STAT 443: Time Series and Forecasting

Lab 10: Spectral density estimation

- The lab must be completed in `R Markdown`. Display all the `R` code used to perform your analysis.
- Create a pdf or html file and use it as your lab submission.
- Please ensure that the file you submit is in good order (e.g., not corrupted and contains the work you intend to submit). No late (re-)submissions will be accepted.

We have met the idea of the *spectral density function,* $f(\omega)$, of a stochastic process, this being the Fourier transform of the autocovariance function for the process. As we will see, there is a tool known as the *periodogram* which can estimate the spectral density function given a sample from the stochastic process. In effect, the periodogram is a histogram, and can be created in R using the command `spec.pgram()`. By default, this command plots the periodogram on the log scale, so in the following use the option `log="no"` for better comparisons. R uses $\omega/2\pi$ as frequency (per unit time), so the horizontal axis in the plot of the periodogram ranges from 0 to 0.5 rather than 0 to π. Moreover, as an estimate of the spectral density, the periodogram from R should be divided by π to be consistent with definitions provided in class. In what follows though it is the shape of the periodogram that is of most interest.

In the following, suppose that $\{Z_t\}_{t\in\mathbb{Z}}$ is white noise with variance 4.

1. Here we will compare the spectral density of white noise with the periodograms obtained from simulated white noise samples.

   (a) Plot the spectral density function for $\{Z_t\}_{t\in\mathbb{Z}}$.

   (b) Use the `arima.sim()` command, or otherwise, to simulate a series of length 100 from $\{Z_t\}_{t\in\mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.

   (c) Now simulate a series of length 1000 from $\{Z_t\}_{t\in\mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.

   (d) Repeat parts (b) and (c) several times. Comment on how the periodogram from R behaves as an estimator of the spectral density function based on what you have observed.

2. Let $\{X_t\}_{t\in\mathbb{Z}}$ be defined by

$$X_t = Z_t - 0.9Z_{t-1}.$$

   (a) Plot the spectral density function for $\{X_t\}_{t\in\mathbb{Z}}$; see in-class activity "Examples of Spectral Densities" (Question 1).

   (b) Use the `arima.sim()` command to simulate a series of length 100 from $\{X_t\}_{t\in\mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.

   (c) Use the `arima.sim` command to simulate a series of length 1000 from $\{X_t\}_{t\in\mathbb{Z}}$.Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.

   (d) Repeat parts (b) and (c) several times. Comment on how the periodogram from R behaves as an estimator of the spectral density function based on what you have observed.

3. The process

$$X_t = 0.8X_{t-1} + Z_t$$

has spectral density function

$$f(\omega) = \frac{2^2}{\pi\left(1 - 1.6\cos(\omega) + 0.8^2\right)}, \qquad \omega \in (0, \pi).$$

   (a) Plot the spectral density function for $\{X_t\}_{t\in\mathbb{Z}}$.

   (b) Use the `arima.sim()` command to simulate a series of length 100 from $\{X_t\}_{t\in\mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.

   (c) Use the `arima.sim()` command to simulate a series of length 1000 from $\{X_t\}_{t\in\mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.

   (d) Repeat parts (b) and (c) several times. Comment on how the periodogram from R behaves as an estimator of the spectral density function, based on what you have observed.