Network Capacity under Traffic Symmetry: Wireline and Wireless Networks

Sudeep Kamath*, Sreeram Kannan†, Pramod Viswanath‡
*ECE Department, University of California, San Diego,
sudeep@eecs.berkeley.edu
†EECS Department, University of California, Berkeley,
kannan1@illinois.edu
‡Electrical and Computer Engineering, University of Illinois Urbana-Champaign,
pramodv@illinois.edu

***Abstract*—The problem of designing near optimal strategies for multiple unicast traffic in wireline networks is wide-open; however, channel symmetry or traffic symmetry can be leveraged to show that routing can achieve with a poly-logarithmic approximation factor of the edge-cut bound. For the same problem, the edge-cut bound is known to only upper bound rates of routing flows and unlike the information theoretic cut-set bound, it does not upper bound (capacity-achieving) information rates with general strategies. In this paper, we demonstrate that under *channel or traffic symmetry*, the edge-cut bound upper-bounds general information rates, thus providing a capacity approximation result. The key technique is a combinatorial result relating edge-cut bounds to generalized network sharing bounds.**

Finally, we generalize the results to wireless networks via an intermediary class of combinatorial graphs known as polymatroidal networks – our main result is that a natural architecture separating the physical and networking layers is near optimal when the traffic is symmetric among source-destination pairs, even when the channel is asymmetric (due to asymmetric power constraints, or prior frequency allocation like frequency division duplexing). This result is complementary to an earlier work of two of the authors proving a similar result under channel symmetry [1].

I. Introduction

The central problem of network information theory is to characterize the capacity region of a general network. Wireline networks are a special class of such networks where the edges between vertices are *unidirectional*, *orthogonal* and *noise-free*. In this class of networks, network coding has the potential to provide significant advantages in comparison to flow (i.e. routing strategies) for multicast problems [2] as well as for multiple unicast problems [3]. Recent results due to Dougherty, Freiling, Zeger and Chan, Grant suggest that characterizing the capacity region of a multiple unicast network is a hard problem [4], [5], [6]. In particular, even coding strategies such as linear codes do not achieve capacity in general [5].

On the other hand, the literature on hardness of cut problems typically deal with edge-cut bounds which are conventional outer bounds on flow. But these bounds are not fundamental bounds on the capacity region [7], i.e. they can often be beaten if network coding is allowed. [3] showed that the capacity region of a k-unicast network can be upto k times larger than edge-cut bounds. Although edge-cut bounds in directed networks are not fundamental, they are combinatorially well-represented. They are however, hard to approximate in general [8], [9].

One class of networks for which edge-cut bounds can be approximated well are undirected networks. Leighton and Rao show that for the problem of k-unicast in undirected networks, flow solutions approach the edge-cut bounds up to a factor of $\Theta(\log k)$ [10], [11]. There has also been discovered a semi-definite programming relaxation approach that allows an approximation of edge-cut bounds up to a factor of $\Theta(\sqrt{\log k})$ [12]. Interestingly, for undirected networks, edge-cut bounds can be derived from the vertex bipartition cutset bound and are hence, fundamental outer bounds on the capacity region. Thus, [10], [11] also characterize up to a factor of $\Theta(\log k)$ the capacity region of k-unicast in undirected networks. It has been conjectured that flow solutions in fact, achieve capacity [13], [14].

Another setting in which edge-cut bounds can be approximated well is the problem of multiple unicast in directed wireline networks with *symmetric demands*, i.e. for each source communicating to its destination at a certain rate, there is an independent message to be communicated from the destination back to the source at the same rate. Klein, Plotkin, Rao, Tardos [15] show under this model that flow solutions achieve within $\Theta\left(\log^3(k+1)\right)$ of the edge-cut bounds. We ask the question: “Are these edge-cut bounds fundamental outer bounds on the capacity region?” Surprisingly, the answer turns out to be yes and the proof of this result is one of the main contributions of this paper. The key tool we use in the proof is the Generalized Network Sharing (GNS) bound that was first developed in [16] for directed wireline networks and was used subsequently also for two-unicast linear deterministic networks [17]. This completes an approximate characterization of the capacity region for multiple-unicast networks with symmetric demands.

Additionally, consider the groupcast problem in directed wireline networks. There is a group of nodes and each node in the group has an independent message to be relayed to every other node in the group. [18] shows that the maximum sum-rate achievable by routing flow for groupcast is at least half the multicut, a simple edge-cut based outer bound. We ask the question: “Is the multicut a fundamental outer bound on the sum-rate?”. We can show that the answer is no but that twice the multicut is indeed a fundamental outer bound. This shows that routing flow is approximately optimal for maximizing sum-rate in groupcast.

When there is some kind of symmetry in the network, either in the underlying graph (undirected or bidirected networks) or in the traffic (directed network with symmetric demands, sum-rate in groupcast), the following picture seems to emerge.

- (Achievability) *Algorithmic Meta-Theorem*: Edge-cut bounds can be well-approximated either by flows [10], [11], [15], [19], [18] or by other means [12].
- (Converse) *Information-Theoretic Meta-Theorem*: Edge-cut bounds are fundamental or close to fundamental outer bounds on the capacity region.
- *Combined Meta-Theorem*: Flows approximately achieve capacity.

In the second half of this paper, we extend these results to the wireless setting. The capacity region of multiple unicast in general wireless networks is an open problem in the general case. Recent work [20] [21] [1] has made progress in this direction by giving an approximate characterization of this capacity region by using the reciprocity in wireless channels, building on flow-cut gaps in undirected wireline networks. It has been shown that simple layered architectures involving local physical-layer schemes combined with global routing can achieve approximately optimal performance in wireless networks.

In many practical scenarios, the channel reciprocity may not hold due to asymmetric power constraints, directional antennas or frequency-duplexing. The question we address in this paper is: "do layered architectures continue to be near-optimal even in this case?" We answer this question in the affirmative under the *symmetric demands* model: there are k specially-marked source-sink pairs of nodes $(s_i, t_i), i = 1, 2, ..., k$ with s_i wanting to communicate an independent message to t_i at rate R_i *and* t_i wanting to communicate an independent message to s_i at rate R_i. This traffic model is relevant in several practical scenarios including voice calls, video calls, and interactive gaming.

Building on our results from wireline networks with symmetric demands, we show an analogous result for wireless networks with symmetric demands if it composed of channels, for which good schemes are known at a local physical layer level. Our results for wireless networks with symmetric demands include

1) Capacity approximations for networks comprised of Gaussian MAC and broadcast channels,
2) Degrees-of-freedom approximation for fixed Gaussian networks, and
3) Capacity approximations for fading Gaussian networks.

At the heart of our achievable scheme is a connection to "polymatroidal networks" for which the symmetric demands problem was recently addressed, and it was shown that flow is within an inverse poly-logarithmic factor of the cut-set bound in [19]. The techniques for proving our achievable scheme closely mirror [1]. We need to derive our outer bound carefully; it is derived based on a suitable extension of the generalized network sharing bound [16] for wireless networks having a certain form. These results demonstrate the power of having the symmetry assumption in solving network capacity problems, which has been hitherto unexplored to the knowledge of the authors.

The rest of this paper is organized as follows. We set up notation and preliminaries in Section II. We discuss k-unicast directed symmetric-demand networks in Section III. We study a special class of Gaussian networks that we call MAC+BC networks in Section IV. Finally, we conclude with a discussion describing how these general results can be extended in two possible ways: to other traffic patterns (groupcast) and to general Gaussian networks (degrees-of-freedom in fixed Gaussian channels and capacity approximation in ergodic Gaussian channels) and broadcast erasure networks with feedback.

II. Preliminaries

In this section, we will setup the wireline multiple-unicast problem with symmetric demands, along with the necessary notation.

Definition. A k-unicast directed wireline network $\mathcal{N}$ for source-destination pairs $\{(s_i; d_i)\}_{i\in\mathcal{I}}$ with $|\mathcal{I}| = k$ (for instance, $\mathcal{I} := \{1, 2, \dots, k\}$) is a tuple $(\mathcal{G}, \underline{\mathrm{C}})$ where

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the underlying directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, with $s_i, d_i \in \mathcal{V}(\mathcal{G})$ for $i \in \mathcal{I}$,
- $\underline{\mathrm{C}} = (C_e : e \in \mathcal{E}(\mathcal{G}))$ is the edge-capacity vector, with $C_e \in \mathbb{R}_{\geq 0} \cup \{\infty\}\ \forall e \in \mathcal{E}(\mathcal{G})$.

For each $i \in \mathcal{I}$, s_i has independent information to be communicated to d_i at rate R_i.

Notation. *For $v \in \mathcal{V}(\mathcal{G})$, let $\mathsf{In}(v)$ and $\mathsf{Out}(v)$ denote the edges entering into and leaving v respectively.*

Definition. Given a k-unicast network $\mathcal{N} = (\mathcal{G}, \underline{\mathrm{C}})$ for source-destination pairs $\{(s_i; d_i)\}_{i\in\mathcal{I}}$, we say that the non-negative rate tuple $(R_i : i \in \mathcal{I})$ is *achievable* if for any $\epsilon > 0$, there exist positive integers N and T (called block length and number of epochs respectively), using notation $H_v := \Pi_{i\in\mathcal{I}:v=s_i}\{0,1\}^{\lceil NTR_i\rceil}$ (with an empty product being the singleton set),

- encoding functions for $1 \leq t \leq T, e = (u, v) \in \mathcal{E}$, $f_{e,t} : H_u \times \Pi_{e'\in\mathsf{In}(u)} \left(\{0,1\}^{\lfloor NC_{e'}\rfloor}\right)^{(t-1)} \mapsto \{0,1\}^{\lfloor NC_e\rfloor}$,
- decoding functions at destinations d_i for $i \in \mathcal{I}$, $f_{d_i} : H_{d_i} \times \Pi_{e'\in\mathsf{In}(d_i)} \left(\{0,1\}^{\lfloor NC_{e'}\rfloor}\right)^T \mapsto \{0,1\}^{\lceil NTR_i\rceil}$,

with the property that under the uniform probability distribution on $\Pi_{i\in\mathcal{I}}\{0,1\}^{\lceil NTR_i\rceil}$,

$$\mathsf{Pr}\left(g(m_1, m_2, \dots, m_k) \neq (m_1, m_2, \dots, m_k)\right) \leq \epsilon,$$

where $g : \Pi_{i\in\mathcal{I}}\{0,1\}^{\lceil NTR_i\rceil} \mapsto \Pi_{i\in\mathcal{I}}\{0,1\}^{\lceil NTR_i\rceil}$ is the global decoding function induced inductively by

- $\{f_{e,t} : e \in \mathcal{E}(\mathcal{G}), 1 \leq t \leq T\}$ and $\{f_{d_i} : i \in \mathcal{I}\}$ in the directed graph case and
- $\{f^u_{e,t}, f^v_{e,t} : e = (u, v) \in \mathcal{E}(\mathcal{G}), 1 \leq t \leq T\}$ and $\{f_{d_i} : i \in \mathcal{I}\}$ in the undirected graph case.

