STAT 443: Assignment 2

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Problem 1

(a)

Given AR(3) process:

$$X_t = \frac{2}{5}X_{t-1} + \frac{1}{4}X_{t-2} - \frac{1}{10}X_{t-3} + Z_t \qquad \{Z_t\}_{t\in N} \sim WN(0, \sigma^2)$$

The characteristic polynomial is given by:

$$1 - \frac{2}{5}B - \frac{1}{4}B^2 + \frac{1}{10}B^3$$

Equating the characteristic polynomial to find the roots, we get

$$B = -2, 2, 2.5$$

Since these roots are outside the unit circle in the complex plane, the AR(3) process is stationary.

(b)

The AR(3) process is given by

$$X_t = \frac{2}{5}X_{t-1} + \frac{1}{4}X_{t-2} - \frac{1}{10}X_{t-3} + Z_t \qquad \{Z_t\}_{t\in N} \sim WN(0, \sigma^2)$$

We already know that the AR(3) process is stationary. Multiplying by X_{t-k} for $k = 1, 2, 3$ we get,

$$X_tX_{t-1} = \frac{2}{5}X_{t-1}X_{t-1} + \frac{1}{4}X_{t-2}X_{t-1} - \frac{1}{10}X_{t-3}X_{t-1} + Z_tX_{t-1}$$
$$X_tX_{t-2} = \frac{2}{5}X_{t-1}X_{t-2} + \frac{1}{4}X_{t-2}X_{t-2} - \frac{1}{10}X_{t-3}X_{t-2} + Z_tX_{t-2}$$
$$X_tX_{t-3} = \frac{2}{5}X_{t-1}X_{t-3} + \frac{1}{4}X_{t-2}X_{t-3} - \frac{1}{10}X_{t-3}X_{t-3} + Z_tX_{t-3}$$

Taking expectation on both sides and dividing by σ_x^2, we get

$$\rho(1) = \frac{2}{5}\rho(0) + \frac{1}{4}\rho(-1) - \frac{1}{10}\rho(-2)$$

$$\rho(2) = \frac{2}{5}\rho(1) + \frac{1}{4}\rho(0) - \frac{1}{10}\rho(-1)$$
$$\rho(3) = \frac{2}{5}\rho(2) + \frac{1}{4}\rho(1) - \frac{1}{10}\rho(0)$$

We know that $\rho(0) = 1$ and $\rho(-k) = \rho(k)$.

So these equations become

$$\rho(1) = \frac{2}{5} + \frac{1}{4}\rho(1) - \frac{1}{10}\rho(2)$$
$$\rho(2) = \frac{2}{5}\rho(1) + \frac{1}{4} - \frac{1}{10}\rho(1)$$
$$\rho(3) = \frac{2}{5}\rho(2) + \frac{1}{4}\rho(1) - \frac{1}{10}$$

Solving the system of equations, we get

$$\rho(1) \approx 0.48077, \rho(2) \approx 0.39423, \rho(3) \approx 0.17788$$

Now, let

$$D^3 - \frac{2}{5}D^2 - \frac{1}{4}D^1 + \frac{1}{10}D^0 = 0$$

Solving the above equation, we get

$$d_1 = \frac{1}{2},\ d_2 = \frac{2}{5},\ d_3 = \frac{-1}{2}$$

Now,

$$\rho(h) = A_1 d_1^{|h|} + A_2 d_2^{|h|} + A_3 d_3^{|h|}$$
$$= A_1 \left(\frac{1}{2}\right)^{|h|} + A_2 \left(\frac{2}{5}\right)^{|h|} + A_3 \left(\frac{-1}{2}\right)^{|h|}$$

