Concentration of Measure

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Review

What is concentration?

"A random variable that depends in a smooth way on many independent random variables (but not too much on any of them) is essentially constant."
- M. Talagrand, 1996.

If Z is a function of many independent variables $X_1, X_2, \ldots, X_n$, under what conditions can we say typical deviations of Z are small?

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- Low variance captured by a general theorem: *tensorization of variance*

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- Low variance captured by a general theorem: *tensorization of variance*
- Today, we'll see the entropy method, a general tool to show sub-Gaussian tails

Entropy

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$$\frac{\mathrm{Ent}(Z)}{\mathbb{E}Z} = D\left(P_{\frac{Z}{\mathbb{E}Z}} || P\right)$$

(where if $Q = P_Y$, then $Q(A) = \mathbb{E}[Y 1_A]$, $\mathbb{E}_Q[W] = \mathbb{E}[YW]$)

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$$X^{(i)} := (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n), \mathrm{Ent}^{(i)}(Z) := \mathrm{Ent}(Z|X^{(i)})$$

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$$\mathrm{Ent}(Z) \leq \sum_{i=1}^{n} \mathbb{E}[\mathrm{Ent}^{(i)}(Z)]$$

Log Moment Generating Functions (log m.g.f.)

With $\mathbb{E}Z = 0$, let $\psi(\lambda) := \log \mathbb{E}e^{\lambda Z}, \quad \lambda \in \mathbb{R}$

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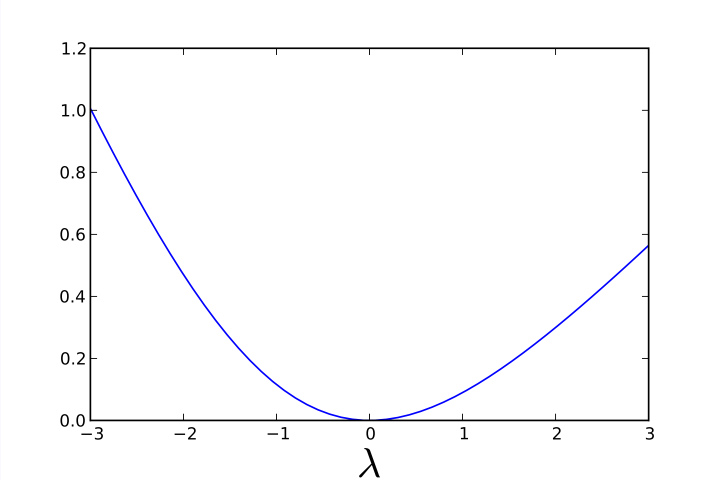

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Herbst argument

$$\text{If } \frac{\mathrm{Ent}(e^{\lambda Y})}{\mathbb{E}[e^{\lambda Y}]} \leq \frac{\lambda^2\sigma^2}{2} \ \forall\ \lambda > 0,$$

$$\text{then } \log \mathbb{E}[e^{\lambda(Y-\mathbb{E}Y)}] \leq \frac{\lambda^2\sigma^2}{2} \ \forall\ \lambda > 0$$

Herbst condition equivalent to sub-Gaussianity

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Note: $\frac{\mathrm{Ent}(e^{\lambda Y})}{\mathbb{E}[e^{\lambda Y}]} = D\left(P_{e^{\lambda Y}/\mathbb{E}[e^{\lambda Y}]} || P\right)$. (Proof on board)

Hoeffding's lemma

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If $Z \in [a, b]$, then Z is $\big((b-a)^2/4\big)$-sub-Gaussian,

i.e. $\psi(\lambda) = \log \mathbb{E} e^{\lambda(Z - \mathbb{E}Z)} \le \frac{\lambda^2}{2} \frac{(b-a)^2}{4} \;\; \forall \;\; \lambda \in \mathbb{R}$

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$$\psi''(\lambda)$$

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$$\psi''(\lambda) = \mathrm{Var}_Q(Z) \leq (b-a)^2/4$$

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Proof 2

$$\frac{\mathrm{Ent}(e^{\lambda Y})}{\mathbb{E}[e^{\lambda Y}]} = \lambda\psi'(\lambda) - \lambda = \int_0^\lambda \theta\psi''(\theta)\, d\theta \leq \frac{\lambda^2}{2} \frac{(b-a)^2}{4}$$

Then, Herbst argument.

Bounded differences inequality

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Bounded differences inequality (McDiarmid's inequality)

Suppose

$$|f(x_1, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n)| \leq c_i.$$

If $X = (X_1, X_2, \ldots, X_n)$ has independent components, then $Z = f(X_1, X_2, \ldots, X_n)$ is σ^2- sub-Gaussian with

$$\sigma^2 = \frac{1}{4} \sum_{i=1}^{n} c_i^2.$$

Therefore, $\log \mathbb{E} e^{\lambda(Z - \mathbb{E}Z)} \leq \frac{\lambda^2 \sigma^2}{2} \ \forall\ \lambda \in \mathbb{R}$ and

$$\mathbb{P}[|Z - \mathbb{E}Z| \geq t] \leq 2e^{-\frac{t^2}{2\sigma^2}} \ \forall t > 0$$

Bounded differences inequality: proof

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Proof of bounded differences inequality

This can be proved by the entropy method as follows.