The closure of the set of achievable rate tuples is called the *capacity region* and is denoted by $\mathcal{C}$.

Definition. Given a k-unicast network $\mathcal{N} = (\mathcal{G}, \underline{\mathrm{C}})$ for source-destination pairs $\{(s_i; d_i)\}_{i\in\mathcal{I}}$, we say that the non-negative rate tuple $(R_i : i \in \mathcal{I})$ is *achievable by routing flow* if there exist for each $i \in \mathcal{I}$ and each $e = (u, v) \in \mathcal{E}(\mathcal{G})$, real numbers $f_{i,e} \geq 0$ in the directed graph case and $f^u_{i,e}, f^v_{i,e} \geq 0$ in the undirected graph case such that $\sum_{i \in \mathcal{I}} f_{i,e} \leq C_e \; \forall \; e \in \mathcal{E}(\mathcal{G})$, and for each $i \in \mathcal{I}$ and each $v \in \mathcal{V}(\mathcal{G})$,

$$\left(\sum_{e \in \mathsf{Out}(v)} f_{i,e} \right) - \left(\sum_{e \in \mathsf{In}(v)} f_{i,e} \right) = \begin{cases} 0 & \text{if } v \neq s_i, d_i, \\ R_i & \text{if } v = s_i, \\ -R_i & \text{if } v = d_i. \end{cases}$$

The closure of the set of rate tuples achievable by routing flow is called the *flow region* and is denoted by $\mathcal{F}$.

Definition. Given a k unicast network $\mathcal{N} = (\mathcal{G}, \underline{\mathrm{C}})$ for source-destination pairs $\{(s_i; d_i)\}_{i \in \mathcal{I}}$, we define the *edge-cut outer bound* denoted by $\mathcal{R}_{\text{edge-cut}}$, to be the set of all non-negative tuples $(R_i : i \in \mathcal{I})$ that satisfy for every $E \subseteq \mathcal{E}(\mathcal{G})$, the inequality $\sum_{i \in J} R_i \leq \sum_{e \in E} C_e$ where index $i \in J \subseteq \mathcal{I}$ if and only if $\mathcal{G} \setminus E$ has no directed paths from s_i to d_i.

While it is clear that $\mathcal{F} \subseteq \mathcal{R}_{\text{edge-cut}}$ and $\mathcal{F} \subseteq \mathcal{C}$, the connection between $\mathcal{C}$ and $\mathcal{R}_{\text{edge-cut}}$ is unclear. It is easy to show examples where $\mathcal{C} \nsubseteq \mathcal{R}_{\text{edge-cut}}$. Thus, simple edge-cut based outer bounds are not in general, *fundamental*, i.e. they are not outer bounds on the capacity region. Indeed, [3] provides a series of k-unicast networks, one for each k with $k = 2^n$ with $F_{\text{sum-rate}} = R_{\text{multicut}} = \frac{1}{k} C_{\text{sum-rate}}$ and $\mathcal{C} \nsubseteq (k - \epsilon)\mathcal{R}_{\text{edge-cut}}$ for any $\epsilon > 0$.

III. k-PAIR UNICAST DIRECTED SYMMETRIC-DEMAND NETWORKS

Definition. A *k-pair unicast directed symmetric-demand network* is a $2k$-unicast directed network $\mathcal{N}$ with $2k$ distinct distinguished nodes (source-destination nodes) $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ with source-destination pairs $\{s_i; d_i\}_{i \in \mathcal{I}}$ where $\mathcal{I} = \{1, 2, \ldots, k\} \cup \{-1, -2, \ldots, -k\}$ and for $i > 0$, $s_i = u_i, d_i = v_i$, while for $i < 0, s_i = v_{-i}, d_i = u_{-i}$. The rate tuple $(R_i : 1 \leq i \leq k)$ is defined to be in the capacity region $\mathcal{C}$, flow region $\mathcal{F}$, edge-cut outer bound $\mathcal{R}_{\text{edge-cut}}$ for the k-pair unicast directed symmetric-demand network if the rate tuple $(R'_i : i \in \mathcal{I})$, given by $R'_i = R_{|i|}$ for $i \in \mathcal{I}$, lies in the capacity region, flow region, edge-cut outer bound respectively of the $2k$-unicast directed network.

Remark 1. *There is no loss of generality in assuming $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ distinct since if they aren't, we can add more nodes and infinite capacity edges to make them distinct while obtaining a network with identical capacity region.*

Definition. Given a directed network $\mathcal{N} = (\mathcal{G}, \underline{\mathrm{C}})$ with a set of $2r$ distinct distinguished vertices $w_1, w_2, \ldots, w_r, w'_1, w'_2, \ldots, w'_r$, with w_i communicating to w'_i at rate R_i, and possibly other sources and destinations with their independent messages. If a set of edges $E \subseteq \mathcal{E}(\mathcal{G})$ has the property that $\mathcal{G} \setminus E$ has no directed paths from w_i to w'_j whenever $\pi(i) \geq \pi(j), 1 \leq i, j \leq r$, for some permutation $\pi : \{1, 2, \ldots, r\} \mapsto \{1, 2, \ldots, r\}$, then we say that E is a *GNS-cut (Generalized Network Sharing cut)* for $\{w_1, w_2, \ldots, w_r; w'_1, w'_2, \ldots, w'_r\}$ with permutation π.

Definition. Given a k-pair unicast directed symmetric-demand network $\mathcal{N} = (\mathcal{G}, \underline{\mathrm{C}})$ with source-destination nodes $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$, we define the *GNS-cut outer bound* denoted by $\mathcal{R}_{\text{GNS-cut}}$, to be the set of all non-negative tuples $(R_i : 1 \leq i \leq k)$ that satisfy for every $E \subseteq \mathcal{E}(\mathcal{G})$, the inequality $\sum_{i \in J} R_i \leq \sum_{e \in E} C_e$ whenever E is a GNS-cut for $\{w_1, w_2, \ldots, w_r; w'_1, w'_2, \ldots, w'_r\}$ with some permutation π where

- $J \subseteq \{1, 2, \ldots, k\}, |J| = r$,
- $w_1, w_2, \ldots, w_r, w'_1, w'_2, \ldots, w'_r$ are distinct,
- for $1 \leq j \leq r$, $(w_j, w'_j) = (u_i, v_i)$ or (v_i, u_i) for some $i \in J$.

We define a *weak edge-cut* outer bound for this class of networks.

Definition. Given a k-pair unicast directed symmetric-demand network $\mathcal{N} = (\mathcal{G}, \underline{\mathrm{C}})$ with source-destination nodes $u_1, u_2, \ldots, u_k,\ v_1, v_2, \ldots, v_k$, we define the *weak edge-cut outer bound* denoted by $\mathcal{R}_{\text{weak-edge-cut}}$, to be the set of all non-negative tuples $(R_i : 1 \leq i \leq k)$ that satisfy for every $E \subseteq \mathcal{E}(\mathcal{G})$, the inequality $\sum_{i \in J} R_i \leq \sum_{e \in E} C_e$ where index $i \in J \subseteq \{1, 2, \ldots, k\}$ if and only if $\mathcal{G} \setminus E$ has no directed paths from either u_i to v_i or v_i to u_i or both.

Remark 2. *For $E \subseteq \mathcal{E}(\mathcal{G})$, if J_1 is the set of indices $i, 1 \leq i \leq k$ for which $\mathcal{G} \setminus E$ has no directed paths from either u_i to v_i or from v_i to u_i but not both and J_2 is the set of indices $i, 1 \leq i \leq k$ for which $\mathcal{G} \setminus E$ has no directed paths from u_i to v_i and from v_i to u_i, then the edge-cut outer bound has the inequality $\sum_{i \in J_1} R_i + 2 \sum_{j \in J_2} R_j \leq \sum_{e \in E} C_e$ while the weak edge-cut outer bound has the inequality $\sum_{i \in J_1} R_i + \sum_{j \in J_2} R_j \leq \sum_{e \in E} C_e$. It is therefore, clear that*

$$\mathcal{R}_{\text{edge-cut}} \subseteq \mathcal{R}_{\text{weak-edge-cut}} \subseteq 2\mathcal{R}_{\text{edge-cut}}.$$

Theorem 1. *There exists a universal constant $\kappa > 0$ such that for any k-pair unicast directed symmetric-demand network,*

$$\frac{\mathcal{R}_{\text{weak-edge-cut}}}{\kappa(\log^3(k+1))} \subseteq \mathcal{F} \ \subseteq \mathcal{C} \subseteq \mathcal{R}_{\text{GNS-cut}} = \mathcal{R}_{\text{weak-edge-cut}}, \tag{1}$$

Remark 3. *Klein, Plotkin, Rao, Tardos showed in [15] that*

$$\frac{\mathcal{R}_{\text{weak-edge-cut}}}{\kappa(\log^3(k+1))} \subseteq \mathcal{F} \ \subseteq \mathcal{R}_{\text{weak-edge-cut}},$$

for a universal constant κ. Furthermore, $\mathcal{C} \subseteq \mathcal{R}_{\text{GNS-cut}}$ follows from the Generalized Network Sharing bound [16]. The main contribution in this section is the observation $\mathcal{R}_{\text{GNS-cut}} = \mathcal{R}_{\text{weak-edge-cut}}$, i.e. that the weak edge cuts correspond precisely to the GNS cuts.

