1) To design a flow network with multiple non-unique max-flow functions, we can use this flow network: V = {s, a, b, t}. E = {(s, a), (s, b), (a, t), (b, t)}. s = source. t = sink

The capacity function $c: V \times V \rightarrow \{0, 1\}$ for the given flow network is as follows:

$c(s, a) = 1$

$c(s, b) = 1$

$c(a, t) = 1$

$c(b, t) = 1$

The given flow network has four vertices: source (s), two intermediate vertices (a and b), and the sink (t). There are four edges in total: (s, a), (s, b), (a, t), and (b, t).

The capacity function c(v, u) assigns a capacity of 1 to all edges in the flow network. This means that each edge can carry a maximum flow of 1 unit.

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  s
 / \
a   b
 \ /
  T
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Max-flow function 1:

Assigning flow values to the edges:

$f(s, a) = 1$

$f(s, b) = 0$

$f(a, t) = 1$

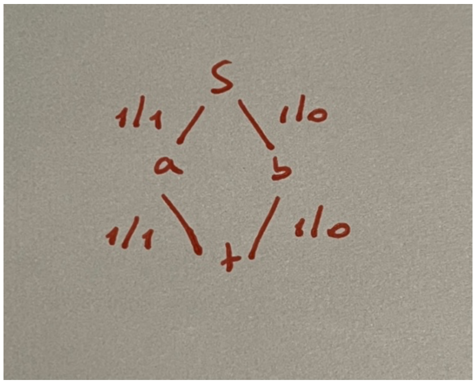
$f(b, t) = 0$

In this case, the value of the max-flow is:

$|f| = f(s, a) = f(a, t) = 1$

In this case, we assign a flow of 1 unit to both edges (s, a) and (a, t). Then, we assign a flow of 0 unit to both edges (s, b) and (b, t).

The value of the max-flow is the sum of the flows entering the sink, which is given by $|f| = f(s, a) = f(a, t) = 1$



Max-flow function 2:

Assigning flow values to the edges:

f(s, a) = 0

f(s, b) = 1

f(a, t) = 0

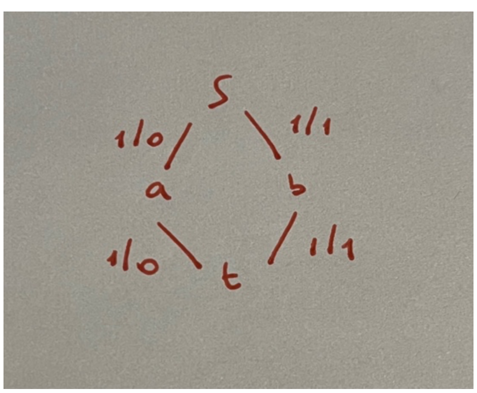
f(b, t) = 1

In this case, the value of the max-flow is:

|f| = f(s, b) = f(b, t) = 1

In this case, we assign a flow of 1 unit to both edges (s, b) and (b, t). However, we assign a flow of 0 units to both edges (s, a) and (a, t).

Again, the value of the max-flow is the sum of the flows entering the sink, which is given by |f| = f(s, b) = f(b, t) = 1



Both max-flow functions have the same value of 1, but they differ in the flow values assigned to the edges (specifically, in the flow values of the edges connected to the sink node t).

2) Claim A is true. There is only one max-flow function for flow networks where each edge has unique capacity.

By contradiction, we may demonstrate this. Assume that the maximum value of the flow in G is achieved by two different max-flow functions, f and g. Let h equal f-g. Since the flow into each node is equal to the flow out of each node, h is a valid flow function in G with value 0.

Additionally, h has the feature that if an edge (u, v) in E has a positive capacity for any two different vertices u and v in V, then h(u, v) = 0. This is due to the fact that if h(u, v) > 0, then either f(u, v) > g(u, v) or g(u, v) > f(u, v) > h(u, v), in which case we might increase the flow along (u, v) by reducing h(u, v). Therefore, h is a legitimate flow function with a value of 0, satisfying the requirement that if an edge (u, v) in E has a positive capacity for any two different vertices u and v in V, then h(u, v) = 0.

The residual graph has a path from s to v with regard to h, and T = V \ S defines the cut (S,T) defined by S = v ∈ V. We know that s ∈ S and t ∈ T since h is a legitimate flow function with a value of 0. Additionally, we need c(u, v) > 0 for any edge (u, v) in E with u ∈ S and v ∈ T, as otherwise h(u, v) = f(u, v) - g(u, v) = 0 - 0 = 0, which is incongruent with the fact that (u, v) is in the cut. As a result, the cut has a positive capacity.

The capacity of any cut is equal to the value of the flow passing that cut, on the other hand, because f and g are both max-flow functions with the same value. Since f and g have similar values, the capacity of the cut (S, T) is also equal to them. However, this disproves our assumption that f and g are distinct max-flow functions. As a result, G can only have one max-flow function.

3) No, F is not guaranteed to be a flow on G.

F should satisfy the constraints of a flow function, which are:

Capacity constraint: For all edges (u, v) in E, 0 ≤ F(u, v) ≤ c(u, v), where c(u, v) is the capacity of edge (u, v).

Skew symmetry: For all u,v in V, F(u,v) = -F(v,u).

Flow conservation: For all u in V-{s,t}, the total flow into u is equal to the total flow out of u.

Consider a flow network G with three nodes: s (source), v (intermediate), and t (sink). Let's have two edges e1=(s,v) and e2=(v,t) with capacities c(e1)=2 and c(e2)=1.

Let's define flow functions f1 and f2 as follows:

f1(s,v) = 2, f1(v,t) = 1

f2(s,v) = 1, f2(v,t) = 2

Both f1 and f2 satisfy the capacity constraint individually since the flow values are within the respective edge capacities.

Now, let's calculate F(u, v) = f1(u, v) + f2(u, v) for all edges (u, v):

F(s,v) = f1(s,v) + f2(s,v) = 2 + 1 = 3

F(v,t) = f1(v,t) + f2(v,t) = 1 + 2 = 3

The values of F(s,v) and F(v,t) both equal 3, which exceeds the capacity of edge e1=(s,v). Therefore, F does not satisfy the capacity constraint for this flow network.

f1 + f2 satisfies skew-symmetry. We know flows f1 and f2 individually satisfy skew-symmetry if they are in N. Therefore for any (u, v), we have (f1 +f2)(u, v) = f_1(u, v)+f_2(u, v) = −f1(v, u)−f2(v, u) = −(f1 +f2)(v, u). It also satisfies flow conservation. Flow conservation for a flow f states that for all u ∈ V \ {s, t}, we have ∑f(u, v) = 0. This holds for individually for f1, f2. Hence, we have ∑(f1 + f2)(u, v) = ∑(f1(u, v) + f2(u, v) = ∑ f1(u, v) + ∑ f2(u, v) = 0 + 0 = 0.