digits starting with 1. Amongst these, the invalid sequences are:

($\otimes$ indicates either 1 or -1 in the location)

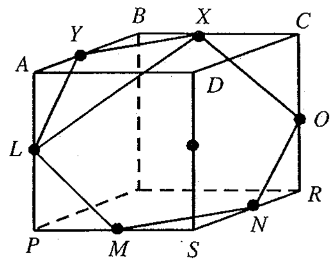
1	-1	-1	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\rightarrow$	16 sequences
1	-1	1	-1	-1	$\otimes$	$\otimes$	$\rightarrow$	4
1	-1	1	-1	1	-1	-1	$\rightarrow$	1
1	-1	1	1	-1	-1	-1	$\rightarrow$	1
1	1	-1	-1	-1	$\otimes$	$\otimes$	$\rightarrow$	4
1	1	-1	-1	1	-1	-1	$\rightarrow$	1
1	1	-1	1	-1	-1	-1	$\rightarrow$	1
1	1	1	-1	-1	-1	-1	$\rightarrow$	1

Giving a total of 29 invalid sequences. Hence there are $64 - 29 = 35$ valid sequences,

hence (A).

27. (Also I27)
Consider the cube $ABCDPQRS$ as shown.
Join the midpoints L, M, N, O, X and Y. This gives a regular hexagon LMNOXY.
Join L and X.

This gives a triangles with $\angle LYX = 120°$. The largest angle obtained in any other triangle is $90°$, so that largest possible angle is $120°$,



hence (C).

28. The solution of the quadratic equation $x^2+bx+2=0$ are

$$x_1=-\frac{b}{2}-\sqrt{\frac{b^2}{4}-2} \text{ and } x_2=-\frac{b}{2}+\sqrt{\frac{b^2}{4}-2}.$$

Solution of the inequality must lie between x_1 and x_2.
If the length of the interval $[x_1, x_2]$ is larger or equal to 4, then there are at least 4 integers in that interval. Therefore

$$x_2-x_1=2\sqrt{\frac{b^2}{4}-2}<4,$$

i.e. $\frac{b^2}{4}-2<4$ and $b^2<24$.

On the other hand, for the existence of x_1, we need $\frac{b^2}{4}-2\geq 0$, that is $b^2>8$.
So, we have exactly 4 possibilities for b, namely $b=-4,-3,\ 3,\ 4$,

$b=-4$	$x_1=2-\sqrt{2}$	$x_2=2+\sqrt{2}$	integer solns: 1, 2, 3
$b=-3$	$x_1=1$	$x_2=2$	integer solns: 1, 2
$b=3$	$x_1=-2$	$x_2=-1$	integer solns: $-2,-1$
$b=4$	$x_1=-2-\sqrt{2}$	$x_2=-2+\sqrt{2}$	integer solns: $-3,-2,-1$

So there are two values of b, -4 and 4 which result in three integer solution,

hence (C).

28. Let A, B, C, D be four points in the coordinate plane with coordinates (0, 0), (1, 0), (3,4), (0, 1) respectively. Also let P be a point with coordinates (x, y). Then

$$\sqrt{x^2+y^2}+\sqrt{(x-3)^2+(y-4)^2}=PA+PC\geq AC$$

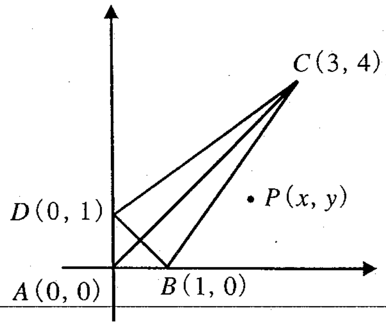
with equality only is P is on AC and

$$\sqrt{(x-1)^2+y^2}+\sqrt{x^2+(y-1)^2}=PB+PD\geq BD$$

Therefore

$$\sqrt{x^2+y^2}+\sqrt{(x-1)^2+y^2}+\sqrt{x^2+(y-1)^2}+\sqrt{(x-3)^2+(y-4)^2}\geq AC+BD$$

with the equality taking place when P is the point where AC and DB meet.



This means that the smallest possible value that

$$\sqrt{x^2+y^2}+\sqrt{(x-1)^2+y^2}+\sqrt{x^2+(y-1)^2}+\sqrt{(x-3)^2+(y-4)^2}$$

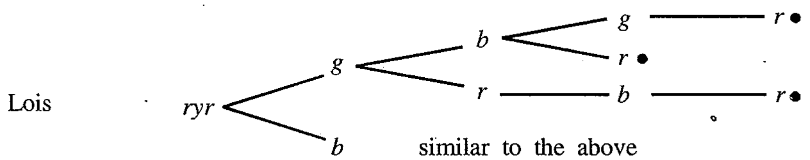
can have is $AC+DB=\sqrt{3^2+4^2}+\sqrt{1^2+1^2}=5+\sqrt{2}$,

hence (D).

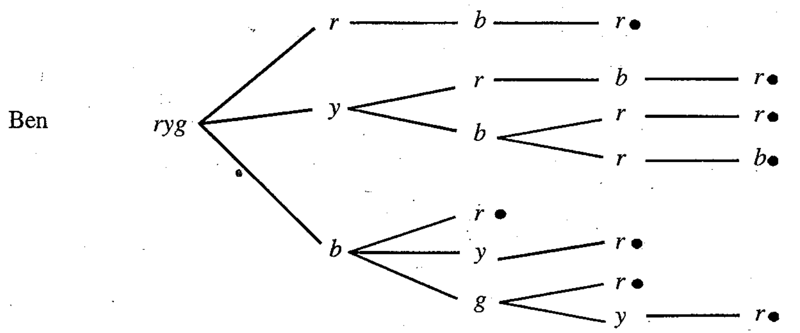
30. (Also I30)

Lois cannot use y again, but either of the other colours will do, so, without loss of generality, she can continue *ryrg*. She can then use b or r. If she chooses b as the fifth letter, she has *ryrgb*. She can then use g or r. If she uses r the sequence ends *ryrgbr*, but is she chooses g then she can add r and the sequence ends *ryrgrbr*. Thus Lois can make a sequence of length 7 but no longer.

This can represented in the following tree diagram, where • means that, under the two given rules, no further addition can be made to the sequence,



Be can add r, y or b. If he chooses r, he cannot use y or g again, and the sequence ends as *rygrbr*. If he chooses y, he cannot use g again, but can continue *rygyr* or *rygyb*. The first one must end as *rygyrbr* and the second as *rygybyr* and *rygybrb*. If he chooses b then the next letter could be r, y or g. r would finish the sequence: *rygbr* and y allows one more r: *rygbyr*. With g we have *rygbg*. A choice of r here finishes the sequence: *rygbgr*. y is the only other possibility giving *rygbgy* and only r can be added: *rygbgyr*. So Ben can also make a sequence of length 7 but no longer.



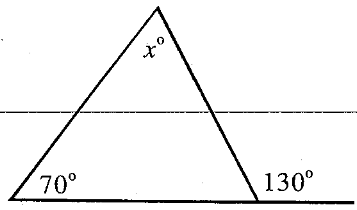
hence (D).

Solutions
2005 Senior Division

1. $(4 \times 5) \div (2 \times 10) = 20 \div 20 = 1$,

hence (E).

2. $x + 70 = 130$ (exterior angle of the triangle),
so $x = 60$,



hence (E).

3. (Also I7)

$$1+\cfrac{1}{3+\cfrac{1}{2}} = 1+\frac{1}{\frac{7}{2}} = 1+\frac{2}{7} = \frac{9}{7},$$

hence (E).

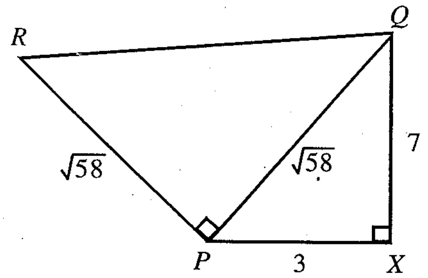
4. The straight line $y = x + g$ passes throught the point (2,3).
Then $3 = 2 + g$ and $g = 1$,

hence (B).

5. (Also I8)
The number $t8u = 100t + 10 \times 8 + u = 100t + 80 + u$,

hence (E).

6. ΔPXQ is a right angled triangle with sides of length 3 and 7 as shown.
So $3^2 + 7^2 = PQ^2$ and $PQ = \sqrt{58}$



Area of $\Delta PQR = \frac{1}{2} \times PQ \times QR = \frac{1}{2} \times \sqrt{58} \times \sqrt{58} = \frac{1}{2} \times 58 = 29$,

hence (B).

7. The total number of marks gained was $70 \times 20 + 80 \times 30 = 1400 + 2400 = 3800$.
The average mark overall is then $\frac{3800}{50} = \frac{380}{5} = 76$,

hence (C).

8. (Also I10)
The wheel with the larger diameter will rotate less than the wheel with the smaller diameter, so the difference in the number of revolutions of each wheel in a journey of 1800km is the number of rotations of the smaller wheel (200 cm diameter) minus the number of rotations of the larger wheel (diameter 225 cm)
This difference

$$\begin{aligned}&= \frac{1800\times1000\times100}{200} - \frac{1800\times1000\times100}{225}\\ &= 900\ 000 - 800\ 000\\ &= 100\ 000,\end{aligned}$$

hence (D).

9. The angle sum of a pentagon is $5 \times 180 - 360 = 540$ degrees.
The remaining angle, in degrees, is then $540 - 400 = 140$,

hence (C).

10. $\sqrt[4]{2}\times\sqrt{32\sqrt{2}} = 2^{\frac{1}{4}}\times2^{\frac{5}{2}}\times2^{\frac{1}{4}} = 2^{\left(\frac{1}{4}+\frac{5}{2}+\frac{1}{4}\right)} = 2^3 = 8$,

hence (A).

11. Let the fraction be t.
Then

$$\left(t+\left(\frac{1}{t}\right)\right)^2 = \left(t-\left(\frac{1}{t}\right)\right)^2 + 4 = \frac{81}{400} + 4 = \frac{1681}{400}$$

Then $t+\frac{1}{t} = \sqrt{\frac{1681}{400}} = \frac{41}{20}$,

hence (B).

12. At $t = 0$,

$$Q = \frac{100}{(1+0)^2} = 100$$

So, half the gas is 50 cubic units.
Then

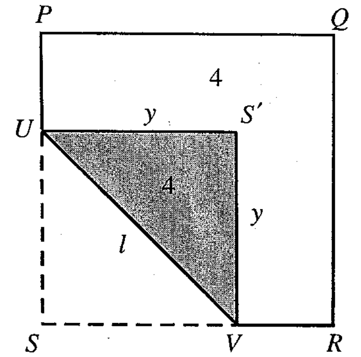
$$\begin{aligned}50 &= \frac{100}{(1+2t)^2}\\ (1+2t)^2 &= 2\\ 1+4t+4t^2 &= 2\\ t &= \frac{-4\pm\sqrt{16+16}}{8}\\ &= \frac{-4\pm4\sqrt{2}}{8}\\ \therefore\ t &= \frac{\sqrt{2}-1}{2},\end{aligned}$$

hence (A).

13. (Also I14)
The outcomes for the digits to be a perfect square are:- (1,6), (2,5), (3,6), (4,6), (5,2), (6,1), (6,3) and (6,4), 8 in all. There are 36 possible outcomes, so the probability is $\frac{8}{36}=\frac{2}{9}$

hence (B).

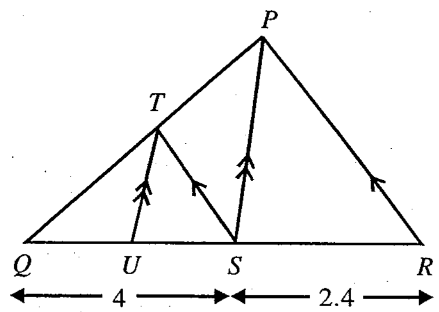
14. (Also I17)
Let the area of the shaded portion be x, so the area of the white portion is x and the part of the square not visible also has area x.
Then $3x = 12$ and $x = 4$.



Let the side of the shaded rectangle be y. Then $\frac{1}{2}\times y^2=4$ and $y^2=8$.
The length of the fold line UV is the hypotenuse l of the isosceles triangle with two equal sides length y.
So $l^2=y^2+y^2=2y^2=16$ and $l=4$,

hence (A).

15. $\frac{QU}{US}=\frac{QT}{TP}$ as $UT\|SP$.
$\frac{QT}{TP}=\frac{QS}{SR}$ as $ST\|RT$.
So
$\frac{QU}{US}=\frac{QS}{SR}$
So $\frac{QU}{US}=\frac{4}{2.4}=\frac{40}{24}=\frac{5}{3}$.
Then $\frac{QU}{QS}=\frac{5}{8}$ and so $QU=\frac{5}{8}\times 4=2.5$,



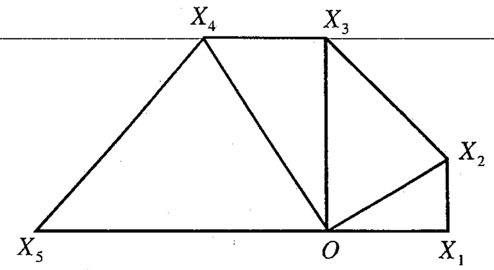
hence (B).

16. (Also I21)
Let the train from Canberra be at a point P at 12:40 pm, when the other train leaves Sydney.

The train from Sydney will reach P at the same time the train from Canberra arrives at Sydney, that is at 3:30 pm, which is 170 minutes later than the first train arrived at P. Since both trains travelled at the same speed, they passed at the mid-point between P and Sydney, 85 minutes after 12:40 pm, at 2:05 pm,

hence (C).

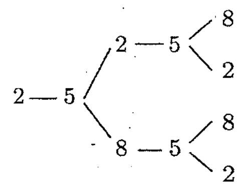
17. Since the angle that each triangle has at the origin is 45°, it will be the hypotenuse of the eighth triangle which will overlap OX_1.



The length of the hypotenuse increases by a factor of $\sqrt{2}$ each time, so the eighth, OX_8, will have length $\left(\sqrt{2}\right)^8 = 16$. Thus X_1X_k (where $k = 8$) = $16 - 1 = 15$,

hence (D).

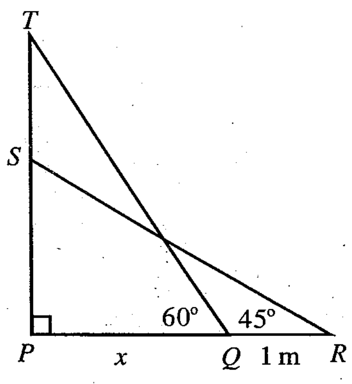
18. (Also I22)
Consider 5-digit numbers starting with 2. The following tree diagram shows there are 4 such numbers starting with 2.



