STAT 321: Assignment 4

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Problem 1

Let $D = 1$ be the event that the rater is diligent and $D = 0$ be the event that the rater is non-diligant. Let $S_i = 1$ be the event that i number of content are labeled as spam, and $S_i = 0$ be the event that i number of content are labeled as non-spam. Now,

$$P(D = 1) = 0.90 \ and \ P(D = 0) = 0.10$$

$$P(S_1 = 1|D = 1) = 0.20 \ and \ P(S_1 = 0|D = 1) = 0.80$$
$$P(S_1 = 1|D = 0) = 0 \ and \ P(S_1 = 1|D = 0) = 1$$

Now, using Bayes Theorem,

$$P(D = 1|S_4 = 0) = \frac{P(S_4 = 0|D = 1) \cdot P(D = 1)}{P(S_4 = 0|D = 1) \cdot P(D = 1) + P(S_4 = 0|D = 0) \cdot P(D = 1)}$$

We also know that,

$$P(S_4 = 1|D = 1) = 0.8^4 \ and \ P(S_4 = 1|D = 0) = 1^4$$

$$\therefore P(D = 1|S_4 = 0) = \frac{0.8^4 \cdot 0.90}{0.8^4 \cdot 0.90 + 1^4 \cdot 0.10}$$
$$\Rightarrow P(D = 1|S_4 = 0) \approx 0.7866$$

Problem 2

Let Y be the number of friends that say it is raining in Vancouver. And let R be the event that it is raining in Vancouver.

$$P(R) = 0.25 \ and \ P(R') = 0.75$$

Using Bayes Theorem,

$$P(R|Y = 3) = \frac{P(Y = 3|R) \cdot P(R)}{P(Y = 3|R) \cdot P(R) + P(Y = 3|R') \cdot P(R')}$$

Now, the probability that each of the friend is lying is $\frac{1}{3}$. So,

$$P(Y = 3|R) = \left(\frac{2}{3}\right)^3 \ and \ P(Y = 3|R') = \left(\frac{1}{3}\right)^3$$

Putting this together, we get,

$$P(R|Y = 3) = \frac{\left(\frac{2}{3}\right)^3 \cdot 0.25}{\left(\frac{2}{3}\right)^3 \cdot 0.25 + \left(\frac{1}{3}\right)^3 \cdot 0.75}$$

$$\Rightarrow P(R|Y = 3) \approx 0.7273$$

Problem 3

(a)

Let E be the event that we win, that is, $\mathbb{HH}$ shows up first. Let H_i and T_i be the event that $\mathbb{H}$ or $\mathbb{T}$ show up for the ith toss. Now,

$$P(E|H_1) = \frac{1}{2}$$
$$P(H_1) = \frac{1}{2}$$

$$And\ P(E|T_1) = 0$$
$$P(T) = \frac{1}{2}$$

Using Law of Total Probability,

$$P(E) = P(E|H_1) \cdot P(H_1) + P(E|T_1) \cdot P(T_1)$$

$$\Rightarrow P(E) = 0.50 \cdot 0.50 + 0 \cdot 0.50 = 0.25$$

So the probability of us winning is 0.25.

(b)

In this case, let E be the event that our friends wins, that is, $\mathbb{HHT}$ shows up first. Let H_i and T_i be the event that $\mathbb{H}$ or $\mathbb{T}$ show up for the ith toss. Now,

$$Let\ P(E) = p$$
$$Then\ P(E|T_1) = p$$
$$Let\ P(E|H_1) = q$$

Using the Law of Total Probability,

$$P(E) = P(E|H_1) \cdot P(H_1) + P(E|T_1) \cdot P(T_1)$$

$$\Rightarrow p = p \cdot \frac{1}{2} + q \cdot \frac{1}{2}$$
$$\Rightarrow p = q$$

Now, considering the first two rolls,

$$P(E|H_1 \cap H_2) = 1$$
$$P(E|H_1 \cap T_2) = \frac{1}{2} \times p$$
$$P(E|T_1 \cap H_2) = q = p$$
$$and P(E|T_1 \cap T_2) = p$$

$$We\ know\ that\ P(H_1 \cap H_2) = 0.25\,, P(T_1 \cap H_2) = 0.25,$$
$$P(H_1 \cap T_2) = 0.25\ and\ P(T_1 \cap T_2) = 0.25$$

Again, using the Law of Total Probability,

$$P(E) = P(E|H_1 \cap H_2) \cdot 0.25 + P(E|H_1 \cap T_2) \cdot 0.25 + P(E|T_1 \cap H_2) \cdot 0.25 + P(E|T_1 \cap T_2) \cdot 0.25$$

$$\therefore p = 0.25 \times [1 + 0.5 \cdot p + p + p]$$
$$\Rightarrow p = 0.25 \times [2.5p + 1]$$
$$\Rightarrow 4p - 2.5p = 1$$
$$\Rightarrow 1.5p = 1$$
$$\Rightarrow p \approx 0.667$$

$$\therefore P(Our\ friend\ winning) = 0.667$$
$$\Rightarrow P(Us\ winning) = 1 - p = 0.333$$

So the probability of us winning is 0.333.

