STAT 321: Assignment 5

Saksham Sudershan

26 March 2022

Problem 1

(a)

Let $\Theta = 1$ denote the case in which Alice is given the first coin whose probability of heads, $p_H = 0.3$. Similarly, let $\Theta = 2$ and $\Theta = 3$ denote the case of Alice being given coin 2 and 3 respectively.
Let $X = 1$ denote the case in which she gets heads on the first flip, and tails on the second one.

$$\hat{\Theta}_{MAP}(x) = \arg\max_{\theta} P(\theta|x)$$

If Alice was given coin 1,

$$P(\Theta = 1|X = 1) = \frac{P(\Theta = 1) \cdot P(X = 1|\Theta = 1)}{P(\Theta = 1) \cdot P(X = 1|\Theta = 1) + P(\Theta = 2) \cdot P(X = 1|\Theta = 2) + P(\Theta = 3) \cdot P(X = 1|\Theta = 3)}$$

$$\Rightarrow P(\Theta = 1|X = 1) = \frac{\frac{1}{3} \cdot (0.3 \times 0.7)}{\frac{1}{3} \cdot (0.3 \times 0.7) + \frac{1}{3} \cdot (0.7 \times 0.3) + \frac{1}{3} \cdot (0.5 \times 0.5)}$$

$$\Rightarrow P(\Theta = 1|X = 1) = \frac{0.07}{0.07 + 0.07 + 0.0833}$$

$$\therefore P(\Theta = 1|X = 1) = 0.3135$$

If Alice was given coin 2,

$$P(\Theta = 2|X = 1) = \frac{P(\Theta = 1) \cdot P(X = 1|\Theta = 1)}{P(\Theta = 1) \cdot P(X = 1|\Theta = 1) + P(\Theta = 2) \cdot P(X = 1|\Theta = 2) + P(\Theta = 3) \cdot P(X = 1|\Theta = 3)}$$

$$\Rightarrow P(\Theta = 2|X = 1) = \frac{\frac{1}{3} \cdot (0.7 \times 0.3)}{\frac{1}{3} \cdot (0.3 \times 0.7) + \frac{1}{3} \cdot (0.7 \times 0.3) + \frac{1}{3} \cdot (0.5 \times 0.5)}$$

$$\Rightarrow P(\Theta = 2|X = 1) = \frac{0.07}{0.07 + 0.07 + 0.0833}$$

$$\therefore P(\Theta = 2|X = 1) = 0.3135$$

If Alice was given coin 3,

$$P(\Theta = 3|X = 1) = \frac{P(\Theta = 1) \cdot P(X = 1|\Theta = 1)}{P(\Theta = 1) \cdot P(X = 1|\Theta = 1) + P(\Theta = 2) \cdot P(X = 1|\Theta = 2) + P(\Theta = 3) \cdot P(X = 1|\Theta = 3)}$$

$$\Rightarrow P(\Theta = 3|X = 1) = \frac{\frac{1}{3} \cdot (0.5 \times 0.5)}{\frac{1}{3} \cdot (0.3 \times 0.7) + \frac{1}{3} \cdot (0.7 \times 0.3) + \frac{1}{3} \cdot (0.5 \times 0.5)}$$

$$\Rightarrow P(\Theta = 3|X = 1) = \frac{0.0833}{0.07 + 0.07 + 0.0833}$$

$$\therefore P(\Theta = 3|X = 1) = 0.3730$$

Since $P(\theta|x)$ is maximzed for the case that Alice was given coin 1,

$$\therefore \hat{\Theta}_{MAP}(X = 1) = 3$$

(b)

Now, $P(\Theta = 1) = 0.6$, $P(\Theta = 2) = 0.4$ and $P(\Theta = 3) = 0$.
She only flips the coin once. Let $X = 1$ denote the case in which she flips a heads and $X = 0$ denote the case in which she flips a tails.

