Concentration of Measure

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PRINCETON UNIVERSITY

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What is concentration?

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Goal: Quantify by bounding for $t > 0$,

$$\mathbb{P}\left[|Z - \mathbb{E}Z| \geq t\right]$$

Applications

Concentration of measure has far-reaching consequences in

- Pure and applied probability,
- High-dimensional statistics,
- Functional analysis,
- Computer science,
- Machine learning,
- Statistical physics,
- Complex graphs and networks,
- Information theory, communication and coding theory.

Approaches for Proving Concentration

- *The martingale approach*: Hoeffding (1963), Azuma (1967), Milman and Schechtman (1986), Shamir and Spencer (1987) and McDiarmid (1989, 1998), Sipser and Spielman (1996), Richardson and Urbanke (2001)
- *Talagrand's inequalities for product measures*: Talagrand (1996).
- *Entropy method and log-Sobolev inequalities*: Ledoux (1996), Massart (1998), Lugosi et al. (1999, 2001)
- *Transportation method*: Ahlswede, Gács and Körner (1976), Marton (1986, 1996, 1997), Dembo (1997), Villani (2003, 2008)
- *Stein's method of exchangeable pairs*: Chatterjee (2007), Chatterjee and Dey (2010), Goldstein et al. (2011, 2014)

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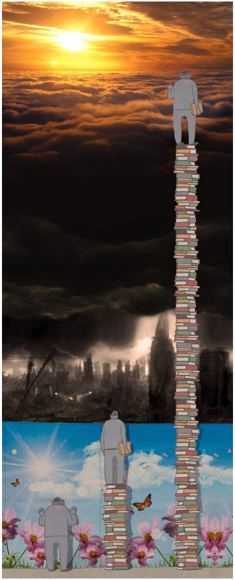
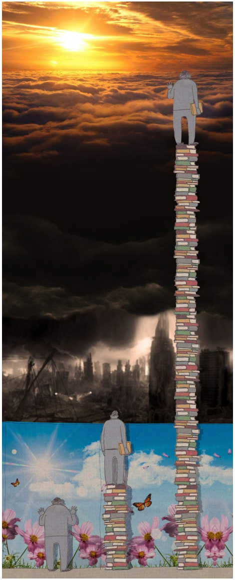
We will focus on the entropy method and transportation method where information theoretic methods shine.





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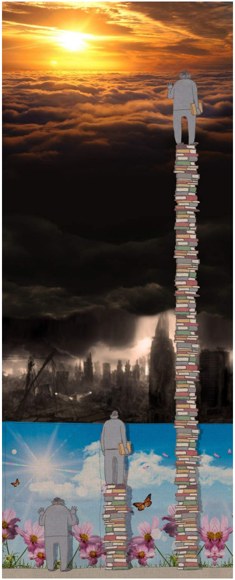

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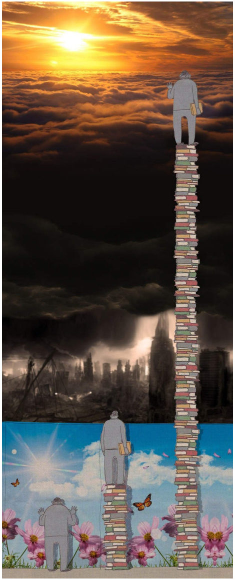

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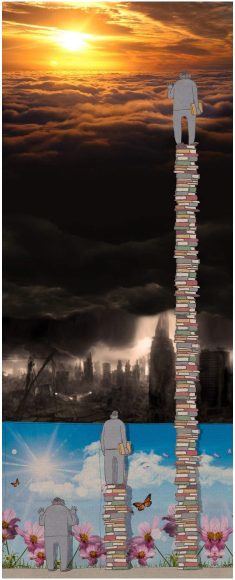