In a similar manner, we get 4 starting with 1, 8 with 3, 4 with 4, 4 with 5, 8 with 6, 4 with 7, 4 with 8 and 5 with 9, giving a total of $4 + 4 + 8 + 4 + 4 + 8 + 4 + 4 + 5 = 45$ such numbers,

hence (D).

19. Let $PQ = x$.



Then

$$\frac{x}{l} = \cos 60 = \frac{1}{2} \quad \text{from the triangle } PQT$$

$$\frac{x+1}{l} = \cos 45 = \frac{1}{\sqrt{2}} \quad \text{from the triangle } PRS$$

Eliminating l we obtain

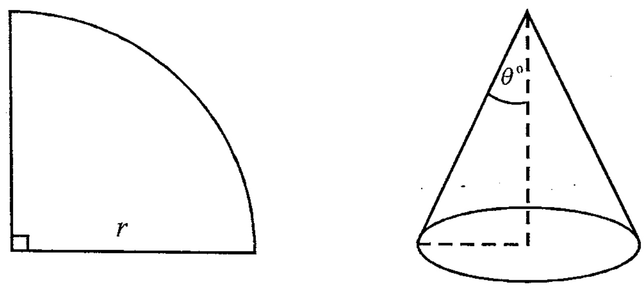
$l = 2x = \sqrt{2}(x+1) = \sqrt{2}x + \sqrt{2}$,

and $x = \dfrac{\sqrt{2}}{2-\sqrt{2}} = \dfrac{1}{\sqrt{2}-1} = \sqrt{2}+1$

So $l = 2x = 2\left(\sqrt{2}+1\right)$,

hence (B).

20. Let r be the radius of the quarter circle.



The length of the arc rolled to form the circumference of the base of the cone is $\dfrac{\pi r}{2}$.

The radius of the base of the cone is

$$\left(\frac{1}{2\pi}\right)\left(\frac{\pi r}{2}\right) = \frac{r}{4},$$

so that

$$\sin\theta^\circ = \left(\frac{r}{4}\right)/r = \frac{1}{4},$$

hence (A).

21.

$$\sqrt{x^2 + \sqrt{x^3+1}} = 1 - x$$
$$x^2 + \sqrt{x^3+1} = 1 - 2x + x^2$$
$$\sqrt{x^3+1} = 1 - 2x$$
$$x^3 + 1 = 1 - 4x + 4x^2$$
$$x^3 - 4x^2 + 4x = 0$$
$$x\left(x^2 - 4x + 4\right) = 0$$
$$x(x-2)(x-2) = 0$$
$$\therefore\ x = 0 \text{ or } 2.$$

As we have squared both sides, there are possibly extraneous roots, so checking for each value:

$x = 0$ gives $\sqrt{1} = 1$ which is true.

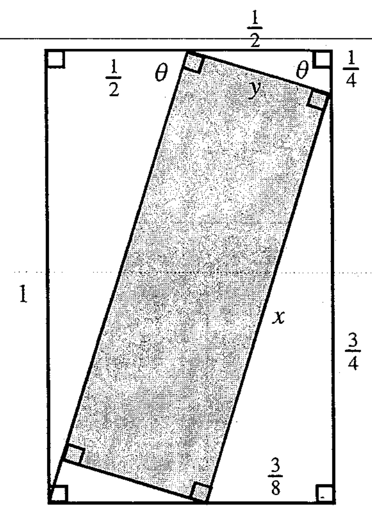
$x = 2$ gives $2+\sqrt{4+\sqrt{9}} = 2+\sqrt{7} \neq 1$ which is false.

So there is one solution, $x = 0$,

hence (B).

22. (Also I23)

Using the fact that the four outer triangles are similar, we obtain the additional dimensions as shown on the figure.



The area of the shaded rectangle is xy.

From the triangle on the top we get

$$y^2 = \frac{1}{4}+\frac{1}{16} = \frac{5}{16}$$

$$y = \frac{\sqrt{5}}{4}$$

From the triangle on the right we get

$$x^2 = \frac{9}{16}+\frac{9}{64} = \frac{45}{64}$$

$$x = \frac{3\sqrt{5}}{8}$$

The area of the rectangle is

$$xy = \frac{\sqrt{5}}{4}\times\frac{3\sqrt{5}}{8} = \frac{15}{32} = \frac{7.5}{16}$$

This is greater than $\frac{7}{16}$ and less than $\frac{8}{16}$,

hence (D).

23. Expanding, $(1-2x^3)(1+kx)^2 = (1-6x+12x^2-8x^3)(1+2kx+kx^2)$

The coefficient of x^2 is then $k^2-12k+12$.

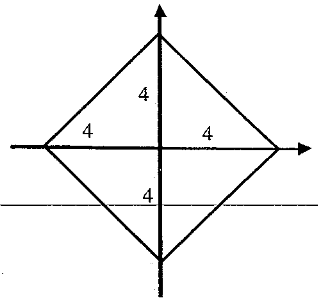
So $k^2-12k+12=40$ and $k^2-12k-28=0$.

The two values of k which satisfy are the sum of the roots of this last equation, that is

$$k_1+k_2 = \frac{-(-12)}{1} = 12,$$

hence (D).

24. The graph of $|x| + |y| = 4$ is that part of the lines $x + y = 4$, $x - y = 4$, $-x + y = 4$ and $-x - y = 4$
for $-4 \le x \le 4$ and $-4 \le y \le 4$



So, the area is $2\times\frac{1}{2}\times 8\times 4 = 32$ square units,

hence (E).

25. (Alternative 1)

$$2^{2005} = \left(2^{10}\right)^{200}\times 32 = (1024)^{200}\times 32 \approx 10^{600}\times 10^{2} = 10^{602}$$

More accurately

$$2^{10} = 1024 > 10^{3}\text{, hence } 2^{2000} > 10^{600}.$$

$$\text{Also } 2005 < 13\times 155 \text{ and } 2^{13} = 8192 < 10^{4}$$

So

$$2^{2005} < \left(10^{4}\right)^{55} \text{ and } 2^{13} = 8192 < 10^{4}$$

Hence

$$2^{2005} < \left(10^{4}\right)^{155} = 10^{620}.$$

So the number of digits in 2^{2005} is closest to 600,

hence (C).

(Alternative 2)

The number of digits in N is the integer $[\log_{10} N]$.

Note that

$$2^{13} = 8192 < 10^{4} \text{ and } 2^{10} = 1024 > 10^{3}$$

whence $4/13 > \log_{10}2 > 3/10$.

Hence

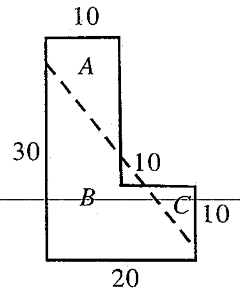
$$601 < \frac{2005\times 3}{10} < \log_{10} 2^{2005} < \frac{2005\times 4}{13} < 617$$

So the number of digits in $\log_{10}\ 2^{2005}$ is closest to 600,

hence (C).

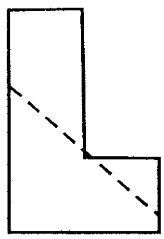
26. (Also S26)

Label the three pieces as shown. Depending on the angle of the cut, A may be bigger than C or vice versa, but whatever the angle, it is clear that one of them is going to be smaller than B.

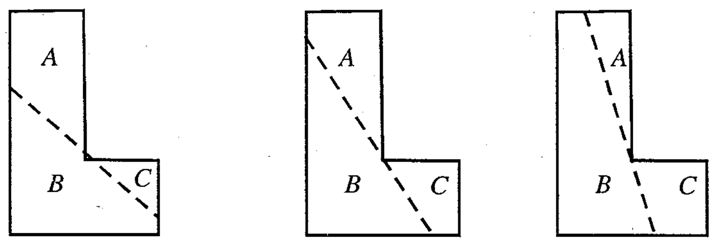


So, if the cut does not go through the inside corner, it will be possible to make my piece bigger by moving the cut a small distance to the left, but parallel to the old one.

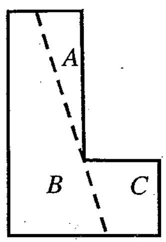
So, for my best solution. I know the cut must go through the corner.



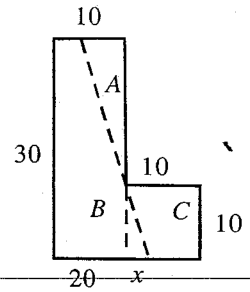
Considering which edge the cut may pass through, there are now three possibilities:-



With the cut of either of the first two types, it is easy to see that my piece will be C and that this can be improved by making a small clockwise adjustment to the direction of the cut. So my optimum solution is of the third type. Start at the limiting position,



rotate the cut clockwise and see what happens. At the starting position, C is smaller than A and so is my piece. As the cut rotates, C gets bigger and A gets smaller (smoothly), approaching a position where A is very small and clearly smaller than C, in which case A is my piece. The best solution is when A and C are the same size. If we write the dimension x as shown:-



We see that the area of A is $20x$ and the area of C is $100 - 5x$. Equating these gives $x = 4$ and my piece is either A or C, of equal minimum area 80 cm^2. So my largest possible piece is 80 cm^2.

27. We have

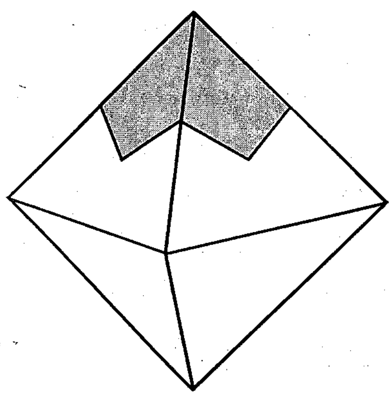
$$f(f(t)) = 6t - 2005 \qquad (1)$$

and

$$f(t) = 6t - 2005. \qquad (2)$$

Thus $6t - 2005 = f(f(t)) = f(6t - 2005)$.
Hence $f(f(t)) = f(6t - 2005) = 6t - 2005$.
But, from (2), LHS $= 6(6t - 2005) - 2005 = 6t - 2005 =$ RHS
Thus $30t = 6 \times 2005$ and $t = 401$.

28. (Also I29)
Using the symmetry of the figure, we can see that the octohedron is made up of 6 shapes which are identical to the one shown.



The volume of that portion is then $\frac{120}{6} = 20$ cm^3.

29. Given

$$x+y+z=5 \qquad (1)$$
$$x^2+y^2+z^2=15 \qquad (2)$$
$$xy=z^2, \qquad (3)$$

From (1) we get

$$x+y=5-z$$
$$x^2+2xy+y^2=25-10z+z^2$$

Substituting (2) and (3) in this gives

$$15-z^2+2z^2=25-10z+z^2$$
$$10z=10$$
$$z=1$$

Then

$$\begin{aligned}\frac{1}{x}+\frac{1}{y}+\frac{1}{x}&=\frac{x+y}{xy}+\frac{1}{z}\\&=\frac{5-z}{z^2}+\frac{1}{z}\\&=4+1=5\end{aligned}$$

Comment

If you were to completely solve the system of equations and obtain the solutions

$$(x,y,z)=\left(2\pm\sqrt{3},2\mp\sqrt{3},1\right)$$

substitution would give the same result.

30. (Also J30 & I30)

The integer cannot be odd. Otherwise all its divisors would be odd and the sum of the four squares of its divisors even, yielding a contradiction. Hence the smallest two divisors must be 1 and 2. The next smallest divisor must be 4 or a prime p. It cannot be 4, otherwise the sum would include exactly two odd squares (to make it even) and would be a number divisible by 2 but not by 4, yielding a contradiction.

Thus, the smallest three divisors are 1, 2 and an odd prime p. Since the sum is even, the remaining divisor is $2p$. Thus the number is equal to

$$1+4+p^2+4p^2=5(1+p^2).$$

Since p does not divide $1+p^2$, it must divide (and so be equal to) 5.

The number is $5\times 26=130=1\times 2\times 5\times 13$, and so the largest prime divisor is 13.

Solutions
2006 Senior Division

1. (Also J4)

$$\frac{6\times25}{3\times5\times2}=\frac{6\times25}{6\times5}=\frac{25}{5}=5,$$

hence (D).

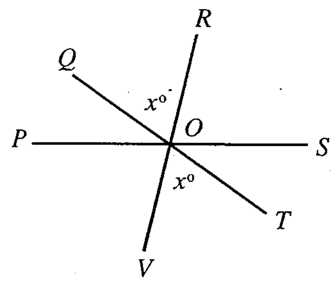
2. (Also I3)

$$a=2b-5$$
$$2b=a+5$$
$$b=\frac{a+5}{2},$$

hence (D).

3. (Also J15 & I8)

In the diagram, $\angle POR=120°$ and $\angle QOS=145°$.



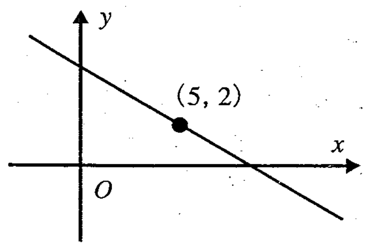
Let $\angle TOV=\angle QOR=x°$.
Then $\angle POR+\angle QOS=\angle POS+x° = 180°+x°$.
So $120 + 145 = 180 + x$ and $x = 85$,

hence (C).

4. $\frac{7}{x^2}=7x^{-2}$,

hence (E).

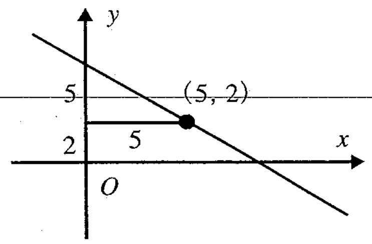
5. In the figure, the line has gradient -1, and it passes through the point (5,2).



(Alternative 1)
Its equation is then $y-2 = -1(x-5)$ which is $y+x-7=0$.
The y-intercept is when $x = 0$ and is 7,

hence (D).

(Alternative 2)
Draw the perpendicular from the point (5,2) to the y—axis,



and it is easy to see that the y—intercept is 7,

hence (D).

6. (Also I9)
If you begin reading at the top of page 13 and finish at the bottom of page 14 you have read $14-13+1=2$ pages, so if you start at the top of page x and read to the bottom of page y you will have read $y-x+1$ pages,

hence (D).

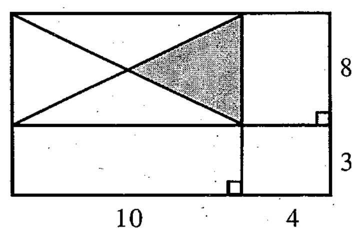
7. Let the dimensions of the box be x, y and z centimetres.
The volume of the box is then xyz cm^3.
We have also that $xy = 35$, $yz = 60$ and $xz = 84$.
So $x^2y^2z^2 = 35 \times 60 \times 84 = 7 \times 5 \times 5 \times 12 \times 7 \times 12$, and then
$xyz = 5 \times 7 \times 12 = 420$ and the volume of the box is 420cm^3,

hence (A).

