STAT 321: Assignment 2

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Problem 1

(a)

We know that $P(A) = 0.01$, $P(B) = 0.005$ and $P(C) = 0.02$. Let S by the event that a person shows symptoms. And we also know $P(S|A) = 1 - 0.1 = 0.9$, $P(S|B) = 1 - 0.05 = 0.95$ and $P(S|C) = 1 - 0.25 = 0.75$. Let E' respresent the complement of event E.

$$P(S) = P(S|A) \cdot P(A) + P(S|B) \cdot P(B) + P(S|C) \cdot P(C)$$
$$P(S) = 0.9 \cdot 0.01 + 0.95 \cdot 0.005 + 0.75 \cdot 0.02$$
$$P(S) = 0.009 + 0.00475 + 0.015$$
$$P(S) = 0.02875$$

Using Bayes Theorem,

$$P(A|S) = \frac{P(S|A) \cdot P(A)}{P(S)} = \frac{0.009}{0.02875}$$
$$P(A|S) = 0.313$$

$$P(B|S) = \frac{P(S|B) \cdot P(B)}{P(S)} = \frac{0.00475}{0.02875}$$
$$P(B|S) = 0.165$$

$$P(C|S) = \frac{P(S|C) \cdot P(C)}{P(S)} = \frac{0.015}{0.02875}$$
$$P(C|S) = 0.522$$

(b)

$$P(S') = 1 - 0.02875 = 0.97125$$

$$P(A|S') = \frac{P(S'|A) \cdot P(A)}{P(S')} = \frac{0.1 \cdot 0.01}{0.97125}$$
$$P(A|S') = 0.00103$$

$$P(B|S') = \frac{P(S'|B) \cdot P(B)}{P(S')} = \frac{0.05 \cdot 0.005}{0.97125}$$
$$P(B|S') = 0.00026$$

$$P(C|S') = \frac{P(S'|C) \cdot P(C)}{P(S')} = \frac{0.25 \cdot 0.02}{0.97125}$$
$$P(C|S') = 0.00515$$

(c)

$$Sensitivity\ :\ P(T_+|A) = 0.95$$
$$Specificity\ :\ P(T_-|A') = 0.90$$

(i)

$$P(A'|T_-) = \frac{P(T_-|A') \cdot P(A')}{P(T_-)}$$
$$P(T_-) = P(T_-|A) \cdot P(A) + P(T_-|A')P(A') = 0.05 \times 0.01 + 0.90 \times 0.99 = .8915$$
$$So\ P(A'|T_-) = \frac{0.90 \cdot 0.99}{0.8915}$$
$$P(A'|T_-) = 0.99944$$

(ii)
For probability of a screening error based on the given test,

$$P(Error) = P(T_-|A) \cdot P(A) + P(T_+|A') \cdot P(A')$$
$$P(Error) = 0.05 \cdot 0.01 + 0.10 \cdot 0.99$$
$$P(Error) = 0.0995$$

Problem 2

(a)

Let R_i represent picking a red ball on the i^{th} draw and B_i represent picking a blue ball on the i^{th} draw. Then using the multiplicative rule,

$$P(X_3 = 4) = P(B_1 \cap B_2 \cap B_3)$$
$$P(X_3 = 4) = P(B_3|B_1 \cap B_2) \cdot P(B_2|B_1) \cdot P(B_1)$$
$$P(X_3 = 4) = \frac{5}{8} \cdot \frac{3}{5} \cdot \frac{1}{2} = 0.1875$$

$$P(X_3 = 5) = P[(R_1 \cap B_2 \cap B_3) \cup (B_1 \cap R_2 \cap B_3) \cup (B_1 \cap B_2 \cap R_3)]$$
$$P(X_3 = 5) = P(B_3|R_1 \cap B_2) \cdot P(B_2|R_1) \cdot P(R_1) + P(B_3|B_1 \cap R_2) \cdot P(R_2|B_1) \cdot P(B_1) + P(R_3|B_1 \cap B_2) \cdot P(B_2|B_1) \cdot (B_1)$$
$$P(X_3 = 5) = \frac{4}{8} \cdot \frac{2}{5} \cdot \frac{1}{2} + \frac{4}{8} \cdot \frac{2}{5} \cdot \frac{1}{2} + \frac{3}{8} \cdot \frac{3}{5} \cdot \frac{1}{2}$$
$$P(X_3 = 5) = 0.3125$$