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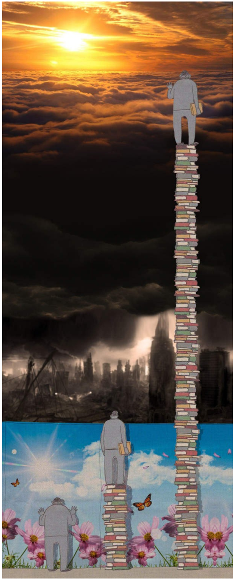

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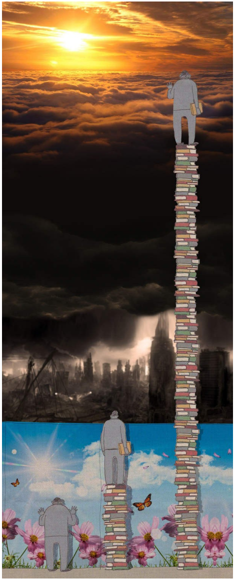

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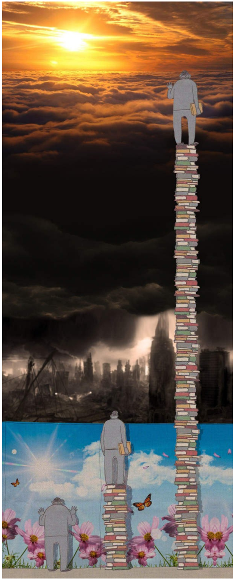

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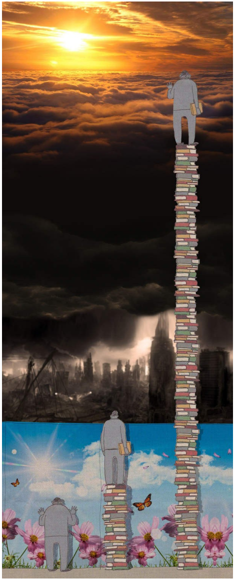

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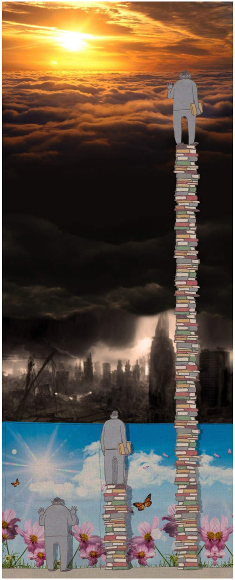

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None	Monotone and dominated convergence theorem
None	Non-measurable subsets of $\mathbb{R}$

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Role of information theory

"The emphasis put on information theoretic methods is one main feature of the exposition and there is considerable benefit in this approach for a number of fundamental results [...]"
- M. Ledoux, foreword to 'Concentration Inequalities' by Boucheron, Lugosi, Massart.

Roadmap

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Slides available on my homepage:
`http://www.princeton.edu/~sukamath/concentration.pdf`

Variance bounds

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Say Z is a function of independent random variables $X_1, X_2, \ldots, X_n$. An upper bound on $\mathrm{Var}(Z)$ gives tail bounds as:

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Trivial example

Let $Z = X_1 + X_2 + \ldots + X_n$ where $\{X_i\}_{i=1}^n$ are independent and identically distributed (i.i.d.) with finite variance. Then,

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Mean $= \Theta(n)$, Standard Deviation $= O(\sqrt{n})$.

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Populate an $m \times n$ matrix A by independent entries, each taking values in $[0, 1]$. The random variable $Z = \|A\|$ satisfies

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Let Z be the estimate of entropy of an unknown distribution defined by the entropy of the empirical distribution from drawing n independent samples.

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$$\mathrm{Var}(Z) \leq \frac{\log^2 n}{n}$$

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- When it is, we say the quantity *tensorizes*
- Quantities that tensorize behave well in high dimension
- Variance is such a quantity!

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Simplest application: Bounded differences inequality

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Tight if $f(X) = \sum_{i=1}^n X_i$ with X_i equiprobable on $\{-1, +1\}$

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Standard deviation $= O(\sqrt{n})$, Mean $= \Theta(n)$.