8.
$$x = 3^n + 3^n + 3^n = 3 \times 3^n = 3^{n+1}.$$
Hence $x^2 = (3^{n+1})^2 = 3^{2n+2}$,

hence (B).

9. (Also I14)
The shaded area is $\frac{1}{4}\times 8\times 10 = 20$ square units (quarter of the rectangle).



The total area is 14×11 square units.

The fraction shaded is then $\frac{20}{14\times11} = \frac{10}{7\times11} = \frac{10}{77}$,

hence (E).

10. (Also I15)

The train takes $\frac{1}{4}$ of a minute to pass a post and $\frac{3}{4}$ of a minute to pass through a 600m tunnel. This means that it takes the front of the train $\frac{1}{2}$ of a minute to pass through the 600m tunnel and that the train is travelling 0.6km every $\frac{1}{2}$ minute and so is travelling at $0.6 \times 120 = 72$ km/h,

hence (D).

11. We can get 2 red balls and 1 white ball in 3 ways, that is by drawing in sequence *RRW*, *RWR* or *WRR*. These ways are mutually exclusive and so the probability of getting 2 *R* and one *W* is the sum of the probability of each of these.

Now

$P(RRW) = \frac{8}{20}\times\frac{7}{19}\times\frac{3}{18}$.

$P(RWR) = \frac{8}{20}\times\frac{3}{19}\times\frac{7}{18}$.

$P(WRR) = \frac{3}{20}\times\frac{8}{19}\times\frac{7}{18}$.

The probability of getting 2 red balls and 1 white ball is then

$$3\times\frac{8}{20}\times\frac{7}{19}\times\frac{3}{18} = \frac{7}{5\times19} = \frac{7}{95},$$

hence (E).

12. $16^8 \times 5^{25} = (2^4)^8 \times 5^{25} = 2^{25} \times 5^{25} \times 2^7 = 2^7 \times 10^{25} = 128 \times 10^{25}$,
so has 3 digits followed by 25 zeros and has 28 digits in total,

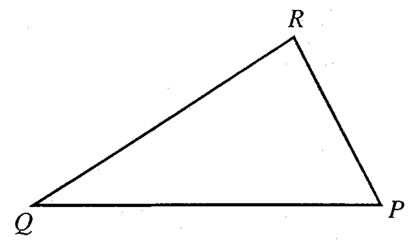
hence (E).

13. We are given $x < y < 0 < z$.
Consider $x + y + z > 0$. $-4 + -3 + 3 = -4 < 0$ so (A) is not always true.
Consider $(x + y)^2 - z > 0$. $(-2 + -1)^2 - 10 = -1 < 0$, so (B) is not always true,
Consider $x + y + z^2 > 0$. $-5 + -1 + 2^2 = -2 < 0$, so (C) is not always true,
Consider $x + y - z > 0$. This is $-ve + -ve + -ve = -ve$, so (D) must be false,
Consider $x + y - z < 0$. This is $-ve + -ve + -ve = -ve$, so must be true,

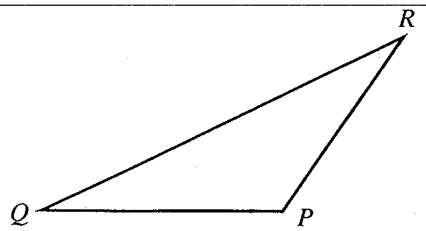
hence (E).

14. Let $0 < A < B < 90°$, with $\sin A = \frac{1}{4}$, $\sin B = \frac{1}{3}$. Then we have 4 cases to consider:

(a) $Q = A$, $P = B$, which is clearly possible.



(b) $Q = 180° - A$, $P = B$ which is not possible since $P + Q = 180° + (B - A) > 180°$.
(c) $Q = A$, $P = 180 - B$ which is possible since $P + Q = 180° + (A - B) < 180°$.



(d) If $Q = 180° - A$ and $P = 180° - B$, there are two obtuse angles, which is impossible.
So $\angle R$ can have two values,

hence (C).

15. (Also I17 & J24)
If a 3-digit number has digits all the same then the number is one of the numbers 111, 222, 333, . . . , 999.
Each of these numbers is divisible by 111 and the prime factors of 111 are 3 and 37.
So, one of the two-digit numbers must be a multiple of 3 and the other must be a multiple of 37.
We cannot obtain the products 111, 222 or 333 multiplying two two-digit numbers. We can get
$(1 \times 37) \times (4 \times 3) = 444$,
$(1 \times 37) \times (5 \times 3) = 555$,
$(1 \times 37) \times (6 \times 3) = 666$,
$(1 \times 37) \times (7 \times 3) = 777$,
$(1 \times 37) \times (8 \times 3) = 888$,
$(1 \times 37) \times (9 \times 3) = 999$,
$(2 \times 37) \times (4 \times 3) = 888$,
with 888 being the only one obtainable in two ways, so the number of pairs is 7,

hence (C).

16. (Also I20)
Let the amount of salt in the original mixture be x grams. This means the fraction of salt in the mix is $\frac{x}{450}$. When this saltiness is reduced by 10% by adding y litres of flour, this fraction becomes $\frac{9}{10} \times \frac{x}{450}$ and so

$$\frac{x}{450+y} = \frac{9}{10}\times\frac{x}{450} = \frac{x}{500},$$

so 50 grams of flour must be added,

hence (A).

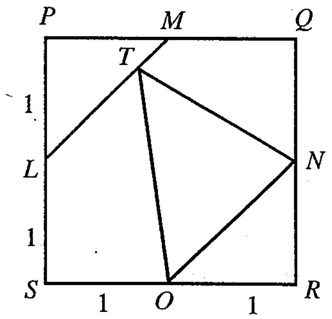
17. (Also I23)
Let the weights of the bales in kilograms be a, b, c, d, e. Then as all pairs of weights are different, the weights of the five bales are different. Assume then that $a < b < c < d < e$.
The lowest two sums have to be $a + b$ and $a + c$ and the highest two sums have to be $c + e$ and $d + e$.
Also, as each bale is weighed in pairs with four others, the sum of the weights of all pairs must be 4 times their combined weights, so $4(a + b + c + d + e) = 110 + 112 + 113 + 114 + 115 + 116 + 117 + 118 + 120 + 121 = 1156$ and $a + b + c + d + e = 289$.
So we have five equations

$$\begin{aligned} a+b &= 110 & (1)\\ a+c &= 112 & (2)\\ c+e &= 120 & (3)\\ d+e &= 121 & (4)\\ a+b+c+d+e &= 289 & (5) \end{aligned}$$

Substituting (1) and (4) in (5) gives $110 + c + 121 = 289$ and so $c = 58$.
Substituting this in (3) gives $e = 62$ so the heaviest bale is 62 kg,

hence (E).

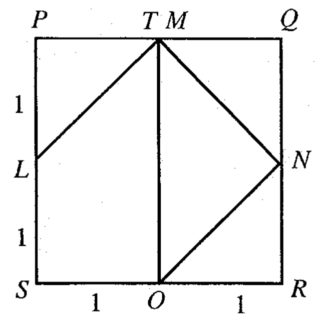
18. (Alternative 1)
$ON = \sqrt{2}$ and the height of ΔTNO is equal to $LO = \sqrt{2}$.



The area of ΔTNO is then $\frac{1}{2}\times\sqrt{2}\times\sqrt{2} = \frac{1}{2}\times 2 = 1$,

hence (B).

(Alternative 2)
Since it does not matter where T lies on LM, let it coincide with M.



The area of $\triangle TON$ is clearly 1,

hence (B).

19.

$$7^{x+1} - 7^{x-1} = 336\sqrt{7}$$
$$7^{x-1}\left(7^2 - 1\right) = 48 \times 7 \times 7^{\frac{1}{2}}$$
$$7^{x-1} \times 48 = 48 \times 7^{1.5}$$
$$x - 1 = 1.5$$
$$x = 2.5,$$

hence (A).

20. (Also J23 & I22)

(Alternative 1)

For each of the three choices to place R in row one, there are two choices to place R in row two and then only one choice in row 3.

So, there are 6 ways of placing the Rs.

R		
		R
	R	

For each of these, there are two choices to place W in row one, then one in row two and one in row three, and for each of these, the placement of B is determined.

Hence the number of different patterns is $6 \times 2 \times 1 = 12$,

hence (D).

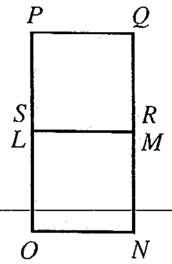
(Alternative 2)

There are 6 ways of placing 3 colours in the top row.

Then there are 2 choices for colours in the second row and the final row is then determined, so there are $6 \times 2 = 12$ ways,

hence (D).

21. (Also I24)
P moves along 3 circular arcs. The first rotation about R is $180°$ with a radius of the diagonal of the square, $\sqrt{2}$. This arc is then of length $\pi\times\sqrt{2}$.

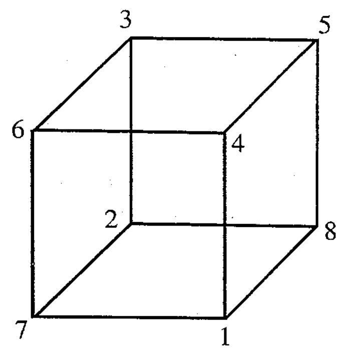


The next rotation about Q is $180°$ with radius 1, so the length is π.
The third rotation is about P so P does not move.
The last rotation is $180°$ about L with radius 1 so has length π.
The total length of the path traced out is then

$$\pi\times\sqrt{2}+\pi+\pi=\pi\left(2+\sqrt{2}\right),$$

hence (A).

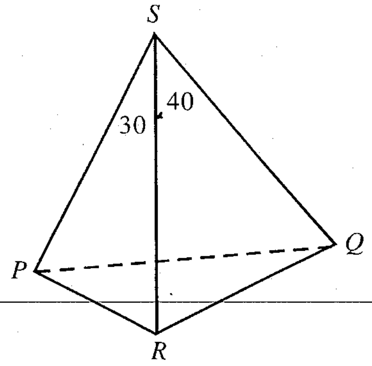
22. (Also J29 & I25)
There are 6 faces and hence 6 face sums. Since each vertex lies on 3 faces, the sum of all the face sums of the cube is $3(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) = 108$, so if there are 6 equal face sums, that sum must be $108 \div 6 = 18$.



The figure gives an example of 6 equal face sums, which is the maximum number possible.

hence (E).

23. Consider the tetrahedron PRQS as shown.



Clearly, $\angle PSQ < \angle PSR + \angle QSR$. Hence $\angle PSQ < 70°$.
Also $\angle QSR < \angle PSR + \angle PSQ$. Hence $\angle PSQ > 10°$.
Therefore the possible sizes of $\angle PSQ$ are 11°, 12°, . . ., 69°. So there are 59 possible sizes,

hence (B).

24. (Alternative 1)
If x is a solution then

$$\begin{aligned} a+x &= a^2 - 2a\sqrt{a-x} + a - x \\ 2a\sqrt{a-x} &= a^2 - 2x \\ 4a^2(a-x) &= a^4 - 4a^2x + 4x^2 \\ 4x^2 &= a^3(4-a) \end{aligned}$$

Consequently $0 \le a \ge 4$.

The possible positive integer solutions are 1, 2, 3, 4.

Check: $a = 1$: $\sqrt{1+x} + \sqrt{1-x} > 1$, not a solution;

$a = 2$, $x = 2$ a solution;

$a = 4$, $x = 0$ a solution.

What about $a = 3$?

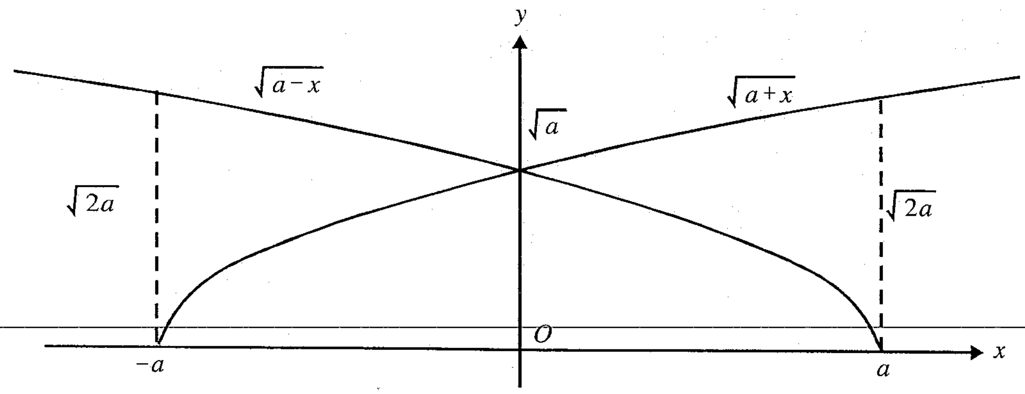
$$\sqrt{3+0} + \sqrt{3-0} = 2\sqrt{3} > 3 > \sqrt{3+3} + \sqrt{3-3} = \sqrt{6}$$

so, by continuity, there is a solution for x between 0 and 3.

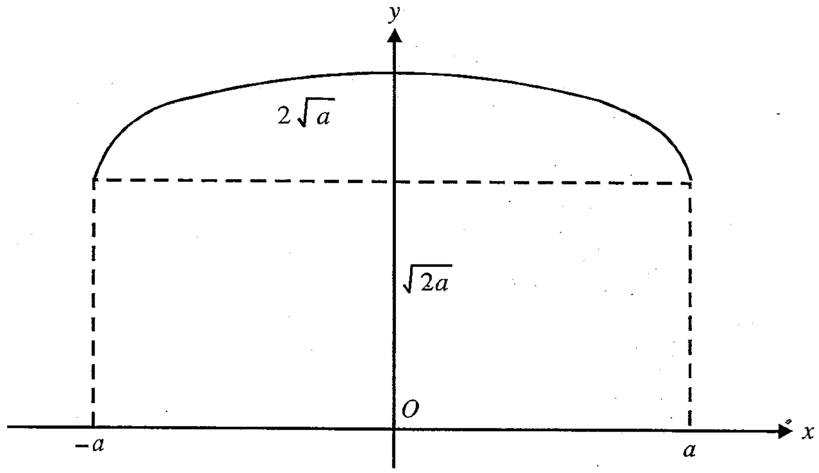
There are then 3 values of a:- 2, 3 and 4,

hence (D).

(Alternative 2)
Sketch the graphs of $\sqrt{a+x}$ and $\sqrt{a-x}$ together.



Adding these two, we have a function only defined between $-a$ and a.



It is clear that the equation $\sqrt{a+x}+\sqrt{a-x}=a$ can only have a solution when $\sqrt{2a}\le a\le 2\sqrt{a}$. Since $a > 0$ is given, we may square throughout with impunity to get $2a\le a^2\le 4a$ and so $2\le a\le 4$, giving three possibilities, 2, 3 and 4,

hence (D).