Remark 4. *The GNS bound is to the capacity region what the edge-cut bound is to the flow region, namely an intuitive outer bound that arises from simple connectivity properties of the underlying graph of the network. While more sophisticated bounds [7], [3], [22] include the GNS bound as a special case, it is the simplicity of the GNS bound that becomes* *useful for showing that weak edge-cuts and GNS-cuts are identical for k-pair directed symmetric-demand networks, i.e.* $\mathcal{R}_{\mathrm{GNS-cut}} = \mathcal{R}_{\mathrm{weak-edge-cut}}$. *We also note that the outer bound* $\mathcal{R}_{\mathrm{GNS-cut}}$ *is strictly tighter than the cutset outer bound in general, and that the capacity region* $\mathcal{C}$ *is not always contained in* $\mathcal{R}_{\mathrm{edge-cut}}$ *although it is always contained in* $\mathrm{R}_{\mathrm{weak-edge-cut}}$.

We conjecture that a statement much stronger than that in Theorem 1 is true:

Conjecture 2. *For k-pair unicast directed symmetric-demand networks,*

$$\mathcal{F} \subseteq \mathcal{C} \subseteq 2\mathcal{F},$$

i.e. network coding can improve rates beyond routing flow by at most a factor 2.

Conjecture 2 can be considered a 'directed symmetric-demand' analog of the Li and Li conjecture [13], [14] which states that the flow region and capacity region are identical for undirected networks.

Proof:

As stated in the earlier remark, [15] shows that

$$\frac{\mathcal{R}_{\mathrm{weak-edge-cut}}}{\kappa(\log^3(k+1))} \subseteq \mathcal{F} \ \subseteq \mathcal{R}_{\mathrm{weak-edge-cut}}, \tag{2}$$

for a universal constant κ.

We first present the proof of $\mathcal{C} \subseteq \mathcal{R}_{\mathrm{GNS-cut}}$. The essential idea is contained in [16] but we provide a proof here for completeness.

Consider a communication scheme with block length N and number of epochs T that achieves for $1 \le i \le r$, rate R_i for the message from w_i to w_i' with error probability at most ϵ. Let E be a GNS-cut for $\{w_1, w_2, \ldots, w_r; w_1', w_2', \ldots, w_r'\}$ which may assumed to be with the identity permutation without loss of generality. For $1 \le i \le r$, let W_i be the source message at w_i that is required to be delivered to w_i'. Let W_0 denote the vector of all other source messages in the network. $W_0, W_1, \ldots, W_r$ are mutually independent and each $W_i, 0 \le i \le r$ has the uniform distribution. Let X_E denote the vector of all symbols transmitted on the edges of E over the duration of the complete scheme. For $1 \le i \le r$, let $\hat{W}_i$ denote the estimate at w_i' of the source message W_i upon completion of the coding scheme. Note that

$$H(W_1, W_2, \ldots, W_r | X_E, W_0) \tag{3}$$

$$= \sum_{i=1}^{r} H\left(W_i | X_E, W_0, \{W_j : 1 \le j < i\}\right) \tag{4}$$

$$= \sum_{i=1}^{r} H\left(W_i | X_E, W_0, \{W_j : 1 \le j < i\}, \hat{W}_i\right) \tag{5}$$

[since $\hat{W}_i$ is a function of $X_E, W_0, \{W_j : 1 \le j < i\}$ from the connectivity properties of $\mathcal{G} \setminus E$]

$$\le \sum_{i=1}^{r} H\left(W_i | \hat{W}_i\right) \tag{6}$$

$$\le \sum_{i=1}^{r} \left(h(\epsilon) + \epsilon\lceil NTR_i \rceil\right) = rh(\epsilon) + \epsilon \sum_{i=1}^{r} \lceil NTR_i \rceil, \tag{7}$$

where $h(\cdot)$ is the binary entropy function. Thus, we have

$$\sum_{i=1}^{r} \lceil NTR_i \rceil = H(W_1, W_2, \ldots, W_r) \tag{8}$$

$$= I(W_1, W_2, \ldots, W_r; X_E, W_0) + H(W_1, W_2, \ldots, W_r | X_E, W_0) \tag{9}$$

$$\le I(W_1, W_2, \ldots, W_r; X_E | W_0) + rh(\epsilon) + \epsilon \sum_{i=1}^{r} \lceil NTR_i \rceil \tag{10}$$

$$\le H(X_E) + rh(\epsilon) + \epsilon \sum_{i=1}^{r} \lceil NTR_i \rceil \tag{11}$$

$$\le \sum_{e \in E} T\lfloor NC_e \rfloor + rh(\epsilon) + \epsilon \sum_{i=1}^{r} \lceil NTR_i \rceil \tag{12}$$

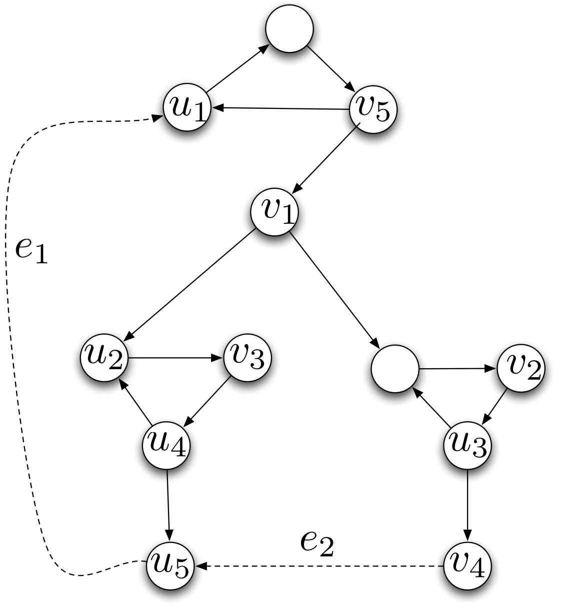
This establishes that $\sum_{i=1}^{r} R_i \le \sum_{e \in E} C_e$. Hence, we have $\mathcal{C} \subseteq \mathcal{R}_{\mathrm{GNS-cut}}$.

Now, we prove the equivalence between weak edge-cuts and GNS-cuts for k-pair unicast directed symmetric-demand networks.

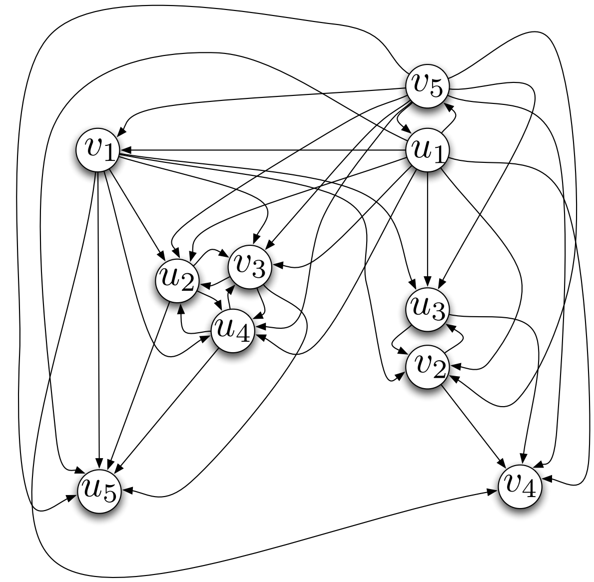
It is easy to see that the inequality obtained from a GNS-cut can always be obtained from a weak edge-cut since a GNS-cut requires stronger disconnections as compared to a weak edge-cut. This gives $\mathcal{R}_{\mathrm{weak-edge-cut}} \subseteq \mathcal{R}_{\mathrm{GNS-cut}}$. To show $\mathcal{R}_{\mathrm{GNS-cut}} \subseteq \mathcal{R}_{\mathrm{weak-edge-cut}}$, we now consider $E \subseteq \mathcal{E}(\mathcal{G})$, and say $i \in J \subseteq \{1, 2, \ldots, k\}$ if and only if $\mathcal{G} \setminus E$ has no directed paths from either u_i to v_i or from v_i to u_i or both. We show that E is a GNS-cut for $\{w_1, w_2, \ldots, w_r; w_1', w_2', \ldots, w_r'\}$ with some permutation π where the $2r$ vertices $w_1, w_2, \ldots, w_r, w_1', w_2', \ldots, w_r'$ are all distinct and for $1 \le j \le r$, $(w_j, w_j') = (u_i, v_i)$ or (v_i, u_i) for some $i \in J$ with $|J| = r$. We will prove this for the case $J = \{1, 2, \ldots, k\}$. The proof for other choices of J is similar.

Define the *connectivity graph* $\mathcal{G}_c$ as a directed graph over $2k$ vertices $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ as follows. For every pair of distinct vertices w and z, there is a directed edge from w to z in $\mathcal{G}_c$ if and only if w has a directed path to z in $\mathcal{G} \setminus E$. See Fig. 1(b) for an example. $\mathcal{G}_c$ is transitively closed, i.e. for three distinct vertices w, z, x, if w has an edge to z and z has an edge to x, then w has an edge to x. Define two distinct vertices u and v in $\mathcal{G}_c$ as *associated*, if u has an edge to v and v has an edge to u. If we define every vertex to be associated with itself, this relation is reflexive and symmetric. As $\mathcal{G}_c$ is transitively closed, this relation is also transitive and so, association is an equivalence relation. Further, for each $i = 1, 2, \ldots, k$, we have that u_i and v_i are not associated since there are no paths in $\mathcal{G} \setminus S$ from either u_i to v_i or from v_i to u_i.

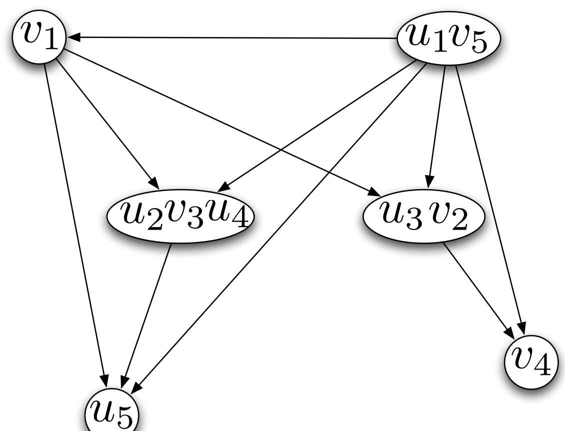
Now, define the *reduced connectivity graph* $\mathcal{G}_r$ as a directed graph with vertices represented by the equivalence classes defined from being associated in $\mathcal{G}_c$. See Fig. 1(c) for an example. There is a directed edge from equivalence class $\mathcal{E}_1$ to $\mathcal{E}_2$ in $\mathcal{G}_r$ if there is a directed edge in $\mathcal{G}_c$ from each vertex in $\mathcal{E}_1$ to each vertex in $\mathcal{E}_2$. By transitive closure of $\mathcal{G}_c$, this happens if and only if there is a directed edge in $\mathcal{G}_c$ from some vertex in $\mathcal{E}_1$ to some vertex in $\mathcal{E}_2$. $\mathcal{G}_r$ has at least two vertices since u_1 and v_1 cannot belong to the same equivalence class.



