STAT 443: Assignment 3

Saksham Sudershan (Student #31339427)

02 April 2022

Problem 1

(a)

We know that,

$$X_t = Z_t + 0.7Z_{t-1} + 0.2Z_{t-2} - 0.1Z_{t-1} \qquad \textit{where } Z_t \sim WN(0,1)$$

Now,

$$\gamma(0) = \sigma_Z^2 \left[\sum_{i=0}^{3} \beta_i^2 \right]$$
$$= 1 \times [1^2 + 0.7^2 + 0.2^2 + 0.1^2]$$
$$= 1 \times [1 + 0.49 + 0.04 + 0.01]$$
$$= 1 \times [1.54]$$
$$= 1.54$$

$$\gamma(1) = \sigma_Z^2 \left[\sum_{i=0}^{2} \beta_i \beta_{i+1} \right]$$
$$= 1 \times [1 \cdot 0.7 + 0.7 \cdot 0.2 - 0.2 \cdot 0.1]$$
$$= 1 \times [0.7 + 0.14 - 0.02]$$
$$= 1 \times [0.82]$$
$$= 0.82$$

$$\gamma(2) = \sigma_Z^2 \left[\sum_{i=0}^{1} \beta_i \beta_{i+2} \right]$$
$$= 1 \times [1 \cdot 0.2 - 0.7 \cdot 0.1]$$
$$= 1 \times [0.2 - 0.07]$$
$$= 1 \times [0.13]$$
$$= 0.13$$

$$\gamma(3) = \sigma_Z^2 \left[\sum_{i=0}^{0} \beta_i \beta_{i+3} \right]$$

$$= 1 \times [-1 \cdot 0.1]$$
$$= 1 \times [-0.1]$$
$$= -0.1$$

$$For\ k > 3, \quad \gamma(k) = 0$$

So now, the spectral density function is given by,

$$f(\omega) = \frac{1}{\pi}\left\{\gamma(0) + 2 \times \sum_{k=1}^{\infty} \gamma(k)\cos(k\omega)\right\}$$

$$= \frac{1}{\pi}\{1.54 + 2[\gamma(1)\cos(\omega) + \gamma(2)\cos(2\omega) + \gamma(3)\cos(3\omega)]\}$$
$$= \frac{1}{\pi}\{1.54 + 2[0.82\cos(\omega) + 0.13\cos(2\omega) - 0.1\cos(3\omega)]\}$$
$$= \frac{1}{\pi}\{1.54 + 1.64\cos(\omega) + 0.26\cos(2\omega) - 0.2\cos(3\omega)\}$$
$$= \frac{1.54 + 1.64\cos(\omega) + 0.26\cos(2\omega) - 0.2\cos(3\omega)}{\pi}$$

(b)