If Alice flips heads,

$$P(\Theta = 1|X = 1) = \frac{P(\Theta = 1) \cdot P(X = 1|\Theta = 1)}{P(\Theta = 1) \cdot P(X = 1|\Theta = 1) + P(\Theta = 2) \cdot P(X = 1|\Theta = 2) + P(\Theta = 3) \cdot P(X = 1|\Theta = 3)}$$

$$\Rightarrow P(\Theta = 1|X = 1) = \frac{0.6 \cdot 0.3}{0.6 \cdot 0.3 + 0.4 \cdot 0.7 + 0 \cdot 0.5}$$

$$\Rightarrow P(\Theta = 1|X = 1) = \frac{0.18}{0.18 + 0.28 + 0}$$

$$\therefore P(\Theta = 1|X = 1) \approx 0.4737$$

$$P(\Theta = 2|X = 1) = \frac{0.28}{0.18 + 0.28}$$

$$\therefore P(\Theta = 2|X = 1) \approx 0.7368$$

And,

$$P(\Theta = 3|X = 1) = 0$$

If Alice flips tails,

$$P(\Theta = 1|X = 0) = \frac{P(\Theta = 1) \cdot P(X = 0|\Theta = 1)}{P(\Theta = 1) \cdot P(X = 0|\Theta = 1) + P(\Theta = 2) \cdot P(X = 0|\Theta = 2) + P(\Theta = 3) \cdot P(X = 0|\Theta = 3)}$$

$$\Rightarrow P(\Theta = 1|X = 0) = \frac{0.6 \cdot 0.7}{0.6 \cdot 0.7 + 0.4 \cdot 0.3 + 0 \cdot 0.5}$$

$$\Rightarrow P(\Theta = 1|X = 0) = \frac{0.42}{0.42 + 0.18 + 0}$$

$$\therefore P(\Theta = 1|X = 0) \approx 0.8$$

$$P(\Theta = 2|X = 0) = \frac{0.12}{0.42 + 0.12}$$

$$\therefore P(\Theta = 2|X = 0) = 0.2$$

And,

$$P(\Theta = 3|X = 0) = 0$$

Thus, $\hat{\Theta}_{MAP}(X = 1) = 2$ and $\hat{\Theta}_{MAP}(X = 0) = 1$.

Now, the overall probability of error is given by,

$$P(\hat{\Theta}_{MAP} \neq \Theta) = P(\Theta = 1, X = 1) + P(\Theta = 3, X = 1) + P(\Theta = 2, X = 0) + P(\Theta = 3, X = 0)$$

$$\therefore P(\hat{\Theta}_{MAP} \neq \Theta) = 0.4737 + 0 + 0.2 + 0 = 0.6737$$

Problem 2

(a)

We know the radioactivity level of "Iocane" has the $Unif[0,1]$ distribution, and that of "Sennari" has the $Exp(1)$ distribution.
Let $\Theta = 1$ denote that the bottle has "Iocane" and $\Theta = 2$ denote that the bottle has "Sennari".
Let $X = x$ denote radioactivity measured.
We know that,

$$\hat{\Theta}_{MAP} = \arg\max_{\theta} \; P(\theta|x)$$

If the bottle contains "Iocane",

$$P(\Theta = 1|X = x) = \frac{P(\Theta = 1) \cdot P(X = x|\Theta = 1)}{P(\Theta = 1) \cdot P(X = x|\Theta = 1) + P(\Theta = 2) \cdot P(X = x|\Theta = 2)}$$

$$P(\Theta = 1|X = x) = \begin{cases} \dfrac{\frac{1}{2} \cdot \frac{1}{1-0}}{\frac{1}{2} \cdot \frac{1}{1-0} + \frac{1}{2} \cdot e^{-x}} & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

$$\therefore P(\Theta = 1|X = x) = \begin{cases} \dfrac{1}{1 + e^{-x}} & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

If the bottle contains "Sennari",

$$P(\Theta = 2|X = x) = \frac{P(\Theta = 2) \cdot P(X = x|\Theta = 2)}{P(\Theta = 1) \cdot P(X = x|\Theta = 1) + P(\Theta = 2) \cdot P(X = x|\Theta = 2)}$$

$$P(\Theta = 2|X = x) = \begin{cases} \dfrac{\frac{1}{2} \cdot e^{-x}}{\frac{1}{2} \cdot \frac{1}{1-0} + \frac{1}{2} \cdot e^{-x}} & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$\therefore P(\Theta = 2|X = x) = \begin{cases} \dfrac{e^{-x}}{1 + e^{-x}} & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

We know that,

$$\frac{e^{-x}}{1 + e^{-x}} \; is \; a \; monotonically \; increasing \; function.$$

At $x = 0$, $P(\Theta = 1|X = x) = 0.5$. And at $x = 1$, $P(\Theta = 1|X = x) = 0.731$. Thus, when $x = 0$, we can choose "Iocane" or "Sennari" arbitrarily; and at any point $0 < x \leq 1$, $\hat{\Theta}_{MAP} = 1$, or we choose "Iocane".