(a) Given network graph $\mathcal{G}$



(b) Connectivity graph $\mathcal{G}_c$



(c) Reduced connectivity graph $\mathcal{G}_r$

Fig. 1. Network graph, connectivity graph and reduced connectivity graph for an instance of a 5-pair unicast directed symmetric-demand network. (a) shows a 5-pair unicast symmetric demand network with 12 nodes. Note that for each i, there is a path from u_i to v_i and vice versa. If $E = \{e_1, e_2\}$ consists of the dashed edges, then E provides a weak-edge-cut since its removal disconnects either u_i from v_i or v_i from u_i for each i.

Now, note that $\mathcal{G}_r$ is a directed acyclic graph. Suppose not, i.e. suppose the equivalence classes $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots, \mathcal{E}_r, \mathcal{E}_1$ in that order describe a directed cycle. Then, in the graph $\mathcal{G}_c$, for vertex w_j chosen from equivalence class $\mathcal{E}_j$ for $j = 1, 2, \ldots, r$, we have w_j has a directed edge to w_{j+1} for $j = 1, 2, \ldots, r-1$ and w_r has a directed edge to w_1. Transitive closure of $\mathcal{G}_c$ implies that there must be a directed edge from w_j to w_k for $j, k = 1, 2, \ldots, r, j \neq k$, leading to a collapse of the $r \geq 2$ equivalence classes into one equivalence class, a contradiction.

We now describe an algorithm $\mathcal{P}$ that fills the cells of a $k \times 2$ table with vertex names from $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ such that the following properties hold:

(α) Each vertex shows up exactly once in the table.

(β) Each row of the table is made up of vertices u_i and v_i for some i.

(γ) In graph $\mathcal{G}_c$, vertex u obtained from the first column of row i does not have an edge to vertex v obtained from the second column of row j whenever $i \geq j$.

A directed acyclic graph has at least one sink vertex, i.e. a vertex with no outgoing edges. This is the proposed algorithm $\mathcal{P}$.

(1) Pick any sink vertex in directed acyclic graph $\mathcal{G}_r$.
(2) List the vertices of $\mathcal{G}_c$ in the equivalence class represented by the chosen sink vertex.
   (a) Pick a vertex w from the list.
   (b) If vertex w has been entered previously in the table, do nothing. Else, add vertex w in the first column of the lowest row in the table not yet filled. Add the destination of vertex w in the second column of the same row, e.g. if v_3 was entered in the first column

of the lowest available row, then fill u_3 in the second column.

(c) Remove w from the list and go back to (a) if the list is still non-empty, else proceed to (3)

(3) Modify graph $\mathcal{G}_r$ by deleting the chosen sink vertex. The modified graph continues to be a directed acyclic graph. If this graph has non-zero number of vertices, go to step (1), else quit.

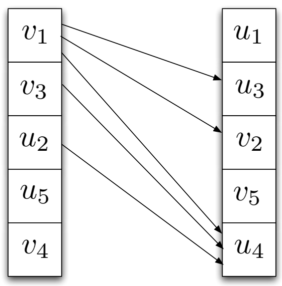


Fig. 2. One of the several 5×2 tables generated by running algorithm $\mathcal{P}$ on the $\mathcal{G}_c, \mathcal{G}_r$ shown in Fig. 1. The order of choosing sinks was v_4, u_5, $u_2v_3u_4$, u_3v_2, v_1, u_1v_5. The arrows show connectivity from the vertices on the left to the vertices on the right in $\mathcal{G}_c$. Note that no arrows go 'horizontally' or go 'upward'. They always go 'downward' which is the desired GNS-cut property.

Let us verify the claimed properties. By step (b), it is clear that each non-empty row of the table is filled with a vertex and its destination, i.e. the vertices u_i and v_i for some i. As the algorithm terminates only when all vertices have been listed and checked for their presence in the table, and as the vertices are added only when they have not been added previously, it follows that each vertex shows up exactly once and the table is completely filled upon termination of the algorithm. This verifies claimed properties (α) and (β). Now, we verify property (γ).

- Consider vertices in row j of the table, say w and w' with w in the first column. These are source-destination pairs u_i, v_i for some i. We claim that w has no edge to w' in $\mathcal{G}_c$. Suppose it did. Then, there would be an edge in $\mathcal{G}_r$ from the equivalence class $\mathcal{E}$ containing w to the equivalence class $\mathcal{E}'$ containing w'. These equivalence classes must be distinct as w and w' are source-destination pairs. This means that the algorithm $\mathcal{P}$ must pick the equivalence class containing w' before picking the equivalence class containing w. When w' is probed in the list of vertices, w must not have been entered in to the table as yet, and thus w' would then be entered in the first column of some row and w in the second column of the same row. This contradicts the assumed structure of the table. Thus, we have no edge from w to w' in $\mathcal{G}_c$.
- Now, consider rows i and j with $i > j$. Let the vertices in row i be w and w' with w in the first column and the vertices in row j be z and z' with z in the first column. We claim that there is no edge from w to z' in $\mathcal{G}_c$. Suppose there is. Then, either w and z' are in the same equivalence class in $\mathcal{G}_c$ or they are not. If they are not, then the equivalence class containing z' has an incoming edge from the equivalence class containing w and thus, the former ought to have been picked by the algorithm before the latter. This is inconsistent with the table which was filled with w in the first column of a row while z' had not yet been filled. Now, if w and z' are in the same equivalence class, then clearly z does not fall in that equivalence class. Moreover, the equivalence class containing z is picked after the equivalence class containing w and z'. The algorithm $\mathcal{P}$, when exhausting the list of vertices in the equivalence class containing w and z' is supposed to have accepted w and added it to the first column of a row and rejected z'. But when z' was probed, we are still in the same equivalence class as w, so z had not been probed yet. Then z' must have been added to the first column of some row, which contradicts the structure of the table. Thus, there is no edge from w to z' in $\mathcal{G}_c$.

Now, if the j^{th} row of the table consists of u_i, v_i, we set $\pi(j) = i$ and $(w_i, w_i') = (u_i, v_i)$ or (v_i, u_i) depending on whether the first entry in the row is u_i or v_i. This shows that S is a GNS-cut for $\{w_1, w_2, \ldots, w_k; w_1', w_2', \ldots, w_k'\}$ with permutation π. This gives $\mathcal{R}_{\text{weak-edge-cut}} \supseteq \mathcal{R}_{\text{GNS-cut}}$ and completes the proof. ■

IV. Gaussian networks composed of Broadcast and Multiple Access Channels with symmetric demands

A simplest model for a wireless network is a network that is comprised only of non-interacting multiple access channels and broadcast channels connected together through common nodes. This model is particularly interesting to study because the component channels are fully understood whereas a network composed of such channels is not well understood. We abbreviate networks composed only of multiple access channels and broadcast channels as MAC+BC networks. The communication network is represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and an edge coloring $\psi : \mathcal{E} \to \Lambda$, where Λ is the set of colors. Each node v has a set of colors $\Lambda(v) \subseteq \Lambda$ on which it operates. Each color can be thought of as an orthogonal resource, so that the broadcast and interference constraints for the wireless channel apply only *within* a given color. The set of edges A_c corresponding to color $c \in \Lambda$ interact with each other and can be said to constitute a channel.

The channel model can therefore be written as,

$$y_i^c(t) = \sum_{j \in \mathsf{In}_c(i)} h_{ji}^c x_j^c(t) + z_i^c(t) \quad \forall c \in \Lambda(i), \tag{13}$$

where $x_i^c(t), y_i^c(t), z_i^c(t)$ are the transmitted vector, received vector, and noise vector on color c at time instant t, h_{ji}^c is the (fixed) channel coefficient between node i and node j on color c and $\mathsf{In}_c(i)$ represents the set of in-neighbors of node i who are operating on color c. We denote by y_i the vector comprised of $\{y_i^c\}$ for all $c \in \Lambda(i)$. We do not assume any symmetry in the channels, so that, in general h_{ij}^c may be different from h_{ji}^c.

We say that a given $c \in \Lambda$ corresponds to a multiple access channel (MAC) of degree d, if the set of edges A_c is of the form $A_c = \{i_1j, i_2j, ..., i_dj\}$, i.e., all edges are directed towards a particular node j. Similarly a channel c is said to correspond to a broadcast channel (BC) of degree d if $A_c = \{ij_1, ij_2, ..., ij_d\}$ for some node i. If A_c is a singleton set, we say that the channel c corresponds to an orthogonal link. A network is said to be a MAC+BC network if the set Λ can be decomposed as $\Lambda = \mathcal{M} \cup \mathcal{B}$, where $\mathcal{M}$ is the set of MAC channels and $\mathcal{B}$ is the set of broadcast channels or orthogonal links. Stated alternately, a network is composed of broadcast and multiple access channels if and only if no edge is involved simultaneously in a broadcast and interference constraint inside the same color. We will call such a network a "Gaussian MAC+BC network". An example of such a network is shown in Fig. 3.

Each node i has an average power constraint P to transmit for each color that it transmits in. If there are distinct power constraints for different nodes, they can be absorbed into the channel co-efficient without loss of generality. We assume that the channel h_{ij}^c is fixed (time-invariant) and is known at all the nodes.

The main contribution in this section is an extension of the Generalized Network Sharing bound to Gaussian networks in Lemma 4. Coupling this with existing results (Theorem 5) in approximation algorithms for polymatroidal networks shown in [19], and the observation that GNS-cuts and weak-edge-cuts are identical for networks with symmetric demands, we prove the main result of this section which is an approximate capacity characterization for MAC+BC Gaussian networks, namely Theorem 3.