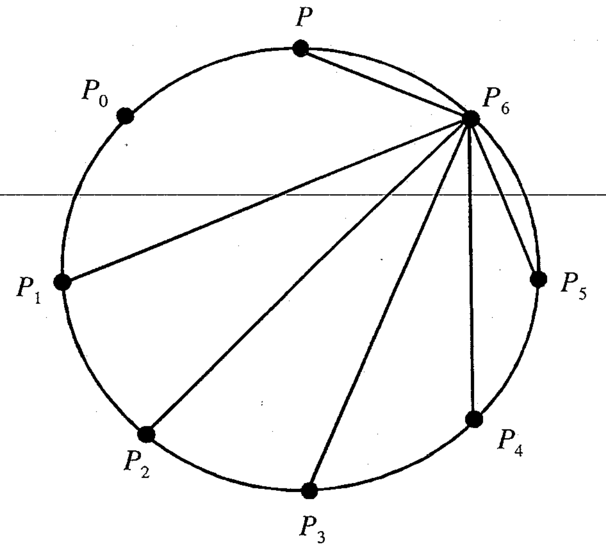
25. We are given 8 points on a circle, one of them P, where all points other than P lie on a different number of chords joining these points. There are two cases:

(1) One point does not lie on any chord. This means that the points other than P must lie on 0, 1, 2, 3, 4, 5 and 6 chords. Label these points P_0, P_1, P_2, P_3, P_4, P_5 and P_6 respectively.

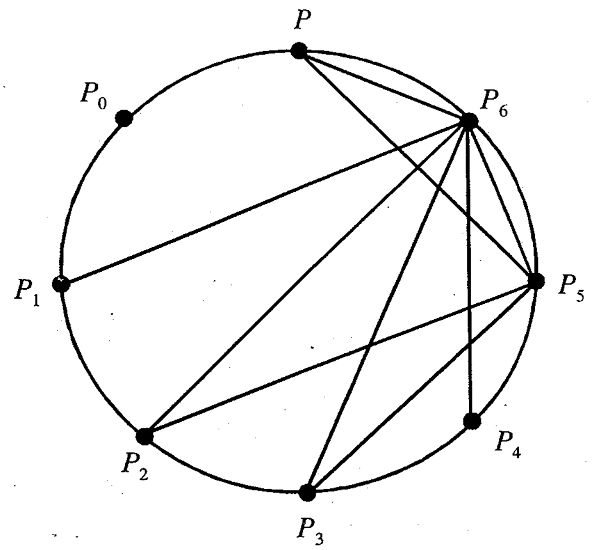
(2) One point is connected to every other point. This means that the points other than P must lie on 1, 2, 3, 4, 5, 6 and 7 chords. Label these points P_1, P_2, P_3, P_4, P_5, P_6 and P_7, respectively.

Case 1.

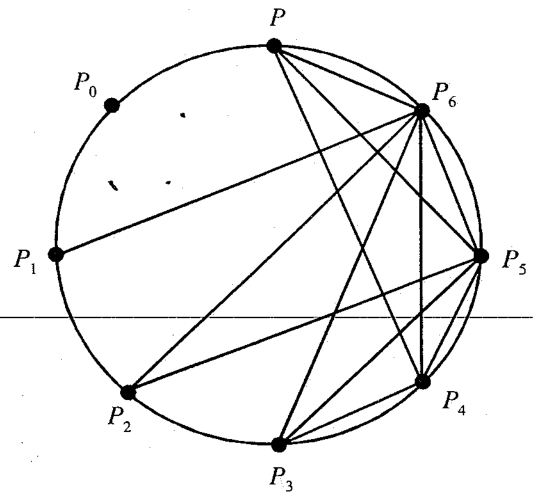
It does not matter in what order the points lie on the circle, so arrange them as shown and since P_6 is connected to every point other than P_0 we get the following diagram:



Now connect P_5. It cannot be connected to P_1 (already connected to P_6), so must connect to every other point other than P_0.

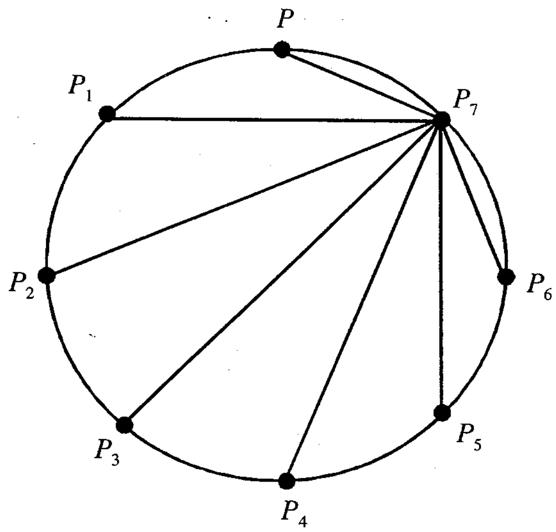


Now, noting that P_0, P_1, P_2, P_5 and P_6 are on their limit, P_4 can only connect to P_3 and P and this completes the diagram as follows:

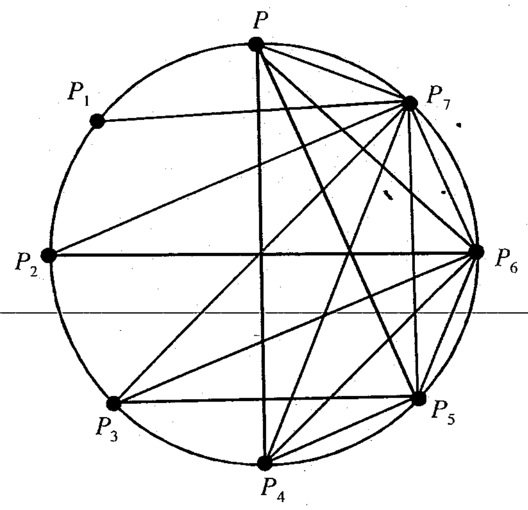


This case shows that P must lie on 3 chords.

Case 2.
As before, it does not matter in what order the points lie on the circle, so arrange them as shown and since P_7 is connected to every point we get the following diagram:



Now, continuing with P_6 and so on in a similar fashion to Case 1, we obtain the following completed diagram:



In this case P lies on 4 chords.
The minimum number of chords on which P lies is then 3 from Case 1,

hence (C).

26. (Also I27 & J27)
There are 18 possible digit sums for two-digit numbers: 1, 2, 3, . . ., 17, 18. There is only one two-digit number with digit sum 1, namely 10; and there is only one two-digit number with digit sum 18, that is 99. Therefore the largest number of two-digit numbers that can be written on the whiteboard, such that no three numbers have the same digit sum, is $1 \times 2 + 2 \times 16 = 34$. Thus the smallest number of students in the class for the teacher to be correct is 35.

27. (Alternative 1)
We are given

$$a+b+c=4 \qquad (1)$$
$$a^2+b^2+c^2=10 \qquad (2)$$
$$a^3+b^3+c^3=22 \qquad (3)$$

Squaring (1) and subtracting (2) we get $2(ab+bc+ca)=6$ and $ab+bc+ca=3$.
So

$$\begin{aligned} a^3+b^3+c^3-3abc &= (a+b+c)(a^2+b^2+c^2-ab-bc-cb) \\ 22-3abc &= 4(10-3) \\ abc &= -2 \end{aligned}$$

So, using the result that for a cubic equation $x^3+px^2+qx+r=0$ with roots a, b, c, that $a+b+c=-p$, $ab+bc+ca=q$ and $abc=-r$, tells us that a, b and c are roots of the equation

$$x^3-4x^2+3x+2=0$$
$$(x-2)(x^2-2x-1)=0.$$
$$x=2,\ 1\pm\sqrt{2}$$

Now $$\left(1+\sqrt{2}\right)^4=1+4\sqrt{2}+6\times2+4\times2\sqrt{2}+4$$
$$\left(1-\sqrt{2}\right)^4=1-4\sqrt{2}+6\times2-4\times2\sqrt{2}+4$$
$$2^4=16$$

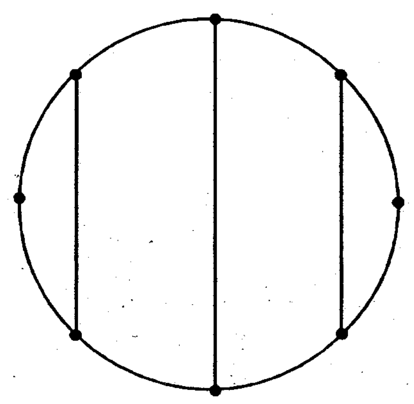
Then $$a^4+b^4+c^4=16+2(1+12+4)$$
$$=50.$$

(Alternative 2)

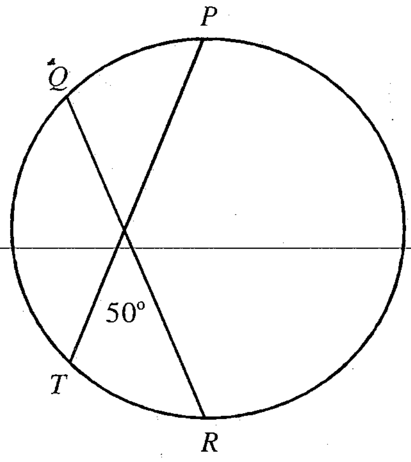
$$\left(a^2+b^2+c^2\right)^2=a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)$$
$$=100$$
$$(ab+bc+ca)^2=a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)$$
$$9=a^2b^2+b^2c^2+c^2a^2-2\times2\times4$$
$$a^2b^2+b^2c^2+c^2a^2=9+16=25$$
$$a^4+b^4+c^4=100-2\times25=50$$

Comment
This solution does not show that there exist real numbers a, b and c which satisfy the given conditions, but it would seem reasonable to assume in the context of the question that they do.

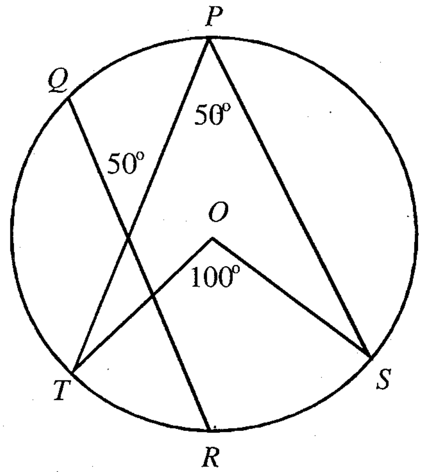
28. (Also I29)
(Alternative 1)
The question does not stipulate that the two diagonals must share a common vertex. However, for each diagonal in a regular polygon with more than 5 sides, there are parallel diagonals through other vertices. An example follows:



Consider the case where a pair of diagonals PT and QR intersect at an angle of $50°$ as in the following diagram.



Then, if arc $PQ <$ arc PT, then there must exist a point S on arc PT such that $PS \parallel QR$. So TS subtends an angle of $50°$ at the circumference of the circle and also an angle of $100°$ at the centre.



It follows that the angle subtended at the centre of the circle by a single side of the polygon must be a divisor of $100°$ and also of $360°$. The highest common factor of 100 and 360 is 20, so the smallest number of edges the polygon can have is $\frac{360}{20} = 18$.

(Alternative 2)

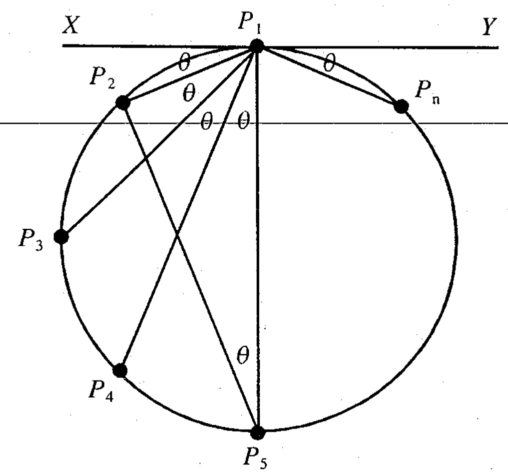
For the angle between two diagonals in a regular $n-$gon to be $50°$, there must be two vertices of this polygon such that they divide the perimeter of the polygon in ratio 50:130.

Hence there exists a postive integer a such that $\frac{a}{n-a} = \frac{5}{13}$.

Therefore $13a = 5n - 5a$, and $18a = 5n$. Since 5 and 18 are relatively prime, n must be

divisible by 18. So $n \geq 18$ and the example of $n = 18$, $a = 5$ shows that $n = 18$ is possible.
Hence the smallest value of n is 18.

(Alternative 3)
Inscribe the polygon $P_1, P_2, P_3, \ldots$ in a circle. Let XY be the tangent at P.



Angle $P_2P_1P_3$, angle $P_3P_1P_4$ and so on are equal (θ) as they are subtended by equal arcs at P_1.

Angle XP_1P_2 and angle YP_1P_n are also equal (to θ) by the alternate segment theorem.

Hence $\angle XP_1Y$ is divided into n equal angles by edges P_1P_2, P_1P_n and by all diagonals drawn from P_1 to the other $n - 1$ vertices of the polygon.

Each of these n angles is equal to $\dfrac{180°}{n}$.

All angles between diagonals are integer multiples of this basic angle as moving from P_1 to P_2 rotates all diagonals by $\dfrac{360}{n} = 2\times\dfrac{180}{n}$; so that each diagonal is parallel to a diagonal, edge or tangent at P_1 and this applies to each vertex.

So, to generate 50° between diagonals, we need to find the smallest value of n which makes $\dfrac{180}{n}$ a factor of 50.

The highest common factor of 180 and 50 is 10, so n is minimum when $\dfrac{180}{n} = 10$ and this is when $n = 18$.

29. (Also I30)
We are looking for a maximum product

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

where $n_1 + n_2 + n_3 + \cdots + n_k = 19$.

If any factor n_i is ≥ 5, it can be replaced by two factors 2 and $n_i - 2$ which leave the sum unchanged, but increases the product since $2 \times (x-2) > x$ for $x \geq 5$. So, every factor in the largest product is ≤ 4.

Similarly, if any factor n_i is equal to 4, it can be replaced by 2×2 with no change to the product, so we shall do this and then every factor is ≤ 3.

If any factor is 1, it can be combined with another factor, replacing $1 \times n_i$ by $(n_i + 1)$ which increases the product, so now all factors in the largest product are 2 or 3.

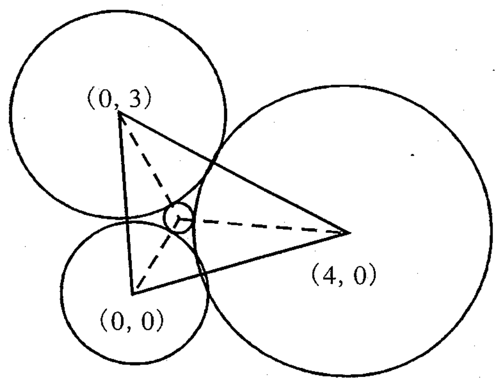
If there are three or more 2s, $2 \times 2 \times 2$ can be replaced by 3×3 to increase the product. So in the largest product, there are at most two 2s.

