

STAT 443: Assignment 2

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Problem 1

(a)

Given AR(3) process:

$$X_t = \frac{2}{5}X_{t-1} + \frac{1}{4}X_{t-2} - \frac{1}{10}X_{t-3} + Z_t \quad \{Z_t\}_{t \in N} \sim WN(0, \sigma^2)$$

The characteristic polynomial is given by:

$$1 - \frac{2}{5}B - \frac{1}{4}B^2 + \frac{1}{10}B^3$$

Equating the characteristic polynomial to find the roots, we get

$$B = -2, 2, 2.5$$

Since these roots are outside the unit circle in the complex plane, the AR(3) process is stationary.

(b)

The AR(3) process is given by

$$X_t = \frac{2}{5}X_{t-1} + \frac{1}{4}X_{t-2} - \frac{1}{10}X_{t-3} + Z_t \quad \{Z_t\}_{t \in N} \sim WN(0, \sigma^2)$$

We already know that the AR(3) process is stationary. Multiplying by X_{t-k} for $k = 1, 2, 3$ we get,

$$X_t X_{t-1} = \frac{2}{5}X_{t-1}X_{t-1} + \frac{1}{4}X_{t-2}X_{t-1} - \frac{1}{10}X_{t-3}X_{t-1} + Z_t X_{t-1}$$

$$X_t X_{t-2} = \frac{2}{5}X_{t-1}X_{t-2} + \frac{1}{4}X_{t-2}X_{t-2} - \frac{1}{10}X_{t-3}X_{t-2} + Z_t X_{t-2}$$

$$X_t X_{t-3} = \frac{2}{5}X_{t-1}X_{t-3} + \frac{1}{4}X_{t-2}X_{t-3} - \frac{1}{10}X_{t-3}X_{t-3} + Z_t X_{t-3}$$

Taking expectation on both sides and dividing by σ_x^2 , we get

$$\rho(1) = \frac{2}{5}\rho(0) + \frac{1}{4}\rho(-1) - \frac{1}{10}\rho(-2)$$

$$\begin{aligned}\rho(2) &= \frac{2}{5}\rho(1) + \frac{1}{4}\rho(0) - \frac{1}{10}\rho(-1) \\ \rho(3) &= \frac{2}{5}\rho(2) + \frac{1}{4}\rho(1) - \frac{1}{10}\rho(0)\end{aligned}$$

We know that $\rho(0) = 1$ and $\rho(-k) = \rho(k)$.

So these equations become

$$\begin{aligned}\rho(1) &= \frac{2}{5} + \frac{1}{4}\rho(1) - \frac{1}{10}\rho(2) \\ \rho(2) &= \frac{2}{5}\rho(1) + \frac{1}{4} - \frac{1}{10}\rho(1) \\ \rho(3) &= \frac{2}{5}\rho(2) + \frac{1}{4}\rho(1) - \frac{1}{10}\end{aligned}$$

Solving the system of equations, we get

$$\rho(1) \approx 0.48077, \rho(2) \approx 0.39423, \rho(3) \approx 0.17788$$

Now, let

$$D^3 - \frac{2}{5}D^2 - \frac{1}{4}D^1 + \frac{1}{10}D^0 = 0$$

Solving the above equation, we get

$$d_1 = \frac{1}{2}, d_2 = \frac{2}{5}, d_3 = \frac{-1}{2}$$

Now,

$$\begin{aligned}\rho(h) &= A_1 d_1^{|h|} + A_2 d_2^{|h|} + A_3 d_3^{|h|} \\ &= A_1 \left(\frac{1}{2}\right)^{|h|} + A_2 \left(\frac{2}{5}\right)^{|h|} + A_3 \left(\frac{-1}{2}\right)^{|h|}\end{aligned}$$

To find A_1, A_2 , and A_3 ,

$$\begin{aligned}\rho(1) &= A_1 \left(\frac{1}{2}\right)^1 + A_2 \left(\frac{2}{5}\right)^1 + A_3 \left(\frac{-1}{2}\right)^1 \\ \rho(2) &= A_1 \left(\frac{1}{2}\right)^2 + A_2 \left(\frac{2}{5}\right)^2 + A_3 \left(\frac{-1}{2}\right)^2 \\ \rho(3) &= A_1 \left(\frac{1}{2}\right)^3 + A_2 \left(\frac{2}{5}\right)^3 + A_3 \left(\frac{-1}{2}\right)^3\end{aligned}$$