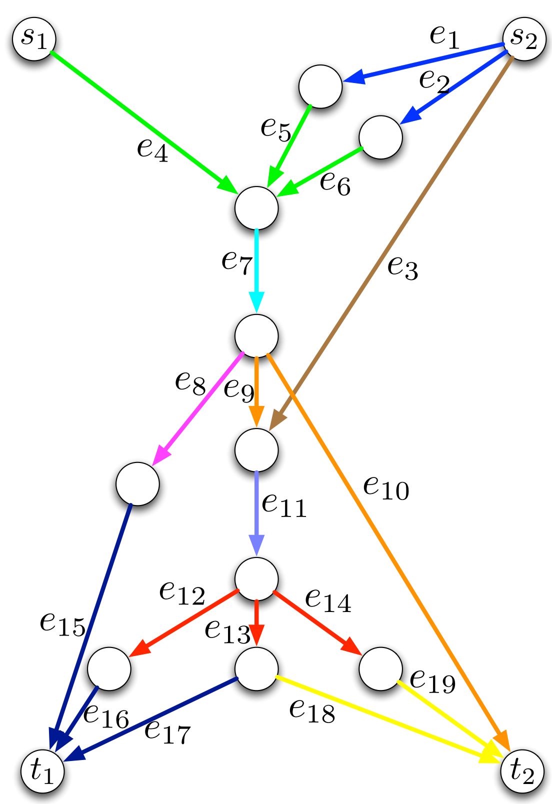


Fig. 3. Example of a MAC+BC Gaussian network

A. Multiple Unicast with symmetric demands in Gaussian MAC+BC Networks

A k-unicast Gaussian network with symmetric demands has k pairs of nodes s_i, d_i, $i = 1, 2, ..., k$, where node s_i has a message to send to d_i and d_i has an independent message to send to s_i, both at rate R_i. We would like to characterize the closure of the set of all achievable rate tuples, called the capacity region $\mathcal{C}$. We will use $\mathcal{R}_{\text{ach}}$ to denote rates achievable by a simple scheme that we will propose. Also, for clarity, we will use $\mathcal{X}_v, \mathcal{Y}_v$ to denote the input and output alphabets at node v, implicitly understanding that in the Gaussian network, $\mathcal{X}_v = \mathbb{R}^a, \mathcal{Y}_v = \mathbb{R}^b$, for suitable integers a and b. Formally, a $(\lceil 2^{TR_1} \rceil, \lceil 2^{TR_2} \rceil, \ldots, \lceil 2^{TR_k} \rceil, T)$ coding scheme for this network which communicates over T time instants is comprised of the following.

1) Independent random variables W_i, W_i' which are distributed uniformly on $\mathcal{W}_i := \{1, \ldots, \lceil 2^{TR_i} \rceil\}$ for $i = 1, \ldots, k$ respectively. W_i, W_i' denote the message intended from s_i to d_i and the message intended from d_i to s_i respectively.
2) The source mappings for time t,
$$f_{s_i,t} : (\mathcal{W}_i, \mathcal{Y}_{s_i}^{t-1}) \to \mathcal{X}_{s_i}^t, \;\; f_{d_i,t} : (\mathcal{W}_i, \mathcal{Y}_{d_i}^{t-1}) \to \mathcal{X}_{d_i}^t. \quad (14)$$
3) The relay mappings for each $v \in \mathcal{V} \backslash \{s_1, d_1, s_2, d_2, \ldots, s_k, d_k\}$ and time t,
$$f_{v,t} : \mathcal{Y}_v^{t-1} \to \mathcal{X}_v^t. \quad (15)$$
4) The decoding map at destinations,
$$g_{d_i} : \mathcal{Y}_{d_i}^T \to \mathcal{W}_i, \qquad g_{s_i} : \mathcal{Y}_{s_i}^T \to \mathcal{W}_i. \quad (16)$$

If $\hat{W}_i$ is the decoded symbol at d_i and $\hat{W}_i'$ is the decoded symbol at s_i, then the probability of error for destinations s_i, d_i under this coding scheme is given by
$$P_e^i \;\; := \;\; \max\{\Pr\{\hat{W}_i \neq W_i\}, \Pr\{\hat{W}_i' \neq W_i'\}\}. \quad (17)$$
A rate tuple $(R_1, R_2, \ldots, R_k)$, where R_i is the rate of communication in bits per unit time from source s_i to destination d_i, is said to be achievable if for any $\epsilon > 0$, there exists a $(\lceil 2^{TR_1} \rceil, \lceil 2^{TR_2} \rceil, \ldots, \lceil 2^{TR_k} \rceil, T)$ scheme that achieves a probability of error less than ϵ for all nodes, i.e., $\max_i P_e^i \leq \epsilon$. The capacity region $\mathcal{C}$ is the closure of the set of all achievable rate tuples.

1) Weak edge-cut bound: Similar to the wireline network case, we define a weak edge-cut bound region for the wireless network with demand symmetry. The weak edge-cut bound region for the Gaussian MAC+BC network is defined by the following: consider any set of edges $F \subseteq \mathcal{E}$, and let $K(F)$ denote the set of $i \in \{1, 2, ..., k\}$ such that either there is no path from s_i to d_i or there is no path from d_i to s_i in $\mathcal{G} \setminus F$. The value of the cut F is defined by $\nu(F) := \sum_c \nu(F^c)$, where $F = \cup_c F^c$ with F^c being the set of edges that participate in color c and $\nu(F^c)$ is the capacity under complete coordination of source nodes in channel c. More formally, if c is a broadcast channel, $\nu(F^c)$ is equal to the sum-capacity of the broadcast channel specified only by edges in F^c, under complete coordination of destination terminals in F^c. Similarly, if c is a MAC channel, $\nu(F^c)$ is equal to the sum-capacity of the MAC channel specified by edges in F^c, under complete coordination of source terminals in F^c.

The weak edge-cut bound region is now given as

$$\mathcal{R}_{\text{weak-edge-cut}} = \{(R_1, ..., R_k) : \sum_{i \in K(F)} R_i \leq \nu(F)\ \forall F \subseteq \mathcal{E}\}.$$

As in the wireline network case, it is not clear if $\mathcal{R}_{\text{weak-edge-cut}}$ is an outer bound to the capacity region $\mathcal{C}$.

Our main result is the following:

Theorem 3. *For the k-unicast problem with symmetric demands in a Gaussian MAC+BC network, the weak edge-cut bound is a fundamental outer bound on the capacity region and a simple separation strategy can achieve $\mathcal{R}_{ach}(P)$ which satisfies,*

$$\frac{\mathcal{R}_{\text{weak-edge-cut}}\left(\frac{P}{d_{\max}}\right)}{\kappa \log^3(k+1)} \subseteq \mathcal{R}_{ach}(P) \subseteq \mathcal{C}(P) \subseteq \mathcal{R}_{\text{weak-edge-cut}}(P), \tag{18}$$

where κ is a universal constant independent of problem parameters and $d_{\max}$ is the maximum degree of any broadcast or MAC component channel c.

Thus, the edge-cut bound is a fundamental upper bound on the capacity region and furthermore the edge-cut bound, scaled down in power by a factor $d_{\max}$ and scaled down in rate by a factor $\frac{1}{\kappa \log^3(k+1)}$, can be achieved by the proposed scheme.

B. Outer bound

We first establish that the weak edge-cut bound is fundamental, i.e., every communication scheme must have rate pairs that lie inside this region: $\mathcal{C} \subseteq \mathcal{R}_{\text{weak-edge-cut}}$. We will prove this result using a GNS bound for Gaussian networks.

Given an ℓ-unicast MAC+BC Gaussian network with source destination pairs $\{s_i, d_i\}_{i=1}^{\ell}$, we define a set of edges $F \subseteq \mathcal{E}$ to be a GNS-cut if there exists a permutation $\pi : \{1, 2, \ldots, \ell\} \to \{1, 2, \ldots, \ell\}$ such that there are no paths from s_i to d_i in $\mathcal{G} \backslash F$, whenever $\pi(i) \geq \pi(j)$.

Lemma 4. *(GNS bound for MAC+BC Gaussian networks) For an ℓ-unicast Gaussian MAC+BC network, every GNS cut F is fundamental, i.e., $\sum_{i \in K(F)} R_i \leq \nu(F)$ for any communication scheme achieving $(R_1, ..., R_\ell)$. Alternately $\mathcal{C} \subseteq \mathcal{R}_{\text{GNS-cut}}$, where*

$$\mathcal{R}_{\text{GNS-cut}} = \cap_{F \subseteq \mathcal{E}} \{(R_1, .., R_\ell) : \sum_{i \in K(F)} R_i \leq \nu(F)\}. \tag{19}$$

An instance of Lemma 4 can be found in Fig. 4.

Proof: Let F be a GNS-cut disconnecting s_i from d_i for $i = 1, 2, \ldots, \ell$ with say, the identity permutation π_{id}. Thus, $K(F) = \{1, 2, \ldots, \ell\}$. A similar proof holds in the case when $K(F) \subset \{1, 2, \ldots, \ell\}$. We first provide a proof of the GNS bound when the network has an acyclic underlying graph $\mathcal{G}$.

Recall that the set of colors $\Lambda = \mathcal{M} \cup \mathcal{B}$ where $\mathcal{M}$ consists of the colors of edges involved in MAC components and $\mathcal{B}$ consists of colors of edges involved in broadcast components or orthogonal links. For $\mu \in \Lambda$, let A_μ denote the set of edges involved in μ. Now, construct a directed graph $\mathcal{G}'$ as follows: for each $\mu \in \Lambda$, there is a node in $\mathcal{G}'$ and add a directed edge from node μ to node ν in $\mathcal{G}'$ if there exists an edge in A_μ that is upstream to some edge in A_ν in the original DAG $\mathcal{G}$. Since the set of all edges with a given color constitute either a MAC or a BC, we have that $\mathcal{G}'$ is a directed acyclic graph. Thus, we can have a total order on the vertices of $\mathcal{G}'$ consistent with the partial order of ancestry in $\mathcal{G}'$. This gives a total order on Λ and therefore also a total order on the subset $\mathcal{D} := \{\mu \in \Lambda : F \cap A_\mu \neq \emptyset\}$, which we will denote by $\mu_1 < \mu_2 < \ldots < \mu_r$, where μ_1 is the most "upstream".