To find A_1, A_2, and A_3,

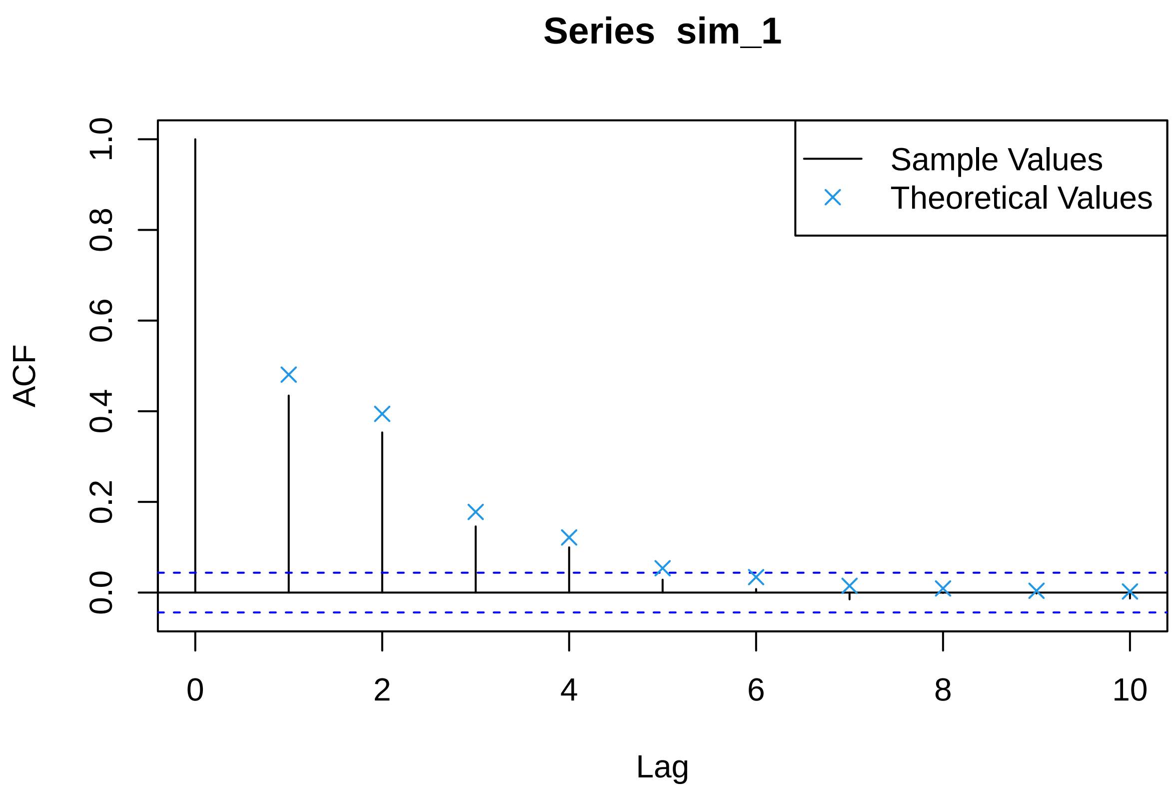
$$\rho(1) = A_1 \left(\frac{1}{2}\right)^1 + A_2 \left(\frac{2}{5}\right)^1 + A_3 \left(\frac{-1}{2}\right)^1$$
$$\rho(2) = A_1 \left(\frac{1}{2}\right)^2 + A_2 \left(\frac{2}{5}\right)^2 + A_3 \left(\frac{-1}{2}\right)^2$$
$$\rho(3) = A_1 \left(\frac{1}{2}\right)^3 + A_2 \left(\frac{2}{5}\right)^3 + A_3 \left(\frac{-1}{2}\right)^3$$

Solving the above system of equations, where $\rho(1) = 0.48077, \rho(2) = 0.39423$ and $\rho(3) = 0.17788$, we get,

$$A_1 \approx 2.42298, A_2 \approx -1.60243, A_3 \approx 0.179496$$
$$\rho(h) = 2.42298 \left(\frac{1}{2}\right)^{|h|} - 1.60243 \left(\frac{2}{5}\right)^{|h|} + 0.179496 \left(\frac{-1}{2}\right)^{|h|}$$