There is only one way that 19 can be written as such a sum: there are five 3s and two 2s.

So the maximum product is $3^5 \times 2^2 = 972$.

30. Joining the centres of the three main circles, we see that we have a 3-4-5 right-angled triangle.

Let us give its vertices coordinates as shown. Let us also say that the small circle in the middle has radius r and its centre has coordinates (x, y).



Now, looking at the lengths of the three dotted lines joining the outer vertices to the centre of the small circle, we have the equations

$$x^2 + y^2 = (1+r)^2 \qquad (1)$$
$$x^2 + (3-y)^2 = (2+r)^2 \qquad (2)$$
$$(4-x)^2 + y^2 = (3+r)^2. \qquad (3)$$

Subtracting the first one from each of the others gives

$$(3-y)^2 - y^2 = (2+r)^2 - (1+r)^2$$
$$(4-x)^2 - x^2 = (3+r)^2 - (1+r)^2,$$

which simplify to $3y = 3 - r$, $2x = 2 - r$.

Substituting from these back into (1) gives

$$\left(1-\frac{r}{2}\right)^2 + \left(1-\frac{r}{3}\right)^2 = (1+r)^2$$

which, upon multiplying by 36 and simplifying, gives

$$23r^2 + 132r - 36 = 0$$

Applying the usual formula, we have

$$r = \frac{-66+\sqrt{5184}}{23} = \frac{-66+72}{23} = \frac{6}{23} = \frac{p}{q}$$

(the other root is negative, which is impossible).
So, $p + q = 6 + 23 = 29$.

Solutions
2007 Senior Division

1. $2(5.61-4.5)=2(1.11)=2.22$,

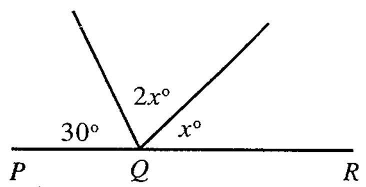
hence (D).

2.

$$2^n+2^n=2^m$$
$$2(2^n)=2^m$$
$$2^{n+1}=2^m$$
$$n+1=m,$$

hence (B).

3. Since PQR is a straight line, $30+2x+x=180$, $3x=150$



and $x=50$,

hence (C).

4. (Also JI0 & I5)

We can see that $\frac{7}{15}$, $\frac{3}{7}$ and $\frac{4}{9}$ are each less than $\frac{1}{2}$ while $\frac{6}{11}>\frac{1}{2}$ so $\frac{6}{11}$ is the largest,

hence (C).

5. (Also I7)

Since $7\times 89=623$, the length of the call was 7 minutes, and 7 minutes from 10:57 am is 11:04 am and so the call finished then,

hence (C).

6. Since $(2, k)$ lies on the 2 lines, we have

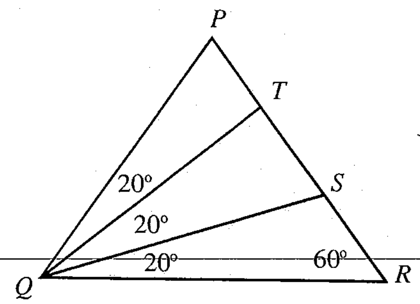
$$4+k=q \quad \text{and} \quad k=2-p.$$

So, $4+(2-p)=q$ and $p+q=6$,

hence (E).

7. (Also J17)

As QT and QS trisect $\angle PQR$ we get the angles as shown in the diagram:



Then $\angle QST = 80°$ (exterior angle) and so, from the angle sum of ΔQTS, $\angle QTS = 180° - 100° = 80°$

hence (C).

8. (Also J20)

The possibilities are:

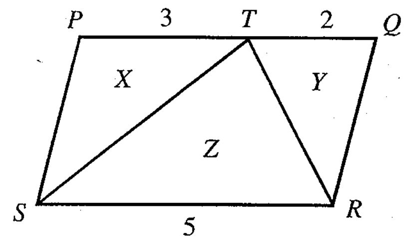
(A) 13 & 14. $14 = 2 \times 7$ which has 4 factors.
(B) 19 & 20. $20 = 2^2 \times 5$ which has 6 factors.
(C) 37 & 38. $38 = 19 \times 2$ which 4 factors.
(D) 43 & 44. $44 = 2^2 \times 11$ which has 6 factors.
(E) 53& 54. $54 = 2 \times 3^3$ which has 8 factors.

Andy's age has 8 factors, so two ages are 53 and 54,

hence (E).

9. (Also I14)

Let the areas of the triangles PTS, TQR and RST be X, Y and Z respectively.



These triangles have the same height h, so $X = \frac{3h}{2}$, $Y = \frac{2h}{2}$ and $Z = \frac{5h}{2}$.

So

$$\frac{X+Z}{X+Y+Z} = \frac{\frac{3h}{2}+\frac{5h}{2}}{\frac{3h}{2}+\frac{2h}{2}+\frac{5h}{2}} = \frac{4}{5},$$

hence (D).

10. Three of the numbers are 5, 8 and 8. Let the other two be x and y. Since the mean is 5,

$$\text{Mean} = \frac{1}{5}(x+y+21) = 5$$
$$\therefore\ x+y+21 = 25$$
$$x+y = 4$$

Since we have one mode of 8, $x \neq y$, the other two must be 1 and 3.
The difference between the largest and smallest is $8-1=7$,

hence (D).

---

11. The recommended dose is 4 drops per litre, so Dad had put in only 2 drops per litre.
After he used the first 2 litres, the remaining 6 litres contained 12 drops.
When he filled it up again, he had 8 litres with the 12 drops, so he needed to add 20 more drops,

hence (A).

12. (Also MP16 & UP 13 & J13 & I12)
Using the rules that each row, column and small square contains 1, 2, 3 and 4, we fill in the square one at a time, then

	2		
			1
	1	3	
4			

→

1	2		
			1
	1	3	
4	3		

→

1	2		
	4		1
2	1	3	
4	3		

→

1	2		3
3	4		1
2	1	3	4
4	3		

→

1	2	4	3
3	4	2	1
2	1	3	4
4	3	1	2

The sum of the numbers in the four corners is $1+4+2+3=10$,

hence (E).

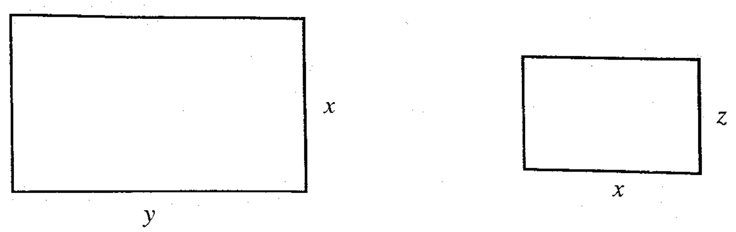
13. Holly writes down 11, 13, 17, 19, 31, 33, 37, 39, 71, 73, 77, 79, 91, 93, 97, 99.

The primes are 11, 13, 17, 19, 31, 37, 71, 73, 79 and 97.

So there are 10 primes in the sample space of 16, and the probability of a prime number is $\frac{10}{16}=\frac{5}{8}$,

hence (A).

14. Let the two rectangles have sides y, x and x, z where $y > x > z$.



As the combined area is 40 cm^2, we have $5 \geq x \geq 2$, since if $x = 6$ the area of the larger rectangle would be > 40 and if $x = 1$ the smallest rectangle would not exist.

Since the perimeter of the larger rectangle is twice that of the smaller, we get

$$x + y = 2(x + z)$$
$$z = \frac{y - x}{2}$$

The combined areas are 40 cm^2, so

$$yx + xz = 40$$
$$xy + \frac{x(y-x)}{2} = 40$$
$$3xy - x^2 = 80$$
$$x(3y - x) = 80$$

Since x divides 80, we consider $x = 2, 4, 5$.
$x = 2$ gives $3y = 42$, $y = 14$, $z = 6$, not possible, as $z = 6 > x$.
$x = 4$ gives $3y = 24$, $y = 8$, possible.
$x = 5$ gives $3y = 21$, $y = 7$, possible.

When $x = 4$, $y = 8$ and $z = 2$, and this results in two similar rectangles.

The only solution is when $x = 5$, $y = 5$ and $z = 1$,

hence (A).

15. Let the two-digit number be $10x + y$.
Reversing the digits gives $10y + x$.
Thus $10x + y + 10y + x = 11(x + y)$ where $1 \leq x + y \leq 18$.

For $11(x + y)$ to be perfect square, $x + y = 11$, and the possible numbers are 29, 38, 47, 45, 56, 65, 74, 83 and 92, eight numbers in all,

hence (D).

16. Number the seats from 1 to 6.

1	2	3	4	5	6	
A		B		C		6 ways
A		B			C	6 ways
A			B		C	6 ways
	A		B		C	6 ways

We can see that are 4 different combinations of three seats possible, and for each of these we can place Ann, Bill and Carol in 6 different ways, giving $4\times6=24$ different ways,

hence (B).

Comment

The question is equivalent to finding k pairwise non-consecutive numbers from $\{1, 2, 3, 4, ..., n\}$, which is $\binom{n-k+1}{k}$.

When $k = 3$ and $n = 6$ we get $\binom{4}{3}=4$, so there are 4 ways of selecting 3 seating positions where no two are alongside each other.

17. Consider the equation $a^b=1$.

There are three cases: case 1 when $b = 0$ and $a \neq 0$; case 2 when $a = 1$; and case 3 when $a=-1$ and b is an even integer.

Case 1: when $x + 1 = 0$, $x=-1$ and $x^2-3x+1=5\neq0$, so $x=-1$ is a solution.

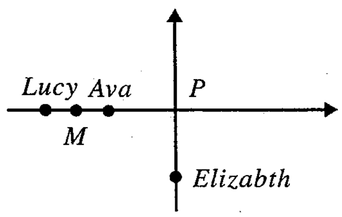
Case 2: when $x^2-3x+1=1$, $x^2-3x=0$ and $x(x-3)=0$, so $x = 0$ or 3, giving 2 solutions.

Case 3: when $x^2-3x+1=-1$, $x^2-3x+2=0$, $(x-2)(x-1)=0$, $x=2$ or 1. But $x+1$ must be even, and it is not for $x=2$ but it is for $x=1$.

So the solutions are $x=-1$, 0, 1 and 3,

hence (D).

18. Since Ava and Lucy jog along the same path at 8 km/h, the mid-point M between



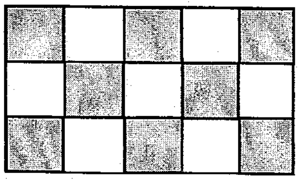
Lucy and Ava moves at 8 km/h and is 56 m from P when Ava is 50 m from P. M travels 56 m whilst Elizabeth travels

$$\frac{6}{8}\times56=42\text{ m},$$

hence (B).

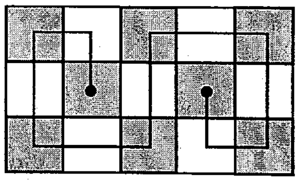
19. (Also J24)

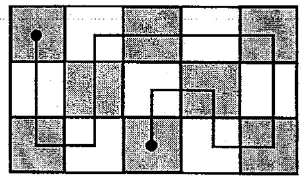
Colour the 15 squares black and white in the usual chessboard fashion with dark squares at all the corners. Then there are 8 black squares and 7 white squares.



Since the squares alternate in colour along the counter's path, the starting square and the end square have to be black. Hence it is impossible for any of the 7 white squares to be starting squares.

There are only three non-equivalent positions of black squares: the corner ones, the interior ones and the ones on the edge but not in a corner. The following two paths show that any black squares could have been a starting square.



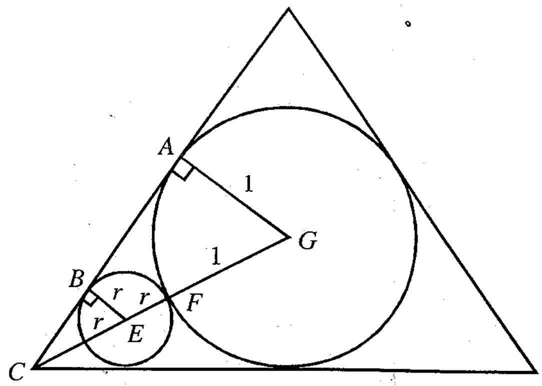


So there are 8 starting squares,

hence (D).

20. (Alternative 1)

Let the radius of the small circle be r. Both ΔAGC and ΔBEC are 90°, 60°, 30° triangles with sides in the ratio $2:\sqrt{3}:1$.

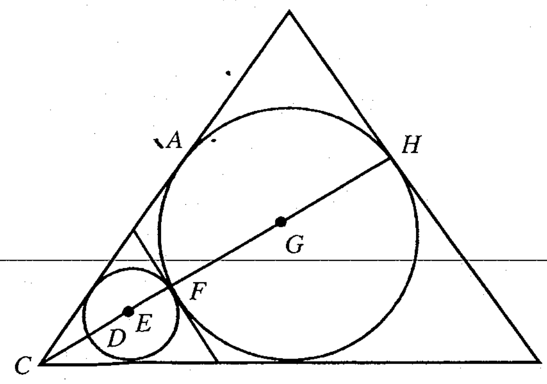


Hence $GC = 2$ and $CE = 2r$, so

$$GC = GF + FE + CE$$
$$2 = 1 + r + 2r$$
$$r = \frac{1}{3},$$

hence (A).

(Alternative 2)
Since G is the centroid of the larger triangle, $CF = FG = GH = 1$.



Since E is the centroid of the smaller triangle, $CD = DE = EF = \frac{1}{3}$.

hence (A).

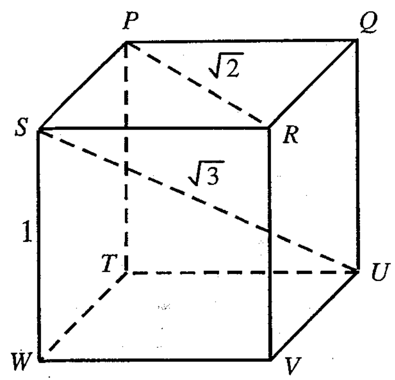
21. (Also J28& I24)
There are twelve lift stops altogether. Each pair of floors has a lift which connects them. Hence, as $\binom{6}{2} = 15 > 2$, there are at most five floors.

This is possible if the first lift stops on floors 1, 4 and 5, the second on 2, 4 and 5, the third on 3, 4 and 5, and the fourth on 1, 2 and 3.

So, the maximum number of floors is 5,

hence (B).

22. (Also I25),
Consider the cube *PQRSTUVW*.

