STAT 443: Time Series and Forecasting

Lab 11: Smoothing the Periodogram

- The lab must be completed in `R Markdown`. Display all the `R` code used to perform your analysis.
- Create a pdf or html file and use it as your lab submission.
- Please ensure that the file you submit is in good order (e.g., not corrupted and contains the work you intend to submit). No late (re-)submissions will be accepted.

We have met the *spectral density function,* $f(\omega)$, of a stochastic process, and the *periodogram,* $I(\omega)$, which can estimate the spectral density function given a sample from the stochastic process. In effect, the periodogram is a histogram, created in R using the command `spec.pgram.` By default this command plots the periodogram on the log scale, the option `log="no"` allowing for direct comparisons. R uses $\omega/2\pi$ as frequency (per unit time), so the horizontal axis in the plot of the periodogram ranges from 0 to 0.5 rather than 0 to π. Moreover, as an estimate of the spectral density, the periodogram from R should be divided by π to be consistent with definitions we have provided. In what follows though it is the shape of the periodogram that is of most interest.

We have seen that the periodogram is in general not a consistent estimator of $f(\omega)$. We need to modify the (raw) periodogram to make it consistent. One approach to this is to smooth the periodogram, applying a moving average filter. This can be performed within `spec.pgram` via the argument `spans`. For instance, `spans=c(m)` applies a smoothing filter of length m to the periodogram values. Filters can be applied in sequence: `spans=c(m,n)` applies two smoothers in turn, of length m and n, respectively.

In the following, suppose that $\{Z_t\}_{t\in\mathbb{Z}}$ is white noise with mean zero and variance 4.

1. Here we will compare the spectral density of white noise with smoothed periodograms obtained from simulated white noise samples.

   (a) Use the `arima.sim` command, or otherwise, to simulate a series of length 100 from $\{Z_t\}_{t\in\mathbb{Z}}$. Use `spec.pgram` to create and plot the periodogram for your sample. Now smooth your periodogram by smoothing with spans (i) `c(5),` (ii) `c(15),` (iii) `c(7,5).` Comment on what you observe, comparing the periodogram with the spectral density function.

   (b) Use the `arima.sim` command, or otherwise, to simulate a series of length 1000 from $\{Z_t\}_{t\in\mathbb{Z}}$. Use `spec.pgram` to create and plot the periodogram for your sample. Now smooth your periodogram by smoothing with spans (i) `c(5),` (ii) `c(15),` (iii) `c(7,5).` Comment on what you observe.

(c) In terms of the shape at least, it should make no difference here whether we take logs of the periodogram of the data from (b). Play around with various choices of (i) taking logs or not and (ii) the spans argument, looking at no less than four special cases. Which choice was "best", and why?

(d) Suppose we know that the relationship between the periodogram $I(\omega)$ and the spectrum $f(\omega)$ is

$$\frac{2I(\omega)}{f(\omega)} \sim \chi_2^2.$$

Think about taking logs of the above relationship. In terms of the difference between the estimator, $I(\omega)$, and the expected value, $f(\omega)$, what impact does taking logs have on the error between the two? Is this consistent with your investigations in (c)?

2. Let $\{X_t\}_{t\in\mathbb{Z}}$ be defined by

$$X_t = Z_t - 0.9Z_{t-1}.$$

Recall that the spectral density function for $\{X_t\}_{t\in\mathbb{Z}}$ is

$$f(\omega) = \frac{7.24}{\pi}\left(1 - \frac{1.8\cos(\omega)}{1.81}\right), \qquad \omega \in (0, \pi).$$

Remind yourself how this function behaves over $(0, \pi)$.

(a) Use the `arima.sim` command to simulate a series of length 100 from $\{X_t\}_{t\in\mathbb{Z}}$. Use `spec.pgram` to create and plot the periodogram for your sample. Now smooth your periodogram by smoothing with spans (i) `c(5),` (ii) `c(15),` (iii) `c(7,5).` Comment on what you observe.

(b) Working upwards from the choice of integers in (ii) and (iii) above, find (if you can!) both single and double smoothers that give periodograms very close to the actual spectrum.

(c) Use the `arima.sim` command to simulate a series of length 1000 from $\{X_t\}_{t\in\mathbb{Z}}$. Use `spec.pgram` to create and plot the periodogram for your sample. Now smooth your periodogram by smoothing with spans (i) `c(5),` (ii) `c(15),` (iii) `c(7,5)`. Comment on what you observe.

(d) Working upwards from the choice of integers in (ii) and (iii) above, find (if you can!) both single and double smoothers that give periodograms very close to the actual spectrum. Comment on your results.