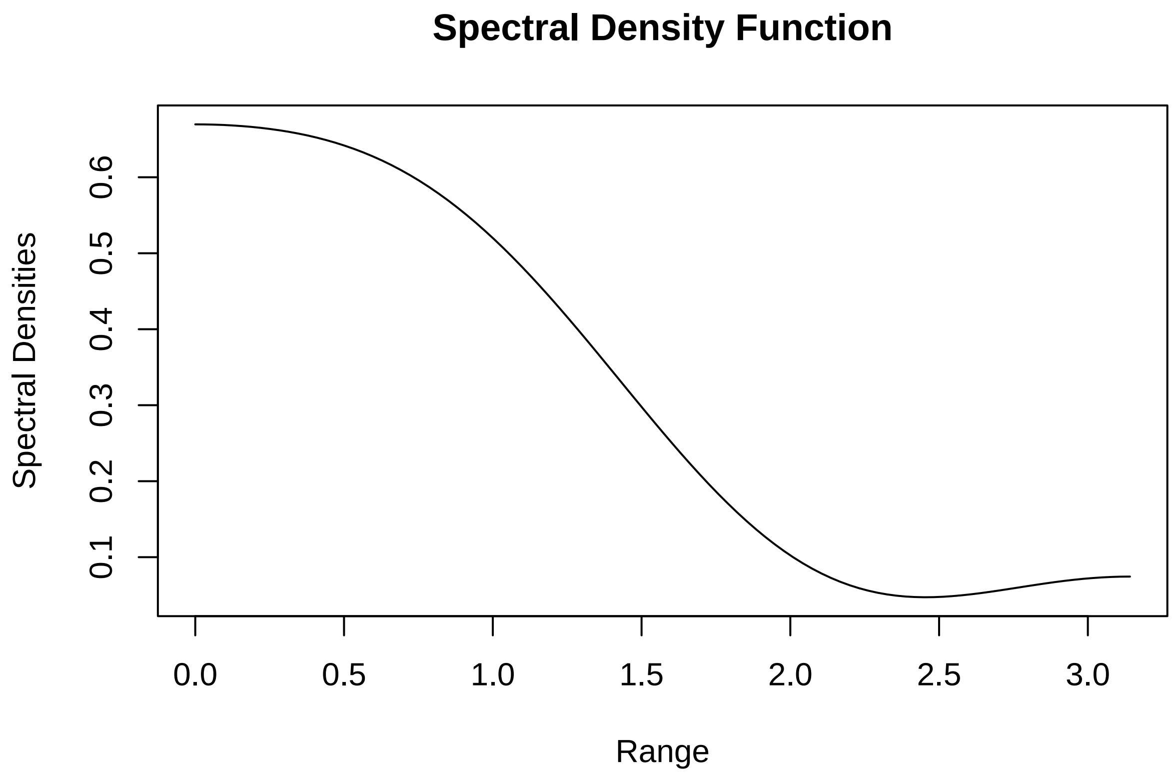
The normalized spectral density function $f^*(\omega)$ is given by,

$$f^*(\omega) = \frac{f(\omega)}{\sigma_X^2} = \frac{f(\omega)}{1.54}$$

$$= \frac{\frac{1.54}{1.54} + \frac{1.64}{1.54}\cos(\omega) + \frac{0.26}{1.54}\cos(2\omega) - \frac{0.2}{1.54}\cos(3\omega)}{\pi}$$
$$= \frac{1 + 1.0649\cos(\omega) + 0.1688\cos(2\omega) - 0.13\cos(3\omega)}{\pi}$$

(c)

```
curve((1+(1.0649*cos(x))+(0.1688*cos(2*x))-(0.130*cos(3*x)))*(1/pi) , from = 0, to = pi, xlab = "Range"
```



The graph of the Spectral Density Function shows that it is dominated by low frequencies. It is mostly a decreasing function till about 2.4, after which it starts increasing.

Problem 2

(a)

We know that the spectral density functions for $\{X_t\}_{t \in Z}$ and $\{Y_t\}_{t \in Z}$ are $f_X(\omega)$ and $f_Y(\omega)$, respectively. We also know that these are independent, and that the process $\{W_t\}_{t \in Z}$ is given by $W_t = X_t + Y_t$.

The spectral density function for $\{W_t\}_{t \in Z}$ is given by,

$$f_W(\omega) = \frac{1}{\pi}\left[\gamma_W(0) + 2\sum_{k=1}^{\infty}\gamma_W(k)cos(k\omega)\right]$$

Or,

$$f_W(\omega) = \frac{1}{\pi}\left[Var(W_t) + 2\sum_{k=1}^{\infty}Cov(W_{t+k}, W_t) \times cos(k\omega)\right]$$

Now,

$$\begin{aligned} Var(W_t) &= Var(X_t + Y_t) \\ &= Var(X_t) + Var(Y_t) + Cov(X_t, Y_t) \\ &= Var(X_t) + Var(Y_t) \quad \textit{since X and Y are independent} \end{aligned}$$

And,

$$Cov(W_{t+k}, W_t) = Cov(X_{t+k} + Y_{t+k}, X_t + Y_t)$$

$$= Cov(X_{t+k}, X_t) + Cov(X_{t+k}, Y_t) + Cov(Y_{t+k}, X_t) + Cov(Y_{t+k}, Y_t)$$
$$= Cov(X_{t+k}, X_t) + 0 + 0 + Cov(Y_{t+k}, Y_t)$$
$$= Cov(X_{t+k}, X_t) + Cov(Y_{t+k}, Y_t)$$