(c)

```r
set.seed(123)
sim_1 <- arima.sim(n = 2000, list(ar = c(2/5,1/4,-1/10)), sd = sqrt(1.96))
theoretical_values <- c(0)
for(i in 1:10){
    theoretical_values[i] <- 2.42298*(1/2)^i-1.60243*(2/5)^i+0.179496*(-1/2)^i
}
acf(sim_1, lag.max = 10)
lines(1:10, theoretical_values, type = "p", col = 4, pch = 4)
legend("topright",legend=c("Sample Values","Theoretical Values"), lty=c(1, NA) ,pch = c(NA ,4),col=c(1,
```



Problem 2

(a)

The ARMA process is given by:

$$X_t = \frac{7}{10}X_{t-1} + Z_t - \frac{1}{10}Z_{t-1}$$

SO,

$$\phi(B) = 1 - \frac{7}{10}B$$

And

$$\theta(B) = 1 - \frac{1}{10}B$$

Equating both the above equations to 0, we can find the roots. Thus, when $\phi(B) = 0$, $B = \frac{10}{7}$ and when $\theta(B) = 0$, $B = 10$.

Since the roots for both equations lie outside the unit circle, the ARMA process is both stationary and invertible.

(b)

$$\psi(B) = \frac{\theta(B)}{\phi(B)} = \frac{1 - \frac{1}{10}B}{1 - \frac{7}{10}B}$$

$$\psi(B) = (1 - .1B)(1 - .7B)^{-1}$$

Now,

$$(1 - .7B)^{-1} = \frac{1}{1 - .7B}$$

This takes the form of a sum $(\frac{a}{1-r})$ of a geometric series which looks like $a + ar + ar^2 \dots$

So, $a = 1$ and $r = .7B$ and the expanded geometric series is,

$$\frac{1}{1 - .7B} = 1 + .7B + .49B^2 + .343B^3 \dots$$

So,

$$\psi(B) = (1 - .1B)(1 + .7B + .49B^2 + .343B^3 \dots)$$
$$\psi(B) = (1 + .7B + .49B^2 + .343B^3 \dots) + (-0.1B - 0.07B^2 - 0.049B^3 \dots)$$
$$\psi(B) = 1 + 0.6B + 0.42B^2 + 0.294B^3 \dots$$

where,

$$\psi_i = 0.6 \times 0.7^{i-1} \qquad for\ i = 1, 2, \dots$$

So, the pure MA process is

$$X_t = Z_t + 0.6Z_{t-1} + 0.42Z_{t-2} + 0.294Z_{t-3} + \dots$$

(c)

$$\pi(B) = \frac{\phi(B)}{\theta(B)} = \frac{1 - \frac{7}{10}B}{1 - \frac{1}{10}B}$$

$$\pi(B) = (1 - .7B)(1 - .1B)^{-1}$$

Now,

$$(1 - .1B)^{-1} = \frac{1}{1 - .1B}$$

This takes the form of a sum $(\frac{a}{1-r})$ of a geometric series which looks like $a + ar + ar^2 \dots$

So, $a = 1$ and $r = .1B$ and the expanded geometric series is,

$$\frac{1}{1-.1B} = 1 + .1B + .01B^2 + .001B^3 \dots$$

So,

$$\pi(B) = (1-.7B)(1+.1B+.01B^2+.001B^3\dots)$$
$$\psi(B) = (1+.1B+.01B^2+.001B^3\dots) + (-0.7B - 0.07B^2 - 0.007B^3\dots)$$
$$\psi(B) = 1 - 0.6B - 0.06B^2 - 0.006B^3\dots$$
$$\pi_i = 0.6 \times 0.1^{i-1} \qquad for\ i = 1, 2, \dots$$

So the pure AR process is,

$$X_t = 0.6X_{t-1} + 0.06X_{t-2} + 0.006X_{t-3} + \dots + Z_t$$

(d)

We know that the ARMA(1,1) process can also be written as an MA process given by:

$$X_t = Z_t + 0.6Z_{t-1} + 0.42Z_{t-2} + 0.294Z_{t-3} + \dots$$

We know that the ACF function for lag h for an MA process is given by,

$$\rho(h) = \frac{\sum_{i=0}^{\infty}\beta_i\beta_{i+h}}{\sum_{i=0}^{\infty}\beta_i^2}$$

