STAT 321: Assignment 1

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Problem 1

$$\mathbb{P}(At\ least\ 2\ students\ share\ a\ birthday) = 1 - \mathbb{P}(No\ students\ share\ a\ birthday) \tag{1}$$

$$Total\ number\ of\ students = 20$$

Let us assume there are 365 days in a year and that all birthdays are equally likely. Then,

$$\begin{aligned}\mathbb{P}(No\ students\ share\ a\ birthday) &= \frac{365}{365} \cdot \frac{365-1}{365} \cdot \frac{365-2}{365} \cdots \frac{365-20}{365} \\ &= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{345}{365} \\ &= \frac{365!}{345! \cdot 365^{20}} \\ &\approx 0.5886\end{aligned}$$

Putting our result back in (1),

$$\mathbb{P}(At\ least\ 2\ students\ share\ a\ birthday) = 1 - 0.5886 = 0.4114$$

So the probability of at least two students sharing a birthday is ≈ 0.4114

Problem 2

(a) FALSE

Let us assume $A \subset B$ such that $P(A) = 0.5$ and $P(B) = 0.6$ and $P(A \cap B) = 0.5$.

Now $P(A) \cdot P(B) = 0.3$.

So $P(A \cap B) = 0.5 \leq P(A) \cdot P(B) = 0.3$ does not hold.

Then *P(A ∩ B)* is not less than or equal to *P(A)P(B)*.

(b) FALSE

Let A be any the events in sample space Ω and $P_E(A) = P(A \cap E)$. We know that $0 < P(E) < 1$.

$E \in \Omega$ so $P_E(\Omega) = P(\Omega \cap E) = P(E)$

But $P(E) < 1$, which violates the Second Axiom of Probability, that $P(\Omega) = 1$

Thus, P_E does not satisfy the axioms of probability.

(c) TRUE

$$\begin{aligned}\mathbb{P}(A \cap B^C) &= \mathbb{P}(A - B)\\ &= \mathbb{P}(A) - \mathbb{P}(A \cap B)\end{aligned}$$

And,

$$\begin{aligned}\mathbb{P}(A \cup B) - \mathbb{P}(B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) - \mathbb{P}(B)\\ &= \mathbb{P}(A) - \mathbb{P}(A \cap B)\end{aligned}$$

Thus,

$$\mathbb{P}(A \cap B^C) = \mathbb{P}(A \cup B) - \mathbb{P}(B)$$

(d) TRUE

We have to prove,

$$P(\cap_{i=1}^{n} A_i) \leq \min_i [P(A_i)] \tag{1}$$

Let $n = 1$. Then,

$$P(\cap_{i=1}^{1} A_i) = P(A_1) \leq \min_i [P(A_1)]$$

Let $n = 2$. Then,

$$P(\cap_{i=1}^{2} A_i) = P(A_1 \cap A_2)$$

If $P(A) > P(B)$, we know that $P(A \cap B) \leq P(B)$. So,

$$P(\cap_{i=1}^{2} A_i) \leq \min_i [P(A_i)]$$

Now, let $n = k$ be true for (1),

$$P(\cap_{i=1}^{k} A_i) \leq \min_i [P(A_i)] \tag{2}$$

For $n = k + 1$, we know that $P(\cap_{i=1}^{k} A_i \cap A_{k+1}) \leq P(A_{k+1})$
So

$$P(\cap_{i=1}^{k+1} A_i) \leq \min_i [P(A_i)]$$

Thus, by induction Equation (1) holds true.

Problem 3

(a)

Computer 1 is connected to the network by 3 independent connections, each of which have a probability p of being broken at any time. For all these 3 connections to be broken,

$$\mathbb{P}(\textit{All 3 connections are broken}) = \mathbb{P}(\textit{Connector n being broken})^3$$

where n is any of the connection between computer 1 and the others. So,

$$\mathbb{P}(\textit{All 3 connections are broken}) = p^3$$

(b)

For $j = 2$,

There are 2 connections shared between Computer 1 and Computer 2. Therefore, for both Computer 1 and Computer 2 to be isolated, 4 wires will have to be broken. So,

$$\mathbb{P}(4 \textit{ connections are broken}) = p^4$$

For $j = 3$, There are no connections shared between Computer 1 and Computer 3. Therefore, for both Computer 1 and Computer 3 to be isolated, all 6 wires will have to be broken. So,

$$\mathbb{P}(\textit{All connections are broken}) = p^6$$

For $j = 4$, There is only 1 connection shared between Computer 1 and Computer 4. Therefore, for both Computer 1 and Computer 4 to be isolated, 5 wires will have to be broken. So,

$$\mathbb{P}(5 \textit{ connections are broken}) = p^5$$

(c)

$$\mathbb{P}(\textit{No computer is isolated}) = 1 - \mathbb{P}(\textit{At least } 1 \textit{ computer is isolated})$$

Now,

$$\begin{aligned}\mathbb{P}(\textit{At least } 1 \textit{ computer is isolated}) &= \mathbb{P}(1 \textit{ computer is isolated} \\ &\cup 2 \textit{ computers are isolated} \cup 3 \textit{ computer are isolated}) \\ &\cup \textit{All computers are isolated}) \\ &= \sum_{1\le i\le 4} \mathbb{P}(A_i) - \sum_{1\le i\le j\le 4} \mathbb{P}(A_i \cap A_j) + \sum_{1\le i\le j\le l\le 4} \mathbb{P}(A_i \cap A_j \cap A_l) \\ &- \sum_{1\le i\le j\le l\le h\le 4} \mathbb{P}(A_i \cap A_j \cap A_l \cap A_h)\end{aligned}$$

where each $A_x(x = i, j, l, h)$ is the event that a computer is isolated.