Thus our decision rule is dependent upon our measure of radioactivity of the bottle.

$$\Rightarrow \hat{\Theta}_{MAP} = \begin{cases} 1 \; or \; 2 \; (chosen \; arbitrarily) & x = 0 \\ 1 & 0 < x \leq 1 \\ 2 & x > 1 \end{cases}$$

(b)

Let's assume we choose "Iocane" always at $x = 0$. Thus, our probability of error is given by,

$$P(\hat{\Theta}_{MAP} \neq \Theta) = P(\Theta = 2, x = 0) + P(\Theta = 2, 0 < x \leq 1) + P(\Theta = 1, x > 1)$$

Or,

$$
\begin{aligned}
P(\hat{\Theta}_{MAP} \neq \Theta) &= P(\Theta = 2, 0 \leq x \leq 1) + P(\Theta = 1, x > 1) \\
&= P(\Theta = 2) \times P(0 \leq x \leq 1 | \Theta = 2) + P(\Theta = 1) \times P(x > 1 | \Theta = 1) \\
&= \frac{1}{2} \cdot [1 - e^{-1}] + \frac{1}{2} \cdot 0 && \textit{Using CDF of Exp}(1) \textit{ distribution} \\
&= \frac{1}{2} \cdot 0.6321 \\
&= 0.316
\end{aligned}
$$

So the associated probability of error for our decision rule is 0.316.

Problem 3

We know that,

$$
S = \begin{cases} -1 & \textit{with probability } \dfrac{1}{3} \\ 0 & \textit{with probability } \dfrac{1}{3} \\ 1 & \textit{with probability } \dfrac{1}{3} \end{cases}
$$

$$
f_Z(z) = \frac{\lambda}{2} e^{-\lambda |z|}
$$

And,

$$
\begin{gathered}
Y = S + Z \\
\Rightarrow Z = Y - S
\end{gathered}
$$

(a)

We have to find $f_{Y|S}(y|s)$ for $s = -1, 0, 1$.

Let $s = -1$.

$$
\begin{aligned}
f_{Y|S}(y|s) &= f_{Z|S}(y - s|s) && \textit{Since } Z = Y - S \\
&= f_Z(y - s) && \textit{Since } Z \textit{ and } S \textit{ are independent} \\
&= f_Z(y + 1) \\
&= \frac{\lambda}{2} e^{-\lambda |y+1|}
\end{aligned}
$$