Solving the above system of equations, where $\rho(1) = 0.48077, \rho(2) = 0.39423$ and $\rho(3) = 0.17788$, we get,

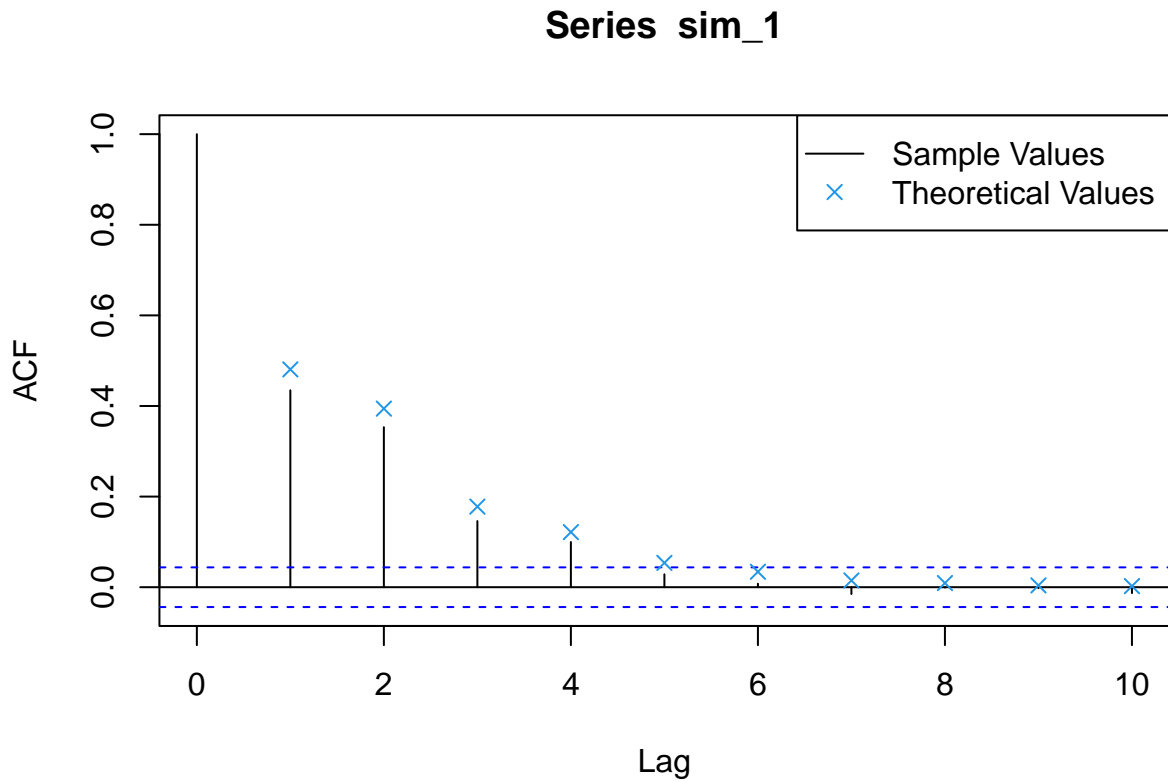
$$\begin{aligned}A_1 &\approx 2.42298, A_2 \approx -1.60243, A_3 \approx 0.179496 \\ \rho(h) &= 2.42298 \left(\frac{1}{2}\right)^{|h|} - 1.60243 \left(\frac{2}{5}\right)^{|h|} + 0.179496 \left(\frac{-1}{2}\right)^{|h|}\end{aligned}$$

(c)

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set.seed(123)
sim_1 <- arima.sim(n = 2000, list(ar = c(2/5,1/4,-1/10)), sd = sqrt(1.96))
theoretical_values <- c(0)
for(i in 1:10){
  theoretical_values[i] <- 2.42298*(1/2)^i-1.60243*(2/5)^i+0.179496*(-1/2)^i
}
acf(sim_1, lag.max = 10)
lines(1:10, theoretical_values, type = "p", col = 4, pch = 4)
legend("topright",legend=c("Sample Values","Theoretical Values"), lty=c(1, NA) ,pch = c(NA ,4),col=c(1,4))