Putting this together for spectral density function of $\{W_t\}_{t\in Z}$,

$$f_W(\omega) = \frac{1}{\pi}\left[Var(X_t) + Var(Y_t) + 2\sum_{k=1}^{\infty}[Cov(X_{t+k}, X_t) + Cov(Y_{t+k}, Y_t)] \times cos(k\omega)\right]$$
$$\Rightarrow f_W(\omega) = \frac{1}{\pi}\left[Var(X_t) + 2\sum_{k=1}^{\infty} Cov(X_{t+k}, X_t) \times cos(k\omega) + Var(Y_t) + \sum_{k=1}^{\infty} Cov(Y_{t+k}, Y_t) \times cos(k\omega)\right]$$
$$\Rightarrow f_W(\omega) = \frac{1}{\pi}\left[Var(X_t) + 2\sum_{k=1}^{\infty} Cov(X_{t+k}, X_t) \times cos(k\omega)\right] + \frac{1}{\pi}\left[Var(Y_t) + \sum_{k=1}^{\infty} Cov(Y_{t+k}, Y_t) \times cos(k\omega)\right]$$
$$\Rightarrow f_W(\omega) = f_X(\omega) + f_Y(\omega)$$

(b)

$$X_t = -0.5X_{t-1} + Z_t$$

$$\Rightarrow (1 + 0.5B)X_t = Z_t$$
$$\Rightarrow X_t = \frac{1}{1 + 0.5B} Z_t$$
$$\Rightarrow X_t = Z_t[1 - 0.5B + 0.25B^2 \dots] = Z_t - 0.5Z_{t-1} + 0.25Z_{t-2} \dots$$

Since Z_t and Y_t are independent, $X_t = \frac{1}{1+0.5B} Z_t$ and Y_t must be independent.

Now,

$$\gamma_Y(0) = \sigma^2 \ and\ \gamma_Y(k) = 0\ for\ k > 0$$

The spectral density function for Y_t is given by,

$$f_Y(\omega) = \frac{1}{\pi}\left[\gamma_Y(0) + 2\sum_{k=1}^{\infty}\gamma_Y(k)cos(k\omega)\right]$$
$$\Rightarrow f_Y(\omega) = \frac{\sigma^2}{\pi}$$

And,

$$\gamma_X(k) = \sigma^2 \sum_{i=0}^{\infty} \beta_i\beta_{i+k}$$
$$\Rightarrow \gamma_X(k) = \sigma^2 \sum_{i=0}^{\infty} \beta_i\beta_i \times -0.5^k \qquad\qquad since\ \beta_j = -0.5^j, j \in [0, \infty)$$
$$\gamma_X(0) = \sigma^2\left[1^2 + 0.5^2 + 0.5^4 \dots\right] = \frac{\sigma^2}{1 - 0.5^2}$$
$$\gamma_X(k) = \frac{\sigma^2}{1 - 0.5^2} \times -0.5^k = \frac{(-0.5)^k\sigma^2}{1 - 0.5^2}$$