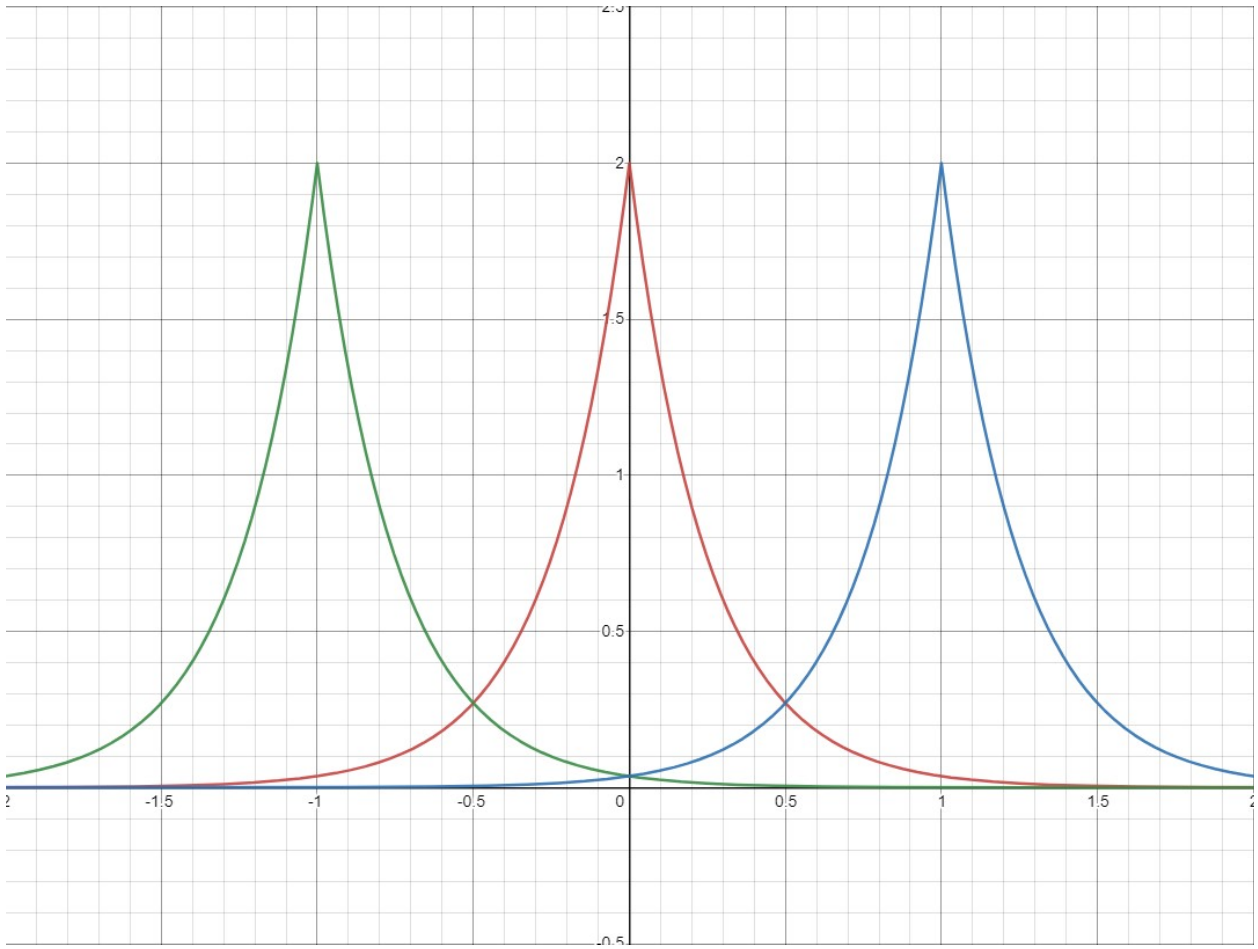
Let $s = 0$. Similarly,

$$
\begin{aligned}
f_{Y|S}(y|s) &= f_Z(y) \\
&= \frac{\lambda}{2} e^{-\lambda |y|}
\end{aligned}
$$

And let $s = -1$. Again,

$$
\begin{aligned}
f_{Y|S}(y|s) &= f_Z(y - 1) \\
&= \frac{\lambda}{2} e^{-\lambda |y-1|}
\end{aligned}
$$

(b)



We can find the critical points for when $\hat{S}_{MAP}(y)$ changes for given value of y by equating:

$$f_Z(y+1) = f_Z(y) \qquad when\ y \in (-1, 0)$$
$$\Rightarrow \frac{\lambda}{2}e^{-\lambda|y+1|} = \frac{\lambda}{2}e^{-\lambda|y|}$$
$$\Rightarrow |y+1| = |y|$$
$$\Rightarrow y+1 = -y$$
$$\therefore y = -0.5$$

And,

$$f_Z(y) = f_Z(y-1) \qquad when\ y \in (0, 1)$$
$$\Rightarrow \frac{\lambda}{2}e^{-\lambda|y|} = \frac{\lambda}{2}e^{-\lambda|y-1|}$$
$$\Rightarrow |y| = |y-1|$$
$$\Rightarrow y = -y+1$$
$$\therefore y = 0.5$$

So,

$$\hat{S}_{MAP} = \begin{cases} -1 & y \leq -0.5 \\ 0 & -0.5 < y \leq 0.5 \\ 1 & y > 0.5 \end{cases}$$