$$P(X_3 = 6) = P[(R_1 \cap R_2 \cap B_3) \cup (R_1 \cap B_2 \cap R_3) \cup (B_1 \cap R_2 \cap R_3)]$$
$$P(X_3 = 6) = P(B_3|R_1 \cap R_2) \cdot P(R_2|R_1) \cdot P(R_1) + P(R_3|R_1 \cap B_2) \cdot P(B_2|R_1) \cdot P(R_1) + P(R_3|B_1 \cap R_2) \cdot P(R_2|B_1) \cdot (B_1)$$
$$P(X_3 = 6) = \frac{3}{8} \cdot \frac{3}{5} \cdot \frac{1}{2} + \frac{4}{8} \cdot \frac{2}{5} \cdot \frac{1}{2} + \frac{4}{8} \cdot \frac{2}{5} \cdot \frac{1}{2}$$
$$P(X_3 = 6) = 0.3125$$

$$P(X_3 = 7) = P(R_1 \cap R_2 \cap R_3)$$
$$P(X_3 = 7) = P(R_3|R_1 \cap R_2) \cdot P(R_2|R_1) \cdot P(R_1)$$
$$P(X_3 = 7) = \frac{5}{8} \cdot \frac{3}{5} \cdot \frac{1}{2} = 0.1875$$

So we have,

$$P(X_3 = 4) = 0.1875$$
$$P(X_3 = 5) = 0.3125$$
$$P(X_3 = 6) = 0.3125$$
$$P(X_3 = 7) = 0.1875$$

(b)

$$E(X_3) = \sum_{i=4}^{7} P(X_3 = i)$$
$$E(X_3) = 4 \times 0.1875 + 5 \times 0.3125 + 6 \times 0.3125 + 7 \times 0.1875$$
$$E(X_3) = 5.5$$

$$Var[X_3] = E(X_3^2) - E(X_3)^2$$
$$Var[X_3] = 31.25 - 30.25$$
$$Var[X_3] = 1$$

(c)

For Conditional Probabilities,

$$P(R_1|X_3 = 5) = \frac{P(R_1 \cap B_2 \cap B_3)}{P(X_3 = 5)}$$
$$P(R_1|X_3 = 5) = \frac{\frac{4}{8} \cdot \frac{2}{5} \cdot \frac{1}{2}}{0.3125}$$
$$P(R_1|X_3 = 5) = \frac{0.1}{0.31250} = 0.32$$

Problem 3

(a)

Let S_T be the value of stock S after T days. Then,

$$P(S_5 = 23|S_{10} = 26) = \frac{P(S_5 = 23 \cap S_{10} = 26)}{P(S_{10} = 26)}$$
$$P(S_5 = 23|S_{10} = 26) = \frac{\binom{5}{4} \cdot p^4(1-p)^1 \times \binom{5}{4} \cdot p^4(1-p)^1}{\binom{10}{8} \cdot p^8(1-p)^2}$$

(b)

$50p$ is the expected number of stocks that go up in 1 day, and $50 \times (1-p)$ is the expected number of stocks that decrease in value in 1 day.

$$Expected\ Gain/Loss\ in\ 1\ Day = 50p \cdot 1\$ + 50(1-p) \cdot -1\$$$

$$Expected\ Gain/Loss\ in\ 1\ Day = 50p - 50 + 50p = 100p - 50$$

Now over T days,

$$Expected\ Gain/Loss\ in\ T\ Days = T \times (100p - 50)$$

$$Expected\ Value\ of\ Portfolio\ in\ T\ Days = 1000 + 100pT - 50T$$

Since only p is a random variable,

$$Var[Portfolio] = 100^2 \cdot Var[p]$$

$$Var[Portfolio] = 100^2 \cdot [p \cdot 1 - p]$$

$$SD[Portfolio] = 100\sqrt{p - p^2}$$

(c)

In finance, it is usually important to decrease the risk by minimizing variance by choosing stocks with negative correlations. However, in this scenario, the stocks are all independent and so it makes no difference whether 50 stocks of the same company are chosen or if stocks of 50 different companies are chosen. The expected value and SD will still be given by,

$$E[Portfolio] = 1000 + 100pT - 50T$$

$$SD[Portfolio] = 100\sqrt{p - p^2}$$

Problem 4

(a)