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Thus, $\mathrm{Var}(Z) \leq \sum_{i=1}^{n} c_i^2/4 = (\log^2 n)/n$

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Thus, $\mathrm{Var}(Z) \leq \sum_{i=1}^{n} c_i^2/4 = (\log^2 n)/n$

But Z is not really concentrated at $H(p)$ unless $n \gtrsim k$.

Plug-in entropy estimation

Entropy of a distribution $p = (p_1, p_2, \dots, p_k)$ is defined as

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For $n << k$, Z is concentrated but somewhere else!

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- Can estimate $\mathbb{E}Z$ from Monte Carlo methods if Z is concentrated

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Variant : "guess functions"

$$\mathrm{Var}(Z) \leq \sum_{i=1}^{n} \mathbb{E}[\mathrm{Var}^{(i)}(Z)] \leq \sum_{i=1}^{n} \mathbb{E}[(Z - Z_i)^2]$$

Convex Lipschitz functions

Suppose $f : [a, b]^n \mapsto \mathbb{R}$ is convex, differentiable and L-Lipschitz, i.e. $|f(x) - f(y)| \leq L\|x - y\|_2 \ \forall \ x, y$.

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Differentiability assumption unnecessary: convolve f with a smooth kernel.

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With $a = 0, b = 1, L = 1$ in previous result, we get

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Largest eigenvalue of a random matrix

Let A be an $n \times n$ symmetric matrix with independent entries $X_{ij}, 1 \leq i \leq j \leq n$ independent, $-1 \leq X_{ij} \leq 1$.

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Let Z_{ij} denote $\lambda_{\max}$ for the matrix $\bar{A}^{ij}$ which is same as the matrix A except $X_{ij} = X_{ji}$ gets replaced by an independent copy $X'_{ij} = X'_{ji}$.

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$$Z - Z_{ij} = u^{\mathrm{T}} A u - \max_{\|w\|=1} w^{\mathrm{T}} \bar{A}^{ij} w$$

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Thus, $\mathrm{Var}(Z) \le 16$.

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- Can a general principle capture superconcentration? Active research area

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If $X \sim \mathscr{N}(0, I_n)$, and $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable, then

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Note: The bounds are tight if f is linear!

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$$\text{Let } S_m = \frac{1}{\sqrt{m}} \left(Y_1 + Y_2 + \ldots + Y_m \right).$$

$$\mathrm{Var}^{(i)}(f(S_m)) = \frac{1}{4} \left(f\left(S_m - \frac{Y_i}{\sqrt{m}} + \frac{1}{\sqrt{m}} \right) - f\left(S_m - \frac{Y_i}{\sqrt{m}} - \frac{1}{\sqrt{m}} \right) \right)^2$$

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&\le \frac{1}{4} \left(\frac{2}{\sqrt{m}} |f'(S_m)| + \frac{2K}{m} \right)^2
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&\le \frac{1}{4}\left(\frac{2}{\sqrt{m}}|f'(S_m)| + \frac{2K}{m}\right)^2 = \frac{1}{m}\left(|f'(S_m)| + \frac{K}{\sqrt{m}}\right)^2
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As $m \to \infty$, we have $S_m \to X \sim \mathcal{N}(0,1)$ in distribution by the Central Limit Theorem.

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Extend to all continuously differentiable functions by

- Truncation of f to $[-M, M]$ and apply dominated convergence theorem as $M \to \infty$
- Smoothen truncated f by convolution with a sharply concentrated twice differentiable kernel with compact support

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Now, if $X \sim \mathscr{N}(0, I_n)$ is an n-dimensional Gaussian vector and $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable, use tensorization of variance again.

$$\mathrm{Var}(f(X)) \leq \sum_{i=1}^{n} \mathbb{E}\left[\mathrm{Var}^{(i)}(f(X))\right]$$

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&\leq \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}^{(i)}\left|\frac{\partial f}{\partial x_i}(X)\right|^2\right]
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&\leq \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}^{(i)}\left|\frac{\partial f}{\partial x_i}(X)\right|^2\right] \\
&= \mathbb{E}\left[\|\nabla f(X)\|^2\right]
\end{aligned}$$