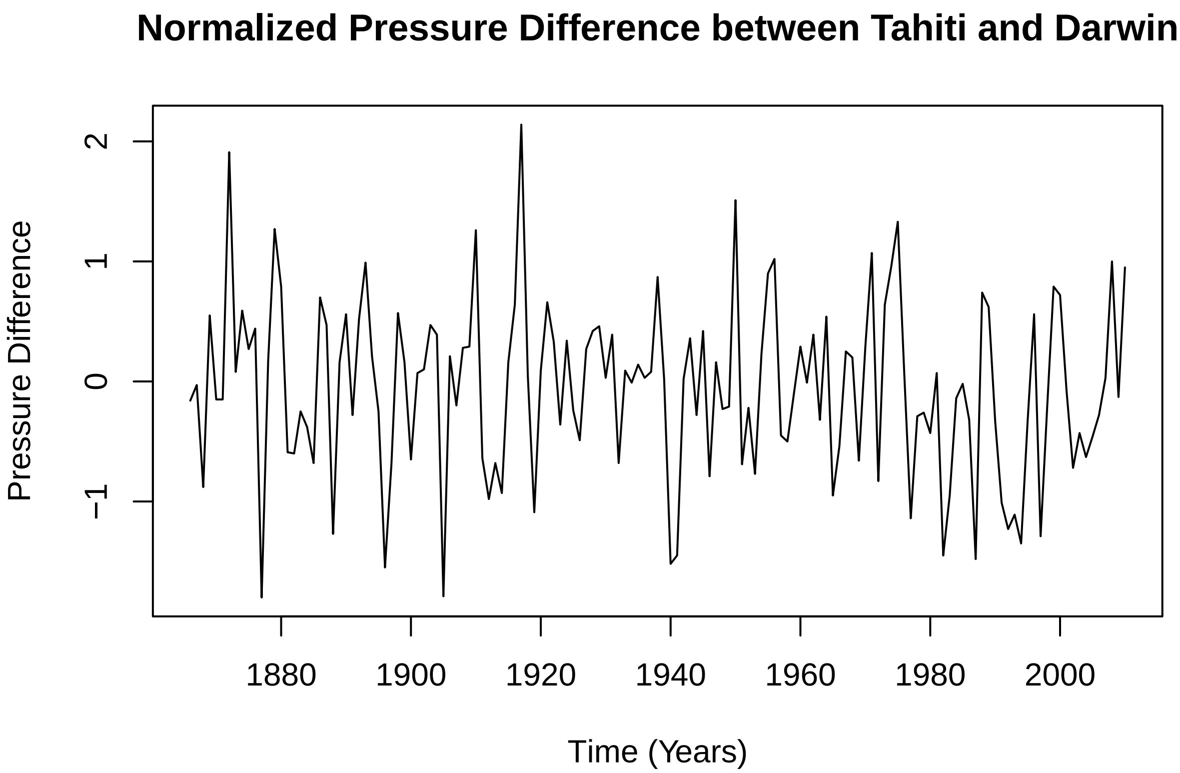
Now, we can find the spectral density function of $W_t = X_t + Y_t$:

$$f_W(\omega) = f_X(\omega) + f_Y(\omega)$$
$$\Rightarrow f_W(\omega) = \frac{1}{\pi}\left[\frac{\sigma^2}{1 - 0.5^2} + \sum_{k=1}^{\infty}\frac{(-0.5)^k\sigma^2}{1 - 0.5^2} \times cos(k\omega)\right] + \frac{\sigma^2}{\pi}$$