Problem 4

(a)

Y is the random variable which denotes the number of balls in bin 1. There are a total of n number of balls. Thus, we can say that $Y \sim Binom(y, p)$, where $p = 1/10$. So the pmf of Y is given by,

$$p_Y(y) = \binom{n}{y} \left(\frac{1}{10}\right)^y \left(\frac{9}{10}\right)^{n-y} \qquad where\ 0 \leq y \leq n$$

(b)

Z is the random variable which denotes the number of balls in bin 6,7,8,9 and 10. There are a total of n number of balls. Thus, we can say that $Z \sim Binom(z, p)$, where $p = 5/10$. So the pmf of Z is given by,

$$p_Z(z) = \binom{n}{z} \left(\frac{5}{10}\right)^z \left(\frac{5}{10}\right)^{n-z} \qquad where\ 0 \leq z \leq n$$

(c)

Given that there are y number of balls in bin 1, there are $n - y$ balls left. We also know that from $n - y$ balls, none go into bin 1. So,

$$p_{Z|Y}(z|y) = \binom{n-y}{z} \left(\frac{5}{9}\right)^z \left(\frac{4}{9}\right)^{n-y-z} \qquad where\ 0 \leq y \leq n\ and\ 0 \leq z \leq n - y$$

(d)

Given that there are z number of balls in bins 6-10, there are $n - z$ balls left. We also know that from $n - z$ balls, none go into bin 6-10. So,

$$p_{Y|Z}(y|z) = \binom{n-z}{y} \left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{n-z-y} \qquad where\ 0 \leq z \leq n\ and\ 0 \leq y \leq n - z$$

Problem 5

Let R_i and B_i represent that the ith ball pulled out is red and blue respectively.

(a)

$$P(R_1) = \frac{1}{2}$$

(b)

$$P(R_2) = P(R_2|R_1) \cdot P(R_1) + P(R_2|B_1) \cdot P(B_1)$$
$$\Rightarrow P(R_2) = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}$$
$$\Rightarrow P(R_2) = \frac{1}{2}$$

(c)

$$P(R_3 \cap R_2 \cap R_1) = P(R_3|R_2 \cap R_1) \cdot P(R_2|R_1) \cdot P(R_1)$$
$$\Rightarrow P(R_3 \cap R_2 \cap R_1) = \frac{3}{4} \cdot \frac{2}{3} \cdot 12$$
$$\Rightarrow P(R_3 \cap R_2 \cap R_1) = \frac{1}{4}$$

(d)

$$P(2\ of\ the\ first\ 3\ balls\ are\ red) = P(R_1 \cap R_2 \cap B_3) + P(R_1 \cap B_2 \cap R_3) + P(B_1 \cap R_2 \cap R_3)$$
$$\Rightarrow P(2\ of\ 3\ are\ red) = \left(\frac{1}{2} \times \frac{2}{3} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{3} \times \frac{2}{4}\right) + \left(\frac{1}{2} \times \frac{1}{3} \times \frac{2}{4}\right)$$
$$\Rightarrow P(2\ of\ 3\ are\ red) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}$$

Problem 6

Using Bayes Theorem,

$$f_{P|X}(p|9) = \frac{P(P = p) \cdot P(X = 9|P = p)}{P(X = 9)}$$

We know that $P \sim$ Unif[0,1], so

$$P(P = p) = \frac{1}{1 - 0} = 1 \qquad where 0 \leq p \leq 1$$

And we also know that $X \sim$ Binom(10,P), so

$$P(X = 9|P = p) = \binom{10}{9} p^9 (1 - p)$$

Now, $P(P = p) = 1$ when $p \in [0, 1]$. So we can integrate $P(X = 9|P = p)$ for all $p \in [0, 1]$ to find $P(X = 9)$. So,

$$P(X = 9) = \int_0^1 \binom{10}{9} p^9 (1 - p)$$
$$\Rightarrow P(X = 9) = \binom{10}{9} \int_0^1 p^9 - p^{10}$$
$$\Rightarrow P(X = 9) = \binom{10}{9} \left[\frac{p^{10}}{10} - \frac{p^{11}}{11}\right]_0^1$$
$$\Rightarrow P(X = 9) = \binom{10}{9} \left[\frac{1}{10} - \frac{1}{11}\right]$$
$$\Rightarrow P(X = 9) = \binom{10}{9} \cdot \frac{1}{110}$$

Putting this all together,

$$f_{P|X}(p|9) = \frac{\binom{10}{9} p^9 (1-p)}{\binom{10}{9} \cdot \frac{1}{110}}$$

$$\Rightarrow f_{P|X}(p|9) = 110 \cdot p^9 (1-p)$$

Sketching this,