The probability of a bit being decoded erroneously is p. Since the occurs independently from bit to bit, we can say that the probability for j bits to be decoded erroneously when n bits are transmitted is given by:

$$P(j\ bits\ erroneous\ |\, n\ bits\ transmitted) = \binom{n}{j} \cdot p^j \cdot (1-p)^{(n-j)}$$

(b)

To calculate the probability that the error-correcting code can correct the errors given n-digits are transmitted and j bits are incorrectly decoded, we can find the number of ways codes can be constructed which leads to the error-correcting code not working. Thus,

$$P(Code\ cannot\ be\ corrected) = \frac{\binom{j}{2} \cdot 2! \cdot (n-1)!}{n!}$$

We choose 2 out of the j incorrect bits, which can be arranged themselves in 2! ways, and arrange the rest $(n-1)$ bits in $(n-1)!$ ways. We divide this by the total number of ways the bits can be arranged, which is $n!$.

Now, the probability that the error-correcting code can correct the errors is given by,

$$P(Code\ can\ be\ corrected) = 1 - P(Code\ cannot\ be\ corrected)$$
$$= 1 - \frac{\binom{j}{2} \cdot 2! \cdot (n-1)!}{n!}$$

(c)

```
decoder_prob <- function( no_of_bits = 8, prob_incorrect = 0.01, runs = 1000){
  count <- c(0)
  for(i in 1:runs){
    decoder_sample <- sample( c(0,1), size = no_of_bits, replace = TRUE,
                              prob= c(prob_incorrect, (1-prob_incorrect)))
    flag <- c(0)
    for(i in 1:(no_of_bits-1)){
      if(decoder_sample[i]==0 && decoder_sample[i+1]==0){
       flag = 1
      }
    }
    if(flag==1){
      count <- count + 1
    }
  }
  (runs-count)/runs
}

prob_1 <- data.frame()
for(i in 1:4){
```

```r
  for(j in 1:4){
    if(j==1){
      prob_1[i,j] <- decoder_prob(i*8, 0.01, 10000)
    }
    if(j>1){
      prob_1[i,j] <- decoder_prob(i*8, (j-1)*0.05, 10000)
    }
  }
}
row.names(prob_1) <- c("n=8","n=16", "n=24", "n=32")
colnames(prob_1) <- c("p=0.01", "p=0.05","p=0.10","p=0.15")
prob_1
```

```
##      p=0.01 p=0.05 p=0.10 p=0.15
## n=8  0.9988 0.9833 0.9347 0.8689
## n=16 0.9990 0.9628 0.8715 0.7385
## n=24 0.9981 0.9474 0.8077 0.6260
## n=32 0.9982 0.9287 0.7577 0.5350
```

(d)

```r
improved_decoder_prob <- function(no_of_bits = 8, prob_incorrect = 0.01, runs = 1000){
  count <- c(0)
  for(i in 1:runs){
    decoder_sample <- sample( c(0,1), size = no_of_bits, replace = TRUE,
                              prob= c(prob_incorrect, (1-prob_incorrect)))
    flag <- c(0)
    for(i in 1:(no_of_bits-2)){
      if(decoder_sample[i]==0 && decoder_sample[i+1]==0 && decoder_sample[i+2]==0){
       flag = 1
      }
    }
    if(flag==1){
      count <- count + 1
    }
  }
  (runs-count)/runs
}

prob_2 <- data.frame()
for(i in 1:4){
  for(j in 1:4){
    if(j==1){
      prob_2[i,j] <- improved_decoder_prob(i*8, 0.01, 10000)
    }
    if(j>1){
      prob_2[i,j] <- improved_decoder_prob(i*8, (j-1)*0.05, 10000)
    }
  }
}
row.names(prob_2) <- c("n=8","n=16", "n=24", "n=32")
```

```
colnames(prob_2) <- c("p=0.01", "p=0.05","p=0.10","p=0.15")
prob_2
```

```
##      p=0.01 p=0.05 p=0.10 p=0.15
## n=8       1 0.9993 0.9936 0.9814
## n=16      1 0.9980 0.9878 0.9597
## n=24      1 0.9972 0.9827 0.9357
## n=32      1 0.9955 0.9729 0.9158
```

Problem 5

(a)

The ball has a 50-50 chance of going to the left or to the right. Thus, for the ball to go into cell 0, it has to take 5 left turns at each peg encountered. So,

$$P(Ball\ goes\ into\ cell\ 0) = .5^5 = 0.03125$$

For cell 1, the ball has to take 4 left turns, and 1 right turn. However, the right turn can take place at different pegs. To take into account the different paths that the ball can take, we can use combination.

