Assignment 1

STAT 321: Stochastic Systems and Signals
Winter Session 2021-2022 (Term 2)

Assignment Instructions

- Assignments are to be completed individually.
- When answering the questions, writing down the final answer will not be sufficient to receive full marks. Please show all calculations unless otherwise specified. Also define any events and notation that you use in your solutions.
- You may use the language of your choice for the numerical parts, but the free statistical software `R` is recommended. Include any code used in your assignment as an Appendix **at the very end of your assignment**.
- *Due date:* January 28 at 11:59pm (Vancouver Time). You must submit an electronic (preferably PDF) version of your assignment on Canvas.

Problem 1 [5 marks]

An introductory course in probability in a high school in Vancouver is comprised of 20 students. What is the probability that at least two students have the same birthday? Assume that there are 365 days in a year, and that all birthdays are equally likely.

Problem 2 [10 marks]

For the following claims, state whether they are TRUE or FALSE. Statements claimed to be TRUE must be accompanied by a proof, and statements claimed to be FALSE must be accompanied by a counterexample.

(a) Let A and B be events in the sample space Ω. Then $P(A \cap B) \leq P(A)P(B)$. **[2 marks]**

(b) Let E be an event with $0 < P(E) < 1$ and define $P_E(A) = P(A \cap E)$ for every event A in the sample space S. Then P_E satisfies the axioms of probability. **[2 marks]**

(c) Let A and B be events in the sample space Ω. Then $P(A \cap B^C) = P(A \cup B) - P(B)$. **[3 marks]**

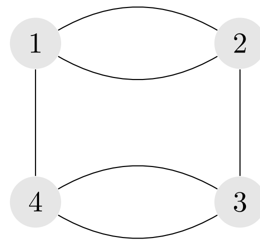
(d) Let $A_1, A_2, \ldots, A_n$ be events in the sample space Ω. Then

$$P\left(\cap_{i=1}^{n} A_i\right) \leq \min_i \{P(A_i)\}.$$

[3 marks]

Problem 3 [15 marks]

Four computers are connected in a network according to the following diagram, where each line represents a connection. Each connection has a probability p of being broken at any time, independent of any other connections. A computer is isolated if all connections to it are broken.



(a) What is the probability that computer 1 is isolated? [**5 marks**]

(b) What is the probability that both computers 1 and j are isolated for $j = 2, 3, 4$? [**5 marks**]

(c) What is the probability that no computer is isolated in this network? [**5 marks**]

Problem 4 [10 marks]

To study the migration of adult Pacific salmons, researchers tag salmons with electronic transmitters that track individual fish throughout their migration. Underwater receivers are built at various locations along the river to receive the signal from these electronic transmitters. At a location where a river diverges into distributaries, two independent receivers A and B are used at to detect and locate these salmons. The probability that receiver A receives the signal from a fish is 0.8, whereas the corresponding probability of receiver B is 0.9.

(a) What is the probability that the signal from a fish will be received? [**5 marks**]

(b) What is the probability that the signal from a fish will be received by only one receiver? [**5 marks**]

Problem 5 [20 marks]

In order to complete this question, we will need to use a fundamental theorem in probabilty theory known as the *(Weak) Law of Large Numbers* (*WLLN*). We will use this important theorem throughout the course to approximate probabilities (or other quantities) that are otherwise difficult to calculate theoretically.

A simplified version of the theorem goes as follows. Assume we are interested in computing the probability of some event A in the sample space Ω that is otherwise hard (or possibly impossible) to calculate theoretically. We may approximate this probability via a simulation in the following way:

- Generate a large number of samples of the data of interest: $x_1, x_2, ..., x_N$
- Let $I_A(x_i) = \begin{cases} 1, x_i \in A \\ 0, \text{otherwise} \end{cases}$

- $P(A) \approx \frac{\sum_{i=1}^{N} I_A(x_i)}{N}$

For example, in the script `Coin Flip Example - R Code.R` on Canvas, we want to compute the probability of flipping heads (H) for a coin with unknown probability $P(H)$. We define $A = \{H\}$ and we approximate this probability by simulating a large number of coin flips $x_i \in \{H, T\}$ for $i = 1, \ldots, N$, and approximating $P(H) \approx \frac{\sum_{i=1}^{N} I_A(x_i)}{N}$. The latter is also known as a *(Simple) Monte Carlo Approximation*. Note that the larger the sample size N is, the more accurate the approximation will be.

Using this simulation approximation approach for a probability of interest, you are required to approximate the probabilities of interest of the following scenario. Assume that you roll a *fair* 8-faced die 20 times. You want to approximate the probability that at least i of the rolls have a value of 6 or greater, given that you know at least j of the rolls have a value of 4 or greater, for every combination of $i = 12, 13, 14$ and $j = 8, 9, 10, 11, 12$.

For notational convenience, define the probability of interest by $p(i, j)$ (i.e., we are interested in $3 \times 5 = 15$ different probabilities).

Your tasks for this question are the following:

(a) Write a simple pseudo-code of the steps to approximate $p(i, j)$ for a fixed i and fixed j. **[5 marks]**

(b) Write your own `R` code (or code in the language of your choice) to implement your pseudo-code in (a) to approximate all probabilities $p(i, j)$ for $i = 12, 13, 14$ and $j = 8, 9, 10, 11, 12$ via simulation. Note that you must include your `R` code at the very end of your assignment as an Appendix. **[10 marks]**

(c) Report your numerical values obtained for the approximation of the probabilities $p(i, j)$ for $i = 12, 13, 14$ and $j = 8, 9, 10, 11, 12$. **[5 marks]**

(Hint: It is recommended to go through the entire script `Coin Flip Example - R Code.R` on Canvas to make sure you understand this concept to approximate probabilities. In particular, note that we want to approximate a conditional probability here.)