$$\textit{So}, \sum_{1\le i\le 4} \mathbb{P}(A_i) = 4p^3$$

$$\textit{And}, \sum_{1\le i\le j\le 4} \mathbb{P}(A_i \cap A_j) = p^4 + p^6 + p^5 + p^5 + p^6 + p^4 =$$

$$\textit{And}, + \sum_{1\le i\le j\le l\le 4} \mathbb{P}(A_i \cap A_j \cap A_l = p^6 + p^6 + p^6 + p^6$$

$$\textit{And}, \sum_{1\le i\le j\le l\le h\le 4} \mathbb{P}(A_i \cap A_j \cap A_l \cap A_h) = p^6$$

Now,

$$\begin{aligned}\mathbb{P}(\textit{At least } 1 \textit{ computer is isolated}) &= 4p^3 - (2p^4 + 2p^5 + 2p^6) + (4p^6) - (p^6) \\ &= 4p^3 - 2p^4 - 2p^5 + p^6\end{aligned}$$

$$\mathbb{P}(\textit{No computer is isolated}) = 1 - (4p^3 - 2p^4 - 2p^5 + p^6)$$

Problem 4

(a)

$$\mathbb{P}(Signal\ from\ a\ fish\ will\ be\ received) = \mathbb{P}(A \cup B)$$
$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Since A and B are independent receivers,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\mathbb{P}(A \cap B) = 0.8 \cdot 0.9 = 0.72$$

$$And,\ \ \mathbb{P}(A \cup B) = 0.8 + 0.9 - 0.72 = 0.98$$

So,

$$\mathbb{P}(Signal\ from\ a\ fish\ will\ be\ received) = 0.98$$

(b)

$$\mathbb{P}(Signal\ from\ a\ fish\ will\ be\ received\ by\ only\ 1\ receiver) = \mathbb{P}(A\ \Delta\ B)$$
$$= \mathbb{P}[(A \cap B^C) \cup (A^C \cap B)]$$
$$= \mathbb{P}(A \cap B^C) + \mathbb{P}(A^C \cap B) - \mathbb{P}(A \cap A^C \cap B \cap B^C)$$

We know that $\mathbb{P}(A \cap A^C \cap B \cap B^C) = 0$.

So,

$$\mathbb{P}(Signal\ from\ a\ fish\ will\ be\ received\ by\ only\ 1\ receiver) = \mathbb{P}(A \cap B^C) + \mathbb{P}(A^C \cap B)$$

$$= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$
$$= 0.8 - 0.72 + 0.9 - 0.72$$

So,

$$\mathbb{P}(Signal\ from\ a\ fish\ will\ be\ received\ by\ only\ 1\ receiver) = 0.26$$

Problem 5

(a)

For a Monte Carlo Approximation, we can simulate a sample of 20 8-sided die and counting the number of times we get i rolls greater than or equal to 6 and j rolls greater than or equal to 4. By counting these we can estimate the probability.

```
# Run number of simulations to estimate probability
  number_sim <- 10000

  # Add conditions to satisfy calculating probability for events
  condition_1 = condition_2 <- c(0)

  # Simulating n= number_sim rolls
  for(k in 1:number_sim){
    sample_rolls <- sample(1:8, size = 20, replace = TRUE)

    # Counting the number of dice rolls greater than or equal to 6 in the sample
    i_check <- sum(sample_rolls[]>=6)
    # Counting the number of dice rolls greater than or equal to 4 in the sample
    j_check <- sum(sample_rolls[]>=4)

    # Counting the samples which satisfy conditions 1, that is there are at least i rolls greater than
    if(i_check >= i){
      condition_1 = condition_1 + 1
    }
    # Counting the samples which satisfy conditions 1, that is there are at least j rolls greater than
    if(j_check >= j){
      condition_2 = condition_2 + 1
    }
  }
  # Estimating conditional probability
  prob <- condition_1/condition_2
```

(b) and (c)

We can create a function with inputs i and j to output the probabilities for the given i and j. Then we can run a loop to create a matrix for the probabilities for each i= [12,14] and j=[8,12].

```
sim_dice <- function(i,j){
  number_sim <- 10000
  condition_1 = condition_2 <- c(0)
  for(k in 1:number_sim){
    sample_rolls <- sample(1:8, size = 20, replace = TRUE)
    if(sum(sample_rolls[]>=6) >= i){
      condition_1 = condition_1 + 1
    }
    if(sum(sample_rolls[]>=4) >= j){
      condition_2 = condition_2 + 1
    }
  }
```

```
  prob <- condition_1/condition_2

  return(prob)
}

output <- data.frame(row.names = c("j=8","j=9","j=10","j=11","j=12"))
for(m in 1:3){
  for(o in 1:5){
    output[o,m] <- sim_dice(m+11,o+7)
  }
}
colnames(output) <- c("i=12","i=13","i=14")

output
```

```
##            i=12       i=13        i=14
## j=8  0.03395654 0.01354630 0.003644094
## j=9  0.03656894 0.01108923 0.003408739
## j=10 0.03432171 0.01144788 0.004779492
## j=11 0.04113267 0.01381249 0.004506150
## j=12 0.05020313 0.01763928 0.004199855
```

The above are the estimated probabilities for every combination of is and js.