This can also be written as,

$$\rho(h) = \frac{\beta_0\beta_h + \sum_{i=1}^{\infty}\beta_i\beta_{i+h}}{\beta_0^2 + \sum_{i=1}^{\infty}\beta_i^2}$$

We also know that our $\beta_1, \beta_2 \dots$ take the form of a geometric series given by,

$$\beta_n = a \cdot r^{n-1} \qquad\qquad n \in [1, \infty)$$

where $a = 0.6$ and $r = 0.7$

We know that $\beta_0 = 1$, so we can write the ACF function as,

$$\rho(h) = \frac{1\cdot\beta_h + \sum_{i=1}^{\infty}\beta_i\beta_{i+h}}{1 + \sum_{i=1}^{\infty}\beta_i^2}$$

$$\implies \rho(h) = \frac{1\cdot 0.6\cdot 0.7^{h-1} + \sum_{i=1}^{\infty} 0.6\cdot 0.7^{i-1} \times 0.6\cdot 0.7^{i+h-1}}{1 + \sum_{i=1}^{\infty}(0.6\cdot 0.7^{i-1})^2}$$
$$\implies \rho(h) = \frac{0.6\cdot 0.7^{h-1} + 0.36\cdot 0.7^h\sum_{i=1}^{\infty} 0.7^{2i-2}}{1 + 0.36\sum_{i=1}^{\infty} 0.7^{2i-2}}$$

Now, $\sum_{i=1}^{\infty} 0.7^{2i-2}$ converges to 1.96078 So, we can see that the ACF then becomes,

$$\rho(h) = \frac{0.6\cdot 0.7^h\cdot 0.7^{-1} + 0.36\cdot 1.96078\cdot 0.7^h}{1 + 0.36\cdot 1.96078}$$
$$\implies \rho(h) \approx \frac{0.857\cdot 0.7^h + 0.706\cdot 0.7^h}{1.706}$$
$$\implies \rho(h) \approx \frac{1.563}{1.706}\cdot 0.7^h$$
$$\implies \rho(h) \approx 0.916\cdot 0.7^h$$

Problem 3

The SARIMA process is of order $(2,1,0) \times (0,1,2)_{12}$

Now,

$$W_t = \nabla^1 \nabla_{12}^1 X_t$$
$$W_t = \nabla_{12}^1 X_t - \nabla_{12}^1 X_{t-1}$$
$$W_t = (X_t - X_{t-12}) - (X_{t-1} - X_{t-1-12})$$
$$W_t = X_t - X_{t-12} - X_{t-1} + X_{t-13}$$

The form that the SARIMA process takes is:

$$\phi(B)\Phi(B^s)W_t = \theta(B)\Theta(B^s)Z_t$$
$$\left(1 - \phi_1(B) - \phi_2(B^2)\right) W_t = \left(1 + \Theta_1(B) + \Theta_2(B^2)\right) Z_t$$
$$\left(1 - \phi_1(B) - \phi_2(B^2)\right) (X_t - X_{t-1} - X_{t-12} + X_{t-13}) = \left(1 + \Theta_1(B) + \Theta_2(B^2)\right) Z_t$$

So the left hand side of the equation becomes

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - X_{t-1} + \phi_1 X_{t-2} + \phi_2 X_{t-3} - X_{t-12} + \phi_1 X_{t-13} + \phi_2 X_{t-14} + X_{t-13} - \phi_1 X_{t-14} - \phi_2 X_{t-15}$$

$$X_t - (\phi_1 + 1)X_{t-1} - (\phi_2 - \phi_1)X_{t-2} - (-\phi_2 X_{t-3}) - X_{t-12} - (-\phi_1 - 1)X_{t-13} - (-\phi_2 + \phi_1)X_{t-14} - \phi_2 X_{t-15}$$

And the right hand side is,

$$Z_t + \Theta_1 Z_{t-12} + \Theta_2 Z_{t-24}$$

Thus, the ARMA process is of the order $ARMA(15, 24)$.