(c)

The decoding error is given by,

$$\begin{aligned}
P(\hat{S}_{MAP} \neq S) &= P(\hat{S}_{MAP} = -1, S = 0 \text{ } or \text{ } 1) \cup P(\hat{S}_{MAP} = 0, S = -1 \text{ } or \text{ } 1) \cup P(\hat{S}_{MAP} = 1, S = -1 \text{ } or \text{ } 0) \\
&= P(y \leq -0.5, S = 0 \text{ } or \text{ } 1) + P(-0.5 < y \leq 0.5, S = -1 \text{ } or \text{ } 1) + P(y > 0.5, S = -1 \text{ } or \text{ } 0) \\
&= \int_{-\infty}^{-0.5} \frac{\lambda}{2} e^{-\lambda|y|} dy + \int_{-0.5}^{0} \frac{\lambda}{2} e^{-\lambda|y+1|} dy + \int_{0}^{0.5} \frac{\lambda}{2} e^{-\lambda|y-1|} dy + \int_{0.5}^{\infty} \frac{\lambda}{2} e^{-\lambda|y|} dy \\
&= \int_{-\infty}^{-0.5} \frac{\lambda}{2} e^{\lambda y} dy + \int_{-0.5}^{0} \frac{\lambda}{2} e^{-\lambda(y+1)} dy + \int_{0}^{0.5} \frac{\lambda}{2} e^{\lambda(y-1)} dy + \int_{0.5}^{\infty} \frac{\lambda}{2} e^{-\lambda y} dy \\
&= \int_{-\infty}^{-0.5} \frac{\lambda}{2} e^{\lambda y} dy + e^{-\lambda} \int_{-0.5}^{0} \frac{\lambda}{2} e^{-\lambda y} dy + e^{-\lambda} \int_{0}^{0.5} \frac{\lambda}{2} e^{\lambda y} dy + \int_{0.5}^{\infty} \frac{\lambda}{2} e^{-\lambda y} dy \\
&= \frac{\lambda}{2} \cdot \left[\frac{e^{\lambda y}}{\lambda}\right]_{-\infty}^{-0.5} + \frac{\lambda}{2} e^{-\lambda} \cdot \left[\frac{-e^{-\lambda y}}{\lambda}\right]_{-0.5}^{0} + \frac{\lambda}{2} e^{-\lambda} \cdot \left[\frac{e^{\lambda y}}{\lambda}\right]_{0}^{0.5} + \frac{\lambda}{2} \cdot \left[\frac{-e^{-\lambda y}}{\lambda}\right]_{0.5}^{\infty} \\
&= \frac{\lambda}{2} \cdot \left[\frac{e^{-0.5\lambda} - 0}{\lambda}\right] + \frac{\lambda}{2} e^{-\lambda} \cdot \left[\frac{e^{0.5\lambda} - 1}{\lambda}\right] + \frac{\lambda}{2} e^{-\lambda} \cdot \left[\frac{e^{0.5\lambda} - 1}{\lambda}\right] + \frac{\lambda}{2} \cdot \left[\frac{e^{0.5\lambda} - 0}{\lambda}\right] \\
&= \frac{e^{-0.5\lambda}}{2} + \frac{e^{-0.5\lambda} - e^{-\lambda}}{2} + \frac{e^{-0.5\lambda} - e^{-\lambda}}{2} + \frac{e^{-0.5\lambda}}{2} \\
&= \frac{4e^{-0.5\lambda} - 2e^{-\lambda}}{2} \\
&= 2e^{-0.5\lambda} - e^{-\lambda}
\end{aligned}$$

$$\therefore P(\hat{S}_{MAP} \neq S) = 2e^{-0.5\lambda} - e^{-\lambda}$$

Problem 4

In normal mode

$$X = \begin{cases} +1 & with \text{ } probability \text{ } \frac{1}{2} \\ -1 & with \text{ } probability \text{ } \frac{1}{2} \end{cases}$$

(a)

If $M = 1$, then $S = X$, and so $Y = X + Z$.