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\begin{aligned}
\mathrm{Ent}(e^{\lambda Z}) &\le \sum_{i=1}^{n} \mathbb{E}[\mathrm{Ent}^{(i)}(e^{\lambda Z})] \\
&\le \sum_{i=1}^{n} \mathbb{E}\left[\frac{\lambda^2}{2}\frac{c_i^2}{4}\mathbb{E}^{(i)}[e^{\lambda Z}]\right] \\
&= \frac{\lambda^2}{2}\left(\sum_{i=1}^{n}\frac{c_i^2}{4}\right)\mathbb{E}[e^{\lambda Z}]
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Herbst argument completes the proof.

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But more: $\mathbb{P}[|Z - \mathbb{E}Z| \geq t] \leq 2e^{-2t^2/n}$

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$$\left| a \log \frac{1}{a} - b \log \frac{1}{b} \right| \leq \frac{\log n}{n} \text{ if } |a - b| = \frac{1}{n}.$$

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Thus, $\mathrm{Var}(Z) \leq \sum_{i=1}^{n} c_i^2/4 = (\log^2 n)/n$

But more: $\mathbb{P}[|Z - \mathbb{E}Z| \geq t] \leq 2e^{-t^2 \log^2 n/(2n)}$

$x^{(i)}$-dependent bounded differences

Suppose

$$|f(x_1, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n)| \leq c_i(x^{(i)}).$$

If $X = (X_1, X_2, \ldots, X_n)$ has independent components, then $Z = f(X_1, X_2, \ldots, X_n)$ is σ^2- sub-Gaussian with

$$\sigma^2 = \frac{1}{4} \sup_x \left(\sum_{i=1}^{n} c_i^2(x^{(i)}) \right).$$

Therefore, $\log \mathbb{E} e^{\lambda(Z - \mathbb{E}Z)} \leq \dfrac{\lambda^2 \sigma^2}{2} \ \forall \ \lambda \in \mathbb{R}$ and

$$\mathbb{P}[|Z - \mathbb{E}Z| \geq t] \leq 2e^{-\frac{t^2}{2\sigma^2}} \ \forall t > 0$$

Proof of $x^{(i)}$-dependent bounded differences inequality

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Herbst argument completes the proof.

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Herbst argument completes the proof.

Proof does not work if $c_i(x^{(i)})$ is replaced by $c_i(x)$.

Proof of $x^{(i)}$-dependent bounded differences inequality

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\operatorname{Ent}(e^{\lambda Z}) &\leq \sum_{i=1}^{n} \mathbb{E}[\operatorname{Ent}^{(i)}(e^{\lambda Z})] \\
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&\leq \sum_{i=1}^{n} \mathbb{E}\left[\frac{\lambda^2}{2}\mathbb{E}^{(i)}\left[\frac{c_i^2(X^{(i)})}{4}e^{\lambda Z}\right]\right] \\
&\leq \frac{\lambda^2}{2}\sigma^2\mathbb{E}[e^{\lambda Z}]
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We will see later that the transportation method works in this case.

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where $\mathscr{E}(f) = \dfrac{1}{2}\mathbb{E}\left[\sum_{i=1}^{n}(f(X) - f(\tilde{X}^{(i)}))^2\right].$

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Proof of Bernoulli log-Sobolev inequality

$n = 1$ case: Calculus exercise

$$\frac{a^2}{2}\log a^2 + \frac{b^2}{2}\log b^2 - \frac{a^2+b^2}{2}\log\frac{a^2+b^2}{2} \leq \frac{1}{2}(a-b)^2$$

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$n > 1$: Tensorization of entropy

By replacing f with $e^{\lambda f/2}$ and using Herbst argument, we can show concentration inequalities on the Boolean hypercube

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Note: tight if f is linear.

Proof of Gaussian concentration

Suppose f is differentiable and apply the Gaussian log-Sobolev inequality to $e^{\lambda f/2}$.

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If not differentiable, convolve f with a smooth kernel.

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Let $y_1, y_2, \ldots, y_n$ be n points in $\mathbb{R}^n$. For every $0 < \epsilon < 1$, there is a linear map $T : \mathbb{R}^n \mapsto \mathbb{R}^k$ where $k \geq \dfrac{24 \log n}{\epsilon^2}$ so that for every $1 \leq i, j \leq n$,

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Then, apply union bound

Summary

- Entropy method: a general tool to show sub-Gaussian tails for functions of many independent random variables
- It is especially powerful in conjunction with log-Sobolev inequalities
- We'll see another general tool, the transportation method next time