$$P(Ball\ goes\ into\ cell\ 1) = \binom{5}{1}(0.5)^4(0.5)^1 = 5 \cdot (.5)^4 \cdot (.5)^1 = 0.15625$$

Similarly, we can calculate other probabilities as well:

$$P(Ball\ goes\ into\ cell\ 2) = \binom{5}{2}(0.5)^3(0.5)^2 = 0.3125$$
$$P(Ball\ goes\ into\ cell\ 3) = \binom{5}{3}(0.5)^2(0.5)^3 = 0.3125$$
$$P(Ball\ goes\ into\ cell\ 4) = \binom{5}{4}(0.5)^1(0.5)^4 = 0.15625$$
$$P(Ball\ goes\ into\ cell\ 5) = \binom{5}{5}(0.5)^0(0.5)^5 = 0.03125$$

(b)

We can make a function that simulates the Galton Board or quincunx for a given number of runs and number of rows of pegs.

```
galton_sim <- function(no_row_of_pegs = 5, no_runs = 100){
  cells <- as.data.frame(matrix(0, nrow = (no_row_of_pegs+1), ncol = 1))
  for(i in 0:(no_row_of_pegs)){
    row.names(cells)[i+1] <- paste("Cell",i)
  }

  for(i in 1:(no_runs)){
      marble_run <- sample((0:1), size = no_row_of_pegs, replace = TRUE)
      cells[(sum(marble_run)+1), 1] <- cells[(sum(marble_run)+1), 1] + 1
```

```
  }
  colnames(cells) <- paste("Number of Balls")
  cells
}
```

We can use this function to simulate a quincunx with 5 rows of pegs, and drop a ball 1000 times, recording the values.

```
sim_1 <- galton_sim(5, 1000)
```

To compare with theoretical values, we can make another row which shows theoretical values and another with expected values for a 1000 runs. We can see that the frequencies calculated through running the experiment quite closely resemble the expected number of balls given by theoretical probabilites.

```
for (i in 1:6) {
  sim_1[i, 2] <- choose(5,(i-1))*((0.5)^(i))*((0.5)^(5-i))
  sim_1[i, 3] <- (sim_1[i,2])*1000
}
colnames(sim_1) <- c("Number of Balls","Theoretical Probabilities",
                     "Expected Number of Balls")
sim_1
```

```
##        Number of Balls Theoretical Probabilities Expected Number of Balls
## Cell 0              25                   0.03125                    31.25
## Cell 1             136                   0.15625                   156.25
## Cell 2             341                   0.31250                   312.50
## Cell 3             313                   0.31250                   312.50
## Cell 4             143                   0.15625                   156.25
## Cell 5              42                   0.03125                    31.25
```

(c)

For each cell, we can generalize the probability given the number of rows of pegs is 100, and there is a 50-50 chance of the ball going left or right. For a given cell $k-1$, where $k \in N$ is given by:

$$P(Ball\ Goes\ Into\ Cell\ k-1) = \binom{100}{k}(0.5)^k(0.5)^{(100-k)}$$

(d)

We can use the same function again with the number of rows set as 100, and do a 1000 runs on this, recording the values.

```
sim_2 <- galton_sim(100, 1000)
```