$$\begin{aligned}
f_{Y|M}(y|1) &= f_{Y|X}(y|x) \\
&= f_{Z|X}(y - x|x) \\
&= f_Z(y - z) \\
&= \begin{cases} \frac{1}{2} & (y - x) \in [-1, 1] \\ 0 & otherwise \end{cases}
\end{aligned}$$

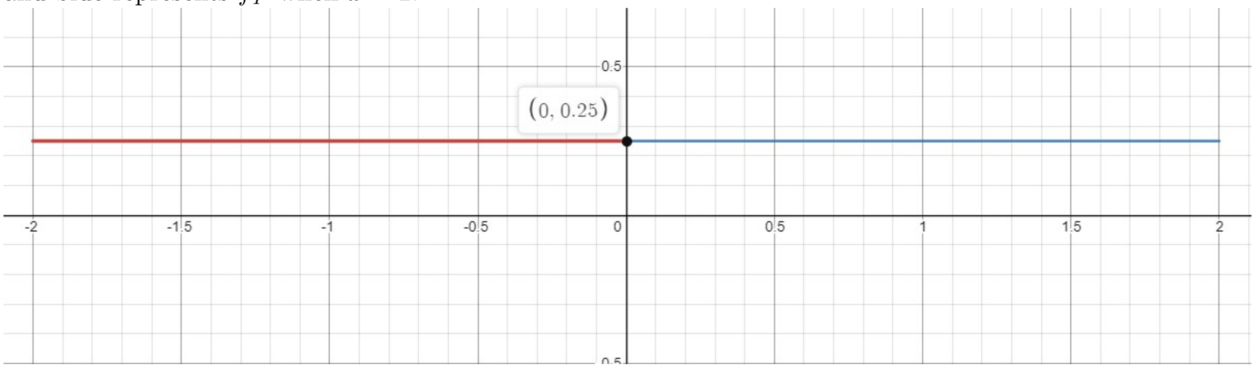
In case $X = +1$,

$$-1 \leq y - 1 \leq 1$$
$$0 \leq < y \leq 2$$

In case $X = -1$,

$$-1 \leq y + 1 \leq 1$$
$$-2 \leq y \leq 0$$

So, if $X = -1$, $Y|M = 1 \sim Unif[-2, 0]$ and if $X = 1$, $Y|M = 1 \sim Unif[0, 2]$.
Now, we know that $P(X = -1) = P(X = 1) = \frac{1}{2}$. So the total probability that $y \in [-2, 0]$ (when $x = -1$) is $0.5 \times \frac{1}{0+2} = 0.25$. It is similar for $x = 1$. So the sketch is given by, where red represents f_Y when $x = -1$ and blue represents f_Y when $x = 1$:

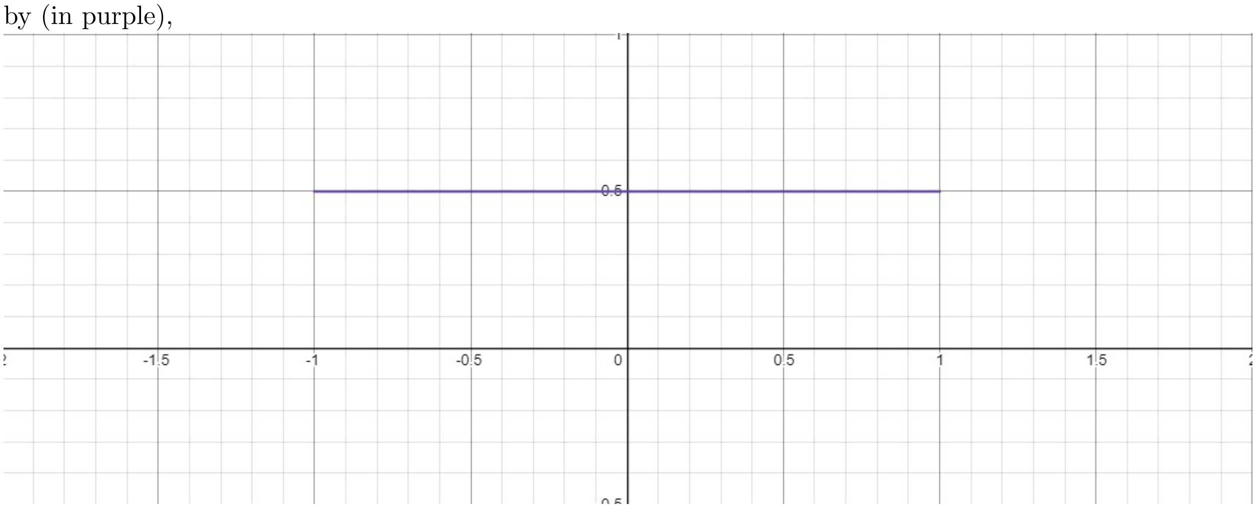


(b)