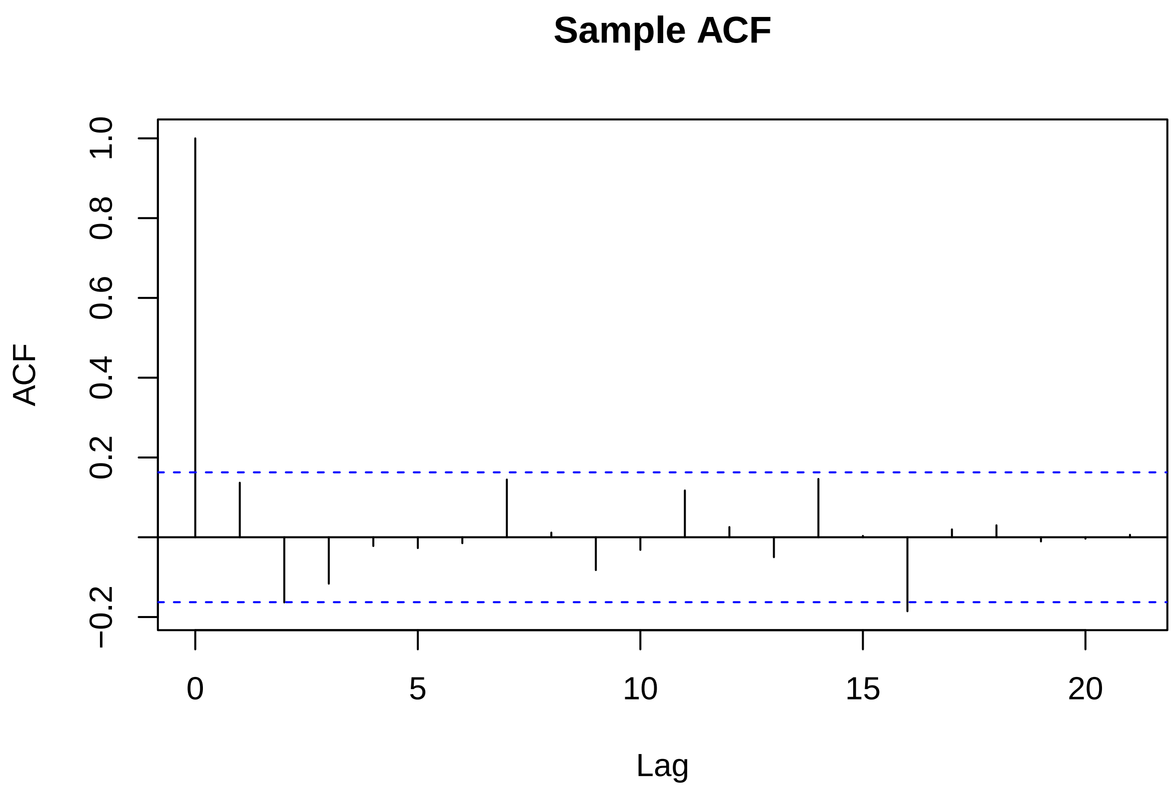
Problem 3

(a)

```
data <- read.table("soi.txt", sep = "\t", header = T)
soi.ts <- ts(data$annual, start = c(1866), end = c(2010))
plot(soi.ts, xlab = "Time (Years)", ylab = "Pressure Difference",
     main = "Normalized Pressure Difference between Tahiti and Darwin")
```



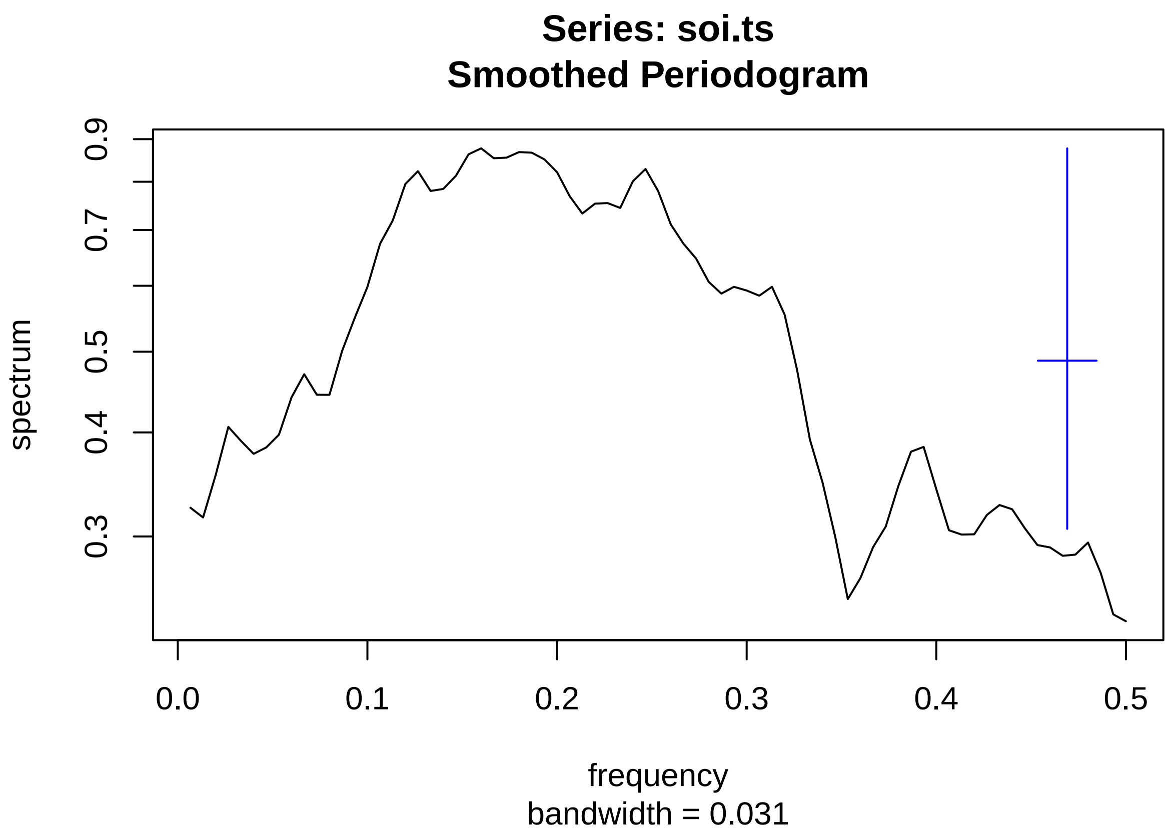

```
acf(soi.ts, main = "Sample ACF")
```



The data shows a large number of fluctuations, and appears quite random from the time series plot. The ACF plot almost resembles that of a white noise.