- For $\mu \in \mathcal{M}$, we denote transmissions along edge e in A_μ by X_e and we denote the reception by Y_μ so that $Y_\mu = \sum_{e \in A_\mu} X_e + Z_\mu$ where Z_μ is Gaussian noise. Further, define $U_\mu := \{X_e : e \in F \cap A_\mu\}$, and $V_\mu := \sum_{e \in F \cap A_\mu} X_e + Z_\mu$. Intuitively, U_μ (V_μ) is the transmission (reception) on channel μ if only edges in F were present in the channel.
- For $\mu \in \mathcal{B}$, we denote the transmission on the broadcast component or orthogonal link by X_μ and the receptions at heads of $e \in A_\mu$ by Y_e so that $Y_e = X_\mu + Z_e$ where $\{Z_e, e \in A_\mu\}$ are independent Gaussian noise random variables. Further define $U_\mu := X_\mu$, and $V_\mu := \{Y_e : e \in F \cap A_\mu\}$.

Define $\tilde{Y}_{d_i} = \{Y_\mu^n : \mathsf{head}(e) = d_i, e \in A_\mu, \mu \in \mathcal{M}\} \cup \{Y_e^n : \mathsf{head}(e) = d_i, e \in A_\mu, \mu \in \mathcal{B}\}$.

$$\begin{aligned}
& n[\sum_{i=1}^{\ell} R_i - \epsilon_n] \leq \sum_{i=1}^{\ell} I(W_i; \tilde{Y}_{d_i}) \\
& \leq \sum_{i=1}^{\ell} I(W_i; \{V_\mu^n : \mu \in \mathcal{D}\}, \{W_j : j < i\}) \\
& \qquad [\text{since } W_i - \{V_\mu^n : \mu \in \mathcal{D}\}, \{W_j : j < i\} - \tilde{Y}_{d_i} \\
& \qquad \text{as } F \text{ is a GNS cut with identity permutation}] \\
& = \sum_{i=1}^{\ell} I(W_i; \{V_\mu^n : \mu \in \mathcal{D}\} | \{W_j : j < i\}) \\
& \qquad [\text{since } W_i \text{ is independent of } \{W_j : j < i\}] \\
& = I(\{W_i : 1 \leq i \leq \ell\}; \{V_\mu^n : \mu \in \mathcal{D}\}) \\
& = h(\{V_\mu^n : \mu \in \mathcal{D}\}) \\
& \quad - h(\{V_\mu^n : \mu \in \mathcal{D}\} | \{W_i : 1 \leq i \leq \ell\}) \\
& \leq \sum_{\mu \in \mathcal{D}} h(V_\mu^n) - h(\{V_\mu^n : \mu \in \mathcal{D}\} | \{W_i : 1 \leq i \leq \ell\}) \\
& =: \sum_{\mu \in \mathcal{D}} h(V_\mu^n) - A.
\end{aligned}$$

Now, we consider the negative term A above.

$$\begin{aligned}
A &= h(\{V_\mu^n : \mu \in \mathcal{D}\} | \{W_i : 1 \leq i \leq \ell\}) \\
&= h(V_{\mu_1}^n, V_{\mu_2}^n, \ldots, V_{\mu_r}^n | \{W_i : 1 \leq i \leq \ell\}) \\
&\geq h(V_{\mu_1}^n, V_{\mu_2}^n, \ldots, V_{\mu_r}^n | \{W_i : 1 \leq i \leq k\}, U_{\mu_1}^n) \\
&= h(V_{\mu_1}^n | \{W_i : 1 \leq i \leq \ell\}, U_{\mu_1}^n) \\
&\quad + h(V_{\mu_2}^n, \ldots, V_{\mu_r}^n | \{W_i : 1 \leq i \leq \ell\}, U_{\mu_1}^n, V_{\mu_1}^n)
\end{aligned}$$

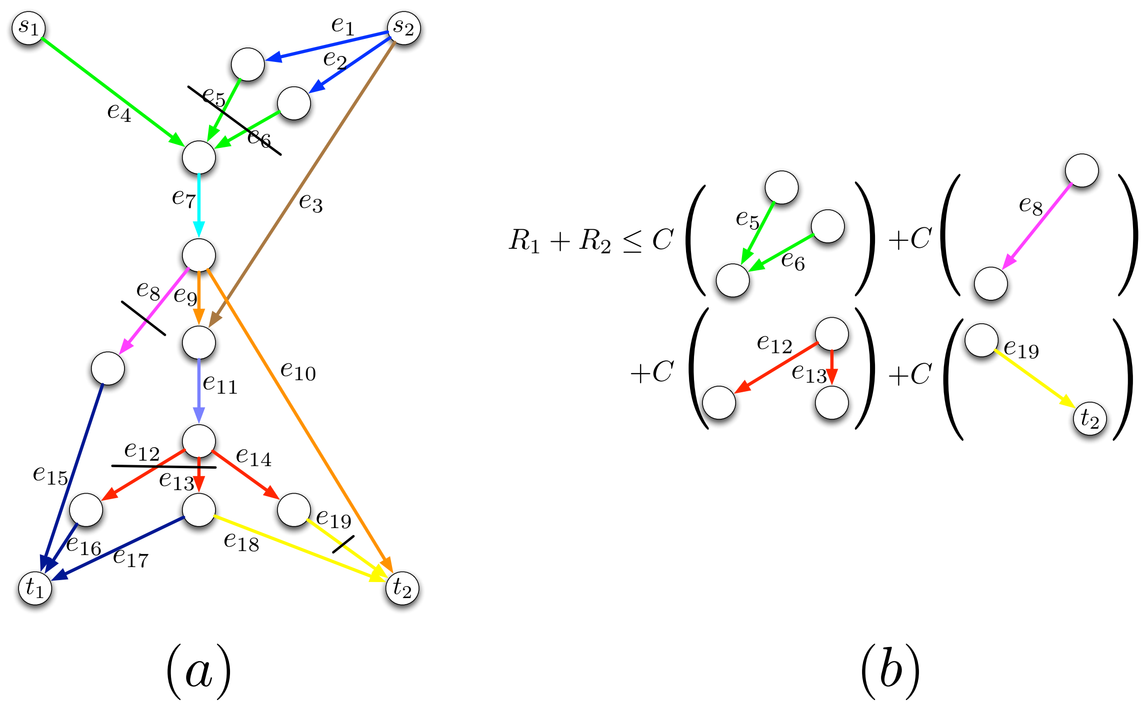


Fig. 4. GNS bound for MAC+BC Gaussian networks. The set of edges $\{e_5, e_6, e_8, e_{12}, e_{13}, e_{19}\}$ forms a GNS-cut in the two-unicast network of (a). The outer bound on the capacity region of the two-unicast network, that can be derived from the GNS-cut is shown in (b).

$$
\begin{aligned}
&= h(V_{\mu_1}^n | U_{\mu_1}^n) \\
&\quad + h(V_{\mu_2}^n, \ldots, V_{\mu_r}^n | \{W_i : 1 \leq i \leq \ell\}, U_{\mu_1}^n, V_{\mu_1}^n) \\
&\qquad\qquad [\text{since } \{W_i : 1 \leq i \leq \ell\} - U_{\mu_1}^n - V_{\mu_1}^n \\
&\qquad\qquad \text{as } \mu_1 \text{ is the most upstream channel}] \\
&\geq h(V_{\mu_1}^n | U_{\mu_1}^n) \\
&\quad + h(V_{\mu_2}^n, \ldots, V_{\mu_r}^n | \{W_i : 1 \leq i \leq \ell\}, U_{\mu_1}^n, V_{\mu_1}^n, U_{\mu_2}^n) \\
&= h(V_{\mu_1}^n | U_{\mu_1}^n) + h(V_{\mu_2}^n | \{W_i : 1 \leq i \leq \ell\}, U_{\mu_1}^n, V_{\mu_1}^n, U_{\mu_2}^n) \\
&\quad + h(V_{\mu_3}^n, \ldots, V_{\mu_r}^n | \{W_i : 1 \leq i \leq \ell\}, U_{\mu_1}^n, V_{\mu_1}^n, U_{\mu_2}^n, V_{\mu_2}^n) \\
&= h(V_{\mu_1}^n | U_{\mu_1}^n) + h(V_{\mu_2}^n | U_{\mu_2}^n) \\
&\quad + h(V_{\mu_3}^n, \ldots, V_{\mu_r}^n | \{W_i : 1 \leq i \leq \ell\}, U_{\mu_1}^n, V_{\mu_1}^n, U_{\mu_2}^n, V_{\mu_2}^n) \\
&\qquad\qquad [\text{since } \{W_i : 1 \leq i \leq \ell\}, U_{\mu_1}^n, V_{\mu_1}^n - U_{\mu_2}^n - V_{\mu_2}^n \\
&\qquad\qquad \text{as only } \mu_1 \text{ could be more upstream than } \mu_2] \\
&\geq \sum_{\mu \in \mathcal{D}} h(V_\mu^n | U_\mu^n),
\end{aligned}
$$

from repeating these steps. Thus, we obtain

$$
\begin{aligned}
n[\sum_{i=1}^{\ell} R_i - \epsilon_n] &\leq \sum_{\mu \in \mathcal{D}} I(U_\mu^n; V_\mu^n) && (20) \\
&\leq n \sum_{\mu \in \mathcal{D}} \nu(F_\mu) && (21) \\
&= n\nu(F), && (22)
\end{aligned}
$$

and therefore it follows that the GNS bound is a fundamental upper bound on the capacity region for acyclic networks.

For a general cyclic network, we can employ a standard time-layering argument in order to complete the proof. While the details of our method and the use of time-layering to deal with cyclic networks are fairly standard, see [2], [23], one key difference is that here we are using time-layering in order to prove an outer-bound, whereas the earlier works utilized time layering to show achievability. We will provide a brief sketch of the method here. Given a cyclic network $\mathcal{G}$ and a coding scheme over n time instants, we construct a time-layered graph $\mathcal{G}^n$ as follows. The nodes in the graph $\mathcal{G}^n$ are arranged in $n+1$ layers $0, 1, ..., n$. For each i, layer i has a copy of all the nodes V in the original graph, we call this $V[i]$ and the copy of node v in layer i is called $v[i]$. Add directed edges in the graph in the following manner.

- For each $(u, v) \in \mathcal{E}$ in the original graph with channel coefficient h_{vu}^c on color c, we add edges $(u[i], v[i+1])$ for $i = 0, 1, ..., n-1$ with channel coefficient $h_{vu}^{c_i}$ on color c_i.
- Create an edge from $v[i]$ to $v[i+1]$ for each v of infinite capacity in an independent channel (in order to model memory of the link).