If the system is in idle mode, $M = 0$, then $S = 0$. So,

$$f_{Y|M}(y|0) = f_Z(y)$$

So, f_y takes the form of the distribution of Z, since there is only the ambient noise. The sketch is given by (in purple),

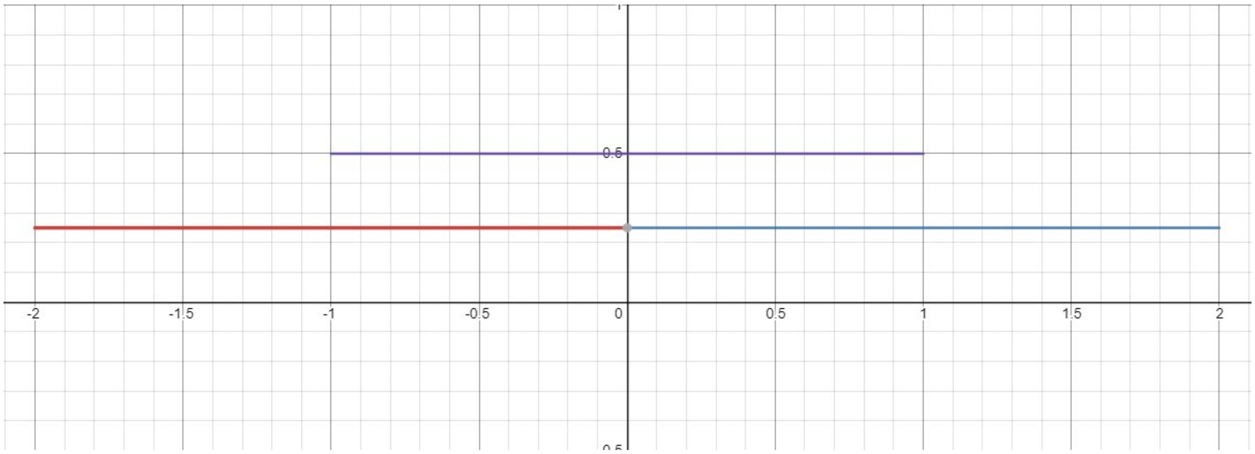


(c)

We know that,

$$\hat{M}_{MAP}(x) = \arg\max_m P(m|y)$$

So, $\hat{M}_{MAP} = 1$ if $f_{Y|M}(y|1) > f_{Y|M}(y|0)$, and $\hat{M}_{MAP} = 0$ if $f_{Y|M}(y|1) < f_{Y|M}(y|0)$. From the sketch of $f_{Y|M}$ for both $M = 1$ (red and blue) and $M = 0$ (purple) given below, we can find the decision rule for optimal decoder $\hat{M}_{MAP}$.



The optimal decoder $\hat{M}_{MAP}(y)$ is given by,

$$\hat{M}_{MAP}(y) = \begin{cases} 1 & y < -1 \\ 0 & -1 < y < 1 \\ 1 & y > 1 \end{cases}$$

(d)

The associated probability of error of our $\hat{M}_{MAP}$ is given by:

$$\begin{aligned}
P(\hat{M}_{MAP} \neq M) &= P(\hat{M}_{MAP} = 1, M = 0) + P(\hat{M}_{MAP} = 0, M = 1) \\
&= P(M = 0, y < -1) + P(M = 0, y > 1) + P(M = 1, -1 < y < 1) \\
&= P(M = 0) \cdot P(y < -1 | M = 0) + P(M = 0) \cdot P(y > 1 | M = 0) + P(M = 1) \cdot P(-1 < y < 1 | M = 1) \\
&= \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \int_{-1}^{1} 0.25 \; dy \\
&= 0 + 0 + \frac{1}{2} \cdot \left[\frac{y}{4} \right]_{-1}^{1} \\
&= \frac{1}{2} \cdot 0.5 \\
&= 0.25
\end{aligned}$$