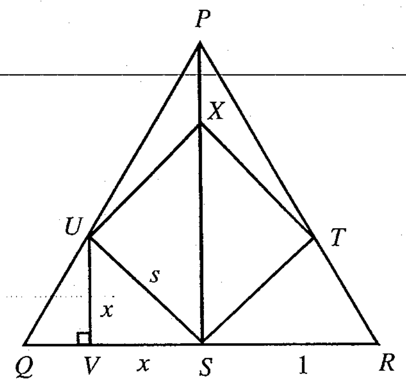


Since the bee flew so that it visited every vertex of the cube without being twice at the same point, the bee's path consists of exactly 7 straight line segments. The length of an edge is 1 unit, the length of a diagonal of a face is $\sqrt{2}$ and the length of a diagonal of the cube is $\sqrt{3}$.

The bee's path cannot have more than one diagonal of the cube as any two of them meet in the centre of the cube. So the largest possible length of such a path is at most $\sqrt{3}+6\sqrt{2}$. The example $PRUWQTSV$ shows that a path of such length does exist,

hence (D).

23. (Alternative 1)
Draw UV perpendicular to QR. Join PS. Let $UV = x$ and the side of the square be s.



Now, as PS bisects $\angle UST$, $\angle USV = \angle VUS = 45°$ and then $VS = x$, so $QV = 1-x$.
Now, the triangle QVU and QSP are similar (equiangular) and so

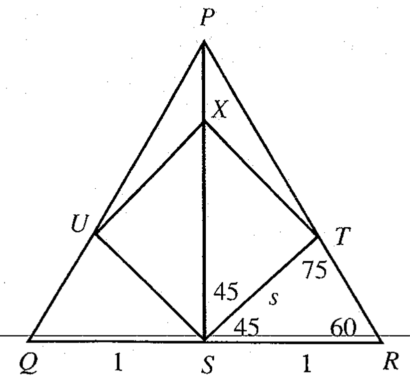
$$\frac{QV}{QS} = \frac{VU}{SP}$$
$$\frac{1-x}{1} = \frac{x}{\sqrt{3}}$$
$$x = \sqrt{3} - x\sqrt{3}$$
$$x = \frac{\sqrt{3}}{1+\sqrt{3}} = \frac{3-\sqrt{3}}{2}$$

Now, from the triangle VUS, we get

$$s^2 = 2x^2 = 2\frac{\left(3-\sqrt{3}\right)^2}{4}$$
$$= \frac{12-6\sqrt{3}}{2} = 6-3\sqrt{3},$$

hence (A).

(Alternative 2)
Join PS. Clearly $\angle TRS = 60°$ and $\angle TSR = 45°$, hence $\angle STR = 75°$.



By the sine rule, $\frac{s}{\sin 60°} = \frac{1}{\sin 75°}$.

Now, $\sin(A+B) = \sin A \cos B + \cos A \sin B$, so

$$\sin 75° = \sin(45° + 30°) = \frac{1}{\sqrt{2}} \times \frac{1}{2} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

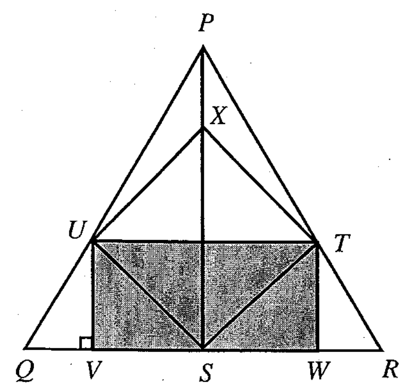
So, $s = \frac{\sin 60°}{\sin 75°} = \frac{\sqrt{3}/2}{(\sqrt{3}+1)/2\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{3}+1}$

and the area $s^2 = \frac{6}{4+2\sqrt{3}} = \frac{3}{2+\sqrt{3}} = \frac{3(2-\sqrt{3})}{4-3} = 6 - 3\sqrt{3}$,

hence (A).

(Alternative 3)

Draw UV and TW perpendicular to QR. Note that the area of $STXU$ is equal to that of the shaded rectangle.



Suppose that $UV = \sqrt{3}$, then $QU = 2$ and $UP = 2\sqrt{3}$, so that $PQ = 2 + 2\sqrt{3}$.

But, we are give that $PQ = 2$, so that $UV = \sqrt{3} \times \frac{2}{2+2\sqrt{3}} = \frac{3-\sqrt{3}}{2}$.

The area of the shaded rectangle is then $2UV^2 = 6 - 3\sqrt{3}$,

hence (A).

24. Given $f(x) = ax^2 + bx + c$ and $f(x)f(-x) = f(x^2)$ for all x, we get

$$(ax^2 + bx + c)(ax^2 - bx + c) \equiv ax^4 + bx^2 + c.$$

This gives $a^2 = a$, $2ac - b^2 = b$ and $c^2 = c$, and $a = 0$ or 1, $c = 0$ or 1.
Case 1: $a = 0$. Then $b = 0, -1$, $c = 0, 1$ and $f(x) = 0, 1, -x, 1-x$.
Case 2: $a = 1$. Then $c = 0$, $b = 0, -1$ and $a = c = 1$, $b = 1, -2$. These give $f(x) = x^2$, $x^2 - x$, and $f(x) = x^2 + x + 1$, $x^2 - 2x + 1$.
So these are 8 such functions,

hence (C).

25. (Alternative 1)

$$(\sqrt{2}+1)^1 = \sqrt{2}+1$$
$$(\sqrt{2}+1)^2 = 2\sqrt{2}+3$$
$$(\sqrt{2}+1)^3 = 5\sqrt{2}+7$$
$$(\sqrt{2}+1)^4 = 12\sqrt{2}+17$$
$$(\sqrt{2}+1)^5 = 29\sqrt{2}+41$$
$$(\sqrt{2}+1)^6 = 70\sqrt{2}+99$$
$$(\sqrt{2}+1)^7 = 169\sqrt{2}+239$$
$$(\sqrt{2}+1)^8 = 408\sqrt{2}+577$$
$$(\sqrt{2}+1)^9 = 595\sqrt{2}+1393$$

Modulo 3 the coefficients behave as
1, 1; 2, 0; 2, 1; 0, 2; 2, 2; 1, 0; 1, 2; 0, 1; 1, 1.
This pattern clearly repeats every 8 powers $(2007 = 8 \times 250 + 7)$.
Hence, if $(\sqrt{2}+1)^{2007} = a + b\sqrt{2}$ then $b \equiv 1 \pmod 3$.
So, the highest common factor of b and 81 is 1,

hence (A).

(Alternative 2)
Since $(\sqrt{2}+1)^{2007} = a + b\sqrt{2}$, we have $(\sqrt{2}-1)^{2007} = b\sqrt{2} - a$.
By multiplying these two equations we obtain

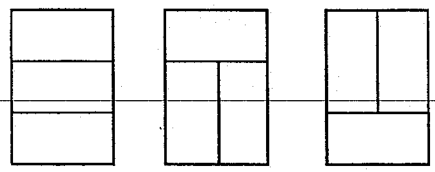
$$(\sqrt{2}+1)^{2007}(\sqrt{2}-1)^{2007} = (a + b\sqrt{2})(b\sqrt{2} - a)$$

Hence $1 = 2b^2 - a^2$.

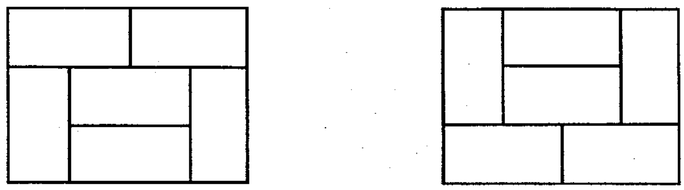
If b is divisible by 3, then a^2 is congruent to -1 modulo 3, which is not possible, hence (A).

26. A 3 by 6 rectangle can be broken down into three 3 by 2 rectangles, one 3 by 4 rectangle and one 3 by 2 rectangle, or one 3 by 6 rectangle.

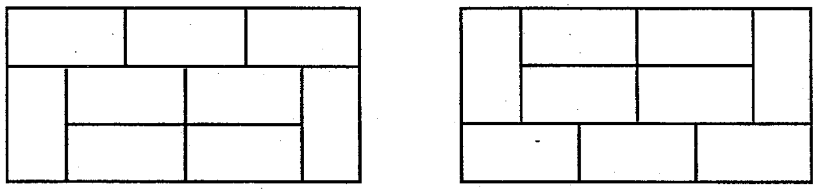
For the 3 by 2 rectangles we have three tilings as shown.



For the 3 by 4 rectangle, we can obtain nine tilings by combining two tilings for the 3 by 2 rectangles. However, there are also 2 solutions which cannot be obtained this way. These are called irreducible tilings and they are shown below.



For the 3 by 6 rectangle, we also have two irreducible tilings as shown below.

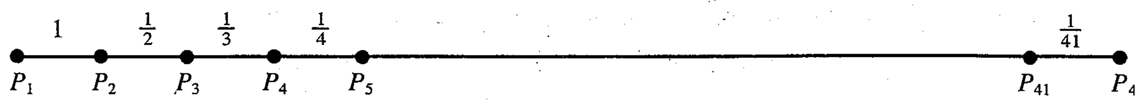


By combining the three different tilings to the 3 by 2 rectangles we get $3\times3\times3=27$ different tilings.

By combining a tiling for the 3 by 2 rectangle with an irreducible 3 by 4 rectangle we get $2\times3\times2=12$ different tilings (we can put the 3 by 2 at either end of the 3 by 4).

Adding the 2 irreducible tilings for the 3 by 6 rectangle, we get $2 + 12 + 27 = 41$ different tilings.

27. Since the distance from P_i to P_{i+1} is $\frac{1}{i}$ we get the distances between successive point to be



P_1P_2 occurs in 1×41 distances: P_1P_2, P_1P_3, P_1P_4 and so on.

P_2P_3 occurs in 2×40 distances: P_1P_3, P_2P_3, P_1P_4, P_2P_4 and so on.
P_3P_4 occurs in 3×39 distances.
$\vdots$
$P_{41}P_{42}$ occurs in 41×1 distances.

Hence the sum of all these distances

$$\begin{aligned} &= 41\times 1\times 1 + 40\times 2\times\frac{1}{2} + 39\times 3\times\frac{1}{3} + \cdots + 41\times 1\times\frac{1}{41} \\ &= 41+40+39+\cdots+3+2+1 \\ &= 21\times 41 = 861. \end{aligned}$$

28. (Also I28)
Let $10a + b$ be a number with at most two digit.

The equation $10a + b = 19(a + b)$ cannot hold unless $a = b = 0$. So all lucky numbers have at least three digits.

Suppose a lucky number has m digits for some $m \geq 4$. Then its digit sum is at most $9m$ while the number is at least 10^{m-1}. Hence $171m \geq 10^{m-1}$.

For $m = 4$, $684 \geq 1000$ is false, so these are no lucky numbers with 4 digits.

For $m \geq 5$, the situation is worse. Hence all lucky numbers have exactly three digits.

Suppose the number is abc.

Then $100a + 10b + c = 19a + 19b + 19c$, so $81a = 9b + 18c$ or $9a = b + 2c$.

For $a = 1$, we have $(b, c) = (1, 4), (3, 3), (5, 2), (7, 1)$ and $(9, 0)$.
For $a = 2$, we have $(b, c) = (0, 9), (2, 8), (4, 7), (6, 6)$ and $(8, 5)$.
For $a = 3$, we have $(b, c) = (9, 9)$, and there are no other solutions.

Hence there are exactly 11 lucky numbers, namely, 114, 133, 152, 171, 190, 209, 228, 247, 266, 285 and 399.

29. (Also I30)
The digits in base 10 which can be read upside down are 0, 1, 2, 5, 6, 8 and 9.

Then, writing numbers which can be read upside down is like writing numbers in base 7 using just those digits.

Writing 2007 in base 7 is $5\times 7^3 + 5\times 7^2 + 6\times 7 + 5 = 5565$.

But, in this pseudo base 7, the 5 is replaced by 8 and the 6 by 9.

So, 5565 is written as 8898 and is the 2007th number to be read upside down. The last three digits are 898.

30. (Alternative 1)
Given

$$x+y=3(z+u) \qquad (1)$$
$$x+z=4(y+u) \qquad (2)$$
$$x+u=5(y+z) \qquad (3)$$

Rewrite as

$$x+y=3z+3u \qquad (4)$$
$$x-4y=-z+4u \qquad (5)$$
$$x-5y=5z-u \qquad (6)$$

Then, $(4)-(5)$, $(5)-(6)$ gives

$$5y=4z-u \qquad (7)$$
$$y=-6z+5u \qquad (8)$$

Hence

$$5(-6z+5u)=4z-u$$
$$26u=34z$$
$$13u=17z.$$

If we let u = 17 and z = 13, the smallest possible values of z and u then, from (8) we find $y=-78+85=7$ and from (4) we find $x=39+51-7=83$. So 83 is the smallest possible value of x.

(Alternative 2)
Given

$$x+y=3(z+u) \qquad (1)$$
$$x+z=4(y+u) \qquad (2)$$
$$x+u=5(y+z) \qquad (3)$$

From (1) it follows that $x+y+z+u=4(z+u)$.

Similarly, from (2) and (3) we have $x+y+z+u=5(y+u)$

and $x+y+z+u=6(y+z)$.

Thus if $S=x+y+z+u, 4|S, 5|S$ and $6|S$ so that $60|S$. Setting $S=4\cdot5\cdot6=120$ we obtain

$$x+y=3(z+u)=\frac{3}{4}S=90$$
$$x+z=4(y+u)=\frac{4}{5}S=96$$
$$x+u=5(y+z)=\frac{5}{6}S=100$$

Furthermore,

$$\begin{aligned} x &= \frac{(x+y)+(x+z)+(x+u)-(x+y+z+u)}{2} \\ &= \frac{90+96+100-120}{2} = \frac{166}{2} = 83\,. \end{aligned}$$

If we set $S = 60 = \text{LCM}[4,5,6]$ we do not obtain an integer value for x, so 83 is the smallest possible value for x.

Comment

This question is a variation of a problem from Iamblichus of Chalcis (c.326).

Solutions
2008 Senior Division

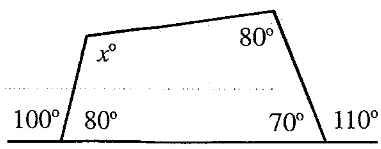
1. $8002 - 2008 = 5994$,©

hence (E).

2. The difference between $\frac{1}{20}$ and $\frac{2}{20}$ is $\frac{2}{10} - \frac{1}{20} = \frac{4-1}{20} = \frac{3}{20}$,

hence (E).

3. Using supplementary angles and the angle sum of a quadrilateral, we get



$x = 360 - 80 - 80 - 70 = 130$,

hence (D).

4. (Also J6 & I4)

$$\frac{200\times 8}{200\div 8} = \frac{200\times 8\times 8}{200} = 8\times 8 = 64,$$

hence (D).