Thus the time-layered graph $\mathcal{G}^n$ is created. This graph defines an instance of a Gaussian MAC+BC network, which is acyclic. For this new graph, we define a communication problem by specifying that sources $s_1[0], ..., s_\ell[0]$ wish to communicate independent information to destinations $d_1[n], d_2[n], ..., d_\ell[n]$. Observe that any scheme on the original network utilizing n time instants gives a valid scheme on this graph $\mathcal{G}^n$. Thus upper bounds on the communication rates in this graph serve as upper bounds to $n(R_1, R_2, ..., R_\ell)$ whenever $(R_1, R_2, \ldots, R_\ell)$ lies in the capacity region of the original network. Now given a GNS cut on the original graph with the identity permutation, defined by a set of edges F, we can define a cut on this graph $\mathcal{G}^n$ by $F^n := \cup_{i\in[n]} F[i]$, where $F[i] = \{(u[i-1], v(i), \ \forall (u,v) \in F\}$. If F disconnected source s_a from destination d_b in the original graph, this cut F^n disconnects $s_a[0]$ from $d_b[n]$ in the time-layered graph because any remaining path from $s_a[0]$ to $d_b[n]$ would imply a path in the original graph from s_a to d_b. This implies that any GNS cut on the original graph can produce a GNS cut on $\mathcal{G}^n$ with n times the value, as each edge occurs n times in F^n. Since the rate is also scaled by n times in this time layered graph, this proves that the GNS bound is a valid upper bound on the rate of an arbitrary (cyclic) graph. ∎

Now that we know that GNS-cut is a valid upper bound on the Gaussian network for any ℓ unicast problem, we will define the GNS-cut for a symmetric demands problem (in exactly the same way as it was defined for wireline networks).

Definition. Given a k-pair unicast directed symmetric-demand MAC + BC Gaussian network with source-destination nodes $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$, we define the *GNS-cut outer bound* denoted by $\mathcal{R}_{\mathrm{GNS-cut}}$, to be the set of all non-negative tuples $(R_i : 1 \leq i \leq k)$ that satisfy for every $F \subseteq \mathcal{E}(\mathcal{G})$, the inequality $\sum_{i\in J} R_i \leq \nu(F)$ whenever F is a GNS-cut for $\{w_1, w_2, \ldots, w_r; w'_1, w'_2, \ldots, w'_r\}$ with some permutation π where

- $J \subseteq \{1, 2, \ldots, k\}, |J| = r$,
- $w_1, w_2, \ldots, w_r, w'_1, w'_2, \ldots, w'_r$ are distinct,
- for $1 \leq j \leq r$, $(w_j, w'_j) = (u_i, v_i)$ or (v_i, u_i) for some $i \in J$.

By Theorem 1, we have that $\mathcal{R}_{\mathrm{weak-edge-cut}} = \mathcal{R}_{\mathrm{GNS-cut}}$. Using this result in conjunction with Lemma 4, gives us the desired result,

$$\mathcal{C} \subseteq \mathcal{R}_{\mathrm{weak-edge-cut}}. \tag{23}$$

We parametrize both $\mathcal{C}$ and $\mathcal{R}_{\mathrm{weak-edge-cut}}$ by power constraint P in order to emphasize its dependence.

C. Coding Scheme

The coding scheme we propose is a separation-based strategy: each component broadcast or multiple access channel is coded for independently creating bit-pipes on which information is routed globally. In order to evaluate the rate region of this scheme, we use polymatroidal networks as an interface for which we can show that the flow region corresponding to routing and the bounding region defined by edge-cuts are close to each other. This coding scheme and the calculation of its achievable rate closely parallel the scheme proposed for networks composed of symmetric MAC and BC channels in [1].

For simplicity of notation, we will assume that all MAC and broadcast channels have degree $d = d_{\max}$. It will be clear from the details that this assumption is not necessary. For a finite set V, a set function $f : 2^V \mapsto \mathbb{R}$ is said to satisfy the *polymatroidal axioms* if

- $f(\emptyset) = 0$,
- $A \subseteq B \implies f(A) \leq f(B)$,
- The function f is submodular, i.e. for any two sets $A, B \subseteq V$, $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

A bounding region $\mathcal{B}$ defined over $\mathbb{R}^{|\mathbb{V}|}$ is said to be a *polymatroidal region* if it is of the form $\mathcal{B} = \{(R_v : v \in V) : R_v \geq 0$ and for any $A \subseteq V, \sum_{v\in A} R_v \leq f(A)\}$ for some functin f that satisfies the polymatroidal axioms.

Let us first consider the coding for the multiple access channel with channel coefficients $h_1, ..., h_d$ and power constraint P at each of the d nodes. Let the rate region achievable on this multiple access channel be denoted by

$$\mathcal{R}^{\mathsf{MAC}}_{\mathsf{ach}}(P) = \{\overline{R} : \sum_{i\in A} R_i \leq \log\left(1 + \sum_{i\in A} |h_i|^2 P\right) \ \forall A\}. \tag{24}$$

This region is known to be polymatroidal. The outer bound for the MAC under arbitrary source cooperation is given by

$$\mathcal{R}^{\mathsf{MAC}}_{\mathsf{cut}}(P) = \{\overline{R} : \sum_{i\in A} R_i \leq \log\left(1 + \left(\sum_{i\in A} |h_i|\right)^2 P\right) \ \forall A\}. \tag{25}$$

The capacity region of a broadcast channel with channel coefficients $h_1, ..., h_k$ and power constraint P is not a polymatroidal region. However, it can be approximated by a polymatroidal region [20]. In particular, the achievable region includes the following polymatroidal region, and we will restrict our broadcast channel scheme to operate inside the following polymatroidal region, as we will show that this is not too far from the cutset outer bound:

$$\mathcal{R}^{\mathsf{BC}}_{\mathsf{ach}}(P) = \{\overline{R} : \sum_{i\in A} R_i \leq \log\left(1 + \sum_{i\in A} |h_i|^2 \frac{P}{d}\right) \ \forall A\}, \tag{26}$$

The cutset bound on the broadcast channel under arbitrary destination cooperation is

$$\mathcal{R}^{\mathsf{BC}}_{\mathsf{cut}}(P) = \{\overline{R} : \sum_{i\in A} R_i \leq \log\left(1 + \sum_{i\in A} |h_i|^2 P\right) \ \forall A\}. \tag{27}$$

It can be easily verified that $\mathcal{R}^{\mathsf{MAC}}_{\mathsf{cut}}(P) \subseteq \mathcal{R}^{\mathsf{MAC}}_{\mathsf{ach}}(dP)$ and $\mathcal{R}^{\mathsf{BC}}_{\mathsf{cut}}(P) \subseteq \mathcal{R}^{\mathsf{BC}}_{\mathsf{ach}}(dP)$. Now, each multiple access or broadcast channel can be replaced by a set of d bit-pipes whose rates are jointly constrained by the corresponding capacity constraints. It turns out that this falls inside a class of networks called polymatroidal networks, that have been already studied [19]. We will now give a short description of polymatroidal networks and some results for these networks.

D. Polymatroidal Networks

Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We have considered networks in the previous chapter with capacity constraints on the edges. A polymatroidal network has more general capacity constraints coupling edges that meet at a node. In the polymatroidal network, for each node v there are two associated submodular functions: ρ^{In}_v and ρ^{Out}_v which impose joint capacity constraints on the edges in $\mathsf{In}(v)$ and $\mathsf{Out}(v)$ respectively. That is, for any set of edges $S \subseteq \mathsf{In}(v)$, the total capacity available on the edges in S is constrained to be at most $\rho^{\mathsf{In}}_v(S)$. Similarly, ρ^{Out}_v constrains edges in $\mathsf{Out}(v)$.

For any subset of edges $F \subseteq \mathcal{E}$, we define the disconnection set $K(F)$ as the set of indices i for which source s_i has no paths to destination d_i in $\mathcal{G} \setminus F$. In standard networks, the value of the cut F is simply $\sum_{e\in F} c(e)$ where $c(e)$ is the capacity constraint on edge e. The value of a cut F in polymatroidal networks is defined as follows: each edge (u,v) in F is first assigned to either u or v; we say that an assignment of edges to nodes $g : F \to V$ is *valid* if it satisfies this restriction. Once this assignment is made, we can compute the value of the cut according to this assignment by evaluating the submodular functions corresponding to the set of edges grouped together. The value of the cut $\nu(F)$ is the minimum over all assignments, that is,

$$\nu(F) := \min_{g:F\to V, g \text{ valid}} \sum_v \left[\rho_v^{\mathsf{In}}(\mathsf{In}(v)\cap g^{-1}(v)) \right. \\ \left. +\rho_v^{\mathsf{Out}}(\mathsf{Out}(v)\cap g^{-1}(v))\right]. \quad (28)$$

A max-flow min-cut theorem for the single unicast problem in directed polymatroidal networks is known in the literature [24], [25]. For the k-unicast problem in a directed polymatroidal network with symmetric demands, the following theorem is proved in [19], which generalizes the result of [15] from edge capacity constraints to polymatroidal capacity constraints. The weak edge-cut bound for symmetric-demand polymatroidal networks is defined similarly as for standard networks.

Theorem 5. *(from [19]) For a symmetric-demand directed polymatroidal network with k source-destination pairs, any rate tuple in the weak edge-cut rate region divided by a factor $\kappa(\log^3(k+1))$ is achievable by routing, where κ is a universal constant.*

If $\mathcal{R}^{\text{poly}}_{\text{weak-edge-cut}}$ stands for the weak edge-cut region in the polymatroidal network and $\mathcal{R}^{\text{poly}}_{\text{ach}}$ stands for the region achievable by flow in the polymatroidal network, then Theorem 5 can be rewritten as:

$$\frac{\mathcal{R}^{\text{poly}}_{\text{weak-edge-cut}}}{\kappa \log^3(k+1)} \subseteq \mathcal{R}^{\text{poly}}_{\text{ach}}. \quad (29)$$

E. Analysis of Achievable Rates in Gaussian MAC+BC Network

The proposed separation strategy converts the Gaussian MAC+BC network into a directed polymatroidal network with symmetric demands. Using the achievable rates for the corresponding MAC and BC channels from (24) and (26), we can see that this polymatroidal network has the following submodular functions at any given node v,

$$\rho_v^{\mathsf{In}}(S) = \sum_c \log\left(1+\sum_{(uv)\in S}|h^c_{vu}|^2 P\right), \quad (30)$$

$$\rho_v^{\mathsf{Out}}(S) = \sum_c \log\left(1+\sum_{(vu)\in S}|h^c_{vu}|^2 \frac{P}{d}\right). \quad (31)$$

This fully defines the polymatroidal network. Now any rate tuple achievable on this polymatroidal network is achievable in the Gaussian MAC+BC network using the proposed separation architecture.