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Problem 2

(a)

The ARMA process is given by:

$$X_t = \frac{7}{10}X_{t-1} + Z_t - \frac{1}{10}Z_{t-1}$$

SO,

$$\phi(B) = 1 - \frac{7}{10}B$$

And

$$\theta(B) = 1 - \frac{1}{10}B$$

Equating both the above equations to 0, we can find the roots. Thus, when $\phi(B) = 0$, $B = \frac{10}{7}$ and when $\theta(B) = 0$, $B = 10$.

Since the roots for both equations lie outside the unit circle, the ARMA process is both stationary and invertible.

(b)

$$\psi(B) = \frac{\theta(B)}{\phi(B)} = \frac{1 - \frac{1}{10}B}{1 - \frac{7}{10}B}$$

$$\psi(B) = (1 - .1B)(1 - .7B)^{-1}$$

Now,

$$(1 - .7B)^{-1} = \frac{1}{1 - .7B}$$

This takes the form of a sum $(\frac{a}{1-r})$ of a geometric series which looks like $a + ar + ar^2 \dots$

So, $a = 1$ and $r = .7B$ and the expanded geometric series is,

$$\frac{1}{1 - .7B} = 1 + .7B + .49B^2 + .343B^3 \dots$$

So,

$$\begin{aligned}\psi(B) &= (1 - .1B)(1 + .7B + .49B^2 + .343B^3 \dots) \\ \psi(B) &= (1 + .7B + .49B^2 + .343B^3 \dots) + (-0.1B - 0.07B^2 - 0.049B^3 \dots) \\ \psi(B) &= 1 + 0.6B + 0.42B^2 + 0.294B^3 \dots\end{aligned}$$

where,

$$\psi_i = 0.6 \times 0.7^{i-1} \quad \text{for } i = 1, 2, \dots$$

So, the pure MA process is

$$X_t = Z_t + 0.6Z_{t-1} + 0.42Z_{t-2} + 0.294Z_{t-3} + \dots$$

(c)

$$\pi(B) = \frac{\phi(B)}{\theta(B)} = \frac{1 - \frac{7}{10}B}{1 - \frac{1}{10}B}$$

$$\pi(B) = (1 - .7B)(1 - .1B)^{-1}$$

Now,

$$(1 - .1B)^{-1} = \frac{1}{1 - .1B}$$

This takes the form of a sum $(\frac{a}{1-r})$ of a geometric series which looks like $a + ar + ar^2 \dots$

So, $a = 1$ and $r = .1B$ and the expanded geometric series is,

$$\frac{1}{1 - .1B} = 1 + .1B + .01B^2 + .001B^3 \dots$$

So,

$$\begin{aligned}\pi(B) &= (1 - .7B)(1 + .1B + .01B^2 + .001B^3 \dots) \\ \psi(B) &= (1 + .1B + .01B^2 + .001B^3 \dots) + (-0.7B - 0.07B^2 - 0.007B^3 \dots) \\ \psi(B) &= 1 - 0.6B - 0.06B^2 - 0.006B^3 \dots \\ \pi_i &= 0.6 \times 0.1^{i-1} \quad \text{for } i = 1, 2, \dots\end{aligned}$$

So the pure AR process is,

$$X_t = 0.6X_{t-1} + 0.06X_{t-2} + 0.006X_{t-3} + \dots + Z_t$$

(d)

We know that the ARMA(1,1) process can also be written as an MA process given by:

$$X_t = Z_t + 0.6Z_{t-1} + 0.42Z_{t-2} + 0.294Z_{t-3} + \dots$$

We know that the ACF function for lag h for an MA process is given by,

$$\rho(h) = \frac{\sum_{i=0}^{\infty} \beta_i \beta_{i+h}}{\sum_{i=0}^{\infty} \beta_i^2}$$

This can also be written as,

$$\rho(h) = \frac{\beta_0 \beta_h + \sum_{i=1}^{\infty} \beta_i \beta_{i+h}}{\beta_0^2 + \sum_{i=1}^{\infty} \beta_i^2}$$