5. *Alternative 1*

As $x^2 - 4x + 3 = (x-3)(x-1)$, the minimum value occurs when $x = 2$ and is $1 \times -1 = -1$,

hence (A).

Alternative 2

$x^2 - 4x + 3 = (x-2)^2 - 1$, so the minimum value is -1,

hence (A).

6. When the money is divided the two shares are \$1.75 and \$1.25.
The ratio of the larger to smaller is $175 : 125 = 7 : 5$,

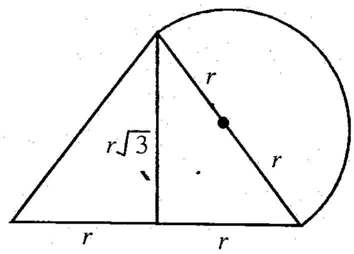
hence (B).

7. (Also I10)

When 1000^{2008} is written as a numeral, it consists of the digit 1 followed by 3×2008 zeros, so there are $1 + 3 \times 2008 = 6025$ digits,

hence (C).

8. Let the radius of the circle be r. Then the base of the triangle is $2r$ and, by Pythagoras, the height of the triangle is $r\sqrt{3}$.



The area of the semicircle is $\frac{\pi r^2}{2}$.

The area of the triangle is $\frac{2r \times r\sqrt{3}}{2} = r^2\sqrt{3}$.

So, ratio of the area of the semicircle to that of the triangle is

$$\frac{\pi r^2}{2} : r^2\sqrt{3} = \pi : 2\sqrt{3}$$

hence (B).

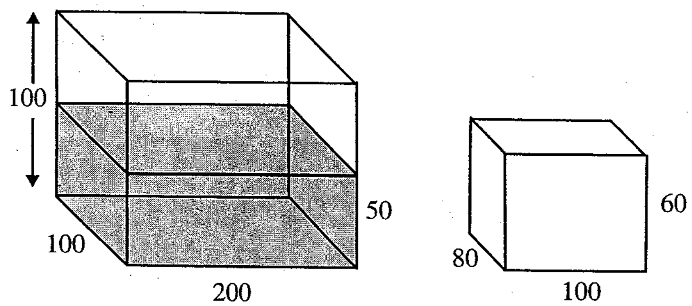
9. As $\cos x = 0.5 = \frac{1}{2}$, $\cos^2 x = \frac{1}{4}$ and so $\sin x = \frac{\sqrt{3}}{2}$. Then $\tan x = \frac{\sqrt{3}}{2} \Big/ \frac{1}{2} = \sqrt{3}$.

So $\tan x = \sqrt{3}$ is the largest,

hence (E).

10. (Also J24 & I15)

The volume of the water is $100 \times 200 \times 50 = 1\,000\,000 \text{ cm}^3$.



The volume of the prism is $80 \times 100 \times 60 = 480\,000 \text{ cm}^3$.

So, when the prism is placed in the tank, the new height of water is

$$\frac{1480000}{20000} = 74 \text{ cm}.$$

The prism is 60 cm high so is covered by 14 centimetres of water,

hence (B).

11. Now $2^{500} = 32^{100}$, $3^{400} = 81^{100}$, $4^{300} = 64^{100}$, $5^{200} = 25^{100}$ and $6^{100} = 6^{100}$.

The largest is $81^{100} = 3^{400}$,

hence (B).

12. In this problem, the distribution is symmetric, unimodal and the centre point is a valid point and so the most likely value and expected value are the same.
The expected outcome is

$$1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=\frac{21}{6}=\frac{7}{2}$$

So, in 100 throws, the expected score would be $100\times\frac{7}{2}=350$.

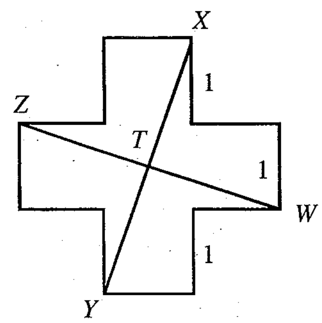
hence (D).

---

13. (Also I16)
Factorising, we get $2008 = 2^3 \times 251$ where 251 is prime. So, to complete to a square, we must multiply by $2 \times 251 = 502$,

hence (D).

14. Draw in the ZW line as shown.



The point where XY and ZW intersect is T and, by symmetry, XT, WT, YT and ZT are the same length.
Since the sides of the rectangle are XY and XT, the ratio is 2 : 1,

hence (C).

15. Given $f(x) = ax^2 + bx + c$, we have

$$f(1) = a + b + c = 2 \qquad (1)$$
$$f(2) = 4a + 2b + c = 3 \qquad (2)$$
$$f(3) = 9a + 3b + c = 1 \qquad (3)$$

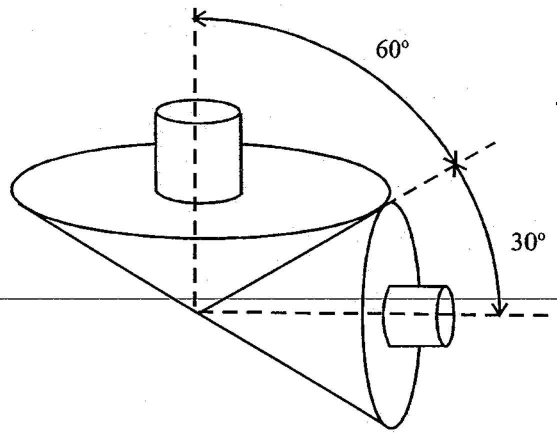
which gives

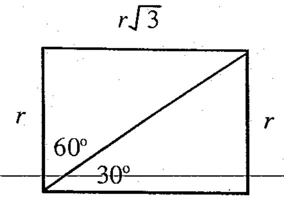
$$3a + b = 1 \qquad (2)-(1) \qquad (4)$$
$$5a + b = -2 \qquad (3)-(2) \qquad (5)$$
$$2a = -3 \qquad (5)-(4) \qquad (6)$$

Thus $a=-\frac{3}{2}$, $b=\frac{11}{2}$ and $c=-2$,

hence (A).

16. Let the radius of the smaller roller be r.





Then the radius of the larger roller is $r\sqrt{3}$.

So, the circumference of the larger roller is $2\pi r\sqrt{3}$ and that of the smaller roller is $2\pi r$.

So, when the larger roller makes 1 revolution, the smaller roller makes $2\pi r\sqrt{3}/2\pi r = \sqrt{3}$ revolutions,

hence (D).

17. Consider the sets with 1, 2 and 3 elements.
1 element: 1, 2, 3, 4, 5, 6 giving 6
2 elements: 1, 3; 1, 4; 1, 5; 1, 6; 2, 4; 2, 5; 2, 6; 3, 5; 3, 6 and 4, 6 giving 10
3 elements 1, 3, 5; 1, 3, 6; 1, 4, 6 and 2, 4, 6 giving 4.
Any set of 4 elements must contain consecutive numbers.

So there are 20 subsets in all,

hence (C).

18. (Also J23 & I21)
The amount of water collected is proportional to the areas of the two roofs. So
volume collected on farmhouse roof : volume collected on barn roof
is 200 : 80 = 5 : 2.

So, if Farmer Taylor is to collect as much water as possible, the empty space in the tanks has to be in the same ratio 5 : 2.

Currently, there is 100 − 35 = 65 kL available in the farmhouse tank and 25 − 13 = 12 kL in the barn tank.

Now, the ratio 65 : 12 is greater than 5 : 2, so we must pump some water from the barn tank into the house tank.

If we pump x kL from the barn tank into the house tank, then the empty space in the house tank is $65 - x$ kL and the empty space in the barn tank is $12 + x$ kL.

So, we want $65 - x : 12 + x = 5 : 2$, which gives $\dfrac{65-x}{5} = \dfrac{12+x}{2}$,

$130 - 2x = 60 + 5x$, $7x = 70$ and $x = 10$.

So, to collect the maximum amount of water possible we must pump 10 kL from the barn tank into the farmhouse tank,

hence (D).

19. More generally, suppose that

$$u_1 = a,\ u_2 = b,\ u_n = u_{n-1} - u_{n-2}\ (n \geq 3),$$

where a and b are fixed real numbers.

Then

$$\begin{array}{ll} u_3 = b - a & u_4 = b - a - b = -a \\ u_5 = -a - (b - a) = -b & u_6 = -b - (-a) = a - b \\ u_7 = a - b - (-b) = a & u_8 = a - (a - b) = b \end{array}$$

and the sequence repeats with period 6.

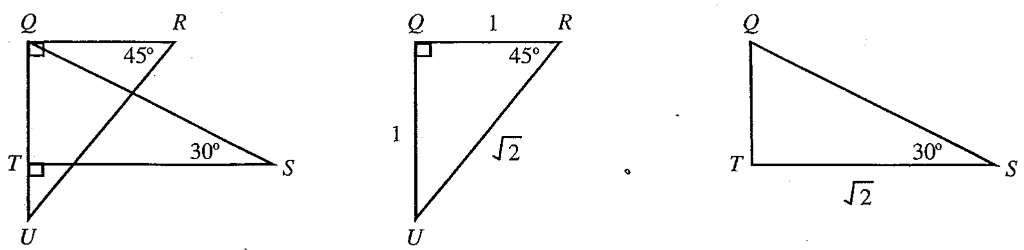
Since $2008 \equiv 4 \pmod 6$, $u_{2008} = u_4 = -a$.

So, in this case, $u_{2008} = u_4 = -\sqrt{2}$,

hence (A).

20. Let $QR = 1$, then $RU = ST = \sqrt{2}$.

From the right-angled triangle QTS, we get $\dfrac{QT}{\sqrt{2}} = \tan 30° = \dfrac{1}{\sqrt{3}}$.



Therefore the area of $\Delta QST = \frac{1}{2} \times \sqrt{2} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

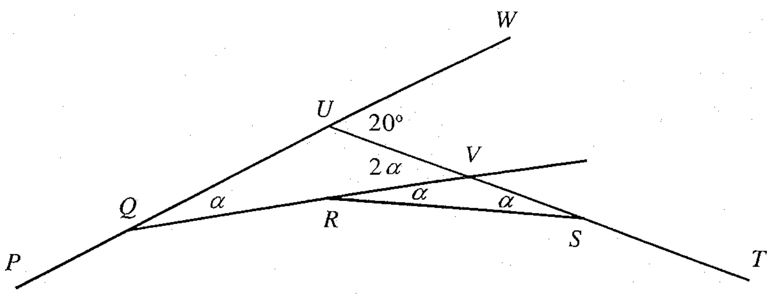
The area of $\Delta QRU = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$.

So, the ratio of area ΔQRU : area $\Delta QST = \frac{1}{2} : \frac{1}{\sqrt{3}} = \sqrt{3} : 2$,

hence (D).

21. *Alternative 1*

The equal sides PQ, QR, RS and ST are as shown. Then the sides PQ and TS extended meet at U at an angle of 20°.



Let α be the exterior angle of the polygon.
Then $\angle QVU = 2\alpha$ (exterior angle of a triangle) and $\angle VUW = 3\alpha = 20°$.
As

$$3\alpha = 20°, \quad \alpha = \frac{20}{3} = \frac{360}{54},$$

so the polygon has 54 sides,

hence (E).

Alternative 2
In the regular polygon, the sides PQ and TS meet at U where $\angle QUS = 160°$.
Consider the regular polygon as being generated by rotating a side n times.
Three steps rotate the side 20°, so $18 \times 3 = 54$ steps return the side to its original position. So, there are 54 sides,

hence (E).

22. If $a^3 \le 2008$ then $a \le 12$, as $12^3 = 1728 < 2008 < 2197 = 13^3$.

We have to count the numbers divisible by 2^3, 3^3, 4^3, 5^3, 6^3, 7^3, 8^3, 9^3, 10^3, 11^3 and 12^3.

Now $\dfrac{2008}{8} = 251$, so there are 251 numbers divisible by 2^3. This count also includes numbers divisible by 4^3, 6^3, 8^3, 10^3 and 12^3.
The notation $\lfloor x \rfloor$ means the integer part of x.

There are $\left\lfloor \dfrac{2008}{27} \right\rfloor = 74$ numbers divisible by 3^3. This includes numbers divisible by 6^3, 9^3 and 12^3.

There are $\left\lfloor \dfrac{2008}{125} \right\rfloor = 16$ numbers divisible by 5^3 which include those divisible by 10^3.

There are $\left\lfloor \dfrac{2008}{343} \right\rfloor = 5$ numbers divisible by 7^3 and 1 number divisible by 11^3.

But we have counted some numbers twice, those divisible by 6^3, 10^3 and 12^3, so we must subtract $\left\lfloor \dfrac{2008}{216} \right\rfloor = 9$ and $\left\lfloor \dfrac{2008}{1000} \right\rfloor = 2$. (Those divisible by 12^3 have been counted in these.)

So the number of cubic factors is $251 + 74 + 16 + 5 + 1 - 9 - 2 = 336$,

hence (B).

23. *Alternative 1*

Suppose a is a 3-digit palindrome, b is a 3-digit palindrome and $a-b$ is a 3-digit palindrome. Then, with x, y, u and v all single digit positive numbers,

$$\begin{aligned} a &= 100x + 10y + x \\ b &= 100u + 10v + u \\ a-b &= 100(x-u) + 10(y-v) + (x-u) \end{aligned}$$

Then it follows that $x-u > 0$ and $y-v \geq 0$.
So, we need the number of pairs (x, u) with $9 \geq x > u > 0$ times the number of pairs (y, v) with $9 \geq y \geq v \geq 0$.

The first number is $\binom{9}{2} = 36$, since $1 \leq x \leq 9$, $1 \leq u \leq 9$ and $x > u$.

The second number is $\binom{10}{2} + 10 = 45 + 10 = 55$, since $y = v$ or $0 \leq y \leq 9$, $0 \leq v \leq 9$ and $y > v$.
So, the total number is $36 \times 55 = 1980$,

hence (B).

Alternative 2

Let the numbers be xyx and uvu, where x, y, u and v are single-digit positive numbers.
Now, for $xyx - uvu$ to be a palindrome which is 3-digit, $x > u$ and $x \leq 9$ and $u \geq 1$, while at the same time $y \geq v$ and $y \leq 9$ and $v \geq 0$.

If $x = 9$ there are 8 possibilities for u.
If $x = 8$ there are 7 possibilities for u.
$\vdots$
If $x = 2$ there is 1 possibility for u.
So, there are $8 + 7 + 6 + \ldots + 2 + 1 = 36$ possible values for x and u.

If $y = 9$, there are 10 possibilities for v (0 to 9).
If $y = 8$, there are 9 possibilities for v (0 to 8).
$\vdots$
If $y = 1$, there are 2 possibilities for v (0 and 1).
If $y = 0$, v must be 0, so there is one possible value for v.
So, there are $10 + 9 + 8 + \ldots + 1 = 55$ values of y and v.