Now we can use Theorem 5 to show that the achieved rate tuple in the Gaussian network is within a poly-logarithmic factor of the weak edge-cut bound in the polymatroidal network, i.e.

$$\frac{\mathcal{R}^{\text{poly}}_{\text{weak-edge-cut}}(P)}{\kappa \log^3(k+1)} \subseteq \mathcal{R}^{\text{poly}}_{\text{flow}}(P) = \mathcal{R}^{\text{g}}_{\text{ach}}(P). \quad (32)$$

Here we have parametrized all the rate regions by the power constraint in order to make this dependence explicit. In order to prove our main result, we still need to connect the weak edge-cut bound in the polymatroidal network to the weak edge-cut bound in the Gaussian network.

We will connect the value of the cut in the polymatroidal network to the value of the cut in the Gaussian network. Let us take a cut derived from a set of edges F in the polymatroidal network and a valid assignment g of F, which yields the minimum among all possible valid assignments. In this assignment, each edge (u,v) of F is assigned either to u or to v. Thus for any node, some incoming edges are assigned together and some outgoing edges are assigned together and the value of the cut is given by

$$\nu(F) = \sum_v [\rho_v^{\mathsf{In}}(\mathsf{In}(v)\cap g^{-1}(v)) + \rho_v^{\mathsf{Out}}(\mathsf{Out}(v)\cap g^{-1}(v))].$$

Note that each of these functions ρ_v^{In} and ρ_v^{Out} corresponds to the constraints in the achievable region of the original MAC and broadcast channels. If we take the cut corresponding to F in the original network, then these functions will be replaced by the functions corresponding to the cut in the MAC and broadcast channels, whose equations are given in (25) and (27). It has been observed earlier that,

$$\mathcal{R}^{\text{MAC}}_{\text{cut}}(P) \subseteq \mathcal{R}^{\text{MAC}}_{\text{ach}}(dP) \quad (33)$$

$$\mathcal{R}^{\text{BC}}_{\text{cut}}(P) \subseteq \mathcal{R}^{\text{BC}}_{\text{ach}}(dP). \quad (34)$$

Note that the value of any cut in the polymatroidal network is a function of the power constraint implicitly, since ρ^{In} and ρ^{Out} are functions of the power constraint. Let us call this function $v(P)$. Now if we look at the same cut in the Gaussian network then the value of this cut here is at most $v(dP)$ because of (33) and (34). Thus the weak edge-cut region in the polymatroidal network and the weak edge-cut region in the Gaussian network can be related to each other as follows,

$$\mathcal{R}^{\text{g}}_{\text{weak-edge-cut}}(P) \subseteq \mathcal{R}^{\text{poly}}_{\text{weak-edge-cut}}(dP), \quad (35)$$

or alternately $\mathcal{R}^{\text{g}}_{\text{weak-edge-cut}}(\frac{P}{d}) \subseteq \mathcal{R}^{\text{poly}}_{\text{weak-edge-cut}}(P)$. This result, when combined with (32) and (23), yields the following relationship,

$$\frac{\mathcal{R}^{\text{g}}_{\text{weak-edge-cut}}(\frac{P}{d})}{\kappa \log^3(k+1)} \subseteq \mathcal{R}^{\text{g}}_{\text{ach}}(P) \subseteq \mathcal{C} \\ \subseteq \mathcal{R}^{\text{g}}_{\text{weak-edge-cut}}(P), \quad (36)$$

which completes the proof of Theorem 3.

V. Discussion and Extensions

This paper has focused on k-pair unicast networks with symmetric demands. Several extensions of these results can be obtained easily. We omit the full details of these extensions to keep the paper focused.

First, we can consider the following instance of traffic symmetry in wireline networks is that of a groupcast network. A *k-groupcast directed network* is a $k(k-1)$-unicast directed network $\mathcal{N}$ with k distinct distinguished nodes (group-nodes) $v_1, v_2, \ldots, v_k$ such that each group-node has an independent message for every other group-node. We can consider the maximum sum-rate for a groupcast directed network. Let $F_{\text{sum}-\text{rate}}, C_{\text{sum}-\text{rate}}$ denote the maximum flow sum-rate and capacity sum-rate respectively. Also let R_{multicut} denote the minimum possible sum of the capacities of a set of edges whose removal disconnects each v_i from each v_j, $i \neq j$. Then.

$$\frac{1}{2} R_{\text{multicut}} \leq F_{\text{sum}-\text{rate}} \leq C_{\text{sum}-\text{rate}} \leq 2R_{\text{multicut}} \qquad (37)$$

The inequality $\frac{1}{2}R_{\text{multicut}} \leq F_{\text{sum}-\text{rate}} \leq R_{\text{multicut}}$ was proved by Naor and Zosin in [18] and $C_{\text{sum}-\text{rate}} \leq 2R_{\text{multicut}}$ is a simple consequence of the GNS bound.

Secondly, these results can be extended to various other network models. They are listed here:

- Degrees-of-freedom approximation in fixed Gaussian channels. For a k-pair unicast directed wireless network with symmetric demands, if the fixed channel coefficients are drawn from a continuous distribution, the DOF region given by $\mathcal{D}_{\text{ach}}$ satisfying

$$\frac{\mathcal{D}_{\text{weak}-\text{edge}-\text{cut}}}{\kappa \log^3(k+1)} \subseteq \mathcal{D}_{\text{ach}} \subseteq \mathcal{D} \subseteq \mathcal{D}_{\text{weak}-\text{edge}-\text{cut}}, \qquad (38)$$

  is achievable with probability 1, where κ is a universal constant.
- Capacity approximation in ergodic fading channels. Consider a symmetric weak-tailed ergodic fading distribution with fading co-efficient h satisfying $\mathbb{E}|h|^2 = 1$, and $a = e^{-\mathbb{E}(\log|h|^2)} < \infty$. For a k-pair directed Gaussian network with symmetric demands and such a symmetric weak-tailed ergodic fading distribution, the rate region given by $\mathcal{R}_{\text{ach}}(P)$ satisfying

$$\frac{\mathcal{R}_{\text{weak}-\text{edge}-\text{cut}}(\frac{P}{ad^3})}{\kappa \log^3(k+1)} \subseteq \mathcal{R}_{\text{ach}}(P) \subseteq \mathcal{C}(P) \subseteq \mathcal{R}_{\text{weak}-\text{edge}-\text{cut}}(P), \qquad (39)$$

  is achievable, where κ is a universal constant, d is the maximum degree of any node.
- Broadcast erasure networks with feedback. A broadcast erasure channel is a packet transmission channel where a transmitter can transmit packets to d receivers each of which either receives the packet or the packet gets erased, the erasures being independent and occurring with probability ϵ. The erasure channel has feedback if the transmitter after a transmission knows the realization of erasures at each of its receivers before the next transmission. A broadcast erasure network is a network comprised of broadcast erasure channels.

  For a k-pair unicast broadcast erasure network with feedback and symmetric demands and with probability of erasure ϵ, a separation strategy can achieve a rate given by $\mathcal{R}_{\text{ach}}$, where

$$\frac{\mathcal{R}_{\text{weak}-\text{edge}-\text{cut}}(\epsilon)}{\kappa\psi(\epsilon, d)\log^3(k+1)} \subseteq \mathcal{R}_{\text{ach}}(\epsilon) \subseteq \mathcal{C}(\epsilon) \subseteq \mathcal{R}_{\text{weak}-\text{edge}-\text{cut}}(\epsilon), \qquad (40)$$

  where κ is a universal constant, d is the maximum degree of any broadcast channel in the network and $\psi(\epsilon, d) = \left[\sum_{i=1}^{d} \frac{\epsilon^{i-1}-\epsilon^i}{1-\epsilon^i}\right]^{-1} \leq \log_2(d+1)$.

Finally, the three results above can all be extended similarly to the sum-rate in the groupcast traffic model.

At a philosophical level, it is intriguing that the kind of symmetry that allows one to prove closeness of flow and edge-cuts (undirected networks, networks with symmetric demands, groupcast networks) also leads to the near-fundamentality of such edge-cuts. It would be interesting to see whether there is a deeper explanation of this phenomenon.

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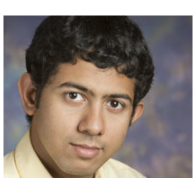
Sudeep Kamath is a postdoctoral scholar at the University of California, San Diego. He received the Bachelor of Technology degree in Electrical Engineering from the Indian Institute of Technology Bombay in 2008. He received the M.S. degree in Electrical Engineering and Computer Science in 2011 and the M.A. degree in Statistics, and Ph.D. degree in Electrical Engineering and Computer Science in 2013, all from the University of California, Berkeley. He is a recipient of the Eliahu Jury Award from the EECS Department of UC Berkeley (2013) and the Institute Silver Medal from the EE Department at IIT Bombay (2008).



Sreeram Kannan is a postdoctoral researcher at the University of California, Berkeley. He received his Ph.D. in Electrical Engineering and M.S. in mathematics from the University of Illinois Urbana Champaign. He is a recipient of the Van Valkenburg research award from UIUC, 2013, a co-recipient of the Qualcomm Cognitive Radio Contest first prize, 2010, a recipient of Qualcomm Roberto Padovani Scholarship, 2010, a recipient of the S.V.C. Aiya medal from the Indian Institute of Science, 2008, and a co-recipient of Intel India Student Research Contest first prize, 2006. His research interests include information theory, wireless networks, approximation algorithms and computational biology.

Pramod Viswanath received the PhD degree in EECS from the University of California at Berkeley in 2000. He was a member of technical staff at Flarion Technologies until August 2001 before joining the ECE department at the University of Illinois, Urbana-Champaign. He is a recipient of the Xerox Award for Faculty Research from the College of Engineering at UIUC (2010), the Eliahu Jury Award from the EECS department of UC Berkeley (2000), the Bernard Friedman Award from the Mathematics department of UC Berkeley (2000), and the NSF CAREER Award (2003). He was an associate editor of the IEEE Transactions on Information Theory for the period 2006-2008.