(b)

```
spc <- spec.pgram(soi.ts, spans = (sqrt(2*length(soi.ts))))
```




```r
dominating_freq <- spc$freq[which.max(spc$spec)]
dominating_freq
```

```
## [1] 0.16
```

The smoothed periodogram estimates that there are high densities in the middle frequencies.

The dominating model frequency is 0.16. The corresponding angular frequency is 1.0053096. The corresponding wavelength is 6.25.

(c)

```r
# Making fourier frequency function
four_freq <- function(p = -1, example.ts){
  if(p>floor(length(example.ts)/2) ){
    stop("P is greater than N/2")
  }
  if(!is(example.ts, "ts")){
    stop("Object is not a time series")
  }
  omega_p <- (2*pi*p)/length(example.ts)
  return(omega_p)
}

four_freq(10, soi.ts)
```

```
## [1] 0.4333231
```

To use the function four_freq, there are 2 arguments that are necessary. The needs to be a time series for which we find the Fourier frequency, and the user needs to input the constant p.

The output of the function at $p = 10$ is 0.4333.

(d)

```r
data$row <- seq.int(nrow(data))
ybar <- mean(data$annual)
count <- 1
signif_freq <- data.frame()
crit <- qf(0.95,2, 142)
for(i in 1:(length(soi.ts)/2) ){
  model <- lm(data$annual ~ cos(four_freq(i, soi.ts)*data$row)+sin(four_freq(i, soi.ts)*data$row) )
  test_stat <- ((1/2)*sum((model$fitted.values-ybar)^2))/((1/142)*sum(model$residuals^2))
  if(test_stat>crit){
    signif_freq[count,1] <- i
    signif_freq[count,2] <- model$coefficients[1]
    signif_freq[count,3] <- model$coefficients[2]
    signif_freq[count,4] <- model$coefficients[3]
    count <- count +1
  }
}

colnames(signif_freq)<- c("p", "a_0", "a_p", "b_p")
signif_freq
```

```
##    p         a_0         a_p         b_p
## 1 16 -0.05682759  0.12266443 -0.22333955
## 2 20 -0.05682759  0.21402222 -0.03789696
## 3 23 -0.05682759 -0.07272298  0.20606985
## 4 25 -0.05682759 -0.15146776 -0.16459522
## 5 41 -0.05682759  0.21868125 -0.14668587
```

The above dataframe gives all values of p that give significant Fourier frequencies at the 5% CI. The other columns are also the esimates of a_0, a_p and b_p for the Fourier Frequency corresponding to the p.

(e)

```r
sum(signif_freq$a_0)
```

```
## [1] -0.2841379
```

Using the dataframe, we can find the esimated coefficients of all significant frequencies. The model is given by,

$$Y_t = -0.284 + 0.123\cos(0.693t) - 0.223\sin(0.693t) + 0.214\cos(0.867t) - 0.039\sin(0.867t)$$
$$-0.073\cos(0.997t) + 0.206\sin(0.997t) - 0.151\cos(1.083t) + -0.165\sin(1.083t)$$
$$+0.219\cos(1.777t) - 0.147\sin(1.776t)$$

(f)

```
for(i in 1: length(data$annual)){
  data$model[i] <- sum(signif_freq$a_0) + signif_freq[1,3]*cos(four_freq(16, soi.ts)*i) + signif_freq[2
}
plot(data$year, data$annual, type = "l", xlab = "Time (In Years)", ylab = "Pressure Difference", main =
lines(data$year, data$model, col = 2)
legend("topright", legend = c("Data", "Estimated Model"), lty = c(1,1), col = c(1,2))
```