We also know that our $\beta_1, \beta_2 \dots$ take the form of a geometric series given by,

$$\beta_n = a \cdot r^{n-1} \quad n \in [1, \infty)$$

where $a = 0.6$ and $r = 0.7$

We know that $\beta_0 = 1$, so we can write the ACF function as,

$$\begin{aligned}\rho(h) &= \frac{1 \cdot \beta_h + \sum_{i=1}^{\infty} \beta_i \beta_{i+h}}{1 + \sum_{i=1}^{\infty} \beta_i^2} \\ \Rightarrow \rho(h) &= \frac{1 \cdot 0.6 \cdot 0.7^{h-1} + \sum_{i=1}^{\infty} 0.6 \cdot 0.7^{i-1} \times 0.6 \cdot 0.7^{i+h-1}}{1 + \sum_{i=1}^{\infty} (0.6 \cdot 0.7^{i-1})^2} \\ \Rightarrow \rho(h) &= \frac{0.6 \cdot 0.7^{h-1} + 0.36 \cdot 0.7^h \sum_{i=1}^{\infty} 0.7^{2i-2}}{1 + 0.36 \sum_{i=1}^{\infty} 0.7^{2i-2}}\end{aligned}$$

Now, $\sum_{i=1}^{\infty} 0.7^{2i-2}$ converges to 1.96078 So, we can see that the ACF then becomes,

$$\begin{aligned}\rho(h) &= \frac{0.6 \cdot 0.7^h \cdot 0.7^{-1} + 0.36 \cdot 1.96078 \cdot 0.7^h}{1 + 0.36 \cdot 1.96078} \\ \Rightarrow \rho(h) &\approx \frac{0.857 \cdot 0.7^h + 0.706 \cdot 0.7^h}{1.706} \\ \Rightarrow \rho(h) &\approx \frac{1.563}{1.706} \cdot 0.7^h \\ \Rightarrow \rho(h) &\approx 0.916 \cdot 0.7^h \quad \text{for } h \in [1, \infty) \\ \rho(h) &= 1 \quad \text{for } h = 0 \\ \text{and } \rho(h) &= \rho(-h) \quad \text{for } h \in (-\infty, 0)\end{aligned}$$

Problem 3

The SARIMA process is of order $(2, 1, 0) \times (0, 1, 2)_{12}$

Now,

$$\begin{aligned} W_t &= \nabla^1 \nabla_{12}^1 X_t \\ W_t &= \nabla_{12}^1 X_t - \nabla_{12}^1 X_{t-1} \\ W_t &= (X_t - X_{t-12}) - (X_{t-1} - X_{t-1-12}) \\ W_t &= X_t - X_{t-12} - X_{t-1} + X_{t-13} \end{aligned}$$

The form that the SARIMA process takes is:

$$\begin{aligned} \phi(B)\Phi(B^s)W_t &= \theta(B)\Theta(B^s)Z_t \\ (1 - \phi_1(B) - \phi_2(B^2)) W_t &= (1 + \Theta_1(B) + \Theta_2(B^2)) Z_t \\ (1 - \phi_1(B) - \phi_2(B^2)) (X_t - X_{t-1} - X_{t-12} + X_{t-13}) &= (1 + \Theta_1(B) + \Theta_2(B^2)) Z_t \end{aligned}$$

So the left hand side of the equation becomes

$$\begin{aligned} X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - X_{t-1} + \phi_1 X_{t-2} + \phi_2 X_{t-3} - X_{t-12} + \phi_1 X_{t-13} + \phi_2 X_{t-14} + X_{t-13} - \phi_1 X_{t-14} - \phi_2 X_{t-15} \\ X_t - (\phi_1 + 1)X_{t-1} - (\phi_2 - \phi_1)X_{t-2} - (-\phi_2 X_{t-3}) - X_{t-12} - (-\phi_1 - 1)X_{t-13} - (-\phi_2 + \phi_1)X_{t-14} - \phi_2 X_{t-15} \end{aligned}$$

And the right hand side is,

$$Z_t + \Theta_1 Z_{t-12} + \Theta_2 Z_{t-24}$$

Thus, the ARMA process is of the order $ARMA(15, 24)$.