Assignment #6
Due: April 10, 2022

1. *Balls and bins revisited.* There are 10 bins numbered $1, 2, \ldots, 10$. n balls are thrown into the 10 bins. For each ball, the probability that it falls into bin i is $\frac{1}{10}$ for $i = 1, 2, \ldots, 10$. Different balls are thrown independently of each other. Let Y be the number of balls in bin 1. Let Z be the total number of balls in bins $6, 7, 8, 9, 10$.

   (a) Find $\mathsf{P}(Y = y|Z = z)$. Please specify the range of y, z.

   (b) Given $Z = z$, find the estimator $\hat{y}(z)$ that minimizes conditional MSE $\mathsf{E}[(\hat{y}(z) - Y)^2|Z = z]$.

   (c) Find the conditional MSE $\mathsf{E}[(\hat{y}(z) - Y)^2|Z = z]$ for the estimator in part (b).

   (d) Find the linear LMS estimator of Y given $Z = z$.

   (e) Find $\mathsf{E}[Z]$ and $\mathrm{Var}[Z]$.

   (f) Find $\mathsf{E}[Y]$ and $\mathrm{Var}[Y]$.

   (g) Find $\mathrm{Cov}(Y, Z)$.

   Hint: Use the law of total expectation. Try to first determine the conditional pmf.

   (h) Find the linear LMS estimator of Z given Y.

   (i) Find the corresponding (overall) MSE for the estimator in part (h).

2. *Estimation vs. detection.* Let the signal

$$X = \begin{cases} +1, & \text{with probability } \frac{1}{2} \\ -1, & \text{with probability } \frac{1}{2}, \end{cases}$$

   and the noise $Z \sim \mathrm{Unif}[-2, 2]$ be independent random variables. Their sum $Y = X + Z$ is observed.

   (a) Find the LMS estimate of X given Y

   (b) Find the (overall) MSE for the estimator you find in part (a).

   (c) Now suppose we use a decoder to decide whether $X = +1$ or $X = -1$ so that the probability of error is minimized. Find the MAP decoder and its probability of error. Compare the MAP decoder's MSE to the least MSE.

3. *Stick breaking.* Given a stick of length 1, break it into two pieces at a location chosen uniform at random. Denote the breaking location by X, then $X \sim \mathrm{Unif}[0, 1]$. Keep the piece corresponding to the interval $[X, 1]$. Break it again into two pieces at a location chosen uniform at random. Denote the second breaking location by Y, then $Y|\{X = x\} \sim \mathrm{Unif}[x, 1]$.

(a) Find the estimator of X given Y that minimizes the MSE $\mathsf{E}[(\hat{X} - X)^2]$.

(b) Find the conditional MSE given $Y = y$ for the estimator you find in part (a).

(c) Find the covariance $\mathrm{Cov}(X, Y)$.

(d) Find the linear LMS estimator of X given Y.

(e) Find the MSE for the estimator you find in part (d)

4. *Estimation based on a function of the observation.* Let Θ be a positive random variable, with known mean μ and variance σ^2, to be estimated on the basis of a measurement X of the form $X = \sqrt{\Theta}W$. We assume that W is independent of Θ with zero mean, unit variance, and known fourth moment $\mathsf{E}[W^4]$. Thus, the conditional mean and variance of X given Θ are 0 and Θ, respectively, so we are essentially trying to estimate the variance of X given an observed value.

   (a) Find the linear LMS estimator of Θ based on $X = x$.

   (b) Let $Y = X^2$. Find the linear LMS estimator of Θ based on $Y = y$.

5. *Neural net.* Let $Y = X + Z$, where the signal $X \sim \mathrm{Unif}[-1, 1]$ and noise $Z \sim N(0, 1)$ are independent. We want to estimate $\mathrm{sgn}(X)$, where

$$\mathrm{sgn}(x) = \begin{cases} -1 & x \leq 0 \\ +1 & x > 0. \end{cases}$$

   (a) Find the function $g(y)$ that minimizes

$$\mathrm{MSE} = \mathsf{E}\big[(\mathrm{sgn}(X) - g(Y))^2\big].$$

   Express your answer in terms of the cumulative distribution function of $N(0, 1)$

$$\Phi(z) \triangleq \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, \mathrm{d}x.$$

   (b) Plot $g(y)$ as a function of y.

6. *Communication in Gaussian noise.* In a communication system, a transmitter wants to send some signal to a receiver over a noisy medium. Suppose the signal Θ is a Gaussian distributed random variable $\Theta \sim \mathcal{N}(0, \sigma_\Theta^2)$. The noise in the medium is modeled as a Gaussian distributed random variable $W \sim \mathcal{N}(0, \sigma_W^2)$ independent of the signal. The receiver observes $X = 2\Theta + W$.

   (a) Find the estimator of Θ given X that minimizes the MSE $\mathsf{E}[(\hat{\Theta} - \Theta)^2]$.

   (b) Find the MSE for the estimator you found in part (a).

   (c) Find the linear LMS estimator of Θ given X.

   (d) Find the MSE for the estimator you found in part (c).

   (e) Find the LMS estimator of Θ^2 given X.

   (f) Find the linear LMS estimator of Θ^2 given X.