Problem 5

We know that,

$$X = \begin{cases} 1 & with \; probability \; \frac{1}{2} \\ 10 & with \; probability \; \frac{1}{2} \end{cases}$$

$$\hat{X}_{MAP} = \arg\max_x P(x|y)$$

For $X = 1$,

$$\begin{aligned}
P(X=1|Y=y) &= \frac{P(Y=y|X=1)\cdot P(X=1)}{P(X=x)} \\
&= \frac{P(Y=y|X=1)\cdot P(X=1)}{P(Y=y|X=1)\cdot P(X=1)+P(Y=y|X=2)\cdot P(X=2)} \\
&= \frac{1^y\cdot e^{-1}}{1^y\cdot e^{-1}+10^y\cdot e^{-10}}
\end{aligned}$$

For $X = 1$,

$$\begin{aligned}
P(X=1|Y=y) &= \frac{P(Y=y|X=1)\cdot P(X=1)}{P(X=x)} \\
&= \frac{P(Y=y|X=1)\cdot P(X=1)}{P(Y=y|X=1)\cdot P(X=1)+P(Y=y|X=2)\cdot P(X=2)} \\
&= \frac{10^y\cdot e^{-10}}{1^y\cdot e^{-1}+10^y\cdot e^{-10}}
\end{aligned}$$

To find the critical point $y*$.

$$\frac{1^{y*}\cdot e^{-1}}{1^{y*}\cdot e^{-1}+10^{y*}\cdot e^{-10}} = \frac{10^{y*}\cdot e^{-10}}{1^{y*}\cdot e^{-1}+10^{y*}\cdot e^{-10}}$$

$$\Rightarrow y* = \frac{9}{log(10} \approx 3.9 \approx 4$$

Now, if $y = 3$, $P(X=1|Y=y) \approx 0.89$. So, our decision rule is given by,

$$\hat{X}_{MAP} = \begin{cases} 1, & y < 4 \\ 10, & otherwise \end{cases}$$

So our $y* = 4$.
Now, the probability of error is given by,

$$\begin{aligned}
P(\hat{X}_{MAP} \neq X) &= P(X=10, Y<4) \cup P(X=1, Y>4) \\
&= P(Y<4|X=10)\cdot P(X=10) + P(Y>4|X=1)\cdot P(X=1) \\
&= P(X=10)\cdot\left[\sum_{i=0}^{3} P(Y=i|X=10)\right] + P(X=1)\cdot\left[1-\sum_{i=0}^{3} P(Y=i|X=1)\right] \\
&= \frac{1}{2}\cdot\left[\sum_{i=0}^{3}\frac{10^i e^{-10}}{i!}\right] + \frac{1}{2}\cdot\left[1-\sum_{i=0}^{3}\frac{1^i e^{-1}}{i!}\right] \\
&\approx \frac{1}{2}\left[0.01034 + 0.01899\right] \\
&= 0.014665
\end{aligned}$$

Problem 6

We know that $\Theta \sim \text{Unif}[101, 200]$ and given Θ, $X \sim \text{Unif}[1, \Theta]$.

$$\hat{\Theta}_{MAP} = \arg\max_{\theta} P(\theta|x)$$

Now,

$$P(\Theta = \theta | X = x) = \frac{P(X = x|\Theta = \theta) \cdot P(\Theta = \theta)}{P(X = x} \quad when\ x\ \in [1, \theta]$$

$$= \frac{\frac{1}{\theta - 1} \cdot \frac{1}{200-101+1}}{P(X = x)}$$

We can see that $P(X = x)$ is independent of θ, so we can maximize,

$$\frac{1}{\theta - 1} \cdot \frac{1}{100}$$

For $\theta > 1$, the above function is monotonically decreasing. So we get two cases,

$$\hat{\Theta}_{MAP} = \begin{cases} 101, & x < 101 \\ x, & x \in [101, 200] \end{cases}$$