To compare easily with theoretical values, we can make another row which shows theoretical values and another with expected values for a 1000 runs. We can see again that the actual frequencies obtained through the experiment closely resemble the theoretical values or the expected number of balls in each cell.

```
for (i in 1:101) {
  sim_2[i, 2] <- round(choose(100,(i-1))*((0.5)^(i))*((0.5)^(100-i)), digits = 5)
  sim_2[i, 3] <- round((sim_2[i,2])*1000, digits = 2)
}
colnames(sim_2) <- c("Number of Balls","Theoretical Probabilities",
                     "Expected Number of Balls")
sim_2
```

```
##         Number of Balls Theoretical Probabilities Expected Number of Balls
## Cell 0                0                   0.00000                     0.00
## Cell 1                0                   0.00000                     0.00
## Cell 2                0                   0.00000                     0.00
## Cell 3                0                   0.00000                     0.00
## Cell 4                0                   0.00000                     0.00
## Cell 5                0                   0.00000                     0.00
## Cell 6                0                   0.00000                     0.00
## Cell 7                0                   0.00000                     0.00
## Cell 8                0                   0.00000                     0.00
## Cell 9                0                   0.00000                     0.00
## Cell 10               0                   0.00000                     0.00
## Cell 11               0                   0.00000                     0.00
## Cell 12               0                   0.00000                     0.00
## Cell 13               0                   0.00000                     0.00
## Cell 14               0                   0.00000                     0.00
## Cell 15               0                   0.00000                     0.00
## Cell 16               0                   0.00000                     0.00
## Cell 17               0                   0.00000                     0.00
## Cell 18               0                   0.00000                     0.00
## Cell 19               0                   0.00000                     0.00
## Cell 20               0                   0.00000                     0.00
## Cell 21               0                   0.00000                     0.00
## Cell 22               0                   0.00000                     0.00
## Cell 23               0                   0.00000                     0.00
## Cell 24               0                   0.00000                     0.00
## Cell 25               0                   0.00000                     0.00
## Cell 26               0                   0.00000                     0.00
## Cell 27               0                   0.00000                     0.00
## Cell 28               0                   0.00000                     0.00
## Cell 29               0                   0.00001                     0.01
## Cell 30               0                   0.00002                     0.02
## Cell 31               0                   0.00005                     0.05
## Cell 32               0                   0.00011                     0.11
## Cell 33               1                   0.00023                     0.23
## Cell 34               1                   0.00046                     0.46
## Cell 35               2                   0.00086                     0.86
## Cell 36               1                   0.00156                     1.56
## Cell 37               2                   0.00270                     2.70
## Cell 38               5                   0.00447                     4.47
## Cell 39              11                   0.00711                     7.11
## Cell 40              11                   0.01084                    10.84
## Cell 41              17                   0.01587                    15.87
## Cell 42              18                   0.02229                    22.29
## Cell 43              28                   0.03007                    30.07
```

```
## Cell 44            39            0.03895            38.95
## Cell 45            38            0.04847            48.47
## Cell 46            71            0.05796            57.96
## Cell 47            64            0.06659            66.59
## Cell 48            77            0.07353            73.53
## Cell 49            77            0.07803            78.03
## Cell 50            91            0.07959            79.59
## Cell 51            70            0.07803            78.03
## Cell 52            57            0.07353            73.53
## Cell 53            63            0.06659            66.59
## Cell 54            73            0.05796            57.96
## Cell 55            46            0.04847            48.47
## Cell 56            37            0.03895            38.95
## Cell 57            28            0.03007            30.07
## Cell 58            27            0.02229            22.29
## Cell 59            12            0.01587            15.87
## Cell 60             9            0.01084            10.84
## Cell 61             9            0.00711             7.11
## Cell 62             4            0.00447             4.47
## Cell 63             5            0.00270             2.70
## Cell 64             3            0.00156             1.56
## Cell 65             0            0.00086             0.86
## Cell 66             3            0.00046             0.46
## Cell 67             0            0.00023             0.23
## Cell 68             0            0.00011             0.11
## Cell 69             0            0.00005             0.05
## Cell 70             0            0.00002             0.02
## Cell 71             0            0.00001             0.01
## Cell 72             0            0.00000             0.00
## Cell 73             0            0.00000             0.00
## Cell 74             0            0.00000             0.00
## Cell 75             0            0.00000             0.00
## Cell 76             0            0.00000             0.00
## Cell 77             0            0.00000             0.00
## Cell 78             0            0.00000             0.00
## Cell 79             0            0.00000             0.00
## Cell 80             0            0.00000             0.00
## Cell 81             0            0.00000             0.00
## Cell 82             0            0.00000             0.00
## Cell 83             0            0.00000             0.00
## Cell 84             0            0.00000             0.00
## Cell 85             0            0.00000             0.00
## Cell 86             0            0.00000             0.00
## Cell 87             0            0.00000             0.00
## Cell 88             0            0.00000             0.00
## Cell 89             0            0.00000             0.00
## Cell 90             0            0.00000             0.00
## Cell 91             0            0.00000             0.00
## Cell 92             0            0.00000             0.00
## Cell 93             0            0.00000             0.00
## Cell 94             0            0.00000             0.00
## Cell 95             0            0.00000             0.00
## Cell 96             0            0.00000             0.00
## Cell 97             0            0.00000             0.00
```

```
## Cell 98                0                   0.00000                    0.00
## Cell 99                0                   0.00000                    0.00
## Cell 100               0                   0.00000                    0.00
```