So, the total number of pairs is $36 \times 55 = 1980$,

hence (B).

Alternative 3

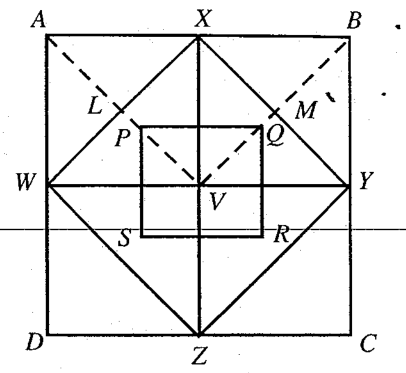
The valid pairs have the form $a = xyx$ with $2 \leq x \leq 9$ and $0 \leq y \leq 9$, and with $b = uvu$ where $1 \leq u \leq x-1$ and $0 \leq v \leq y$.
This gives $(x-1)(y-1)$ pairs with $a = xyx$.
So, for each x in the range 2 to 9 there are 55 choices for y making

$$(1 + 2 + 3 \cdots + 8) \times 55 = 36 \times 55 = 1980$$

pairs in all,

hence (B).

24. Consider the view of the figure through one face of the cube (which is a projection of all the points and lines onto this face of the cube).



$ABCD$ is one face of the larger cube.
X, Y, Z, W and V are five of the six vertices of the octahedron.
PQ is an edge of the smaller cube. (This corresponds to a projection of all points onto the face $ABCD$. However, the projected PQ has the same length as the original PQ as PQ is parallel to the face $ABCD$.)

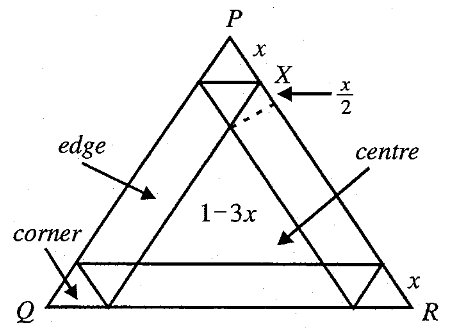
P is the centroid (centre) of the $\triangle VXW$, hence $PV : LP = 2 : 1$.

Also $AL = LV$, so $PV : AV = 2 : (2 + 1 + 3) = 1 : 3$.

This means that $PQ : AB = 1 : 3$,

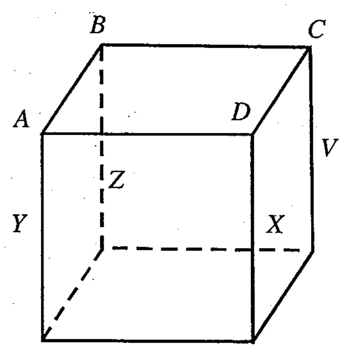
hence (D).

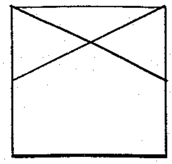
25. There are only three different areas, corner, edge and centre. Clearly the edge area is bigger than the corner area. So we are trying to maximise the smaller of the corner area and the central area.
If $PX = x$, then the two areas are an equilateral triangle of side x and another of side $1 - 3x$.
When these two areas are equal $x = 1 - 3x$
and so $PX = \frac{1}{4}$,



hence (C).

26. Assume the cube has an edge length 1. X, Y, V and Z are mid-points of edges.
Four lines AV, BX, DZ and CY, from the top face, produce 1 internal point inside the top half of the cube, which from symmetry, will be $\frac{1}{4}$ of a unit from the centre of the face.

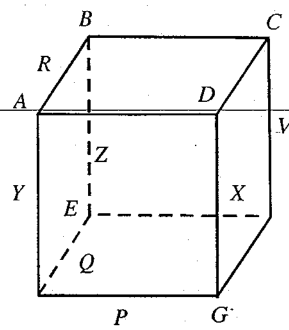


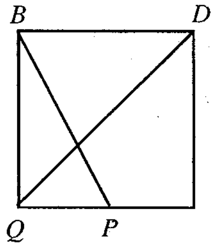


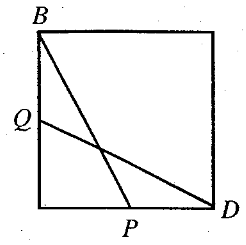
In a similar fashion the other 5 faces will produce a point to give 6 such points.

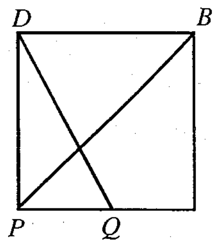
Also BP and DQ produce an internal point equidistant from the sides of the cube.

When we look at the front view, top view and side view of the cube (projections), we can see that this point of intersection is $\frac{1}{3}$ of a unit









from each face near the corner of the cube (from similar triangles with side ratio 2 : 1 in each case). There are 8 corners and so 8 such points.

Thus the number of points of intersection found so far is $6 + 8 = 14$.

To show that there are no more points, consider AV. There are three possible lines that join B with the midpoints of the edges and do not lie on the surface of the cube. For BX and AV we get one of the 14 points that are described above. For BP and AV, consider their projections on the face ADG. Since their projections have only one common point at a corner of the square, they cannot intersect inside the cube. Checking all other pairs of lines in a similar fashion, we see that there are no more points of intersection except the ones discussed above, and the answer is therefore 14.

27. (Also J29 & I29)

We need to find the biggest sums we can get for a given number of terms. Some experiments:

1 term	$1 = 1$
2 terms	$2 = 1 + 1$
3 terms	$4 = 1 + 2 + 1$
4 terms	$6 = 1 + 2 + 2 + 1$
5 terms	$9 = 1 + 2 + 3 + 2 + 1$
6 terms	$12 = 1 + 2 + 3 + 3 + 2 + 1$

We can see that we get different formulas for odd or even numbers of terms.

For $2n$ terms, the maximum sum is $n(n + 1)$.
For $2n + 1$ terms, the maximum sum is $(n + 1)^2$.

Now $44 \times 45 = 1980$ and $45 \times 45 = 2025$, so 88 terms will not get us there, but 89 looks as though it should.

We can show it does, by starting with the biggest 88 term sum:

$$1980 = 1 + 2 + 3 + \ldots + 88 + 88 + 87 + \ldots + 3 + 2 + 1$$

This is 28 short of what we want, so put in a term of 28 next to one of the two 28s already there and we have a sum of 2008 with a minimum of 89 terms.

28. Now

$$3x^2 - 8y^2 + 3x^2y^2 = 2008 \quad \Leftrightarrow \quad 3x^2y^2 + 3x^2 - 8y^2 = 2008$$
$$3x^2y^2 + 3x^2 - 8y^2 - 8 = 2008 - 8 \quad \Leftrightarrow \quad 3x^2(y^2 + 1) - 8(y^2 + 1) = 2000$$
$$\Leftrightarrow \quad (3x^2 - 8)(y^2 + 1) = 2000 = 2^4 \times 5^3.$$

The factors of 2000, arranged in corresponding pairs, are

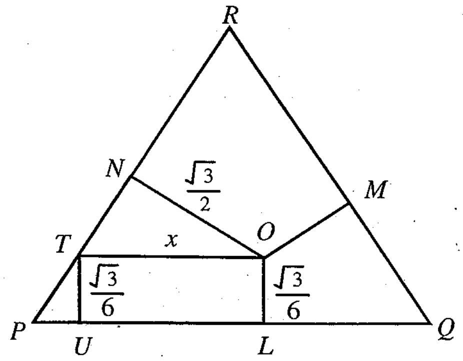
(1,2000), (2,1000), (4,500), (5,400), (8,250),
(10,200), (16,125), (20,100), (25,80), (40,50).

The only pair containing one number of the form $3x^2 - 8$ and the other of form $y^2 + 1$ is (40,50), where

$$3x^2 - 8 = 40 \text{ gives } 3x^2 = 48 \text{ and } x^2 = 16, x = 4.$$
$$y^2 + 1 = 50 \text{ gives } y^2 = 49 \text{ and } y = 7,$$

noting that x and y are positive integers.
Hence $xy = 4 \times 7 = 28$.

29. *Alternative 1*



Let the side of the equilateral triangle PQR be 2 units. Then the altitude of the triangle is $\sqrt{3}$ units and the area of the triangle is $\sqrt{3}$ square units.
Draw OT parallel to QP to meet RP at T and draw TU perpendicular to PQ. Now, by considering the areas of the triangles POQ, QOR and ROP, which combine to give the area of $\triangle PQR$, their altitudes combine to give that of $\triangle PQR$, which is $\sqrt{3}$.

So, $OL = \dfrac{\sqrt{3}}{6}$ and $ON = \dfrac{\sqrt{3}}{2}$.

Triangles ONT and TPU are 90°, 60° and 30° triangles.

So, $\dfrac{\sqrt{3}}{2} \div x = \dfrac{\sqrt{3}}{2}$ and $x = 1$.

So, the area of $\triangle ONT$ is $\frac{1}{2}\times\frac{\sqrt{3}}{2}\times\frac{1}{2}=\frac{\sqrt{3}}{8}$.

From $\triangle TPU$ we get $\frac{\sqrt{3}/6}{PU}=\sqrt{3}$, $PU=\frac{1}{6}$ and the area of $\Delta TPU=\frac{1}{2}\times\frac{1}{6}\times\frac{\sqrt{3}}{6}=\frac{\sqrt{3}}{72}$.

The area of rectangle $OTUL$ is $1\times\frac{\sqrt{3}}{6}=\frac{\sqrt{3}}{6}$.

So, the area of $LONP$ is

$$\frac{\sqrt{3}}{8}+\frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{72}=\frac{22\sqrt{3}}{72}=\frac{11\sqrt{3}}{36}$$

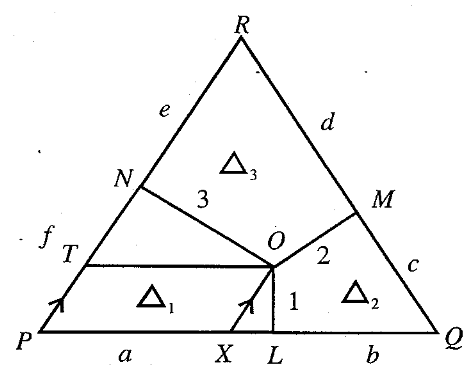
So, the ratio of the area of $LONP$ to the area of $\triangle PQR$ is

$$\frac{11\sqrt{3}}{36}:\sqrt{3}=\frac{11}{36}.$$

There are no common factors, so the sum of the numerator and denominator is $11+36=47$.

Alternative 2
Let Δ_1, Δ_2 and Δ_3 be the areas of the quadrilaterals.

$$2\Delta_1 = a + 3f$$
$$2\Delta_2 = b + 2c$$
$$2\Delta_3 = 2d + 3e$$



So, to calculate $\frac{\Delta_1}{\Delta_1+\Delta_2+\Delta_3}$

we only have to calculate the lengths of a, b, c, d, e and f.

Construct $XO \parallel PR$ with X on PL. $XL=\tan 30°=\frac{1}{\sqrt{3}}$.

$\frac{ON}{TO}=\cos 30°$ so $TO=XP=\frac{3}{\cos 30°}=\frac{6}{\sqrt{3}}$. Hence $a=PL=\frac{7}{\sqrt{3}}$.

Similarly, let $OT \parallel PL$ with T on PN. $\frac{TN}{3}=\tan 30°$, $TN=\frac{3}{\sqrt{3}}$.

$\frac{1}{XO}=\cos 30°=\frac{\sqrt{3}}{2}$. So, $PT=XO=\frac{2}{\sqrt{3}}$ and $f=\frac{3}{\sqrt{3}}+\frac{2}{\sqrt{3}}=\frac{5}{\sqrt{3}}$.

Similarly we can obtain $b=\frac{5}{\sqrt{3}}$, $c=\frac{4}{\sqrt{3}}$, $d=\frac{8}{\sqrt{3}}$ and $e=\frac{7}{\sqrt{3}}$.

Hence

$$\frac{\Delta_1}{\Delta_1+\Delta_2+\Delta_3} = \frac{a+3f}{a+3f+2d+3e+b+2c}$$
$$= \frac{(7+15)/\sqrt{3}}{(7+15+16+21+5+8)/\sqrt{3}}$$
$$= \frac{22}{7} = \frac{11}{36}.$$

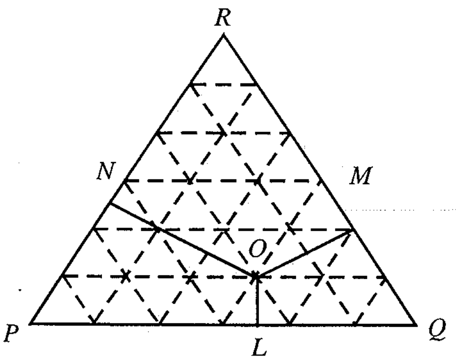
The sum of the numerator and denominator is 47.

Alternative 3

Let h be the altitude of $\triangle PQR$.
Considering the areas of the triangles OPQ, OPR and OQR, the sum of their altitudes is h, so $OL + OM + ON = 6OL$.

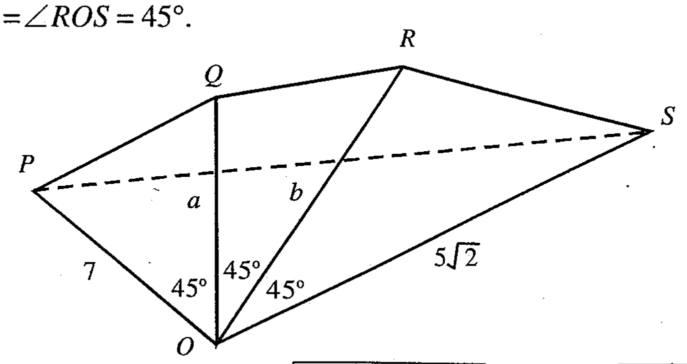
So $OL = \frac{h}{6}$, $OM = \frac{h}{3}$ and $ON = \frac{h}{2}$.

This places O on a triangular grid as shown.



So, counting the areas of the small triangles, we find that the area of $OLPN$ is $\frac{11}{36}$ of the area of $\triangle PQR$.
So the required number is $11 + 36 = 47$.

30. Consider a pentagon $OPQRS$ such that $OP = 7$, $OQ = a$, $OR = b$, $OS = 5\sqrt{2}$, and $\angle POQ = \angle QOR = \angle ROS = 45°$.



Then by the cosine theorem, $PQ = \sqrt{49 + a^2 - 7\sqrt{2}a}$, $QR = \sqrt{a^2 + b^2 - \sqrt{2}ab}$ and $RS = \sqrt{50 + b^2 - 10b}$.

By the triangle inequality, the smallest value that $PQ + QR + RS$ can have is PS, and by the cosine theorem it equals

$$\sqrt{OP^2 + OS^2 + \sqrt{2} \times OP \times OS} = \sqrt{49+50+70} = 13.$$

So the minimum value is 13.